

Meet Precontinuous Posets

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Abstract

In this paper, we introduce the concept of meet precontinuous posets, a generalization of meet continuous lattices to posets. The main results are: (1) A poset P is meet precontinuous iff its normal completion is a meet continuous lattice iff a certain system $\gamma(P)$ which is, in the case of complete lattices, the lattice of all Scott-closed sets is a complete Heyting algebra; (2) A poset P is precontinuous iff the way below relation is the smallest approximating auxiliary relation iff P is meet precontinuous and there is a smallest approximating auxiliary relation on P . Finally, given a poset P and an auxiliary relation on P , we characterize those join-dense subsets of P whose way-below relation agrees with the given auxiliary relation.

Keywords: Precontinuous poset, meet precontinuous poset, meet continuous lattice, normal completion, auxiliary relation.

1 Introduction

Domain theory was introduced by Scott in the late 60s for the denotational semantics of programming languages. It provides the mathematical foundation for the design, definition, and implementation of programming languages, and for systems for the specification and verification of programs. From both the computer science side and the purely mathematical side, one of important aspects of domain theory is to carry as much as possible of the theory of continuous domains to as general an ordered structure as possible. Due to their strong connections to computer science, general topology and topological algebra, continuous domains have been

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extensively studied by people coming from various areas [1,7]. There are several different equivalent ways to define continuous domains, the most straightforward one is formulated by using the way below relation. We say x way below y , denoted by $x \ll y$, iff $x \in \Downarrow y := \bigcap \{\downarrow D : D \text{ is directed and } y \leq \vee D\}$. A dcpo P is called a *continuous domain* if each of the sets $\Downarrow y$ is directed and has join y . This definition has turned out to be very fruitful for many categorical and topological developments generalizing the theory of continuous lattices, but it is still rather restrictive, taking into consideration only dcpos. So, there are more and more occasions to study posets which miss suprema of directed sets (see [5,10,12,15,18]). Though continuous posets inherit some good properties of continuous domains, they also fail to satisfy some hopeful properties, for example, they are not *completion-invariant* [3], i.e. the normal completion of a continuous poset is not always a continuous lattice (see [2]). In [2], Ern  introduced a new way below relation and the concept of precontinuous posets by taking Frink ideals [6] instead of directed lower sets, which are not restricted to dcpos and has the desired completion-invariant property.

A meet continuous lattice is a complete lattice in which binary meets distribute over directed suprema (see [5]). This algebraic notion has a purely topological characterization that can be generalized to the setting of directed complete partial orders (dcpos) in [7,9]: A dcpo P is *meet continuous* if for any $x \in P$ and any directed subset D with $x \leq \sup D$, one has $x \in cl_{\sigma(P)}(\downarrow x \cap \downarrow D)$, where $cl_{\sigma(P)}(\downarrow x \cap \downarrow D)$ is the Scott closure of the set $\downarrow x \cap \downarrow D$. In [11], Mao and Xu generalized the concept of meet continuous dcpos to general posets. Though meet continuous posets inherit some good properties of meet continuous dcpos, they fail to be *completion-invariant*. A simple counterexample will be given in Section 2.

In this paper, we introduce the concept of meet precontinuous posets. It is proved that a poset P is meet precontinuous iff its normal completion is a meet continuous lattice; Considering the operator Γ and the system $\gamma(P)$ in [2], we show that a poset P is meet precontinuous iff the system $\gamma(P)$ is a complete Heyting algebra, and P is a precontinuous poset and \ll has the interpolation property iff P is a meet precontinuous poset and for all $x \in P$, $U \in \gamma(P)^c$ which is the family of complements of elements of $\gamma(P)$, $x \in U$ implies that there are finite $F \subseteq P$ and $V \in \gamma(P)^c$ such that $x \in V \subseteq \uparrow F \subseteq U$ and Γ is idempotent. We also locate the way below relation within auxiliary relations. It is proved that a poset P is precontinuous iff the way below relation is the smallest approximating auxiliary relation iff P is meet precontinuous and there is a smallest approximating auxiliary relation on P . Finally, we show how to construct all precontinuous posets whose way-below relation agrees with the given auxiliary relation.

2 Preliminaries

Let P be a poset. For all $x \in P$, $A \subseteq P$, let $\uparrow x = \{y \in P : x \leq y\}$ and $\uparrow A = \bigcup_{a \in A} \uparrow a$; $\downarrow x$ and $\downarrow A$ are defined dually. Let $D(P) = \{A \subseteq P : A = \downarrow A\}$. A^\uparrow and A^\downarrow denote the sets of all upper and lower bounds of A , respectively. Let $A^\delta = (A^\uparrow)^\downarrow$ and $\delta(P) = \{A^\delta : A \subseteq P\}$. $(\delta(P), \subseteq)$ is called the normal completion,

or the Dedekind-MacNeille completion of P . By a *completion-invariant property* we mean a property that holds for P iff it holds for any normal completion of P . A subset $I \subseteq P$ is called a Frink ideal if for all finite subsets $Z \subseteq I$, we have $Z^\delta \subseteq I$. Let $\text{Fid}(P)$ denote the set of all Frink ideals of P .

A subset A of a poset P is *Scott closed* if $\downarrow A = A$ and for any directed set $D \subseteq A$, $\sup D \in A$ whenever $\sup D$ exists. The complement of a Scott closed set is a Scott open set, and the family of these sets forms a topology, called the *Scott topology*, denoted $\sigma(P)$. The topology generated by the complements of all principal ideals $\downarrow x$ is called upper topology and denoted $\nu(P)$.

The following lemma is well-known (see [3]).

Lemma 2.1 *Let P be a poset.*

- (1) *The maps $(-)^{\uparrow} : (2^P)^{op} \rightarrow 2^P$, $A \mapsto A^{\uparrow}$ and $(-)^{\downarrow} : 2^P \rightarrow (2^P)^{op}$, $A \mapsto A^{\downarrow}$ are order preserving.*
- (2) *$((-)^{\uparrow}, (-)^{\downarrow})$ is a Galois connection between $(2^P)^{op}$ and 2^P , that is, for all $A, B \subseteq P$, $B^{\uparrow} \supseteq A \Leftrightarrow B \subseteq A^{\downarrow}$. Thus both $\delta : 2^P \rightarrow 2^P$, $A \mapsto A^{\delta} = (A^{\uparrow})^{\downarrow}$ and $\delta^* : 2^P \rightarrow 2^P$, $A \mapsto (A^{\downarrow})^{\uparrow}$ are closure operators.*
- (3) *For all $\{C_j : j \in J\} \subseteq 2^P$, $(\bigcup_{j \in J} C_j)^{\uparrow} = \bigcap_{j \in J} C_j^{\uparrow}$, $(\bigcup_{j \in J} C_j)^{\downarrow} = \bigcap_{j \in J} C_j^{\downarrow}$.*
- (4) *Let $L = \delta(P)$. For all $\{A_i^{\delta} : i \in I\} \subseteq L$, $\bigwedge_L \{A_i^{\delta} : i \in I\} = \bigcap \{A_i^{\delta} : i \in I\}$, $\bigvee_L \{A_i^{\delta} : i \in I\} = (\bigcup \{A_i^{\delta} : i \in I\})^{\delta} = (\bigcup_{i \in I} A_i)^{\delta}$.*

Corollary 2.2 *Let P be a poset. Then the map $j : P \rightarrow \delta(P)$, $x \mapsto \downarrow x$ is an order embedding of P in normal completion $\delta(P)$ and*

- (1) *j preserves all existing joins and meets;*
- (2) *For all $A^{\delta} \in \delta(P)$, $A^{\delta} = \bigvee_{a \in A} j(a) = \bigvee_{a \in A^{\delta}} j(a)$.*

3 Meet precontinuous posets

In [2], Ern  introduced the operators: $\Gamma : 2^P \rightarrow 2^P$, $Y \mapsto \bigcup \{I^{\delta} : I \in \text{Fid}(P), I \subseteq \downarrow Y\}$. By $\gamma(P)$, we denote the system of all fixed points of Γ , i.e. $\gamma(P) = \{Y \subseteq P : \Gamma Y = Y\} = \{Y : I \subseteq Y \text{ and } I \in \text{Fid}(P) \text{ implies } I^{\delta} \subseteq Y\}$. Obviously, $\gamma(P)^c = \{P \setminus Z : Z \in \gamma(P)\} = \{U \subseteq P : I \in \text{Fid}(P) \text{ and } I^{\delta} \cap U \neq \emptyset \text{ implies } I \cap U \neq \emptyset\}$. In a complete lattice, $\gamma(P)$ is the system of all *Scott closed* sets and $\gamma(P)^c$ is the system of all *Scott open* sets. In general, Γ preserves arbitrary intersections, but it need not preserve finite unions (see [1, Example 1]). Therefore, $\gamma(P)$ need not form a topology.

It is easy to get the following

Proposition 3.1 *Let P be a poset. Then for all $\{Y_i : i \in I\} \subseteq \gamma(P)$, $\bigwedge_{\gamma(P)} \{Y_i : i \in I\} = \bigcap_{i \in I} Y_i$ and $\bigvee_{\gamma(P)} \{Y_i : i \in I\} = \bigcap \{A \in \gamma(P) : \bigcup_{i \in I} Y_i \subseteq A\} = \{x \in P : \text{for all } U \in \gamma(P)^c \text{ with } x \in U, U \cap (\bigcup_{i \in I} Y_i) \neq \emptyset\}$.*

In [11], Mao and Xu generalized the concept of meet continuous dcpos to general posets and introduced the following

Definition 3.2 Let P be a poset. Then P is called a *meet continuous poset* if for all $x \in P$ and all directed subsets D , if $\sup D$ exists and $x \leq \sup D$, then $x \in cl_{\sigma(P)}(\downarrow D \cap \downarrow x)$, where $cl_{\sigma(P)}(\downarrow x \cap \downarrow D)$ is the Scott closure of the set $\downarrow D \cap \downarrow x$.

Meet continuity is not completion-invariant, as the following example shows.

Example 3.3 Let $P = \{0, 1, 2, \dots\} \cup \{a\}$. Define a partial order " \leq " on P such that $0 < 1 < 2 < \dots$ and $0 < a$. Obviously, P is meet continuous. Since $A = \{0, 1, 2, \dots\}$ implies $A^\uparrow = \emptyset$, we have $A^\delta = P$, so it follows that $\delta(P) = \{\{0\}, \{0, 1\}, \{0, 1, 2\}, \dots\} \cup \{\{0, a\}\} \cup \{\{P\}\}$. Take $\mathcal{D} = \{\{0\}, \{0, 1\}, \{0, 1, 2\}, \dots\}$ and $x = \{0, a\}$. Then $x \wedge \bigvee \mathcal{D} = x$, but $\bigvee x\mathcal{D} = \{0\}$. Therefore, $\delta(P)$ is not a meet continuous lattice.

Now we shall introduce a new concept of meet precontinuity which is proved to be completion-invariant.

Definition 3.4 A poset P is called *meet precontinuous* if for each $x \in P$ and $I \in \text{Fid}(P)$, $x \in I^\delta$ implies $x \in (\downarrow x \cap I)^\delta$.

Example 3.5 Let $S = \{0, 1, 2, \dots\}$ with a partial order " \leq " on S such that $0 < 1 < 2 < \dots$. If a and b are two incomparable upper bounds of S , then the resulting poset $P = S \cup \{a, b\}$ is a meet precontinuous poset.

Example 3.6 The poset $P = \{0, 1, 2, \dots\} \cup \{a\}$ constructed in Example 3.3 is meet continuous. Let $D = \{0, 1, 2, \dots\}$, we have $a \in D^\delta = P$, but $a \notin (\downarrow a \cap \downarrow D)^\delta = \{0\}$, thus it is not meet precontinuous.

By Proposition 3.1, we can get the following

Theorem 3.7 For a poset P , the following conditions are equivalent:

- (1) P is meet precontinuous;
- (2) Let $I \in \text{Fid}(P)$ and let $x \in I^\delta$. Then for each $U \in \gamma(P)^c$ with $x \in U$ we have $U \cap I \cap \downarrow x \neq \emptyset$.
- (3) For any $x \in P$ and any $U \in \gamma(P)^c$, $\uparrow(U \cap \downarrow x) \in \gamma(P)^c$.
- (4) $\gamma(P)$ is a complete Heyting algebra.

Proposition 3.8 For a poset P , the following conditions are equivalent:

- (1) P is meet precontinuous;
- (2) $\delta : \text{Fid}(P) \rightarrow \delta(P)$ preserves finite intersections, i.e., for all $\{A_i \mid 1 \leq i \leq n\} \subseteq \text{Fid}(P)$, $\bigcap_{i=1}^n A_i^\delta = (\bigcap_{i=1}^n A_i)^\delta$;
- (3) For all $A, B \in \text{Fid}(P)$, $A^\delta \cap B^\delta = (A \cap B)^\delta$;
- (4) For all $x \in P$, $A \in \text{Fid}(P)$, $\downarrow x \cap A^\delta = (\downarrow x \cap A)^\delta$.

Proof. (2) \Leftrightarrow (3): Trivial.

(1) \Rightarrow (3): For all $A, B \in \text{Fid}(P)$, it is clear that $(A \cap B)^\delta \subseteq A^\delta \cap B^\delta$. Let $x \in A^\delta \cap B^\delta$. By (1), $x \in \downarrow x = (\downarrow x \cap A)^\delta = (\downarrow x \cap B)^\delta$. Denote $M = \downarrow x \cap A$ and $N = \downarrow x \cap B$. Then $n \in N$ implies that $n \in \downarrow x = M^\delta$, which in turn yields $n \in \downarrow n = (\downarrow n \cap M)^\delta$. Therefore, by Lemma 2.1, $x \in N^\delta = (\bigcup_{n \in N} \downarrow n)^\delta = (\bigcup_{n \in N} (\downarrow n \cap M)^\delta)^\delta \subseteq ((M \cap N)^\delta)^\delta = (M \cap N)^\delta \subseteq (A \cap B)^\delta$.

(3) \Rightarrow (4): Trivial.

(4) \Rightarrow (1): For $x \in P$, $A \in \text{Fid}(P)$, if $x \in A^\delta$, then by (4), $x \in \downarrow x \cap A^\delta = (\downarrow x \cap A)^\delta$. \square

By Theorem 3.7 and Proposition 3.8, we can get the following

Theorem 3.9 *For a poset P , the following two conditions are equivalent:*

- (1) P is meet precontinuous;
- (2) $(\delta(P), \subseteq)$ is a meet continuous lattice.

4 Representations of precontinuous posets by $\gamma(P)$

Definition 4.1 Let P be a poset.

(1) Given two elements $x, y \in P$, we say x is *way-below* y (in symbols: $x \ll y$) if $x \in \downarrow y := \bigcap \{I \in \text{Fid}(P) : y \in I^\delta\}$.

(2) P is called *precontinuous* if for each $x \in P$, $x \in (\downarrow x)^\delta$.

Definition 4.2 ([7]) For a complete lattice L , define a relation \prec on L by $x \prec y \Leftrightarrow y \in \text{int}_{v(L)} \uparrow x$. L is called *hypercontinuous* if $x = \sup\{u \in L : u \prec x\}$ for all $x \in L$.

Definition 4.3 ([14]) Let P be a poset. We define a binary relation ρ on the set of subsets of P as follows: $A \rho B$ if and only if whenever S is a subset of P for which $\sup S$ exists and is in $\uparrow B$, then $S \cap \uparrow A \neq \emptyset$. Let $\rho(x) = \{A \subseteq P : A \text{ is finite and } A \rho \{x\}\}$.

Definition 4.4 ([14]) A complete lattice L is called a *generalized completely distributive lattice* if and only if for all $x \in L$, $\uparrow x = \bigcap \{\uparrow A : A \in \rho(x)\}$.

Lemma 4.5 ([13,16]) *For a complete lattice L , the following two conditions are equivalent:*

- (1) L is a hypercontinuous lattice;
- (2) For all $x, y \in L$ with $x \not\leq y$ implies that there are $u \in L$ and finite $F \subseteq L$ such that:
 - (i) $x \not\leq \downarrow F$, $u \not\leq \uparrow y$, and
 - (ii) For all $z \in L$, $z \in \downarrow F$ or $z \in \uparrow u$.

By Lemma 4.5, we can get the following

Proposition 4.6 *For a poset P , the following two conditions are equivalent:*

- (1) $(\gamma(P)^c, \subseteq)$ is a hypercontinuous lattice;

- (2) For all $x \in P$, $U \in \gamma(P)^c$ with $x \in U$ implies that there are finite $F \subseteq P$ and $V \in \gamma(P)^c$ such that $x \in V \subseteq \uparrow F \subseteq U$.

Lemma 4.7 ([17]) *A complete lattice L is a generalized completely distributive lattice if and only if L^{op} is a hypercontinuous lattice.*

Lemma 4.8 ([17]) *For a complete lattice L . The following conditions are equivalent:*

- (1) L is a complete distributive lattice;
- (2) L is both Heyting algebra and a generalized completely distributive lattice.

Lemma 4.9 ([2]) *For a poset P , the following conditions are equivalent:*

- (1) P is a precontinuous poset and \ll has the interpolation property;
- (2) Γ is a closure operator and preserves arbitrary intersections of lower sets;
- (3) $\gamma(P) = \Gamma(2^P)$ is a completely distributive closure system (and Γ is the corresponding closure operator).

By Theorem 3.7, Proposition 4.6, Lemma 4.7, Lemma 4.8 and Lemma 4.9, we can get the following

Theorem 4.10 *For a poset P , the following two conditions are equivalent:*

- (1) P is a precontinuous poset and \ll has the interpolation property;
- (2) P is a meet precontinuous poset, and for all $x \in P$, $U \in \gamma(P)^c$ with $x \in U$ implies that there are finite $F \subseteq P$ and $V \in \gamma(P)^c$ such that $x \in V \subseteq \uparrow F \subseteq U$ and Γ is idempotent.

5 Locating the way below relation \ll within $\text{Aux}(P)$

In this section, we investigate the relationships between meet precontinuous posets and precontinuous posets, and given a poset P and an auxiliary relation on P , we characterize those join-dense subsets of P whose way-below relation agrees with the given auxiliary relation.

Definition 5.1 ([7]) We say that a binary relation \prec on a poset P is an *auxiliary relation*, if it satisfies the following conditions for all $u, x, y, z \in P$,

- (i) $x \prec y$ implies $x \leq y$;
- (ii) $u \leq x \prec y \leq z$ implies $u \prec z$;
- (iii) if a smallest element 0 exists, then $0 \prec x$.

The set of all auxiliary relations on P will be denoted by $\text{Aux}(P)$. As $\text{Aux}(P)$ is closed under arbitrary intersections in $2^{P \times P}$, it is therefore a complete lattice.

Definition 5.2 An auxiliary relation \prec on poset P is *approximating* if for each $x \in P$, $s_{\prec}(x) = \{y \in P : y \prec x\} \in \text{Fid}(P)$ and $x \in (s_{\prec}(x))^{\delta}$. The set of all approximating auxiliary relations is denoted by $\text{App}(P)$.

Proposition 5.3 ([7,8]) *Let P be a poset and let M be the set of all monotone functions $s : P \rightarrow D(P)$ satisfying $s(x) \subseteq \downarrow x$ for all $x \in P$ - considered as a poset relative to the ordering $s \leq t$ iff $s(x) \subseteq t(x)$ for all $x \in P$. Then the mapping*

$$\phi : \text{Aux}(L) \rightarrow M, \phi(\prec) = s_{\prec} = (x \mapsto \{y : y \prec x\})$$

is a well-defined isomorphism from $\text{Aux}(P)$ onto M , whose inverse associates to each function $s \in M$ the relation \prec_s given by $x \prec_s y \Leftrightarrow x \in s(y)$.

From the definition of way below relation \ll , we have $s_{\ll}(x) = \bigcap \{I \in \text{Fid}(P) : x \in I^{\delta}\}$ for all $x \in P$.

Lemma 5.4 *Let P be a poset. For each $I \in \text{Fid}(P)$, we define the function $m_I : P \rightarrow D(P)$ by*

$$m_I(x) = \begin{cases} \downarrow x \cap I, & \text{if } x \in I^{\delta}, \\ \downarrow x, & \text{otherwise.} \end{cases}$$

Then $m_I \in M$ for all $I \in \text{Fid}(P)$ and $\ll = \bigcap \{\prec_{m_I} : I \in \text{Fid}(P)\}$.

Proof. It is obvious that $m_I \in M$ for all $I \in \text{Fid}(P)$. From Proposition 5.3, we need only to prove that $\phi(\bigcap \{\prec_{m_I} : I \in \text{Fid}(P)\}) = \phi(\ll) = s_{\ll}$. Since ϕ is an isomorphism, $\phi(\bigcap \{\prec_{m_I} : I \in \text{Fid}(P)\}) = \bigwedge_{I \in \text{Fid}(P)} \phi(\prec_{m_I}) = \bigwedge_{I \in \text{Fid}(P)} s_{\prec_{m_I}}$. From the definitions of \prec_{m_I} and $s_{\prec}(x)$, $s_{\prec_{m_I}}(x) = m_I(x)$. Now, we prove that $(\bigwedge_{I \in \text{Fid}(P)} s_{\prec_{m_I}})(x) = s_{\ll}(x)$ for each $x \in P$. In fact,

$$\begin{aligned} \left(\bigwedge_{I \in \text{Fid}(P)} s_{\prec_{m_I}} \right)(x) &= \bigwedge \{s_{\prec_{m_I}}(x) : I \in \text{Fid}(P)\} \\ &= \bigcap \{m_I(x) : I \in \text{Fid}(P)\} \\ &= \left(\bigcap \{\downarrow x \cap I : x \in I^{\delta}\} \right) \cap \left(\bigcap \{\downarrow x : x \notin I^{\delta}\} \right) \\ &= \bigcap \{I \in \text{Fid}(P) : x \in I^{\delta}\} \\ &= s_{\ll}(x) \end{aligned}$$

□

Lemma 5.5 *In a meet precontinuous poset P , all relations $s_{\prec_{m_I}}$ belonging to the functions m_I for $I \in \text{Fid}(P)$ are approximating. This holds, in particular, for precontinuous posets, as these are meet precontinuous.*

Proof. For each $x \in P$,

$$s_{\prec_{m_I}}(x) = m_I(x) = \begin{cases} \downarrow x \cap I, & \text{if } x \in I^{\delta}, \\ \downarrow x, & \text{otherwise.} \end{cases}$$

If $x \in I^\delta$, then $x \in (\downarrow x \cap I)^\delta = (s_{\prec_{m_I}}(x))^\delta$ since P is meet precontinuous; If $x \notin I^\delta$, then $x \in (\downarrow x)^\delta = (m_I(x))^\delta = (s_{\prec_{m_I}}(x))^\delta$. Obviously, $\downarrow x \in \text{Fid}(P)$ and $\downarrow x \cap I \in \text{Fid}(P)$. Therefore, \prec_{m_I} is approximating for all $I \in \text{Fid}(P)$. \square

Lemma 5.6 *In a poset P , the way below relation \ll is contained in all approximating auxiliary relations, and is equal to their intersection, if P is meet precontinuous.*

Proof. Suppose $x, y \in P$ with $y \ll x$, and \prec is an approximating auxiliary relation. Then $s_{\prec}(x) = \{u \in P : u \prec x\} \in \text{Fid}(P)$ and $x \in (s_{\prec}(x))^\delta$. It follows from the definition of \ll that $y \in s_{\prec}(x)$, that is, $y \prec x$. Thus $\ll \subseteq \prec$ and $\ll \subseteq \bigcap \{\prec : \prec \in \text{App}(P)\}$. If P is meet precontinuous, then $\prec_{m_I} \in \text{App}(P)$ by Lemma 5.5. Thus $\bigcap \{\prec : \prec \in \text{App}(P)\} \subseteq \bigcap \{\prec_{m_I} : I \in \text{Fid}(P)\}$. By Lemma 5.4, $\ll = \bigcap \{\prec_{m_I} : I \in \text{Fid}(P)\}$. Whence $\bigcap \{\prec : \prec \in \text{App}(P)\} \subseteq \ll$. Hence $\ll = \bigcap \{\prec : \prec \in \text{App}(P)\}$. \square

Theorem 5.7 *For a poset P , the following conditions are equivalent:*

- (1) P is precontinuous;
- (2) \ll is the smallest approximating auxiliary relation on P ;
- (3) P is meet precontinuous and there is a smallest approximating auxiliary relation on P .

Proof. (1) \Leftrightarrow (2): By definition, P is precontinuous iff \ll is an approximating auxiliary relation. Thus the equivalence of (1) and (2) follows from the first part in Lemma 5.5.

(2) \Rightarrow (3): Trivial.

(3) \Rightarrow (1): By Lemma 5.5, \ll is the intersection of all approximating auxiliary relations. Thus, if there is a smallest approximating auxiliary relation, this has to be \ll , and we see that (3) implies (1). \square

To conclude the paper, we construct posets which are precontinuous and whose way below relation \ll agrees with auxiliary relation \prec . In cases of possible ambiguity, we shall use superscripts to distinguish between concepts related to two posets P and Q , here \ll^P from \ll^Q .

Definition 5.8 ([3]) Let P be a poset. We say that $S \subseteq P$ is *join-dense* in P if $x \in (\downarrow x \cap S)^\delta$ for all $x \in P$.

Theorem 5.9 *Let S be join-dense in a poset P and \prec an auxiliary relation on S . If for each $x \in S$, $s_{\prec}(x) = \{y \in S : y \prec x\} \in \text{Fid}(P)$, then the following statements are equivalent:*

- (1) P is precontinuous and \prec agrees with \ll^P on S .
- (2) For all $x, y \in S$, $x \prec y$ implies $x \ll^P y$ and $x \in (s_{\prec}(x))^\delta$.

Proof. (1) \Rightarrow (2): We show $x \in (s_{\prec}(x))^\delta$ for all $x \in S$. If there exists $x \in S$ with $x \notin (s_{\prec}(x))^\delta$, then there exists $y \in P$ with $y \in (s_{\prec}(x))^\uparrow$ such that $x \not\leq y$. Since P is precontinuous, $\downarrow x \not\leq \downarrow y$, i.e., there exists $m \in \downarrow x$ and $m \not\leq y$. Since S is join-dense

in P , there exists $z \in S$ such that $z \leq m$ and $z \not\leq y$. Whence $z \leq m \ll^P x$. Hence $z \ll^P x$. By (1), $z \prec x$, a contradiction to $y \in (s_{\prec}(x))^{\uparrow}$.

(2) \Rightarrow (1):

CLAIM. If (2) holds then, for all $a, b \in P$, $a \ll^P b$ if and only if there exists a finite $X \subseteq S$ such that $a \in X^{\delta}$ and for each $x \in X$ there is $p \in S$ with $x \prec p \leq b$.

Proof of Claim. Assume $a \ll^P b$. If $b = 0$, then $X = \emptyset$ works. So we shall assume $b > 0$. Let $\mathcal{H}_b = \{X \subseteq S : X \text{ is finite and for each } x \in X \text{ there is } p \in S \text{ such that } x \prec p \leq b\}$. Let $\mathcal{D}_b = \{X^{\delta} : X \in \mathcal{H}_b\}$. Since \mathcal{H}_b is closed under finite unions, $\bigcup \mathcal{D}_b \in \text{Fid}(P)$. We show $b \in (\bigcup \mathcal{D}_b)^{\delta}$. If $b \notin (\bigcup \mathcal{D}_b)^{\delta}$, then there exists $m \in P$ with $\bigcup \mathcal{D}_b \subseteq \downarrow m$ such that $b \not\leq m$. Since S is join-dense, there exists $p \in S$ such that $p \leq b$ and $p \not\leq m$. Since $p \in (s_{\prec}(p))^{\delta}$ by (2), there exists $z \in S$ with $z \prec p$ such that $z \not\leq m$. Let $X = \{z\}$, then $X \in \mathcal{H}_b$. Thus $z \in \{z\}^{\delta} \subseteq \bigcup \mathcal{D}_b \subseteq \downarrow m$, contradicting $z \not\leq m$. Whence $b \in (\bigcup \mathcal{D}_b)^{\delta}$. Since $a \ll^P b$, $a \in \bigcup \mathcal{D}_b$. Thus there exists $X \in \mathcal{H}_b$ such that $a \in X^{\delta}$.

Conversely, assume $a \in X^{\delta}$ where $X \subseteq S$ is finite and for each $x \in X$ there exists $p_x \in S$ with $x \prec p_x \leq b$. We show $a \ll^P b$. Let $I \in \text{Fid}(P)$ with $b \in I^{\delta}$. By (2), $x \ll^P p_x \leq b \in I^{\delta}$. Since I^{δ} is a lower set, $p_x \in I^{\delta}$. Whence $x \in I$. Hence $X \subseteq I$. Since $I \in \text{Fid}(P)$, $a \in X^{\delta} \subseteq I$. So $a \ll^P b$.

Firstly, we show P is precontinuous. Let $b \in P$. In the course of proving the Claim, we have shown that $b \in (\bigcup \mathcal{D}_b)^{\delta}$. By Claim, $\bigcup \mathcal{D}_b \subseteq \downarrow b$. So $b \in (\bigcup \mathcal{D}_b)^{\delta} \subseteq (\downarrow b)^{\delta}$.

Then we show $\prec = \ll^P$. By (2), $\prec \subseteq \ll^P$. Conversely, let $a \ll^P b$. By Claim, there exists a finite $X \subseteq S$ such that $a \in X^{\delta}$ and for each $x \in X$, $x \prec b$. By hypothesis $s_{\prec}(b) \in \text{Fid}(P)$, we have $X^{\delta} \subseteq s_{\prec}(b)$. So $a \prec b$. \square

6 Summary

In this paper, based on cut operator, the concept of meet precontinuous posets is introduced as a generalization of meet continuous lattices. Some properties and characterizations of meet precontinuous posets are investigated. In particular, we discuss the relationships between meet precontinuous posets and precontinuous posets. The important contribution of this paper is that we show that a poset is meet precontinuous iff its normal completion is a meet continuous lattice. Therefore, from the sense of Ern  in [2], our generalization is a good generalization.

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