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Novel linear programming models based on distance measure of IFSs and modified TOPSIS method for portfolio selection

Xue Deng *, Chuangjie Chen

School of Mathematics, South China University of Technology, Guangzhou 510640, China

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ABSTRACT

Due to the unavailability of time series data for newly listed stocks or products, it is a challenge for investors to make rational portfolio selection under uncertain circumstances. To solve the problem, a new approach is put forward in this paper. Firstly, the problem is considered as a multi-criteria decision making (MCDM) problem based on the assumption that the assessments are given in intuitionistic fuzzy set (IFS) form, which can better describe the uncertainty. Then, the TOPSIS method, which is widely used in MCDM problem, has been modified in two aspects. On the one hand, the new definitions of Absolute Positive Ideal Solution (APIS) and Absolute Negative Ideal Solution (ANIS) are proposed to represent the best returns and the greatest cost/risk in portfolio selection. They are more meaningful than the definitions of Positive Ideal Solution (PIS) and Negative Ideal Solution (NIS), which could only express the best and the worst case of portfolio but cannot achieve a balance between returns and risk. On the other hand, the weighted closeness coefficient is refined to offer the more appropriate results that are consistent with investors' demands or preferences. In addition, based on distance measure of IFSs, several novel linear programming models with different constraints are proposed to allocate the investment ratios according to investors' demands. The models can make up for the disadvantage of TOPSIS method which only considers the ranking of investments but neglects the investment ratios. Finally, compared with the conventional TOPSIS method and the IF ELECTRE Method in a numerical example, our new approach is demonstrated to be more effective and more flexible. Particularly, it can provide a more appropriate strategy in accordance with investors' preferences and demands.

1. Introduction

Modern portfolio selection theory and the mean–variance probabilistic model, which were proposed by Markowitz [1], established the relationship between portfolio selection and mathematics. This theory revealed that investors aim to gain more returns while trying to avoid risks as far as possible, and then the model provided a way for investors to balance the returns and risks in accordance with their risk preferences. On this basis, portfolio selection has developed rapidly. With considerable researches held in this field, more and more achievements have sprung up. These achievements extended the portfolio selection theory and transformed the model into kinds of new forms with different constraints according to investors' demands. For instance, Davis and Norman [2] studied an investment model which contains linear transaction costs; Zhou and Li [3] paid attention to a continuous-time mean–variance portfolio selection model; Ledoit and Wolf [4] put forward the improved estimation of the co-variance matrix of stock returns; Liu [5] did a research on portfolio selection in stochastic environments; Tofighian et al. [6] dealt with multi-period project portfolio selection problem and presented a mathematical model considering risks, stochastic incomes, and possibility of investing extra budget in each time period.

However, it should be noticed that the conventional model for portfolio selection is becoming more and more unavailable and useless due to the complexity of portfolio circumstance which contains both objective and subjective uncertainties. Zhou and Xu [7] pointed out that time series data may be unavailable for newly

* Corresponding author.
E-mail address: dxue@scut.edu.cn (X. Deng).

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listed stocks or products. Besides, Yazdi [8] mentioned that the actual circumstance is not definitive or predictable because of the existing uncertainties. More generally, it is difficult to obtain the precise probability distributions of returns as well as enough data just by relying on the past statistics due to the ever-changing economic environment [9]. Thus, it has become a top priority to look for a more appropriate way to deal with portfolio selection problem under the uncertain circumstances.

Fuzzy set theory, which is proposed by Zadeh [10] in 1965 to describe uncertainties in mathematical language, plays an important role in the modern society that is full of all kinds of uncertainty. As a powerful tool to deal with uncertainty, it has received a great concern and has been studied by many researchers for some relative practical applications. Intuitionistic fuzzy set (IFS), which presented by Atanassov [11], is doubtless one of the greatest extensions of fuzzy set theory. Since it can describe uncertainties from positive and negative perspectives by using the membership degree and non-membership degree, respectively, it is more effective than fuzzy set. In intuitionistic fuzzy set theory, distance measure, the representation of the deviation between IFSs, is widely used in practice, such as pattern recognition [12,13], decision making [14,15,16] and medical diagnosis [17]. It was first defined by Szmidt and Kacprzyk [18] in 2000. And then, in order to pursue higher accuracy and more practical significance, various distance measures have been proposed. In this paper, we will adopt H-max distance measure [15] to quantify the differences between investments.

The advantage of intuitionistic fuzzy set in dealing with uncertainties implies that it can effectively solve the portfolio selection problem in uncertain environment. Yue et al. [19] proposed a fuzzy linear higher order moment portfolio selection model; Guo et al. [9] studied the fuzzy multi-period portfolio selection problem with V-Shaped transaction cost; and Chen et al. [20] put forward a mean–variance-skewness fuzzy portfolio selection model based on intuitionistic fuzzy optimization. To help investors make a selection among several newly listed investments and then allocate the investment ratios for each selected investment, in this paper, the portfolio selection under uncertain circumstances is regarded as a multi-criteria decision making (MCDM) problem. This kind of problem requires the decision maker to rank the provided alternatives according to the assessments with respect to several criteria [21]. In order to describe the uncertain circumstances better, all these assessments will be given in IFS form.

For the sake of ranking the investments conveniently, the method called technique for order preference by similarity to ideal solution (TOPSIS) is introduced. TOPSIS is presented by Hwang and Yoon [22] in 1981, which points out the best alternative is one that has not only the shortest distance from the Positive Ideal Solution (PIS) but also the farthest distance from the Negative Ideal Solution (NIS) [23]. In recent years, TOPSIS has been extended to fuzzy environment and widely applied to solve MCDM problems [8,16,21,23]. However, there is still not substantially improvement on its model. As pointed out by Dwivedi et al. [16], one of the limitations of the conventional TOPSIS method is that it does not take care of the weights of PIS and NIS distances. Although there have been some modifications on closeness coefficient, which plays a significant role in TOPSIS method, it is a pity that these modifications are not perfect enough. Besides, for portfolio selection, the conventional TOPSIS method cannot give investors a hand to allocate investment ratios for each selected investment, which cannot effectively spread the risk and is fatal in portfolio selection.

To pursue more adaptability and flexibility, the conventional TOPSIS method is modified and refined in this paper. Firstly, the new definitions of Absolute Positive Ideal Solution (APIS) and Absolute Negative Ideal Solution (ANIS) are proposed to represent the best returns and the greatest cost/risk in portfolio selection. Compared with PIS and NIS, APIS and ANIS can achieve a balance between returns and risk, which is important for a successful investment. Then, considering that investment is a kind of decision making activity which largely depends on the decision maker's preference, the weighted closeness coefficient is refined to offer the more appropriate results that are consistent with the investor's demand or preference. Finally, the modified TOPSIS method is combined with linear programming model to allocate the investment ratios according to the investor's demand, which could spread risk to some extent.

The rest of this paper is organized as follows. In Section 2, some basic conceptions of IFS and TOPSIS method are introduced for preparation. The conventional TOPSIS method is modified in Section 3. In Section 4, three kinds of linear portfolio optimization programming models based on distance measure are given. Section 5 presents a numerical application to check the effectiveness and practicality of the models and then a sensitivity analysis is held. Finally, the conclusion is drawn in Section 6.

2. Preliminaries

This section will introduce some basic concepts of IFS and TOPSIS method.

2.1. The intuitionistic fuzzy set theory

Definition 1. . [10] Let X be a universe of discourse. Then set A is called a fuzzy set in X if it has a form as:

$$A = \{\langle x,\ \mu_A(x)\rangle | x \in X\} \tag{1}$$

where the function $\mu_A(x)$ represents the membership degree of the element x to the set A with the range of $[0,\ 1]$.

Definition 2. . [11] Let X be a universe of discourse. Then set A is called an intuitionistic fuzzy set (IFS) in X if it has a form as:

$$A = \{\langle x,\ \mu_A(x),\ \nu_A(x)\rangle | x \in X\} \tag{2}$$

where the function $\mu_A(x)$ represents the membership degree and $\nu_A(x)$ represents the non-membership degree of the element x to the set A with the range of $[0,\ 1]$. Meanwhile, they should satisfy $0 \leqslant \mu_A(x) + \nu_A(x) \leqslant 1, \forall x \in X$.

Let $\pi_A(x) = 1 - (\mu_A(x) + \nu_A(x))$, $\pi_A(x)$ is called the hesitancy degree of x to the IFS A and determined by both $\mu_A(x)$ and $\nu_A(x)$. We can easily get $0 \leqslant \pi_A(x) \leqslant 1,\ \forall x \in X$. Then, IFS A can be described as the following form:

$$A = \{\langle x, \mu_A(x),\ \nu_A(x),\ \pi_A(x)\rangle | x \in X\} \tag{3}$$

Hesitancy degree plays an important role in many applications of IFS, such as measuring the distance of two IFSs [13] and reflecting the uncertainty of IFS [24]. It is also the major difference between IFS and fuzzy set. Considering that the membership degree $\mu_A(x)$ represents the acceptance degree and the non-membership degree $\nu_A(x)$ represents the rejection degree of a decision maker, hesitancy degree could be regarded as the neutrality degree of the decision maker, which may affect the decision making results to some extent.

For convenience, $IFS(X)$ is used to denote the collection of all IFSs in X in this paper.

Entropy acts as an important role in the intuitionistic fuzzy theory for the reason that it represents the degree of uncertainty to IFS. Thus, it is also applied in many fields. In this paper, we will use entropy to assign weights to each criterion in TOPSIS method [25]. The greater the entropy, the more the uncertainty and the less important the IFS is. Considering the simplicity of the formula and the effective utilization of $\mu_A(x)$, $\nu_A(x)$ and $\pi_A(x)$, the intuitionistic fuzzy entropy proposed by Wei et al. [24] is adopted in this paper:

Definition 3. . [24] Suppose $X = \{x_1, x_2, \cdots, x_n\}$ is a finite fixed set, A is an IFS in$X$, where$A = \{\langle x, \mu_A(x), \nu_A(x), \pi_A(x)\rangle | x \in X\}$, and $E(A)$ is a mapping from $IFS(X)$ to $[0, 1]$, which is expressed as follows:

$$E(A) = \frac{1}{n}\sum_{i=1}^{n} \cos\frac{\mu_A(x_i) - \nu_A(x_i)}{2(1+\pi_A(x_i))}\pi \tag{4}$$

$E(A)$ is called an intuitionistic fuzzy entropy over X.

We will use the H-max distance measures [15] to describe the deviations of IFSs. This distance measure adopts a proper inclusion relation of IFS and has many great properties. It is given as follows:

Definition 4. . [15] Suppose $X = \{x_1, x_2, \cdots, x_n\}$ is a finite fixed set, A and B are two IFSs in X, where$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\}$, and $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle | x \in X\}$. The H-max distance $d(A, B)$ between A and B is expressed as follows:

$$d(A, B) = \frac{1}{3n}\sum_{i=1}^{n}\left(\begin{array}{c} |\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)| + \\ |\max\{\mu_A(x_i), \nu_B(x_i)\} - \max\{\mu_B(x_i), \nu_A(x_i)\}| \end{array}\right) \tag{5}$$

2.2. The conventional TOPSIS method

2.2.1. The steps of the conventional TOPSIS method

The steps of the conventional TOPSIS method are briefly shown as follows [22]:

Step 1: Acquire the assessments of the alternatives with respect to each criterion.
Step 2: Define the PIS and the NIS for comparison.
Step 3: Assign weights to each criterion.
Step 4: Calculate the weighted distance between every alternative and PIS, NIS.
Step 5: Calculate the closeness coefficient of every alternative.
Step 6: Rank the alternatives by the closeness coefficient.

2.2.2. PIS, AIS and closeness coefficient

In the intuitionistic fuzzy environment, assume that there is a set $A = \{A_1, A_2, \cdots, A_m\}$ consisting of m alternatives which wait for ranking and a set $C = \{C_1, C_2, \cdots, C_n\}$ consisting of n criteria used for assessment. In order to deal with the values of assessment more conveniently, we denote the assessments as an IFS matrix $D = (D_{ij})_{m\times n}$, where $D_{ij} = (\mu_{ij}, \nu_{ij})$ represents the assessment of the alternative A_i in intuitionistic fuzzy set form. This form uses the membership degree μ_{ij} to describe the decision maker's satisfaction while using the non-membership degree ν_{ij} to describe the decision maker's dissatisfaction with respect to the criterion C_j, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$. In some cases, the criteria set C is considered to comprise two incompatible sets C^+ and C^-, where C^+ consists only of benefit/returns criteria and C^- consists only of cost/risk criteria, respectively. Then, the PIS $A^+ = (D_1^+, D_2^+, \cdots, D_n^+)$ and the NIS $A^- = (D_1^-, D_2^-, \cdots, D_n^-)$ are usually defined as follows:

$$D_j^+ = (\mu_j^+, \nu_j^+) = \begin{cases} \left(\max\limits_{1\leqslant i\leqslant m}\{\mu_{ij}\}, \min\limits_{1\leqslant i\leqslant m}\{\nu_{ij}\}\right), & \text{if } C_j \in C^+, \\ \left(\min\limits_{1\leqslant i\leqslant m}\{\mu_{ij}\}, \max\limits_{1\leqslant i\leqslant m}\{\nu_{ij}\}\right), & \text{if } C_j \in C^-. \end{cases} \tag{6}$$

$$D_j^- = (\mu_j^-, \nu_j^-) = \begin{cases} \left(\min\limits_{1\leqslant i\leqslant m}\{\mu_{ij}\}, \max\limits_{1\leqslant i\leqslant m}\{\nu_{ij}\}\right), & \text{if } C_j \in C^+, \\ \left(\max\limits_{1\leqslant i\leqslant m}\{\mu_{ij}\}, \min\limits_{1\leqslant i\leqslant m}\{\nu_{ij}\}\right), & \text{if } C_j \in C^-. \end{cases} \tag{7}$$

where D_j^+ and D_j^- represent the assessments of A^+ and A^- under the criterion C_j in intuitionistic fuzzy set form, respectively, $1 \leqslant j \leqslant n$.

After assigning the weights to each criterion and adopting a suitable distance measure in accordance with actual demand, we can get the weighted distances $d_i^+ = \sum_{j=1}^n w_j d_{ij}^+$ and $d_i^- = \sum_{j=1}^n w_j d_{ij}^-$ between the alternative A_i and PIS, A_i and NIS, respectively. Here, w_j denotes the weight assigned to criterionC_j, and d_{ij}^+, d_{ij}^- denote the distances between A_i and PIS, A_i and NIS with respect to the criterion C_j, respectively, $\sum_{j=1}^n w_j = 1$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$. Then, the closeness coefficient CC_i of alternative A_i is expressed in the following way:

$$CC_i = \frac{d_i^-}{d_i^+ + d_i^-} \tag{8}$$

Since the greater the d_i^- and the smaller the d_i^+, the better the alternative A_i, we have the conclusion that the greater the closeness coefficient, the better the alternative.

3. The modified TOPSIS method

Multi-criteria decision making (MCDM) problem requires decision makers to rank the provided alternatives according to the assessments under several given criteria [21]. Thus, it is rational to regard the portfolio selection problem, which is based on the assessments under several criteria rather than the past statistical data, as a kind of MCDM problem. As one of the common methods applied in the ranking of alternatives, TOPSIS works on a simple but logical principle that the best alternative should have the shortest distance from the Positive Ideal Solution (PIS) and the farthest distance from the Negative Ideal Solution (NIS) [23].

However, the conventional TOPSIS method can only provide the decision maker with a reasonable ranking of the alternatives, but cannot offer an appropriate result according to the decision maker's preference, thus it is inflexible in the portfolio selection problem with different preferences. In order to make the method more flexible and make up for the drawbacks of the closeness coefficient, two improvements are made on the TOPSIS method from the following aspects:

(i) Two new definitions called Absolute Positive Ideal Solution (APIS) and Absolute Negative Ideal Solution (ANIS) are proposed;
(ii) The new definition called weighted closeness coefficient is presented.

3.1. The Absolute Positive Ideal Solution (APIS) and Absolute Negative Ideal Solution (ANIS)

Let us consider the following question: is the word "dislike" equivalent to the word "hate"? Obviously, the answer is no, at least, the former is less emotional than the latter. Now, there is another similar question: can we always transform the risk criteria into the returns type? This answer is the same as that of the former question. Thus, it is inappropriate to put the returns criteria and risk criteria into together to define the ideal solution when we use TOPSIS method.

From this point of view, the following new definition is presented:

Definition 5. Let $A = \{A_1, A_2, \cdots, A_m\}$ be the set of alternatives that wait for ranking and $C = \{C_1, C_2, \cdots, C_n\}$ be the set that consists of n criteria used for assessment. Consider the set C comprises two incompatible sets C^+ and C^-, where $C^+ = \left(C^+_{n_1}, C^+_{n_2}, \cdots, C^+_{n_p}\right)$ consists only of returns criteria and $C^- = \left(C^-_{n_1}, C^-_{n_2}, \cdots, C^-_{n_q}\right)$ consists only of cost/risk criteria, respectively, $n_p + n_q = n$. If $D_{ij} = \left(\mu_{ij}, v_{ij}\right)$ represents the assessment of the alternative A_i with respect to the criterion C_j in intuitionistic fuzzy set form, where membership degree μ_{ij} describes the decision maker's satisfaction and non-membership degree v_{ij} describes the decision maker's dissatisfaction, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$, the Absolute Positive Ideal Solution (APIS) $A'^+ = \left(D'^+_{n_1}, D'^+_{n_2}, \cdots, D'^+_{n_p}\right)$ and the Absolute Negative Ideal Solution (ANIS) $A'- = \left(D'^-_{n_1}, D'^-_{n_2}, \cdots, D'^-_{n_q}\right)$ are defined as follows:

$$D'^+_{n_j} = \left(\mu'^+_{n_j}, v'^+_{n_j}\right) = \left(\max_{1\leqslant i\leqslant m} \mu_{in_j}, \min_{1\leqslant i\leqslant m} v_{in_j}\right) \tag{9}$$

$$D'^-_{n_k} = \left(\mu'^-_{n_k}, v'^-_{n_k}\right) = \left(\max_{1\leqslant i\leqslant m} \mu_{in_k}, \min_{1\leqslant i\leqslant m} v_{in_k}\right) \tag{10}$$

where $D'^+_{n_j}$ represents the assessment of A'^+ under the criterion $C^+_{n_j}$ and $D'^-_{n_k}$ represents the assessment of A'^- under the criterion $C^-_{n_k}$ in intuitionistic fuzzy set form, $1 \leqslant n_j \leqslant n_p$, $1 \leqslant n_k \leqslant n_q$,$n_p + n_q = n$.

Then, the weighted distances d'^+_i and d'_i- between the alternative A_i and APIS, A_i and ANIS are defined as follows:

$$d'^+_j = \sum_{j=1}^{p} w_{n_j} d^+_{in_j}, \tag{11}$$

$$d'^-_j = \sum_{k=1}^{q} w_{n_k} d^-_{in_k}, \tag{12}$$

where w_{n_j}, w_{n_k} denote the weights assigned to criteria $C^+_{n_j}$ and$C^-_{n_k}$, respectively, and $d^+_{in_j}$ denotes the distances between A_i and APIS with respect to the criterion$C^+_{n_j}$, $d^-_{in_k}$ denotes the distances between A_i and ANIS with respect to the criterion $C^-_{n_k}$, respectively, $\sum_{j=1}^{p} w_{n_j} = 1$,$\sum_{k=1}^{q} w_{n_k} = 1$, $1 \leqslant n_j \leqslant n_p$, $1 \leqslant n_k \leqslant n_q$, $n_p + n_q = n$.

From the definition, we know that APIS represents an imaginary alternative which has the best attributes corresponding to each returns criterion but ignoring all the cost/risk criteria, while ANIS represents an imaginary alternative which has the worst attributes corresponding to each cost/risk criterion but ignoring all the returns criteria. Thus, it is practicable to consider that APIS is better than PIS and ANIS is worse than NIS.

In this paper, we would use APIS and ANIS to replace PIS and NIS for the following reasons:

(i) APIS is better than PIS and ANIS is worse than NIS, which is mentioned above, can make the comparison more distinct as to the results of different preferences, which is proved in the sensitivity analysis of the numerical application in Section 5.
(ii) In the conventional TOPSIS method, PIS and NIS represent the best and the worst alternatives, respectively; in our modified TOPSIS method, APIS and ANIS represent the greatest returns and the highest cost/risk, respectively. In terms of the demand of portfolio selection, the significance of APIS and ANIS is more evident and realistic than PIS and NIS.
(iii) Compared with PIS and NIS, it can be found that the APIS and ANIS are more flexible than them, because the constraints on APIS and ANIS are less than those on PIS and AIS (they must take both returns and risk criteria into account), which will be illustrated in Example 1.

Remark 1. *If the returns criteria set is empty, it implies that the investor only considers risks but ignores returns, i.e. the investor is completely risk averse. Under this condition, ANIS is equivalent to NIS but APIS is non-existent; correspondingly, if the cost/risk criteria set is empty, it also implies that the investor is completely profit-seeking. Under this condition, APIS is equivalent to PIS but ANIS is non-existent.*

It is obvious that the definition of APIS and ANIS can be extended to other conditions, whether the environment is uncertain or not. A simple numerical example is given to illustrate the flexibility of APIS and ANIS by comparing with PIS and NIS in a certain environment.

Example 1. *Let A_1 and A_2 be two alternatives which wait for ranking, and $C = \{C_1, C_2, C_3, C_4\}$ be the set of criteria, where C_1 and C_2 are returns type and the rest belongs to risk type. The assessments of alternatives under each criterion are listed in Table 1:*

Without losing generality, suppose each criterion is allocated the same weight. Then, according to Section 2.2.2 and Table 1, we can find out the PIS, NIS are (0.9, 0.8, 0.3, 0.6) and (0.7, 0.7, 0.5, 0.7), respectively. Similarly, according to Definition 5, we get the APIS and ANIS (0.9, 0.8) and (0.5, 0.7) respectively. On these bases, the differences between the alternatives and PIS, NIS, APIS, ANIS can be obtained and is showed in Table 2.

Remark 2. *It should be noticed that according to Formula (11) and Formula (12), the weight of each criterion is* 0.5 *while we calculating the differences between the alternatives and APIS, ANIS.*

From Table 2, we can find that the closeness coefficients of the alternatives are the same whether we use PIS, NIS or APIS, ANIS (Which is one of the drawbacks of the closeness coefficient and will be addressed in Section 3.2). Hence, it is difficult for a neutral decision maker to make a ranking of these two alternatives.

But the superiority of the APIS, ANIS lies in: if the decision maker is profit-seeking, he/she would prefer A_1 because it is closer to APIS than other. Conversely, if the decision maker is risk averse, A_2 will be a better choice. Nevertheless, if the decision maker adopts PIS and NIS to make decision, he/she will be in a dilemma of selection, whether he/she is profit-seeking or not.

In fact, by the definitions of PIS and NIS, it is easy to draw the conclusion that in most cases, the less the distance between the alternative and PIS, the larger the distance between it and NIS. This conclusion points out that TOPSIS method may be unavailable in some cases (such as Example 1).

3.2. The drawbacks of conventional closeness coefficient

Dwivedi et al. [16] pointed out that the closeness coefficient in the conventional TOPSIS method does not take care of weights of

Table 1
The assessments of alternatives under each criterion.

assessments	C_1	C_2	C_3	C_4
A_1	0.9	0.7	0.5	0.6
A_2	0.7	0.8	0.3	0.7

Table 2
The differences between the alternatives and PIS, NIS, APIS, ANIS.

Assessments	PIS	NIS	APIS	ANIS
A_1	0.075	0.075	0.05	0.05
A_2	0.075	0.075	0.10	0.10

PIS and NIS distances. This limitation is also revealed in the analysis of Table 2. Example 2 points it out from another perspective.

Example 2. *Let A_1 and A_2 be two investments which wait for selection and assume that the investor is highly profit-seeking rather than risk averse. The weighted distances d_i^+ and d_i^- between investment A_i and PIS, A_i and NIS are shown as follows*:

$$d_1^+ = 0.2,\ d_1^- = 0.3.$$

$$d_2^+ = 0.3,\ d_2^- = 0.5.$$

According to Formula (8), we have $CC_1 = 0.6$ and $CC_2 = 0.625$, i.e. $CC_1 < CC_2$. Thus, we may choose A_2 to invest. Nevertheless, the investor is likely to select A_1 because of his/her preference for greater returns, which may be a contradiction to our result.

The examples stress that taking different weights of PIS and NIS distances into account is necessary, which is also mentioned by Dwivedi et al. [16]. In addition, Dwivedi proved that the general linear weighting method is invalid due to the essence of the closeness coefficient — it is always determined by the value of $\frac{d_i^+}{d_i^-}$, whether it is linear weighted or not. To solve the problem, Dwivedi put forward a new form of closeness coefficient CC_i' that can be written as follows [16]:

$$CC_i' = \frac{(d_i^-)^{w^-}}{(d_i^+)^{w^+} + (d_i^-)^{w^-}}, \tag{13}$$

where w^+ and w^- are the exponential weights of PIS and NIS distances, respectively,$0 \leqslant w^+ \leqslant 1, 0 \leqslant w^- \leqslant 1$.

It is doubtless that the new closeness coefficient is more flexible and useful, yet it still has some shortcomings if we apply it into the modified TOPSIS method. On the one hand, it neglects the relationship between w^+ and w^-, thus it will result in the loss of preciseness to some extent. On the other hand, it allows the weights to be zero, and Dwivedi explained that under this condition, the results will only depend on one distance. However, this is the reason for it lacks the resistance to extreme data, so that it cannot be applied to all conditions. The following instance is presented to illustrate these shortcomings distinctly.

Example 3. *Let A_1 and A_2 be two investments which wait for selection and assume that the investor is highly profit-seeking. The weighted distances $d_i'^+$ and $d_i'^-$ between investment A_i and APIS, A_i and ANIS are shown as follows*:

$$d_1'^+ = 0,\ d_1'^- = 0.3,$$

$$d_2'^+ = 0,\ d_2'^- = 0.5.$$

It is apparent that A_2 is better than A_1. Now, according to Formula (13), let $w^+ = 1$ and $w^- = 1$, i.e. let the new closeness coefficient regress to the conventional one, we have $CC_1' = CC_2' = 1$, which is not in line with our intuition. To make the result consistent with the investor's preference, we adjust the weight of NIS distance and try to set $w^- = 0.5$. However, we still have $CC_1' = CC_2' = 1$, which implies that the new closeness coefficient is not precise enough and lacks the resistance to extreme data. Even if we let $w^- = 0$, the result will not change at all. The only way to avoid the inconsistent result is to make $w^+ = 0$, then the closeness coefficient will only rely on the ANIS distance. Nonetheless, under this condition, the weight of ANIS distance will lose its significance no matter how we change its value.

In summary, both the conventional closeness coefficient and its weighting form have some problems. Hence, it is imperative to refine them to fix these problems.

3.3. The new definition of weighted closeness coefficient

As mentioned above, there are some drawbacks in the conventional closeness coefficient and its new form proposed by Dwivedi [16]. To make up for these drawbacks, the new definition called weighted closeness coefficient is put forward.

3.3.1. The weighted closeness coefficient

Definition 6. *Suppose $A = \{A_1, A_2, \cdots, A_m\}$ is the set of alternatives which wait for ranking, $d_i'^+$, $d_i'^-$ are the weighted distances between alternative A_i and APIS, A_i and ANIS, respectively,$i = 1, 2, \cdots, m$. For any index$i$, if there are always $d_i'^+ \leqslant 1$ and $d_i'^- \leqslant 1$, then, the weighted closeness coefficient C_i^w is defined as follows*:

$$C_i^w = \frac{\left(\frac{1+d_i'^-}{2}\right)^{1-\lambda}}{\left(\frac{1+d_i'^+}{2}\right)^{1+\lambda} + \left(\frac{1+d_i'^-}{2}\right)^{1-\lambda}} = \frac{4^\lambda \times \left(1+d_i'^-\right)^{1-\lambda}}{\left(1+d_i'^+\right)^{1+\lambda} + 4^\lambda \times \left(1+d_i'^-\right)^{1-\lambda}} \tag{14}$$

where λ is the parameter to adjust the weights of APIS and ANIS distances,$-0.5 \leqslant \lambda \leqslant 0.5, 1 \leqslant i \leqslant m$.

To transform Formula (14) into a simpler form, denote $\hat{d}_i^+ = \frac{1+d_i'^+}{2}$,$\hat{d}_i^- = \frac{1+d_i'^-}{2}$, then, C_i^w can be written as:

$$C_i^w = \frac{\left(\hat{d}_i^-\right)^{1-\lambda}}{\left(\hat{d}_i^+\right)^{1+\lambda} + \left(\hat{d}_i^-\right)^{1-\lambda}} \tag{15}$$

Remark 3. *If there is an index i that satisfies $d_i'^+ \geqslant 1$ or $d_i'^- \geqslant 1$, we can normalize all the distances at first by making $\bar{d}_i^+ = \frac{d_i'^+}{\max\limits_{1\leqslant i\leqslant m}\left\{\max\left\{d_i'^+, d_i'^-\right\}\right\}}$ and $\bar{d}_i^- = \frac{d_i'^-}{\max\limits_{1\leqslant i\leqslant m}\left\{\max\left\{d_i'^+, d_i'^-\right\}\right\}}$. Then, denote $\hat{d}_i^+ = \frac{1+\bar{d}_i^+}{2}$, $\hat{d}_i^- = \frac{1+\bar{d}_i^-}{2}$,$1 \leqslant i \leqslant m$, Formula (15) is still effective.*

Remark 4. *Using $\hat{d}_i^+ = \frac{1+d_i'^+}{2}$ and $\hat{d}_i^- = \frac{1+d_i'^-}{2}$ instead of $d_i'^+$ and $d_i'^-$ respectively can increase the resistance to extreme data. And then, let the molecular and denominator (in left expression) multiply $2^{1+\lambda}$ at the same time, we can obtain the right expression.*

The constraint $-0.5 \leqslant \lambda \leqslant 0.5$ will be explained in Remark 5 of Section 3.3.2.

3.3.2. Some properties of the weighted closeness coefficient

Now, let us analyze the properties and the role of λ in the weighted closeness coefficient.

Theorem 1. *The weighted closeness coefficient C^w is an increasing function of λ.*

Proof. *Since $0 \leqslant d'^+ \leqslant 1$, $0 \leqslant d'^- \leqslant 1$, $\hat{d}^+ = \frac{1+d'^+}{2}$ and $\hat{d}^- = \frac{1+d'^-}{2}$, we have $\frac{1}{2} \leqslant \hat{d}^+ \leqslant 1$ and $\frac{1}{2} \leqslant \hat{d}^- \leqslant 1$, i.e. $0 < \hat{d}^+ \times \hat{d}^- \leqslant 1$. Then, $\left(\hat{d}^+ \times \hat{d}^-\right)^\lambda$ will decrease if λ increases.* Since

$$C^w = \frac{\left(\hat{d}^-\right)^{1-\lambda}}{\left(\hat{d}^+\right)^{1+\lambda} + \left(\hat{d}^-\right)^{1-\lambda}} = \frac{1}{1 + \frac{\left(\hat{d}^+\right)^{1+\lambda}}{\left(\hat{d}^-\right)^{1-\lambda}}} = \frac{1}{1 + \frac{\hat{d}^+}{\hat{d}^-} \times \left(\hat{d}^+ \times \hat{d}^-\right)^\lambda},$$

then, C^w will increase if $\left(\hat{d}^{+} \times \hat{d}^{-}\right)^{\lambda}$ decreases, i.e. C^w will increase if λ increases.

Theorem 2. *The weighted closeness coefficient C^w is an increasing function of $d^{'-}$ and a decreasing function of $d^{'+}$.*

Proof. *Since $\hat{d}^{+} = \frac{1+d^{'+}}{2}$ and $\hat{d}^{-} = \frac{1+d^{'-}}{2}$, then, $\hat{d}^{+}$ will increase if $d^{'+}$ increases and $\hat{d}^{-}$ will increase if $d^{'-}$ increases.* Since

$$C_i^w = \frac{\left(\hat{d}^{-}\right)^{1-\lambda}}{\left(\hat{d}^{+}\right)^{1+\lambda} + \left(\hat{d}^{-}\right)^{1-\lambda}} = \frac{1}{1 + \frac{\left(\hat{d}^{+}\right)^{1+\lambda}}{\left(\hat{d}^{-}\right)^{1-\lambda}}},$$

and $-0.5 \leqslant \lambda \leqslant 0.5$, i.e.$1+\lambda > 0, 1-\lambda > 0$, and $\hat{d}^{+} > 0$, $\hat{d}^{-} > 0$, then, C^w will increase if $\hat{d}^{+}$ decreases or $\hat{d}^{-}$ increases, i.e. C^w will increase if $d^{'+}$ decreases or $d^{'-}$ increases.

Theorem 2 discloses that the weighted closeness coefficient is consistent with the definitions of APIS and ANIS distances and the principle that "the greater the closeness coefficient, the better the alternative".

Let us pay attention to the following example.

Example 4. *Suppose A_1 and A_2 are two investments which wait for selection and assume that the investor is highly profit-seeking. The weighted distances $d_i^{'+}$ and $d_i^{'-}$ between investment A_i and APIS, A_i and ANIS are shown as follows*:

$$d_1^{'+} = 0.1,\ d_1^{'-} = 0.1,$$
$$d_2^{'+} = 0.3,\ d_2^{'-} = 0.3.$$

It is apparent that A_1 is better than A_2 according to the investor's preference. But by using the conventional closeness coefficient, we have $CC_1 = CC_2 = 0.5$. Then, by using the weighted closeness coefficient and making $\lambda = 0$, we can get

$$\hat{d}_1^{+} = 0.55,\ \hat{d}_1^{-} = 0.55,$$
$$\hat{d}_2^{+} = 0.65,\ \hat{d}_2^{-} = 0.65,$$

i.e. $C_1^w = C_2^w = 0.5$. It is easy to find that the result is caused by $\frac{\hat{d}_1^{+}}{\hat{d}_1^{-}} = \frac{\hat{d}_2^{+}}{\hat{d}_2^{-}}$ and still not meet our demand. Thus, considering the condition that $\lambda = 0.5$, we have $C_1^w = 0.6452 > 0.6061 = C_2^w$, which is consistent with the investor's preference.

In fact, we have the following conclusion:

Theorem 3. Suppose A_1 and A_2 are two investments which wait for selection and $d_i^{'+}$, $d_i^{'-}$ are the weighted distances between investment A_i and APIS, A_i and ANIS, respectively, $i = 1, 2$. Let $\hat{d}_i^{+} = \frac{1+d_i^{'+}}{2}$, $\hat{d}_i^{-} = \frac{1+d_i^{'-}}{2}$, $i = 1, 2$. If $\hat{d}_1^{+} < \hat{d}_2^{+}$, $\hat{d}_1^{-} < \hat{d}_2^{-}$, and $\frac{\hat{d}_1^{+}}{\hat{d}_1^{-}} = \frac{\hat{d}_2^{+}}{\hat{d}_2^{-}}$, i.e. the alternative A_1 can bring the investor more returns while it also brings more risks, we have: If $\lambda = 0$, then $C_1^w = C_2^w$; If $\lambda < 0$, then $C_1^w < C_2^w$; If $\lambda > 0$, then $C_1^w > C_2^w$.

Proof. *Let $m = \frac{\hat{d}_1^{+}}{\hat{d}_1^{-}} = \frac{\hat{d}_2^{+}}{\hat{d}_2^{-}}$, i.e. $\hat{d}_1^{+} = m \times \hat{d}_1^{-}$ and $\hat{d}_2^{+} = m \times \hat{d}_2^{-}$, $m > 0$.*

Since $0 < \hat{d}_1^{-} < \hat{d}_2^{-}$, $m > 0$, then,

$$C_i^w = \frac{\left(\hat{d}_i^{-}\right)^{1-\lambda}}{\left(\hat{d}_i^{+}\right)^{1+\lambda} + \left(\hat{d}_i^{-}\right)^{1-\lambda}} = \frac{1}{1 + \frac{\left(\hat{d}_i^{+}\right)^{1+\lambda}}{\left(\hat{d}_i^{-}\right)^{1-\lambda}}} = \frac{1}{1 + \frac{\left(m\hat{d}_i^{-}\right)^{1+\lambda}}{\left(\hat{d}_i^{-}\right)^{1-\lambda}}} = \frac{1}{1 + m^{1+\lambda}\left(\hat{d}_i^{-}\right)^{2\lambda}}$$

(i) If $\lambda = 0$, since $m^{1+\lambda}\left(\hat{d}_1^{-}\right)^{2\lambda} = m = m^{1+\lambda}\left(\hat{d}_2^{-}\right)^{2\lambda}$, then $C_1^w = C_2^w$;
(ii) If $\lambda < 0$, since $m^{1+\lambda}\left(\hat{d}_1^{-}\right)^{2\lambda} > m^{1+\lambda}\left(\hat{d}_2^{-}\right)^{2\lambda}$, then $C_1^w < C_2^w$;
(iii) If $\lambda > 0$, since $m^{1+\lambda}\left(\hat{d}_1^{-}\right)^{2\lambda} < m^{1+\lambda}\left(\hat{d}_2^{-}\right)^{2\lambda}$, then $C_1^w > C_2^w$.

Let us look at Formula (15) again. First, suppose $\lambda = 0$. Since $-0.5 < \lambda \leqslant 0.5$, $0 < \hat{d}^{+} \leqslant 1$ and$0 < \hat{d}^{-} \leqslant 1$, if we increase the value of λ, $\left(\hat{d}^{-}\right)^{1-\lambda}$ will increase while $\left(\hat{d}^{+}\right)^{1+\lambda}$ will decrease, which is similar to increasing the weight of $\hat{d}^{-}$ while decreasing the weight of $\hat{d}^{+}$, i.e. increasing the weighted ANIS distance while decreasing the weighted APIS distance. Thus, it represents that the investor prefers more returns. Combined with Theorem 3, we have

(i) If the investor is profit-seeking, there should be $\lambda > 0$; If the investor wants to avoid risk, there should be$\lambda < 0$; If the investor wants to keep neutral between returns and risk, there should be$\lambda = 0$.
(ii) The bigger or the smaller the λ, the more obvious the investor's preference.

Based on the above conclusions, λ is called the preference factor of the weighted closeness coefficient.

Remark 5. *For the range of λ, we stipulate it is $-0.5 \leqslant \lambda \leqslant 0.5$. The reason lies in*: *if an alternative has the same APIS and ANIS distances*, *i.e. we have $\hat{d}^{+} = \hat{d}^{-}$, then, when $\lambda = 0.5$, we get the weighted closeness coefficient $C^w = \frac{\left(\hat{d}^{-}\right)^{0.5}}{\left(\hat{d}^{+}\right)^{1.5} + \left(\hat{d}^{-}\right)^{0.5}} = \frac{1}{1+\hat{d}^{+}\left(\frac{\hat{d}^{+}}{\hat{d}^{-}}\right)^{0.5}} = \frac{1}{1+\hat{d}^{+}}$, which is totally dependent on the primitive APIS distance ($\hat{d}^{+}$ is obtained from APIS distance for improving the resistance to extreme data), and implies the decision maker is completely profit-seeking. It is in line with our judgments mentioned above. Similarly, when $\lambda = -0.5$, we have $C^w = \frac{\left(\hat{d}^{-}\right)^{1.5}}{\left(\hat{d}^{+}\right)^{0.5} + \left(\hat{d}^{-}\right)^{1.5}} = \frac{\hat{d}^{-}}{\left(\frac{\hat{d}^{+}}{\hat{d}^{-}}\right)^{0.5} + \hat{d}^{-}} = \frac{\hat{d}^{-}}{1+\hat{d}^{-}}$, which shows that the decision maker is completely risk averse.*

3.3.3. Validity check of the weighted closeness coefficient

Let us apply the weighted closeness coefficient to Example 2 and Example 3 to verify its effectiveness. According to the investor's preference, we set $\lambda > 0$ in both two instances, and then the results are shown in Table 3 and Fig. 1, Table 4 and Fig. 2, respectively.

From Table 3 and Fig. 1, Table 4 and Fig. 2, we can find that:

(i) All the values of weighted closeness coefficient are increasing while the value of λ is increasing, which conforms to Theorem 4.
(ii) The value of C_1^w is overtaking the value of C_2^w while λ is increasing in Table 3 and Fig. 1, even when $\lambda \geqslant 0.4$, we have $C_1^w > C_2^w$. The reason for this phenomenon is that $d_1^{'+} < d_2^{'+}$,

Table 3
The values of C_1^w and C_2^w with different values of preference factor λ in Example 2.

λ	0.1	0.2	0.3	0.4	0.5
C_1^w	0.5434	0.5667	0.5879	0.6122	0.6343
C_2^w	0.5535	0.5712	0.5887	0.6060	0.6230

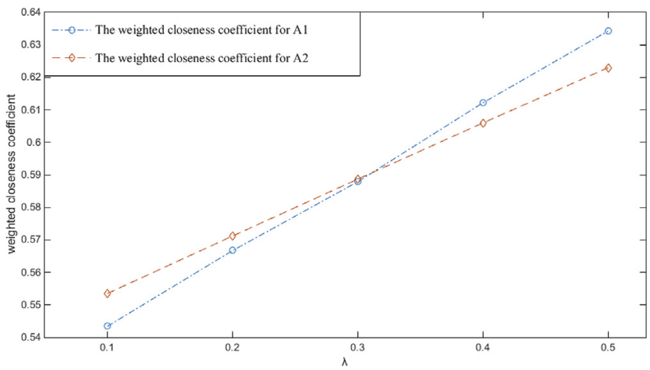

Fig. 1. The values of C_1^w and C_2^w with different values of preference factor λ in Example 2.

Table 4
The values of C_1^w and C_2^w with different values of preference factor λ in Example 3.

λ	0.1	0.2	0.3	0.4	0.5
C_1^w	0.5926	0.6194	0.6456	0.6708	0.6952
C_2^w	0.6233	0.6460	0.6681	0.6895	0.7101
$C_2^w - C_1^w$	0.0307	0.0266	0.0226	0.0187	0.0149

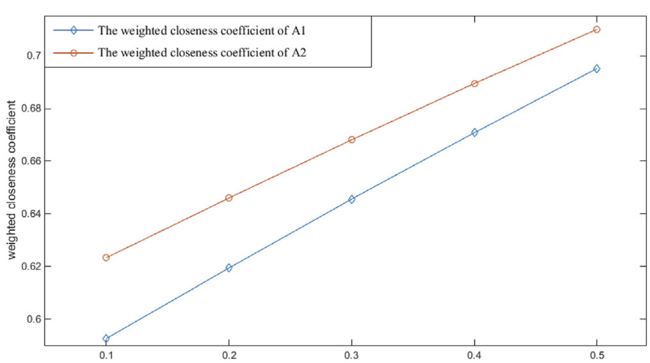

Fig. 2. The values of C_1^w and C_2^w with different values of preference factor λ in Example 3.

i.e. in Example 2, the alternative A_1 can bring the investor more returns. Then, if we increase the value of λ, which is equivalent to increasing the investor's preference for returns, the advantage of A_1 will become more and more obvious. It is consistent with our intuition.

(iii) In Example 3, there is $C_1^w < C_2^w$ when $\lambda > 0$, which is in line with our judgment. Besides, the value of $C_2^w - C_1^w$ is decreasing while the value of λ is increasing due to $d_1'^+ = d_2'^+$. It can be expressed that with the increase of λ, the investor's preference for returns is increasing and the weight of risk is gradually decreasing. Thereby, in the perspective of the investor, the difference between two alternatives is reduced, which is realistic and reasonable.

4. Several novel linear programming models based on distance measure for portfolio selection

Consider such a problem: If $A = \{A_1, A_2, \ldots, A_m\}$ is a set of investments which wait for selection and the investor wants to choose several of them rather than only one to invest. How to choose the investments and allocate the investment ratios can help the investor obtain great returns while avoiding the risk to some extent?

In fact, conventional MCDM problem only takes the ranking of the alternatives into account and the investor will choose the best alternative for investment, which is imperfect for the portfolio selection problem mentioned above. Portfolio selection problem not only focus on a satisfactory returns but also focus on an acceptable risk. However, choosing one alternative for investment cannot spread risk well. Next, we will solve the problem by combining the modified TOPSIS method and the linear portfolio optimization programming models based on the distance measure of IFSs. To meet investors' different demands, three models with different constraints are given in this section.

4.1. The general model for portfolio selection

Firstly, the steps to solve the aforementioned problem are listed as follows:

Step 1: Gain the assessment matrix for investments under the given criteria.
Step 2: Define the APIS and ANIS, and then denote them as the greatest returns and the ceiling of the highest risk, respectively.
Step 3: Assign the weights to each criterion by using the entropy weight method [25].
Step 4: Use the H-max distance measure to calculate the weighted distances between each investment and APIS, ANIS.
Step 5: Calculate the weighted closeness coefficient of each investment.
Step 6: Rank the investments by their weighted closeness coefficients and pick out a certain quantity of them from priority to inferiority.
Step 7: Allocate the investment ratios for each selected investment by the linear portfolio optimization programming model which will be mentioned in the following analysis.

Step 1 to Step 6 have been analyzed in the above sections, thus this section will focus on Step 7. For convenience, let $A' = \{A_1', A_2', \cdots, A_k'\}$ be the set of the selected investments and $d_i'^+, d_i'^-$ be the weighted distances between investment A_i' and APIS, A_i' and ANIS, respectively, $i = 1, 2, \cdots, k$. In consideration of the investment intention, we should make the returns of portfolio as great as possible while making the risk as small as possible. Since the APIS and ANIS are regarded as the greatest returns and the ceiling of the highest risk, respectively, it is taken for granted to pursue the maximization of total weighted ANIS distance and the minimization of total weighted APIS distance. In addition, each selected investment should be invested so that all the investment ratios should be positive. Hence, a linear portfolio optimization programming model based on the weighted distances is proposed as follows:

$$\begin{aligned}
&\min D^+(X) = \sum_{i=1}^{k} x_i d_i'^+ \\
&\max D^-(X) = \sum_{i=1}^{k} x_i d_i'^- \\
&\text{s.t.} \begin{cases} \sum_{i=1}^{k} x_i = 1, \\ l_i \leqslant x_i \leqslant u_i. \end{cases}
\end{aligned} \tag{16}$$

where $d_i'^+$, $d_i'^-$ are the weighted distances between the selected investment A_i' and APIS, A_i' and ANIS, respectively, and x_i is the investment ratio allocated to the selected investment A_i', l_i, u_i represent the minimum investment ratio and the maximal investment ratio for A_i', respectively, $0 \leqslant u_i \leqslant l_i \leqslant 1$, $i = 1, 2, \cdots, k$.

Model (16) is quite a simple linear programming model. Obviously, it can be transformed into the following simpler model:

$$\min D(X) = \sum_{i=1}^{k} x_i (d_i'^{+} - d_i'^{-}) \quad \text{s.t.} \begin{cases} \sum_{i=1}^{k} x_i = 1, \\ l_i \leqslant x_i \leqslant u_i. \end{cases} \tag{17}$$

If the investor is satisfied with the model, we just need to solve the model. Otherwise, he/she may put forward some other requirements, such as desiring more returns in spite of greater risks, or asking for a return which is higher than a certain limit while keeping the risk does not exceed a set ceiling. Under this condition, we can add some constraints to Model (17) and get the more suitable models.

4.2. The weighted model for portfolio selection

If the investor desires more returns in spite of greater risks, or prefers to avoid the risk as far as possible even if he/she would get fewer returns, it is feasible to modulate the weights of the total weighted APIS distance and the total weighted ANIS distance. Then, Model (17) can be refined as the following form:

$$\min D(X) = \sum_{i=1}^{k} x_i (\alpha d_i'^{+} - \beta d_i'^{-}) \quad \text{s.t.} \begin{cases} \sum_{i=1}^{k} x_i = 1, \\ l_i \leqslant x_i \leqslant u_i, \ \alpha \geqslant 0, \ \beta \geqslant 0. \end{cases} \tag{18}$$

where $d_i'^{+}$, $d_i'^{-}$ are the weighted distances between the selected investment A_i' and APIS, A_i' and ANIS, respectively, α and β denote the weights of APIS, ANIS distances, x_i is the investment ratio allocated to the selected investment A_i', l_i, u_i represent the minimum investment ratio and the maximal investment for ratio A_i', respectively, $0 \leqslant u_i \leqslant l_i \leqslant 1$,$i = 1, 2, \cdots, k$.

It is evident that when $\alpha = \beta = 1$, Model (18) regresses to Model (17).

If the investor is profit-seeking, he/she can set $\alpha > \beta$, or else, he/she can set $\alpha \leqslant \beta$. In order to make the model more standardized, it is better to stipulate a rational bound on α and β. Thus, some constraints can be imposed on the model and then transformed into the form as follows:

$$\min D(X) = \sum_{i=1}^{k} x_i (\alpha d_i'^{+} - \beta d_i'^{-}) \quad s.t. \begin{cases} \alpha + \beta = 2, \\ \sum_{i=1}^{k} x_i = 1, \\ l_i \leqslant x_i \leqslant u_i, \ \alpha \geqslant 0, \ \beta \geqslant 0. \end{cases} \tag{19}$$

4.3. The baseline model for portfolio selection

If the investor asks for a return higher than a certain limit or he/she wants to avoid the risk which may exceed a set ceiling, we can add the corresponding constraints on the model and then get the following model:

$$\min D(X) = \sum_{i=1}^{k} x_i (d_i'^{+} - d_i'^{-}) \quad \text{s.t.} \begin{cases} \sum_{i=1}^{k} x_i d_i'^{+} \leqslant D_\Delta^{+}(\theta_1), \\ \sum_{i=1}^{k} x_i d_i'^{-} \geqslant D_\Delta^{-}(\theta_2), \\ \sum_{i=1}^{k} x_i = 1, \\ l_i \leqslant x_i \leqslant u_i, \\ D_\Delta^{+}(\theta_1) = \max\limits_{1\leqslant i\leqslant n}\{d_i'^{+}\} - \theta_1 \times \left(\max\limits_{1\leqslant i\leqslant n}\{d_i'^{+}\} - \min\limits_{1\leqslant i\leqslant n}\{d_i'^{+}\}\right), \\ D_\Delta^{-}(\theta_2) = \min\limits_{1\leqslant i\leqslant n}\{d_i'^{-}\} + \theta_2 \times \left(\max\limits_{1\leqslant i\leqslant n}\{d_i'^{-}\} - \min\limits_{1\leqslant i\leqslant n}\{d_i'^{-}\}\right). \end{cases} \tag{20}$$

where $d_i'^{+}$, $d_i'^{-}$ are the weighted distances between the selected investment A_i' and APIS, A_i' and ANIS, respectively, and x_i is the investment ratio allocated to the selected investment A_i', $D_\Delta^{+}(\theta_1)$, $D_\Delta^{-}(\theta_2)$ represent the requirements for returns and risk and satisfy that $0 \leqslant \theta_1 \leqslant 1, 0 \leqslant \theta_2 \leqslant 1$, l_i, u_i represent the minimum investment and the maximal investment for A_i', respectively, $0 \leqslant u_i \leqslant l_i \leqslant 1$, $i = 1, 2, \cdots, k$.

It is evident that when $\theta_1 = \theta_2 = 0$, Model (20) regresses to Model (17).

Note that comparing with Model (19), we cannot ensure that there is a feasible solution for Model (20). To avoid the problem as far as possible, it is judicious to let $\theta_1 = 0$ or $\theta_2 = 0$ to reduce the restriction. Then, the model will be transformed into one of the following two types:

a. The risks baseline type:

$$\min D^{+}(X) = \sum_{i=1}^{k} x_i d_i'^{+} \quad s.t. \begin{cases} \sum_{i=1}^{k} x_i d_i'^{-} \geqslant D_\Delta^{-}(\theta), \\ \sum_{i=1}^{k} x_i = 1, \\ l_i \leqslant x_i \leqslant u_i, \\ D_\Delta^{-}(\theta) = \min\limits_{1\leqslant i\leqslant n}\{d_i'^{-}\} + \theta \times \left(\max\limits_{1\leqslant i\leqslant n}\{d_i'^{-}\} - \min\limits_{1\leqslant i\leqslant n}\{d_i'^{-}\}\right). \end{cases} \tag{21}$$

b. The returns baseline type:

$$\max D^{-}(X) = \sum_{i=1}^{k} x_i d_i'^{-} \quad s.t. \begin{cases} \sum_{i=1}^{k} x_i d_i'^{+} \leqslant D_\Delta^{+}(\theta), \\ \sum_{i=1}^{k} x_i = 1, \\ l_i \leqslant x_i \leqslant u_i, \\ D_\Delta^{+}(\theta) = \max\limits_{1\leqslant i\leqslant n}\{d_i'^{+}\} - \theta \times \left(\max\limits_{1\leqslant i\leqslant n}\{d_i'^{+}\} - \min\limits_{1\leqslant i\leqslant n}\{d_i'^{+}\}\right). \end{cases} \tag{22}$$

Although we still cannot ensure that both Model (21) and Model (22) have at least one feasible solution, we increase the probability. Moreover, the models have the advantage that they can ensure the risks will not exceed a certain ceiling or the returns will be higher than a certain baseline, which is exactly the shortcoming of Model (19).

If Model (21) or Model (22) is adopted, we can set the value of θ within the range $[0, 1]$ to obtain a result that is consistent with the investor's preference. It is apparent that in Model (21), the bigger the value of θ, the more the investor wants to avoid risk; whereas in Model (22), the bigger the value of θ, the greater the investor's preference for returns. Therefore, we have

(i) If the investor is highly risk averse, we can make $\theta > 0.7$ in Model (21) or $\theta < 0.3$ in Model (22);
(ii) If the investor is slightly risk averse, we can make $0.7 \geqslant \theta > 0.5$ in Model (21) or $0.3 \leqslant \theta < 0.5$ in Model (22);
(iii) If the investor is neutral, we can use Model (19) or make $\theta = 0.5$ in Model (21) or Model (22);
(iv) If the investor is slightly profit-seeking, we can make $0.3 \leqslant \theta < 0.5$ in Model (21) or $0.7 \geqslant \theta > 0.5$ in Model (22);
(v) If the investor is highly profit-seeking, we can make $\theta < 0.3$ in Model (21) or $\theta > 0.7$ in Model (22).

In next section, we will use a numerical example to show how the results vary with different values of θ in Model (21) or Model (22).

5. Numerical application

To verify the validity and applicability of the linear programming models proposed in Section 4, an example of portfolio selection in the intuitionistic fuzzy environment is given in this section.

Example 5. *Assuming that ten companies which hold different newest projects are looking for investment, and an investor is interested in the projects so that he/she intends to choose three of them to invest. As these projects are new, there is no historical data available for reference. In order to make a sensible investment, the investor assesses the capabilities of each company according to the criteria listed as follows*:

- The company is in a large scale (C_1).
- The company has a promising prospect (C_2).
- The company has a great influence in the market (C_3).
- The project is important for the company (C_4).
- The company had suffered a huge economic blow (C_5).
- The company has many failed projects (C_6).
- The company had a terrible reputation (C_7).

For the sake of convenience, $A_1, A_2, \cdots, A_{10}$ are used to denote the companies and $C_1, C_2, \cdots, C_7$ are used to denote the above criteria. We can find that C_1, C_2, C_3, C_4 are returns criteria and the rest is risk criteria. The investor can express his/her acceptance and rejection degree of each company under different criteria, so that it is easy to get the assessment matrix in intuitionistic fuzzy form.

5.1. The ranking of the investments

According to the steps proposed in Section 4, an intuitionistic fuzzy assessment matrix $D = (D_{ij})_{10\times7} = \left(\left(\mu_{ij}, \nu_{ij}\right)\right)_{10\times7}$ is obtained from the investor and shown in Table 5.

From Table 5, we can identify the PIS A^+, the NIS A^-, the APIS A'^+, the ANIS A'^-, respectively. They are presented as follows:

The PIS: $A^+ = ((0.7, 0.1), (0.6, 0.1), (0.9, 0), (0.8, 0), (0.1, 0.8), (0, 0.8), (0.2, 0.8))$,
The NIS: $A^- = ((0.1, 0.6), (0.1, 0.5), (0.1, 0.7), (0, 0.5), (0.5, 0.1), (0.5, 0.2), (0.7, 0.1))$,
The APIS: $A'^+ = ((0.7, 0.1), (0.6, 0.1), (0.9, 0), (0.8, 0))$,
The ANIS: $A'^- = ((0.5, 0.1), (0.5, 0.2), (0.7, 0.1))$.

The next step is to assign weights to each criterion by using the entropy weight method [25]. Firstly, we calculate the entropy $E(C_j)$ of each criterion according to Formula (4). It should be noticed that if we use the conventional TOPSIS method, the weight w_j of criterion C_j should be assigned according to the following formula [25]:

$$w_j = \frac{1 - E(C_j)}{\sum_{t=1}^{7}(1 - E(C_j))} \tag{23}$$

However, if we use the modified TOPSIS method, the weight w'_j of criterion C_j should be assigned according to the following formula:

$$w'_j = \begin{cases} \frac{1-E(C_j)}{\sum_{t=1}^{4}(1-E(C_j))}, & \text{if } 1 \leqslant j \leqslant 4, \\ \frac{1-E(C_j)}{\sum_{t=5}^{7}(1-E(C_j))}, & \text{if } 5 \leqslant j \leqslant 7. \end{cases} \tag{24}$$

where $1 \leqslant j \leqslant 7$. The results are shown in Table 6.

After assigning the weights, we use the H-max distance measure shown in Formula (5) to calculate the weighted distances between investment A_i and PIS A^+, A_i and NIS A^-, A_i and APIS A'^+, A_i and ANIS A'^-. Then, the results are shown in Table 7.

From Table 7, we can find that the data of APIS distances is little different from that of PIS distances while the data of ANIS distances is little different from that of NIS distances.

Let us rank the investments by their weighted closeness coefficients with various values of λ according to Formula (15). To be convenient, we denote the weighted closeness coefficient of PIS and NIS distances as $C^w(A^+, A^-)$, the weighted closeness coefficient of APIS and ANIS distances as $C^w(A'^+, A'^-)$. Then, the results with various values of λ are shown in Table 8.

From Table 8, we can find that:

(i) When $\lambda = 0$ or $\lambda = 0.5$, there is little difference in the ranking of $C^w(A^+, A^-)$ and the ranking of$C^w(A'^+, A'^-)$, which shows that APIS and ANIS are as effective as PIS and NIS.
(ii) When $\lambda = -0.5$, the ranking of $C^w(A^+, A^-)$ is hugely different from the ranking of $C^w(A'^+, A'^-)$. According to the assessment matrix shown in Table 5, it is not difficult to find that the assessments of A_4 are comparatively better than those of

Table 5
The intuitionistic fuzzy assessment matrix provided by the investor.

Assessments	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A_1	(0.3, 0.4)	(0.4, 0.4)	(0.6, 0.3)	(0.5, 0.5)	(0.1, 0.7)	(0.4, 0.2)	(0.2, 0.5)
A_2	(0.2, 0.6)	(0.3, 0.4)	(0.2, 0.3)	(0.7, 0.1)	(0.1, 0.2)	(0.1, 0.3)	(0.2, 0.1)
A_3	(0.2, 0.1)	(0.3, 0.3)	(0.1, 0.4)	(0.2, 0.1)	(0.3, 0.1)	(0.2, 0.4)	(0.3, 0.3)
A_4	(0.5, 0.2)	(0.4, 0.2)	(0.5, 0.1)	(0.5, 0.3)	(0.3, 0.2)	(0.2, 0.6)	(0.2, 0.7)
A_5	(0.4, 0.3)	(0.4, 0.4)	(0.6, 0.2)	(0.6, 0.1)	(0.4, 0.5)	(0.3, 0.3)	(0.2, 0.8)
A_6	(0.7, 0.1)	(0.5, 0.2)	(0.9, 0.0)	(0.5, 0.2)	(0.5, 0.3)	(0.5, 0.5)	(0.3, 0.6)
A_7	(0.1, 0.6)	(0.3, 0.5)	(0.1, 0.7)	(0.8, 0.0)	(0.4, 0.4)	(0.2, 0.3)	(0.2, 0.4)
A_8	(0.3, 0.3)	(0.6, 0.3)	(0.4, 0.4)	(0.5, 0.5)	(0.1, 0.8)	(0.1, 0.7)	(0.7, 0.2)
A_9	(0.5, 0.3)	(0.4, 0.1)	(0.4, 0.2)	(0.6, 0.2)	(0.2, 0.5)	(0.0, 0.8)	(0.5, 0.4)
A_{10}	(0.1, 0.1)	(0.1, 0.3)	(0.1, 0.4)	(0.0, 0.5)	(0.2, 0.1)	(0.4, 0.2)	(0.3, 0.1)

Table 6
The entropy and weights of each criterion.

Criterion	C_1	C_2	C_3	C_4	C_5	C_6	C_7
$E(C_j)$	0.9278	0.9710	0.8537	0.8659	0.9105	0.8973	0.8885
w_j	0.1053	0.0423	0.2135	0.1957	0.1306	0.1498	0.1627
w_j'	0.1892	0.0760	0.3834	0.3515	0.2947	0.3382	0.3672

Table 7
The weighted H-max distances between each investment and PIS, NIS, APIS, ANIS.

Distances	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}
A^+	0.3055	0.3836	0.4528	0.2491	0.2316	0.1994	0.3887	0.3165	0.2414	0.5595
A^-	0.3088	0.3175	0.2666	0.3909	0.3920	0.4034	0.2671	0.2926	0.3944	0.1629
A'^+	0.3310	0.3698	0.4912	0.2646	0.2290	0.1013	0.4264	0.3759	0.2716	0.6077
A'^-	0.3510	0.3010	0.2519	0.3689	0.3356	0.2126	0.2871	0.3469	0.3755	0.1794

Table 8
The weighted closeness coefficients of each investment with various values of λ.

λ	$\lambda = 0$		$\lambda = 0.5$		$\lambda = -0.5$	
C^w	$C^w(A^+, A^-)$	$C^w(A'^+, A'^-)$	$C^w(A^+, A^-)$	$C^w(A'^+, A'^-)$	$C^w(A^+, A^-)$	$C^w(A'^+, A'^-)$
A_1	0.5027	0.5147	0.7670	0.7568	0.2369	0.2655
A_2	0.4529	0.4487	0.7034	0.7093	0.2241	0.2136
A_3	0.3706	0.3390	0.6289	0.5931	0.1698	0.1528
A_4	0.6108	0.5823	0.8341	0.8169	0.3287	0.3034
A_5	0.6286	0.5944	0.8489	0.8409	0.3377	0.2889
A_6	0.6692	0.6773	0.8770	0.9346	0.3646	0.2355
A_7	0.4073	0.4024	0.6808	0.6580	0.1813	0.1907
A_8	0.4804	0.4799	0.7523	0.7188	0.2196	0.2500
A_9	0.6203	0.5803	0.8411	0.8124	0.3352	0.3063
A_{10}	0.2255	0.2279	0.4909	0.4720	0.0808	0.0888
Ranking	$A_6 \succ A_5 \succ A_9 \succ A_4 \succ A_1 \succ A_8 \succ A_2 \succ A_7 \succ A_3 \succ A_{10}$	$A_6 \succ A_5 \succ A_4 \succ A_9 \succ A_1 \succ A_8 \succ A_2 \succ A_7 \succ A_3 \succ A_{10}$	$A_6 \succ A_5 \succ A_9 \succ A_4 \succ A_1 \succ A_8 \succ A_2 \succ A_7 \succ A_3 \succ A_{10}$	$A_6 \succ A_5 \succ A_4 \succ A_9 \succ A_1 \succ A_8 \succ A_2 \succ A_7 \succ A_3 \succ A_{10}$	$A_6 \succ A_5 \succ A_9 \succ A_4 \succ A_1 \succ A_2 \succ A_8 \succ A_7 \succ A_3 \succ A_{10}$	$A_9 \succ A_4 \succ A_5 \succ A_1 \succ A_8 \succ A_6 \succ A_2 \succ A_7 \succ A_3 \succ A_{10}$

A_5 with respect to the risk criteria, while the assessments of A_5 are comparatively better than those of A_6 with respect to the risk criteria. Thus, it is more rational to think A_4 is better than A_5 and A_5 is better than A_6 when the investor is extremely risk averse. Moreover, it can be found that A_8 is a little better than A_6 under the same condition. In consequence, the ranking of $C^w(A'^+, A'^-)$ should be regarded as more rational than the ranking of $C^w(A^+, A^-)$, i.e. APIS and ANIS are better than PIS and NIS in this case.

5.2. Comparison with IF ELECTRE method

ELECTRE approach, proposed by Benayoun et al. [26], is also a MCDM method widely used [27]. This method is initially applied in some MCDM problems with crisp assessments. And then, to solve uncertainty problems, Wu and Chen [28] developed the IF ELECTRE method.

In this part, we will adopt IF ELECTRE to rank the alternatives $A_1, A_2, \cdots, A_{10}$ and then make a comparison with our modified TOPSIS method. According to the steps given by Wu and Chen [28], the ranking process is shown as follows:

Step 1: Determine the decision matrix. This step is finished in Section 5.1, please see Table 5.
Step 2: Determine the concordance and discordance sets. After the pairwise comparison, we obtain the results shown in Tables 9–14.

Remark 6. In Table 9, the value in 1st (horizontal) row (A_1) and 3rd (vertical) column (A_3) is "3, 5, 7". It means that compared with A_3, the strong concordance set of A_1 is $\{C_3, C_5, C_7\}$. Similarly, the values in Tables 10–14 can also be described in this way. Note that "—" denotes an empty set.

Remark 7. *Since the criteria C_5, C_6, C_7 are risk criteria, when we compare the investments, we will transform the corresponding assessments into returns type: if the assessment is denoted by an intuitionistic fuzzy number (μ, ν), then its value will be converted to (ν, μ) (refer to the definition of complementary set $\bar{A}$ in* [28]).

Step 3: Calculate the concordance matrix. Here, the investor gives the relative weights for the concordance sets as $[W(C_S), W(C_M), W(C_W)] = [1, \frac{2}{3}, \frac{1}{3}]$ (refer to the weights in [28]), and then we have the concordance matrix shown in Table 15.

For instance, in 3rd (horizontal) row (A_3) and 5th (vertical) column (A_5), the value is

$$Conc(3, 5) = W(C_S) \times \sum_{i \in C_S(3, 5)} w_i + W(C_M) \times \sum_{i \in C_M(3, 5)} w_i + W(C_W) \times \sum_{i \in C_W(3, 5)} w_i$$

Table 9
The strong concordance set C_S.

Assessments	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}
A_1	–	–	3, 5, 7	5	–	–	3	3	5	3, 5, 7
A_2	–	–	–	–	–	4	–	–	–	4
A_3	–	–	–	–	–	–	–	–	–	–
A_4	6	3	3, 7	–	6	–	–	1	–	2, 3, 4, 6, 7
A_5	–	3	3, 7	–	–	5, 7	–	7	7	3, 4, 7
A_6	1	3	2, 3	1, 3	1, 3	–	1, 3	1, 3	3	2, 3, 4
A_7	–	–	4	–	4	4	–	–	–	4, 7
A_8	2, 6	2	5, 6	5	2, 6	5	2, 5, 6	–	5	5, 6
A_9	1, 6	3, 6	3, 5, 6	5	6	–	1, 6	–	–	2, 3, 4, 6
A_{10}	–	–	–	–	–	–	–	–	–	–

Table 10
The moderate concordance set C_M.

Assessments	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}
A_1	–	1	–	–	5	5	1, 2, 5	7	7	–
A_2	4, 6	–	3, 5	4, 5	6	–	2, 3, 6	4	4	3, 5, 6, 7
A_3	6	1, 2	–	–	6	–	1, 2, 3	7	–	4, 6
A_4	1, 2, 4	1, 2	2	–	1, 2	6, 7	1, 2, 3	3, 4, 7	1, 3, 7	–
A_5	1, 3, 4, 6	1	–	4	–	4	1, 2, 3	3, 4	4	6
A_6	2, 3, 4	1, 2	–	4	2	–	2	4, 7	1, 7	–
A_7	4, 6	4	7	4	6	5	–	4, 7	4, 7	6
A_8	1	1	–	6	5	6	1, 3	–	–	–
A_9	2, 4	1, 2	2	2, 4, 6	2, 5	5, 6	2, 3, 5	3, 4, 6, 7	–	–
A_{10}	–	–	5	–	–	–	1, 3	–	–	–

Table 11
The weak concordance set C_W.

Assessments	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}
A_1	–	2, 3, 5, 7	1, 2, 4	2, 3,	2, 3	4	7	1, 4	2, 3	1, 2, 4, 6
A_2	–	–	1, 2, 4	–	4	–	1	–	–	1, 2
A_3	–	6, 7	–	–	–	–	6	–	–	1, 2, 3, 5, 7
A_4	7	5, 6, 7	1, 4, 5, 6	–	–	4	6, 7	–	2	1, 5
A_5	2, 7	2, 5, 6, 7	1, 2, 4, 5	2, 3, 5, 7	–	–	5, 6, 7	1	2, 3, 5	1, 2, 5
A_6	6, 7	5, 6, 7	1, 4, 5, 6, 7	2, 5	6	–	6, 7	–	2	1, 5, 6, 7
A_7	–	2, 5, 6, 7	2, 3, 5	5	–	–	–	–	–	1, 2, 3, 5
A_8	4, 5	3, 5, 6, 7	1, 2, 3, 4	2, 4	–	2, 4	–	–	2, 3	1, 2, 3, 4, 7
A_9	–	5, 7	1, 4, 7	1	1, 4	4	7	1	–	1, 5, 7
A_{10}	6	7	3	–	–	–	–	–	–	–

Table 12
The strong discordance set D_S.

Assessments	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}
A_1	–	–	6	1, 6, 7	1, 6, 7	1, 3	–	2, 5, 6	1, 6	–
A_2	2, 3, 5, 7	–	–	3, 7	2, 3, 7	1, 2, 3	4, 7	2, 5, 6	1, 3, 6	–
A_3	3, 5, 7	3, 4	–	2, 3, 5, 6, 7	3, 4, 7	1, 2, 3, 7	4, 7	2, 3, 5, 6	3, 5, 6	–
A_4	5	4	–	–	7	1, 2, 3	4	5, 6	4, 5, 6	–
A_5	–	4	6	1, 6	–	1, 3	4	2, 5, 6	1, 6	–
A_6	5	4	–	7	4, 5, 7	–	4, 5	5	4	–
A_7	1, 2, 3, 5, 7	1	6	1, 6, 7	1, 2, 3, 5, 7	1, 3	–	2, 3, 5, 6	1, 6	–
A_8	3	–	–	1, 7	1, 3, 7	1, 3, 7	–	–	1, 6, 7	–
A_9	5	4	–	3, 7	3, 7	1, 3, 7	4	5	–	–
A_{10}	3, 4, 5, 7	3, 4, 5	1, 2, 6, 7	2, 3, 4, 6, 7	3, 4, 6, 7	1, 2, 3, 4, 7	4, 7	2, 3, 4, 5, 6	2, 3, 4, 5, 6	–

$$= 1 \times 0 + \frac{2}{3} \times w_6 + \frac{1}{3} \times 0 = \frac{2}{3} \times 0.1498 = 0.0999,$$

and, in 6th (horizontal) row (A_6) and 9th (vertical) column (A_9), the value is

$$Conc(6,\ 9) = W(C_S) \times \sum_{i \in C_S(6,\ 9)} w_i + W(C_M) \times \sum_{i \in C_M(6,\ 9)} w_i + W(C_W) \times \sum_{i \in C_W(6,\ 9)} w_i$$

$$= 1 \times w_3 + \frac{2}{3} \times (w_1 + w_7) + \frac{1}{3} \times w_2$$
$$= 1 \times 0.2135 + \frac{2}{3} \times 0.2680 + \frac{1}{3} \times 0.0423 = 0.4063.$$

Where w_i is the weight of the criterion C_i and is mentioned in Table 6.

Remark 8. *Since IF ELECTRE does not take the returns criteria and risk criteria apart, thus, we use the formula (23) to calculate the weights of criteria. In addition, note that the formula (23) is irrelevant to our modified TOPSIS method, thus it is also the difference between our method and IF ELECTRE.*

Table 13
The moderate discordance set D_M.

Assessments	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}
A_1	–	4, 6	–	–	4	2	4, 6	–	4	–
A_2	1	–	–	1, 2	1	–	–	1	2	–
A_3	–	5	–	–	–	–	–	–	2	–
A_4	–	–	–	–	4	–	–	–	–	–
A_5	5	–	–	–	–	2	–	–	–	–
A_6	–	–	–	6	–	–	–	6	5, 6	–
A_7	–	3	1	2, 3	–	2	–	1	2, 3, 5	–
A_8	7	4	7	3	4	–	4, 7	–	4	–
A_9	7	–	–	–	–	–	–	–	–	–
A_{10}	–	6	4	–	–	–	6	–	–	–

Table 14
The weak discordance set D_W.

Assessments	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}
A_1	–	–	–	–	–	6, 7	–	–	–	–
A_2	–	–	6, 7	6	5	5, 6, 7	5	3, 7	5, 7	–
A_3	1, 2, 4	–	–	1, 4	1, 2, 5	4, 5, 6	5	1, 4	1, 4, 7	–
A_4	3	–	–	–	3, 5	5	5	2	–	–
A_5	–	–	–	–	–	6	–	–	–	–
A_6	–	–	–	–	–	–	–	2	–	–
A_7	–	–	–	–	–	6, 7	–	–	–	–
A_8	–	–	–	–	–	–	–	–	–	–
A_9	3	–	–	–	–	2	–	2	–	–
A_{10}	1, 2	1, 2	–	1, 5	1, 2, 5	5, 6	2, 5	1, 7	1, 7	–

Table 15
The concordance matrix*Conc* .

Assessments	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}
A_1	–	0.2532	0.6212	0.2811	0.1723	0.1523	0.4532	0.4223	0.3243	0.6712
A_2	0.2303	–	0.3438	0.2175	0.1651	0.1957	0.3055	0.1305	0.1305	0.6826
A_3	0.0999	0.2026	–	0.0000	0.0999	0.0000	0.2907	0.1085	0.0000	0.4485
A_4	0.4329	0.4596	0.5982	–	0.2482	0.2736	0.3449	0.4866	0.3351	0.8426
A_5	0.5112	0.4455	0.5342	0.3135	–	0.4238	0.3884	0.4706	0.4220	0.7645
A_6	0.5105	0.4596	0.5038	0.5069	0.3969	–	0.4512	0.5577	0.4063	0.6343
A_7	0.2303	0.2923	0.4330	0.1740	0.2956	0.2828	–	0.2389	0.2389	0.6222
A_8	0.3711	0.3314	0.4660	0.3098	0.2792	0.3098	0.5352	–	0.2159	0.5202
A_9	0.4138	0.5595	0.6767	0.4242	0.3654	0.2522	0.5669	0.5162	–	0.7342
A_{10}	0.0499	0.0542	0.1582	0.0000	0.0000	0.0000	0.2125	0.0000	0.0000	–

Table 16
The discordance matrix *Disc*.

Assessments	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}
A_1	–	0.5333	0.4286	0.8333	0.8571	0.9167	0.5641	1.0000	1.0000	0.0000
A_2	1.0000	–	0.1000	1.0000	1.0000	1.0000	0.5556	1.0000	1.0000	0.0000
A_3	1.0000	1.0000	–	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0000
A_4	1.0000	0.5455	0.0000	–	0.3810	1.0000	0.5625	0.9333	0.7778	0.0000
A_5	0.6667	0.1538	0.2727	1.0000	–	1.0000	0.3333	0.5882	1.0000	0.0000
A_6	1.0000	0.2941	0.0000	0.3704	0.6250	–	0.3478	1.0000	0.6111	0.0000
A_7	1.0000	0.6667	0.5641	0.7500	1.0000	1.0000	–	0.9167	0.9231	0.0000
A_8	0.6667	0.4444	0.3750	1.0000	1.0000	1.0000	0.6667	–	0.8571	0.0000
A_9	0.3125	0.2727	0.0000	1.0000	0.8462	1.0000	0.4615	1.0000	–	0.0000
A_{10}	1.0000	1.0000	0.6667	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	–

Remark 9. *The value "—" means that A_i does not need to be compared with itself. In the next steps, this value will have the same meaning.*

Step 4: Calculate the discordance matrix. Here, the investor gives the relative weights for the discordance sets as $[W(D_S),\ W(D_M),\ W(D_W)] = [1,\ \frac{2}{3},\ \frac{1}{3}]$ (refer to the weights in [28]), and then we have the discordance matrix shown in Table 16.

For instance, in 7th (horizontal) row (A_7) and 8th (vertical) column (A_8), the value is

$$Disc(7,\ 8) = \frac{\max\left(W(D_S)\times \max_{i\in D_S(7,\ 8)} d(D_{7i},D_{8i}), W(D_M)\times \max_{i\in D_M(7,\ 8)} d(D_{7i},D_{8i}), W(D_W)\times \max_{i\in D_W(7,\ 8)} d(D_{7i},D_{8i})\right)}{\max_{i=1,\ 2,\ \dots\ ,\ 7} d(D_{7i},D_{8i})}$$
$$= \frac{\max\left(1\times\max\left(d(D_{72},D_{82}),\ d(D_{73},D_{83}),\ d(D_{75},D_{85}),\ d(D_{76},D_{86})\right),\ \frac{2}{3}\times d(D_{71},D_{81})\right)}{\max\left(0.2667,\ 0.2667,\ 0.3000,\ 0.3667,\ 0.3667,\ 0.3000,\ 0.4000\right)}$$
$$= \frac{\max\left(1\times\max\left(0.2667,\ 0.3000,\ 0.3667,\ 0.3000\right),\ \frac{2}{3}\times 0.2667\right)}{\max\left(0.2667,\ 0.2667,\ 0.3000,\ 0.3667,\ 0.3667,\ 0.3000,\ 0.4000\right)} = \frac{0.3667}{0.4000} = 0.9167,$$

Similarly, in 9th (horizontal) row (A_9) and 1st (vertical) column (A_1), the value is.

$$Disc(9,\ 1) = \frac{\max\left(1\times d(D_{95},D_{15}),\ \frac{2}{3}\times d(D_{97},D_{17}),\ \frac{1}{3}\times d(D_{93},D_{13})\right)}{\max\left(0.1667,\ 0.1000,\ 0.1667,\ 0.1667,\ 0.1667,\ 0.5333,\ 0.1667\right)}$$
$$= \frac{\max\left(1\times 0.1667,\ \frac{2}{3}\times 0.1667,\ \frac{1}{3}\times 0.1667\right)}{\max\left(0.1667,\ 0.1000,\ 0.1667,\ 0.1667,\ 0.1667,\ 0.5333,\ 0.1667\right)} = \frac{0.1667}{0.5333} = 0.3125,$$

where D_{ij} is the assessment of A_i under the criterion C_j (Please see Table 5), and $d(D_{7i},D_{8i})$ is the distance between D_{7i} and D_{8i}, which is defined by Definition 4.

Step 5: Determine the concordance dominance matrix. Since $\bar{C} = \frac{\sum_{i\neq j} Conc(i,\ j)}{10\times(10-1)} = 0.3428$, the concordance dominance matrix is obtained and shown in Table 17.

Step 6: Determine the discordance dominance matrix. Since $\bar{D} = \frac{\sum_{i\neq j} Disc(i,\ j)}{10\times(10-1)} = 0.6849$, the concordance dominance matrix is obtained and shown in Table 18.

Step 7: Determine the aggregate dominance matrix. The result is shown in Table 19.

Step 8: Eliminate the less favorable alternatives. According to Table 19, the over-ranking relationship is illustrated in Fig. 3.

From Fig. 3 and Table 19, we can easily find that A_6 is the only best investment because no investment dominates it. And correspondingly, A_{10} is the only worst investment because it is dominated by all other investments. By eliminate the best and the worst investments recursively, the ranking of these investments is $A_6 \succ A_4,\ A_5,\ A_9 \succ A_2,\ A_8 \succ A_1 \succ A_7 \succ A_3 \succ A_{10}$, which is mostly similar to the ranking of $C^w(A'^+,\ A'^-)$ in Table 8 (when $\lambda = 0$, i.e. the investor is risk neutral).

However, according to Fig. 3 and Table 19, we cannot judge which of $A_4,\ A_5,\ A_9$ is the best. It is the same for A_2 and A_8. To solve this problem, Wu and Chen [28] utilized TOPSIS index to rank them. The next steps are:

Table 17
The concordance dominance matrix *Cond*.

Assessments	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}
A_1	–	0	1	0	0	0	1	1	0	1
A_2	0	–	1	0	0	0	0	0	0	1
A_3	0	0	–	0	0	0	0	0	0	1
A_4	1	1	1	–	0	0	1	1	0	1
A_5	1	1	1	0	–	1	1	1	1	1
A_6	1	1	1	1	1	–	1	1	1	1
A_7	0	0	1	0	0	0	–	0	0	1
A_8	1	0	1	0	0	0	1	–	0	1
A_9	1	1	1	1	1	0	1	1	–	1
A_{10}	0	0	0	0	0	0	0	0	0	–

Table 18
The discordance dominance matrix *Disd*.

Assessments	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}
A_1	–	1	1	0	0	0	1	0	0	1
A_2	0	–	1	0	0	0	1	0	0	1
A_3	0	0	–	0	0	0	0	0	0	1
A_4	0	1	1	–	1	0	1	0	0	1
A_5	1	1	1	0	–	0	1	1	0	1
A_6	0	1	1	1	1	–	1	0	1	1
A_7	0	1	1	0	0	0	–	0	0	1
A_8	1	1	1	0	0	0	1	–	0	1
A_9	1	1	1	0	0	0	1	0	–	1
A_{10}	0	0	1	0	0	0	0	0	0	–

Table 19
The aggregate dominance matrix *AGG*.

Assessments	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}
A_1	–	0	1	0	0	0	1	0	0	1
A_2	0	–	1	0	0	0	0	0	0	1
A_3	0	0	–	0	0	0	0	0	0	1
A_4	0	1	1	–	0	0	1	0	0	1
A_5	1	1	1	0	–	0	1	1	0	1
A_6	0	1	1	1	1	–	1	0	1	1
A_7	0	0	1	0	0	0	–	0	0	1
A_8	1	0	1	0	0	0	1	–	0	1
A_9	1	1	1	0	0	0	1	0	–	1
A_{10}	0	0	0	0	0	0	0	0	0	–

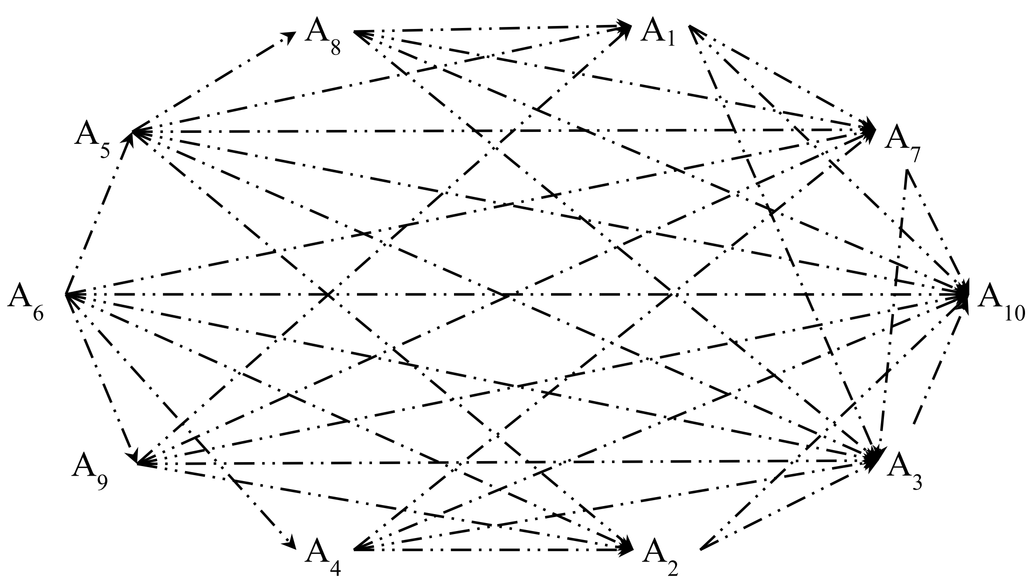

Fig. 3. The over-ranking relationships of the investments.

Step 5b: Determine the concordance dominance matrix *Conc′*. It use the positive-ideal solution of TOPSIS, if C^* is the biggest value in the concordance matrix, then calculate

$$Conc'(i,\ j) = C^* - Conc(i,\ j) \tag{25}$$

Step 6b: Determine the discordance dominance matrix*Donc′*. Let D^* is the biggest value in the discordance matrix, then calculate

$$Donc'(i,\ j) = D^* - Donc(i,\ j) \tag{26}$$

Step 7b: Determine the aggregate dominance matrix *AGG′*, where

$$Agg'(i,\ j) = \frac{Donc'(i,\ j)}{Conc'(i,\ j) + Donc'(i,\ j)} \tag{27}$$

Step 8b: Rank the investments and determine the best investment based on the following formula:

$$P(A_i) = \frac{1}{m-1}\sum_{j \neq i}^{m} Agg'(i,\ j) \tag{28}$$

The bigger the value of $P(A_i)$ is, the better the investment is.

The results are shown in Tables 20–22, where $C^* = 0.8426$, $D^* = 1$.

From Table 22, the ranking of these investments is $A_6 \succ A_9 \succ A_5 \succ A_4 \succ A_1 \succ A_8 \succ A_7 \succ A_2 \succ A_3 \succ A_{10}$. And then, let's compare the process and results between IF ELECTRE and our proposed method. To be convenient, we denote the ranking obtained from Fig. 3 and Table 19 as R', the ranking obtained from Table 22 as R'', and the ranking of $C^w(A'^+, A'^-)$ in Table 8 (when $\lambda = 0$) as R, i.e. $R' : A_6 \succ A_4,\ A_5,\ A_9 \succ A_2,\ A_8 \succ A_1 \succ A_7 \succ A_3 \succ A_{10}$, $R'' : A_6 \succ A_9 \succ A_5 \succ A_4 \succ A_1 \succ A_8 \succ A_7 \succ A_2 \succ A_3 \succ A_{10}$, and $R : A_6 \succ A_5 \succ A_4 \succ A_9 \succ A_1 \succ A_8 \succ A_2 \succ A_7 \succ A_3 \succ A_{10}$.

(i) Obviously, R' is inconsistent with R'', such as the ranking of $A_1,\ A_8$. It means that IF ELECTRE may provide investors with unreliable results that contradict the facts.

(ii) From the assessments in Table 5 and the weights of criteria in Table 6, we can find that A_2 is better than A_7, which is pointed out by R', R and the conventional TOPSIS method (Please see the ranking of $C^w(A^+, A^-)$ in Table 8, when $\lambda = 0$). However, R'' reverses the ranking of A_2 and A_7. From this perspective, we can conclude that R'' is not effective enough.

(iii) Similarly, we can find that A_1 is better than A_2, which is pointed out by R'', R and the conventional TOPSIS method. However, R' reverses the ranking of A_1 and A_2. From this perspective, we can conclude that R' is also not effective enough.

(iv) Whether we use R, R', R'' or the conventional TOPSIS method, we can always get the same result: A_6 is the best investment and A_3, A_{10} have the terrible performances. Thus, our method is in line with the conventional TOPSIS method and IF ELECTRE to some extent.

(v) According to the steps of IF ELECTRE, we can find that it is more complicate than our modified method. It needs to calculate three concordance sets and three discordance sets for the construction of the concordance matrix and discordance matrix by pairwise comparison. In contrast, our method just needs to compare each investment with APIS and ANIS, which is more time-saving.

(vi) In IF ELECTRE, investors need to give the weights for concordance sets and discordance sets, respectively. However, Wu and Chen [28] did not provide a method to determine the weights of concordance sets and discordance sets. In addition, the practical significance of the weights of strong concordance/discordance set, moderate concordance/ discordance set and weak concordance/ discordance set is

Table 20
The concordance dominance matrix $Conc'$.

Assessments	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}
A_1	–	0.5894	0.2214	0.5615	0.6703	0.6903	0.3894	0.4203	0.5183	0.1714
A_2	0.6123	–	0.4988	0.6251	0.6775	0.6469	0.5371	0.7121	0.7121	0.1600
A_3	0.7427	0.6400	–	0.8426	0.7427	0.8426	0.5519	0.7341	0.8426	0.3941
A_4	0.4097	0.3830	0.2444	–	0.5944	0.5690	0.4977	0.3560	0.5075	0.0000
A_5	0.3314	0.3971	0.3084	0.5291	–	0.4188	0.4542	0.3720	0.4206	0.0781
A_6	0.3321	0.3830	0.3388	0.3357	0.4457	–	0.3914	0.2849	0.4363	0.2083
A_7	0.6123	0.5503	0.4096	0.6686	0.5470	0.5598	–	0.6037	0.6037	0.2204
A_8	0.4715	0.5112	0.3766	0.5328	0.5634	0.5328	0.3074	–	0.6267	0.3224
A_9	0.4288	0.2831	0.1659	0.4184	0.4772	0.5904	0.2757	0.3264	–	0.1084
A_{10}	0.7927	0.7884	0.6844	0.8426	0.8426	0.8426	0.6301	0.8426	0.8426	–

Table 21
The discordance dominance matrix $Donc'$.

Assessments	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}
A_1	–	0.4667	0.5714	0.1667	0.1429	0.0833	0.4359	0.0000	0.0000	1.0000
A_2	0.0000	–	0.9000	0.0000	0.0000	0.0000	0.4444	0.0000	0.0000	1.0000
A_3	0.0000	0.0000	–	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
A_4	0.0000	0.4545	1.0000	–	0.6190	0.0000	0.4375	0.0667	0.2222	1.0000
A_5	0.3333	0.8462	0.7273	0.0000	–	0.0000	0.6667	0.4118	0.0000	1.0000
A_6	0.0000	0.7059	1.0000	0.6296	0.3750	–	0.6522	0.0000	0.3889	1.0000
A_7	0.0000	0.3333	0.4359	0.2500	0.0000	0.0000	–	0.0833	0.0769	1.0000
A_8	0.3333	0.5556	0.6250	0.0000	0.0000	0.0000	0.3333	–	0.1429	1.0000
A_9	0.6875	0.7273	1.0000	0.0000	0.1538	0.0000	0.5385	0.0000	–	1.0000
A_{10}	0.0000	0.0000	0.3333	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	–

Table 22
The aggregate dominance matrix Agg' and the value of $P(A_i)$.

Assessments	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}	$P(A_i)$	Ranking
A_1	–	0.4419	0.7207	0.2289	0.1757	0.1077	0.5282	0.0000	0.0000	0.8537	3.0568	5
A_2	0.0000	–	0.6434	0.0000	0.0000	0.0000	0.4528	0.0000	0.0000	0.8621	1.9583	8
A_3	0.0000	0.0000	–	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.7173	0.7173	9
A_4	0.0000	0.5427	0.8036	–	0.5101	0.0000	0.4678	0.1578	0.3045	1.0000	3.7865	4
A_5	0.5014	0.6806	0.7022	0.0000	–	0.0000	0.5948	0.5254	0.0000	0.9276	3.9320	3
A_6	0.0000	0.6483	0.7469	0.6522	0.4569	–	0.6250	0.0000	0.4713	0.8276	4.4282	1
A_7	0.0000	0.3772	0.5156	0.2722	0.0000	0.0000	–	0.1213	0.1130	0.8194	2.2186	7
A_8	0.4141	0.5208	0.6240	0.0000	0.0000	0.0000	0.5202	–	0.1857	0.7562	3.0210	6
A_9	0.6159	0.7198	0.8577	0.0000	0.2437	0.0000	0.6614	0.0000	–	0.9022	4.0007	2
A_{10}	0.0000	0.0000	0.3275	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	–	0.3275	10

not clear enough. Most important of all, IF ELECTRE cannot provide investors with more flexible results according to their risk appetite, which is just an advantage of our method.

In summary, our modified TOPSIS method is more effective and more flexible than IF ELECTRE.

5.3. The ratios of each selected investment

Now, stipulate each investment ratio should be higher than 0.1 and then choose the investments according to the ranking of $C^w(A'^+, A'^-)$ shown in Table 8, and we can allocate the investment ratios in line with the following conditions:

(i) If the investor is risk neutral ($\lambda = 0$), A_4, A_5, A_6 will be the selected investments for the investor. Let x_i be the investment ratio allocated to the investmentA_i,$i = 4, 5, 6$.

Case 1. *By using Model (19) and setting*$\alpha = 1$, *we have*

$$\min D(X) = -0.1043x_4 - 0.1066x_5 - 0.1113x_6 \quad s.t.\begin{cases} x_4 + x_5 + x_6 = 1, \\ x_i \geqslant 0.1, \ i = 4, 5, 6. \end{cases} \tag{29}$$

Then, we can get the optimal investment ratios $x_4 = x_5 = 0.1$, $x_6 = 0.8$, while the total weighted APIS distance and the total weighted ANIS distance are $D^+ = 0.1304$ and$D^- = 0.2405$, respectively.

Case 2. *By using Model (21) and setting* $\theta = 0.5$, *we have*

$$\min D^+(X) = 0.2646x_4 + 0.2290x_5 + 0.1013x_6 \quad s.t.\begin{cases} 0.3689x_4 + 0.3356x_5 + 0.2126x_6 \geqslant 0.2908, \\ x_4 + x_5 + x_6 = 1, \\ x_i \geqslant 0.1, \ i = 4, 5, 6. \end{cases} \tag{30}$$

Then, we can get the optimal investment ratios $x_4 = 0.1000$, $x_5 = 0.5087$, $x_6 = 0.3913$, while the total weighted APIS distance and the total weighted ANIS distance are $D^+ = 0.1826$ and$D^- = 0.2908$, respectively.

Case 3. *By using Model (22) and setting*$\theta = 0.5$, *we have*

$$\max D^-(X) = 0.3689x_4 + 0.3356x_5 + 0.2126x_6 \quad s.t.\begin{cases} 0.2646x_4 + 0.2290x_5 + 0.1013x_6 \leqslant 0.1830, \\ x_4 + x_5 + x_6 = 1, \\ x_i \geqslant 0.1, \ i = 4, 5, 6. \end{cases} \tag{31}$$

Then, we can get the optimal investment ratios $x_4 = 0.1000$, $x_5 = 0.5119$, $x_6 = 0.3881$, while the total weighted APIS distance and the total weighted ANIS distance are $D^+ = 0.1830$ and $D^- = 0.2912$, respectively.

(ii) If the investor is highly profit-seeking ($\lambda = 0.5$), A_4, A_5, A_6 will be the selected investments for the investor. Let x_i be the investment ratio allocated to the investment A_i,$i = 4, 5, 6$.

Case 1. *By using Model (19) and setting*$\alpha = 1.5$, *we have*

$$\min D(X) = 0.2125x_4 + 0.1757x_5 + 0.0456x_6 \quad s.t.\begin{cases} x_4 + x_5 + x_6 = 1, \\ x_i \geqslant 0.1, \ i = 4, 5, 6. \end{cases} \tag{32}$$

Then, we can get the optimal investment ratios $x_4 = x_5 = 0.1$, $x_6 = 0.8$, while the total weighted APIS distance and the total weighted ANIS distance are $D^+ = 0.1304$ and$D^- = 0.2405$, respectively. This result is the same as that of Case 1 in condition (i).

Case 2. *By using Model (21) and setting*$\theta = 0.2$, *we have*

$$\min D^+(X) = 0.2646x_4 + 0.2290x_5 + 0.1013x_6 \quad s.t.\begin{cases} 0.3689x_4 + 0.3356x_5 + 0.2126x_6 \geqslant 0.2439, \\ x_4 + x_5 + x_6 = 1, \\ x_i \geqslant 0.1, \ i = 4, 5, 6. \end{cases} \tag{33}$$

Then, we can get the optimal investment ratios $x_4 = 0.1000$, $x_5 = 0.1274$, $x_6 = 0.7726$, while the total weighted APIS distance and the total weighted ANIS distance are $D^+ = 0.1339$ and $D^- = 0.2439$, respectively.

Case 3. *By using Model (22) and setting*$\theta = 0.8$, *we have*

$$\max D^-(X) = 0.3689x_4 + 0.3356x_5 + 0.2126x_6 \quad s.t.\begin{cases} 0.2646x_4 + 0.2290x_5 + 0.1013x_6 \leqslant 0.1340, \\ x_4 + x_5 + x_6 = 1, \\ x_i \geqslant 0.1, \ i = 4, 5, 6. \end{cases} \tag{34}$$

Then, we can get the optimal investment ratios $x_4 = 0.1000$, $x_5 = 0.1282$, $x_6 = 0.7718$, while the total weighted APIS distance and the total weighted ANIS distance are $D^+ = 0.1340$ and $D^- = 0.2440$, respectively.

(iii) If the investor is highly risk averse ($\lambda = -0.5$), A_4, A_5, A_9 will be the selected investments for the investor. Let x_i be the investment ratio allocated to the investment A_i, $i = 4, 5, 9$.

Case 1. *By using Model (19) and setting*$\alpha = 0.5$, *we have*

$$\min D(X) = -0.4211x_4 - 0.3889x_5 - 0.4275x_9 \quad s.t.\begin{cases} x_4 + x_5 + x_9 = 1, \\ x_i \geqslant 0.1, \ i = 4, 5, 9. \end{cases} \tag{35}$$

Then, we can get the optimal investment ratios $x_4 = x_5 = 0.1$, $x_9 = 0.8$, while the total weighted APIS distance and the total weighted ANIS distance are $D^+ = 0.2666$ and$D^- = 0.3708$, respectively.

Case 2. *By using Model (21) and setting*$\theta = 0.8$, *we have*

$$\min D^+(X) = 0.2646x_4 + 0.2290x_5 + 0.2716x_9 \quad s.t.\begin{cases} 0.3689x_4 + 0.3356x_5 + 0.3755x_9 \geqslant 0.3675, \\ x_4 + x_5 + x_9 = 1, \\ x_i \geqslant 0.1, \ i = 4, 5, 9. \end{cases} \tag{36}$$

Then, we can get the optimal investment ratios $x_4 = 0.1000$, $x_5 = 0.1840$, $x_9 = 0.7160$, while the total weighted APIS distance and the total weighted ANIS distance are $D^+ = 0.2631$ and $D^- = 0.3675$, respectively.

Case 3. *By using Model (22) and setting* $\theta = 0.2$, *we have*

$$\max D^-(X) = 0.3689x_4 + 0.3356x_5 + 0.3755x_9$$
$$s.t.\begin{cases} 0.2646x_4 + 0.2290x_5 + 0.2716x_9 \leqslant 0.2631, \\ x_4 + x_5 + x_9 = 1, \\ x_i \geqslant 0.1,\ i = 4,\ 5,\ 9. \end{cases} \tag{37}$$

Then, we can get the optimal investment ratios $x_4 = 0.1000$, $x_5 = 0.1831$, $x_9 = 0.7169$, while the total weighted APIS distance and the total weighted ANIS distance are $D^+ = 0.2631$ and $D^- = 0.3675$, respectively.

Remark 10. *In fact, although it seems that there is little difference in the results of Case* 2 *and Case* 3 *in condition* (*iii*), *their essences are largely different. After calculating in more detail, we can find that the accurate total weighted APIS distance is higher than* 0.2631 *in Case* 2, *but it is lower than* 0.2631 *in Case* 3, *which is due to their different constraints.*

Now, let us analyze the above results:

(i) From Table 23, it can be found that when the investor is neutral, the results of three cases are different, especially the difference between Case 1 and Case 2 or Case 3. As a matter of fact, when $\alpha = 1$, Model (19) regresses to Model (17), which is completely neutral for the reason that the investor does not take any preference of returns or risks into account. Whereas, Model (21) can ensure that the risk will not exceed a certain ceiling and Model (22) can ensure that the returns will not fall below a certain baseline. Therefore, to the investor, these two models still indicate his/her preferences to some extent.

(ii) Comparing the data of each condition in Table 23, we can find that for Model (21), the bigger the value of θ, the greater the total weighted ANIS distance and the more the investor wants to avoid risk; for Model (22), the bigger the value of θ, the greater the investor's preference for returns. These results are consistent with the conclusions mentioned at the end of Section 4.

(iii) From Table 23, we can also find the greater the total weighted APIS distance, the greater the total weighted ANIS distance, which means that the greater the returns, the greater the risks. It is rational and realistic in portfolio selection.

(iv) From Fig. 4, we can find that for all the conditions, the value of $D^- - D^+$ is always more than 0.1. According to the actual meanings of APIS distance and ANIS distance, it is evident that the greater the value of $D^- - D^+$, the better the portfolio selection. However, Fig. 5 shows that not all the investments can satisfy that the value of $d_i'^- - d_i'^+$ is more than zero, not to mention whether their value of $d_i'^- - d_i'^+$ is more than 0.1. Here, $d_i'^-$ represents the ANIS distance while $d_i'^+$ represents the APIS distance of investment A_i, $i = 1,\ 2,\ \cdots,\ 10$. In summary, we can ensure that the results given by the proposed models are effective to a large extent.

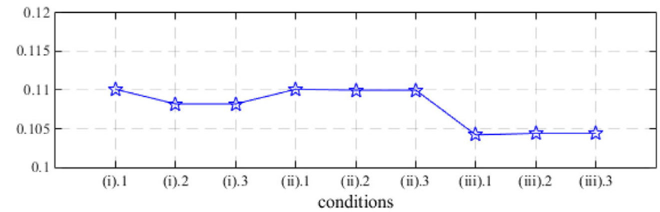

Fig. 4. The values of $D^- - D^+$ under different conditions.

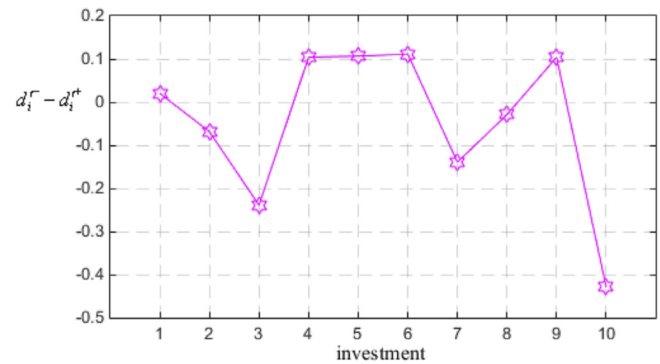

Fig. 5. The value of $d_i'^- - d_i'^+$ of each investment.

What kind of models should be used to obtain the optimal investment ratios depends on the investor's preference. The above models are just presented to offer a more flexible selection.

5.4. Sensitivity analysis

To illustrate the superiority of the modified TOPSIS method better, a sensitivity analysis is carried out to study the influence of λ in the ranking of weighted closeness coefficients. In Section 3, it had been pointed out that the parameter λ indicates the decision maker's preference as to returns and cost/risk, which is the reason that it is called the preference factor of the weighted closeness coefficient. Thus, we will also do an analysis with reference to the relationship between its value and decision makers' preferences.

To be convenient, we denote the weighted closeness coefficient of APIS distance and ANIS distance as*CA*, the weighted closeness coefficient of PIS distance and NIS distance as*Cc*. By using the data shown in Table 7, the weighted closeness coefficients of various values of λ are calculated and the results are presented in Fig. 6 and Fig. 7.

Table 23
The total weighted APIS distances and ANIS distances in example 5.

Condition	(i)			(ii)			(iii)		
	Case 1	Case 2	Case 3	Case 1	Case 2	Case 3	Case 1	Case 2	Case 3
APIS distance	0.1304	0.1826	0.1830	0.1304	0.1339	0.1340	0.2666	0.2631	0.2631
ANIS distance	0.2405	0.2908	0.2912	0.2405	0.2439	0.2440	0.3708	0.3675	0.3675

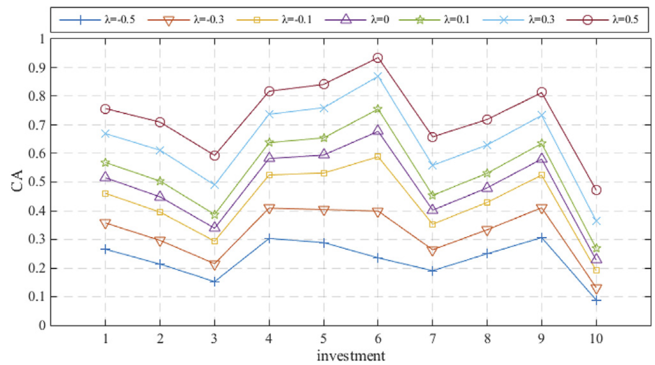

Fig. 6. *CA* with different values ofλ.

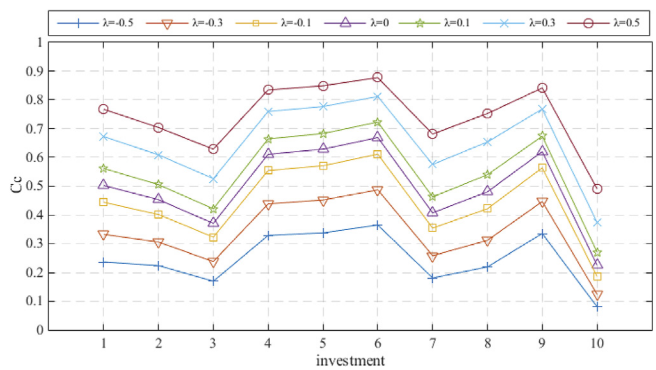

Fig. 7. *Cc* with different values ofλ.

From Fig. 6 and Fig. 7, we can find that:

(i) All the values of weighted closeness coefficients increase with the increase of the value ofλ, which is in line with Theorem 1.
(ii) The shapes of lines in Fig. 7 are similar, whereas in Fig. 6, they are obviously different. It implies that APIS and ANIS can fit in the requirement of the investor's different preferences, while PIS and NIS cannot provide enough flexibility to meet the demand because they can only offer the same results. In addition, according to the analysis of Table 8 in Section 5, it is pointed out that the rankings of $C^w(A'^+, A'^-)$ are more rational than those of $C^w(A^+, A^-)$. In summary, APIS and ANIS are better than PIS and AIS.
(iii) In Fig. 6, the influence of λ is evident in the change of the rankings, especially in the rankings of A_5 andA_6. With the analysis of Table 5, we can find that the assessments of A_5 and A_6 with respect to returns criteria are much better than other investments while their assessments with respect to risk criteria are much worse. It means that their differences between advantage and disadvantage are great. Comparatively, the differences of assessments of A_3 and A_{10} between returns criteria and risk criteria are smaller so that it can be found that their rankings change little in Fig. 6. Hence, we can point out that the ranking of the investment may be more sensitive to the change of the value of λ if the difference in its assessments of returns criteria and risk criteria is greater.

5.5. The benefits and goals of our approach

To make the benefits and goals of our approach more clear, a solid discussion is provided as follows:

(a) The modified TOPSIS can help investors to properly estimate the returns and risks of the stocks when the historical data is insufficient or unavailable. That is important, because the estimation result will directly affect the effectiveness and rationality of the portfolio strategy offered by the model. In fact, without the modified TOPSIS method or other effective evaluation method to quantify the returns and risks as input data, investors cannot construct the corresponding portfolio model, not to mention to make rational portfolio selection.
(b) The weighted closeness coefficient can provide a more appropriate evaluation result for investors. It should be noted that different investors have different risk preferences. If the approach can only provide the same result for different investors, it is not enough flexible and cannot be widely used. However, if the approach provides risk-averse investors with results that the best stock has the highest risk, or it provides profit-seeking investors with results that the best stock has the least return, doubtlessly, that will be more terrible. Thus, an approach that can effectively provide investors with suitable evaluation results, which are in line with their preferences, is necessary.
(c) The new kind of intuitionistic fuzzy portfolio selection model is put forward to make up for the disadvantage of TOPSIS method. As we known, TOPSIS or most of other MCDM methods can help people to find out the best alternative or rank the alternatives according to multiple criteria. Although these methods can quantitatively evaluate the alternatives, they cannot tell people how to combine these alternatives and distribute their resource to make profit as much as possible while avoiding the potential risk. Comparatively, portfolio selection can finish this task well. In addition, referring to the classical M–V model proposed by Markowitz, we construct a new kind of bi-objective model, which utilizes the APIS distance and ANIS distance as the expected return and risks of stocks, rather than the single objective model based on the weighted closeness coefficient. The advantage is that we can better balance the return and risk of portfolio (if we use the weighted closeness coefficient, all the resource will be occupied by the best stock, which will lead to a poor ability to disperse risk).

6. Conclusion

In this paper, the portfolio selection problem under intuitionistic fuzzy circumstance is regarded as a multi-criteria decision making (MCDM) problem, and then the TOPSIS method is used to resolve the problem. Nevertheless, traditional TOPSIS method merely provides decision makers a ranking of alternatives which is not completely in line with decision markers' preferences. Therefore, it is not effective or flexible enough for some portfolio selection problems.

In order to make the methods more flexible to meet the demand of investors with different preferences, the Absolute Positive Ideal Solution (APIS) and the Absolute Negative Ideal Solution (ANIS) are defined for TOPSIS method to represent the returns ceiling and the risk upper limit in portfolio selection, respectively. Compared with PIS and NIS, APIS and ANIS can better achieve the balance between returns and risk. In addition, the closeness coefficient, which decides the ranking of alternatives, is also refined to become more effective to provide the results that are consistent with investors' preferences. Meanwhile, its resistance to extreme data is also improved.

To allocate investment ratios and spread risk, three different kinds of linear programming models based on the distance measure of IFSs are presented. These models can make up for the disadvantage of TOPSIS method that it only considers the ranking of investments but neglects the investment ratios. To demonstrate the effectiveness of our models, a numerical example is provided and the results are analyzed. Finally, a sensitivity analysis is held to study the influence of the parameter called preference factor of the weighted closeness coefficient. Compared with the conventional one, the analysis also illustrates the advantage of the modified TOPSIS method in the ranking with investors' preferences.

In the future work, we will study a new kind of intuitionistic fuzzy portfolio selection programming with complex constraints, and then research for a novel algorithm to solve the model more effectively.

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Compliance with ethical standards

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Ethical approval: This article does not contain any studies with human participants or animals performed by any of the authors.

References

[1] Markowitz HM. Portfolio selection. J Fin 1952;7(1):77–91.
[2] Davis MHA, Norman AR. Portfolio selection with transaction costs. Math OR 1990;15(4):676–713.
[3] Zhou XY, Li D. Continuous-time mean-variance portfolio selection: A stochastic LQ framework. Appl Math Optim 2000;42(1):19–33.
[4] Ledoit O, Wolf M. Improved estimation of the covariance matrix of stock returns with an application to portfolio selection. J Emp Fin 2003;10(5):603–21.
[5] Liu J. Portfolio selection in stochastic environments. Rev Finan Stud 2007;20(1):1–39.
[6] Tofighian AA, Moezzi H, Barfuei MK, Shafiee M. Multi-period project portfolio selection under risk considerations and stochastic income. J Indus Eng Int 2018;14(3):571–84.
[7] Zhou W, Xu ZS. Portfolio selection and risk investment under the hesitant fuzzy environment. Knowl-Based Syst 2017;144:21–31.
[8] Yazdi M. Risk assessment based on novel intuitionistic fuzzy-hybrid-modified TOPSIS approach. Saf Sci 2018;110:438–48.
[9] Guo S, Yu L, Li X, Kar S. Fuzzy multi-period portfolio selection with different investment horizons. Eur J Oper Res 2016;254(3):1026–35.
[10] Zadeh LA. Fuzzy sets. Inf Control 1965;8(3):338–53.
[11] Atanassov KT. Intuitionistic fuzzy sets. Fuzzy Sets Syst 1986;20(1):87–96.
[12] Chen SM, Chang CH. A novel similarity measure between atanassovs intuitionistic fuzzy sets based on transformation techniques with applications to pattern recognition. Inf Sci 2015;291(C):96–114.
[13] Park JH, Lim KM, Kwun YC. Distance measure between intuitionistic fuzzy sets and its application to pattern recognition. Korean Inst Intell Syst 2009;19(4):556–61.
[14] Shen F, Ma X, Li Z, Xu Z, Cai D. An extended intuitionistic fuzzy TOPSIS method based on a new distance measure with an application to credit risk evaluation. Inf Sci 2017;428:105–19.
[15] Ngan RT, Le HS, Cuong BC, Ali M. H-max distance measure of intuitionistic fuzzy sets in decision making. Appl Soft Comput 2018;69:393–425.
[16] Dwivedi G, Srivastava RK, Srivastava SK. A generalised fuzzy TOPSIS with improved closeness coefficient. Expert Syst Appl 2017;96:185–95.
[17] Luo M, Zhao R. A distance measure between intuitionistic fuzzy sets and its application in medical diagnosis. Artif Intell Med 2018;89:34–9.
[18] Szmidt E, Kacprzyk J. Distances between intuitionistic fuzzy sets. Fuzzy Sets Syst 2000;114(3):505–18.
[19] Yue W, Wang Y. A new fuzzy multi-objective higher order moment portfolio selection model for diversified portfolios. Phys A 2017;465:124–40.
[20] Chen SM, Cheng SH, Lan TC. A novel similarity measure between intuitionistic fuzzy sets based on the centroid points of transformed fuzzy numbers with applications to pattern recognition. IEEE Int Conf Syst 2016;27:1125–9.
[21] Joshi D, Kumar S. Intuitionistic fuzzy entropy and distance measure based TOPSIS method for multi-criteria decision making. Egypt Inf J 2014;15(2):97–104.
[22] Hwang CL, Yoon K. Multiple attribute decision making-methods and application: a state of the art survey. Berlin, Heidelberg: Springer; 1981.
[23] Onat NC, Gumus S, Kucukvar M, Tatari O. Application of the TOPSIS and intuitionistic fuzzy set approaches for ranking the life cycle sustainability performance of alternative vehicle technologies. Sustain Prod Consump 2016;6:12–25.
[24] Wei CP, Gao ZH, Guo TT. An intuitionistic fuzzy entropy measure based on trigonometric function. Control Decision 2012;27(4):571–4.
[25] Wu JH, Li PY, Qian H, Chen J. On the sensitivity of entropy weight to sample statistics in assessing water quality: statistical analysis based on large stochastic samples. Environ Earth Sci 2015;74(3):2185–95.
[26] R Benayoun, B Roy, BSussman. Manuel de référence du programme ELECTRE. Documentation Inria Mai, 1966.
[27] Wang X, Triantaphyllou E. Ranking irregularities when evaluating alternatives by using some ELECTRE methods. Omega 2008;36(1):45–63.
[28] MC Wu, TY Chen. The ELECTRE multicriteria analysis approach based on intuitionistic fuzzy sets. IEEE, 2009.