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# Total Domination in Regular Graphs

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**Abstract**

We find new upper bounds on the size of a minimum totally dominating set for random regular graphs and for regular graphs with large girth. These bounds are obtained through the analysis of a local algorithm using a method due to Hoppen and Wormald [17].

*Keywords:* Total domination, Random regular graphs, Large girth.

## 1 Introduction and Main Results

This paper is concerned with totally dominating sets in graphs. As usual, a graph $G = (V, E)$ consists of a *vertex set* $V$ and of an *edge set* $E \subseteq \{\{u, v\}: u, v \in V, u \neq v\}$. Even though we use standard graph-theoretical notation and terminology, we define concepts that appear in the statement of the main results of this paper. For other definitions, we refer the reader to [1].

There is a multitude of parameters that are related with the general notion of domination in graphs. The most studied version is the *domination number* $\gamma(G)$ of a graph $G = (V, E)$. A set $S \subseteq V$ is a *dominating set* of $G$ if every vertex in $V \setminus S$ is adjacent to some vertex in $S$. The *domination number* is the minimum size of a dominating set of $G$, that is,

$$\gamma(G) = \min\{|S|: S \text{ is a dominating set of } G\}.$$

The following related notion has been introduced by Cockayne, Dawes and Hedetniemi [7]. A *totally dominating set* $S \subseteq V$ is a set such that every vertex $v \in V$ is

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adjacent to a vertex in $S$. Clearly, any totally dominating set is also a dominating set, and there is a totally dominating set in a graph if and only if it does not have isolated vertices. Naturally, the *total domination number* $\gamma_t(G)$ of a graph $G$ with no isolated vertices is defined as

$$\gamma_t(G) = \min\{|S| \colon S \text{ is a totally dominating set of } G\}.$$

Computing the value of $\gamma(G)$ is a notoriously hard problem, which appears on the original list of NP-complete problems provided by Karp [19]. Pfaff, Laskar and Hedetniemi [20] proved that computing $\gamma_t(G)$ is also NP-complete. For results and references about domination and total domination, we refer to Haynes, Hedetniemi and Slater [13] and to Henning and Yeo [15], respectively.

There has been a large number of upper and lower bounds on the size of a minimum totally dominating set in an $n$-vertex graph $G$. Since the addition of a vertex to a set may only dominate its neighbors, it is clear that $\gamma_t(G) \geq n/\Delta(G)$, where $\Delta(G)$ denotes the maximum degree of $G$. On the other hand, Henning and Yeo [15, Theorem 5.1] proved that $\gamma_t(G) \leq \left(\frac{1+\ln(\delta)}{\delta}\right) n$, where $\delta(G) \geq 1$ is the minimum degree of $G$. A natural setting for comparing upper and lower bounds of this type are *d-regular* graphs, namely graphs where every vertex is incident with $d$ edges, so that $\delta(G) = \Delta(G) = d$ (we shall always assume that $nd$ is even).

In general, the size of a minimum totally dominating set may still vary considerably among $n$-vertex $d$-regular graphs. For instance, if $G$ is a collection of disjoint complete bipartite graphs $K_{d,d}$, we have $\gamma_t(G) = n/d$, as every component is totally dominated by one vertex of each side of the bipartition. This shows that the trivial lower bound on $\gamma_t(G)$ mentioned above is sharp for all $d$. On the other hand, if $G$ is a collection of disjoint complete graphs $K_{d+1}$, we have $\gamma_t(G) = 2n/(d+1)$, which is substantially larger. Table 1 gives upper bounds $\Gamma_0(G)$ on the size of a minimum totally dominating set in a $d$-regular graph $G$ (actually, these bounds have been obtained for $\delta(G) = d$). As it turns out, the upper bounds in the table are sharp for $d$-regular graphs for $d \in \{2, 3, 4\}$.

Two traditional ways to investigate the behavior of a graph-theoretical parameter on $d$-regular graphs are to consider its *typical value*, namely its value for a randomly chosen $d$-regular graph, and to consider its value on graphs with *large girth*. The *girth* of a graph is the length of a shortest cycle in the graph. Properties of random regular graphs have been intensively studied (see [24] for a survey of results in such probabilistic models). The effect of large girth on graph parameters has also been of interest at least since Erdős [11] showed that, for any given positive integers $k$ and $g$, there is a graph with girth at least $g$ and chromatic number at least $k$, which shows the global nature of the chromatic number of a graph.

Regarding the effect of large girth on the total domination number of a $d$-regular graph $G$, Henning and Yeo [16] showed that, if $G$ is an $n$-vertex graph with $\delta(G) \geq 2$ and girth $g \geq 3$, then

$$\gamma_t(G) \leq \left(\frac{1}{2} + \frac{1}{g}\right) n.$$

| $d$ | $\Gamma_0(G)/\lvert V(G)\rvert$ | Source | $\Gamma_d^g$ |
|---|---|---|---|
| 2 | 2/3 | See footnote [4] | 1/2 [16] |
| 3 | 1/2 | [2] (2004) | 0.4883 |
| 4 | $3/7 \simeq 0.4285$ | [23] (2007) | 0.4136 |
| 5 | $17/44 \simeq 0.3863$ | [8] (2015) | 0.3656 |
| 6 | 0.4653 | [15, Theorem 5.1] | 0.3256 |

Table 1
Upper bounds on the size of a minimum totally dominating set in a $d$-regular graph $G$, the corresponding references and approximations of the upper bound $\Gamma_d^g$ on the parameter $\gamma_t^g(d,\infty)$ obtained in this paper.

In particular, this shows that the trivial lower bound is asymptotically optimal for 2-regular graphs as $g \to \infty$. (This fact can also be proved directly by looking at totally dominating sets of long cycles.) We study this parameter for general $d$. Precisely, for $d \geq 2$ and $g_0 \geq 3$, let

$$\gamma_t^g(d,g_0) := \sup\{\gamma_t(G)/|V(G)|\colon G \text{ is } d\text{-regular with girth } g \geq g_0\}, \tag{1}$$

that is, $\gamma_t^g(d,g_0)$ is the smallest possible upper bound on $d$-regular graphs with girth at least $g_0$. This produces a monotone sequence as $g_0$ increases, and we consider the parameter

$$\gamma_t^g(d,\infty) = \lim_{g_0\to\infty} \gamma_t^g(d,g_0). \tag{2}$$

Henning's result shows that $\gamma_t^g(2,\infty) = 1/2$. One of the main results in our paper, which will be described in detail below (Theorem 1.3), implies the following upper bounds on $\gamma_t^g(d,\infty)$ for some fixed values of $d$.

**Theorem 1.1** *For integers $3 \leq d \leq 6$, we have $\gamma_t^g(d,\infty) \leq \Gamma_d^g$ for the values of $\Gamma_d^g$ given in Table 1.*

We now consider the typical value of the total domination number on a large $d$-regular graph. To this end, let $\mathbb{G}_{n,d}$ be the set of (labelled) $n$-vertex $d$-regular graphs and, for an integer $d \geq 2$ and a constant $\varepsilon > 0$, consider

$$\gamma_t^R(d,\varepsilon) = \inf_{\substack{\mathcal{A}\subseteq\mathbb{G}_{n,d}, n\in\mathbb{N},\\ |\mathcal{A}|\geq(1-\varepsilon)|\mathbb{G}_{n,d}|}} \sup\left\{\frac{\gamma(G)}{n}\colon G\in\mathcal{A}\right\}. \tag{3}$$

Note that, for fixed $d$, $\gamma_t^R(d,\varepsilon)$ is bounded and increases as $\varepsilon$ decreases, so that this limit is well-defined:

$$\gamma_t^R(d) = \lim_{\varepsilon\to 0^+} \gamma_t^R(d,\varepsilon). \tag{4}$$

Let $\mathcal{G}_{n,d}$ denote the probability space with sample space $\mathbb{G}_{n,d}$ and uniform probability distribution. In the language of probability, finding an upper bound $\Gamma_d^r$ on $\gamma_t^R(d)$ means that a random $d$-regular graph *asymptotically almost surely* (a.a.s.) has a minimum totally dominating set of size at most $\Gamma_d^r$.

A well-known construction, which uses the fact that random $d$-regular graphs a.a.s. have a small number of cycles of bounded length [5,25], allows us to prove that

$$\gamma_t^R(d) \leq \gamma_t^g(d, \infty), \tag{5}$$

so that any deterministic upper bound on the total domination number of $d$-regular graphs with large girth gives us an upper bound on the total domination number of a typical $d$-regular graph. This leads to the following.

**Theorem 1.2** *For integers $3 \leq d \leq 6$, a graph $G \in \mathcal{G}_{n,d}$ asymptotically almost surely satisfies $\gamma_t(G)/n \leq \Gamma_d^g$ for the values of $\Gamma_d^g$ given in Table 1.*

In fact, the connection between the behavior of graph parameters for graphs with large girth and for random regular graphs given in (5) in the context of total domination actually holds for many different parameters, and it is a significant open question whether (5) holds with equality (see Backhausz and Szegedy [3] for a detailed description of problems in this line of research).

Recently, Wormald and the first author [17] proved that an upper bound on $\gamma_t^R(d)$ implies an upper bound on $\gamma_t^g(d, \infty)$ as long as it is obtained through the analysis of a *local algorithm*, as described in their paper. (Again, the previous sentence would hold for a host of parameters other than total domination.) In this paper, this result by Hoppen and Wormald plays a fundamental role as the tool that allows us to transfer a result obtained in the random graph context (Theorem 1.2) to the large-girth context (Theorem 1.1).

We should also mention that the ability of local algorithms to approximate the value of graph parameters for graphs with large girth has attracted a lot of attention. Recently, Gamarnik and Sudan [12] showed that, for sufficiently large $d$, local algorithms cannot approximate the size of the largest independent set in a $d$-regular graph of large girth with an arbitrarily small multiplicative error. The approximation gap was improved by Rahman and Virág [21].

The proof of our results use the approach described in [17]. We use a method due to Wormald [26], known as the differential equation method, to analyse the performance of a specific local algorithm that produces a totally dominating set in an input graph $G$ when this algorithm is applied to a random regular graph $G \in \mathcal{G}_{n,d}$. We then translate this result to all graphs with sufficiently large girth using [17, Theorem 8.1]. The differential equation method is a concentration-type result that has proven to be very successful in the analysis of random processes. In the particular case of random regular graphs, it has already been used to study parameters related with domination, see [9,10], and results for graphs with large girth using the general approach described above have also been proved in [18].

To be more precise, Theorems 1.1 and 1.2 follow from the following technical result, which holds for any fixed $d \geq 3$.

**Theorem 1.3** *For any $d \geq 3$ and $\varepsilon > 0$, a random graph $G \in \mathcal{G}_{n,d}$ asymptotically almost surely contains a totally dominating set $D_T \subseteq V(G)$ such that*

$$|D_T| \leq n\left(q(x^*) + \varepsilon\right),$$

*where* $z_0(x)$ *and* $q(x)$ *are solutions to the initial value problem (7) and* $x^* = \inf\{x > 0 : z_0(x) = 0\}$.

The system of differential equations mentioned in the statement of the theorem arises naturally as we analyse the algorithm described in Section 2. It is not easy to compute the value of $q(x^*)$ analytically, and the values of $\Gamma_d^g$ in Table 1 are numerical approximations of this quantity for some values of $d$ (the fourth decimal place has been rounded up). This immediately leads to the conclusion of Theorem 1.2. To derive Theorem 1.1, we just apply [17, Theorem 8.1], which requires that we check that the algorithm satisfies a particular set of rules described in [17].

The remainder of the paper is structured as follows. In Section 2, we present our algorithm and we describe the setting in which the analysis is carried out. Section 3 contains information about the proof of our main results.

## 2 A heuristic to produce small totally dominating sets

In addition to many other applications, the differential equation method has been used to analyze the typical performance of a large number algorithms on random regular graphs. As in most applications of this method to random regular graphs, instead of working directly with regular graphs, we use the approach of Bollobás [5], known as the *configuration model*, which considers the following probability space whose elements may be generated by the following simple randomized procedure. Start with $nd$ points in $n$ buckets labelled $1, \dots, n$, with $d$ points in each bucket, and choose uniformly at random (u.a.r.) a *pairing* $P = a_1, \dots, a_{dn/2}$ of the points such that each $a_i$ is an unordered pair of points, and each point is in precisely one pair $a_i$. As usual $\mathcal{P}_{n,d}$ denotes the probability space of pairings. By collapsing each bucket into a single vertex, we see that each pairing corresponds to a $d$-regular pseudograph (loops and multiple edges permitted) with vertex set $1, \dots, n$ and with an edge for each pair. A pair with points in buckets $i$ and $j$ gives rise to an edge joining vertices $i$ and $j$. A straightforward calculation shows that any two simple $d$-regular graphs (i.e. with no loops or multiple edges) on $n$ vertices are produced with the same probability. For fixed $d$, a crucial property is that the probability that a random pairing produces a $d$-regular graph tends to the positive constant $e^{(1-d^2)/4}$ as $n$ tends to infinity (Bender and Canfield [4]), and so results that hold a.a.s. for random pairings in $\mathcal{P}_{n,d}$ must also hold a.a.s. for random $d$-regular graphs.

We choose the pairs sequentially: the first point in a pair can be selected using any rule that depends only on the choices made so far, as long as the second is chosen u.a.r. from the remaining points. We call this *exposing* the pair, and this property is the *independence property* of the model. For simplicity, we shall refer to graphs and vertices even though we mean pairings and buckets. Here, we consider the following heuristic.

We start with an $n$-vertex graph $G_0$ where no edges have been exposed and with a set $D_0 = \emptyset$. The idea is to generate a random $d$-regular graph by sequentially exposing its edges and, at the same time, produce a totally dominating set $D$. Our construction proceeds by rounds that are labeled by a discrete parameter $t$. For each $t \in \{0, \ldots, T_C\}$, where $T_C$ satisfies a termination condition, we produce $G_{t+1}$ and $D_{t+1}$ from $G_t$ and $D_t$, respectively, according to the following rules:

(1) Choose a vertex $v_t$ u.a.r. among all vertices of degree 0 in $G_t$ and expose a vertex $u_t$ to be adjacent to $v_t$.

(2) If the degree of $u_t$ in $G_t$ is 0, then expose all remaining neighbors of both $u_t$ and $v_t$ to produce $G_{t+1}$ and define $D_{t+1} = D_t \cup \{v_t, u_t\}$.

(3) If the degree of $u_t$ in $G_t$ is not 0, then expose all remaining neighbors of $u_t$ to produce $G_{t+1}$ and define $D_{t+1} = D_t \cup \{u_t\}$.

Note that, if $G_t$ generates a simple graph, the set of vertices of degree 0 in $G_t$ is precisely the set of vertices that are not dominated by vertices in $D_t$. As a consequence, this sequence of steps should be performed until $G_t$ does not contain any isolated vertices. However, for technical reasons associated with the analysis, we need to consider an additional parameter $\varepsilon > 0$ and we perform the algorithm until the number of vertices of degree 0 in $G_t$ falls below $\varepsilon n$. Then the set $D_{T_C}$ obtained at the end of the heuristic is not yet a totally dominating set, but may be turned into a totally dominating set by adding a neighbor of each vertex that has not yet been dominated.

The relevant variables associated with this heuristic are $Q(t) = |D_t|$ and $Y_i(t)$, the number of vertices of degree $i$ in $G_t$, for $i \in \{0, \ldots, d\}$. In fact, since vertices of degree $d$ do not affect the remainder of the application of the algorithm, we ignore the variable $Y_d(t)$. We write $h_t = (G_0, \ldots, G_t)$ to denote the history of the process to time $t$ (that is, the results obtained in an actual application of the heuristic up to round $t$). The basic idea of the differential equation method is to keep track of the expected value of each variable at each round. If some technical conditions are satisfied, powerful results by Wormald, see for instance [26, Theorem 5.1], imply that the actual values of the variables are a.a.s. close to their expected value *for all* $t \in \{0, \ldots, T_C\}$. To achieve them, we shall prove that the following conditions are met (we observe that some of them are stronger than what is actually needed for [26, Theorem 5.1]):

(i) There is an absolute constant $\beta = \beta(d)$ such that

$$1 \leq Q(t+1) - Q(t) \leq 2 \text{ and } \max_{0 \leq j \leq d-1} |Y_j(t+1) - Y_j(t)| \leq \beta$$

for all $j \in \{0, \ldots, d-1\}$ and all $t \in \{0, \ldots, T_D\}$.

(ii) There exist functions $f_0, f_1, \ldots, f_{d-1}, f_d : \mathbb{R}^{d+1} \to \mathbb{R}$ and $\lambda_1 = \lambda_1(n) = o(1)$ such that, for all $0 \leq j \leq d-1$,

$$|\mathbb{E}[Y_j(t+1) - Y_j(t) | h_t] - f_j(t/n, Y_0(t)/n, \ldots, Y_{d-1}(t)/n)| \leq \lambda_1(n)$$

and

$$|\mathbb{E}[Q(t+1) - Q(t) | h_t] - f_d(t/n, Y_0(t)/n, \ldots, Y_{d-1}(t)/n)| \leq \lambda_1(n)$$

for all $t < T_D$.

(iii) The functions $f_j$ defined in (ii) are Lipschitz continuous in a domain

$$D \cap \{(t, z_0, \dots, z_{d-1}) : t \geq 0\},$$

where $D$ is an open, connected and bounded set containing the point $= (x_0, z_0, \dots, z_{d-1}) = (0, 1, 0, \dots, 0)$, which represents that, at the beginning of the algorithm, $Y_i(0) = z_i n$ for $1 \leq i \leq d-1$.

Roughly speaking, condition (i) tells us that the variables cannot vary substantially in a single round of the heuristic, condition (ii) tells us that the expected change in the variables (conditional on the history of the process) may be calculated, while condition (iii) tells us that these expected changes are described by well-behaved functions. If these conditions are met, Theorem 5.1 [26] establishes that the system of differential equations associated with the functions $f_j$ has a unique solution $(z_0(x), \dots, z_{d-1}(x), q(x))$ with initial conditions $z_0(0) = 1$, $z_i(0) = 0$ for $i \geq 1$ and $q(0) = 0$.

Moreover, the variables $Q(t)$ and $Y_i(t)$ are approximated throughout the process by the solutions of a system of differential equations involving the functions defined in (ii). More precisely, for $\lambda > \lambda_1$, there is an absolute constant $C$ such that, with probability $1 - O\left(\frac{\beta}{\lambda} \exp\left(-\frac{n\lambda^3}{\beta^3}\right)\right)$, we have

$$Y_j(t)/n = z_j(t/n) + O(\lambda), \quad Q(t)/n = q(t/n) + O(\lambda) \tag{6}$$

for all $j$ and all $0 \leq t \leq \sigma n$, where $\sigma = \sigma(n)$ is the supremum of all $x$ such that the solution to the system of differential equations may be extended up to distance at most $C\lambda$ from the boundary of $D$.

## 3 Proving our main results

In this section, we argue that the conditions (i), (ii) and (iii) described in the previous section are satisfied for our heuristic. In particular, we compute the functions $f_0, \dots, f_{d-1}, f_d$ that give rise to the system of differential equations that mentioned in the statement of Theorem 1.3.

We first note that $\beta = 2d$ is a trivial bound for (i), as we expose at most $2d-1$ pairings at each round, involving at most $2d$ vertices. To verify (ii), we first compute $\mathbb{E}[X(t+1) - X(t)|h_t]$ for each variable $X$. In fact, the conditional expectations in our process may be computed based on $G_t$, rather than in the full history $h_t$. We are able to prove that

$$\begin{aligned}
&\mathbb{E}[Y_j(t+1) - Y_j(t)|G_t] \\
&= \sum_{i=1}^{d-1} \frac{(d-i)Y_i(t)}{S(t)} \left[-\delta_{i,j} + \delta_{1,j} + (d-i-1)\left(\frac{(d-j+1)Y_{j-1}(t)}{S(t)} - \frac{(d-j)Y_j(t)}{S(t)}\right)\right] \\
&\quad -\delta_{0,j} + \frac{dY_0(t)}{S(t)} \left[-\delta_{0,j} + (2d-2)\left(\frac{(d-j+1)Y_{j-1}(t)}{S(t)} - \frac{(d-j)Y_j(t)}{S(t)}\right)\right] + O\left(\frac{1}{\sqrt{n}}\right)
\end{aligned}$$

and that

$$\mathbb{E}[Q(t+1) - Q(t)|G_t] = 1 + \frac{dY_0(t)}{S(t)} + O\left(\frac{1}{\sqrt{n}}\right),$$

where $S(t) = \sum_{m=0}^{d-1}(d-m)Y_m(t)$ and $\delta_{i,j}$ is the function with values in $\{0,1\}$ such that $\delta_{i,j}=1$ if and only if $i=j$. The variation in the number of vertices of degree $j$ depends basically on the number of vertices of degree $j$ and $j-1$ in the previous step.

Normalizing the quantities involved in the recurrence relations by

$$x := t/n, \quad y_i(x) := Y_i(xn)/n, \quad q(x) := Q(xn)/n,$$

and letting $n \to \infty$, we may see the recurrence relations as a discretization of the following system of differential equations and initial conditions:

$$\begin{cases} z_j'(x) \;=\; f_j(x, z_0, z_1, \ldots, z_{d-1}) \quad \text{for all } 0 \le j \le d-1 \\ q'(x) \;=\; f_d(x, z_0, z_1, \ldots, z_{d-1}) \\ z_0(0) \;=\; 1,\ z_j(0) = 0 \text{ for } 1 \le j \le d-1,\ q(0) = 0, \end{cases} \tag{7}$$

where the functions $f_j$ are defined by

$$\begin{aligned} &f_j(x, z_0, z_1, \ldots, z_{d-1}) \\ &= \sum_{i=1}^{d-1} \frac{(d-i)z_i(x)}{s(x)}\left[-\delta_{i,j} + \delta_{1,j} + (d-i-1)\left(\frac{(d-j+1)z_{j-1}(x)}{s(x)} - \frac{(d-j)z_j(x)}{s(x)}\right)\right] \\ &\quad -\delta_{0,j} + \frac{dz_0(x)}{s(x)}\left[-\delta_{0,j} + (2d-2)\left(\frac{(d-j+1)z_{j-1}(x)}{s(x)} - \frac{(d-j)z_j(x)}{s(x)}\right)\right] \end{aligned}$$

for $j \in \{0, \ldots, d-1\}$ and $s(x) = \sum_{m=0}^{d-1}(d-m)z_m(x)$. Moreover,

$$f_d(x, z_0, z_1, \ldots, z_{d-1}) = 1 + \frac{dz_0(x)}{s(x)}.$$

At this point, we have found the functions $f_j$ that verify (ii) with $\lambda_1 = O(1/\sqrt{n})$.

Next we need to define a domain $D \subseteq \mathbb{R}^{d+1}$ for which (iii) is satisfied. For $\varepsilon > 0$, let $D_\varepsilon$ contain all tuples $(x, z_0, z_1, \ldots, z_{d-1}) \in \mathbb{R}^{d+1}$ such that $-\varepsilon < x < 1$, $\varepsilon < z_0 < 1+\varepsilon$ and $-\varepsilon/d < z_1, \ldots, z_{d-1} < 1+\varepsilon$.

**Proposition 3.1** *For any $\varepsilon > 0$, each function $f_j$ is Lipschitz continuous in $D_\varepsilon$.*

**Sketch of the proof.** Note that each $f_j$ is a rational function of the form $p/s^2$, where $p$ and $s$ are multivariate polynomials on $d+1$ variables such that $s$ does not contain roots in $D_\varepsilon$. □

As mentioned in the previous section, using $\beta = 2d$, we may find $A > 0$ sufficiently large so that $\lambda(n) = A/n^{1/4}$ satisfies (6) with probability at least

$$1 - O\left(n\gamma + \frac{\beta}{\lambda}\exp\left(-\frac{n\lambda^3}{\beta^3}\right)\right) = 1 - O\left(\frac{2d}{n^{1/4}}\exp\left(-\frac{An^{1/4}}{8d^3}\right)\right).$$

This expression converges to 1 as $n \to \infty$. In particular, we conclude that $Q(t)/n = q(t/n) + O(\lambda))$ holds with high probability up to step $\sigma n$.

We still need to prove that step $\sigma n$ occurs in a region where $z_0$ is small, which implies that we may carry out the analysis of the process up to a point where almost all vertices of the input graph have been totally dominated. To prove this, we shall establish properties of the solutions to the system of differential equations.

The results below ensure that the solutions $z_j(x)$ and $q(x)$ lie within the interval $[0, 1]$ for all values of $x \geq 0$ such that $z_0(x) > 0$. This implies that the reason why the vector of solutions approaches the boundary of the closure of $D_\varepsilon$ is that $z_0(x)$ approaches 0.

**Proposition 3.2** *There exists $\delta > 0$ such that, for all $x \in (0, \delta]$ and $0 \leq j \leq d-1$, we have $z_j(x) > 0$.*

**Sketch of the proof.** This is trivial for $z_0(x)$, as it is continuous and $z_0(0) = 1$. To show that the statement is true for $j \geq 1$, it suffices to verify that the first nonzero derivative of $z_j(x)$ is positive at $x = 0$, which is easily done by induction on $j$. □

The next proposition gives upper bounds on $z_j(x)$ and $s(x)$ provided that some conditions are satisfied.

**Proposition 3.3** *If $z_j(x) \geq 0$ and $z_0(x) > \varepsilon_0$ for all $x \in [0, \theta]$ and all $j \in \{1, \ldots, d-1\}$, where $\varepsilon_0 > 0$ and $\theta > 0$, then $s(x) \leq d$ and $z_j(x) \leq 1$ for all $\in [0, \theta]$ and all $j \in \{0, \ldots, d-1\}$.*

**Sketch of the proof.** Let $Z(x) = z_0(x) + z_1(x) + \cdots + z_{d-1}(x)$. Observe that $Z'(x) = z_0'(x) + z_1'(x) + \cdots + z_{d-1}'(x)$, which, using the differential equations in (7), lead to

$$Z'(x) = -1 - \frac{dz_0(x)}{s(x)} - \left[\frac{(d-(d-1))z_{d-1}(x)}{s(x)}\right]\left[(2d-2)\frac{dz_0(x)}{s(x)} + \sum_{i=1}^{d-1}\frac{(d-i)z_i(x)}{s(x)}(d-i-1)\right] \leq -1,$$

for $x \in [0, \theta]$. Note that the last inequality follows from the fact that $s(x) \geq dz_0(x) \geq d\varepsilon_0$. This proves that $Z(x)$ is decreasing in $[0, \theta]$. The conclusion that $s(x) \leq d$ comes from $dZ(0) = dz_0(0) = d = s(0)$ and $dZ(x) \geq s(x)$.

To bound $z_j(x)$, observe that, for all $j \in \{0, 1, \ldots, d-1\}$, we have $1 \geq Z(x) \geq z_j(x)$ and $Z(0) = 1 \geq z_j(0)$. □

We know that there is $\theta > 0$ satisfying the hypotheses of Proposition 3.3, namely $\theta = \delta$ of Proposition 3.2. Henceforth, when we refer to $\delta$ and $\theta$ we always mean a values of $\delta$ and $\theta$ for which the hypotheses of Propositions 3.2 and 3.3 hold. The next result, whose elementary (but slightly technical) proof is omitted, implies that, after the point where all functions $z_j(x)$ are positive, for $i \leq d-2$, a function $z_{i+1}(x)$ cannot approach 0 unless $z_i(x)$ approaches 0.

**Proposition 3.4** *Assume that $z_j(x) \geq 0$ for all $x \in [\delta, \theta]$ and $j \in \{0, 1, \ldots, d-1\}$. Moreover, assume that there is $i \in \{0, 1, \ldots, d-2\}$ and $\varepsilon_i > 0$ such that $\varepsilon_i \leq \min\{z_j(\delta)/2 : 0 \leq j \leq d-1\}$ and $z_i(x) > \varepsilon_i$. Then there is $\varepsilon_{i+1} > 0$ such that $z_{i+1}(x) > \varepsilon_{i+1}$ in $[\delta, \theta]$.*

Let $\varepsilon_0 > 0$ be an arbitrary real number less than $\min\{z_j(\delta)/2 : 0 \leq j \leq d-1\}$. For suitable values of $\theta$, we may apply Proposition 3.4 repeatedly in order to obtain $\{\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{d-1}\}$ such that $z_j(x) > \varepsilon_j$ for all $j \in \{0, 1, \ldots, d-1\}$.

Define $\theta_0 := \inf\{x > \delta : z_j(x) = \varepsilon_j, \text{for some } j\}$. This is well-defined because $z_j(\delta) > \varepsilon_j$ for all $j$ and $z_0'(x) \leq -1$ in any interval $[\delta, \theta]$ for which $z_j(x) \geq 0$ for all $j$. We claim that $z_0(\theta_0) = \varepsilon_0$. Indeed, assume that $z_j(\theta_0) = \varepsilon_j$ for som $j \geq 0$. Proposition 3.4 and the continuity of $z_j$ ensure that $z_{j-1}(\theta_0) = \varepsilon_{j-1}$. Repeating the argument, we get to $z_0(\theta_0) = \varepsilon_0$.

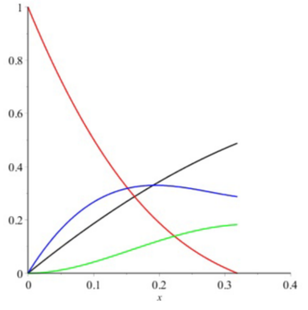


Fig. 1. $z_0(x)$ is red, $z_1(x)$ is blue, $z_2(x)$ is green, $q(x)$ is black. Note that vertices with degree 3 that are not in the totally dominating set produced by the algorithm are not depicted in the graphs.

This leads to the following conclusion.

**Proposition 3.5** *For all* $\varepsilon_0 > 0$ *less than* $\min\{z_j(\delta_j)/2 : 0 \leq j \leq d-1\}$*, there exists* $\theta_0 > \delta$ *such that* $z_0(\theta_0) = \varepsilon_0$ *and, for all* $x \in [\delta, \theta_0]$ *and* $j \in \{0, 1, \ldots, d-1\}$*,* $z_j(x) > 0$.

Putting together Propositions 3.2, 3.3 and 3.5, since $z_0(x)$ is decreasing and no other variable may approach the boundary of $D_{\varepsilon_0}$ until $z_0$ approaches $\varepsilon_0$, we derive the following.

**Proposition 3.6** *For* $d \geq 3$*, consider the initial value problem given in (7). Fix* $\varepsilon > 0$ *and let* $\theta_\varepsilon := \inf\{x > 0 : z_0(x) = \varepsilon\}$*. Then, for all* $x \in [0, \theta_\varepsilon]$*, we have* $\varepsilon \leq z_0(x) \leq 1$*,* $0 \leq q(x) \leq 1$ *and* $0 \leq z_j(x) \leq 1$*, for* $1 \leq j \leq d-1$.

**Proof.** It suffices to check that $0 \leq q(x) \leq 1$, as the other statements are consequences of the previous propositions. To see why this holds, let $\ell(x) = -z_0(x) + 1$, so that $q(0) = \ell(0) = 0$. Moreover, for $x \in [0, \theta_\varepsilon]$, we have $1 \leq 1 + dz_0(x)/s(x) = q'(x) \leq -z_0'(x) = \ell'(x)$. As a consequence, we have $0 \leq q(x) \leq \ell(x) \leq 1 - \varepsilon$. □

Proposition 3.6 tells us that, as $\varepsilon \to 0^+$, $z_0$ tends to 0, since, for all $\varepsilon > 0$, its derivative is less than $-1$ for all $x \in [0, \theta_\varepsilon]$. As a consequence, the constant $x^* = \inf\{x > 0 : z_0(x) = 0\}$ is well-defined and we have proved Theorem 1.3.

To illustrate the behavior of the solutions to (7), we show a computational approximation of the solutions for $d = 3$ in Figure 1.

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