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ORIGINAL ARTICLE

Generalized covering approximation space and near concepts with some applications

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Abstract In this paper, we shall integrate some ideas in terms of concepts in topology. First, we introduce some new concepts of rough membership relations and functions in the generalized covering approximation space. Second, we introduce some topological applications namely "near concepts" in the generalized covering approximation space. Accordingly, several types of fuzzy sets are constructed. The basic notions of near approximations are introduced and sufficiently illustrated. Near concepts are provided to be easy tools to classify the sets and to help for measuring exactness and roughness of sets. Many proved results, examples and counter examples are provided. Finally, we give two practical examples to illustrate our approaches.

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1. Introduction

Rough set theory, a mathematical tool to deal with inexact or uncertain knowledge in information systems, has originally described the indiscernibility of elements by equivalence relations. Covering rough sets [1–9,11,12] is a natural extension of classical rough sets by relaxing the partitions arising from equivalence relations to coverings. In our work [6], we have introduced a framework to generalize covering approximation space that was introduced by Zhu [11]. In fact, we have introduced the generalized covering approximation space $\mathcal{G}_n - \boldsymbol{CAS}$ as a generalization to rough set theory and covering approximation space. The $\mathcal{G}_n - \boldsymbol{CAS}$ is defined by the triple $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$, where $U \neq \emptyset$ be a finite set, $\mathcal{R}$ be a binary relation on U and $\mathcal{C}_n$ be n-cover of U associated to $\mathcal{R}$, where $n \in \{r, l\}$ (*for more details see* [6]).

The main works in this paper are divided into three parts. In the beginning of work, we introduce some new generalized definitions to rough membership relations (resp. functions) and new types of fuzzy sets in $\mathcal{G}_n - \boldsymbol{CAS}$. Second part aims to introduce one of an important topological concepts which are called "near concepts" in rough context (specially, in $\mathcal{G}_n - \boldsymbol{CAS}$). In fact, we apply near concepts in $\mathcal{G}_n - \boldsymbol{CAS}$ to define different tools for modifying the original operations. The suggested methods in this paper represent easy mathematical tools to approximate the rough sets and removing the uncertainty (vagueness) of sets. In addition, comparisons between the suggested methods are obtained and many examples (resp. counter examples) to illustrate these connections are provided. Hence, we can say that our approaches are very useful in rough context namely, in information analysis and in decision making. Finally, in the end of paper, simple practical examples are provided to illustrate the suggested methods and to show the importance of these methods in rough context namely in information system and in multi-valued information system. In addition, we give some comparisons between our approaches and others approaches such as Pawlak and Lin approaches.

2. *j*-Rough membership relations, *j*-rough membership functions and *j*-fuzzy sets

The present section is devoted to introduce new definitions for rough membership relations and functions as easy tools to classify the sets and help for measuring exactness and roughness of sets. These rough membership functions allow us to define four different fuzzy sets in $\mathcal{G}_n - \boldsymbol{CAS}$. Moreover, the suggested rough membership relations (resp. functions) are more accurate than classical rough membership function that was given by Lin [10] and the other types.

Definition 2.1. Let $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$ be a $\mathcal{G}_n - \boldsymbol{CAS}$, and $A \subseteq U$. Then we say that:

(i) x is "j-surely" belongs to A, written $x \underline{\in}_j A$, if $x \in \underline{\mathcal{R}_j}(A)$.
(ii) x is "j-possibly" belongs to X, written $x \bar{\in}_j A$, if $x \in \overline{\mathcal{R}}_j(A)$.

These two rough membership relations are called "j-strong" and "j-weak" membership relations respectively.

Lemma 2.1. *Let $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$ be a $\mathcal{G}_n - \boldsymbol{CAS}$, and $A \subseteq U$. Then the following statements are true in general:*

(i) $x \underline{\in}_j A$ implies $x \in A$.	(ii) $x \in A$ implies $x \overline{\in}_j A$.

Proof. Straightforward. □

The converse of the above lemma is not true in general, as the following example illustrates:

Example 2.1. Let $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$ be a $\mathcal{G}_n - \boldsymbol{CAS}$, where $U = \{a, b, c, d\}$ and $\mathcal{R} = \{(a,a), (b,b), (c,c), (c,b), (c,d), (d,a)\}$. We will show the above remark in case of $j = r$ and the other cases similarly: Suppose that $A = \{a, b, d\}$, then we get $\underline{\mathcal{R}}_r(A) = \{a, b\}$ and $\overline{\mathcal{R}}_r(A) = U$. Clearly $d \in A$ but $d \,\underline{\notin}_r\, A$ and $c \,\overline{\in}_r\, A$ but $c \notin A$.

The following proposition is very interesting since it is give the relationships between different types of membership relations $\underline{\in}_j$ and $\overline{\in}_j$. Accordingly, we will illustrate the importance of using these different types of these membership relations.

Proposition 2.1. *Let $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$ be a $\mathcal{G}_n - \boldsymbol{CAS}$, and $A \subseteq U$. Then the following are true generally*

(i) *If* $x \underline{\in}_i A \Rightarrow x \underline{\in}_r A \Rightarrow x \underline{\in}_u A$.	(iii) *If* $x \overline{\in}_u A \Rightarrow x \overline{\in}_r A \Rightarrow x \overline{\in}_i A$.
(ii) *If* $x \underline{\in}_i A \Rightarrow x \underline{\in}_l A \Rightarrow x \underline{\in}_u A$.	(iv) *If* $x \overline{\in}_u A \Rightarrow x \overline{\in}_l A \Rightarrow x \overline{\in}_i A$.

Proof. We will prove the first statement and the others similarly:

(i) If $x \underline{\in}_i A \Rightarrow x \in \underline{\mathcal{R}}_i(A) \Rightarrow x \in \underline{\mathcal{R}}_r(A) \Rightarrow x \underline{\in}_r A$.

Also, if $x \underline{\in}_r A \Rightarrow x \in \underline{\mathcal{R}}_r(A) \Rightarrow x \in \underline{\mathcal{R}}_u(A) \Rightarrow x \underline{\in}_u A$. □

The converse of the above proposition is not true in general as the following example illustrates.

Example 2.2. Let $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$ be a $\mathcal{G}_n - \boldsymbol{CAS}$, where $U = \{a, b, c, d\}$ and $\mathcal{R} = \{(a,a), (a,b), (b,c), (b,d), (c,a), (d,a)\}$. Suppose that $A = \{b, c, d\}$. Then $\underline{\mathcal{R}}_u(A) = \emptyset$, $\underline{\mathcal{R}}_r(A) = \{c, d\}$, $\underline{\mathcal{R}}_l(A) = \{b\}$ and $\underline{\mathcal{R}}_i(A) = \{b, c, d\}$. Accordingly, $c \underline{\in}_r A$ and $b \underline{\in}_l A$ but $b \,\underline{\notin}_u\, A$ and $c \,\underline{\notin}_u\, A$. Also $b \underline{\in}_i A$ and $c \underline{\in}_i A$ but $b \,\underline{\notin}_r\, A$ and $c \,\underline{\notin}_l\, A$. By similar way, we can illustrate the others cases.

Definition 2.2. Let $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$ be a $\mathcal{G}_n - \boldsymbol{CAS}$, and $A \subseteq U$. Then for each $j \in \{r, l, i, u\}$ and $x \in U$:

The j- rough membership functions on U for subset A are $\mu_A^j : U \to [0, 1]$ where $\mu_A^j(x) = \frac{|N_j(x) \cap A|}{|N_j(x)|}$ and $|A|$ denotes the cardinality of A.

The rough j-membership function expresses conditional probability that x belongs to A given $\mathcal{R}$ and can be interpreted as a degree that x belongs to A in view of information about x expressed by $\mathcal{R}$. Moreover, in case of infinite universe, the above membership function μ_A^j can be use for spaces having locally finite minimal neighborhoods for each point.

Lemma 2.2. *Let $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$ be a $\mathcal{G}_n - \boldsymbol{CAS}$, and $A \subseteq U$. Then for every $x \in U$*

(i) $\mu_A^u(x) = 1 \Rightarrow \mu_A^r(x) = 1 \Rightarrow \mu_A^i(x) = 1$.	(ii) $\mu_A^u(x) = 0 \Rightarrow \mu_A^r(x) = 0 \Rightarrow \mu_A^i(x) = 0$.
(iii) $\mu_A^u(x) = 1 \Rightarrow \mu_A^l(x) = 1 \Rightarrow \mu_A^i(x) = 1$.	(iv) $\mu_A^u(x) = 0 \Rightarrow \mu_A^l(x) = 0 \Rightarrow \mu_A^i(x) = 0$.

Proof. We will prove first statement and the others similarly:

(i) $\mu_A^u(x) = 1 \Rightarrow x \in_u A \Rightarrow x \in_r A \Rightarrow \mu_A^r(x) = 1$. Also, $\mu_A^r(x) = 1 \Rightarrow x \in_r A \Rightarrow x \in_i A \Rightarrow \mu_A^i(x) = 1$. □

Remark 2.1.

(i) According to the above results, we can prove that μ_A^i is more accurate than the others types that is:
  (1) If $x \in A \Rightarrow \mu_A^u(x) \leqslant \mu_A^r(x) \leqslant \mu_A^i(x)$ and if $x \in A \Rightarrow \mu_A^u(x) \leqslant \mu_A^l(x) \leqslant \mu_A^i(x)$.
  (2) If $x \notin A \Rightarrow \mu_A^i(x) \leqslant \mu_A^r(x) \leqslant \mu_A^u(x)$ and if $x \in A \Rightarrow \mu_A^i(x) \leqslant \mu_A^l(x) \leqslant \mu_A^u(x)$.
(ii) The converse of the above lemma is not true in general.

The following example illustrates Remark 2.1.

Example 2.3. According to Example 2.2, consider the subset $A = \{b, c, d\}$. Then we get

$\mu_A^r(a) = \frac{\|\{a\} \cap A\|}{\|\{a\}\|} = 0$.	$\mu_A^l(a) = \frac{\|\{a\} \cap A\|}{\|\{a\}\|} = 0$.
$\mu_A^r(b) = \frac{\|\{a,b\} \cap A\|}{\|\{a,b\}\|} = \frac{1}{2}$.	$\mu_A^l(b) = \frac{\|\{b\} \cap A\|}{\|\{b\}\|} = 1$.
$\mu_A^r(c) = \frac{\|\{c,d\} \cap A\|}{\|\{c,d\|} = 1$.	$\mu_A^l(c) = \frac{\|\{a,c,d\} \cap A\|}{\|\{a,c,d\}\|} = \frac{2}{3}$.
$\mu_A^r(d) = \frac{\|\{c,d\} \cap A\|}{\|\{c,d\}\|} = 1$.	$\mu_A^l(d) = \frac{\|\{a,c,d\} \cap A\|}{\|\{a,c,d\}\|} = \frac{2}{3}$.

$\mu_A^i(a) = \frac{\lvert\{a\}\cap A\rvert}{\lvert\{a\}\rvert} = 0.$	$\mu_A^u(a) = \frac{\lvert\{a\}\cap A\rvert}{\lvert\{a\}\rvert} = 0.$
$\mu_A^i(b) = \frac{\lvert\{b\}\cap A\rvert}{\lvert\{b\}\rvert} = 1.$	$\mu_A^u(b) = \frac{\lvert\{a,b\}\cap A\rvert}{\lvert\{a,b\}\rvert} = \frac{1}{2}.$
$\mu_A^i(c) = \frac{\lvert\{c,d\}\cap A\rvert}{\lvert\{c,d\}\rvert} = 1.$	$\mu_A^u(c) = \frac{\lvert\{a,c,d\}\cap A\rvert}{\lvert\{a,c,d\}\rvert} = \frac{2}{3}.$
$\mu_A^i(d) = \frac{\lvert\{c,d\}\cap A\rvert}{\lvert\{c,d\}\rvert} = 1.$	$\mu_A^u(d) = \frac{\lvert\{a,c,d\}\cap A\rvert}{\lvert\{a,c,d\}\rvert} = \frac{2}{3}.$

One of the key issues in all fuzzy sets is how to determine fuzzy membership functions. A membership functions provides a measure of the degree of similarity of element to fuzzy set. The following definition uses the j-rough membership functions μ_A^j to define four different types of fuzzy sets in $\mathcal{G}_n - \boldsymbol{CAS}$.

Definition 2.3. Let $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$ be a $\mathcal{G}_n - \boldsymbol{CAS}$, and $A \subseteq U$. Then the j-fuzzy sets in U is the set of ordered pairs:

$$\widetilde{A}_j = \left\{ \left(x, \mu_A^j(x)\right) \middle| x \in U \right\}, \quad \forall j \in \{r, l, i, u\}.$$

Example 2.4. According to Example 2.3, consider the subset $A = \{b, c, d\}$. Then we get

$$\widetilde{A}_r = \left\{ (a,0), \left(b, \frac{1}{2}\right), (c,1), (d,1) \right\}, \quad \widetilde{A}_l = \left\{ (a,0), (b,1), \left(c, \frac{2}{3}\right), \left(d, \frac{2}{3}\right) \right\},$$

$$\widetilde{A}_u = \left\{ (a,0), \left(b, \frac{1}{2}\right), \left(c, \frac{2}{3}\right), \left(d, \frac{2}{3}\right) \right\}, \quad \text{and} \quad \widetilde{A}_i = \{(a,0), (b,1), (c,1), (d,1)\}.$$

3. Near rough concepts in the generalized covering approximation space $\mathcal{G}_n$ $\boldsymbol{CAS}$

The main goal of this section is to introduce one of the important topological applications which are named "near concepts" in $\mathcal{G}_n - \boldsymbol{CAS}$. Moreover, we introduce the new concepts "j-near approximations" (resp. j-near boundary regions and j-near accuracy measures) to generalize the j-approximations (resp. j-boundary regions and j-accuracy measures). In addition, we introduce near exactness and near roughness by applying near concepts to make more accuracy for definability of sets in $\mathcal{G}_n - \boldsymbol{CAS}$.

Definition 3.1. Let $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$ be a $\mathcal{G}_n - \boldsymbol{CAS}$, and $A \subseteq U$. Thus we define "near rough" sets in U as follows: For each $j \in \{r, l, i, u\}$, the subset A is called

(i) j-Pre rough set (briefly p_j) if $A \subseteq \underline{\mathcal{R}}_j\left(\overline{\mathcal{R}}_j(A)\right)$.
(ii) j-Semi rough set (briefly s_j) if $A \subseteq \overline{\mathcal{R}}_j\left(\underline{\mathcal{R}}_j(A)\right)$.
(iii) γ_j-rough set if $A \subseteq \left[\left(\underline{\mathcal{R}}_j\left(\overline{\mathcal{R}}_j\right)\right) \cup \overline{\mathcal{R}}_j\left(\underline{\mathcal{R}}_j(A)\right)\right]$.

The above sets are called "j-near rough sets" and the families of j-near rough sets of U denotes by $K_j(U)$, for each $K \in \{P, S, \gamma\}$.

Remark 3.1. The family of j-pre rough sets and the family of j-semi rough sets are not comparable as the following example illustrates.

Example 3.1. Let $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$ be a $\mathcal{G}_n - \boldsymbol{CAS}$, where $U = \{a, b, c, d\}$ and

$\mathcal{R} = \{(a,a), (a,b), (b,a), (b,b), (c,a), (c,b), (c,c), (c,d), (d,d)\}$.

We will show the above remark in case of $j = r$ and the other cases similarly as follows:

$N_r(a) = \{a, b\} = N_r(b), \quad N_r(c) = U, \quad N_r(d) = \{d\}$.

Thus, we compute the j-near rough sets for $j = r$ as follows:

The family of r-pre rough sets is: $P_r(U) = \{U, \emptyset, \{a\}, \{b\}, \{d\}, \{a,b\}, \{a,d\}, \{b,d\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}\}$.

The family of r-semi rough sets is: $S_r(U) = \{U, \emptyset, \{d\}, \{a,b\}, \{c,d\}, \{a,b,c\}, \{a,b,d\}\}$.

The main goal of the following definitions is to introduce the new approximation operators (j-near approximations) which modify and generalize the j-approximations.

Definition 3.2. Let $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$ be a $\mathcal{G}_n - \boldsymbol{CAS}$, and $A \subseteq U$. Then we define the j-near approximations of any subset A as follows: For each $j \in \{r, l, i, u\}$

(i) The j-pre lower and the j-pre upper approximations of A are defined respectively by

$$\underline{\mathcal{R}}_j^p(A) = \left\{x \in A | N_j(x) \subseteq \overline{\mathcal{R}}_j(A)\right\} \quad \text{and} \quad \overline{\mathcal{R}}_j^p(A) = A \cup \{x \in A^c | N_j(x) \cap \underline{\mathcal{R}}_j(A) \neq \emptyset\}$$

(ii) The j-semi lower and the j-semi upper approximations of A are defined respectively by

$$\underline{\mathcal{R}}_j^s(A) = \left\{x \in A \middle| N_j(x) \cap \underline{\mathcal{R}}_j(A) \neq \emptyset\right\} \text{ and } \overline{\mathcal{R}}_j^s(A) = A \cup \left\{x \in \overline{\mathcal{R}}_j(A) \middle| N_j(x) \subseteq \overline{\mathcal{R}}_j(A)\right\}$$

(iii) The γ_j-lower and the γ_j-upper approximations of A are defined respectively by

$$\underline{\mathcal{R}}_j^\gamma(A) = \underline{\mathcal{R}}_j^p(A) \cup \underline{\mathcal{R}}_j^s(A) \quad \text{and} \quad \overline{\mathcal{R}}_j^\gamma(A) = \overline{\mathcal{R}}_j^p(A) \cap \overline{\mathcal{R}}_j^s(A)$$

Definition 3.3. Let $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$ be a $\mathcal{G}_n - \boldsymbol{CAS}$, and $A \subseteq U$. Then we define the j-near boundary, j-near positive and j-near negative regions of A are defined respectively as follows:

$$\forall j \in \{r, l, i, u\}, \quad k \in \{p, s, \gamma\} : B_j^k(A) = \overline{\mathcal{R}}_j^k(A) - \underline{\mathcal{R}}_j^k(A), \quad POS_j^k(A) = \underline{\mathcal{R}}_j^k(A) \quad \text{and} \quad NEG_j^k(A) = U - \overline{\mathcal{R}}_j^k(A).$$

Definition 3.4. Let $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$ be a $\mathcal{G}_n - \boldsymbol{CAS}$, and $A \subseteq U$. Then, for each $j \in \{r, l, i, u\}$, $k \in \{p, s, \gamma\}$, the *j*-near accuracy of the *j*-near approximations of $A \subseteq U$ is defined by

$$\delta_j^k(A) = \frac{\left|\underline{\mathcal{R}}_j^k(A)\right|}{\left|\overline{\mathcal{R}}_j^k(A)\right|}, \quad \text{where } \left|\overline{\mathcal{R}}_j^k(A)\right| \neq 0. \qquad \text{Obviously } 0 \leqslant \delta_j^k(A) \leqslant 1.$$

Definition 3.5. Let $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$ be a $\mathcal{G}_n - \boldsymbol{CAS}$, and $A \subseteq U$. Then, for each $j \in \{r, l, i, u\}$, $k \in \{p, s, \gamma\}$, the subset A is called "*j*-near definable (briefly k_j-exact) set" if $\underline{\mathcal{R}}_j^k(A) = \overline{\mathcal{R}}_j^k(A) = A$. Otherwise, it is called *j*-near rough (briefly k_j-rough). It is clear that A is k_j-exact if $\delta_j^k(A) = 1$ and $B_j^k(A) = \emptyset$. Otherwise, it is k_j-rough.

Remark 3.2. In the $\mathcal{G}_n - \boldsymbol{CAS}$, $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$, we can compute the *j*-near approximations of any subset $A \subseteq U$, directly by using the *j*-approximations, as the following lemma illustrates.

Lemma 3.1. *Let $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$ be a $\mathcal{G}_n - \boldsymbol{CAS}$, and $A \subseteq U$. Then, for each $j \in \{r, l, i, u\}$:*

(i) $\underline{\mathcal{R}}_j^p(A) = A \cap \underline{\mathcal{R}}_j(\overline{\mathcal{R}}_j(A))$	(ii) $\overline{\mathcal{R}}_j^p(A) = A \cup \overline{\mathcal{R}}_j(\underline{\mathcal{R}}_j(A))$
(iii) $\underline{\mathcal{R}}_j^s(A) = A \cap \overline{\mathcal{R}}_j(\underline{\mathcal{R}}_j(A))$	(iv) $\overline{\mathcal{R}}_j^s(A) = A \cup \underline{\mathcal{R}}_j(\overline{\mathcal{R}}_j(A))$

Proof. From Definition 3.2, the proof is obvious. □

Lemma 3.2. *Let $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$ be a $\mathcal{G}_n - \boldsymbol{CAS}$, and $A \subseteq U$. Then, for each $j \in \{r, l, i, u\}$, $k \in \{p, s, \gamma\}$:*

The subset A is k_j-rough set if $A = \underline{\mathcal{R}}_j^k(A)$.

The following proposition introduces the fundamental properties of the j-near approximations.

Proposition 3.1. *Let $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$ be a $\mathcal{G}_n - \boldsymbol{CAS}$, and $A, B \subseteq U$. Then, $\forall j \in \{r, l, i, u\}$, $k \in \{p, s, \gamma\}$:*

(1) $\underline{\mathcal{R}}_j^k(A) \subseteq A \subseteq \overline{\mathcal{R}}_j^k(A)$	(7) $\underline{\mathcal{R}}_j^k(A \cup B) \supseteq \underline{\mathcal{R}}_j^k(A) \cup \underline{\mathcal{R}}_j^k(B)$.
(2) $\underline{\mathcal{R}}_j^k(U) = \overline{\mathcal{R}}_j^k(U) = U$ *and* $\underline{\mathcal{R}}_j^k(\emptyset) = \overline{\mathcal{R}}_j^k(\emptyset) = \emptyset$.	(8) $\overline{\mathcal{R}}_j^k(A \cup B) \supseteq \overline{\mathcal{R}}_j^k(A) \cup \overline{\mathcal{R}}_j^k(B)$.
(3) *If* $A \subseteq B$ *then* $\underline{\mathcal{R}}_j^k(A) \subseteq \underline{\mathcal{R}}_j^k(B)$.	(9) $\underline{\mathcal{R}}_j^k(A) = \left[\overline{\mathcal{R}}_j^k(A^c)\right]^c$ *and* $\overline{\mathcal{R}}_j^k(A) = \left[\underline{\mathcal{R}}_j^k(A^c)\right]^c$, *where* A^c *is the complement of* A.
(4) *If* $A \subseteq B$ *then* $\overline{\mathcal{R}}_j^k(A) \subseteq \overline{\mathcal{R}}_j^k(B)$.	(10) $\underline{\mathcal{R}}_j^k\left(\underline{\mathcal{R}}_j^k(A)\right) = \underline{\mathcal{R}}_j^k(A)$.
(5) $\underline{\mathcal{R}}_j^k(A \cap B) \subseteq \underline{\mathcal{R}}_j^k(A) \cap \underline{\mathcal{R}}_j^k(B)$.	(11) $\overline{\mathcal{R}}_j^k\left(\overline{\mathcal{R}}_j^k(A)\right) = \overline{\mathcal{R}}_j^k(A)$.
(6) $\overline{\mathcal{R}}_j^k(A \cap B) \subseteq \overline{\mathcal{R}}_j^k(A) \cap \overline{\mathcal{R}}_j^k(B)$.	

Proof. Firstly, the proof of (1), (2), (10) and (11) is obvious directly from Definition 3.2.

Now, we will prove the left properties for case $k = p$ and the other cases similarly.

(3) If $A \subseteq B$ then $\underline{\mathcal{R}}_j^p(A) = \{x \in A | N_j(x) \subseteq \overline{\mathcal{R}}_j(A)\} \subseteq \{x \in B | N_j(x) \subseteq \overline{\mathcal{R}}_j(B)\} = \underline{\mathcal{R}}_j^p(B)$.

The proof of (4)–(8), by similar way as (3).

(9) From Lemma 3.1, we get

$$[\underline{\mathcal{R}}_j^p(A^c)]^c = [A \cap \underline{\mathcal{R}}_j(\overline{\mathcal{R}}_j(A))]^c = A^c \cup [\underline{\mathcal{R}}_j(\overline{\mathcal{R}}_j(A))]^c = A^c \cup \overline{\mathcal{R}}_j(\underline{\mathcal{R}}_j(A)) = \overline{\mathcal{R}}_j^p(A).$$

Similarly, $\underline{\mathcal{R}}_j^p(A) = [\overline{\mathcal{R}}_j^p(A^c)]^c$. □

The following results introduce the relationships between the j-approximations and the j-near approximations. Moreover, these results show the importance of applying near concepts in $\mathcal{G}_n - \boldsymbol{CAS}$.

Theorem 3.1. *Let $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$ be a $\mathcal{G}_n - \boldsymbol{CAS}$, and $A \subseteq U$. Then, $\forall j \in \{r, l, i, u\},\ k \in \{p, s, \gamma\}$:*

$$\underline{\mathcal{R}}_j(A) \subseteq \underline{\mathcal{R}}_j^k(A) \subseteq A \subseteq \overline{\mathcal{R}}_j^k(A) \subseteq \overline{\mathcal{R}}_j(A)$$

Proof. We will prove the proposition in case of $k = p$ and the other cases similarly:

Let $x \in \underline{\mathcal{R}}_j(A)$, then $x \in A$ such that $N_j(x) \subseteq A$. Thus $x \in A$ such that $N_j(x) \subseteq \overline{\mathcal{R}}_j(A)$ and this implies $x \in \underline{\mathcal{R}}_j^p(A)$. By duality, we get $\overline{\mathcal{R}}_j^p(A) \subseteq \overline{\mathcal{R}}_j(A)$. □

Corollary 3.1. *Let $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$ be a $\mathcal{G}_n - \boldsymbol{CAS}$, and $A \subseteq U$. Then $\forall j \in \{r, l, i, u\},\ k \in \{p, s, \gamma\}$:*

(1) $B_j^k(A) \subseteq B_j(A)$.	(2) $\delta_j(A) \leqslant \delta_j^k(A)$.

Remark 3.3. The main goals of the following example are:

(i) The converse of the above results is not true in general.
(ii) Using near concepts in rough context is very useful for removing the vagueness of sets and accordingly, these approaches is very useful in decision making.

Example 3.2. Let $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$ be a $\mathcal{G}_n - \boldsymbol{CAS}$, where $U = \{a, b, c, d\}$ and

$\mathcal{R} = \{(a, a), (a, b), (b, a), (b, b), (c, a), (c, b), (c, c), (c, d), (d, d)\}$.

Thus, we get $N_r(a) = \{a, b\} = N_r(b)$, $N_r(c) = U$, $N_r(d) = \{d\}$.

By using Definitions 3.2 and 3.4, the following table gives comparisons between the j-accuracy of approximations and the k_j-accuracy of approximations of the all subsets of U, where $j = r, \forall k \in \{p, s, \gamma\}$:

From Table 3.1, we notice that:

(i) Using γ_r in constructing the approximations of sets is more accurate than others types, since for any subset $A \subseteq U$, $\delta_r(A) \leqslant \delta_r^{\gamma}(A)$ and $\delta_r^k(A) \leqslant \delta_r^{\gamma}(A)$, $\forall k \in \{p, s\}$. Thus, these approaches will helps to extract and discovery the hidden information in data that collected from real-life applications. For example, all shaded sets in Table 3.1.

(ii) Every r-exact set is r-near exact, but the converse is not true. For example, shaded sets in Table 3.1.

Remark 3.4. The following result is very interesting because it is prove that the j-near approaches are more accurate than the j-approaches. Moreover, it is illustrates the importance of j-near concepts in exactness of sets.

Proposition 3.2. *Let $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$ be a $\mathcal{G}_n - \boldsymbol{CAS}$, and $A \subseteq U$. Then $\forall j \in \{r, l, i, u\}$, $k \in \{p, s, \gamma\}$, the following is true in general: If A is j-exact set, then it is k_j-exact.*

Table 3.1 Comparisons between the j-accuracy and the k_j-accuracy of approximations of the all subsets of U.

$\wp(U)$	$\delta_r(A)$	$\delta_r^p(A)$	$\delta_r^s(A)$	$\delta_r^{\gamma}(A)$
$\{a\}$	0	1	0	1
$\{b\}$	0	1	0	1
$\{c\}$	0	0	0	0
$\{d\}$	1/2	1/2	1	1
$\{a, b\}$	2/3	2/3	1	1
$\{a, c\}$	0	1/2	0	1/2
$\{a, d\}$	1/4	2/3	1/4	2/3
$\{b, c\}$	0	1/2	0	1/2
$\{b, d\}$	1/4	2/3	1/4	2/3
$\{c, d\}$	1/2	1/2	1	1
$\{a, b, c\}$	2/3	2/3	1	1
$\{a, b, d\}$	3/4	3/4	3/4	3/4
$\{a, c, d\}$	1/4	1	1/2	1
$\{b, c, d\}$	1/4	1	1/2	1
U	1	1	1	1
$\emptyset$	0	0	0	0

Proof. If A is j-exact set, then $B_j(A) = \emptyset$. Thus, by Corollary 3.1, $B_j^k(A) = \emptyset$ and accordingly A is k_j-exact. □

The converse of the above proposition is not true in general as Example 3.2 illustrates.

The main goal of the following results is to introduce the relationships between different types of j-near approximations, j-near boundary, j-near accuracy and j-near exactness respectively.

Proposition 3.3. *Let* $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$ *be* $\mathcal{G}_n - \boldsymbol{CAS}$ *and* $A \subseteq U$. *Then,* $\forall j \in \{r, l, i, u\}$, *the following statements are true in general:*

(i) $\underline{\mathcal{R}}_j^p(A) \subseteq \underline{\mathcal{R}}_j^\gamma(A)$.	(iii) $\overline{\mathcal{R}}_j^\gamma(A) \subseteq \overline{\mathcal{R}}_j^p(A)$.
(ii) $\underline{\mathcal{R}}_j^s(A) \subseteq \underline{\mathcal{R}}_j^\gamma(A)$.	(iv) $\overline{\mathcal{R}}_j^\gamma(A) \subseteq \overline{\mathcal{R}}_j^s(A)$.

Proof. By using Lemmas 3.1 and 3.2, the proof is obvious. □

Corollary 3.2. *Let* $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$ *be a* $\mathcal{G}_n - \boldsymbol{CAS}$, *and* $A \subseteq U$. *Then* $\forall j \in \{r, l, i, u\}$, *the following statements are true in general:*

(i) $\delta_j^p(A) \leqslant \delta_j^\gamma(A)$.	(iii) $B_j^\gamma(A) \subseteq B_j^p(A)$.
(ii) $\delta_j^S(A) \leqslant \delta_j^\gamma(A)$.	(iv) $B_j^\gamma(A) \subseteq B_j^s(A)$.

Corollary 3.3. *Let* $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$ *be a* $\mathcal{G}_n - \boldsymbol{CAS}$, *and* $A \subseteq U$. *Then* $\forall j \in \{r, l, i, u\}$, *the following statements are true in general:*

(i) *A is p_j-exact* $\Rightarrow$ *A is γ_j-exact.*	(ii) *A is s_j-exact* $\Rightarrow$ *A is γ_j-exact.*

Remark 3.5.

(i) The converse of the above results is not true in general as the following example illustrates.
(ii) $\forall j \in \{r, l, i, u\}$, $\delta_j^\gamma(A) = max\left(\delta_j^p(A),\ \delta_j^S(A)\right)$, where ***max*** represents the maximum of two quantities.

Example 3.3. According to Example 3.2, it is clear that A is γ_r-exact but it is not s_r-exact. In addition, the subset B is γ_r-exact but it is not p_r-exact.

The relationships between different types of j-near approximations (for each $j \in \{r, l, i, u\}$) are not comparable (no it is not like to the j-approximations as in [6]) as the following remark illustrates.

Remark 3.6. Let $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$ be a $\mathcal{G}_n - \boldsymbol{CAS}$, and $A \subseteq U$. Then $\forall k \in \{p, s, \gamma\}$, the following statements are not true in general:

(i) $\underline{\mathcal{R}}^k_u(A) \subseteq \underline{\mathcal{R}}^k_r(A) \subseteq \underline{\mathcal{R}}^k_i(A)$.	(iii) $\overline{\mathcal{R}}^k_i(A) \subseteq \overline{\mathcal{R}}^k_r(A) \subseteq \overline{\mathcal{R}}^k_u(A)$.
(ii) $\underline{\mathcal{R}}^k_u(A) \subseteq \underline{\mathcal{R}}^k_l(A) \subseteq \underline{\mathcal{R}}^k_i(A)$.	(iv) $\overline{\mathcal{R}}^k_i(A) \subseteq \overline{\mathcal{R}}^k_l(A) \subseteq \overline{\mathcal{R}}^k_u(A)$.

The following example illustrates this remark.

Example 3.4. Let $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$ be a $\mathcal{G}_n - \boldsymbol{CAS}$, where $U = \{a, b, c, d\}$ and

$\mathcal{R} = \{(a, a), (b, b), (b, a), (c, a), (c, d), (d, a), (d, c), (d, a)\}$.

Now suppose that $A = \{b\}$, $B = \{a, d\}$ and $C = \{a, c\}$. Thus, we get $\underline{\mathcal{R}}^p_r(A) = \emptyset$, but $\underline{\mathcal{R}}^p_u(A) = A$ and $\underline{\mathcal{R}}^p_l(B) = \{d\}$, but $\underline{\mathcal{R}}^p_u(B) = \{a, d\}$. In addition, we have $\underline{\mathcal{R}}^s_i(C) = \{a\}$, but $\underline{\mathcal{R}}^s_r(C) = A$.

4. *j*-near rough membership relations, *j*-near rough membership functions and *j*-near fuzzy sets in $\mathcal{G}_n$ CAS

By considering j-near concepts, the new concepts "j-near rough membership relations" (resp. "j-near rough membership functions") are provided to modify and generalize the j-membership relations (resp. j-membership functions) in $\mathcal{G}_n - \boldsymbol{CAS}$. The near rough membership functions are considered as easy tools to classify the sets and help for measuring near exactness and near roughness of sets. The existence of near rough membership functions made us introduce the concept of near fuzzy sets.

Definition 4.1. Let $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$ be a $\mathcal{G}_n - \boldsymbol{CAS}$, and $A \subseteq U$. Then $\forall j \in \{r, l, i, u\}, k \in \{p, s, \gamma\}$, we say:

(i) x is "j-near surely" (briefly k_j-surely) belongs to A, written $x \underline{\in}^k_j A$, if $x \in \underline{\mathcal{R}}^k_j(A)$.
(ii) x is "j-near possibly" (briefly k_j-possibly) belongs to X, written $x \,\overline{\in}^k_j A$, if $x \in \overline{\mathcal{R}}^k_j(A)$.

Lemma 4.1. *Let* $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$ *be a* $\mathcal{G}_n - \boldsymbol{CAS}$*, and* $A \subseteq U$*. Thus* $\forall j \in \{r, l, i, u\},\ k \in \{p, s, \gamma\}$*, the following statements are true in general:*

(i) *If* $x \underline{\in}_j^k A$ *implies to* $x \in A$.	(ii) *If* $x \in A$ *implies to* $x \bar{\in}_j A$.

Proof. Straightforward. □

The converse of the above lemma is not true in general, as the following example illustrates:

Example 4.1. Consider $U = \{a, b, c, d\}$ and $\mathcal{R} = \{(a, a), (b, b), (b, a), (c, a), (c, d), (d, a), (d, c), (d, a)\}$.

Thus, we get $N_r(a) = \{a\},\ N_r(b) = \{a, b\}, N_r(c) = \{a, c, d\},\ N_r(d) = \{d\}$.

We will show the above remark in case of ($j = r$ and $k = p$) and the other cases similarly:

Suppose that $A = \{b, d\}$, then we get $\underline{\mathcal{R}}_r^p(A) = \{d\}$ and $\overline{\mathcal{R}}_r^p(A) = \{b, c, d\}$.

Clearly $d \in A$ but $d \underline{\notin}_r^p A$ and $c \bar{\in}_r^p A$ but $c \notin A$.

Remark 4.1. We can redefine the *j*-near approximations by using $\underline{\in}_j^k$ and $\bar{\in}_j^k$ as follows:

For any $A, B \subseteq U$: $\underline{\mathcal{R}}_j^k(A) = \left\{x \in U \middle| x \underline{\in}_j^k A\right\}$ and $\overline{\mathcal{R}}_j^k(A) = \left\{x \in U \middle| x \bar{\in}_j^k A\right\}$.

The following proposition is very interesting since it is give the relationships between the *j*-rough membership relations and *j*-near rough membership relations. Accordingly, we will show the importance of using these different types of *j*-near rough membership relations.

Proposition 4.1. *Let* $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$ *be a* $\mathcal{G}_n - \boldsymbol{CAS}$*, and* $A \subseteq U$*. Then* $\forall j \in \{r, l, i, u\},\ k \in \{p, s, \gamma\}$*, the following statements are true in general:*

(i) $x \underline{\in}_j A \ \Rightarrow\ x \underline{\in}_j^k A$.	(ii) $x \bar{\in}_j^k A \ \Rightarrow\ x \overline{\in}_j A$.

Proof. We will prove first statement and the other similarly:

(i) $x \underline{\in}_j A \ \Rightarrow\ x \in \underline{\mathcal{R}}_j(A) \ \Rightarrow\ x \in \underline{\mathcal{R}}_j^k(A) \ \Rightarrow\ x \underline{\in}_j^k A$. □

The converse of the above proposition is not true in general as the following example illustrates.

Example 4.2. Let $U = \{a, b, c, d\}$ and $\mathcal{R} = \{(a,a),(b,b),(b,a),(c,a),(c,d),(d,a),(d,c),(d,a)\}$.

We will show the above remark in case of ($j = r$ and $k = s$) and the other cases similarly:

Suppose that $A = \{a, c\}$ and $B = \{b, d\}$, then we get $\underline{\mathcal{R}}_r(A) = \{a\}$ and $\underline{\mathcal{R}}_r^s(A) = \{a, c\}$.

Clearly $c \underline{\in}_r^s A$, but $c \underline{\notin}_r A$ although $c \in A$.

Also $\overline{\mathcal{R}}_r(B) = \{b, c, d\}$ and $\overline{\mathcal{R}}_r^s(B) = \{b, d\}$. Clearly $c \bar{\notin}_r^s B$, but $c \bar{\in}_r B$ although $c \notin B$.

Definition 4.2. Let $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$ be a $\mathcal{G}_n - \boldsymbol{CAS}$, and $A \subseteq U$. Thus we can define the j-near rough membership functions for $\mathcal{G}_n - \boldsymbol{CAS}$ as follows: For each $j \in \{r, l, i, u\}$, $k \in \{p, s, \gamma\}$ and $x \in U$, the j-near rough membership functions on U for subset A are $\mu_A^{k_j} : U \to [0, 1]$, where

$$\mu_A^{k_j}(x) = \begin{cases} 1 & \text{if } 1 \in \Psi_A^{k_j}(x). \\ min\left(\Psi_A^{k_j}(x)\right) & \text{Otherwise.} \end{cases}$$

and $\Psi_A^{k_j}(x) = \frac{|k_j(x) \cap A|}{|k_j(x)|}$ such that $k_j(x)$ is a j-near rough set in that contains x.

The following result is very interesting since it gives the relation between the rough j-membership functions and j-near rough membership functions. Moreover, it illustrates the importance of j-near rough membership functions.

Lemma 4.2. *Let $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$ be a $\mathcal{G}_n - \boldsymbol{CAS}$, and $A, B \subseteq U$. Then, for each $j \in \{r, l, i, u\}$ and $k \in \{p, s, \gamma\}$, the following is true in general:*

(i) $\mu_A^j(x) = 1 \Rightarrow \mu_A^{k_j}(x) = 1, \quad \forall x \in U.$	(ii) $\mu_A^j(x) = 0 \Rightarrow \mu_A^{k_j}(x) = 0, \quad \forall x \in U.$

Proof. (i) If $\mu_A^j(x) = 1$, then $N_j(x) \subseteq A, \quad \forall x \in U$. Thus $x \in \underline{\mathcal{R}}_j(A)$ and this implies $x \in \underline{\mathcal{R}}_j^k(A)$ which is a j-near rough set contained in A. Accordingly, $\mu_A^{k_j}(x) = 1, \quad \forall x \in U$.

(ii) If $\mu_A^j(x) = 0$, then $N_j(x) \cap A = \emptyset, \quad \forall x \in U$. But $N_j(x)$ is a j-near rough set that contains x. $\square$

Accordingly $0 \in \Psi_A^{k_j}(x)$ and this means that $min\left(\Psi_A^{k_j}(x)\right) = 0$. Hence $\mu_A^{k_j}(x) = 0, \quad \forall x \in U$.

Remark 4.2.

(i) According to the above results, we can prove that $\mu_A^{k_j}$ is more accurate than μ_A^j, this means that:

(1) If $x \in A \Rightarrow \mu_A^j(x) \leqslant \mu_A^{k_j}(x)$. (2) If $x \notin A \Rightarrow \mu_A^{k_j}(x) \leqslant \mu_A^j(x)$.

(ii) The converse of Lemma 4.2 is not true in general.

The following example illustrates Remarks 4.2.

Example 4.3. Let $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$ be a $\mathcal{G}_n - \boldsymbol{CAS}$, where $U = \{a, b, c, d\}$ and

$\mathcal{R} = \{(a,a), (a,b), (b,a), (b,b), (c,a), (c,b), (c,c), (c,d), (d,d)\}$.

We will show the above result in case of $j = r$ and $k = s$ the other cases similarly as follows:

First we have $N_r(a) = \{a,b\} = N_r(b)$, $N_r(c) = U$, $N_r(d) = \{d\}$. Thus we can get.

The family of all r-semi rough sets is: $S_r(U) = \{U, \emptyset, \{a\}, \{d\}, \{a,b\}, \{a,d\}, \{a,c\}, \{c,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}\}$.

Now consider the subset $A = \{a, c\}$, then the r-rough membership functions of $A, x \in U$ are

$\mu_A^r(a) = \frac{\lvert\{a\}\cap A\rvert}{\lvert\{a\}\rvert} = 1.$	$\mu_A^r(c) = \frac{\lvert\{a,c,d\}\cap A\rvert}{\lvert\{a,c,d\}\rvert} = \frac{2}{3}.$
$\mu_A^r(b) = \frac{\lvert\{a,b\}\cap A\rvert}{\lvert\{a,b\}\rvert} = \frac{1}{2}.$	$\mu_A^r(d) = \frac{\lvert\{d\}\cap A\rvert}{\lvert\{d\}\rvert} = 0.$

But the r-semi rough membership functions of $A, x \in U$ are

$$\Psi_A^{s_r}(a) = \left\{\frac{|\{a\} \cap A|}{|\{a\}|} = 1, \ \frac{|\{a,b\} \cap A|}{|\{a,b\}|} = \frac{1}{2}, \ldots\right\} \Rightarrow \mu_A^{s_r}(a) = 1.$$

$$\Psi_A^{s_r}(b) = \left\{\frac{|\{a,b\} \cap A|}{|\{a,b\}|} = \frac{1}{2}, \ \frac{|\{a,b,c\} \cap A|}{|\{a,b,c\}|} = \frac{2}{3}, \ \frac{|\{a,b,d\} \cap A|}{|\{a,b,d\}|} = \frac{1}{3}\right\} \Rightarrow \mu_A^{s_r}(b) = \frac{1}{3}.$$

$$\Psi_A^{s_r}(c) = \left\{ \frac{|\{a,c\} \cap A|}{|\{a,c\}|} = 1, \ \frac{|\{c,d\} \cap A|}{|\{c,d\}|} = \frac{1}{2}, \ldots \right\} \Rightarrow \mu_A^{s_r}(c) = 1.$$

$$\Psi_A^{s_r}(d) = \left\{ \frac{|\{d\} \cap A|}{|\{d\}|} = 0, \ \frac{|\{a,d\} \cap A|}{|\{a,d\}|} = \frac{1}{2}, \ldots \right\} \Rightarrow \mu_A^{s_r}(a) = 0.$$

The j-near rough membership functions $\mu_A^{k_j}$ allow us to define twelve different types of fuzzy sets in $\mathcal{G}_n - \boldsymbol{CAS}$ as the following definition illustrates.

Definition 4.3. Let $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$ be a $\mathcal{G}_n - \boldsymbol{CAS}$, and $A \subseteq U$. Then for each $j \in \{r, l, i, u\}$ and $k \in \{p, s, \gamma\}$, the j-near fuzzy set in U is a set of ordered pairs: $\widetilde{A}_j^k = \{(x, \mu_A^{k_j}(x)) | x \in U\}$.

Example 4.4. According to Example 4.3, the r-semi fuzzy set of a subset $A = \{a, c\}$ is

$$\widetilde{A}_r^s = \left\{ (a,1), \left(b, \frac{1}{3}\right), (c,1), (d,0) \right\}.$$

But the r-fuzzy set of a subset $A = \{a, c\}$ is $\widetilde{A}_r = \left\{(a,1), \left(b, \frac{1}{2}\right), \left(c, \frac{2}{3}\right), (d,0)\right\}$.

5. Illustrative examples

The main goal of this section is to introduce two practical examples in order to illustrate the importance of applying near concept in rough context. In the first example we use an equivalence relation that induced from an information system and hence we compare between our approaches and Pawlak approach. In the second example, we apply our approaches in a multi-valued information system (**MVIS**) [14]. This type of information system is generalization to information system which uses an arbitrary binary relation and thus Pawlak approach does not fit in this type. Lin [10] introduced general rough membership function depending on an arbitrary binary relation, these rough membership function coincide with our j-rough membership function in the case of $j = r$ only. But, the other types j of our j-rough membership functions are more accurate than $j = r$, so we can see that our approaches are the appropriate tools for these types and very useful in information analysis. Finally, in the second example we introduce a comparison between our approaches and Lin method.

Example 5.1. Consider the following information system as in Table 5.1 that represents the data about 6 students, as shown below.

From Table 5.1, we have.

The set of universe: $U = \{1, 2, 3, 4, 5, 6\}$,

Table 5.1 Information system.

Student	Analysis	Algebra	Statistics	Decision
1	Bad	Good	Medium	Accept
2	Good	Bad	Medium	Accept
3	Good	Good	Good	Accept
4	Bad	Good	Bad	Reject
5	Good	Bad	Medium	Reject
6	Bad	Good	Good	Accept

The set of attributes: $AT = \{\text{Analysis, Algebra, Statistics}\} = C \cup \{\text{Decision}\} = D$,

The sets of values: $V_{\text{Analysis}} = \{\text{Bad, Good}\}$, $V_{\text{Algebra}} = \{\text{Bad, Good}\}$, $V_{\text{Statistics}} = \{\text{Bad, Medium, Good}\}$ and $V_{\text{Decision}} = \{\text{Accept, Reject}\}$.

But we take the set of condition attributes, $C = \{\text{Analysis, Algebra, Statistics}\}$.

Thus we have: $U/C = \{\{1\},\{2,5\},\{3\},\{4\},\{6\}\}$ and the set of r-pre rough set is

$P_r(U) = \wp(U)$ (*set of all subsets in* U).

Suppose that $X(\text{Decision} : \text{Accept}) = \{1,2,3,6\}$. Thus we compute the rough membership function with respect to Pawlak [13,14] and with respect to our approaches as follows:

Pawlak Definition [13,14] (rough membership function):

For $x = 1$, then $\mu_X^C(1) = \frac{\lvert\{1\}\cap X\rvert}{\lvert\{1\}\rvert} = 1$.	For $x = 3$, then $\mu_X^C(3) = \frac{\lvert\{3\}\cap X\rvert}{\lvert\{3\}\rvert} = 1$.
For $x = 2$, then $\mu_X^C(2) = \frac{\lvert\{2,5\}\cap X\rvert}{\lvert\{2,5\}\rvert} = \frac{1}{2}$.	For $x = 6$, then $\mu_X^C(6) = \frac{\lvert\{6\}\cap X\rvert}{\lvert\{6\}\rvert} = 1$.

Our Definition (r-pre rough membership function):

$$\Psi_X^{p_r}(1) = \left\{\frac{|\{1\}\cap A|}{|\{1\}|} = 1,\ \frac{|\{1,2\}\cap A|}{|\{1,2\}|} = 1, \ldots\right\} \Rightarrow \mu_X^{p_r}(1) = 1,$$

$$\Psi_X^{p_r}(2) = \left\{\frac{|\{2\}\cap A|}{|\{2\}|} = 1,\ \frac{|\{2,6\}\cap A|}{|\{2,6\}|} = 1, \ldots\right\} \Rightarrow \mu_X^{p_r}(2) = 1,$$

$$\Psi_X^{p_r}(3) = \left\{\frac{|\{3\}\cap A|}{|\{3\}|} = 1,\ \frac{|\{3,5\}\cap A|}{|\{3,5\}|} = \frac{1}{2}, \ldots\right\} \Rightarrow \mu_X^{p_r}(3) = 1 \text{ and}$$

$$\Psi_X^{p_r}(6) = \left\{ \frac{|\{6\} \cap A|}{|\{6\}|} = 1,\ \frac{|\{6,5\} \cap A|}{|\{6,5\}|} = \frac{1}{2}, \ldots \right\} \Rightarrow \mu_X^{p_r}(6) = 1.$$

Moreover, for some elements that has decision (Reject) such that 5 we get:

In Pawlak: $\mu_X^C(5) = \frac{|\{2,5\} \cap X|}{|\{2,5\}|} = \frac{1}{2}$, that is 5 may be belongs to the set X(Decision : Accept),

$X = \{1, 2, 3, 6\}$ and this contradicts to Table 5.1.

But in our definition: we have $\Psi_X^{p_r}(5) = \left\{ \frac{|\{5\} \cap A|}{|\{5\}|} = 0,\ \frac{|\{5,6\} \cap A|}{|\{5,6\}|} = \frac{1}{2}, \ldots \right\} \Rightarrow \mu_X^{p_r}(5) = 0.$

This means that 5 does not belongs to the set X(Decision : Accept) $= \{1, 2, 3, 6\}$ which is coincide with Table 5.1. Hence, our approaches are more accurate than Pawlak definition.

Example 5.2. Consider the following multi-valued information system (**MVIS**) as in Table 5.2. Suppose we are given data about 5 persons, as shown below.

Where $\mathcal{R}_1 =$ Languages $= \{$English, German, Arabic$\}$, $\mathcal{R}_2 =$ Sports $= \{$Handball, Basketball, Tennis$\}$ and $\mathcal{R}_3 =$ Skills $= \{$Swimming, Running, Fishing$\}$ such that $x\mathcal{R}_n y,\ \forall n = 1, 2, 3$.

We will use the case of $j = r$ and $k = \gamma$ as follows:

$a\mathcal{R}_1 = \{a, b\}$, $b\mathcal{R}_1 = \{b\}$, $c\mathcal{R}_1 = \{b, c, d\}$, $d\mathcal{R}_1 = \{d\}$, $e\mathcal{R}_1 = \{d, e\}$,

$a\mathcal{R}_2 = \{a, b, c\}$, $b\mathcal{R}_2 = \{a, b, c\}$, $c\mathcal{R}_2 = \{c\}$, $d\mathcal{R}_2 = \{c, d\}$, $e\mathcal{R}_2 = \{e\}$ and

$a\mathcal{R}_3 = \{a, b\}$, $b\mathcal{R}_3 = \{a, b\}$, $c\mathcal{R}_3 = \{c, d\}$, $d\mathcal{R}_3 = \{d\}$, $e\mathcal{R}_3 = \{d, e\}$.

In order to represent the set of all condition attributes, we generate the following relation from all above relations as follows: $x\mathcal{R} = \bigcap_{n=1}^{3} x\mathcal{R}_n$. Thus we get $a\mathcal{R} = \{a, b\}$, $b\mathcal{R} = \{b\}$, $c\mathcal{R} = \{c, d\}$, $d\mathcal{R} = \{d\}$, $e\mathcal{R} = \{e\}$.

Clearly, this relation is symmetry relation (reflexive and symmetric) but is not transitive and thus it is not equivalence relation. Hence, Pawlak approach does not fit in this case, so we use Lin definition and our approaches as follows:

Table 5.2 Multi-valued information system (**MVIS**).

Person	Languages	Sports	Skills	Decision
A	{E}	{H}	{S}	Accept
B	{E,G}	{H}	{S}	Reject
C	{G}	{H,B}	{R}	Accept
D	{G,A}	{B}	{R,F}	Accept
E	{A}	{T}	{F}	Reject

Suppose that $X(\text{Decision} : \text{Accept}) = \{a, c, d\}$. Thus.

Lin Definition [10] **(rough membership function):**

For $x = a$, then $\mu_X^{\mathcal{R}}(b) = \frac{|\{a,b\} \cap X|}{|\{a,b\}|} = 1$.
For $x = c$, then $\mu_X^{\mathcal{R}}(c) = \frac{|\{c,d\} \cap X|}{|\{c,d\}|} = \frac{1}{2}$.
For $x = d$, then $\mu_X^{\mathcal{R}}(d) = \frac{|\{d\} \cap X|}{|\{d\}|} = 1$.

Our approaches ($\mathcal{G}_n - \boldsymbol{CAS}$):

First, the relation $\mathcal{R}$ is $\mathcal{R} = \{(a, a), (a, b), (b, b), (c, c), (c, d), (d, d), (e, e)\}$. Thus we get.

The r-neighborhoods of all elements are: $N_r(a) = \{a, b\}$, $N_r(b) = \{b\}$, $N_r(c) = \{c, d\}$, $N_r(d) = \{d\}$, $N_r(e) = \{e\}$.

The l-neighborhoods of all elements are: $N_l(a) = \{a\}$, $N_l(b) = \{a, b\}$, $N_l(c) = \{c\}$, $N_l(d) = \{c, d\}$, $N_l(e) = \{e\}$.

The i-neighborhoods of all elements are: $N_i(a) = \{a\}$, $N_i(b) = \{b\}$, $N_i(c) = \{c\}$, $N_i(d) = \{d\}$, $N_i(e) = \{e\}$.

1. ***r*-rough membership function:**
For $x = a$, then $\mu_X^r(a) = \frac{|\{a,b\} \cap X|}{|\{a,b\}|} = 1$.
For $x = c$, then $\mu_X^r(c) = \frac{|\{c,d\} \cap X|}{|\{c,d\}|} = \frac{1}{2}$.
For $x = d$, then $\mu_X^r(d) = \frac{|\{d\} \cap X|}{|\{d\}|} = 1$.

2. ***l*-rough membership function:**
For $x = a$, then $\mu_X^l(a) = \frac{|\{a\} \cap X|}{|\{a\}|} = 1$.
For $x = c$, then $\mu_X^l(c) = \frac{|\{c\} \cap X|}{|\{c\}|} = 1$.
For $x = d$, then $\mu_X^l(d) = \frac{|\{c,d\} \cap X|}{|\{c,d\}|} = 1$.

3. ***i*-rough membership function:**
For $x = a$, then $\mu_X^i(a) = \frac{|\{a\} \cap X|}{|\{a\}|} = 1$.
For $x = c$, then $\mu_X^i(c) = \frac{|\{c\} \cap X|}{|\{c\}|} = 1$.
For $x = d$, then $\mu_X^i(d) = \frac{|\{d\} \cap X|}{|\{d\}|} = 1$.

It is clear that Lin rough membership function is the same as r-rough membership function. Moreover, our approaches l-rough (resp. i-rough) membership function is more accurate than r-rough membership function and Lin rough membership function. Finally, we can also apply j-near rough membership function as in Example 5.1.

6. Conclusions and future works

In this work, we introduced one of an important topological application that named "near concepts" in rough context. Accordingly, different types of approximations (resp. rough membership relations and functions) were provided to be easy mathematical tools to classify the sets and help for measuring exactness and roughness of sets. These tools are more accurate than other types that were defined by others authors. Consequently, our approaches are very interesting in decision making. We believe that these structures are useful in the applications and thus these techniques open the way for more topological applications in rough context and help in formalizing many applications from real-life data. In our future works, we will apply the suggested methods in this paper in real life applications and problems.

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References

[1] B. Chen, J. Li, On topological covering-based rough spaces, Int. J. Phys. Sci. 6 (17) (2011) 4195–4202.
[2] G. Liu, Y. Sai, A comparison of two types of rough sets induced by coverings, Int. J. Approx. Reason. 50 (2009) 521–528.
[3] J.A. Pomykala, Approximation operators in approximation space, Bull. Polish Acad. Sci. Math. 36 (1988) 193–210.
[4] Lijuan Wang, Xibei Yang, Chen Wu, Multi-covering based rough set model, rough sets, fuzzy sets, data mining, and granular computing, Lect. Notes Comput. Sci. 8170 (2013) 236–244.
[5] M.E. Abd El-Monsef, O.A. Embaby, M.K. El-Bably, New Approach to covering rough sets via relations, Int. J. Pure Appl. Math. 91 (3) (2014) 329–347.
[6] M.E. Abd El-Monsef, A.M. Kozae, M.K. El-Bably, On generalizing covering approximation space, J. Egypt. Math. Soc. (2015), http://dxdoi.org/10.1016/j.joems.2014.12.007.
[7] M. Restrepo, J. Gómez, Covering based rough sets and relation based rough sets, rough sets and intelligent systems paradigms, Lect. Notes Comput. Sci. 8537 (2014) 143–152.
[8] P. Zhu, Covering rough sets based on neighborhoods: an approach without using neighborhoods, Int. J. Approx. Reason. 52 (3) (2011) 461–472.
[9] R. Slowinski, D. Vanderpooten, A generalized definition of rough approximations based on similarity, IEEE Trans. Knowl. Data Eng. 12 (2) (2000) 331–336.
[10] T.Y. Lin, Granular computing on binary relation I: Data mining and neighborhood systems, in: Rough Sets in Knowledge Discovery, Physica-Verlag, 1998, pp. 107–121.
[11] W. Zhu, Topological approaches to covering rough sets, Int. J. Comput. Inform. Sci. 177 (2007) 1499–1508.
[12] Y.Y. Yao, B. Yao, Covering based rough set approximations, Inform. Sci. 200 (2012) 91–107.
[13] Z. Pawlak, Rough sets, Int. J. Comput. Inform. Sci. 11 (1982) 341–356.
[14] Z. Pawlak, Rough Sets, Theoretical Aspects of Reasoning about Data, Kluwer Academic Publishers, Boston, 1991.