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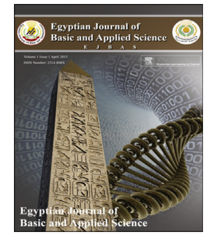

Analytical approximations of two and three dimensional time-fractional telegraphic equation by reduced differential transform method

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A R T I C L E I N F O

Article history:
Received 28 October 2013
Received in revised form
16 December 2013
Accepted 2 January 2014
Available online 5 February 2014

Keywords:
Two and three dimensional TFTEs
Fractional calculus
Reduced differential transform
method (RDTM)
Analytical solutions

A B S T R A C T

In this article, an analytical solution based on the series expansion method is proposed to solve the time-fractional telegraph equation (TFTE) in two and three dimensions using a recent and reliable semi-approximate method, namely the reduced differential transformation method (RDTM) subjected to the appropriate initial condition. Using RDTM, it is possible to find exact solution or a closed approximate solution of a differential equation. The accuracy, efficiency, and convergence of the method are demonstrated through the four numerical examples.

1. Introduction

Several real phenomena emerging in engineering and science fields can be demonstrated successfully by developing models using the fractional calculus theory. Fractional differential theory has gained much more attention as the fractional order system response ultimately converges to the integer order equations. The applications of the fractional differentiation for the mathematical modeling of real world physical problems such as the earthquake modeling, the traffic flow model with fractional derivatives, measurement of viscoelastic material properties, etc., have been widespread in this modern era. Before the nineteenth century, no analytical solution method was available for such type of equations even for the linear fractional differential equations. Recently, Keskin and Oturanc [1] developed the reduced differential transform

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Peer review under responsibility of Mansoura University.

Production and hosting by Elsevier

2314-808X/$ – see front matter
http://dx.doi.org/10.1016/j.ejbas.2014.01.002

method (RDTM) for the fractional differential equations and showed that RDTM is the easily useable semi analytical method and gives the exact solution for both the linear and nonlinear differential equations.

Let us assume that $u(x,y,t)$ and $i(x,y,t)$ denote the electric voltage and the current in a double conductor, then the time-fractional telegraphic equations (TFTEs) in the two dimension (2D) aregiven as

$$\left.\begin{aligned}\frac{\partial^{2\alpha}u}{\partial t^{2\alpha}}+2p\frac{\partial^{\alpha}u}{\partial t^{\alpha}}+q^2u=\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}+f_1(x,y,t),\\ \frac{\partial^{2\alpha}i}{\partial t^{2\alpha}}+2p\frac{\partial^{\alpha}i}{\partial t^{\alpha}}+q^2i=\frac{\partial^2i}{\partial x^2}+\frac{\partial^2i}{\partial y^2}+f_2(x,y,t),\end{aligned}\right\},(x,y,t)\in\Omega;\ p>0,q>0 \tag{1}$$

where $\Omega = [a,b] \times [c,d] \times [t>0]$. The initial conditions are assumed to be

$$\left.\begin{aligned}u(x,y,0)&=\phi_1(x,y),\\ u_t(x,y,0)&=\phi_2(x,y),\\ i(x,y,0)&=\chi_1(x,y),\\ i_t(x,y,0)&=\chi_2(x,y),\end{aligned}\right\},(x,y)\in\Omega \tag{2}$$

Similarly, the three dimensional (3D) time-fractional order telegraphic equation (TFTE) can be given as

$$\left.\begin{aligned}\frac{\partial^{2\alpha}u}{\partial t^{2\alpha}}+2p\frac{\partial^{\alpha}u}{\partial t^{\alpha}}+q^2u=\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}+\frac{\partial^2u}{\partial z^2}+f_1(x,y,z,t),\\ \frac{\partial^{2\alpha}i}{\partial t^{2\alpha}}+2p\frac{\partial^{\alpha}i}{\partial t^{\alpha}}+q^2i=\frac{\partial^2i}{\partial x^2}+\frac{\partial^2i}{\partial y^2}+\frac{\partial^2i}{\partial z^2}+f_2(x,y,z,t),\end{aligned}\right\},(x,y,z,t)\in\Omega;\ p>0,q>0 \tag{3}$$

where $\Omega = [a,b] \times [c,d] \times [e,f] \times [t > 0]$,with initial conditions

$$\left.\begin{aligned}u(x,y,z,0)&=\xi_1(x,y,z),\\ u_t(x,y,z,0)&=\xi_2(x,y,z),\\ i(x,y,z,0)&=\psi_1(x,y,z),\\ i_t(x,y,z,0)&=\psi_2(x,y,z)\end{aligned}\right\},(x,y,z)\in\Omega \tag{4}$$

In Eqs. (1) and (3) p and q denote constants. For $p > 0$, $q = 0$, (1) and (3) represent time-fractional order damped wave equations in two and three dimensions respectively.

It has been observed that telegraph equation is more suitable than ordinary diffusion equation in modeling reaction diffusion. The hyperbolic partial differential equations model the vibrations of structures (e.g. machines, buildings and beams) and they are the basis for fundamental equations of atomic physics. The telegraph equation is an important equation for modeling several relevant problems in engineering and science such as wave propagation [2], random walk theory [3], signal analysis [4] etc. In recent years, from the literature it can be seen that much attention has been given to the development of analytical and numerical schemes for the one dimensional and two dimensional hyperbolic fractional and non-fractional TFTE [5–22]. To the best of our knowledge till now no one has applied the RDTM to solve the time-fractional order telegraphic equations in two and three dimensions.

In this paper, we propose an analytical scheme namely the reduced differential transformation method based on series solution method to find analytical solutions of the time-fractional telegraph equation (TFTE) in two and three dimensions. The accuracy and efficiency of the proposed method are demonstrated by the four test examples. The main advantage of the method is that it solves the telegraph equation directly without using linearization, transformation, discretization or restrictive assumptions. Also, the RDTM scheme is very easy to implement for the multidimensional time-fractional order physical problems emerging in various fields of engineering and science.

2. Fractional calculus

In this section, we demonstrate some notations and definitions that will be used further in the study. Fractional calculus theory is almost more than two decades' old in the literature. Several definitions of fractional integrals and derivatives have been proposed but the first major contribution to give proper definition is due to Liouville as follows.

Definition 2.1. *A real function $f(x), x > 0$ is said to be in the space $C_\mu, \mu \in \mathbb{R}$ if there exists a real number $q(>\mu)$, such that $f(x) = x^q g(x)$, where $g(x) \in C[0,\infty)$, and it is said to be in the space C_μ^m if $f^{(m)} \in C_\mu, m\in \mathbb{N}$.*

Definition 2.2. *For a function f, the Riemann–Liouville fractional integral operator [23] of order $\alpha \geq 0$, is defined as*

$$\begin{cases} J^\alpha f(x)=\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)dt,\ \alpha>0,x>0,\\ J^0f(x)=f(x)\end{cases} \tag{5}$$

The Riemann–Liouville derivative has certain disadvantages when trying to model real world problems with fractional differential equations. To overcome this discrepancy, Caputo and Mainardi [24] proposed a modified fractional differentiation operator D^α in his work on the theory of viscoelasticity. The Caputo fractional derivative allows the utilization of initial and boundary conditions involving integer order derivatives, which havte clear physical interpretations.

Definition 2.3. *The fractional derivative of f in the Caputo sense [25] can be defined as*

$$D^\alpha f(x)=J^{m-\alpha}D^m f(x)=\frac{1}{\Gamma(m-\alpha)}\int_0^x (x-t)^{m-\alpha-1}f^{(m)}(t)dt, \tag{6}$$

for $m-1<\alpha s\leq m$, $m\in\mathbb{N}$, $x>0$, $f\in C_{-1}^m$.

The fundamental basic properties of the Caputo fractional derivative are given as.

Lemma. *If $m-1 < \alpha \leq m, m \in \mathbb{N}$ and $f\in C_\mu^m, \mu \geq -1$, then*

$$\begin{cases} D^\alpha J^\alpha f(x)=f(x),x>0,\\ D^\alpha J^\alpha f(x)=f(x)-\sum_{k=0}^{m} f^{(k)}(0^+)\frac{x^k}{k!},x>0,\end{cases} \tag{7}$$

Table 1 – Fundamental operations of the reduced differential transform method.

Original function	Reduced differential transformed function
$R_D[u(x,y,z,t)v(x,y,z,t)]$	$U_k(x,y,z) \otimes V_k(x,y,z) = \sum_{r=0}^{k} U_r(x,y,z)V_{k-r}(x,y,z)$
$R_D[\alpha u(x,y,z,t) \pm \beta v(x,y,z,t)]$	$\alpha U_k(x,y,z) \pm \beta V_k(x,y,z)$
$R_D[\partial^{N\alpha}/\partial t^{N\alpha} u(x,y,z,t)]$	$\Gamma(k\alpha + N\alpha + 1)/\Gamma(k\alpha + 1)U_{k+N}(x,y,z)$
$R_D[\partial^{m+n+p+s}/\partial x^m \partial y^n \partial z^p \partial t^s u(x,y,z,t)]$	$(k + s)!/k!\partial^{m+n+p}/\partial x^m \partial y^n \partial z^p U_{k+s}(x,y,z)$
$R_D[x^m y^n z^p t^q]$	$\{x^m y^n z^p, k = q0$, otherwise
$R_D[e^{\lambda t}]$	$\lambda^k/k!$
$R_D[\sin(\alpha x + \beta y + \gamma z + \omega t)]$	$w^k/k!\sin(\pi k/2! + \alpha x + \beta y + \gamma z)$
$R_D[\cos(\alpha x + \beta y + \gamma z + \omega t)]$	$w^k/k!\cos(\pi k/2! + \alpha x + \beta y + \gamma z)$

In this study, the Caputo fractional derivative is taken since it allows traditional initial and boundary conditions to be included in the derivation of the problem. Some other properties of fractional derivative can be found in [25,26].

3. Reduced differential transform method (RDTM)

In this section, we introduce the basic definitions of the reduced differential transformations.

Consider a function of four variables $w(x,y,z,t)$, and assume that it can be represented as a product $w(x,y,z,t) = F(x,y,z)G(t)$. On extending the basis of the properties of the one-dimensional differential transformation [26,27], the function $w(x,y,z,t)$ can be represented as

$$\begin{aligned} w(x,y,z,t) &= \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{i_3=0}^{\infty} F(i_1,i_2,i_3)x^{i_1}y^{i_2}z^{i_3}\sum_{j=0}^{\infty} G(j)t^j \\ &= \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{i_3=0}^{\infty}\sum_{j=0}^{\infty} W(i_1,i_2,i_3)x^{i_1}y^{i_2}z^{i_3}t^j, \end{aligned} \tag{8}$$

where $W(i_1,i_2,i_3) = F(i_1,i_2,i_3)G(j)$ is called the spectrum of $w(x,y,z,t)$.

Let R_D denotes the reduced differential transform operator and R_D^{-1} the inverse reduced differential transform operator. The basic definition and operation of the RDTM method is described below.

Definition 2.1. *If $w(x,y,z,t)$ is analytic and continuously differentiable with respect to space variables x,y and time variable t in the domain of interest, then the spectrum function* [28,29]

$$R_D[w(x,y,z,t)] \approx W_k(x,y,z) = \frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^k}{\partial t^k}w(x,y,z,t)\right]_{t=t_0} \tag{9}$$

is the reduced transformed function of $w(x,y,z,t)$.

In this article, (lowercase) $w(x,y,z,t)$ represents the original function while (uppercase) $W_k(x,y,z)$ stands for the reduced transformed function. The differential inverse reduced transform of $W_k(x,y,z)$ is defined as

$$R_D^{-1}[W_k(x,y,z)] \approx w(x,y,z,t) = \sum_{k=0}^{\infty} W_k(x,y,z)(t-t_0)^{k\alpha} \tag{10}$$

Combining Eqs. (9) and (10), *we get*

$$w(x,y,z,t) = \sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^k}{\partial t^k}w(x,y,z,t)\right]_{t=t_0}(t-t_0)^{k\alpha} \tag{11}$$

When $t = 0$, Eq. (11) *reduces to*

$$w(x,y,z,t) = \sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^k}{\partial t^k}w(x,y,z,t)\right]_{t=t_0} t^{k\alpha} \tag{12}$$

From the Eq. (11), *it can be seen that the concept of the reduced differential transform is derived from the power series expansion of the function.*

Definition 2.2. *If* $u(x,y,z,t) = R_D^{-1}[U_k(x,y,z)]$,$v(x,y,z,t) = R_D^{-1}[V_k(x,y,z)]$, *and the convolution* $\otimes$ *denotes the reduced differential transform version of the multiplication, then the fundamental operations of the reduced differential transform are shown in the* Table 1.

In Table 1, *Γ represents the Gama function, which is defined as*

$$\Gamma(\gamma) := \int_0^{\infty} e^{-t}t^{\gamma-1}dt, \gamma \in \mathbb{C} \tag{13}$$

4. RDTM for two dimensional TFTE

Applying the RDTM to the two dimensional TFTE (1), we have the following relation

$$\left.\begin{aligned} &\tfrac{\Gamma(k\alpha+2\alpha+1)}{\Gamma(k\alpha+1)}U_{k+2}(x,y) + 2p\tfrac{\Gamma(k\alpha+\alpha+1)}{\Gamma(k\alpha+1)}U_{k+1}(x,y) + q^2U_k(x,y) = \tfrac{\partial^2}{\partial x^2}U_k(x,y) + \tfrac{\partial^2}{\partial y^2}U_k(x,y) + R_D[f_1(x,y,t)]; \\ &\tfrac{\Gamma(k\alpha+2\alpha+1)}{\Gamma(k\alpha+1)}I_{k+2}(x,y) + 2p\tfrac{\Gamma(k\alpha+\alpha+1)}{\Gamma(k\alpha+1)}I_{k+1}(x,y) + q^2I_k(x,y) = \tfrac{\partial^2}{\partial x^2}I_k(x,y) + \tfrac{\partial^2}{\partial y^2}I_k(x,y) + R_D[f_2(x,y,t)], \end{aligned}\right\}. \tag{14}$$

Now applying the method to the initial conditions (2), we get

$$\left.\begin{aligned} U_0(x,y) &= \phi_1(x,y), \\ U_1(x,y) &= \phi_2(x,y), \\ I_0(x,y) &= \chi_1(x,y), \\ I_1(x,y) &= \chi_2(x,y), \end{aligned}\right\}, (x,y) \in \Omega. \tag{15}$$

From above two equations we get the values of $U_k(x,y)$,$I_k(x,y)$,$k = 2,3,4,\ldots$ etc. Using the differential inverse reduced transform of $U_k(x,y)$;$I_k(x,y)$,$k = 0,1,2,3,\ldots$, we get the approximate solution for $u(x,y,t)$ and $i(x,y,t)$ as

$$\left.\begin{array}{l} u(x,y,t)=\sum\limits_{k=0}^{\infty}U_k(x,y)t^{k\alpha}=U_0(x,y)+U_1(x,y)t^{\alpha}+U_2(x,y)t^{2\alpha}+U_3(x,y)t^{3\alpha}+\ldots, \\ i(x,y,t)=\sum\limits_{k=0}^{\infty}I_k(x,y)t^{k\alpha}=I_0(x,y)+I_1(x,y)t^{\alpha}+I_2(x,y)t^{2\alpha}+I_3(x,y)t^{3\alpha}+\ldots \end{array}\right\}. \tag{16}$$

5. RDTM for three dimensional TFTE

Applying the RDTM to the three dimensional TFTE (4), we have the following relation

$$\left.\begin{array}{l} \frac{\Gamma(k\alpha+2\alpha+1)}{\Gamma(k\alpha+1)}U_{k+2}(x,y,z)+2p\frac{\Gamma(k\alpha+\alpha+1)}{\Gamma(k\alpha+1)}U_{k+1}(x,y,z)+q^2U_k(x,y,z)=\frac{\partial^2}{\partial x^2}U_k(x,y,z)+\frac{\partial^2}{\partial y^2}U_k(x,y,z)+\frac{\partial^2}{\partial z^2}U_k(x,y,z)+R_D\left[f_1(x,y,z,t)\right]; \\ \frac{\Gamma(k\alpha+2\alpha+1)}{\Gamma(k\alpha+1)}I_{k+2}(x,y,z)+2p\frac{\Gamma(k\alpha+\alpha+1)}{\Gamma(k\alpha+1)}I_{k+1}(x,y,z)+q^2I_k(x,y,z)=\frac{\partial^2}{\partial x^2}I_k(x,y,z)+\frac{\partial^2}{\partial y^2}I_k(x,y,z)+\frac{\partial^2}{\partial z^2}I_k(x,y,z)+R_D\left[f_2(x,y,z,t)\right], \end{array}\right\}. \tag{17}$$

Now applying the method to the initial conditions (4), we get

$$\left.\begin{array}{l} U_0(x,y,z)=\xi_1(x,y,z), \\ U_1(x,y,z)=\xi_2(x,y,z), \\ I_0(x,y,z)=\psi_1(x,y,z), \\ I_1(x,y,z)=\psi_2(x,y,z), \end{array}\right\}, (x,y,z)\in\Omega. \tag{18}$$

Applying the same procedure as in the case of 2D TFTE, we get the approximate solution for $u(x,y,z,t)$ and $i(x,y,z,t)$ as

$$\left.\begin{array}{l} u(x,y,z,t)=\sum\limits_{k=0}^{\infty}U_k(x,y,z)t^{k\alpha}=U_0(x,y,z)+U_1(x,y,z)t^{\alpha}+U_2(x,y,z)t^{2\alpha}+\ldots, \\ i(x,y,z,t)=\sum\limits_{k=0}^{\infty}I_k(x,y,z)t^{k\alpha}=I_0(x,y,z)+I_1(x,y,z)t^{\alpha}+I_2(x,y,z)t^{2\alpha}+\ldots \end{array}\right\}. \tag{19}$$

6. Numerical examples

In this section, we describe the method explained in the Sections 4 and 5 by taking four examples of both linear and nonlinear 2D and 3D TFTEs to validate the efficiency and reliability of the RDTM scheme.

Example 6.1. *Consider the 2D linear TFTE*

$$\frac{\partial^{2\alpha}u}{\partial t^{2\alpha}}+2\frac{\partial^{\alpha}u}{\partial t^{\alpha}}+u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} \tag{20}$$

subject to the initial conditions

$$\left.\begin{array}{l} u(x,y,0)=e^{x+y}, \\ u_t(x,y,0)=-3e^{x+y}, \end{array}\right\}. \tag{21}$$

Applying the RDTM to Eq. (20), we obtain the following recurrence relation

$$\frac{\Gamma(k\alpha+2\alpha+1)}{\Gamma(k\alpha+1)}U_{k+2}(x,y)+2\frac{\Gamma(k\alpha+\alpha+1)}{\Gamma(k\alpha+1)}U_{k+1}(x,y) =\frac{\partial^2}{\partial x^2}U_k(x,y)+\frac{\partial^2}{\partial y^2}U_k(x,y)-U_k(x,y). \tag{22}$$

Using the RDTM to the initial conditions (21), we have

$$U_0(x,y)=e^{x+y};\ U_1(x,y)=-3e^{x+y}. \tag{23}$$

From Eq. (23) into Eq. (22), we get the following $U_k(x,y)$ values successively

$$U_k(x,y)=\frac{(-3)^k}{\Gamma\left(\frac{k}{\lambda}+1\right)}\Gamma\left(\frac{\lambda+1}{\lambda}\right)e^{x+y};k\geq 2. \tag{24}$$

where $\alpha=1/\lambda,\lambda>0$. Using the differential inverse reduced transform of $U_k(x,y)$, we get

$$\begin{aligned} u(x,y,t)&=\sum_{k=0}^{\infty}U_k(x,y)t^{k\alpha}=\sum_{k=0}^{\infty}U_k(x,y)t^{k/\lambda} \\ &=U_0(x,y)+U_1(x,y)t^{1/\lambda}+U_2(x,y)t^{2/\lambda}+U_3(x,y)t^{3/\lambda}+\ldots \\ &=e^{x+y}\left[1+(-3)t^{1/\lambda}+\Gamma\left(\tfrac{\lambda+1}{\lambda}\right)\left\{\frac{(-3)^2}{\Gamma\left(\frac{2}{\lambda}+1\right)}t^{2/\lambda}+\frac{(-3)^3}{\Gamma\left(\frac{3}{\lambda}+1\right)}t^{3/\lambda}+\ldots\right\}\right]. \end{aligned} \tag{25}$$

Eq. (25) represents the solution of the TFTE (20). When $\lambda=1$, i.e. $\alpha=1$, we get

$$\begin{aligned} u(x,y,t)&=e^{x+y}\left(1+(-3)t+\frac{(-3)^2}{2!}t^2+\frac{(-3)^3}{3!}t^3+\ldots\ldots+\frac{(-3)^k}{k!}t^k+\ldots\ldots\right) \\ &=e^{x+y-3t}. \end{aligned} \tag{26}$$

Example 6.2. *Consider the following 3D linear TFTE*

$$\frac{\partial^{2\alpha}u}{\partial t^{2\alpha}}+2\frac{\partial^{\alpha}u}{\partial t^{\alpha}}+u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2} \tag{27}$$

subject to initial conditions

$$\left.\begin{array}{l} u(x,y,z,0)=\sinh(x)\sinh(y)\sinh(z), \\ u_t(x,y,z,0)=-\sinh(x)\sinh(y)\sinh(z), \end{array}\right\}. \tag{28}$$

Applying the RDTM to Eq. (27), we obtain the following recurrence relation

$$\begin{aligned} &\frac{\Gamma(k\alpha+2\alpha+1)}{\Gamma(k\alpha+1)}U_{k+2}(x,y,z)+2\frac{\Gamma(k\alpha+\alpha+1)}{\Gamma(k\alpha+1)}U_{k+1}(x,y,z) \\ &=\frac{\partial^2}{\partial x^2}U_k(x,y,z)+\frac{\partial^2}{\partial y^2}U_k(x,y,z)+\frac{\partial^2}{\partial z^2}U_k(x,y,z)-U_k(x,y,z). \end{aligned} \tag{29}$$

Using the RDTM to the initial conditions (28), we have

$$\begin{aligned} &U_0(x,y,z)=\sinh(x)\sinh(y)\sinh(z); \\ &U_1(x,y,z)=-\sinh(x)\sinh(y)\sinh(z). \end{aligned} \tag{30}$$

From Eq. (30) into Eq. (29), we get the following $U_k(x,y,z)$ values successively

$$U_k(x,y,z)=\frac{(-1)^k}{\Gamma\left(\frac{k}{\lambda}+1\right)}\Gamma\left(\frac{\lambda+1}{\lambda}\right)\sinh(x)\sinh(y)\sinh(z);k\geq 2. \tag{31}$$

Using the differential inverse reduced transform of $U_k(x,y,z)$, we get

$$\begin{aligned} &u(x,y,z,t)=\sum_{k=0}^{\infty}U_k(x,y,z)t^{k\alpha}=\sum_{k=0}^{\infty}U_k(x,y,z)t^{k/\lambda} \\ &=U_0(x,y,z)+U_1(x,y,z)t^{1/\lambda}+U_2(x,y,z)t^{2/\lambda}+U_3(x,y,z)t^{3/\lambda}+\ldots \\ &=\sinh(x)\sinh(y)\sinh(z)\left[1+(-1)t^{1/\lambda}+\Gamma\left(\tfrac{\lambda+1}{\lambda}\right)\left\{\tfrac{(-1)^2}{\Gamma\left(\frac{2}{\lambda}+1\right)}t^{2/\lambda}+\tfrac{(-1)^3}{\Gamma\left(\frac{3}{\lambda}+1\right)}t^{3/\lambda}+\ldots\right\}\right]. \end{aligned} \tag{32}$$

Eq. (32) represents the solution of the TFTE (27). When $\lambda=1$, i.e. $\alpha=1$, we get

$$u(x,y,z,t)=e^{-t}\sinh(x)\sinh(y)\sinh(z), \tag{33}$$

which is the closed form solution of the non-fractional form of the TFTE (27).

Example 6.3. *Consider the following 2D nonlinear TFTE*

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^{2\alpha}u}{\partial t^{2\alpha}}+2\frac{\partial^{\alpha}u}{\partial t^{\alpha}}+u^2-e^{2(x+y)-4t}+e^{(x+y)-2t} \tag{34}$$

under the initial conditions

$$\left.\begin{array}{l} u(x,y,0)=e^{x+y}, \\ u_t(x,y,0)=-2e^{x+y}, \end{array}\right\}. \tag{35}$$

Applying the RDTM technique to Eq. (34), we obtain the following iterative formula:

$$\begin{aligned} &\frac{\Gamma(k\alpha+2\alpha+1)}{\Gamma(k\alpha+1)}U_{k+2}(x,y)+2\frac{\Gamma(k\alpha+\alpha+1)}{\Gamma(k\alpha+1)}U_{k+1}(x,y) \\ &=\frac{\partial^2}{\partial x^2}U_k(x,y)+\frac{\partial^2}{\partial y^2}U_k(x,y)-\sum_{r=0}^{k}U_r(x,y)U_{k-r}(x,y) \\ &\quad+e^{2(x+y)}\left(\frac{(-4)^k}{k!}\right)-e^{(x+y)}\left(\frac{(-2)^k}{k!}\right). \end{aligned} \tag{36}$$

Using the RDTM to the initial conditions (34), we get

$$U_0(x,y)=e^{x+y};\ U_1(x,y)=-2e^{x+y}. \tag{37}$$

Using Eq. (37) in Eq. (36), we get the following $U_k(x,y)$ values successively

$$U_k(x,y)=\frac{(-2)^k}{\Gamma\left(\frac{k}{\lambda}+1\right)}\Gamma\left(\frac{\lambda+1}{\lambda}\right)e^{x+y};k\geq 2. \tag{38}$$

Using the differential inverse reduced transform of $U_k(x,y)$, we get

$$\begin{aligned} &u(x,y,t)=\sum_{k=0}^{\infty}U_k(x,y)t^{k\alpha}=\sum_{k=0}^{\infty}U_k(x,y)t^{k/\lambda} \\ &=U_0(x,y)+U_1(x,y)t^{1/\lambda}+U_2(x,y)t^{2/\lambda}+U_3(x,y)t^{3/\lambda}+\ldots \\ &=e^{x+y}\left[1+(-1)t^{1/\lambda}+\Gamma\left(\tfrac{\lambda+1}{\lambda}\right)\left\{\tfrac{(-2)^2}{\Gamma\left(\frac{2}{\lambda}+1\right)}t^{2/\lambda}+\tfrac{(-2)^3}{\Gamma\left(\frac{3}{\lambda}+1\right)}t^{3/\lambda}+\ldots\right\}\right]. \end{aligned} \tag{39}$$

When $\lambda=1$, we get the exact solution of the non-fractional form of the TFTE (34) as

$$u(x,y,t)=e^{(x+y)-2t}. \tag{40}$$

Example 6.4. *Consider the 3D nonlinear TFTE given as*

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{\partial^{2\alpha}u}{\partial t^{2\alpha}}+2\frac{\partial^{\alpha}u}{\partial t^{\alpha}}+u^2-e^{2(x-y-z)-4t}+e^{(x-y-z)-2t} \tag{41}$$

subject to the initial conditions

$$\left.\begin{array}{l} u(x,y,z,0)=e^{x-y-z}, \\ u_t(x,y,z,0)=-e^{x-y-z}, \end{array}\right\}. \tag{42}$$

Applying the RDTM technique to Eq. (41), we obtain the following iterative formula:

$$\begin{aligned}
&\frac{\Gamma(k\alpha+2\alpha+1)}{\Gamma(k\alpha+1)}U_{k+2}(x,y,z)+2\frac{\Gamma(k\alpha+\alpha+1)}{\Gamma(k\alpha+1)}U_{k+1}(x,y,z)\\
&=\frac{\partial^2}{\partial x^2}U_k(x,y,z)+\frac{\partial^2}{\partial y^2}U_k(x,y,z)+\frac{\partial^2}{\partial z^2}U_k(x,y,z)\\
&\quad-\sum_{r=0}^{k}U_r(x,y,z)U_{k-r}(x,y,z)\ +e^{2(x+y+z)}\left(\frac{(-4)^k}{k!}\right)\\
&\quad-e^{(x+y+z)}\left(\frac{(-2)^k}{k!}\right).
\end{aligned} \tag{43}$$

Using the RDTM to the initial conditions (42), *we get*

$$U_0(x,y,z)=e^{x-y-z};U_1(x,y,z)=-e^{x-y-z}. \tag{44}$$

Using Eq. (44) *in* Eq. (43), *we get the following* $U_k(x,y,z)$ *values successively*

$$U_k(x,y,z)=\frac{(-1)^k}{\Gamma\left(\frac{k}{\lambda}+1\right)}\Gamma\left(\frac{\lambda+1}{\lambda}\right)e^{x-y-z};k\geq 2.$$

$$U_k(x,y,z)=\frac{(-1)^k}{\Gamma\left(\frac{k}{\lambda}+1\right)}\Gamma\left(\frac{\lambda+1}{\lambda}\right)e^{x-y-z};k\geq 2. \tag{45}$$

Using the differential inverse reduced transform of $U_k(x,y,z)$, *we get*

$$\begin{aligned}
&u(x,y,z,t)=\sum_{k=0}^{\infty}U_k(x,y,z)t^{k\alpha}=\sum_{k=0}^{\infty}U_k(x,y,z)t^{k/\lambda}\\
&=U_0(x,y,z)+U_1(x,y,z)t^{1/\lambda}+U_2(x,y,z)t^{2/\lambda}+U_3(x,y,z)t^{3/\lambda}+\ldots\\
&=e^{x-y-z}\left[1+(-1)t^{1/\lambda}+\Gamma\left(\tfrac{\lambda+1}{\lambda}\right)\left\{\tfrac{(-1)^2}{\Gamma\left(\frac{2}{\lambda}+1\right)}t^{2/\lambda}+\tfrac{(-1)^3}{\Gamma\left(\frac{3}{\lambda}+1\right)}t^{3/\lambda}+\ldots\right\}\right].
\end{aligned} \tag{46}$$

When $\alpha=1$, *the exact solution of the non-fractional form of the nonlinear TFTE* (41) *is obtained as*

$$u(x,y,z,t)=e^{x-y-z-t}. \tag{47}$$

7. Conclusions

In the present study, we have illustrated the reduced differential transform method for the analytical solution of two and three dimensional second order hyperbolic linear and nonlinear TFTEs. The method is applied in a direct way without using linearization, transformation, discretization or restrictive assumptions. The effectiveness of the method is shown from the computational results. These results show that the RDTM technique is highly accurate, rapidly converge and is very easily implementable mathematical tool for the multidimensional physical problems emerging in various fields of engineering and sciences.

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