

# Proper Multi-Type Display Calculi for Rough Algebras

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## Abstract

In the present paper, we endow the logics of topological quasi Boolean algebras, topological quasi Boolean algebras 5, intermediate algebras of types 1-3, and pre-rough algebras with proper multi-type display calculi which are sound, complete, conservative, and enjoy cut elimination and subformula property. Our proposal builds on an algebraic analysis and applies the principles of the multi-type methodology in the design of display calculi.

*Keywords:* Rough sets, topological quasi Boolean algebras, topological quasi Boolean algebras 5, pre-rough algebras, intermediate algebras, canonical extensions, multi-type calculi, proper display calculi.

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## 1 Introduction

Rough algebras and related structures arise in tight connection with formal models of imperfect information [24], and have been investigated for more than twenty years using techniques from universal algebra and algebraic logic, giving rise to a

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rich theory which develops the algebraic semantics, duality, representation and proof theory of their associated logics (cf. e.g. [1,20,4,25,26]). In particular, sound and complete sequent calculi have been introduced for these logics in [25,26]. However, the cut rule in these calculi is not eliminable. Very recently, sequent calculi with cut elimination and a non-standard version of subformula property have been introduced in [22] for some of these logics, but not for the logic of the so-called *intermediate algebras of type 3* (cf. [25], Definition 2.11). In these calculi, the subformula property is non-standard because each logical connective has four introduction rules, two of which are non-standard and introduce the given logical connective under the scope of negation.

In the present paper, we introduce a family of proper display calculi for the logics associated with the classes of ‘*rough algebras*’<sup>2</sup> discussed in [25]; namely, topological quasi Boolean algebras (tqBa), topological quasi Boolean algebras 5 (tqBa5), intermediate algebras of types 1-3 (IA1, IA2, IA3), and pre-rough algebras (pra) (cf. Definition 2.1).

We apply the methodology of *multi-type calculi*, the main feature of which is the systematic use of notions and insights from algebraic logic, duality and representation to solve problems in structural proof theory. Multi-type calculi are introduced in [10,8,9], motivated by [14,11], and have proven effective in endowing a wide range of diverse and differently motivated logical systems, spanning from basic lattice logic [18] to inquisitive logic [12], with calculi for which soundness, completeness, conservativity, cut elimination and subformula property are guaranteed uniformly by the general theory. This methodology has contributed to create an overarching environment in which the algebraic proof theory of paraconsistent logics such as semi De Morgan logic [15] and bilattice logic [16,21] can be studied in connection with very well known and well behaved logics such as linear logic [19] and first order logic [27]. Multi-type calculi also allow to capture a wide class of axiomatic extensions of given logics [17], and therefore provide a powerful and flexible environment for the design of new families of logics, such as those introduced in [2] to reason about agents’ resources and capabilities, which pave the way to novel applications of algebraic and proof-theoretic methods in non-classical logics to formalization problems in fields ranging from artificial intelligence to the social sciences.

The first contribution of the present paper is an equivalent presentation of rough algebras, based on so-called *heterogeneous algebras* [3]. Intuitively, heterogeneous algebras are algebras with more than one domain, and their operations might span across different domains. The classes of heterogeneous algebras corresponding to rough algebras have three domains, respectively corresponding to (abstract representations of) *general* sets and upper and lower *definable* sets of an approximation space. Each of these three domains corresponds to a distinct *type*. The modal operators capturing the lower and upper definable approximations of a general set are then modeled as heterogeneous maps from the general type to one of the two definable types. The equivalent heterogeneous presentations of rough algebras come

<sup>2</sup> Although the name ‘rough algebras’ has a specific meaning in this literature (reported in Definition 2.1), in the present paper we find it convenient to use it as the generic name for the class of topological quasi Boolean algebras and its subclasses.

naturally equipped with a multi-type logical language, and are characterized by axiomatizations which can be readily recognized to be *analytic inductive* (cf. [17, Definition 55]), and hence, by the general theory of multi-type calculi, can be effectively captured by *proper* multi-type display calculi which are sound, complete, conservative, and enjoy cut elimination and *standard* subformula property, given that the introduction rules for all connectives are standard. The introduction of these calculi is the second contribution of the present paper.

Compared with [22], the multi-type methodology allows for more modularity, which not only has made it possible to account for the logic of IA3 (which could not be encoded in an analytic rule otherwise, see Footnote 6), but will also make it possible to extend the present theory so as to cover weaker versions of rough algebras based on e.g. semi De Morgan algebras [15], or even general lattices [5,13], which will account for the proof-theoretic aspects of the logics of rough concepts. More generally, thanks to the multi-type methodology, the logics of rough algebras have been embedded into the wider context of the logics which are properly displayable. Properly displayable logics are studied as a class, and several metatheoretic results, such as semantic cut elimination, finite embeddability property, finite model property, can be given uniformly for large subclasses. Moreover, the modularity of the proof theoretic environment of properly displayable logics makes it possible to make different logics interact in a systematic way, so as to obtain e.g. dynamic epistemic logics based on the logics of rough algebras. This opens new interesting possibilities to enrich the theory of the logics of rough algebras.

## 2 Preliminaries

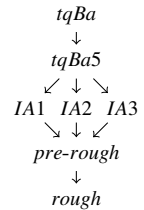
### 2.1 Varieties of rough algebras

**Definition 2.1** (cf. Section 2 [26])  $\mathbb{T} = (\mathbb{L}, I)$  is a *topological quasi-Boolean algebra* (tqBa) if  $\mathbb{L} = (\mathbb{L}, \vee, \wedge, \neg, \top, \perp)$  is a De Morgan algebra and for all  $a, b \in \mathbb{L}$ ,

$$\text{T1. } I(a \wedge b) = Ia \wedge Ib, \quad \text{T2. } I Ia = Ia, \quad \text{T3. } Ia \leq a, \quad \text{T4. } I\top = \top.$$

For any  $a \in \mathbb{T}$ , let  $Ca := \neg I \neg a$ . We consider the subclasses of tqBas defined as in the following table.

| Algebras                                   | Acronyms | Axioms   |
|--|----------|--|
| <i>topological quasi Boolean algebra 5</i> | tqBa5    | T5: $CIa = Ia$   |
| <i>intermediate algebra of type 1</i>      | IA1      | T5, T6: $Ia \vee \neg Ia = \top$                       |
| <i>intermediate algebra of type 2</i>      | IA2      | T5, T7: $Ia \vee Ib = I(a \vee b)$                     |
| <i>intermediate algebra of type 3</i>      | IA3      | T5, T8: $Ia \leq Ib$ and $Ca \leq Cb$ imply $a \leq b$ |
| <i>pre-rough algebra</i>                   | pra      | T5, T6, T7, T8.  |



A *rough algebra* is a complete and completely distributive pre-rough algebra.

**Lemma 2.2** Any tqBa  $\mathbb{T} = (\mathbb{L}, \vee, \wedge, \neg, I, \top, \perp)$  satisfies the following equalities:

$$(i) \ I(Ia \vee Ib) = Ia \vee Ib \qquad (ii) \ C(Ca \wedge Cb) = Ca \wedge Cb.$$

**Proof.** (i)  $I(Ia \vee Ib) \leq Ia \vee Ib$  is a straightforward consequence of T3. As to the

converse direction, it is enough to show that  $Ia \leq I(Ia \vee Ib)$  and  $Ib \leq I(Ia \vee Ib)$ . Let us show the first of these inequalities. From T1 it immediately follows that  $I$  is monotone. Hence  $Ia \leq Ia \vee Ib$  implies  $IIa \leq I(Ia \vee Ib)$ . Hence, by T2,  $Ia \leq IIa \leq I(Ia \vee Ib)$ . Analogously one proves  $Ib \leq I(Ia \vee Ib)$ . The proof for (ii) is dual.  $\square$

Below, we use the abbreviated names of the algebras written in “blackboard bold” (e.g. **TQBA**, etc.) to indicate their corresponding classes. When it is unambiguous, we will use *rough algebras* as the generic name for these classes.

## 2.2 The logics of rough algebras

Fix a denumerable set **Atprop** of propositional variables, let  $p$  denote an element in **Atprop**. The logics of rough algebras share the language  $\mathcal{L}$  which is defined recursively as follows:

$$A ::= p \mid \top \mid \perp \mid \neg A \mid IA \mid CA \mid A \wedge A \mid A \vee A.$$

**Definition 2.3** The logic **H.TQBA** of the class **TQBA** is defined by adding the following axioms to De Morgan logic:

$$\begin{aligned} IA \vdash A, \quad IA \vdash IIA, \quad I(A \wedge B) \vdash IA \wedge IB, \quad IA \wedge IB \vdash I(A \wedge B), \quad \top \vdash I\top \\ CA \vdash \neg I\neg A, \quad \neg I\neg A \vdash CA, \quad \neg C\neg A \vdash IA, \quad IA \vdash \neg C\neg A. \end{aligned}$$

We consider the following extensions of **H.TQBA** corresponding to the subclasses of **TQBA** reported above:

| Class of algebras | name of logic  | Axioms/Rules   |
|-------------------|----------------|--|
| <b>TQBA5</b>      | <b>H.TQBA5</b> | 1: $CA \vdash IA$  |
| <b>IA1</b>        | <b>H.IA1</b>   | 1, 2: $\top \vdash IA \vee \neg IA$                        |
| <b>IA2</b>        | <b>H.IA2</b>   | 1, 3: $I(A \vee B) \vdash IA \vee IB$                      |
| <b>IA3</b>        | <b>H.IA3</b>   | 1, 4: $\frac{IA \vdash IB \quad CA \vdash CB}{A \vdash B}$ |
| <b>PRA</b>        | <b>H.PRA</b>   | 1, 2, 3, 4   |

Let **H** denote any of the logics in the table above (second column), and **A** denote its corresponding class of algebras in the table above (first column, same row as **H**).

**Theorem 2.4 (Completeness)** *H is sound and complete with respect to A, that is, if  $A \vdash B$  is an  $\mathcal{L}$ -sequent, then  $A \vdash B$  is derivable in H iff  $h(A) \leq h(B)$  for all  $\mathbb{T} \in \mathbb{A}$  and every interpretation  $h : \mathcal{L} \rightarrow \mathbb{T}$ .*

## 3 Towards a multi-type presentation: algebraic analysis

In this section, we equivalently represent rough algebras as heterogeneous algebras.

### 3.1 The kernels of rough algebras

For any tqBa  $\mathbb{T}$  (cf. Definition 2.1), we let  $K_I := \{Ia \mid a \in L\}$  and  $K_C := \{Ca \mid a \in L\}$ , and let  $\iota : L \rightarrow K_I$  and  $\gamma : L \rightarrow K_C$  be defined by the assignments  $a \mapsto Ia$  and  $a \mapsto Ca$ , respectively. Let  $e_I : K_I \hookrightarrow L$  and  $e_C : K_C \hookrightarrow L$  denote the natural embeddings. Axioms T1, T2, and T3 imply that  $I : L \rightarrow L$  is an interior operator and  $C : L \rightarrow L$

is a closure operator on  $L$  seen as a poset. Hence, by general order-theoretic facts (cf. [7, Chapter 7]),  $e_I$  (resp.  $e_C$ ) is the left (resp. right) adjoint of  $\iota$  (resp.  $\gamma$ ), in symbols:  $e_I \dashv \iota$  and  $\gamma \dashv e_C$ , i.e. for any  $\alpha \in K_I$ ,  $\xi \in K_C$  and  $a \in L$ ,

$$e_I \alpha \leq a \quad \text{iff} \quad \alpha \leq \iota a \qquad \gamma \xi \leq a \quad \text{iff} \quad \xi \leq e_C a. \quad (1)$$

The following equations are straightforward consequences of the definitions of the maps and (1):

$$\iota(e_I \alpha) = \alpha \qquad e_I(\iota a) = I a \qquad \gamma(e_C \xi) = \xi \qquad e_C(\gamma a) = C a. \quad (2)$$

**Definition 3.1** For any tqBa  $\mathbb{T} = (L, \vee, \wedge, I, \neg, \top, \perp)$ , the *left-kernel*  $\mathbb{K}_I = (K_I, \cup, \cap, 1_I, 0_I)$  and the *right-kernel*  $\mathbb{K}_C = (K_C, \sqcup, \sqcap, 1_C, 0_C)$  are such that, for all  $\alpha, \beta \in K_I$ , and all  $\xi, \chi \in K_C$ ,

$$\begin{array}{llll} \text{K1. } \alpha \cup \beta := \iota(e_I \alpha) \vee e_I \beta & \text{K3. } 1_I := \iota \top & \text{K'1. } \xi \sqcup \chi := \gamma(e_C \xi \vee e_C \chi) & \text{K'3. } 1_C := \gamma \top \\ \text{K2. } \alpha \cap \beta := \iota(e_I \alpha \wedge e_I \beta) & \text{K4. } 0_I := \iota \perp & \text{K'2. } \xi \sqcap \chi := \gamma(e_C \xi \wedge e_C \chi) & \text{K'4. } 0_C := \gamma \perp \end{array}$$

If  $\mathbb{T}$  is a tqBa5, we define  $\sim: K_I \rightarrow K_I$  and  $-: K_C \rightarrow K_C$  by the following equation:

$$\text{K5. } \sim \alpha := \iota \neg e_I \alpha \qquad \text{K'5. } -\xi := \gamma \neg e_C \xi$$

The next lemma captures the relationship between a tqBa and its kernels via the properties of their connecting maps:

**Lemma 3.2** For any tqBa  $\mathbb{T}$ ,

1.  $\iota: \mathbb{T} \rightarrow \mathbb{K}_I$  and  $\gamma: \mathbb{T} \rightarrow \mathbb{K}_C$  are surjective maps which satisfy the following equations: for all  $a, b \in L$ ,
  - (a)  $\iota a \cap \iota b = \iota(a \wedge b)$ ,  $\iota \top = 1_I$ ,  $\iota \perp = 0_I$ ;
  - (b)  $\gamma a \cup \gamma b = \gamma(a \vee b)$ ,  $\gamma \top = 1_C$ ,  $\gamma \perp = 0_C$ .
2.  $e_I: \mathbb{K}_I \rightarrow \mathbb{T}$  and  $e_C: \mathbb{K}_C \rightarrow \mathbb{T}$  are injective maps which satisfy the following equations:
  - (a)  $e_I \alpha \wedge e_I \beta = e_I(\alpha \cap \beta)$ ,  $e_I \alpha \vee e_I \beta = e_I(\alpha \cup \beta)$ ;
  - (b)  $e_C \xi \wedge e_C \chi = e_C(\xi \sqcap \chi)$ ,  $e_C \xi \vee e_C \chi = e_C(\xi \sqcup \chi)$ ;
  - (c)  $e_I 1_I = \top$ ,  $e_I 0_I = \perp$ ,  $e_C 1_C = \top$ ,  $e_C 0_C = \perp$ .

**Proof.** We only prove 1(a) and 2(a), the arguments for 1(b) and 2(b) being dual. The identities in 2(c) easily follow using K3, K4, K'3, K'4 and the definition of  $\mathbb{T}$ . The surjectivity of  $\iota$  is an immediate consequence of the definition of  $K_I$  (cf. beginning of Section 3.1). In what follows, we show that  $\iota$  satisfies 1(a).

$$\begin{aligned} \iota a \cap \iota b &= \iota(e_I a \wedge e_I b) & \text{K2} \\ &= \iota(I a \wedge I b) & (2) \\ &= \iota(a \wedge b) & \text{T1} \\ &= \iota e_I(a \wedge b) & (2) \\ &= \iota(a \wedge b) & (2) \end{aligned}$$

The remaining identities in 1(a) can be shown analogously using K3 and K4. Let us show that  $e_I$  satisfies 2(a) and 2(c). For any  $\alpha, \beta \in K_I$ , let  $a, b \in L$  be such that  $\alpha = \iota a$  and  $\beta = \iota b$ .

|                        |     |                                 |                                       |                        |     |                               |                                       |
|------------------------|-----|---------------------------------|---------------------------------------|------------------------|-----|-------------------------------|---------------------------------------|
| $e(\alpha \cap \beta)$ | $=$ | $e(\iota a \cap \iota b)$       | $(\alpha = \iota a, \beta = \iota b)$ | $e(\alpha \cup \beta)$ | $=$ | $e(\iota a \cup \iota b)$     | $(\alpha = \iota a, \beta = \iota b)$ |
|                        | $=$ | $e(e \iota a \wedge e \iota b)$ | K2                                    |                        | $=$ | $e(e \iota a \vee e \iota b)$ | by K1                                 |
|                        | $=$ | $I(Ia \wedge Ib)$               | (2)                                   |                        | $=$ | $I(Ia \vee Ib)$               | (2)                                   |
|                        | $=$ | $Ia \wedge Ib$                  | T1                                    |                        | $=$ | $Ia \vee Ib$                  | Corollary 2.2                         |
|                        | $=$ | $Ia \wedge Ib$                  | T2                                    |                        | $=$ | $e \iota a \vee e \iota b$    | (2)                                   |
|                        | $=$ | $e \iota a \wedge e \iota b$    | (2)                                   |                        | $=$ | $e \alpha \vee e \beta$       | $(\alpha = \iota a, \beta = \iota b)$ |
|                        | $=$ | $e \alpha \wedge e \beta$       | $(\alpha = \iota a, \beta = \iota b)$ |                        |     |                               |                                       |

□

The next propositions and lemmas provide a ‘multi-type’ characterization of defining properties of axiomatic extensions of tqBas in terms of additional properties of the kernels or of their connecting maps:

**Proposition 3.3** *If  $\mathbb{T}$  is a tqBa5, then  $\mathbb{K}_I \cong \mathbb{K}_C$ .*

**Proof.** Let  $f : K_I \rightarrow K_C$  be defined as  $f := \gamma e_I$ . To show that  $f$  is surjective, let  $\xi \in K_C$ , and let  $\xi = \gamma a$  for some  $a \in L$ ,

|            |     |                           |                     |
|------------|-----|---------------------------|---------------------|
| $\gamma a$ | $=$ | $\gamma e_C \gamma a$     | adjunction          |
|            | $=$ | $\gamma C a$              | (2)                 |
|            | $=$ | $\gamma I C a$            | T5                  |
|            | $=$ | $\gamma e_I e_C \gamma a$ | (2)                 |
|            | $=$ | $\gamma e_I e_C \xi$      | $(\xi = \gamma a)$  |
|            | $=$ | $f e_C \xi$               | $(f := \gamma e_I)$ |

Since both  $\gamma$  and  $e_I$  are monotone, so is  $f := \gamma e_I$ . To finish the proof, we need to show that for all  $\alpha, \beta \in K_I$ , if  $\gamma e_I(\alpha) \leq \gamma e_I(\beta)$ , then  $\alpha \leq \beta$ . Since  $e_C$  is an order embedding, the assumption can be equivalently rewritten as  $e_C \gamma e_I(\alpha) \leq e_C \gamma e_I(\beta)$ . Let  $a, b \in L$  such that  $\alpha = \iota a$  and  $\beta = \iota b$ . Then we can equivalently rewrite the assumption as  $e_C \gamma e_I \iota a \leq e_C \gamma e_I \iota b$ . Since  $I := e_I \iota$  and  $C := e_C \gamma$ , we can again equivalently rewrite the assumption as  $C I a \leq C I b$ , and hence, by T5, as  $I a \leq I b$ , that is,  $e_I \iota a \leq e_I \iota b$ . Since  $e_I$  is an order-embedding, this yields  $\iota a \leq \iota b$ , that is,  $\alpha \leq \beta$ , as required. This finishes the proof that  $\mathbb{K}_I \cong \mathbb{K}_C$  as lattices. Finally, we need to show that  $f(\sim \alpha) = \neg f(\alpha)$  for any  $\alpha \in K_I$ . For such an  $\alpha$ , let  $a \in L$  s.t.  $\alpha = \iota(a)$ .

|                 |     |                                     |                 |     |                                      |
|-----------------|-----|-------------------------------------|-----------------|-----|--------------------------------------|
| $f \sim \alpha$ | $=$ | $\gamma e_I \sim \alpha$            | $\neg f \alpha$ | $=$ | $\gamma \neg e_C f \alpha$           |
|                 | $=$ | $\gamma e_I \sim \iota a$           |                 | $=$ | $\gamma \neg e_C \gamma e_I \alpha$  |
|                 | $=$ | $\gamma e_I \iota \neg e_I \iota a$ |                 | $=$ | $\gamma \neg e_C \gamma e_I \iota a$ |
|                 | $=$ | $\gamma e_I \iota \neg I a$         |                 | $=$ | $\gamma \neg C e_I \iota a$          |
|                 | $=$ | $\gamma I \neg I a$                 |                 | $=$ | $\gamma \neg C I a$                  |
|                 | $=$ | $\gamma I C \neg a$                 |                 | $=$ | $\gamma I \neg I a$                  |
|                 | $=$ | $\gamma C \neg a$                   |                 | $=$ | $\gamma I C \neg a$                  |
|                 |     |                                     |                 | $=$ | $\gamma C \neg a$                    |

□

By the proposition above, we can drop the subscripts in  $\mathbb{K}_I$  (or  $\mathbb{K}_C$ ) and in  $e_I$  and  $e_C$ , and refer to  $\mathbb{K}$  as the *kernel* of  $\mathbb{T}$ . The following lemma has a straightforward proof which uses K5:

**Lemma 3.4** (1) *If  $\mathbb{T}$  is a tqBa5, then  $e \sim \alpha = \neg e \alpha$ ;*  
 (2) *If  $\mathbb{T}$  is an IA1, then  $\iota(a \vee b) = \iota a \cup \iota b$ .*

**Proposition 3.5** *If  $\mathbb{T}$  is a tqBa5, then  $\mathbb{K}$  is a De Morgan algebra. Moreover, if  $\mathbb{T}$  is an IA1, then  $\mathbb{K}$  is a Boolean algebra.*

**Proof.** For any  $\alpha, \beta \in K_I$ , let  $a, b \in L$  such that  $\alpha = \iota a$  and  $\beta = \iota b$ . Let us show that  $\sim \sim \alpha = \alpha$  and  $\sim(\alpha \cup \beta) = \sim \alpha \cap \sim \beta$ .

|                  |     |                                     |                   |                         |     |  |  |
|------------------|-----|-------------------------------------|-------------------|-------------------------|-----|--|--|
| $\sim\sim\alpha$ | $=$ | $\iota\sim e\iota\sim e\alpha$      | K5                | $\sim(\alpha\cup\beta)$ | $=$ | $\iota\sim e(\alpha\cup\beta)$                                 | K5   |
| $=$              |     | $\iota\sim e\iota\sim e\iota\alpha$ | (i)               | $=$                     |     | $\iota\sim e\iota(e\alpha\vee e\beta)$                         | K1   |
| $=$              |     | $\iota\sim I\sim I\alpha$           | (2)               | $=$                     |     | $\iota\sim e\iota(e\iota\alpha\vee e\iota\beta)$               | $(\alpha = \iota a, \beta = \iota b)$                |
| $=$              |     | $\iota C I \alpha$                  | $C = \neg I \neg$ | $=$                     |     | $\iota\sim I(I\alpha\vee I\beta)$                              | (2)  |
| $=$              |     | $\iota I \alpha$                    | T5                | $=$                     |     | $\iota\sim(I\alpha\vee I\beta)$                                | Lemma 2.2  |
| $=$              |     | $\iota e \iota \alpha$              | (2)               | $=$                     |     | $\iota(\sim I \alpha \wedge \sim I \beta)$                     | definition of $\mathbb{L}$                           |
| $=$              |     | $\iota \alpha$                      | (2)               | $=$                     |     | $\iota(C\sim\alpha \wedge C\sim\beta)$                         | $C = \neg I \neg$ and definition of $\mathbb{L}$     |
| $=$              |     | $\alpha$                            | (i)               | $=$                     |     | $\iota(IC\sim\alpha \wedge IC\sim\beta)$                       | T5, $C = \neg I \neg$ and definition of $\mathbb{L}$ |
|                  |     |                                     |                   | $=$                     |     | $\iota(I\sim I\alpha \wedge I\sim I\beta)$                     | $C = \neg I \neg$ and definition of $\mathbb{L}$     |
|                  |     |                                     |                   | $=$                     |     | $\iota(e\iota\sim e\iota\alpha \wedge e\iota\sim e\iota\beta)$ | (2)  |
|                  |     |                                     |                   | $=$                     |     | $\iota(e\iota\sim e\alpha \wedge e\iota\sim e\beta)$           | $(\alpha = \iota(a), \beta = \iota(b))$              |
|                  |     |                                     |                   | $=$                     |     | $\iota(e \sim \alpha \wedge e \sim \beta)$                     | K5   |
|                  |     |                                     |                   | $=$                     |     | $\sim\alpha \cap \sim\beta$                                    | K2   |

Using K5, K3, (2) and T7, one can show the identities  $\sim 1_I = \iota\sim e 1_I = \iota\sim e \iota \top = \iota\sim I \top = \iota \perp = 0_I$ . The argument for  $\sim 0_I = 1_I$  can be given dually. Hence,  $\mathbb{K}_I$  is a De Morgan algebra. If  $\mathbb{T}$  is an IA1, in order to show that  $\mathbb{K}_I$  is a Boolean algebra, we only need to show  $\sim \alpha \cup \alpha = 1_I$ .

|                           |     |  |                   |
|---------------------------|-----|--|-------------------|
| $\sim \alpha \cup \alpha$ | $=$ | $\iota\sim e\alpha \cup \alpha$                    | K5                |
| $=$                       |     | $\iota(e\iota\sim e\alpha \vee e\alpha)$           | K1                |
| $=$                       |     | $\iota(e\iota\sim e\iota\alpha \vee e\iota\alpha)$ | (i)               |
| $=$                       |     | $\iota(I\sim I\alpha \vee I\alpha)$                | (2)               |
| $=$                       |     | $\iota(\sim C I \alpha \vee I \alpha)$             | $C = \neg I \neg$ |
| $=$                       |     | $\iota(\sim I \alpha \vee I \alpha)$               | T5                |
| $=$                       |     | $\iota \top$                                       | T6                |
| $=$                       |     | $1_I$  | K3                |

□

### 3.2 Heterogeneous rough algebras

In the present subsection, we show that the properties that we have verified to hold for rough algebras, their kernels and connecting maps yield an equivalent presentation of rough algebras

**Definition 3.6** A *heterogeneous tqBa* (htqBa) is a tuple  $\mathbb{H} = (\mathbb{D}, \mathbb{L}_I, \mathbb{L}_C, e_I, e_C, \iota, \gamma)$  such that:

H1  $\mathbb{D} = (\mathbb{D}, \vee, \wedge, \neg, \top, \perp)$  is a De Morgan algebra;

H2  $\mathbb{L}_I = (\mathbb{L}_I, \cup, \cap, \sim, 0_I, 1_I)$  and  $\mathbb{L}_C = (\mathbb{L}_C, \sqcup, \sqcap, -, 0_C, 1_C)$  are bounded distributive lattices;

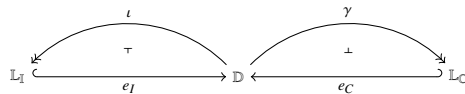
H3  $e_I : \mathbb{L}_I \hookrightarrow \mathbb{D}$  and  $e_C : \mathbb{L}_C \hookrightarrow \mathbb{D}$  are lattice homomorphisms;

H4  $\iota : \mathbb{D} \rightarrow \mathbb{L}_I$  and  $\gamma : \mathbb{D} \rightarrow \mathbb{L}_C$  satisfy the following identities:

$$\iota(a \wedge b) = \iota a \cap \iota b \quad \iota \top = 1 \quad \iota \perp = 0 \quad \gamma(a \vee b) = \gamma a \sqcup \gamma b \quad \gamma \top = 1 \quad \gamma \perp = 0;$$

H5  $e_I \dashv \iota \quad \gamma \dashv e_C \quad \iota e_I \alpha = \alpha \quad \gamma e_C \xi = \xi$ ; <sup>3</sup>

H6  $e_C \gamma a = \neg e_I \iota \neg a$ .



The heterogeneous algebras corresponding to the subclasses of tqBas considered in Section 2.1 are defined as follows:

<sup>3</sup> Condition H5 implies that  $\iota$  is surjective and  $e$  is injective.

| Algebra             | Acronym | Conditions  |
|---------------------|---------|---|
| heterogeneous tqBa5 | htqBa5  | H7: $L_I = L_C = L$ is a De Morgan algebra, $e_I = e_C = e$ is a De Morgan homomorphism |
| heterogeneous IA1   | hIA1    | H7, H8: $L$ is a Boolean algebra  |
| heterogeneous IA2   | hIA2    | H7, H9: $\iota(a \vee b) = \iota a \cup \iota b$  |
| heterogeneous IA3   | hIA3    | H7, H10: $\iota a \leq \iota b$ and $\gamma a \leq \gamma b$ imply $a \leq b$           |
| heterogeneous pra   | hpra    | H7, H8, H9, H10   |

In what follows, we use the abbreviated names of the heterogeneous algebras written in “blackboard bold” (e.g. **HTQBA**, etc.) to indicate their corresponding classes. A heterogeneous algebra  $\mathbb{H}$  is *perfect* if every lattice reduct in the signature of  $\mathbb{H}$  is perfect,<sup>4</sup> and every join (resp. meet) preserving (resp. reversing) map in the signature of  $\mathbb{H}$  is completely join (resp. meet) preserving (resp. reversing).

**Definition 3.7** If  $\mathbb{T} = (\mathbb{L}, I)$  is a tqBa, we let  $\mathbb{T}^+ := (\mathbb{L}, \mathbb{K}_I, \mathbb{K}_C, e_I, e_C, \iota, \gamma)$ , where:

- $\mathbb{K}_I$  and  $\mathbb{K}_C$  are the left and right kernels of  $\mathbb{T}$  (cf. Definition 3.1);
- $e_I : \mathbb{K}_I \hookrightarrow \mathbb{L}$  and  $e_C : \mathbb{K}_C \hookrightarrow \mathbb{L}$  are defined as the embeddings of the domains of  $\mathbb{K}_I$  and  $\mathbb{K}_C$  into the domain of  $\mathbb{L}$ ;
- $\iota : \mathbb{L} \rightarrow \mathbb{K}_I$  and  $\gamma : \mathbb{L} \rightarrow \mathbb{K}_C$  are defined by  $\iota(a) = Ia$  and  $\gamma(a) = Ca$  respectively.

If  $\mathbb{T} = (\mathbb{L}, I)$  is a tqBa5,  $\mathbb{K}_I$  and  $\mathbb{K}_C$  can be identified and also  $e_I$  and  $e_C$  can, hence we write  $\mathbb{T}^+ := (\mathbb{L}, \mathbb{K}, e, \iota, \gamma)$ .

**Definition 3.8** If  $\mathbb{H} = (\mathbb{D}, L_I, L_C, e_I, e_C, \iota, \gamma)$  is an htqBa, we let  $\mathbb{H}_+ := (\mathbb{D}, I, C)$  where the unary operations  $I$  and  $C$  on  $\mathbb{D}$  are defined by the assignments  $a \mapsto e_I \iota a$  and  $a \mapsto e_C \gamma a$  respectively.

Let  $\mathbb{A}$  denote a class of rough algebras (cf. Section 2.1), and  $\mathbb{HA}$  its corresponding class of heterogeneous algebras.

**Proposition 3.9** (i) If  $\mathbb{T} \in \mathbb{A}$ , then  $\mathbb{T}^+ \in \mathbb{HA}$ ;

(ii) If  $\mathbb{H} \in \mathbb{HA}$ , then  $\mathbb{H}_+ \in \mathbb{A}$ ;

(iii)  $\mathbb{T} \cong (\mathbb{T}^+)_+$  and  $\mathbb{H} \cong (\mathbb{H}_+)^+$ .

### 3.3 Canonical extensions of heterogeneous algebras

As discussed in other papers adopting the multi-type methodology (cf. e.g. [15,19]), canonicity in the multi-type environment serves both to provide complete semantics for the analytic extensions of the basic logic (i.e. extensions obtained by adding analytic inductive axioms) and to prove the conservativity of their associated display calculi. In what follows, we let  $\mathbb{D}^\delta$ ,  $\mathbb{L}_I^\delta$ , and  $\mathbb{L}_C^\delta$  denote the canonical extensions of the algebras  $\mathbb{D}$ ,  $\mathbb{L}_I$ , and  $\mathbb{L}_C$  respectively, and  $e_I^\delta$ ,  $e_C^\delta$ ,  $\iota^\pi$ , and  $\gamma^\pi$  denote the extensions of  $e_I$ ,  $e_C$ ,  $\iota$ , and  $\gamma$  respectively.<sup>5</sup>

<sup>4</sup> A distributive lattice is *perfect* if it is complete, completely distributive and completely join-generated by the collection of its completely join-prime elements.

<sup>5</sup> The order-theoretic properties of  $e_I, e_C, \iota$  and  $\gamma$  guarantee that they are *smooth*, that is, for each of them,  $\sigma$ -extension and  $\pi$ -extension coincide. However, the different notations in the superscripts are meant to emphasize that while the smoothness of the embeddings is used in the canonicity proofs, it is not needed in the case of  $\iota^\pi$  and  $\gamma^\pi$ .



**Definition 3.10** If  $\mathbb{H} = (\mathbb{D}, \mathbb{L}_I, \mathbb{L}_C, e_I, e_C, \iota, \gamma) \in \mathbf{HA}$  is an htqBa, then the *canonical extension* of  $\mathbb{H}$  is the heterogeneous algebra  $\mathbb{H}^\delta = (\mathbb{D}^\delta, \mathbb{K}_I^\delta, \mathbb{K}_C^\delta, e_I^\delta, e_C^\delta, \iota^\pi, \gamma^\sigma)$ .

The defining conditions of the heterogeneous algebras of Definition 3.6 can be expressed as analytic inductive inequalities (cf. Definition 4.3), and each such inequality is canonical. Hence:

**Proposition 3.11** *If  $\mathbb{H} \in \mathbf{HA}$ , then  $\mathbb{H}^\delta$  is a perfect element of  $\mathbf{HA}$ .*

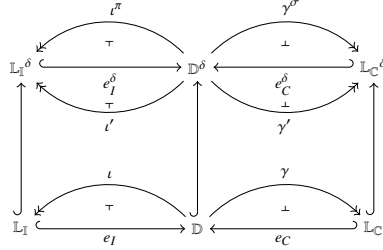


Fig. 1. Extending heterogeneous algebras to canonical extensions

In Section 6.1, soundness of each multi-type calculus will be proven w.r.t. perfect HAs.

## 4 Multi-type language for heterogeneous rough algebras

Heterogeneous algebras provide a natural interpretation for the following multi-type language  $\mathcal{L}_{\text{MT}}$  consisting of terms of types  $\mathbb{D}$ ,  $\mathbb{K}_I$  and  $\mathbb{K}_C$  (the kernel-type negations apply to the language of H.TQBA5 and its extensions).

$$\mathbb{D} \ni A ::= p \mid \circ_I \alpha \mid \circ_C \xi \mid \top \mid \perp \mid A \wedge A \mid A \vee A \mid \neg A$$

$$\mathbb{K}_I \ni \alpha ::= \blacksquare_I A \mid 1_I \mid 0_I \mid \alpha \cup \alpha \mid \alpha \cap \alpha \mid \sim \alpha \quad \mathbb{K}_C \ni \xi ::= \blacklozenge_C A \mid 1_C \mid 0_C \mid \xi \sqcup \xi \mid \xi \sqcap \xi \mid \neg \xi$$

The logic H.TQBA5 can be also captured in a simpler language consisting of the two types  $\mathbb{D}$  as above and  $\mathbb{K}$  as follows:

$$\mathbb{K} \ni \alpha ::= \blacksquare_I A \mid \blacklozenge_C A \mid 1 \mid 0 \mid \sim \alpha \mid \alpha \cup \alpha \mid \alpha \cap \alpha.$$

### 4.1 Analytic inductive $\mathcal{L}_{\text{MT}}$ -inequalities

In the present section, we specialize the definition of *analytic inductive inequalities* (cf. [17, Definition 55]) to the multi-type language  $\mathcal{L}_{\text{MT}}$ . This definition also applies to the algebraic language of htqBas that interprets it, so that we will talk about analytic inductive term-inequalities. We will make use of the following auxiliary definition: an *order-type* over  $n \in \mathbb{N}$  is an  $n$ -tuple  $\epsilon \in \{1, \partial\}^n$ . For every order type  $\epsilon$ , we denote its *opposite* order type by  $\epsilon^\partial$ , that is,  $\epsilon^\partial(i) = 1$  iff  $\epsilon(i) = \partial$  for every  $1 \leq i \leq n$ . The connectives of the language above are grouped into the following families  $\mathcal{F} := \mathcal{F}_{\mathbb{D}} \cup \mathcal{F}_{\text{MT}} \cup \mathcal{F}_{\mathbb{K}_I} \cup \mathcal{F}_{\mathbb{K}_C}$  and  $\mathcal{G} := \mathcal{G}_{\mathbb{D}} \cup \mathcal{G}_{\text{MT}} \cup \mathcal{G}_{\mathbb{K}_I} \cup \mathcal{G}_{\mathbb{K}_C}$ :

$\mathcal{F}_D := \{\top, \wedge, \neg\}$  $\mathcal{F}_{K_I} := \{1_I, \cap, \sim\}$  $\mathcal{G}_D := \{\perp, \vee, \neg\}$  $\mathcal{G}_{K_I} := \{0_I, \cup, \sim\}$

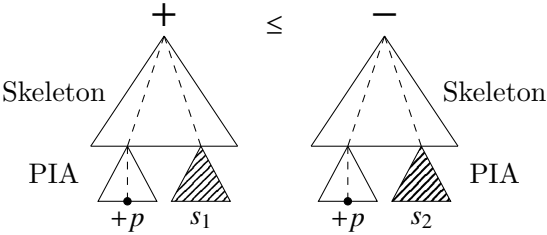
$\mathcal{F}_{MT} := \{\blacklozenge_I, \blacklozenge_C, \circ_I, \circ_C\}$  $\mathcal{F}_{K_C} := \{1_C, \sqcap, -\}$  $\mathcal{G}_{MT} := \{\blacksquare_I, \blacksquare_C, \circ_I, \circ_C\}$  $\mathcal{G}_{K_C} := \{0_C, \sqcup, -\}$

For any  $f \in \mathcal{F}$  (resp.  $g \in \mathcal{G}$ ), we let  $n_f \in \mathbb{N}$  (resp.  $n_g \in \mathbb{N}$ ) denote the arity of  $f$  (resp.  $g$ ), and the order-type  $\epsilon_f$  (resp.  $\epsilon_g$ ) on  $n_f$  (resp.  $n_g$ ) indicate whether the  $i$ th coordinate of  $f$  (resp.  $g$ ) is positive ( $\epsilon_f(i) = 1$ ,  $\epsilon_g(i) = 1$ ) or negative ( $\epsilon_f(i) = \partial$ ,  $\epsilon_g(i) = \partial$ ). The order-theoretic motivation for this grouping is that the algebraic interpretations of  $\mathcal{F}$ -connectives (resp.  $\mathcal{G}$ -connectives) preserve finite joins (resp. meets) in each positive coordinate and reverse finite meets (resp. joins) in each negative coordinate. For any term  $s(p_1, \dots, p_n)$ , any order type  $\epsilon$  over  $n$ , and any  $1 \leq i \leq n$ , an  $\epsilon$ -critical node in a signed generation tree of  $s$  is a leaf node  $+p_i$  with  $\epsilon(i) = 1$  or  $-p_i$  with  $\epsilon(i) = \partial$ . An  $\epsilon$ -critical branch in the tree is a branch ending in an  $\epsilon$ -critical node. For any term  $s(p_1, \dots, p_n)$  and any order type  $\epsilon$  over  $n$ , we say that  $+s$  (resp.  $-s$ ) agrees with  $\epsilon$ , and write  $\epsilon(+s)$  (resp.  $\epsilon(-s)$ ), if every leaf in the signed generation tree of  $+s$  (resp.  $-s$ ) is  $\epsilon$ -critical. We will also write  $+s' < *s$  (resp.  $-s' < *s$ ) to indicate that the subterm  $s'$  inherits the positive (resp. negative) sign from the signed generation tree  $*s$ . Finally, we will write  $\epsilon(s') < *s$  (resp.  $\epsilon^\partial(s') < *s$ ) to indicate that the signed subtree  $s'$ , with the sign inherited from  $*s$ , agrees with  $\epsilon$  (resp. with  $\epsilon^\partial$ ).

**Definition 4.1 [Signed Generation Tree]** The *positive* (resp. *negative*) *generation tree* of any  $\mathcal{L}_{MT}$ -term  $s$  is defined by labelling the root node of the generation tree of  $s$  with the sign  $+$  (resp.  $-$ ), and then propagating the labelling on each remaining node as follows: For any node labelled with  $\ell \in \mathcal{F} \cup \mathcal{G}$  of arity  $n_\ell \geq 1$ , and for any  $1 \leq i \leq n_\ell$ , assign the same (resp. the opposite) sign to its  $i$ th child node if  $\epsilon_\ell(i) = 1$  (resp. if  $\epsilon_\ell(i) = \partial$ ). Nodes in signed generation trees are *positive* (resp. *negative*) if are signed  $+$  (resp.  $-$ ).

**Definition 4.2 [Good branch]** Nodes in signed generation trees are called  $\Delta$ -adjoints, *syntactically left residual* (SLR), *syntactically right residual* (SRR), and *syntactically right adjoints* (SRA), according to the specification given in Table 1. A branch in a signed generation tree  $*s$ , with  $* \in \{+, -\}$ , is a *good branch* if it is the concatenation of two paths  $P_1$  and  $P_2$ , one of which may possibly be of length 0, such that  $P_1$  is a path from the leaf consisting (apart from variable nodes) only of PIA-nodes (for explanations on terminology, we refer to [23, Remark 3.24]), and  $P_2$  consists (apart from variable nodes) only of Skeleton-nodes.

| Skeleton  | PIA   |
|---|---|
| $\Delta$ -adjoints $\left\{ \begin{array}{l} + \quad \vee \quad \cup \quad \sqcup \quad \perp \quad 0_I \quad 0_C \\ - \quad \wedge \quad \cap \quad \sqcap \quad \top \quad 1_I \quad 1_C \end{array} \right.$   | SRA $\left\{ \begin{array}{l} + \quad \perp \quad 0_I \quad 0_C \quad \wedge \quad \cap \quad \sqcap \quad \circ_I \quad \circ_C \quad \blacksquare_I \quad \neg \quad - \\ - \quad \top \quad 1_I \quad 1_C \quad \vee \quad \cup \quad \sqcup \quad \circ_I \quad \circ_C \quad \blacklozenge_C \quad \neg \quad - \end{array} \right.$ |
| SLR $\left\{ \begin{array}{l} + \quad \top \quad 1_I \quad 1_C \quad \wedge \quad \cap \quad \sqcap \quad \circ_I \quad \circ_C \quad \blacklozenge_C \quad \neg \quad - \\ - \quad \perp \quad 0_I \quad 0_C \quad \vee \quad \cup \quad \sqcup \quad \circ_I \quad \circ_C \quad \blacksquare_C \quad \neg \quad - \end{array} \right.$ | SRR $\left\{ \begin{array}{l} + \quad \vee \quad \cup \quad \sqcup \\ - \quad \wedge \quad \cap \quad \sqcap \end{array} \right.$   |



**Definition 4.3** [Analytic inductive inequalities, cf. [17, Definition 55]] For any order type  $\epsilon$  and any irreflexive and transitive relation  $<_{\Omega}$  on  $p_1, \dots, p_n$ , the signed generation tree  $*s$  ( $* \in \{-, +\}$ ) of a term  $s(p_1, \dots, p_n)$  is *analytic*  $(\Omega, \epsilon)$ -*inductive* if

- (i) every branch of  $*s$  is good (cf. Definition 4.2);
- (ii) for all  $1 \leq i \leq n$ , every  $m$ -ary SRR-node occurring in any  $\epsilon$ -critical branch with leaf  $p_i$  is of the form  $\oplus(s_1, \dots, s_{j-1}, \beta, s_{j+1}, \dots, s_m)$ , where for any  $h \in \{1, \dots, m\} \setminus j$ :
  - (a)  $\epsilon^{\partial}(s_h) < *s$  (cf. discussion before Definition 4.2), and
  - (b)  $p_k <_{\Omega} p_i$  for every  $p_k$  occurring in  $s_h$  and for every  $1 \leq k \leq n$ .

An inequality  $s \leq t$  is *analytic*  $(\Omega, \epsilon)$ -*inductive* if the signed generation trees  $+s$  and  $-t$  are analytic  $(\Omega, \epsilon)$ -inductive. An inequality  $s \leq t$  is *analytic inductive* if it is analytic  $(\Omega, \epsilon)$ -inductive for some  $\Omega$  and  $\epsilon$ .

In each setting in which they are defined, analytic inductive inequalities are a subclass of inductive inequalities (cf. [17]). In their turn, inductive inequalities are *canonical* (that is, preserved under canonical extensions, as defined in each setting, cf. [6]). Hence, the following is an immediate consequence of the general canonicity of inductive inequalities.

**Theorem 4.4** *Analytic inductive  $\mathcal{L}_{\text{MT}}$ -inequalities are canonical.*

#### 4.2 Translating the original language of rough algebras into the multi-type language

The toggle between the single-type algebras and their corresponding heterogeneous algebras is reflected syntactically by the translation  $(\cdot)^t : \mathcal{L} \rightarrow \mathcal{L}_{\text{MT}}$  defined as follows:

$$\begin{array}{llll}
 p^t & = & p & \quad \quad \quad \perp^t & = & \perp & \quad \quad \quad \top^t & = & \top & \quad \quad \quad (\neg A)^t & = & \neg A^t \\
 (A \wedge B)^t & = & A^t \wedge B^t & \quad \quad \quad (A \vee B)^t & = & A^t \vee B^t & \quad \quad \quad (IA)^t & = & \circ_C \blacksquare_I A^t & \quad \quad \quad (CA)^t & = & \circ_I \blacklozenge_C A^t
 \end{array}$$

Recall that  $\mathbb{T}^+$  denotes the heterogeneous algebra associated with the given algebra  $\mathbb{T}$  (cf. Definition 3.7). The following proposition is proved by a routine induction on  $\mathcal{L}$ -formulas.

**Proposition 4.5** *For all  $\mathcal{L}$ -formulas  $A$  and  $B$  and every  $\mathcal{L}$ -algebra  $\mathbb{T}$ ,  $\mathbb{T} \models A \leq B$  iff  $\mathbb{T}^+ \models A^t \leq B^t$ .*

We are now in a position to translate the axioms and rules of any logic  $\mathbf{H}$  defined in Section 2.2 into  $\mathcal{L}_{\text{MT}}$ .

| original language                                    |                    | translation  |                    | interpretation   |        |
|--|--------------------|--|--------------------|--|--------|
| $IA \vdash A$  | $\rightsquigarrow$ | $\circ_I \blacksquare_I A \vdash A$  | $\rightsquigarrow$ | $e_I a \leq a$   | (i)    |
| $\top \vdash I\top$                                  | $\rightsquigarrow$ | $\top \vdash \circ_I \blacksquare_I \top$  | $\rightsquigarrow$ | $\top \leq e_I \top$   | (ii)   |
| $I(A \wedge B) \vdash IA \wedge IB$                  | $\rightsquigarrow$ | $\circ_I \blacksquare_I (A \wedge B) \vdash \circ_I \blacksquare_I A \wedge \circ_I \blacksquare_I B$  | $\rightsquigarrow$ | $e_I \iota(a \wedge b) \leq e_I a \wedge e_I b$                            | (iii)  |
| $IA \wedge IB \vdash I(A \wedge B)$                  | $\rightsquigarrow$ | $\circ_I \blacksquare_I A \wedge \circ_I \blacksquare_I B \vdash \circ_I \blacksquare_I (A \wedge B)$  | $\rightsquigarrow$ | $e_I a \wedge e_I b \leq e_I \iota(a \wedge b)$                            | (iv)   |
| $IA \vdash IIA$                                      | $\rightsquigarrow$ | $\circ_I \blacksquare_I A \vdash \circ_I \blacksquare_I \circ_I \blacksquare_I A$  | $\rightsquigarrow$ | $e_I a \leq e_I e_I a$   | (v)    |
| $CIA \vdash IA$                                      | $\rightsquigarrow$ | $\circ_C \blacklozenge_C \circ_I \blacksquare_I A \vdash \circ_I \blacksquare_I A$   | $\rightsquigarrow$ | $e_C \gamma e_I a \leq e_I a$  | (vi)   |
| $I(A \vee B) \vdash IA \vee IB$                      | $\rightsquigarrow$ | $\circ_I \blacksquare_I (A \vee B) \vdash \circ_I \blacksquare_I A \vee \circ_I \blacksquare_I B$  | $\rightsquigarrow$ | $e_I \iota(a \vee b) \leq e_I a \vee e_I b$                                | (vii)  |
| $IA \vee IB \vdash I(A \vee B)$                      | $\rightsquigarrow$ | $\circ_I \blacksquare_I A \vee \circ_I \blacksquare_I B \vdash \circ_I \blacksquare_I (A \vee B)$  | $\rightsquigarrow$ | $e_I a \vee e_I b \leq e_I \iota(a \vee b)$                                | (viii) |
| $\top \vdash IA \vee \neg IA$                        | $\rightsquigarrow$ | $\top \vdash \circ_I \blacksquare_I A \vee \neg \circ_I \blacksquare_I A$  | $\rightsquigarrow$ | $\top \leq e_I a \vee \neg e_I a$  | (ix)   |
| $\frac{IA \vdash IB \quad CA \vdash CB}{A \vdash B}$ | $\rightsquigarrow$ | $\circ_I \blacksquare_I A \vdash \circ_I \blacksquare_I B$ and $\circ_C \blacklozenge_C A \vdash \circ_C \blacklozenge_C B$ implies $A \vdash B$ | $\rightsquigarrow$ | $e_I a \leq e_I b$ and $e_C \gamma a \leq e_C \gamma b$ implies $a \leq b$ | (x)    |

Since  $e_I$  and  $e_C$  are order-embeddings,  $e_I \iota(a) \leq e_I \iota(b)$  and  $e_C \gamma(a) \leq e_C \gamma(b)$  are respectively equivalent to  $\iota(a) \leq \iota(b)$  and  $\gamma(a) \leq \gamma(b)$ , and hence the quasi-inequality (x) can be equivalently rewritten as the following quasi-inequality, which defines the class  $\mathbf{HIA3}$ :

$$\iota(a) \leq \iota(b) \text{ and } \gamma(a) \leq \gamma(b) \text{ implies } a \leq b.$$

By applying adjunction, the inequalities in the antecedent can be equivalently rewritten as  $a = e_C(\gamma(b)) \wedge a$  and  $b = b \vee e_I(\iota(a))$ . Hence, the initial quasi-inequality can be equivalently rewritten as the following  $\mathcal{L}_{\text{MT}}$ -inequality:

$$a \wedge e_C \gamma(b) \leq e_I \iota(a) \vee b. \quad (3)$$

The inequality above is analytic inductive<sup>6</sup>, and hence it can be used, together with the other axioms of heterogeneous algebras, which, as observed in Section 3.3, are analytic inductive, to generate the analytic structural rules of the calculi introduced in Section 5, with a methodology analogous to the one introduced in [17]. As we will discuss in Section 6.2, the inequalities (i)-(ix) are derivable in the appropriate calculi obtained in this way.

## 5 Proper display calculi for the logics of rough algebras

In this section, we introduce proper multi-type display calculi  $\mathbf{D.A}$  for the logics associated with each class of algebras  $\mathbf{A}$  mentioned in Section 2.1. The language of these calculi has types  $\mathbf{D}$  and  $\mathbf{K}_I$  and  $\mathbf{K}_C$ , and is built up from structural and operational (aka logical) connectives. Heterogeneous connectives  $\circ_I, \circ_C, \blacksquare_I, \blacklozenge_C$  are interpreted as  $e_I, e_C, \iota, \gamma$  in heterogeneous algebras respectively. Each structural connective is denoted by decorating its corresponding logical connective with  $\wedge$  (resp.  $\vee$  or  $\neg$ ). Below, we adopt the convention that unary connectives bind more strongly than binary ones.

### 5.1 Language

- Structural and operational terms:

<sup>6</sup> Notice that applying similar steps in the single-type environment would give rise to the inequality  $A \wedge C^\# CB \leq B \vee I^\flat IB$  (where  $C^\#$  and  $I^\flat$  respectively denote the right adjoint of  $C$  and the left adjoint of  $I$ ) which is inductive but not analytic.

$$\begin{aligned}
& \mathbf{D} \begin{cases} A ::= p \mid \top \mid \perp \mid \circ_I \alpha \mid \circ_C \xi \mid \neg A \mid A \wedge A \mid A \vee A \\ X ::= A \mid \hat{\perp} \mid \hat{\top} \mid \delta_C \Pi \mid \delta_I \Gamma \mid \neg X \mid X \hat{\wedge} X \mid X \hat{\vee} X \mid X \hat{\supset} X \mid X \hat{\supset} X \end{cases} \\
& \mathbf{K}_I \begin{cases} \alpha ::= \blacksquare_I A \mid 1_I \mid 0_I \mid \alpha \cap \alpha \mid \alpha \cup \alpha \mid (\sim \alpha) \\ \Gamma ::= \alpha \mid \blacksquare_I X \mid \blacklozenge_I X \mid \delta_I \Gamma \mid \hat{1}_I \mid \Gamma \hat{\wedge} \Gamma \mid \Gamma \hat{\vee} \Gamma \mid \Gamma \hat{\supset} \Gamma \mid \Gamma \hat{\supset} \Gamma \mid (\hat{\supset} \Gamma) \end{cases} \\
& \mathbf{K}_C \begin{cases} \xi ::= \blacklozenge_C A \mid 1_C \mid 0_C \mid \xi \cap \xi \mid \xi \cup \xi \mid (\sim \xi) \\ \Pi ::= \xi \mid \blacklozenge_C X \mid \blacksquare_C X \mid \delta_C \Pi \mid \hat{1}_C \mid \Pi \hat{\wedge} \Pi \mid \Pi \hat{\vee} \Pi \mid \Pi \hat{\supset} \Pi \mid \Pi \hat{\supset} \Pi \mid (\hat{\supset} \xi) \end{cases}
\end{aligned}$$

The formulas and structures in brackets in the table above pertain to the language of D.TQBA5 and its extensions.

- Interpretation of structural connectives as their logical counterparts<sup>7</sup>

(i) structural and operational pure D-type connectives:

|                       |              |               |                |              |                 |                 |
|-----------------------|--------------|---------------|----------------|--------------|-----------------|-----------------|
| structural operations | $\hat{\top}$ | $\hat{\perp}$ | $\hat{\wedge}$ | $\hat{\vee}$ | $\hat{\supset}$ | $\hat{\supset}$ |
| logical operations    | $\top$       | $\perp$       | $\wedge$       | $\vee$       | $\supset$       | $(\supset)$     |

(ii) structural and operational pure  $\mathbf{K}_I$ -type and  $\mathbf{K}_C$ -type connectives:

|                       |             |            |              |              |                 |                 |             |            |              |              |                 |                 |
|-----------------------|-------------|------------|--------------|--------------|-----------------|-----------------|-------------|------------|--------------|--------------|-----------------|-----------------|
| structural operations | $\hat{1}_I$ | $\delta_I$ | $\hat{\cap}$ | $\hat{\cup}$ | $\hat{\supset}$ | $\hat{\supset}$ | $\hat{1}_C$ | $\delta_C$ | $\hat{\cup}$ | $\hat{\cap}$ | $\hat{\supset}$ | $\hat{\supset}$ |
| logical operations    | $1_I$       | $0_I$      | $\cap$       | $\cup$       | $(\supset)$     | $(\supset)$     | $1_C$       | $0_C$      | $\cup$       | $\cap$       | $(\supset)$     | $(\supset)$     |

(iii) As mentioned above, the language of D.TQBA5 and its extensions includes the following structural and operational pure  $\mathbf{K}_I$ -type and  $\mathbf{K}_C$ -type connectives:

|                       |              |              |
|-----------------------|--------------|--------------|
| structural operations | $\hat{\sim}$ | $\hat{\sim}$ |
| logical operations    | $\sim$       | $\sim$       |

(iv) structural and operational multi-type connectives, and their algebraic counterparts:

|                        |                                       |                    |                                       |                     |                                       |                                       |
|------------------------|---------------------------------------|--------------------|---------------------------------------|---------------------|---------------------------------------|---------------------------------------|
| types                  | $\mathbf{D} \rightarrow \mathbf{K}_I$ |                    | $\mathbf{D} \rightarrow \mathbf{K}_C$ |                     | $\mathbf{K}_I \rightarrow \mathbf{D}$ | $\mathbf{K}_C \rightarrow \mathbf{D}$ |
| structural operations  | $\blacklozenge_I$                     | $\blacksquare_I$   | $\blacklozenge_C$                     | $\blacksquare_C$    | $\delta_I$                            | $\delta_C$                            |
| logical operations     | $(\blacklozenge_I)$                   | $(\blacksquare_I)$ | $(\blacklozenge_C)$                   | $(\blacksquare_C)$  | $\circ_I$                             | $\circ_C$                             |
| algebraic counterparts | $\iota'$                              | $\iota''$          | $\gamma^{\sigma'}$                    | $\gamma^{\sigma''}$ | $e_I^{\delta}$                        | $e_C^{\delta}$                        |

## 5.2 Rules

In what follows, we will use  $X, Y, W, Z$  as structural D-variables,  $\Gamma, \Delta, \Lambda$  as structural  $\mathbf{K}_I$ -variables, and  $\Pi, \Sigma, \Omega$  as structural  $\mathbf{K}_C$ -variables. The proper multi-type display calculus D.TQBA includes the following axiom and rules:<sup>8</sup>

- Identity and Cut:

$$\begin{aligned}
& Id_D \frac{p \vdash p}{p \vdash p} \quad Cut_D \frac{X \vdash A \quad A \vdash Y}{X \vdash Y} \quad Cut_{K_I} \frac{\Gamma \vdash \alpha \quad \alpha \vdash \Delta}{\Gamma \vdash \Delta} \quad Cut_{K_C} \frac{\Pi \vdash \xi \quad \xi \vdash \Sigma}{\Pi \vdash \Sigma}
\end{aligned}$$

- Pure D-type display rules:

$$\begin{aligned}
& res_D \frac{X \hat{\wedge} Y \vdash Z}{Y \vdash X \hat{\supset} Z} \quad \frac{X \vdash Y \hat{\vee} Z}{Y \hat{\supset} X \vdash Z} res_D \quad gal_D \frac{\neg X \vdash Y}{Y \vdash \neg X} \quad \frac{X \vdash \neg Y}{Y \vdash \neg X} gal_D
\end{aligned}$$

- Pure  $\mathbf{K}_I$ -type and  $\mathbf{K}_C$ -type display rules:

$$\begin{aligned}
& res_{K_I} \frac{\Gamma \hat{\wedge} \Delta \vdash \Lambda}{\Delta \vdash \Gamma \hat{\supset} \Lambda} \quad \frac{\Gamma \vdash \Delta \hat{\cup} \Lambda}{\Delta \hat{\supset} \Gamma \vdash \Lambda} res_{K_I} \quad res_{K_C} \frac{\Pi \hat{\wedge} \Sigma \vdash \Omega}{\Sigma \vdash \Pi \hat{\supset} \Omega} \quad \frac{\Pi \vdash \Sigma \hat{\cup} \Omega}{\Sigma \hat{\supset} \Pi \vdash \Omega} res_{K_C}
\end{aligned}$$

- Multi-type display rules:

<sup>7</sup> In the synoptic table, the operational symbols which occur only at the structural level will appear between round brackets.

<sup>8</sup> For the sake of conciseness, we adopt the convention that the *position* of the name of rules – on the left or on the right of the inference line – is relevant to correctly identify each given rule. For instance, the name of each logical rule is placed on the right or on the left of the inference line, depending on whether the given rule is a right- or a left-introduction rule. Some structural rules have a double inference line, meaning that the rule is an abbreviation of two rules (one to be read top-down and the other bottom-up). In this case, we use one and the same name for both rules.

$$ad_{DK_I} \frac{\delta_I \Gamma \vdash Y}{\Gamma \vdash \blacksquare_I Y} \quad \frac{Y \vdash \delta_I \Gamma}{\blacklozenge_I Y \vdash \Gamma} ad_{DK_I} \quad ad_{DK_C} \frac{Y \vdash \delta_C \Pi}{\blacklozenge_C Y \vdash \Pi} \quad \frac{\delta_C X \vdash \Pi}{X \vdash \blacksquare_C \Pi} ad_{DK_C}$$

- Pure-type structural rules: these include standard Weakening (W), Contraction (C), Commutativity (E) and Associativity (A) in each type. We do not report on them.<sup>9</sup>

$$\frac{X \vdash Y}{\neg Y \vdash \neg X} cont \quad \uparrow \frac{X \vdash Y}{X \hat{\wedge} \uparrow \vdash Y} \quad \frac{X \vdash Y}{X \vdash Y \check{\vee} \perp} \perp \quad \hat{I}_I \frac{\Gamma \vdash \Delta}{\Gamma \hat{\wedge} \hat{I}_I \vdash \Delta} \quad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta \check{\vee} \hat{O}_I} \check{O}_I \quad \hat{I}_C \frac{\Pi \vdash \Sigma}{\Pi \hat{\wedge} \hat{I}_C \vdash \Sigma} \quad \frac{\Pi \vdash \Sigma}{\Pi \vdash \Sigma \check{\vee} \hat{O}_C} \check{O}_C$$

- Multi-type structural rules:

$$\begin{aligned} \delta_I \hat{I}_I & \frac{\uparrow \vdash Y}{\delta_I \hat{I}_I \vdash Y} \quad \frac{\Gamma \vdash \blacksquare_I \perp}{\Gamma \vdash \hat{O}_I} \blacksquare_I \uparrow \quad \blacklozenge_C \uparrow \frac{\blacklozenge_C \uparrow \vdash \Pi}{\hat{I}_C \vdash \Pi} \quad \frac{X \vdash \perp}{X \vdash \delta_C \hat{O}_C} \delta_C \check{O}_C \\ \check{O}_I & \frac{\delta_I \Gamma \vdash \delta_I \Delta}{\Gamma \vdash \Delta} \quad \frac{\delta_C \Pi \vdash \delta_C \Sigma}{\Pi \vdash \Sigma} \delta_C \quad \hat{\blacklozenge}_C \blacksquare_C \frac{\blacklozenge_C X \vdash \blacksquare_C Y}{X \vdash Y} \quad \frac{\blacklozenge_I X \vdash \blacksquare_I Y}{X \vdash Y} \hat{\blacklozenge}_I \blacksquare_I \\ IC & \frac{X \vdash \delta_I \blacksquare_I \neg Y}{X \vdash \neg \delta_C \blacklozenge_C Y} \quad \frac{X \vdash \delta_C \blacksquare_C \neg Y}{X \vdash \neg \delta_I \blacklozenge_I Y} IC \quad CI \frac{\delta_C \blacklozenge_C \neg X \vdash Y}{\neg \delta_I \blacksquare_I X \vdash Y} \quad \frac{\delta_I \blacklozenge_I \neg X \vdash Y}{\neg \delta_C \blacksquare_C X \vdash Y} CI \end{aligned}$$

- Logical rules: those for the pure-type connectives are standard and omitted; those for multi-type connectives:

$$\begin{aligned} \blacklozenge_I & \frac{\blacklozenge_I A \vdash \Gamma}{\blacklozenge_I A \vdash \Gamma} \quad \frac{\Gamma \vdash A}{\blacklozenge_I \Gamma \vdash \blacklozenge_I A} \blacklozenge_I \quad \circ_I \frac{\delta_I \alpha \vdash X}{\circ_I \alpha \vdash X} \quad \blacksquare_I \frac{A \vdash \Gamma}{\blacksquare_I A \vdash \blacksquare_I \Gamma} \quad \frac{\Gamma \vdash \blacksquare_I A}{\Gamma \vdash \blacksquare_I A} \blacksquare_I \quad \circ_C \frac{\delta_C \xi \vdash X}{\circ_C \xi \vdash X} \\ \blacklozenge_C & \frac{\blacklozenge_C A \vdash \Pi}{\blacklozenge_C A \vdash \Pi} \quad \frac{\Pi \vdash A}{\blacklozenge_C \Pi \vdash \blacklozenge_C A} \blacklozenge_C \quad \circ_I \frac{X \vdash \delta_I \alpha}{X \vdash \circ_I \alpha} \circ_I \quad \blacksquare_C \frac{A \vdash \Pi}{\blacksquare_C A \vdash \blacksquare_C \Pi} \quad \frac{\Pi \vdash \blacksquare_C A}{\Pi \vdash \blacksquare_C A} \blacksquare_C \quad \circ_C \frac{X \vdash \delta_C \xi}{X \vdash \circ_C \xi} \circ_C \end{aligned}$$

The calculus D.TQBA5 is obtained by adding the following rules to D.TQBA:

- Display rules:

$$gal_{K_I} \frac{\sim \Gamma \vdash \Delta}{\sim \Delta \vdash \Gamma} \quad \frac{\Gamma \vdash \sim \Delta}{\Delta \vdash \sim \Gamma} gal_{K_I} \quad gal_{K_C} \frac{\sim \Pi \vdash \Sigma}{\sim \Sigma \vdash \Pi} \quad \frac{\Pi \vdash \sim \Sigma}{\Sigma \vdash \sim \Pi} gal_{K_C}$$

- Pure  $K_I$ -type and  $K_C$ -type structural rules:

$$cont_I \frac{\Gamma \vdash \Delta}{\sim \Delta \vdash \sim \Gamma} \quad \frac{\Pi \vdash \Sigma}{\sim \Sigma \vdash \sim \Pi} cont_C$$

- Multi-type structural rules:

$$\frac{X \vdash \delta_I \blacksquare_I Y}{X \vdash \delta_C \blacksquare_C Y} \delta_I \blacksquare_I \quad \frac{X \vdash \delta_I \sim \Gamma}{X \vdash \sim \delta_I \Gamma} \delta_I \sim \quad \frac{X \vdash \delta_C \sim \Pi}{X \vdash \sim \delta_C \Pi} \delta_C \sim$$

- Logical rules for  $\sim$  and  $-$ :

$$\sim \frac{\sim \alpha \vdash \Gamma}{\sim \alpha \vdash \Gamma} \quad \frac{\Gamma \vdash \sim \alpha}{\Gamma \vdash \sim \alpha} \sim \quad - \frac{\sim \alpha \vdash \xi}{-\alpha \vdash \xi} \quad \frac{\Pi \vdash \sim \xi}{\Pi \vdash -\xi} -$$

The proper display calculi for the axiomatic extensions of H.TQBA5 discussed in Section 2.2 are obtained by adding the analytic structural rules indicated in the following table to the calculus D.TQBA5.

| Name of logic | Display Calculus | Rules  |
|---------------|------------------|--|
| H.IA1         | D.IA1            | $cgri \frac{\Gamma \hat{\wedge} \Delta \vdash \Lambda}{\Delta \vdash \sim \Gamma \check{\vee} \Lambda} \quad \frac{\Pi \hat{\wedge} \Sigma \vdash \Omega}{\Sigma \vdash \sim \Pi \check{\vee} \Omega} cgri$  |
| H.IA2         | D.IA2            | $\hat{\blacklozenge}_C \hat{\Pi} \frac{\blacklozenge_C (X \hat{\wedge} Y) \vdash \Pi}{\blacklozenge_C X \hat{\wedge} \blacklozenge_C Y \vdash \Pi} \quad \frac{\Gamma \vdash \blacksquare_I (X \check{\vee} Y)}{\Gamma \vdash \blacksquare_I X \check{\vee} \blacksquare_I Y} \blacksquare_I \check{\vee}$ |
| H.IA3         | D.IA3            | $ia3 \frac{X \vdash Y \quad W \vdash Z}{X \hat{\wedge} \delta_C \blacklozenge_C W \vdash \delta_I \blacksquare_I Y \check{\vee} Z}$  |
| H.PRA         | D.PRA            | $cgri, \hat{\blacklozenge}_C \hat{\Pi}, \blacksquare_I \check{\vee}, ia3.$   |

<sup>9</sup> In what follows, we use subscripts (indicating the type) to distinguish the rules for lattice operators in different type rules.

## 6 Properties

Throughout this section, we let  $\mathbf{H}$  denote any of the logics defined in Section 2.2; let  $\mathbf{A}$  and  $\mathbf{HA}$  denote its corresponding class of single-type and heterogeneous algebras, respectively, and let  $\mathbf{D.A}$  denote the display calculus for  $\mathbf{H}$ . The verification of the properties of every  $\mathbf{D.A}$  follows very closely those of analogous properties of other calculi designed using the general methodology of multi-type calculi (cf. e.g. [19,15,18,16]). For this reason, we only sketch them.

### 6.1 Soundness on perfect $\mathbf{HA}$ algebras

In the present subsection, we outline the verification of the soundness of the rules of  $\mathbf{D.A}$  w.r.t. the semantics of *perfect* elements of  $\mathbf{HA}$  (see Definition 3.6). The first step consists in interpreting structural symbols as logical symbols according to their (precedent or succedent) position, as indicated at the beginning of Section 5. This makes it possible to interpret sequents as inequalities, and rules as quasi-inequalities. For example, the rules on the left-hand side below are interpreted as the quasi-inequalities on the right-hand side:

$$\frac{X \vdash Y \quad W \vdash Z}{X \hat{\wedge} \hat{\circ}_C \hat{\blacklozenge}_C W \vdash \hat{\circ}_I \hat{\blacksquare}_I Y \check{\vee} Z} \text{ia3} \quad \rightsquigarrow \quad \forall a \forall b \forall c \forall d [(a \leq c \ \& \ b \leq d) \Rightarrow a \wedge e_C \gamma(b) \leq e_I(c) \vee d].$$

The verification of the soundness of the rules of  $\mathbf{D.A}$  then consists in verifying the validity of their corresponding quasi-inequalities in any perfect element of  $\mathbf{HA}$ . The verification of the soundness of pure-type rules and of the introduction rules following this procedure is routine, and is omitted. The soundness of the rule *ia3* above is verified by the following ALBA-reduction, which shows that the quasi-inequality above is equivalent to the inequality (3), which, as discussed in Section 4, is valid on every  $\mathbf{H} \in \mathbf{HIA3}$ .

$$\begin{aligned} & \forall p \forall q [p \wedge e_C \gamma(q) \leq e_I(p) \vee q] \\ \text{iff} & \quad \forall p \forall q \forall a \forall b \forall c \forall d [(a \leq p \ \& \ b \leq q \ \& \ p \leq c \ \& \ q \leq d) \Rightarrow a \wedge e_C \gamma(b) \leq e_I(c) \vee d] \\ \text{iff} & \quad \forall a \forall b \forall c \forall d [(a \leq c \ \& \ b \leq d) \Rightarrow a \wedge e_C \gamma(b) \leq e_I(c) \vee d]. \end{aligned}$$

The validity of the quasi-inequalities corresponding to the remaining structural rules follows in an analogous way.

### 6.2 Completeness

Let  $A^\tau \vdash B^\tau$  be the translation of any  $\mathcal{L}$ -sequent  $A \vdash B$  into the language of  $\mathbf{D.A}$  which composes the translation introduced in Section 4 with the correspondence between algebraic operations and logical connectives indicated in table (iv) of Section 5.1.

**Proposition 6.1** *For every  $\mathbf{H}$ -derivable sequent  $A \vdash B$ , the sequent  $A^\tau \vdash B^\tau$  is derivable in  $\mathbf{D.A}$ .*

We only show the derivations of rule T8, axioms T6, and one direction of axiom T7.

T8.  $IA \vdash IB$  and  $CA \vdash CB$  imply  $A \vdash B \rightsquigarrow \circ_I \blacksquare_I A \vdash \circ_I \blacksquare_I B$  and  $\circ_C \blacklozenge_C A \vdash \circ_C \blacklozenge_C B$  imply  $A \vdash B$





## 6.4 Cut elimination and subformula property

In the present section, we briefly sketch the proof of cut elimination and subformula property for D.A. As hinted to earlier on, proper display calculi have been designed so that the cut elimination and subformula property can be inferred from a meta-theorem, following the strategy introduced by Belnap for display calculi. The meta-theorem to which we will appeal for each D.A was proved in [9].

All conditions in [9, Theorem 4.1] except  $C'_8$  are readily satisfied by inspecting the rules. Condition  $C'_8$  requires to check that reduction steps are available for every application of the cut rule in which both cut-formulas are principal, which either remove the original cut altogether or replace it by one or more cuts on formulas of strictly lower complexity. In what follows, we only show  $C'_8$  for the unary connectives.

### Pure D-type connectives:

$$\frac{\frac{\frac{\vdots \pi_1}{X \vdash \neg A} \quad \neg \quad \frac{\vdots \pi_2}{\neg A \vdash Y}}{X \vdash \neg A} \quad \neg A \vdash Y}{X \vdash Y} \text{Cut}_D \rightsquigarrow \frac{\frac{\frac{\vdots \pi_2}{\neg A \vdash Y} \quad \text{gal}_D \quad \frac{\vdots \pi_1}{X \vdash \neg A}}{\neg Y \vdash A} \quad \frac{X \vdash \neg A}{A \vdash \neg X} \text{gal}_D}{\neg Y \vdash \neg X} \text{Cut}_D \rightsquigarrow \frac{\neg Y \vdash \neg X}{X \vdash Y} \text{cont}$$

The cases for  $\sim \alpha$  and  $-\xi$  of D.TQBA5 and its extensions are standard and similar to the one above.

### Multi-type connectives:

$$\frac{\frac{\frac{\vdots \pi_1}{\Gamma \vdash \check{\neg}_I A} \quad \check{\neg}_I \quad \frac{\vdots \pi_2}{A \vdash Y}}{\Gamma \vdash \check{\neg}_I A} \quad \check{\neg}_I \quad \frac{A \vdash Y}{\check{\neg}_I A \vdash \check{\neg}_I Y}}{\Gamma \vdash \check{\neg}_I Y} \rightsquigarrow \frac{\frac{\frac{\vdots \pi_1}{\Gamma \vdash \check{\neg}_I A} \quad \text{ad}_{DK_I}}{\check{\neg}_I \Gamma \vdash A} \quad \frac{\vdots \pi_2}{A \vdash Y}}{\text{ad}_{DK_I} \quad \check{\neg}_I \Gamma \vdash Y} \rightsquigarrow \frac{\check{\neg}_I \Gamma \vdash Y}{\Gamma \vdash \check{\neg}_I Y}$$

$$\frac{\frac{\frac{\vdots \pi_1}{X \vdash \check{\neg}_I \alpha} \quad \circ_I \quad \frac{\vdots \pi_2}{\check{\neg}_I \alpha \vdash Y}}{X \vdash \check{\neg}_I \alpha} \quad \circ_I \quad \frac{\check{\neg}_I \alpha \vdash Y}{\circ_I \alpha \vdash Y}}{X \vdash Y} \rightsquigarrow \frac{\frac{\frac{\vdots \pi_1}{X \vdash \check{\neg}_I \alpha} \quad \text{ad}_{DK_I}}{\hat{\neg}_I X \vdash \alpha} \quad \text{ad}_{DK_I} \quad \frac{\vdots \pi_2}{\check{\neg}_I \alpha \vdash Y}}{\hat{\neg}_I X \vdash \check{\neg}_I Y} \quad \hat{\neg}_I \check{\neg}_I$$

The cases for  $\blacklozenge C A$  and  $\circ_C \xi$  are analogous.

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