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Decomposing Split Graphs into Locally Irregular Graphs [1,2]

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Abstract

A graph is locally irregular if any pair of adjacent vertices have distinct degrees. A locally irregular decomposition of a graph G is a decomposition of G into locally irregular subgraphs. A graph is said to be decomposable if it admits a locally irregular decomposition. In this paper we prove that any decomposable split graph whose clique has at least 10 vertices can be decomposed into at most three locally irregular subgraphs. Furthermore, we characterize those whose decomposition can be into one or two locally irregular subgraphs.

Keywords: Locally irregular, graph decomposition, split graphs.

1 Introduction

All graphs considered in this text are simple and finite, i.e., contain neither loops nor parallel edges and their vertex and edge sets are finite, and we follow standard notation and terminology (see [6,7]).

Given a graph G, a collection $\mathcal{D} = \{H_1, H_2, \ldots, H_k\}$ of subgraphs of G is a *decomposition* of G if $\{E(H_1), E(H_2), \ldots, E(H_k)\}$ is a partition of $E(G)$. A graph is *locally irregular* if any pair of adjacent vertices have distinct degrees. A *locally irregular decomposition* of a graph G is a decomposition $\mathcal{D}$ of G such that every

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subgraph $H \in \mathcal{D}$ is locally irregular. Not all graphs admit a locally irregular decomposition; take for example the K_3 (complete graph with three vertices). We say that a graph is *decomposable* if it admits a locally irregular decomposition. Given a decomposable graph G, the *irregular chromatic index* of G, denoted by $\chi'_{\mathrm{irr}}(G)$, is the smallest size of a locally irregular decomposition of G. Alternatively, one can see a locally irregular decomposition of a graph G as being an edge coloring of G such that each color induces a locally irregular graph. We call such coloring of a *locally irregular edge coloring*.

This problem arose from the question "what is the antonym of regular graphs?". The natural answer, which is "graphs for which all vertices have distinct degrees" is immediately infeasible, since it is well-known that all graphs have at least two vertices with same degree. The notion of locality is thus considered. It is worth mentioning that there are other attempts to answer the question, such as the class of highly irregular graphs [1]. Also, the locally irregular edge coloring is closely related to the 1-2-3 Conjecture, the 1-2 Conjecture, and to the concepts of distinguishing colorings and irregularity strength [5,2].

Locally irregular decomposition as defined above was introduced by Baudon, Bensmail, Przybyło, and Woźniak [2]. They characterized all graphs that are decomposable and another main result is that d-regular graphs with $d \geq 10^7$ have a locally irregular 3-edge coloring. They also showed a locally irregular 3-edge coloring for trees that are not an odd-length path, and for K_n with $n \geq 4$, and showed a locally irregular 2-edge coloring for regular bipartite graphs with minimum degree at least 3, and hypercubes Q_n with $n \geq 2$. Furthermore, the authors showed that $\chi'_{\mathrm{irr}}(G) \leq \lfloor |E(G)|/2 \rfloor$ for all decomposable graphs G and proposed the following conjecture.

Conjecture 1.1 (Baudon, Bensmail, Przybyło, and Woźniak, 2015 [2]) *If G is a decomposable graph, then $\chi'_{\mathrm{irr}}(G) \leq 3$.*

Although we can check in polynomial time whether a graph G is locally irregular, deciding if there exists a locally irregular 2-edge coloring of G is NP-complete, even when restricted to planar graphs with maximum degree at most 6 [3]. Note that proving Conjecture 1.1 would show that deciding whether there exists a locally irregular k-edge coloring for $k \geq 3$ is in P.

A result from Przybyło [9] shows that every graph with minimum degree at least 10^{10} has a locally irregular 3-edge coloring. Bensmail, Merker, and Thomassen [4] gave the first constant upper bound on $\chi'_{\mathrm{irr}}(G)$, showing that $\chi'_{\mathrm{irr}}(G) \leq 328$ for any decomposable graph G. They also showed that every 16-edge-connected bipartite graph G has $\chi'_{\mathrm{irr}}(G) \leq 2$ and every bipartite graph G has $\chi'_{\mathrm{irr}}(G) \leq 10$. Lužar, Przybyło, and Soták [8] then improved the previous results showing that $\chi'_{\mathrm{irr}}(G) \leq 220$ for any decomposable graph G and that $\chi'_{\mathrm{irr}}(G) \leq 7$ for any bipartite graph G. They also showed that if G is subcubic, then $\chi'_{\mathrm{irr}}(G) \leq 4$.

Our focus is to characterize decomposable split graphs. A graph G is *split* if there exists a partition $\{X, Y\}$ of $V(G)$ such that $G[X]$ is a complete graph and Y is a stable set. When defining a split graph G for the first time, we usually write

$G(X,Y)$ to also define a partition $\{X,Y\}$ of $V(G)$, where X is a maximal clique and $Y = V(G) \setminus X$ is a stable set.

Let $G(X,Y)$ be a split graph with $X = \{v_1, \dots, v_n\}$. For any $v_i \in X$, we denote by $d_G(v_i, Y)$ the number of neighbors of v_i in Y, i.e., $d_G(v_i, Y) = |N_G(v_i) \cap Y|$. For simplicity we just write d_i for $d_G(v_i, Y)$ whenever the graph G and the stable set Y are clear from the context.

It is easy to verify that for split graphs $G(X,Y)$ with $X = \{v_1, \dots, v_n\}$, we have $\chi'_{\text{irr}}(G) = 1$ if and only if $d_1 > \cdots > d_n$. In fact, since X is a maximal clique, $d_G(y) \leq n-1$ for all $y \in Y$, which implies that $d_G(y)$ is smaller than the degree of all its neighbors in X. Therefore, G is locally irregular if and only if the vertices in X have distinct degrees in G, which is possible if and only if $d_1 > \cdots > d_n$, and hence the result follows. We state this in the following fact.

Fact 1.2 *Let $G(X,Y)$ be a split graph with $X = \{v_1, \dots, v_n\}$ where $d_1 \geq \cdots \geq d_n$ and $n \geq 2$. We have $\chi'_{\text{irr}}(G) = 1$ if and only if $d_1 > \cdots > d_n$.*

Our main result is Theorem 1.3, which describes $\chi'_{\text{irr}}(G)$ for all split graphs $G(X,Y)$ such that X is a clique with at least 10 vertices.

Theorem 1.3 *Let $G(X,Y)$ be a split graph with $X = \{v_1, \dots, v_n\}$ where $d_1 \geq \cdots \geq d_n$. If $n \geq 10$, then the following holds.*

(i) $\chi'_{\text{irr}}(G) \leq 2$ if and only if $d_1 \geq \lfloor n/2 \rfloor$ or $d_2 \geq 1$;

(ii) $\chi'_{\text{irr}}(G) = 3$ if and only if $d_1 < \lfloor n/2 \rfloor$ and $d_2 = 0$.

Theorem 1.3 is proved in Section 2.

In the remainder of the paper, given a graph G and a coloring $\varphi\colon E(G) \to \{\text{red}, \text{blue}\}$, we denote the two edge-disjoint spanning monochromatic subgraphs under φ by $G_{\text{red},\varphi}$ and $G_{\text{blue},\varphi}$. Formally,

$$G_{\text{red},\varphi} = (V(G), \varphi^{-1}(\text{red})) \quad \text{and} \quad G_{\text{blue},\varphi} = (V(G), \varphi^{-1}(\text{blue})) \ .$$

We may omit the term φ from $G_{\text{red},\varphi}$ and $G_{\text{blue},\varphi}$ whenever φ is clear from the context. This notation naturally extends to colorings that use more than two colors.

2 Decomposing split graphs with a large maximal clique

In this section we give a characterization of the irregular chromatic index of all split graphs with a maximal clique that has at least 10 vertices.

Let $G(X,Y)$ with $X = \{v_1, \dots, v_n\}$ with $n \geq 10$. In Lemma 2.3 we prove that $d_1 < \lfloor n/2 \rfloor$ and $d_2 = 0$ implies $\chi'_{\text{irr}}(G) = 3$. We also prove that if $d_1 \geq \lfloor n/2 \rfloor$ or $d_2 \geq 1$, then $\chi'_{\text{irr}}(G) \leq 2$, which follows directly from Lemmas 2.4 and 2.7. Therefore, note that Theorem 1.3 follows from Lemmas 2.3, 2.4 and 2.7.

In Section 2.1 we prove Lemmas 2.3 and 2.4. The starting point for proving these lemmas is a specific coloring of $E(K_n)$, which we call *normal*, given in Definition 2.1. In Section 2.2 we prove Lemma 2.7. For proving this result, we start with an intricate coloring of $E(K_n)$, which we call *strange* (see Definition 2.5).

Given a graph G, we say that the edge $uv \in E(G)$ is a *conflicting edge* if $d_G(u) = d_G(v)$.

2.1 Normal colorings of complete graphs

We start this section by defining *normal colorings* of complete graphs. See Figure 1 for example.

Definition 2.1 (*Normal colorings*) *Given a complete graph G with n vertices and a sequence $\boldsymbol{V} = (v_1, \ldots, v_n)$ of $V(G)$, the* normal coloring for $\boldsymbol{V}$ *is the 2-edge coloring $\varphi: E(G) \to \{\text{red}, \text{blue}\}$ defined as follows, where $X_1 = \{v_1, \ldots, v_{\lceil n/2 \rceil}\}$ and $X_2 = V(G) \setminus X_1$:*

(*i*) *$G_{\text{red}}[X_1]$ is a complete graph;*

(*ii*) *$G_{\text{red}}[X_2]$ contains no edges;*

(*iii*) *$N_{G_{\text{red}}}(v_i) = \{v_1, \ldots, v_{n-i+1}\}$ for $\lceil n/2 \rceil + 1 \leq i \leq n$.*

Note that in a normal coloring of a complete graph G for a sequence $\boldsymbol{V} = (v_1, \ldots, v_n)$, we have

- $d_{G_{\text{red},\varphi}}(v_i) = n - i$, for $1 \leq i \leq \lceil n/2 \rceil$;
- $d_{G_{\text{red},\varphi}}(v_i) = n - i + 1$, for $\lceil n/2 \rceil + 1 \leq i \leq n$.

Therefore, we know that, for a normal coloring φ of G,

$$\text{the only vertices with same degree in } G_{\text{red},\varphi} \text{ (and } G_{\text{blue},\varphi}) \text{ are } v_{\lceil n/2 \rceil} \text{ and } v_{\lceil n/2 \rceil + 1}\ . \tag{1}$$

From the definition of normal colorings and (1), since there is a (unique) conflicting edge $v_{\lceil n/2 \rceil} v_{\lceil n/2 \rceil + 1}$, we know that if n is even (resp. odd), then G_{blue} (resp. G_{red}) is locally irregular and G_{red} (resp. G_{blue}) is not locally irregular.

The following proposition will be useful for proving Lemma 2.3.

Proposition 2.2 *Let G be a connected graph with $V(G) = \{u_1, u_2, \ldots, u_n\}$ and $d_G(u_1) \geq \cdots \geq d_G(u_n)$. If G contains only one pair of vertices u, v with $d_G(u) = d_G(v)$, then the following holds:*

(*i*) *$d_G(u) = d_G(v) = \lfloor n/2 \rfloor$;*

(*ii*) *$u = u_{\lceil n/2 \rceil}$ and $v = u_{\lceil n/2 \rceil + 1}$;*

(*iii*) *$X = \{u_1, \ldots, u_{\lceil n/2 \rceil - 1}\}$ is a clique and $Y = \{u_{\lceil n/2 \rceil + 2}, \ldots, u_n\}$ is a stable set;*

(*iv*) *$X \subseteq N_G(u) \cap N_G(v)$;*

(*v*) *$(N_G(u) \cup N_G(v)) \cap Y = \emptyset$;*

(*vi*) *$uv \in E(G)$ if and only if n is even.*

Proof The proof follows by induction on n. If $n = 2$, then $G \simeq K_2$, and if $n = 3$, then $G \simeq P_3$. In both cases, (*i*)-(*vi*) hold. Thus, we may assume that $n \geq 4$.

Since G is a connected graph with n vertices and u and v are the only vertices of G with the same degree, there are $n - 1$ distinct values of degrees in G. Moreover,

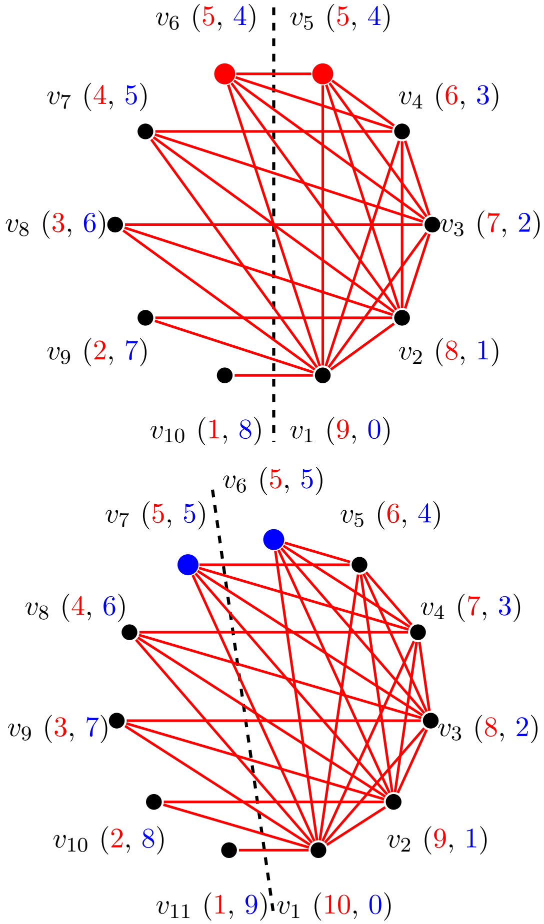

Figure 1. Normal colorings of $E(K_n)$ with sequence $(v_1, \dots, v_n)$, for $n = 10, 11$. The vertices of the conflicting edge are highlighted. For better visualization, we omit all blue edges.

since G is connected, for any vertex w of G we have $1 \le d_G(w) \le n-1$, and as a result of this, we know that the set of degrees of all vertices in G is $\{1, 2, \dots, n-1\}$. Therefore, $d_G(u_1) = n-1$ and $d_G(u_n) = 1$. Let $G' = G - \{u_1, u_n\}$. Note that $d_{G'}(w) = d_G(w) - 1$ for all $w \in V(G) \setminus \{u_1, u_n\}$.

We will show that G' is a connected graph with only one pair of vertices with the same degree. If $d_G(u) \in \{1, n-1\}$, then the vertices of G' have distinct degrees. In particular, there exists a non-trivial component of G' where all vertices have distinct degrees, which is an absurd. Thus, we may assume that $d_G(u) \notin \{1, n-1\}$, and hence G' has precisely two vertices with the same degree. Now note that graph G' has no trivial components, since u_n is the only vertex of G with degree 1. Also, if G' had more than one component, then it would contain a component where all vertices have distinct degrees, which is an absurd. Therefore, the graph G' is connected and contains precisely two vertices with the same degree (u and v), and hence, by induction hypothesis, (i)-(vi) hold for G'.

For clarity, let $V(G') = \{u'_1, u'_2, \dots, u'_{n'}\} = \{u_2, u_3, \dots, u_{n-1}\}$. Since (i) holds for G', and $d_{G'}(w) = d_G(w) - 1$ for all $w \in V(G) \setminus \{u_1, u_n\}$, we have $d_G(u) = d_G(v) = d_{G'}(u) + 1 = \lfloor n'/2 \rfloor = \lfloor (n-2)/2 \rfloor + 1 = \lfloor n/2 \rfloor$. Thus, *(i)* holds for G. Since *(ii)* holds for G', the vertices $u = u'_{\lceil n'/2 \rceil} = u_{\lceil (n-2)/2 \rceil + 1} = u_{\lceil n/2 \rceil}$ and $v = u'_{\lceil n'/2 \rceil + 1} = u_{(\lceil (n-2)/2 \rceil + 1) + 1} = u_{\lceil n/2 \rceil + 1}$ have the same degree in G', and consequently in G, and hence *(ii)* holds for G. Since *(iii)* holds for G', the set $X' = \{u_2, \ldots, u_{\lceil n/2 \rceil - 1}\}$ is a clique of G', the set $Y' = \{u_{\lceil n/2 \rceil + 2}, \ldots, u_{n-1}\}$ is a stable set of G', and hence $X = \{u_1\} \cup X'$ is a clique of G and $Y = \{u_n\} \cup Y'$ is a stable set of G, from where we conclude that *(iii)* holds for G. Since *(iv)* holds for G', and $X' \subseteq N_{G'}(u) \cap N_{G'}(v)$, and since $d_G(u_1) = n - 1$, we have $X \subseteq N_G(u)$ and $X \subseteq N_G(v)$. Therefore, *(iv)* holds for G. Since *(v)* holds for G', and $(N_{G'}(u) \cup N_{G'}(v)) \cap Y' = \emptyset$, and since u_n has degree 1 in G and $u_1 u_n \in E(G)$, we have $N_G(u) \cap Y = \emptyset$ and $N_G(v) \cap Y = \emptyset$. Therefore, *(v)* holds for G. Finally, since *(vi)* holds for G', and G' has $n - 2$ vertices, *(vi)* holds for G, which finishes the proof. □

Lemma 2.3 *Let $G(X,Y)$ be a split graph with $X = \{v_1, \ldots, v_n\}$ where $d_1 \geq \cdots \geq d_n$ and $n \geq 4$. If $d_1 < \lfloor n/2 \rfloor$ and $d_2 = 0$, then $\chi'_{\mathrm{irr}}(G) = 3$.*

Proof *Sketch.* Let $G(X,Y)$ be a split graph with $X = \{v_1, \ldots, v_n\}$, $d_1 \geq \cdots \geq d_n$, $n \geq 4$, $d_1 < \lfloor n/2 \rfloor$, and $d_2 = 0$. We start by proving that $\chi'_{\mathrm{irr}}(G) \geq 3$. This is shown by supposing that there exists a locally irregular 2-edge coloring of G, $\varphi\colon E(G) \to \{\mathrm{red}, \mathrm{blue}\}$, which allows us to show that G_{red} satisfies Proposition 2.2. Then we find a contradiction by showing that the degree of v_1 in G_{red} is the same as one of its neighbors, contradicting the fact that G_{red} was locally irregular in the first place. To finish the proof, we exhibit a locally irregular 3-edge coloring of G. □

In the proof of Lemma 2.4 below the reader may find it useful to refer to Figure 1.

Lemma 2.4 *Let $G(X,Y)$ be a split graph with $X = \{v_1, \ldots, v_n\}$ where $d_1 \geq \cdots \geq d_n$ and $d_{\lfloor n/2 \rfloor} \geq 1$. If $n \geq 3$, then, $\chi'_{\mathrm{irr}}(G) \leq 2$.*

Proof There are two cases to consider depending on the parity of n, but the only difference in the proofs is the coloring we give to $E(G[X])$, which are symmetric. For n even, we start with a normal coloring $\varphi\colon E(G[X]) \to \{\mathrm{red}, \mathrm{blue}\}$ for the sequence $(v_1, \ldots, v_{n/2}, v_n, v_{n-1}, \ldots, v_{n/2+1})$. In case n is odd we consider a normal coloring of $E(G[X])$ for sequence $(v_{\lceil n/2 \rceil}, \ldots, v_n, v_{\lfloor n/2 \rfloor}, \ldots, v_1)$. Thus, for the rest of this proof, we may assume, without loss of generality, that n is even.

Let $X_1 = \{v_1, v_2, \ldots, v_{n/2}\}$ and $X_2 = X \setminus X_1$. From (1) we know that the only vertices with the same degree in $G_{\mathrm{red}}[X]$ or $G_{\mathrm{blue}}[X]$ are $v_{n/2}$ and v_n.

We will obtain a locally irregular 2-edge coloring of G from φ. We start by extending φ to a coloring φ' of $E(G)$ with colors red and blue in the following way. For all edges xy between X_1 and Y let $\varphi'(xy) = \mathrm{red}$, and for all edges xy between X_2 and Y let $\varphi'(xy) = \mathrm{blue}$.

Let us first analyze the graph $G_{\mathrm{red},\varphi'}$. Since $d_{n/2} \geq 1$, for every vertex $x \in X_1$ we have $d_{G_{\mathrm{red},\varphi'}}(x) > d_{G_{\mathrm{red},\varphi}}(x) \geq n/2$, and since $d_1 \geq \cdots \geq d_{n/2}$, the degree of any two vertices of X_1 remain different in $G_{\mathrm{red},\varphi'}$. Also, since there are no red edges between v_n and Y, we have $d_{G_{\mathrm{red},\varphi'}}(v_n) = n/2 < d_{G_{\mathrm{red},\varphi'}}(x)$ for every $x \in X_1$. The red degree of vertices $v_{n/2+1}, \ldots, v_n$ are the same in φ and φ', and the red degree of any vertex $y \in Y$ is at most $n/2$, since there are no red edges between X_2 and Y. Therefore, since the degrees of the vertices of X_1 in $G_{\text{red},\varphi'}$ are at least $n/2+1$, we conclude that $G_{\text{red},\varphi'}$ is locally irregular.

It remains to show that $G_{\text{blue},\varphi'}$ is locally irregular. Since φ is a normal coloring and n is even, we know that $G_{\text{blue},\varphi'}[X]$ is locally irregular. If there is no $y \in Y$ that has a neighbor $x \in X_2$ with $d_{G_{\text{blue},\varphi'}}(y) = d_{G_{\text{blue},\varphi'}}(x)$, then the result follows. Thus we may assume the opposite, i.e., there exist $y \in Y$ and $x \in X_2$ with the same degree in $G_{\text{blue},\varphi'}$. Since the maximum possible degree of a vertex of Y in $G_{\text{blue},\varphi'}$ is $n/2$ and the minimum degree of a vertex of X_2 in $G_{\text{blue},\varphi'}[X]$ is $n/2-1$, we conclude that

$$d_{G_{\text{blue},\varphi'}}(y) = d_{G_{\text{blue},\varphi'}}(v_n) = n/2.$$

Therefore, because of the pair y, v_n, the graph $G_{\text{blue},\varphi'}$ is not locally irregular. In this case, we can change the color of one or two edges in φ' to obtain a locally irregular 2-edge coloring φ'' for G, as we explain next.

If $d_{n/2} \geq 2$, then let φ'' be the coloring obtained from φ' by changing the color of yv_n from blue to red. We claim that the graphs $G_{\text{red},\varphi''}$ and $G_{\text{blue},\varphi''}$ are locally irregular. In fact, this holds since v_n has degree $n/2+1$ in $G_{\text{red},\varphi''}$ and every vertex in X_1 has degree at least $n/2+2$ in $G_{\text{red},\varphi''}$.

Now assume that $d_{n/2} = 1$ and let $z \in Y$ be the only neighbor of $v_{n/2}$ in Y. In this case consider the coloring φ'' obtained from φ' by changing the color of yv_n from blue to red and the color of $zv_{n/2}$ from red to blue. Although $d_{G_{\text{blue},\varphi''}}(y) = d_{G_{\text{blue},\varphi''}}(v_n) = (n/2) - 1$, they are not neighbors in $G_{\text{blue},\varphi''}$. Also, any vertex in $X_2 \setminus \{v_n\}$ has degree at least $(n/2)+1$ in $G_{\text{blue},\varphi''}$, so there are no conflicts in $G_{\text{blue},\varphi''}$ involving y or v_n. Note that we have $d_{G_{\text{red},\varphi''}}(v_n) = (n/2)+1$, but since $d_{G_{\text{red},\varphi''}}(v_{n/2}) = n/2$ and every vertex in $X_1 \setminus \{v_{n/2}\}$ has red degree at least $(n/2)+2$ in $G_{\text{red},\varphi''}$, there are no conflicts in $G_{\text{red},\varphi''}$ involving v_n. This also implies that, since $d_{G_{\text{red},\varphi''}}(v_{n/2}) = n/2$, there are no conflicts involving $v_{n/2}$ in $G_{\text{red},\varphi''}$. Furthermore, since $d_{G_{\text{blue},\varphi''}}(v_{n/2}) = n/2$ and any vertex in $X_2 \setminus \{v_n\}$ has degree at least $(n/2)+1$ in $G_{\text{blue},\varphi''}$, we conclude that there are no conflicts involving $v_{n/2}$ in $G_{\text{blue},\varphi''}$. Therefore, $G_{\text{red},\varphi''}$ and $G_{\text{blue},\varphi''}$ are locally irregular, and the result follows. □

2.2 Strange colorings of complete graphs

As in Section 2, we start by defining the colorings of complete graphs that are the starting point for proving the results in this section. The following definition is technical, so we refer the reader to Figure 2 for a better understanding of it.

Definition 2.5 (Strange coloring) *Given a complete graph G with n vertices and a sequence $\boldsymbol{V} = (v_1, \ldots, v_n)$ of $V(G)$, first consider a coloring $\varphi' \colon E(G) \to \{\text{red}, \text{blue}\}$ defined as follows, where $X_1 = \{v_1, \ldots, v_{\lceil n/2 \rceil}\}$ and $X_2 = V(G) \setminus X_1$:*

(i) $G_{\text{red}}[X_1]$ *is a complete graph;*

(ii) $G_{\text{red}}[X_2]$ *contains no edges;*

(iii) $N_{G_{\text{red},\varphi'}}(v_i) = \{v_1, \ldots, v_{n-i}\}$ *for* $\lceil n/2 \rceil + 1 \leq i \leq n-1$*;*

(*iv*) $\varphi'(v_1v_n) = \text{red}$;

(*v*) $\varphi'(v_{\lceil n/2\rceil+1}v_{\lfloor n/2\rfloor}) = \text{red}$;

(*vi*) *All other edges are blue.*

The strange coloring *of G for $\boldsymbol{V}$ is the coloring φ obtained from φ' by changing the color of the following edges, which we call* strange edges*:*

- $v_{\lfloor n/2\rfloor}v_{\lfloor n/2\rfloor-1}$ *becomes blue;*
- $v_{\lfloor n/2\rfloor-1}v_{n-1}$ *becomes red;*
- $v_1v_{\lceil n/2\rceil+1}$ *for $\lceil n/2\rceil$ even becomes blue;*
- $v_1v_{\lfloor n/2\rfloor+1}$ *for $\lceil n/2\rceil$ odd becomes blue;*
- $v_{n-2}v_{n-3},\ v_{n-4}v_{n-5},\ \ldots,\ v_{n/2+4}v_{n/2+3}$ *for $n = 0 \pmod 4$ become red;*
- $v_{n-1}v_{\lfloor n/2\rfloor-1},\ v_{n-2}v_{\lfloor n/2\rfloor},\ v_{n-3}v_{\lfloor n/2\rfloor+1},\ v_{n-4}v_{n-5},\ v_{n-6}v_{n-7},\ \ldots,\ v_{\lfloor n/2\rfloor+3}v_{\lfloor n/2\rfloor+2}$ *for $n = 1 \pmod 4$ become red;*
- $v_{n-2}v_{n-3},\ v_{n-4}v_{n-5},\ \ldots,\ v_{n/2+3}v_{n/2+2}$ *for $n = 2 \pmod 4$ become red;*
- $v_{n-2}v_{n-3},\ v_{n-4}v_{n-5},\ \ldots,\ v_{\lfloor n/2\rfloor+4}v_{\lfloor n/2\rfloor+3}$ *for $n = 3 \pmod 4$ become red.*

Note that in a strange coloring of a complete graph G for a sequence $\boldsymbol{V} = (v_1, \ldots, v_n)$, we have

$$d_{G_{\text{red},\varphi}}(v_i) = n - i + 1, \text{ for } 3 \le i \le n \ ,$$

$$d_{G_{\text{red},\varphi}}(v_1) = n - \lfloor n/2\rfloor - 1 \ ,$$

and

$$d_{G_{\text{red},\varphi}}(v_2) = \begin{cases} n - \lfloor n/2\rfloor - 2 & \text{if } \lceil n/2\rceil \text{ is even} \\ n - \lfloor n/2\rfloor - 1 & \text{otherwise} \end{cases} .$$

Therefore, we know that, for a strange coloring φ of G,

$$\text{the only vertices with same degree in } G_{\text{red},\varphi} \text{ are } v_1 \text{ and } v_{\lfloor n/2\rfloor+2} \ , \tag{2}$$

and

the only vertices with same degree in

$$G_{\text{blue},\varphi} \text{ are } \begin{cases} v_2 \text{ and } v_{\lfloor n/2\rfloor+3} & \text{if } \lceil n/2\rceil \text{ is even} \\ v_2, v_1, \text{ and } v_{\lfloor n/2\rfloor+1} & \text{otherwise.} \end{cases}$$

From the definition of strange coloring and by (2) and (3), we conclude that G_{red} has exactly one conflicting edge $v_1v_{\lfloor n/2\rfloor+2}$ while

$$G_{\text{blue}} \text{ has exactly } \begin{cases} \text{one conflicting edge } v_2v_{\lfloor n/2\rfloor+3} & \text{if } \lceil n/2\rceil \text{ is even} \\ \text{two conflicting edges } v_2v_{\lfloor n/2\rfloor+2} \text{ and } v_2v_1 & \text{otherwise.} \end{cases}$$

Given a graph G and a 2-edge coloring $\varphi\colon E(G) \to \{\text{red}, \text{blue}\}$, we say an even cycle $C = (u_1, \ldots, u_k, u_{k+1} = u_1)$ is an *alternating cycle in* φ if $\varphi(u_iu_{i+1}) \neq \varphi(u_{i+1}u_{i+2})$, for any $1 \le i < k$, and $d_{G_\gamma}(u_i) \neq d_{G_\gamma}(u_{i+1})$, for any $1 \le i \le k$

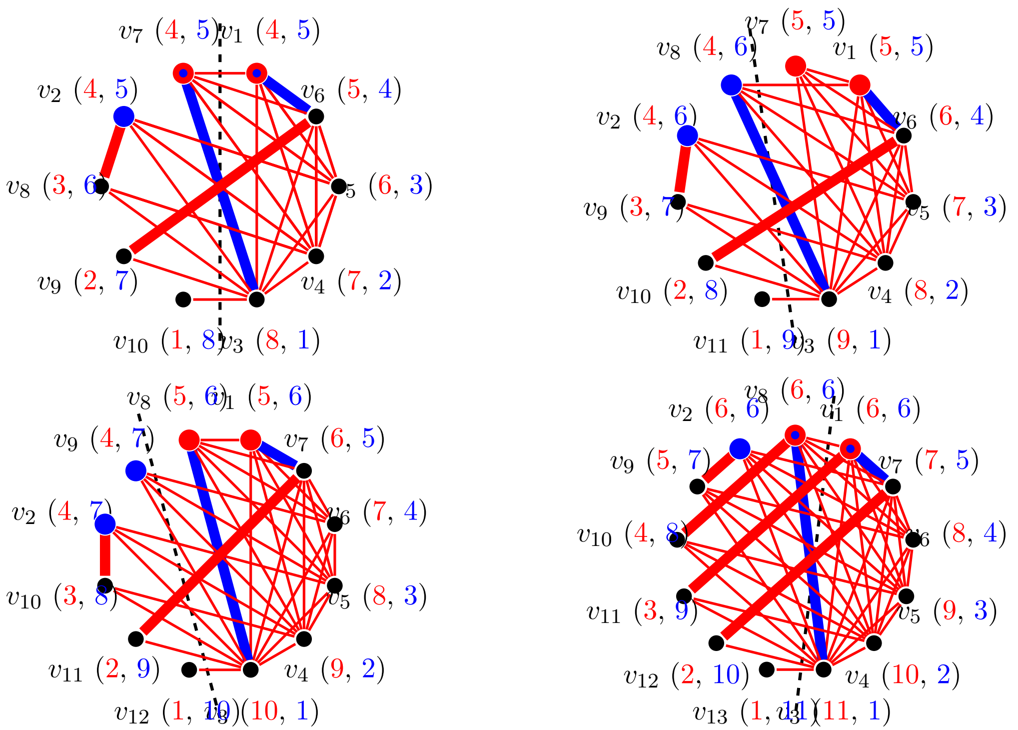

Figure 2. Strange colorings of $E(K_n)$ with $n \in \{10, 11, 12, 13\}$, where $V(K_n) = \{v_1, \ldots, v_n\}$ and the sequence is (v_3, …, $v_{\lfloor n/2 \rfloor+1}$, v_1, $v_{\lfloor n/2 \rfloor+2}$, $v_{\lfloor n/2 \rfloor+3}$, v_2, $v_{\lfloor n/2 \rfloor+4}$, …, v_n) if $\lceil n/2 \rceil$ is even, and it is (v_3, …, $v_{\lfloor n/2 \rfloor+1}$, v_1, $v_{\lfloor n/2 \rfloor+2}$, v_2, $v_{\lfloor n/2 \rfloor+3}$, …, v_n) otherwise. Strange edges are highlighted, as well as the vertices of the conflicting edges. For better visualization, we omit the other blue edges.

and $\gamma \in \{\text{red}, \text{blue}\} \setminus \varphi(u_i u_{i+1})$. In other words, an alternating cycle has incident edges with different colors and the endpoints of a red edge (resp. blue edge) have different blue degrees (resp. red degrees). Let φ' be the 2-edge coloring of $E(G)$ such that $\varphi'(uv) \in \{\text{red}, \text{blue}\} \setminus \varphi(uv)$ if $uv \in E(C)$ and $\varphi'(uv) = \varphi(uv)$ otherwise. We say φ' is *obtained from* φ *by inverting* C.

Lemma 2.6 shows that an inversion on an alternating cycle does not create conflicting edges. It will be used in the proof of Lemma 2.7.

Lemma 2.6 *Let G be a graph, $\varphi\colon E(G) \to \{\text{red}, \text{blue}\}$ be a 2-edge coloring of G. Let C be an alternating cycle in φ and let φ' be obtained from φ by inverting C. The set of conflicting edges of $G_{\gamma,\varphi'}$ is a subset of the conflicting edges of $G_{\gamma,\varphi}$, for any $\gamma \in \{\text{red}, \text{blue}\}$.*

Proof First note that, in φ, every vertex of $V(C)$ has one incident blue edge and one incident red edge, which remains valid in φ'. No other edge has its color changed. Thus, $d_{G_{\gamma,\varphi}}(v) = d_{G_{\gamma,\varphi'}}(v)$ for any $v \in V(G)$ and $\gamma \in \{\text{red}, \text{blue}\}$.

Let $\gamma \in \{\text{red}, \text{blue}\}$, $\bar{\gamma} \in \{\text{red}, \text{blue}\} \setminus \{\gamma\}$, and consider any edge $uv \in E(G)$. If $uv \notin E(C)$, then uv is conflicting in $G_{\gamma,\varphi}$ (resp. $G_{\bar{\gamma},\varphi}$) if and only if it is conflicting in $G_{\gamma,\varphi'}$ (resp. $G_{\bar{\gamma},\varphi'}$). So let $uv \in E(C)$ and let $\varphi(uv) = \gamma$. From the definition of alternating cycle, $d_{G_{\bar{\gamma},\varphi}}(u) \neq d_{G_{\bar{\gamma},\varphi}}(v)$. This means that uv is not conflicting in $G_{\bar{\gamma},\varphi}$, no matter if it is conflicting in $G_{\gamma,\varphi}$ or not. □

Lemma 2.7 *Let $n \geq 10$ and let $G(X,Y)$ be a split graph with $X = \{v_1, \dots, v_n\}$ where $d_1 \geq \cdots \geq d_n$ and $d_{\lfloor n/2 \rfloor} = 0$. If $d_1 \geq \lfloor n/2 \rfloor$ or $d_2 \geq 1$, then $\chi'_{\mathrm{irr}}(G) = 2$.*

Proof *Sketch.* First note that since $d_{\lfloor n/2 \rfloor} = 0$ we have $\chi'_{\mathrm{irr}}(G) \geq 2$ due to Fact 1.2.

Let $\varphi' \colon E(G[X]) \to \{\mathrm{red}, \mathrm{blue}\}$ be the strange coloring of $E(G[X])$ for the sequence

$$\pi = \begin{cases} (v_3, \dots, v_{\lfloor n/2 \rfloor+1}, v_1, v_{\lfloor n/2 \rfloor+2}, v_{\lfloor n/2 \rfloor+3}, v_2, v_{\lfloor n/2 \rfloor+4}, \dots, v_n) & \text{if } \lceil n/2 \rceil \text{ is even} \\ (v_3, \dots, v_{\lfloor n/2 \rfloor+1}, v_1, v_{\lfloor n/2 \rfloor+2}, v_2, v_{\lfloor n/2 \rfloor+3}, \dots, v_n) & \text{otherwise} \end{cases}. \tag{3}$$

We show how to obtain a locally irregular 2-edge coloring of G from φ'. We start by extending φ' to a coloring φ of $E(G)$ with colors red and blue. Let $X_{\mathrm{red}} = \{v_1, v_3, v_4, \dots, v_{\lfloor n/2 \rfloor - 1}\}$. We give color red to all edges between X_{red} and Y, and color blue to all edges between v_2 and Y. Note that this is a coloring of $E(G)$ because $d_{\lfloor n/2 \rfloor} = 0$.

We first show that if there is a conflicting edge in $G_{\mathrm{red},\varphi}$, then it is unique and there exists an alternating cycle that removes it. Then we show the same for $G_{\mathrm{blue},\varphi}$. □

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