POVMs and Naimark's Theorem Without Sums

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Abstract

We provide a definition of POVM in terms of abstract *tensor structure* only. It is justified in two distinct manners. **i.** At this abstract level we are still able to prove Naimark's theorem, hence establishing a bijective correspondence between abstract POVMs and abstract projective measurements (cf. [12]) on an extended system, and this proof is moreover *purely graphical*. **ii.** Our definition coincides with the usual one for the particular case of the Hilbert space tensor product. We also provide a very useful *normal form* result for the classical object structure introduced in [12].

Keywords: POVM, Naimark's theorem, †-compact category, classical object, CPM-construction.

1 Introduction

The work presented in this paper contributes to a line of research which aims at recasting the quantum mechanical formalism in purely *category-theoretic terms* [2,3,12,21], providing it with compositionality, meaningful types, additional degrees of axiomatic freedom, a comprehensive operational foundation, and in particular,

high-level mechanisms for reasoning i.e. *logic*. The computational motivation for this line of research, if not immediately obvious to the reader, can be found in earlier papers e.g. [2]. Particularly informal physicist-friendly introductions to this program are available [7,8,9]. This program originates in a paper by Samson Abramsky and one of the authors [2], and an important contribution was made by Peter Selinger, establishing an abstract definition of mixed state and completely positive map in purely multiplicative terms [21]. The starting point of this paper is a recent category-theoretic definition for projective quantum measurements which does not rely on any additive structure, due to Dusko Pavlovic and one of the authors [12]. We refer to this manner of defining quantum measurements as *coalgebraically*. We show that the usual notion of POVM (e.g. [6,13,19]) admits a purely multiplicative category-theoretic counterpart, in the sense that it is supported both by a *Naimark-type* argument with respect to the coalgebraically defined 'projective' quantum measurements, and by the fact that we recover the usual notion of POVM when we consider the category of Hilbert spaces and linear maps.

Recall that a *projective measurement* is characterised by a set of projectors $\{P_i : \mathcal{H} \to \mathcal{H}\}_i$, i.e. for all i we have $P_i \circ P_i = P_i = P_i^\dagger$, such that $\sum_i P_i = 1_\mathcal{H}$, which implicitly implies that for $i \neq j$ we have $P_i \circ P_j = 0$. To each i we assign an *outcome probability* $\mathrm{Tr}(P_i \circ \rho)$. More generally, a *POVM* is a set of positive operators $\{F_i : \mathcal{H} \to \mathcal{H}\}_i$, i.e. $F_i = f_i^\dagger \circ f_i$ for some linear operator f_i, such that $\sum_i F_i = 1_\mathcal{H}$, and to each i we now assign an *outcome probability* $\mathrm{Tr}(F_i \circ \rho)$. By positivity and by cyclicity of the trace we can rewrite this outcome probability as $\mathrm{Tr}(f_i \circ \rho \circ f_i^\dagger)$. While in the case of projective measurements the state of the system undergoes a change $\rho \mapsto P_i \circ \rho \circ P_i$, for a POVM one typically is only concerned with the probabilities of outcomes, so the *type* of a POVM is

$$\mathsf{POVM} : quantum\ (mixed)\ n\text{-}states\ \to\ classical\ (mixed)\ n\text{-}states\,.$$

Using the fact that classical n-states can be represented by $[0,1]$-valued diagonal $n \times n$-matrices with trace one we can write

$$\mathsf{POVM} :: \rho \mapsto \sum_i \mathrm{Tr}(f_i \rho f_i^\dagger)|i\rangle\langle i|$$

where we used standard Dirac notation to represent the canonical projectors $\{|i\rangle\langle i|\}_i$ with respect to the computational base $\{|i\rangle\}_i$.

2 Abstract CPMs and projective measurements

For the basic definitions of †-compact categories and their interpretation as semantics for quantum mechanics we refer to the existing literature [3,12,21] and references therein. The connection between such categories and graphical calculi is in [1,4,5,14,15,16,17,18,20,21] and references therein. We recall here the CPM-construction due to Selinger [21] and the coalgebraic characterisation of projective measurements due to Pavlovic and one of the authors [12]. This coalgebraic characterisation of projective measurements comprises the definition of *classical object* which captures the behavioral properties of classical data by making explicit the ability to copy and delete this data.

2.1 Mixed states and completely positive maps

A morphism $f : A \to A$ is *positive* if there exists an object B and a morphism $g : A \to B$ such that $f = g^\dagger \circ g$. Graphically this means that we have the following decomposition:

A morphism $f : A \otimes A^* \to B \otimes B^*$ is *completely positive* if there exists an object C and morphisms $g : A \otimes C \to B$ and/or $h : A \to B \otimes C$ such that f is equal to

A *mixed state* $\rho : I \otimes I^* \to A \otimes A^*$, which is a special case of a completely positive map, is the *name* of a positive map (for some $h = g^\dagger$):

Name of f Mixed state

— note that we rely here on the canonical isomorphism $I \simeq I \otimes I^*$. Given any †-compact category, define **CPM(C)** as the category with the same objects as **C**, whose morphisms $f : A \to B$ are the completely positive morphism $f : A \otimes A^* \to B \otimes B^*$ in **C**, and with composition inherited from **C**. As shown in [21], if **C** is †-compact then so is **CPM(C)**, and the morphisms of **CPM(FdHilb)** are the usual completely positive maps and mixed states.

Remark 2.1 It is worth noting that this *purely multiplicative* definition of completely positive maps (i.e. it relies on tensor structure alone) incarnates the *Kraus representation* [19], where the usual summation is now implicitly captured by the *internal trace-* and/or *cotrace*-structure on **CPM(C)** [10], i.e. the half-circles in the pictures representing completely positive maps.

There also is a canonical 'almost' embedding of **C** into **CPM(C)** defined as

$$Pure :: \mathbf{C} \to \mathbf{CPM(C)} : f \mapsto f \otimes f_* \,.$$

From now on, we will omit $(-)^*$ on the objects and $(-)_*$ on the morphisms in the "symmetric image" which is induced by the **CPM**-construction.

2.2 Classical objects

The type we are after for a quantum measurement is

$$A \to X \otimes A$$

expressing that we have as input a quantum state of type A, and as output a measurement outcome of type X together with the collapsed quantum state still of type A. We distinguish between *quantum data A* and *classical data X* by our ability to freely copy and delete the latter. Hence a classical object $\langle X, \delta, \epsilon \rangle$ is defined to be an object X together with a *copying operation* $\delta : X \to X \otimes X$ and a *deleting operation* $\epsilon : X \to I$, which satisfy some obvious behavioral constraints that capture the particular nature of these operations. Let $\lambda_X : X \simeq I \otimes X$ be the natural isomorphism of the monoidal structure and let $\eta_X : I \to X^* \otimes X$ be the *unit* of the †-compact structure for object X.

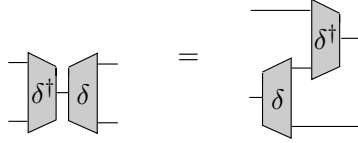**Theorem 2.2** [12] *Classical objects can be equivalently defined as:*

(i) *special †-compact Frobenius algebras $\langle X, \delta, \epsilon \rangle$ which realise*

$$\eta_X = \delta \circ \epsilon^\dagger,$$

where specialness means $1_X = \delta^\dagger \circ \delta$ and the †-Frobenius identity

$$\delta \circ \delta^\dagger = (1_X \otimes \delta) \circ (\delta^\dagger \otimes 1_X)$$

depicts as

(ii) *special X-self-adjoint internal commutative comonoids $\langle X, \delta, \epsilon \rangle$, where X-self-adjointness stands for*

$$\delta = (1_X \otimes \delta^\dagger) \circ (\eta_X \otimes 1_X) \circ \lambda_X \qquad \text{and} \qquad \epsilon = \eta_X^\dagger \circ (1_X \otimes \epsilon^\dagger).$$

which are graphically represented as

   In particular do we have self-duality of X i.e. η_X realises $X^* := X$, and also δ and ϵ prove to be self-dual i.e. $\delta_* = \delta$ and $\epsilon_* = \epsilon$.

2.3 Coalgebraically defined projective measurements

Classical objects, being internal commutative comonoids, canonically induce commutative comonads, so we can consider the Eilenberg-Moore coalgebras with respect to these. This results in the following characterization of quantum spectra as the *X-self-adjoint coalgebras* for those comonads. Given a classical object $\langle X, \delta, \epsilon \rangle$, a *projector-valued spectrum* is a morphism $\mathcal{P} : A \to X \otimes A$ which is X-complete

i.e. $(\epsilon \otimes 1_A) \circ \mathcal{P} = \lambda_A$, and which also satisfies

to which we respectively refer as X-*idempotence* and X-*self-adjointness* and are respectively depicted as

Remark 2.3 It is most definitely worth noting that X-idempotence exactly incarnates *von Neumann's projection postulate*, in a strikingly resource-sensitive fashion: repeating a quantum measurement has the same effect as merely copying the data obtained in the first measurement.

As shown in [12], in **FdHilb** these projector-valued spectra are in bijective correspondence with the usual projector spectra defined in terms of self-adjoint linear operators. In particular, the classical object

$$\left\langle \mathbb{C}^{\oplus n}, \, |\,i\rangle \mapsto |\,ii\rangle, \, |\,i\rangle \mapsto 1 \right\rangle$$

yields the projector spectra of all n-outcome measurements on a Hilbert space of dimension $k \geq n$, where X-idempotence assures projectors to be idempotent ($\mathrm{P}_i^2 = \mathrm{P}_i$) and mutually orthogonal ($\mathrm{P}_i \circ \mathrm{P}_{j \neq i} = \mathbf{0}$), X-self-adjointness assures them to be self-adjoint ($\mathrm{P}_i^\dagger = \mathrm{P}_i$), and X-completeness assures $\sum_{i=1}^{i=n} \mathrm{P}_i = 1_\mathcal{H}$ i.e. probabilities arising from the Born-rule add up to 1.

   Given this representation theorem, and the fact that such a projector-valued spectrum already admits the correct type of a quantum measurement, one might think that projector-valued spectra are in fact quantum measurements. Unfortunately this is not the case: a projector-valued spectrum preserves the relative phases encoded in the initial state. In other words, the off-diagonal elements of the density matrix of the initial state expressed in the measurement basis do not vanish. But this can be easily fixed. In [12] it was shown that these redundant phases can be eliminated by first embedding **C** into **CPM(C)** and then post-composing the image $\mathcal{P} \otimes \mathcal{P}_*$ of a projector-valued spectrum $\mathcal{P}$ under *Pure* with $1_A \otimes \text{Decohere} \otimes 1_A$ where

$$\text{Decohere} := (1_X \otimes \eta_X^\dagger \otimes 1_X) \circ (\delta_X \otimes \delta_X) : X \otimes X \to X \otimes X$$

or, graphically,
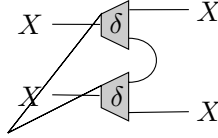
Note that Decohere is indeed a morphism in **CPM(C)**. One also verifies that equivalently one can set Decohere $= \delta \circ \delta^\dagger$. Conclusively, a projective measurement is a composite

$$\mathcal{M} := (1_A \otimes \text{Decohere} \otimes 1_A) \circ (\mathcal{P} \otimes \mathcal{P}_*)$$

where X carries a classical object structure and $\mathcal{P}$ is a corresponding projector-valued spectrum, and is of type $A \to X \otimes A$ in **CPM(C)**.

We will slightly relax this measurement notion by dropping X-completeness, something which is quite standard in quantum information literature where rather than $\sum_i F_i = 1_{\mathcal{H}}$ one regularly only requires $\sum_i F_i \leq 1_{\mathcal{H}}$ for POVMs. The same relaxation applies to our definition of projector-valued spectra.

3   Normalisation

We now provide a normal form result for expressions involving the structural witnesses of †-compactness and classical objects.

A *classical network* is a morphism obtained by composing, tensoring and taking adjoints of δ, ϵ (and hence also of η and identities) and the natural isomorphisms of the symmetric monoidal structure.
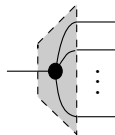
Depicting δ and ϵ as
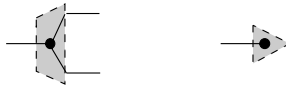
a classical network is connected if its pictorial representation forms a topologically connected whole of dots and lines, which means that there is a *path* from any input, output, or dot to any other input, output, or dot.

Set $\delta_0 := \epsilon^\dagger$ and $\delta_1 := 1_X$ and, for $n \geq 2$,

$$\delta_n := (\delta \otimes 1_{X^{\otimes n-2}}) \circ (\delta \otimes 1_{X^{\otimes n-3}}) \circ ... \circ (\delta \otimes 1_X) \circ \delta .$$

For $n > 2$, δ_n is depicted as
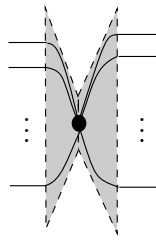

where there are n output wires.

Classical networks of the form

$$\delta_n \circ \delta_m^\dagger : X^{\otimes m} \to X^{\otimes n}.$$

are completely determined by their number of inputs and outputs. For instance, the pair $(0, 1)$ defines $\epsilon^\dagger$, the pair $(1, 2)$ defines δ, the pair $(2, 2)$ defines $\delta \circ \delta^\dagger$ etc. We can depict the classical network $\delta_n \circ \delta_m^\dagger$ as

where the number of wires going in is m and the number of wires going out is n, except for $\delta_1 \circ \delta_1^\dagger = 1_X$ which we depict by a wire without a dot.

We introduce *rewriting rules* which will realise the normalisation process:

Fusion rule: We direct Frobenius identity:

$$(1 \otimes \delta^\dagger) \circ (\delta \otimes 1) \;\overset{=}{\rightsquigarrow}\; \delta \circ \delta^\dagger \;\overset{=}{\leftsquigarrow}\; (\delta^\dagger \otimes 1) \circ (1 \otimes \delta)$$

1st Annihilation rule: We direct (co)monoid (co)unit laws:

$$(\epsilon \otimes 1) \circ \delta \;\overset{=}{\rightsquigarrow}\; 1 \;\overset{=}{\leftsquigarrow}\; (1 \otimes \epsilon) \circ \delta \qquad \delta^\dagger \circ (\epsilon^\dagger \otimes 1) \;\overset{=}{\rightsquigarrow}\; 1 \;\overset{=}{\leftsquigarrow}\; \delta^\dagger \circ (1 \otimes \epsilon^\dagger)$$

**2nd Annihilation rule:** We direct specialness:

$$\delta^\dagger \circ \delta \;\overset{=}{\rightsquigarrow}\; 1$$

Note that each of these rules reduces the number of dots in classical networks.

Lemma 3.1 [normalisation] *Each connected classical network admits a normal form $\delta_n \circ \delta_m^\dagger$ which only depends on its number of inputs and outputs, and is realised using the above described rewriting rules.*

Proof: We sketch the 'proof by rewriting' and illustrate each rewriting step on a

generic example, namely the connected classical network
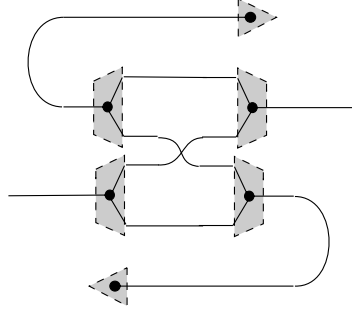
Step 1: Replace all occurrences of η ($\eta^\dagger$) by $\delta \circ \epsilon^\dagger$ ($\epsilon \circ \delta^\dagger$). This substitution does not affect connectedness. Let the resulting number of dots be N.
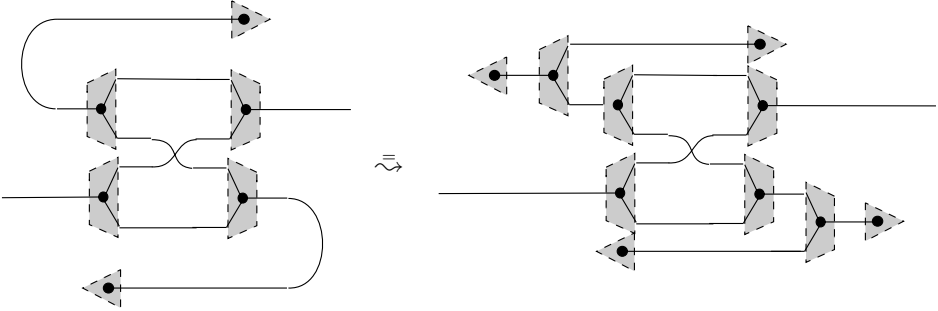
Step 2: Use bifunctoriality to move all ϵ's and $\epsilon^\dagger$'s out of the 'main body of the expression' in order to obtain a composition of the form

$$E_\epsilon \circ E_{\delta,\delta^\dagger} \circ E_{\epsilon^\dagger}$$

where E_ϵ is a tensor product of identities and ϵ's, $E_{\delta,\delta^\dagger}$ a classical network without ϵ's nor $\epsilon^\dagger$'s, and $E_{\epsilon^\dagger}$ a tensor products of identities and $\epsilon^\dagger$'s.
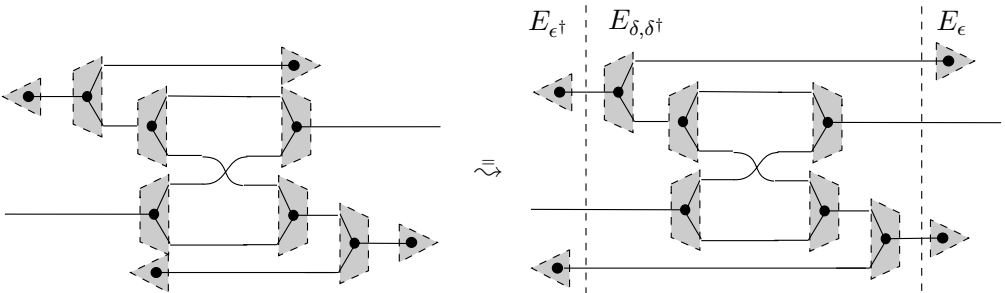

Step 3: Since the components $E_{\epsilon^\dagger}$ and E_ϵ are completely disconnected, the component $E_{\delta,\delta^\dagger}$ has to be connected. Induction on $E_{\delta,\delta^\dagger}$ using the fusion rule to 'move $\delta^\dagger$'s before δ's', using the 1st annihilation rule to cancel out components of the form $\delta^\dagger \circ \delta$, and using (co)associativity and (co)commutativity of δ and $\delta^\dagger$ results in an expression of the form $\delta_k \circ \delta_l^\dagger$ with $k, l > 0$. Indeed, confluence is witnessed by the fact that:

• both rules reduce the total number of dots with at least one,

- as long as the number of dots is at least two we will always be able to apply one of the rules at least one more time due to connectedness,

- we start with a finite number N of dots so rewriting terminates,

- a classical network with either one or no dots can always be rewritten in the normal form by (co)associativity and (co)commutativity.
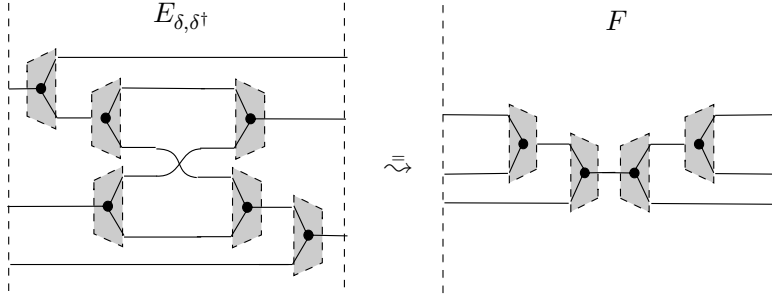
Step 4: In $E_\epsilon \circ (\delta_k \circ \delta_l^\dagger) \circ E_{\epsilon^\dagger}$, by connectedness, all ϵ's ($\epsilon^\dagger$'s) can be cancelled out by the 2nd annihilation rule.
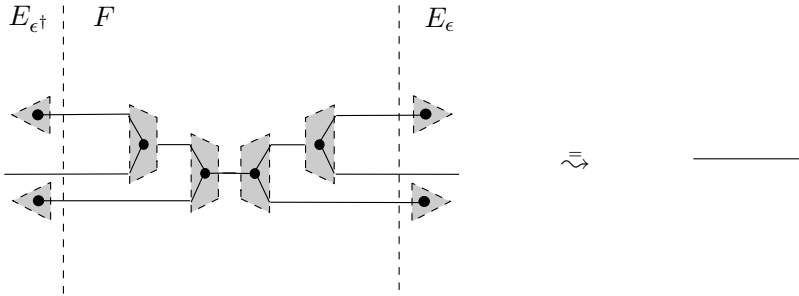
Hence we obtain the desired normal form. $\square$

It is easy to see that this lemma induces a rewriting scheme for the 'classical component of more general expressions', i.e. the part only involving classical object structure, simply by normalising all (maximal) classical networks it comprises while considering the 'boundary' of the classical component as its inputs and outputs. We will make this more precise in future writings.

4 Abstract POVMs

In the same vein as the notions of X-self-adjointness, X-idempotence, and also X-unitarity introduced in [12], we now define the appropriate generalisations of scalars, their inverses, isometries, and positivity of morphisms. This means that we will introduce new classes of morphisms whose types include X, which we interpret as a X-indexed family of morphisms. Most generally, an X-*morphism* is any morphism of type $f : X \otimes A \to B$ where X is a classical object. A more general high-level treatment will be in [11].
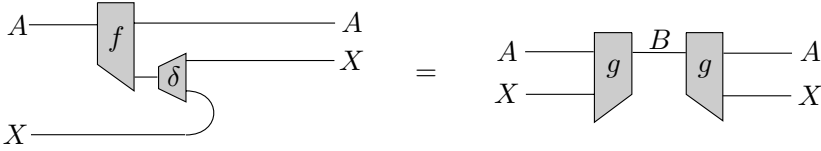
Definition 4.1 An X-*isometry* is a morphism $\mathcal{V} : X \otimes A \to B$ for which

$$\mathcal{V}_\delta := (1_X \otimes \mathcal{V}) \circ (\delta \otimes 1_A) : X \otimes A \to X \otimes B$$

is an isometry i.e. it satisfies

$$\mathcal{V}_\delta^\dagger \circ \mathcal{V}_\delta = 1_{X \otimes A}\,.$$

**Definition 4.2** A morphism $f : A \to A \otimes X$ is X-*positive* if there exists an X-morphism $g : B \to A \otimes X$ such that

In the second picture, the fact that the trapezoid on the left points with its sharp corner to the left, as compared to trapezoid on the right of which the sharp corner points to the right, indicates that it is "daggered" as compared to the one on the right. This graphical convention will be reused in what follows.

Recall that a *polar decomposition* of a linear operator M is a factorisation of $M = V \circ H$ where V an isometry and H is positive.

Definition 4.3 We say that an X-morphism $f : A \to B \otimes X$ is X-*polar decomposable* if there exists an X-positive morphism $g : A \to X \otimes A$ and an X-isometry $\mathcal{V} : X \otimes A \to B$ such that $f = \mathcal{V}_\delta \circ g$ i.e. f can be depicted as
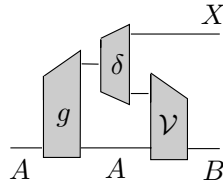
Definition 4.4 An X-*scalar* is a morphism $f : I \to X$. An X-scalar $t : I \to X$ is an X-*inverse* of $s : I \to X$ iff, setting $\lambda_I : I \simeq I \otimes I$, we have

$$\delta^\dagger \circ (s \otimes t) \circ \lambda_I = \epsilon^\dagger\,.$$

In **FdHilb** X-scalars are n-tuples of complex numbers. An X-scalar's X-inverse in **FdHilb** is the n-tuple consisting of the component-wise inverses to the given n-tuple. In our context, X-scalars will arise when tracing out A in a morphism $f : A \to A \otimes X$, yielding the X-scalar $\mathrm{Tr}_{I,X}^A(f) : I \to X$. Graphically an X-scalar is represented as
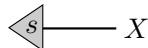
From now on, we will work within **CPM(C)**. Classical objects will however always be defined in **C**, and then embedded in **CPM(C)** via *Pure*.

Definition 4.5 [POVM] Let $\langle X, \delta, \epsilon \rangle$ be a classical object. A *POVM* on a system of type A which produces outcomes in X is a morphism
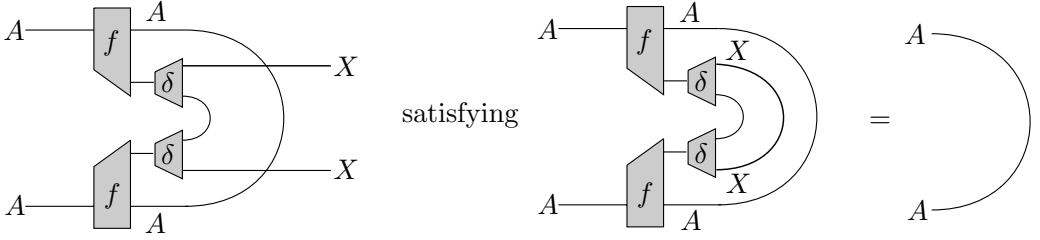
and for which $f \in \mathbf{C}(A, X \otimes A)$ is X-polar-decomposable.

Hence, within **CPM(C)** the type of such a POVM is indeed $A \to X$. In **FdHilb** the requirement on X-polar-decomposability is of course trivially satisfied since any linear map admits a polar decomposition.

Theorem 4.6 *In the category* **FdHilb** *the abstract POVMs of Definition 4.5 exactly coincide with the assignments $\rho \mapsto \sum_i \mathrm{Tr}(g_i \rho g_i^\dagger)|i\rangle\langle i|$ corresponding to POVMs defined in the usual manner* (cf. Section 1).

Proof. Consider a POVM as in Definition 4.5. In **FdHilb** a classical object is of the form $\mathbb{C}^{\oplus n}$ and induces canonical base vectors $|i\rangle : \mathbb{C} \to \mathbb{C}^{\oplus n}$. Set

$$\hat{f}_i := (\langle i | \otimes 1_A) \circ f : A \to A \quad \text{and} \quad f_i := (|i\rangle\langle i| \otimes 1_A) \circ f : A \to X \otimes A.$$

In particular do we have $f = \sum_{i=1}^{i=n} f_i$. Hence, we can rewrite the POVM as

$$\mathrm{tr}^A\Big[\mathrm{Decohere} \circ \Big(\sum_i f_i \otimes \sum_j f_{j*}\Big) \circ -\Big] = \mathrm{tr}^A\Big[\mathrm{Decohere} \circ \sum_{i,j}(f_i \otimes f_{j*}) \circ -\Big]$$

$$= \mathrm{tr}^A\Big[\sum_i (f_i \otimes f_{i*}) \circ -\Big].$$

Passing from **CPM(C)** to standard Dirac notation, i.e. from $|i\rangle \otimes |i\rangle_*$ to $|i\rangle\langle i|$ and from $(f \otimes f_*) \circ -$ to $f(-)f^\dagger$, also using $f_i = (|i\rangle \otimes 1_A) \circ \hat{f}_i$, we obtain

$$\sum_i \mathrm{Tr}(\hat{f}_i(-)\hat{f}_i^\dagger)|i\rangle\langle i|.$$

Using the polar decomposition of $\hat{f}_i$ and cyclicity of the trace we get

$$\sum_i \mathrm{Tr}(\hat{f}_i(-)\hat{f}_i^\dagger)|i\rangle\langle i| = \sum_i \mathrm{Tr}(U_i g_i(-)g_i^\dagger U_i^\dagger)|i\rangle\langle i|$$

$$= \sum_i \mathrm{Tr}(g_i(-)g_i^\dagger)|i\rangle\langle i|$$

which is the intended result. Finally, the abstract normalisation condition tells us that indeed, $f^\dagger \circ f = 1_A$ and so $g^\dagger \circ g = 1_A$. The converse direction constitutes analogous straightforward translation in the graphical language. □

Theorem 4.7 [Abstract Naimark theorem] *Given an abstract POVM, there exists an abstract projective measurement on an extended system which realises this POVM. Conversely, each abstract projective measurement on an extended system yields an abstract POVM.*

Proof: We need to show that there exists a projective measurement $h : C \otimes A \to C \otimes A \otimes X$ in **C** together with an auxiliary input $\rho : I \to C$ in **CPM(C)** such that they produce the same probability as a given POVM defined via $f : A \to A \otimes X$, as in Definition 4.5, provided we trace out the extended space after the measurement. Graphically this boils down to
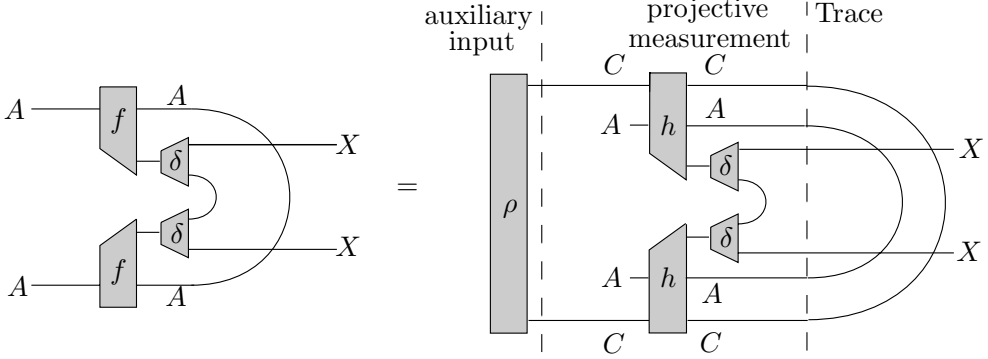
i.e. an equality between two morphisms of type $A \to X$ in **CPM(C)**. Firstly we exploit X-polar-decomposability of f. Factoring out f graphically yields
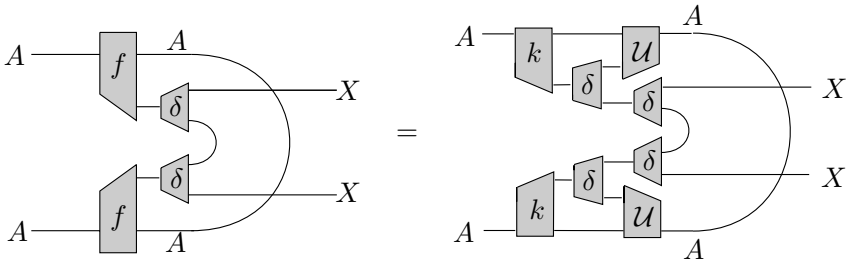

which by graphical manipulation and coassociativity of δ rearranges as
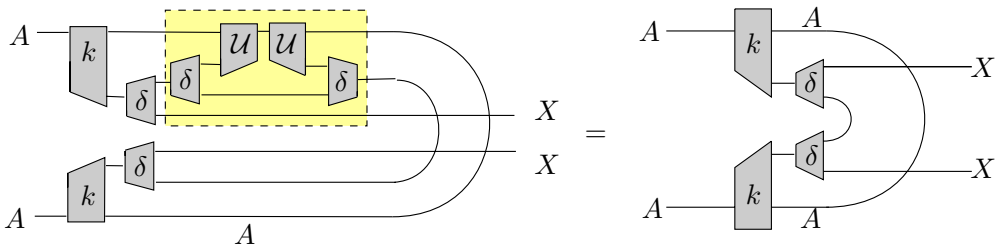
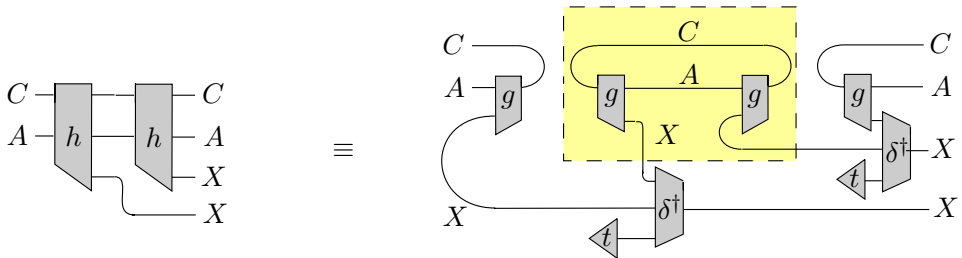The pale square on the right-hand-side vanishes $\mathcal{U}$ being an X-isometry. Set

where $k = g^\dagger \circ g$ by X-positivity as in Definition 4.2 (we will consider g to be fixed

for the reminder of the proof), where $t := \left(\mathrm{Tr}^A(f)\right)^{-1}$ is an X-scalar, [5] and where the $\delta^\dagger$ with three input wires is $(1_X \otimes \delta^\dagger) \circ \delta^\dagger$ — which is meaningful by associativity of the comultiplication. Let
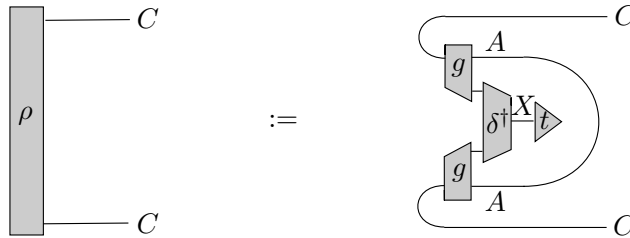
We now check X-idempotence of h. We have

Via X-positivity of f, the pale square on the previous picture becomes $\delta \circ s$ where $s := \mathrm{Tr}^A(f)$ is an X-scalar which is inverse to the X-scalar t. Factoring out the X-scalars, using normalisation and cancelling relative inverse X-scalars, we obtain the following equality between the pale squares below
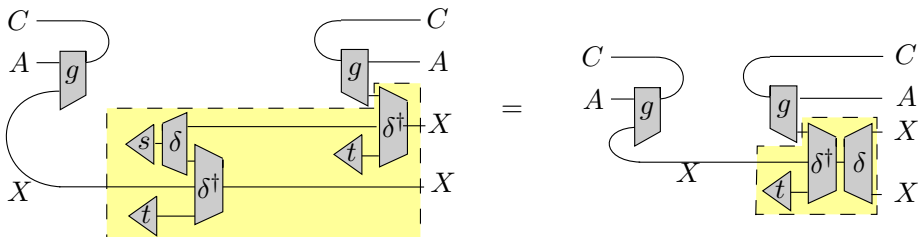

so we indeed obtain X-idempotence for h. It should be obvious that h is also X-self-adjoint by construction, so h defines a (not necessarily X-complete) projector-valued spectrum, and hence defines a projective measurement by adjoining the Decohere-morphism. Next we show that this projective measurement indeed realises the given POVM when feeding-in the mixed state ρ, as defined above, to its C-input, and when tracing-out the A-output. In the following, we will ignore the Decohere-morphism

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[5] It was observed by Pavlovic and one of the authors that every †-compact category **C** admits a universal *localization* $L\mathbf{C}$ together with a †-compact functor $\mathbf{C} \to L\mathbf{C}$, which is initial for all †-compact categories with †-compact functors from **C**, and where a †-compact category is *local* iff all of its positive scalars are either divisors of zero, or invertible, where *zero* is multiplicatively defined in the obvious manner. These considerations extend to X-scalars. This result will appear in a forthcoming paper.
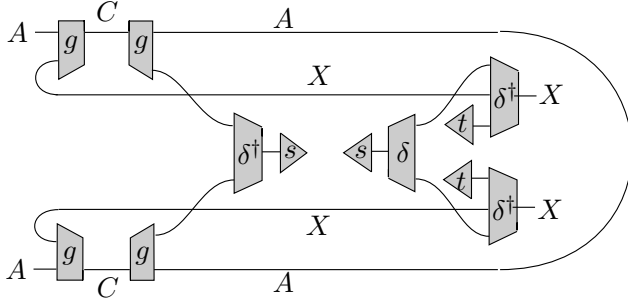
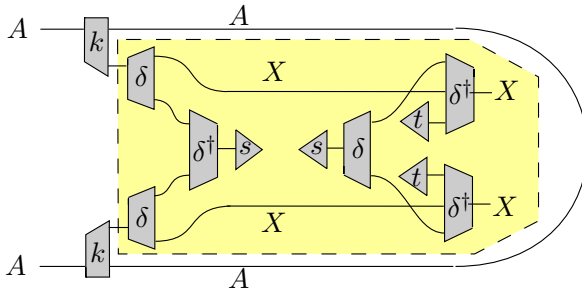since, as we will see later, it will cancel as it is idempotent. Now, in
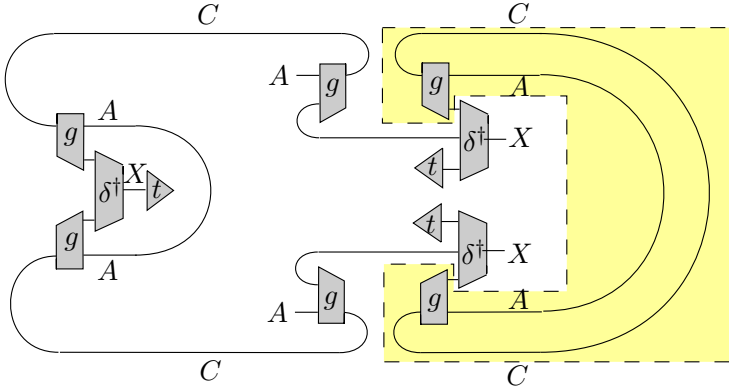
the pale square is $\delta \circ s$ by X-positivity of f. Hence we then obtain
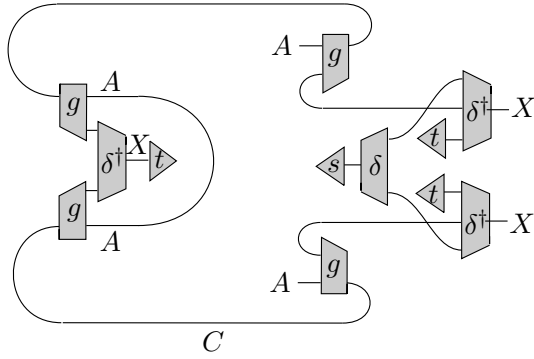

Via an obvious graph isomorphism we get

Again, by X-positivity of f, we obtain

The pale square in the previous picture reduces to the Decohere-morphism if first,

we factor out the X-scalars, we apply normalisation and cancel out the relative inverse X-scalars. Re-adjoining the Decohere-morphism which we omitted, which now cancels out by Decohere's idempotence, we finally obtain
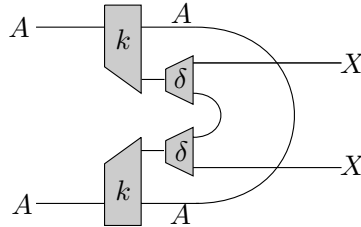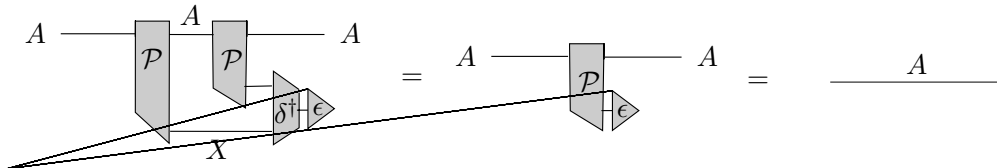

Conversely, we need to show that each projective measurement on an extended system yields a POVM. A projector-valued spectrum is X-positive since its X-idempotence and X-self-adjointness yield
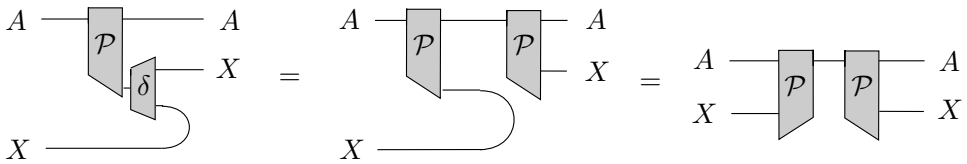
Next, observe that for an X-complete projector-valued spectrum we always have $\mathcal{P}^\dagger \circ \mathcal{P} = 1_A$ since
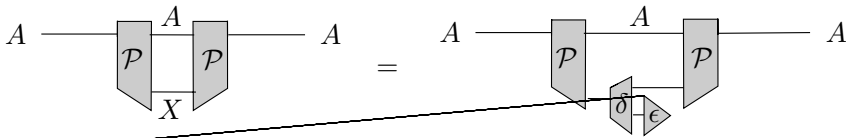
and by X-self-adjointness of $\mathcal{P}$ and δ we get

where the first equality uses X-idempotence of $\mathcal{P}$ and $\delta^\dagger \circ \delta = 1_A$. The second equality is obtained from the definition of X-completeness. Now, when considering a projective measurement on an extended system, using this fact together with $\delta^\dagger \circ \delta = 1_X$ we obtain
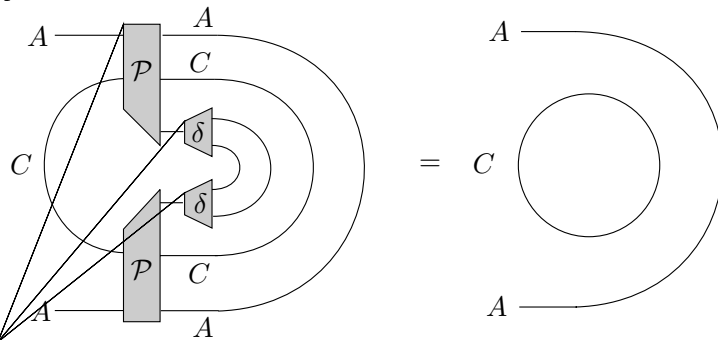
thence satisfying the normalisation condition up to a C-dependent scalar. The POVM which we obtain is

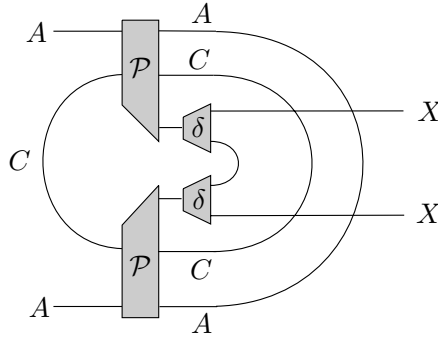what completes the proof.   □

Remark 4.8 Manipulation of classical data in the above proof is extremely simplified by the normalisation lemma. A more refined version of this result together with its consequences will be given and discussed in a forthcoming paper [11].

Remark 4.9 While POVMs are not concerned with the state after the measurement, our analysis does produce an obvious candidate for non-destructive generalised measurements, sometimes referred to as PMVMs in the literature [13]. We postpone a discussion to forthcoming writings.

Remark 4.10 Notice the delicate role which X-completeness and normalisation of the POVMs plays in all this, on which, due to lack of space, we cannot get into. We postpone this discussion to an extended version of the present paper, which is also forthcoming.

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