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The Chromatic Index of Proper Circular-arc Graphs of Odd Maximum Degree which are Chordal

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Abstract

The complexity of the edge-coloring problem when restricted to chordal graphs, listed in the famous D. Johnson's NP-completeness column of 1985, is still undetermined. A conjecture of Figueiredo, Meidanis, and Mello, open since the late 1990s, states that all chordal graphs of odd maximum degree Δ have chromatic index equal to Δ. This conjecture has already been proved for proper interval graphs (a subclass of proper circular-arc $\cap$ chordal graphs) of odd Δ by a technique called pullback. Using a new technique called multi-pullback, we show that this conjecture holds for all proper circular-arc $\cap$ chordal graphs of odd Δ. We also believe that this technique can be used for further results on edge-coloring other graph classes.

Keywords: Pullback, circular-arc, chromatic index, edge-coloring, chordal

1 Introduction

Circular-arc graphs are the intersection graphs of a finite set of arcs on a circle. If no arc properly contains another, the graph is said to be a *proper circular-arc graph*. If all the arcs have the same length, the graph is said to be a *unit circular-arc graph*. Although the class of the circular-arc graphs is well studied, very little is known about deciding the chromatic index of these graphs, except for the subclass consisting of the n-vertex proper circular-arc graphs of odd maximum degree Δ

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which have $n \not\equiv 1, \Delta \pmod{(\Delta+1)}$ and a maximal clique of size two, or which have $n \equiv 0 \pmod{(\Delta+1)}$ [1].

Circular-arc graphs form a superclass of interval graphs. An important difference between these two classes is that interval graphs have a linear number of maximal cliques (in the number of vertices), while circular-arc graphs may have an exponential number of maximal cliques. This may suggest why some problems are more difficult for circular-arc graphs than for interval graphs. For instance, vertex-coloring is polynomial for interval graphs, but NP-hard for circular-arc graphs [4].

The NP-hard [5] edge-coloring problem is the problem of determining the minimum amount of colors needed to color the edges of a graph such that no two adjacent edges receive the same color. This amount is called the chromatic index of G, denoted $\chi'(G)$. By definition, $\chi'(G) \geq \Delta(G)$ for any graph G. The celebrated Vizing's Theorem brings that $\chi'(G) \leq \Delta(G) + 1$ [10]. Therefore, graphs which satisfy $\chi'(G) = \Delta(G)$ are referred to as *Class 1* graphs, and those satisfying $\chi'(G) = \Delta(G) + 1$ are referred to as *Class 2*. For instance, a complete graph K_n is *Class 1* if n is even, and *Class 2* otherwise.

We solve the edge-coloring problem in the class of *proper circular-arc* $\cap$ *chordal* (PCAC) graphs of odd maximum degree, that is, we prove that all these graphs are *Class 1* and our proof yields a polynomial-time exact edge-coloring algorithm for these graphs. It is important to remark that even for proper interval graphs (often referred to as *indifference* graphs in the literature), which form an important subclass of PCAC graphs, the problem is solved only for graphs with odd maximum degree Δ, by a technique called *pullback* [2]. Later, this technique was also used to solve the edge-coloring problem for all dually chordal graphs (which form a superclass of interval graphs) of odd Δ [3]. The complexity of determining the chromatic index of chordal graphs is one of the problems in the famous D. Johnson's NP-*completeness column* [6] which are still open, even restricted to graphs of odd Δ.

To solve the problem for the PCAC graphs of odd maximum degree, we design a new technique called *multi-pullback*, which we suspect that can be used for other graph classes.

This paper is organized as follows: the remaining of this section is dedicated to some preliminary definitions; in Section 2 we discuss the pullback functions introduced in [2] and present our multi-pullback functions; then, in Section 3 we present our results on PCAC graphs using the multi-pullback functions introduced in Section 2.

Preliminary definitions

In this paper, graph-theoretical definitions follow their usual meanings in the literature. In particular, $G = (V(G), E(G))$ is a graph, $V(G)$ is the set of vertices of G and $E(G)$ is the set of edges of G. An edge uv is said to be *incident* to the vertices u and v, and the vertices u and v are said to be *neighbors*. The *degree* of a vertex u, denoted $d_G(u)$, is the number of edges that are incident to the vertex u. The *maximum degree* of G is $\Delta(G) := \max\{d_G(u) : u \in V(G)\}$. The *open neighborhood* of u is the set $N_G(u) := \{v : uv \in E(G)\}$. The *closed neighborhood* of u is the set

$N_G[u] := N_G(u) \cup \{u\}$.

If $N_G[u] = V(G)$, then the vertex u is said to be *universal* in G. We say that a graph H is a *subgraph* of G if $V(H) \subset V(G)$ and $E(H) \subset E(G)$. Let $U \subset V(G)$. The subgraph of G *induced* by U is defined by $G[U] := (U, \{uv \in E(G) : u, v \in U\})$. Let $F \subset E(G)$. The subgraph of G *induced* by F is defined by $G[F] := (\{u : uv \in F$ for some $v \in V(G), F)$. The *core* of a graph is the subgraph induced by its vertices of maximum degree. The *semi-core* of a graph is the subgraph induced by the vertices of maximum degree and their neighbors.

A k-*edge-coloring* of G is a proper edge-coloring of G with k colors, that is, an assignment of colors to the edges of a graph in such a way that no two adjacent edges receive the same color and that at most k colors are used. A set $U \subset V(G)$ is said to be a *clique* if it induces a complete graph in G. A clique is said to be *maximal* if it is not properly contained in any other clique. A *simplicial vertex* in G is a vertex that belongs to only one maximal clique of G.

2 Pullback and multi-pullback functions

A function $f\colon V(G) \to V(G')$ is said to be a *pullback* if it is a homomorphism (i.e. for all $uv \in E(G)$ we have $f(u)f(v) \in E(G')$), and if f is injective when restricted to $N_G[u]$ for all $u \in V(G)$.

Lemma 2.1 ([2,3]) *If f is a pullback from G to G' and λ' is an edge-coloring of G', then the function $\lambda(uv) := \lambda'(f(u)f(y))$ is an edge-coloring of G.*

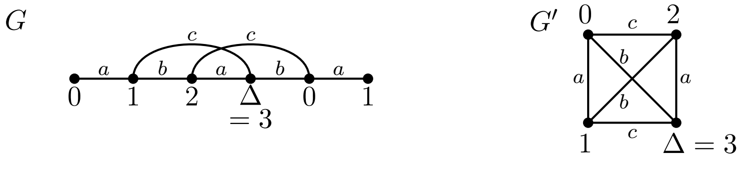

Figure 1. Example of a pullback from an indifference graph G to $G' := K_4$

Definition 2.2 Let $G = (V, E)$ be a graph with $E \neq \emptyset$ and let $\{E_1, \ldots, E_t\}$ be a partition of E. A *multi-pullback* F from G to a collection of t graphs $\{G'_1, \ldots, G'_t\}$ is a collection of t functions $\{f_1, \ldots, f_t\}$ such that:

(i) f_i is a pullback from $G[E_i]$ to G'_i;

(ii) there is some positive integer k and some collection of k-edge-colorings $\lambda'_1, \ldots, \lambda'_t$ of $G'_1, \ldots, G'_t$, respectively, such that the edge-colorings obtained from $\lambda'_1, \ldots, \lambda'_t$ and the pullbacks $f_1, \ldots, f_t$ do not create any color conflict on the edges of G, that is, the function defined by

$\lambda(uv) := \lambda'_i(f_i(u)f_i(v))$, being E_i *the* set of the partition to which uv belongs,

is a proper k-edge-coloring of G.

Observe the necessity of including (ii) in Definition 2.2, otherwise the pullbacks $f_1, \ldots, f_t$ could define *non-compatible* edge-colorings (that is, color conflicts could be created when assembling all edge-colorings in order to construct the edge-coloring of G). Also in the definition, observe that disjointness is assumed only among the sets of the partition $\{E_1, \ldots, E_t\}$, but not among the domains of the functions in F, which are sets of vertices, not edges. This means that a single vertex u can be mapped to a vertex v of G'_i by a pullback f_i and to a different vertex w of G'_j by a pullback f_j, depending on which *role* we want u to assume in order to color each edge incident to u.

Figure 2 shows an example of a collection of functions $\{f_1, f_2, f_3\}$ which can be verified to be a multi-pullback from a PCAC graph G with $\Delta = 5$ to the K_6, under the 5-edge-colorings $\lambda'_1 = \lambda'_2 = \lambda'_3 =: \lambda'$ of the K_6 defined by

$$\lambda'(uv) = \begin{cases} (u+v) \bmod \Delta\,, & \text{if neither } u \text{ nor } v \text{ is } \Delta; \\ (2v) \bmod \Delta\,, & \text{if } u = \Delta. \end{cases} \tag{1}$$

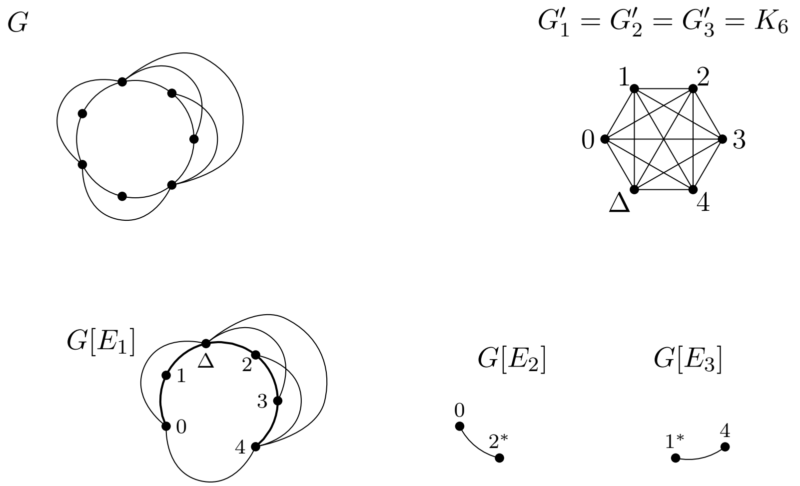

Figure 2. Example of a multi-pullback from G to the K_6 under the 5-edge-coloring defined in (1). Observe that the vertex marked with an asterisk is mapped to two distinct vertices by f_2 and by f_3, with no color conflict being created.

3 The result

Proper circular-arc graphs have the consecutive 1's property [9], i.e. there is a circular order for the vertices in such a way that for every edge $\overrightarrow{uv}$ under the clockwise orientation of the edges along this order, all the vertices clockwise between u and v induce a complete graph in the original undirected graph. This order is called a *proper circular-arc order.*

Lemma 3.1 *Let G be a PCAC graph of odd maximum degree. If G has a universal vertex, or if the semi-core of G is an indifference graph, then G is* Class 1.

Proof Observe first that if G has a universal vertex, then G is a subgraph of $K_{\Delta(G)+1}$ and hence *Class 1*. On the other hand, if the semi-core of G is an indifference graph, then G is also *Class 1* because the chromatic index of a graph is equal to the chromatic index of its semi-core [7], and because all indifference graphs of odd maximum degree are *Class 1* [2]. □

Lemma 3.2 below provides a full characterization of the structure of proper circular-arc ∩ chordal graphs which do not satisfy Lemma 3.1.

Lemma 3.2 *If G is a PCAC graph of odd maximum degree with no universal vertex such that the semi-core S of G is not an indifference graph, then $S = G$ and there is a 6-partition $\{Y_A, Y_{AB}, Y_B, Y_{BC}, Y_C, Y_{AC}\}$ of $V(G)$ which splits any proper circular-arc order σ of G into six contiguous subsequences of σ in a manner that, being the cardinality of each set in the partition denoted by lowercase y with the corresponding subscript:*

- (i) *the graph G has exactly four maximal cliques, which can be given by $X_A \coloneqq \{Y_{AB} \cup Y_A \cup Y_{AC}\}$, $X_B \coloneqq \{Y_{BC} \cup Y_B \cup Y_{AB}\}$, $X_C \coloneqq \{Y_{AC} \cup Y_C \cup Y_{BC}\}$, and $Z = \{Y_{AB} \cup Y_{AC} \cup Y_{BC}\}$,wherein X_A is assumed without loss of generality to be of maximum size among the cliques X_A, X_B, and X_C, which are the cliques which appear contiguously in σ (that is, all the vertices in each of these cliques appear consecutively in σ);*
- (ii) *all the vertices in Y_{AB} and in Y_{AC} have degree $\Delta(G)$ in G;*
- (iii) *$\Delta(G) = y_A + y_B + y_{AB} + y_{BC} + y_{AC} - 1 = y_A + y_C + y_{AB} + y_{BC} + y_{AC} - 1$;*
- (iv) *$y_A \geq y_B = y_C$;*

Proof Let σ be a proper circular-arc order of G and let $(X_0, X_1, \cdots, X_{t-1})$ be the maximal cliques that appear contiguously in σ. We must have $t \geq 3$, otherwise it can be straightforwardly shown that G is an indifference graph.

We claim that there is no X_i such that $X_i \subset X_{(i-1) \bmod t} \cup X_{(i+1) \bmod t}$. If this claim holds, an induced cycle of size t is easily obtained by choosing one vertex from each $X_i \cap X_{(i+1) \bmod t}$. Because G is chordal and it is not an indifference graph, we have $t = 3$. These three maximal cliques of G that appear contiguously in σ are X_A, X_B, and X_C, respectively.

Since G is not an indifference graph, we have that the intersection of two consecutive cliques in σ is not empty (otherwise in any circular-arc model of G there would be a point on the circumference which would be uncovered by any arc). We define the sets $Y_{AB} \coloneqq X_A \cap X_B$, $Y_{BC} \coloneqq X_B \cap X_C$, and $Y_{AC} \coloneqq X_A \cap X_C$, and also $Y_A \coloneqq X_A \setminus (Y_{AB} \cup Y_{AC})$, $Y_B \coloneqq X_B \setminus (Y_{AB} \cup Y_{BC})$, and $Y_C \coloneqq X_C \setminus (Y_{AC} \cup Y_{BC})$. As all the vertices in $Y_{AB} \cup Y_{BC} \cup Y_{AC}$ are neighbors of each other, there is a fourth maximal clique $Z \coloneqq Y_{AB} \cup Y_{BC} \cup Y_{AC}$ that does not appear contiguously in σ.

Up to this point, we have proven that if the claim holds then there are at least three maximal cliques (X_A, X_B, and X_C) which appear contiguously in σ, as well as the fourth clique Z. We have also proven that the sets Y_{AB}, Y_{BC}, and Y_{AC} are not empty. We can further demonstrate that the sets Y_A, Y_B, and Y_C are non-empty, which is equivalent to prove that each of the cliques X_A, X_B, and X_C has a simplicial vertex. If $Y_A = \emptyset$, then every vertex of Y_{BC} is universal (see Figure 3), contradicting the hypothesis. The non-emptiness of Y_B and Y_C follows analogously.

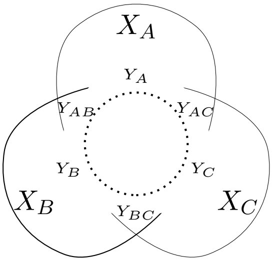

Figure 3. Structure of a PCAC graph according to Lemma 3.2.

Now we shall prove the claim and that X_A, X_B, and X_C are the only maximal cliques contiguously in σ. Assume for the sake of contradiction that there is a fourth maximal clique X_D contiguously in σ. Since the intersections Y_{AB}, Y_{BC}, and Y_{AC} are all non-empty, the clique X_D must be contained in the union of two cliques from $\{X_A, X_B, X_C\}$. Without loss of generality, $X_D \subset X_A \cup X_B$. By the same arguments presented above, the four sets $(X_D \cap X_A) \setminus (X_B \cup X_C)$, $(X_D \cap X_B) \setminus (X_A \cup X_C)$, $(X_B \cap X_C) \setminus X_D$, and $(X_C \cap X_A) \setminus X_D$ are all non-empty. Ergo, by choosing four vertices, one from each of these sets, we obtain an induced cycle of size four, contradicting the fact that G is chordal. Hence, we have proven that X_D cannot exist and also that the claim holds. Furthermore, since all the vertices in Y_A, Y_B, and Y_C are simplicial, the only maximal clique which can be formed not contiguously in σ is the clique Z (recall Figure 3).

Assuming without loss of generality that X_A is of maximum size among X_A, X_B, and X_C, it remains to demonstrate (ii)–(iv). Clearly, the vertices of maximum degree in G are in $Y_{AB} \cup Y_{BC} \cup Y_{AC}$. We shall demonstrate that either Y_{AB} and Y_{AC}, or all the sets from $\{Y_{AB}, Y_{AC}, Y_{BC}\}$ have vertices of maximum degree (this proves (ii)). If only one set I from $\{Y_{AB}, Y_{AC}, Y_{BC}\}$ has vertices of maximum degree in G, then surely $I \neq Y_{BC}$, because of the assumption on the cardinality of X_A. If $I = Y_{AB}$, then the semi-core of G is an indifference graph, because the order $Y_B, Y_{BC}, Y_{AB}, Y_{AC}, Y_A$ is an indifference order[7]. The case $I = Y_{AC}$ follows analogously. Remark that this also proves that the semi-core of G equals G.

Notice that vertices which belong to the same set from $\{Y_A, Y_{AB}, Y_B, Y_{BC}, Y_C, Y_{AC}\}$ have the same closed neighborhood and hence the same degree. Let u be a vertex in Y_{AB}, v a vertex in Y_{AC}, and w a vertex in Y_{BC}. We know that $\Delta(G) = d_G(u) = d_G(v) \geq d_G(w)$ and also that:

$$
\begin{aligned}
d_G(u) &= y_{BC} + y_B + y_{AB} + y_A + y_{AC} - 1\,;\\
d_G(v) &= y_{AB} + y_A + y_{AC} + y_C + y_{BC} - 1\,;\\
d_G(w) &= y_{AB} + y_B + y_{BC} + y_C + y_{AC} - 1\,.
\end{aligned}
$$

[7] An indifference order of an indifference graph is a linear order of the vertices so that vertices belonging to the same maximal clique appear consecutively in this order [8].

From these equations, we have (iii) and also that $y_B = y_C$ and $y_A \geq y_B$, completing the proof of (iv). $\square$

Theorem 3.3 *Every proper circular-arc $\cap$ chordal graph with odd maximum degree is Class 1.*

Proof In view of Lemma 3.1, let G be a PCAC graph of odd maximum degree with no universal vertex such that the semi-core of G is not an indifference graph. Let also $\{Y_A, Y_{AB}, Y_B, Y_{BC}, Y_C, Y_{AC}\}$ be a partition of $V(G)$ as in Lemma 3.2 (recall Figure 3). Let $\{E_1, E_2, E_3, E_4\}$ be the partition of $E(G)$ defined by:

$$
\begin{aligned}
E_1 &\coloneqq E(G[Y_A \cup Y_{AB} \cup Y_B \cup Y_{BC} \cup Y_{AC}])\,;\\
E_2 &\coloneqq \{uv : u \in Y_{AC} \text{ and } v \in C\}\,;\\
E_3 &\coloneqq \{uv : u \in Y_{BC} \text{ and } v \in C\}\,;\\
E_4 &\coloneqq E(G[C])\,.
\end{aligned}
$$

Let $V(K_{\Delta(G)+1}) = \{0, \ldots, \Delta(G)\}$ and $V(K_{y_C}) = \{0, \ldots, c-1\}$. We shall construct a multi-pullback $\{f_1, f_2, f_3, f_4\}$ with $f_i \colon V_i \to G'_i$, for all $i \in \{1, \ldots, 4\}$, being

$$
\begin{aligned}
V_1 &\coloneqq Y_A \cup Y_{AB} \cup Y_B \cup Y_{BC} \cup Y_{AC}\,,\\
V_2 &\coloneqq Y_{AC} \cup Y_C\,,\\
V_3 &\coloneqq Y_{BC} \cup Y_C\,,\\
V_4 &\coloneqq Y_C\,,
\end{aligned}
$$

and being $G'_1 \coloneqq G'_2 \coloneqq G'_3 \coloneqq K_{\Delta(G)+1}$ and $G'_4 = K_{y_C}$, under the edge-colorings $\lambda'_1, \lambda'_2, \lambda'_3, \lambda'_4$ defined by $\lambda'_1 \coloneqq \lambda'_2 \coloneqq \lambda'_3 \coloneqq \lambda'$, wherein λ' is the $\Delta(G)$-edge-coloring of $K_{\Delta(G)+1}$ defined in (1), and λ'_4 is the $\Delta(G)$-edge-coloring of K_{y_C} defined by

$$f'_4(uv) = (2y_{AC} + y_A + y_C + y_{BC} + u + v) \bmod \Delta(G)\,.$$

Remark that λ' is an optimal edge-coloring of $K_{\Delta(G)+1}$ (which is *Class 1* since $\Delta(G)$ is odd) and that λ_4 is surely not optimal, since $c < \Delta(G) - 2$.

Remark by Lemma 3.2 that $|V_1| = \Delta(G) + 1$. In order to define f_1, take any bijective labeling function satisfying:

$$
\begin{aligned}
f_1(Y_{AC}) &= \{0, \ldots, y_{AC} - 1\}\,;\\
f_1(Y_A) &= \{y_{AC}, \ldots, y_{AC} + y_A - 1\}\,;\\
f_1(Y_B) &= \{y_{AC} + y_A, \ldots, y_{AC} + y_A + y_B - 1\}\,;\\
f_1(Y_{BC}) &= \{y_{AC} + y_A + y_B, \ldots, y_{AC} + y_A + y_B + y_{BC} - 1\}\,;\\
f_1(Y_{AB}) &= \{y_{AC} + y_A + y_B + y_{BC}, \ldots, y_{AC} + y_A + y_B + y_{BC} + y_{AB} - 1\}\,.
\end{aligned}
$$

Here, we use $f_1(Z)$ to denote $\bigcup_{z \in Z}\{f_1(z)\}$. Notice that we have used $\Delta(G) + 1$ distinct labels, from 0 to $\Delta(G)$, and it is easy to realize that this labeling is a *pullback* from $G[E_1]$ to the G'_1.

It remains to color the edges incident to the vertices of Y_C, that is, it remains to define $f_2, \ldots, f_4$. Remark that $G[E_2 \cup E_3]$ is a bipartite graph, with parts Y_C and

$Y_{BC} \cup Y_{AC}$, and $G[E_4]$ is a complete graph. Figure 4 represents the sets Y_{BC}, Y_C, and Y_{AC}.

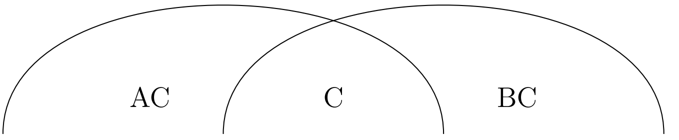

Figure 4. The sets Y_{BC}, Y_C, and Y_{AC}

Recall that $G[E_2]$ is the bipartite graph induced by the edges between Y_{AC} and Y_C, and notice that the edges incident to vertices in Y_{AC} are not incident to vertices in Y_B. This is why we can define f_2 by assigning to the vertices of Y_{AC} the same labels which they have been assigned by f_1, and to the vertices of Y_C the same labels assigned to the vertices of Y_B by f_1, in the manner that we clarify in the sequel. As $y_B = y_C$, there will be enough labels for all the vertices of Y_C.

Analogously, the graph $G[E_3]$ is the bipartite graph induced by the edges between Y_{BC} and Y_C. Notice that vertices in Y_{BC} are not neighbors of vertices in Y_A, therefore, the labels assigned by f_1 to the vertices in Y_A can be reused by f_3 to the vertices in Y_C (in the manner that we clarify in the sequel), if the vertices in Y_{BC} are assigned by f_3 the same labels which they have been assigned by f_1. Recall that $y_A \geq y_C$, so there will be enough labels.

To complete the proof, it remains only to define which are the three labels assigned to each vertex in Y_C by f_2, f_3, and f_4, and to show that the edge-coloring obtained through these pullbacks do not create color conflicts in G. Let $Y_C = \{u_0, \ldots, u_{y_C-1}\}$. We define for each $u_i \in Y_C$ the triplet $(f_2(u_i), f_3(u_i), f_4(u_i)) := (y_{AC} + y_A + i, y_{AC} + i, i)$. Let λ be the $\Delta(G)$-edge-coloring of G as in Definition 2.2. We show that λ is a proper edge-coloring, for which it suffices to show that all the colors of the edges incident to the same vertex u_i in Y_C are different.

The colors of the edges incident to u_i can be verified to be as follows (all the colors listed below are $\bmod \Delta(G)$, but this information is omitted for a clear description):

- the colors of the edges of $G[E_2]$ that are incident to u_i are the y_{AC} colors from the set
$$\{y_{AC} + y_A + i, \ldots, 2y_{AC} + y_A + i - 1\};$$
- the colors of the edges of $G[E_3]$ that are incident to u_i are the y_{BC} colors from the set
$$\{2y_{AC} + y_A + y_B + i, \ldots, 2y_{AC} + y_A + y_B + y_{BC} + i - 1\};$$
- the colors of the edges of $G[E_4]$ that are incident to u_i are the $y_C - 1$ colors from the set
$$\{2y_{AC}+y_A+y_B+y_{BC}+i, \ldots, 2y_{AC}+y_A+2y_B+y_{BC}+i\} \setminus \{2y_{AC}+y_B+y_B+y_{BC}+2i\};$$

Notice that, at the edges incident to u_i, the y_B colors between $(2y_{AC} + y_A + i) \bmod \Delta(G)$ and $(2y_{AC} + y_A + y_B + i - 1) \bmod \Delta(G)$ are not used, as well as the color

$(2y_{AC} + y_A + y_B + y_{BC} + 2i) \bmod \Delta(G)$. As $y_{AC} + y_B + y_{BC} + y_C \leq \Delta(G) = y_{AC} + y_{BC} + y_{AB} + y_A + y_C - 1$, there is no color conflict at u_i.

Since we have shown that there is no color conflict at any vertex $u_i \in Y_C$, we conclude that G is *Class 1*. □

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