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On the Effective Existence of Schauder Bases (Extended Abstract)

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Abstract

We construct a computable Banach space which possesses a Schauder basis, but does not possess any computable Schauder basis.

Keywords: Computable analysis, Schauder bases

1 Introduction

Let X be an infinite-dimensional Banach space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. A sequence (x_i) of elements of X is called a *Schauder basis* (or simply a *basis*) of X if for every $x \in X$ there is a unique sequence (α_i) of elements of $\mathbb{F}^\omega$ such that x is the limit of the norm-convergent series $\sum_{i=0}^{\infty} \alpha_i x_i$. If X is a finite-dimensional vector space then a finite sequence $(x_1, \dots, x_n) \in \mathbb{F}^{<\omega}$ is a (Schauder) basis of X if for every $x \in X$ there are unique $\alpha_1, \dots, \alpha_n$ such that $x = \sum_{i=1}^{n} \alpha_i x_i$. A finite or infinite sequence is called *basic* if it is a basis of the closure of its linear span. (Finite sequences are hence basic if, and only if, they are linearly independent.)

The theory of Schauder bases is a central area of research and also an important tool in functional analysis. Background information can be found in e.g. [1,8,10,11].

In computable analysis [4,13], Brattka and Dillhage [3] have shown that computable versions of a number of classical theorems on compact operators on Banach spaces can be proved under the assumption that the computable Banach spaces under consideration possess computable bases (with certain additional properties).

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The restriction to spaces with computable bases does not seem to be too costly in terms of generality because virtually all of the separable Banach spaces that are important for applications are known to possess a computable basis.

Complete orthonormal sequences in Hilbert spaces are particularly well-behaved examples of Schauder bases. It is a fundamental fact that every separable Hilbert space contains such a complete orthonormal sequence. It is furthermore known (see [5, Lemma 3.1]) that every computable Hilbert space contains a computable complete orthonormal sequence and hence a computable basis. Can this be generalized to arbitrary computable Banach spaces with bases? More precisely: If a computable Banach space possesses a basis, does it necessarily possess a computable basis? The aim of the present note is to show that the answer is “no” in general. Our example will be a subspace of the space of zero-convergent sequences whose terms are in *Enflo’s space* – a famous example of a separable Banach space that lacks the *approximation property* (see below). The construction will proceed by direct diagonalization.

Some remarks on notation: As we will never consider more than one norm on the same linear space, we will denote every norm by $\|\cdot\|$; which norm is meant will be clear from what it is applied to. If $x_1, x_2, \ldots$ are elements of a Banach space, denote by $[x_1, x_2, \ldots]$ the closure of their linear span; analogously, let $[x_1, \ldots, x_n]$ denote the linear span of $x_1, \ldots, x_n$. Whenever we speak of the *rational span* of a set of vectors, we mean all their finite linear combinations with coefficients taken from $\mathbb{Q}$ (if $\mathbb{F} = \mathbb{R}$) or $\mathbb{Q}[i]$ (if $\mathbb{F} = \mathbb{C}$), respectively. If X is a normed space, put $B_X := \{x \in X \ : \ \|x\| \leq 1\}$. Denote by $\mathcal{K}(X)$ the hyperspace of compact subsets of X.

As far as computable analysis is concerned we shall adopt the notation from [3]. In particular, see [3] and the references cited therein for the definition of *computable Banach spaces*. Computability of points, sequences, continuous functions, compact subsets, etc. shall always be understood as computability with respect to the representations considered in [3].

2 Some properties of Enflo’s space

The question whether every separable Banach space has a basis was posed by Banach in 1932; forty years later, it was answered in the negative by Enflo [7]. Enflo in fact constructed a Banach space that lacks the *approximation property* (AP). A Banach space is said to have AP if the identity operator can be approximated uniformly on every compact subset by finite-rank operators (see [8, Definition 3.4.26, Theorem 3.4.32]). A Banach space with a basis necessarily has AP (see [8, Theorem 4.1.33]).

Enflo’s example was simplified by Davie [6]. It is easy to verify (when looking at Davie’s proof) and not surprising that the space defined by Davie is computable:

Lemma 2.1 *There is a computable (complex) Banach space* $(Z, \|\cdot\|, (e_i))$ *without AP.*

Associated with every basic sequence (x_i) of elements of an infinite-dimensional

Banach space X is the sequence (P_n) of *natural projections* $P_n : [x_0, x_1, \ldots] \to [x_0, \ldots, x_n]$, which are defined by $P_n(\sum_{i=0}^{\infty} \alpha_i x_i) := \sum_{i=0}^{n} \alpha_i x_i$. The P_n are bounded linear mappings, and it is a fundamental fact that $\sup_n \|P_n\| < \infty$ (see [8, Corollary 4.1.17]). The value $\sup_n \|P_n\|$ is called the *basis constant* $\mathrm{bc}((x_i))$ of (x_i). Basis constants can also be defined (in an analogous way) for finite basic sequences, and hence also if X is only finite-dimensional. Finally, the basis constant of a space X that possesses a basis is defined as

$$\mathrm{bc}(X) := \inf\{\mathrm{bc}((x_i)) \;:\; (x_i) \text{ is a basis of } X\}.$$

A Banach space X is said to have *local basis structure* if there is a constant C such that for every finite-dimensional subspace $V \subseteq X$ there is a finite-dimensional space W with $V \subseteq W \subseteq X$ and $\mathrm{bc}(W) \leq C$. This notion was introduced in [9] (under a different name). A sufficient criterion given by Szarek [12] can be used to prove:

Lemma 2.2 *Z has local basis structure.*

Via exhaustive search techniques, the following effective statement can be derived:

Lemma 2.3 *There is a constant $C > 0$, a computable linearly independend sequence (c_i) of elements of Z, and a strictly increasing computable function $\sigma : \mathbb{N} \to \mathbb{N}$ such that $[c_0, c_1, \ldots] = Z$ and $\mathrm{bc}([c_0, \ldots, c_{\sigma(n)}]) < C$ for every $n \in \mathbb{N}$.*

3 The construction

Let Y be the Banach space of all zero-convergent sequences of elements of Z equipped with the sup-norm, that is

$$\|(z_i)\| := \sup_{i\in\mathbb{N}} \|z_i\| \text{ for every } (z_i) \in Y.$$

For $m \in \mathbb{N}$ let $\pi^{(m)} : Y \to Z$ be the projection $\pi^{(m)}((z_i)) := z_m$; on the other hand, let $\eta^{(m)} : Z \to Y$ be the isometric embedding defined by

$$\pi^{(i)}(\eta^{(m)}(z)) := \begin{cases} z & \text{if } m = i, \\ 0 & \text{otherwise.} \end{cases}$$

For all $i, m \in \mathbb{N}$, put $g_{\langle i,m\rangle} := \eta^{(m)}(e_i)$. It is easy to verify that $(Y, \|\cdot\|, (g_i))$ is a computable Banach space and that $(\eta^{(m)})$ and $(\pi^{(m)})$ are computable sequences of functions.

The facts that Z lacks AP and that AP is inherited by complemented subspaces [3] yield:

[3] Recall that a subspace F of X is called *complemented* if there is a continuous linear mapping $P : X \to X$ with $P^2 = P$ and $P(X) = F$. (Remark: It is known that having a basis is *not* inherited by complemented subspaces in general; see [12].)

Lemma 3.1 *Let X be a closed subspace of Y such that there is an $m \in \mathbb{N}$ with $\eta^{(m)}(Z) \subseteq X$. Then X does not have a basis.*

Let (c_i) and σ be as in Lemma 2.3. Put $X_m := [c_0, \dots, c_{\sigma(m)}]$. The idea of the proof of the following lemma is to construct a subspace of Y which does not include any space of the form $\eta^{(m)}(Z)$, but includes infinitely many spaces of the form $\eta^{(m)}(X_k)$. In view of Lemma 3.1 and the fact that $\mathrm{clo}\,(\bigcup_m X_m) = Z$, this means, loosely speaking, that the constructed space is "close to not having a basis". Using diagonalization, we will ensure that the space is "close enough to not having a basis" such that no computable sequence is a basis of it.

Lemma 3.2 *There is a computably enumerable set $L \subseteq \mathbb{N}$ such that the following holds:*

(i) *$\{k \;:\; \langle n, j, k\rangle \in L\}$ is finite for all $n, j \in \mathbb{N}$.*

(ii) *Put*

$$\tau(\langle n, j\rangle) := \max\big(\{k \;:\; \langle n, j, k\rangle \in L\} \cup \{0\}\big),\; n, j \in \mathbb{N}.$$

No computable sequence of elements of Y is a basis of the subspace

$$X := \{(z_i) \in Y \;:\; z_m \in X_{\tau(m)} \text{ for all } m \in \mathbb{N}\}.$$

Next, we have to make sure that the space from Lemma 3.2 *has* a basis. This can be proved based on the uniform boundedness of the basis constants of the X_m:

Lemma 3.3 *The space X constructed in Lemma 3.2 possesses a basis.*

We are now ready to complete the construction: Let X be the Banach space constructed in Lemma 3.2. We have just seen that X has a basis. Let (h_i) be a computable enumeration of the set

$$\{\eta^{(m)}(c_s) \;:\; m \in \mathbb{N},\, 0 \leq s \leq \sigma(\tau(m))\}.$$

Then $[h_0, h_1, \dots] = X$. Reference [2, Proposition 3.10] yields that $(X, \|\cdot\|, (h_i))$ is a computable Banach space and that the embedding $X \hookrightarrow Y$ is computable. We conclude that $(X, \|\cdot\|, (h_i))$ cannot have a computable basis, because this basis would be a computable sequence in Y in contradiction to Lemma 3.2. The foregoing results are subsumed under the following theorem:

Theorem 3.4 *The computable Banach space $(X, \|\cdot\|, (h_i))$ as constructed above possesses a basis, but does not possess any computable basis.* □

Remark. One of the additional properties of bases considered by Brattka and Dillhage [3] is the property of being *shrinking*. A basis (x_i) of X is shrinking if the following holds for every element f of the topological dual X^*:

$$\lim_{n\to\infty} \sup\{|f(x)| \;:\; x \in B_{[x_{n+1}, x_{n+2}, \dots]}\} = 0.$$

It is well known that the dual c_0^* of c_0 can be identified with ℓ_1 (see [8, Example 1.10.4]); here $(q_i) \in \ell_1$ applied to $(p_i) \in c_0$ is defined as $\sum_i q_i p_i$. This easily leads to the fact that the unit vector basis $((1,0,0,\dots),(0,1,0,\dots),\dots)$ of c_0 is shrinking. In a completely analogous way, one can show that the basis, constructed in Lemma 3.3, of the space X, constructed in Lemma 3.2, is shrinking: In fact, the dual of X can be identified with the space of all sequences (f_i) with $f_i \in X^*_{\tau(i)}$ and $\sum_i \|f_i\| < \infty$, where (f_i) applied to x is $\sum_{i=0}^{\infty} f_i(\pi^{(i)}(x))$. $\square$

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