Available online at www.sciencedirect.com
ScienceDirect
Electronic Notes in Theoretical Computer Science
Electronic Notes in Theoretical Computer Science 175 (2007) 75–86
www.elsevier.com/locate/entcs

# General Reversibility

Vincent Danos

*CNRS & Université Paris 7*

Jean Krivine

*INRIA Rocquencourt & Université Paris 6*

Paweł Sobociński[1]

*Computer Laboratory, University of Cambridge*

**Abstract**

The first and the second author introduced reversible CCS (RCCS) in order to model concurrent computations where certain actions are allowed to be reversed. Here we show that the core of the construction can be analysed at an abstract level, yielding a theorem of pure category theory which underlies the previous results. This opens the way to several new examples; in particular we demonstrate an application to Petri nets.

*Keywords:* Reversible computation, CCS, reversible calculus of communicating systems (RCCS), Petri net

## 1 Introduction

The reversible calculus of communicating systems (RCCS) [1] is essentially Milner's CCS [9] with the caveat that some observable actions in the standard labelled transition system (LTS) semantics are understood to be reversible. Technically, the theoretical development involved the engineering of explicit syntax for keeping track of a computation history. Such a history, together with a CCS term, forms the configuration of a given process. Appropriate new structural operational semantics (SOS) rules allowed the reversible components of a given state's history to be undone. Phillips and Ulidowski [10] proposed a different approach to keeping the record of a computation's history; instead of keeping an explicit representation of

[1] Research partially supported by EPSRC grant EP/D066565/1.


doi:10.1016/j.entcs.2006.07.036

history together with an unevaluated term, they keep the structure of terms essentially unaltered by making the SOS rules *static*. Causality is kept track of by tagging actions with so-called communication keys.

In [2], it was argued that a calculus such as RCCS (or CCSK of [10]) is suited for modelling *transactions* – ie computations where several agents interact in order to agree on a common irreversible action; see [3] for example. Indeed, it seems that guaranteeing the soundness of such transactions is easy enough since policies are normally specified by requiring the local states of the participants to satisfy certain criteria. On the other hand, completeness seems to be more difficult, since the existence of a possible computation leading to all of the agents having the required state does not guarantee that such a state will be reached – for instance, the agents may deadlock while racing to obtain the necessary shared resources. If we stipulate that the actions leading to transactions are reversible and enrich the participants with histories, meaning that the intermediate actions can be undone, the irreversible computations are "essentially" the transactions. More concretely, it was shown in [2] that the LTS where the labels are taken to be the transactions and the LTS of processes with histories and reversible actions, where the reversible actions are equated with $\tau$s, are weakly bisimilar.

In this paper we show that the design of a calculus such as RCCS involves an underlying abstract construction of the history category from a category of computations. The fact that the computations agree essentially with the causal (irreversible) computations in the original category is captured by an equivalence of categories.

The main contributions of this paper are:

(i) the observation that subcategories $\mathcal{R}$ of reversible and $\mathcal{I}$ of causal computations form a factorisation system $\langle \mathcal{I}, \mathcal{R} \rangle$ on the category of computations $\mathbf{C}$ (cf §3);

(ii) given a factorisation system $\langle \mathcal{I}, \mathcal{R} \rangle$ on $\mathbf{C}$, an explicit construction of the "category of histories" $h_\star(\mathbf{C}, \mathcal{R})$ (cf Definition 4.3);

(iii) a proof that $h_\star(\mathbf{C}, \mathcal{R})$ essentially follows from a free construction; concretely we prove that $h_\star(\mathbf{C}, \mathcal{R})$ is equivalent to a certain category of fractions (cf Theorem 4.5);

(iv) an equivalence of categories $h_\star(\mathbf{C}, \mathcal{R}) \simeq \mathcal{I}$ (cf Theorem 4.4) – this is the main result of the paper and guarantees that in order to capture the causal computations it is enough to keep the reversible parts of a computation along as part of the state and allow these histories to be undone;

(v) a direct application of Theorem 4.4 to the categories of computations induced by Petri nets;

(vi) an explanation of how Theorem 4.4 relates to the previous work [2] concerning RCCS. In particular, a weak bisimulation that relates the LTS of transactions to the LTS of reversible histories where the reversible actions are treated as internal (cf Theorem 5.3).

### Structure of the paper

In §2 we recall the basic concepts of categories of fractions and factorisation systems. In §3 we introduce several examples, including Petri nets, and show that the sets of causal and reversible computations form factorisation systems. The construction of the history category together with our main Theorem 4.4 is given in §4. Finally, in §5 we explore the connections with labelled transition systems. The paper assumes a basic acquaintance with the categorical notions of adjunctions and symmetric monoidal (SM) categories.

## 2 Categories of fractions and factorisation systems

### Categories of fractions

Given a category $\mathbf{C}$ and an arbitrary class of morphisms $\Sigma$, we denote by $\mathbf{C}[\Sigma^{-1}]$ the *category of fractions* obtained by "freely" adding formal inverses to the arrows of $\Sigma$ (see, for instance [5]).

The category of fractions is characterised by a universal property: the existence of a functor $\Phi\colon \mathbf{C} \to \mathbf{C}[\Sigma^{-1}]$ which sends each arrow in $\Sigma$ to an isomorphism, and moreover, given a category $\mathbf{D}$ and a functor $F\colon \mathbf{C} \to \mathbf{D}$ which takes each arrow in $\Sigma$ to an isomorphism, the existence of a unique functor $F'\colon \mathbf{C}[\Sigma^{-1}] \to \mathbf{D}$ such that $F'\Phi = F$.

$$\begin{tikzcd} \mathbf{C} \arrow[r, "\Phi"] \arrow[dr, "F"'] & \mathbf{C}[\Sigma^{-1}] \arrow[d, "F'"] \\ & \mathbf{D} \end{tikzcd}$$

### Factorisation systems

Given a category $\mathbf{C}$ and two arrows $f, g \in \mathbf{C}$ we shall write $f \perp g$ if $f$ and $g$ satisfy the following property: given a commutative diagram with $p, q$ arbitrary morphisms of $\mathbf{C}$ there exists a unique morphism $h\colon C \to B$ such that $gh = q$ and $hf = p$, as illustrated. Notice that $\perp$ is not symmetric. Given an arbitrary set $\mathcal{X}$ of arrows of $\mathbf{C}$ there are two closure operations which use $\perp$:

$$\begin{tikzcd} A \arrow[r, "p"] \arrow[d, "f"'] & B \arrow[d, "g"] \\ C \arrow[r, "q"'] \arrow[ur, dotted, "h"] & D \end{tikzcd}$$

$$\mathcal{X}^{\perp} = \{\, y \text{ in } \mathbf{C} \mid \forall x \in \mathcal{X}.\, x \perp y \,\} \text{ and } \mathcal{X}^{\top} = \{\, y \text{ in } \mathbf{C} \mid \forall x \in \mathcal{X}.\, y \perp x \,\}.$$

If we let $\mathrm{Iso}(\mathbf{C})$ ($\mathrm{Ar}(\mathbf{C})$) be the class of all isomorphisms (morphisms) of $\mathbf{C}$ then it's immediate that $\mathrm{Iso}(\mathbf{C})^{\perp} = \mathrm{Ar}(\mathbf{C}) = \mathrm{Iso}(\mathbf{C})^{\top}$.

The following are standard properties enjoyed by the closure operations:

**Proposition 2.1**

*(i)* $\mathcal{X}^{\perp\top\perp} = \mathcal{X}^{\perp}$;

*(ii)* $\mathcal{X}^{\top\perp\top} = \mathcal{X}^{\top}$;

*(iii)* $\mathcal{X} \subseteq \mathcal{X}' \Rightarrow \mathcal{X}'^{\perp} \subseteq \mathcal{X}^{\perp}$

*(iv)* $\mathcal{X} \subseteq \mathcal{X}' \Rightarrow \mathcal{X}'^{\top} \subseteq \mathcal{X}^{\top}$.

Following [4], we define a *prefactorisation system* as follows:

**Definition 2.2** [Prefactorisation system] A prefactorisation system for a category $\mathbf{C}$ consists of two classes $\mathcal{I}$, $\mathcal{R}$ of arrows of $\mathbf{C}$ such that $\mathcal{I}^{\perp} = \mathcal{R}$ and $\mathcal{R}^{\top} = \mathcal{I}$.

By the first two parts of Proposition 2.1 it is immediate that for any class of arrows $\mathcal{X}$ of $\mathbf{C}$, $\langle \mathcal{X}^{\top}, \mathcal{X}^{\top\perp} \rangle$ and $\langle \mathcal{X}^{\perp\top}, \mathcal{X}^{\perp} \rangle$ are prefactorisation systems.

The following are some of the well-known properties of prefactorisation systems [4]:

**Proposition 2.3** *Suppose that $\langle \mathcal{I}, \mathcal{R} \rangle$ is a prefactorisation system on $\mathbf{C}$. Then:*

*(i)* $\mathrm{Iso}(\mathbf{C}) \subseteq \mathcal{I}$, $\mathrm{Iso}(\mathbf{C}) \subseteq \mathcal{R}$ *and* $\mathcal{I} \cap \mathcal{R} = \mathrm{Iso}(\mathbf{C})$*;*

*(ii)* $\mathcal{I}$ *and* $\mathcal{R}$ *are closed under composition.*

The conclusion of Proposition 2.3 implies that $\mathcal{I}$ and $\mathcal{R}$ are actually subcategories of $\mathbf{C}$ since they contain the identities and are closed under composition. We shall take advantage of this by often confusing $\mathcal{I}$ and $\mathcal{R}$ with the subcategories they form the arrows of.

**Definition 2.4** [Factorisation system] A prefactorisation system $\langle \mathcal{I}, \mathcal{R} \rangle$ on $\mathbf{C}$ is a factorisation system if every arrow $p$ in $\mathbf{C}$ can be written $p = g \circ f$ for some $f$ in $\mathcal{I}$ and $g$ in $\mathcal{R}$.

**Example 2.5** Clearly $\langle \mathbf{C}, \mathrm{Iso}(\mathbf{C}) \rangle$ and $\langle \mathrm{Iso}(\mathbf{C}), \mathbf{C} \rangle$ are factorisation systems in any category. Probably the most well-known factorisation system is of course $\langle \mathcal{E}, \mathcal{M} \rangle$ in the category of sets $\mathbf{Set}$, where $\mathcal{E}$ is the class of surjections and $\mathcal{M}$ is the class of injections.

The following is a well-known property of factorisation systems:

**Lemma 2.6** *$\langle \mathcal{I}, \mathcal{R} \rangle$-factorisation is unique up to isomorphism: if $p\colon A \to B$ in $\mathbf{C}$ can be factorised $p = g_1 f_1$ and also $p = g_2 f_2$ where $f_i\colon A \to C_i$ is in $\mathcal{I}$ and $g_i\colon C_i \to B$ is in $\mathcal{R}$ for $i = 1, 2$, then there exists a unique isomorphism $h\colon C_1 \to C_2$ such that $h f_1 = f_2$ and $g_2 h = g_1$*

## 3 Reversibility

Following the theoretical exposition, we give a number of motivating examples of factorisation systems. We shall consider categories of computations which decompose into an underlying set of atomic actions, some of which are a priori specified as reversible. Given a computation which consists of both types of actions, it should be possible to break it up into a *causal* (non-reversible) component followed by a maximal *reversible* component. If we denote the causal computations by $\mathcal{I}$ and the reversible computations by $\mathcal{R}$, it turns out that $\langle \mathcal{I}, \mathcal{R} \rangle$ usually forms a factorisation system on the category of computations.

**Example 3.1** [Single-threaded reversibility] Consider an alphabet $\Sigma = I + R$ for some sets $I$ and $R$; we think of $I$ as a set of irreversible atomic actions and $R$ as a set of reversible atomic actions. Let $\Sigma^*$ denote the free monoid over $\Sigma$ considered as a one-object category.

Let $\mathcal{R} = R^*$ and let $\mathcal{I} = \mathcal{R}^\top = \Sigma^* I + \epsilon$ – the set of all strings which end with an irreversible action, together with the empty string. Then $\langle \mathcal{I}, \mathcal{R} \rangle$ is a factorisation system on $\Sigma^*$.

**Example 3.2** [Multi-threaded reversibility] Let $\mathbf{C}$ be the free SM category on a graph $G$ – ie one first forms the free category on $G$ and then the free SM category on the resulting category. We think of the vertices of $G$ as representing the states of a particular thread of computation, and the edges as possible actions. Then, following this intuition, the arrows of $\mathbf{C}$ represent multithreaded computations of finitely many non-communicating processes, with the tensor product $\otimes$ representing parallel composition.

Suppose that the edges of $G$ are partitioned into two sets, $I$ and $R$. Let $G_R$ denote the graph with the same nodes as $G$ but with the edges restricted to the members of $R$.

Let $\mathcal{R}$ be the free SM category on $G_R$. Clearly $\mathcal{R}$ is a subcategory of $\mathbf{C}$ in a canonical way. Let $\mathcal{I} = \mathcal{R}^\top$. It is easy to verify that $\mathcal{I}$ is the smallest subcategory of $\mathbf{C}$ which contains the isomorphisms of $\mathbf{C}$, arrows of the form $i\alpha$ with $i \in I$ and whose arrows are closed under $\otimes$. Then $\langle \mathcal{I}, \mathcal{R} \rangle$ is a factorisation system on $\mathbf{C}$.

It is instructive to consider a more substantial example in order to illustrate the theory. Here we shall consider Petri nets as SM categories in the tradition of [8]. Note, however, that we do not deal with *strict* symmetric monoidal categories. We shall first need to recall the notion of a *tensor scheme* [6] and the associated notion of a free SM category on a tensor scheme; indeed, as we shall see, tensor schemes are very closely related to Petri nets. Note that tensor schemes can also be used in order to construct ordinary (ie non-symmetric) free monoidal categories.

**Definition 3.3** [Tensor scheme]A tensor scheme $\mathcal{S}$ consists of a set $V$ of vertices, a set $E$ of edges, and functions $s, t : E \to V^*$, where $V^*$ is the free monoid (the set of finite words) on $V$. Every tensor scheme leads to a free SM category $\mathbf{C}$ – see [6] for details. Intuitively, the objects of $\mathbf{C}$ can be seen as finite words (ie the product in $V^*$ is interpreted as $\otimes$ in $\mathbf{C}$) in $V$ and the arrows of $\mathbf{C}$ are generated freely from the basic edges in $E$. Concretely, the arrows can be seen as certain equivalence classes or as certain string diagrams; see [11]. Notice that the procedure described in Example 3.2 can be seen as a special case of a tensor scheme (where all the edges have one letter words as sources and targets).

**Definition 3.4** [Petri net] A *Petri net* $\mathcal{N}$ with a set of states $S$ and set of transitions $T$ is a graph $s, t \colon T \to S^\oplus$ where $S^\oplus$ is the free commutative monoid on $S$. A *Petri category* $\mathbf{C}_\mathcal{N}$ is the free SM category on $\mathcal{N}$, considered as a tensor scheme. [2]

The Petri category $\mathbf{C}_\mathcal{N}$ can be thought of as the category with arrows the (truly) concurrent computations of a net $\mathcal{N}$.

**Example 3.5** [Petri net reversibility] Suppose that the set $T$ of transitions $\mathcal{N}$ can be partitioned $T = I + R$, where the set $I$ contains the transitions which are deemed

[2] One fixes a particular ordering of places for the source and the target of each transition, the order is immaterial.

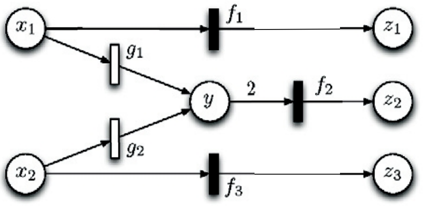


Fig. 1. A simple Petri net, the filled transitions are irreversible.

irreversible and $R$ the transitions deemed reversible. We obtain a factorisation system $\langle \mathcal{I}, \mathcal{R} \rangle$ as in our previous examples.

Let $\mathcal{R}$ be the free SM category on $\mathcal{N}_R$, the Petri net with the same places as $\mathcal{N}$ and with $R$ as its set of transitions, considered as a tensor scheme; it is clearly a subcategory of $\mathbf{C}_{\mathcal{N}}$ in a canonical way. Let $\mathcal{I} = \mathcal{R}^{\top}$ – the arrows of $\mathcal{I}$ can be described roughly as in Example 3.2. The pair $\langle \mathcal{I}, \mathcal{R} \rangle$ forms a factorisation system on $\mathbf{C}_{\mathcal{N}}$.

Consider the concrete example of a net illustrated in Figure 1, where precisely the unfilled transitions ($g_1$ and $g_2$) are taken to be reversible. Suppose that places $x_1$ and $x_2$ initially contain one token each; intuitively, we can consider places $x_1$ and $x_2$ as agents which each have an option of committing to two transactions: $x_1$ can commit to either $f_1$ or $f_2$ while $x_2$ can commit to $f_2$ or $f_3$. In terms of $\mathbf{C}_{\mathcal{N}}$ this amounts to the fact that there are arrows $f_1 \colon x_1 \to z_1$, $f_3 \colon x_2 \to z_3$ and $f_2.g_1 \otimes g_2 \colon x_1 \otimes x_2 \to z_2$. Notice that if $x_1$ chooses to perform $g_1$ and $x_2$ commits to $f_3$ then the computation begun by $x_1$ is stuck unless the $g_1$ transition can be reversed and $f_1$ chosen instead.

Consider the effect of adding new transitions $g_{1\star}$ and $g_{2\star}$ to act as the inverses of $g_1$ and $g_2$ respectively. If we deem that reversed computations are the same as doing nothing then the resulting Petri category is just $\mathbf{C}_{\mathcal{N}}[\mathcal{R}^{-1}]$. However, this setting is clearly unsuitable to model the expected behaviour of the net: consider starting with a single token in $x_2$ and performing the $g_2$ transition. Since now $g_{1\star}$ is enabled, we can perform $g_{1\star}$ and then $f_1$, thus arriving at a behaviour which is not in the specification – $x_2$ being able to commit to action $f_1$.

## 4 Histories

A key technical feature of RCCS is that histories are kept as part of the state, which allows reversible moves to be backtracked correctly. Here we repeat the construction at a higher level of abstraction, assuming only the presence of a factorisation system.

**Definition 4.1** [Category $h(\mathbf{C}, \mathcal{R})$ of histories] Suppose that $\langle \mathcal{I}, \mathcal{R} \rangle$ is a factorisation system on $\mathbf{C}$. Let $h(\mathbf{C}, \mathcal{R})$ be the category with:

- objects: arrows $g$ in $\mathcal{R}$;
- arrows: commutative diagrams, as illustrated, where $f$ is in $\mathbf{C}$ and $f' \in \mathcal{I}$.

$$\begin{tikzcd} P_1 \arrow[r, "f'"] \arrow[d, "g_1"'] & P_2 \arrow[d, "g_2"] \\ Q_1 \arrow[r, "f"'] & Q_2 \end{tikzcd}$$

Notice that given an object $g_1 \colon P_1 \to Q_1$ in $h(\mathbf{C}, \mathcal{R})$ and an arbitrary arrow $f \colon Q_1 \to Q_2$, there exists a unique up-to-isomorphism object $g_2$ of $h(\mathbf{C}, \mathcal{R})$ and arrow $f' \colon P_1 \to P_2$ in $\mathcal{I}$ such that $\langle f, f' \rangle : g_1 \to g_2$ is in $h(\mathbf{C}, \mathcal{R})$ – here $g_2 \circ f'$ is just the $\langle \mathcal{I}, \mathcal{R} \rangle$-factorisation of $f \circ g_1$. Notice that if $f \in \mathcal{R}$, then using the fact that arrows of $\mathcal{R}$ compose and uniqueness of factorisation (Lemma 2.6), we have that $f' \in \mathrm{Iso}(\mathcal{C})$.

Recall from Proposition 2.3 that we can consider $\mathcal{I}$ to be a category. There is an obvious functor $M \colon h(\mathbf{C}, \mathcal{R}) \to \mathcal{I}$ which takes an object $g_1 \colon P_1 \to Q_1$ to $P_1$ and the diagram above to the arrow $f' \colon P_1 \to P_2$. Returning to our intuitions, this functor takes a computation to its causal (non-reversible) component. Using the final remark of the previous paragraph, $M$ sends arrows which have a lower component in $\mathcal{R}$ to isomorphisms.

$$\begin{tikzcd} P_1 \arrow[r, "f"] \arrow[d] & P_2 \arrow[d] \\ P_1 \arrow[r, "f"'] & P_2 \end{tikzcd}$$

There is also a (full and faithful) functor $N \colon \mathcal{I} \to h(\mathbf{C}, \mathcal{R})$, which takes an object $P_1 \in \mathcal{I}$ to the identity on $P_1$ (null history) and a morphism $f \colon P_1 \to P_2$ to the illustrated diagram.

**Proposition 4.2** *$N$ is left adjoint to $M$.*

**Proof.** Given $g_1 \colon P_1 \to Q_1 \in h(\mathbf{C}, \mathcal{R})$, consider the illustrated morphism $\epsilon_{g_1} = \langle g_1, \mathrm{id} \rangle : NM(g_1) \to g_1$. It is easy to verify that $\epsilon$ defines a natural transformation $NM \Rightarrow \mathrm{id}_{h(\mathbf{C},\mathcal{R})}$ – it is the counit of the adjunction. The unit is trivial as $MN = \mathrm{id}_{\mathcal{I}}$, and the triangle identities are easily checked. □

$$\begin{tikzcd} P_1 \arrow[r] \arrow[d] & P_1 \arrow[d, "g_1"] \\ P_1 \arrow[r, "g_1"'] & Q_1 \end{tikzcd}$$

Recall that our intuition is that the objects of $h(\mathbf{C}, \mathcal{R})$ represent (reversible) histories. We shall now extend $h(\mathbf{C}, \mathcal{R})$ with "reversed" computations with the effect that such histories can be undone.

**Definition 4.3** [Category $h_\star(\mathbf{C}, \mathcal{R})$ of reversible histories ] Suppose that $\langle \mathcal{I}, \mathcal{R} \rangle$ is a factorisation system. Let $\Phi \colon \mathbf{C} \to \mathbf{C}[\mathcal{R}^{-1}]$ be the canonical functor to the category of fractions. Let $h_\star(\mathbf{C}, \mathcal{R})$ denote the category with:

- objects: arrows $g$ in $\mathcal{R}$;
- arrows: formal diagrams, as illustrated, with $f \in \mathcal{I}$, $f_\star \in \mathbf{C}[\mathcal{R}^{-1}]$, such that $f_\star \Phi(g_1) = \Phi(g_2 f)$ in $\mathbf{C}[\mathcal{R}^{-1}]$.

$$\begin{tikzcd} P_1 \arrow[r, "f"] \arrow[d, "g_1"'] & P_2 \arrow[d, "g_2"] \\ Q_1 \arrow[r, "f_\star"'] & Q_2 \end{tikzcd}$$

There is an evident functor $\Psi \colon h(\mathbf{C}, \mathcal{R}) \to h_\star(\mathbf{C}, \mathcal{R})$ which maps the lower com-

$$\begin{tikzcd}
h(\mathbf{C},\mathcal{R}) \arrow[rr, bend left, "\Psi"] \arrow[dr, bend left, "M"] & & h_\star(\mathbf{C},\mathcal{R}) \arrow[dl, bend left, "M_\star"] \\
& \mathcal{I} \arrow[ul, bend left, "N"] \arrow[ur, bend left, "N_\star"] &
\end{tikzcd}$$

Fig. 2. Histories and causal computations.

ponent of a history morphism from $\mathbf{C}$ to $\mathbf{C}[\mathcal{R}^{-1}]$ via $\Phi$:

$$\begin{tikzcd}
P_1 \arrow[r, "f'"] \arrow[d, "g_1"'] & P_2 \arrow[d, "g_2"] \\
Q_1 \arrow[r, "f"'] & Q_2
\end{tikzcd}
\quad\longmapsto\quad
\begin{tikzcd}
P_1 \arrow[r, "f'"] \arrow[d, "g_1"'] & P_2 \arrow[d, "g_2"] \\
Q_1 \arrow[r, "\Phi f"'] & Q_2
\end{tikzcd}$$

Let $M_\star \colon h_\star(\mathbf{C},\mathcal{R}) \to \mathcal{I}$ be the functor which takes an arrow of $h_\star(\mathbf{C},\mathcal{R})$ to its upper component. Clearly $M_\star \Psi = M$.

**Theorem 4.4** *$M_\star$ is an equivalence of categories.*

**Proof.** Let $N_\star = \Psi N \colon \mathcal{I} \to h_\star(\mathbf{C},\mathcal{R})$ (see Fig 2) – clearly $M_\star N_\star = \mathrm{id}_{\mathcal{I}}$, we shall show that there exists a natural isomorphism $N_\star M_\star \Rightarrow \mathrm{id}_{h_\star(\mathbf{C},\mathcal{R})}$.

Indeed, since $\Psi$ is the identity on objects, we have $N_\star M_\star g = \Psi N M \Psi g = NMg$, and thus it suffices to show that $\Phi\epsilon$ is a natural isomorphism, where $\epsilon$ is the counit of the adjunction $N \dashv M$. We illustrate $\Psi\epsilon_g$, clearly it is an invertible morphism of $h_\star(\mathbf{C},\mathcal{R})$. Naturality is straightforward.

$$\begin{tikzcd}
P \arrow[r] \arrow[d] & P \arrow[d, "g"] \\
P \arrow[r, "\Phi g"'] & Q
\end{tikzcd}$$

□

Recall from Example 2.5 that $\langle \mathbf{C}, \mathrm{Iso}(\mathbf{C})\rangle$ and $\langle \mathrm{Iso}(\mathbf{C}), \mathbf{C}\rangle$ are trivial factorisation systems in any category $\mathbf{C}$. The conclusion of Theorem 4.4 implies immediately that $h_*(\mathbf{C}, \mathrm{Iso}(\mathbf{C})) \simeq \mathbf{C}$ and $h_*(\mathbf{C},\mathbf{C}) \simeq \mathrm{Iso}(\mathbf{C})$.

We shall now show that $h_\star(\mathbf{C},\mathcal{R})$ essentially follows from a free construction. Let $\mathcal{R}' = \{\, \langle g, \varphi\rangle \in \mathrm{Ar}(h(\mathbf{C},\mathcal{R})) \mid g \in \mathcal{R}\,\}$, the set of those arrows of $h(\mathbf{C},\mathcal{R})$ where the lower component is in $\mathcal{R}$ (cf paragraph following Definition 4.1).

**Theorem 4.5** *There is an equivalence of categories $h_\star(\mathbf{C},\mathcal{R}) \simeq h(\mathbf{C},\mathcal{R})[\mathcal{R}'^{-1}]$.*

**Proof.** Let $\mathbf{X} = h(\mathbf{C},\mathcal{R})[\mathcal{R}'^{-1}]$. Since we know that $h_\star(\mathbf{C},\mathcal{R}) \simeq \mathcal{I}$, it is enough to show that also $\mathbf{X} \simeq \mathcal{I}$. Let $\Phi' \colon h(\mathbf{C},\mathcal{R}) \to \mathbf{X}$ be the canonical quotienting functor. Since $M \colon h(\mathbf{C},\mathcal{R}) \to \mathcal{I}$ sends every member of $\mathcal{R}'$ to an isomorphism, we have a unique functor $M' \colon \mathbf{X} \to \mathcal{I}$ such that $M'\Phi' = M$. Let $N' = \Phi' N \colon \mathcal{I} \to \mathbf{X}$. Then $M'N' = M'\Phi'N = MN = \mathrm{id}_{\mathcal{I}}$.

Let $\epsilon \colon NM \to \mathrm{id}_{h(\mathbf{C},\mathcal{R})}$ be the counit of the adjunction $N \dashv M$. Clearly $\Phi'$ sends each component of $\epsilon$ to an isomorphism in $\mathbf{X}$. Since $\Phi'$ is the identity on objects, we have that for each object $g \in \mathbf{X}$, $\Phi'\epsilon_g \colon N'M'g \to g$ is an isomorphism. It remains to check that $\Phi'\epsilon$ defines a natural transformation $N'M' \to \mathrm{id}_{\mathbf{X}}$. To do this we need to check that the commutativity of an arbitrary square, as illustrated, where $h$ is in $\mathbf{X}$.

It is well-known that arrows in $\mathbf{X}$ are equivalence classes of zig-zags in $h(\mathbf{C},\mathcal{R})$ where each of the reverse arrows is in $\mathcal{R}'$. Using the functoriality of $N'M'$ and the fact that $\epsilon$ is a natural transformation, we can "fill in" the diagram below at each point, and since $h = (\Phi'\gamma_n)^{-1}\Phi'\alpha_n \ldots (\Phi'\gamma_1)^{-1}\Phi'\alpha_1$, naturality easily follows by a straightforward diagram chase.

$$
\begin{tikzcd}
N'M'g \arrow[r, "\Phi' NM\alpha_1"] \arrow[d, "\Phi'\epsilon_g"'] & N'M'g_1 \arrow[d, dotted, "\Phi'\epsilon_{g_1}"] & N'M's_1 \arrow[l, "\Phi' NM\gamma_1"'] \arrow[d, dotted, "\Phi'\epsilon_{s_1}"] \arrow[r, "\Phi' NM\alpha_2"] & {} & \cdots & {} \arrow[r, "\Phi' NM\alpha_n"] & M'N'g_n \arrow[d, dotted, "\Phi'\epsilon_{g_n}"'] & N'M'g' \arrow[l, "\Phi' NM\gamma_n"'] \arrow[d, "\Phi'\epsilon_{g'}"] \\
g \arrow[r, "\Phi'\alpha_1"'] & g_1 & s_1 \arrow[l, "\Phi'\gamma_1"] \arrow[r, "\Phi'\alpha_2"'] & {} & \cdots & {} \arrow[r, "\Phi'\alpha_n"'] & g_n & g' \arrow[l, "\Phi'\gamma_n"]
\end{tikzcd}
$$

□

Considering Examples 3.1, 3.2 and 3.5, Theorem 4.4 states that to understand the structure of causal computations it is enough to remember the maximal reversible component of a given computation and allow these histories to be backtracked.

$$
\begin{tikzcd}
N'M'g \arrow[r, "N'M'h"] \arrow[d, "\Phi'\epsilon_g"'] & N'M'g' \arrow[d, "\Phi'\epsilon_{g'}"] \\
g \arrow[r, "h"'] & g'
\end{tikzcd}
$$

Returning to the discussion concerning the net of Figure 1, the missing ingredient was clearly the explicit keeping track of the history of the current computation – ie instead of working in $\mathbf{C}_{\mathcal{N}}[\mathcal{R}^{-1}]$ we work in the history category $h_\star(\mathbf{C}_{\mathcal{N}},\mathcal{R})$. (cf Definition 4.3). Our main result, Theorem 4.4, establishes that the categories $h_\star(\mathbf{C}_{\mathcal{N}},\mathcal{R})$ and $\mathcal{I}$ are equivalent, which confirms that the computations of nets with histories are essentially the same as the causal computations of the original net. Of course, the main result is clearly more general than this particular example, for instance it underlies the previous work on RCCS [1,2].

Indeed, it is interesting to compare the concrete implementation of a reversible process algebra, like RCCS, with the abstract construction we present in this paper. Roughly, the definition of RCCS in [1] can be summed up as the development of a correct syntactic presentation of the category of reversible histories $h_\star(\mathbf{C},\mathcal{R})$, where $\mathbf{C}$ is the category of computations of CCS.

## 5 Free categories as transition systems

The categories of Examples 3.2 and 3.5 can be thought of as a transition systems as well as categories; indeed, since the categories are generated freely, their arrows can be seen as (equivalence classes of) traces in the transition systems. Here we shall elucidate the consequences of our main Theorem 4.4 for the underlying transition systems, obtaining a direct generalisation of the main result of [2]. Notice however that the results of §4 are more general, since the underlying categories are not assumed to be free; indeed, the only assumption is the presence of a factorisation system.

Let $\mathcal{S} = \langle V, E\rangle$ be a tensor scheme with edges $E$ partitioned into sets of irreversible actions $I$ and reversible actions $R$. Let $\mathbf{C}$ be a freely generated SM category over $\mathcal{S}$. Let $\mathcal{R}$ be the subcategory of $\mathbf{C}$ generated by $\mathcal{S}_R = \langle V, R\rangle$. Let $\mathcal{I} = \mathcal{R}^\top$. Then $\langle \mathcal{I}, \mathcal{R}\rangle$ is a factorisation system. [3]

**Definition 5.1** Let TS($\mathbf{C}$) be defined as follows:

- states are isomorphism classes of objects of $\mathbf{C}$;
- transitions are labelled with elements of $E$ and arise as follows

$$\frac{P_1 \otimes P_2 \xrightarrow{\alpha\otimes P_2} P_1' \otimes P_2 \text{ in } \mathbf{C}, \quad \alpha \in E}{[P_1 \otimes P_2] \xrightarrow{\alpha} [P_1' \otimes P_2]}$$

Using the fact that $\mathbf{C}$ is freely generated, any non-invertible arrow of $\mathbf{C}$ generates a finite set of traces in TS($\mathbf{C}$). We shall refer to each possible trace of an arbitrary morphism $f$ in $\mathbf{C}$ as a *serialisation* of $f$.

A trace $\sigma$ is said to be *causal* if it is a serialisation of an arrow $f$ in $\mathcal{I}$. A trace $\sigma$ is an *i-transaction* if it is causal and contains precisely one action $i \in I$ (and arbitrarily many actions from $R$). Let CTS($\mathbf{C}$) be the LTS with the same states as TS($\mathbf{C}$), but with transitions

$$\frac{[P] \xrightarrow{\sigma} [P'] \ \text{ in } \ \mathrm{TS}(\mathbf{C}), \quad \sigma \text{ an } i\text{-transaction}}{[P] \xrightarrow{i} [P'] \ \text{ in } \ \mathrm{CTS}(\mathbf{C})}$$

Thus CTS($\mathbf{C}$) is the LTS of transactions. Correspondingly, we shall now define the history LTS, where states are enriched with a history, and the transitions are those of $TS(\mathbf{C})$ as well as new transitions which allow backtracking.

**Definition 5.2** Let RTS($\mathbf{C}$) be defined as follows:

- states: isomorphism classes of objects $h(\mathbf{C}, \mathcal{R})$ (structural isomorphisms);
- transitions labelled with elements of $E \cup R_*$ where $R_\star = \{ r_\star \mid r \in R\}$. They are derived from morphisms in $h_\star(\mathbf{C}, \mathcal{R})$, as illustrated below:

$$\frac{\begin{array}{ccc} P_1 & \xrightarrow{f} & P_2 \\ {\scriptstyle g_1}\downarrow & & \downarrow{\scriptstyle g_2} \\ Q_1 \otimes Q_2 & \xrightarrow{\alpha\otimes Q_2} & Q_1' \otimes Q_2 \end{array}, \quad \alpha \in E}{\begin{bmatrix} P_1 \\ {\scriptstyle g_1}\downarrow \\ Q_1 \otimes Q_2 \end{bmatrix} \xrightarrow{\alpha} \begin{bmatrix} P_2 \\ \downarrow{\scriptstyle g_2} \\ Q_1' \otimes Q_2 \end{bmatrix}} \qquad \frac{\begin{array}{ccc} P_1 & \xrightarrow{f} & P_2 \\ {\scriptstyle g_1}\downarrow & & \downarrow{\scriptstyle g_2} \\ Q_1 \otimes Q_2 & \xrightarrow{r^{-1}\otimes Q_2} & Q_1' \otimes Q_2 \end{array}, \quad r \in R}{\begin{bmatrix} P_1 \\ {\scriptstyle g_1}\downarrow \\ Q_1 \otimes Q_2 \end{bmatrix} \xrightarrow{r_\star} \begin{bmatrix} P_2 \\ \downarrow{\scriptstyle g_2} \\ Q_1' \otimes Q_2 \end{bmatrix}}$$

It is clear from the construction of $h_\star(\mathbf{C}, \mathcal{R})$ that any morphism in $h_\star(\mathbf{C}, \mathcal{R})$ induces a set of serialisations (traces) in RTS($\mathbf{C}$).

**Theorem 5.3** *Consider a free* SM *category* $\mathbf{C}$ *generated from a tensor scheme* $\mathcal{S} = \langle V, E\rangle$ *with* $E = I + R$, *together with an induced factorisation system* $\langle \mathcal{I}, \mathcal{R}\rangle$ *where*

[3] We leave it as future work to determine sufficient conditions on a subcategory which ensure that $\mathcal{R} = \mathcal{R}^{\top\perp}$.

*$\mathcal{R}$ is the subcategory of $\mathbf{C}$ freely generated by $\mathcal{S}_R = \langle V, R\rangle$. Let* $\mathrm{CTS}(\mathbf{C})$ *be the* LTS *of transactions (cf Definition 5.1) and* $\mathrm{RTS}(\mathbf{C})$ *be the reversible* LTS *(cf Definition 5.2) where the reversible actions are considered to be silent. Then* $\mathrm{CTS}(\mathbf{C}) \approx \mathrm{RTS}(\mathbf{C})$.

**Proof.** We shall show that the (object part of the) functor $M_\star\colon h_\star(\mathbf{C}, \mathcal{R}) \to \mathcal{I}$ is actually a functional weak bisimulation.

Recall that $M_\star(P \xrightarrow{g} Q) = P$. Clearly $M_\star$ is well-defined as a function from states of $\mathrm{RTS}(\mathbf{C})$ to states of $\mathrm{CTS}(\mathbf{C})$. Suppose that there is a transition

$$[P \xrightarrow{g} Q] \overset{\alpha}{\longrightarrow} [P' \xrightarrow{g'} Q'].$$

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "g"'] & P' \arrow[d, "g'"] \\ Q \arrow[r, "\alpha\otimes X"'] & Q' \end{tikzcd}$$

$$\begin{tikzcd} P \arrow[r] \arrow[d, "g"'] & P \arrow[r, "f_i"] \arrow[d] \arrow[dr, phantom, "(\dagger)"] & P' \arrow[d] \\ Q \arrow[r, "g_*"'] & P \arrow[r, "f_i"'] & P' \end{tikzcd}$$

Then either $\alpha \in R$, in which case the transition is silent – we have $P' \cong P$ so we can counter with the empty trace.

If $\alpha \notin R$ then we have the first diagram where $f$ is in $\mathcal{I}$. Since we are in a free category, any serialisation of $f$ must contain $\alpha$ as a unique action from $I$. Thus $f$ leads to a trace in $TS(\mathbf{C})$ which is an $\alpha$-transaction – ie we have a labelled transition $[P] \overset{\alpha}{\longrightarrow} [P']$ in $\mathrm{CTS}(\mathbf{C})$.

Now consider an arbitrary transition $[P] \overset{i}{\longrightarrow} [P']$. Let $P \xrightarrow{f_i} P'$ be the corresponding arrow in $\mathcal{I}$. Then in particular we have the square $(\dagger)$ in $h_*(\mathbf{C}, \mathcal{R})$, as illustrated in the second diagram. Let $g_*$ be the inverse to $g$ in $\mathbf{C}[\mathcal{R}^{-1}]$. Clearly $i$ is the only irreversible action in any serialisation (in $\mathrm{RTS}(\mathbf{C})$) of the combined second diagram, so we have a weak transition $[p \xrightarrow{g} q] \longrightarrow^* \overset{i}{\longrightarrow} [p' \to p']$. □

# 6 Conclusion

The main contribution of this paper is the development of the underlying abstract concepts which become apparent when designing "reversed" versions of known formalisms, such as Petri nets or CCS. In particular, we show that the problem reduces to developing the particular syntactic representations (such as the concrete syntactic representation of histories in RCCS) of the reversible history category $h_*(\mathbf{C}, \mathcal{R})$. The fact that the resulting computations capture the intended causal behaviour can then be seen as a consequence of our Theorem 4.4, which is formalism independent. We hope that this conceptual clarification will be of use to designers of reversible formalisms.

Another contribution is the observation that breaking up a computation into irreversible-reversible components naturally leads to a factorisation system on the category of computations. As part of future work, we plan to study such factorisation systems in more detail. We also plan to explore connections with previous work on factorisation systems in rewriting theory [7].

# References

[1] V. Danos and J. Krivine. Reversible communicating systems. In *Proceedings of Concur'04*, volume 3170 of *LNCS*, pages 292–307. Springer, 2004.

[2] V. Danos and J. Krivine. Transactions in RCCS. In *Proceedings of Concur'05*, volume 3653 of *LNCS*, pages 398–412. Springer, 2005.

[3] V. Danos, J. Krivine, and F. Tarissan. Self-assembling trees. In *SOS'06*, ENTCS. Elsevier, July 2006. To appear.

[4] P. J. Freyd and G. M. Kelly. Categories of continuous functors, I. *Journal of Pure and Applied Algebra*, 2:169–191, 1972.

[5] P. Gabriel and M. Zisman. *Calculus of fractions and homotopy theory*. Springer-Verlag, 1967.

[6] A. Joyal and R. Street. The geometry of tensor calculus, I. *Advances in Mathematics*, 88:55–112, 1991.

[7] P.-A. Melliès. A factorisation theorem in rewriting theory. In *Proceedings of CTCS '97*, volume 1290 of *LNCS*, pages 49–68. Springer, 1997.

[8] J. Meseguer and U. Montanari. Petri nets are monoids. *Information and computation*, 88:105–155, 1990.

[9] R. Milner. *A Calculus of Communicating Systems*, volume 92 of *LNCS*. Springer, 1980.

[10] I. Phillips and I. Ulidowski. Reversing algebraic process calculi. In *Proceedings of FoSSaCS '06*, volume 3921 of *LNCS*, pages 246–260. Springer, 2006.

[11] R. Street. Higher categories, strings, cubes and simplex equations. *Applied Categorical Structures*, 3:29–77, 1995.