Term Equational Systems and Logics

(Extended Abstract)

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Abstract

We introduce an abstract general notion of system of equations between terms, called Term Equational System, and develop a sound logical deduction system, called Term Equational Logic, for equational reasoning. Further, we give an analysis of algebraic free constructions that together with an internal completeness result may be used to synthesise complete equational logics. Indeed, as an application, we synthesise a sound and complete nominal equational logic, called Synthetic Nominal Equational Logic, based on the category of Nominal Sets.

Keywords: Equational systems, algebraic theories, free algebras, equational logic, soundness, completeness, Nominal Sets, Schanuel topos.

Introduction

Formal reasoning is fundamental in computer science, and frameworks for supporting it abound. Those that are most relevant to this work are equational logic [10] and algebraic theories [16] together with their extensions, *e.g.* to rewriting [18], higher-order [15,19], and nominal [8,4] settings. The common conceptual core of these frameworks consists of syntactic structure providing a notion of term together with a logical system of equational judgements. Ideally, this is further equipped with a model theory for which soundness and completeness hold. A main general aim of the paper is to formally articulate this scenario by means of a general and practical mathematical theory. Indeed, we provide a mathematical theory for the development of term equational systems and logics.

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Our Term Equational System framework is given by a semantic universe together with an abstract notion of syntax on it. From it, the following are induced.

1. A notion of term in context, and therefore notions of equation and equational theory.

2. A model theory, providing a satisfaction relation between models and equational judgements.

3. A sound logical deduction system, called Term Equational Logic, for equational reasoning.

4. A construction of free algebras that, in certain important and common scenarios, provides a sound and complete system of equations that aids the development of a complete logic.

We outline the technical development of the paper introducing it here in restricted form so as to more easily convey the basic ideas and intuitions. The more general enriched universes of discourse considered in the paper (Definition 2.2) are indispensable to accommodate multi-sorted theories.

For the purpose of this introduction, thus, we consider a symmetric monoidal closed category $(\mathscr{C}, I, \otimes, [-, =])$ as universe of discourse and a monad $\mathbb{T} = (T, \eta, \mu)$ on it, equipped with a strength τ, as notion of syntax. For $C \in \mathscr{C}$, the object $TC \in \mathscr{C}$ intuitively represents terms with variables in C. A global term corresponds thus to a map $I \to TC$. More generally, however, as it is well-known from categorical logic (see *e.g.* [17]), one should consider generalised terms of the form $A \to TC$. For these, the pair (A, C) provides a notion of arity, with C being the covariant arity of the term and A the contravariant arity. Intuitively, one may think of generalised terms of arity (A, C) as an A-parameterised family of terms with variables in C. A notion of generalised equation between terms of the same arity, denoted as $t \equiv t'$, and hence of generalised theory, is thus induced. (See Section 2.)

A natural model theory arises. As expected, models for the monad are given by Eilenberg-Moore algebras. The interpretation of a generalised term $t : A \to TC$ in such a model $(X, s : TX \to X)$ is a map

$$[\![t]\!]_{(X,s)} : [C, X] \otimes A \to X \ ,$$

intuitively evaluating the A-parameterised family of terms t according to the algebra structure (X, s) with respect to assignments in X for the variables in C. Formally, the interpretation map $[\![t]\!]_{(X,s)}$ is given by the composite

$$[C, X] \otimes A \xrightarrow{[C,X] \otimes t} [C, X] \otimes TC \xrightarrow{\tau_{[C,X],C}} T\big([C, X] \otimes C\big) \xrightarrow{T\epsilon} TX \xrightarrow{s} X \ .$$

Consequently, a model (X, s) is said to satisfy the equation $t \equiv t' : A \to TC$ whenever $[\![t]\!]_{(X,s)} = [\![t']\!]_{(X,s)} : [C, X] \otimes A \to X$. (Again see Section 2.)

Based on the model theory, a sound equational logic, called Term Equational Logic, is introduced. The logic has the rules Ref, Sym, Trans of equivalence relations; the rule Axiom stating that the equations of a theory are derivable; the rule Subst stating that substitution is a congruence; the rule Ext stating that an operation

of context extension is a congruence; and the rule LocChar expressing the local character of entailment (see *e.g.* [17]). (See Section 3.)

In the presence of free models for a given generalised theory, we show an internal soundness and completeness result: A generalised equation is satisfied by all models of a generalised theory iff it is satisfied in a particular free model (see Section 4). This, in view of an explicit construction of free models (see [6,7] and Section 1), may be used to synthesise a sound and complete deduction system which, in turn, may suggest how to extend the aforementioned equational logic to make it complete, and allow the use of equational term rewriting as an alternative for establishing equational judgements.

Overall, thus, we advocate the following general methodology for developing term equational systems and logics.

1. Select a universe of discourse $\mathscr{C}$ and consider within it a notion of signature such that every signature Σ gives rise to a strong monad $\mathbb{T}_\Sigma = (T_\Sigma, \tau, \eta, \mu)$ on $\mathscr{C}$.

   We do not insist on an a priori prescription for the definition of signature, but rather consider it as being domain specific. Of course, standard notions of signature (*e.g.*, as they arise in the context of (enriched) algebraic theories [14,20]) may be considered. However, one may also need to go beyond them—see [5].

2. Select a class of arities (A, C) and give a syntactic description of the generalised terms $A \to T_\Sigma C$. This yields a syntactic notion of equational theory with an associated model theory arising from that of Term Equational Systems.

   We are ultimately interested in the existence of free algebras for the signature Σ satisfying the equations of a theory. In cocomplete universes of discourse, these may be constructed according to the theory of free constructions of [6], *e.g.* by considering functors T_Σ that preserve colimits of λ-chains and equations with λ-compact arities (see Definition 2.13) for some limit ordinal λ.

   For applications to finitary theories, however, one need only examine the case $\lambda = \omega$; even under the further conditions that T_Σ preserves epimorphisms and the arities of equations are projective (see Definition 2.13). This guarantees a simple inductive construction of free algebras, well-suited to the extraction of explicit descriptions.

3. Synthesise a deduction system for equational reasoning on syntactic terms with rules arising as syntactic counterparts of the rules from the Term Equational Logic associated to the underlying Term Equational System. By construction, thus, soundness will be guaranteed.

   The analysis of the rule Subst will typically involve the consideration of a syntactic substitution operation corresponding to the Kleisli composition of generalised terms.

4. In view of the internal completeness result, analyse the construction of free algebras so as to either establish the completeness of the synthesised equational logic, or get insight into how to extend it to make it complete.

This methodology in the universe of Sets with Σ a finitary signature, $\mathbb{T}_\Sigma$ the induced term monad, and theories with equations of arity $(1, C)$, for C a finite set of variables, leads to equational logic. In fact, the instantiation of our framework for Term Equational Systems and Logics in universes of Sorted Sets (see Examples 2.6, 2.11, and 3.2) essentially yields the framework of equational theories and logics for monads on categories of Sorted Sets of Climent and Soliveres [3].

In Section 5, as a substantial case study providing evidence for the practicality of the mathematical theory, the above methodology is applied within the universe of Nominal Sets (which is equivalent to the Schanuel topos). In this development, terms are generated from signatures of nominal sets of finitary operators together with meta-variables carrying nominal-variables; the context of meta-variables arises as the covariant arity of a term, whilst the context of nominal-variables as its contravariant arity. The equational theories thus arising naturally specify nominal algebraic structures (as *e.g.* that of $\alpha\beta\eta$-equivalence of λ-terms). For these, a sound deduction system for equational reasoning is synthesised and, furthermore, shown to be complete. Our nominal equational logic is logically equivalent to those recently introduced by Gabbay and Mathijssen [8], and Clouston and Pitts [4]. (We note that a similar development can be carried out in the universe $\boldsymbol{Set}^{\mathbb{I}}$, for $\mathbb{I}$ the category of finite sets of names and injections between them, and that this leads to the Binding Equational Logic of Hamana [11, Section 4].)

In a subsequent paper, we will also apply the methodology in the context of second-order abstract syntax as developed in [5] to synthesise an equational logic for second-order algebraic theories.

Our mathematical theory generalises to incorporate *rewriting modulo equations*. However, we do not dwell on this here.

1 Equational systems and free algebras

Equational systems.   We recall the notion of equational system introduced by the authors [6].

Definition 1.1 (Algebra) A Σ-*algebra* (X, s) for an endofunctor Σ on a category $\mathscr{C}$ is given by an object $X \in \mathscr{C}$, called the *carrier*, together with a morphism $s : \Sigma X \rightarrow X$ in $\mathscr{C}$, called the *structure map*. *Homomorphisms* from a Σ-algebra (X, s) to another one (Y, t) are morphisms $h : X \rightarrow Y$ such that $h \circ s = t \circ \Sigma h$.

The category Σ-**Alg** has objects given by Σ-algebras and morphisms given by homomorphisms. The *forgetful functor* $U_\Sigma : \Sigma$-**Alg** $\rightarrow \mathscr{C}$ maps Σ-algebras to their carrier objects.

Definition 1.2 (Functorial term) A *functorial signature* Σ on a category $\mathscr{C}$ is an endofunctor on it. A *functorial term* T in a *functorial context* Γ for the functorial signature Σ on $\mathscr{C}$, denoted $\mathscr{C} : \Sigma \rhd \Gamma \vdash T$, is given by an endofunctor Γ on $\mathscr{C}$ and a functor $T : \Sigma$-**Alg** $\rightarrow \Gamma$-**Alg** such that $U_\Gamma \circ T = U_\Sigma$.

Definition 1.3 (Equational system) An *Equational System* (ES)

$$\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L \equiv R)$$

consists of a category $\mathscr{C}$ together with a pair of functorial terms $\mathscr{C} : \Sigma \rhd \Gamma \vdash L$ and $\mathscr{C} : \Sigma \rhd \Gamma \vdash R$, referred to as a *functorial equation*.

Definition 1.4 (ES algebra) For an ES $\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L \equiv R)$, an $\mathbb{S}$-*algebra* (X, s) is a Σ-algebra satisfying the equation; that is, such that $L(X, s) = R(X, s) : \Gamma X \rightarrow X$.

The category $\mathbb{S}$-**Alg** is the full subcategory of Σ-**Alg** consisting of $\mathbb{S}$-algebras, and the *forgetful functor* $U_\mathbb{S} : \mathbb{S}$-**Alg** $\rightarrow \mathscr{C}$ maps $\mathbb{S}$-algebras to their carrier objects.

Example 1.5 Every monad $\mathbb{T} = (T, \eta, \mu)$ on a category $\mathscr{C}$ with binary coproducts has an associated ES defined as $\overline{\mathbb{T}} = (\mathscr{C} : T \rhd \Gamma \vdash L \equiv R)$ where $\Gamma X = X + T^2 X$ and $L(X, s) = [\mathrm{id}_X, s \circ Ts]$, $R(X, s) = [s \circ \eta_X, s \circ \mu_X]$. The category $\overline{\mathbb{T}}$-**Alg** is (isomorphic to) the category of Eilenberg-Moore algebras $\mathscr{C}^\mathbb{T}$.

For more examples of ESs and their categories of algebras see Sections 2 and 5.

Free algebras. We outline a basic theory for the *inductive construction* of free algebras, see [6,7].

Definition 1.6 An ES $\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L \equiv R)$ is said to be λ-*finitary*, for λ a limit ordinal, if the category $\mathscr{C}$ is cocomplete, and both the functors Σ and Γ preserve colimits of λ-chains. Such an ES is said to be λ-*inductive* if furthermore both functors Σ and Γ preserve epimorphisms.

As it is customary, we refer to the above notions in the case of the ordinal ω simply as finitary and inductive.

Theorem 1.7 ([6,7]) *Let $\mathbb{S}$ be an ES. If $\mathbb{S}$ is λ-finitary then the forgetful functor $\mathbb{S}$-**Alg** $\rightarrow \mathscr{C}$ has a left adjoint.*

Consider an *inductive* ES $\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L \equiv R)$. Since the forgetful functor $U_\mathbb{S}$ decomposes as $\mathbb{S}$-**Alg** $\overset{J_\mathbb{S}}{\hookrightarrow} \Sigma$-**Alg** $\overset{U_\Sigma}{\rightarrow} \mathscr{C}$, its left adjoint can be described in two stages as the composition of a left adjoint to U_Σ followed by a left adjoint to $J_\mathbb{S}$.

The inductive construction of a left adjoint to U_Σ, or equivalently that of free Σ-algebras, is well-known (see *e.g.* [1,2]). The free Σ-algebra (TX, τ_X) on an object $X \in \mathscr{C}$ and the universal map $\eta_X : X \rightarrow TX$ are constructed as follows. The object TX is a colimit of the ω-chain $\{ f_n : X_n \rightarrow X_{n+1} \}_{n \geq 0}$ inductively defined by setting $X_0 = 0$ to be initial and $X_{n+1} = X + \Sigma(X_n)$, for all for $n \geq 0$; and letting $f_0 = u$ be the unique map and $f_{n+1} = X + \Sigma(f_n)$, for all $n \geq 0$. Since the functor $X + \Sigma(-)$ preserves colimits of ω-chains, the object $X + \Sigma(TX)$ is a colimit of the ω-chain $\{ f_n : X_n \rightarrow X_{n+1} \}_{n \geq 1}$. The map $[\eta_X, \tau_X]$ is the unique mediating map as

follows:

$$
\begin{array}{ccccc}
X + \Sigma 0 & \xrightarrow{X+\Sigma u} & X + \Sigma(X + \Sigma 0) & \cdots\cdots & X + \Sigma(TX) \\
\| & & \| & & {\scriptstyle \exists!\,[\eta_X,\tau_X]}\Big\downarrow \cong \\
0 \xrightarrow{\;u\;} X + \Sigma 0 & \xrightarrow{X+\Sigma u} & X + \Sigma(X + \Sigma 0) & \cdots\cdots & TX \quad \text{colim}
\end{array}
\tag{1}
$$

The intuition behind this construction of TX, in which Σ represents a signature and X an object of variables, is that of taking the union of the sequence of objects X_n of terms of depth at most n built from the operators in Σ and the variables in X.

To give a left adjoint to $J_{\mathbb{S}}$, we construct a free $\mathbb{S}$-algebra $(\widetilde{X}, \widetilde{s})$ on a Σ-algebra (X, s) and a universal homomorphism $q : (X, s) \to (\widetilde{X}, \widetilde{s})$ as in the diagram (2) below. The intuition behind the construction of X_1 in there is that of quotienting X according to the equation $L \equiv R$. For $n \geq 1$, the construction of X_{n+1} from X_n is intuitively quotienting the object X_n by congruence rules. Therefore, the construction of free ES algebras formalises that of quotienting by equations and congruence rules.

$$
\begin{array}{ccccccccc}
& \Sigma X & \xrightarrow{\;\Sigma q_0\;} & \Sigma X_1 & \xrightarrow{\;\Sigma q_1\;} & \Sigma X_2 & \xrightarrow{\;\Sigma q_2\;} & \Sigma X_3 & \cdots\cdots & \Sigma \widetilde{X} \\
& {\scriptstyle s}\downarrow & {\scriptstyle s_0}\searrow & \text{po} & {\scriptstyle s_1}\searrow & \text{po} & {\scriptstyle s_2}\searrow & & & \Big\downarrow{\scriptstyle \exists!\,\widetilde{s}} \\
\Gamma X \;\overset{L(X,s)}{\underset{R(X,s)}{\rightrightarrows}}\; X & \xrightarrow[\text{coeq}]{\;q_0\;} & X_1 & \xrightarrow{\;q_1\;} & X_2 & \xrightarrow{\;q_2\;} & X_3 & \cdots\cdots & \widetilde{X} \\
& & & & & & & & \text{colim}
\end{array}
\tag{2}
$$

More precisely, the map q_0 is the coequaliser of the parallel pair $L(X, s), R(X, s)$. The map s_0 is set to be $q_0 \circ s$ and the maps q_i and s_i, for $i \geq 1$, are inductively defined by letting X_{i+1} with q_i and s_i be a pushout of s_{i-1} and $\Sigma(q_{i-1})$. The carrier object $\widetilde{X}$ is given as a colimit of the ω-chain $\{\, q_i \,\}_{i \geq 0}$, and the structure map $\widetilde{s}$ as the unique mediating map from the colimit $\Sigma \widetilde{X}$, induced by the assumption that Σ preserves colimits of ω-chains. The universal homomorphism $q : (X, s) \to (\widetilde{X}, \widetilde{s})$ is the component $q : X \to \widetilde{X}$ of the colimiting cone. Using that Σ preserves epimorphisms, one can inductively show that each q_i and $\Sigma(q_i)$ are epimorphisms, and thus that so is q. Furthermore, we have $q : \big(X, L(X, s)\big) \to \big(\widetilde{X}, L(\widetilde{X}, \widetilde{s})\big)$ and $q : \big(X, R(X, s)\big) \to \big(\widetilde{X}, R(\widetilde{X}, \widetilde{s})\big)$ in Γ-**Alg**, and since q coequalises $L(X, s), R(X, s)$ it follows that Γq equalises $L(\widetilde{X}, \widetilde{s}), R(\widetilde{X}, \widetilde{s})$. Hence, as Γq is an epimorphism, $L(\widetilde{X}, \widetilde{s}) = R(\widetilde{X}, \widetilde{s})$ and $(\widetilde{X}, \widetilde{s})$ is an $\mathbb{S}$-algebra. (Note that the overall argument does not rely on Γ preserving colimits of ω-chains.)

Thus we see that the requirement in inductive ESs that Σ and Γ preserve epimorphisms plays a central role in allowing the construction of free algebras by the simple inductive process in (2). Further interest in this preservation property resides in the results below.

Definition 1.8 For ESs $\mathbb{S}_i = (\mathscr{C} : \Sigma \rhd \Gamma_i \vdash L_i \equiv R_i)$, $i = 1, 2$, where $\mathscr{C}$ has binary coproducts, the ES $\mathbb{S}_1 \& \mathbb{S}_2$ is defined as $\big(\mathscr{C} : \Sigma \rhd (\Gamma_1 + \Gamma_2) \vdash [L_1, L_2] \equiv [R_1, R_2]\big)$.

Proposition 1.9 *Let* $\mathbb{S}_i = (\mathscr{C} : \Sigma \rhd \Gamma_i \vdash L_i \equiv R_i)$, *for* $i = 1, 2$, *be* λ-*inductive*

ESs. For $F \dashv U_\Sigma : \Sigma\text{-}\mathbf{Alg} \longrightarrow \mathscr{C}$ *and* $F_i \dashv J_{\mathbb{S}_i} : \mathbb{S}_i\text{-}\mathbf{Alg} \hookrightarrow \Sigma\text{-}\mathbf{Alg}$, *for* $i = 1, 2$, *the* Σ-*algebra* $F_2 F_1 F X$ *is a free* $(\mathbb{S}_1 \& \mathbb{S}_2)$-*algebra on* X.

Corollary 1.10 *For a* λ-*inductive ES* $\mathbb{S} = (\mathscr{C} : T \rhd \Gamma \vdash L \equiv R)$ *and a monad* $\mathbb{T} = (T, \eta, \mu)$ *on* $\mathscr{C}$, *the free* $\mathbb{S}$-*algebra* $(\widetilde{TX}, \widetilde{\mu_X})$ *on the* T-*algebra* (TX, μ_X) *is a free* $(\mathbb{S} \& \overline{\mathbb{T}})$-*algebra on* X.

2 Term Equational Systems

We introduce a more concrete, yet abstract, notion of equational system, called Term Equational System (TES).

Enriched universes. The notion of TES (Definition 2.5) is given in the context of certain enriched universes of discourse (Definition 2.2). So as to place these within the theory of enriched categories, we need recall the following.

For a $\mathscr{V}$-category $\mathscr{C}$, with $\mathscr{V}$ monoidal closed, the notions of tensors $V \otimes A$ and powers $[V, B]$ for $V \in \mathscr{V}$ and $A, B \in \mathscr{C}$, also referred to in the literature as copowers and cotensors, respectively arise from isomorphisms

$$\mathscr{C}(V \otimes A, B) \cong \mathscr{V}[V, \mathscr{C}(A, B)] \cong \mathscr{C}(A, [V, B]) \quad \text{in } \mathscr{V}$$

for which the former is $\mathscr{V}$-natural in B and the latter $\mathscr{V}$-natural in A, where $\mathscr{V}[-, =]$ denotes the internal hom-functor of $\mathscr{V}$ (see [13] for details). Further, in the important and quite common case in which $\mathscr{V}$ is symmetric monoidal closed, a $\mathscr{V}$-category $\mathscr{C}$ with tensors and powers can be equivalently presented as an ordinary category $\mathscr{C}_0$ equipped with a $\mathscr{V}$-action $(-) \otimes (=) : \mathscr{V} \times \mathscr{C}_0 \longrightarrow \mathscr{C}_0$ such that the functors $(-) \otimes C : \mathscr{V} \longrightarrow \mathscr{C}_0$ and $V \otimes (-) : \mathscr{C}_0 \longrightarrow \mathscr{C}_0$ have right adjoints for all $C \in \mathscr{C}_0$ and $V \in \mathscr{V}$ (see [12] for details).

Remark 2.1 For clarity, and as a notational convention, we will henceforth highlight $\mathscr{V}$-enriched structures by underlining them. Corresponding structures that are not underlined will denote ordinary categorical notions. Thus, for instance, for a $\mathscr{V}$-category $\underline{\mathscr{C}}$, its underlying ordinary category is denoted $\mathscr{C}$.

Definition 2.2 (Enriched universe) An *enriched universe* $(\mathscr{V}, \underline{\mathscr{C}})$ is given by a symmetric monoidal closed category $\mathscr{V}$ and a $\mathscr{V}$-category $\underline{\mathscr{C}}$ with tensors and powers.

As a basic example of enriched universe, note that every symmetric monoidal closed category $\mathscr{V}$ gives rise to the enriched universe $(\mathscr{V}, \underline{\mathscr{V}})$. Another class of examples is obtained from a family of enriched universes $\{ (\mathscr{V}, \underline{\mathscr{C}}_i) \}_{i \in I}$ for a small set I when $\mathscr{V}$ has I-indexed products; as $(\mathscr{V}, \prod_{i \in I} \underline{\mathscr{C}}_i)$ is an enriched universe. (This construction is needed for specifying I-sorted TESs.)

Term Equational Systems. A Term Equational System (TES) consists of a theory of equations between terms.

Definition 2.3 (Generalised term) For an endofunctor T on a category $\mathscr{C}$ and objects $A, C \in \mathscr{C}$, a *generalised term of type* T *with arity* (A, C) is a morphism $A \rightarrow TC$ in $\mathscr{C}$.

Definition 2.4 (Generalised theory) A pair of generalised terms $t \equiv t' : A \to TC$ is called a *generalised equation*. A *generalised theory of type T* is a set of generalised equations consisting of generalised terms of type T.

Definition 2.5 (Term Equational System) A *Term Equational System* (TES) $\mathbb{S} = (\mathcal{V}, \mathscr{C}, \mathbb{T}, \mathcal{A})$ is given by an enriched universe $(\mathcal{V}, \mathscr{C})$, a $\mathcal{V}$-monad $\mathbb{T} = (\underline{T}, \underline{\eta}, \underline{\mu})$ on $\mathscr{C}$, and a generalised theory $\mathcal{A}$ of type T in $\mathscr{C}$.

Example 2.6 As a basic example, we show how multi-sorted equational theories [10,3] arise as TESs.

Recall that an S-sorted signature Σ, for S a set of sorts, is specified by a family of sets of operators $\{\, \Sigma(\sigma, \mathsf{s}) \,\}_{\sigma \in \mathsf{S}^*, \mathsf{s} \in \mathsf{S}}$, where the elements of $\Sigma(\mathsf{s}_1 \ldots \mathsf{s}_n, \mathsf{s})$ stand for operators of arity $\mathsf{s}_1, \ldots, \mathsf{s}_n \to \mathsf{s}$. Recall further that an S-sorted equational theory on such a signature is given by a set of tuples $(\Gamma \vdash t \equiv t' : \mathsf{s})$ for Γ a context of S-sorted variables and t, t' terms of sort s built from the operators in Σ and variables in Γ.

Every S-sorted signature Σ induces the signature endofunctor F_Σ on $\boldsymbol{Set}^\mathsf{S}$ given as follows:

$$(F_\Sigma X)(\mathsf{s}) = \coprod_{\sigma \in \mathsf{S}^*} \Sigma(\sigma, \mathsf{s}) \times \prod_{i \in |\sigma|} X(\sigma_i) \qquad (X \in \boldsymbol{Set}^\mathsf{S}, \mathsf{s} \in \mathsf{S}) \ .$$

The forgetful functor $F_\Sigma\text{-}\mathbf{Alg} \to \boldsymbol{Set}^\mathsf{S}$ is monadic, and the induced monad $\mathbb{T}_\Sigma$ on $\boldsymbol{Set}^\mathsf{S}$ abstractly embodies the notions of terms, variables, and substitution for the signature Σ.

Moreover, multi-sorted equational theories on an S-sorted signature Σ are in bijective correspondence with TESs $(\boldsymbol{Set}, \boldsymbol{Set}^\mathsf{S}, \mathbb{T}_\Sigma, \mathcal{A})$ for which the equations in $\mathcal{A}$ have arity (A_s, C_Γ), for s an S-sort and Γ an S-sorted context, where $A_\mathsf{s}(\mathsf{t}) = \{\, () \mid \mathsf{s} = \mathsf{t} \,\}$ and $C_\Gamma(\mathsf{t}) = \{\, x \mid (x : \mathsf{t}) \in \Gamma \,\}$.

The $\mathcal{V}$-monad $\underline{\mathbb{T}} = (\underline{T}, \underline{\eta}, \underline{\mu})$ of a TES is typically given, and henceforth considered, in the equivalent form of a strong monad (T, τ, η, μ) on $\mathscr{C}$ with strength

$$\tau_{V,C} : V \otimes TC \ \cdot\!\!\to T(V \otimes C) : \mathcal{V} \times \mathscr{C} \ \to \mathscr{C} \ .$$

Strong monads for TESs commonly arise as free monads on strong endofunctors as in the proposition below.

Proposition 2.7 *For $\mathcal{V}$ symmetric monoidal closed, let $\mathscr{C}$ be a $\mathcal{V}$-category with tensors and powers. Further, for an endofunctor F on $\mathscr{C}$ with strength $\tau_{V,C} : V \otimes FC \ \cdot\!\!\to F(V \otimes C)$, assume that the forgetful functor $F\text{-}\mathbf{Alg} \ \to \mathscr{C}$ has a left adjoint and let $\mathbb{T} = (T, \eta, \mu)$ be the induced monad on $\mathscr{C}$.*

Then, $\mathbb{T}$ becomes a strong monad, with the components of the strength $\overline{\tau}$ given by the unique maps such that

$$V \otimes FTC \xrightarrow{\tau_{V,TC}} F(V \otimes TC) \xrightarrow{F\bar{\tau}_{V,C}} FT(V \otimes C)$$

where (TX, φ_X) is a free F-algebra on X.

Model theory. The model theory of TESs is developed.

Let $\mathbb{S} = (\mathcal{V}, \underline{\mathcal{C}}, \mathbb{T}, \mathcal{A})$ be a TES. Every generalised term $t : A \to TC$ in $\mathcal{C}$ induces a functorial term $[\![t]\!] : T\text{-}\mathbf{Alg} \to (\underline{\mathcal{C}}(C, -) \otimes A)_0\text{-}\mathbf{Alg}$ over $\mathcal{C}$, mapping $s : TX \to X$ to the composite

$$\underline{\mathcal{C}}(C,X) \otimes A \xrightarrow{\underline{\mathcal{C}}(C,X)\otimes t} \underline{\mathcal{C}}(C,X) \otimes TC \xrightarrow{\tau_{\underline{\mathcal{C}}(C,X),C}} T\big(\underline{\mathcal{C}}(C,X) \otimes C\big) \xrightarrow{T\epsilon} TX \xrightarrow{s} X \,,$$

which is the transpose of $\underline{\mathcal{C}}(C,X) \xrightarrow{T_{C,X}} \underline{\mathcal{C}}(TC,TX) \xrightarrow{\underline{\mathcal{C}}(t,s)} \underline{\mathcal{C}}(A,X)$.

The functorial interpretation of terms induces a model-theoretic notion of equality between generalised terms for algebras. We formalise it by introducing a *satisfaction relation*: for a T-algebra (X,s),

$$(X,s) \models u \equiv v : A \to TC \quad \text{iff} \quad [\![u]\!](X,s) = [\![v]\!](X,s) : \underline{\mathcal{C}}(C,X) \otimes A \to X \,.$$

More generally, for a set of T-algebras $\mathcal{A}$, we set $\mathcal{A} \models u \equiv v$ iff $(X,s) \models u \equiv v$ for all $(X,s) \in \mathcal{A}$.

It is an important fact that the algebras satisfying an equation are closed under powers. Indeed, every algebra structure $s : TX \to X$ induces an algebra structure s_V on $[V,X]$ given by the transpose of the composite

$$V \otimes T[V,X] \xrightarrow{\tau_{V,[V,X]}} T(V \otimes [V,X]) \xrightarrow{T\epsilon} TX \xrightarrow{s} X$$

and we have the following result.

Lemma 2.8 *For (X,s) a T-algebra, $(X,s) \models u \equiv v$ iff $([V,X], s_V) \models u \equiv v$ for all $V \in \mathcal{V}$.*

Definition 2.9 (TES algebra) An $\mathbb{S}$-*algebra* for a TES $\mathbb{S} = (\mathcal{V}, \underline{\mathcal{C}}, \mathbb{T}, \mathcal{A})$ is an Eilenberg-Moore algebra (X,s) for the monad $\mathbb{T}$ satisfying the equations in $\mathcal{A}$; that is, such that $(X,s) \models u \equiv v$ for all $(u \equiv v) \in \mathcal{A}$.

The category $\mathbb{S}\text{-}\mathbf{Alg}$ is the full subcategory of the category of Eilenberg-Moore algebras $\mathcal{C}^{\mathbb{T}}$ consisting of the $\mathbb{S}$-algebras. We thus have the following situation

$$\mathbb{S}\text{-}\mathbf{Alg} \hookrightarrow \mathcal{C}^{\mathbb{T}} \hookrightarrow T\text{-}\mathbf{Alg}$$

The following remark indicates why, in general, a single equation is not sufficient for the specification of categories of TES algebras.

Proposition 2.10 *Assume that $\mathscr{C}$ has I-indexed coproducts for a small set I. For (X, s) a T-algebra, if $(X, s) \models t_i \equiv t_i' : A_i \to TC_i$ for all $i \in I$ then $(X, s) \models t \equiv t'$ for*

$$
\begin{array}{c}
\coprod_{i \in I} A_i \xrightarrow{\quad\quad\quad t \quad\quad\quad} \\
\coprod_{i \in I} t_i \searrow \quad \coprod_{i \in I} TC_i \xrightarrow{[T\iota_i]_{i \in I}} T\big(\coprod_{i \in I} C_i \big) \\
\coprod_{i \in I} t_i' \nearrow \\
\coprod_{i \in I} A_i \xrightarrow{\quad\quad\quad t' \quad\quad\quad}
\end{array}
$$

On the other hand, the converse need not hold.

Example 2.11 Let $\mathbb{S}_{(\Sigma, \mathcal{A})}$ be the TES associated to an equational theory $\mathcal{A}$ on an S-sorted signature Σ as in Example 2.6.

The functorial term induced by a map $A_{\mathsf{s}} \to T_\Sigma(C_\Gamma)$ in $\boldsymbol{Set}^{\mathsf{S}}$, or equivalently a term t of sort s in context Γ, maps a $\mathbb{T}_\Sigma$-algebra (X, s), or equivalently an F_Σ-algebra (X, s^*), to a map

$$
[\![\Gamma \vdash t : \mathsf{s}]\!]_{(X, s^*)} : \boldsymbol{Set}^{\mathsf{S}}(C_\Gamma, X) \otimes A_{\mathsf{s}} \to X \quad \text{in } \boldsymbol{Set}^{\mathsf{S}} \tag{3}
$$

where, for $S \in \boldsymbol{Set}$ and $P \in \boldsymbol{Set}^{\mathsf{S}}$, the tensor $S \otimes P \in \boldsymbol{Set}^{\mathsf{S}}$ is given pointwise by setting $(S \otimes P)(\mathsf{t}) = S \times P(\mathsf{t})$ for all $\mathsf{t} \in \mathsf{S}$. Thus, to give (3) is equivalent to give a single function

$$
[\![\Gamma \vdash t : \mathsf{s}]\!]_{(X, s^*)} : \textstyle\prod_{\mathsf{t} \in \mathsf{S}} \boldsymbol{Set}\big(C_\Gamma(\mathsf{t}), X(\mathsf{t})\big) \to X(\mathsf{s}) .
$$

The explicit description of this function is given by the following inductive definition.

- For $(x : \mathsf{s}) \in \Gamma$,

$$
[\![\Gamma \vdash x : \mathsf{s}]\!]_{(X, s^*)}(\rho) = \rho_{\mathsf{s}}(x) .
$$

- For $\mathsf{f} : \mathsf{s}_1, \ldots, \mathsf{s}_n \to \mathsf{s}$ in Σ,

$$
[\![\Gamma \vdash \mathsf{f}(t_1, \ldots, t_n) : \mathsf{s}]\!]_{(X, s^*)}(\rho) = s_{\mathsf{f}}^*\big([\![\Gamma \vdash t_1 : \mathsf{s}_1]\!]_{(X, s^*)}(\rho), \ldots, [\![\Gamma \vdash t_n : \mathsf{s}_n]\!]_{(X, s^*)}(\rho)\big)
$$

where s_{f}^* denotes the f-component of the structure map s^*.

It follows that $\mathbb{S}_{(\Sigma, \mathcal{A})}$-**Alg** is (isomorphic to) the category of $(\Sigma, \mathcal{A})$-algebras.

Remark 2.12 We note that the model theory of TESs can be easily recast in the framework of ESs. Indeed, for $\mathscr{C}$ with small coproducts, every TES $\mathbb{S} = (\mathscr{V}, \mathscr{C}, \mathbb{T}, \mathcal{A})$ induces an ES $\overline{\mathbb{S}} = (\mathscr{C} : T \rhd \Gamma_{\mathcal{A}} \vdash \overline{L} \equiv \overline{R})$ such that $(\overline{\mathbb{S}} \& \overline{\mathbb{T}})$-**Alg** $\cong \mathbb{S}$-**Alg**, where

$$
\Gamma_{\mathcal{A}}(X) = \coprod_{(t \equiv t' : A \to TC) \in \mathcal{A}} \mathscr{C}(C, X) \otimes A
$$

$$
\overline{L}(X, s) = \big[[\![t]\!](X, s) \big]_{(t \equiv t') \in \mathcal{A}} , \quad \overline{R}(X, s) = \big[[\![t']\!](X, s) \big]_{(t \equiv t') \in \mathcal{A}}
$$

Furthermore, when the strong monad $\mathbb{T}$ arises as the free monad on a strong endofunctor F (as in Proposition 2.7), the TES $\mathbb{S}$ induces a simpler ES $\widehat{\mathbb{S}}$ with $\widehat{\mathbb{S}}$-**Alg** $\cong \mathbb{S}$-**Alg**. Indeed, $\widehat{\mathbb{S}} = (\mathscr{C} : F \rhd \Gamma_{\mathcal{A}} \vdash \widehat{L} \equiv \widehat{R})$, where

$$\widehat{L}(X, s^*) = \big[\, [\![t]\!](X, s) \,\big]_{(t \equiv t') \in \mathcal{A}} \,, \quad \widehat{R}(X, s^*) = \big[\, [\![t']\!](X, s) \,\big]_{(t \equiv t') \in \mathcal{A}}$$

for (X, s) the $\mathbb{T}$-algebra corresponding to the F-algebra (X, s^*).

Definition 2.13 Let $(\mathscr{V}, \mathscr{C})$ be an enriched universe. A pair of objects (A, C) in $\mathscr{C}$ is respectively said to be λ-*compact*, for λ a limit ordinal, and *projective* if the endofunctor $(\mathscr{C}(C, -) \otimes A)_0$ on $\mathscr{C}$ respectively preserves colimits of λ-chains and epimorphisms.

Proposition 2.14 *Let* $\mathbb{S} = (\mathscr{V}, \mathscr{C}, \mathbb{T}, \mathcal{A})$ *be a TES. If the category* $\mathscr{C}$ *is cocomplete, the endofunctor* T *on* $\mathscr{C}$ *preserves colimits of* λ-*chains (and epimorphisms), and every arity of an equation in* $\mathcal{A}$ *is* λ-*compact (and projective), then the ES* $\overline{\mathbb{S}} \& \overline{\mathbb{T}}$ *is* λ-*finitary* (λ-*inductive*).

Thus, the theory of Section 1 may be applied to TESs.

3   Term Equational Logic

We introduce a sound deduction system for reasoning about equality between generalised terms in the context of TESs.

For a TES $\mathbb{S} = (\mathscr{V}, \mathscr{C}, \mathbb{T}, \mathcal{A})$, we consider equality judgements of the form

$$\mathcal{A} \vdash u \equiv v : A \rightarrow TC$$

where u, v are generalised terms of arity (A, C) in $\mathscr{C}$. The associated Term Equational Logic (TEL) consists of the following rules.

- Equality rules.

$$\mathsf{Ref} \ \ \mathcal{A} \vdash u \equiv u \qquad \mathsf{Sym} \ \ \frac{\mathcal{A} \vdash u \equiv v}{\mathcal{A} \vdash v \equiv u} \qquad \mathsf{Trans} \ \ \frac{\mathcal{A} \vdash u \equiv v \quad \mathcal{A} \vdash v \equiv w}{\mathcal{A} \vdash u \equiv w}$$

- Axioms.

$$\mathsf{Axiom} \ \ \frac{(u \equiv v) \in \mathcal{A}}{\mathcal{A} \vdash u \equiv v}$$

- Congruence of substitution.

$$\mathsf{Subst} \ \ \frac{\mathcal{A} \vdash u_1 \equiv v_1 : A \ \rightarrow TB \qquad \mathcal{A} \vdash u_2 \equiv v_2 : B \ \rightarrow TC}{\mathcal{A} \vdash u_1[u_2] \equiv v_1[v_2] : A \ \rightarrow TC}$$

where $w_1[w_2]$ denotes the Kleisli composite $A \ \overset{w_1}{\rightarrow} TB \ \overset{Tw_2}{\rightarrow} T(TC) \ \overset{\mu_C}{\rightarrow} TC$.

- Congruence of tensor extension.

$$\mathsf{Ext} \ \ \frac{\mathcal{A} \vdash u \equiv v : A \ \rightarrow TC}{\mathcal{A} \vdash \langle V \rangle u \equiv \langle V \rangle v : V \otimes A \ \rightarrow T(V \otimes C)} \ (V \in \mathscr{V})$$

where $\langle V \rangle w$ denotes the composite $V \otimes A \overset{V \otimes w}{\rightarrow} V \otimes TC \overset{\tau_{V,C}}{\rightarrow} T(V \otimes C)$.

- Local character.

$$\mathsf{LocChar} \; \frac{\mathcal{A} \vdash u \circ e_i \equiv v \circ e_i : A_i \to TC \quad (i \in I)}{\mathcal{A} \vdash u \equiv v : A \to TC} \; (\{\, e_i \,\}_{i \in I} \text{ jointly epi})$$

Remark 3.1 One can also consider the following theory-dependent rule:

$$\frac{\mathcal{A} \vdash T(e) \circ u \equiv T(e) \circ v : A \to TD}{\mathcal{A} \vdash u \equiv v : A \to TC} \; (e : C \to D \; \mathbb{S}\text{-extendable})$$

where $e : C \to D$ in $\mathscr{C}$ is $\mathbb{S}$-*extendable* iff $\underline{\mathscr{C}}(e, X) : \underline{\mathscr{C}}(D, X) \to \underline{\mathscr{C}}(C, X)$ in $\mathscr{V}$ is an epimorphism for all $\mathbb{S}$-algebra carriers X.

However, we know of no concrete example where this rule is of real significance.

Example 3.2 The TEL associated to the TES of a multi-sorted equational theory (see Example 2.6) conservatively extends its multi-sorted equational logic [10]. Note for instance that the usual congruence rule for substitution is derivable from the TEL rules Subst and LocChar. Details are left to the interested reader (see also [3]).

The soundness of TEL is the following result.

Theorem 3.3 (External soundness) *For a TES* $\mathbb{S} = (\mathscr{V}, \underline{\mathscr{C}}, \mathbb{T}, \mathcal{A})$,

$$\mathcal{A} \vdash u \equiv v \quad \text{implies} \quad \mathbb{S}\text{-}\mathbf{Alg} \models u \equiv v\,.$$

Interestingly, the soundness of the congruence rule for tensor extension essentially amounts to Lemma 2.8.

4   Internal completeness

One cannot expect to obtain an external completeness result at the level of generality of the previous section. However, in the presence of free constructions, an internal completeness result does hold. Its importance in relationship to external completeness is illustrated in the application of the following section.

Let $\mathbb{S} = (\mathscr{V}, \underline{\mathscr{C}}, \mathbb{T}, \mathcal{A})$ be a TES for which the forgetful functor $\mathbb{S}\text{-}\mathbf{Alg} \to \mathscr{C}$ has a left adjoint $\mathscr{C} \to \mathbb{S}\text{-}\mathbf{Alg} : X \mapsto (\widetilde{T}X, \sigma_X : T\widetilde{T}X \to \widetilde{T}X)$, and let $\widetilde{\mathbb{T}} = (\widetilde{T}, \widetilde{\eta}, \widetilde{\mu})$ be the associated monad on $\mathscr{C}$. Then, the embedding $\mathbb{S}\text{-}\mathbf{Alg} \hookrightarrow \mathscr{C}^{\mathbb{T}}$ induces a strict monad morphism $q : \mathbb{T} \to \widetilde{\mathbb{T}}$, with each component $q_X : TX \to \widetilde{T}X$ given by the unique homomorphism $(TX, \mu_X) \to (\widetilde{T}X, \sigma_X)$ extending $\widetilde{\eta}_X : X \to \widetilde{T}X$ along $\eta_X : X \to TX$.

Lemma 4.1 *The monad* $\widetilde{\mathbb{T}}$ *has a strength* $\widetilde{\tau}$ *induced from* τ, *with components* $\widetilde{\tau}_{V,X}$ *given by the unique map such that*

$$V \otimes T\widetilde{T}X \xrightarrow{\tau_{V,\widetilde{T}X}} T(V \otimes \widetilde{T}X) \xrightarrow{T(\widetilde{\tau}_{V,X})} T\widetilde{T}(V \otimes X)$$

$$V \otimes \widetilde{T}X \quad \xdashrightarrow{\exists! \, \widetilde{\tau}_{V,X}} \quad \widetilde{T}(V \otimes X)$$

with arrows $V \otimes \sigma_X$, $\sigma_{V \otimes X}$, $V \otimes \widetilde{\eta}_X$, $\widetilde{\eta}_{V \otimes X}$, $V \otimes X$

and making $q : T \xrightarrow{\cdot} \widetilde{T}$ a map of strengths in the sense that

$$
\begin{array}{ccc}
V \otimes TX & \xrightarrow{V \otimes q_X} & V \otimes \widetilde{T}X \\
\tau_{V,X} \downarrow & & \downarrow \widetilde{\tau}_{V,X} \\
T(V \otimes X) & \xrightarrow{q_{V \otimes X}} & \widetilde{T}(V \otimes X)
\end{array}
$$

The equivalence of the first two statements below is a form of strong completeness; it states that an equation is satisfied by all models iff it is satisfied in a freely generated, hence somewhat syntactic, one.

Theorem 4.2 (Internal soundness and completeness) *For a TES $\mathbb{S}$ such that the forgetful functor $\mathbb{S}\text{-}\mathbf{Alg} \to \mathscr{C}$ has a left adjoint, the following are equivalent:*

(i) $\mathbb{S}\text{-}\mathbf{Alg} \models u \equiv v : A \to TC$

(ii) $(\widetilde{T}C, \sigma_C) \models u \equiv v : A \to TC$

(iii) $q_C \circ u = q_C \circ v : A \to \widetilde{T}C$

Proof. (i) $\Rightarrow$ (ii) Vacuously.

(ii) $\Rightarrow$ (iii) Because, for $u : A \to TC$ in $\mathscr{C}$ and $\widetilde{n}_C : I \to \underline{\mathscr{C}}(C, \widetilde{T}C)$ in $\mathscr{V}$ the transpose of $I \otimes C \cong C \xrightarrow{\widetilde{\eta}_C} \widetilde{T}C$ in $\mathscr{C}$, we have that

$$
\begin{array}{ccc}
A \cong I \otimes A & \xrightarrow{\widetilde{n}_C \otimes A} & \underline{\mathscr{C}}(C, \widetilde{T}C) \otimes A \\
u \downarrow & & \downarrow [\![u]\!](\widetilde{T}C, \sigma_C) \\
TC & \xrightarrow{q_C} & \widetilde{T}C
\end{array}
$$

(iii) $\Rightarrow$ (i) Because, for all $(X, s) \in \mathbb{S}\text{-}\mathbf{Alg}$, we have

$$
\begin{array}{ccc}
\underline{\mathscr{C}}(C, X) \otimes TC & \xrightarrow{\underline{\mathscr{C}}(C,X) \otimes q_C} & \underline{\mathscr{C}}(C, X) \otimes \widetilde{T}C \\
\tau_{\underline{\mathscr{C}}(C,X),C} \downarrow & & \downarrow \widetilde{\tau}_{\underline{\mathscr{C}}(C,X),C} \\
T(\underline{\mathscr{C}}(C,X) \otimes C) & \xrightarrow{q_{\underline{\mathscr{C}}(C,X) \otimes C}} & \widetilde{T}(\underline{\mathscr{C}}(C,X) \otimes C) \\
T\epsilon \downarrow & & \downarrow \widetilde{T}\epsilon \\
TX & \xrightarrow{q_X} & \widetilde{T}X
\end{array}
$$

with s, s' into X

where s' denotes the unique homomorphism $(\widetilde{T}X, \sigma_X) \to (X, s)$ extending id_X along $\widetilde{\eta}_X : X \to \widetilde{T}X$. $\square$

Importantly for applications, we remark that when a TES is represented by an inductive ES, the strong-monad morphism $q : \mathbb{T} \dashrightarrow \widetilde{\mathbb{T}}$ has an inductive description. For instance, for a TES $\mathbb{S} = (\mathscr{V}, \underline{\mathscr{C}}, \mathbb{T}, \mathcal{A})$ with $\mathscr{C}$ cocomplete, $\underline{\mathbb{T}}$ arising as a free monad on a strong endofunctor F that preserves colimits of ω-chains and epimorphisms, and $\mathcal{A}$ with equations of compact and projective arity, the ES $\widehat{\mathbb{S}}$ (see Remark 2.12) is inductive and the quotient map $q_X : TX \twoheadrightarrow \widetilde{T}X$ is inductively constructed as follows:

$$
\begin{array}{c}
\xymatrix{
FTX \ar@{->>}[r]^{Fq_0} \ar[d]_{\mu_X^*} & FY_1 \ar@{->>}[r]^{Fq_1} \ar[dl] & FY_2 \ar@{->>}[r] \ar[dl] & \cdots \ar@{->>}[r] & F(\widetilde{T}X) \ar@{-->}[d]^{\exists! \, \sigma_X^*} \\
TX \ar@{->>}[r]_{q_0} & Y_1 \ar@{->>}[r]_{q_1} & Y_2 \ar@{->>}[r] & \cdots \ar@{->>}[r] & \widetilde{T}X
}
\end{array}
\qquad (4)
$$

where (TX, μ_X^*) and $(\widetilde{T}X, \sigma_X^*)$ are the F-algebras respectively corresponding to the $\mathbb{T}$-algebras (TX, μ_X) and $(\widetilde{T}X, \sigma_X)$, and where q_0 is the universal map that coequalises every pair $[\![u]\!](TX, \mu_X)$ and $[\![v]\!](TX, \mu_X)$ with $(u \equiv v) \in \mathcal{A}$.

5   Synthetic Nominal Equational Logic

We recall the basic structure of the category $\mathbf{\mathcal{N}om}$ of Nominal Sets [9, Section 6] (which is equivalent to the Schanuel topos [17, Section III.9]) and present it as an enriched universe.

On $\mathbf{\mathcal{N}om}$ we consider a class of TESs which we call NESs (Nominal Equational Systems). A syntactic notion of nominal theory arises, and its model theory is derived from that of NESs. An equational logic, called SNEL (Synthetic Nominal Equational Logic), for nominal theories is synthesised from the TEL associated to NESs. The logic SNEL is sound by construction, and we establish its completeness by means of the internal completeness result through an analysis of the inductive construction of free algebras.

Gabbay and Mathijssen [8], on the one hand, and Clouston and Pitts [4], on the other, have recently introduced sound and complete equational deduction systems for nominal algebraic structures. These logics are logically equivalent to our SNEL. However, our novel top-down development—from general mathematical theory to concrete application—leads to new syntax and proof of completeness.

Nominal sets.   For a fixed countably infinite set A of atoms, the group $\mathfrak{S}_0(\mathsf{A})$ of finite permutations of atoms consists of the bijections on A that fix all but finitely many elements of A. A $\mathfrak{S}_0(\mathsf{A})$-*action* $X = (\,|X|, \cdot\,)$ consists of a set $|X|$ equipped with an action $(-) \cdot (=) \;:\; \mathfrak{S}_0(\mathsf{A}) \times |X| \;\to\; |X|$ satisfying $\mathrm{id}_\mathsf{A} \cdot x = x$ and $\pi' \cdot (\pi \cdot x) = (\pi'\pi) \cdot x$ for all $x \in |X|$ and $\pi, \pi' \in \mathfrak{S}_0(\mathsf{A})$. $\mathfrak{S}_0(\mathsf{A})$-actions form a category with morphisms $X \to Y$ given by *equivariant functions*; that is, functions $f : |X| \to |Y|$ such that $f(\pi \cdot x) = \pi \cdot (fx)$ for all $\pi \in \mathfrak{S}_0(\mathsf{A})$ and $x \in |X|$.

For a $\mathfrak{S}_0(\mathsf{A})$-action X, a finite subset S of A is said to *support* $x \in X$ if for all atoms $a, a' \notin S$, we have that $(a\ a') \cdot x = x$, where the *transposition* $(a\ a')$ is

the bijection that swaps a and a', and fixes all other atoms. A *nominal set* is a $\mathfrak{S}_0(\mathsf{A})$-action in which every element has finite support. As an example, the set of atoms A becomes the *nominal set of atoms* $\mathbb{A}$ when equipped with the evaluation action $\pi \cdot a = \pi(a)$. The category $\boldsymbol{\mathcal{N}om}$ is the full subcategory of the category of $\mathfrak{S}_0(\mathsf{A})$-actions consisting of nominal sets.

The supports of an element of a nominal set are closed under intersection, and we write $\mathsf{supp}_X(x)$, or simply $\mathsf{supp}(x)$, for the intersection of the supports of x in the nominal set X. For elements x and y of two, possibly distinct, nominal sets X and Y, we write $x \mathrel{\#} y$ whenever $\mathsf{supp}_X(x)$ and $\mathsf{supp}_Y(y)$ are disjoint. Thus, for $a \in \mathbb{A}$ and $x \in X$, $a \mathrel{\#} x$ stands for $a \notin \mathsf{supp}_X(x)$; that is, *a is fresh for x*.

The category $\boldsymbol{\mathcal{N}om}$ is complete and cocomplete. In particular, for a family of nominal sets $\{\, X_i \,\}_{i \in I}$, the coproduct $\coprod_{i \in I} X_i$ is given by $|\coprod_{i \in I} X_i| = \coprod_{i \in I} |X_i|$ with action $\pi \cdot \iota_i(x) = \iota_i(\pi \cdot x)$; whilst the product $\prod_{i \in I} X_i$, for finite I, is given by $|\prod_{i \in I} X_i| = \prod_{i \in I} |X_i|$ with action $\pi \cdot (x_i)_{i \in I} = (\pi \cdot x_i)_{i \in I}$. Further, $\boldsymbol{\mathcal{N}om}$ carries a symmetric monoidal structure $(I, \#)$. The *unit* I is the singleton set consisting of the empty tuple equipped with the unique action. The *separating tensor* $X \mathrel{\#} Y$ is the nominal subset of $X \times Y$ with underlying set given by $\{\, (x, y) \in |X| \times |Y| \mid x \mathrel{\#} y \,\}$. As usual, we write $X^{\#n}$ for $X \mathrel{\#} \ldots \mathrel{\#} X$ (n times). For instance, $\mathbb{A}^{\#n}$ consists of n-tuples of distinct atoms equipped with the pointwise action. Note that $X^{\#0}$ is I.

Henceforth, we write $\boldsymbol{a}^n$, or simply $\boldsymbol{a}$ when n is clear from the context, as a shorthand for a tuple $a_1, \ldots, a_n$ of distinct atoms. A *multi transposition* $(\boldsymbol{a}^n \, \boldsymbol{b}^n)$ denotes a fixed bijection on A satisfying $(\boldsymbol{a}^n \, \boldsymbol{b}^n)(a_i) = b_i$, for $i = 1, \ldots, n$, and $(\boldsymbol{a}^n \, \boldsymbol{b}^n)(c) = c$ for $c \notin \{\boldsymbol{a}^n\} \cup \{\boldsymbol{b}^n\}$.

The separating tensor $\#$ is closed and the corresponding internal-hom functor is denoted $[-, =]$. In particular, the functor $[\mathbb{A}^{\#n}, -]$ provides a notion of atom multi abstraction. For a nominal set X, the nominal set $[\mathbb{A}^{\#n}, X]$ has underlying set given by the quotient set $|\mathbb{A}^{\#n} \times X|/_{\approx}$ determined by the α-*equivalence relation* $\approx$ defined as $(\boldsymbol{a}, x) \approx (\boldsymbol{b}, x')$ iff there exists a fresh $\boldsymbol{c} \in \mathbb{A}^{\#n}$ (*i.e.*, a tuple satisfying $\boldsymbol{c} \mathrel{\#} \boldsymbol{a}, x, \boldsymbol{b}, x'$) such that $(\boldsymbol{a} \, \boldsymbol{c}) \cdot x = (\boldsymbol{b} \, \boldsymbol{c}) \cdot x'$. We write $\langle \boldsymbol{a} \rangle x$ for the equivalence class $[(\boldsymbol{a}, x)]_{\approx}$. Note that $\mathsf{supp}(\langle \boldsymbol{a}^n \rangle x)$ is $\mathsf{supp}(x) \setminus \{\boldsymbol{a}^n\}$.

Thus, the structure $((\boldsymbol{\mathcal{N}om}, I, \#), \boldsymbol{\mathcal{N}om})$ is an enriched universe.

We now urge the reader to read the remaining of the section in the context of the methodology for developing term equational systems and logics presented in the Introduction.

Nominal Equational Systems.   A *nominal signature* Σ is given by a family of nominal sets $\{\, \Sigma(n) \,\}_{n \in \mathbb{N}}$, each of which consists of *operators* of arity n. To each such signature, we associate the endofunctor F_Σ on $\boldsymbol{\mathcal{N}om}$ defined by $F_\Sigma(X) = \coprod_{n \in \mathbb{N}} \Sigma(n) \times X^n$ with strength $\tau_{X,Y} : F_\Sigma(X) \mathrel{\#} Y \;\rightarrowtail\; F_\Sigma(X \mathrel{\#} Y)$ for $X, Y \in \boldsymbol{\mathcal{N}om}$ given by

$$\tau_{X,Y}\big(\iota_n(\mathsf{o}, x_1, \ldots, x_n), y\big) = \iota_n\big(\mathsf{o}, (x_1, y), \ldots, (x_n, y)\big)$$

for $n \in \mathbb{N}, \mathsf{o} \in \Sigma(n), x_1, \ldots, x_n \in X, y \in Y$.

Nominal Equational Systems (NESs) are of the form $(\boldsymbol{Nom}, \underline{\boldsymbol{Nom}}, \mathbb{T}_\Sigma, \mathcal{A})$ where $\mathbb{T}_\Sigma$ is the free monad on F_Σ (with strength defined as in Proposition 2.7) and where every term in $\mathcal{A}$ has an arity of the form $(\mathbb{A}^{\# n}, \coprod_{i=1}^{\ell} \mathbb{A}^{\# n_i})$ for $n, \ell, n_1, \ldots, n_\ell \in \mathbb{N}$. It follows that NESs satisfy all the conditions in Proposition 2.14 and hence induce inductive ESs.

Nominal syntax.   We proceed to analyse the syntactic structure underlying NESs. To this end, first note that, since the functor F_Σ preserves colimits of ω-chains and the category $\boldsymbol{Nom}$ is cocomplete, the free monad $\mathbb{T}_\Sigma$ on F_Σ is constructed as in (1). Thus, we have the following inductive definition of $T_\Sigma X$:

$$
\begin{aligned}
t \in T_\Sigma X ::= \ & x \qquad\qquad\quad (\, x \in X \,) \\
& |\ \ \mathsf{o}\ t_1 \ldots t_k \quad (\, \mathsf{o} \in \Sigma(k), t_1, \ldots, t_k \in T_\Sigma X \,)
\end{aligned}
$$

with action given by $\pi \cdot x = \pi \cdot_X x$ and $\pi \cdot (\mathsf{o}\ t_1 \ldots t_k) = (\pi \cdot \mathsf{o})\,(\pi \cdot t_1) \ldots (\pi \cdot t_k)$.

More generally, we need to consider generalised terms $\mathbb{A}^{\# n} \to \coprod_{i=1}^{\ell} \mathbb{A}^{\# n_i}$. From the bijections

$$
\begin{aligned}
& \{\, t : \mathbb{A}^{\# n} \to T_\Sigma \big(\textstyle\coprod_{i=1}^{\ell} \mathbb{A}^{\# n_i} \big) \,\} \\
& \cong\ \{\, t : I \to [\mathbb{A}^{\# n}, T_\Sigma \big(\textstyle\coprod_{i=1}^{\ell} \mathbb{A}^{\# n_i} \big)] \,\} \\
& \cong\ \{\, t \in [\mathbb{A}^{\# n}, T_\Sigma \big(\textstyle\coprod_{i=1}^{\ell} \mathbb{A}^{\# n_i} \big)] \mid \mathsf{supp}(t) = \emptyset \,\} \\
& =\ \{\, \langle \boldsymbol{a} \rangle\, t \in [\mathbb{A}^{\# n}, T_\Sigma \big(\textstyle\coprod_{i=1}^{\ell} \mathbb{A}^{\# n_i} \big)] \mid \mathsf{supp}(t) \subseteq \{\boldsymbol{a}\} \,\}
\end{aligned}
$$

it follows that every such generalised term can be described by pairs

$$
(\boldsymbol{a}, t) \in \mathbb{A}^{\# n} \times T_\Sigma(\textstyle\coprod_{i=1}^{\ell} \mathbb{A}^{\# n_i}) \text{ with } \mathsf{supp}(t) \subseteq \{\,\boldsymbol{a}\,\} \tag{5}
$$

via the abstraction quotient map $\mathbb{A}^{\# n} \times T_\Sigma(\coprod_{i=1}^{\ell} \mathbb{A}^{\# n_i}) \ \twoheadrightarrow\ [\mathbb{A}^{\# n}, T_\Sigma(\coprod_{i=1}^{\ell} \mathbb{A}^{\# n_i})]$ and the above bijections.

We thus obtain a representation of generalised terms of arity $(\mathbb{A}^{\# n}, \coprod_{i=1}^{\ell} \mathbb{A}^{\# n_i})$ as syntactic *open terms* built up from ℓ variables respectively of *valence* $n_1, \ldots, n_\ell$ in the context of n atoms. It is therefore convenient to introduce a syntactic notion of context that reflects this structure. To this end, fix a countably infinite set $\mathbf{V}$ of variables and define a *context* $[\boldsymbol{a}]V$ as consisting of an *atom context* given by a tuple of distinct atoms $\boldsymbol{a}$ and a *variable context* given by a valence function V from a finite subset $|V| \subseteq \mathbf{V}$ of variables to $\mathbb{N}$. Every context $[\boldsymbol{a}]V$ determines the arity $(\mathbb{A}^{\# |\boldsymbol{a}|}, \widetilde{V})$ with $\widetilde{V} = \coprod_{x \in |V|} \mathbb{A}^{\# V(x)}$.

The *open terms* t in context $[\boldsymbol{a}]V$, for which we use the notation $[\boldsymbol{a}]V \vdash t$, corresponding to the descriptions (5) are thus given by the following rules:

$$
\frac{}{[\boldsymbol{a}]V \vdash x(\boldsymbol{b})} \ \ (x(\boldsymbol{b}) \in \widetilde{V} \text{ and } \{\boldsymbol{b}\} \subseteq \{\boldsymbol{a}\})
$$

$$
\frac{[\boldsymbol{a}]V \vdash t_i \quad (1 \le i \le k)}{[\boldsymbol{a}]V \vdash \mathsf{o}\ t_1 \ldots t_k} \ \ (\mathsf{o} \in \Sigma(k) \text{ and } \mathsf{supp}(\mathsf{o}) \subseteq \{\boldsymbol{a}\})
$$

where we write $x(\boldsymbol{b})$ for the element $\iota_x(\boldsymbol{b})$ of $\widetilde{V}$ and, when convenient, further abbreviate $x()$ as x. The generalised term of arity $(\mathbb{A}^{\#|\boldsymbol{a}|}, \widetilde{V})$ associated to the open term $[\boldsymbol{a}]V \vdash t$ is denoted $\mathcal{T}([\boldsymbol{a}]V \vdash t)$. Hence, $\mathcal{T}([\boldsymbol{a}]V \vdash t)(\boldsymbol{b}) = (\boldsymbol{a}\,\boldsymbol{b}) \cdot t$.

Nominal theories. A *nominal theory* consists of a nominal signature and a set of equations of the form $[\boldsymbol{a}]V \vdash t \equiv t'$ where $[\boldsymbol{a}]V \vdash t$ and $[\boldsymbol{a}]V \vdash t'$ are open terms. We give the canonical example.

Example 5.1 (*cf.* [8,4]) The nominal signature Σ_λ for the untyped λ-calculus is given by the nominal sets of operators $\Sigma_\lambda(0) = \{V_a \mid a \in \mathbb{A}\}$, $\Sigma_\lambda(1) = \{L_a \mid a \in \mathbb{A}\}$, $\Sigma_\lambda(2) = \{A\}$ with action $\pi \cdot V_a = V_{\pi(a)}$, $\pi \cdot L_a = L_{\pi(a)}$, $\pi \cdot A = A$. The nominal theory $\mathbb{S}_\lambda$ for $\alpha\beta\eta$-equivalence of untyped λ-terms consists of the following equations:

(α) $[a,b]\{\,x:1\,\} \vdash L_a\ x(a) \equiv L_b\ x(b)$

(β_κ) $[a]\{\,x:0, y:1\,\} \vdash A\ (L_a\ x)\ y(a) \equiv x$

(β_V) $[a]\{\,x:1\,\} \vdash A\ (L_a\ V_a)\ x(a) \equiv x(a)$

(β_L) $[a,b]\{\,x:2, y:1\,\} \vdash A\ \big(L_a\ (L_b\ x(a,b))\big)\ y(a) \equiv L_b\ \big(A\ (L_a\ x(a,b))\ y(a)\big)$

(β_A) $[a]\{\,x:1, y:1, z:1\,\}$

$\qquad\qquad \vdash A\ \big(L_a\ (A\ x(a)\ y(a))\big)\ z(a) \equiv A\ \big(A\ (L_a\ x(a))\ z(a)\big)\ \big(A\ (L_a\ y(a))\ z(a)\big)$

(β_ε) $[a,b]\{\,x:1\,\} \vdash A\ \big(L_a\ x(a)\big)\ V_b \equiv x(b)$

(η) $[a]\{\,x:0\,\} \vdash L_a\ (A\ x\ V_a) \equiv x$

where a variable context V with $|V| = \{\,x_1, \ldots, x_\ell\,\}$ and $V(x_i) = n_i$ $(i = 1, \ldots, \ell)$ is denoted $\{\,x_1:n_1, \ldots, x_\ell:n_\ell\,\}$.

Model theory. Every nominal theory $\mathbb{S} = (\Sigma, \mathcal{A})$ induces the NES $\widetilde{\mathbb{S}} = (\boldsymbol{Nom}, \boldsymbol{Nom}, \mathbb{T}_\Sigma, \widetilde{\mathcal{A}})$, with

$$\widetilde{\mathcal{A}} = \{\,\mathcal{T}([\boldsymbol{a}]V \vdash t) \equiv \mathcal{T}([\boldsymbol{a}]V \vdash t') \mid ([\boldsymbol{a}]V \vdash t \equiv t') \in \mathcal{A}\,\}\,,$$

that provides its model theory. This we now spell out in elementary terms.

Every open term $[\boldsymbol{a}]V \vdash t$ has an interpretation as a functorial term

$$[\![[\boldsymbol{a}]V \vdash t]\!] : F_\Sigma\text{-}\mathbf{Alg} \to F_{[\boldsymbol{a}]V}\text{-}\mathbf{Alg}\,,$$

where $F_{[\boldsymbol{a}]V}(X) = \mathbb{A}^{\#|\boldsymbol{a}|} \# \prod_{y \in |V|}[\mathbb{A}^{\#V(y)}, X]$, given as follows.

For $(X, \xi) \in F_\Sigma\text{-}\mathbf{Alg}$ and $\big(\boldsymbol{b}, (\langle \boldsymbol{c}_y \rangle s_y)_{y \in |V|}\big) \in F_{[\boldsymbol{a}]V}X$,

$$[\![[\boldsymbol{a}]V \vdash x(\boldsymbol{c})]\!]_{(X,\xi)}\big(\boldsymbol{b}, (\langle \boldsymbol{c}_y \rangle s_y)_y\big) = (\boldsymbol{c}_x\ (\boldsymbol{a}\,\boldsymbol{b}) \cdot \boldsymbol{c}) \cdot s_x$$

$$[\![[\boldsymbol{a}]V \vdash \mathsf{o}\ t_1 \ldots t_k]\!]_{(X,\xi)}\big(\boldsymbol{b}, (\langle \boldsymbol{c}_y \rangle s_y)_y\big) = \xi_k(\mathsf{o}', t_1', \ldots, t_k')$$

where $\xi_k : \Sigma(k) \times X^k \to X$ is the k-component of the structure map ξ and where

$$\mathsf{o}' = (\boldsymbol{a}\,\boldsymbol{b}) \cdot \mathsf{o}\,, \quad t_i' = [\![[\boldsymbol{a}]V \vdash t_i]\!]_{(X,\xi)}\big(\boldsymbol{b}, (\langle \boldsymbol{c}_y \rangle s_y)_y\big)\,.$$

$$\text{Ref } \frac{[\boldsymbol{a}]V \vdash t}{[\boldsymbol{a}]V \vdash t \equiv t} \qquad \text{Sym } \frac{[\boldsymbol{a}]V \vdash t \equiv t'}{[\boldsymbol{a}]V \vdash t' \equiv t} \qquad \text{Trans } \frac{[\boldsymbol{a}]V \vdash t \equiv t' \quad [\boldsymbol{a}]V \vdash t' \equiv t''}{[\boldsymbol{a}]V \vdash t \equiv t''}$$

$$\text{Axiom } \frac{([\boldsymbol{a}]V \vdash t \equiv t') \in \mathcal{A}}{[\boldsymbol{a}]V \vdash t \equiv t'} \qquad \text{Elim } \frac{[\boldsymbol{a}, \boldsymbol{b}]V \vdash t \equiv t'}{[\boldsymbol{a}]V \vdash t \equiv t'} \ (\boldsymbol{b} \ \# \ t, t')$$

$$\text{Intro } \frac{[\boldsymbol{a}]V \vdash t \equiv t'}{[\boldsymbol{a}, \boldsymbol{b}]V^{\langle \boldsymbol{b} \rangle} \vdash t[x(\boldsymbol{c}_x) \mapsto x(\boldsymbol{c}_x, \boldsymbol{b})]_{x \in |V|} \equiv t'[x(\boldsymbol{c}_x) \mapsto x(\boldsymbol{c}_x, \boldsymbol{b})]_{x \in |V|}}$$

$$\text{where } |V^{\langle \boldsymbol{b} \rangle}| = |V| \text{ with } \forall_{x \in |V|} \ V^{\langle \boldsymbol{b} \rangle}(x) = V(x) + |\boldsymbol{b}|$$

$$\text{Subst } \frac{[\boldsymbol{a}]U \vdash t \equiv t' \qquad \{\, [\boldsymbol{b}_x{}^{U(x)}]V \vdash s_x \equiv s'_x \,\}_{x \in |U|}}{[\boldsymbol{a}]V \vdash t[x(\boldsymbol{b}_x) \mapsto s_x]_{x \in |U|} \equiv t'[x(\boldsymbol{b}_x) \mapsto s_x]_{x \in |U|}}$$

Fig. 1. Rules of SNEL.

Then, for an F_Σ-algebra (X, ξ) and a judgement $[\boldsymbol{a}]V \vdash t \equiv t'$ the *satisfaction* relation $(X, \xi) \models ([\boldsymbol{a}]V \vdash t \equiv t')$ holds whenever

$$[\![[\boldsymbol{a}]V \vdash t]\!]_{(X,\xi)} = [\![[\boldsymbol{a}]V \vdash t']\!]_{(X,\xi)} : F_{[\boldsymbol{a}]V}X \to X \,.$$

An $\mathbb{S}$-*algebra* is an F_Σ-algebra satisfying the equations in $\mathcal{A}$. The category $\mathbb{S}$-**Alg** is the full subcategory of F_Σ-**Alg** consisting of $\mathbb{S}$-algebras.

Example 5.2 For the nominal theory of Example 5.1, an $\mathbb{S}_\lambda$-algebra has a carrier $X \in \boldsymbol{\mathcal{N}om}$ with structure maps $[\![V]\!] : \mathbb{A} \to X$, $[\![L]\!] : \mathbb{A} \times X \to X$, $[\![A]\!] : X \times X \to X$ satisfying the equations of the theory. For instance, according to the equation (α) we have that

$$[\![L]\!](a, (c\,a) \cdot x) = [\![L]\!](b, (c\,b) \cdot x) \quad \text{for all } (a, b, \langle c \rangle x) \in \mathbb{A} \# \mathbb{A} \# [\mathbb{A}, X]$$

and according to the equation (η) we have that

$$[\![L]\!](a, [\![A]\!](x, [\![V]\!]a)) = x \quad \text{for all } (a, x) \in \mathbb{A} \# X \,.$$

The initial $\mathbb{S}_\lambda$-algebra given by the construction (4) on the initial term algebra $T_{\Sigma_\lambda}0$ has as carrier the nominal set of $\alpha\beta\eta$-equivalence classes of λ-terms with the appropriate action.

Synthetic Nominal Equational Logic.   We now introduce *SNEL* (Synthetic Nominal Equational Logic) for equational reasoning with nominal theories $\mathbb{S} = (\Sigma, \mathcal{A})$ according to the above model theory. The logic deals with judgements $[\boldsymbol{a}]V \vdash t \equiv t'$ for open terms $[\boldsymbol{a}]V \vdash t$ and $[\boldsymbol{a}]V \vdash t'$, and is described in Figure 1. The operation of *substitution* used therein maps open terms

$$[\boldsymbol{a}]U \vdash t \ , \ \big\{ \, [\boldsymbol{b}_y{}^{U(y)}, \boldsymbol{c}]V \vdash s_y \, \big\}_{y \in |U|} \qquad \text{with } \boldsymbol{a} \mathbin{\#} \boldsymbol{c}$$

to the open term

$$[\boldsymbol{a}, \boldsymbol{c}]V \vdash t[y(\boldsymbol{b}_y) \mapsto s_y]_{y \in |U|}$$

defined as follows:

$$\big(x(\boldsymbol{b})\big)[y(\boldsymbol{b}_y) \mapsto s_y]_{y \in |U|} \ = \ (\boldsymbol{b}_x \ \boldsymbol{b}) \cdot s_x$$

$$(\mathsf{o} \ t_1 \dots t_k)[y(\boldsymbol{b}_y) \mapsto s_y]_{y \in |U|} \ = \ \mathsf{o} \ (t_1[y(\boldsymbol{b}_y) \mapsto s_y]_{y \in |U|}) \dots (t_k[y(\boldsymbol{b}_y) \mapsto s_y]_{y \in |U|}) \, .$$

Note that the definition of substitution models that of parameterised Kleisli composition, in that

$$\mathcal{T}\big([\boldsymbol{a}, \boldsymbol{c}]V \vdash t[y(\boldsymbol{b}_y) \mapsto s_y]_{y \in |U|}\big) \ = \ \mathcal{T}([\boldsymbol{a}]U \vdash t)^{\langle \boldsymbol{c} \rangle}\big[[\mathcal{T}([\boldsymbol{b}_y{}^{U(y)}, \boldsymbol{c}]V \vdash s_y)]_{y \in |U|}\big]$$

where $\mathcal{T}([\boldsymbol{a}]U \vdash t)^{\langle \boldsymbol{c} \rangle}$ is the composite

$$\mathbb{A}^{|\boldsymbol{a}|} \mathbin{\#} \mathbb{A}^{|\boldsymbol{c}|} \quad\xrightarrow{\mathcal{T}([\boldsymbol{a}]U \vdash t) \,\#\, \mathbb{A}^{|\boldsymbol{c}|}}\quad T_\Sigma(\widetilde{U}) \mathbin{\#} \mathbb{A}^{|\boldsymbol{c}|} \quad\xrightarrow{\tau_{\widetilde{U}, \mathbb{A}^{|\boldsymbol{c}|}}}\quad T_\Sigma(\widetilde{U} \mathbin{\#} \mathbb{A}^{|\boldsymbol{c}|}) \cong T_\Sigma\big(\widetilde{U^{\langle \boldsymbol{c} \rangle}}\big) \, .$$

The logic SNEL is sound because it has actually been induced from the TEL for the NES $\widetilde{\mathbb{S}}$. Indeed, if the judgement $[\boldsymbol{a}]V \vdash t \equiv t'$ is derivable in SNEL then the judgement $\widetilde{\mathcal{A}} \vdash \mathcal{T}([\boldsymbol{a}]V \vdash t) \equiv \mathcal{T}([\boldsymbol{a}]V \vdash t')$ is derivable in TEL. More precisely, the rule Elim arises from the TEL rule LocChar with respect to the epimorphic projection map $\mathbb{A}^{\#(|\boldsymbol{a}| + |\boldsymbol{b}|)} \longrightarrow\!\!\!\!\!\rightarrow \mathbb{A}^{\# |\boldsymbol{a}|}$; the rule Intro arises from the TEL rule Ext extended with $\mathbb{A}^{\# |\boldsymbol{b}|}$; and the rule Subst arises from the TEL rule Subst together with the rule LocChar with respect to the jointly epi family of maps $\{ \, \iota_x : \mathbb{A}^{\# \, U(x)} \to \widetilde{U} \, \}_{x \in |U|}$.

Remark 5.3 Since the category of sets embeds in that of nominal sets, every classical equational theory is a nominal theory and for them SNEL restricted to contexts with empty atom context and variables of valence zero reduces to classical equational logic.

Completeness. Consider a nominal theory $\mathbb{S} = (\Sigma, \mathcal{A})$. Since, for the associated NES $\widetilde{\mathbb{S}} = (\boldsymbol{\mathcal{N}om}, \boldsymbol{\mathcal{N}om}, \mathbb{T}_\Sigma, \widetilde{\mathcal{A}})$, the monad $\mathbb{T}_\Sigma$ is free on the endofunctor F_Σ and the induced ES $\widehat{\widetilde{\mathbb{S}}}$ (see Remark 2.12) is inductive, the free $\mathbb{S}$-algebra monad $\mathbb{T}_\mathbb{S}$ exists and the quotient monad morphism $q_X : T_\Sigma X \ \cdot\!\!\!\twoheadrightarrow T_\mathbb{S} X$ is constructed as in (4).

By the internal completeness result, we have that the satisfaction relation $\mathbb{S}\text{-}\mathbf{Alg} \models ([\boldsymbol{a}]V \vdash t \equiv t')$ is equivalent to $q_{\widetilde{V}} \circ \mathcal{T}([\boldsymbol{a}]V \vdash t) = q_{\widetilde{V}} \circ \mathcal{T}([\boldsymbol{a}]V \vdash t') :$ $\mathbb{A}^{\# |\boldsymbol{a}|} \ \cdot\!\!\!\!> T_\mathbb{S}(\widetilde{V})$. Moreover, by the equivariance of $\mathcal{T}([\boldsymbol{a}]V \vdash t)$ and $\mathcal{T}([\boldsymbol{a}]V \vdash t')$, this is further equivalent to $q_{\widetilde{V}}(t) = q_{\widetilde{V}}(t')$ in $T_\mathbb{S}(\widetilde{V})$.

The construction of the nominal set $T_\mathbb{S}\widetilde{V}$ and the map $q_{\widetilde{V}}$ as in (4) is as follows:

$$\mathsf{Ref}^n \; \frac{}{t \approx^n t} \qquad\qquad \mathsf{Sym}^n \; \frac{t \approx^n t'}{t' \approx^n t} \qquad\qquad \mathsf{Trans}^n \; \frac{t \approx^n t' \quad t' \approx^n t''}{t' \approx^n t}$$

$$\mathsf{Axiom}^1 \; \frac{\big([\boldsymbol{a}^n]U \vdash t \equiv t'\big) \in \mathcal{A}}{\big((\boldsymbol{a}^n \; \boldsymbol{b}^n) \cdot t\big)\{\langle \boldsymbol{c}_x \rangle \, s_x\}_x \approx^1 \big((\boldsymbol{a}^n \; \boldsymbol{b}^n) \cdot t'\big)\{\langle \boldsymbol{c}_x \rangle \, s_x\}_x}$$

$$\text{where } \boldsymbol{b}^n \# \langle \boldsymbol{c}_x \rangle \, s_x \in [\mathbb{A}^{\# \, U(x)}, T_\Sigma \widetilde{V}] \text{ for all } x \in |U|$$

$$\mathsf{Cong}^n \; \frac{t_i \approx^{n-1} t_i' \quad (1 \le i \le k)}{\mathsf{o}\, t_1 \dots t_k \approx^n \mathsf{o}\, t_1' \dots t_k'} \; \big(n \ge 2, \, \mathsf{o} \in \Sigma(k)\big)$$

$$\mathsf{Inc}^n \; \frac{t \approx^{n-1} t'}{t \approx^n t'} \; \big(n \ge 2\big)$$

Fig. 2. Rules for $\approx^n$.

Since the forgetful functor $|-| : \boldsymbol{\mathcal{N}om} \to \boldsymbol{\mathcal{S}et}$ creates colimits, we have the following explicit description of the above construction.

The nominal set Y_n $(n \ge 1)$ has underlying set $|Y_n| = |T_\Sigma \widetilde{V}|/_{\approx^n}$ with action $\pi \cdot [t]_{\approx^n} = [\pi \cdot t]_{\approx^n}$ for $\approx^n$ the equivalence relation given by the rules in Figure 2. The *instantiation operation* used in Axiom^1 is given by:

$$\big(y(\boldsymbol{c})\big)\{\langle \boldsymbol{c}_x \rangle \, s_x\}_x = (\boldsymbol{c}_y \; \boldsymbol{c}) \cdot s_y \;,$$

$$\big(\mathsf{o}\, t_1 \dots t_k\big)\{\langle \boldsymbol{c}_x \rangle \, s_x\}_x = \mathsf{o} \, \big(t_1\{\langle \boldsymbol{c}_x \rangle \, s_x\}_x\big) \dots \big(t_k\{\langle \boldsymbol{c}_x \rangle \, s_x\}_x\big) \;.$$

The map q_0 sends t to $[t]_{\approx^1}$, and the map q_n $(n \ge 1)$ sends $[t]_{\approx^n}$ to $[t]_{\approx^{n+1}}$.

The nominal set $T_\mathbb{S}\widetilde{V}$, being the colimit of the ω-chain $\langle Y_n \rangle_{n \ge 0}$, is given by $|T_\mathbb{S}\widetilde{V}| = |T_\Sigma \widetilde{V}|/_{\approx^\omega}$ with action $\pi \cdot [t]_{\approx^\omega} = [\pi \cdot t]_{\approx^\omega}$ where $\approx^\omega$ is the equivalence relation generated by the relation $\approx$ of Figure 3.

Since the map $q_{\widetilde{V}} : T_\Sigma \widetilde{V} \to T_\mathbb{S}\widetilde{V}$ sends t to $[t]_{\approx^\omega}$, we have that the satisfaction relation $\mathbb{S}\text{-}\mathbf{Alg} \models ([\boldsymbol{a}]V \vdash t \equiv t')$ holds iff $t \approx^\omega t'$. Furthermore, since for $\big([\boldsymbol{a}^n]U \vdash t \equiv t'\big) \in \mathcal{A}$ and $\boldsymbol{b}^n \# \langle \boldsymbol{c}_x \rangle \, s_x \in [\mathbb{A}^{\# \, U(x)}, T_\Sigma \widetilde{V}]$ for all $x \in |U|$, the judgement

$$[\boldsymbol{d}]V \vdash \big((\boldsymbol{a}^n \; \boldsymbol{b}^n) \cdot t\big)\{\langle \boldsymbol{c}_x \rangle \, s_x\}_x \equiv \big((\boldsymbol{a}^n \; \boldsymbol{b}^n) \cdot t'\big)\{\langle \boldsymbol{c}_x \rangle \, s_x\}_x$$

with $\{\, \boldsymbol{d} \,\} \supseteq \boldsymbol{b}^n \cup \bigcup_x \mathsf{supp}(s_x) \backslash \{\boldsymbol{c}_x\}$ is derivable in SNEL, it follows that, for all $[\boldsymbol{a}]V \vdash t$ and $[\boldsymbol{a}]V \vdash t'$, if $t \approx^\omega t'$ in $T_\Sigma \widetilde{V}$ then $[\boldsymbol{a}]V \vdash t \equiv t'$ is derivable in SNEL. Hence, SNEL is complete.

$$\text{Axiom} \; \dfrac{\big([\boldsymbol{a}^n]U \vdash t \equiv t'\big) \in \mathcal{A}}{\big((\boldsymbol{a}^n\,\boldsymbol{b}^n)\cdot t\big)\{\langle \boldsymbol{c}_x\rangle\,s_x\}_x \approx \big((\boldsymbol{a}^n\,\boldsymbol{b}^n)\cdot t'\big)\{\langle \boldsymbol{c}_x\rangle\,s_x\}_x}$$

$$\text{where } \boldsymbol{b}^n \,\#\, \langle \boldsymbol{c}_x\rangle\,s_x \in [\mathbb{A}^{\#\,U(x)}, T_\Sigma \widetilde{V}] \text{ for all } x \in |U|$$

$$\text{Cong} \; \dfrac{t_i \approx t_i' \quad (1 \le i \le k)}{\mathsf{o}\,t_1 \dots t_k \approx \mathsf{o}\,t_1' \dots t_k'} \; \big(\mathsf{o} \in \Sigma(k)\big)$$

Fig. 3. Rules for $\approx$.

We conclude with two corollaries of completeness.

(i) Since $[\boldsymbol{a}]V \vdash t \equiv t'$ is derivable in SNEL iff $q_{\widetilde{V}}(t) = q_{\widetilde{V}}(t')$ and the map $q_{\widetilde{V}}$ is equivariant, the rule

$$\dfrac{[\boldsymbol{a}^n]V \vdash t \equiv t'}{[\boldsymbol{b}^n]V \vdash (\boldsymbol{a}^n\,\boldsymbol{b}^n)\cdot t \equiv (\boldsymbol{a}^n\,\boldsymbol{b}^n)\cdot t'}$$

is admissible in SNEL.

(ii) Since $[\boldsymbol{a}]V \vdash t \equiv t'$ is derivable in SNEL iff there exist $t = t_0, \dots, t_i, \dots, t_n = t'$ in $T_\Sigma \widetilde{V}$ such that $t_i \approx t_{i+1}$ or $t_{i+1} \approx t_i$ for all $0 \le i < n$, one may use equational term rewriting to establish nominal equational judgements.

Acknowledgement

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