# On Computable Metrization

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**Abstract**

Every second-countable regular topological space $X$ is metrizable. For a given "computable" topological space satisfying an axiom of computable regularity M. Schröder [9] has constructed a computable metric. In this article we study whether this metric space $(X, d)$ can be considered computationally as a subspace of some computable metric space [13]. While Schröder's construction is "pointless", i.e., only sets of a countable base but no concrete points are known, for a computable metric space a concrete dense set of computable points is needed. By partial completion we extend $(X, d)$ to a metric space $(\tilde{X}, \tilde{d})$ with computable metric and canonical representation. We construct a computable sequence $(x_i)_{i \in \mathbb{N}}$ of points which is dense in $(\tilde{X}, \tilde{d})$. The isometric embedding of $X$ into $\tilde{X}$ is computable. Its inverse is computable if some further computability axiom holds true. The space $(\tilde{X}, \tilde{d})$ can be embedded computationally into the computable metric space generated by the sequence $(x_i)_{i \in \mathbb{N}}$ of points. The inverse of this embedding is continuous.

*Keywords:* Computable Analysis, TTE, computable metrization, computable embedding, computable metric space

# 1 Introduction

Computable Analysis connects Computability/Computational Complexity with Analysis/Numerical Computation by combining concepts of approxima-

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tion and of computation. During the last 70 years various mutually non-equivalent models of real number computation have been proposed ([11], Chap. 9 in [13]). Among these models the representation approach (Type-2 Theory of Effectivity, TTE) [13] introduced for real functions by Grzegorczyk and Lacombe [6,7] seems to be particularly realistic, flexible and expressive. So far the study of computability on sets of points, sets (open, closed, compact) and continuous functions has developed mainly bottom-up, i.e., from the real numbers to Euclidean space and metric spaces [17,2,15,13,18,1,19]. But often generalisations to more general spaces are needed (locally compact Hausdorff spaces [3], non-metrizable spaces [16], second countable $T_0$-spaces [9,5]). In this article we study computability in basic topology using the representation approach to computable analysis.

Schröder [9] has shown that every "computably regular" space has a computable metric generating the same topology. In his proof he effectivizes a classical proof where the metric is defined from a sequence of pseudo-metics each of which is defined via an Urysohn function [4] which can be interpreted as a contour map on the space. On the other hand, in TTE *computable metric spaces* are defined axiomatically [13] by a dense sequence of points with computable distances. So the question arises whether the "computably regular" space with computable metric can be considered as a computable metric space. Schröder's top-down approach is pointless, that is, only basic open sets "are known" and handled via code names. Concrete points, in particular computable ones, are not considered. It is even possible that the space contains no computable points at all. Since every computable metric space must have a dense set of computable points a computably metrizable space cannot be a computable metric spaces in general. In this article we study whether Schröder's space with computable metric can be embedded into a computable metric space.

It has turned out that the concept of a computable topological space from [13] is not optimal for a top-down study of computability in topology. Therefore we start from a slightly different concept of "computable $T_0$-spaces" (already considered in [5]). In Section 3 we discuss the new definition, introduce (standard) representations of the points and the open sets and show that they meet our intuition very well by considering robustness and a basic intuitive idea of "$x \in O$" for an open set $O$.

Since for our construction of computable points we need details of Schröder's proof of computable metrization [9] (which as part of a TR is not easily accessible) in Section 4 we outline his proof in a slightly modified terminology. Most of the proofs from Sections 3 and 4 are omitted.

In Section 5 we show that the represented metric space $(X, \delta_{\mathbf{X}})$ has a

"weak completion" $(\tilde{X}, \tilde{\delta})$ in which we can find a computable dense set of points. The embedding from $(X, \delta_{\mathbf{X}})$ to $(\tilde{X}, \tilde{\delta_{\mathbf{X}}})$ is computable, its inverse is continuous and even computable in some cases. The computable dense set induces a Cauchy representation $\delta_C$ of the space. The embedding from $(\tilde{X}, \tilde{\delta})$ to $(\tilde{X}, \delta_C)$ is computable, its inverse is at least continuous.

## 2  Preliminaries

In this article we use the framework of TTE (Type-2 theory of effectivity), see [13] for more details. A partial function from $X$ to $Y$ is denoted by $f :\subseteq X \to Y$. We assume that $\Sigma$ is a fixed finite alphabet containing the symbols 0 and 1 and consider computable functions on finite and infinite sequences of symbols $\Sigma^*$ and $\Sigma^\omega$, respectively, which can be defined, for example, by Type-2 machines, i.e., Turing machines reading from and writing on finite or infinite tapes. We use the "wrapping function" $\iota : \Sigma^* \to \Sigma^*$, $\iota(a_1 a_2 \ldots a_k) := 110a_1 0a_2 0 \ldots a_k 011$ for coding words such that $\iota(u)$ and $\iota(v)$ cannot overlap properly. We consider standard functions for finite or countable tupling on $\Sigma^*$ and $\Sigma^\omega$ denoted by $\langle \cdot \rangle$. By "$\lhd$" we denote the subword relation.

We use the concept of multi-functions. A *multi-valued partial function*, or *multi-function* for short, from $A$ to $B$ is a triple $f = (A, B, \mathrm{R}_f)$ such that $\mathrm{R}_f \subseteq A \times B$ (the *graph* of $f$). Usually we will denote a multi-function $f$ from $A$ to $B$ by $f :\subseteq A \rightrightarrows B$. For $X \subseteq A$ let $f[X] := \{b \in B \mid (\exists a \in X)(a, b) \in \mathrm{R}_f\}$ and for $a \in A$ define $f(a) := f[\{a\}]$. Notice that $f$ is well-defined by the values $f(a) \subseteq B$ for all $a \in A$. We define $\mathrm{dom}(f) := \{a \in A \mid f(a) \neq \emptyset\}$. In the applications we have in mind, for a multi-function $f :\subseteq A \rightrightarrows B$, $f(a)$ is interpreted as the set of all results which are "acceptable" on input $a \in A$. Any concrete computation will produce on input $a \in \mathrm{dom}(f)$ some element $b \in f(a)$, but usually there is no method to select a specific one. In accordance with this interpretation the "functional" composition $g \circ f :\subseteq A \rightrightarrows D$ of $f :\subseteq A \rightrightarrows B$ and $g :\subseteq C \rightrightarrows D$ is defined by $\mathrm{dom}(g \circ f) := \{a \in A \mid a \in \mathrm{dom}(f) \text{ and } f(a) \subseteq \mathrm{dom}(g)\}$ and $g \circ f(a) := g[f(a)]$ (in contrast to "non-deterministic" or "relational" composition $gf$ defined by $g\, f(a) := g[f(a)]$ for *all $a \in A$*).

Notations $\nu :\subseteq \Sigma^* \to M$ and representations $\delta :\subseteq \Sigma^\omega \to M$ are used for introducing relative continuity and computability on "abstract" sets $M$. For a representation $\delta :\subseteq \Sigma^\omega \to M$, if $\delta(p) = x$ then the point $x \in M$ can be identified by the "name" $p \in \Sigma^\omega$.

For naming systems $\gamma_i :\subseteq Y_i \to M_i$ $(i = 0, \ldots, k)$, a function $h :\subseteq Y_1 \times \ldots \times Y_k \to Y_0$ is a $(\gamma_1, \ldots, \gamma_k, \gamma_0)$-realization of $f :\subseteq M_1 \times \ldots \times M_k \rightrightarrows M_0$, if $\gamma_0 \circ h(p_1, \ldots, p_k) \in f(\gamma_1(p_1), \ldots, \gamma_k(p_k))$ whenever $f(\gamma_1(p_1), \ldots, \gamma_k(p_k))$ exists.

The multi-function $f$ is $(\gamma_1, \ldots, \gamma_k, \gamma_0)$-continuous (–computable), if it has a continuous (computable) $(\gamma_1, \ldots, \gamma_k, \gamma_0)$-realization.

For naming systems $\gamma : \subseteq Y \rightrightarrows M$ and $\gamma' : \subseteq Y' \to M'$ $(Y, Y' \in \{\Sigma^*, \Sigma^\omega\})$, let $\gamma \leq_t \gamma'$ (t-reducible) and $\gamma \leq \gamma'$ (reducible), iff the identity id $: a \mapsto a$ $(a \in M)$ is $(\gamma, \gamma')$-continuous and $(\gamma, \gamma')$–computable, respectively. Define $t$-equivalence and equivalence as follows: $\gamma \equiv_t \gamma' \iff (\gamma \leq_t \gamma'$ and $\gamma' \leq_t \gamma)$ and $\gamma \equiv \gamma' \iff (\gamma \leq \gamma'$ and $\gamma' \leq \gamma)$, respectively.

Two representations induce the same continuity or computability, iff they are $t$-equivalent or equivalent, respectively. If multi-functions on represented sets have realizations, then their composition is realized by the composition of the realizations. In particular, the computable multi-functions on represented sets are closed under composition. Much more generally, the computable multi-functions on represented sets are closed under flowchart programming with indirect addressing [14]. This result allows convenient informal construction of new computable functions on multi-represented sets from given ones.

Let $\nu_\mathbb{N} : \subseteq \Sigma^* \to \mathbb{N}$ be some standard notation of the natural numbers, let $\rho, \rho_<$ and $\rho_>$ be the Cauchy representation, the lower representation and the upper representation of the real numbers, respectively. For representations $\delta : \subseteq \Sigma^\omega \to M$ and $\delta' : \subseteq \Sigma^\omega \to M'$ define $[\delta, \delta']\langle p, p' \rangle := (\delta(p), \delta(p'))$, $\delta \wedge \delta'\langle p, p'\rangle = x$, iff $\delta(p) = \delta'(p') = x$ and $[\delta]^\omega \langle p_0, p_1, p_2, \ldots \rangle := \delta(p_0) \times \delta(p_1) \times \delta(p_2) \times \ldots$ [13].

# 3   Representations of Points and Open Sets

In this section we define computable $T_0$-spaces and summarize arguments showing that the standard representations of points and open subsets are very natural.

A topological space $\mathbf{X} = (X, \tau)$ is a $T_0$-space, if for all $x, y \in X$ such that $x \neq y$ there is an open set $U \in \tau$ such that $x \in U \iff y \notin U$. In a $T_0$-space every point can be identified by the set of its neighbourhoods $U \in \tau$. A space $\mathbf{X}$ is called *second-countable*, if it has a countable base [4]. In the following we consider only second countable $T_0$-spaces. For introducing concepts of effectivity we assume that some notation $\nu$ of a base $\beta$ with recursive domain is given.

**Definition 3.1** [effective/computable $T_0$-space]

(i) An effective $T_0$-space is a tuple $\mathbf{X} = (X, \tau, \beta, \nu)$ such that $(X, \tau)$ is a second countable $T_0$-space and $\nu : \subseteq \Sigma^* \to \beta$ is a notation of a base $\beta$ of $\tau$ with recursive domain. We assume that $U \neq \emptyset$ for $U \in \beta$.

(ii) We call an effective $T_0$-space $\mathbf{X}$ a computable $T_0$-space, iff it has com-

putable intersection, that is, if there is a computable function $h : \subseteq \Sigma^* \times \Sigma^* \to \Sigma^\omega$ such that for all $u, v \in \text{dom}(\nu)$,

(1)  $\nu(u) \cap \nu(v) = \bigcup \{\nu(w) \mid w \in \text{dom}(\nu) \text{ and } \iota(w) \lhd h(u, v)\}$ .

Notice that for every effective $T_0$-space there is a function $h : \subseteq \Sigma^* \times \Sigma^* \to \Sigma^\omega$, which is *continuous* since $\Sigma^*$ is discrete, such that (1) holds true. Therefore, Part (ii) in Def. 3.1 does not restrict the topology. In the definition of a computable $T_0$-space the computable function $h$ can be replaced by an r.e. set: **X** is computable, iff there is an r.e. set $B \subseteq (\text{dom}(\nu))^3$ such that for all $u, v \in \text{dom}(\nu)$,

(2)  $\nu(u) \cap \nu(v) = \bigcup \{\nu(w) \mid (u, v, w) \in B\}$ .

Starting from a notation $\nu$ with r.e. domain is not more general since then there is an equivalent notation $\tilde\nu$ with recursive domain such that the representations of points and sets introduced below by $\nu$ and $\tilde\nu$ become equivalent.

In [13] Def. 3.2.1 an "effective topological space" is a triple $\mathbf{S} = (X, \sigma, \nu')$ such that $\nu' : \subseteq \Sigma^* \to \sigma$ is a notation of a countable *subbase* $\sigma$ of a $T_0$-topology $\tau$ on $X$. **S** is called "computable", if its equivalence problem $\{(u, v) \in (\text{dom}(\nu'))^2 \mid \nu'(u) = \nu'(v)\}$ is recursively enumerable. For a foundation of Computable Topology, this new definition 3.1 seems to be more useful than the former definition from [13]. We give a short review of the differences:

While $\nu$ now is a notation of a base $\beta$, $\nu'$ is a notation of a subbase $\sigma$. Of course, $\beta$ is also a subbase. On the other hand, from $\nu'$ the canonical notation $\bar\nu$ of the finite intersections of subbase elements is a notation of a base. But generally empty intersections cannot be excluded computationally.

We assume now that $\text{dom}(\nu)$ is recursive, but $\text{dom}(\nu')$ may be not even r.e. for an effective topological space but must be r.e. for a computable topological space.

While for an effective topological space the notation $\bar\nu$ satisfies (1) trivially, we require now (1) explicitly for a computable $T_0$-space. Finally, for a computable topological space the equivalence problem of $\nu'$ must be r.e. while for the equivalence problem of $\nu$ there is no restriction.

In the following let $\mathbf{X} = (X, \tau, \beta, \nu)$ be an effective $T_0$-space. For technical reasons we introduce

(3)  $D_\nu := \{q \in \Sigma^\omega \mid w \in \text{dom}(\nu) \text{ if } \iota(w) \lhd q\}$ .

First, we define the standard representation of $X$.

**Definition 3.2** [standard representation of $X$]
Define the standard representation $\delta_{\mathbf{X}} : \subseteq \Sigma^\omega \to X$ of **X** as follows: $\text{dom}(\delta_{\mathbf{X}}) \subseteq D_\nu$ and for all $x \in X$ and $p \in D_\nu$,

(4)   $\delta_{\mathbf{X}}(p) = x :\iff \{w \in \mathrm{dom}(\nu) \mid x \in \nu(w)\} = \{w \mid \iota(w) \triangleleft p\}$ .

A $\delta_{\mathbf{X}}$-name $p$ of an element $x \in X$ is a list of *all* words $w$ such that $x \in \nu(w)$. The definition of $\delta_{\mathbf{X}}$ corresponds to the definition of $\delta'_{\mathbf{S}}$ in [13], Lemma 3.2.3, in particular, $\delta_{\mathbf{X}}$ is *admissible* with final topology $\tau$ (Sec. 3.2 in [13]).

The definition of $\delta_{\mathbf{X}}$ is "robust".

**Definition 3.3** [recursively related spaces]
We call two effective $T_0$-spaces $\mathbf{X}_1 = (X, \tau, \beta_1, \nu_1)$ and $\mathbf{X}_2 = (X, \tau, \beta_2, \nu_2)$ *recursively related*, iff there are computable functions $g, g' : \subseteq \Sigma^* \to \Sigma^\omega$ auch that

(5)   $\nu_1(u) = \bigcup_{\iota(w) \triangleleft g(u)} \nu_2(w)$   and   $\nu_2(v) = \bigcup_{\iota(w) \triangleleft g'(v)} \nu_1(w)$ .

Equivalently, $\mathbf{X}_1$ and $\mathbf{X}_2$ are recursively related, iff there are r.e. sets $C, C' \subseteq \Sigma^* \times \Sigma^*$ such that

(6)   $\nu_1(u) = \bigcup_{(u,w) \in C} \nu_2(w)$   and   $\nu_2(v) = \bigcup_{(v,w) \in C'} \nu_1(w)$ .

In particular, $\mathbf{X}_1$ and $\mathbf{X}_2$ are recursively related, if $\nu_1 \equiv \nu_2$.

**Lemma 3.4 ($\delta_{\mathbf{X}}$ is robust)**
*For effective* $\mathrm{T}_0$*-spaces* $\mathbf{X}_1 = (X, \tau, \beta_1, \nu_1)$ *and* $\mathbf{X}_2 = (X, \tau, \beta_2, \nu_2)$,

(i)  $\delta_{\mathbf{X}1} \equiv_t \delta_{\mathbf{X}2}$ ;

(ii) $\delta_{\mathbf{X}1} \equiv \delta_{\mathbf{X}2}$  *if* $\mathbf{X}_1$ *and* $\mathbf{X}_2$ *are recursively related.*

For the representation $\delta_{\mathbf{X}}$ the relation $x \in U$ for $x \in X$ and $U \in \beta$ becomes recursively enumerable (more precisely $[\delta_{\mathbf{X}}, \nu]$-r.e., Definitions 3.1.3 and 3.3.3 in [13], see Lemma 3.7). We introduce the inner representation of the the set $\tau$ of the open and the outer representation of the set $\tau^c := \{O^c \mid O \in \tau\}$ of closed sets.

**Definition 3.5** [representations of open and closed sets]

(i) Define the inner representation $\theta^< : \subseteq \Sigma^\omega \to \tau$ by  $\mathrm{dom}(\theta^<) := D_\nu$ and

(7)   $\theta^<(p) := \bigcup_{\iota(w) \triangleleft p} \nu(w)$ .

(ii) Define the outer reperesentation $\psi^> : \subseteq \Sigma^\omega \to \tau^c$ by

$$\psi^>(p) := X \setminus \theta^<(p) .$$

Thus, $\theta^<(p)$ is the union of all $\nu(w)$ such that $w$ is listed by $p$. The outer representation of the closed sets is defined by the inner representation of their

complements. In [13] for subsets of the Euclidean space, $\theta^<$ is called $\theta^{\mathrm{en}}_<$ and $\psi^>$ is called $\psi^{\mathrm{en}}_>$. In [2,1] for computable metric spaces, $\psi^>$ is called $\delta_{\mathrm{union}}$.

The computable version of "$\beta \subseteq \tau$" is $\nu \leq \theta^<$. In fact, the computable function $h : v \mapsto \iota(v)000\ldots$ translates $\nu$ to $\theta^<$. Recursively related spaces induce equivalent set representations:

**Lemma 3.6** ($\theta^<$ **and** $\psi^>$ **are robust**)
*For effective* $\mathrm{T}_0$-*spaces* $\mathbf{X}_1 = (X, \tau, \beta_1, \nu_1)$ *and* $\mathbf{X}_2 = (X, \tau, \beta_2, \nu_2)$,

(i) $\theta^<_1 \equiv_t \theta^<_2$ *and* $\psi^>_1 \equiv_t \psi^>_2$ ;

(ii) $\theta^<_1 \equiv \theta^<_2$ *and* $\psi^>_1 \equiv \psi^>_2$, *if* $\mathbf{X}_1$ *and* $\mathbf{X}_2$ *are recursively related.*

The representation $\tilde{\theta}$ of the open sets where a name of $O$ lists *all* $u \in \mathrm{dom}(\nu_1)$ such that $\nu_1(u) \subseteq O$ is not useful since it is not robust in general (Theorem 5.1.14 in [13]).

As a fundamental idea about "open" the property $x \in O$ can be verified by looking close enough to (zooming in on) $x$ and $O$. In terms of computability the relation should be recursively enumerable: $x \in O$ is true, iff we will know this fact after a finite computation. For a discussion of the concept of open see [10]. By the following lemma the representations $\delta_{\mathbf{X}}$ and $\theta^<$ can be characterized up to equivalence as maximal elements among representations for which the element relation is open or r.e., respectively. It substantiates again that the representations $\delta_{\mathbf{X}}$ and $\theta^<$ are very "natural". Let "$x \in U$":= $\{(x,U) \in X \times \beta \mid x \in U\}$ and "$x \in O$":= $\{(x,O) \in X \times \tau \mid x \in O\}$.

**Lemma 3.7** *Let* $\mathbf{X} = (X, \tau, \beta, \nu)$ *be an effective* $\mathrm{T}_0$-*space. Then for every representation* $\delta : \subseteq \Sigma^\omega \to X$ *and* $\theta : \subseteq \Sigma^\omega \to \tau$,

(8)  "$x \in U$" *is* $(\delta, \nu)$ *-open* $\iff$ "$x \in O$" *is* $(\delta, \theta^<)$ *-open* $\iff$ $\delta \leq_t \delta_{\mathbf{X}}$ ,

(9)  "$x \in U$" *is* $(\delta, \nu)$ *-r.e.* $\iff$ "$x \in O$" *is* $(\delta, \theta^<)$ *-r.e.* $\iff$ $\delta \leq \delta_{\mathbf{X}}$ ,

(10) "$x \in O$" *is* $(\delta_{\mathbf{X}}, \theta)$ *-open* $\iff$ $\theta \leq_t \theta^<$ ,

(11) "$x \in O$" *is* $(\delta_{\mathbf{X}}, \theta)$ *-r.e.* $\iff$ $\theta \leq \theta^<$ *(for computable* $\mathbf{X}$).

Finite and countable union and finite intersection on open sets are computable, more precisely:

**Lemma 3.8**
*Let* $\mathbf{X} = (X, \tau, \beta, \nu)$ *be an effective* $\mathrm{T}_0$-*space. Then*

(i) *countable union on* $\tau$ *is* $([\theta^<]^\omega, \theta^<)$–*computable,*

(ii) *intersection is* $(\theta^<, \theta^<, \theta^<)$–*computable, if* $\mathbf{X}$ *is computable.*

Instead of listing *all* basic open sets containing $x$ we could list "sufficiently many" basic neighbourhoods.

**Definition 3.9**
Define a repereresentation $\delta_{\mathbf{X}}^b : \subseteq \Sigma^\omega \to X$ of $\mathbf{X}$ as follows:   $\operatorname{dom}(\delta_{\mathbf{X}}^b) \subseteq D_\nu$ and
for all $x \in X$ and $p \in D_\nu$,

(12) $\delta_{\mathbf{X}}^b(p) = x : \Longleftrightarrow \{\nu(u) \mid \iota(u) \lhd p\}$ is a neighborhood base of $x$.

**Lemma 3.10**

(i) $\delta_{\mathbf{X}} \leq \delta_{\mathbf{X}}^b$,

(ii) $\delta_{\mathbf{X}}^b \leq_t \delta_{\mathbf{X}}$,

(iii) *in general neither* $\delta_{\mathbf{X}}^b \leq \delta_{\mathbf{X}}$ *nor the relation* $x \in U$ *is* $[\delta_{\mathbf{X}}^b, \nu]$*-r.e.*

**Proof:** i. Obvious.

   ii. Let $q \in \Sigma^\omega$ be a(n encoded) list of all $(u, v)$ such that $\nu(u) \subseteq \nu(v)$. Then
for some computable function $h : \subseteq \Sigma^\omega \times \Sigma^\omega \to \Sigma^\omega$, $p \mapsto h(p, q)$ translates $\delta_{\mathbf{X}}^b$
to $\delta_{\mathbf{X}}$

   iii. Omitted.                                                                    $\square$

# 4   Effective Metrization

In [9] Schröder has constructed a computable metric for a given "computable"
regular space. We will extend this space to a bigger one by adding a dense
set of computable points. Since for this purpose we return to the details of
Schröder's proof we outline it in this section using a slightly modified termi-
nology.

   A $T_0$-space $\mathbf{X} = (X, \tau)$ is *regular*, if for every $x \in X$ and every neighbour-
hood $V$ of $x$ there exists a neighbourhood $U$ of $x$ such that $x \in \bar{U} \subseteq V$ ([4]).
By a simple argument, every regular space is a a $T_2$-space (Hausdorff space).
A $T_0$-space $\mathbf{X} = (X, \tau)$ is *normal*, if for every pair $(A, A')$ of disjoint closed
sets there is a pair $(O, O')$ of disjoint open sets such that $A \subseteq O$ and $A' \subseteq O'$.
The following definition is equivalent to that from [9].

**Definition 4.1** [computably regular space]
A *computably regular space* is a computable $T_0$-space $\mathbf{X} = (X, \tau, \beta, \nu)$ for
which there is a computable function t3 $: \subseteq \Sigma^* \times \Sigma^* \to \Sigma^\omega$ such that for
$R := \operatorname{dom}(\text{t3})$,

(13) $(\forall v \in \operatorname{dom}(\nu)), \ \nu(v) = \displaystyle\bigcup_{(u,v) \in R} \nu(u),$   and

(14)    $(\forall (u, v) \in R) \, \nu(u) \subseteq \psi^>(\text{t3}\,(u, v)) \subseteq \nu(v)$

   Every computably regular space is regular. Every countably based regular

space is normal [4]. We prove a computable version.

**Definition 4.2** [computably normal]
A *computably normal* space is a computable $T_0$-space such that the multifunction $T4 : \subseteq \tau^c \times \tau^c \rightrightarrows \tau \times \tau$ defined by

$$\mathrm{dom}(T4) := \{(A, B) \in \tau^c \times \tau^c \mid A \cap B = \emptyset\},$$

$$(O_A, O_B) \in T4\,(A, B) : \Longleftrightarrow A \subseteq O_A \wedge B \subseteq O_B \wedge O_A \cap O_B = \emptyset,$$

is $(\psi^>, \psi^>, \theta^<, \theta^<)$–computable.

**Theorem 4.3 (computably regular implies computably normal)**
*Every computably regular space is computably normal.*

In the metrization proof in [9,4] a sequence of pseudometrics is defined each of which is constructed via a sequence of sets inducing contour levels on the space $X$.

**Definition 4.4** (i) Let $r : \mathbb{N} \to \{q \in \mathbb{Q} \mid 0 \le q \le 1\}$ be a bijective standard numbering with $r_0 = 0$ and $r_1 = 1$

(ii) and define the set of *contour maps* on **X** by

$$\mathrm{CM} := \{(V_i, C_i)_{i \in \mathbb{N}} \mid V_i \in \tau, C_i \in \tau^c, V_i \subseteq C_i \subseteq V_j \text{ if } r_i < r_j, \text{ for } i, j \in \mathbb{N}\}.$$

For any sequence $(V_i, C_i)_{i \in \mathbb{N}} \in \mathrm{CM}$, the boundary of each set $V_i$ can be considered as a contour line of level $r_i \in \mathbb{Q}$. Since these lines do not cross, the sequence $(V_i, C_i)_{i \in \mathbb{N}} \in \mathrm{CM}$ can be considered as a contour map on the space **X** with a hollow $V_0$ of level 0 and the top of a hill $C_1$ of level 1. For a computably normal space, from $V_0$ and $V_1$ a contour map $(V_i, C_i)_{i \in \mathbb{N}} \in \mathrm{CM}$ can be computed.

**Lemma 4.5** *Let* $\mathbf{X} = (X, \tau, \beta, \nu)$ *be computably normal space. Define* $\mathrm{SEQ} : \subseteq \tau^c \times \tau^c \rightrightarrows (\tau \times \tau^c)^{\mathbb{N}}$ *by*

$$\mathrm{dom}(\mathrm{SEQ}) := \{(A, B) \in \tau^c \times \tau^c \mid A \cap B = \emptyset\},$$

$$(V_i, C_i)_{i \in \mathbb{N}} \in \mathrm{SEQ}(A, B) : \Longleftrightarrow (V_i, C_i)_{i \in \mathbb{N}} \in \mathrm{CM}, \ A \subseteq V_0 \ \ and \ \ V_1 = B^c.$$

*Then* $\mathrm{SEQ}$ *is* $(\psi^>, \psi^>, [\theta^<, \psi^>]^\omega)$*–computable.*

A contour map $(V_i, C_i)_{i \in \mathbb{N}} \in \mathrm{CM}$ can be "extended" to a continuous real function on **X**.

**Lemma 4.6** *Let* $\mathbf{X} = (X, \tau, \beta, \nu)$ *be computably normal space and define* SF $: \subseteq (\tau \times \tau^c)^{\mathbb{N}} \to \mathrm{C}(X, [0, 1])$ *by*

$$\mathrm{dom}(\mathrm{SF}) = \mathrm{CM},$$

$$\mathrm{SF}((V_i, C_i)_{i \in \mathbb{N}})(x) = \begin{cases} \inf\{r_s \mid x \in V_s\} & \text{if } x \in V_1 \\ 1 & \text{otherwise} \end{cases}.$$

*Then* SF *is* $([\theta^<, \psi^>]^\omega, [\delta_{\mathbf{X}} \to \rho])$*–computable,* $\mathrm{SF}((V_i, C_i)_{i \in \mathbb{N}})[V_0] = \{0\}$ *and* $\mathrm{SF}((V_i, C_i)_{i \in \mathbb{N}})[X \setminus V_1] = \{1\}$.

As a by-product we get a computable Urysohn theorem.

**Corollary 4.7 (computable Urysohn)**
*Let* $\mathbf{X} = (X, \tau, \beta, \nu)$ *be computably normal space. Then the multi-function* UR $:=$ SF $\circ$ SEQ *mapping every pair* $(A, B)$ *of disjoint closed sets to all continuous functions* $f : X \to [0; 1]$ *such that* $f[A] = 0$ *and* $f[B] = 1$ *is* $(\psi^>, \psi^>, [\delta_{\mathbf{X}} \to \rho])$*—computable.*

For a definition of the representation $[\delta_{\mathbf{X}} \to \rho]$ see [13]. Let $\mathbf{X} = (X, \tau, \beta, \nu)$ be a computably regular space and let $k \mapsto (u_k, v_k)$ be a canonical numbering of the r.e. set $R = \mathrm{dom}(t3)$ from Definition 4.1.

**Lemma 4.8** *There is a sequence* $((V_i^k, C_i^k)_{i \in \mathbb{N}})_{k \in \mathbb{N}}$ *of contour maps such that*

(i) $\nu(u_k) \subseteq V_0^k$ *and* $\nu(v_k) = V_1^k$,

(ii) $(k, i) \mapsto (V_i^k, C_i^k)$ *is* $(\nu_{\mathbb{N}}, \nu_{\mathbb{N}}, [\theta^<, \psi^>])$*–computable.*

**Proof:** For each $k$ compute some element

$$(V_i^k, C_i^k)_{i \in \mathbb{N}} \in \mathrm{SEQ}(\psi^>(t3(u_k, v_k), \nu(v_k)^c)).$$

Notice that $U \mapsto U^c$ is $(\nu, \psi^>)$–computable. $\qquad\qquad\square$

**Lemma 4.9** *For the computable sequence of contour maps from Lemma 4.8 define sequences of functions* $(f_k)_k$ *and* $(d_k)_k$ *of real functions on* $X$ *by*

(15) $\qquad f_k := \mathrm{SF}((V_i^k, C_i^k)_{i \in \mathbb{N}})$,

(16) $d_k(x, y) := \mid f_k(x) - f_k(y) \mid .$

*Then for each* $k$, $d_k$ *is a continuous (w.r.t.* $\tau$*) pseudometric on* $X$ *and*

$$(k, x) \mapsto f_k(x) \text{ is } (\nu_{\mathbb{N}}, \delta_{\mathbf{X}}, \rho)\text{--computable, and}$$
$$(k, x, y) \mapsto d_k(x, y) \text{ is } (\nu_{\mathbb{N}}, \delta_{\mathbf{X}}, \delta_{\mathbf{X}}, \rho)\text{--computable.}$$

The proof is obvious. From the sequenc $(d_k)_k$ of pseudometrics a computable metric can be defined.

**Theorem 4.10 (effective metrization)**
*The function $d : X \times X \to \mathbb{R}$ defined by*

$$(17)\ d(x,y) := \sum_{k=0}^{\infty} 2^{-k} d_k(x,y)$$

*is a $(\delta_{\mathbf{X}}, \delta_{\mathbf{X}}, \rho)$–computable metric on $X$ inducing the topology $\tau$.*

# 5 Computable Dense Subsets and Embedding to Computable Metric Spaces

A computable metric space [12,13] is a tuple $\mathbf{M} = (M, d, A, \alpha)$ such that $(M, d)$ is a metric space and $\alpha : \subseteq \Sigma^* \to A$ is a notation with recursive domain of a (countable) dense subset $A \subseteq M$ such that the distance on $A$, $(a, b) \mapsto d(a, b)$, is $(\alpha, \alpha, \rho)$–computable. The canonical representation of a computable metric space is the Cauchy representaion defined by $\delta_C(p) = x$, iff $p = \iota(u_0)\iota(u_1)\iota(u_2)\dots$ such that $(\forall i)d(x, \alpha(u_i)) \leq 2^{-i}$. In particular, the elments $a \in A$ are $\delta_C$–computable.

In the previous section for a computably regular space $\mathbf{X} = (X, \tau, \beta, \nu)$ we have constructed a $(\delta_{\mathbf{X}}, \delta_{\mathbf{X}}, \rho)$–computable metric (Definition 4.10) which generates the topology $\tau$. But this does not mean that we have found a *computable metric space* since we do not have a dense set of computable points where *computable* means $\delta_{\mathbf{X}}$–computable such that the Cauchy representation is equivalent to $\delta_{\mathbf{X}}$. In general, a computably regular $T_0$-space $\mathbf{X}$ with computable metric $d$ may have no $\delta_{\mathbf{X}}$–computable points at all. But there is hope to find a dense set of computable points in its metric completion.

**Example 5.1** Consider the computable $T_0$-space $\mathbf{X} = (\mathbb{R}, \tau_{\mathbb{R}}, \text{Int}, I)$ (Definition 3.1) where $\tau_{\mathbb{R}}$ is the Euclidean topology and $I$ is a standard notation of the set Int of open intervals with rational end-points. Let $\delta_{\mathbf{X}}$ be the "topological" standard representation of $\mathbf{X}$ (Definition 3.2). On the other hand, for the real numbers there is the computable metric space $\mathbf{M} = (\mathbb{R}, d, \mathbb{Q}, \nu_{\mathbb{Q}})$ with Cauchy representation $\rho$. Since $\delta_{\mathbf{X}} \equiv \rho$, $\nu_Q$ is a notation of a set of $\delta_{\mathbf{X}}$–computable points.

Let $\mathbf{X}' = (\mathbb{R}', \tau_{\mathbb{R}}', \text{Int}', I')$ be the space obtained from $\mathbf{X}$ by deleting all the $\rho$–computable points. Since $\nu(u) \subseteq \nu(v) \iff \nu'(u) \subseteq \nu'(v)$, $\mathbf{X}'$ is a computable $T_0$-space. The embedding $x \mapsto x$ from $\mathbf{X}'$ to $\mathbf{X}$ is $(\delta_{\mathbf{X}'}, \delta_{\mathbf{X}})$–computable and its inverse is $(\delta_{\mathbf{X}}, \delta_{\mathbf{X}'})$–computable. Therefore, $X'$ cannot contain any $\delta_{\mathbf{X}'}$–

computable point, since the computable embedding would map it to a $\delta_{\mathbf{X}}$–computable, hence $\rho$–computable point.

There are various methods of metric completion. Embedding $\mathbf{X}$ into the (complete) Banach space $B(X)$ of bounded functions from $X$ into $\mathbb{R}$ (4.41 in [8]) requires a computability concept on $B(X)$ which is a new problem. For effective completion via Cauchy sequences on a countable dense set (19.33 in [8], [13]) we need to know (computable) points, but we do not have such points. A third method can be derived from Cauchy's intersection property: A metric space is complete, iff $\bigcap_n S_n \neq \emptyset$ for *every* descending sequence of nonempty closed sets $S_n$ such that $\mathrm{dm}(S_n) \leq 2^{-n}$. A completion of our metric space $(X, d)$ can be constructed by considering all (or sufficiently many) nested sequences of closed sets, introducing a pseudometric on these sequences and factorizing. It turns out that in our case we can compute *sufficiently many* sequences in order to find a dense set of computable points in a "partial completion" of $(X, d)$.

Notice, that so far our approach to computable topology was pointless. We have handled the notation $\nu$ of basic sets but have not considered any concrete points of the space $\mathbf{X}$. Points occur only hypothetically. For example, if a function $f$ is $(\delta_{\mathbf{X}}, \delta_{\mathbf{X}})$–computable, then there is a Tpe-2 machine, which from a name $p \in \Sigma^\omega$ of some point $x$ computes some name of $f(x)$. By our concept of realization, the machine works correctly for correct inputs but may behave arbitrarily otherwise. However, the machine is not able to check whether an input sequence is a correct name. It is the "user" who must "understand" the representation $\delta_{\mathbf{X}}$ and who is "responsible" himself for supplying the machine with correct names of points.

In this section we extend our computably regular space $\mathbf{X} = (X, \tau, \beta, \nu)$ in such a way that the extension contains a dense set of computable points with computable distances. We start from the computable sequence $(f_k)_{k \in \mathbb{N}}$ of contour functions from Lemma 4.9. In the following we consider implicitly $\nu_{\mathbb{N}}$ and some standard notation of the finite sequences of rational numbers. We begin with a technical lemma.

**Lemma 5.2**
*There is a computable function $h : (v, p, m) \mapsto (u, (a^0, b^0, \ldots, a^m, b^m))$ mapping every $v \in \mathrm{dom}(\nu)$, every $p$ such that $\delta_{\mathbf{X}}(p) \in \nu(v)$ and every $m \in \mathbb{N}$ to some word $u \in \mathrm{dom}(\nu)$ and a sequence of rational numbers such that for all $k \leq m$,*

(18) $\delta_{\mathbf{X}}(p) \in \nu(u) \subseteq \overline{\nu(u)} \subseteq \nu(v)$,

(19)      $0 < b^k - a^k \leq 2^{-m}$,

(20)      $\nu(u) \subseteq f_k^{-1}[a^k; b^k]$.

**Proof:** First, from $v$ and $p$ some $u'$ can be computed such that $\delta_{\mathbf{X}}(p) \in \nu(u')$ and $(u', v) \in R$ (Definition 4.1). Then $\overline{\nu(u')} \subseteq \nu(v)$. By Lemma 4.9, the function

(21) $(m, a, b) \mapsto f_m^{-1}(a; b)$ is $(\nu_{\mathbb{N}}, \nu_{\mathbb{Q}}, \nu_{\mathbb{Q}}, \theta^<)$–computable

$(0 \leq a < b \leq 1)$. There are rational numbers $a^0, b^0, \ldots, a^m, b^m$ such that $(\forall k \leq m)\, 0 < b^k - a^k \leq 2^{-m}$ and

$$\delta_{\mathbf{X}}(p) \in \nu(u') \cap f_0^{-1}(a^0; b^0) \cap \ldots \cap f_m^{-1}(a^m; b^m) \neq \emptyset\,.$$

Applying exhaustive search, by Lemma 3.8 and (21) some word $u$ and rational numbers $a^0, b^0, \ldots, a^m, b^m$ can be computed such that $(\forall k \leq m)\, 0 < b^k - a^k \leq 2^{-m}$ and

$$\delta_{\mathbf{X}}(p) \in \nu(u) \subseteq \nu(u') \cap f_0^{-1}(a^0; b^0) \cap \ldots \cap f_m^{-1}(a^m; b^m)\,.$$

(18-20) follow immediately. $\qquad\square$

The following similar lemma can be proved accordingly.

**Lemma 5.3**
*There is a computable function $h' : (v, m) \mapsto (u, (a^0, b^0, \ldots, a^m, b^m))$ mapping every $v \in \mathrm{dom}(\nu)$ and every $m \in \mathbb{N}$ to some word $u \in \mathrm{dom}(\nu)$ and a sequence of rational numbers such that for all $k \leq m$,*

(22)  $\nu(u) \subseteq \overline{\nu(u)} \subseteq \nu(v)\,,$
(23) $0 < b^k - a^k \leq 2^{-m}\,,$
(24)  $\nu(u) \subseteq f_k^{-1}[a^k; b^k]\,.$

By the following lemma in every base element $\nu(v)$ we can compute a nested sequence $(U_i)_{i \in \mathbb{N}}$ of base elements and a sequence of real numbers $(y_k)_{k \in \mathbb{N}}$ such that for every pseudometric $d_k$, the diameters $\mathrm{dm}_k(U_i)$ and the distances $d_k(U_i, y_k)$ converge to 0 rapidly.

**Lemma 5.4**
*There is a $(\mathrm{I\!I}_{\Sigma^*}, [\nu^\omega, \rho^\omega])$–computable function $H : w \mapsto ((U_i)_{i \in \mathbb{N}}, (y_k)_{k \in \mathbb{N}})$ mapping each $w \in \mathrm{dom}(\nu)$ to some sequence of base elements and some sequence of real numbers such that for all $i$ and for all $k \leq i$,*

(25)   $U_0 = \nu(w), \quad (\forall i)\ \overline{U_{i+1}} \subseteq U_i\,,$
(26)         $\mathrm{dm}_k(U_i) \leq 2^{-i}\,,$
(27) $\sup\{|f_k(x) - y_k| \mid x \in U_i\} \leq 2^{-i}.$

**Proof:** We compute a sequence $(u_i)_i$ of (names of) base elements and for each $k$ a sequence $((a_i^k; b_i^k))_{i \in \mathbb{N}}$ of intervals with rational endpoints inductively as

follows.

$i = 0$ : $u_0 := w$, $a_0^k = 0$ and $b_0^k = 1$ for all $k$.

$i > 0$ : With the function $h'$ from Lemma 5.3 define

$$(28)(u_i, (a_i^0, b_i^0, \ldots, a_i^i, b_i^i)) := h'(u_{i-1}, i), \quad \text{and} \quad a_i^k = 0 \quad \text{and} \quad b_i^k = 1 \quad \text{for} \quad k > i.$$

Define $U_i := \nu(u_i)$. Then (25) follows from (22). By (24), $U_i \subseteq f_k^{-1}[a_i^k; b_i^k]$ for all $i, k$. Therefore by (25), $\bigcap_{i \le m}[a_i^k; b_i^k] \neq \emptyset$ for all $k, m$. By (28) and (23), $0 \le b_i^k - a_i^k \le 2^{-i}$ for $k \le i$. Therefore, $y_k$ such that $\{y_k\} = \bigcap_{i \in \mathbb{N}}[a_i^k; b_i^k]$ exists. (27) follows from $y_k \in [a_i^k; b_i^k]$, $0 \le b_i^k - a_i^k \le 2^{-i}$ for $k \le i$ and $U_i \subseteq f_k^{-1}[a_i^k; b_i^k]$. Since $0 \le b_i^k - a_i^k \le 2^{-i}$ for $k \le i$, the sequence of intervals converges rapidly such that the real numbers $y_k$ can be computed.

(26) follows from $U_i \subseteq f_k^{-1}[a_i^k; b_i^k]$ and $0 \le b_i^k - a_i^k \le 2^{-i}$ for $k \le i$.      □

The following lemma can be proved accordingly by using Lemma 5.2 instead of Lemma 5.3

**Lemma 5.5**

*There is a $(\mathrm{I\!I}_{\Sigma^\omega}, [\nu^\omega, \rho^\omega])$–computable function $H' : p \mapsto ((U_i)_{i \in \mathbb{N}}, (y_k)_{k \in \mathbb{N}})$ mapping each $p \in \mathrm{dom}(\delta_{\mathbf{X}})$ to some sequence of base elements and some sequence of real numbers such that for all $i$ and for all $k \le i$,*

(29) $U_0 = \nu(w)$, $\quad (\forall i)\ \delta_{\mathbf{X}}(p) \in \overline{U_{i+1}} \subseteq U_i$,

(30) $\quad\quad\quad \mathrm{dm}_k(U_i) \le 2^{-i}$,

(31) $\quad \sup\{|f_k(x) - y_k| \mid x \in U_i\} \le 2^{-i}$.

Now we extend our computably regular space $\mathbf{X} = (X, \tau, \beta, \nu)$ with computable metric $d$ from Theorem 4.10 to some space containing the limits of the sequences computed above. It is a "partial" completion via some nested sequences of closed balls. For a metric $d$ on $M$ and $A, B \subseteq M$ we define, as usual, $d(A, B) := \inf_{a \in A, b \in B} d(a, b)$.

**Definition 5.6** [partial completion]

(i) Define a pseudometric space $(S, d_S)$ by

$$S := \{(U_i)_{i \in \mathbb{N}} \mid U_i \in \beta,\ \mathrm{dm}(U_i) < 2^{-i},\ \overline{U_{i+1}} \subseteq U_i \ \text{ for all } i \in \mathbb{N}\},$$

$$d_S((U_i)_{i \in \mathbb{N}}, (V_i)_{i \in \mathbb{N}}) := \sup_{i \in \mathbb{N}} d(U_i, V_i).$$

(ii) Let the metric space $(\tilde{X}, \tilde{d})$ be the factorization of the pseudometric space $(S, d_S)$ with respect to $\sim$, that is, $(\tilde{X}, \tilde{d}) = (S/_\sim, d_S/_\sim)$, where

$$(U_i)_{i \in \mathbb{N}} \sim (V_i)_{i \in \mathbb{N}} \iff \sup_{i \in \mathbb{N}} d(U_i, V_i) = 0.$$

(iii) Let $\tilde{\tau}$ be the topology generated by the metric $\tilde{d}$.

**Lemma 5.7**
The spaces $(S, d_S)$ and $(\tilde{X}, \tilde{d})$ are well-defined.

**Proof:** Obviously, $d(s, s') \geq 0$, $d(s, s) = 0$ and $d(s, s') = d(s', s)$ for all $s, s' \in S$. It remains to prove the tiangle inequality for $(U_i)_{i \in \mathbb{N}}, (V_i)_{i \in \mathbb{N}}, (W_i)_{i \in \mathbb{N}} \in S$. For all $i \in \mathbb{N}$ and all $x \in U_i, y', y'' \in V_i, z \in W_i$,

$$d(x, z) \leq d(x, y') + d(y', y'') + d(y'', z)$$
$$\leq d(x, y') + 2^{-i} + d(y'', z)$$

since $\mathrm{dm}(V_i) < 2^{-i}$. This implies

$$d(U_i, W_i) = \inf_{\substack{x \in U_i \\ z \in W_i}} d(x, z) \leq d(x, y') + 2^{-i} + d(y'', z).$$

Since this is true for all $x, y', y'', z$,

$$d(U_i, W_i) \leq \inf_{\substack{x \in U_i \\ y' \in V_i}} d(x, y') + 2^{-i} + \inf_{\substack{y'' \in V_i \\ z \in W_i}} d(y'', z) = d(U_i, V_i) + 2^{-i} + d(V_i, W_i).$$

We obtain

$$d(U_i, W_i) \leq \sup_j d(U_j, V_j) + 2^{-i} + \sup_j d(V_j, W_j)$$

and therefore,

$$\sup_j d(U_j, W_j) = \sup_j (d(U_j, W_j) - 2^{-j}) \leq \sup_j d(U_j, V_j) + \sup_j d(V_j, W_j),$$

hence

$$d_S((U_i)_{i \in \mathbb{N}}, (W_i)_{i \in \mathbb{N}}) \leq d_S((U_i)_{i \in \mathbb{N}}, (V_i)_{i \in \mathbb{N}}) + d_S((V_i)_{i \in \mathbb{N}}, (W_i)_{i \in \mathbb{N}}).$$

The metric space $(\tilde{X}, \tilde{d})$ is obtained from the pseudometric space by identifying points of distance 0. □

We introduce a canonical representation for the space $(\tilde{X}, \tilde{d})$ as follows.

**Definition 5.8**
Define a representation $\tilde{\delta}$ of $\tilde{X}$ by

$$\mathrm{dom}(\tilde{\delta}) := \{\iota(u_0)\iota(u_1)\ldots \mid ((\nu(u_0), \nu(u_1), \ldots) \in S\},$$
$$\tilde{\delta}(\iota(u_0)\iota(u_1)\ldots) := (\nu(u_0), \nu(u_1), \ldots)/_\sim.$$

Since the set $S$ in Definition 5.6 does not contain all descending sequences of closed sets (or balls) in $(X, d)$, the space $(\tilde{X}, \tilde{d})$ may be a proper subspace

of the completion of $(X, d)$ (up to isometric mappings). But the space $(\tilde{X}, \tilde{d})$ already has a computable dense sequence of computable points. Furthermore, the space $(X, d)$ can be embedded isometrically into $(\tilde{X}, \tilde{d})$. The inverse of this embedding is continuous. For computability we introduce some further axiom.

**Definition 5.9** [embedding]
Define a function $\mathrm{emb} : X \to \tilde{X}$ by

$$\mathrm{emb}(x) := (U_i)_i/_\sim \;\; \text{such that} \;\; (U_i)_i \in S \;\; \text{and} \;\; (\forall i) x \in U_i \,.$$

**Theorem 5.10**

(i) *The embedding function* $\mathrm{emb}$ *is well-defined, isometric and* $(\delta_{\mathbf{X}}, \tilde{\delta})$–
   *computable.*

(ii) *Its inverse is* $(\tilde{\delta}, \delta_{\mathbf{X}})$-*continuous.*

**Proof:**
**Prop. 1:** For each $p \in \mathrm{dom}(\delta_{\mathbf{X}})$ a sequence $(V_j)_j \in S$ can be computed such that $(\forall j) \delta_{\mathbf{X}}(p) \in V_j$.
**Proof 1:** Let $(U_i)_i$ be the sequence computed by $H'$ from $p$ (Lemma 5.5. Define $V_j := U_{j+2}$. By (29), $(\forall j) \delta_{\mathbf{X}}(p) \in V_j$. It remains to show $\mathrm{dm}(V_j) \le 2^{-j}$. Suppose $x, y \in V_j$. Since $x, y \in U_{i+2}$ by (30),

$$
\begin{aligned}
d(x,y) &= \sum_{k \le j+2} 2^{-k} d_k(x,y) + \sum_{k > j+2} 2^{-k} d_k(x,y) \\
&\le \sum_{k \le j+2} 2^{-k} 2^{-j-2} + \sum_{k > j+2} 2^{-k} \\
&\le 2 \cdot 2^{-j-2} + 2^{-j-2} \;\; < \;\; 2^{-j} \,.
\end{aligned}
$$

Therefore, $\mathrm{dm}(V_j) \le 2^{-j}$.
**Prop. 2:** If $(V_j)_j, (W_j)_j \in S$, $(\forall j) x \in V_j$ and $(\forall j) y \in W_j$.
Then $d_S((V_j)_j, (W_j)_j) = d(x,y)$.
**Proof 2:** For all $j$, $d(V_j, W_j) \le d(x,y)$ (since $x \in V_j$ and $y \in W_j$), hence $d_S((V_j)_j, (W_j)_j) = \sup_j d(V_j, W_j) \le d(x,y)$.
Furthermore, for all $j$, all $x_j \in V_j$ and $y_j \in W_j$,

$$
\begin{aligned}
d(x,y) &\le d(x, x_j) + d(x_j, y_j) + d(y_j, y) \\
&\le 2^{-j} + d(x_j, y_j) + 2^{-j}
\end{aligned}
$$

since $\mathrm{dm}(V_j), \mathrm{dm}(W_j) \le 2^{-j}$. Since this is true for all $x_j \in V_j$ and $y_j \in W_j$,

$$d(x,y) \le 2 \cdot 2^{-j} + d(V_j . W_j) \le 2^{-j} + d_S((V_j)_j, (W_j)_j)$$

Since this is true for all $j$, $\;\; d(x,y) \le d_S((V_j)_j, (W_j)_j)$.

Let $\delta_{\mathbf{X}}(p) = \delta_{\mathbf{X}}(q)$ and let $(V_j)_j \in S$ and $(W_j)_j \in S$ be the sequences computed from $p$ and $q$, repectively, according to Prop. 1. Then by Prop. 2, $d_S((V_j)_j, (W_j)_j = d(\delta_{\mathbf{X}}(p), \delta_{\mathbf{X}}(q)) = 0$. Therefore, the function emb is well-defined and $(\delta_{\mathbf{X}}, \tilde{\delta})$–computable.

Immediately from Prop. 2 we conclude $\tilde{d}(\text{emb}(x), \text{emb}(y)) = d(x, y)$. Therefore, the embedding is isometric.

If $\tilde{\delta}(p) = \text{emb}(x) \in \tilde{X}$, then $p$ lists a neighbourhood base of $x$. Therefore the identity is a computable $(\tilde{\delta}, \delta_{\mathbf{X}}^b)$-realization of $\text{emb}^{-1}$ and $\text{emb}^{-1}$ is $(\tilde{\delta}, \delta_{\mathbf{X}})$-continuous by Lemma 3.10. $\qquad\square$

Possibly neither $\text{emb}^{-1}$ is $(\tilde{\delta}, \delta_{\mathbf{X}})$–computable nor the metric $\tilde{d}$ is $(\tilde{\delta}, \tilde{\delta}, \rho)$–computable on the set $\text{emb}(X) \subseteq \tilde{X}$. By the next theorem, however, there is a notation $\alpha$ of a dense subset of $\tilde{X}$ such that the distance $\tilde{d}$ is $(\alpha, \alpha, \rho)$–computable on its range.

**Theorem 5.11**
*There is a notation $\alpha : \subseteq \Sigma^* \to A$ with recursive domain of some dense set $A$ in $(\tilde{X}, \tilde{d})$ such that $\alpha \leq \tilde{\delta}$ and $\tilde{d}$ is $(\alpha, \alpha, \rho)$–computable on $A$.*

**Proof:** We use the function $H$ from Lemma 5.4. Let $\text{dom}(\alpha) := \text{dom}(\nu)$. For each $w \in \text{dom}(\nu)$ let

(32) $\alpha(w) := (U_{j+2})_j / \sim$ wherer $(U_i)_i = \text{pr}_1 H(w)$.

As in the proof of Proposition 1 in the proof of Theorem 5.10, we can show $(U_{j+2})_j \in S$. Therefore, $\alpha$ is well-defined and $\alpha \leq \tilde{\delta}$.

We show that $\text{range}(\alpha)$ is dense in $\tilde{X}$. Let $\tilde{x} \in \tilde{X}$. Then $\tilde{x} = (\nu(w_i))_i / \sim$ for some $w_0, w_1, \ldots \in \text{dom}(\nu)$. For $k \in \mathbb{N}$, $\alpha(w_k) = (V_i)_i / \sim$ for some sequence $(V_i)_i \in S$ such that $V_i \subseteq \nu(w_k)$ for all $i$. Then

(33) $\tilde{d}((\nu(w_i))_i / \sim, (V_i)_i / \sim) = \sup_i d(\nu(w_i), V_i) \leq \text{dm}(\nu(w_k)) \leq 2^{-k}$.

We show computability of $\tilde{d}$ on $A$. For $H$ from Lemma 5.4 and $u, v \in \text{dom}(\nu) = \text{dom}(\alpha)$ let

$$H(u) = ((U_i)_i, (y_j)_j),$$
$$H(v) = ((V_i)_i, (z_j)_j).$$

For $y \in U_i$ and $z \in V_i$ (since in general $||a - b| - |c - d|| \leq |a - c| + |b - d|$),

$$\left| d(y, z) - \sum_{k \geq 0} 2^{-k} |y_k - z_k| \right|$$

$$= \left| \sum_{k \leq i} 2^{-k} d_k(y, z) + \sum_{k > i} 2^{-k} d_k(y, z) - \sum_{k \leq i} 2^{-k} |y_k - z_k| - \sum_{k > i} 2^{-k} |y_k - z_k| \right|$$

$$\leq \left| \sum_{k \leq i} 2^{-k} |f_k(y) - f_k(z)| - \sum_{k \leq i} 2^{-k} |y_k - z_k| \right| + 2^{-i+1}$$

$$\leq \sum_{k \leq i} 2^{-k} \left| |f_k(y) - f_k(z)| - |y_k - z_k| \right| + 2^{-i+1}$$

$$\leq \sum_{k \leq i} 2^{-k} \left( |f_k(y) - y_k| + |f_k(z) - z_k| \right) + 2^{-i+1}$$

$$\leq \sum_{k \leq i} 2^{-k} (2^{-i} + 2^{-i}) + 2^{-i+1} \quad \text{by (27)}$$

$$\leq 6 \cdot 2^{-i}$$

For $s := \sum_{k \geq 0} 2^{-k} |y_k - z_k|$, $s - 2^{-i+3} \leq d(y, z) \leq s + 2^{-i+3}$ and therefore,

$$s - 2^{-i+3} \leq d(U_i, V_i) \leq s + 2^{-i+3}$$

since $d(U_i, V_i) = \inf\{d(y, z) \mid y \in U_i, z \in V_i\}$. Since $d(U_i, V_i) \leq d(U_{i+1}, V_{i+1})$, for all $i$,

$$s - 2^{-i+3} \leq \sup_j d(U_j, V_j) \leq s + 2^{-i+3}.$$

We obtain

$$s = \sup_j d(U_j, V_j) = d_S((U_{i+2})_i, (V_{i+2})_i) = \tilde{d}(\alpha(u), \alpha(v)).$$

Since the sequences $(y_i)_i$ and $(z_i)_i$ of real numbers can be computed from $u$ and $v$ and since $y_i, z_i \leq 2$ by (27), $s = \sum_{k \geq 0} 2^{-k} |y_k - z_k| = \tilde{d}(\alpha(u), \alpha(v))$ can be computed from $u$ and $v$. $\qquad\square$

So far for a computably regular space $\mathbf{X} = (X, \tau, \beta, \nu)$ with standard representation $\delta_{\mathbf{X}}$ we have constructed a computable metric $d$ generating $\tau$ and a metric space $\tilde{\mathbf{X}} = (\tilde{X}, \tilde{d})$ with representation $\tilde{d}$ such that

(i) there is a $(\delta_{\mathbf{X}}, \tilde{\delta})$–computable isometric embedding emb $: X \to \tilde{X}$,

(ii) emb$^{-1}$ is $(\tilde{\delta}, \delta_{\mathbf{X}})$-continuous,

(iii) there is a notation $\alpha$ of a dense subset $A \subseteq \tilde{X}$ with recursive domain such that $\alpha \leq \tilde{\delta}$ and $\tilde{d}$ is $(\alpha, \alpha, \rho)$–computable on $A$.

For proving $\delta_{\mathbf{X}}^b \leq_t \delta_{\mathbf{X}}$, that is, continuous reducibility of $\delta_{\mathbf{X}}^b$ to $\delta_{\mathbf{X}}$ in Lemma 3.10 we have used the set $\{(u, v) \mid \nu(u) \subseteq \nu(v)\}$ as an "oracle". In many applications there is some similar r.e. oracle for this purpose. In such a case, $\delta_{\mathbf{X}}^b \equiv \delta_{\mathbf{X}}$. Since emb$^{-1}$ is $(\tilde{\delta}, \delta_{\mathbf{X}}^b)$–computable (proof of Theorem 5.10.ii) the same oracles can be used to prove that emb$^{-1}$ is $(\tilde{\delta}, \delta_{\mathbf{X}})$-continuous or –computable.

Notice that $(\tilde{X}, \tilde{d}, A, \alpha)$ is a computable metric space [13]. The following Theorem summarizes these results.

**Theorem 5.12**
*Let* $\mathbf{X} = (X, \tau, \beta, \nu)$ *be a computably regular space such that* $\{(u, v) \mid \nu(u) \subseteq \nu(v)\}$ *is r.e. Then there is a* $(\delta_{\mathbf{X}}, \tilde{\delta})$–*computable isometric embedding to the computable metric space* $(\tilde{X}, \tilde{d}, A, \alpha)$ *and the inverse of the embedding is* $(\tilde{\delta}, \delta_{\mathbf{X}})$–*computable.*

Given a computable metric space we have to consider the associated Cauchy representation (see [13]).

**Definition 5.13**
Let $\delta_C$ be the Cauchy representation of the computable metric space $(\tilde{X}, \tilde{d}, A, \alpha)$.

The representations $\tilde{\delta}$ and $\delta_C$ induce two computability concepts on $\tilde{X}$. They are related as follows.

**Theorem 5.14**
$\tilde{\delta} \le \delta_C, \quad \delta_C \le_t \tilde{\delta}.$

**Proof:** $\tilde{\delta} \le \delta_C$: In the proof of Theorem 5.11 we have already proved that $A$ is dense. Suppose $\tilde{\delta}(p) = x$ Then by Definition 5.8, $p = \iota(w_0)\iota(w_1)\ldots$ such that $x = (\nu(w_0), \nu(w_1), \ldots)/_\sim$. By (33) for all $k$, $\tilde{d}(x, \alpha(w_k)) \le 2^{-k}$, hence $x = \delta_C(\iota(w_0)\iota(w_1)\ldots)$. Therefore, the identity translates $\tilde{\delta}$ to $\delta_C$.
    $\delta_C \le_t \tilde{\delta}$: (Omitted) $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$


**Corollary 5.15**
*The distance* $\tilde{d}$ *on* $\tilde{X}$ *is* $(\tilde{\delta}, \tilde{\delta}, \rho)$–*computable.*

**Proof:** This is true since $\tilde{d}$ is $(\delta_C, \delta_C, \rho)$–computable and $\tilde{\delta} \le \delta_C$. $\qquad \square$


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