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A Cartesian Closed Category of Domains with Almost Algebraic Bases

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Abstract

In this paper, we investigate the properties of almost algebraic domains introduced by G. Hamrin and V. Stoltenberg-Hansen in 2006. We introduce a notion of $\mathcal{M}$-closed basis and define a new class of domains, called ωAML-domains, which are continuous L-domains endowed with countable, almost algebraic, and $\mathcal{M}$-closed bases. The main result of this paper is: the class of ωAML-domains is closed under function spaces and finite cartesian products. Hence, the category of ωAML-domains together with Scott continuous functions is cartesian closed. The results of this paper give an answer to an open problem posed by G. Hamrin and V. Stoltenberg-Hansen in "Two categories of effective continuous cpos, Theoretical Computer Science, 365(2006), 216-236".

Keywords: almost algebraic, L-domain, $\mathcal{M}$-closed basis, cartesian closed category

1 Introduction

Domain theory was introduced by Dana Scott [14, 15] in the late sixties as a mathematical tool to model the denotational semantics of programming languages. For the sake of defining the semantics of higher-order functions in programming, one must consider its counterpart in the formation of function spaces of domains that correspond to the language features of interest. So it is a natural requirement that the considered category of domains should be cartesian closed, and hence seeking various cartesian closed categories of domains is one of the most important problems

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in domain theory. Many well-known kinds of domains had been found forming cartesian closed categories together with Scott continuous functions as morphisms, such as continuous (algebraic) lattices, bounded complete domains, Scott domains [15] and SFP domains [16], and so on. In 1990, the maximal cartesian closed full subcategories of the category of continuous (algebraic) domains were fully classified by Achim Jung in [8, 9].

In 2006, G. Hamrin and V. Stoltenberg-Hansen [6] investigated a class of special continuous domains, within which each domain has an *almost algebraic* basis. An inspiration for the notion of an almost algebraic basis is due to [17]. In that paper a basis with the inverse approximation property is considered, resulting in a cartesian closed subcategory of the bounded complete domains. The notion of an almost algebraic basis is also related to Tang [18], who considered conditions on a basis in order to obtain a cartesian closed category of continuous lattices. These extra conditions on a basis are under suitable assumptions equivalent to a basis being almost algebraic.

In [6], Hamrin and Stoltenberg showed that the class of continuous domains with countable, closed and almost algebraic bases is closed under function spaces and finite cartesian products. Hence, the category consisting of such objects is cartesian closed. In particular, they showed that the notion of almost algebraic is beneficial to study effective domains. Here, a basis B of a domain is called closed provided that it is closed under the least upper bounds of all bounded finite subsets of B, thus a domain with a closed basis is exactly bounded complete. As mentioned above, Jung [8,9] showed that the maximal cartesian closed subcategories of the category of continuous domains consist of continuous L-domains or FS-domains. It is possible that there are some new cartesian closed subcategories of continuous domains with the property almost algebraic. To obtain a new cartesian closed category of domains, the preservation of the required properties by function spaces is the critical factor and also the essential difficulty. For example, see [4, 7, 9, 13, 16, 19]. So Hamrin and Stoltenberg [6] posed the following open problem:

Find a further condition on the basis weaker than being closed but which is preserved and preserves almost algebraicity under the function space construction.

In this paper, we will answer this question. We introduce a new notion for a base of a continuous L-domain, called $\mathcal{M}$*-closed*, which is strictly weaker than being closed. We show that the class of all continuous L-domains endowed with countable, almost algebraic and $\mathcal{M}$-closed bases is closed under function spaces and finite cartesian products. Hence, the category consisting of such L-domains is cartesian closed.

The paper is organized as follows. Section 2 introduces some notions and definitions we need. Section 3 discusses the properties of almost algebraic domains. Section 4 investigates the properties of continuous L-domains endowed with countable, almost algebraic and $\mathcal{M}$-closed bases and the function spaces among them. Section 5 gives a cartesian closed category of continuous L-domains with almost algebraic bases.

2 Preliminaries

We first review some basic knowledge of domain theory. The reader can find more details and proofs in [1] and the textbooks [2, 5]. Let A and B be sets. As usual, we let $A \subseteq B$ denote that A is a subset of B. We also let $A \subseteq_f B$ denote that A is a finite subset of B. A partially ordered set (poset) $(D; \leq)$ is a set D with a reflexive, transitive and antisymmetric relation $\leq$. For $A \subseteq D$, we set

$$\downarrow A = \{x \in D : \exists a \in A,\ x \leq a\},\ \uparrow A = \{x \in D : \exists a \in D,\ a \leq x\},$$

and A is called a lower or upper set, if $A = \downarrow A$ or $A = \uparrow A$ respectively. For an element $a \in D$, we use $\downarrow a$ or $\uparrow a$ instead of $\downarrow\{a\}$ or $\uparrow\{a\}$. A subset A of D is called directed if it is nonempty and every nonempty finite subset of A has an upper bound in A. Particularly, we say that D is a *dcpo* if every directed subset A of D has a least upper bound (denoted by $\bigvee A$) in D. In this case, when D has a least element (denoted by $\perp$), we call it a *cpo*.

Let D be a dcpo. For $x, y \in D$, we say that x is way-below y, denoted by $x \ll y$, if for any directed subset A of D, $y \leq \bigvee A$ implies $x \leq a$ for some $a \in A$. For $x \in D$, we set

$$\twoheaddownarrow x = \{y \in D : y \ll x\},\ \twoheaduparrow x = \{y \in D : x \ll y\}.$$

Definition 2.1 Let D be a dcpo. A subset $B \subseteq D$ is a basis (or a base) for D if for each $x \in D$, $B \cap \twoheaddownarrow x$ is directed and $x = \bigvee(B \cap \twoheaddownarrow x)$. Particularly, when B is countable we say that B is an ω-base of D.

A basis B is *reduced* if for all $b \in B$ we have $\twoheaduparrow b \neq \emptyset$. If B is a basis for a dcpo D then $\{b \in B : \twoheaduparrow b \neq \emptyset\}$ is also a basis for D. A basis B is *closed* if for all $b, c \in B$ such that $\uparrow b \cap \uparrow c \neq \emptyset$ we have that $b \vee c$ exists and $b \vee c \in B$.

A dcpo D is continuous if it has a basis, it is also said to be a continuous domain when it is a cpo. An element $d \in D$ is called compact if $d \ll d$. Let $K(D)$ be the set of all compact elements of D. A dcpo D is called algebraic if $K(D)$ is a basis.

Lemma 2.2 *Let D be a continuous dcpo with a base B.*

(1) For a finite $M \subseteq D$ and $x \in D$,

$$M \ll x \Rightarrow \exists b \in B,\ M \ll b \ll x,$$

where $M \ll x$ means $m \ll x$ for all $m \in M$.

(2) If B is a ω-base, then for each $x \in D$ there exists a sequence $(x_n)_{n\in\omega} \subseteq B$ such that

$$x_1 \ll x_2 \ll \cdots \ll x_n \ll x_{n+1} \ll \cdots \ll x$$

and $x = \bigvee_{n\in\omega} x_n$.

Next, we describe the continuous functions between dcpos or cpos.

Definition 2.3 Let D and E be two dcpos.

(1) A function $f : D \to E$ is called (Scott) *continuous* if it is monotone and for all directed sets $A \subseteq D$ we have $f(\bigvee A) = \bigvee f(A)$.

(2) The *function space*, denoted by $[D \to E]$, is the set of all Scott continuous functions from D into E ordered by the pointwise order, i.e., $f \leq g$ iff $f(x) \leq g(x)$ in E for all $x \in D$.

Let **DCPO** be the category of all dcpos with Scott continuous functions. Then **DCPO** is cartesian closed. Moreover, a full subcategory of **DCPO** is cartesian closed iff it is closed under continuous function spaces and finite products [1, 16].

The simplest continuous functions are the step functions $(a \searrow b)$.

Definition 2.4 The step function $(a \searrow b) : D \to E$, for $a \in D$ and $b \in E$, is defined by

$$(a \searrow b)(x) = \begin{cases} b, & \text{if } a \ll x, \\ \bot, & \text{otherwise.} \end{cases}$$

It is straightforward to see that $(a \searrow b)$ is continuous. We have the following relationship between step functions and continuous functions.

Lemma 2.5 [5, Exercise II-2.31] *Let D and E be cpos and let $a \in D$ and $b \in E$.*

(1) Suppose $f \in [D \to E]$. Then

$$b \ll f(a) \Longrightarrow (a \searrow b) \ll f.$$

(2) If D and E are continuous cpos with bases B_D and B_E and $f \in [D \to E]$ then

$$f = \bigvee \{(a \searrow b) : a \in B_D, b \in B_E, b \ll f(a)\}.$$

Definition 2.6 Let D be a cpo.

(1) We say that D is an L-domain if every nonempty bounded subset of D has a greatest lower bound; equivalently, for any $x \in D$, $\downarrow x$ is a complete lattice in the induced order. For $a \in D$ and $A \subseteq \downarrow a$, we use $\bigvee_{\downarrow a} A$ to denote the least upper bound of A in $\downarrow a$.

(2) We say D is bounded complete if every nonempty bounded subset of D has a least upper bound.

Easily one sees that every bounded complete dcpo is an L-domain and the reverse does not hold. We also have the following useful lemma for L-domains and function spaces.

Lemma 2.7 *Let D, E be two L-domains.*

(1) For $a, b \in D$ and $A \subseteq \downarrow a$, $a \leq b$ implies $\bigvee_{\downarrow a} A = \bigvee_{\downarrow b} A$.

(2) $[D \to E]$ is an L-domain such that for $f, g, h \in [D \to E]$, if $f, g \leq h$ then $(f \vee_{\downarrow h} g)(x) = f(x) \vee_{\downarrow h(x)} g(x)$ for all $x \in D$. Moreover, if $f, g \ll h$ then $f \vee_{\downarrow h} g \ll h$.

(3) $[D \to E]$ *is continuous (resp. algebraic) whenever* D *and* E *are continuous (resp. algebraic).*

Since the class of continuous (resp. algebraic) L-domains is also closed under finite cartesian products, the above lemma, whose complete proof appears in [8, 9], shows that the category of continuous (resp. algebraic) L-domains with continuous functions is cartesian closed. Moreover, it is one of the maximal cartesian closed subcategory of continuous domains.

Definition 2.8 Let D be a continuous cpo. We say that a basis B of D has property $\mathcal{M}$ if for all $a, b, c, d \in B$ with $a \ll c, b \ll d$, there exists a finite subset $F \subseteq D$ such that

$$\uparrow c \cap \uparrow d \subseteq \uparrow F \subseteq \uparrow a \cap \uparrow b.$$

We say that a continuous cpo has property $\mathcal{M}$ if it has a basis with property $\mathcal{M}$.

A continuous cpo D has property $\mathcal{M}$ iff its Lawson topology is compact iff $\uparrow a \cap \uparrow b$ is Scott compact for all $a, b \in D$ [5, Theorem III-5.8]. The following lemma is easy to be proved.

Lemma 2.9 *Let* D *be a continuous cpo with property* $\mathcal{M}$*. Then for all* $a, b \in D$*,* $\uparrow a \cap \uparrow b = \emptyset$ *implies that there exist* $x \in \twoheaddownarrow a$ *and* $y \in \twoheaddownarrow b$ *such that* $\uparrow x \cap \uparrow y = \emptyset$*.*

3 Almost algebraic dcpos

In this section, we introduce the notion of an almost algebraic dcpo and the related open problem posed by Hamrin and Stoltenberg-Hansen in [6].

Definition 3.1 Let D be a continuous cpo.

(1) Given $a \in D$, a sequence $(a_n)_{n\in\omega} \subseteq D$ is said to be an almost algebraic sequence of a if

$$a \ll \cdots \ll a_{n+1} \ll a_n \ll \cdots \ll a_1 \ll a_0$$

and for each $b \in D$, $a \ll b$ implies $a_n \ll b$ for some $n \in \omega$.

(2) A basis B of D is called almost algebraic if the following holds:
  (a) Each $a \in B$ has an almost algebraic sequence $(a_n)_{n\in\omega} \subseteq B$.
  (b) For all $a, b \in B$, if $\twoheaduparrow a \subseteq \twoheaduparrow b$ then $b \leq a$.

We say that D is almost algebraic if it has an almost algebraic basis B_D.

Obviously, all algebraic cpos are almost algebraic by taking constant sequences of compact elements. The following result is obvious.

Lemma 3.2 *Let* D *be a continuous cpo with an almost algebraic basis* B*. Let* $A = \{a_i : i \in K\} \subseteq_f B$*. Then there is* $\hat{A} = \{\hat{a}_i : i \in K\} \subseteq_f B$ *such that*

(1) $a_i \ll \hat{a}_i$ *for all* $i \in K$*.*

(2) For any $i, j \in K$*,* $a_i \leq a_j \Longrightarrow \hat{a}_i \leq \hat{a}_j$*.*

Proposition 3.3 *Let* B_D *be an almost algebraic basis of a dcpo* D *and* $a \in B_D$*. If* $(a_n)_{n\in\omega} \subseteq B$ *is an almost algebraic sequence of* a*, then* $a = \bigwedge_{n\in\omega} a_n$*.*

Proof. Let b be an arbitrary lower bound of $(a_n)_{n\in\omega}$. We need to show $b \le a$. For any $x \in \Uparrow a$, there exists $n \in \omega$ such that $a_n \ll x$. Since $b \le a_n$, we get $b \ll x$, i.e., $\Uparrow a \subseteq \Uparrow b$. From the assumption that B_D is almost algebraic we obtain $b \le a$. □

Intuitively, the motivation of the notion "almost algebraic" is that an element in a basis can be approximated by an increasing way-below sequence and a decreasing way-below sequence like the real numbers. In the following, we will see that a well-known domain, namely the formal closed ball domain of a complete metric space, is almost algebraic but not algebraic.

Let (X, d) be a complete metric space. Set $\mathbb{R}^+ = [0, +\infty)$ to be the set of all non-negative real numbers and set $\mathbb{Q}_+$ to be the set of all positive rational numbers. Let

$$\mathbf{B}X = X \times \mathbb{R}^+$$

ordered as follows: $\forall (x, r), (y, s) \in \mathbf{B}X$,

$$(x, r) \le (y, s) \Leftrightarrow d(x, y) \le r - s.$$

It was showed in [3, 10, 11] that $(\mathbf{B}X, \le)$ is a continuous dcpo such that for $(x, r), (y, s) \in \mathbf{B}X$, $(x, r) \ll (y, s)$ iff $d(x, y) < r - s$. Moreover, set $B_X = X \times \mathbb{Q}_+$, then B_X is a basis of $\mathbf{B}X$.

Proposition 3.4 *For a complete metric space (X, d), B_X is an almost algebraic basis of $\mathbf{B}X$. Moreover, if (X, d) has a countable dense subset Y, i.e., (X, d) is a Polish space, then the set $B_X^Y = Y \times \mathbb{Q}_+$ is an almost algebraic ω-basis of $\mathbf{B}X$.*

Proof. Let (X, d) be a complete metric space. Then B_X is a basis of $(\mathbf{B}X, \le)$. For each $(x, r) \in B_X$, we set $r_n = r - \frac{r}{2^n}$ and $a_n = (x, r_n)$ for all $n < \omega$. Then

$$(x, r) \ll \cdots \ll a_{n+1} \ll a_n \ll \cdots \ll a_1.$$

Suppose $(x, r) \ll (y, s)$ for some $(y, s) \in \mathbf{B}X$. Then $d(x, y) < r - s$. There exists an enough large $n_0 < \omega$ such that $d(x, y) + s < r - \frac{r}{2^{n_0}} < r$. Thus, $a_{n_0} = (x, r_{n_0}) \ll (y, s)$. It means that $(a_n)_\omega$ is an almost algebraic sequence of (x, r). Suppose $(z, k) \in B_X$ with $\Uparrow(z, k) \subseteq \Uparrow(x, r)$. Assume $(x, r) \not\le (z, k)$. Then $d(x, z) + k > r$. Since $k > 0$, there exists an enough small $\varepsilon \in \mathbb{Q}_+$ such that $k - \varepsilon > 0$ and $d(x, z) + k - \varepsilon \ge r$. Set $k' = k - \varepsilon$. Then $(z, k) \ll (z, k')$ and $(x, r) \not\ll (z, k')$. It is a contradiction. Hence we have $(x, r) \le (z, k)$. Therefore, B_X is an almost algebraic basis of $\mathbf{B}X$ by Definition 3.1. Similarly, one can show that $B_X^Y = Y \times \mathbb{Q}_+$ is an almost algebraic ω-basis of $\mathbf{B}X$ when X is a Polish space. □

Also, there are continuous cpos which are not almost algebraic. The following is such an example.

Example 3.5 Let $D = [0, 1] \times [0, 1]$ ordered as follows: $\forall (x_1, y_1), (x_2, y_2) \in D$, $(x_1, y_1) \le (x_2, y_2)$ iff one of the followings conditions holds:

- $x_1 \le x_2$ and $y_1 = y_2$,

- $x_1 = 0$ and $y_1 \le y_2$.

We see that $(D, \sqsubseteq)$ is a bounded complete cpo and for all $(x_1, y_1), (x_2, y_2) \in D$,

$$(x_1, y_1) \ll (x_2, y_2) \Leftrightarrow (x_1 < x_2 \ \& \ y_1 = y_2) \text{ or } (x_1 = 0 \ \& \ y_1 < y_2).$$

Hence, D is continuous. Let $y \in (0,1)$ and let $((x_n, y_n))_{n\in\omega} \subseteq D$ be a sequence such that $(0, y) = \bigwedge_{n=1}^{\infty} (x_n, y_n)$ and

$$(0, y) \ll \cdots \ll (x_{n+1}, y_{n+1}) \ll (x_n, y_n) \ll \cdots \ll (x_2, y_2) \ll (x, y_1).$$

Then one of the follow two statements holds:

(1) $\forall n \in \omega$, $y_n = y$ and $x_{n+1} < x_n$.

(2) $\exists n_0 \in \omega$, $x_n = 0$ and $y < y_{n+1} < y_n$ whenever $n > n_0$.

For (1), take $y' > y$, then $(0, y) \ll (0, y')$ but $(x_n, y_n) \not\le (0, y')$ for all $n \in \omega$. For (2), take $x > 0$, then $(0, y) \ll (x, y)$ but $(x_n, y_n) \not\le (x, y)$ for all $n \in \omega$. Hence, $(0, y)$ has no almost algebraic sequences in D for any $y \in (0, 1)$. Note that for any basis B of D and $(x, y'') \in B \cap \Downarrow(0, y)$, we have $x = 0$ and $y'' \le y$. Hence, D has no almost algebraic bases.

Therefore, Even the property *almost algebraic* seems strange, it is also very natural (see Proposition 3.4), and the class of all almost algebraic cpos strictly intervenes between algeraic cpos and continuous cpos. It also leads a natural question: which cpos with almost algebraic bases can form a cartesian closed category? Of course, such a category should contain at least one non-algebraic object. Hamrin and Stoltenberg-Hansen [6] proved the following result.

Theorem 3.6 *Let D and E be bounded complete domains with countable closed and almost algebraic bases B_D and B_E respectively. Then the function space $[D \to E]$ has a closed, countable and almost algebraic basis.*

This theorem says that the category of bounded complete domains with countable closed and almost algebraic bases is cartesian closed. The proof of the above theorem in [6] is seriously dependent on the closedness of bases. Recall that, a base is closed if it is closed under the suprema of its bounded finite subsets. Is it easy that a bounded complete domain has a countable closed basis? Let see an example.

Example 3.7 Let $I = [0, 1]$ and $\mathbb{I} = \{[a, b] : 0 \le a \le b \le 1\}$ endowed with the reverse-inclusion relation, that is, $[a, b] \le [c, d] \Leftrightarrow [a, b] \supseteq [c, d]$. Then $(\mathbb{I}, \le)$ is a bounded complete domain and is called the unit interval domain. Set

$$B_{\mathbb{I}} = \{[a, b] : a, b \in \mathbb{Q} \cap I \ \& \ a < b\}.$$

It is easy to see that $B_{\mathcal{I}}$ is a countable almost algebraic basis of $\mathbb{I}$ (the proof is similar to Proposition 3.4). However, $B_{\mathbb{I}}$ is not closed. For example, $[0, \frac{1}{2}], [\frac{1}{2}, 1] \in B_{\mathbb{I}}$ but $[0, \frac{1}{2}] \vee [\frac{1}{2}, 1] = \{\frac{1}{2}\} \notin B_I$.

For seeking a new cartesian closed subcategory of almost algebraic domains, Hamrin and Stoltenberg-Hansen in [6] posed the following open problem:

- Find a further condition on the basis weaker than being closed but which is preserved and preserves almost algebraicity under the function space construction.

In this paper, we will answer this question. In the next section, a notion called $\mathcal{M}$-closed basis is defined on a continuous L-domain which is very advantageous to help us to handle the above open problem.

4 L-domains with countable, almost algebraic and $\mathcal{M}$-closed bases

In this section, we introduce a $\mathcal{M}$-closed base on a continuoous L-domain. A series of properties of continuous L-domains with countable $\mathcal{M}$-closed and almost algebraic bases are obtained.

Proposition 4.1 [6, Proposition 22] *Let D,E be continuous cpos and suppose that D has an almost algebraic basis B_D and E has a countable basis B_E. Let $a, c \in B_D$ and $b, d \in B_E$, where $b \neq \perp$. Then the following hold:*

(1) $(a \searrow b) \leq (c \searrow d) \Longleftrightarrow c \leq a \ \&\ b \leq d$.

(2) $(a \searrow b) \ll (c \searrow d) \Longleftrightarrow c \ll a \ \&\ b \ll d$.

(3) If $(c_n)_{n\in\omega}$ is an almost algebraic sequence for c and $(d_n)_{n\in\omega}$ an approximating sequence for d, then $\bigvee_{n\in\omega}(c_n \searrow d_n) = (c \searrow d)$.

The following is a crucial lemma which is very useful to investigate the way below relation between continuous functions of almost algebraic L-domains. Note that it is a generalization of [6, Lemma 23].

Lemma 4.2 *Let D, E be two continuous L-domains endowed with an almost algebraic basis B_D and a countable basis B_E respectively. For all $a \in B_D, b \in B_E$ and $f \in [D \to E]$, we have*

$$(a \searrow b) \ll f \Longleftrightarrow b \ll f(a).$$

Proof. The implication from right to left follows from Lemma 2.5(1).

We now show the other direction. Set

$$Step(f) = \{(c \searrow d) : (c, d) \in B_D \times B_E \ \&\ d \ll f(c)\}.$$

By Lemma 2.5(2), $f = \bigvee Step(f) = \bigvee\{\bigvee_{\downarrow f} A : A \subseteq_f Step(f)\}$. Since $(a \searrow b) \ll f$, there is a finite subset $\{(c_i \searrow d_i) : i \in I_0\} \subseteq Step(f)$ such that

$$(a \searrow b) \ll \bigvee_{\downarrow f}\{(c_i \searrow d_i) : i \in I_0\} \qquad (*).$$

Without losing generality, we assume $b \neq \perp$. From the assumption that B_D is almost algebraic, we get almost algebraic sequences (a_n) for a and $(c_i^j)_j$ for each

c_i, where $i \in I_0$. Since B_E is a countable basis we choose sequences $(d_i^j)_j$ from B_E increasing with respect to $\ll$ such that $\bigvee_{j\in\omega} d_i^j = d_i$ for each $i \in I_0$. From this it follows that $(c_i^j \searrow d_i^j) \ll (c_i \searrow d_i)$ for all $i \in I_0$ and $j \in \omega$. Hence $(c_i \searrow d_i) = \bigvee_j (c_i^j \searrow d_i^j)$ for all $i \in I_0$ and

$$\bigvee_{\downarrow f}\{(c_i^j \searrow d_i^j) : i \in I_0\} \ll \bigvee_{\downarrow f}\{(c_i \searrow d_i) : i \in I_0\}$$

for all $j \in \omega$. Furthermore, we have

$$\bigvee_{\downarrow f}\{(c_i^j \searrow d_i^j) : i \in I_0\} \ll \bigvee_{\downarrow f}\{(c_i^{j'} \searrow d_i^{j'} : i \in I_0\}$$

for $j < j'$. Therefore

$$\begin{aligned}\bigvee_{\downarrow f}\{(c_i \searrow d_i) : i \in I_0\} &= \bigvee_{\downarrow f}\bigvee_{j\in\omega}\{(c_i^j \searrow d_i^j) : i \in I_0\}\\ &= \bigvee_{j\in\omega}\bigvee_{\downarrow f}\{(c_i^j \searrow d_i^j) : i \in I_0\} \quad (**).\end{aligned}$$

From $(*)$ and $(**)$, there exists $j_0 \in \omega$ such that $(a \searrow b) \ll \bigvee_{\downarrow f}\{(c_i^{j_0} \searrow d_i^{j_0}) : i \in I_0\}$. For each a_n in the descending sequence towards a we have

$$\begin{aligned} b &= (a \searrow b)(a_n)\\ &\le \bigvee_{\downarrow f(a_n)} \{(c_i^{j_0} \searrow d_i^{j_0})(a_n) : i \in I_0\}\\ &= \bigvee_{\downarrow f(a_n)} \{d_i^{j_0} : c_i^{j_0} \ll a_n\}.\end{aligned}$$

Let $I_n = \{i \in I_0 : c_i^{j_0} \ll a_n\}$. Note that $I_n \supseteq I_{n+1}$ and I_0 is finite, hence there exists n_0 such that if $n \ge n_0$ then $I_n = I_{n_0}$ holds. From the assumption $b \ne \perp$ it follows that $I_{n_0} \ne \emptyset$. For each $i \in I_{n_0}$, $c_i^{j_0} \ll a_n$ holds for all $n \ge n_0$. Hence $a = \bigwedge_{n\in\omega} a_n \ge c_i^{j_0} \gg c_i$ holds for all $i \in I_{n_0}$. Thus

$$\begin{aligned} f(a) &\ge \bigvee_{\downarrow f(a)} \{(c_i \searrow d_i) : i \in I_0\}(a)\\ &= \bigvee_{\downarrow f(a)} \{d_i : c_i \ll a, i \in I_0\}\\ &\ge \bigvee_{\downarrow f(a)} \{d_i : i \in I_{n_0}\}\\ &= \bigvee_{\downarrow f(a_{n_0})} \{d_i : i \in I_{n_0}\} \quad (\text{because of } f(a) \le f(a_{n_0}))\\ &\gg \bigvee_{\downarrow f(a)} \{d_i^{j_0} : i \in I_{n_0}\}\\ &\ge b.\end{aligned}$$

□

Next, we introduce an important property of a base, called $\mathcal{M}$-closed, which is a generalization of a closed base.

Let D be a cpo. Given a finite subset F of D, we set

$$\mathrm{mub}F = \min\{a \in D : F \leq a\}$$

to be the set of all minimal upper bounds of F. If D is an L-domain, then for each upper bound b of F, there exists $a \in \mathrm{mub}F$ such that $a \leq b$ (It is called mub-complete in [1]).

Definition 4.3 Let D be a continuous L-domain. A basis B_D of D is said to be $\mathcal{M}$-*closed*, if for any nonempty finite subset $\{(a_i, b_i) : i \in K\} \subseteq B_D \times B_D$ with $b_i \ll a_i$ for all $i \in K$, there exists $\{c_i : i \in K\} \subseteq_f B_D$ such that

(1) $b_i \ll c_i \ll a_i$ for each $i \in K$.

(2) For any $I \subseteq K$, if $\bigcap_{i\in I} \Uparrow a_i \neq \emptyset$ then $\mathrm{mub}\{c_i : i \in I\}$ is finite and contained in B_D, i.e., $\mathrm{mub}\{c_i : i \in I\} \subseteq_f B_D$.

By the definition of property $\mathcal{M}$, the following result is obvious.

Lemma 4.4 *Let B_D be a basis of a continuous L-domain D. If B_D is $\mathcal{M}$-closed, then it has property $\mathcal{M}$.*

Proposition 4.5 *Let D be a continuous L-domain. A basis B_D of D is $\mathcal{M}$-closed if and only if for any nonempty finite subset $\{(a_i, b_i) : i \in K\} \subseteq B_D \times B_D$ with $b_i \ll a_i$ for all $i \in K$, there exists $\{c_i : i \in K\} \subseteq_f B_D$ such that*

(1) $b_i \ll c_i \ll a_i$ for each $i \in K$.

(2) For any $I \subseteq K$, $\mathrm{mub}\{c_i : i \in I\}$ is finite and contained in B_D, i.e., $\mathrm{mub}\{c_i : i \in I\} \subseteq_f B_D$.

Proof. The "if" part is obvious. Suppose that B_D is $\mathcal{M}$-closed and $\{(a_i, b_i) : i \in K\} \subseteq B_D \times B_D$ is nonempty finite with $b_i \ll a_i$ for all $i \in K$. For any nonempty $I \subseteq K$, if $\bigcap_{i\in I} \uparrow a_i = \emptyset$, then we can find $\{b_i^I : i \in I\} \subseteq B_D$ such that $b_i \ll b_i^I \ll a_i$ for all $i \in I$ and $\bigcap_{i\in I} \uparrow b_i^I = \emptyset$ by Lemma 2.9. Set

$$\mathcal{I} = \{I \subseteq K : \bigcap_{i\in I} \uparrow a_i = \emptyset\}.$$

For each $i \in K$, we set

$$B_i = \{b_i\} \cup \{b_i^I : i \in I \in \mathcal{I}\}.$$

B_i is nonempty and $B_i \subseteq \downdownarrows a_i$. Set $b_i' = \bigvee_{\downarrow a_i} B_i$. Then $b_i' \ll a_i$. For each $i \in K$, pick $b_i'' \in B_D$ such that $b_i' \ll b_i'' \ll a_i$. Then the following conditions hold:

(i) $\forall I \subseteq K$, $\bigcap_{i\in I} \uparrow a_i = \emptyset \Longleftrightarrow \bigcap_{i\in I} \uparrow b_i' = \bigcap_{i\in I} \uparrow b_i'' = \emptyset$,

(ii) $\forall I \subseteq K$, $\bigcap_{i\in I} \uparrow a_i \neq \emptyset \Longleftrightarrow \bigcap_{i\in I} \Uparrow b_i'' \neq \emptyset$.

Set $B = \{(b'_i, b''_i) : i \in K\}$. Since B_D is $\mathcal{M}$-closed, we can find $c_i \in B_D$ for each $i \in K$ such that $b'_i \ll c_i \ll b''_i$ and for any $I \subseteq K$, $\bigcap_{i\in I} \Uparrow b''_i \neq \emptyset$ implies $\mathrm{mub}\{c_i : i \in I\} \subseteq_f B_D$. Hence, from the above statements (i) and (ii), all c_i's satisfy the conditions of this proposition. □

In the following, we define a new class of domain called ωAML-domains.

Definition 4.6 A continuous cpo is said to be an ωAML-domain if it is an L-domain endowed with a basis B_D satisfying the following conditions:

(a) B_D is countable,

(b) B_D is almost algebraic,

(c) B_D is $\mathcal{M}$-closed.

From now on, we always use a pair like (D, B_D) to be an ωAML-domain with a required basis B_D.

Since a continuous L-domain has property $\mathcal{M}$ if and only if it is an FS-domain (a retract of a *bifinite domain* further, see [12]), every ωAML-domain is an FS-domain.

Let's see some examples. (1) An algebraic L-domain is an ωAML-domain if and only if it has an ω-base with property $\mathcal{M}$. (2) The unit interval domain $\mathbb{I}$ in Example 3.7 is an ωAML-domain, i.e., the base $B_{\mathbb{I}}$ is countable, almost algebraic and $\mathcal{M}$-closed. (3) Given an algebraic L-domain L with an ω-base with property $\mathcal{M}$, the function space $[\mathbb{I} \to L]$ is an ωAML-domain. Particularly, if L is not bounded complete, then $[\mathbb{I} \to L]$ is neither bounded complete nor algebraic.

Proposition 4.7 *Let (D, B_D) be an ωAML-domain, $\{a_i : i \in K\} \subseteq_f B_D$. Then we have*

(1) There are decreasing sequences $(a_i^n)_{n\in\omega} \subseteq B_D$, $i \in K$ such that
- *(i) For each $i \in K$, $(a_i^n)_{n\in\omega}$ is an almost algebraic sequence of a_i.*
- *(ii) $\forall I \subseteq K, \forall n \in \omega$, $\bigcap_{i\in I} \Uparrow a_i = \emptyset \Longleftrightarrow \bigcap_{i\in I} \uparrow a_i^n = \emptyset \Longleftrightarrow \bigcap_{i\in I} \Uparrow a_i^n = \emptyset$.*
- *(iii) $\forall I \subseteq K, \forall n \in \omega$, $\mathrm{mub}\{a_i^n : i \in I\} \subseteq_f B_D$.*

(2) There are increasing sequences $(\hat{a}_i^n)_{n\in\omega} \subseteq B_D$, $i \in K$ such that
- *(i) For each $i \in K$, $\hat{a}_i^1 \ll \hat{a}_i^2 \ll \cdots \ll \hat{a}_i^n \ll \hat{a}_i^{n+1} \ll \cdots \ll a_i$ in D and $\bigvee_{n\in\omega} \hat{a}_i^n = a_i$.*
- *(ii) $\forall I \subseteq K, \forall n \in \omega$, $\bigcap_{i\in I} \uparrow a_i = \emptyset \Longrightarrow \bigcap_{i\in I} \uparrow \hat{a}_i^n = \emptyset$;*
- *(iii) $\forall I \subseteq K, \forall n \in \omega$, $\mathrm{mub}\{\hat{a}_i^n : i \in I\} \subseteq_f B_D$.*

Proof. (1) Since B_D is almost algebraic, we choose almost algebraic sequences $(\tilde{a}_i^j)_j$ for each a_i, where $i \in K$. Set

$$\mathcal{F} = \{I \subseteq K : \bigcap_{i\in I} \Uparrow a_i \neq \emptyset\}.$$

For any $I \in \mathcal{F}$, we have $\bigcap_{i\in I} \Uparrow a_i \neq \emptyset$. Suppose $x \in \bigcap_{i\in I} \Uparrow a_i$, that is, $a_i \ll x$ holds for all $i \in I$. For a fixed $i \in I$, since $(\tilde{a}_i^j)_j$ is an almost algebraic sequence of a_i, there exists $n_i \in \omega$ such that $\tilde{a}_i^{n_i} \ll x$ holds. Set

$$n_I = \max\{n_i : i \in I\}.$$

Then $\tilde{a}_i^{n_I} \ll x$ for all $i \in I$, i.e., $\bigcap_{i\in I} \uparrow\!\!\uparrow \tilde{a}_i^{n_i} \neq \emptyset$. Set

$$n_0 = \max\{n_I : I \in \mathcal{F}\},$$

we have

$$\forall I \subseteq K, \forall n \geq n_0, \bigcap_{i\in I} \uparrow\!\!\uparrow a_i \neq \emptyset \Longleftrightarrow \bigcap_{i\in I} \uparrow\!\!\uparrow \tilde{a}_i^n \neq \emptyset.$$

We now consider a finite set $\{(\tilde{a}_i^n, \tilde{a}_i^{n+1}) : i \in K\}$ for all $n \geq n_0$. Since B_D is $\mathcal{M}$-closed, there exist $\{\bar{a}_i^n : i \in K\} \subseteq B_D$ such that $\tilde{a}_i^{n+1} \ll \bar{a}_i^n \ll \tilde{a}_i^n$ for each $i \in K$ and $\mathrm{mub}\{\bar{a}_i^n : i \in I\} \subseteq_f B_D$ for all $I \in \mathcal{F}$ by Proposition 4.5.

Next, we define $a_i^n = \bar{a}_i^{n+n_0}$ for each $n \in \omega$ and each $i \in K$. It is easy to see that sequences $(a_i^n)_{n\in\omega}$ for all $i \in K$ satisfy the required conditions.

(2) Since B_D is a countable basis of D, we can choose sequences $(a_i^j)_j$ from B_D increasing with respect to $\ll$ such that $\bigvee_j a_i^j = a_i$ for each $i \in K$. Now we consider a finite set $\{(a_i^{j+1}, a_i^j) : i \in K\}$ for all $j \in \omega$. From the assumption that B_D is $\mathcal{M}$-closed, there exist $\{\bar{a}_i^j : i \in K\} \subseteq B_D$ such that

$$a_i^j \ll \bar{a}_i^j \ll a_i^{j+1}$$

and

$$\mathrm{mub}\{\bar{a}_i^j : i \in I\} \subseteq_f B_D$$

for $I \subseteq K$ and $j \in \omega$ by Proposition 4.5. Clearly, $(\bar{a}_i^j)_{j\in\omega}$ is an increasing sequence with respect to $\ll$ such that $\bigvee_j \bar{a}_i^j = a_i^j$ for all $i \in K$. Set

$$\mathcal{E} = \{I \in K : \bigcap_{i\in I} \uparrow\!\!\uparrow a_i = \emptyset\}.$$

Then $\mathcal{E}$ is finite and for each $I \in \mathcal{E}$, there is $n_I \in \omega$ such that $\bigcap_{i\in I} \uparrow\!\!\uparrow \bar{a}_i^{n_I} = \emptyset$ by Lemma 2.9. Next, set

$$n_0 = \max\{n_I : I \in \mathcal{E}\}$$

and let $\hat{a}_i^n = \bar{a}_i^{n+n_0}$. It is easy to check that sequences $(\hat{a}_i^n)_n$ $(i \in K)$ are increasing and satisfy the required conditions. □

Definition 4.8 Let (D, B_D) be an ωAML-domain, $\{a_i : i \in K\} \subseteq_f B_D$.

(1) The family $\{(a_i^n)_{n\in\omega} : i \in K\}$ of the decreasing sequences in Proposition 4.7.(1) is called a $\mathcal{M}$-closed almost algebraic family of $\{a_i : i \in K\}$.

(2) The family $\{(\hat{a}_i^n)_{n\in\omega} : i \in K\}$ of the increasing sequences in Proposition 4.7.(2) is called a $\mathcal{M}$-closed approximating family of $\{a_i : i \in K\}$.

Let D be a continuous L-domain and $\{a_i : i \in I\} \subseteq D$ be a nonempty finite subset with $\bigcap_{i\in I} \uparrow a_i \neq \emptyset$. For each $i \in I$, let $(a_i^n)_{n\in\omega} \subseteq D$ be an increasing sequence with $\bigvee_{n\in\omega} a_i^n = a_i$ and $a_i^n \ll a_i^{n+1}$ for all $n \in \omega$. Set

$$M_{\langle a,I\rangle} = \mathrm{mub}\{a_i : i \in I\}$$
$$M_{\langle a^n,I\rangle} = \mathrm{mub}\{a_i^n : i \in I\}$$

for all $n \in \omega$. We define a map $r_I^n : M_{\langle a,I\rangle} \longrightarrow M_{\langle a^n,I\rangle}$ as follows: $\forall m \in M_{\langle a,I\rangle}$,

$$r_I^n(m) = \bigvee_{\downarrow m}\{a_i^n : i \in I\}$$

for each $n \in \omega$. Easily one sees that r_I^n is well defined.

Lemma 4.9 *Let (D, B_D) be an ωAML-domain, $\{a_i : i \in I\} \subseteq_f B_D$. Let $\{(a_i^n)_{n\in\omega} : i \in I\}$ be a $\mathcal{M}$-closed approximating family of $\{a_i : i \in I\}$. Then we have*

(1) If $\bigcap_{i\in I} \uparrow a_i \neq \emptyset$ and $M_{\langle a,I\rangle} \subseteq_f B_D$, then

(i) For any $m \in M_{\langle a,I\rangle}$, $\bigvee_{n\in\omega} r_I^n(m) = m$ and

$$r_I^1(m) \ll r_I^2(m) \ll \cdots \ll r_I^n(m) \ll r_I^{n+1}(m) \ll \cdots \ll m.$$

(ii) For each $n \in \omega$, there exists $\hat{n} > n$ such that $M_{\langle a^{\hat{n}},I\rangle} \subseteq \uparrow r_I^n(M_{\langle a,I\rangle})$.

(2) If $\bigcap_{i\in I} \uparrow a_i = \emptyset$, then $\bigcap_{i\in I} \uparrow a_i^n = \emptyset$ for all $n \in \omega$.

Proof. (1) For any $m \in M_{\langle a,I\rangle}$, it is easy to check that $(r_I^n(m))_{n\in\omega}$ is an increasing sequence with respect to $\ll$ such that $\bigvee_{n\in\omega} r_I^n(m) = m$. Suppose that (ii) is not true, that is, there is $n_0 \in \omega$ such that

$$M_{\langle a^n,I\rangle} \not\subseteq \uparrow r_I^{n_0}(M_{\langle a,I\rangle})$$

for all $n > n_0$. Set

$$F_n = M_{\langle a^n,I\rangle} \setminus \uparrow r_I^{n_0}(M_{\langle a,I\rangle})$$

for all $n > n_0$. Then $F_n \neq \emptyset$ for all $n > n_0$. We claim that $F_{n+1} \subseteq \uparrow F_n$ holds for all $n > n_0$. In fact, for any $y \in F_{n+1}$, there is

$$x_n = \bigvee_{\downarrow y}\{a_i^n : i \in I\} \in M_{\langle a^n,I\rangle}$$

such that $x_n \leq y$. This implies that $x_n \notin \uparrow r_I^{n_0}(M_{\langle a,I\rangle})$ as $y \notin \uparrow r_I^{n_0}(M_{\langle a,I\rangle})$. Hence, $x_n \in F_n$, i.e., $F_{n+1} \subseteq \uparrow F_n$. Consider the family $\{F_n : n > n_0\}$. By the Rudin's Lemma [5, Lemma III-3.3], there is a directed subset $H \subseteq \bigcup_{n>n_0} F_n$ such that $H \cap F_n \neq \emptyset$ for all $n > n_0$. It means that $\bigvee H \in \bigcap_{i\in I} \uparrow a_i$, hence there is $m \in \mathrm{mub}\{a_i : i \in I\}$ such that $m \leq \bigvee H$. We have $r_I^{n_0}(m) \ll m \leq \bigvee H$. Hence, there is $x \in H$ such that $r_I^{n_0}(m) \leq x$, and therefore $x \in \uparrow r_I^{n_0}(M_{\langle a,I\rangle})$. On the other hand, $H \subseteq \bigcup_{n>n_0} F_n$ implies that x is in F_n for some $n > n_0$, hence x is not in $\uparrow r_I^{n_0}(M_{\langle a,I\rangle})$ - a contradiction.

(2) It follows directly from the definition of $\mathcal{M}$-closed approximating family. □

Theorem 4.10 *Let (D, B_D) be an ωAML-domain. Let $\{a_i : i \in K\} \subseteq_f B_D$ satisfy the following conditions:*

(i) $\forall I \subseteq K$, $\bigcap_{i\in I} \Uparrow a_i = \emptyset \Longrightarrow \bigcap_{i\in I} \uparrow a_i = \emptyset$;

(ii) $\forall I \subseteq K$, $\bigcap_{i\in I} \Uparrow a_i \neq \emptyset \Longrightarrow M_{\langle a,I\rangle} \subseteq_f B_D$.

Then there exists a $\mathcal{M}$-closed approximating family $\{(a_i^n)_{n\in\omega} : i \in K\}$ of $\{a_i : i \in K\}$ such that

(1) For all $I \subseteq K$, we have

(i) $\bigcap_{i\in I} \Uparrow a_i \neq \emptyset \Longrightarrow \forall n \in \omega,\ M_{\langle a^{n+1},I\rangle} \subseteq \uparrow r_I^n(M_{\langle a,I\rangle})$;

(ii) $\bigcap_{i\in I} \Uparrow a_i = \emptyset \Longrightarrow \forall n \in \omega,\ \bigcap_{i\in I} \uparrow a_i^n = \emptyset$.

(2) For any $I \subseteq K$, if $\bigcap_{i\in I} \Uparrow a_i \neq \emptyset$, then for any $m \in M_{\langle a,I\rangle}$,

$$\Uparrow m \backslash \bigcup_{j\in K\backslash I} \Uparrow a_j \neq \emptyset \Longrightarrow \forall n \in \omega,\ \Uparrow r_I^n(m) \backslash \bigcup_{j\in K\backslash I} \Uparrow a_j^n \neq \emptyset.$$

(3) For all $I, J \subseteq K$, if $I \subseteq J$, then for all $(m_1, m_2) \in M_{\langle a,I\rangle} \times M_{\langle a,J\rangle}$,

$$m_1 \not\leq m_2 \Longrightarrow \forall n \in \omega,\ r_I^n(m_1) \not\leq r_J^n(m_2).$$

Proof. By Proposition 4.7.(2), there are increasing sequences $(\hat{a_i}^n)_{n\in\omega} \subseteq B_D$ for all $i \in K$ which form a weakly $\mathcal{M}$-closed approximating family of $\{a_i : i \in K\}$.

First of all, set

$$\mathcal{F} = \{I \subseteq K : \bigcap_{i\in I} \Uparrow a_i \neq \emptyset\} = \{F_1, F_2, \cdots, F_l\}.$$

For F_1, by Lemma 4.9 (1) we have an increasing infinite sequence $(n_j^1)_{j\in\omega} \subseteq \omega$ such that

$$M_{\langle \hat{a}^{n_{j+1}^1}, F_1\rangle} \subseteq \uparrow r_{F_1}^{n_j^1}(M_{\langle a,F_1\rangle})$$

holds for all $j \in \omega$. For F_2, it is obvious that $\{(\hat{a}_i^{n_j^1})_{j\in\omega} : i \in F_2\}$ is a weakly $\mathcal{M}-$closed approximating family of $\{a_i : i \in F_2\}$. By Lemma 4.9 (1) again, there is an increasing infinite subsequence $(n_j^2)_j$ of $(n_j^1)_j$ such that

$$M_{\langle \hat{a}^{n_{j+1}^2}, F_2\rangle} \subseteq \uparrow r_{F_2}^{n_j^2}(M_{\langle a,F_2\rangle}).$$

By induction, we can get an increasing sequence $(n_j)_{j\in\omega} \subseteq \omega$ such that

$$M_{\langle \hat{a}^{n_{j+1}}, F_i\rangle} \subseteq \uparrow r_{F_i}^{n_j}(M_{\langle a,F_i\rangle})$$

holds for each i with $1 \leq i \leq l$ and $j \in \omega$. Easily one sees that, $\{(\hat{a}_i^{n_j})_{j\in\omega} : i \in K\}$ is a $\mathcal{M}$-closed approximating family of $\{a_i : i \in K\}$ such that condition (1) holds.

Secondly, we set

$$\mathcal{M} = \{m : \exists I \in \mathcal{F},\ \ m \in M_{\langle a,I\rangle}\ \&\ \Uparrow m \backslash \bigcup_{j\in K\backslash I} \Uparrow a_j \neq \emptyset\}.$$

Clearly, $\mathcal{M}$ is finite. For any $m \in \mathcal{M}$, there is $I \in \mathcal{F}$ such that $m \in M_{\langle a,I\rangle}$ with $\Uparrow m \backslash \bigcup_{j\in K\backslash I} \Uparrow a_j \neq \emptyset$. Then $a_j \not\lesssim m$ for all $j \in K\backslash I$. Given $j \in K\backslash I$, since $a_j = \bigvee_{p\in\omega} \hat{a}_j^{n_p}$, there is $p_j \in \omega$ such that $\hat{a}_j^{n_{p_j}} \not\lesssim m$. Set

$$p_m = \max\{p_j : j \in K\backslash I\}.$$

Then $\hat{a}_j^{n_{p_m}} \not\lesssim m$ holds for all $j \in K\backslash I$. It means that $\hat{a}_j^{n_p} \not\lesssim m$ holds for all $j \in K\backslash I$ and $p \geq p_m$, which is equivalent to

$$\Uparrow m \backslash \bigcup_{j\in K\backslash I} \Uparrow \hat{a}_j^{n_p} \neq \emptyset$$

for all $p \geq p_m$. Because $r_I^{n_p}(m) \leq m$, we have that

$$\Uparrow r_I^{n_p}(m) \backslash \bigcup_{j\in K\backslash I} \Uparrow \hat{a}_j^{n_p} \neq \emptyset$$

for all $p \geq p_m$. Next we set

$$p_0 = \max\{p_m : m \in \mathcal{M}\},$$

then we get that $\Uparrow r_I^{n_p}(m) \backslash \bigcup_{j\in K\backslash I} \Uparrow \hat{a}_j^{n_p} \neq \emptyset$ for all $p \geq p_0$ and $m \in \mathcal{M}$.

Finally, set

$$\mathcal{O} = \{(x,y) : \exists I, J \subseteq K,\ I \subseteq J \ \&\ (x,y) \in M_{\langle a,I\rangle} \times M_{\langle a,J\rangle} \ \&\ x \not\lesssim y\}.$$

Clearly, $\mathcal{O}$ is finite. For any given $(x,y) \in \mathcal{O}$, there are $I, J \subseteq K$ with $I \subseteq J$ and we denote x and y by $m_I \in M_{\langle a,I\rangle}$ and $m_J \in M_{\langle a,J\rangle}$ respectively. Since $m_I \not\lesssim m_J$, there is $p_{(x,y)} \in \omega$ with $p_{(x,y)} \geq p_0$ such that

$$r_I^{n_{p_{(x,y)}}}(m_I) \not\lesssim m_J.$$

Therefore, $r_I^{n_p}(m_I) \not\lesssim r_J^{n_p}(m_J)$ for all $p \geq p_{(x,y)}$. Now, we set

$$p_1 = \max\{p_{(x,y)} : (x,y) \in \mathcal{O}\}.$$

Then $r_I^{n_p}(m_I) \not\lesssim r_J^{n_p}(m_J)$ for $p \geq p_1$ and $(m_I, m_J) \in \mathcal{O}$. In the end, we define $a_i^j = \hat{a}_i^{n_{j+p_1}}$ for each $i \in K$ and $j \in \omega$. It is easy to check that $\{(a_i^j)_{j\in\omega} : i \in K\}$ satisfy the all required conditions of this proposition. □

Definition 4.11 Let $(D, B_D), (E, B_E)$ be two ωAML-domains. A nonempty finite subset $\{(a_i \searrow b_i) : i \in K\}$ of step functions of the function space $[D \to E]$ is called a *good step function family*, a *g.s.f-family* for short, if

(1) $(a_i, b_i) \in B_D \times B_E$ for all $i \in K$ and $\bigcap_{i\in K} \Uparrow(a_i \searrow b_i) \neq \emptyset$.

(2) $\forall I \subseteq K$, $\bigcap_{i\in I} \Uparrow a_i = \emptyset \Longrightarrow \bigcap_{i\in I} \uparrow a_i = \emptyset$.

(3) $\forall I \subseteq K$, $\bigcap_{i\in I} \Uparrow a_i \neq \emptyset \Longrightarrow M_{\langle a,I\rangle} \subseteq_f B_D \ \&\ M_{\langle b,I\rangle} \subseteq_f B_E$.

Let $(D, B_D), (E, B_E)$ be two ωAML-domains, let $\{(a_i \searrow b_i) : i \in K\}$ be a $g.s.f$-family. For any nonempty $I \subseteq K$, set

$$\begin{aligned}\overline{M}_{\langle a,I\rangle} &= \{m \in M_{\langle a,I\rangle} : \forall j \in K\backslash I,\ a_j \not\leq m\},\\ \mathrm{Mub}_{\langle a,K\rangle} &= \bigcup\{M_{\langle a,I\rangle} : \emptyset \neq I \subseteq K\},\\ \mathrm{Mub}_{\langle b,K\rangle} &= \bigcup\{M_{\langle b,I\rangle} : \emptyset \neq I \subseteq K\}.\end{aligned}$$

Easily, we have that $\{\overline{M}_{\langle a,I\rangle} : \emptyset \neq I \subseteq K\}$ is a decomposition of $\mathrm{Mub}_{\langle a,K\rangle}$.

Lemma 4.12 *(1) For any $m \in \overline{M}_{\langle a,I\rangle}$, $\uparrow m \setminus \bigcup_{j\in K\backslash I} \uparrow a_j \neq \emptyset$.*

(2) For any $x \in \bigcup_{i\in K} \uparrow a_i$, there are a unique subset $I \subseteq K$ and a unique $m \in \overline{M}_{\langle a,I\rangle}$ such that $x \in \uparrow m \setminus \bigcup_{j\in K\backslash I} \uparrow a_j$.

Proof. (1) Assume that there exists some $m \in \overline{M}_{\langle a,I\rangle}$ such that

$$\uparrow m \setminus \bigcup_{j\in K\backslash I} \uparrow a_j = \emptyset.$$

Then $\uparrow m \subseteq \bigcup_{j\in K\backslash I} \uparrow a_j$. Since B_D is almost algebraic, there is an almost algebraic sequence $(m_n)_{n\in\omega}$ of m. We claim that there exists $j_0 \in K\backslash I$ such that $m_n \in \uparrow a_{j_0}$ for all $n \in \omega$. Otherwise, for each $j \in K\backslash I$, there is $n_j \in \omega$ such that $m_{n_j} \notin \uparrow a_j$; set

$$n_0 = \max\{n_j : j \in K\backslash I\},$$

we have $m_{n_0} \notin \bigcup_{j\in K\backslash I} \uparrow a_j$. It is a contradiction. Hence, $\uparrow m \subseteq \uparrow a_{j_0}$ for some $j_0 \in K\backslash I$. Note that since $m, a_{j_0} \in B_D$, it follows that $a_{j_0} \leq m$ holds, which is contradictory to $m \in \overline{M}_{\langle a,I\rangle}$.

(2) It follows from that for a given $x \in \bigcup_{i\in K} \uparrow a_i$, we set $I = \{i \in K : a_i \ll x\}$ and $m = \bigvee_{\downarrow x}\{a_i : i \in I\}$. □

Definition 4.13 A map $T_K : \mathrm{Mub}_{\langle a,K\rangle} \longrightarrow \mathrm{Mub}_{\langle b,K\rangle}$ is called *consistent* if

(1) T_K is monotone under the induced orders on $\mathrm{Mub}_{\langle a,K\rangle}$ and $\mathrm{Mub}_{\langle b,K\rangle}$.

(2) For any nonempty $I \subseteq K$, for any $m \in \overline{M}_{\langle a,I\rangle}$, $T_K(m) \in M_{\langle b,I\rangle}$.

Suppose $T_K : \mathrm{Mub}_{\langle a,K\rangle} \longrightarrow \mathrm{Mub}_{\langle b,K\rangle}$ is a consistent map. We define a function $f_{T_K} : D \longrightarrow E$ as follows: $\forall x \in D$,

$$f_{T_K}(x) = \begin{cases} T_K(m), \text{ if } \exists \emptyset \neq I \subseteq K,\ m \in \overline{M}_{\langle a,I\rangle},\ x \in \uparrow m \backslash \bigcup_{j\in K\backslash I} \uparrow a_j, \\ \bot, \qquad \text{otherwise.} \end{cases}$$

It is easy to check by Lemma 4.12 that f_{T_K} is well defined.

Proposition 4.14 *f_{T_K} is a Scott continuous function with $f_{T_K} \in \mathrm{mub}\{(a_i \searrow b_i) : i \in K\}$.*

Proof. First of all, we show that f_{T_K} is monotone. Let $x, y \in D$ with $x \le y$. If $x \notin \bigcup_{i\in K} \Uparrow a_i$, then $f_{T_K}(x) = \perp \le f_{T_K}(y)$. If $x \in \bigcup_{i\in K} \Uparrow a_i$, then there is $I \subseteq K$ with $m_x \in \overline{M}_{\langle a,I\rangle}$ such that $x \in \Uparrow m_x \backslash \bigcup_{j\in K\backslash I} \Uparrow a_j$ from Lemma 4.12 (2). By the same reason, there is $J \subseteq K$ with $I \subseteq J$ such that there is $m_y \in \overline{M}_{\langle a,J\rangle}$ with $y \in \Uparrow m_y \backslash \bigcup_{j\in K\backslash J} \Uparrow a_j$. Then

$$m_y = \bigvee_{\downarrow y}\{a_i : i \in J\} \ge \bigvee_{\downarrow y}\{a_i : i \in I\} = \bigvee_{\downarrow x}\{a_i : i \in I\} = m_x.$$

So we have that $f_{T_K}(x) = T_K(m_x) \ge T_K(m_y) = f_{T_K}(y)$.

Next we show that f_{T_K} preserves the supremum of every directed subset. Let H be a directed set of D. We need to consider two cases for $\bigvee H$. (1) $\bigvee H \notin \bigcup_{i\in K} \Uparrow a_i$. It implies that $h \notin \bigcup_{i\in K} \Uparrow a_i$ for all $h \in H$. Thus we have that $f_{T_K}(\bigvee H) = \perp = \bigvee_{h\in H} f_{T_K}(h)$. (2) $\bigvee H \in \bigcup_{i\in K} \Uparrow a_i$. It implies that there is $I \subseteq K$ such that $\bigvee H \in \Uparrow m \backslash \bigcup_{i\in K\backslash I} \Uparrow a_i$ for some $m \in \overline{M}_{\langle a,I\rangle}$ from Lemma 4.12(2). It follows that there is $h_0 \in H$ with $m \ll h_0$ and $h_0 \notin \bigcup_{i\in K\backslash I} \Uparrow a_i$. Then

$$f_{T_K}(\bigvee H) = T_K(m) = f_{T_K}(h_0) = \bigvee_{h\in H} f_{T_K}(h)$$

as f_{T_K} is monotone.

Clearly, $(a_i \searrow b_i) \le f_{T_K}$ holds for each $i \in K$. For any $x \in \bigcup_{i\in K} \Uparrow a_i$, by Lemma 4.12 (2), there is $I \subseteq K$ such that there exists $m \in \overline{M}_{\langle a,I\rangle}$ with $x \in \Uparrow m \backslash \bigcup_{j\in K\backslash I} \Uparrow a_j$. Then

$$\begin{aligned}
\bigvee_{\downarrow f_{T_K}} \{(a_i \searrow b_i) : i \in K\}(x) &= \bigvee_{\downarrow f_{T_K}(x)} \{b_i : a_i \ll x\} = \bigvee_{\downarrow f_{T_K}(x)} \{b_i : i \in I\} \\
&= \bigvee_{\downarrow T_K(m)} \{b_i : i \in I\} = T_K(m) \\
&= f_{T_K}(x).
\end{aligned}$$

Therefore, $f_{T_K} \in \mathrm{mub}\{(a_i \searrow b_i) : i \in K\}$. □

The following is a characterization of the minimal upper bounds of a good step function family in function spaces.

Theorem 4.15 *Let $(D, B_D), (E, B_E)$ be two ωAML-domains, let $\{(a_i \searrow b_i) : i \in K\}$ be a g.s.f-family, $f \in [D \longrightarrow E]$. The following two conditions are equivalent to each other:*

(1) $f \in \mathrm{mub}\{(a_i \searrow b_i) : i \in K\}$.

(2) There exists a consistent map $T_K : \mathrm{Mub}_{\langle a,K\rangle} \longrightarrow \mathrm{Mub}_{\langle b,K\rangle}$ such that $f = f_{T_K}$.

Proof. (2) $\Longrightarrow$ (1): It follows directly from Proposition 4.14.

(1) $\Longrightarrow$ (2): For any $x \in \bigcup_{i\in K} \Uparrow a_i$, by Lemma 4.12 (2), there is a subset $I \subseteq K$ such that there exists $m \in \overline{M}_{\langle a,I\rangle}$ with $x \in \Uparrow m \setminus \bigcup_{j\in K\backslash I} \Uparrow a_j$. Next, we show that f

is constant on $\Uparrow m \setminus \bigcup_{j\in K\setminus I} \Uparrow a_j$. We choose an almost algebraic sequence $(m_j)_{j\in\omega}$ of m, then there is some $m_j \ll x$. For any $y \in \Uparrow m \setminus \bigcup_{i\in K\setminus I} \Uparrow a_i$, there is some $m_l \ll y$. Let n be the maximum of $j$and l. Then $m \ll m_n \ll x, y$. Hence, we have

$$\begin{aligned} f(x) &= (\bigvee_{\downarrow f}\{(a_i \searrow b_i) : i \in K\})(x) \\ &= \bigvee_{\downarrow f(x)} \{b_i : a_i \ll x\} = \bigvee_{\downarrow f(x)} \{b_i : i \in I\} \\ &= \bigvee_{\downarrow f(m_n)} \{b_i : i \in I\} \\ &= f(y) \quad (*). \end{aligned}$$

Then we define a map $T_K : Mub_{\langle a,K\rangle} \to Mub_{\langle b,K\rangle}$ as follows: $\forall m \in \overline{M}_{\langle a,I\rangle}$, $T_K(m) = f(x)$ for any $x \in \Uparrow m \setminus \bigcup_{i\in K\setminus I} \Uparrow a_i$. From $(*)$, T_K is well defined. It is easy to check that T_K is a consistent map with $f = f_{T_K}$. □

Proposition 4.16 *Let $(D, B_D), (E, B_E)$ be two ωAML-domains, let $\{(a_i \searrow b_i) : i \in K\}$ be a g.s.f-family. For $g \in [D \longrightarrow E]$ and for a consistent map $T_K : \mathrm{Mub}_{\langle a,K\rangle} \longrightarrow \mathrm{Mub}_{\langle b,K\rangle}$, we have that*

$$f_{T_K} \ll g \Longleftrightarrow \forall m \in \mathrm{Mub}_{\langle a,K\rangle},\ T_K(m) \ll g(m).$$

Proof. If $f_{T_K} \ll g$, then $(a_i \searrow b_i) \ll g$ for all $i \in K$ form Theorem 4.15. By Lemma 4.2, we have that $b_i \ll g(a_i)$ for all $i \in K$. For any $m \in Mub_{\langle a,K\rangle}$, there is $I \subseteq K$ with $m \in \overline{M}_{\langle a,K\rangle}$. By Lemma 4.12, $\Uparrow m \setminus \bigcup_{i\in K\setminus I} \Uparrow a_i \neq \emptyset$. Pick $x \in \Uparrow m \setminus \bigcup_{i\in I} \Uparrow a_i$. Then

$$g(x) \geq f_{T_K}(x) = T_K(m) \in \mathrm{mub}\{b_i : i \in I\}.$$

On the other hand, by $b_i \ll g(a_i)$ for all $i \in I$ we have that

$$g(m) \gg \bigvee_{\downarrow g(m)} \{b_i : i \in I\} = \bigvee_{\downarrow g(x)} \{b_i : i \in I\} = T_K(m).$$

Conversely, let $(g_j)_j$ be a directed collection of Scott continuous functions from (D, B_D) to (E, B_E) with $g \leq \bigvee_j g_j$. For any $m \in \mathrm{Mub}_{\langle a,K\rangle}$, by the assumption $T_K(m) \ll g(m)$, there exists j_m with $T_K(m) \ll g_{j_m}(m)$. Pick j_0 with $g_{j_0} \geq g_{j_m}$ for all $m \in \mathrm{Mub}_{\langle a,K\rangle}$. It is easy to check that $f_{T_K} \leq g_{j_0}$. Hence, $f_{T_K} \ll g$. □

5 The category of ωAML-domains

In this section, we are ready for showing that the category of ωAML-domains is cartesian closed. The main difficulty is to show the the function spaces are almost algebraic.

Definition 5.1 The category ω**AML** is given by:

- Objects are all ωAML-domains
- Morphisms are Scott continuous functions

Let $(D, B_D), (E, B_E)$ be two ωAML-domains. We set $GSF_{[D\to E]}$ to be the set of all $g.s.f$ families of $[D \to E]$. Let

$$B_{[D\to E]} = \bigcup\{\text{mub}\{(a_i \searrow b_i) : i \in K\} : \{(a_i \searrow b_i) : i \in K\} \in GSF_{[D\to E]}\}.$$

Obviously, $\{(a \searrow b) : a \in B_D,\ b \in B_E\} \subseteq B_{[D\to E]}$.

The following result is straightforward from Definition 4.8 and 4.11.

Lemma 5.2 *Let $\{(a_i \searrow b_i) : (a_i, b_i) \in B_D \times B_E \ \&\ i \in K\}$ be a finite set of step functions.*

(1) Let $\{(a_i^n)_{n\in\omega} : i \in K\}$ (resp. $\{(b_i^n)_{n\in\omega} : i \in K\}$) be a $\mathcal{M}$-closed almost algebraic family of $\{a_i : i \in K\}$ (resp. a $\mathcal{M}$-closed approximating family of $\{b_i : i \in K\}$). If $\bigcap_{i\in K} \uparrow\!(a_i \searrow b_i) \neq \emptyset$, then $(a_i^n \searrow b_i^n) \ll (a_i^{n+1} \searrow b_i^{n+1}) \ll (a_i \searrow b_i)$ for all $n \in \omega$ and $\{(a_i^n \searrow b_i^n) : i \in I\}$ is a g.s.f. family for all nonempty $I \subseteq K$.

(2) Let $\{(\hat{a}_i^n)_{n\in\omega} : i \in K\}$ (resp. $\{(\hat{b}_i^n)_{n\in\omega} : i \in K\}$) be a $\mathcal{M}$-closed approximating family of $\{a_i : i \in K\}$ (resp. a $\mathcal{M}$-closed almost algebraic family of $\{b_i : i \in K\}$). If $\{(a_i \searrow b_i) : i \in K\}$ is a g.s.f family, then $(\hat{a}_i \searrow \hat{b}_i) \ll (\hat{a}_i^{n+1} \searrow \hat{b}_i^{n+1}) \ll (\hat{a}_i^n \searrow \hat{b}_i^n)$ for all $n \in \omega$ and $\{(\hat{a}_i^n \searrow \hat{b}_i^n) : i \in I\}$ is a g.s.f. family for all nonempty $I \subseteq K$.

Next, we will show the category ω**AML** is cartesian closed through several propositions.

Proposition 5.3 *$B_{[D\to E]}$ is a countable basis of $[D \to E]$.*

Proof. First of all, we show that $B_{[D\to E]}$ is a countable set. Because B_D and B_E are countable, we have that $GSF_{[D\to E]}$ is countable. For any given $g.s.f$ family $\{(a_i \searrow b_i) : i \in K\}$ of $[D \to E]$, mub$\{(a_i \searrow b_i) : i \in K\}$ is finite from Theorem 4.15. Hence $B_{[D\to E]}$ is countable.

Next we show that $B_{[D\to E]}$ is a basis of $[D \to E]$. For any $f \in [D \to E]$,

$$f = \bigvee\{(a \searrow b) : a \in B_D,\ b \in B_E,\ b \ll f(a)\}.$$

We only need to show that $\bigvee_{\downarrow f}\{(a_i \searrow b_i) : a_i \in B_D,\ b_i \in B_E,\ b_i \ll f(a_i),\ i \in K\}$ for a nonempty finite set K can be approximated by elements of $B_{[D\to E]}$. In fact, by Proposition 4.7 we get decreasing sequences $(a_i^n)_{n\in\omega}$ for all $i \in K$ which form a $\mathcal{M}$-closed almost algebraic family of $\{a_i : i \in K\}$ and increasing sequences $(b_i^n)_{n\in\omega}$ for all $i \in K$ which form a $\mathcal{M}$-closed approximating family of $\{b_i : i \in K\}$. For any $n \in \omega$,

$$\bigvee_{\downarrow f}\{(a_i^n \searrow b_i^n) : i \in K\} \ll \bigvee_{\downarrow f}\{(a_i \searrow b_i) : i \in K\} \ll f.$$

By Lemma 5.2 (1), $\{(a_i^n \searrow b_i^n) : i \in K\}$ is a g.s.f family for all $n \in \omega$. Thus, $\bigvee_{\downarrow f}\{(a_i^n \searrow b_i^n) : i \in I\} \in B_{[D\to E]}$ with $\bigvee_{\downarrow f}\{(a_i^n \searrow b_i^n) : i \in K\} \ll \bigvee_{\downarrow f}\{(a_i \searrow b_i) :$

$i \in K\}$. Note that since

$$\bigvee_{\downarrow f}\{(a_i^n \searrow b_i^n) : i \in K\} \ll \bigvee_{\downarrow f}\{(a_i^{n+1} \searrow b_i^{n+1}) : i \in K\}$$

and

$$\bigvee_{n\in\omega}\bigvee_{\downarrow f}\{(a_i^n \searrow b_i^n) : i \in K\} = \bigvee_{\downarrow f}\{(a_i \searrow b_i) : i \in K\},$$

we finish the proof. □

Proposition 5.4 *$B_{[D\to E]}$ is $\mathcal{M}$-closed.*

Proof. Let $\{(f_j, g_j) : j \in K\}$ be a finite set of $B_{[D\to E]} \times B_{[D\to E]}$ with $g_j \ll f_j$ for all $j \in K$. It follows that for any given $j \in K$, there is a $g.s.f$ family $\{(a_{ji} \searrow b_{ji}) : i \in I_j\}$ with $f_j \in \mathrm{mub}\{(a_{ji} \searrow b_{ji}) : i \in I_j\}$. Set $P_j = \{j\} \times I_j$ and $P = \bigcup_{j\in K} P_j$. By Proposition 4.7, we obtain a $\mathcal{M}$-closed almost algebraic family $\{(a_p^n)_{n\in\omega} : p \in P\}$ of $\{a_p : p \in P\}$ and a $\mathcal{M}$-closed approximating family $\{(b_p^n)_{n\in\omega} : p \in P\}$ of $\{b_p : p \in P\}$. Clearly,

$$\begin{aligned} f_j &= \bigvee_{\downarrow f_j}\{(a_p \searrow b_p) : p \in P_j\} \\ &= \bigvee_{\downarrow f_j}\{\bigvee_{n\in\omega}(a_p^n \searrow b_p^n) : p \in P_j\} \\ &= \bigvee_{n\in\omega}\bigvee_{\downarrow f_j}\{(a_p^n \searrow b_p^n) : p \in P_j\} \end{aligned}$$

holds for each $j \in K$. From the assumption $g_j \ll f_j$ we obtain an n_j such that if $n \geq n_j$ then $g_j \ll \bigvee_{\downarrow f_j}\{(a_p^n \searrow b_p^n) : p \in P_j\}$ for any given $j \in K$. We set n_0 =max$\{n_j : j \in K\}$, then $g_j \ll \bigvee_{\downarrow f_j}\{(a_p^n \searrow b_p^n) : i \in I_j\}$ holds for all $j \in K$ and $n \geq n_0$.

Next, we set $c_p = a_p^{n_0+1}$ and $d_p = b_p^{n_0+1}$ for all $p \in P$. Then for any $p \in P$,

$$a_p \ll c_p \ll a_p^{n_0},\ b_p^{n_0} \ll d_p \ll b_p.$$

For each $j \in K$, set

$$h_j = \bigvee_{\downarrow f_j}\{(c_p \searrow d_p) : p \in P_j\}.$$

By Lemma 5.2 (1), $\{(c_p \searrow d_p) : p \in P_j\}$ is a g.s.f. family. Hence $h_j \in B_{[D\to E]}$ and $g_j \ll h_j \ll f_j$ for all $j \in K$. For any nonempty $J \subseteq K$ such that $\bigcap_{j\in J} \Uparrow f_j \neq \emptyset$, applied Lemma 5.2 (1) again, $\{(c_p \searrow d_p) : p \in \bigcup_{j\in J} P_j\}$ is also a g.s.f. family; thus

$$\mathrm{mub}\{h_j : j \in J\} \subseteq \mathrm{mub}\{(c_p \searrow d_p) : p \in \bigcup_{j\in J} P_j\} \subseteq_f B_{[D\to E]}.$$

□

Proposition 5.5 *$B_{[D\to E]}$ is almost algebraic.*

Proof. First of all, we show that each $f \in B_{[D\to E]}$ has an almost algebraic sequence $(f_n)_{n\in\omega} \subseteq B_{[D\to E]}$.

Given $f \in B_{[D\to E]}$, there is a $g.s.f$ family $\{(a_i \searrow b_i) : i \in K\}$ with $f \in \mathrm{mub}\{(a_i \searrow b_i) : i \in K\}$. By Theorem 4.15, there exists a consistent map

$$T_K : \mathrm{Mub}_{\langle a,K\rangle} \longrightarrow \mathrm{Mub}_{\langle b,K\rangle}$$

such that $f = f_{T_K}$. Applying Lemma 3.2 to $\{T_K(m) : m \in \mathrm{Mub}_{\langle a,K\rangle}\}$, we get $\{\overline{T_K}(m) : m \in \mathrm{Mub}_{\langle a,K\rangle}\} \subseteq B_E$ such that $T_K(m) \ll \overline{T_K}(m)$ and $\overline{T_K}(m_1) \leq \overline{T_K}(m_2)$ whenever $T_K(m_1) \leq T_K(m_2)$ for all $m, m_1, m_2 \in \mathrm{Mub}_{\langle a,K\rangle}$.

From Theorem 4.10, there exists a $\mathcal{M}$-closed approximating family $\{(a_i^n)_{n\in\omega} : i \in K\}$ of $\{a_i : i \in K\}$, which satisfies properties (1), (2) and (3) of Theorem 4.10. By Proposition 4.7 (1), there is a $\mathcal{M}$-closed almost algebraic family $(\{(b_i^n)_{n\in\omega} : i \in K\})$ of $\{b_i : i \in K\}$. From Lemma 5.2 (2), $\{(a_i^n \searrow b_i^n) : i \in K\}$ is a g.s.f. family for each $n \in \omega$.

For any $m \in \mathrm{Mub}_{\langle a,K\rangle}$, there is $I_m \subseteq K$ with $m \in \overline{M}_{\langle a,I_m\rangle}$. Since

$$\overline{T_K}(m) \gg T_K(m) \in M_{\langle b,I_m\rangle},$$

we have an n_m such that if $n \geq n_m$ then $\overline{T_K}(m) \in \bigcap_{i\in I_m} \Uparrow b_i^n$. Set

$$n_0 = \max\{n_m : m \in \mathrm{Mub}_{\langle a,K\rangle}\}.$$

Pick a fixed $n > n_0$. For any $x \in \mathrm{Mub}_{\langle a^n,K\rangle}$, there is $I_x \subseteq K$ with $x \in \overline{M}_{\langle a^n,I_x\rangle}$. This means that $\bigcap_{i\in I_x} \uparrow a_i^n \neq \emptyset$. Hence, $\bigcap_{i\in I_x} \Uparrow a_i \neq \emptyset$ and there exists a unique $m_x \in M_{\langle a,I_x\rangle}$ such that $r_I^{n-1}(m_x) \ll x$ by Theorem 4.10. Now, for each $n > n_0$, we define a map

$$T_K^n : \mathrm{Mub}_{\langle a^n,K\rangle} \longrightarrow \mathrm{Mub}_{\langle b^n,K\rangle}$$

as follows: $\forall x \in Mub_{\langle a^n,K\rangle}$,

$$T_K^n(x) = \bigvee_{\downarrow\overline{T_K}(m_x)} \{b_i^n : i \in I_x\}.$$

Claim 1: For each $n \geq n_0$, T_K^n is consistent.

By the definition of T_K^n, it is sufficient to show that T_K^n is monotone. Let $x, y \in \mathrm{Mub}_{\langle a^n,K\rangle}$ with $x \leq y$. There are $I, J \subseteq K$ such that $x \in \overline{M}_{\langle a^n,I\rangle}$ and $y \in \overline{M}_{\langle a^n,J\rangle}$. Clearly, $I \subseteq J$. By Theorem 4.10, there is $m_1 \in M_{\langle a,I\rangle}$ and $m_2 \in M_{\langle a,J\rangle}$ such that $r_I^{n-1}(m_1) \ll x$ and $r_J^{n-1}(m_2) \ll y$ hold respectively. Then

$$
\begin{aligned}
r_I^{n-1}(m_1) &= \bigvee_{\downarrow x}\{a_i^{n-1} : i \in I\} \\
&= \bigvee_{\downarrow y}\{a_i^{n-1} : i \in I\} \\
&\leq \bigvee_{\downarrow y}\{a_i^{n-1} : i \in J\} \\
&= r_J^{n-1}(m_2).
\end{aligned}
$$

It follows that $m_1 \leq m_2$ by Theorem 4.10 (3). Hence

$$
\begin{aligned}
T_K^n(x) &= \bigvee_{\downarrow \overline{T_K}(m_1)} \{b_i^n : i \in I\} \\
&= \bigvee_{\downarrow \overline{T_K}(m_2)} \{b_i^n : i \in I\} \\
&\leq \bigvee_{\downarrow \overline{T_K}(m_2)} \{b_i^n : i \in J\} \\
&= T_K^n(y).
\end{aligned}
$$

Therefore, $f_{T_K^n}$ is a Scott continuous function with $f_{T_K^n} \in \mathrm{mub}\{(a_i^n \searrow b_i^n) : i \in K\}$ for all $n \geq n_0$ by Proposition 4.14. Next we show the following claim.

Claim 2: $(f_{T_K^{n+n_0}})_{n\in\omega}$ is an almost algebraic sequence of f.

We show it through three steps.

Step 1: For each $n > n_0$, $f \ll f_{T_K^n}$.

By Proposition 4.16, we only need to show that $T_K(m) \ll f_{T_K^n}(m)$ for all $m \in \mathrm{Mub}_{\langle a,K\rangle}$. For any $m \in \mathrm{Mub}_{\langle a,K\rangle}$, there is $I \subseteq K$ such that $m \in \overline{M}_{\langle a,I\rangle}$. It means that $\bigcap_{i\in I} \Uparrow a_i \neq \emptyset$ and $\Uparrow m \backslash \bigcup_{i\in K\backslash I} \Uparrow a_i \neq \emptyset$ by Lemma 4.12. So $\Uparrow r_I^n(m) \backslash \bigcup_{i\in K\backslash I} \Uparrow a_i^n \neq \emptyset$ for all $n > n_0$ by Theorem 4.10 (2). It follows that $r_I^n(m) \in \overline{M}_{\langle a^n,I\rangle}$. As $r_I^{n-1}(m) \ll r_I^n(m) \ll m$ by Lemma 4.9 (1), we have that

$$
f_{T_K^n}(m) \geq T_K^n(r_I^n(m)) = \bigvee_{\overline{T_K}(m)} \{b_i^n : i \in I\} \gg T_K(m).
$$

Step 2: For each $n > n_0$, $f_{T_K^{n+1}} \ll f_{T_K^n}$.

The proof is similar to Step 1. By Proposition 4.16, we only need to show that $T_K^{n+1}(x) \ll f_{T_K^n}(x)$ holds for all $x \in \mathrm{Mub}_{\langle a^{n+1},K\rangle}$. Given $x \in \mathrm{Mub}_{\langle a^{n+1},K\rangle}$, there is $I \subseteq K$ such that $x \in \overline{M}_{\langle a^{n+1},I\rangle}$. By Theorem 4.10, $\bigcap_{i\in I} \Uparrow a_i \neq \emptyset$ and there is $m \in M_{\langle a,I\rangle}$ such that $r_I^n(m) \ll x$.

Case 1: $\Uparrow r_I^n(m) \backslash \bigcup_{i\in K\backslash I} \Uparrow a_i^n \neq \emptyset$.

We have that

$$
\begin{aligned}
f_{T_K^n}(x) \geq T_K^n(r_I^n(m)) &= \bigvee_{\downarrow \overline{T_K}(m)} \{b_i^n : i \in I\} \\
&\gg \bigvee_{\downarrow \overline{T_K}(m)} \{b_i^{n+1} : i \in I\} \\
&= T_K^{n+1}(x).
\end{aligned}
$$

Case 2: $\Uparrow r_I^n(m) \setminus \bigcup_{i \in K \setminus I} \Uparrow a_i^n = \emptyset$.

Clearly, there is $J \subseteq K$ with $I \subseteq J$ such that $r_I^n(m) \in \overline{M}_{\langle a^n, J \rangle}$. By Theorem 4.10 and Lemma 4.9, $\bigcap_{j \in J} \Uparrow a_j^n \neq \emptyset$ implies that $\bigcap_{j \in J} \Uparrow a_j \neq \emptyset$ and there is $m_1 \in M_{\langle a, J \rangle}$ such that $r_J^{n-1}(m_1) \ll r_I^n(m) \ll x$. Because

$$
r_J^{n-1}(m_1) = \bigvee_{\downarrow r_I^n(m)} \{a_i^{n-1} : i \in J\} \geq \bigvee_{\downarrow r_I^n(m)} \{a_i^{n-1} : i \in I\} = r_I^{n-1}(m),
$$

we have that $m_1 \geq m$ by Theorem 4.10 (3). Therefore,

$$
\begin{aligned}
f_{T_K^n}(x) \geq T_K^n(r_I^n(m) &= \bigvee_{\downarrow \overline{T_K}(m_1)} \{b_i^n : i \in J\} \\
&\geq \bigvee_{\downarrow \overline{T_K}(m_1)} \{b_i^n : i \in I\} \\
&= \bigvee_{\downarrow \overline{T_K}(m)} \{b_i^n : i \in I\} \\
&\gg \bigvee_{\downarrow \overline{T_K}(m)} \{b_i^{n+1} : i \in I\} \\
&= T_K^{n+1}(x).
\end{aligned}
$$

Step 3: For any $g \in [D \to E]$ with $f \ll g$, there exists $n > n_0$ such that $f_{T_K^n} \ll g$.

For any given $m \in \mathrm{Mub}_{\langle a, K \rangle}$, there exists $I \subseteq K$ such that $m \in \overline{M}_{\langle a, I \rangle}$. By Proposition 4.16 we have that $T_K(m) \ll g(m)$. Set

$$
A_m = \{J \subseteq K : m \in M_{\langle a, J \rangle}\}.
$$

Since $\bigvee_{n \in \omega} r_J^n(m) = m$ for each $J \in A_m$, there is n_J such that if $n \geq n_J$ then $g(r_J^n(m)) \gg T_K(m)$. Set

$$
N_m = \max\{n_J : J \in A_m\}.
$$

Then $g(r_J^n(m)) \gg T_K(m)$ for all $J \in A_m$ and for all $n \leq n_0$. By the assumption that B_E is almost algebraic, there exists $y_m \in B_E$ such that $T_K(m) \ll y_m \ll \overline{T_K}(m)$ and $y_m \ll g(r_J^{N_m}(m))$ for all $J \in A_m$. From $T_K(m) \ll y_m$, there is $p_m > n_0$ such that that if $p \geq p_m$ then $y_m \in \bigcap_{i \in I} \Uparrow b_i^p$. Set

$$
N_0 = \max\{\max\{N_m : m \in \mathrm{Mub}_{\langle a, K \rangle}\}, \max\{p_m : m \in \mathrm{Mub}_{\langle a, K \rangle}\}\}.
$$

We claim that $f_{T_K^{N_0+1}} \ll g$. In fact, for any $x \in \mathrm{Mub}_{\langle a^{N_0+1},K\rangle}$, there is $I_x \subseteq K$ such that $x \in \overline{M}_{\langle a^{N_0+1},I_x\rangle}$. By Theorem 4.10 (1), there exists $m \in M_{\langle a,I_x\rangle}$ with $r_{I_x}^{N_0}(m) \ll x$. Then we have

$$\begin{aligned} T_K^{N_0+1}(x) &= \bigvee_{\downarrow \overline{T_K}(m)} \{b_i^{N_0+1} : i \in I_x\} \\ &= \bigvee_{\downarrow y_m} \{b_i^{N_0+1} : i \in I_x\} \leq y_m \\ &\ll g(r_{I_x}^{N_0}(m)) \\ &\leq g(x). \end{aligned}$$

Hence, by Proposition 4.16, $f_{T_K^{N_0+1}} \ll g$. Combining all of above, we have shown that Claim 2.

Next, we show that for any $f, g \in B_{[D\to E]}$, $\Uparrow f \subseteq \Uparrow g \Rightarrow g \leq f$.

Claim 3: If $g = (a \searrow b)$ with $(a,b) \in B_D \times B_E$, then $\Uparrow f \subseteq \Uparrow g \Rightarrow g \leq f$.

Suppose that $f \in \mathrm{mub}\{(a_i \searrow b_i) : i \in K\}$. There exists a consistent map $T_K : \mathrm{Mub}_{\langle a,K\rangle} \longrightarrow \mathrm{Mub}_{\langle b,K\rangle}$ such that $f = f_{T_K}$. From the above proof, we get the almost algebraic sequence $((f_{T_K^n})_{n>n_0})$ of f as above. Since $\Uparrow f \subseteq \Uparrow g$, we have that $(a \searrow b) \ll f_{T_K^n}$ for all $n \geq n_0$; thus $b \ll f_{T_K^n}(a)$ for all $n \geq n_0$ by Lemma 4.2. Without losing generality, we assume $b \neq \perp$. By the definition of $f_{T_K^n}$, there are a nonempty $I \subseteq K$ and an $x_a \in \overline{M}_{\langle a,I\rangle}$ with $a \in \Uparrow x_a \setminus \bigcup_{i\in K\setminus I} \Uparrow a_i^n$ such that $f_{T_K^n}(a) = T_K^n(x_a)$. Hence, we have that

$$b \ll f_{T_K^n}(a) = \bigvee_{\downarrow f_{T_K^n}(a)} \{b_i^n : a_i^n \ll a\}.$$

Set

$$I_n = \{i \in K : a_i^n \ll a\}.$$

Clearly, $I_n \neq \emptyset$ and $I_{n+1} \subseteq I_n$ hold for all $n \geq n_0$. Hence there exists $n_1 \geq n_0$ such that if $n \geq n_1$ then $I_n = I_{n_1}$. This implies that $a \gg a_i^n$ for all $i \in I_{n_1}$ and $n \geq n_0$. So, we have $a \geq a_i$ for all $i \in I_{n_1}$ and $a \not\geq a_i$ for all $i \in K\setminus I_{n_1}$. It follows that $a \in \bigcap_{i\in I_{n_1}} \uparrow a_i$ and there is $m \in M_{\langle a,I\rangle}$ with $m \leq a$ and $\bigcap_{i\in I_{n_1}} \Uparrow a_i \neq \emptyset$. Obviously, $m \in \overline{M}_{\langle a,I_{n_1}\rangle}$ and $a \in \Uparrow r_{I_{n_1}}^n(m) \setminus \bigcup_{i\in K\setminus I_{n_1}} \Uparrow a_i^n$. Therefore,

$$f_{T_K^n}(a) = \bigvee_{\downarrow \overline{T_K}(m)} \{b_i^n : i \in I_{n_1}\} \gg b.$$

It is easy to see that $(f_{T_K^n}(a))_{n\geq n_0}$ is an almost algebraic sequence of $T_K(m)$. So we have

$$T_K(m) = \bigwedge_{n\geq n_0} f_{T_K^n}(a) \geq b.$$

For any $x \in \uparrow a$, we have $x \in \uparrow m$. Note that since $\uparrow m \setminus \bigcup_{i \in K \setminus I_{n_1}} \uparrow a_i \neq \emptyset$, by the assumption that B_D is almost algebraic we obtain $\hat{x} \in \uparrow m \setminus \bigcup_{i \in K \setminus I_{n_1}} \uparrow a_i$ such that $\hat{x} \ll x$. Hence,

$$f(x) \geq f(\hat{x}) = f_{T_K}(x) = T_K(m) \geq b.$$

This implies that $(a \searrow b) \leq f$. So we have proved claim 3.

Next, suppose that $g, f \in B_{[D \to E]}$ with $\uparrow f \subseteq \uparrow g$. Then there is a g.s.f. family $\{(a_i \searrow b_i) : i \in K\}$ such that $g \in \mathrm{mub}\{(a_i \searrow b_i) : i \in K\}$. Clearly, $\uparrow f \subseteq \uparrow g \subseteq \uparrow(a_i \searrow b_i)$ for all $i \in K$. By Claim 3, $(a_i \searrow b_i) \leq f$ for all $i \in K$. Pick $\hat{f} \in \uparrow f$, then $g \ll \hat{f}$. Hence, we have

$$\begin{aligned} g &= \bigvee_{\downarrow g} \{(a_i \searrow b_i) : i \in K\} \\ &= \bigvee_{\downarrow \hat{f}} \{(a_i \searrow b_i) : i \in K\} \\ &= \bigvee_{\downarrow f} \{(a_i \searrow b_i) : i \in K\} \\ &\leq f. \end{aligned}$$

□

From the above three propostions, the class of all ωAML-domains is closed under function spaces. Let L be an algebraic L-domain such that (1) the set of all compact elements is countable and has property $\mathcal{M}$, (2) L is not bounded complete. Then $[\mathbb{I} \to L]$ is an ωAML-domains, which is neither algebraic nor bounded complete. Hence, the notion of a $\mathcal{M}$-closed is strictly weaker than being closed. Since the finite cartesian products of ωAML-domains are ωAML-domains, we have the following result.

Theorem 5.6 *The category* ω**AML** *of* ωAML*-domains is cartesian closed.*

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