Available online at www.sciencedirect.com

SciVerse ScienceDirect

AASRI Procedia 1 (2012) 63 – 67

AASRI Procedia

www.elsevier.com/locate/procedia

2012 AASRI Conference on Computational Intelligence and Bioinformatics

Asymptotic Stability of a Class of Fourth Order Nonlinear Systems

Zhigang Zhang[a], Cong Wang[b] *

[a]*School of Mathematics and Physics, USTB, Beijing 100083, China*
[b]*School of Economics and Management USTB, Beijing 100083, China*

Abstract

In this paper, two asymptotic stability criterions of the trivial solutions to a class of forth order nonlinear systems are improved, and an error which exists in the proof of Theorem 2 of the paper [1] is pointed out.

Keywords: nonlinear system, Liapunov function, asymptotic stability

Introduction

Lyapunov stability is named after Aleksandr Lyapunov, a Russian mathematician who published his book "The General Problem of Stability of Motion" in 1892[1]. Lyapunov was the first to consider the modifications necessary in nonlinear systems to the linear theory of stability based on linearizing near a point of equilibrium. His work, initially published in Russian and then translated to French, received little attention

* Corresponding author. Tel.: 13601392801, 15210600309
E-mail address: zzgcyf@263.net , wangcong-1990@163.com.

doi:10.1016/j.aasri.2012.06.012

for many years. Interest in it started suddenly during the Cold War (1953-1962) period when the so-called "Second Method of Lyapunov" was found to be applicable to the stability of aerospace guidance systems which typically contain strong nonlinearities not treatable by other methods. A large number of publications appeared then in the control and systems literature. More recently the concept of Lyapunov exponent (related to Lyapunov's First Method of discussing stability) has received wide interest in connection with chaos theory. Lyapunov stability methods have also been applied to finding equilibrium solutions in traffic assignment problems following the work by MJ Smith and MB Wisten.

Various types of stability may be discussed for the solutions of differential equations describing dynamical systems. The most important type is that concerning the stability of solutions near to a point of equilibrium. This may be discussed by the theory of Lyapunov. In simple terms, if all solutions of the dynamical system that start out near an equilibrium point x_e and stay near x_e forever, then x_e is Lyapunov stable. More strongly, if x_e is Lyapunov stable and all solutions that start out near x_e and converge to x_e, then $x_e$is asymptotically stable. The notion of exponential stability guarantees a minimal rate of decay, i.e., an estimate of how quickly the solutions converge. The idea of Lyapunov stability can be extended to infinite-dimensional manifolds, where it is known as structural stability, which concerns the behavior of different but "nearby" solutions to differential equations. Input-to-state stability (ISS) applies Lyapunov notions to systems with inputs.

In the paper[3], Liapunov function for a fourth order linear system with constant coefficients is given by means of the Barbashin's formula, and then stability of the trivial solutions to a class of forth order nonlinear systems is studied by applying the comparison method. But, we think it necessary to point out the following two problems:

First, the two stable theorems in [2] are treaded $yz > 0$ or $(az+u)\operatorname{sgn} y \geq 0, (dx+cy)\operatorname{sgn} z \geq 0$ as stability condition, therefore the given conditions in [2] cannot ensure that the function $V(x, y, z, u)$ is positive in arbitrarily small neighborhood of the origin.

Next, an error exists in the proof of theorem 2 in [2]. Because we cannot assert that

$$(a(\frac{y}{z})^2 + c)|z| \leq (ah^2 + c)l$$

by $a > 0, c > 0$ and $k = nl$ in the region $|y| \leq k, |z| \leq l$, the paper [2] hasn't finished the proving of Theorem 2.

In this Paper, we improve upon the two asymptotic stability criterions in [2] and obtain the following conclusions.

Theorem 1. Consider the fourth order nonlinear equation

$$x^{(4)} + ax^{(3)} + bx^{(2)} + cx^{(1)} + f(x) = 0 \tag{1}$$

or the equivalent system of (1)

$$\begin{cases} x^{(1)} = y, y^{(1)} = z, z^{(1)} = u, \\ u^{(1)} = -f(x) - cy - bz - au \end{cases} \tag{2}$$

where constants $a > 0, b > 0, c > 0$ and $f(0) = 0$.

If there exists a constant $h > 0$, the function $f(x)$ which has continuous derivative of second order satisfies the following conditions

$$abc - c^2 - a^2 f'(x) \geq \varepsilon > 0, f'(x) \geq \varepsilon_1 > 0, |f''(x)| \leq \frac{2\theta\varepsilon}{ah}$$

where $0 < \theta < 1$ in the neighborhood $|x| \leq h$, then the trivial solution of the system(2) is asymptotically stable.

Proof. By the given conditions we easily obtain $xf(x) > 0$ and

$$\int_0^x [abc - c^2 - a^2 f'(\eta)] f(\eta) d\eta > 0 \text{, when } 0 < |x| \leq h.$$

Form the function

$$V(x, y, z, u) = \frac{a^3}{c}[f(x) + cy + \frac{c}{a}z]^2 + [(ab - c)y + a^2 z + au]^2$$
$$+ \frac{2a}{c}\int_0^x [abc - c^2 - a^2 f'(\eta)] f(\eta) d\eta + [abc - c^2 - a^2 f'(x)] y^2$$

Obviously, $V(x, y, z, u)$ is positive in the neighborhood $\|(x, y, z, u)\| \leq h$, the total derivative of V for the system (2) is

$$\begin{aligned} \frac{dV}{dt} &= -2a[abc - c^2 - a^2 f'(x)] y^2 - a^2 f''(x) y^3 \\ &= -2a[abc - c^2 - a^2 f'(x) + \frac{a}{2} f''(x) y] y^2 \\ &\leq -2a[\varepsilon - \frac{ah}{2}|f''(x)|] y^2 \leq -2a(1 - \theta)\varepsilon y^2 \leq 0 \end{aligned}$$

and $dV/dt = 0$ can only be achieved by $(x, y, z, u) = (0,0,0,0)$ .Hence the trivial solution of (2) is asymptotically stable by the BarBashin-Krasovskii's theorem.

Theorem 2. Consider the fourth order nonlinear equation

$$x^{(4)} + ax^{(3)} + \varphi'(x^{(1)})x^{(2)} + cx^{(1)} + dx = 0 \tag{3}$$

or the equivalent system of (3)

$$\begin{cases} x^{(1)} = y, y^{(1)} = z, z^{(1)} = u, \\ u^{(1)} = -dx - cy - \varphi'(y)z - au \end{cases} \tag{2}$$

where constants $a > 0, c > 0$ and $d > 0$.

If there exists a constant $h > 0$, the function $\varphi(x)$ which has continuous derivative of second order satisfies the following conditions

$$ac\varphi'(y) - c^2 - a^2 d \ge \varepsilon > 0, |\varphi'(y)| \le \frac{2\theta\varepsilon}{ch}$$

where $0 < \theta < 1$ in the neighborhood $|y| \le h$, then the trivial solution of the system(4) is asymptotically stable.

Proof . Form the function

$$\begin{aligned} V(x, y, z, u) &= a[dx + cy + \frac{c}{a}z]^2 + \frac{a^2}{c}[dy + cz + \frac{c}{a}u]^2 \\ &+ \frac{2d}{c}\int_0^y [ac\varphi'(\eta) - c^2 - a^2 d]\eta d\eta + \frac{1}{a}[ac\varphi'(\eta) - c^2 - a^2 d]z^2 \end{aligned}$$

It is easy to see that $V(x, y, z, u)$ is positive in the neighborhood $\|(x, y, z, u)\| \le h$, the total derivative of V for the system (4) is

$$\begin{aligned} \frac{dV}{dt} &= -2[ac\varphi'(y) - c^2 - a^2 d]z^2 + c\varphi''(y)z^3 \\ &= -2[ac\varphi'(\eta) - c^2 - a^2 d - \frac{c}{2}\varphi''(y)z]z^2 \\ &\le -2[\varepsilon - \frac{ch}{2}|\varphi''(y)|]z^2 \le -2(1-\theta)\varepsilon z^2 \le 0 \end{aligned}$$

and $dV/dt = 0$ can only be achieved by $(x, y, z, u) = (0,0,0,0)$ .Therefore, the trivial solution of (4) is asymptotically stable by the BarBashin-Krasovskii's theorem.

References

[1] Lyapunov A.M. *The General Problem of the Stability of Motion* (In Russian), Doctoral dissertation, Univ. Kharkov 1892

[2] Liang Zaizhong. A study of the construction of Lyapunov function for a class of forth order nonlinear systems.[J], Applied Mathematics and Mechanics, 16:2(1995):181~188,(in Chinese)

[3]Wang Lian,Wang Muqiu.Qualitative analysis of nonlinear ordinary differential equations.Haerbin.[J], Haerbin Polytechnic University Press(1987):381~422,(in Chinese)