Available online at www.sciencedirect.com

ScienceDirect

Electronic Notes in Theoretical Computer Science

Electronic Notes in Theoretical Computer Science 346 (2019) 171–183

www.elsevier.com/locate/entcs

On Tuza's Conjecture for Triangulations and Graphs with Small Treewidth

F. Botler[2]

Programa de Engenharia de Sistemas e Computação
Universidade Federal do Rio de Janeiro, Brazil

C.G. Fernandes[3] J. Gutiérrez[4]

Departamento de Ciência da Computação
Universidade de São Paulo, Brazil

Abstract

Tuza (1981) conjectured that the cardinality $\tau(G)$ of a minimum set of edges that intersects every triangle of a graph G is at most twice the cardinality $\nu(G)$ of a maximum set of edge-disjoint triangles of G. In this paper we present three results regarding Tuza's Conjecture. We verify it for graphs with treewidth at most 6; and we show that $\tau(G) \leq \frac{3}{2}\,\nu(G)$ for every planar triangulation G different from K_4; and that $\tau(G) \leq \frac{9}{5}\,\nu(G) + \frac{1}{5}$ if G is a maximal graph with treewidth 3.

Keywords: Triangle transversal, triangle packing, treewidth, triangulation.

1 Introduction

In this paper, we use the term *graph* to refer to a simple graph and the term *multigraph* to refer to a graph that might contain parallel edges. Other than this, the notation and terminology are standard. A *triangle transversal* of a graph G is a set of edges of G whose removal results in a triangle-free graph; and a *triangle packing* of G is a set of edge-disjoint triangles of G. We denote by $\tau(G)$ (resp. $\nu(G)$)

[1] This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001. C. G. Fernandes was partially supported by Conselho Nacional de Desenvolvimento Científico e Tecnológico - CNPq (Proc. 456792/2014-7, 308116/2016-0, and 423833/2018-9). J. Gutiérrez was supported by FAPESP (Proc. 2015/08538-5).

[2] Email:fbotler@cos.ufrj.br

[3] Email:cris@ime.usp.br

[4] Email:juanguti@ime.usp.br

https://doi.org/10.1016/j.entcs.2019.08.016
1571-0661/

the cardinality of a minimum triangle transversal (resp. maximum triangle packing) of G. Tuza [14] posed the following conjecture.

Conjecture 1.1 (Tuza, 1981) *For every graph G, we have $\tau(G) \leq 2\,\nu(G)$.*

This conjecture was verified for many classes of graphs. In particular, Tuza [15] verified it for planar graphs, and Haxell and Kohayakawa [9] proved that if G is a tripartite graph, then $\tau(G) \leq 1.956\,\nu(G)$. The reader may refer to [1,5,6,8,10,11] for more results concerning Tuza's Conjecture. In this paper we present three results regarding Tuza's Conjecture. We verify it for graphs with treewidth at most 6; and we show that $\tau(G) \leq \frac{3}{2}\,\nu(G)$ for every planar triangulation G different from K_4; and that $\tau(G) \leq \frac{9}{5}\,\nu(G)$ if G is a 3-*tree*, i.e., a graph obtained from K_3 by successively choosing a triangle in the graph and adding a new vertex adjacent to its three vertices.

Puleo [13] introduced a set of tools for dealing with graphs that contain vertices of small degree (Lemma 3.3), and verified Tuza's Conjecture for graphs with maximum average degree less than 7, i.e., for graphs in which every subgraph has average degree less than 7. In this paper, we extend Puleo's technique (Lemma 3.4) in order to prove Tuza's Conjecture for graphs with treewidth at most 6 (Theorem 3.5). Note that there are graphs with treewidth at most 6 and maximum average degree at least 7 (Figure 1).

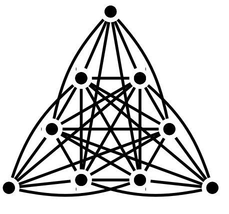

Figure 1. A graph with treewidth 6 and average degree 22/3.

In another direction, as suggested in [9], we show that, for certain classes of graphs, the ratio $\tau(G)/\nu(G)$ can be bounded by a constant smaller than 2. Specifically, we show that, if G is a planar triangulation different from K_4, then $\tau(G) \leq \frac{3}{2}\,\nu(G)$ (Theorem 4.5) and, if G is a 3-tree, then $\tau(G) \leq \frac{9}{5}\,\nu(G) + \frac{1}{5}$ (Theorem 5.2).

This paper is organized as follows. In Section 2, we establish the notation and present some auxiliary results used throughout the paper. In Section 3, we verify Tuza's Conjecture for graphs with treewidth at most 6 and, in Sections 4 and 5, we study planar triangulations and 3-trees, respectively. Finally, in Section 6 we present some concluding remarks. Due to space limitations, we present only the sketch of some proofs.

2 Rooted tree decompositions

In this section we present most of the notation and some auxiliary results regarding tree decompositions. A *rooted tree* is a pair (T, r), where T is a tree and r is a node of T. Given $t \in V(T)$, if t' is a node in the (unique) path in T that joins r and t, then we say that t' is an *ancestor* of t. Every vertex in T that has t as its ancestor is called a *descendant* of t. If $t \neq r$, then the *parent* of t, denoted by $p(t)$, is the ancestor of t that is adjacent to t. The *successors* of t are the nodes whose parent is t, and we denote the set of successors of t by $S_T(t)$. A node of T with no successor is a *leaf* of T. The *height* of t, denoted by $h_T(t)$, is the number of edges of a longest path that joins t to a descendant of t. When T is clear from the context, we simply write $S(t)$ and $h(t)$.

Definition 2.1 A *tree decomposition* of a graph G is a pair $\mathcal{D} = (T, \mathcal{V})$ consisting of a tree T and a collection $\mathcal{V} = \{V_t \subseteq V(G) : t \in V(T)\}$, satisfying the following conditions:

(i) $\bigcup\{V_t : t \in V(T)\} = V(G)$;

(ii) for every $uv \in E(G)$, there exists a $t \in V(T)$ such that $u, v \in V_t$;

(iii) if a vertex v is in $V_{t_1} \cap V_{t_2}$, then $v \in V_t$ for every t in the path of T that joins t_1 and t_2.

The *width* of $\mathcal{D}$ is the number $\max\{|V_t| - 1 : t \in V(T)\}$, and the *treewidth* $tw(G)$ of G is the width of a tree decomposition of G with minimum width. Let G be a graph with treewidth k. If $|V_t| = k + 1$ for every $t \in V(T)$, and $|V_t \cap V_{t'}| = k$ for every $tt' \in E(T)$, then we say that $(T, \mathcal{V})$ is a *full tree decomposition* of G.

The following result was proved by Bodlaender [2] (see also Gross [7]).

Proposition 2.2 *Every graph admits a full tree decomposition.*

A triple $(\mathcal{V}, T, r)$ is a *rooted tree decomposition* of a graph G if $(\mathcal{V}, T)$ is a full tree decomposition of G, (T, r) is a rooted tree, and $V_t \cap V_{p(t)} \neq V_t \cap V_{t'}$ for every $t \in V(T) \setminus \{r\}$ and $t' \in S(t)$. Every full tree decomposition can be modified into a rooted tree decomposition with a root r. So the following proposition comes naturally.

Proposition 2.3 *Every graph admits a rooted tree decomposition.*

Given a rooted tree decomposition $(\mathcal{V}, T, r)$ of a graph G and a node $t \in V(T) \setminus \{r\}$, we say that the (unique) vertex in $V_t \setminus V_{p(t)}$ is the *representative* of t. We leave undefined the representative of r.

We denote by $N_G(u)$ the set of neighbors of u for a vertex $u \in V(G)$. When G is clear from the context, we simply write $N(u)$. In what follows, we denote by $N[u]$ the closed neighborhood $N(u) \cup \{u\}$ of u, by $d(u)$ the degree of u, and by $\Delta(G)$ the maximum degree of G.

Remark 2.4 If y is the representative of a leaf t of a rooted tree decomposition of a graph G, then $N_G(y) \subseteq V_t$.

3 Graphs with treewidth at most 6

In this section, we verify Tuza's Conjecture for graphs with treewidth at most 6 by extending the technique introduced by Puleo [13], which explores the neighborhood of vertices of small degree in order to decrease the size of the graph studied. For that, we use the following definitions (see also [13]).

Definition 3.1 For a graph G, a nonempty set $V_0 \subseteq V(G)$ is called *reducible* if there is a set $X \subseteq E(G)$ and a set Y of edge-disjoint triangles in G such that the following conditions hold:

(i) $|X| \leq 2|Y|$;

(ii) $X \cap E(A) \neq \emptyset$ for every triangle A in G that contains a vertex of V_0; and

(iii) if $u, v \notin V_0$ and $uv \in E(A)$ for some $A \in Y$, then $uv \in X$.

In this case, we say that (V_0, X, Y) is a *reducing triple for* G. If G has no reducible set, G is called *irreducible*.

The following lemma comes naturally (see [13, Lemma 2.2]).

Lemma 3.2 *Let* (V_0, X, Y) *be a reducing triple for a graph* G*, and put* $G' = (G - X) - V_0$*. If* $\tau(G') \leq 2\,\nu(G')$*, then* $\tau(G) \leq 2\,\nu(G)$*.*

We say that a graph G is *robust* if, for every $v \in V(G)$, each component of $G[N(v)]$ has order at least 5. We often use the following result (see [13, Lemma 2.7]).

Lemma 3.3 *Let* G *be an irreducible robust graph and let* $x, y \in V(G)$*. The following statements hold:*

(a) *if* $d(x) \leq 6$*, then* $\Delta(\overline{G[N(x)]}) \leq 1$ *and* $|E(\overline{G[N(x)]})| \neq 2$*;*

(b) *if* $d(x) \leq 6$ *and* $d(y) \leq 6$ *then* $xy \notin E(G)$*;*

(c) *if* $d(x) = 7$ *and* $d(y) = 6$ *then* $N[y] \not\subseteq N[x]$*; and*

(d) *if* $d(x) \leq 8$ *and* $d(y) = 5$*, then* $N[y] \not\subseteq N[x]$*.*

We extend the result above in the following lemma.

Lemma 3.4 *Let* G *be an irreducible robust graph and let* $x, y \in V(G)$*. If* $d(x)$*,* $d(y) \leq 6$ *and* $|N(x) \cup N(y)| \leq 7$*, then* (i) $d(x) = d(y) = 5$*;* (ii) $|N(x) \cap N(y)| = 3$*; and* (iii) $G[N(x)] \simeq G[N(y)] \simeq K_5$*.*

Proof The proof consists of the analysis of a series of cases. In each of these cases, we find a reducing triple for G, which contradicts the irreducibility of G. In this extended abstract, we present only the proof of (i).

By Lemma 3.3(b), we have that $xy \notin E(G)$. Assume, without loss of generality, that $d(x) \geq d(y)$. Because G is robust, $d(y) \geq 5$. For a contradiction, suppose that $d(x) = 6$, and let $N(x) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$. In what follows, we analyze the case in which $d(x) = d(y) = 6$; the case in which $d(x) = 6$ and $d(y) = 5$ is similar.

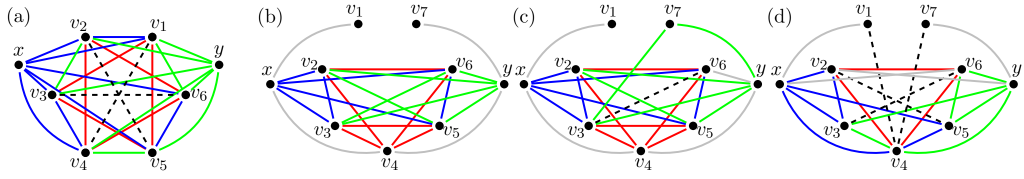

Figure 2. Illustrations for the proof of Lemma 3.4. The triangles in Y containing x and y are illustrated, respectively, in blue and green, while the triangles in Y containing neither x nor y are illustrated in red. Dashed edges illustrate edges that may not exist.

Because $|N(x) \cup N(y)| \leq 7$, we may assume that $N(y) \supseteq \{v_2, v_3, v_4, v_5, v_6\}$. By Lemma 3.3(a), without loss of generality, we may assume that $E(\overline{G[N(x)]}) \subseteq \{v_1v_4, v_2v_5, v_3v_6\}$.

Case 1. $N(y) = N(x)$.

Let $X = E(G[\{v_1, v_2, v_3, v_4, v_5, v_6\}])$ and $Y = \{v_1v_3v_5, v_2v_4v_6, xv_1v_2, yv_2v_3, xv_3v_4, yv_4v_5, xv_5v_6, yv_1v_6\}$. Then $|X| \leq 15 \leq 16 = 2|Y|$ and $(\{x, y\}, X, Y)$ is a reducing triple for G (Figure 2(a)), a contradiction.

Case 2. $N(y) \neq N(x)$.

Then $N(y) = \{v_2, v_3, v_4, v_5, v_6, v_7\}$ for some $v_7 \in \overline{V(G)} \setminus \{v_1\}$. By Lemma 3.3(a), without loss of generality, we may assume that $E(\overline{G[N(y)]}) \subseteq \{v_2v_5, v_3v_6, v_4v_7\}$. Let $H = G[\{v_2, v_3, v_4, v_5, v_6\}]$. We have three cases.

Subcase 2.1. $|E(\overline{H})| = 0$.

Let $X = \{xv_1,\ yv_7\} \cup E(H)$ and $Y = \{v_2v_4v_6,\ v_3v_4v_5,\ xv_2v_3,\ xv_5v_6,\ yv_2v_5,\ yv_3v_6\}$. Therefore $|X| = 2 + |E(H)| = 12 = 2|Y|$ and $(\{x, y\}, X, Y)$ is a reducing triple for G (Figure 2(b)), a contradiction.

Subcase 2.2. $|E(\overline{H})| = 1$.

Without loss of generality, we may assume that $v_2v_5 \in E(H)$. Let $X = \{xv_1, yv_7, v_3v_7\} \cup E(H)$ and $Y = \{v_2v_4v_6, v_3v_4v_5, xv_2v_3, xv_5v_6, yv_2v_5, yv_3v_7\}$. Hence $|X| = 3 + |E(H)| = 12 = 2|Y|$ and $(\{x, y\}, X, Y)$ is a reducing triple for G (Figure 2(c)), a contradiction.

Subcase 2.3. $E(\overline{H}) = \{v_2v_5, v_3v_6\}$.

Let $X = \{xv_1, yv_7\} \cup E(H)$ and $Y = \{v_2v_4v_6, xv_2v_3, yv_3v_4, xv_4v_5, yv_5v_6\}$. So $|X| = 2 + |E(H)| = 10 = 2|Y|$ and $(\{x, y\}, X, Y)$ is a reducing triple for G (Figure 2(d)), a contradiction. □

Using Lemmas 3.3 and 3.4, we verify Tuza's Conjecture for graphs with treewidth at most 6.

Theorem 3.5 *If G is a graph with treewidth at most 6, then $\tau(G) \leq 2\,\nu(G)$.*

Proof Suppose, for a contradiction, that the statement does not hold, and let G be a graph with treewidth at most 6 and $\tau(G) > 2\,\nu(G)$, that minimizes $|V(G)|$. We claim that G is irreducible. Indeed, suppose that G has a reducing triple (V_0, X, Y), and let $G' = (G-X)-V_0$. Note that G' has treewidth at most 6, and $\tau(G') \leq 2\,\nu(G')$ by the minimality of G. By Lemma 3.2, $\tau(G) \leq 2\,\nu(G)$, a contradiction. One can check that G is robust by deriving a contradiction to the minimality of G if there is a vertex v for which $G[N(v)]$ contains a component with at most four vertices. So G is irreducible and robust.

Suppose that $|V(G)| \leq 7$. Then $\Delta(G) \leq 6$, and $E(G) = \emptyset$ by Lemma 3.3(b). Hence $\tau(G) \leq 2\,\nu(G)$, a contradiction. Therefore $|V(G)| \geq 8$. By Proposition 2.3, G has a rooted tree decomposition $(T, \mathcal{V}, r)$. Because $|V(G)| \geq 8$, we have that $|V(T)| > 1$, and hence there is a node $t \in V(T)$ with $h(t) = 1$. If $t \neq r$, let v_t be the representative of t; otherwise, let v_t be an arbitrary vertex of V_t.

First, suppose that $S(t) = \{t'\}$ and let x be the representative of t'. We have that $d(x) \in \{5, 6\}$ by Remark 2.4 and because G is robust. Remark 2.4 applied to $G - x$ and t implies that $N_{G-x}(v_t) \subseteq V_t$, therefore $d(v_t) = |N(v_t)| \leq |N_{G-x}(v_t)| + 1 \leq |V_t| \leq 7$. If $d(v_t) = 7$, then v_t must be adjacent to x and every vertex in $V_t \setminus \{v_t\}$. Because $V_{t'} \subseteq V_t \cup \{x\}$, we have that $N[x] \subseteq N[v_t]$, a contradiction either to Lemma 3.3(c) or to Lemma 3.3(d). So $d(v_t) \leq 6$ and, by Lemma 3.3(b), $v_t x \notin E(G)$. Thus $N(x) \cup N(v_t) \subseteq V_t \setminus \{v_t\}$, and hence $|N(x) \cup N(v_t)| \leq |V_t| - 1 \leq 6$. On the other hand, Lemma 3.4 implies that $d(x) = d(v_t) = 5$ and $|N(x) \cap N(v_t)| = 3$, which implies that $|N(x) \cup N(v_t)| = 7$, a contradiction.

Therefore $|S(t)| > 1$. Now suppose that $|S(t)| \geq 3$, and let t_1, t_2, $t_3 \in S(t)$. Let z_i be the representative of t_i for $i = 1, 2, 3$. By Remark 2.4, $|N(z_i) \cup N(z_j)| \leq |V_t| = 7$, for $i, j \in \{1, 2, 3\}$ and $i \neq j$. Thus, by Lemma 3.4, $d(z_i) = 5$ and $G[N(z_i)] \simeq K_5$ for every $i \in \{1, 2, 3\}$, and $|N(z_i) \cap N(z_j)| = 3$, for $i, j \in \{1, 2, 3\}$ and $i \neq j$. Let $N(z_1) = \{v_1, v_2, v_3, v_4, v_5\}$. Because $|N(z_1) \cap N(z_2)| = 3$, we may assume, without loss of generality, that $N(z_2) = \{v_3, v_4, v_5, v_6, v_7\}$. It is not hard to check that $N(z_3)$ contains exactly one vertex in $N(z_1) \cap N(z_2)$, because $|N(z_1) \cap N(z_3)| = |N(z_2) \cap N(z_3)| = 3$ and $|N(z_3)| = 5$. So we may assume, without loss of generality, that $N(z_3) = \{v_1, v_2, v_3, v_6, v_7\}$. Let $H = G[\{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}]$ and note that every pair of vertices of H is contained in at least one $N(z_i)$ for $i \in \{1, 2, 3\}$. Thus, since $G[N(z_i)] \simeq K_5$ for $i \in \{1, 2, 3\}$, we have $H \simeq K_7$. Let $X = E(H)$ and note that $|X| = 21$. Put

$$Y_1 = \{z_1v_1v_4, z_1v_2v_5, z_2v_3v_4, z_2v_5v_6, z_3v_1v_6, z_3v_2v_3\}$$

and

$$Y_2 = \{v_1v_2v_7, v_2v_4v_6, v_3v_6v_7, v_4v_5v_7, v_1v_3v_5\},$$

and $Y = Y_1 \cup Y_2$. Note that $|Y| = 11$ and thus $|X| \leq 2|Y|$, hence $(\{z_1, z_2, z_3\}, X, Y)$ is a reducing triple for G (Figure 3), a contradiction. We conclude that $|S(t)| = 2$.

Let $t_1, t_2 \in S(t)$ and let x, y be the representatives of t_1, t_2, respectively. Again, $d(x) = d(y) = 5$, $|N(x) \cap N(y)| = 3$, and $G[N(x)], G[N(y)] \simeq K_5$ by Remark 2.4 and Lemma 3.4. Hence $V_t \subseteq N(x) \cup N(y)$. Note that t is a leaf of $(T', \mathcal{V}', r)$, where $T' = T - t_1 - t_2$ and $\mathcal{V}' = \mathcal{V} \setminus \{V_{t_1}, V_{t_2}\}$. Therefore $d_{G-x-y}(v_t) \leq 6$ by Remark 2.4, and $d_G(v_t) \leq 8$. Note that $v_t \in N(x) \cup N(y)$. Without loss of generality, assume that $v_t \in N(x)$. As $G[N(x)]$ is a complete graph, $N[x] \subseteq N[v_t]$, a contradiction to Lemma 3.3(d). This concludes the proof. $\square$

Because a K_8-free chordal graph has treewidth at most 6, Theorem 3.5 generalizes a result of Tuza regarding chordal graphs [15, Proposition 3].

Corollary 3.6 *If G is a K_8-free chordal graph, then $\tau(G) \leq 2\,\nu(G)$.*

4 Planar triangulations

All new definitions in this section are taken from the book of Bondy and Murty [3, Chapter 10]. A graph is *planar* if it can be drawn in the plane so that its edges intersect only at their ends. We refer to such a drawing of a planar graph as a *plane graph*. A plane graph partitions the plane into arcwise-connected open sets, which we call *faces*. The boundary of a face f in a plane graph is the boundary of the open set f in the usual topological sense. Two faces are adjacent if their boundaries have an edge in common. The *dual graph* G^* of a given plane graph G is the multigraph whose vertex set is the set of faces of G, and in which two vertices are joined by k edges if the boundaries of the corresponding faces have k edges in common. Note that if G^* has a bridge, then G contains a loop. Therefore, the dual of a simple plane graph has no bridges. A simple connected plane graph in which all faces have three edges is called a *plane triangulation*. A *planar triangulation* is a graph which admits a plane triangulation. It is not hard to check that a simple connected plane graph is a triangulation if and only if its dual is a bridgeless cubic multigraph. Moreover, the dual of a triangulation has parallel edges only if the triangulation is a single triangle. Also, every planar triangulation has a unique plane triangulation, thus we will refer to its faces and its dual. A *facial triangle* in a plane graph is a triangle that is the boundary of one of its faces.

In this section, we prove that $\tau(G)/\nu(G) \leq 3/2$ for every planar triangulation G different from K_4. It is not hard to check that if $G = K_5 - e$, then $\tau(G)/\nu(G) = 3/2$. Therefore, this bound is tight. If G is a triangle, then $\tau(G) = \nu(G) = 1$. So we may assume that G is not K_3. The proof is divided in two parts. In Lemma 4.2, we give an upper bound for $\tau(G)$ and, in Lemma 4.4, we give a lower bound for $\nu(G)$. For the proof of Lemma 4.2, we need the following theorem of Petersen (see [12]).

Theorem 4.1 (Petersen, 1981) *Every bridgeless cubic graph contains a perfect matching.*

Lemma 4.2 *If G is a planar graph, then $\tau(G) \leq n - 2$.*

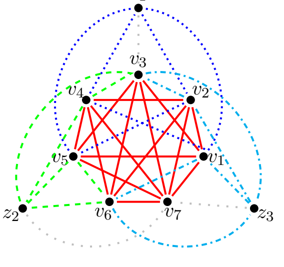

Figure 3. Illustration for the proof of Theorem 3.5. The triangles in Y_1 containing z_1, z_2, and z_3 are illustrated, respectively, in blue (dotted), green (dashed), and cyan (dash-dotted), while the triangles in Y_2 are illustrated in red (solid).

Proof We may assume $n \geq 4$. Let H be a maximal plane graph containing G and note that $\tau(G) \leq \tau(H)$. By Euler's formula, H has $2n-4$ faces. Because H is a plane triangulation, H^* is a bridgeless cubic graph on $2n-4$ vertices. Thus, by Theorem 4.1, H^* contains a perfect matching M^*. Let M be the edges of H corresponding to the edges in M^*. Note that $|M| = |M^*| = n-2$. Since $H^* - M^*$ contains only vertices with even degree, $H - M$ is a bipartite graph. Therefore $H - M$ has no triangles, and hence $\tau(H) \leq |M| = n-2$. □

To prove a lower bound on $\nu(G)$, we use the well-known Brook's Theorem (see [4]). The chromatic number of a graph G is denoted by $\chi(G)$.

Theorem 4.3 (Brooks, 1941) *If G is a connected graph, then $\chi(G) \leq \Delta(G)$ or G is either an odd cycle or a complete graph.*

Lemma 4.4 *If G is a planar triangulation different from K_4, then $\nu(G) \geq \frac{2}{3}(n-2)$.*

Proof We may assume $G \neq K_3$. The dual G^* of G is a cubic graph different from K_4. By Theorem 4.3, there is an independent set Y^* of G^* with at least $\frac{|V(G^*)|}{3}$ vertices. Let Y be the facial triangles of G corresponding to the vertices in Y^*. As Y^* is an independent set in G^*, Y is a triangle packing in G. Thus $\nu(G) \geq |Y| = |Y^*| \geq \frac{|V(G^*)|}{3} = \frac{2}{3}(n-2)$. □

The main result of this section comes directly from Lemmas 4.2 and 4.4.

Theorem 4.5 *If G is a planar triangulation different from K_4, then $\tau(G) \leq \frac{3}{2}\,\nu(G)$.*

We were also able to obtain the following slightly stronger version of Lemma 4.4 for a (finite) class of planar 3-trees. This result will be used in the next section. Given a copy H of K_4 in a planar 3-tree G, we denote by $g(H)$ the pair (i,j) where i (resp. j) is the maximum (resp. minimum) number of vertices of G inside a face of H. We say that a planar 3-tree G is *restricted* if G contains a copy H of K_4 such that $g(H) = (2,0)$. In this case, we always consider the face of H with no vertices of G inside as the external face of G, that is, the vertices of G not in H are drawn in the internal faces of H. If, additionally, H has two faces with precisely one vertex of G inside, then we say that G is *super restricted*.

Proposition 4.6 *If G is a restricted (resp. super restricted) planar 3-tree with f faces, then there is a triangle packing $\mathcal{P}$ of facial triangles of G containing the external face, and such that $|\mathcal{P}| \geq \lceil f/3 \rceil$ (resp. $|\mathcal{P}| = 5$).*

Proof Let v^* be the vertex of G^* corresponding to the external face of G, and let $G' = G^* - N[v^*]$. Since G^* is a cubic graph, $G' \not\simeq K_4$. By Theorem 4.3, G' contains an independent set I' with $\lceil (f-4)/3 \rceil$ vertices, and hence the set $\mathcal{P}$ of facial triangles of G corresponding to the vertices in $I' \cup \{v^*\}$ is a triangle packing of facial triangles of G containing the external face, and such that $|\mathcal{P}| \geq \lceil (f-4)/3 \rceil + 1$. Hence, if $f - 1 \not\equiv 0 \pmod 3$, then $\lceil (f-4)/3 \rceil + 1 = \lceil f/3 \rceil$, and $\mathcal{P}$ is the desired triangle packing. Thus we may assume that $f - 1 \equiv 0 \pmod 3$ and, because G is restricted, G is one of the ten first graphs in Figure 4, which have either 10 faces and at least 4 independent faces (counting the external face), or 16 faces and at least 6 independent faces (counting the external face). If G is super restricted, then G is one of the two last graphs in Figure 4. □

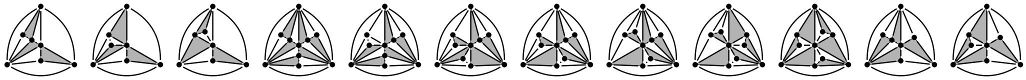

Figure 4. All 10 restricted planar 3-trees with $f \equiv 1 \pmod 3$ faces and the two super restricted ones.

In the full version of the paper, we show that Proposition 4.6 cannot be extended for general planar 3-trees.

5 3-Trees

In this section, we prove that $\tau(G) \leq \frac{9}{5}\,\nu(G) + \frac{1}{5}$ for every 3-tree G. Given a graph G, a transversal X and a triangle packing Y of G, we say that the pair (X, Y) is a $\frac{9}{5}$-*TP* of G if $|X| \leq \frac{9}{5}|Y| + \frac{1}{5}$. Note that if G has a $\frac{9}{5}$-TP, then $\tau(G) \leq \frac{9}{5}\,\nu(G) + \frac{1}{5}$. Let $(T, \mathcal{V}, r)$ be a rooted tree decomposition of a graph G. Given a node $t \in V(T) \setminus \{r\}$, we denote by $R(t)$ the set of all representatives of all descendants of t. Note that the representative of t is also in $R(t)$. Recall that $S(t)$ is the set of successors of t. For every triple of vertices $\Delta \subseteq V_t$, let $S^{\Delta}(t) = \{t' \in S(t) : V_{t'} \cap V_t = \Delta\}$. When t is clear from the context, we simply write S^{Δ}.

Our proof relies on the analysis of nodes with small height, which guarantees that a minimal counterexample has a particular configuration. For nodes with height 1 we present the following lemma.

Lemma 5.1 *Let G be a 3-tree such that $\tau(G) > \frac{9}{5}\,\nu(G) + \frac{1}{5}$, and G is minimal over all such graphs. Let $(T, \mathcal{V}, r)$ be a rooted tree decomposition of G of width 3, and let $t \in V(T) \setminus \{r\}$. If $h(t) = 1$, then $|S(t)| = 1$.*

Proof Let $V_t = \{a, b, c, d\}$. Suppose, for a contradiction, that $|S(t)| > 1$. Without loss of generality, assume that $V_t \cap V_{p(t)} = \{b, c, d\}$. Let $S(t) = \{t_1, t_2, \ldots, t_k\}$. For every $i \in [k]$, let v_i be the representative of t_i. Since $h(t) = 1$, at least one of Δ in $\{abc, abd, acd\}$ is such that $S^{\Delta} \neq \emptyset$. Suppose that exactly one triangle Δ in $\{abc, abd, acd\}$, say $\Delta = abc$, is such that $S^{\Delta} \neq \emptyset$. Let $G' = G - R(t)$. Note that G' is a 3-tree. By the minimality of G, there exists a $\frac{9}{5}$-TP of G', say (X', Y'). Thus, $(X' \cup \{ab, bc, ac\}, Y' \cup \{acv_1, abv_2\})$ is a $\frac{9}{5}$-TP of G, a contradiction. Indeed, $\tau(G) \leq \tau(G') + 3 \leq \frac{9}{5}\,\nu(G') + \frac{16}{5} \leq \frac{9}{5}(\nu(G) - 2) + \frac{16}{5} \leq \frac{9}{5}\,\nu(G) + \frac{1}{5}$ (Figure 5(a)). Thus, we may assume that at most one triangle Δ in $\{abc, abd, acd\}$ is such that $S^{\Delta} = \emptyset$. Suppose that $|S(t)| = 2$ and assume, without loss of generality, that $t_1 \in S^{abc}$ and $t_2 \in S^{abd}$. Let $G' = G - R(t)$. Again G' is a 3-tree. By the minimality of G, there exists a $\frac{9}{5}$-TP of G', say (X', Y'). Let $e \in X' \cap E(bcd)$. Without loss of generality, we have two cases. If $e = bc$, then we let $X = X' \cup \{ad, av_1, bv_2\}$. If $e = cd$, then we let $X = X' \cup \{ab, cv_1, dv_2\}$. In both cases, $(X, Y' \cup \{acv_1, abv_2\})$ is a $\frac{9}{5}$-TP of G, a contradiction (Figure 5(b)). Finally suppose that $|S(t)| \geq 3$.

Without loss of generality, assume that either $t_1 \in S^{abc}$ and $t_2, t_3 \in S^{abd}$, or $t_1 \in S^{abc}$, $t_2 \in S^{abd}$, and $t_3 \in S^{acd}$. Let $G' = G - R(t)$. By the minimality of G, there exists a $\frac{9}{5}$-TP of G', say (X', Y'). Note that $E(bcd) \cap X' \neq \emptyset$. Hence $(X' \cup \{ab, bc, cd, ac, bd, ad\}, Y' \cup \{acv_1, abv_2\, adv_3\})$ is a $\frac{9}{5}$-TP of G, a contradiction (Figure 5(c)). This concludes the proof. $\square$

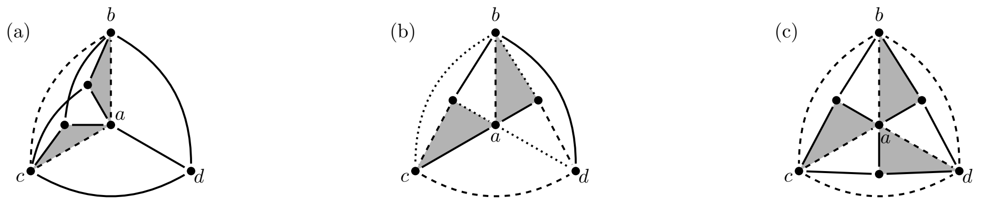

Figure 5. Illustrations for the proof of Lemma 5.1.

In what follows, we prove the main theorem of this section.

Theorem 5.2 *If G is a 3-tree, then $\tau(G) \leq \frac{9}{5}\,\nu(G) + \frac{1}{5}$.*

Proof Suppose, for a contradiction, that the statement does not hold. Let G be a minimal counterexample over all 3-trees with $\tau(G) > \frac{9}{5}\,\nu(G) + \frac{1}{5}$. We claim that if $(T, \mathcal{V}, r)$ is a rooted tree decomposition of G of width 3, then $|V(T)| \geq 4$. Indeed, if $|V(T)| = 1$, then $|V(G)| = 4$, $\tau(G) = 2$, and $\nu(G) = 1$. If $|V(T)| = 2$, then $|V(G)| = 5$, $\tau(G) = 3$, and $\nu(G) = 2$. If $|V(T)| = 3$, then $|V(G)| = 6$, $\tau(G) \leq 4$, and $\nu(G) = 3$. In each case, we contradict $\tau(G) > \frac{9}{5}\,\nu(G) + \frac{1}{5}$. Due to space limitations, we omit the proof of the next claim.

Claim 5.3 *There exists a rooted tree decomposition $(T, \mathcal{V}, r)$ of G of width 3 with a node $t \in V(T) \setminus \{r\}$ such that $h(t) = 2$.*

Let $(T, \mathcal{V}, r)$ be a rooted tree decomposition of G given by Claim 5.3, and let t be a node in T with $h(t) = 2$. Let $L = \{l_1, \ldots, l_m\}$ be the set of successors of t that are leaves, and let $Q = S(t) \setminus L = \{q_1, \ldots, q_k\}$. For every $i \in [m]$, let u_i be the representative of l_i. By Lemma 5.1, $|S(q_i)| = 1$ for every $i \in [k]$. For every such i, let $S(q_i) = \{q_i'\}$, and let v_i be the representative of q_i and v_i' be the representative of q_i' (Figure 6). Let $Q' = \{q_1', \ldots, q_k'\}$. Let $V_t = \{a, b, c, d\}$ and assume, without loss of generality, that $V_t \cap V_{p(t)} = \{b, c, d\}$.

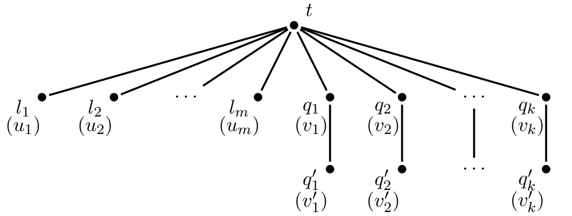

Figure 6. The subtree of T rooted at t.

Claim 5.4 *Let Δ be a triple of vertices in V_t with $\Delta \neq bcd$. Then* (i) *$|S^{\Delta}(t)| \leq 2$; and* (ii) *either $S^{\Delta}(t) \cap L = \emptyset$ or $S^{\Delta}(t) \cap Q = \emptyset$.*

Proof Due to space limitations, we omit the proof of (i). For (ii), suppose, for a contradiction, that $\Delta = abd$ and $l_1, q_1 \in S^{\Delta}(t)$. Say $V_{q_1} \cap V_{q_1'} = \{x_0, x_1, v_1\}$ and let $G' = G - v_1'$. Since G is minimal, there exists a $\frac{9}{5}$-TP, say (X', Y'), of G'. Without loss of generality, assume that $yy' \in X' \cap E(x_0x_1v_1)$, let $z \in \{x_0, x_1, v_1\} \setminus \{y, y'\}$, and let $X = X' \cup \{zv_1'\}$. Note that X is a transversal of G. Suppose $ww' \in E(x_0x_1v_1) \setminus E(Y')$. Hence $(X, Y' \cup \{ww'v_1'\})$ is a $\frac{9}{5}$-TP of G, a contradiction. Thus, every edge of $x_0x_1v_1$ is in $E(Y')$. Since $d_{G'}(v_1) = 3$, we must have $x_0x_1v_1 \in Y'$ and let $x_2v_1 \notin E(Y')$. Suppose that both x_0u_1 and x_1u_1 are not in $E(Y')$. Then $(X, Y' \setminus \{x_0x_1v_1\} \cup \{x_0v_1v_1', x_0x_1u_1\})$ is a $\frac{9}{5}$-TP of G, a contradiction. So at least one between x_0u_1 and x_1u_1, say x_0u_1, is in $E(Y')$. Since $d_{G'}(u_1) = 3$ and $x_0x_1v_1 \in Y'$, we have that $x_0x_2u_1 \in Y'$ and that $x_1u_1 \notin E(Y')$. But then $(X, Y' \setminus \{x_0x_1v_1, x_0x_2u_1\} \cup \{x_1v_1v_1', x_0x_2v_1, x_0x_1u_1\})$ is a $\frac{9}{5}$-TP of G, a contradiction. □

We proceed with the proof of the theorem. Abusing notation, for every triple of vertices $\Delta \subseteq V_t$ such that $\Delta \neq bcd$, let $h(\Delta) = 0$ if $S^{\Delta}(t) = \emptyset$, and $h(\Delta) = 1 + \max\{h(t') : t' \in S^{\Delta}(t)\}$ otherwise. Let $k' = |\{\Delta \subseteq V_t : h(\Delta) = 2\}|$, and $m' = |\{\Delta \subseteq V_t : h(\Delta) = 1\}|$. Let G^+ be the graph obtained from $G[V_t \cup R(t)]$ by removing, for each triple of vertices $\Delta \subseteq V_t$ such that $\Delta \neq bcd$ and $|S^{\Delta}(t)| = 2$, the vertices in $R(x)$, for precisely one $x \in S^{\Delta}(t)$. One can check that, by Claim 5.4, G^+ is a restricted planar 3-tree. Also, G^+ has $f = 4 + 4k' + 2m'$ faces. Let Q^+ be the set $\{q_i \in Q : v_i \notin V(G^+)\}$. By Proposition 4.6, G^+ contains a triangle packing $\mathcal{P}^+$ of facial triangles containing the face bcd and such that $|\mathcal{P}^+| \geq \lceil f/3 \rceil$, and if G^+ is super restricted, then $|\mathcal{P}^+| = 5$. For each $q_i \in Q^+$, let $T_i = v_iv_i'w_i$, where w_i is a vertex adjacent to both v_i and v_i', and let $\mathcal{P} = \mathcal{P}^+ \cup \{T_i : q_i \in Q^+\}$. Note that $|\mathcal{P}| = |\mathcal{P}^+| + (k - k')$.

Claim 5.5 *Let $G' = G - R(t)$. If X' is a transversal of G', then* (i) *there exists a transversal of G with size $|X'| + 1 + 2k + m$; and* (ii) *there exists a transversal of G with size at most $|X'| + 5 + k$.*

Proof (i) Because X' is a transversal of G' and bcd is a triangle in G', $X' \cap E(bcd) \neq \emptyset$. Without loss of generality, assume that $bc \in X'$. Given a 4-clique K in G and a set $X \subseteq E(G)$ such that $E(K) \cap X = \{e\}$, denote by $K \otimes X$ the only edge in K that does not share any vertex with e. For instance, $V_t \otimes \{bc\} = ad$. For every $i \in [m]$, let $e_i = V_{l_i} \otimes \{bc, ad\}$. For every $i \in [k]$, let $f_i = V_{q_i} \otimes \{bc, ad\}$ and $f_i' = V_{q_i'} \otimes \{bc, ad, f_i\}$. Note that $X = X' \cup \{ad\} \cup \{e_i : i \in [m]\} \cup \{f_i : i \in [k]\} \cup \{f_i' : i \in [k]\}$ is a transversal of G and $|X| = |X'| + 1 + 2k + m$. (ii) Let $X = X' \cup E(G[V_t]) \cup \{v_iv_i' : i \in [k]\}$. Note that X is a transversal of G. Because X' is a transversal of G' and bcd is a triangle in G', $X' \cap E(bcd) \neq \emptyset$, and hence $|X' \cap E(G[V_t])| \geq 1$. Thus $|X| = |X'| + |E(G[V_t])| - |X' \cap E(G[V_t])| + k \leq |X'| + 5 + k$. □

Let $G' = G - R(t)$ and let (X', Y') be a $\frac{9}{5}$-TP of G'. The only triangle in $\mathcal{P}$ containing edges of G' is bcd. Therefore $Y = Y' \cup (\mathcal{P} \setminus \{bcd\})$ is a triangle packing of G of size $|Y'| + |\mathcal{P}^+| - 1 + (k - k')$. By Claim 5.5, there exists a transversal X of G with size $|X'| + \min\{1 + 2k + m, 5 + k\}$. Note that $k' \geq 1$ because $h(t) = 2$.

Also $k' + m' \leq 3$. If $m' = 2$, then $k' = 1$ and G^+ is super restricted. In this case, because $5(5+k) < 9(3+k)$ we have that $|X| \leq |X'| + 5 + k < \frac{9}{5}(|Y'| + 3 + k) + \frac{1}{5} = \frac{9}{5}(|Y'| + |\mathcal{P}^+| - 1 + (k - k')) + \frac{1}{5} \leq \frac{9}{5}|Y| + \frac{1}{5}$. So we may assume that $m' \leq 1$. Note that, for all possible values of k' and m', we have $\lceil f/3 \rceil - 1 + (k - k') = \lceil (4+4k'+2m')/3 \rceil - 1 + k - k' = 1 + k + \lceil (k'-2+2m')/3 \rceil = 1 + k + \lfloor k'/3 \rfloor + \lceil m'/2 \rceil$.

Suppose that $k \leq 2$. Then we have that $10k \leq 9k + 2$ and $5m \leq 9\lceil m'/2 \rceil + 2$ because $m' \leq m \leq 2m' \leq 2$, and hence $5(1 + 2k + m) = 5 + 10k + 5m \leq 5 + 9k + 2 + 9\lceil m'/2 \rceil + 2 = 9(1 + k + \lceil m'/2 \rceil)$. Moreover, we have

$$|X| \leq |X'| + 1 + 2k + m \leq \frac{9}{5}|Y'| + \frac{1}{5} + \frac{9}{5}\left(1 + k + \left\lceil \frac{m'}{2} \right\rceil\right) \leq \frac{9}{5}|Y| + \frac{1}{5}.$$

Now suppose that $k' \leq 2$ and $m' = 0$. Since $k' \geq \lceil k/2 \rceil$, we have that $k \leq 4$, and hence $10k \leq 9k + 4$. Moreover, we have that $m = 0 = \lceil m'/2 \rceil$, and hence $5(1 + 2k + m) = 5 + 10k + 5m \leq 5 + 9k + 4 + 9\lceil m'/2 \rceil = 9(1 + k + \lceil m'/2 \rceil)$. Analogously to the case above, we have $|X| \leq |X'| + 1 + 2k + m \leq \frac{9}{5}|Y| + \frac{1}{5}$. Hence we may assume that $k \geq 3$, and either $k' = 3$ or $m' \neq 0$. This implies that $5k = 9k - 4k \leq 9k - 12$, and therefore $5(5 + k) = 25 + 5k \leq 9k + 13 \leq 9(k + 2)$. Moreover, we have that $\lfloor k'/3 \rfloor + \lceil m'/2 \rceil \geq 1$. So, the proof is complete because

$$\begin{aligned} |X| \leq |X'| + 5 + k &\leq \frac{9}{5}|Y'| + \frac{1}{5} + \frac{9}{5}(k + 2) \\ &\leq \frac{9}{5}\left(|Y'| + 1 + k + \left\lfloor \frac{k'}{3} \right\rfloor + \left\lceil \frac{m'}{2} \right\rceil\right) + \frac{1}{5} = \frac{9}{5}|Y| + \frac{1}{5}. \end{aligned}$$

□

6 Concluding remarks

In this paper we present three results related to Tuza's Conjecture. In Section 3, we obtained a lemma (Lemma 3.4) that extends Puleo's tools [13], and allowed us to verify Tuza's Conjecture for graphs with treewidth at most 6. Since any minimal counterexample to Tuza's Conjecture is an irreducible robust graph, Lemma 3.2 may be used in further results regarding this problem. In Sections 4 and 5, we obtained stronger versions of Tuza's Conjecture for specific classes of graphs. We believe that the techniques used here may also be useful to deal with other classes of graphs, perhaps by introducing new ingredients.

References

[1] J. D. Baron and J. Kahn. Tuza's conjecture is asymptotically tight for dense graphs. *Combin. Probab. Comput.*, 25(5):645–667, 2016.

[2] H. L. Bodlaender. A partial k-arboretum of graphs with bounded treewidth. *Theoret. Comput. Sci.*, 209(1-2):1–45, 1998.

[3] J. A. Bondy and U. S. R. Murty. *Graph Theory*, volume 244 of *Graduate Texts in Mathematics*. Springer, New York, 2008.

[4] R. L. Brooks. On colouring the nodes of a network. *Math. Proc. Cambridge Philos. Soc.*, 37(2):194–197, 1941.

[5] X. Chen, Z. Diao, X. Hu, and Z. Tang. Sufficient conditions for Tuza's conjecture on packing and covering triangles. In *Combinatorial Algorithms*, volume 9843 of *Lecture Notes in Comput. Sci.*, pages 266–277. Springer, 2016.

[6] Q. Cui, P. Haxell, and W. Ma. Packing and covering triangles in planar graphs. *Graphs and Combin.*, 25(6):817–824, 2009.

[7] J. L. Gross. Embeddings of graphs of fixed treewidth and bounded degree. *Ars Math. Contemp.*, 7(2):379–403, 2014.

[8] P. E. Haxell. Packing and covering triangles in graphs. *Discrete Math.*, 195(1):251–254, 1999.

[9] P. E. Haxell and Y. Kohayakawa. Packing and covering triangles in tripartite graphs. *Graphs and Combin.*, 14(1):1–10, 1998.

[10] P. E. Haxell, A. Kostochka, and S. Thomassé. Packing and covering triangles in K_4-free planar graphs. *Graphs and Combin.*, 28(5):653–662, 2012.

[11] M. Krivelevich. On a conjecture of Tuza about packing and covering of triangles. *Discrete Math.*, 142(1–3):281–286, 1995.

[12] J. Petersen. Die Theorie der regulären Graphs. *Acta Math.*, 15(1):193–220, 1891.

[13] G. J. Puleo. Tuza's conjecture for graphs with maximum average degree less than 7. *European J. of Combin.*, 49:134–152, 2015.

[14] Z. Tuza. Conjecture, finite and infinite sets. In *Proceedings of the Colloquia Mathematica Societatis Janos Bolyai, Eger, Hungary, North-Holland, Amsterdam*, page 888, 1981.

[15] Z. Tuza. A conjecture on triangles of graphs. *Graphs and Combin.*, 6(4):373–380, 1990.