Generic Axiomatized Digital Surface-Structures

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Abstract

In digital topology, Euclidean n-space $\mathbb{R}^n$ is usually modeled either by the set of points of a discrete grid, or by the set of n-cells in a convex cell complex whose union is $\mathbb{R}^n$. For commonly used grids and complexes in the cases $n = 2$ and 3, certain pairs of adjacency relations (κ, λ) on the grid points or n-cells (such as (4,8) and (8,4) on $\mathbb{Z}^2$) are known to be "good pairs". For these pairs of relations (κ, λ), many results of digital topology concerning a set of grid points or n-cells and its complement (such as Rosenfeld's digital Jordan curve theorem) have versions in which κ-adjacency is used to define connectedness on the set and λ-adjacency is used to define connectedness on its complement. At present, results of 2D and 3D digital topology are usually proved for one good pair of adjacency relations at a time — so for each result there are different (but analogous) theorems for different good pairs of adjacency relations. In this paper we take the first steps in developing an alternative approach to digital topology based on very general axiomatic definitions of "well-behaved digital spaces". This approach gives the possibility of stating and proving results of digital topology as single theorems which apply to all spaces of the appropriate dimensionality that satisfy our axioms. Specifically, this paper introduces the notion of a *generic axiomatized digital surface-structure* (GADS) — a general, axiomatically defined, type of discrete structure that models subsets of the Euclidean plane and of other surfaces. Instances of this notion include GADS corresponding to all of the good pairs of adjacency relations that have previously been used (by ourselves or others) in digital topology on planar grids and boundary surfaces. We define basic concepts for a GADS (such as homotopy of paths and the intersection number of two paths), give a discrete definition of *planar* GADS (which are GADS that model subsets of the Euclidean plane) and present some fundamental results including a Jordan curve theorem for planar GADS.

1   Introduction

In digital topology, Euclidean n-space $\mathbb{R}^n$ is usually modeled either by the set of points of a discrete grid, or by the set of n-cells in a convex cell complex whose union is $\mathbb{R}^n$. Connectedness in Euclidean n-space is usually modeled by graph-theoretic notions of connectedness derived from adjacency relations defined on the grid points or n-cells.

For commonly used grids and complexes in the cases $n = 2$ and 3, certain pairs of adjacency relations (κ, λ) on the grid points or n-cells are known to be "good pairs". For these pairs of relations (κ, λ), many results of digital topology concerning a set of grid points or n-cells and its complement have versions in which κ-adjacency is used to define connectedness on the set and λ-adjacency is used to define connectedness on its complement.

For example, $(4, 8)$ and $(8, 4)$ are good pairs of adjacency relations on $\mathbb{Z}^2$. Thus Rosenfeld's digital Jordan curve theorem [10] is valid when one of 4- and 8-adjacency is used to define the sense in which a digital simple closed curve is connected and the other of the two adjacency relations is used to define connected components of the digital curve's complement. The theorem is not valid if the same one of 4- or 8-adjacency is used for both purposes: $(4, 4)$ and $(8, 8)$ are not good pairs on $\mathbb{Z}^2$.

Some adjacency relations form good pairs with themselves. An example of such a good pair is the pair $(6, 6)$ on the grid points of a 2D hexagonal grid. (The grid points are the centers of the hexagons in a tiling of the Euclidean plane by regular hexagons, and two points are 6-adjacent if they are the centers of hexagons that share an edge.) Another example is the good pair (κ_2, κ_2) on $\mathbb{Z}^2$, where κ_2 is Khalimsky's adjacency relation [6] on $\mathbb{Z}^2$, which is defined as follows: Say that a point of $\mathbb{Z}^2$ is *pure* if its coordinates are both even or both odd, and *mixed* otherwise. Then two points of $\mathbb{Z}^2$ are κ_2-adjacent if they are 4-adjacent, or if they are pure points and are 8-adjacent.

In three dimensions, $(6, 26)$, $(26, 6)$, $(6, 18)$, $(18, 6)$ are good pairs of adjacency relations on $\mathbb{Z}^3$. A different example of a good pair on $\mathbb{Z}^3$ is (κ_3, κ_3), where κ_3 is the 3D analog of κ_2: Two points of $\mathbb{Z}^3$ are κ_3-adjacent if they are 6-adjacent, or if they are 26-adjacent and at least one of the two is a pure point, where a *pure* point is a point whose coordinates are all odd or all even. $(12, 12)$, $(12, 18)$ and $(18, 12)$ are good pairs of adjacency relations [6] on the points of a 3D face-centered cubic grid (e.g., on $\{(x, y, z) \in \mathbb{Z}^3 \mid x + y + z \equiv 0 (\text{mod } 2)\}$) and $(14, 14)$ is a good pair on the points of a 3D body-centered cubic grid (e.g., on $\{(x, y, z) \in \mathbb{Z}^3 \mid x \equiv y \equiv z (\text{mod } 2)\}$).

At present, results of 2D and 3D digital topology are usually proved for one good pair of adjacency relations at a time, and the details of the proof

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[6] If α is an irreflexive symmetric binary relation on the set G of all points of a Cartesian or non-Cartesian grid, then α is referred to as the *k-adjacency* relation on G, and is denoted by the positive integer k, if for all $p \in G$ the set $\{q \in G \mid p \, \alpha \, q\}$ contains just k points and they are all strictly closer to p (in Euclidean distance) than is any other point of $G \setminus \{p\}$.

may be significantly different for different good pairs. In the case of 3D grids, even if we consider only the nine good pairs of adjacency relations mentioned above, a result such as a digital Jordan surface theorem would be expected to have nine different versions with nine separate proofs!

This state of affairs seems to us to be unsatisfactory. We have begun to consider an alternative approach to digital topology, in which "well-behaved" digital spaces are defined axiomatically, using axioms that are general enough to admit digital spaces which correspond to the good pairs of adjacency relations mentioned above. This approach allows a result of 2D or 3D digital topology to be proved as a *single* theorem for all well-behaved spaces that satisfy appropriate hypotheses. (Our Jordan curve theorem, Theorem 4.7 below, illustrates this.)

In this paper we confine our attention to digital spaces that model subsets of the Euclidean plane and other surfaces, and give an axiomatic definition of a very general class of such spaces, which includes spaces corresponding to all of the good pairs of adjacency relations that have been used in the literature on 2D digital topology (both in the plane and on boundary surfaces). A space that satisfies our axiomatic definition is called a GADS. As will be seen in Section 2.5, a substantial part of the mathematical framework used in our definition of a GADS has previously been used by the third author [4,5].

As first steps in the development of digital topology for these spaces, we define the intersection number of two paths on a GADS, and outline a proof that the number is invariant under homotopic deformation of the two paths. This is mostly a generalization, to arbitrary GADS, of definitions and theorems given by the first author and Malgouyres in [1,2,3]. We also give a (discrete) definition of *planar* GADS, which model subsets of the Euclidean plane, and present a Jordan curve theorem for such GADS. In contrast to some earlier work by the second author (e.g., [7,8,9]), this paper does not use any arguments that are based on polyhedral continuous analogs of digital spaces, but uses only discrete arguments.

2   GADS and pGADS

2.1   *Basic Concepts and Notations*

For any set P we denote by $P^{\{2\}}$ the set of all unordered pairs of distinct elements of P (equivalently, the set of all subsets of P with exactly two elements). Let P be any set and let $\rho \subseteq P^{\{2\}}$.[7] Two elements a and b of P [respectively, two subsets A and B of P] are said to be *ρ-adjacent* if $\{a, b\} \in \rho$ [respectively, if there exist $a \in A$ and $b \in B$ with $\{a, b\} \in \rho$]. If $x \in P$ we denote by $N_\rho(x)$ the set of elements of P which are ρ-adjacent to x; these elements are also called the *ρ-neighbors* of x. We call $N_\rho(x)$ the *punctured*

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[7] ρ can be viewed as a binary, symmetric and irreflexive relation on P, and (P, ρ) as an undirected simple graph.

ρ-*neighborhood* of x.

A ρ-*path* from $a \in P$ to $b \in P$ is a finite sequence $(x_0, \ldots, x_l)$ of one or more elements of P such that $x_0 = a$, $x_l = b$ and, for all $i \in \{0, \ldots, l-1\}$, $\{x_i, x_{i+1}\} \in \rho$. The nonnegative integer l is the *length* of the path. A ρ-path of length 0 is called a *one-point path*. For all integers m, n, $0 \le m \le n \le l$, the subsequence $(x_m, \ldots, x_n)$ of $(x_0, \ldots, x_l)$ is called an *interval* or *segment* of the path. For all $i \in \{1, \ldots, l\}$ we say that the elements x_{i-1} and x_i are *consecutive* on the path, and also that x_{i-1} *precedes* x_i and x_i *follows* x_{i-1} on the path. Note that consecutive elements of a ρ-path can never be equal.

A ρ-path $(x_0, \ldots, x_l)$ is said to be *simple* if $x_i \ne x_j$ for all distinct i and j in $\{0, \ldots, l\}$. It is said to be *closed* if $x_0 = x_l$, so that x_0 follows x_{l-1}. It is called a ρ-*cycle* if it is closed and $x_i \ne x_j$ for all distinct i and j in $\{1, \ldots, l\}$. One-point paths are the simplest ρ-cycles. Two ρ-cycles $c_1 = (x_0, \ldots, x_l)$ and $c_2 = (y_0, \ldots, y_l)$ are said to be *equivalent* if there exists an integer k, $0 \le k \le l-1$, such that $x_i = y_{(i+k) \bmod l}$ for all $i \in \{0, \ldots, l\}$.

If $S \subseteq P$, two elements a and b of S are said to be ρ-*connected in S* if there exists a ρ-path from a to b that consists only of points in S. ρ-connectedness in S is an equivalence relation on S; its equivalence classes are called the ρ-*components of S*. The set S is said to be ρ-*connected* if there is just one ρ-component of S.

Given two sequences $c_1 = (x_0, \ldots, x_m)$ and $c_2 = (y_0, \ldots, y_n)$ such that $x_m = y_0$, we denote by $c_1.c_2$ the sequence $(x_0, \ldots, x_m, y_1, \ldots, y_n)$, which we call the *catenation of c_1 and c_2*. Whenever we use the notation $c_1.c_2$, we are also implicitly saying that the last element of c_1 is the same as the first element of c_2. It is clear that if c_1 and c_2 are ρ-paths of lengths l_1 and l_2, then $c_1.c_2$ is a ρ-path of length $l_1 + l_2$.

For any sequence $c = (x_0, \ldots, x_m)$, the *reverse of c*, denoted by c^{-1}, is the sequence $(y_0, \ldots, y_m)$ such that $y_k = x_{m-k}$ for all $k \in \{0, \ldots, m\}$. It is clear that if c is a ρ-path of length l then so is c^{-1}.

A *simple closed ρ-curve* is a nonempty finite ρ-connected set C such that each element of C has exactly two ρ-neighbors in C. (Note that a simple closed ρ-curve must have at least three elements.) A ρ-cycle c of length $|C|$ that contains every element of a simple closed ρ-curve C is called a ρ-*parameterization of C*. Note that if c and c' are ρ-parameterizations of a simple closed ρ-curve C, then c' is equivalent to c or to c^{-1}.

If x and y are ρ-adjacent elements of a simple closed ρ-curve C, then we may say that x and y are ρ-*consecutive* on C. If x and y are distinct elements of a simple closed ρ-curve C that are not ρ-consecutive on C, then each of the two ρ-components of $C \setminus \{x, y\}$ is called a ρ-*cut-interval* (of C) associated with x and y.

2.2 Definition of a GADS

Definition 2.1 (2D digital complex) *A* 2D digital complex *is an ordered triple* $(V, \pi, \mathcal{L})$, *where*

- *V is a set whose elements are called* vertices *or* spels,
- *$\pi \subseteq V^{\{2\}}$, and the pairs of vertices in π are called* proto-edges,
- *$\mathcal{L}$ is a set of simple closed π-curves whose members are called* loops,

and the following four conditions hold:

(i) *V is π-connected and contains more than one vertex.*

(ii) *For any two distinct loops L_1 and L_2, $L_1 \cap L_2$ is either empty, or consists of a single vertex, or is a proto-edge.*

(iii) *No proto-edge is included in more than two loops.*

(iv) *Each vertex belongs to only a finite number of proto-edges.*

When specifying a 2D digital complex whose vertex set is the set of points of a grid in $\mathbb{R}^n$, a positive integer k (such as 4, 8 or 6) may be used to denote the set of all unordered pairs of k-adjacent vertices. We write $\mathcal{L}_{2\times 2}$ to denote the set of all unit lattice squares in $\mathbb{Z}^2$. The triple $(\mathbb{Z}^2, 4, \mathcal{L}_{2\times 2})$ is a simple example of a 2D digital complex.

Definition 2.2 (GADS) *A* generic axiomatized digital surface-structure, *or* GADS, *is a pair* $\mathcal{G} = ((V, \pi, \mathcal{L}), (\kappa, \lambda))$ *where* $(V, \pi, \mathcal{L})$ *is a 2D digital complex (whose vertices, proto-edges and loops are also referred to as vertices, proto-edges and loops of $\mathcal{G}$) and where κ and λ are subsets of $V^{\{2\}}$ that satisfy Axioms 1, 2 and 3 below. The pairs of vertices in κ and λ are called κ-edges and λ-edges, respectively. $(V, \pi, \mathcal{L})$ is called the* underlying complex *of $\mathcal{G}$.*

Axiom 1 *Every proto-edge is both a κ-edge and a λ-edge: $\pi \subseteq \kappa \cap \lambda$.*

Axiom 2 *For all $e \in (\kappa \cup \lambda) \setminus \pi$, some loop contains both vertices of e.*

Axiom 3 *If $x, y \in L \in \mathcal{L}$, but x and y are not π-consecutive on L, then*

(a) *$\{x, y\}$ is a λ-edge if and only if $L \setminus \{x, y\}$ is not κ-connected.*

(b) *$\{x, y\}$ is a κ-edge if and only if $L \setminus \{x, y\}$ is not λ-connected.*

Regarding Axiom 2, note that if $e \in (\kappa \cup \lambda) \setminus \pi$ (i.e., e is a κ- or λ-edge that is not a proto-edge) then there can only be one loop that contains both vertices of e, by condition (ii) in the definition of a 2D digital complex.

As illustrations of Axiom 3, observe that both $((\mathbb{Z}^2, 4, \mathcal{L}_{2\times 2}), (4, 8))$ and $((\mathbb{Z}^2, 4, \mathcal{L}_{2\times 2}), (8, 4))$ satisfy Axiom 3, but $((\mathbb{Z}^2, 4, \mathcal{L}_{2\times 2}), (4, 4))$ violates the "if" parts of the axiom, while $((\mathbb{Z}^2, 4, \mathcal{L}_{2\times 2}), (8, 8))$ violates the "only if" parts of the axiom.

A GADS is said to be *finite* if it has finitely many vertices; otherwise it is said to be *infinite*. The set of all GADS can be ordered as follows:

Definition 2.3 ($\subseteq$ order, subGADS) *Let $\mathcal{G} = ((V, \pi, \mathcal{L}), (\kappa, \lambda))$ and $\mathcal{G}' =$*

$((V', \pi', \mathcal{L}'), (\kappa', \lambda'))$ *be* GADS *such that*

- $V \subseteq V'$, $\pi \subseteq \pi'$ *and* $\mathcal{L} \subseteq \mathcal{L}'$.
- *For all* $L \in \mathcal{L}$, $\kappa \cap L^{\{2\}} = \kappa' \cap L^{\{2\}}$ *and* $\lambda \cap L^{\{2\}} = \lambda' \cap L^{\{2\}}$.

Then we write $\mathcal{G} \subseteq \mathcal{G}'$ *and say that* $\mathcal{G}$ *is a* subGADS *of* $\mathcal{G}'$. *We also refer to* $\mathcal{G}$ *as* the subGADS *of* $\mathcal{G}'$ *induced by* $(V, \pi, \mathcal{L})$. *We write* $\mathcal{G} \subsetneq \mathcal{G}'$ *to mean* $\mathcal{G} \subseteq \mathcal{G}'$ *and* $\mathcal{G} \neq \mathcal{G}'$. *We write* $\mathcal{G} < \mathcal{G}'$ *to mean* $\mathcal{G} \subsetneq \mathcal{G}'$ *and* $\mathcal{L} \neq \mathcal{L}'$.

The following simple but important property of GADS is an immediate consequence of the symmetry of Axioms 1, 2 and 3 with respect to κ and λ:

Property 2.4 *If* $((V, \pi, \mathcal{L}), (\kappa, \lambda))$ *is a* GADS *then* $((V, \pi, \mathcal{L}), (\lambda, \kappa))$ *is also a* GADS. *So any statement which is true of every* GADS $((V, \pi, \mathcal{L}), (\kappa, \lambda))$ *remains true when* κ *is replaced by* λ *and* λ *by* κ.

2.3 Interior Vertices and pGADS

We are particularly interested in those GADS that model a surface without boundary. The next definition gives a name for any such GADS.
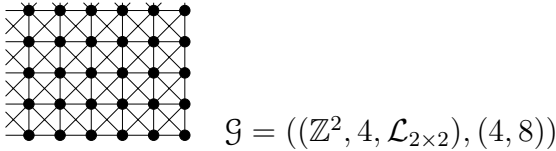
Definition 2.5 (pGADS) *A* pGADS *is a* GADS *in which every proto-edge is included in two loops. (The* p *in* pGADS *stands for* pseudomanifold.*)*

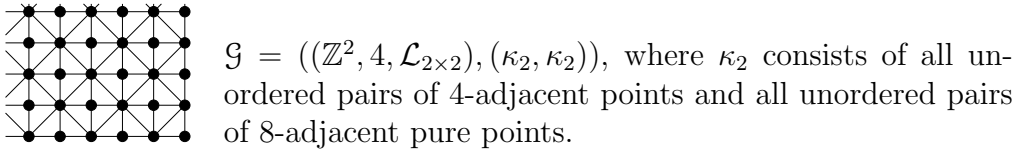A finite pGADS models a closed surface. A pGADS that models the Euclidean plane must be infinite.

A vertex v of a GADS $\mathcal{G}$ is called an *interior vertex* of $\mathcal{G}$ if every proto-edge of $\mathcal{G}$ that contains v is included in two loops of $\mathcal{G}$. It follows that a GADS $\mathcal{G}$ is a pGADS if and only if every vertex of $\mathcal{G}$ is an interior vertex.

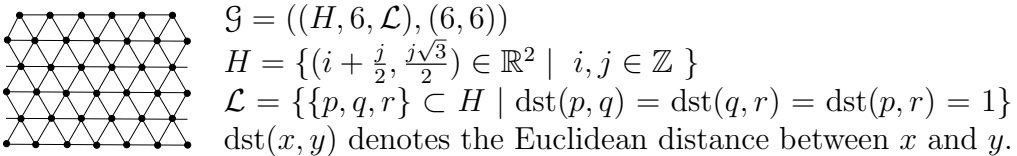Below are pictures of some pGADS.

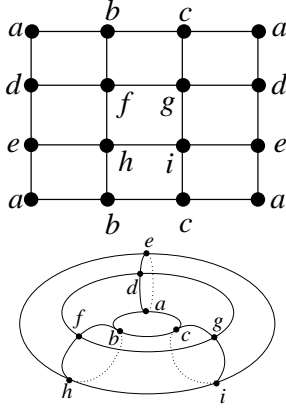**Example 2.6** $\mathbb{Z}^2$ *with the 4- and 8-adjacency relations*

$\mathcal{G} = ((\mathbb{Z}^2, 4, \mathcal{L}_{2 \times 2}), (4, 8))$

Example 2.7 $\mathbb{Z}^2$ *with Khalimsky's adjacency relation*

$\mathcal{G} = ((\mathbb{Z}^2, 4, \mathcal{L}_{2 \times 2}), (\kappa_2, \kappa_2))$, where κ_2 consists of all unordered pairs of 4-adjacent points and all unordered pairs of 8-adjacent pure points.

Example 2.8 *The hexagonal grid with the 6-adjacency relation*

$\mathcal{G} = ((H, 6, \mathcal{L}), (6, 6))$
$H = \{(i + \frac{j}{2}, \frac{j\sqrt{3}}{2}) \in \mathbb{R}^2 \mid i, j \in \mathbb{Z}\}$
$\mathcal{L} = \{\{p, q, r\} \subset H \mid \mathrm{dst}(p, q) = \mathrm{dst}(q, r) = \mathrm{dst}(p, r) = 1\}$
$\mathrm{dst}(x, y)$ denotes the Euclidean distance between x and y.

Example 2.9 *A torus-like* pGADS

$$\mathcal{G} = ((V, \kappa, \mathcal{L}), (\kappa, \lambda))$$
$$V = \{a, b, c, d, e, f, g, h, i\}$$
$$\kappa = \{\{a,b\}, \{b,c\}, \{c,a\}, \{d,f\}, \{f,g\}, \{g,d\},$$
$$\{e,h\}, \{h,i\}, \{i,e\}, \{b,f\}, \{c,g\}, \{a,d\},$$
$$\{f,h\}, \{g,i\}, \{d,e\}, \{h,b\}, \{i,c\}, \{e,a\}\}$$
$$\lambda = \{\{x,y\} \mid \exists L \in \mathcal{L}, \ x, y \in L\} \quad \text{(not shown)}$$
$$\mathcal{L} = \{\{a,b,f,d\}, \{d,f,h,e\}, \{e,h,b,a\},$$
$$\{b,c,g,f\}, \{f,g,i,h\}, \{h,i,c,b\},$$
$$\{c,a,d,g\}, \{g,d,e,i\}, \{i,e,a,c\}\}$$

2.4 Strong Connectedness and Singularities

Let $\mathcal{G} = ((V, \pi, \mathcal{L}), (\kappa, \lambda))$ be a GADS. Two loops L and L' of $\mathcal{G}$ are said to be *adjacent* if $L \cap L'$ is a proto-edge of $\mathcal{G}$. A subset $\mathcal{L}'$ of $\mathcal{L}$ is said to be *strongly connected* if for any two loops L and L' in $\mathcal{L}'$, there exists a sequence $L_0, \ldots, L_n$ of loops in $\mathcal{L}'$ such that $L_0 = L$, $L_n = L'$ and, for all $i \in \{0, \ldots, n-1\}$, L_i and L_{i+1} are adjacent. $\mathcal{G}$ is said to be *strongly connected* if $\mathcal{L}$ is strongly connected. (So whether or not $\mathcal{G}$ is strongly connected depends only on the underlying complex of $\mathcal{G}$.)

A vertex x of $\mathcal{G}$ is said to be a *singularity* of $\mathcal{G}$ if the set of all loops of $\mathcal{G}$ that contain x is not strongly connected. Vertices that are not singularities are said to be *nonsingular*. Again, whether or not x is a singularity of $\mathcal{G}$ depends only on the underlying complex of $\mathcal{G}$.

Even a strongly connected pGADS may have a singularity. For example, the pGADS obtained from the torus-like pGADS of Example 2.9 above by identifying the vertices a, b and c has a singularity at $a = b = c$ but is strongly connected.

2.5 Relationship to the Mathematical Framework of [4,5]

Here we briefly discuss the relationship between our concept of a GADS and digital structures previously studied by the third author in [4,5].

If $((V, \pi, \mathcal{L}), (\kappa, \lambda))$ is a GADS, then, in the terminology of [5], (V, π) is a *digital space*, π is the *proto-adjacency* of that space, and each of κ and λ is a *spel-adjacency* of the space. The principal new ingredients in our concept of a GADS are the set of loops $\mathcal{L}$ and Axioms 2 and 3. In a GADS $((V, \pi, \mathcal{L}), (\kappa, \lambda))$ with the property that every simple closed π-path of length 4 is a loop of the GADS, the "if" parts of Axiom 3 make $\{\kappa, \lambda\}$ a *normal* pair of spel-adjacencies.

An important difference between our theory and that of [4,5] is that our theory is restricted to spaces that model subsets of surfaces (though only because of condition (iii) in the definition of a 2D digital complex).

3   Homotopic Paths and Simple Connectedness

In this section $\mathcal{G} = ((V, \pi, \mathcal{L}), (\kappa, \lambda))$ is a GADS, ρ satisfies $\pi \subseteq \rho \subseteq \kappa \cup \lambda$, and X is a ρ-connected subset of V. (We are mainly interested in the cases where $\rho = \kappa, \lambda$ or π.)

Loosely speaking, two ρ-paths in X with the same initial and the same final vertices are said to be ρ-*homotopic* within X in $\mathcal{G}$ if one of the paths can be transformed into the other by a sequence of small local deformations within X. The initial and final vertices of the path must remain fixed throughout the deformation process. The next two definitions make this notion precise.

Definition 3.1 (elementary $\mathcal{G}$-deformation) *Two finite vertex sequences c and c' of $\mathcal{G}$ with the same initial and the same final vertices are said to be* the same up to an elementary $\mathcal{G}$-deformation *if there exist vertex sequences c_1, c_2, γ and γ' such that $c = c_1.\gamma.c_2$, $c' = c_1.\gamma'.c_2$, and either there is a proto-edge $\{x, y\}$ for which one of γ and γ' is (x) and the other is (x, y, x), or there is a loop of $\mathcal{G}$ that contains all of the vertices in γ and γ'.*

Definition 3.2 (homotopic ρ-paths) *Two ρ-paths c and c' in X with the same initial and the same final vertices are ρ-homotopic within X in $\mathcal{G}$ if there exists a sequence of ρ-paths $c_0, \ldots, c_n$ in X such that $c_0 = c$, $c_n = c'$ and, for $0 \leq i \leq n - 1$, c_i and c_{i+1} are the same up to an elementary $\mathcal{G}$-deformation. Two ρ-paths with the same initial and the same final vertices are said to be ρ-homotopic in $\mathcal{G}$ if they are ρ-homotopic within V in $\mathcal{G}$.*

The next proposition states a useful characterization of ρ-homotopy that is based on a more restrictive kind of local deformation than was considered above, which allows only the insertion or removal of either a "ρ-back-and-forth" or a cycle that parameterizes a simple closed ρ-curve in a loop of $\mathcal{G}$.

Definition 3.3 (minimal ρ-deformation) *Two ρ-paths c and c' with the same initial and the same final vertices are said to be* the same up to a minimal ρ-deformation in $\mathcal{G}$ *if there exist ρ-paths c_1, c_2 and γ such that one of c and c' is $c_1.\gamma.c_2$, the other of c and c' is $c_1.c_2$, and either $\gamma = (x, y, x)$ for some ρ-edge $\{x, y\}$ or γ is a ρ-parameterization of a simple closed ρ-curve whose vertices are contained in a single loop of $\mathcal{G}$.*

This concept of deformation is particularly simple when $\rho = \pi$, because a simple closed π-curve whose vertices are contained in a single loop of $\mathcal{G}$ must in fact be a loop of $\mathcal{G}$, since a loop of $\mathcal{G}$ is a simple closed π-curve.

Proposition 3.4 *Two ρ-paths c and c' in X with the same initial and the same final vertices are ρ-homotopic within X in $\mathcal{G}$ if and only if there is a sequence of ρ-paths $c_0, \ldots, c_n$ in X such that $c_0 = c$, $c_n = c'$ and, for $0 \leq i \leq n - 1$, c_i and c_{i+1} are the same up to a minimal ρ-deformation in $\mathcal{G}$.*

The proof of this proposition is not particularly difficult, and we leave it to the interested reader.

Definition 3.5 (reducible closed path) *Let $c = (x_0, \ldots, x_n)$ be a closed*

ρ-path in X (so $x_n = x_0$). Then c is said to be ρ-reducible within X in $\mathcal{G}$ if c and the one-point path (x_0) are ρ-homotopic within X in $\mathcal{G}$. We say c is ρ-reducible in $\mathcal{G}$ if c is ρ-reducible within V in $\mathcal{G}$.

Definition 3.6 (simple connectedness) *The set X is said to be ρ-simply connected in $\mathcal{G}$ if every closed ρ-path in X is ρ-reducible within X in $\mathcal{G}$. The* GADS *$\mathcal{G}$ is said to be* simply connected *if V is π-simply connected in $\mathcal{G}$.*

Whether or not a GADS is simply connected depends only on its underlying complex. If $\mathcal{G}$ is simply connected then V is ρ-simply connected in $\mathcal{G}$ for any ρ such that $\pi \subseteq \rho \subseteq \kappa \cup \lambda$. This is because $\pi \subseteq \rho \subseteq \kappa \cup \lambda$ implies that for any ρ-path c there is a π-path c' such that c and c' are ρ-homotopic in $\mathcal{G}$, and $\pi \subseteq \rho$ implies that a π-reducible π-path is also a ρ-reducible ρ-path.

The final result in this section gives a useful sufficient condition for a GADS to have no singularities:

Proposition 3.7 *Let $\mathcal{G}$ be a* GADS *that is both simply connected and strongly connected. Then $\mathcal{G}$ has no singularities.*

Proof: Let $\mathcal{G} = ((V, \pi, \mathcal{L}), (\kappa, \lambda))$ and suppose x is a singularity of $\mathcal{G}$. Then there exist two nonempty sets of loops of $\mathcal{G}$, $\alpha_1 = \{L_1, \ldots, L_i\}$ and $\alpha_2 = \{L_{i+1}, \ldots, L_l\}$, such that $\{L_1, \ldots, L_l\}$ is the set of all loops of $\mathcal{G}$ that contain x, and such that $L \cap L' = \{x\}$ for all L in α_1 and L' in α_2.

For any π-path $c = (c_0, c_1, \ldots, c_n)$, let $\nu(c, x)$ be the number of pairs (c_i, c_{i+1}) for which c_i belongs to a loop in α_1 and $c_{i+1} = x$, minus the number of pairs (c_i, c_{i+1}) for which $c_i = x$ and c_{i+1} belongs to a loop in α_1. It is easy to verify that if c' and c'' are two π-paths which are the same up to a minimal π-deformation in $\mathcal{G}$ then $\nu(c', x) = \nu(c'', x)$. So, since $\mathcal{G}$ is simply connected, $\nu(c, x) = 0$ for every closed π-path c (by Proposition 3.4).

Now let y be a π-neighbor of x that belongs to a loop in α_1, and let z be a π-neighbor of x that belongs to a loop in α_2. Since $\mathcal{G}$ is strongly connected, there must be a π-path c' from z to y that does not contain x. But the closed π-path $c = (x, z).c'.(y, x)$ would satisfy $\nu(c, x) = 1$, a contradiction. $\square$

4 Planar GADS and a Jordan Curve Theorem

In this section we define a class of GADS that are discrete models of subsets of the Euclidean plane. The definition depends on two concepts which we now present:

Definition 4.1 (Euler number of a GADS) *Let $\mathcal{G} = ((V, \pi, \mathcal{L}), (\kappa, \lambda))$ be a* finite GADS. *Then the integer $|V| - |\pi| + |\mathcal{L}|$ is called the* Euler number *of $\mathcal{G}$, and is denoted by $\chi(\mathcal{G})$.*

Note that the Euler number of a GADS depends only on the underlying complex, and that it is not defined for an infinite GADS.

Definition 4.2 (limit of an increasing GADS sequence) *For all $i \in \mathbb{N}$*

let $\mathcal{G}_i = ((V_i, \pi_i, \mathcal{L}_i), (\kappa_i, \lambda_i))$ be a GADS and let $\mathcal{G}_0 \subseteq \mathcal{G}_1 \subseteq \mathcal{G}_2 \subseteq \ldots$. Then $\bigcup_{i \in \mathbb{N}} \mathcal{G}_i$ denotes $((\bigcup_{i \in \mathbb{N}} V_i, \bigcup_{i \in \mathbb{N}} \pi_i, \bigcup_{i \in \mathbb{N}} \mathcal{L}_i), (\bigcup_{i \in \mathbb{N}} \kappa_i, \bigcup_{i \in \mathbb{N}} \lambda_i))$, which is a GADS if each element of $\bigcup_{i \in \mathbb{N}} V_i$ is contained in only finitely many distinct members of $\bigcup_{i \in \mathbb{N}} \pi_i$.

We are now in a position to define a planar GADS. Whether or not a GADS is planar depends only on its underlying complex, as can be deduced quite easily from the following definition.

Definition 4.3 (planar GADS) *A* pGADS $\mathcal{PG}$ *is said to be* planar *if* $\mathcal{PG} = \bigcup_{i \in \mathbb{N}} \mathcal{G}_i$ *for some infinite sequence of finite* GADS $\mathcal{G}_0 < \mathcal{G}_1 < \mathcal{G}_2 < \ldots$ *such that* $\mathcal{G}_i$ *is strongly connected and* $\chi(\mathcal{G}_i) = 1$ *for all* $i \in \mathbb{N}$. *A* GADS $\mathcal{G}$ *is said to be* planar *if there exists a planar* pGADS $\mathcal{PG}$ *such that* $\mathcal{G} \subseteq \mathcal{PG}$.

It is evident that all planar pGADS are infinite and strongly connected. A somewhat less obvious property of planar pGADS is that they are all simply connected. This follows quite easily from:

Proposition 4.4 *Let* $\mathcal{G}$ *be a strongly connected* GADS *and let* $\mathcal{G}'$ *be a finite* GADS *such that* $\mathcal{G}' < \mathcal{G}$ *and* $\chi(\mathcal{G}') = 1$. *Then* $\mathcal{G}'$ *is simply connected.*

Sketch of proof: Let $\mathcal{G}' = ((V', \pi', \mathcal{L}'), (\kappa', \lambda'))$. In the case where $\mathcal{L}' = \emptyset$, $|V'| - |\pi'| = \chi(\mathcal{G}') = 1$ and so (V', π') is a tree. In this case the result is easily proved by induction on the number of proto-edges. To prove the result in the case where $\mathcal{G}'$ has at least one loop, we use induction on the number of loops. [The induction step is based on the easily established fact that, since $\mathcal{G}' < \mathcal{G}$ and $\mathcal{G}$ is strongly connected, there must exist a proto-edge e of $\mathcal{G}'$ that belongs to just one loop of $\mathcal{G}'$, L say. Readily, $\mathcal{G}'$ is simply connected if the subGADS of $\mathcal{G}$ induced by $(V', \pi' \setminus \{e\}, \mathcal{L}' \setminus \{L\})$ is simply connected.] □

Corollary 4.5 *A planar* pGADS *is simply connected.*

Proof: Let $\mathcal{PG} = ((V^*, \pi^*, \mathcal{L}^*), (\kappa^*, \lambda^*))$ be a planar pGADS, and suppose c^* is a π^*-path that is not π^*-reducible in $\mathcal{PG}$. By the definition of a planar pGADS, there exists a GADS $\mathcal{G}' = ((V', \pi', \mathcal{L}'), (\kappa', \lambda'))$ which satisfies the hypotheses of the above proposition when $\mathcal{G} = \mathcal{PG}$, such that c^* is a π'-path of $\mathcal{G}'$. Since c^* is not π^*-reducible in $\mathcal{PG}$, c^* is not π'-reducible in $\mathcal{G}'$, which contradicts the proposition. □

As a consequence of this corollary and Proposition 3.7, we deduce:

Proposition 4.6 *A planar* pGADS *has no singularities.*

The next theorem is our main result concerning planar GADS. It generalizes Rosenfeld's digital Jordan curve theorem [10] (for $\mathbb{Z}^2$ with (4,8) or (8,4) adjacencies) to every planar GADS. We will outline a proof of this theorem in Section 8.

Theorem 4.7 (Jordan curve theorem) *Let* $\mathcal{PG} = ((V, \pi, \mathcal{L}), (\kappa, \lambda))$ *be a planar* GADS. *Let* C *be a simple closed* κ-*curve that is not included in any loop of* $\mathcal{PG}$, *and which consists entirely of interior points of* $\mathcal{PG}$. *Then* $V \setminus C$

has exactly two λ-components, and, for each vertex $x \in C$, $N_\lambda(x)$ intersects both λ-components of $V \setminus C$.

5 Local Orientations and Orientability

5.1 Definitions

Let L_1 and L_2 be adjacent loops of a GADS $\mathcal{G} = ((V, \pi, \mathcal{L}), (\kappa, \lambda))$ and let $\{x, y\} = L_1 \cap L_2$. Then π-parameterizations c_1 of L_1 and c_2 of L_2 are said to be *coherent* if x precedes y in one of c_1 and c_2 but x follows y in the other of c_1 and c_2. A *coherent π-orientation* of a set of loops $\mathcal{L}' \subseteq \mathcal{L}$ is a function Ω with domain $\mathcal{L}'$ such that:

(i) For each loop L in $\mathcal{L}'$, $\Omega(L)$ is a π-parameterization of L.

(ii) For all pairs of adjacent loops L and L' in $\mathcal{L}'$, the π-parameterizations $\Omega(L)$ and $\Omega(L')$ of L and L' are coherent.

Two coherent π-orientations Ω_1 and Ω_2 of $\mathcal{L}'$ are said to be *equivalent* if, for every L in $\mathcal{L}'$, $\Omega_1(L)$ and $\Omega_2(L)$ are equivalent π-parameterizations of L.

A *coherent orientation of $\mathcal{G}$* is a coherent π-orientation of the set $\mathcal{L}$ of all loops of $\mathcal{G}$. The GADS $\mathcal{G}$ is said to be *orientable* if it has a coherent orientation. Evidently, if $\mathcal{G}'$ and $\mathcal{G}$ are GADS such that $\mathcal{G}' \subseteq \mathcal{G}$ and $\mathcal{G}$ is orientable, then $\mathcal{G}'$ is also orientable. Note that whether or not a GADS is orientable depends only on its underlying complex. It is easy to verify that the four pGADS shown in the diagrams of Section 2.3 are all examples of orientable GADS.

5.2 The Cycle $\mathcal{N}_{\Omega, y}(x)$ Around a Nonsingular Interior Vertex x of a GADS

Let $\mathcal{G} = ((V, \pi, \mathcal{L}), (\kappa, \lambda))$ be a (not necessarily orientable) GADS, and let x be a nonsingular interior vertex of $\mathcal{G}$.

A *loop-circuit* of $\mathcal{G}$ is a sequence $(L_0, \ldots, L_{l-1})$ of loops of $\mathcal{G}$ such that, for all $i \in \{0, \ldots, l-1\}$, L_i is adjacent to $L_{(i+1) \bmod l}$. A *loop-circuit of $\mathcal{G}$ at x* is a loop-circuit of $\mathcal{G}$ that is an enumeration of the set of loops of $\mathcal{G}$ that contain x (with each of those loops occurring just once). Thus if $(L_0, \ldots, L_{l-1})$ is a loop-circuit of $\mathcal{G}$ at x then, for each $i \in \{0, \ldots l-1\}$, $L_i \cap L_{(i+1) \bmod l}$ is a proto-edge of $\mathcal{G}$ that contains x (by condition (ii) in the definition of a 2D digital complex).

The set of loops of $\mathcal{G}$ that contain x is strongly connected (since x is nonsingular in $\mathcal{G}$), and it is easy to show that each loop in the set is adjacent to exactly two others (since x is an interior vertex of $\mathcal{G}$). Therefore a loop-circuit of $\mathcal{G}$ at x exists.

A *coherent local orientation of $\mathcal{G}$ at x* is a coherent π-orientation of the loops of $\mathcal{G}$ that contain x. Let $\Lambda = (L_0, \ldots, L_{l-1})$ be a loop-circuit of $\mathcal{G}$ at x. Then the *coherent local orientation of $\mathcal{G}$ at x induced by Λ*, denoted by Ω_x^Λ, is defined as follows. For $0 \le i \le l-1$ let c_i be a π-parameterization of L_i that begins and ends at x, in which the second vertex is the vertex of $L_i \cap$

$L_{(i-1) \bmod l} \setminus \{x\}$, and the second-last vertex is the vertex of $L_i \cap L_{(i+1) \bmod l} \setminus \{x\}$. Then Ω_x^Λ is defined by $\Omega_x^\Lambda(L_i) = c_i$ for $0 \le i \le l - 1$.

Now let Ω_x' be any coherent local orientation of $\mathcal{G}$ at x. Then $\Omega_x'(L_0)$ is equivalent either to $\Omega_x^\Lambda(L_0)$ or to $(\Omega_x^\Lambda(L_0))^{-1}$. It is readily confirmed that Ω_x' must be equivalent to Ω_x^Λ in the former case and to $\Omega_x^{\Lambda^{-1}}$ in the latter case.

For any vertex v of $\mathcal{G}$, the *punctured loop neighborhood* of v in $\mathcal{G}$, denoted by $N_{\mathcal{L}}(v)$, is defined to be the union of all the loops of $\mathcal{G}$ which contain v, minus the vertex v itself.

Let Ω_x be a coherent local orientation of $\mathcal{G}$ at x. For each vertex y of $N_{\mathcal{L}}(x)$, we now define a π-cycle $\mathcal{N}_{\Omega_x,y}(x)$ with the following properties:

(i) The vertices of $\mathcal{N}_{\Omega_x,y}(x)$ are exactly the vertices of $N_{\mathcal{L}}(x)$.

(ii) $\mathcal{N}_{\Omega_x,y}(x)$ begins and ends at y.

Let $\Lambda = (L_0, \ldots, L_{l-1})$ be a loop-circuit of $\mathcal{G}$ at x such that Ω_x^Λ is equivalent to Ω_x. For $i \in \{0, \ldots, l-1\}$ let p_i be the π-path obtained from $\Omega_x^\Lambda(L_i)$ by removing its first and last vertices (both of which are the vertex x). Then we define $\mathcal{N}_{\Omega_x,y}(x)$ to be the π-cycle that is equivalent to the π-cycle $p_0.p_1.\ldots.p_{l-1}$ and which begins and ends at y. It is readily confirmed that this π-cycle is independent of our choice of the loop-circuit Λ (provided that Ω_x^Λ is equivalent to Ω_x). If $\mathcal{G}$ is orientable, and Ω is a coherent orientation of $\mathcal{G}$, then we write $\mathcal{N}_{\Omega,y}(x)$ for $\mathcal{N}_{\Omega_x,y}(x)$, where Ω_x is the coherent local orientation of $\mathcal{G}$ at x that is given by the restriction of Ω to the loops of $\mathcal{G}$ that contain x. The definition of $\mathcal{N}_{\Omega,y}(x)$ is illustrated by Figure 1.
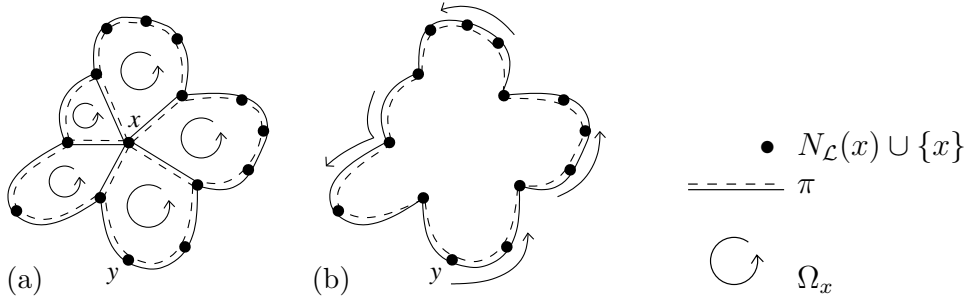

Fig. 1. (a) The set of loops which contain a vertex x, and a coherent local orientation Ω_x of $\mathcal{G}$ at x. (b) The corresponding π-cycle $\mathcal{N}_{\Omega_x,y}(x)$.

5.3 Simply Connected GADS are Orientable

In this section we outline a proof of the following result:

Proposition 5.1 *Let $\mathcal{G}$ be a* GADS *that is a subGADS of a simply connected* GADS. *Then $\mathcal{G}$ is orientable.*

Sketch of proof: Let $\mathcal{G} = ((V, \pi, \mathcal{L}), (\kappa, \lambda))$ be a subGADS of the simply connected GADS $\mathcal{G}' = ((V', \pi', \mathcal{L}'), (\kappa', \lambda'))$. Suppose $\mathcal{G}$ is not orientable. It is not hard to show that this implies $\mathcal{G}$ has a loop-circuit $(L_0, \ldots, L_{l-1})$ whose

set of loops is not π-orientable, such that no two L's are equal and, for all $i, j \in \{0, \ldots, l-1\}$, L_i is not adjacent to L_j unless $j = (i \pm 1) \bmod l$.

The idea now is to construct a π'-path in $\bigcup_{0 \le i \le l-1} L_i$ that cannot be π'-reducible in $\mathcal{G}'$, and so contradict the simple connectedness of $\mathcal{G}'$. For $i \in \{0, \ldots, l-1\}$, let $a_i, b_i \in V$ be vertices such that $\{a_i, b_i\}$ is the π-edge that is shared by L_i and $L_{(i+1) \bmod l}$, and such that, for $i \in \{1, \ldots, l-1\}$, a_{i-1} and a_i belong to the same π-component of $L_i \setminus \{b_{i-1}, b_i\}$. (It is possible that $a_{i-1} = a_i$ or $b_{i-1} = b_i$.) Then it is straightforward to verify that, since $\{L_0, \ldots, L_{l-1}\}$ is not π-orientable, a_{l-1} and b_0 must belong to the same π-component of $L_0 \setminus \{b_{l-1}, a_0\}$. For $i \in \{1, \ldots, l-1\}$, let c_i be the simple π-path in $L_i \setminus \{b_{i-1}, b_i\}$ from a_{i-1} to a_i. Also, let c_l be the simple π-path in $L_0 \setminus \{b_{l-1}, a_0\}$ from a_{l-1} to b_0. Let γ be the π-path $c_1.c_2.\ldots.c_l.(b_0, a_0)$.

Define the *parity* of a π'-path $(x_0, \ldots, x_n)$ to be 0 or 1 according to whether an even or an odd number of terms in its sequence of π'-edges $(\{x_i, x_{i+1}\} \mid 0 \le i \le n-1)$ lie in the set $\{\{a_i, b_i\} \mid 0 \le i \le l-1\}$. It is readily confirmed that π'-paths which are the same up to a minimal π'-deformation in $\mathcal{G}'$ have the same parity. But γ has parity 1 whereas a one-point path has parity 0. Hence γ is not π'-reducible in $\mathcal{G}'$, a contradiction. $\square$

Since every planar pGADS is simply connected (Corollary 4.5), a special case of the above proposition is:

Corollary 5.2 *Every planar* GADS *is orientable.*

6   The Structure of Loops in a GADS

Let $\mathcal{G} = ((V, \pi, \mathcal{L}), (\kappa, \lambda))$ be a GADS and let L be an arbitrary loop of $\mathcal{G}$. In this section we present some properties that $\kappa \cap L^{\{2\}}$ and $\lambda \cap L^{\{2\}}$ must have. These properties will be used in the next section.

Theorem 6.1 *Let C be a simple closed $(\kappa \cap \lambda)$-curve in the loop L. Then C has one of the following properties:*

  (i) *For all distinct $x, y \in C$, $\{x, y\} \in \kappa$.*

 (ii) *For all distinct $x, y \in C$, $\{x, y\} \in \lambda$.*

One way to prove this is to use the following lemma, whose proof we leave to the reader. Assertion (ii) of this lemma is illustrated by Figure 2.

Lemma 6.2 *Let C be a simple closed $(\kappa \cap \lambda)$-curve in the loop L. Then:*

  (i) *Assertions (a) and (b) of Axiom 3 hold with C in place of L whenever $x, y \in C$ but x and y are not $(\kappa \cap \lambda)$-consecutive on C.*

 (ii) *Let $\rho = \kappa$ or λ and let $a, b \in C$ be two vertices which are ρ-adjacent but not $(\kappa \cap \lambda)$-consecutive on C. Let I_1 and I_2 be the two $(\kappa \cap \lambda)$-cut-intervals of C associated with a and b. Then if $x \in I_1$ and $y \in I_2$ are ρ-adjacent, we have $\{x, a\} \in \rho$, $\{x, b\} \in \rho$, $\{y, a\} \in \rho$ and $\{y, b\} \in \rho$.*
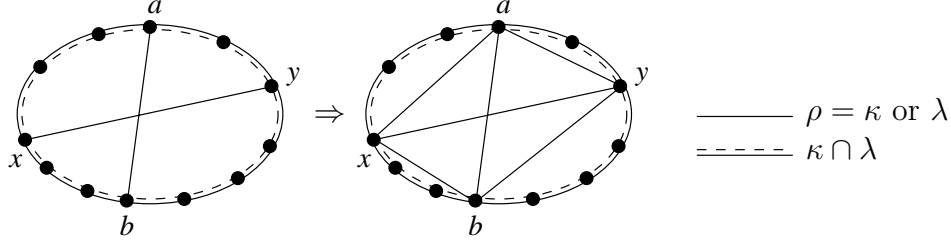
Fig. 2. Illustration of Lemma 6.2(ii).

Sketch of proof of Theorem 6.1: Let x be a vertex on C that belongs to a $(\lambda \setminus \kappa)$- or $(\kappa \setminus \lambda)$-edge of $C^{\{2\}}$. (If no such x exists then $|C| = 3$ by Lemma 6.2(i), and the theorem holds.) We first show that if a and b are vertices of C then it is impossible for $\{x, a\} \in \kappa \setminus \lambda$ and $\{x, b\} \in \lambda \setminus \kappa$ to both be true. This can be established by induction on the size of the $(\kappa \cap \lambda)$-cut-interval of C associated with a and b that does not contain x, using Lemma 6.2. (We begin by verifying that $\{x, a\} \in \kappa \setminus \lambda$ and $\{x, b\} \in \lambda \setminus \kappa$ cannot both be true if a and b are $(\kappa \cap \lambda)$-consecutive on C; otherwise we could deduce a contradiction of Lemma 6.2.) Next, we deduce from Lemma 6.2 that, if y is a $(\kappa \cap \lambda)$-neighbor of x on C and $\{x, a\} \in \kappa \setminus \lambda$ for some $a \in C$, then either y is also a vertex of a $(\kappa \setminus \lambda)$-edge in $C^{\{2\}}$, or else there is a vertex a', in the $(\kappa \cap \lambda)$-cut-interval of C associated with x and a that contains y, such that $\{x, a'\} \in \kappa \setminus \lambda$. It follows from this (by induction on the size of the $(\kappa \cap \lambda)$-cut-interval of C associated with x and a that contains y) that if $\{x, a\} \in \kappa \setminus \lambda$ for some $a \in C$, then each $(\kappa \cap \lambda)$-neighbor y of x on C is also a vertex of a $(\kappa \setminus \lambda)$-edge in $C^{\{2\}}$ — whence (by induction) every vertex of C is a vertex of a $(\kappa \setminus \lambda)$-edge in $C^{\{2\}}$. This would imply that no vertex of C is a vertex of a $(\lambda \setminus \kappa)$-edge in $C^{\{2\}}$, so that there are no $(\lambda \setminus \kappa)$-edges in $C^{\{2\}}$, whence (by Lemma 6.2(i)) every pair of vertices of C are κ-adjacent. Symmetrically, if x is a vertex of a $(\lambda \setminus \kappa)$-edge in $C^{\{2\}}$ then all pairs of vertices of C are λ-adjacent. □

Any loop of $\mathcal{G}$ can be "subdivided" into simple closed $(\kappa \cap \lambda)$-curves, and by Axiom 3 two vertices of the loop cannot be κ- or λ-adjacent unless one of the simple closed $(\kappa \cap \lambda)$-curves contains both vertices. (Figure 3 shows a loop that can be subdivided into three simple closed $(\kappa \cap \lambda)$-curves.) So the following lemma is a straightforward consequence of Theorem 6.1:

Lemma 6.3 *Let $(\rho, \widetilde{\rho}) = (\kappa, \lambda)$ or (λ, κ), and let C be any simple closed ρ-curve included in the loop L such that $|C| \neq 3$. Then C is a simple closed $(\kappa \cap \lambda)$-curve. Moreover, $\{x, y\} \in \widetilde{\rho}$ for all $x, y \in C$.*

The reader can verify that Theorem 6.1 and Lemma 6.3 hold in Figure 3.

The final result of this section implies that for any ρ satisfying $\pi \subseteq \rho \subseteq \kappa \cup \lambda$, a ρ-parameterization of a simple closed ρ-curve whose vertices are contained in a loop L must be a subsequence of a π-parameterization of L — loosely speaking, it must proceed in a single direction around L, and cannot
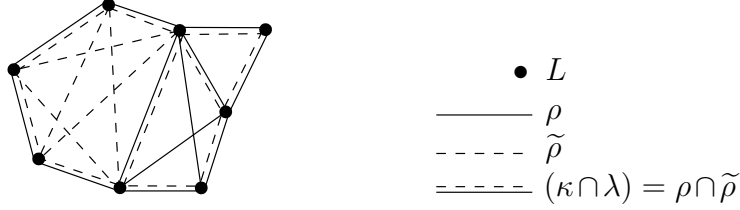
Fig. 3. Illustration of Theorem 6.1 and Lemma 6.3. This is a possible loop in a GADS, if the eight $(\kappa \cap \lambda)$-edges that belong to just one of the three simple closed $(\kappa \cap \lambda)$-curves are proto-edges and the other two $(\kappa \cap \lambda)$-edges are not.

reverse direction at some vertex.

Lemma 6.4 *Let ρ satisfy $\pi \subseteq \rho \subseteq \kappa \cup \lambda$. Let C be a simple closed ρ-curve whose vertices are contained in the loop L. Let x and y be two vertices of C which are ρ-consecutive in C but not π-consecutive in L. Then either $C \setminus \{x, y\} \subseteq I_1$ or $C \setminus \{x, y\} \subseteq I_2$ where I_1 and I_2 are the two π-cut-intervals of L associated with the vertices x and y.*

Proof: If $|C| = 3$ the result is immediate. If $|C| > 3$ then, by Lemma 6.3, C is a simple closed $(\kappa \cap \lambda)$-curve and therefore $\{x, y\} \in \kappa \cap \lambda$, so the result follows from Axiom 3. $\square$

7 The Intersection Number

Let $\mathcal{G} = ((V, \pi, \mathcal{L}), (\kappa, \lambda))$ be an orientable GADS and let Ω be a coherent orientation of $\mathcal{G}$. In this section we define an *intersection number* of a $(\kappa \cup \lambda)$-path p with a closed $(\kappa \cup \lambda)$-path c, which we denote by $I_{c,p}^{\Omega}$. The intersection number is defined only if every common vertex of the two paths is a nonsingular interior vertex of $\mathcal{G}$. Loosely speaking, it is the number of times the path p crosses from the right of the closed path c to its left, minus the number of times p crosses c from left to right.

Our intersection number is a generalization to GADS of the intersection number between paths of surfels in digital boundaries that was defined and used in [1,2], except that we only define the intersection number when one of the two paths is closed.[8] Our main result about the intersection number (Theorem 7.7) is that in an orientable GADS the intersection number of a λ-path with a closed κ-path is invariant under λ-homotopic deformations of the λ-path, assuming that all vertices of the closed κ-path are nonsingular interior vertices of $\mathcal{G}$. As we shall see in the next section, this fact can be used to prove our Jordan curve theorem for planar GADS (Theorem 4.7 above).

The definition of the intersection number is based on the idea that, for each three-vertex segment (x_0, x_1, x_2) of a $(\kappa \cup \lambda)$-path in which x_1 is a nonsingular interior vertex of $\mathcal{G}$, we can partition the set $N_{\mathcal{L}}(x_1) \setminus \{x_0, x_2\}$ into a "left" side and a "right" side with respect to the segment (x_0, x_1, x_2), using the π-cycle

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[8] It is quite easy to extend our definition to two paths that are not closed.

$\mathcal{N}_{\Omega,x_0}(x_1)$ defined in Section 5.2. The details of this are given in the next definition. Note that since $\{x_0, x_1\}, \{x_1, x_2\} \in \kappa \cup \lambda$, Axiom 2 implies that $x_0, x_2 \in N_{\mathcal{L}}(x_1)$, so that x_2 lies on the π-cycle $\mathcal{N}_{\Omega,x_0}(x_1)$.

Definition 7.1 *Let (x_0, x_1, x_2) be a segment of a $(\kappa \cup \lambda)$-path, where x_1 is a nonsingular interior vertex of $\mathcal{G}$, and let $\mathcal{N}_{\Omega,x_0}(x_1) = (v_0, \dots, v_n)$, so that $v_0 = v_n = x_0$. Let $h \in \{0, \dots, n\}$ be the integer such that $v_h = x_2$. Then we define $R_{\Omega}(x_0, x_1, x_2) = \{v_i \mid 0 < i < h\}$ and $L_{\Omega}(x_0, x_1, x_2) = \{v_i \mid h < i < n\}$.*

Let $c = (x_0, \dots, x_l)$ be a closed $(\kappa \cup \lambda)$-path. If x_i is a nonsingular interior vertex of $\mathcal{G}$, we write $\mathrm{Right}_{\Omega}^c(i)$ and $\mathrm{Left}_{\Omega}^c(i)$ for $R_{\Omega}(x_{(i-1)\bmod l}, x_i, x_{(i+1)\bmod l})$ and $L_{\Omega}(x_{(i-1)\bmod l}, x_i, x_{(i+1)\bmod l})$, respectively. Now let $\{y, z\}$ be a $(\kappa \cup \lambda)$-edge. If one of y and z is not an interior vertex of $\mathcal{G}$ or is a singularity of $\mathcal{G}$, and that vertex is also a vertex of c, then $W_{(y,z)}^c$ is undefined. Otherwise, we define $W_{(y,z)}^c = \sum_{i=0}^{l-1} W_{(y,z)}^c(i)$, where:

(i) $W_{(y,z)}^c(i) = -0.5$ if $y = x_i$ and $z \in \mathrm{Right}_{\Omega}^c(i)$, or if $z = x_i$ and $y \in \mathrm{Left}_{\Omega}^c(i)$.

(ii) $W_{(y,z)}^c(i) = +0.5$ if $y = x_i$ and $z \in \mathrm{Left}_{\Omega}^c(i)$, or if $z = x_i$ and $y \in \mathrm{Right}_{\Omega}^c(i)$.

(iii) $W_{(y,z)}^c(i) = 0$ otherwise.

Definition 7.2 (intersection number) *Let $p = (y_0, \dots, y_h)$ be a $(\kappa \cup \lambda)$-path, and c a closed $(\kappa \cup \lambda)$-path, such that every common vertex of c and p is a nonsingular interior vertex of $\mathcal{G}$. Then the* intersection number *of p with c, denoted by $I_{c,p}^{\Omega}$, is defined to be $\sum_{i=0}^{h-1} W_{(y_i, y_{i+1})}^c$.*

The next two lemmas state fundamental properties of the intersection number that follow without much difficulty from this definition.

Lemma 7.3 *Let c be a closed $(\kappa \cup \lambda)$-path, and let p', p_1 and p_2 be $(\kappa \cup \lambda)$-paths such that $p' = p_1.p_2$. Suppose further that every common vertex of c and p' is a nonsingular interior vertex of $\mathcal{G}$. Then $I_{c,p'}^{\Omega} = I_{c,p_1}^{\Omega} + I_{c,p_2}^{\Omega}$.*

Lemma 7.4 *If c_1 and c_2 are closed $(\kappa \cup \lambda)$-paths and every common vertex of c_1 and c_2 is a nonsingular interior vertex of $\mathcal{G}$, then $I_{c_1,c_2}^{\Omega} = -I_{c_2,c_1}^{\Omega}$.*

The next Lemma can be proved using Lemma 6.4: It is a consequence of Axiom 3 and the fact that a λ-parameterization of a simple closed λ-curve that lies in a loop of $\mathcal{G}$ must proceed in a single direction around that loop.

Lemma 7.5 *Let C be a simple closed λ-curve whose vertices all lie in a single loop of $\mathcal{G}$. Let $c = (x_0, \dots, x_l)$ be a λ-parameterization of C. Then C has one of the following two properties:*

(i) *For each i such that x_i is a nonsingular interior vertex of $\mathcal{G}$,*
    (a) $N_{\kappa}(x_i) \setminus C \subseteq \mathrm{Right}_{\Omega}^c(i)$, *and*
    (b) $N_{\kappa}(x_i) \cap C \setminus \{x_{(i-1)\bmod l}, x_{(i+1)\bmod l}\} \subseteq \mathrm{Left}_{\Omega}^c(i)$.

(ii) *For each i such that x_i is a nonsingular interior vertex of $\mathcal{G}$,*
    (a) $N_{\kappa}(x_i) \setminus C \subseteq \mathrm{Left}_{\Omega}^c(i)$, *and*
    (b) $N_{\kappa}(x_i) \cap C \setminus \{x_{(i-1)\bmod l}, x_{(i+1)\bmod l}\} \subseteq \mathrm{Right}_{\Omega}^c(i)$.

This lemma can be used to prove the following important result:

Proposition 7.6 *Let c be a λ-parameterization of a simple closed λ-curve whose vertices all lie in a single loop of $\mathcal{G}$, and let c' be a closed κ-path such that every common vertex of c and c' is a nonsingular interior vertex of $\mathcal{G}$. Then $I^{\Omega}_{c,c'} = 0$.*

Sketch of proof: Let $c = (x_0, \ldots, x_l)$, let C be the simple closed λ-curve parameterized by c, and let $c' = (y_0, \ldots, y_h)$. (Thus $x_l = x_0$ and $y_h = y_0$.) For all j such that y_j and y_{j+1} both lie on C, $W^c_{(y_j,y_{j+1})} = 0$. (Indeed, $W^c_{(y_j,y_{j+1})}(i) = 0$ except possibly at the two values of i in $0, \ldots, l-1$ for which $x_i \in \{y_j, y_{j+1}\}$. $W^c_{(y_j,y_{j+1})}(i) = 0$ for both of those values of i if y_j and y_{j+1} are λ-consecutive on C, and by Lemma 7.5 $W^c_{(y_j,y_{j+1})}(i)$ is $+0.5$ for one value of i and -0.5 for the other if y_j and y_{j+1} are not λ-consecutive on C.) Lemma 7.5 also implies that $W^c_{(y_j,y_{j+1})}$ has one nonzero value (± 0.5) for all j such that $y_j \in C$ and $y_{j+1} \notin C$, and has the opposite nonzero value for all j such that $y_j \notin C$ and $y_{j+1} \in C$. Since c' is a closed λ-curve, there are exactly as many values of j in $0, \ldots, h-1$ for which $y_j \in C$ and $y_{j+1} \notin C$ as there are values of j for which $y_j \notin C$ and $y_{j+1} \in C$. Hence $I^{\Omega}_{c,c'} = \sum_{j=0}^{h-1} W^c_{(y_j,y_{j+1})} = 0$. $\square$

Using this proposition and Proposition 3.4, we now prove:

Theorem 7.7 *Let $\mathcal{G} = ((V, \pi, \mathcal{L}), (\kappa, \lambda))$ be an orientable GADS, and let Ω be a coherent orientation of $\mathcal{G}$. Let c be a closed κ-path all of whose vertices are nonsingular interior vertices of $\mathcal{G}$, and let p and q be two λ-paths which are λ-homotopic in $\mathcal{G}$. Then $I^{\Omega}_{c,p} = I^{\Omega}_{c,q}$.*

Corollary 7.8 *Under the hypotheses of Theorem 7.7, $I^{\Omega}_{c,c'} = 0$ for any closed λ-path c' that is λ-reducible in $\mathcal{G}$.*

Proof of Theorem 7.7: By Proposition 3.4, it is sufficient to prove Theorem 7.7 when p and q are the same up to a minimal λ-deformation in $\mathcal{G}$. There are two cases. First suppose $p = p_1.(x, y, x).p_2$ and $q = p_1.p_2$, where $\{x, y\} \in \lambda$. Then (by Lemma 7.3) $I^{\Omega}_{c,p} = I^{\Omega}_{c,p_1} + I^{\Omega}_{c,(x,y)} + I^{\Omega}_{c,(y,x)} + I^{\Omega}_{c,p_2}$. But it is immediate from Definition 7.2 that $I^{\Omega}_{c,(x,y)} + I^{\Omega}_{c,(y,x)} = 0$, so $I^{\Omega}_{c,p} = I^{\Omega}_{c,p_1} + I^{\Omega}_{c,p_2} = I^{\Omega}_{c,p_1.p_2} = I^{\Omega}_{c,q}$. Next, suppose $p = p_1.\gamma.p_2$ and $q = p_1.p_2$, where γ is a simple closed λ-curve included in a loop of $\mathcal{G}$. Now $I^{\Omega}_{c,p_1.\gamma.p_2} = I^{\Omega}_{c,p_1} + I^{\Omega}_{c,\gamma} + I^{\Omega}_{c,p_2}$. But Proposition 7.6 implies that $I^{\Omega}_{\gamma,c} = 0$ and so, by Lemma 7.4, $I^{\Omega}_{c,\gamma} = 0$. Hence $I^{\Omega}_{c,p} = I^{\Omega}_{c,p_1.\gamma.p_2} = I^{\Omega}_{c,p_1} + I^{\Omega}_{c,p_2} = I^{\Omega}_{c,p_1.p_2} = I^{\Omega}_{c,q}$. $\square$

Note that, by Property 2.4, this theorem, Lemma 7.5 and Proposition 7.6 all remain true when κ is replaced by λ and λ by κ.

8 A Proof of the Jordan Curve Theorem

As an application of the intersection number, we now outline a proof of the Jordan curve theorem for planar GADS (Theorem 4.7 above). Since a planar pGADS is orientable (Corollary 5.2), has no singularities (Proposition 4.6), and

is simply connected (Corollary 4.5), this theorem follows from the following more general result:

Theorem 8.1 *Let $\mathcal{G} = ((V, \pi, \mathcal{L}), (\kappa, \lambda))$ be a GADS that is a subGADS of an orientable pGADS $\mathcal{G}' = ((V', \pi', \mathcal{L}'), (\kappa', \lambda'))$ which has no singularities. Let c be a κ-parameterization of a simple closed κ-curve C of $\mathcal{G}$ such that*

(i) *C is not included in any loop of $\mathcal{G}$.*

(ii) *Every vertex in C is an interior vertex of $\mathcal{G}$.*

(iii) *c is κ'-reducible in $\mathcal{G}'$.*

Then $V \setminus C$ has exactly two λ-components, and, for each vertex $x \in C$, $N_\lambda(x)$ intersects both λ-components of $V \setminus C$.

It is perhaps worth mentioning that in this theorem the hypothesis that $\mathcal{G}'$ is orientable is not really necessary, but is included because we wish to give a proof of the theorem that uses the intersection number (which is only defined in orientable GADS).

Regarding condition (ii), note that an interior vertex v of $\mathcal{G}$ cannot be a vertex of a $(\pi' \setminus \pi)$-edge, and cannot be a singularity of $\mathcal{G}$, for in both cases v would be a singularity of $\mathcal{G}'$, contrary to the hypothesis that $\mathcal{G}'$ has no singularities.

A first step in proving Theorem 8.1 is to prove:

Lemma 8.2 *Under the hypotheses of Theorem 8.1, let Ω be a coherent orientation of $\mathcal{G}'$, and let $c = (x_0, \ldots, x_l)$, so that $x_l = x_0$. Then, for all $i \in \{0, \ldots, l\}$, each of the sets $\mathrm{Left}_\Omega^c(i) \setminus C$ and $\mathrm{Right}_\Omega^c(i) \setminus C$ is nonempty and λ-connected, and contains at least one vertex in $N_\lambda(x)$.*

Note that $\mathrm{Left}_\Omega^c(i)$ and $\mathrm{Right}_\Omega^c(i)$ are sets of vertices of $\mathcal{G}$, since the vertices of c are interior vertices of $\mathcal{G}$. This lemma can be proved using Theorem 6.1. (Subdivide the loops of $\mathcal{G}$ into simple closed $(\kappa \cap \lambda)$-curves.) We omit the details here. Using this lemma, it is not hard to prove the following result:

Proposition 8.3 *Under the hypotheses of Theorem 8.1, $V \setminus C$ has at least two λ-components.*

Proof: Suppose $V \setminus C$ is λ-connected. Since V' is π'-connected (by condition (i) in the definition of a 2D digital complex), for each vertex of $V' \setminus C$ there is a shortest π'-path from that vertex to a vertex in C, and the second-last vertex on such a path must be in $V \setminus C$ because all vertices of C are interior vertices of $\mathcal{G}$. So the fact that $V \setminus C$ is λ-connected implies $V' \setminus C$ is λ'-connected.

Let Ω be a coherent orientation of $\mathcal{G}'$, let $c = (x_0, \ldots, x_l)$ (so that $x_l = x_0$), and pick $i \in \{0, \ldots, l\}$. By Lemma 8.2 there exist vertices $y \in \mathrm{Left}_\Omega^c(i) \cap N_\lambda(x_i) \setminus C$ and $z \in \mathrm{Right}_\Omega^c(i) \cap N_\lambda(x_i) \setminus C$. As $V' \setminus C$ is λ'-connected, there is a λ'-path α in $V' \setminus C$ from y to z. The closed λ'-path $\alpha' = \alpha.(z, x_i, y)$ satisfies $I_{c,\alpha'}^\Omega = 1$. But c is κ'-reducible in $\mathcal{G}'$, so $I_{\alpha',c}^\Omega = 0$ by Theorem 7.7 (with κ replaced by λ' and λ by κ') and, by Lemma 7.4, $I_{c,\alpha'}^\Omega = 0$, a contradiction. $\square$

The next proposition will be used to prove that the set $V \setminus C$ in Theorem 8.1 cannot have more than two λ-components. For any set ρ of unordered pairs of vertices of a GADS, we say that a set A of vertices of the GADS is a ρ-*arc* if A is a singleton set, or if A is a finite ρ-connected set with the following property: There are two (and only two) elements of A that each have just one ρ-neighbor in A, and all other elements of A have exactly two ρ-neighbors in A. Note that if C is any simple closed ρ-curve and $p \in C$ then $C - \{p\}$ is a ρ-arc. Each element of a ρ-arc A that does not have two ρ-neighbors in A is called an *extremity* of A.

Proposition 8.4 *Let* $\mathcal{G} = ((V, \pi, \mathcal{L}), (\kappa, \lambda))$ *be a* GADS. *Let* A *be a* κ-*arc such that every vertex in* A *is an interior vertex of* $\mathcal{G}$ *and no vertex in* A *is a singularity of* $\mathcal{G}$. *Then* $V \setminus A$ *is* λ-*connected.*

Sketch of proof: Our first step is to prove that $N_{\mathcal{L}}(x) \setminus A$ is nonempty and λ-connected if x is an extremity of A. This assertion, like Lemma 8.2, can be proved using Theorem 6.1. Again, we omit the details.

Having established this assertion, we prove the proposition by induction on $|A|$. When $|A| = 1$, the result follows from the assertion, since V is π-connected. Assume the result holds when $|A| = k$, and suppose $|A| = k + 1$. Let x be an extremity of A, and let $A' = A \setminus \{x\}$. Let v be any vertex in $V \setminus A$. By the induction hypothesis v is λ-connected in $V \setminus A'$ to x, and hence to some vertex of $N_{\lambda}(x) \setminus A'$. A shortest λ-path in $V \setminus A'$ from v to $N_{\lambda}(x) \setminus A'$ does not pass through x. Hence v is λ-connected even in $V \setminus A$ to some vertex of $N_{\lambda}(x) \setminus A' \subseteq N_{\mathcal{L}}(x) \setminus A$. Since v is an arbitrary vertex in $V \setminus A$ and $N_{\mathcal{L}}(x) \setminus A$ is λ-connected (by the above assertion), $V \setminus A$ is λ-connected. $\square$

Proof of Theorem 8.1: From Proposition 8.3 we know that $V \setminus C$ has at least two λ-components. Now let x be a vertex of C, and let A be the κ-arc $C \setminus \{x\}$. Let v be any vertex in $V \setminus C$. By Proposition 8.4, v is λ-connected in $V \setminus A$ to x, and hence to some vertex in $N_{\lambda}(x) \setminus A$. A shortest λ-path in $V \setminus A$ from v to $N_{\lambda}(x) \setminus A$ does not pass through x, so v is λ-connected even in $V \setminus C$ to some vertex in $N_{\lambda}(x) \setminus A$. Since this applies to any vertex v in $V \setminus C$, every λ-component of $V \setminus C$ intersects $N_{\lambda}(x) \setminus A$.

Moreover, since $N_{\lambda}(x) \setminus A \subseteq N_{\mathcal{L}}(x) \setminus C$, we can deduce that $V \setminus C$ has no more λ-components than $N_{\mathcal{L}}(x) \setminus C$. But if $c = (x_0, \ldots, x_l)$, so that $x_l = x_0$, and i is the integer in $\{0, \ldots, l - 1\}$ such that $x = x_i$, then $N_{\mathcal{L}}(x) \setminus C = (\text{Left}_{\Omega}^c(i) \cup \text{Right}_{\Omega}^c(i)) \setminus C$ does not have more than two λ-components, by Lemma 8.2. Hence $V \setminus C$ cannot have more than two λ-components. $\square$

9 Concluding Remarks

A new approach to 2D digital topology (including the digital topology of boundary surfaces), based on an axiomatization of the spaces being studied, has been presented. A space that satisfies our axioms is called a GADS. In-

stances of this very general concept include GADS corresponding to all of the good pairs of adjacency relations that have previously been used (by ourselves or others) in digital topology on planar grids and boundary surfaces.

Some results that have been established in the literature for certain specific digital spaces have been generalized to GADS (e.g., a homotopy invariance theorem for intersection numbers of digital paths, and a digital Jordan curve theorem). There are many other results of digital topology for which this could be done, such as results about simple points and boundary tracking. The problem of developing a 3D version of this theory seems more challenging.

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