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The space of maximal elements in a compact domain

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Abstract

In this paper we try to improve the current state of understanding concerning models of spaces with Scott domains. The main result given is that any developable space which has a model by a Scott domain must be Čech-complete. An important consequence is that any metric space homeomorphic to the maximal elements of a Scott domain must be completely metrizable.

1 Introduction

By now it is well-known that the maximal elements of a countably based Scott domain are Polish [7]. More recently, it was shown that all Polish spaces may be represented in this manner [2]. Thus, a space has a model by a countably based Scott domain iff it is a second countable Čech-complete space. (For a survey on much of what is known in this area see [10]).

Locally compact Hausdorff spaces also have models by Scott domains, another case when the space at the top is Čech-complete. In addition, a space modeled by a Scott domain comes embedded in a natural compact Hausdorff space, the domain itself in its Lawson topology, which is something like the way a Čech-complete space is always a G_δ subset of its Stone-Čech compactification.

However, in the author's opinion, all of this is not enough to conclude that the spaces with models by Scott domains are the Čech-complete spaces. There are *many* notions of topological completeness which capture the nice spaces from analysis that in general are not equivalent. How are we to tell which of these capture the notion of completeness offered by the top of a Scott domain? At this point, we do not have enough intuition about arbitrary Scott domains to make such a conjecture.

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But if we return to the case of a countably based Scott domain, we see that there is another property we have failed to take into account. Any space modeled by a countably based Scott domain is metrizable and hence *developable.* This is a subtle topological property that the maximal elements of a Scott domain need not have in general. The author conjectures that a developable space is homeomorphic to the maximal elements of a Scott domain iff it is Čech-complete. In this paper, we establish one direction of this conjecture by proving that a developable space with a model by a Scott domain is Čech-complete.

2 Background

2.1 Domain theory

A *poset* is a partially ordered set [1].

Definition 2.1 Let $(P, \sqsubseteq)$ be a partially ordered set. A nonempty subset $S \subseteq P$ is *directed* if $(\forall x, y \in S)(\exists z \in S)\, x, y \sqsubseteq z$. The *supremum* of a subset $S \subseteq P$ is the least of all its upper bounds provided it exists. This is written $\bigsqcup S$. A *dcpo* is a poset in which every directed subset has a supremum.

Definition 2.2 For a subset X of a dcpo D, set

$$\uparrow\!X := \{y \in D : (\exists x \in X)\, x \sqsubseteq y\} \ \ \& \ \ \downarrow\!X := \{y \in D : (\exists x \in X)\, y \sqsubseteq x\}.$$

We write $\uparrow\!x = \uparrow\!\{x\}$ and $\downarrow\!x = \downarrow\!\{x\}$ for elements $x \in X$. The set of *maximal elements* in a dcpo D is $\max D = \{x \in D : \uparrow\!x = \{x\}\}$.

By the Hausdorff maximality principle, every dcpo has at least one maximal element.

Definition 2.3 In a dcpo $(D, \sqsubseteq)$, $a \ll x$ iff for all directed subsets $S \subseteq D$, $x \sqsubseteq \bigsqcup S \Rightarrow (\exists s \in S)\, a \sqsubseteq s$. We set $\twoheaddownarrow\!x = \{a \in D : a \ll x\}$. An element $x \in D$ is *compact* if $x \ll x$. The set of compact elements in D is written $K(D)$.

Definition 2.4 A subset B of a dcpo D is a *basis* for D if $B \cap \twoheaddownarrow\!x$ contains a directed subset with supremum x, for each $x \in D$.

Definition 2.5 A dcpo D is *continuous* if it has a basis. A *domain* is a continuous dcpo.

Definition 2.6 A dcpo is *algebraic* if its compact elements form a basis. A dcpo is *ω-continuous* if it has a countable basis.

The order-theoretic structure of a domain allows for the derivation of several intrinsically defined topologies. The topology of interest in the study of models is the Scott topology.

Definition 2.7 A subset U of a dcpo D is *Scott open* if

(i) U is an upper set: $x \in U \ \&\ x \sqsubseteq y \Rightarrow y \in U$, and

(ii) U is inaccessible by directed suprema: For every directed $S \subseteq D$,

$$\bigsqcup S \in U \Rightarrow S \cap U \neq \emptyset.$$

The collection of all Scott open sets on D is called the Scott topology. It is denoted σ_D.

A basis for the Scott topology on a *domain* is the collection $\{\twoheaduparrow x : x \in D\}$, where $\twoheaduparrow x = \{y \in D : x \ll y\}$. Unless explicitly stated otherwise, all topological statements about dcpo's are made with respect to the Scott topology.

Proposition 2.8 *A function $f : D \to E$ between dcpo's is continuous iff*

(i) *f is monotone: $x \sqsubseteq y \Rightarrow f(x) \sqsubseteq f(y)$.*

(ii) *f preserves directed suprema: For every directed $S \subseteq D$,*

$$f(\bigsqcup S) = \bigsqcup f(S).$$

Definition 2.9 The *Lawson topology* on a continuous dcpo D has as a basis all sets of the form $\twoheaduparrow x \setminus \uparrow F$ where $x \in D$ and $F \subseteq D$ is finite.

Proposition 2.10 (Jung [6]) *The Lawson topology on a continuous dcpo is compact iff it is Scott compact and the intersection of any two Scott compact upper sets is Scott compact.*

Definition 2.11 A *Scott domain* is a continuous dcpo with least element $\perp$ in which each pair of elements bounded from above has a supremum.

Notice that we have not required Scott domains to be ω-algebraic.

Proposition 2.12 *Every Scott domain has compact Lawson topology.*

On *Scott domains* we have an interest in the Lawson topology because it coincides with the Scott topology on the set of maximal elements.

Proposition 2.13 *If D is a domain with compact Lawson topology, then the relative Scott and Lawson topologies on the set of maximal elements agree.*

2.2 Models of spaces

Definition 2.14 A *model* of a space X is a continuous dcpo D together with a homeomorphism

$$\phi : X \to \max D$$

where $\max D$ carries its relative Scott topology inherited from D. If in addition the domain D is ω-continuous, then $(D, \phi : X \simeq \max D)$ is called a *countably based model.*

The observation that the space of maximal elements in a countably based Scott domain is Polish is due to Lawson [7]. However, there is now a more general result available with a much simpler proof that we will use instead.

Theorem 2.15 (Martin [12]) *The maximal elements of an ω-continuous dcpo are regular iff Polish.*

There is no simple way known to explain the proof of the next result.

Theorem 2.16 (Ciesielski, Flagg & Kopperman [2]) *Every Polish space is homeomorphic to the maximal elements of an ω-continuous Scott domain.*

Thus, a space is Polish iff it has a countably based model by a Scott domain.

Example 2.17 A model of the Cantor set. The collection of functions

$$\Sigma^\infty = \{\, s \mid s : \{1, \ldots, n\} \to \{0, 1\}, 0 \le n \le \infty \,\}$$

is also an ω-algebraic dcpo under the extension order

$$s \sqsubseteq t \Leftrightarrow |s| \le |t| \;\&\; (\,\forall\, 1 \le i \le |s|\,)\; s(i) = t(i),$$

where $|s|$ is written for the cardinality of $\operatorname{dom} s$. The supremum of a directed set $S \subseteq \Sigma^\infty$ is $\bigcup S$, while the approximation relation is

$$s \ll t \Leftrightarrow s \sqsubseteq t \;\&\; |s| < \infty.$$

The extension order in this special case is usually called the *prefix* order. The elements $s \in \Sigma^\infty$ are called *strings* over $\{0, 1\}$. The quantity $|s|$ is called the *length* of a string s. The *empty string* ε is the unique string with length zero. It is the least element $\perp$ of Σ^∞.

We call Σ^∞ the *Cantor set model* since $\max \Sigma^\infty = \{s : |s| = \infty\}$ is homeomorphic to the Cantor set.

More generally, any locally compact Hausdorff space has a model by a Scott domain.

Example 2.18 A model for locally compact Hausdorff spaces. If X is a locally compact Hausdorff space, its *upper space* [5]

$$\mathbf{U}X = \{\emptyset \neq K \subseteq X : K \textit{ is compact}\}$$

ordered under reverse inclusion

$$A \sqsubseteq B \Leftrightarrow B \subseteq A$$

is a continuous dcpo. The supremum of a directed set $S \subseteq \mathbf{U}X$ is $\bigcap S$ and the approximation relation is $A \ll B \Leftrightarrow B \subseteq \operatorname{int}(A)$. The *upper space* is a model of X because $\max \mathbf{U}X = \{\{x\} : x \in X\} \simeq X$.

2.3 Topology

The following results can all be found in [3].

Definition 2.19 A topological space X is *Tychonoff* if it is a subspace of a compact Hausdorff space.

Tychonoff spaces are sometimes also called *completely regular*. They are the spaces which have compactifications. The largest of all such compactifications βX is known as the Stone-Čech compactification.

Definition 2.20 A topological space X is *Čech-complete* if it is Tychonoff and a G_δ in its Stone-Čech compactification βX.

A locally compact Hausdorff space is Čech-complete since it is an open subset of a compact Hausdorff space. Complete metric spaces provide another example.

Theorem 2.21 *A topological space is completely metrizable iff it is a Čech-complete metric space.*

There is the following characterization of Čech-completeness.

Theorem 2.22 *A Tychonoff space X is Čech-complete iff there is a countable family $\{\mathcal{R}_i\}_{i=1}^{\infty}$ of open covers of X with the following property: For any family $\mathcal{F}$ of closed sets such that*

(i) *$\mathcal{F}$ has the finite intersection property, and*

(ii) $(\forall i)(\exists F_i \in \mathcal{F})(\exists U_i \in \mathcal{R}_i)\ F_i \subseteq U_i$

we have $\bigcap \mathcal{F} \neq \emptyset$.

Definition 2.23 A sequence of open covers $\{\mathcal{U}_n\}_{n=1}^{\infty}$ of a space X is called a *development* provided that $\{\mathrm{St}(x, \mathcal{U}_n) : n \geq 1\}$ is a basis at x where

$$\mathrm{St}(x, \mathcal{U}_n) = \bigcup\{A : x \in A \in \mathcal{U}_n\}.$$

A space with a development is termed *developable.* A *Moore space* is a regular space with a development.

Proposition 2.24 *Every metric space is developable.*

3 Models by Scott domains

We review results known about Scott domains in general.

Lemma 3.1 *The space of maximal elements in a Scott domain is Tychonoff.*

Proposition 3.2 (Martin [9]) *A space has a model by a Scott domain iff it has a model by an FS-domain iff it has a model by a Lawson compact domain.*

In order to really have a chance at solving the model problem for Scott domains, we have to consider something simpler that can help us build intuition. First, an observation.

Lemma 3.3 (Martin [12]) *If D is an ω-continuous dcpo and* $\max D$ *is regular, then there is a decreasing sequence (U_n) of Scott open sets in D such that*

$$\max D = \bigcap_{n \geq 1} U_n.$$

Thus, if D is a countably based Scott domain, then the set of maximal elements is a G_δ subset with respect to the Scott topology (this property of Scott domains was first pointed out by Lawson in [7]).

Then it makes sense to consider Scott domains for which the maximal elements form a G_δ set. The rest of the results in this section can be found in the fifth chapter of [8].

Theorem 3.4 (Martin [8]) *For a space $X \subseteq \max D$ embedded as a G_δ subset of a Scott domain D, the notions of paracompactness, metrizability and complete metrizability are all equivalent.*

Paracompactness is that topological idea which allows us to extend the local to the global within a topological space. For example, a locally metrizable paracompact space is metrizable (think of a manifold). Our two favorite examples of paracompacta are metric spaces and compact Hausdorff spaces.

Corollary 3.5 *A compact Hausdorff space is metrizable iff it is a G_δ subset of a Scott domain.*

First, the nicest of all paracompact spaces may sit at the top of a Scott domain without being a G_δ.

Example 3.6 Let $X = [0,1]^2$ with the dictionary order. Then X is a first countable, hereditarily separable, compact Hausdorff space which is not metrizable. Its upper space $\mathbf{U}X$ is a Scott domain whose maximal elements

$$X \simeq \max \mathbf{U}X$$

cannot form a G_δ set with respect to the Scott topology: By the previous corollary, if $\max \mathbf{U}X$ were a G_δ in $\mathbf{U}X$, X would have to be metrizable.

On the other hand, being a G_δ in a Scott domain does not even imply the space is normal, let alone paracompact! We give an example of this next, but first have to define a few terms.

Definition 3.7 The *μ topology* on a continuous dcpo D has as a basis all sets of the form $\twoheaduparrow x \cap \downarrow y$ for $x, y \in D$.

The domain of nonnegative reals in their dual order is denoted $[0,\infty)^*$.

Definition 3.8 A Scott continuous map $\mu : D \to [0,\infty)^*$ on a continuous dcpo D *induces the Scott topology near* $X \subseteq D$ if for all $x \in X$ and all sequences (x_n) with $x_n \ll x$,

$$\lim_{n\to\infty} \mu x_n = \mu x \Rightarrow \bigsqcup_{n\in\mathbb{N}} x_n = x,$$

and this supremum is directed. This is written $\mu \to \sigma_X$.

The definition of $\mu \to \sigma_X$ has several equivalent formulations [8].

Definition 3.9 A *measurement* on a domain D is a Scott continuous map $\mu : D \to [0,\infty)^*$ with $\mu \to \sigma_{\ker \mu}$ where $\ker \mu = \{x \in D : \mu x = 0\}$.

A simple introduction to measurement and its basic applications in computation is given in [11], where a proof of the following can be found.

Lemma 3.10 (Martin [8]) *If μ is a measurement on a domain D, then* $\ker\mu \subseteq \max D$ *is a G_δ subset of D.*

Now the example: A G_δ subset of an algebraic Scott domain which is *not normal.*

Example 3.11 Let X be the Cantor set model Σ^∞ regarded as a space in its μ topology. The space X is locally compact Hausdorff and zero-dimensional so its upper space $\mathbf{U}X$ is an algebraic Scott domain with $X \simeq \mathbf{U}X$.

In [8] it is shown that $\mathbf{U}X$ admits a measurement $\lambda : \mathbf{U}X \to [0,\infty)^*$ with $\ker\lambda = \max \mathbf{U}X$. However, X is a first countable, separable Hausdorff space which is *not normal*: It contains an uncountable closed relatively discrete set, namely, the set $\max\Sigma^\infty$!

This led the author to ask the following: If the kernel of a measurement on a domain is normal, is it metrizable? It is now believed that this question is equivalent to the Moore space problem in topology. (In fact, for continuous posets, it is.)

The last example also has another telling property: The set of finite strings form a dense subset of X which is completely metrizable in its relative topology. It is interesting that this always happens.

Theorem 3.12 (Martin [8]) *If D is a Scott domain and $X \subseteq \max D$ is a G_δ subset of D, then there exists $Y \subseteq X$ such that*

(i) *Y is dense in X.*

(ii) *Y is a G_δ subset of D.*

(iii) *Y is completely metrizable in its relative Scott topology.*

Then G_δ subsets of Scott domains are spaces in which a completely metrizable space has been densely embedded.

4 Completeness

Another approach that one can take to studying arbitrary Scott domains is to assume that the space of maximal elements is developable, since this is also a property that always holds in the countably based case.

Theorem 4.1 *Let D be a Scott domain and $X = \max D$ denote the space of maximal elements. If X is developable, then X is Čech-complete.*

Proof. Let $\{\mathcal{U}_n\}_{n=1}^\infty$ be a development for X. Define a sequence of open covers $\{\mathcal{R}_n\}_{n=1}^\infty$ by

$$\mathcal{R}_n = \{\, \twoheaduparrow b \cap X : (\exists\, U \in \mathcal{U}_n)\, \twoheaduparrow b \cap X \subseteq \uparrow b \cap X \subseteq U \,\},$$

which is possible using interpolation and the fact that the sets $\twoheaduparrow b \cap X$ are a basis for the Scott topology on X.

We now use Theorem 2.22 to prove that X is Čech-complete. Let $\mathcal{F}$ be a collection of closed subsets of X with the finite intersection property such that

$$(\forall i \geq 1)(\exists F_i \in \mathcal{F})(\exists \twoheaduparrow b_i \cap X \in \mathcal{R}_i)\, F_i \subseteq \twoheaduparrow b_i \cap X.$$

We want to show that the elements of $\mathcal{F}$ have nonempty intersection.

Let $C_n = \bigcap_{i=1}^n F_i$. This is a decreasing sequence of nonempty closed sets in X, as $\mathcal{F}$ has the finite intersection property. Let $x_n \in C_n$ for $n \geq 1$. We claim that this sequence converges. First,

$$(\forall n \geq 1)\, x_n \in C_n \subseteq \twoheaduparrow b_n \cap X \subseteq \uparrow b_n \cap X \subseteq U_n,$$

where $U_n \in \mathcal{U}_n$.

Consider the sets $K_n = \bigcap_{i=1}^n \uparrow b_i$, for $n \geq 1$. Each is nonempty since $x_n \in K_n$. Further, D is a *Scott domain*, so we can write $K_n = \uparrow y_n$, where

$$y_n = \bigsqcup_{i=1}^{n} b_i.$$

But (y_n) is increasing so it has a supremum. Thus, $\bigcap K_n = \uparrow \bigsqcup y_n \neq \emptyset$. In particular, there is a maximal element $x \in X$ contained in $\bigcap K_n$.

We claim that $x_n \to x$. First, by definition,

$$x \in \uparrow b_n \cap X \subseteq U_n \;\&\; x \in U_n \subseteq \mathrm{st}(x, \mathcal{U}_n),$$

which means $\twoheaduparrow b_n \cap X \subseteq \uparrow b_n \cap X \subseteq \mathrm{st}(x, \mathcal{U}_n)$, for all $n \geq 1$. Now let $b \ll x$. Then by developability,

$$(\exists n \geq 1)\, x \in \mathrm{st}(x, \mathcal{U}_n) \subseteq \twoheaduparrow b \cap X.$$

However, the sets C_n are decreasing, so

$$(\forall k \geq n)\, x_k \in C_n \subseteq \twoheaduparrow b_n \cap X \subseteq \mathrm{st}(x, \mathcal{U}_n) \subseteq \twoheaduparrow b \cap X.$$

Thus, $x_n \to x$.

But since the sets C_n are closed and each contains a tail of (x_n), we have $x \in C_n$ for all $n \geq 1$. Then $x \in F_n$ for all $n \geq 1$. In fact, because $\{\mathrm{st}(x, \mathcal{U}_n) : n \geq 1\}$ is a basis at x,

$$\{x\} \subseteq \bigcap_{n \geq 1} F_n \subseteq \bigcap_{n \geq 1} \mathrm{st}(x, \mathcal{U}_n) = \{x\},$$

Hence, $\bigcap_{n \geq 1} F_n = \{x\}$.

Now suppose we have *any* $F \in \mathcal{F}$. Then $\{F \cap F_n : n \geq 1\}$ is just another sequence of nonempty closed sets satisfying $F \cap F_n \subseteq \twoheaduparrow b_n \cap X$. The very same argument used above proves that their intersection must be nonempty. But then

$$\emptyset \neq \bigcap_{n \geq 1} F \cap F_n = F \cap \bigcap_{n \geq 1} F_n = F \cap \{x\},$$

which puts $x \in F$. Then $x \in \bigcap \mathcal{F} \neq \emptyset$, finishing the proof. □

In view of Prop. 3.2, the same theorem holds for Lawson compact domains and FS-domains. Likewise for the other results of this section.

Corollary 4.2 *If the maximal elements of a Scott domain are metrizable, they are completely metrizable.*

While we don't have a proof that *all* complete metric spaces have models by Scott domains, we do have a simple and elegant construction for *zero-dimensional* metric spaces. The details are as follows.

Example 4.3 Let X be completely metrizable and zero-dimensional. Then its topology is given by a bounded and complete *ultrametric* $d : X^2 \to [0,1]$. In [4], it is shown that ordering the sets

$$\mathbf{A}X = \{C_\varepsilon(x) : x \in X \ \& \ \varepsilon = 1/2^n, n \in \mathbb{N}^\infty\}$$

under reverse inclusion, where $C_\varepsilon(x) = \{y \in X : d(x,y) \leq \varepsilon\}$ and $1/2^\infty = 0$, gives rise to an algebraic Scott domain with $\max \mathbf{A}X \simeq X$.

Corollary 4.4 *A metric space has a model by an algebraic Scott domain iff it is completely metrizable and zero-dimensional.*

5 Conclusion

The author believes that modeling complete metric spaces with Scott domains is a straightforward mathematics problem that has probably already been solved (in different terminology). However, it seems unlikely that a simple and elegant model outside the zero-dimensional case will ever be found. I am waiting to be proven wrong.

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