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Electronic Notes in Theoretical Computer Science 283 (2012) 153–157

Electronic Notes in Theoretical Computer Science

www.elsevier.com/locate/entcs

Some Conjectures Concerning Complexity of PL subdivisions

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Abstract

The main purpose of this note is to formulate a few conjectures in the field of computational PL topology concerning the asymptotic number of iterations of barycentric subdivisions needed to embed one geometric simplicial complex into another one.

Keywords: Stellar subdivision, barycentric subdivision, combinatorial algebraic topology, PL topology, geometric simplicial complex.

The computational aspects of PL topology are important in applications. For example, one has intensively studied the question whether a given simplicial complex has a geometric realization in $\mathbb{R}^d$, for small d, see e.g., [6,7], and references therein. In comparison, studying computational aspects of barycentric subdivisions is relatively new. Because of this, we felt it might be a good idea to bring some of the central open questions to the readers attention. We begin this note by recalling one of the standard results of PL topology, where we refer to [2] for terminology of PL topology, and to [4] for terminology of combinatorial algebraic topology.

Theorem 0.1 ([2, Theorem I.2])
Given two finite geometric simplicial complexes K and L in $\mathbb{R}^d$, such that $|L| \subseteq |K|$, there exists a positive integer t and a subcomplex $\widetilde{L}$ of $\mathrm{bd}^t K$, such that $\widetilde{L}$ subdivides the complex L.

Here, as in the rest of the note, $\mathrm{bd}\, K$ denotes *generalized barycentric subdivision* of K, i.e., the one where barycenters can be chosen arbitrarily. Let $\eta(K, L)$ denote the minimal possible t for which there exists a subcomplex $\widetilde{L}$ of $\mathrm{bd}^t K$, such that $\widetilde{L}$ subdivides the complex L. Clearly, it is a function of the geometric simplicial

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1571-0661
doi:10.1016/j.entcs.2012.05.009

complexes K and L only, however it can be very hard to determine this number exactly, even in special cases. In [5] we started the study of the case $\dim L = 0$. This turned out to be more sophisticated than initially realized. While many open questions remain, we list some of the known facts.

Theorem 0.2 *Let Δ^d be a geometric d-simplex, $d \geq 1$, and let L be a collection of n points in int Δ^d in general position. There exists a point a, such that the interior of each of the d-simplices of sd (Δ^d, a) contains precisely $\left\lfloor \frac{n}{d+1} \right\rfloor$ points from L.*

Here, sd (Δ^d, a) denotes the *stellar subdivision* of Δ^d with apex in a. Furthermore, for a simplicial complex X, we let $\Sigma(X)$ denote the number of simplices of X.

Theorem 0.3 *Assume K and L are geometric simplicial complexes such that $\dim L = 0$, $\Sigma(L) = n$, $\dim K = d \geq 1$, and $|L| \subset |K|$. Then there exists an iterated barycentric subdivision $\mathrm{bd}^t K$, where $t = \left\lceil \log_{(d+1)!} n \right\rceil + 3$, such that L is a subcomplex of $\mathrm{bd}^t K$.*

The Theorems 0.2 and 0.3 were proved in [5]. While Theorem 0.2 was the main result of [5], it is the Theorem 0.3 which we would like to generalize here. For arbitrary $d \geq m \geq 0$ we set

$$\varphi_{d,m}(n) = \max_{(K,L)} \eta(K,L),$$

where the maximum is taken over all pairs of geometric simplicial complexes (K, L), such that $\Sigma(L) = n$, $\dim L = m$, $\dim K = d$, and $|L| \subseteq |K|$.

Conjecture 0.4 *Asymptotically, we have $\varphi_{d,m}(n) \in \Theta(\log n)$, where $d \geq m \geq 0$ are fixed.*

Here we use the Bachmann-Landau notation $\Theta(\_)$, see [3, Section 1.2.11], to describe the asymptotic behavior of functions.

It is easy to see that Theorem 0.3 implies $\varphi_{d,0}(n) \in \Theta(\log n)$, for all $d \geq 0$. The asymptotics of $\varphi_{d,m}(n)$ for $m > 0$ is currently only conjectured. However, a few observations can be made immediately, for example, we have $\varphi_{d,d}(n) = \varphi_{d,d-1}(n)$ for all $d \geq 1$. It is slightly more technical to verify the conjecture for $d = 2$, $m = 1$, which we now proceed to do.

Proof of Conjecture 0.4 for $d = 2$, $m = 1$. As a first step, we replace K with an iterated barycentric subdivision, such that all the vertices of L are embedded in the 0-dimensional skeleton of that subdivision. According to our previous results, we can do this using $\Theta(\log v)$ barycentric subdivisions, where v is the number of vertices of L. Now we construct a special barycentric subdivision of K. To start with, consider an arbitrary edge e of K. Let $v_1, \dots, v_m$ be the points of intersection of the interior of e with edges from L, listed in the order in which they appear along the edge. Now, pick the barycenter b_e of e as follows:

$$b_e = \begin{cases} v_{\frac{m+1}{2}}, & \text{if } m \text{ is odd;} \\ \text{any point between } v_{\frac{m}{2}} \text{ and } v_{\frac{m}{2}+1}, & \text{if } m \text{ is even, and } m \geq 2; \\ \text{any point in the interior of } e, & \text{if } m = 0. \end{cases}$$

Let σ be an arbitrary triangle of K, we describe now how to place an appropriate barycenter b_σ. Let L_σ denote the set of intervals which constitute the intersection of $int\,\sigma$ with the geometric realization of L. We set $\tilde{n} := |L_\sigma|$. After possible relabeling we are reduced to one of the three cases shown on Figure 1, where a, b, and c denote the numbers of elements of L_σ of each type.

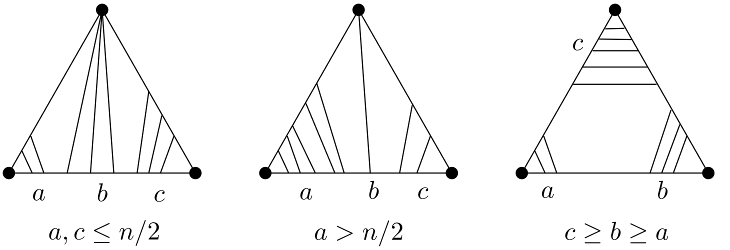

Fig. 1. Possible configurations of intervals in σ.

The corresponding placements of the barycenter b_σ are shown on Figure 2. There it is illustrated in each case which of the edge barycenters b_e is chosen. After that the barycenter b_σ is chosen in the direct vicinity of b_e, so that the edge (b_e, b_σ) does not intersect any of the edges from L_σ. If b_e itself is an endpoint of $L \in L_\sigma$, then b_σ is also chosen to lie on that L.

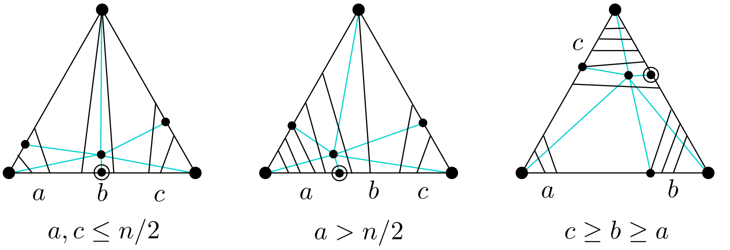

Fig. 2. Placements of the barycenter b_σ.

We invite the reader to verify that in each case, each of the six new triangles intersects at most $\tilde{n}/2$ of the intervals from L_σ. Iterating this type of subdivision will now yield the result. $\square$

There is an interesting special case of Conjecture 0.4 which we would like to single out in a separate conjecture.

Conjecture 0.5 *Let $K = \Delta^d$ be a d-dimensional simplex, $K \subset \mathbb{R}^d$. Assume $\mathcal{L}$ is a collection of m-dimensional subspaces $\mathcal{L} = \{L_1, \dots, L_s\}$, where $0 \leq m \leq d-1$,* *such that*

$$L_i \cap int\, K \neq \emptyset, \qquad \text{for all } i = 1, \dots, s;$$

$$L_i \cap L_j \cap int\, K = \emptyset, \quad \text{for all } i, j = 1, \dots, s, \ i \neq j,$$

in other words, the sets $L_1 \cap int\, K$, ..., $L_s \cap int\, K$ are non-empty disjoint subsets of K.

Then, there exists a barycentric subdivision of K, which we denote B, and a positive number $\varepsilon_{m,d} < 1$ ($\varepsilon_{m,d}$ does not depend on the specific complexes K and L, only on their dimensions), such that for every d-dimensional simplex σ of B, the following inequality holds

$$(1) \quad |\{i \in [s] \mid int\, \sigma \cap L_i \neq 0\}| \leq \varepsilon_{m,d} \cdot s.$$

The special case of Conjecture 0.5 when $m = 0$ was proved in [5]. The case $m = d - 1$ is settled in the next proposition.

Proposition 0.6 *Conjecture 0.5 is true for $m = d-1$ with the value $\varepsilon_{d-1,d} = \frac{d}{d+1}$.*

Proof. As mentioned earlier, we may assume $d \geq 2$. When $m = d-1$ the subspaces L_i are hyperplanes. We can associate to the pair $(K, \mathcal{L})$ a certain graph T as follows: the vertices of T are the regions into which hyperplanes from $\mathcal{L}$ divide K, and two vertices are connected by an edge if and only if the corresponding regions share a codimension 1 face. Clearly, the graph T is a tree, whose edges are in one-to-one correspondence with the elements of $\mathcal{L}$. The tree T has at most $d+1$ leaves, since each region corresponding to a leaf contains at least one vertex of K. In particular, the maximal valency of T is at most $d+1$. Before proceeding we need the following technical fact.

Claim. *Given a tree T with n edges and maximal valency m, there exists an edge e, such that both connected components of $T \setminus \{e\}$ have at most $\frac{m-1}{m}n$ edges.*

Proof of the claim. For an arbitrary $e \in E(T)$, let T_1 and T_2 denote the connected components of $T \setminus \{e\}$, and set $f(e) := \min(|E(T_1)|, |E(T_2)|)$, $g(e) := \max(|E(T_1)|, |E(T_2)|)$. Fix $e \in E(T)$ for which $g(e)$ achieves its maximum. Since $f(e) + g(e) = n - 1$, we have that $f(e)$ achieves its minimum, and we can assume without loss of generality that $|E(T_1)| \geq |E(T_2)|$. We denote $e = (v_1, v_2)$, where $v_i \in T_i$, for $i = 1, 2$. Assume furthermore that the valency of v_1 is $k+1$, with $k \geq 1$, denote the edges of T_1 adjacent to e by e_1, ..., e_k, and assume that removing e_1, ..., e_k from T_1 will produce trees with edge set cardinalities a_1, ..., a_k, see Figure 3.

We have $g(e) = k + a_1 + \ldots + a_k$, and $f(e) = |E(T_2)|$. By the choice of e, we have $a_i \leq f(e)$, for all $i = 1, \dots, k$. Indeed, removing the edge e_i instead of e would yield a subdivision into two trees with the edge set cardinalities a_1 and $f(e) + a_2 + \ldots + a_k + k$. Either $a_1 \leq f(e) + a_2 + \ldots + a_k + k$, in which case $a_1 \leq f(e)$ by the choice of e, or $a_1 > f(e) + a_2 + \ldots + a_k + k$, in which case the inequality $f(e) + a_2 + \ldots + a_k + k > f(e)$ yields a contradiction to the optimality of the choice of e. Thus, we have

$$g(e) = k + a_1 + \ldots + a_k \leq k + kf(e) = k + k(n - 1 - g(e)),$$

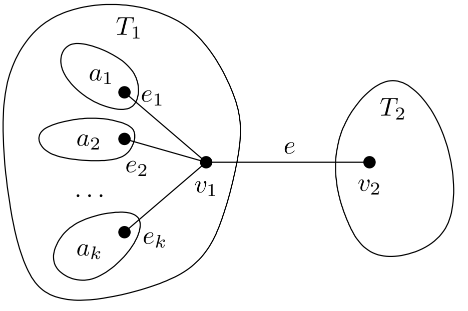

Fig. 3. The tree decomposition in the proof of claim in Proposition 0.6.

hence $(k+1)g(e) \leq nk$, which implies $g(e) \leq \frac{k}{k+1}n \leq \frac{m-1}{m}n$. $\square$

Applied to our situation, we find a hyperplane $L \in \mathcal{L}$, such that L divides the set $\mathcal{L}$ into two parts, each one of cardinality at most $\frac{d}{d+1}s$. On the other hand, $L \cap K$ can be embedded into the $(d-1)$-dimensional skeleton of b$d\,K$. To do this, for every face σ of K, such that $L \cap \sigma \neq \emptyset$, we pick the barycenter b_σ to lie in the intersection $L \cap \sigma$, and we pick arbitrary barycenters b_σ for those σ for which $\sigma \cap L = \emptyset$. In the resulting b$d\,K$ we have a barycentric subdivision of $L \cap K$ embedded in the $(d-1)$-dimensional skeleton. Each d-dimensional simplex σ of b$d\,K$ will lie on one of the sides of $L \cap K$, hence i$nt\,\sigma$ will intersect at most $\frac{d}{d+1}s$ elements of $\mathcal{L}$. $\square$

The next open case of Conjecture 0.5 is $m = 1$, $d = 3$. It would be interesting to settle this low-dimensional explicit case: one has s lines intersecting a tetrahedron and would like to find a barycentric subdivision of that tetrahedron such that every of the 24 obtained tetrahedra intersects at most a certain fixed fraction of these lines. [2]

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