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Neighborhood-Sheaf Semantics for First-Order Modal Logic

Kohei Kishida[1,2]

Department of Artificial Intelligence
University of Groningen
Groningen, The Netherlands

Abstract

This paper extends neighborhood semantics for propositional modal logic to the first-order case, by unifying topological-sheaf semantics (in [2]) for first-order **S4** and Kripke-sheaf semantics (see [11] and [8], just for instance) for quantified **K**. It will be shown how to take a sheaf-like structure over a neighborhood frame, and the resulting semantics properly generalizes the two preceding sheaf semantics; it has a weaker modal logic (in which the rule N fails) sound and complete, while accommodating classical, full first-order logic with equality and function symbols.

Keywords: First-order modal logic, neighborhood semantics, sheaf semantics.

1 Introduction

The goal of this paper is to extend neighborhood semantics for propositional modal logic, which subsumes and unifies both Kripke and topological semantics, to first-order modal logic, and in particular to a sheaf semantics. This is achieved by extending the neighborhood subsumption on the propositional level to the sheaf level, using the notion of a neighborhood sheaf that generalizes and unifies Kripke-sheaf semantics and topological-sheaf semantics.

The semantics laid out in this paper unifies several frameworks of semantics for propositional and quantified modal logic. The relations of extension or subsumption among these frameworks can be summarized by the following diagram; the labels with alphabets indicate semantic ideas explained in the following, and the dotted

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[2] Email: kishidakohei@gmail.com

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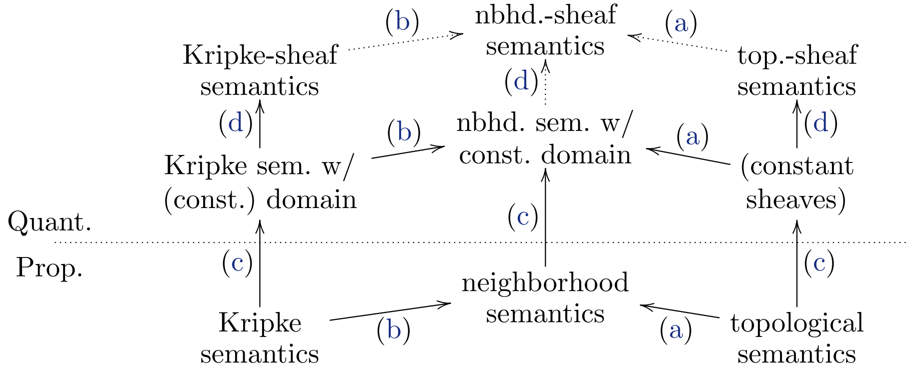
arrows indicate the unification offered in this paper.



Among frameworks of semantics for propositional modal logic—that is, on the bottom level in the diagram—*neighborhood semantics* generalizes *Kripke semantics* and *topological semantics* by the following ideas, respectively:

(a) It generalizes topological semantics by considering interior operations that are more general than topological ones.

(b) It generalizes accessibility relations in Kripke semantics with a neighborhood notion of accessibility.

We briefly review these ideas in Section 2 of this article, to prepare ourselves to extend them to the level of first-order modal logic.

To extend his semantics for propositional modal logic to the level of quantified modal logic, Kripke [12] took advantage of the following idea:

(c) Interpret first-order vocabulary with a domain D of possible individuals; in particular, interpret the "transworld identity" of individuals with the identity of elements of D.

This idea gives rise to *Kripke semantics with domains*, and in particular with *constant* domains, bringing up Kripke semantics from the bottom to the middle level in the diagram above.

Neighborhood semantics with constant domains was given by Arló-Costa and Pacuit [1], who showed how to combine the ideas (b) and (c). Their semantics has constant domains of all possible individuals, but interprets modal operators in terms of neighborhoods rather than accessibility relations.

This extension to the quantified case via (c), however, is not general enough for treating the necessity and contingency of identity of individuals; in particular, it forces the identity and non-identity of individuals to always be necessary. By contrast, *Kripke-sheaf semantics* [8,9,11] can model the contingency of non-identity, by extending the idea (c) further with

(d) Interpret first-order vocabulary, and in particular the transworld identity, with the structure of a sheaf over a set of possible worlds.

A sheaf over a set X (a Kripke frame, for instance) can be regarded as a family of local copies of X nicely patched together, and each such copy interprets the transworld identity. The sheaf point of view subsumes constant domains as constant sheaves, bringing semantics up to the top level in the diagram above.

This idea, (d), was also employed by Awodey and Kishida's [2] *topological-sheaf semantics*. They showed that sheaves over topological spaces provided semantics for modal logic **S4** combined with classical first-order logic with equality and function symbols. We review these sheaf semantics in Section 3.

In Section 4, we generalize topological-sheaf semantics of [2] by applying (a). Employing the same notations and the same types of semantic structures as in [2], we identify a right formulation of the notion of a sheaf that works not just over a topological space but also over a more general neighborhood structures. The resulting semantics subsumes neighborhood semantics with constant domains as a subclass (namely, of constant sheaves), bringing it to the sheaf level via (d).[3] Moreover, it properly subsumes Kripke-sheaf semantics as well, since the Kripke sheaves are just the neighborhood sheaves with relational accessibility. Thus, on the sheaf level, neighborhood-sheaf semantics subsumes and unifies not only topological-sheaf semantics via (a) but also Kripke-sheaf semantics via (b), in just the same way that neighborhood semantics subsumes and unifies topological and Kripke semantics on the propositional level.

2 Reviewing the Propositional Case

This section introduces notations we use for Kripke, topological, and neighborhood semantics, and reviews how the third semantics subsumes the others.

2.1 Possible-World Semantics Basics

Fix a propositional modal language $\mathcal{L}$ with a unary sentential operator $\Box$. Then, by a possible-world semantics for $\mathcal{L}$, we mean a class of *models* that consist of a set $X \neq \varnothing$ and a map $[\![\cdot]\!] : \mathcal{L} \to \mathcal{P}X$, among other things. We may call points in X *possible worlds*, and subsets of X *propositions*, so that we can read $w \in [\![\varphi]\!]$, for a sentence φ and a world $w \in X$, as meaning that φ is true at w. Accordingly, we define the notion of validity so that a sentence φ is valid in $(X, [\![\cdot]\!])$ iff $[\![\varphi]\!] = X$, and an inference is valid iff it preserves validity of sentences. It is also convenient for our purpose to define validity of binary sequents, so that a sequent $\varphi \vdash \psi$ is valid in $(X, [\![\cdot]\!])$ iff $[\![\varphi]\!] \subseteq [\![\psi]\!]$.

We extend $[\![\cdot]\!]$ to interpret sentential operators: For each n-ary operator $\otimes$, we have $[\![\otimes]\!] : (\mathcal{P}X)^n \to \mathcal{P}X$ and $[\![\otimes]\!]([\![\varphi_1]\!], \ldots, [\![\varphi_n]\!]) = [\![\otimes(\varphi_1, \ldots, \varphi_n)]\!]$. Since we discuss modal logic with classical base, we assume $[\![\neg]\!] = X \setminus -$ (so $[\![\neg\varphi]\!] = X \setminus [\![\varphi]\!]$), $[\![\wedge]\!] = \cap$ (so $[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \cap [\![\varphi]\!]$), and so on.

Kripke, topological, and neighborhood semantics are all possible-world semantics in this sense; their difference consists in how they interpret $\Box$. Kripke semantics,

[3] Neighborhood-sheaf semantics fails to entirely subsume neighborhood semantics with constant domains, because the former requires certain conditions on neighborhood structures, which the latter does not.

for instance, equips X with a binary relation R, called an *accessibility relation*, and lets it define $[\![\Box]\!] : \mathcal{P}X \to \mathcal{P}X$ so that $w \in [\![\Box]\!](A)$ iff $u \in A$ for all $u \in X$ such that Rwu.

2.2 Topological Semantics

Topological semantics, which originated with McKinsey and Tarski's [18] result, interprets $\Box$ by equipping $X \neq \varnothing$ with a topology $\mathcal{O}X \subseteq \mathcal{P}X$. Any topology $\mathcal{O}X$ gives rise to an *interior operation* $\mathbf{int} : \mathcal{P}X \to \mathcal{P}X$, by setting

$$w \in \mathbf{int}(A) \iff \text{there is } U \in \mathcal{O}X \text{ such that } w \in U \subseteq A. \tag{1}$$

Then $\Box$ can be interpreted topologically with $[\![\Box]\!] = \mathbf{int}$.

The following conditions characterize topological interior operations; any $\mathbf{int} : \mathcal{P}X \to \mathcal{P}X$ is an interior operation of some topology iff it satisfies

$$A_0 \subseteq A_1 \implies \mathbf{int}(A_0) \subseteq \mathbf{int}(A_1), \tag{M$_{\mathbf{int}}$}$$
$$\mathbf{int}(X) = X, \tag{N$_{\mathbf{int}}$}$$
$$\mathbf{int}(A_0) \cap \mathbf{int}(A_1) \subseteq \mathbf{int}(A_0 \cap A_1), \tag{C$_{\mathbf{int}}$}$$
$$\mathbf{int}(A) \subseteq A, \tag{T$_{\mathbf{int}}$}$$
$$\mathbf{int}(A) \subseteq \mathbf{int}(\mathbf{int}(A)). \tag{4$_{\mathbf{int}}$}$$

Because these correspond respectively to the following rules and axioms of modal logic, the modal logic obtained by adding them to classical propositional logic, namely **S4**, is sound with respect to topological semantics.

$$\varphi \vdash \psi \;/\; \Box\varphi \vdash \Box\psi \tag{M}$$
$$\vdash \varphi \;/\; \vdash \Box\varphi \tag{N}$$
$$\Box\varphi \wedge \Box\psi \vdash \Box(\varphi \wedge \psi) \tag{C}$$
$$\Box\varphi \vdash \varphi \tag{T}$$
$$\Box\varphi \vdash \Box\Box\varphi \tag{4}$$

S4 is indeed complete with respect to topological semantics, as implied by the representation theorem in [18], in the following strong form.

Theorem 2.1 (McKinsey-Tarski [18]) *Every consistent theory $\mathbb{T}$ of propositional modal logic containing* **S4** *has a topological model $(X, [\![\cdot]\!])$ that validates all and only theorems of $\mathbb{T}$.*

2.3 Neighborhood Semantics

Scott [21] and Montague [20] introduced neighborhood semantics, generalizing topological semantics by not assuming the constraints (M$_{\mathbf{int}}$)–(4$_{\mathbf{int}}$) on the interior operation $\mathbf{int} : \mathcal{P}X \to \mathcal{P}X$. Such a general operation $\mathbf{int}$ is associated with the following generalization of the notion of a topological space.

Given a topological space X, we say that A is a *neighborhood* of w, and write $A \in \mathcal{N}(w)$, if the equivalents of (1) hold; so we have

$$w \in \mathbf{int}(A) \iff A \in \mathcal{N}(w). \tag{2}$$

This equivalence gives us a map $\mathcal{N} : X \to \mathcal{PP}X$, called a *neighborhood function* on X, that is just equivalent to $\mathbf{int} : \mathcal{P}X \to \mathcal{P}X$.[4] We call a set X paired with a neighborhood function on it a *neighborhood frame*. Moreover, along (2), the constraints ($\mathrm{M}_{\mathbf{int}}$)–($4_{\mathbf{int}}$) translate respectively to

$$\begin{aligned} A \subseteq B \subseteq X \text{ and } A \in \mathcal{N}(w) &\implies B \in \mathcal{N}(w), && (\mathrm{M}_{\mathcal{N}})\\ & \quad\; X \in \mathcal{N}(w), && (\mathrm{N}_{\mathcal{N}})\\ A, B \in \mathcal{N}(w) &\implies A \cap B \in \mathcal{N}(w), && (\mathrm{C}_{\mathcal{N}})\\ A \in \mathcal{N}(w) &\implies w \in A, && (\mathrm{T}_{\mathcal{N}})\\ A \in \mathcal{N}(w) &\implies \mathbf{int}(A) \in \mathcal{N}(w). && (4_{\mathcal{N}}) \end{aligned}$$

In other words, taking $\mathcal{N} : X \to \mathcal{PP}X$ instead of $\mathcal{O}X \in \mathcal{PP}X$ gives us freedom to drop, or keep, any of the constraints ($\mathrm{M}_{\mathbf{int}}$)–($4_{\mathbf{int}}$) on $\mathbf{int} : \mathcal{P}X \to \mathcal{P}X$ by dropping or keeping the corresponding ones of ($\mathrm{M}_{\mathcal{N}}$)–($4_{\mathcal{N}}$) on $\mathcal{N}$.

This freedom is straightforwardly reflected in logic. Neighborhood semantics, which keeps the interpretation $[\![\Box]\!] = \mathbf{int}$, validates the rule

$$\varphi \vdash \psi,\ \ \psi \vdash \varphi \ /\ \Box\varphi \vdash \Box\psi \tag{E}$$

since, trivially, $[\![\varphi]\!] = [\![\psi]\!]$ implies $\mathbf{int}([\![\varphi]\!]) = \mathbf{int}([\![\psi]\!])$; the semantics has sound and complete the logic $\mathbf{E}$ obtained by adding E to classical logic.

Theorem 2.2 (Scott [21], Montague [20], Segerberg [22]) *Every consistent theory $\mathbb{T}$ of propositional modal logic containing* $\mathbf{E}$ *has a neighborhood model $(X, \mathcal{N}, [\![\cdot]\!])$ that validates all and only theorems of $\mathbb{T}$.*

Beyond this weak logic $\mathbf{E}$, however, M, N, C, T, 4 are rules and axioms that could be added by assuming the corresponding ones of ($\mathrm{M}_{\mathcal{N}}$)–($4_{\mathcal{N}}$).

Theorem 2.3 (Segerberg [22]) *On neighborhood semantics,* M, N, C, T, 4 *correspond respectively to* ($\mathrm{M}_{\mathcal{N}}$)–($4_{\mathcal{N}}$).

2.4 Neighborhood Unification, the Propositional Case

Let us close this section with a brief observation of how neighborhood semantics subsumes Kripke and topological semantics. We just saw in Subsection 2.3 how a topology $\mathcal{O}X$ gives rise to a neighborhood function $\mathcal{N}$ on X that has the same interior operation $\mathbf{int}$ (and hence interprets $\Box$ the same way), and that satisfies ($\mathrm{M}_{\mathcal{N}}$)–($4_{\mathcal{N}}$). On the other hand, given any neighborhood function $\mathcal{N}$ on X satisfying

[4] Writing $Y = \mathcal{P}X$ makes it obvious that $\mathcal{N} : X \to (Y \to 2)$ and $\mathbf{int} : Y \to (X \to 2)$ are just the "transposes" of the neighborhood relation $: X \times Y \to 2$.

($\mathrm{M}_{\mathcal{N}}$)–($4_{\mathcal{N}}$), we can obtain a topology $\mathcal{O}X$ that has the same interior operation $\mathbf{int}$, by setting $\mathcal{O}X = \{\, U \subseteq X \mid \mathbf{int}(U) = U \,\}$. Thus, topological semantics is just neighborhood semantics with ($\mathrm{M}_{\mathcal{N}}$)–($4_{\mathcal{N}}$).

Let us say that a neighborhood frame $(X, \mathcal{N})$ is Kripke if there is a map $\overrightarrow{R} : X \to \mathcal{P}X$ such that $A \in \mathcal{N}(w)$ iff $\overrightarrow{R}(w) \subseteq A$. This equivalence implies

$$w \in \mathbf{int}(A) \iff \overrightarrow{R}(w) \subseteq A \iff u \in A \text{ for all } u \in X \text{ such that } Rwu$$

by (2), where we set $u \in \overrightarrow{R}(w)$ iff Rwu for $\overrightarrow{R} : X \to (X \to 2)$ and $R : X \times X \to 2$. Thus, Kripke semantics is just Kripke neighborhood semantics.

In Kripke semantics, $\overrightarrow{R}(w)$ is *the* neighborhood of w such that $w \in [\![\Box\varphi]\!]$ means $\overrightarrow{R}(w) \subseteq [\![\varphi]\!]$. By contrast, in topological or neighborhood semantics, w may have no such privileged neighborhood; with ($\mathrm{M}_{\mathcal{N}}$), $w \in [\![\Box\varphi]\!]$ only means that $U \subseteq [\![\varphi]\!]$ for *some* $U \in \mathcal{N}(w)$. This contrast between "the" and "some" turns out crucial for the purpose of this article, as we will see in Subsection 4.3. It may also be worth noting that, as a consequence of this contrast, the infinitary version $\bigwedge_i \Box\varphi_i \vdash \Box \bigwedge_i \varphi_i$ of C (though this article does not formally treat infinitary $\bigwedge$) would be valid in Kripke semantics but not in the others (even with ($\mathrm{C}_{\mathcal{N}}$)); under the reading of $\Box$ as for provability or verifiability in general, infinitary C precludes the finite character of proof or verification.

3 Reviewing Existent Sheaf Semantics

This section reviews Kripke- and topological-sheaf semantics. The notations and the types of semantic structures we employ are just those employed in [2]; we refer the reader to [2] for more detailed expositions of them.

3.1 Semantics for First-Order Logic

In Subsection 2.1, we reviewed the common features of Kripke, topological, and neighborhood semantics for propositional modal logic. In this subsection, we do the same for first-order logic. To interpret propositional modal logic, possible-world semantics takes a set X; we extend this, to interpret first-order modal logic, by taking instead the category $\mathbf{Sets}/X$ of sets over X.

The semantic ideas behind taking $\mathbf{Sets}/X$ can be put as follows. Consider an object of $\mathbf{Sets}/X$, that is, any map $\pi : D \to X$. Each $w \in X$ has its inverse image $D_w = \pi^{-1}[\{w\}]$, called the *fiber* over w. D is then the "bundle" of all the fibers taken over X, written $D = \sum_{w \in X} D_w$, meaning that D is the disjoint union of all D_w. We can regard each $w \in X$ as a possible world, and the fiber D_w as the domain of individuals that live in w. Then D is the set of "possible individuals" that live in some world or other. Indeed, each individual $a \in D$ lives in a unique world $\pi(a) \in X$; in this sense, we call π a *residence* map.[5] The n-fold product of

[5] David Lewis's [15] counterpart theory has such a domain of possible individuals, though he did not formulate it in terms of a residence map. See Sections 10 and 11 of [4].

$\pi : D \to X$ in $\mathbf{Sets}/X$ is a map $\pi^n : D^n \to X$, where $D^n = \sum_{w \in X} D^n_w$ is the n-fold *fibered* product of D over X, that is, the bundle of n-fold Cartesian products of D_w; in terms of the residence idea, D^n is the set of n-tuples of individuals that live in the same world, and π^n sends $(a_1, \dots, a_n) \in D^n_w$ to their residence, w. We interpret n-ary formulas by subsets of n-fold fibered, rather than Cartesian, products. An arrow in $\mathbf{Sets}/X$ from $\pi_D : D \to X$ to $\pi_E : E \to X$ is a map $f : D \to E$ over X, that is, such that $\pi_E \circ f = \pi_D$, or equivalently, of the form $\sum_{w \in X}(f_w : D_w \to E_w)$, a bundle of maps from D_w to E_w. We use such a map $f : D^n \to D$ to interpret n-ary function symbols. In this way, we regard $\mathbf{Sets}/X$ as the category of domains of, sets of tuples of, and functions among, possible individuals, over the set X of possible worlds. Each such structure, restricted to fibers over a world $w \in X$, is a usual structure for first-order logic: a domain of, sets of tuples of, and functions among, individuals that live in w.

Accordingly, we extend the usual semantics of first-order logic by bundling it up over X. Fix a first-order language $\mathcal{L}$; then, for each $w \in X$, take a usual $\mathcal{L}$-structure $\mathfrak{M}_w$ and, given a sentence φ with no free variables other than $\bar{x} = (x_1, \dots, x_n)$, let $[\![\, \bar{x} \mid \varphi \,]\!]_w$ be the subset of D^n_w defined by φ. Now, we bundle up these $(\mathfrak{M}_w, [\![\cdot]\!]_w)$ over X, by having $\mathfrak{M} = \sum_{w \in X} \mathfrak{M}_w$ and $[\![\, \bar{x} \mid \varphi \,]\!] = \sum_{w \in X} [\![\, \bar{x} \mid \varphi \,]\!]_w \subseteq \sum_{w \in X} D^n_w = D^n$; that is, $\bar{a} = (a_1, \dots, a_n) \in D^n$ lies in $[\![\, \bar{x} \mid \varphi \,]\!]$ iff φ is true of $\bar{a}$, with each a_i in place of free x_i, in $\mathfrak{M}_{\pi^n(\bar{a})}$. We can similarly interpret a term t with n variables by $[\![\, \bar{x} \mid t \,]\!] : D^n \to D$. This is intuitively how the semantics in $\mathbf{Sets}/X$ works; formally, the definition goes directly, without $\sum$, as follows.

Definition 3.1 Given a first-order (modal) language $\mathcal{L}$, we say that $\mathfrak{M}$ is a *bundle model* for $\mathcal{L}$ if it consists of the following, among other things.

- a surjection $\pi : D \to X$ of some domain D and codomain X;[6]
- for each n-ary primitive predicate F, a subset $F^{\mathfrak{M}} \subseteq D^n$ of the n-fold fibered product of D over X; in particular, $=^{\mathfrak{M}} = \{\, (a, a) \mid a \in D \,\} \subseteq D^2$;
- for each n-ary function symbol f, a map $f^{\mathfrak{M}} : D^n \to D$ over X; and
- for each constant c, a map $c^{\mathfrak{M}} : D^0 \to D$ over X, that is, a map $c^{\mathfrak{M}} : X \to D$ such that $\pi \circ c^{\mathfrak{M}} = 1_X$.

By a *bundle interpretation*, we mean a pair of a bundle model $\mathfrak{M}$ and a map $[\![\cdot]\!]$ that satisfies suitable constraints such as

$$
\begin{aligned}
[\![\, \bar{x} \mid F\bar{x} \,]\!] &= F^{\mathfrak{M}} && \text{for } n\text{-ary primitive predicate } F; \\
[\![\, \bar{x} \mid \neg\varphi \,]\!] &= D^n \setminus [\![\, \bar{x} \mid \varphi \,]\!] && (\text{that is, } [\![\neg]\!] = D^n \setminus -); \\
[\![\, \bar{x} \mid \varphi \wedge \psi \,]\!] &= [\![\, \bar{x} \mid \varphi \,]\!] \cap [\![\, \bar{x} \mid \psi \,]\!] && (\text{that is, } [\![\wedge]\!] = \cap); \\
[\![\, \bar{x} \mid \exists y\, \varphi \,]\!] &= p_n[[\![\, \bar{x}, y \mid \varphi \,]\!]], && \text{for the projection } p_n;^{7} \\
[\![\, \bar{x}, y \mid \varphi(\bar{x}) \,]\!] &= {p_n}^{-1}[[\![\, \bar{x} \mid \varphi(\bar{x}) \,]\!]];
\end{aligned}
$$

and so on. We say that, in such $(\mathfrak{M}, [\![\cdot]\!])$, φ is valid iff $[\![\, \bar{x} \mid \varphi \,]\!] = D^n$; $\varphi \vdash \psi$ is valid

[6] We require π to be surjective so that $D_w \neq \varnothing$ for every $w \in X$.

[7] For each n, the projection p_n is $p_n : D^{n+1} \to D^n :: (\bar{a}, b) \mapsto \bar{a}$.

iff $[\![\, \bar{x} \mid \varphi \,]\!] \subseteq [\![\, \bar{x} \mid \psi \,]\!]$; and an inference is valid iff it preserves validity.

Both Kripke- and topological-sheaf semantics, as well as the semantics we offer in Section 4, are given by certain classes of bundle interpretations. They differ in what structures they add to X, D, π, etc., to interpret $\Box$.

3.2 Topological-Sheaf Semantics

Topological semantics interprets propositional modal logic by adding a topology and $[\![\Box]\!] = \mathbf{int}$ to a set X of worlds. Extending this, topological-sheaf semantics interprets first-order modal logic by adding topologies and $[\![\Box]\!] = \mathbf{int}$ to the semantic structures we reviewed in Subsection 3.1, namely sets over X; specifically, instead of taking objects, products, and arrows in $\mathbf{Sets}/|X|$,[8] it takes those in $\mathbf{LH}/X$, the category of sheaves over a topological space X.

Recall that, given topological spaces X and Y, a map $f : Y \to X$ is a *homeomorphism* if it is a continuous bijection with a continuous inverse. Then the topological notion of a sheaf is defined as follows.

Definition 3.2 A continuous map $\pi : D \to X$ is called a *local homeomorphism* if every $a \in D$ has some $U \in \mathcal{O}D$ such that $a \in U$, $\pi[U] \in \mathcal{O}X$, and the restriction $\pi{\restriction}U : U \to \pi[U]$ of π to U is a homeomorphism. We say that such a pair (D, π) is a *sheaf over the space* X, and call π its *projection.*[9]

Given sheaves (D, π_D) and (E, π_E) over a space X, we say that a map $f : D \to E$ is a *map of sheaves over* X (from (D, π_D) to (E, π_E)) if f is continuous and $\pi_E \circ f = \pi_D$ (that is, over $|X|$). We can prove that maps of sheaves are themselves local homeomorphisms,[10] which implies that the category of sheaves and maps of sheaves is just $\mathbf{LH}/X$, the category $\mathbf{LH}$ of topological spaces and local homeomorphisms over X.

Given a sheaf (D, π) over a space X, let us write D^n for the n-fold fibered product $|D|^n$ of $|D|$ with the following topology: Let $D \times \cdots \times D$ be the n-fold product space of D. Then, since $|D|^n$ is a subset of the n-fold Cartesian product $|D \times \cdots \times D|$ of $|D|$, we let D^n simply be the subspace on $|D|^n$ of $D \times \cdots \times D$. Then (D^n, π^n) is a sheaf over X, and is the n-fold product of (D, π) in $\mathbf{LH}/X$. Moreover, any projection $p : D^n \to D^m$ $(m \leqslant n)$ is a map of sheaves (and hence a local homeomorphism).

Now, topological-sheaf semantics of [2] consists in equipping the bundle semantics of Subsection 3.1 with topologies, by using structures from $\mathbf{LH}/X$ rather than from $\mathbf{Sets}/|X|$. We enter:

Definition 3.3 By a *topological-sheaf model* for a given first-order modal language $\mathcal{L}$, we mean a bundle model $\mathfrak{M} = (\pi, F^{\mathfrak{M}}, f^{\mathfrak{M}}, c^{M})$ for $\mathcal{L}$ such that $\pi : D \to X$ is a

[8] We write $|X|$ for the underlying set of a topological space X.

[9] The notion of a sheaf is often defined in terms of the notion of a functor, in which case the version used here is called an étale space. The functorial notion is equivalent (in the category-theoretical sense) to the version here. See, for instance, [16].

[10] Exercise II.10(b) in [16], 105.

local homeomorphism and $f^{\mathfrak{M}} : D^n \to D$, $c^{\mathfrak{M}} : X \to D$ are all maps of sheaves. By a *topological-sheaf interpretation* for $\mathcal{L}$, we mean a bundle interpretation $(\mathfrak{M}, [\![\cdot]\!])$ for $\mathcal{L}$ such that $\mathfrak{M}$ is a topological-sheaf model and $[\![\Box]\!] : \mathcal{P}(D^n) \to \mathcal{P}(D^n) :: [\![\, \bar{x} \mid \varphi \,]\!] \mapsto [\![\, \bar{x} \mid \Box\varphi \,]\!]$ is $\mathbf{int}_{D^n}$.

It has to be emphasized that $\Box$ is interpreted by not just $\mathbf{int}_X$ but by the family of $\mathbf{int}_{D^n}$, corresponding to the arity of $[\![\, \bar{x} \mid \varphi \,]\!]$ to which $[\![\Box]\!]$ is applied.

Topological-sheaf semantics unifies the semantics in $\mathbf{Sets}/X$ (for first-order logic) and topological semantics (for propositional **S4**) naturally, in the sense that it gives rise to a logic that is a simple union of classical first-order logic and **S4**. Let **FOS4** be the first-order modal logic that consists of

1) all the rules and axioms of classical, full first-order logic, "full" meaning that it has equality and function symbols, and
2) the rules and axioms of propositional modal logic **S4**.

In this logic, schemes of first-order rules and axioms do *not* distinguish sentences containing $\Box$ from ones not. In the axiom $x = y \vdash [x/z]\varphi \to [y/z]\varphi$ of identity, for instance, φ may contain modal operators. Also, modal rules and axioms are insensitive to the first-order structure of sentences. Hence we call **FOS4** a simple union of first-order logic and **S4**. It can be checked straightforwardly that **FOS4** is sound with respect to topological-sheaf semantics. It is moreover complete, in the strong form that exactly extends Theorem 2.1.

Theorem 3.4 (Awodey-Kishida [3]) *Every consistent theory $\mathbb{T}$ of first-order modal logic containing* **FOS4** *has a topological-sheaf interpretation $(\pi, [\![\cdot]\!])$ that validates all and only theorems of $\mathbb{T}$.* [11]

To give examples of theorems of **FOS4**, $x = y \vdash \Box(x = y)$ is provable, because $x = y \vdash \Box(x = x) \to \Box(x = y)$ is an instance of the above-mentioned axiom of identity (with $\Box(x = z)$ for φ), while $\vdash x = x$ implies $\vdash \Box(x = x)$ by N. Also, $\varphi \vdash \exists x\, \varphi$ implies $\Box\varphi \vdash \Box\exists x\, \varphi$ by M, from which $\exists x\, \Box\varphi \vdash \Box\exists x\, \varphi$ follows (because x is not free in $\Box\varphi$); similarly, **FOS4** proves $\Box\forall x\, \varphi \vdash \forall x\, \Box\varphi$. By contrast, **FOS4** proves neither $x \neq y \vdash \Box(x \neq y)$, $\Box\exists x\, \varphi \vdash \exists x\, \Box\varphi$, nor $\forall x\, \Box\varphi \vdash \Box\forall x\, \varphi$. (The same goes for the logic **FOK** of Kripke-sheaf semantics.)

3.3 Kripke-Sheaf Semantics

Kripke semantics interprets propositional modal logic by adding an accessibility relation to a set X of worlds. Extending this, Kripke-sheaf semantics interprets first-order modal logic by adding accessibility relations to structures in $\mathbf{Sets}/X$; that is, instead of sets, it takes Kripke frames.

Let us say that, given Kripke frames (D, R_D) and (X, R_X), a map $f : D \to X$ is a p-morphism if it satisfies the following conditions.

- f is monotone, that is, if $R_D ab$ then $R_X f(a)f(b)$.

[11] Awodey and Kishida's [3] completeness proof is inspired by those of McKinsey and Tarski [18], Segerberg [22], and Butz and Moerdijk [5,6,19].

- If $R_X f(a) w$ then there is $b \in D$ such that $R_D ab$ and $w = f(b)$.

Then, among various (mostly equivalent) versions of definition of Kripke sheaves, one goes as follows. [12]

Definition 3.5 A p-morphism $\pi : D \to X$ is called a *Kripke sheaf over* (X, R_X) if such $b \in D$ as in the definition of p-morphism above is unique—that is, if every pair $u, w \in X$ such that $R_X uw$ has a map $C_{uw} : D_u \to D_w$ such that, for any pair of $a \in D_u$ and $b \in D_w$, $R_D ab$ iff $C_{uw}(a) = b$.

Then, analogously to the case of topological sheaves, given Kripke sheaves $\pi_D : D \to X$ and $\pi_E : E \to X$, we call a map $f : D \to E$ a *map of (Kripke) sheaves over* X (from π_D to π_E) if f is monotone and over X. We can prove that maps of sheaves are themselves Kripke sheaves; hence, writing **KrSh** for the category of Kripke frames (as objects) and Kripke sheaves (as arrows), the category of Kripke sheaves and maps of sheaves over a Kripke frame (X, R_X) is just $\mathbf{KrSh}/(X, R_X)$. Moreover, the n-fold product in $\mathbf{KrSh}/(X, R_X)$ of a Kripke sheaf π over (X, R_X) is $\pi^n : D^n \to X$, where D^n has the accessibility relation R_{D^n} such that $R_{D^n}(\bar{a}, \bar{b})$ iff $R_D a_i b_i$ for every $i \leqslant n$.

Then Kripke-sheaf semantics adds accessibility relations to the bundle semantics by using structures from $\mathbf{KrSh}/(X, R_X)$ rather than from $\mathbf{Sets}/|X|$.

Definition 3.6 By a *Kripke-sheaf model* for a given first-order modal language $\mathcal{L}$, we mean a bundle model $\mathfrak{M} = (\pi, F^{\mathfrak{M}}, f^{\mathfrak{M}}, c^M)$ for $\mathcal{L}$ such that $\pi : D \to X$ is a Kripke-sheaf and $f^{\mathfrak{M}} : D^n \to D$, $c^{\mathfrak{M}} : X \to D$ are all maps of sheaves. By a *Kripke-sheaf interpretation* for $\mathcal{L}$, we mean a bundle interpretation $(\mathfrak{M}, [\![\cdot]\!])$ for $\mathcal{L}$ such that $\mathfrak{M}$ is a Kripke-sheaf model and

$$\bar{a} \in [\![\, \bar{x} \mid \Box\varphi \,]\!] \iff \bar{b} \in [\![\, \bar{x} \mid \varphi \,]\!] \text{ for all } \bar{b} \in D^n \text{ such that } R_{D^n}(\bar{a}, \bar{b}).$$

Again, $\Box$ is interpreted by not just R_X but by the family of R_{D^n}. This semantics unifies the semantics in $\mathbf{Sets}/X$ and Kripke semantics naturally by having sound and complete the simple union **FOK** of first-order logic and **K** (which is given by M, N, and C).

4 Neighborhood-Sheaf Semantics

This section, finally, generalizes and unifies topological- and Kripke-sheaf semantics, by providing a notion of a sheaf over a neighborhood frame.

4.1 Why Sheaves are Needed

Let us analyze topological and Kripke sheaves and identify an aspect of them that is essential for unifying first-order and modal logics, so that we can preserve it as we move to a more general notion of sheaves. In Subsection 3.2, we reviewed a

[12] See, for instance, [23].

standard definition of local homeomorphisms; yet it is helpful for our purpose to rewrite it in terms more closely connected to logic. Recall that a continuous map $f : Y \to X$ is said to be *open* if $f[V] \in \mathcal{O}X$ for every $V \in \mathcal{O}Y$. Then topological sheaves can be characterized in terms of openness of maps, by the following fact.

Fact 4.1 *For any topological spaces X and D and any map $\pi : D \to X$, the following are equivalent:*[13]

- *π is a local homeomorphism (as defined in Subsection 3.2).*
- *π satisfies (i) and (ii) below.*
- *π satisfies (i) and (iii) below.*

(i) *π is an open map.*

(ii) *For every $a \in D$ there is $U \in \mathcal{O}D$ such that $a \in U$ and $\pi{\restriction}U : U \to \pi[U]$ is bijective.*

(iii) *The diagonal map $\Delta : D \to D^2 :: a \mapsto (a, a)$ is an open map.*

An analogous fact holds for Kripke sheaves.

Fact 4.2 *For any Kripke frames (X, R_X) and (D, R_D) and any map $\pi : D \to X$, the following are equivalent:*

- *π is a Kripke sheaf (as defined in Subsection 3.3).*
- *π satisfies (iv) and (v) below.*
- *π satisfies (iv) and (vi) below.*

(iv) *π is a p-morphism.*

(v) *For every $a \in D$, $\pi{\restriction}\left(\overrightarrow{R_D}(a)\right)$ is bijective.*

(vi) *The diagonal map Δ is a p-morphism.*

Recall the facts that maps of sheaves are local homeomorphisms or Kripke sheaves. Therefore we can summarize the facts above by saying that, in sheaf semantics, all the maps we use to interpret the first-order part of first-order modal logic—projections π and p_n, interpretations $[\![\, \bar{y} \mid t \,]\!]$ of terms, and the diagonal map Δ—are open maps or p-morphisms, and indeed that, in order for this to be the case, we must take a sheaf.

The logical reason we need all the relevant maps to be open maps or p-morphisms is the following. Any map $f : Y \to X$ is continuous or monotone iff $f^{-1}[\mathbf{int}_X(B)] \subseteq \mathbf{int}_Y(f^{-1}[B])$ for all $B \subseteq X$, and moreover it is an open map or a p-morphism iff $f^{-1}[\mathbf{int}_X(B)] = \mathbf{int}_Y(f^{-1}[B])$ for all $B \subseteq X$ (in the Kripke case, we take Kripke neighborhood frames and their interior operations). This equality, in case $f = \Delta$

[13] The equivalence between the first and third clauses is Exercise II.10(a) in [16], 104.

for instance, amounts to commutativity of

$$\begin{tikzcd}
\llbracket y, z \mid \varphi(y,z) \rrbracket \arrow[r, mapsto, "\mathbf{int}_{D^2}"] \arrow[d, mapsto, "\Delta^{-1}"'] \arrow[dr, phantom, "/\!\!/"] & \llbracket y, z \mid \Box\varphi(y,z) \rrbracket \arrow[d, mapsto, "\Delta^{-1}"] \\
\llbracket y \mid \varphi(y,y) \rrbracket \arrow[r, mapsto, "\mathbf{int}_D"'] & \llbracket y \mid \Box\varphi(y,y) \rrbracket
\end{tikzcd}$$

In other words, as long as we identify the two sentences

- $\Box([y/z]\varphi)$, obtained by first substituting y for z in φ and then applying $\Box$,
- $[y/z](\Box\varphi)$, obtained by first applying $\Box$ to φ and then substituting t for z,

the well-definedness of $\llbracket y \mid \Box\varphi(y,y) \rrbracket$ requires Δ be an open map. Similar arguments go with other relevant maps;[14] therefore, on certain assumptions regarding the syntax, the well-definedness of the semantics requires that all relevant maps, including π and Δ, be open maps, and hence, by Facts 4.1 and 4.2, that π (and other relevant maps) be sheaves.

4.2 Sheaves over a Neighborhood Frame

Our analysis in Subsection 4.1 points to the idea of generalization that we should define a sheaf over a neighborhood frame in terms of $f^{-1}[\mathbf{int}_X(B)] = \mathbf{int}_Y(f^{-1}[B])$ being satisfied by all the relevant maps f.

Let us say that a map $f : Y \to X$ between neighborhood frames is continuous iff $f^{-1}[\mathbf{int}_X(B)] \subseteq \mathbf{int}_Y(f^{-1}[B])$ for all $B \subseteq X$, and moreover that f is an open map iff $f^{-1}[\mathbf{int}_X(B)] = \mathbf{int}_Y(f^{-1}[B])$ for all $B \subseteq X$—so this definition subsumes the topological case, as well as the Kripke case of monotone maps and p-morphisms. Also, let us say that a neighborhood frame is *MC* if it satisfies ($\mathrm{M}_{\mathcal{N}}$) and ($\mathrm{C}_{\mathcal{N}}$); topological spaces and Kripke frames are MC neighborhood frames. Then we enter

Definition 4.3 Given MC neighborhood frames X and D, we say that a map $\pi : D \to X$ is a *local isomorphism* if

(vii) π is an open map, and

(viii) for every $a \in D$ such that $\mathcal{N}_D(a) \neq \varnothing$, there is $U \in \mathcal{N}_D(a)$ such that $\pi{\restriction}U : U \to \pi[U]$ is bijective.

We call such a pair (D, π) a *neighborhood sheaf* over X.

While (vii) subsumes (i) and (iv), (viii) also clearly subsumes (ii) and (v); so, neighborhood sheaves generalize topological and Kripke ones. Moreover, extending the definitions for the topological and Kripke cases, we define maps of sheaves over X to be continuous maps over X. Then

Fact 4.4 *Maps of sheaves are local isomorphisms. Therefore the category of neighborhood sheaves over a given MC neighborhood frame X is* $\mathbf{LI}/X$*, the category of*

[14] See [2] for more details.

local isomorphisms over X.

In addition, we also have

Fact 4.5 *For any MC neighborhood frames X and D and any open map $\pi : D \to X$, (viii) is the case iff*

(ix) *The diagonal map Δ is an open map.*

That is, in the same way as we did with topological and Kripke sheaves, we have all the relevant maps open if and only if we take sheaves, which is the desideratum in generalizing the topological and Kripke notions of a sheaf.

$\mathbf{LI}/X$ has a finite product, but we should explicitly define what an n-fold product of a neighborhood sheaf is. It goes in essentially the same way as in $\mathbf{LH}/X$. Let (D, π) be a neighborhood sheaf over an MC neighborhood frame X; then its n-fold product in $\mathbf{LI}/X$ is (D^n, π^n), where D^n has the following neighborhood function $\mathcal{N}_{D^n}$: For $i \leqslant n$, write $p_i : D^n \to D :: (a_1, \ldots, a_n) \mapsto a_i$. Given $\bar{a} \in D^n$, define $\mathcal{B} : X \to \mathcal{PP}X$ so that $\mathcal{B}(\bar{a}) = \{ \bigcap_{i \leqslant n} {p_i}^{-1}[U_i] \mid U_i \in \mathcal{N}_D(a_i) \text{ for each } i \leqslant n \}$, and then set $A \in \mathcal{N}_{D^n}(\bar{a})$ iff there is $B \in \mathcal{B}(\bar{a})$ such that $B \subseteq A$. Then (D^n, π^n) is a sheaf and each projection is a map of sheaves.

In this way, all the definitions and all the nice properties we reviewed for $\mathbf{LH}/X$ and $\mathbf{KrSh}/(X, R)$ are naturally generalized for the category $\mathbf{LI}/X$ of neighborhood sheaves. Indeed, the neighborhood notion of a sheaf subsumes the topological and Kripke ones, in the following strong form.

Fact 4.6 *Given any topological space X, $\mathbf{LH}/X \cong \mathbf{LI}/X$, where X is regarded as a neighborhood frame. Also, given any Kripke frame (X, R), $\mathbf{KrSh}/(X, R) \cong \mathbf{LI}/(X, R)$, where (X, R) is regarded as a neighborhood frame.*

4.3 Neighborhood-Sheaf Semantics for First-Order Modal Logic

In Subsection 4.2, we saw that all the relevant facts on the category $\mathbf{LH}/X$ of topological sheaves extended to the category $\mathbf{LI}/X$ of neighborhood sheaves. Therefore neighborhood-sheaf semantics is obtained by simply substituting "neighborhood" and "local isomorphism" for "topological" and "local homeomorphism" in Definition 3.3. It generalizes and unifies the other two sheaf semantics, because the neighborhood notion of a sheaf subsumes and unifies the topological and Kripke notions.

The following conceptual observation is worth making on (v) and (viii): The Kripke version, (v), states that $b \in \overrightarrow{R_D}(a)$ correspond one-to-one to their residences $\pi(b)$; in other words, $\overrightarrow{R_D}(a)$ is *the* "transworld identification" of a, so that a satisfies $\Box\varphi(x)$ iff all its "counterparts" $b \in \overrightarrow{R_D}(a)$ satisfy $\varphi(x)$. By contrast, the neighborhood version, (viii), states that $\mathcal{N}_D$ gives $a \in X$ *some* transworld identification *or other* $U \in \mathcal{N}_D(a)$. (Recall the contrast between "the" and "some" we discussed in Subsection 2.4.)

Finally, the new semantics unifies classical first-order logic and the modal logic corresponding to ($\mathrm{M}_{\mathcal{N}}$) and ($\mathrm{C}_{\mathcal{N}}$). Let **FOMC** be the first-order modal logic that consists of 1) all axioms and rules of classical first-order logic, and 2) the rule M and axiom C of modal logic. This logic is obviously weaker than **FOS4**, but also properly weaker than the logic **FOK** of Kripke-sheaf semantics, since it lacks the rule N. For instance, **FOK** does, but **FOMC** does not, prove $x = y \vdash \Box(x = y)$, whose proof requires N. (We can use M in place of N to prove $\Box\varphi \wedge (x = y) \vdash \Box(x = y)$, a theorem stating that "If anything is necessary, identity is necessary", though it may be that nothing is necessary.) Again, it is straightforward that **FOMC** is sound with respect to neighborhood-sheaf semantics. Moreover, it is complete in the form extending Theorem 3.4. Although we do not provide it here, the proof extends that of Theorem 3.4 (sketched in [2] and fully given in [3]) rather straightforwardly.

Theorem 4.7 *For any consistent theory* $\mathbb{T}$ *of first-order modal logic containing* **FOMC***, there exists a neighborhood-sheaf interpretation* $(\pi, [\![\cdot]\!])$ *that validates all and only theorems of* $\mathbb{T}$*.*

References

[1] Arló-Costa, H. and E. Pacuit, *First-order classical modal logic*, Studia Logica **84** (2006), pp. 171–210.

[2] Awodey, S. and K. Kishida, *Topology and modality: the topological interpretation of first-order modal logic*, Review of Symbolic Logic **1** (2008), pp. 146–66.

[3] Awodey, S. and K. Kishida, *Topological completeness of first-order modal logic* (2008), manuscript.

[4] Braüner, T. and S. Ghilardi, *First-order modal logic*, in: P. Blackburn, J. van Benthem and F. Wolter, editors, "Handbook of Modal Logic", Elsevier, Oxford, 2007, pp. 549–620.

[5] Butz, C., *A topological completeness theorem*, Archive for Mathematical Logic **38** (1999), pp. 79–101.

[6] Butz, C., and I. Moerdijk, *An elementary definability theorem for first order logic*, Journal of Symbolic Logic **64** (1999), pp. 1028–36.

[7] Fourman, M. and D. Scott, *Sheaves and logic*, in: M. Fourman, C. Mulvey and D. Scott, editors, "Applications of Sheaves", Springer, Berlin, 1979, pp. 302–401.

[8] Gabbay, D. M., V. Shehtman and D. Skvortsov, "Quantification in Nonclassical Logic", Volume 1, Elsevier, Oxford, 2009.

[9] Ghilardi, S., *Presheaf semantics and independence results for some non classical first order logics*, Archive for Mathematical Logic **29** (1989), pp. 125–36.

[10] Ghilardi, S. and G. Meloni, *Modal and tense predicate logic: models in presheaves and categorical conceptualization*, in: F. Borceux, editor, "Categorical Algebra and Its Applications", Springer, Berlin, 1988, pp. 130–42.

[11] Goldblatt, R., "Topoi: The Categorial Analysis of Logic", North-Holland, Amsterdam and New York, 1979.

[12] Kripke, S., "Semantical considerations on modal logic", Acta Philosophica Fennica **16** (1963), pp. 83–94; reprinted in L. Linsky, editor, "Reference and Modality", Oxford University Press, London, 1971, pp. 63–72.

[13] Lawvere, W. F., *Adjointness in foundations*, Dialectica **23** (1969), pp. 281–95.

[14] Lawvere, W. F., *Quantifiers and sheaves*, Actes du Congrès International des Mathémathciens **1** (1970), pp. 329–34.

[15] Lewis, D., *Counterpart theory and quantified modal logic*, Journal of Philosophy **65** (1968), pp. 113–26; reprinted with postscripts in his "Philosophical Papers", Volume I, Oxford University Press, New York and Oxford, 1983, pp. 26–46.

[16] Mac Lane, S. and I. Moerdijk, "Sheaves in Geometry and Logic: A First Introduction to Topos Theory", Springer, Berlin, 1992.

[17] Makkai, M. and G. E. Reyes, *Completeness results for intuitionistic and modal logic in a categorical setting*, Annals of Pure and Applied Logic **72** (1995), pp. 25–101.

[18] McKinsey, J. C. C. and A. Tarski, *The algebra of topology*, Annals of Mathematics **45** (1944), pp. 141–91.

[19] Moerdijk, I., *Some topological spaces which are universal for intuitionistic predicate logic*, Indagationes Mathematicae **44** (1982), pp. 227–35.

[20] Montague, R., *Universal grammar*, Theoria **36** (1970), pp. 373–98.

[21] Scott, D., *Advice on modal logic*, in: K. Lambert, editor, "Philosophical Problems in Logic", D. Reidel, Dordrecht, 1970, pp. 143–73.

[22] Segerberg, K., "An Essay in Classical Modal Logic", No. 13 in Filosofiska Studier, Uppsala University, 1971.

[23] Shehtman, V. and D. Skvortsov, *Semantics of non-classical first-order predicate logics*, in: P. P. Petkov, editor, "Mathematical Logic", Plenum Press, New York, 1990, pp. 105–16.