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ORIGINAL ARTICLE

Improving the Cosine Consistency Index for the analytic hierarchy process for solving multi-criteria decision making problems

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Abstract Analytic Hierarchy Process (AHP) is one of the popular decision support systems for multi-criteria decision making problems. The AHP has different theories for prioritization, consistency evaluation and consistency improvement, a review of which is presented in this study before diving deep into the core contribution. Consistency evaluation is one of the key computations while using the AHP. This paper describes a method that can be employed to improve the consistency of the judgment matrix utilized by using the Cosine Consistency Index (CCI). The approach described uses a cosine maximization method to revise the entries in the judgment matrix on an iterative basis until the CCI is improved. The recommended method entails that it is possible to modify any judgment matrix to achieve CCI of desired level. Finally, the proposed algorithm is tested with numerical examples and improved CCI values are validated through paired sample *t*-test. The results of this study showed that the algorithm significantly improves CCI values with the inclusion of proposed approach.

1. Introduction

The Analytic Hierarchy Process (AHP) is one of the more popular decision-making techniques that are widely utilized to address Multi-Criteria Decision-Making (MCDM) problems. This method breaks down the problem into a hierarchy of sub-problems. Then from the elicited judgments from experts on the comparative performance or criticality of the sub-problems, priorities are computed. These priorities enable the decision making related to sorting, ranking or selecting the most suitable alternative in MCDM problems [1]. One of the biggest advantages of a AHP approach is that it helps decision

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makers to dissect a complex issue into its constituent parts in a manner that is more simplistic [2–6]. However, as a MCDM tool, it does have inherent disadvantages and the way in which criteria are aggregated is often criticized as potentially risking a loss of information, for example, in situations in which trade-offs between good and bad scores occur. Furthermore, AHP involves a large amount of pairwise comparisons [4], which could sometimes become tiring during judgment elicitation. Moreover, some of the studies adopt fuzzy set theory [7] and analytical network process [8,9] to offset the limitations in traditional AHP. Also various theories exist as to which decision making processes can effectively help a group of people to mutually agree on problems and opportunities. Techniques such as structuring, ordering, grading and evaluating have been comprehensively explored across a wide variety of studies relating to group decision making processes [10]. Previous research into AHP as a MCDM tool has indicated that it can be very effective when applied to a group decision problem because it allows the priorities of each participant to be accurately estimated [11–15] and subsequently improved through quantitative methods [16–18] before being aggregated into a set of preferences that reflect the requirements of each participant [19–24].

In order to ensure that AHP is implemented in an effective manner, it is important to ensure that the judgment matrix upon which it is based has a Cosine Consistency Index (CCI) that is approximately equal to 1. According to the literature [25], it is acceptable for a CCI to be above 0.90, but anything below 0.90 is unacceptable. However, while their insights are useful, they failed to extend how CCI can be improved. Constructing a judgment matrix that delivers an acceptable CCI is extremely challenging because it is very difficult to compare the various elements of the matrix, and the human capacity to do so is limited. Moreover, some recent studies on decision making in hierarchical collaborative production planning [26], knowledge discovery [27] and service-oriented enterprise architecture [28] have failed to address and statistically validate [29,30] consistency improvement in preferences. This becomes all the more critical for the experts in more complex problems and in the presence of incomplete or subjective information.

One method of improving the CCI of matrices that demonstrate inconsistency (CCI < 0.90) could be to return them to experts who have the ability to restructure them via a series of relevant judgments in a manner that ensures increased CCI. Although such an approach may yield reliable and accurate results, it is largely impractical because of both the longer time requirements of such an approach and the availability of experts for subsequent rounds of judgment elicitation. As such, there is a need to develop a method of improving the consistency of judgment matrices that demonstrate CCI < 0.90, so that the revised matrix achieves an acceptable consistency (CCI $\geqslant 0.90$). Once such a matrix has been developed, it would then be possible to derive the reasonable priority vector of the first matrix by applying the Cosine Maximization Method (CMM). The CMM calculates CCI value by calculating average similarity between priority vector and each column of AHP matrix with an objective of maximizing the CCI value. Previous research have examined this type of approach [15,31–35] and highlighted the need for methods of consistency improvement through convergence focused iterative approaches.

This study proposes an approach that can be used to improve the cosine consistency of a given judgment matrix, so that more priorities can be evaluated for prioritization and empirical findings from data. The research will describe the use of an algorithm on a matrix that exhibits inconsistent CCI to develop a consistent judgment matrix that yields acceptable CCI (CCI $\geqslant 0.90$). A numerical example that demonstrates the effectiveness and accuracy of the proposed algorithm will also be presented followed by validation through *t*-test.

2. Theory and methods

Before moving on to the actual contribution in the current study, it is important to review the background of developments in the methods of AHP. So we first evaluate the different methods within AHP and subsequently the different methods for consistency evaluation within AHP. Subsequently we narrow down our discussion to the Cosine Maximization (CM) and CCI approach developed by Kou and Lin [25] and how the current study extends it.

2.1. Prioritization methods using AHP

The prioritization method provides a process by which the reliable priority vector can be obtained from expert judgments. In recent years, a number of prioritization methods have emerged. However, the performance and suitability of these decision support methods have met with a lot of academic controversy and often it has been proposed to try out hybrid approaches for improving results, from the classic literature to recent studies [25,36,37]. A review of 20 popular prioritization methods within AHP has been summarized in table provided in the supplementary materials (i.e. Eigen vector method [38], weighted least squares method [39], additive normalization method [15], least squares method [40], gradient Eigen weight and least distance method [41], geometric mean method [42], geometric least squares method [43], logarithmic least squares method [44], goal programming method [45], logarithmic goal programming method [46], fuzzy preference programming method [47], unusual and false observations [48], singular value decomposition method [49], interval priority method [50], linear programming method [51], data envelopment analysis method [52], correlation coefficient maximization [53], Bayesian prioritization procedure [54], weight estimation with evolutionary strategy [55], and heuristics and re-evaluation based method [56]). Since there are so many approaches for prioritization under different constraints and contexts, Srdjevic [57] argued that a better priority vector can be derived when various prioritization methods are combined. The availability of so many methods within AHP also highlights the difference of outcome in comparable methods, due to which there is a need to explore methods on improving consistency of priorities.

2.2. Methods for consistency evaluation using AHP

In view of such focused studies on challenges of priority estimation when information and judgments may be imprecise and less clear to the experts, the need for measuring consistency of such contexts was established. Several researchers have identified methods of measuring the extent to which PCM is consistent. Seven common methods and indexes for consistency evaluation have been reviewed in table provided in the supplementary materials (i.e. logarithmic-least squares [58], geometric consistency index [11], random index method [59], the induced matrix method [60], statistical consistency test [61,62], consistency ratio measure [63] and harmonic consistency index [64]). Vargas [65] employed a statistical approach to develop a statistical methodology for the consistency test. Consistency index in previous theories has been used as a reliable source to validate the final solution and to interpret weights for each expert in consensus models [66,67]. Further, group decision making problems can be effectively described by multiplicative preference relations using consensus degree [68]. Some of the previous studies have tried to assess the vulnerability pairwise comparison matrix using dynamic changes to criteria importance by focusing on preference at one or more times [69]. In this study, we focus on the Cosine Maximization Method, which is one of the emerging approaches. The reason for focusing on this method is because it provides high flexibility and efficiency based on multiple performance criteria such as Euclidean distance and minimum violation for improving the consistency of a judgment matrix [25]. Further it develops the same ordinal stability for prioritization as multiple other methods such as Eigen vector based methods and additive normalization methods, while it performs better than weighted least square methods and logarithmic least square methods. Further in terms of Euclidean distance based error measures, CMM has the lowest error reported as compared to other methods such as Eigen vector based methods, additive normalization methods performs, weighted least square methods and logarithmic least square methods. This is why, this study focuses on the CMM and attempts to address some of its existing limitations as discussed in forthcoming sections.

2.3. Methods for consistency improvement in AHP

A range of approaches associated with consistency improvements was investigated by many authors, such as Peters and Zelewski [70] and Ishizaka and Lusti [71]. Moreover, one of the recent studies proposed a set of properties that describe a family of functions for representing inconsistency indices [72]. Seventeen common consistency improvement methods have been reviewed in table provided in the supplementary material (i.e. Eigen value improvement [73], convergent iterative algorithm [33], least square method [74], triplet selection [75], heuristic algorithm [70], controlled error consistent matrix development [71], weak transitivity [76], Gower plot and linear programming [77], auto generate consistent matrix [16], controlled linguistic preference deviation [78], adaptive AHP method [79], missing value multi-layer perceptron [80], orthogonal projection and linearization [81], integer programming [82], consistency and consensus improvement [67], consistency optimization [83] and ordinal consistency improvement methods [84]). Among such approaches, a CMM provides an efficient and valid means of identifying a priority vector in the AHP. CMM offers a number of advantages over other prioritization methods: it enables derivation of a consistency index for the PCM, removes the need for statistical modeling and facilitates the calculation and interpretation of the CCI. Over the past few decades a significant body of academic work has explored many facets of pairwise comparison methodology, but it is only in recent years that studies by Koczkodaj and Szwarc [85] and Brunelli and Fedrizzi [86] have begun to address the key issue of calculating the most viable inconsistency indices. Within the AHP, the most effective methodology for identification of the priority vector has long been debated, and seminal works were published by Cook and Kress [87], Fichtner [88] and Barzilai [89]. However, Ishizaka and Lusti [90] demonstrated by statistical analysis that, in the majority of cases, the variations between the different methods were not statistically significant, and did not materially affect the outcome of the AHP. In addition, a recent study has found that properties 3 and 4 from a proposed list of 6 did not correspond to the Cosine Consistency Index [72].

2.4. Cosine Maximization Method (CMM)

Before moving to the technical use of CMM for the purposes of deriving priority vectors, a fundamental understanding of the associated definitions and theorems is required. These are elaborated as follows:

Definition 1. Matrix $A = (a_{ij})_{n\times n}$ is positive reciprocal if $a_{ij} > 0$, $a_{ii} = 1$ and $a_{ij} = 1/a_{ji}$ for all $i,j \in \{1,2,\ldots,n\}$.

Definition 2. A positive matrix $A = (a_{ij})_{n\times n}$ is perfectly consistent if $a_{ij} = a_{ik}a_{kj}$ for all $i,j,k \in \{1,2,\ldots,n\}$.

Definition 3. The similarity measure between two vectors, t_i and t_j, $SM(t_i,t_j)$, in a n dimensional vector space, V, is a mapping from $V \times V$ to the range $[0,1]$. Thus $SM(t_i,t_j) \in (0,1)$.

Property 1. *The similarity measure in Definition 3 exhibits the following well-known characteristics*:

(1) $\forall t_i \in V$, $SM(t_i,t_i) = 1$;
(2) $\forall t_i,\ t_j \in V$, $SM(t_i,t_j) = 0$ *if* t_i *and* t_j *are dissimilar*;
(3) $\forall t_i,\ t_j,\ t_k \in V$,$SM(t_i,t_j) < SM(t_i,t_k)$ *if* t_i *is more like* t_k *than it is like* t_j.

The objective of the use of the similarity measure was to produce similarity mapping that identifies more similar vectors that have a higher similarity value. Further, the vectors of R^n *are considered column vectors.*

Theorem 1. *If two vectors were* $t_i = (t_{i1},t_{i2},\ldots,t_{in})^T$ *and* $t_j = (t_{j1},t_{j2},\ldots,t_{jn})^T$, *the cosine similarity measure between two vectors* t_i *and* t_j *would be as follows* [25]:

$$\mathrm{CSM}(t_i,t_j) = \left(\sum_{k=1}^{n} t_{ik}t_{jk}\right) \Big/ \left(\sqrt{\sum\nolimits_{k=1}^{n} t_{ik}^2}\sqrt{\sum\nolimits_{k=1}^{n} t_{jk}^2}\right)$$

where $t_i \neq t_j \neq 0$

Several common similarity measures are currently in wide use such as, Dice, Jaccard, overlap and cosine similarity measures [91].

A PCM will result in a set of priority vectors, and these can be used to produce a similarity measure. The hierarchy that results from the use of AHP represents the complex decision problem. Within this, the cosine similarity measure represents the similarity between the priority vector and each column vector of the PCM. This type of measurement system has been applied to both information retrieval [92,93] and AHP [94] models.

The cosine similarity of Theorem 1 can be utilized to derive a reliable priority vector from a given PCM. Let $A = (a_{ij})_{n\times n}$ as a positive reciprocal PCM and $w = (\omega_1, \omega_2, \ldots, \omega_n)^T$ as a weight vector with $\sum_{i=1}^{n}\omega_i = 1$ and $\omega_i \geqslant 0, (i = 1, 2, \ldots, n)$ be a priority vector derived from A through the application of the prioritization method.

If A is perfectly consistent [15]:

$$a_{ij} = \omega_i/\omega_j, \quad i,j \in \{1, 2, \ldots, n\} \tag{1}$$

From (1), A can be precisely characterized by the following:

$$A = \begin{bmatrix} \omega_1/\omega_1 & \omega_1/\omega_2 & \cdots & \omega_1/\omega_n \\ \omega_2/\omega_1 & \omega_2/\omega_2 & \cdots & \omega_2/\omega_n \\ \vdots & \vdots & \ldots & \vdots \\ \omega_n/\omega_1 & \omega_n/\omega_2 & \cdots & \omega_n/\omega_n \end{bmatrix} \tag{2}$$

According to (2), A consists of the following n column vectors:

$$(\omega_1, \omega_2, \ldots, \omega_n)^T/\omega_i, \quad i = 1, 2, \ldots, n \tag{3}$$

Let C_j be the cosine similarity measure between the priority vector w and the jth column vector a_j of A, where $w = (\omega_1, \omega_2, \ldots, \omega_n)^T$ and $a_j = (a_{1i}, a_{2j}, \ldots, a_{nj})^T$.

The application of Theorem 1 results in the following:

$$C_j = \mathrm{CSM}(\omega, a_j) = \left(\sum_{k=1}^{n}\omega_k a_{kj}\right) \Big/ \left(\sqrt{\sum_{k=1}^{n}\omega_k^2}\sqrt{\sum_{k=1}^{n}a_{kj}^2}\right)$$

$$j = 1, 2, \ldots, n \tag{4}$$

Since $a_{ij} = \omega_i/\omega_j$, $i,j \in \{1, 2, \ldots, n\}$, we have

$$C_j = \left(\sum_{k=1}^{n}\omega_k^2/\omega_j\right) \Big/ \left(\sqrt{\sum_{k=1}^{n}\omega_k^2}\sqrt{\sum_{k=1}^{n}(\omega_k/\omega_j)^2}\right) = 1$$

$$j = 1, 2, \ldots, n \tag{5}$$

As such, it is only in the event that A is perfectly consistent that it is possible for the cosine similarity measure between the derived priority vector and each column vector of A to be equal to 1. If this is not the case,

$$0 \leqslant C_j < 1 \tag{6}$$

This means that the derived priority vector and each column vector of A need to be equal to 1 as much as possible for the priority vector to be reliable. The optimization model can be represented via the following equations:

$$\begin{aligned} \mathit{Maximize}\ C &= \sum_{j=1}^{n} C_j \\ &= \sum_{j=1}^{n}\sum_{i=1}^{n}(\omega_i a_{ij}) \Big/ \left(\sqrt{\sum_{k=1}^{n}\omega_k^2}\sqrt{\sum_{k=1}^{n}a_{kj}^2}\right) \end{aligned}$$

Subject to

$$\begin{cases} \sum_{i=1}^{n}\omega_i = 1, \\ \omega_i \geqslant 0, \quad i = 1, 2, \ldots, n \end{cases} \tag{7}$$

We set

$$\hat{\omega}_i = \omega_i \Big/ \sqrt{\sum_{k=1}^{n}\omega_k^2} \geqslant 0, \quad i = 1, 2, \ldots, n \tag{8}$$

and

$$b_{ij} = a_{ij} \Big/ \sqrt{\sum_{k=1}^{n}a_{kj}^2} > 0, \quad i,j = 1, 2, \ldots, n \tag{9}$$

As such, we then have:

$$\sum_{i=1}^{n}\hat{\omega}_i^2 = 1 \tag{10}$$

and

$$\sum_{i=1}^{n}b_{ij}^2 = 1 \tag{11}$$

Therefore, this optimization model (7) can be equally developed into a further optimization model as follows:

$$\mathit{Maximize}\, C = \sum_{j=1}^{n}C_j = \sum_{j=1}^{n}\sum_{i=1}^{n}(b_{ij}\hat{\omega}_i) = \sum_{i=1}^{n}(\sum_{j=1}^{n}b_{ij})\hat{\omega}_i)$$

Subject to

$$\begin{cases} \sum_{i=1}^{n}\hat{\omega}_i^2 = 1, \\ \hat{\omega}_i \geqslant 0, \quad i = 1, 2, \ldots, n \end{cases} \tag{12}$$

In terms of the optimization model (12), the following theorems [25] are of interest:

Theorem 2. *Let $\hat{w}^* = (\hat{\omega}_1^*, \hat{\omega}_2^*, \ldots, \hat{\omega}_n^*)^T$ be the optimal solution to optimization model (12) and C^* be the optimal objective function value of it [25]. Then,*

$$\hat{\omega}_i^* = \sum_{j=1}^{n}b_{ij} \Big/ \sqrt{\sum_{k=1}^{n}\left(\sum_{j=1}^{n}b_{kj}\right)^2}, \quad i = 1, 2, \ldots, n \text{ and}$$

$$C^* = \sqrt{\sum_{i=1}^{n}\left(\sum_{j=1}^{n}b_{ij}\right)^2}$$

Theorem 3. *Let PCM $A = (a_{ij})_{n\times n}$ be perfectly consistent, the CMmethod can precisely derive the optimal objective function value $C^* = n$ and the priorities $\omega_j^* = 1/\sum_{i=1}^{n}a_{ij}(j = 1, 2, \ldots, n)$* [25].

2.5. Cosine Consistency Index

The consistency of PCM is an important issue in the application of AHP to derive a priority vector. In one piece of research, Saaty [38] developed the use of a CI that was correlated with the use of the eigenvector method (EV) and this was represented as follows:

$$\mathrm{CI} = (\lambda - n)/(n - 1) \tag{13}$$

where n is the dimension of the PCM, λ is the principal eigenvalue of the PCM, the Perron root [95] and approximate priority vector of A. According to this approach, a PCM needs to be perfectly consistent for $CI = 0$. However, while a large number of different methods and approaches have been presented in the existing literature, there is a lack of consensus on their effectiveness and reliability [32,96–98]. As such, there is a distinct requirement to produce a new consistency index related to the CM method that measures the inconsistency level of a PCM in a standard and reliable manner.

According to the rules presented in Theorem 3, bearing in mind the fact that C^* is the optimal objective function value of the optimization model (12), for PCM to be perfectly consistent, we require the following:

$$C^* = n \tag{14}$$

Otherwise,

$$0 < C^* < n \tag{15}$$

The influence of the size of a PCM can be eliminated by dividing the objective function value C^* by n. This results in C^*/n, which is the CCI of the PCM and takes on values in the interval $(0, 1]$. The following emerges:

$$\mathrm{CCI} = C^*/n \tag{16}$$

In the event the PCM is perfectly consistent:

$$\mathrm{CCI} = 1 \tag{17}$$

Otherwise

$$0 < \mathrm{CCI} < 1 \tag{18}$$

and this condition entails that the PCM demonstrates relative consistency.

This study has not addressed methods by which CCI thresholds can be identified, nor has it examined the relationship between the consistency of CCI and PCM. While there is no benchmark cutoff measure for CCI, given its implications for PCM, it is pertinent to expect a CCI of at least 90% [25].

Through considering the practical application of CM, it becomes apparent that this method does offer some distinct advantages over prioritization approaches. It is easy to compute, provides a consistent measurement method and is unique. However, the method will only be effective if it is applied to a complete and precise PCM that offers the required

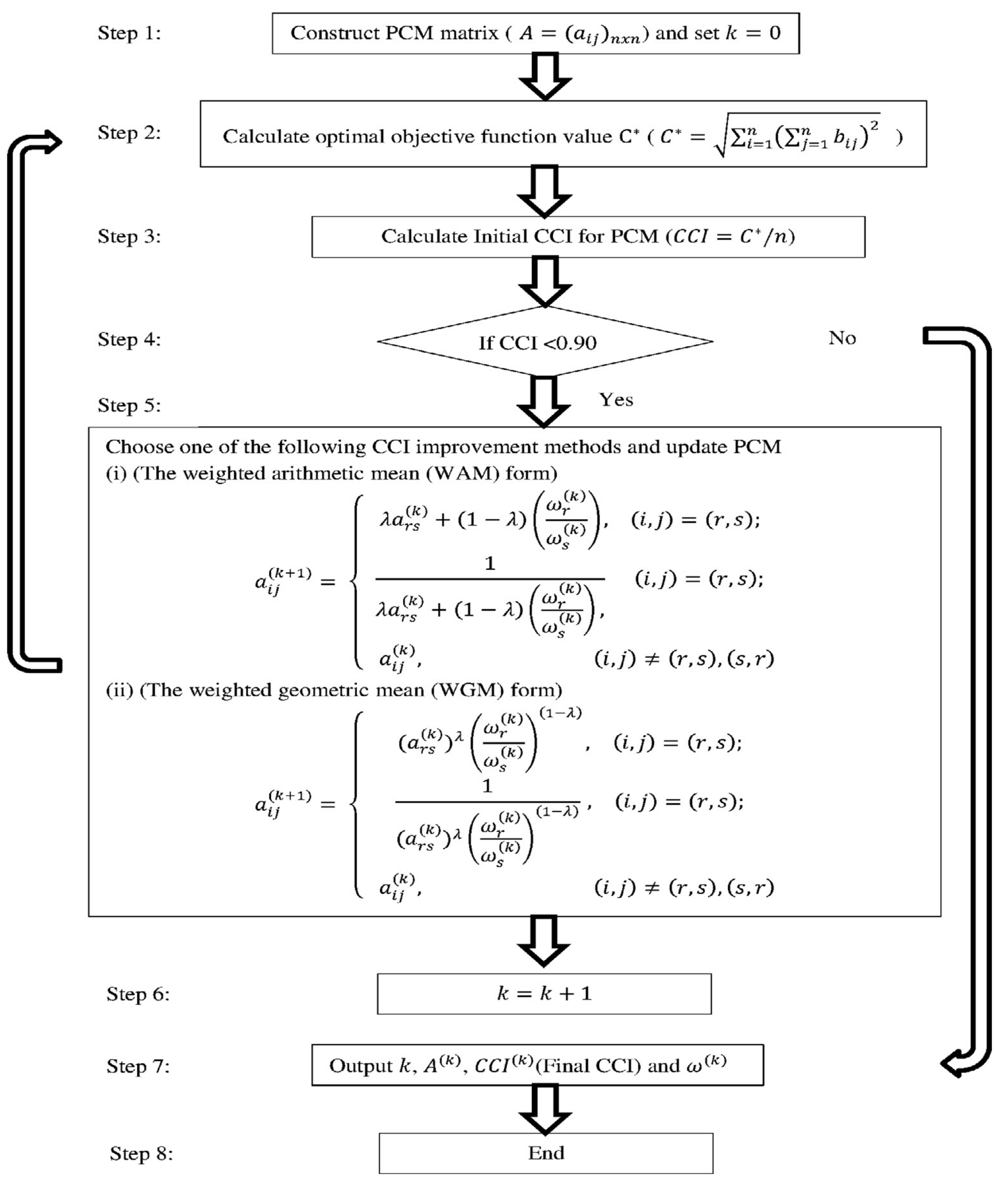

Figure 1 Diagrammatic representation of the algorithm.

reliability and legitimacy and if the CCI level is related to each decision-maker.

3. Improvements to the Cosine Consistency Index approach

Let $N = \{1, 2, \ldots, n\}$. Recall that a judgment matrix $A = (a_{ij})$ is an $n \times n$ matrix, all of whose entries are positive such that $a_{ji} = 1/a_{ij}$, for all $i, j \in N$, especially $a_{ii} = 1$, $i \in N$. The judgment matrix always represents a positive reciprocal matrix. An $n \times n$ judgment matrix is consistent if $a_{ij} = a_{ik}a_{kj}$, for all $i, j, k \in N$

Lemma 2.1. *If an $n \times n$ judgment matrix $A = (a_{ij})$ is a consistent matrix, and $w = (\omega_1, \omega_2, \ldots \omega_n)^T$ is its principal right eigenvector, then $a_{ij} = (\omega_i/\omega_j)$, for all $i, j \in N$. Let $A = (a_{ij})$ be an $n \times n$ matrix, and $w = (\omega_1, \omega_2, \ldots \omega_n)^T$ be the principalright eigenvector of A. From this lemma, we know that if A is a consistent matrix, then $a_{ij} = (\omega_i/\omega_j)$, for all $i, j \in N$, namely,*

$$a_{ij}(\omega_j/\omega_i) = 1, \quad i, j \in N \tag{19}$$

However, this approach does not take into consideration the fact that the people's perceptions and the decisions they make are likely to vary in response to their psychological states and the information to which they have access. As such, Eq.(19) does not hold. Hence, we can take the comparison matrix A as a perturbed matrix of the consistent matrix $W = \omega_i/\omega_j$ namely, set

$$a_{ij}(\omega_j/\omega_i) = \varepsilon_{ij} \quad i, j \in N \tag{20}$$

where ε_{ij} is perturbation variable, $\varepsilon_{ij} > 0$, and $\varepsilon_{ji} = 1/\varepsilon_{ij}$.

Eq. (20) can be expressed as $\varepsilon_{ij} = a_{ij}(\omega_j/\omega_i) i, j \in N$ in this case, we set $\varepsilon_{rs} = \max_{i,j}\{\varepsilon_{ij}\} = \max_{i,j}\{a_{ij}(\omega_j/\omega_i)\}$ and thus, a_{rs} related to ε_{rs} is an entry that has the largest deviation in matrix A. Judgment matrix A demonstrates an unacceptable CCI ($CCI < 0.90$) and it is natural that any attempts to improve it will involve first attempting to modify the entry a_{rs}. It is important that the corresponding entry a_{sr} is also modified in order to ensure that a positive reciprocal matrix is produced. The following algorithm (illustrated in Fig. 1) can be employed to modify the judgment matrices.

Algorithm

For $n \times n$ judgment matrix $A = (a_{ij})$, let k represent the k times of the iteration, and $\lambda \in (0, 1)$. The following represents the approximation method:

Step 1 Let $A^{(0)} = (a_{ij}^{(0)}) = (a_{ij})$ and $k = 0$;

Step 2 Calculate the optimal objective function value C^* (from Theorem 2)

$$Maximize\ C = \sum_{j=1}^{n} C_j = \sum_{j=1}^{n}\sum_{i=1}^{n}(\omega_i a_{ij}) \Big/ \left(\sqrt{\sum\nolimits_{k=1}^{n}\omega_k^2}\sqrt{\sum\nolimits_{k=1}^{n}a_{kj}^2}\right)$$

$$\text{Subject to } \begin{cases} \sum_{i=1}^{n}\omega_i = 1, \\ \omega_i \geqslant 0, i = 1, 2, \ldots, n \end{cases};$$

Step 3 Use Eq. (16) to calculate the CCI;

Step 4 If the CCI is inconsistent; i.e., it is less than 0.90, continue to the next step, otherwise, go to Step 7;

Step 5 Determine the numbers r and s, such that $\varepsilon_{rs} = \max_{i,j}\{a_{ij}^{(k)}(\omega_j^{(k)}/\omega_i^{(k)})\}$, and let $A^{(k+1)} = (a_{ij}^{(k+1)})$, where $a_{ij}^{(k+1)}$. The following formulas can be used:

(i) (The WAM form)

$$a_{ij}^{(k+1)} = \begin{cases} \lambda a_{rs}^{(k)} + (1-\lambda)\left(\frac{\omega_r^{(k)}}{\omega_s^{(k)}}\right), & (i,j) = (r,s); \\ \frac{1}{\lambda a_{rs}^{(k)} + (1-\lambda)\left(\frac{\omega_r^{(k)}}{\omega_s^{(k)}}\right)}, & (i,j) = (r,s); \\ a_{ij}^{(k)}, & (i,j) \neq (r,s), (s,r) \end{cases}$$

(ii) (The WGM form)

$$a_{ij}^{(k+1)} = \begin{cases} (a_{rs}^{(k)})^{\lambda}\left(\frac{\omega_r^{(k)}}{\omega_s^{(k)}}\right)^{(1-\lambda)}, & (i,j) = (r,s); \\ \frac{1}{(a_{rs}^{(k)})^{\lambda}\left(\frac{\omega_r^{(k)}}{\omega_s^{(k)}}\right)^{(1-\lambda)}}, & (i,j) = (r,s); \\ a_{ij}^{(k)}, & (i,j) \neq (r,s), (s,r) \end{cases}$$

Step 6 Let $= k + 1$, and return to step 2;

Step 7 Output k, $A^{(k)}$, $CCI^{(k)}$ and $\omega^{(k)}$, then $A^{(k)}$ is the modified judgment matrix and is the priority vector $\omega^{(k)}$.

Step 8 End

In the next step, this approach of improving the CCI needed empirical validation with real datasets. The validation with actual data is elaborated in the subsequent section.

4. Method validation

This section will present the application of the approach described above within two numerical examples in order to practically demonstrate the recommended approach and highlight the advantages of the CCI improvement approach. The improvements in the CCI values through the application of the WAM and WGM form are respectively shown in Tables 1 and 4. Further, Tables 2 and 5 respectively highlight the average number of iterations required for WAM and geometric mean form to achieve CCI >= 0.90 for 20 different datasets. These datasets are PCMs that were collected as primary responses for prioritization among alternatives for the problem, namely information search channel selection [99]. The users were asked to select preference between the pair for seven different criteria that influence consumers' search for information on Internet.

(1) The WAM form: First, it is necessary to construct a PCM

Table 1 The improvement in the CCI values through the application of the WAM form with $\lambda = 0.5$.

Iteration (k)	0	10	20	30	40	47
CCI value	0.7282	0.7713	0.8057	0.8691	0.8873	0.9022

$$\begin{pmatrix} 1.000 & 5.000 & 0.143 & 5.000 & 7.000 & 0.200 & 0.143 \\ 0.200 & 1.000 & 0.143 & 5.000 & 3.000 & 0.111 & 3.000 \\ 7.000 & 7.000 & 1.000 & 9.000 & 0.111 & 7.000 & 0.143 \\ 0.200 & 0.200 & 0.111 & 1.000 & 9.000 & 7.000 & 0.111 \\ 0.143 & 0.333 & 9.000 & 0.111 & 1.000 & 9.000 & 0.200 \\ 5.000 & 9.000 & 0.143 & 0.143 & 0.111 & 1.000 & 0.111 \\ 7.000 & 0.333 & 7.000 & 9.000 & 5.000 & 9.000 & 1.000 \end{pmatrix}$$

Subsequently, the optimal objective function value C^* should be calculated by following step 2 of the algorithm. CCI should then be calculated using Eq. (16).

$$C^* = 5.0974, \mathrm{CCI} = 0.7282 \text{ and}$$

$$\omega = (0.1138, 0.1287, 0.1871, 0.1007, 0.1204, 0.1020, 0.2472)^T$$

If CCI < 0.90, the error matrix should be calculated and the element that needs to be modified in order to improve the CCI identified. Each of these steps should be repeated until CCI $\geqslant$ 0.90.

The final transformed matrix is as follows with $C^* = 6.3154$, $\mathrm{CCI} = 0.9022$, $k = 47$ and priority vector $\omega^{(k)} = (0.1466, 0.1372, 0.3727, 0.0507, 0.0700, 0.0210, 0.2019)^T$

$$\begin{pmatrix} 1.000 & 5.000 & 0.143 & 5.000 & 7.000 & 3.167 & 0.175 \\ 0.200 & 1.000 & 0.143 & 5.000 & 3.000 & 7.221 & 3.000 \\ 7.000 & 7.000 & 1.000 & 9.000 & 7.843 & 7.000 & 4.453 \\ 0.200 & 0.200 & 0.111 & 1.000 & 0.522 & 7.000 & 0.111 \\ 0.143 & 0.333 & 0.128 & 1.922 & 1.000 & 9.000 & 0.200 \\ 0.316 & 0.139 & 0.143 & 0.143 & 0.111 & 1.000 & 0.111 \\ 5.729 & 0.333 & 0.225 & 9.000 & 5.000 & 9.000 & 1.000 \end{pmatrix}$$

The variables FinalCCI and InitialCCI are the paired variables with a sample size of 20. The improvement in CCI values can be validated by paired sample t-test with following hypothesis:

H01: There is no significant improvement in CCI with the WAM approach.
Ha1: There is a significant improvement in CCI with the WAM approach.

The summary statistics such as mean, standard deviation, and standard error along with their confidence limits of difference for paired variables are displayed in Table 3. The test is significant (t = 26.415, p = 0.000), indicating that there is a significant improvement in CCI with the WAM approach.

Further, one of the assumptions of paired t-test is that the difference between paired observations is assumed to be normally distributed. The authors have used Q–Q plot of difference between FinalCCI and InitialCCI values as a tool to verify the assumption and it can be observed from Fig. 2 that Q–Q plot of CCI improvement values shows no obvious deviations from normality for WAM approach.

(2) The WGM form

Construct a PCM

$$\begin{pmatrix} 1.000 & 0.143 & 5.000 & 0111 & 0.200 & 0.143 & 5.000 \\ 7.000 & 1.000 & 5.000 & 0.143 & 5.000 & 0.200 & 7.000 \\ 0.200 & 0.200 & 1.000 & 7.000 & 0.143 & 5.000 & 5.000 \\ 9.000 & 7.000 & 0.143 & 1.000 & 5.000 & 5.000 & 5.000 \\ 5.000 & 0.200 & 7.000 & 0.200 & 1.000 & 5.000 & 0.111 \\ 7.000 & 5.000 & 0.200 & 0.200 & 0.200 & 1.000 & 7.000 \\ 0.200 & 0.143 & 0.200 & 0.200 & 9.000 & 0.143 & 1.000 \end{pmatrix}$$

First, the optimal objective function value C^* should be calculated by following step 2 of the algorithm. CCI should then be calculated using equation (16).

$$C^* = 5.0921, \quad \mathrm{CCI} = 0.7274 \text{ and}$$

$$\omega = (0.0799, 0.1664, 0.1646, 0.2346, 0.1393, 0.1399, 0.0751)^T$$

If CCI < 0.90, the error matrix should be calculated and the element needs to be modified in order to improve the CCI identified. Each of these steps should be repeated until CCI $\geqslant$ 0.90.

The final transformed matrix is as follows with $C^* = 6.4134$, $\mathrm{CCI} = 0.9162$, $k = 52$ and priority vector $\omega^{(k)} = (0.0318, 0.2816, 0.2214, 0.2665, 0.0972, 0.0767, 0.0249)^T$

$$\begin{pmatrix} 1.000 & 0.143 & 0.145 & 0111 & 0.200 & 0.474 & 0.850 \\ 7.000 & 1.000 & 3.241 & 0.501 & 5.000 & 6.185 & 7.000 \\ 6.885 & 0.309 & 1.000 & 1.716 & 3.723 & 5.000 & 5.000 \\ 9.000 & 1.995 & 0.583 & 1.000 & 5.000 & 5.000 & 5.000 \\ 5.000 & 0.200 & 0.269 & 0.200 & 1.000 & 2.046 & 5.322 \\ 2.110 & 0.162 & 0.200 & 0.200 & 0.489 & 1.000 & 7.000 \\ 1.176 & 0.143 & 0.200 & 0.200 & 0.188 & 0.143 & 1.000 \end{pmatrix}$$

The variables FinalCCI and InitialCCI are the paired variables with a sample size of 20. Similarly, the improvement in CCI values can be validated by paired sample t-test with following hypothesis:

H02: There is no significant improvement in CCI with the WGM approach.
Ha2: There is a significant improvement in CCI with the WGM approach.

The summary statistics such as mean, standard deviation, and standard error along with their confidence limits of difference for paired variables are displayed in Table 6. The test is significant (t = 26.172, p = 0.000), indicating that there is a significant improvement in CCI with the WGM approach.

Table 2 The average number of iterations required to achieve CCI $\geqslant$ 0.90 for 20 different datasets using WAM.

S. No.	Initial CCI	Final CCI	CCI improvement	Average number of iterations
1	0.6761	0.9023	0.2262	49.34
2	0.7552	0.9095	0.1543	42.89
3	0.7844	0.9087	0.1243	37.67
4	0.7696	0.9021	0.1325	39.20
5	0.7188	0.9028	0.1840	47.56
6	0.7377	0.9031	0.1654	45.32
7	0.7374	0.9065	0.1691	45.13
8	0.7265	0.9097	0.1832	44.16
9	0.6938	0.9080	0.2142	48.20
10	0.7280	0.9038	0.1758	46.54
11	0.7243	0.9054	0.1811	45.24
12	0.6794	0.9041	0.2247	49.47
13	0.7288	0.9077	0.1789	46.93
14	0.6872	0.9011	0.2139	48.41
15	0.7305	0.9079	0.1774	45.63
16	0.6942	0.9088	0.2146	46.53
17	0.7121	0.9044	0.1923	46.57
18	0.6776	0.9083	0.2307	49.37
19	0.6856	0.9049	0.2193	48.45
20	0.6672	0.9049	0.2377	49.80

Table 3 Paired samples test.

		Paired differences					t	df	Sig. (2-tailed)
		Mean	Std. deviation	Std. error mean	95% Confidence interval of the difference				
					Lower	Upper			
Pair 1	FinalCCI–InitialCCI	.1899800	.0321643	.0071922	.1749266	.2050334	26.415	19	.000

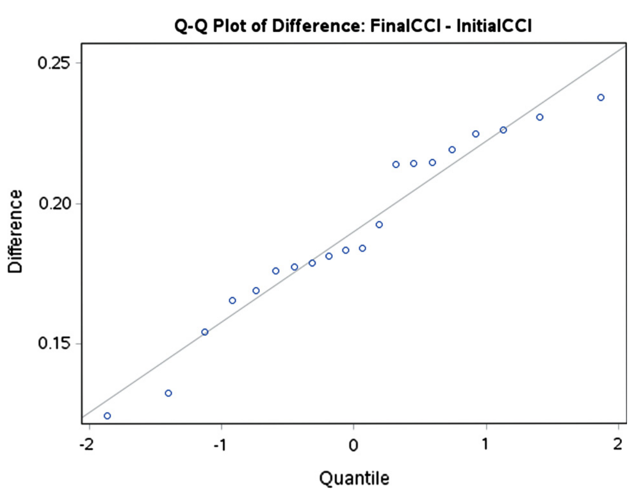

Figure 2 Q–Q plot to assess the normality assumption for paired t-test for WAM approach.

Further, it can be observed from Fig. 3 that Q–Q plot of CCI improvement values shows no obvious deviations from normality for WGM approach.

Thus the improvement in outcome is established based on the proposed method of improvement on the CCI. The average number of iterations in such improvement is also an indication of the low computational complexity of the method, for improving judgments to provide priorities with significantly higher consistencies.

5. Conclusion

This paper described the use of a corrective model that utilizes cosine maximization to produce a comparison matrix that exhibits consistent CCI. The CM technique is used to maximize the sum of the cosine angle between each column vector and derived priority vector of a PCM. An algorithm has been suggested for determining transformed PCM and the weight vector. The cosine maximization was employed to amend a pair of entries that exhibited maximum errors, thus ensuring that the resulting matrix maintained all major information that was present in the original matrix. Through applying either the WAM form or the WGM form detailed in Step 5 of the approach, it was possible to revise the matrix in an effective manner. As such, the approach recommended is viable and can be applied to inconsistent CCI ratings of <0.90 in order to create a positive reciprocal matrix that demonstrates CCI $\geqslant$ 0.90. Further, for given scenario it was possible to establish that average number of iterations required in achieving a CCI $\geqslant$ 0.90 for WAM form and the WGM form is almost similar. The algorithm converges to CCI = 1 but this study limits its improvement scope to near approximate value

Table 4 The improvement in the CCI values through the application of the WGM form with $\lambda = 0.5$.

Iteration (k)	0	10	20	30	40	52
CCI value	0.7274	0.7654	0.7992	0.8477	0.8826	0.9162

Table 5 The average number of iterations required to achieve CCI $\geqslant$ 0.90 for 20 different datasets using WGM.

S. No.	Initial CCI	Final CCI	CCI improvement	Average number of iterations
1	0.6761	0.9079	0.2318	52.62
2	0.7552	0.9138	0.1586	51.62
3	0.7844	0.9070	0.1226	46.38
4	0.7696	0.9013	0.1317	48.01
5	0.7188	0.9031	0.1843	51.34
6	0.7377	0.9031	0.1654	49.53
7	0.7374	0.9048	0.1674	53.14
8	0.7265	0.9093	0.1828	52.43
9	0.6938	0.9086	0.2148	51.29
10	0.7280	0.9093	0.1813	49.45
11	0.7243	0.9045	0.1802	49.72
12	0.6794	0.9086	0.2292	52.67
13	0.7288	0.9045	0.1757	50.82
14	0.6872	0.9091	0.2219	51.42
15	0.7305	0.9091	0.1786	50.39
16	0.6942	0.9048	0.2106	52.78
17	0.7121	0.9023	0.1902	52.49
18	0.6776	0.9039	0.2263	53.12
19	0.6856	0.9008	0.2152	53.93
20	0.6672	0.9036	0.2364	54.87

Table 6 Paired Samples Test.

		Paired differences					t	df	Sig. (2-tailed)
		Mean	Std. deviation	Std. error mean	95% Confidence interval of the difference				
					Lower	Upper			
Pair 1	FinalCCI–InitialCCI	.1902500	.0325086	.0072692	.1750355	.2054645	26.172	19	.000

of 0.90 to address the research gap of Kou and Lin [25]. Moreover, the number of iterations of WAM and WGM to achieve desired consistency level depends on context specific requirements and computational time and cost constraints. Finally, the algorithm was tested with numerical example and improved CCI values were validated through paired sample t-test. It can be concluded that algorithm significantly improved CCI values with the inclusion of proposed approach.

The study successfully carried out in the present research has considerable consequences for managers. Notably, it is possible to exploit the proposed soft computing technique as a crucial decision-making instrument for managers. This is advantageous with respect to the way it can optimize the matrix consistency to desired level. In the context of optimization, it is possible for managers to utilize the technique to select different alternatives without being subject to bias toward a certain alternative or criteria. In terms of applicability, a major challenge of using AHP for empirical research is getting ample number of consistent responses which may facilitate generalizability of results. This approach will ensure that more

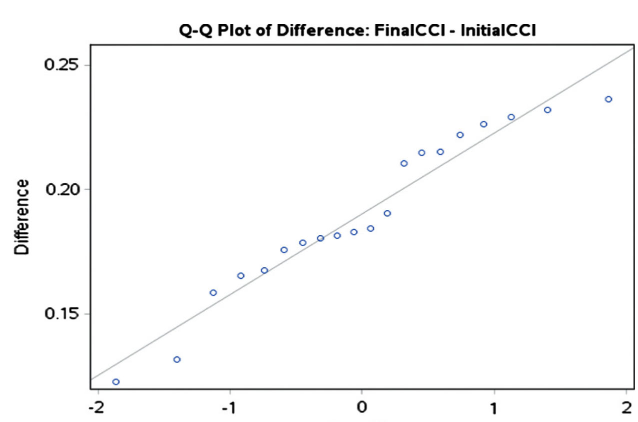

Figure 3 Q–Q plot to assess the normality assumption for paired t-test for WGM approach.

responses may be modified systematically, so that the consistency challenge may be addressed, and more priorities are used for empirical data collection and analysis.

The next steps of the research should involve the practical application of the proposed approach to real-life case studies. Through using these case studies, it will be possible to compare the effectiveness of the proposed methodology with similar existing approaches. This work can potentially be extended in the future to include the amalgamation of statistical measures that can precisely describe optimal consistency levels. Further, some future research can be in direction of improving the algorithm which reduces the number of iterations for reaching the optimal consistency index using cosine maximization method. Moreover, the purpose of study was to develop a CCI improvement method rather to check the significant deviation of new matrix priority vector from original matrix priority vector. However, future studies can focus on adding constraint that checks for significant deviation of new matrix from original matrix for achieving optimum consistency level. Some possible limitations for proposed method are that the CM for priority vector derivation method fails on imprecise and incomplete matrix. The scope of conversion of imprecise information and incomplete judgments from experts to priorities that satisfy all requirements surrounding consistency (or even consensus) has not been explored. Finally, threshold for CCI for achieving desirable value is yet to be derived using relationship between PCM consistency and CCI, and could be taken forward in future exploration of the method.

Appendix A. Supplementary material

Supplementary data associated with this article can be found, in the online version, at http://dx.doi.org/10.1016/j.aci.2016.05.001.

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