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Strongly Semicontinuous Lattices ⋆

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Abstract

In this paper, we introduce and study strongly semicontinuous lattices, a new class of complete lattices lying between the class of semicontinuous lattices and that of continuous lattices. It is shown that a complete lattice L is strongly semicontinuous iff L is semicontinuous and meet semicontinuous. Some versions of strong interpolation properties for the semiway-below relation in (strongly) semicontinuous lattices are established. Characterization theorems by some distributivity and approximate identities for strongly semicontinuous lattices are given, which reveals that strongly semicontinuous lattices indeed share similar properties with continuous lattices. A subtle counterexample is constructed to show that semicontinuous lattices need not be strongly semicontinuous lattices.

Keywords: semicontinuous lattice; meet semicontinuous lattice; strongly semicontinuous lattice; distributivity; finitely separated; approximate identity

1 Introduction

Replacing ideals with semiprime ideals posed by Rav [9], Zhao in [12] introduced the concept of semicontinuous lattices and showed that semicontinuous lattices have many properties similar to that of continuous lattices [10]. As generalizations of continuous lattices, semicontinuous lattices enrich topics in domain theory [2]. So far, much work related to semicontinuous lattices has been done. Bi and Xu [1] introduced and studied the semi-Scott topology and the semi-Lawson topology on semicontinuous lattices. Li and Wu [8] investigated relationships between semicontinuous lattices and completely distributive lattices. Jiang [5] discussed pseudo-primes in semicontinuous lattices and gave some characterizations for semicontinuous lattices. Li in [7] introduced semiprime sets in dcpos and generalized semicontinuous lattices to semicontinuous dcpos. He and Xu in [3] explored relationships between

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semicontinuous lattices and their distributive reflections, and gave negative answers to the problems posed by Zhao in [12]. Wu and Li in [11] introduced meet semicontinuous lattices, which is naturally defined and is a counterpart of meet continuous lattices. It is well-known that continuous lattices are all meet continuous. However, there is a counterexample example which will be constructed in the sequel to show that a semicontinuous lattice may not be meet semicontinuous. This is an unexpected situation for semicontinuous lattices. Note that in the definition of a semicontinuous lattice L, the condition used is that for every element $x \in L$, $\bigvee \Downarrow x \geq x$. If one changes the inequality to equality in the definition, then one gets the concept of strongly continuous lattices [12], rather than the strongly counterpart of semicontinuous lattices. So, it is significant to find some suitable type of semicontinuity which is weaker than continuity while stronger than meet semicontinuity in the realm of complete lattices.

With this motivation, in this paper we intend to define a new semicontinuity, called strongly semicontinuity, by the condition that for a complete lattice L, $\bigvee(\Downarrow x \cap \downarrow x) = x$ for each $x \in L$. We will see that the new semicontinuity implies ordinary semicontinuity. And similar to the case of continuity, in a strongly semicontinuous lattice, an element can be approximated by elements semiway-below (and below) it. It is shown that strongly semicontinuity is equal to semicontinuity plus meet semicontinuity and is weaker than continuity. In terms of some distributivity and approximate identities of complete lattices, characterization theorems for strongly semicontinuous lattices are given.

We will organize our paper as follows: Section 2 gives preliminaries; Section 3 introduces strongly semicontinuous lattices, investigates their properties, proves some strong interpolation properties for the semiway-below relation in (strongly) semicontinuous lattices, and constructs some counterexamples; Section 4 establishes characterizations for strongly semicontinuous lattices.

2 Preliminaries

We recall some basic concepts and results which will be used in the sequel. Most of them come from [2] and [12]. For other unstated concepts please refer to [9].

Let L be a poset. The set of all ideals (resp., filters) in L is denoted by $\mathrm{Id}(L)$ (resp., $\mathrm{Filt}(L)$). Let $P \in \mathrm{Id}(L)$, P is said to be *prime* if $L \setminus P = \emptyset$ or $L \setminus P \in \mathrm{Filt}(L)$. It is easy to see that in a semilattice L, an ideal P is prime iff for all $a, b \in L$, $a \wedge b \in P$ implies $a \in P$ or $b \in P$. We denote the family of all prime ideals of L with $\mathrm{PI}(L)$. For a lattice L and $A \subseteq L$, we use $\bigvee A$ (resp., $\bigwedge A$) to denote the supremum (resp., infimum) of A if it exists.

Definition 2.1 [9] *Let L be a lattice. An ideal I of L is said to be semiprime if for any $x, y, z \in L$, $x \wedge y \in I$ and $x \wedge z \in I$ imply $x \wedge (y \vee z) \in I$. The family of all semiprime ideals of L is denoted by $Rd(L)$.*

Lemma 2.2 [12, Lemma 1.2] *An ideal P of a lattice L is semiprime iff there exist prime ideals $P_j (j \in J)$ of L such that $P = \bigcap_{j \in J} P_j$.*

Thus, $\mathrm{PI}(L) \subseteq \mathrm{Rd}(L)$ and $\mathrm{Rd}(L)$ is closed under intersections.

In a dcpo L, we say that x is *way-below* y, or x *approximates* y, written as $x \ll y$, if whenever D is directed with $\bigvee D \geq y$, then $x \leq d$ for some $d \in D$. Equivalently, $x \ll y$ iff $x \in I$ for every ideal I of L such that $y \leq \bigvee I$. We use $\mathord{\downarrow\!\!\!\downarrow} x$ to denote the set $\{a \in L : a \ll x\}$. If for every element $x \in L$, the set $\mathord{\downarrow\!\!\!\downarrow} x$ is directed and $\bigvee \mathord{\downarrow\!\!\!\downarrow} x = x$, then L is called a domain. A complete lattice which is a domain is called a continuous lattice.

For complete lattices, replacing ideals with semiprime ideals, Zhao in [12] defined a weak form of the way-below relation.

Definition 2.3 [12] *Let L be a complete lattice. Define the relation $\Leftarrow$ on L as follows: for $x, y \in L$, $x \Leftarrow y$ if for any semiprime ideal I of L, $y \leq \bigvee I$ implies $x \in I$. An element $x \in L$ is said to be $\Leftarrow$–compact if $x \Leftarrow x$. Write for each $x \in L$,*

$$\Downarrow x = \{y \in L : y \Leftarrow x\}, \qquad \Uparrow x = \{y \in L : x \Leftarrow y\},$$

and write $\mathrm{SK}(L) = \{x \in L : x \Leftarrow x\}$.

Definition 2.4 [12] *A complete lattice L is said to be semicontinuous, if for any $x \in L$, $x \leq \bigvee(\Downarrow x)$. A complete lattice L is said to be strongly continuous, if for any $x \in L$, $x = \bigvee(\Downarrow x)$.*

Lemma 2.5 [12] *Strongly continuous lattices are continuous lattices, and continuous lattices are semicontinuous. For distributive complete lattices, strongly continuity, continuity, semicontinuity are equivalent to each other.*

Particularly, finite lattices are all semicontinuous lattices.

Remark 2.6 *It is easy to check that for a complete lattice L and $a, b, c, d \in L$,*
(1) $\Downarrow a$ is a semiprime ideal for each $a \in L$ and $a \ll b$ implies $a \Leftarrow b$;
(2) $a \Leftarrow b$ does not imply $a \leq b$, the typical modular lattice M_5 is a counterexample;
(3) (Transitive properties) if $a \leq b \Leftarrow c \leq d$, then $a \Leftarrow d$. If $a \Leftarrow b \Leftarrow c$, then $a \Leftarrow c$.
(4) if $a, b \Leftarrow c$, then $a \vee b \Leftarrow c$.

Lemma 2.7 [12, Theorem 1.8] *If L is a semicontinuous lattice, then $x \Leftarrow y$ implies the existence of a $z \in L$ such that $x \Leftarrow z \Leftarrow y$.*

Lemma 2.8 [12, Theorem 1.9] *Let L be a complete lattice, then the following conditions are equivalent:*

(1) L is semicontinuous;

(2) for any family $\{P_j\}_{j \in J}$ of semiprime ideals P_j of L, $\bigwedge_{j \in J}(\bigvee P_j) = \bigvee(\bigcap_{j \in J} P_j)$ holds;

(3) for any family $\{Q_j\}_{j \in J}$ of prime ideals Q_j of L, $\bigwedge_{j \in J}(\bigvee Q_j) = \bigvee(\bigcap_{j \in J} Q_j)$ holds.

3 Strongly semicontinuous lattices

In this section we define strongly semicontinuous lattices, and construct a subtle counterexample to show that a semicontinuous lattice may not be strongly semicontinuous.

Definition 3.1 *Let L be a complete lattice. If for each $x \in L$, the property*

$$\text{(A):} \qquad \bigvee(\Downarrow x \cap \downarrow x) = x$$

holds, then L is said to be strongly semicontinuous.

Clearly, every strongly semicontinuous lattice is a semicontinuous lattice, and a semicontinuous lattice satisfying the property (A) is a strongly semicontinuous lattice.

It is easy to see that if L is a complete lattice without proper prime ideals, then every pair of elements in L are in the relation $\Leftarrow$, and L is strongly semicontinuous.

Proposition 3.2 *Every continuous lattice is strongly semicontinuous. Especially, every distributive semicontinuous lattice is a strongly semicontinuous lattice.*

Proof. It follows from Remark 2.6(1) that for any $x \in L$, $\twoheaddownarrow x \subseteq \Downarrow x \cap \downarrow x$. Then $x = \bigvee \twoheaddownarrow x \leq \bigvee(\Downarrow x \cap \downarrow x) \leq x$. Thus $\bigvee(\Downarrow x \cap \downarrow x) = x$ and L is a strongly semicontinuous lattice.

By Lemma 2.5, distributive semicontinuous lattices are continuous and strongly semicontinuous. □

The next example shows that strongly semicontinuous lattices may not be continuous.

Example 3.3 *Let $L_{xy} = L \cup \{x, y\}$ (cf. Fig.1), where $L = \{\bot, \top\} \cup \{x_n : n = 0, 1, \cdots\}$. The partial order on L_{xy} is defined as $\bot \leq x_0 \leq \cdots \leq x_n \cdots \leq \top$, $\bot \leq x, y \leq \top$. Obviously, L_{xy} has no proper prime ideals, and $\mathrm{SK}(L_{xy}) = L_{xy}$. And for any $a, b \in L_{xy}$, we have that $a \Leftarrow b$ holds. By Definition 3.1, L_{xy} is a strongly semicontinuous lattice. It is clear that L_{xy} is not continuous.*

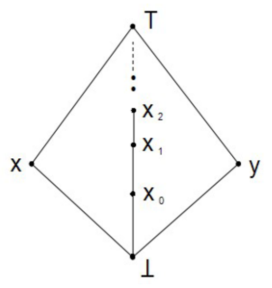

Fig. 1. L_{xy} is strongly semicontinuous but not continuous

In the sequel, we write $xI = x \wedge I = \{x \wedge y : y \in I\}$ for $x \in L$ and $I \subseteq L$.

Proposition 3.4 *For a complete lattice L, the following statements are equivalent:*

(1) *L is a strongly semicontinuous lattice;*

(2) *for any $x, y \in L$, if $x \not\leq y$, then there exists $z \in \Downarrow x \cap \downarrow x$ such that $z \not\leq y$;*

(3) *for each $x \in L$, there exists $B \subseteq \Downarrow x \cap \downarrow x$ such that $x = \bigvee B$;*

(4) *L is semicontinuous and $\bigvee(x \wedge \Downarrow x) = x \wedge (\bigvee \Downarrow x)$.*

Proof. Straightforward. □

Definition 3.5 [11] *A complete lattice L is said to be meet semicontinuous if for any $x \in L$ and $I \in \mathrm{Rd}(L)$, $x \wedge (\bigvee I) = \bigvee (x \wedge I)$.*

Recall that a *meet continuous lattice* means a complete lattice which satisfies the condition (MC): $x \wedge (\bigvee D) = \bigvee (x \wedge D)$.

By Definition 3.5, every meet continuous lattice is meet semicontinuous. Notice that for a distributive lattice, every ideal is semiprime. So, every distributive meet semicontinuous lattice is meet continuous.

Proposition 3.6 *Every strongly semicontinuous lattice is meet semicontinuous.*

Proof. Suppose that L is a strongly semicontinuous lattice. For any $x \in L$ and $I \in \mathrm{Rd}(L)$, to show $x \wedge (\bigvee I) = \bigvee (x \wedge I)$, it suffices to show that $x \wedge (\bigvee I) \leq \bigvee (x \wedge I)$. Let $t = x \wedge (\bigvee I)$. Since $t = \bigvee (\Downarrow t \cap \downarrow t)$, for any $r \in \Downarrow t \cap \downarrow t$ we have that $r \leq t \leq x$, $r \Leftarrow t \leq \bigvee I$ and $r \in I$. Thus, there exists $p_0 \in I$ such that $r = p_0$ and $r \leq x \wedge p_0 \leq \bigvee (x \wedge I)$. Therefore, by the arbitrariness of $r \in \Downarrow t \cap \downarrow t$, we have that $x \wedge (\bigvee I) = t = \bigvee (\Downarrow t \cap \downarrow t) \leq \bigvee (x \wedge I)$. □

Note that every topology is a distributive meet continuous lattice. Consider the usual topology $\tau(\mathbb{Q})$ of the rationals $\mathbb{Q}$ induced from $\mathbb{R}$. It is easy to show that $\tau(\mathbb{Q})$ is not continuous, and thus not strongly semicontinuous. However, $\tau(\mathbb{Q})$ is clearly meet semicontinuous. So, a meet semicontinuous lattice may not be strongly semicontinuous.

Theorem 3.7 *A complete lattice L is strongly semicontinuous iff L is semicontinuous and meet semicontinuous.*

Proof. $\Rightarrow$: By Proposition 3.6.

$\Leftarrow$: For each $x \in L$, by the semicontinuity of L, we have that $\bigvee \Downarrow x \geq x$. It follows from Remark 2.6(1) and the meet semicontinuity of L that $x = x \wedge (\bigvee \Downarrow x) = \bigvee (\Downarrow x \cap \downarrow x) \leq x$, and $\bigvee (\Downarrow x \cap \downarrow x) = x$. By Definition 3.1, L is strongly semicontinuous. □

The next example shows that semicontinuous lattices need not be strongly semicontinuous, nor meet semicontinuous.

Example 3.8 *Let $L = \{\bot, a, b, x, \top\} \cup \{x_n : n = 0, 1, \cdots\}$ (cf. Fig.2).*

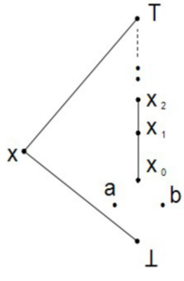

Fig. 2. L is semicontinuous but not strongly semicontinuous

The partial order on L is defined by: $\perp \leq a, b \leq x_0 \leq \cdots \leq x_n \cdots \leq \top$, $\perp \leq x \leq \top$. It is clear that the prime ideals of L are $L \setminus \uparrow x$ and L. We observe that for any $t \in L$, we have $\Downarrow t = L \setminus \uparrow x$ and $t \leq \bigvee \Downarrow t$. Thus, L is a semicontinuous lattice. However, for the prime ideal $L \setminus \uparrow x$, we have $x = x \wedge \bigvee (L \setminus \uparrow x) \neq \bigvee (x \wedge (L \setminus \uparrow x)) = \perp$, revealing that L is not meet semicontinuous, let alone strongly semicontinuous by Theorem 3.7.

Note that in this example, x is not $\Leftarrow$–compact and $\bigvee (\Downarrow x \cap \downarrow x) = \perp < x$, which yields also that L is not a strongly semicontinuous lattice.

Proposition 3.9 *If L is a strongly semicontinuous lattice, then for any family $\{x_\alpha\}_{\alpha \in \Gamma}$ of L, the following* (SSD) *holds:*

(SSD) $\quad \bigwedge_{\alpha \in \Gamma} \bigvee (\Downarrow x_\alpha \cap \downarrow x_\alpha) = \bigvee \bigcap_{\alpha \in \Gamma} (\Downarrow x_\alpha \cap \downarrow x_\alpha)$.

Proof. For convenience, let lhs denote the left hand side of (SSD) and rhs denote the right hand side. It is obvious that in any complete lattice $lhs \geq rhs$. To show the reverse inequality, let $t = lhs = \bigwedge_{\alpha \in \Gamma} x_\alpha$. Then $t \leq x_\alpha$ ($\forall \alpha \in \Gamma$). So, for any $k \in \Downarrow t \cap \downarrow t$, $k \in \Downarrow x_\alpha \cap \downarrow x_\alpha$ for all $\alpha \in \Gamma$ and $k \in \bigcap_{\alpha \in \Gamma} (\Downarrow x_\alpha \cap \downarrow x_\alpha)$, which then implies that $rhs \geq k$ for all $k \in \Downarrow t \cap \downarrow t$. By the strongly semicontinuity of L and the arbitrariness of k, we see that $rhs \geq \bigvee (\Downarrow t \cap \downarrow t) = t$. So, (SSD) holds. $\square$

The next example shows that the distributivity (SSD) of a complete lattice L need not imply the semicontinuity of L, let alone the strongly semicontinuity. So, (SSD) is not sufficient for the (strong) semicontinuity.

Example 3.10 *Let $L = \{\perp, x, \top\} \cup \{x_n : n = 0, 1, \cdots\}$ (cf. Fig.3). The partial*

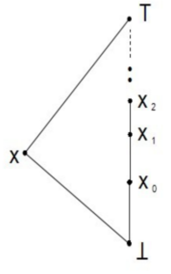

Fig. 3. L satisfies distributivity (SSD) but is not semicontinuous

order on L is defined by: $\perp \leq x_0 \leq \cdots \leq x_n \cdots \leq \top$, $\perp \leq x \leq \top$. Clearly, the prime ideals of L are $\downarrow x$, $L \setminus \uparrow x$ and L. Since $\Downarrow x = \{\perp\}$, L is not a semicontinuous lattice. However, notice that $\Downarrow \perp = \Downarrow x = \{\perp\}$, $\Downarrow \top = L \setminus \uparrow x = \Downarrow x_n (n = 0, 1, \cdots)$. So, for any family $\{x_\alpha\}_{\alpha \in \Gamma}$ of L, we have that the rhs of (SSD)

$$\bigvee \bigcap_{\alpha \in \Gamma} (\Downarrow x_\alpha \cap \downarrow x_\alpha) = \begin{cases} \perp, & \textit{either } \perp \textit{ or } x \in \{x_\alpha\}_{\alpha \in \Gamma}, \\ \top, & x_\alpha = \top \textit{ for all } \alpha \in \Gamma, \\ \bigwedge_{\alpha \in \Gamma} x_\alpha, & \textit{otherwise} \end{cases}$$

which is easily seen to be equal to $\bigwedge_{\alpha \in \Gamma} \bigvee (\Downarrow x_\alpha \cap \downarrow x_\alpha)$, the lhs of (SSD), *showing that* (SSD) *holds in L.*

Further, it can be checked in a similar way that the distributivity (SSD) of the complete lattice in Example 3.8 also holds, showing that (SSD) and semicontinuity together need not imply the strongly semicontinuity of a complete lattice.

In the next section, we will define a stronger distributivity (SSD*) of complete lattices. It turns out that (SSD*) is equivalent to the strongly semicontinuity of complete lattices.

To sum up, we have the following diagram of implication relations for complete lattices. None of the implication relations "⇒" is invertible.

$$\begin{tikzcd}
\mathrm{DistCont} \arrow[r, Rightarrow] & \mathrm{StroCont} \arrow[r, Rightarrow] & \mathrm{Cont} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \mathrm{StroSemi} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \mathrm{Semi} \arrow[d, "\mathrm{MeetSemi}"'] \\
 & & \mathrm{MeetCont} \arrow[r, shift right] & \mathrm{MeetSemi} \arrow[l, shift right, "\mathrm{Dist}"'] \arrow[r, "\mathrm{Semi}"'] & \mathrm{StroSemi},
\end{tikzcd}$$

where, Dist = distributive, Cont = continuous, Semi = semicontinuous, DistCont = distributive continuous, StroCont = strongly continuous, and so on.

Recall that in [12], for a complete lattice, one can define a map $T : L \to L$ by $T(x) = \bigvee(\Downarrow x)$ for all $x \in L$. Let L^T denote the subposet of L consisting of all $x \in L$ with $T(x) = x$. By [12, Theorem 3.4], if L is semicontinuous, then L^T is a continuous lattice. Similarly, for a complete lattice L, we can define a map $\xi : L \to L$ by

$$\xi(x) = \bigvee(\Downarrow x \cap \downarrow x), \quad x \in L.$$

Let $L_\xi = \{x \in L : \xi(x) = x\}$ be a subposet of L. Then the least element $\bot \in L_\xi$, $L^T \subseteq L_\xi$ and $\mathrm{SK}(L) \subseteq L_\xi$. If L is semicontinuous, then the largest element $\top \in L_\xi$. Moreover, L_ξ is closed under arbitrary unions of L, and hence a complete lattice in the hereditary order of L. If L is a strongly semicontinuous lattice, then $L_\xi = L$. However, we do not know even for a given semicontinuous lattice L, whether L_ξ is a (strongly) semicontinuous lattice or not. So, we leave the following problem.

Problem For a semicontinuous lattice L, is L_ξ a (strongly) semicontinuous lattice?

For semicontinuous lattices, they exhibit the well-known interpolation property (Lemma 2.7). Actually, (strongly) semicontinuous lattices have some strong interpolation properties as in the following theorem and corollary.

Theorem 3.11 *If L is a (strongly) semicontinuous lattice, then L satisfies the strong interpolation property: for all $x, y \in L$,*

(SI) $\quad$ *$x \Leftarrow y$ and $x \neq y$ together imply $(\exists z \in L)$ $(x \Leftarrow z \Leftarrow y$ and $x \neq z)$.*

Proof. If x is not $\Leftarrow$–compact, then by Lemma 2.7 there exists a $z \in L$ such that $x \Leftarrow z \Leftarrow y$ and $z \neq x$, the proof completes.

If x is $\Leftarrow$–compact, then $x \Leftarrow x \Leftarrow y$. For $y < x$, we have $y < x \Leftarrow y$ and by Remark 2.6(3), $y \Leftarrow y$. So, $x \Leftarrow y \Leftarrow y$ with $x \neq y$, (SI) holds. For $y \not\leq x$, by the semicontinuity of L, there exists $z_1 \in \Downarrow y$ such that $z_1 \not\leq x$. Set $z = z_1 \vee x$, then

$z \neq x$ and $x \Leftarrow x \leq z \Leftarrow y$. Therefore, (SI) holds. □

By the proof of the above theorem, we have immediately the following corollary.

Corollary 3.12 *In a semicontinuous lattice L, for all $x, y \in L$ with $x \leq y$, one has*

$$(\mathrm{SI}^{\leq}) \qquad x \Leftarrow y \text{ implies } (\exists z \in L)(x \Leftarrow z \Leftarrow y \text{ and } x \leq z);$$

for all $x, y \in L$ with $x < y$, one has

$$(\mathrm{SI}^{<}) \qquad x \Leftarrow y \text{ implies } (\exists z \in L)(x \Leftarrow z \Leftarrow y \text{ and } x < z).$$

4 Characterization theorems

In this section, we will characterize strongly semicontinuous lattices by some distributivity and finitely separated structures.

Theorem 4.1 *For a complete lattice L, the following conditions are equivalent:*

(1) L is strongly semicontinuous;

(2) for any family $\{P_\alpha\}_{\alpha\in\Gamma}$ of semiprime ideals and $x \in L$, the following (SSD*) *holds:*

$$(\mathrm{SSD}^*) \qquad \bigwedge_{\alpha\in\Gamma}((\bigvee P_\alpha) \wedge x) = \bigvee \bigcap_{\alpha\in\Gamma}(P_\alpha \wedge x).$$

Proof. (2) $\Rightarrow$ (1): Suppose that for any family $\{P_\alpha\}_{\alpha\in\Gamma}$ of semiprime ideals P_α of L, $\bigwedge_{\alpha\in\Gamma}((\bigvee P_\alpha) \wedge x) = \bigvee \bigcap_{\alpha\in\Gamma}(P_\alpha \wedge x)$ holds. If one takes x to be the largest element of L, then one obtains Condition (2) in Lemma 2.8 and L is semicontinuous. If one takes P_α to be $\Downarrow x$ for all $\alpha \in \Gamma$, then by the proved semicontinuity of L, one has that $x = (\bigvee \Downarrow x) \wedge x = \bigvee(\Downarrow x \wedge x) = \bigvee(\Downarrow x \cap \downarrow x)$ for all $x \in L$. So, by Definition 3.1 L is strongly semicontinuous.

(1) $\Rightarrow$ (2): By Proposition 3.6, we see that L is meet semicontinuous. By Lemma 2.8 and the meet semicontinuity, we have that

$$\begin{aligned}\bigwedge_{\alpha\in\Gamma}((\bigvee P_\alpha) \wedge x) &= (\bigwedge_{\alpha\in\Gamma}(\bigvee P_\alpha)) \wedge x = (\bigvee(\bigcap_{\alpha\in\Gamma} P_\alpha)) \wedge x \\ &= \bigvee((\bigcap_{\alpha\in\Gamma} P_\alpha) \wedge x) = \bigvee \bigcap_{\alpha\in\Gamma}(P_\alpha \wedge x),\end{aligned}$$

where $\bigcap_{\alpha\in\Gamma} P_\alpha$ as an intersection of semiprime ideals is still a semiprime ideal. □

Corollary 4.2 *In complete lattices,* (SSD*) $\Longrightarrow$ (SSD)*.*

Proof. By Proposition 3.9 and Theorem 4.1. □

The implication in the above corollary can not be reverse. Actually, for a distributive complete lattice L, the continuity of L is equivalent to the distributivity (SSD*) of L. The following example shows that, even L is a topology, the distributivity (SSD) need not imply the continuity of L, nor imply (SSD*).

Example 4.3 *Consider the rationals $\mathbb{Q}$ with the usual topology $\tau(\mathbb{Q})$ induced from $\mathbb{R}$. It is easy to show that $\Downarrow\mathbb{Q} = \{\emptyset\}$, that is, for each $\emptyset \neq V \in \tau(\mathbb{Q})$, $V \ll Q$ does not hold. So, $\tau(\mathbb{Q})$ is not continuous. Noticing that $\forall U \in \tau(\mathbb{Q})$, $\Downarrow U \subseteq \Downarrow\mathbb{Q}$, we have that $\forall U \in \tau(\mathbb{Q})$, $\Downarrow U = \{\emptyset\}$ and the distributivity* (SSD) *trivially holds on $\tau(\mathbb{Q})$.*

We now establish our second main characterization for strongly semicontinuous lattices, which is given by a version of finitely separated structures on complete lattices.

Definition 4.4 [3] *Let L, M be complete lattices. A function $f : L \to M$ is said to be weakly semicontinuous, if for any $I \in \mathrm{Rd}(L)$,*

$$f(\bigvee I) = \bigvee f(I).$$

Let L, M be complete lattices. We use $[L \rightharpoonup M]$ to denote all the order-preserving weakly semicontinuous functions from L to M in the pointwise order, i.e., $f \leq g$ iff $f(x) \leq g(x)$ for all $x \in L$. It is known that the function space $[L \rightharpoonup M]$ is a complete lattice by [8, Lemma 5.1].

Definition 4.5 *Let L be a complete lattice. A directed family $\mathcal{D} \subseteq [L \rightharpoonup L]$ is called an* approximate identity *for L if it satisfies $\bigvee \mathcal{D} = \mathrm{Id}_L$, the identity on L.*

Proposition 4.6 *If a complete lattice L has an approximate identity $\mathcal{D}$ such that $\delta(x) \Leftarrow x$ for all $\delta \in \mathcal{D}$ and for all $x \in L$, then L is a strongly semicontinuous lattice.*

Proof. For each $x \in L$, x is the supremum of the directed set $\{\delta(x) : \delta \in \mathcal{D}\} \subseteq \Downarrow x \cap \downarrow x$. By Proposition 3.4(3), L is a strongly semicontinuous lattice. $\square$

Definition 4.7 [6] *Let L be a dcpo. A function $\delta : L \to L$ on L is finitely separating if there is a finite set F_δ such that for each $x \in L$, there exists $y \in F_\delta$ such that $\delta(x) \leq y \leq x$.*

Proposition 4.8 *Let L be a complete lattice. If there is an approximate identity $\mathcal{D} \subseteq [L \rightharpoonup L]$ for L consisting of finitely separating functions, then $\delta(x) \Leftarrow x$ for all $\delta \in \mathcal{D}$ and for all $x \in L$, and thus L is a strongly semicontinuous lattice.*

Proof. Let $\delta \in \mathcal{D}$. For each $x \in L$, let $I \in \mathrm{Rd}(L)$ such that $\bigvee I \geq x$. Since δ is a finitely separating function, there exists a finite set F_δ such that for each $d \in I$ there exists $y_d \in F_\delta$ with $\delta(d) \leq y_d \leq d$. Let $F'_\delta = \{y_d \in F_\delta : d \in I\}$, then F'_δ is a nonempty finite subset of F_δ. And for each $y \in F'_\delta$, we can get $d_y \in I$ such that $\delta(d_y) \leq y \leq d_y$. As I is a semiprime ideal, there exists $d_0 \in I$ such that $y \leq d_y \leq d_0$ for all $y \in F'_\delta$. Hence for each $d \in I$, $\delta(d) \leq y_d \leq d_0$. Thus, by weakly semicontinuity of δ, we have $\delta(x) \leq \delta(\bigvee I) = \bigvee_{d \in I} \delta(d) \leq d_0$. Therefore, $\delta(x) \in I$. This shows that $\delta(x) \Leftarrow x$. By Proposition 4.6, L is a strongly semicontinuous lattice. $\square$

For a set U, $\mathcal{P}_{\mathrm{fin}}(U)$ denotes the set of all nonempty finite subsets of U.

Proposition 4.9 *If L is a strongly semicontinuous lattice, then there is an approximate identity $\mathcal{D} \subseteq [L \rightharpoonup L]$ for L consisting of finitely separating functions.*

Proof. Let L be a strongly semicontinuous lattice. For each $x \in L$, $S \in \mathcal{P}_{\mathrm{fin}}(L)$, define $\delta_S : L \to L$ by $\delta_S(x) = \bigvee\{y \in S \cap \downarrow x : y \Leftarrow x\}$. Note that if $\{y \in S \cap \downarrow x : y \Leftarrow x\} = \emptyset$, then $\delta_S(x) = \perp$, the least element of L. So, $\delta_S(x)$ is well-defined. It is easy to see that δ_S is order-preserving with $\delta_S(x) \leq x$ for all

$x \in L$. To show δ_S is weakly semicontinuous, for each $P \in Rd(L)$, it suffices to show that $\delta_S(\bigvee P) = \delta_S(k) \leq \bigvee_{p\in P} \delta_S(p)$, where $k = \bigvee P$. If $\{y \in S \cap \downarrow k : y \Leftarrow k\} = \emptyset$, then by the definition of δ_S, we see that $\delta_S(k) = \bot \leq \bigvee_{p\in P} \delta_S(p)$. Let $\{y_1, \cdots, y_l\} = \{y \in S \cap \downarrow k : y \Leftarrow k\} \neq \emptyset$ and $m = \bigvee_{i=1}^{l} y_i$. Then by the definition of δ_S, $\delta_S(k) = m \leq k$. Since $y_i \Leftarrow k$ for $i = 1, 2, \cdots, l$, we have that $m \Leftarrow k$. By (SI$^{\leq}$) in Corollary 3.12, there is a m^* such that $m \Leftarrow m^* \Leftarrow k$ with $m \leq m^*$. It follows from $m^* \Leftarrow k = \bigvee P$ that $m, m^* \in P$. Noticing that $m \leq m^*$ and $y_i \leq m \Leftarrow m^* \in P$, we have $\{y_1, \cdots, y_l\} \subseteq \{y \in S \cap \downarrow m^* : y \Leftarrow m^*\}$ which yields that $\bigvee_{p\in P} \delta_S(p) \geq \delta_S(m^*) \geq \delta_S(k)$. So, δ_S is a weakly semicontinuous function.

Suppose that $S, T \in \mathcal{P}_{\text{fin}}(L)$ with $S \subseteq T$. It is easy to see that $\delta_S \leq \delta_T$ and $\mathcal{D} = \{\delta_S\}_{S\in\mathcal{P}_{\text{fin}}(L)}$ is directed. Notice that the images $\text{im}(\delta_S) \subseteq \{\bigvee F : F \subseteq S\}$ is finite by the finiteness of S. So, δ_S is a finitely separating function. Since L is strongly semicontinuous, for each $x \in L$ and $\delta_S \in \mathcal{D}$, $\delta_S(x) = \bigvee\{y \in S \cap \downarrow x : y \Leftarrow x\} \leq x$,

$$\begin{aligned} \bigvee_{\delta_S \in \mathcal{D}} \delta_S(x) &= \bigvee_{\delta_S \in \mathcal{D}} \left(\bigvee\{y \in S \cap \downarrow x : y \Leftarrow x\}\right) \\ &\geq \bigvee_{z \Leftarrow x} \left(\bigvee\{y \in \{z\} \cap \downarrow x : y \Leftarrow x\}\right) \\ &= \bigvee(\Downarrow x \cap \downarrow x) = x. \end{aligned}$$

Therefore, $\mathcal{D} \subseteq [L \rightharpoonup L]$ is an approximate identity for L consisting of finitely separating functions. □

Since continuous lattices are strongly semicontinuous, we immediately have

Corollary 4.10 *If L is a continuous lattice, then there is an approximate identity $\mathcal{D} \subseteq [L \rightharpoonup L]$ for L consisting of finitely separating functions.*

Now we arrive at our final characterization theorem.

Theorem 4.11 *Let L be a complete lattice. Then L is a strongly semicontinuous lattice if and only if there is an approximate identity $\mathcal{D} \subseteq [L \rightharpoonup L]$ for L consisting of finitely separating functions.*

Proof. It follows from Propositions 4.8 and 4.9. □

With the property (A) in Definition 3.1, the concept of semi-FS domains, the counterpart of FS-domains in the setting of semicontinuous lattices, is posed in [4]. It follows from Proposition 4.9 that a strongly semicontinuous lattice is a semi-FS domain, a situation similar to that a continuous lattice is an FS-domain. In the past 16 years since Zhao's work [12] on semicontinuous lattices, the difficulty in defining suitable semi-FS domains is probably lack of the concept of strongly semicontinuity.

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