A Model for the Delay Management Problem based on Mixed-Integer-Programming

Anita Schöbel [1]

Department of Mathematics
Universität Kaiserslautern
and
Institut für Techno- und Wirtschaftsmathematik
Germany

Abstract

Dealing with delayed vehicles is a necessary issue in the dispositive work of a public transportation company. If a vehicle arrives at some station with a delay, it has to be decided if the connecting vehicles should wait for changing passengers or if they should depart in time. A possible objective function is to minimize the sum of all delays over all customers using the transportation network. In this paper the *delay management problem* is formulated as a mixed integer linear program, and solution approaches based on this formulation are indicated.

1 Introduction

Unfortunately, delays are a main subject of complaints in public transportation, and it does not seem possible to avoid them completely. Customers using a delayed vehicle will probably arrive at their destination later as planned. Even worse, if some passengers want to change from a delayed vehicle into another bus or train, they might miss their connection and maybe have to wait a long time for the next one. In this paper we analyze these effects and will present a model how to deal with delays.

Suppose that a train arrives at a station k with a delay of some minutes. At station k there is a bus ready to depart. What should this connecting bus do? There are two alternatives:

- The bus either can **wait** and therefore cause delay for the customers within the bus, but also for the customers who want to get on this bus later on, and possibly for subsequent other buses which will have to wait for its delay.

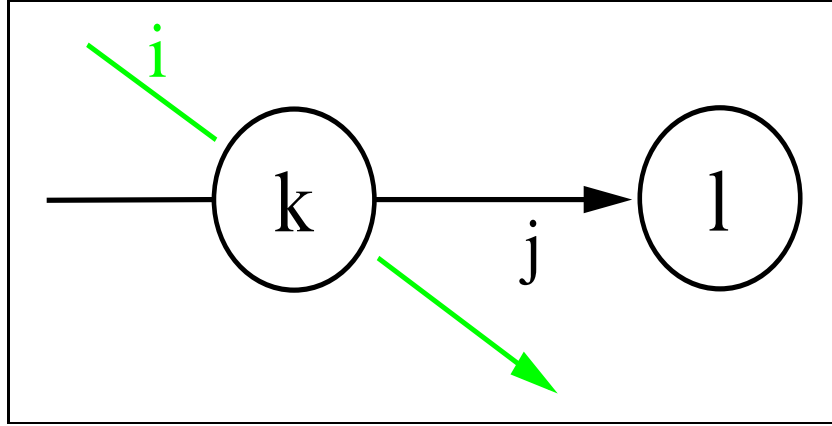---

[1] Email: schoebel@mathematik.uni-kl.de

Fig. 1. A connection (i, j, k) from vehicle i to vehicle j at station k.

- On the other hand, if the bus **departs** in time, all the customers who planned to change from the delayed train into the bus will miss their connection.

The *delay management problem* is to find wait-depart decisions, not only for one single bus, but for all vehicles in the public transportation network, such that the inconvenience over all customers is minimized. As inconvenience for a customer, we consider the amount of his delay when he reaches his destination.

Since in the delay management problem new departure times for each vehicle at each station have to be determined, it is closely related to the problem of finding timetables in public transportation. In this field, a lot of research has been done, as an example for a combinatorial and a graph-theoretical approach and for various heuristics and metaheuristics, see [9] and references therein. As objective functions, the total waiting time or the traveling time have mainly been considered, sometimes also the costs for operating the timetable. Other models which have also been considered mainly in scheduling and timetabling, but that can be useful when dealing with delays are the max-plus-algebra (e.g., used by [6]) or the heaps-of-pieces-approach, see [13]. The main difference to the delay management problem, besides the slightly different objective functions, is that in the delay management problem the connections are not given in advance. In fact, the main decision that must be made is to decide which connections should be maintained and which can be dropped.

Delays and their consequences have been addressed in a few papers before, see e.g., [8,3] for analyzing the occurrence of delays. [11,1,7,12] dealt with delays in railway systems. Mainly simulation and expert systems are used when dealing with the reaction to initial delays. In contrast to these approaches, the delay management problem will be modeled as a mixed-integer program in this paper, such that optimization methods can be used for finding good decisions.

2 Mixed-integer programming formulation

In this paper we will deal with the **delay management problem** defined as follows.

Given a set of vehicles running within a transportation network, and an initial delay V of one of these vehicles, how should the schedules of the vehicles be adapted, such that the total amount of the delays of all customers within the network is minimized?

We first introduce some notation. To describe the transportation network, let $\mathcal{V}$ be the set of stations and $\mathcal{F}$ be the set of vehicles (the fleet). For each vehicle, we introduce $\mathcal{E}^i \subseteq \mathcal{V} \times \mathcal{V}$ as the set of its driving edges within the transportation network. If it is possible to change from vehicle i to vehicle j at a station k, then (i, j, k) will be called a **connection** and the whole set of connections is given by

$$\mathcal{U} \subseteq \mathcal{F} \times \mathcal{F} \times \mathcal{V}.$$

We also need parameters p_i^k, p_i^{kl}, and p_{ij}^k representing the slack times for the waiting time of vehicle i at station k, for the driving time of vehicle i on its way from station k to station l, and for the customers changing from vehicle i to vehicle j at station k, respectively.

Since the objective is to minimize the sum over all delays of all customers in the network, we need to specify the data about the customers. To this end, let $\mathcal{A}$ be a set of paths through the public transportation network, and w_a be the number of customers using a specific path $a \in \mathcal{A}$. To describe the paths in $\mathcal{A}$ we use the parameters h_{ijk}^a defined as follows.

$$h_{ijk}^a = \begin{cases} 1 & \text{if station } k \text{ is reached by vehicle } i \text{ and left by } j \text{ in path } a \\ 0 & \text{otherwise} \end{cases}$$

To avoid a time-dependent representation of the stations we assume that each vehicle $i \in \mathcal{F}$ and each customer path $a \in \mathcal{A}$ contains any station at most once.

Furthermore, we assume that T is the (common) time period for all vehicles, and that in the next time period all vehicles are in time. The initial delay is given as an amount of time, V, of a specified vehicle arriving at a specified station, and we assume $V \leq T$.

As variables we define

$x_i^k =$ departure delay of vehicle i at station k

$y_i^k =$ arrival delay of vehicle i at station k

$$z_a = \begin{cases} 1 & \text{if all connections on path } a \text{ are maintained} \\ 0 & \text{otherwise.} \end{cases}$$

To calculate the sum of delays of all passengers using a specific path $a \in \mathcal{A}$ we distinguish two cases.

Case 1: All connections on path a have been maintained. Then the delay for passengers using this path simply is the arrival delay of their last vehicle i_a at their destination station k_a, given by $w_a y_{i_a}^{k_a}$. Note that these passengers might have had a larger delay during their trip, but only the arrival delay at their destination is relevant.

Case 2: At least one connection on path a is not maintained. Then we assume that the passengers using path a have to wait the whole time period T for the next vehicle going towards their destination, and the delay is hence given by $w_a T$.

Now the model (DM) for the delay management problem can be given.

$$\min \sum_{a \in \mathcal{A}} w_a \left((1 - z_a)T + z_a y_{i_a}^{k_a} \right)$$

such that

(1) $y_1^1 = V$

(2) $y_i^k \leq x_i^k + p_i^k \quad \forall i \in \mathcal{F}, k \in \mathcal{V}$

(3) $x_i^k \leq y_i^l + p_i^{kl} \quad \forall k, l \in \mathcal{V}, i \in \mathcal{F} : (k, l) \in \mathcal{E}^i$

(4) $z_a h_{ijk}^a \leq \dfrac{x_j^k + p_{ij}^k - y_i^k}{M} + 1 \quad \forall (i, j, k) \in \mathcal{U}, a \in \mathcal{A}$

(5) $0 \leq y_i^k, x_i^k \leq T,$

(6) $z_a \in \{0, 1\}$

where i_a denotes the last vehicle on path a and k_a the last station on path a, and M should be chosen such that

$$M \geq \max_{(i,j,k) \in \mathcal{U}} p_{ij}^k + T.$$

Note that constraint (5) ensures that delays greater than the common time period need not be considered, since in this case the customers can take the non-delayed vehicle of the next period.

Constraint (1) contains the source delay, e.g., of vehicle 1 at station 1. Note that this single constraint is the crucial one in the formulation; a relaxation of constraint (1) would lead to the trivial optimal solution without any delays, i.e., $y_i^k = x_i^k = 0$ for all $i \in \mathcal{F}, k \in \mathcal{V}$ and $z_a = 1$ for all $a \in \mathcal{A}$. It is of course also possible to use a set of source delays, i.e.,

$$y_i^k = V_i^k \text{ for all } (i, k) \in \mathcal{D},$$

where $\mathcal{D}$ describes the set of initial delays based on the arrival times of the vehicles.

Inequalities (2) make sure that a vehicle arriving with a delay at some station k will also depart with this delay, where a possibly positive slack time can be subtracted, and constraints (3) ensure that a delay is kept correctly
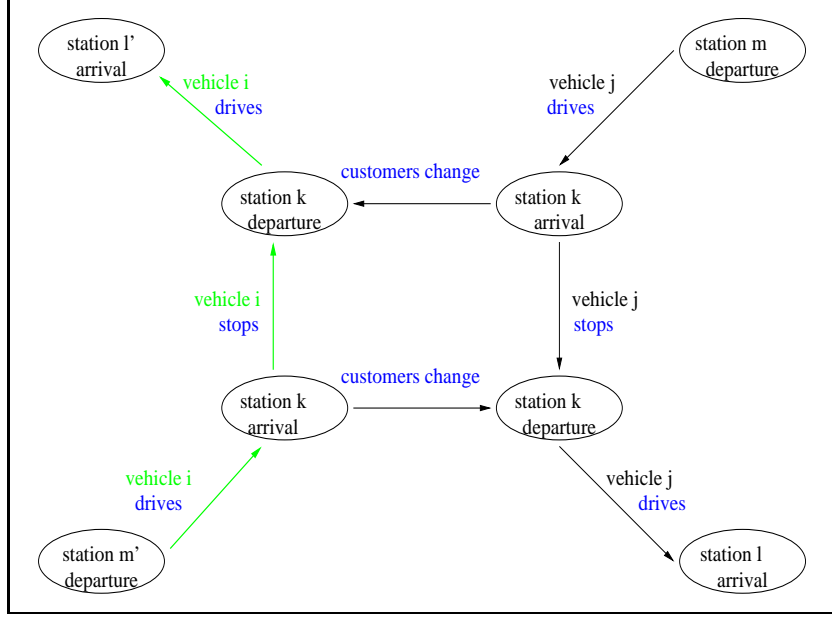
4

Fig. 2. The event activity network

along the driving edges of the vehicles.

In (4) the x_i^k and y_i^k variables are coupled with the data about the customers. To understand the meaning of constraints (4), take a connection $(i, j, k) \in \mathcal{U}$ from a vehicle i to a vehicle j at some station k and a path $a \in \mathcal{A}$ containing this specific connection, i.e. with $h_{ijk}^a = 1$. Note that

$$|x_j^k + p_{ij}^k - y_i^k| \le |x_j^k - y_i^k| + |p_{ij}^k| \le T + \max_{i,j,k} p_{ij}^k \le M.$$

Hence, if the connection is not kept, i.e.,

$$y_i^k - p_{ij}^k > x_j^k$$

we get that $-M \le x_j^k + p_{ij}^k - y_i^k < 0$ and consequently,

$$0 \le \frac{x_j^k + p_{ij}^k - y_i^k}{M} + 1 < 1$$

such that constraint (4) is satisfied if and only if $z_a = 0$. On the other hand, if the connection is maintained,

$$\frac{x_j^k + p_{ij}^k - y_i^k}{M} + 1 \ge 1$$

such that z_a is not restricted by constraints (4) and can hence be chosen as $z_a = 1$ due to the minimization of the objective function.

Based on event-activity networks (see, e.g., [9]), the formulation can be interpreted graph-theoretically as follows. We construct an event-activity network $G = (V, E)$, where the set of nodes $V = V_{arr} \cup V_{dep} \subseteq \mathcal{F} \times \mathcal{V}$ represents all arrivals $(i, k)_{arr}$ of a vehicle i at some station k and all departures $(i, k)_{dep}$ of some vehicle i at some station k. Then three types of edges can be

distinguished (see Figure 2).

- waiting edges $((i,k)_{arr}, (i,k)_{dep})$
- driving edges $((i,k)_{dep}, (i,l)_{arr})$ with $(k,l) \in \mathcal{E}^i$
- changing edges $((i,k)_{arr}, (j,k)_{dep})$ with $(i,j,k) \in \mathcal{U}$

In the event-activity network, the weights of the edges are the given slack times, i.e. p_i^k for waiting edges, p_i^{kl} for driving edges, and p_{ij}^k for changing edges. The node weights y_i^k, x_i^k correspond to the decision variables representing the delay of the arrival and departure events $(i,k)_{arr}$ and $(i,k)_{dep}$, respectively. The goal is to decide which of the changing edges can be dropped and which should be maintained to minimize the overall delay over all customers.

The given formulation of model (DM) can be linearized (and weakened) by substituting the quadratic term $y_{i_a}^{k_a} z_a$ by a new variable u_a, leading to the following model (DM').

$$\min \sum_{a \in \mathcal{A}} w_a \left((1 - z_a)T + u_a \right)$$

such that

$(7) \qquad y_1^1 = V$

$(8) \qquad y_i^k \leq x_i^k + p_i^k \quad \forall i \in \mathcal{F}, k \in \mathcal{V}$

$(9) \qquad x_i^k \leq y_i^l + p_i^{kl} \quad \forall k, l \in \mathcal{V}, i \in \mathcal{F} : (k,l) \in \mathcal{E}^i$

$(10) \; z_a h_{ijk}^a \leq \dfrac{x_j^k + p_{ij}^k - y_i^k}{M} + 1 \quad \forall (i,j,k) \in \mathcal{U}, a \in \mathcal{A}$

$(11) \qquad u_a \geq y_{i_a}^{k_a} - T(1 - z_a) \quad \forall a \in \mathcal{A}$

$(12) \qquad 0 \leq y_i^k, x_i^k \leq T,$

$(13) \qquad u_a \geq 0,$

$(14) \qquad z_a \in \{0, 1\}$

Lemma 2.1 *The linearization is correct.*

Proof.

(DM) $\implies$ **(DM'):** Let y_i^k, x_i^k, z_a be a feasible solution of (DM). Define for all $a \in \mathcal{A}$ $u_a = y_{i_a}^{k_a} z_a$. Since

$$y_{i_a}^{k_a} \leq y_{i_a}^{k_a} z_a + T(1 - z_a) \quad \forall a \in \mathcal{A}$$

y_i^k, x_i^k, z_a, u_a are feasible for (DM'), and both solutions have the same objective value.

(DM) $\implies$ **(DM):** Let y_i^k, x_i^k, z_a, u_a be a feasible solution of (DM'). Then y_i^k, x_i^k, z_a is also feasible for (DM). From (11) we conclude that

$\quad u_a \geq y_{i_a}^{k_a}$ if $z_a = 1$

$\quad u_a \geq 0$ otherwise.

Consequently, $u_a \geq y_{i_a}^{k_a} z_a$ and since we minimize, the objective function does not increase.

□

3   Upper and lower bounds

To calculate upper bounds, we construct feasible solutions by fixing the integer variables $z_a, a \in \mathcal{A}$. Let $\bar{\mathcal{A}}$ denote the set of variables z_a which have been fixed to 1, and let

$$\bar{\mathcal{U}} = \{(i, j, k) \in \mathcal{U} : \text{ there exists } a \in \bar{\mathcal{A}} \text{ such that } h^a_{ijk} = 1\}$$

be the set of connections on paths from $\bar{\mathcal{A}}$. Then constraints (4) reduce to

$$1 \le \frac{x^k_j + p^k_{ij} - y^k_i}{M} + 1 \quad \forall (i, j, k) \in \bar{\mathcal{U}}$$

which is equivalent to

$$x^k_j + p^k_{ij} - y^k_i \ge 0 \quad \forall (i, j, k) \in \bar{\mathcal{U}}.$$

Hence, the the mixed integer program of model (DM) reduces to the following linear program.

$$\min \sum_{a \in \bar{\mathcal{A}}} w_a y^{k_a}_{i_a}$$

such that
$$
\begin{aligned}
y^1_1 &= V \\
-x^k_i + y^k_i &\le p^k_i \quad \forall i \in \mathcal{F}, k \in \mathcal{V} \\
x^k_i - y^l_i &\le p^{kl}_i \quad \forall k, l \in \mathcal{V}, i \in \mathcal{F} : (k, l) \in \mathcal{E}^i \\
-x^k_j + y^k_i &\le p^k_{ij} \quad \forall (i, j, k) \in \bar{\mathcal{U}} \\
0 &\le y^k_i, x^k_i \le T
\end{aligned}
$$

This formulation can be solved efficiently by linear programming, or by using the event-activity network described in the previous section, and performing the critical path methods for activity networks (e.g., described in [4]), see [5]. To find values for the z_a-variables, the following strategies are possible.

All vehicles depart in time: Then the upper bound can be calculated by

$$\sum_{a \in \mathcal{A}: i_a = 1} w_a y^{k_a}_{i_a} + \sum_{a \in \mathcal{A}_1} w_a T$$

where the first term is bounded by $V \sum_{a \in \mathcal{A}: i_a = 1} w_a$ and

$$\mathcal{A}_1 = \{a \in \mathcal{A} : \exists\, j \in \mathcal{F}, k \in \mathcal{V} \text{ such that } (1, j, k) \in \mathcal{U}, h^a_{1jk} = 1\}.$$

All vehicles wait: this leads to the trivial bound of $\sum_{a \in \mathcal{A}} w_a V$, which can be strengthened if the slack times are greater than zero by solving the linear program above.

Heuristic approach: Construct $\bar{\mathcal{A}}$ out of paths which are used by many customers and do not care for the others.

Metaheuristic approach: Finding a good set $\bar{\mathcal{A}}$ can also be solved by metaheuristics such as genetic algorithms or simulated annealing.

7

Note that on the other hand, for any feasible set of values for the x_i^k, y_i^k, $i \in \mathcal{F}, k \in \mathcal{V}$, the optimal values of z_a, $a \in \mathcal{A}$ can be calculated as follows. Let $\mathcal{U}_a = \{(i, j, k) \in \mathcal{U} : h_{ijk}^a = 1\}$ be the set of connections on path $a \in \mathcal{A}$.

Lemma 3.1 *Given y_i^k, x_i^k, the optimal solution for (DM) is given by*

$$z_a^* = \begin{cases} 0 & \text{if } \min_{(i,j,k)\in\mathcal{U}_a} x_j + p_{ij}^k - y_i^k < 0 \\ 1 & \text{otherwise.} \end{cases}$$

Proof. In the case that the x and y-variables are given, (DM) can be reduced to

$$\min \sum_{a \in \mathcal{A}} w_a \left((1 - z_a)T + z_a y_{i_a}^{k_a} \right)$$

such that

$$z_a h_{ijk}^a \leq g_{ij}^k \quad \forall (i, j, k) \in \mathcal{U}, a \in \mathcal{A}$$
$$z_a \in \{0, 1\},$$

where $g_{ij}^k = \frac{x_j^k + p_{ij}^k - y_i^k}{M} + 1$. Since we want to minimize, and since $y_{i_a}^{k_a} \leq T$ for all $a \in \mathcal{A}$, we conclude that $z_a^* = 1$ if this is feasible, i.e. if $g_{ij}^k \geq 1$ for all $(i, j, k) \in \mathcal{U}_a$. Using that

$$g_{ij}^k < 1 \iff x_j^k + p_{ij}^k - y_i^k < 0$$

the result follows. □

A lower bound for (DM) can be obtained by solving the linear programming relaxation of (DM'), where $z_a \in \{0, 1\}$ is replaced by $z_a \leq 1$. To strengthen the formulation, constraint (10) can be replaced by

$$z_a h_{ijk}^a \leq \frac{x_j^k + p_{ij}^k - y_i^k}{M_a} + 1 \quad \forall (i, j, k) \in \mathcal{U}, a \in \mathcal{A},$$

where

$$M_a = \max_{(i,j,k)\in\mathcal{U}_a} p_{ij}^k + V.$$

In this relaxation, the contribution of the missed connections to the objective function is proportional to $x_j^k + p_{ij}^k - y_i^k$. This means that a connection (i, j, k) that has been only just missed because of a few minutes, increases the objective function not so much as if vehicle j would have departed a long time before the arrival of vehicle i.

4 Application and Outlook

The project of this paper was brought up by two large traffic associations serving the states Rheinland-Pfalz and the Saarland (both in Germany). They are interested in analyzing the consequences of delays, or, more general, also of (small) changes in the schedule. When analyzing such changes, it is important not to look only at one single transportation company, but to take into
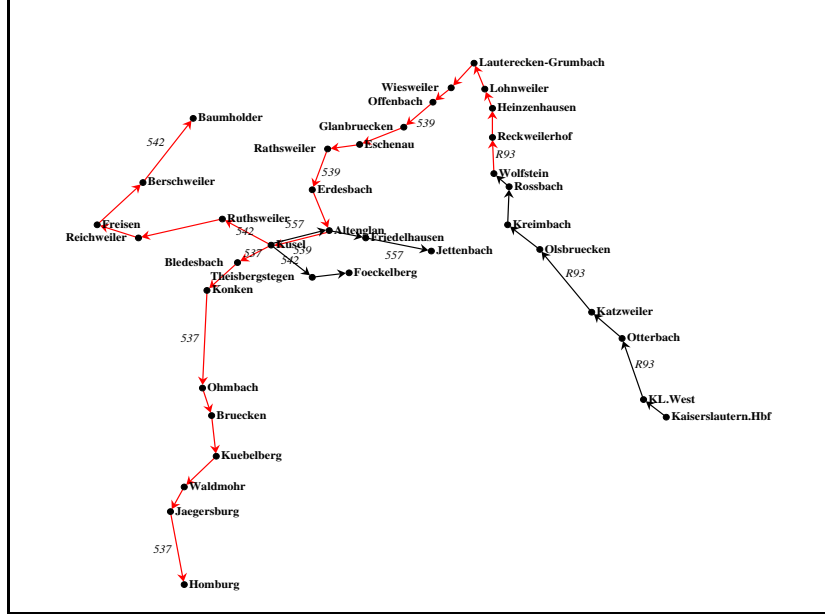
Fig. 3. Calculating all delayed vehicles (red) in the area of the Lautertalbahn in Rheinland-Pfalz, Germany.

account also connections between different public transportation companies. In practice, there are other effects which need to be considered when dealing with the delay management problem. Two of them are mentioned below.

- On single track lines the oncoming traffic has to be taken into consideration. This effect can be modeled by introducing some new connections (i, j, k) representing that vehicle j is not allowed to depart before the oncoming train has been arrived. With artificial paths a whose variables z_a are fixed to 1 it is possible to ensure that all these new connections are maintained.

- Also the vehicle schedules and the drivers schedules may have an effect to the model, since a delayed vehicle or a delayed driver cannot start his next piece of duty in time. These effects can also be modeled by introducing new connections, that may be maintained or dropped, but make sure that a delay is taken over to the next piece of duty.

Together with our project partners we defined two representative test areas and collected the data about stations, edges and the schedules. Some first numerical results based on this data show that solving the linear program to determine the y_i^k and x_i^k can be done within seconds on a PII/500 computer, also for the data of a complete day with more than 1000 vehicles. In Figure 3 vehicles which will have a delay (within a given solution of the z_a, $a \in \mathcal{A}$) are indicated in red. Currently, we are collecting real data about the customers' behavior and are implementing and testing the heuristics discussed in Section 3 (see [2]). A branch-and-bound approach, where the z_a are the branching variables, and which makes use of the bounds developed in the previous section is also under consideration. Furthermore, [10] gives some

additional assumptions ensuring that the branch-and-bound approach finds an optimal solution within polynomial time.

References

[1] Adenso-Díaz, B., M. O. González and P. González-Torre, *On-line timetable re-scheduling in regional train services*, Transportation Research **33B** (1999), pp. 387–398.

[2] Castoldi, L., A. Schöbel and T. Sonneborn, *Heuristic approaches for the delay management problem*, Technical report, Institut für Techno- und Wirtschaftsmathematik, Kaiserslautern (2001), working paper.

[3] Chen, B. and P. Harker, *Two moments estimation of the delay on single-track rail lines with scheduled traffic*, Transportation Science **24** (1990), pp. 261–275.

[4] Elmaghraby, S., "Activity Networks," Wiley Interscience Publication, 1977.

[5] Ginkel, A., "Event-activity networks in delay management," Master's thesis, University of Kaiserslautern (2001 (expected)), currently under research.

[6] Goverde, R., *The max-plus algebra approach to railway timetable design*, in: *Computers in Railways VI: Proceedings of the 6th international conference on computer aided design, manufacture and operations in the railway and other advanced mass transit systems, Lisbon, 1998*, 1998, pp. 339–350.

[7] Hendrickson, C. and G. Kocur, *Schedule delay and departure time decision in a deterministic model*, Transportation Science **15** (1981), pp. 62–77.

[8] Higgins, A. and E. Kozan, *Modeling train delays in urban networks*, Transportation Science **32** (1998), pp. 346–357.

[9] Nachtigall, K., "Periodic Network Optimization and Fixed Interval Timetables," Deutsches Zentrum für Luft– und Raumfahrt, Institut für Flugführung, Braunschweig, 1998, habilitationsschrift.

[10] Scholl, S., "Anschlusssicherung bei Verspätungen im ÖPNV," Master's thesis, University of Kaiserslautern (2001).

[11] Suhl, L. and T. Mellouli, *Requirements for, and design of, an operations control system for railways*, in: *Computer-Aided Transit Scheduling* (1999).

[12] Suhl, L. and T. Mellouli, *Managing and preventing delays in railway traffic by simulation and optimization*, in: *Mathematical Methods on Optimization in Transportation Systems* (2001), pp. 3–16.

[13] van Egmond, R., *An algebraic approach for scheduling train movements*, in: *Proceedings of the 8th international conference on Computer-Aided Transit Scheduling, Berlin, 2000*, 2001.