



The Countable Character of Uncountable Graphs

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Abstract

We show that a graph can always be decomposed into edge-disjoint subgraphs of countable cardinality in which the edge-connectivities and edge-separations of the original graph are preserved up to countable cardinals. We also show that the vertex set of any graph can be endowed with a well-ordering which has a certain compactness property with respect to edge-separation.

Keywords: Graphs, decompositions, connectivity, uncountable cardinals.

1 Introduction

The analysis of processes modelling physical systems is of growing interest in computer science, for example in control systems theory. A physical system, such as a moving particle or the temperature of a room, involves continuous parameters and hence may require a model with uncountably many states. Hybrid systems theory (see e.g. [2]) and Labelled Markov Processes theory ([4]) are concerned with their analysis. Even if, in practice, one generally wants to discretize a system before using it, and even before reasoning about it, a theory of continuous systems allows one to argue that a discretization of a process is indeed a faithful model of the process. Many techniques for analyzing finite systems have been developed in combinatorics but very little

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has been done to study infinite ones. The present paper can be considered as a step in that direction.

In hybrid system theory, the analysis of physical systems is made with finite transition systems, where one state may encode an uncountable number of parameter values which are equivalent with respect to the observed behavior of the system. On the other hand, it is shown in [4] that there are some Labelled Markov processes that have uncountably many states but cannot be reduced to finite or even countable state-space processes. In the latter case, there is an essential ingredient, we believe, to prevent this reduction. The infinite character of Markov processes is strongly linked with the way uncertainty and non determinism are represented, that is, with *probability distributions*; the richness of continuous probability theory ensures that we cannot get out of uncountability without a loss of information of some sort.

For non-probabilistic processes, the common intuition is that there is no substantial gain in considering transition systems with uncountably many states. This paper makes this idea more precise, by showing that the underlying graph of an uncountable transition system can always be decomposed into countable fragments, each of which preserving the edge-connectivity of the original graph. This result is about unoriented graphs, its generalization to the oriented case seems to be more difficult and is still open.

Our convention is that a decomposition is an equivalence relation on $E(G)$ such that every fragment (i.e., subgraph induced by the edges of an equivalence class) is connected. We are interested in finding decompositions whose fragments inherit as far as possible the edge-connectivity of the original graph in the sense that for a given infinite cardinal α , the fragments of the decomposition are all of order at most α and are such that no bond (i.e. cocycle) of cardinality $\leq \alpha$ is split into pieces belonging to different fragments: such decompositions will be called bond-faithful α -decompositions.

The main result of the paper (Theorem 4.6) is that for any graph G one can always construct a bond-faithful ω -decomposition.

It is interesting to note that the results in this paper have no pendant in the finite case. In fact, in the finite case, the following is still a conjecture (see DeVos, Johnson and Seymour [5]) even if, in the infinite case (i.e., the case where a and b are infinite), it is a weakening of Proposition 4.8.

Let G be an $(a + b + 2)$ -edge-connected graph. Does there exist a partition $\{A, B\}$ of $E(G)$ so that (V, A) is a -edge-connected and (V, B) is b -edge-connected?

Theorem 4.6 is used in [9] to carry out a reduction to the countable case which is considerably easier to handle. This application was the first motiva-

tion for introducing the concepts and proving the main result of the present paper.

In the last section of the paper we show that the vertex set of any graph can be endowed with a well-ordering which has a certain compactness property with respect to edge-separation, in the sense that given any (order-)bounded subset $X \subseteq V(G)$ and any upper bound u , if X cannot be separated from u by the removal of a finite number of edges, then the same is true for some finite subset of X . It follows from this result that infinite graphs can always be constructed, vertex after vertex, in such a way that at each step there is always a finite set of edges that separates the new added vertex from all the previous ones that are not infinitely edge-connected to it. In general, there always exists a finite set of edges that separate a vertex x from any other vertex y that is not infinitely edge-connected to it. The particular interest of that construction is that it guarantees that a single finite set of edges will do the job uniformly for all the y at the same time.

This result provides an interesting tool if one wishes to make a recursive construction on infinite graphs and does not want the first steps of the construction to interfere “too much” with the rest. In particular, this might open new perspectives in the theory of automatic structures.

All the results in the present paper rely on decomposition into countable fragments but, under the Generalized Continuum Hypothesis, they can be extended to any uncountable cardinal, see [8] for all the proofs in the general case, and for other related results.

2 Definitions and preliminaries

For the purposes of this paper, we assume all graphs to be unoriented, without loops or multiple edges unless otherwise stated. The symbol G will always denote a graph. A *circuit* is a 2-regular connected graph and a *cycle* is a finite circuit. A *block* of G is a 2-vertex-connected subgraph of G which is maximal with respect to inclusion; in particular a subgraph consisting of a bridge or a loop is a block. If $L \subseteq E(G)$ then $G \setminus L$ denotes the graph obtained from G by removing all edges in L (retaining all vertices). If $X \subseteq V(G)$ then $G[X]$ denotes the induced subgraph of G on X . If $x \in V(G)$ and A, B denote subgraphs of G , we write $G - x = G[V(G) \setminus x]$, $G - A = G[V(G) \setminus V(A)]$, $G \setminus A = G \setminus E(A)$ and $[A, B]_G$ denotes the set of edges of G which join vertices of A to vertices of B . When no confusion is likely we shall write \bar{A} for $G - A$.

A *cut* of G is a set of edges of the form $[A, \bar{A}]_G$. Unless otherwise stated, A will be an induced subgraph of G . An *odd* (resp. *even*) cut is a cut whose

cardinality is odd (resp. even). A *bond* is non-empty cut which is minimal with respect to inclusion. Observe that a cut $[A, \overline{A}]_G$ of a *connected* graph G is a bond if and only if both A and \overline{A} are connected.

Remark 2.1 A cut $[A, \overline{A}]_G$ of a (connected or disconnected) graph is the union of a family of edge-disjoint bonds. It is easy to see that if A or \overline{A} is connected then the family is unique. If both A and \overline{A} are disconnected then the uniqueness does not hold, as illustrated by the example of Figure 1, where $[A, \overline{A}]_G$ is the union of the three bonds which consist respectively of the set of edges incident with each of the vertices of A , and also the union of the bonds symmetrically defined with respect to the three vertices of \overline{A} . In general, given a cut $[A, \overline{A}]_G$ of G , we can easily construct a suitable family \mathcal{F} of bonds as follows: let $(A_i)_{i \in I}$ be the set of all components of the induced subgraph A ; for each $i \in I$, let \mathcal{F}_i be the unique family of bonds of G whose union is $[A_i, \overline{A}_i]_G$, and then put $\mathcal{F} := \bigcup_{i \in I} \mathcal{F}_i$.

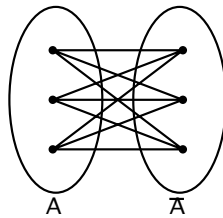


Fig. 1.

Remark 2.2 Each bond of G is contained in some block of G . To see this, suppose that x is a cut-vertex of G and $[A, \overline{A}]_G$ is a bond of G . If $x \in V(\overline{A})$ then the connected subgraph \overline{A} of $G - x$ must be contained in a single component of $G - x$ and so x cannot separate edges of $[A, \overline{A}]_G$; and a similar argument applies if $x \in V(\overline{A})$.

For any two distinct vertices $x, y \in V(G)$, we denote by $\gamma_G(x, y)$ the edge-connectivity between x and y . By the weak version of Menger's Theorem, $\gamma_G(x, y)$ can be equivalently defined as the maximal cardinality of a set of edge-disjoint xy -paths of G or as the minimal cardinality of a cut of G that separates x from y . Thus, $\gamma_G(x, y) = 0$ if and only if x and y belong to different connected components of G . Observe that, assuming that every vertex is κ -edge-connected to itself, κ -edge-connectivity, unlike κ -vertex-connectivity, induces an equivalence relation on $V(G)$ since, for each cardinal κ ,

$$\gamma_G(x, y) \geq \kappa \text{ and } \gamma_G(y, z) \geq \kappa \implies \gamma_G(x, z) \geq \kappa.$$

The equivalence classes of this relation are called the κ -edge-connectivity classes

or simply κ -classes of G . A graph that has exactly one κ -class is said to be κ -edge-connected. In this paper we only consider edge-connectivity up to the cardinal ω . Thus we will mostly use this truncated definition of edge-connectivity:

Definition 2.3 Let x and y be two vertices of G , we define $\tilde{\gamma}_G(x, y)$ as follows

- $\tilde{\gamma}_G(x, y) := \infty$ if $\gamma_G(x, y) \geq \omega$;
- $\tilde{\gamma}_G(x, y) = \gamma_G(x, y)$ otherwise.

In the first case we will say that x and y are *infinitely edge-connected*.

A *decomposition* of G is an equivalence relation on $E(G)$ such that the subgraph induced by the edges of any equivalence class is *connected*. The subgraphs induced in this way are called the *fragments* of the decomposition. Thus, a decomposition of G may be considered as a family of edge-disjoint connected subgraphs of G whose union is the graph G minus its isolated vertices. Among the most frequently studied decompositions are decompositions whose fragments are cycles (i.e., cycle decompositions) and decompositions whose fragments are cycles, rays or double rays. For results on the existence of such decompositions for infinite graphs, see Nash-Williams [10], Sabidussi [11], Thomassen [13], or Laviolette [7] and [9]. The main theorem of the present paper relies on what we will refer to as Nash-Williams's Theorem:

Theorem (Nash-Williams [10]) *A graph has a cycle decomposition if and only if it does not contain any odd cut.*

A decomposition whose fragments are all κ -edge-connected for some (finite or infinite) cardinal κ , is said to be κ -edge-connected, and a decomposition whose fragments are all of cardinality less than or equal to α for some infinite cardinal α , is called an α -decomposition.

In this paper we look for ω -decompositions, and particularly ω -decomposition whose fragments inherit the edge-connectivity of the original graph. More precisely, we consider the following type of decompositions:

Definition 2.4 An ω -decomposition Δ of G is said to be *bond-faithful* if

- (i) any countable bond of G is contained in some fragment of Δ ;
- (ii) any finite bond of a fragment of Δ is also a bond in G .

In a bond-faithful ω -decomposition Δ of G , any countable bond B of G is by (i), contained in some fragment H and hence is a cut of H . Moreover,

if B is finite, then this cut is always a bond of H since otherwise there is a bond B' of H , strictly contained in B , which because of (ii), must also be a bond of G , contradicting the fact that B is a bond of G . Hence, the following properties are always satisfied for any set of edges $B \subseteq E(G)$:

- (1) if B is finite, then B is a bond of G if and only if it is a bond of some fragment of Δ ;
- (2) if B is a countable bond of G , then B is a cut of some fragment of Δ ;
- (3) if B uncountable, and B is a bond of G , then in any fragment H containing edges of B , $B \cap E(H)$ is a countable cut of H .

Note, moreover, that since a cut is an edge-disjoint union of bonds, and because of condition (i) of the definition of bond-faithfulness, we can equivalently replace condition (ii) of that definition by:

(ii') any finite cut of cardinality of a fragment of Δ is also a cut in G .

A fundamental property of bond-faithful ω -decompositions, relating the local edge-connectivities of G to those of the fragments of the decomposition, is expressed in the following proposition.

Proposition 2.5 *If H is a fragment of a bond-faithful ω -decomposition of G and x, y any two vertices of H then*

$$\tilde{\gamma}_H(x, y) = \tilde{\gamma}_G(x, y).$$

Proof. Since $H \subseteq G$ we must have $\tilde{\gamma}_H(x, y) \leq \tilde{\gamma}_G(x, y)$. Hence if $\tilde{\gamma}_H(x, y) = \infty$, there is nothing to show. On the other hand, if $\tilde{\gamma}_H(x, y) = k < \omega$, then there exists a bond of H of cardinality k separating x and y . By property (ii) of a bond-faithful ω -decomposition this implies that

$$\tilde{\gamma}_G(x, y) \leq k = \tilde{\gamma}_H(x, y) \leq \tilde{\gamma}_G(x, y).$$

□

Remark 2.6 It follows from Proposition 2.5 that if G is β -edge-connected, where $\beta \leq \omega$, then every fragment of a bond-faithful ω -decomposition of G is likewise β -edge-connected.

Since a decomposition of G is an equivalence relation on $E(G)$ we have the following natural partial order on decompositions of a graph G .

Definition 2.7 A decomposition Δ_2 is *coarser* than Δ_1 (denoted by $\Delta_2 \succeq \Delta_1$) if each fragment of Δ_1 is contained in some fragment of Δ_2 .

With respect to this order, any (finite or infinite) family of decompositions has a supremum and an infimum. Since fragments have to be connected, the infimum does not always coincide with the infimum in the set of all equivalence relations. However, for the supremum (denoted by $\bigvee_{i \in I} \Delta_i$), the “connected” supremum coincides with the equivalence-supremum, as stated in the next lemma.

Lemma 2.8 *Let $(\Delta_i)_{i \in I}$ be a family of decompositions of a graph G . Then $\bigvee_{i \in I} \Delta_i$ is the transitive closure of the union of the equivalence relations Δ_i .*

Proof. Since the transitive closure Δ of the union of the Δ_i ’s is already the supremum of the Δ_i ’s in the set of all equivalence relations on $E(G)$, one only has to show that every Δ -equivalence class edge-induces a connected graph. This is straightforward and left to the reader. \square

The countable supremum respects ω -decompositions and even preserves bond-faithfulness in a strong way.

Lemma 2.9 *Let $(\Delta_i)_{i \in I}$ be a countable family of ω -decompositions of G . Then $\Delta = \bigvee_{i \in I} \Delta_i$ is an ω -decomposition; moreover, if the family contains at least one bond-faithful ω -decomposition, then Δ will also be bond-faithful.*

Proof. The first assertion follows from the fact that any fragment of Δ is the union of at most ω fragments all of cardinality at most ω . Suppose now that the family contains a bond-faithful ω -decomposition Δ_0 . Then since $\Delta_0 \preceq \Delta$, any countable bond of G is contained in a fragment of Δ . Moreover, if B is a finite bond of a fragment H of Δ , then for any edge $e \in B$ the intersection of B with the fragment H_0 of Δ_0 containing e is a cut of H_0 . Hence B contains a bond B_0 of H_0 . Since Δ_0 is bond-faithful, B_0 is a bond of G . Since B_0 is a bond of G and $B_0 \subseteq B \subseteq E(H)$, it follows that B_0 is a non-empty cut of H contained in the bond B of H and so $B = B_0$, which is a bond of G . \square

3 ω -covers and 2-edge-connected decompositions

Given a cardinal κ , a κ -cover of a graph G is a family $(H_i)_{i \in I}$ of subgraphs of G such that each edge of G belongs to exactly κ members of the family. Hence a decomposition is a 1-cover with all members connected. The case which has received the most attention is $\kappa = 2$ with Seymour’s Double Cover Conjecture, which says that every 2-edge-connected graph admits a cycle 2-cover (i.e. a 2-cover all of whose members are cycles); see Seymour [12] or Bondy [1] for a survey. The following result is a (substantial) weakening of that conjecture.

Theorem 3.1 *Every 2-edge-connected graph has a cycle ω -cover.*

Proof. Let $x_0 \in V(G)$ and for each $i > 0$, let D_i be the set of edges of a 2-edge-connected graph G having one endpoint at distance $i - 1$ from x_0 and the other at distance i . Let D_0 be the set of edges of G whose endpoints are at the same distance from x_0 . Note that the D_i 's form a partition of $E(G)$ into possibly empty sets and that for $i \geq 1$,

$$D_i = [A_i, \overline{A_i}], \text{ where } A_i = \{y \in V(G) : \text{dist}_G(x_0, y) \leq i - 1\}.$$

We will now construct for each $i \geq 0$ a family \mathcal{F}_i of cycles of G such that each edge of D_i belongs to at least one cycle of \mathcal{F}_i , and such that no edge of G belongs to more than ω cycles of \mathcal{F}_i . To obtain \mathcal{F}_0 (the simplest case) we proceed as follows. Form a multigraph G_0 by replacing each edge in $G \setminus D_0$ by ω edges having the same endpoints. Note that G_0 is ω -edge-connected since for any $x \in V(G_0) (= V(G))$ no edge of an x_0x -geodesic will belong to D_0 ; in other words, all edges of the geodesic will have been duplicated ω times. Hence G_0 has no finite cut and therefore no odd cut, implying by Nash-Williams's Theorem stated in Section 2 that G_0 has a decomposition into cycles, say Δ_0 . Any cycle of Δ_0 canonically induces either a cycle in G or an edge in $E(G) \setminus D_0$, the latter case occurring only if the cycle of Δ_0 is of length 2. Let \mathcal{F}_0 be the family of all the cycles of G canonically induced by the cycles of Δ_0 . Then \mathcal{F}_0 will have the desired properties since any edge in D_0 must belong to exactly one cycle in Δ_0 and there are at most ω cycles of \mathcal{F}_0 that may contain a given edge.

Let us now construct \mathcal{F}_i for $i > 0$. Since D_i is a cut of G , it is the disjoint union of bonds (say $D_i = \bigcup_{j \in J_i} B_{ij}$). Given $j \in J_i$, fix two arbitrary distinct edges e_j^1 and e_j^2 of B_{ij} (note that $|B_{ij}| \geq 2$ since by hypothesis G is 2-edge-connected). In the same way as in the construction of G_0 , let us construct $G_i^k, k = 1, 2$, by replacing in G each edge of $E(G) \setminus D_i$ and each e_j^k ($j \in J_i$) by ω edges having the same endpoints. Note that the G_i^k 's, $i > 0, k = 1, 2$, are all ω -edge-connected since $V(G_i^k) = V(G)$ and the edges of G which are being ω -duplicated (i.e., the edges in $(E(G) \setminus D_i) \cup \{e_j^k : j \in J_i\}$) $E(G) \setminus \Delta_i \cup \{E_j^k : j \in J_i\}$) form a connected spanning subgraph of G .

Hence as we have done for \mathcal{F}_0 , we can construct two families of cycles \mathcal{F}_i^k ($k = 1, 2$) of G , obtained from a cycle decomposition of G_i^k , such that any edge of G belongs to at most ω cycles of \mathcal{F}_i^k and any edge of $D_i \setminus \{e_j^k : j \in J_i\}$ belongs to at least one cycle of \mathcal{F}_i^k . Since $\{e_j^1 : j \in J_i\}$ is disjoint from $\{e_j^2 : j \in J_i\}$, $\mathcal{F}_i := \mathcal{F}_i^1 \cup \mathcal{F}_i^2$ will have the desired two properties (a cycle is allowed to appear more than once in the family).

Finally it is easy to see that the family consisting of ω copies of every cycle in $\bigcup_{i \geq 0} \mathcal{F}_i$ is an ω -cover of G . \square

The theorem of Nash-Williams used in this proof is based on a highly

non-trivial transfinite induction. However, as will be seen later, Theorem 3.1 implies Corollary 4.2, which allows a reduction of the proof of Nash-Williams's Theorem to the countable case which is easy to handle (see Remark 4.3). Hence any direct proof of Theorem 3.1 will give rise to a direct proof of Nash-Williams's Theorem. Moreover, Theorem 3.1 gives some partial answer to the Cycle 2-Cover Conjecture in the infinite case.

Corollary 3.2 *Every bridgeless graph admits a 2-edge-connected ω -decomposition.*

Proof. Let G be such a graph. We may clearly suppose that G is connected, i.e. 2-edge-connected. Let Φ be a cycle ω -cover of G given by Theorem 3.1 and Δ the equivalence relation defined as the transitive closure of the relation Θ on $E(G)$, where $e\Theta e'$ if and only if Φ contains a cycle containing both e and e' .

Claim: Δ is a 2-edge-connected ω -decomposition. Let H be a fragment of Δ .

(1) H is connected, since for any two edges $e, e' \in E(H)$ there exist $e_1, \dots, e_n \in E(H)$ such that $e = e_1, e' = e_n$ and $e_i\Theta e_{i+1}$ for any $i = 1, 2, \dots, n-1$. Let $C_i \in \Phi$ be a cycle containing both e_i and e_{i+1} and note that $\bigcup_{i=1}^n C_i$ is a connected subgraph of H containing e and e' .

(2) H is trivially 2-edge-connected since any edge $e \in E(H)$ is contained in a cycle of Φ which belongs to H .

(3) H is at most countable since any edge $e \in E(H)$ is Θ -related to at most ω other edges, and Δ is the transitive closure of Θ . \square

4 Bond-faithful ω -decompositions

The aim of this section is to show that every graph has a bond-faithful ω -decomposition.

Lemma 4.1 *Let Δ_0 be an ω -decomposition of G . Then there exists an ω -decomposition Δ which is coarser than Δ_0 and has the property that for any fragment H of Δ_0 , the only countable bonds of H which are bonds of the corresponding fragment of Δ are those which are bonds of G .*

Thus Δ “purifies” the fragments of Δ_0 of all countable bonds that are not bonds in G .

Proof. For each fragment H of Δ_0 , let $(B_\beta^H)_{\beta \in \gamma_H}$ be any well-ordering of the set of all the countable bonds of H that are not bonds of G . Then for each $\beta \in \gamma_H$, fix an edge e_β^H in B_β^H . Since $|E(H)| \leq \omega$, H has at most ω bonds of cardinality $< \omega$. Thus, without loss of generality, we may suppose that, for

each H , the ordinal γ_H is either finite or the ordinal ω . Let

$$G_\beta := G \setminus \bigcup \{B_\beta^H \setminus e_\beta^H : H \text{ is a fragment of } \Delta_0 \text{ and } \gamma_H > \beta\}$$

for any $\beta < \omega$. Given any fragment K of Δ_0 and any $\beta < \gamma_K$, e_β^K is an edge of G_β because the fragments of Δ_0 are pairwise edge-disjoint. We claim that e_β^K is however not a bridge of G_β . Otherwise, e_β^K will still be a bridge in $G \setminus (B_\beta^K \setminus e_\beta^K)$ because $G \setminus (B_\beta^K \setminus e_\beta^K)$ can be obtained from G_β by putting back every $B_\beta^H \setminus e_\beta^H$ except $B_\beta^K \setminus e_\beta^K$ itself, and because $H \setminus (B_\beta^H \setminus e_\beta^H)$ is a connected subgraph of G for every fragment H of Δ_0 . Hence B_β^K will be a cut of G , and since it is a bond of K , it will therefore be a bond of G , a contradiction.

Now, for each $\beta < \omega$, apply Corollary 3.2 and choose a 2-edge-connected ω -decomposition Γ_β of $G_\beta \setminus \{e \in E(G_\beta) : e \text{ is a bridge of } G_\beta\}$. Then let Φ_β be the ω -decomposition of G obtained from Γ_β by adding every bridge of G_β and every edge of $G \setminus G_\beta$ as an equivalence class of one element. Moreover, for each edge e_β^H , fix a cycle C_β^H that contains e_β^H and is contained in the fragment of Φ_β that contains e_β^H . Hence $B_\beta^H \cap E(C_\beta^H) = \{e_\beta^H\}$ for any fragment H of Δ_0 and any $\beta < \gamma_H$.

Let us show that $\Delta := \Delta_0 \vee (\bigvee_{\beta < \omega} \Phi_\beta)$, is the desired ω -decomposition. Clearly, $\Delta_0 \preceq \Delta$, and it follows from Lemma 2.9 that Δ is an ω -decomposition. Denote by L_H the fragment of Δ that contains H (and hence all the e_β^H 's). Since C_β^H is contained in L_H for any H , $C_\beta^H \setminus e_\beta^H$ is therefore a path (edge-disjoint from B_β^H) that connects (in L_H) the two components which are separated by B_β^H in H . Thus no B_β^H can be a bond of L_H . \square

Applying the preceding lemma ω times we will obtain an ω -decomposition satisfying condition (ii) of the bond-faithfulness definition. This is the content of the following corollary.

Corollary 4.2 *Let Δ_0 be an ω -decomposition of G . Then there exists an ω -decomposition Δ such that $\Delta_0 \preceq \Delta$ and any countable bond of a fragment of Δ is also a bond in G .*

Proof. By Lemma 2.9 we can inductively construct an increasing sequence $(\Delta_\beta)_{\beta < \omega}$ of ω -decompositions as follows:

- Δ_0 is the decomposition given in the hypothesis;
- $\Delta_{\beta+1}$ is an ω -decomposition such that $\Delta_\beta \preceq \Delta_{\beta+1}$ and has the property of Lemma 4.1 with Δ_0, Δ replaced by $\Delta_\beta, \Delta_{\beta+1}$ respectively.

We claim that $\Delta = \bigvee_{\beta < \omega} \Delta_\beta$ is an ω -decomposition having the desired properties. First note that $\Delta_0 \preceq \Delta$ and that, by Lemma 2.9, Δ is an ω -decomposition. Now, by way of contradiction, let B be any finite bond of a fragment

H of Δ which is not a bond of G . If K is the component of G that contains H , then $K \setminus B$ is still a connected graph, and hence no subset of B can be a bond of G . Fix an edge $e \in B$ and, for any ordinal $\beta < \omega$, denote by H_β the fragment of Δ_β that contains e . It is easy to see that $(H_\beta)_{\beta < \omega}$ is a nested sequence of subgraphs of G whose union is H , and that $B \cap E(H_\beta)$ is a cut of H_β . Cuts being edge-disjoint unions of bonds, there is a bond $[A_\beta, \bar{A}_\beta]_{H_\beta}$ of H_β that is contained in B . Since no subset of $B \cap E(H_\beta)$ is a bond of G , $[A_\beta, \bar{A}_\beta]_{H_\beta}$ is not a bond of $H_{\beta+1}$. This and the fact that $H_\beta \setminus [A_\beta, \bar{A}_\beta]_{H_\beta}$ is composed of exactly two connected components (viz. A_β and \bar{A}_β), implies that $H_{\beta+1} \setminus [A_\beta, \bar{A}_\beta]_{H_\beta}$ is connected. Hence there exists an $A_\beta \bar{A}_\beta$ -path that is totally contained in $H_{\beta+1} \setminus H_\beta$, and therefore

$$B \cap (E(H_{\beta+1}) \setminus E(H_\beta)) \neq \emptyset \text{ for any } \beta < \omega.$$

It follows that B is infinite, a contradiction. \square

Remark 4.3 Let G be a graph without any odd cut. Clearly, any decomposition Δ of G given by Corollary 4.2 (with Δ_0 , the decomposition all of whose fragments are single edges) will only consist in countable fragments with no odd cut. Thus, as stated before, Corollary 4.2 allows a reduction of the proof of the Nash-William's Theorem to the countable case.

Before proceeding to our main theorem we need one last result which shows that a vertex of “high” degree in an uncountable graph is either “highly” connected to some other vertex or is a cut-vertex.

Theorem 4.4 *Let G be a connected graph (possibly with and multiple edges), $x \in V(G)$ and μ be a regular uncountable cardinal. If $\deg_G(x) \geq \mu$, then x is a cut-vertex of G or is μ -vertex-connected to some vertex $y \neq x$.*

Proof. Suppose that x is not a cut-vertex of G . Hence $G - x$ is still connected; choose a spanning tree T of $G - x$ and let J be the union of all cycles of $T \cup A$, where A is the subgraph of G induced by all the edges incident with x . Since T is a tree any cycle of $T \cup A$ must contain x . Hence J is connected. Moreover, since T is connected, any two edges e_1, e_2 of G incident with x must be contained in some cycle of $T \cup A$, implying that $A \subseteq J$ and that $J_1 = J - x$ is a tree.

We claim that some $y \in V(J_1)$ has degree at least μ in J_1 . By way of contradiction, suppose this is not the case. Let u be any vertex of J_1 . By a straightforward inductive argument one can show that the sets

$$D_i := \{v \in V(J_1) : \text{dist}_{J_1}(u, v) = i\}$$

are all of cardinality less than μ because μ is regular and $|D_i| \leq \sum_{v \in D_{i-1}} \deg_{J_1}(v)$

for any $i > 0$. This gives rise to a contradiction since $V(J_1) \subseteq \bigcup_{i \in \omega} D_i$, $|J_1| \geq \mu$ and μ is a regular cardinal.

Note that $J - y$ is connected because as already stated, every cycle of $T \cup A$ must contain x . However, since $J_1 = J - x$ is a tree, $J - \{x, y\}$ will break into at least μ components, and from each of these components together with x and y one can construct an xy -path. In this way we obtain at least μ internally vertex-disjoint xy -paths. \square

Corollary 4.5 *If a connected graph G (possibly with loops and multiple edges) contains no two distinct ω_1 -edge-connected vertices, then every block of G are countable.*

Proof. By way of contradiction, suppose B is an uncountable block of G . Since ω_1 is a regular uncountable cardinal, some vertex must have degree at least ω_1 in B and so, by Theorem 4.4 either B has a cut-vertex (contradicting the definition of a block) or two distinct vertices are ω_1 -vertex-connected in B and therefore ω_1 -edge-connected in G (contradicting the hypothesis). \square

The following is our main theorem.

Theorem 4.6 *Every graph has a bond-faithful ω -decomposition.*

Proof. Let G be a graph, by Corollary 4.2, we only have to show that G has an ω -decomposition that satisfies the property (i) of the definition of bond-faithfulness. Clearly we may consider a connected graph G , and we may moreover assume that G is uncountable since otherwise we can take the decomposition having G as its only fragment.

Let σ be the equivalence relation on $V(G)$ induced by ω_1 -edge-connectivity, i.e.,

$$x \sigma y \quad \text{if and only if} \quad x = y \text{ or } \gamma_G(x, y) > \omega.$$

Let G/σ be the quotient graph modulo σ , in other words, the graph obtained from G by identifying the vertices of each σ -class without identifying any edge. Thus G/σ may have loops and multiple edges. Since there is a canonical bijection between $E(G)$ and $E(G/\sigma)$, we will suppose for convenience that $E(G) = E(G/\sigma)$. We shall also use the following notation: given a subgraph H of G/σ , we denote by \hat{H} the lifted subgraph of G corresponding to H (i.e., the subgraph formed by the edges of H , considered as edges of G , together with their incident vertices).

By Corollary 4.5, the blocks of G/σ are countable. Hence by Remark 2.2 so are the bonds of G/σ . Since these bonds are also bonds of G and since a countable bond of G cannot separate two ω_1 -edge-connected vertices, it

follows that the bonds of G/σ are exactly the countable bonds of G .

Let Δ_1 be the decomposition of G/σ whose fragments are its blocks. Clearly Δ_1 is a bond-faithful ω -decomposition of G/σ but unfortunately not necessarily a decomposition of G , because the subgraph of G induced by the edges of a block of G/σ is not necessarily connected.

The existence of such a decomposition of G is a consequence of the following:

Claim: *From the set $(H_i)_{i \in I}$ of all blocks of G/σ , one can construct a family $(K_i)_{i \in I}$ of connected subgraphs of G such that*

- (1) $\hat{H}_i \subseteq K_i$ for any $i \in I$;
- (2) K_i is countable for any $i \in I$;
- (3) each edge $e \in E(G)$ belongs to at most finitely many different K_i 's.

Indeed, assuming the claim to be true, it is easy to see that a suitable ω -decomposition of G is the equivalence relation defined as the transitive closure of the relation Θ given by:

$$e \Theta e' \iff e, e' \in E(K_i) \text{ for some } i \in I.$$

Proof of the claim: Suppose $0 \in I$ and consider the partial order \leq on the index set I arising from the block-cutpoint tree of G/σ , i.e.,

$$i < j \iff i \neq j \text{ and some (and hence any) path of } G/\sigma \text{ joining a vertex of } H_0 \text{ to a vertex of } H_j \text{ contains an edge of } H_i.$$

(See Figure 2 for an example.)

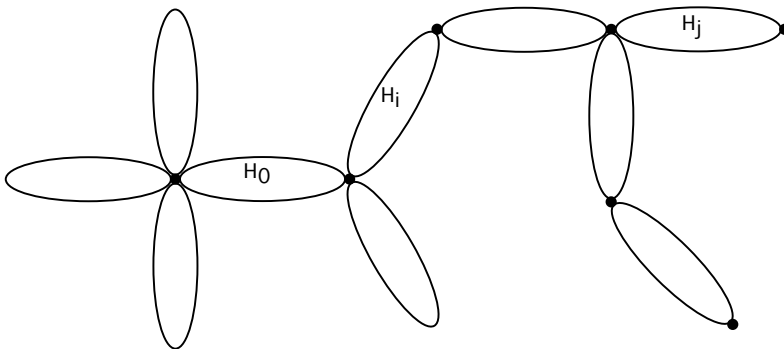


Fig. 2. In this example, $i < j$.

We have chosen to define strict inequality on I because in the case where H_i is a loop, no path of G/σ joining a vertex of H_0 to a vertex of H_i contains an edge of H_i , and even if H_i is not a loop, then some but not all of those paths

contain such an edge.

For each $i \in I$ let

$$L_i := \bigcup_{j \geq i} H_j.$$

Since any $i \in I$ has only finitely many predecessors in the order \leq defined above, it follows that any edge $e \in E(G/\sigma)$ belongs to at most finitely many L_j 's, namely those for which $j \leq i_e$, where i_e is the subscript of the unique H_i that contains e .

Clearly L_i is connected; let us prove that so is \widehat{L}_i . If $i = 0$, then $\widehat{L}_i = G$ which is connected by assumption. If H_i is a loop, then i is \leq -maximal which implies that $H_i = L_i$ and hence that \widehat{L}_i is connected (indeed, a single edge). If $i \neq 0$ and H_i is not a loop, then let q_i be the unique cut-vertex of G/σ belonging to H_i that separates the edges of L_i from those of H_0 . Observe that any two σ -equivalent vertices $x, y \in V(\widehat{L}_i) \subseteq V(G)$, which do not belong to the σ -class Q_i of G corresponding to q_i are connected in G by ω_1 edge-disjoint paths. At most ω of these paths can meet Q_i because otherwise x and y would belong to Q_i . Thus, x and y are connected (in fact ω_1 -edge-connected) in \widehat{L}_i . This, together with the fact that $L_i - q_i$ is connected, implies that $\widehat{L}_i - Q_i$ (the lifted graph corresponding to $L_i - q_i$) is connected. Hence if \widehat{L}_i is not connected, all but one of its components (namely the one that contains $\widehat{L}_i - Q_i$) have all their vertices in Q_i . Any such component corresponds in L_i to a union of loops at q_i . Being blocks contained in L_i , these loops are among the H_j 's with $j \geq i$. But by the definition of the order, any loop at q_i is either H_i itself (which is excluded by assumption) or has a subscript which is incomparable with i . Thus we have reached a contradiction, i.e., \widehat{L}_i is connected.

It is not hard to see (but not needed for the rest of the proof) that the \widehat{L}_i 's satisfy conditions (1) and (3). Their cardinality, however, may be uncountable. To overcome this difficulty, choose a spanning tree T_i of \widehat{L}_i ($i \in I$) and define K_i to be the union \widehat{H}_i and all paths in T_i that connect two vertices of \widehat{H}_i . Clearly, K_i is a connected subgraph of \widehat{L}_i (and hence of G). To finish the proof of the claim, let us show that the family $(K_i)_{i \in I}$ has the required properties (1), (2), (3).

(1) $\widehat{H}_i \subseteq K_i$ is trivially true for any $i \in I$ because $\widehat{H}_i = K_i^0$.

(2) $|E(K_i)| \leq \omega$ for any $i \in I$, because so is $|E(H_i)|$ which is equal to $|E(\widehat{H}_i)|$, and because K_i is the union of \widehat{H}_i and at most ω^2 paths of T_i .

(3) This is a consequence of the fact that any edge $e \in E(G)$ can belong to at most finitely many \widehat{L}_j 's, because as has been shown earlier e (viewed as an edge of G/σ) can belong to at most finitely many L_j 's. \square

Theorem 4.6 implies the following apparently stronger result.

Theorem 4.7 *Let $(H_i)_{i \in I}$ be a family of edge-disjoint connected countable subgraphs of G . Then G has a bond-faithful ω -decomposition Δ such that each H_i and each non-isolated vertex of degree $\leq \omega$ in G is contained in one and only one fragment of Δ .*

Proof. Let Δ_1 be any bond-faithful ω -decomposition of G , Δ_2 the ω -decomposition of G whose fragments are the H_i 's and each of the edges of G which do not belong to any H_i , and Δ_3 the ω -decomposition which is the transitive and reflexive closure of the following binary relation Θ :

$$e\Theta e' \iff \begin{array}{l} \text{both } e, e' \text{ are incident to } x \text{ for} \\ \text{some vertex } x \text{ of degree } \leq \omega \text{ in } G. \end{array}$$

By Lemma 2.9, $\Delta := \Delta_1 \vee \Delta_2 \vee \Delta_3$ is the desired decomposition. \square

Bond-faithful ω -decompositions provide a way of splitting a graph into edge-disjoint subgraphs, each of which preserves the “small” edge-connectivities of the original graph. In this section, we will show that a graph can also be split into edge-disjoint subgraphs which preserve the “high” edge-connectivities of the original graph.

Proposition 4.8 *Every graph G is the edge-disjoint union of two (not necessarily connected) spanning subgraphs, say K and L , such that*

$$\gamma_K(x, y) = \gamma_L(x, y) = \gamma_G(x, y)$$

for each pair x, y of infinitely edge-connected vertices of G .

Proof. We leave it to the reader to show that this is true for countable graphs. So suppose G is uncountable. By Theorem 4.6, there exists a bond-faithful ω -decomposition $\Delta = (H_i)_{i \in I}$ of G . Since we assume the proposition to be proved in the countable case, and each H_i is countable, H_i is the union of two edge-disjoint subgraphs K_i and L_i such that any pair of vertices $x, y \in V(H_i)$ which are infinitely edge-connected in H_i are also infinitely edge-connected in both K_i and L_i . Let $K := \bigcup_{i \in I} K_i$ and $L := \bigcup_{i \in I} L_i$, and let us prove that they both preserve α -edge-connectivity for any $\alpha \geq \omega$ or, in other words, that for any $x, y \in V(G)$ with $\gamma_G(x, y) = \alpha$, we have $\gamma_K(x, y) = \alpha$ and $\gamma_L(x, y) = \alpha$. Note that by symmetry, we only have to show that $\gamma_K(x, y) = \alpha$.

Take a set $\mathcal{P} = (P_\beta)_{\beta < \alpha}$ of edge-disjoint xy -paths of G and subdivide each P_β into edge-disjoint consecutive subpaths $P_\beta^1, P_\beta^2, \dots, P_\beta^{j_\beta}$ such that

- x is an end-vertex of P_β^1 and y of $P_\beta^{j_\beta}$;

- the end-edges of each P_β^j belong to the same fragment of Δ ;
- no edge of $P_\beta^{j+1} \cup P_\beta^{j+2} \cup \dots \cup P_\beta^{j_\beta}$ belongs to the fragment of Δ that contains the end-edges of P_β^j , for any j .

To finish the proof we will show that there exists a set $\mathcal{Q} = (Q_\beta)_{\beta < \alpha}$ of edge-disjoint xy -paths of K such that for each $\beta < \alpha$, Q_β can be subdivided into $Q_\beta^1 \cup Q_\beta^2 \cup \dots \cup Q_\beta^{j_\beta}$ such that

- P_β^j and Q_β^j have the same end-vertices ;
- Q_β^j is contained in K_l where H_l is the fragment of Δ that contains the two end-edges of P_β^j .

Such a family \mathcal{Q} exists if for each fragment H_i of Δ the set \mathcal{P}_i of all the P_β^j 's whose end-edges belong to H_i is in one-to-one correspondence with some set of edge-disjoint paths of K_i such that each P_β^j corresponds to a path having the same end-vertices. Since H_i , and hence \mathcal{P}_i , is countable, we only have to show that the two end-vertices of each subpath in \mathcal{P}_i are infinitely edge-connected in K_i . By way of contradiction suppose there exists some P_β^j in \mathcal{P}_i whose end-vertices u, v satisfy $\gamma_{K_i}(u, v) < \omega$ and suppose that j is the least integer for which there exists such a P_β^j . By the choice of K_i we have $\gamma_{H_i}(u, v) < \omega$, and so some finite bond B of H_i separates u from v in H_i . Since Δ is bond-faithful, B is also a bond of G . Moreover, since $P_\beta^{j+1} \cup \dots \cup P_\beta^{j_\beta}$ is edge-disjoint from H_i , it is edge-disjoint from B , implying that B not only separates u from v in G , but also u from y . Thus, $\gamma_G(u, y) < \omega$. On the other hand, $\gamma_G(x, u) \geq \omega$ by the minimality of j ; therefore x and y cannot be infinitely edge-connected in G , a contradiction. \square

5 A special well-ordering on vertices of graphs

Theorem 4.4 has the following consequence:

Theorem 5.1 *Let G be a graph (possibly with loops and multiple edges) and α an uncountable regular cardinal. Then for any α -class X of G , $[X, \overline{X}]_G$ is a union of bonds of cardinality less than α .*

Proof. Consider G/X , the graph obtained from G by identifying the vertices of X and denote by x the new vertex so obtained. If $[X, \overline{X}]_G$ contains a bond of G of cardinality $\geq \alpha$ then G/X contains a block in which x has degree $\geq \alpha$, contradicting Theorem 4.4 applied to that block. \square

This result, interesting in its own right, also has the striking consequence that it is always possible to well-order the ω -classes of a graph in such a way

that the union of all the ω -classes that precede any given one is separable from it by a finite cut. Since it is always possible to separate a finite set of ω -classes from any other one, it is easy to construct such a well-ordering when there are at most countably many ω -classes. The real problem occurs when there are uncountably many. The existence of such a well-ordering can be a very useful tool for constructions on infinite graphs.

The next theorem establishes this result and generalizes it to any infinite regular cardinal.

Theorem 5.2 *Let \mathcal{W} the set of all ω -classes of G . Then there exists a well-ordering on \mathcal{W} (say $\mathcal{W} = ([x_\delta])_{\delta < \beta}$) such that each $[x_\mu] \in \mathcal{W}$ can be separated from $\bigcup_{\delta < \mu} [x_\delta]$ by a finite cut of G .*

Proof. *Case 1.* G is countable. We claim that in this case any well-ordering $([x_\delta])_{\delta < \beta}$ of \mathcal{W} with $\beta \leq \omega$ has the desired property.

Let $[x_\mu] \in \mathcal{W}$ and for each $\delta < \mu$ let $[A_\delta, \overline{A_\delta}]_G$ be a finite cut such that $[x_\delta] \subseteq V(A_\delta)$ and $[x_\mu] \subseteq V(\overline{A_\delta})$. Now observe that

$$B := \left[\bigcup_{\delta < \mu} A_\delta, \bigcup_{\delta < \mu} \overline{A_\delta} \right]_G$$

is a cut separating $[x_\mu]$ from all the $[x_\delta]$'s, $\delta < \mu$, and moreover that B is finite, because so is μ and all the $[A_\delta, \overline{A_\delta}]_G$'s, and because

$$|B| \leq \sum_{\delta < \mu} |[A_\delta, \overline{A_\delta}]_G|.$$

Case 2. G is an uncountable connected graph. Let \tilde{G} be the quotient graph of G modulo its ω -classes. \tilde{G} may have loops and multiple edges. It is clear that any well-ordering \leq_Θ on $V(\tilde{G})$ such that each $x \in V(\tilde{G})$ can be separated from $\{y \in V(\tilde{G}) : y <_\Theta x\}$ by a finite cut of \tilde{G} , when interpreted in G , is a well-ordering with the required properties.

Since no two vertices of \tilde{G} are infinitely edge-connected, and since ω_1 is regular and uncountable, it follows by Theorem 5.1 that no block of \tilde{G} contains vertices of uncountable degree. Therefore all blocks are countable. Let Δ be the set of all blocks of \tilde{G} . Note that Δ is a decomposition of \tilde{G} . Fix $H_0 \in \Delta$ and define a partial order \leq_1 on Δ by:

$$H \leq_1 K \iff H = K \text{ or } H = H_0 \text{ or every path of } \tilde{G} \text{ joining a vertex of } H_0 \text{ to a vertex of } K \text{ contains an edge of } H.$$

In other words, as in the proof of the Claim of Theorem 4.6, \leq_1 is the partial order induced by the block-cutpoint tree of \tilde{G} rooted at H_0 . We leave it to

the reader to show that, because of this tree structure, \leq_1 can be refined to a well-ordering, say \leq_Δ .

For $x \in V(\tilde{G})$, denote by $\delta(x)$ the \leq_Δ -smallest element of Δ that contains x , and let $\phi : \Delta \rightarrow \lambda$ be any injective function whose codomain is an ordinal and which satisfies $H \leq_\Delta K \Leftrightarrow \phi(H) \leq \phi(K)$ for any $H, K \in \Delta$. For any $H \in \Delta \setminus \{H_0\}$, there exists a unique vertex $x_H \in V(H)$ such that $\delta(x_H) \neq H$. Let x_{H_0} be any vertex of $V(H_0)$.

For each $H \in \Delta$ define an enumeration $\psi_H : V(H) \rightarrow \omega$ of $V(H)$ such that x_H is the first element (i.e.: $\psi_H(x_H) = 0$). Finally define

$$\begin{aligned} \theta : V(\tilde{G}) &\rightarrow \lambda \times \omega \\ x &\mapsto (\phi(\delta(x)), \psi_{\delta(x)}(x)), \end{aligned}$$

where $\lambda \times \omega$ is the well-ordered set obtained by the lexicographic order on the cartesian product.

Clearly θ is injective, and a well-ordering \leq_θ on $V(\tilde{G})$ is defined by

$$x \leq_\theta y \Leftrightarrow \theta(x) \leq \theta(y).$$

We will show that \leq_θ has the required separation property.

Because of the lexicographic structure which induces \leq_θ and because of the choice of the x_H 's, \leq_θ restricted to any $V(H)$ coincides with \leq_H . Since every H is countable, by the claim of Case 1, \leq_H must have the property stated in the proposition.

Let $x \in V(\tilde{G})$ and $S := \{y \in V(\tilde{G}) : y <_\theta x\}$, and suppose by way of contradiction that all cuts separating x from S are infinite. The remark in the preceding paragraph implies that $x \notin V(H_0)$ since otherwise $S = \{y \in V(H_0) : y <_{H_0} x\}$.

Put $K := \delta(x)$ and $S_K := S \cap V(K)$. Observe that $H_0 <_\Delta K$, and that $y <_K x$ for any $y \in S_K$. Moreover, $x_K \in S_K$, since otherwise $x = x_K$, which contradicts the fact that $\delta(x_K) <_\Delta K = \delta(x)$. Thus, \leq_K having the required separation property, there exists a finite cut $C = [A, \overline{A}]_K$ of K such that $x \in V(A)$ and $S_K \subseteq V(\overline{A})$. If A_x is the component of A containing x , then $[A_x, \overline{A}_x]_K \subseteq [A, \overline{A}]_K$; hence without loss of generality we may assume A to be connected. C must be non-empty because K is connected and $S_K \neq \emptyset$ and, moreover, since K is a block of \tilde{G} , C is also a cut of \tilde{G} . Since \tilde{G} is connected, there is a unique induced subgraph B of \tilde{G} such that $A \subseteq B$, $\overline{A} \subseteq \overline{B}$ and $[B, \overline{B}]_{\tilde{G}} = C$. Moreover, since A is connected, so also is B .

To finish the proof, let us show that $S \subseteq V(\overline{B})$. By way of contradiction, suppose there exists $z \in S \cap V(B)$. Being connected, B must contain an xz -path P . Since $S_K \subseteq V(\overline{A}) \subseteq V(\overline{B})$, z cannot belong to S_K , and hence

$z \notin V(K)$, i.e., $\delta(z) \neq K$. Moreover, $\delta(z) <_{\Delta} K$ because of the lexicographic structure of \leq_{Θ} . This implies that P contains x_K , contradicting the fact that $P \subseteq B$ and $x_K \in S_K \subseteq V(\overline{A}) \subseteq V(\overline{B})$.

Case 3. G is uncountable and not connected. Left to the reader. \square

Any well-ordering of the set of ω -classes of G can be extended to a well-ordering on $V(G)$, and it is easy to see that, if the well-ordering on the ω -classes has the property of Theorem 5.2, then any such extension has the compactness property stated in the following corollary. We shall extend the notion of infinite edge-connectivity between two vertices introduced in Section 2, by saying that a set $X \subseteq V(G)$ is *infinitely edge-connected to a vertex* $x \in V(G)$ if there exist ω edge-disjoint xX -paths, or equivalently if X cannot be separated from u by a cut of G of cardinality $< \omega$.

Corollary 5.3 *The set of vertices of any graph G can be well-ordered in such a way that for each pair $X \subseteq V(G)$, $u \in V(G)$ such that u is an upper bound of X , the set X is infinitely edge-connected to u if and only if some finite subset of X is infinitely edge-connected to u . (The finite subset can be chosen to be a singleton.)*

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