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Reduced differential transform method for solving $(1+n)-$ Dimensional Burgers' equation

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ABSTRACT

This paper discusses a recently developed semi-analytic technique so called the reduced differential transform method (RDTM) for solving the $(1+n)-$ dimensional Burgers' equation. The method considers the use of the appropriate initial or boundary conditions and finds the solution without any discretization, transformation, or restrictive assumptions. Four numerical examples are provided in order to validate the efficiency and reliability of the method and furthermore to compare its computational effectiveness with other analytical methods available in the literature.

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1. Introduction

Let us take the following $(1+n)-$ dimensional Burgers' equation

$$\frac{\partial u}{\partial t}=\alpha_1\frac{\partial^2 u}{\partial x_1^2}+\alpha_2\frac{\partial^2 u}{\partial x_2^2}+\alpha_3\frac{\partial^2 u}{\partial x_3^2}+\ldots\ldots+\alpha_n\frac{\partial^2 u}{\partial x_n^2}+\beta u\frac{\partial u}{\partial x_1}. \tag{1}$$

Subject to the initial conditions

$$u(x_1,x_2,x_3,\ldots,x_n,0)=u_0(x_1,x_2,x_3,\ldots,x_n), \tag{2}$$

where $\alpha_i, i=1,2,3,\ldots,n$, and β are constants.

Eq. (1) is used in the study of cellular automata, and interacting particle systems. Eq. (1) describes the flow pattern of the particle in a lattice fluid past an impenetrable obstacle [1,2]; it can be also used as a model to describe the water flow in soils.

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The one dimensional Burgers' equation is quite popular in wave theory, which has wide applications such as in gas dynamics [3], plasma physics etc. Due to its broad range of applications, various studies have been made to generalize it to higher dimension. Richard's equation (1) is a nonlinear PDE and as we know that obtaining the solutions of nonlinear PDEs are more difficult than those of linear differential equations. In most of cases, these nonlinear PDEs do not admit analytical solutions, so these equations should be solved by using special methods. In recent years, some researchers used new techniques for solving these types of problems [4–8]. In the past few decades, traditional integral transform methods such as the Fourier and the Laplace transforms have commonly been used to solve engineering problems. These methods transform differential equations into algebraic equations which are easier to deal with. However, these integral transform methods are more complex and difficult when applying to nonlinear problems. Recently, the $(1+n)-$ dimensional Burgers' equation has been solved by Srivastava and Awasthi [9] using Homotopy perturbation method (HPM), Adomian decomposition method (ADM) and Differential transform method (DTM). However we notice that these methods involve complex computations. In recent years, Keskin et al. introduced a reduced form of DTM as reduced DTM (so called RDTM) and they applied it to approximate some PDE [10] and factional PDEs [11]. More recently, Abazari and Ganji [12] extended RDTM to study the PDE with proportional delay. Futhermore, Gupta [13] used RDTM to fractional Benney–Lin equation, and Abazari and Abazari [14] applied the RDTM on generalized Hirota–Satsuma coupled KdV equation, while Srivastava et al. [15–18] used RDTM to solve the various problems arising in telecommunication systems and biological population model. The reduced differential transform recursive equations produce exactly all the Poisson series coefficients of solutions, whereas the differential transform recursive equations produce exactly all the Taylor series coefficients of solutions.

In this work, RDTM method is employed to solve the $(1+n)-$ dimensional Burgers' equation. Four numerical examples are carried out to validate and illustrate the proposed method. As an important observation, RDTM overcomes the demerit of complex calculation of classical DTM, capable of reducing the size of calculation.

2. Reduced differential transform method

Consider a function $u(x_1,x_2,\ldots,x_n,t)$ of $(n+1)\ -$ variables and assume that it can be represented as a product of $(n+1)$ single-variable function, i.e. $u(x_1,x_2,\ldots,x_n,t)=F_1(x_1)F_2(x_2)\ldots F_n(x_n)F_m(t)$. On the basis of the properties of the one-dimensional differential transform, the function $u(x_1,x_2,\ldots,x_n,t)$ can be represented as

$$u(x_1,x_2,\ldots,x_n,t)=\sum_{k_1=0}^{\infty}F_1(k_1)x_1^{k_1}\sum_{k_2=0}^{\infty}F_2(k_2)x_2^{k_2}\ldots\sum_{k_n=0}^{\infty}F_n(k_n)x_n^{k_n}\times\sum_{k_m=0}^{\infty}F_m(k_m)t^{k_m}=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\ldots\sum_{k_n=0}^{\infty}\sum_{k_m=0}^{\infty}U(k_1,k_2,\ldots,k_n,k_m)x_1^{k_1}x_2^{k_2}\ldots x_n^{k_n}t^{k_m}, \tag{3}$$

where $U(k_1,k_2,\ldots,k_n,k_m)=F_1(k_1)\times F_2(k_2)\times\ldots\times F_n(k_n)\times F_m(k_m)$ is called the spectrum of $u(x_1,x_2,\ldots,x_n,t)$.

Let R_D denotes the reduced differential transform operator and R_D^{-1} the inverse reduced differential transform operator. The basic definition and operation of the RDTM method is described below. The basic definitions and operations of reduced differential transform are introduced below.

Definition 2.1 If $u(x_1,x_2,\ldots,x_n,t)$ is analytic and continuously differentiable with respect to space and time in the domain of interest, then the spectrum function

$$R_D[u(x_1,x_2,\ldots,x_n,t)]\approx U_k(x_1,x_2,\ldots,x_n)=\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x_1,x_2,\ldots,x_n,t)\right]_{t=t_0}, \tag{4}$$

is the reduced transformed function of $u(x_1,x_2,\ldots,x_n,t)$. Here the lowercase $u(x_1,x_2,\ldots,x_n,t)$ represents the original function while the uppercase $U_k(x_1,x_2,\ldots,x_n)$ stands for the reduced transformed function.

We notice that the relationship introduced in (4) is the Poisson series form of the input expression $u(x_1,x_2,\ldots,x_n,t)$ with respect to the variables $x_1,x_2,\ldots,x_n,t$, to order N, using the variable weights $U_k(x_1,x_2,\ldots,x_n)$.

The differential inverse reduced transform of $U_k(x_1,x_2,\ldots,x_n)$ is defined as

$$R_D^{-1}[U_k(x_1,x_2,\ldots,x_n)]\approx u(x_1,x_2,\ldots,x_n,t)=\sum_{k=0}^{\infty}U_k(x_1,x_2,\ldots,x_n)(t-t_0)^k. \tag{5}$$

Combining Eqs. (4) and (5), it can be seen that

$$u(x_1,x_2,\ldots,x_n,t)=\sum_{k=0}^{\infty}\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x_1,x_2,\ldots,x_n,t)\right]_{t=t_0}(t-t_0)^k. \tag{6}$$

From the above definition it can be seen that the concept of the reduced differential transform is derived from the two-dimensional differential transform method.

Definition 2.2 If $u(x_1,x_2,\ldots x_n,t)=R_D^{-1}[U_k(x_1,x_2,\ldots x_n)]$, $v(x_1,x_2,\ldots x_n,t)=R_D^{-1}[V_k(x_1,x_2,\ldots x_n)]$, and the convolution $\otimes$ denotes the reduced differential transform version of multiplication, then the fundamental operations of the reduced differential transform are expressed as follows:

(i).

$$R_D[u(x_1,x_2,\ldots,x_n,t)v(x_1,x_2,\ldots,x_n,t)]=U_k(x_1,x_2,\ldots,x_n)\otimes V_k(x_1,x_2,\ldots,x_n)=\sum_{r=0}^{k}U_k(x_1,x_2,\ldots,x_n)V_{k-r}(x_1,x_2,\ldots,x_n). \tag{7}$$

(ii).

$$R_D[\alpha u(x_1,x_2,\ldots,x_n,t)\pm\beta v(x_1,x_2,\ldots,x_n,t)]=\alpha U_k(x_1,x_2,\ldots,x_n)\pm\beta V_k(x_1,x_2,\ldots,x_n). \tag{8}$$

(iii).

$$R_D\left[\frac{\partial^r}{\partial x_i^r}u(x_1,x_2,\ldots,x_n,t)\right]=\frac{\partial^r}{\partial x_i^r}U_k(x_1,x_2,\ldots,x_n),\quad i=1,2,\ldots,n;\ r=1,2,3,\ldots \tag{9}$$

(iv).

$$R_D\left[\frac{\partial^r}{\partial t^r}u(x_1,x_2,\ldots,x_n,t)\right]=(k+1)(k+2)\ldots(k+r)U_{k+r}(x_1,x_2,\ldots,x_n)=\frac{(k+r)!}{k!}U_{k+r}(x_1,x_2,\ldots,x_n),\ r=1,2,3,\ldots \tag{10}$$

(v).

$$R_D\left[\frac{\partial^{r_1+r_2+\ldots+r_n+r_m}}{\partial x_1^{r_1}\partial x_2^{r_2}\partial x_3^{r_3}\ldots\partial x_n^{r_n}\partial t^{r_m}}u(x_1,x_2,\ldots,x_n,t)\right]=\frac{(k+r_m)!}{k!}\frac{\partial^{r_1+r_2+\ldots+r_n}}{\partial x_1^{r_1}\partial x_2^{r_2}\partial x_3^{r_3}\ldots\partial x_n^{r_n}}U_{k+r_m}(x_1,x_2,\ldots,x_n). \tag{11}$$

(vi).

$$R_D\left[x_1^{a_1}x_2^{a_2}x_3^{a_3}\ldots x_n^{a_n}t^{a_m}\right]=x_1^{a_1}x_2^{a_2}x_3^{a_3}\ldots x_n^{a_n}\delta(k_m-a_m)=\begin{cases}x_1^{a_1}x_2^{a_2}x_3^{a_3}\ldots x_n^{a_n}, & k_m=a_m\\ 0, & \text{otherwise}\end{cases}. \tag{12}$$

3. Numerical examples

In this section, we describe the method explained in the previous sections by the following four examples to validate the efficiency of the proposed method.

Example 3.1 Consider the $(1+n)$ – dimensional Burgers' equation

$$\frac{\partial u}{\partial t}=\left(\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}+\frac{\partial^2 u}{\partial x_3^2}+\ldots+\frac{\partial^2 u}{\partial x_n^2}\right)+u\frac{\partial u}{\partial x_1}, \tag{13}$$

subject to the initial condition

$$u(x_1,x_2,x_3,\ldots,x_n,0)=u_0(x_1,x_2,x_3,\ldots,x_n)=x_1+x_2+x_3+\ldots+x_n. \tag{14}$$

According to the RDTM, we construct the following recurrence equation for the Eq. (13)

$$(k+1)U_{k+1}=\frac{\partial^2}{\partial x_1^2}U_k+\frac{\partial^2}{\partial x_2^2}U_k+\frac{\partial^2}{\partial x_3^2}U_k+\ldots+\frac{\partial^2}{\partial x_n^2}U_k+\sum_{r=0}^{k}U_r\frac{\partial}{\partial x_1}U_{k-r}. \tag{15}$$

From the initial condition (14), we can write

$$U_0=x_1+x_2+\ldots+x_n. \tag{16}$$

Substituting the initial condition (16) in Eq. (15), we obtain the following U_k values successively:

$$U_1=(x_1+x_2+x_3+\ldots+x_n),U_2=(x_1+x_2+x_3+\ldots+x_n),\ldots,U_n=(x_1+x_2+x_3+\ldots+x_n).$$

Then, using the differential inverse transformation we get

$$u(x_1,x_2,\ldots,x_n,t)=\sum_{k=0}^{\infty}U_kt^k=(x_1+x_2+x_3+\ldots+x_n)+(x_1+x_2+x_3+\ldots+x_n)t+(x_1+x_2+x_3+\ldots+x_n)t^2+\ldots \tag{17}$$

The exact solution, in closed form, is given by

$$u(x_1,x_2,x_3,\ldots,x_n,t)=\frac{(x_1+x_2+x_3+\ldots+x_n)}{1-t},\ \text{provided}\ 0\le t<1, \tag{18}$$

which is the same solution as obtained by HPM, ADM and DTM [9].

Example 3.2 Let us take the $(1+3)$ – dimensional Burgers' equation

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}+u\frac{\partial u}{\partial x}, \tag{19}$$

with the initial condition

$$u(x,y,z,0)=u_0(x,y,z)=x+y+z. \tag{20}$$

Applying the RDTM to Eq. (19), we obtain the following recurrence relation

$$(k+1)U_{k+1}=\frac{\partial^2}{\partial x^2}U_k+\frac{\partial^2}{\partial y^2}U_k+\frac{\partial^2}{\partial z^2}U_k+\sum_{r=0}^{k}U_r\frac{\partial}{\partial x}U_{k-r}. \tag{21}$$

From the initial condition (20), we can write

$$U_0=x+y+z. \tag{22}$$

Using Eq. (22) into Eq. (21), we get the following U_k values successively

$$U_1=(x+y+z),U_2=x+y+z,\ldots,U_n=x+y+z.$$

Now, using the differential inverse transformation, we get

$$u(x,y,z,t)=\sum_{k=0}^{\infty}U_kt^k=(x+y+z)+(x+y+z)t+(x+y+z)t^2+\ldots+(x+y+z)t^n+\ldots \tag{23}$$

In closed form, the exact solution is given by

$$u(x,y,z,t)=\frac{(x+y+z)}{1-t},\ \text{provided}\ 0\le t<1, \tag{24}$$

which is same as obtained by HPM, ADM and DTM [9].

Example 3.3 Consider the $(1+2)$ – dimensional Burgers' equation

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+u\frac{\partial u}{\partial x}, \tag{25}$$

under the initial condition

$$u(x,y,0)=u_0(x,y)=x+y. \tag{26}$$

According to the RDTM, we construct the following iteration formula to Eq. (25)

$$(k+1)U_{k+1} = \frac{\partial^2}{\partial x^2}U_k + \frac{\partial^2}{\partial y^2}U_k + \sum_{r=0}^{k} U_r \frac{\partial}{\partial x}U_{k-r}. \tag{27}$$

From the initial condition (26), we can write

$$U_0 = x + y. \tag{28}$$

Using Eq. (28) in Eq. (27), we obtain the following U_k values successively

$$U_1 = x+y, U_2 = x+y, \ldots, U_n = x+y.$$

Then, using the differential inverse transformation, we get

$$u(x,y,t) = \sum_{k=0}^{\infty} U_k t^k = (x+y) + (x+y)t + (x+y)t^2 + \ldots + (x+y)t^n + \ldots \tag{29}$$

The exact solution, in closed form, is given by

$$u(x,y,t) = \frac{(x+y)}{1-t}, \text{ provided } 0 \le t < 1, \tag{30}$$

which is the same result as obtained by HPM, ADM and DTM [9].

Example 3.4 Consider the $(1+1)-$ dimensional Burgers' equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2}, \tag{31}$$

Subject to the initial condition

$$u(x,0) = u_0(x) = 2x. \tag{32}$$

Using the RDTM to Eq. (31), we get the following iteration formula

$$(k+1)U_{k+1} = -\sum_{r=0}^{k} U_r \frac{\partial}{\partial x}U_{k-r} + \frac{\partial^2}{\partial x^2}U_k. \tag{33}$$

From the initial condition (32), we can write

$$U_0 = 2x. \tag{34}$$

Using Eq. (34) in Eq. (33), we get the following U_k values successively

$$U_1 = (-4x), U_2 = (8x), U_3 = (-16x), \ldots$$

Then, using the differential inverse transformation, we get

$$u(x,y,t) = \sum_{k=0}^{\infty} U_k t^k = (2x) + (-4x)t + (8x)t^2 + (-16x)t^3 + \ldots \tag{35}$$

The exact solution, in closed form, is given by

$$u(x,t) = \frac{2x}{1+2t}, \tag{36}$$

which is the same solution as obtained by HPM, ADM and DTM [9].

4. Conclusions

In this work, the reduced form of differential transform method is introduced for solving the $(1+n)-$ dimensional Burgers' equation. The solutions obtained by the method are an infinite power series for appropriate initial condition, which can, in turn, be expressed in a closed form, the exact solution. The results reveal that the proposed method is very effective, convenient and quite accurate mathematical tools for solving the $(1+n)-$ dimensional Burgers' equation. It can be observed that the solution approach of RDTM is much simpler than differential transform method (DTM) and it needs less computational effort than DTM. In other word, RDTM is an alternative approach to overcome the demerit of complex calculation of DTM, capable of reducing the size of calculation. As a special advantage of RDTM rather than DTM, the reduced differential transform recursive equations produce exactly all the Poisson series coefficients of solutions, whereas the differential transform recursive equations produce exactly all the Taylor series coefficients of solutions. We notice that the RDTM technique is highly accurate, rapidly converge and is very easily implementable mathematical tool for the multidimensional physical problems emerging in various domains of engineering and allied sciences.

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