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The Traveling Salesman Problem in Circulant Weighted Graphs With Two Stripes

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Abstract

The Symmetric Circulant Traveling Salesman Problem asks for the minimum cost of a Hamiltonian cycle in a circulant weighted undirected graph. The computational complexity of this problem is not known. Just a constructive upper bound, and a good lower bound have been determined.
This paper provides a characterization of the two stripe case. Instances where the minimum cost of a Hamiltonian cycle is equal either to the upper bound, or to the lower bound are recognized. A new construction providing Hamiltonian cycles, whose cost is in many cases lower than the upper bound, is proposed for the remaining instances.

Keywords: Traveling Salesman Problem, Circulant weighted undirected graphs, computational complexity.

1 Introduction

An $n \times n$ matrix $M = (m_{i,j})$ is called *circulant* if $m_{i,j} = m_{0,(j-i) \bmod n}$, for any $0 \le i, j \le n-1$ (for more details on circulant matrices, see [6]). An undirected (directed) graph is said to be circulant if its adjacency matrix is circulant. Similarly, a weighted undirected (directed) graph is said to be circulant if its weighted adjacency matrix is circulant.

In the last years, graph theoretic properties of circulant graphs have been analyzed. In particular, it was investigated if a known graph problem becomes easier

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when the general instance is forced to be a circulant graph. Codenotti, Gerace and Vigna [5] have shown that Maximum Clique, and Minimum Graph Coloring remain Np-hard, and not approximable within a constant factor, even if restricted to circulant undirected graphs. Muzychuk [12] has, instead, proved that Graph Isomorphism restricted to circulant undirected graphs is in P, while the general case is, probably, harder.

As well as we know, it is an open question whether Hamiltonian Circuit, and Traveling Salesman Problem (for short, TSP) restricted to circulant directed graphs remains Np-hard, or not. Some special cases are solved in [7], [14], [11] and [2].

The undirected case is less difficult. As shown by Burkard, and Sandholzer [4], Hamiltonian Circuit, and Bottleneck TSP are polynomial time solvable on the circulant weighted undirected graphs. Unfortunately, a similar result is not known for TSP (see [3], for a survey on the well solvable special cases).

In this paper we study TSP in the circulant weighted undirected graph case. In §2, and in §3, some definitions, and preliminaries are introduced. In §4, the not Hamiltonian case is solved. For the Hamiltonian case, an upper bound, and a lower bound are presented. More results appear independently in [13], and in [9]. In §5, the two stripe case is analyzed. In the last theorem we link the minimum cost of a Hamiltonian cycle to a set A_G. In particular, we prove that such cost is equal to the upper bound if A_G is empty, and is equal to the lower bound if A_G contains a suitably bounded integer. In the remaining cases, we determine a new Hamiltonian cycle whose cost is in many cases lower than the upper bound. §6 completes the paper by presenting open problems, conclusions, and remarks.

2 The Traveling Salesman Problem

A *weighted undirected graph* (shortly, w.u. graph) $G = (\mathbb{Z}_n, E, c)$ consists of a node set $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$, for some integer $n \geq 2$, of a collection E of 2-subsets of $\mathbb{Z}_n$ called edges, and of a cost function $c : E \to \mathbb{N}$.

The *weighted adjacency matrix* of G is an $n \times n$ matrix $M = (m_{i,j})$, whose general entry is $c(\{i, j\})$, if $\{i, j\} \in E$, or ∞, otherwise. Note that $E = \{\{i, j\} : m_{i,j} \neq \infty\}$, and that M is symmetric, as G is undirected.

A *path* P in $G = (\mathbb{Z}_n, E, c)$ is a node sequence $[v_0, v_1, \ldots, v_m]$ such that $\{v_{k-1}, v_k\} \in E$, for any $k = 1, \ldots, m$; v_0, and v_m are called, respectively, the *starting point*, and the *ending point* of P. If they coincide, then P is a *cycle*. The positive integer m is called the *length* of P. Any path of length 1 is called an *arc*. We attach to P the *cost* $c(P) = \sum_k c(\{v_{k-1}, v_k\})$.

The *inverse* path $-P$ corresponds to the node sequence $[v_m, v_{m-1}, \ldots, v_0]$. Clearly, $c(P) = c(-P)$. Finally, given a path $Q = [u_0, u_1, \ldots, u_{m'}]$ such that $u_0 = v_m$, the *composed* path $P \cdot Q$ corresponds to the node sequence $[v_0, v_1, \ldots, v_m, u_1, \ldots, u_{m'}]$.

A path is *elementary* if any two nodes of its node sequence are distinct. The path $[v_0, v_1, \ldots, v_m]$ is an *Hamiltonian path* for $A \subset \mathbb{Z}_n$, if it is elementary, and $A =$

$\{v_0, v_1, \dots, v_m\}$. It is a *Hamiltonian cycle* for G, if $v_m = v_0$, and $[v_0, v_1, \dots, v_{m-1}]$ is a Hamiltonian path for $\mathbb{Z}_n$.

G is said to be *Hamiltonian* if there exists a Hamiltonian cycle for it. If G is Hamiltonian, we denote by $c^*(G)$ the minimum cost of a Hamiltonian cycle for it. Otherwise, we set $c^*(G) = \infty$. Any Hamiltonian cycle C such that $c(C) = c^*(G)$ is said to be *minimal.*

TSP asks for finding $c^*(G)$, given a w.u. graph G. TSP is an NP-hard problem, and no performance guarantee polynomial time approximation algorithms for it are known. In this paper we study the case in which G is a circulant w.u. graph. As suggested in [13], we call such problem SYMMETRIC CIRCULANT TRAVELING SALESMAN PROBLEM, shortly SCTSP.

3 Definitions on Circulant Graphs

Throughout this paper $a \equiv_m b$ denotes the relation $a \equiv b \bmod m$, and $\langle a \rangle_m$ denotes the integer $a \bmod m$, for any a, $b \in \mathbb{Z}$, and $m \in \mathbb{N}$.

A w.u. graph $G = (\mathbb{Z}_n, E, c)$ is *circulant* if its weighted adjacency matrix M is a circulant one. The set $S_G = \{a : a \in \mathbb{Z}_n, \{0, a\} \in E, a \leq n/2\}$ is called the *stripe set* of G. An element of S_G is called a *stripe.* Clearly, $E = \{ \{u, v\} : u, v \in \mathbb{Z}_n,\ (v - u) \equiv_n \pm a, a \in S_G\}$.

For any $a \in S_G$, an edge $\{u, v\}$ such that $(v - u) \equiv_n \pm a$ is called an *edge of stripe* a. As M is a circulant, and symmetric matrix, it follows that any edge of stripe a has cost $c(\{0, a\})$. This integer is called the *cost of stripe* a. An arc $[v_0, v_1]$ is called a $+a$*-arc* (respectively, a $-a$*-arc*), if $(v_1 - v_0) \equiv_n +a$ (respectively, $(v_1 - v_0) \equiv_n -a$).

Let $s = |S_G|$, and let $(\{c_t\}_{t=1}^s)$ be the s-tuple obtained by sorting in non decreasing order the multiset $\{c(\{0, a\}) : a \in S_G\}$. The integer c_t is called the *t-th cost* of G.

A permutation $(\{a_t\}_{t=1}^s)$ of the set S_G satisfying $c(\{0, a_t\}) = c_t$, for any $1 \leq t \leq s$, is called a *presentation* for S_G. Clearly, for any $1 \leq t < t' \leq s$, $a_t, a_{t'} \in [1, n/2]$, $a_t \neq a_{t'}$, and $c(\{0, a_t\}) = c_t \leq c_{t'} = c(\{0, a_{t'}\})$. It can be easily shown that there exists a unique presentation for S_G if and only if the costs of any two different stripes are different.

If $\pi = (\{a_t\}_{t=1}^s)$ is any presentation for S_G, let $g_0^\pi = n$, and let $g_t^\pi = \gcd(g_{t-1}^\pi, a_t)$, for any $1 \leq t \leq s$. Clearly, $g_t^\pi = \gcd(n, a_1, \dots, a_t)$, for any $1 \leq t \leq s$. It follows from the next theorem that g_s^π represents the number of connected components of G.

Theorem 3.1 *(Boesch, and Tindell, [1]) Let $G = (\mathbb{Z}_n, E, c)$ be a circulant w.u. graph, and let $S_G = \{a_1, \dots, a_s\}$. Then, G has $\gcd(n, a_1, \dots, a_s)$ connected components.*

We say that a (circulant) w.u. graph G is *presentable* as $G(n; \pi; \{c_t\}_{t=1}^s)$, if $\mathbb{Z}_n$ is its node set, s is the cardinality of S_G, c_t is the t-th cost of G, for any $1 \leq t \leq s$, and, finally, π is a presentation for S_G.

Example 3.2 The w.u. graph presentable as $G(8; 2, 1; 1, 6)$ is depicted in *Figure* 1.

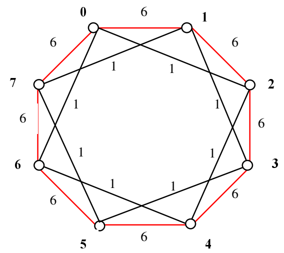

Fig. 1. The circulant w.u. graph $G(8; 2, 1; 1, 6)$

As the costs of the two stripes are distinct, $\pi = (2, 1)$ is the unique presentation for G. We note that $g_1^\pi = 2$, $g_2^\pi = 1$, and that a Hamiltonian cycle for G containing only edges of stripe 2 does not exist.

We end this section by stating a result of Bach, Luby, Goldwasser [10].

Theorem 3.3 *Let G be the w.u. graph presentable as $G(n; \pi; \{c_t\}_{t=1}^s)$. If G is connected, the shortest Hamiltonian path for G costs*

$$SHP(G) \;=\; \sum_{t=1}^{s} (g_{t-1}^\pi - g_t^\pi)\, c_t.$$

4 Bounds for SCTSP

Any Hamiltonian w.u. graph is connected, while the converse is not true. *Proposition* 3.5 in [4] proves that any connected circulant w.u. graph is also Hamiltonian. As a consequence, the following statement holds.

Proposition 4.1 *Let G be the w.u. graph presentable as $G(n; \pi; \{c_t\}_{t=1}^s)$. Then, $c^*(G) = \infty$ if and only if $g_s^\pi > 1$.*

Definition 4.2 Let G be the w.u. graph presentable as $G(n; \pi; \{c_t\}_{t=1}^s)$, and let $\pi = (\{a_t\}_{t=1}^s)$. If G is connected, let us define

$$\begin{aligned} r(\pi) &= \min\{t : 0 \le t \le s,\ g_t^\pi = 1\}; \\ q(\pi) &= \min\{t : 0 \le t < r(\pi),\ g_t^\pi = g_{r(\pi)-1}^\pi\}. \end{aligned}$$

According to *Proposition 4.1*, we may consider SCTSP just in the connected case. In this case, Van der Veen [13] has proposed a recursive procedure for constructing Hamiltonian cycles. $UB(G, \pi)$, that is, the cost of the Hamiltonian cycle so obtained, given in input a w.u. graph G, and a presentation π for S_G, is an upper bound for $c^*(G)$.

An explicit calculus of $UB(G, \pi)$ will be given in [8]. The next theorem, due to Van der Veen [13], gives its expression in some cases including the two stripe one.

Theorem 4.3 *Let G be the w.u. graph presentable as $G(n; \pi; \{c_t\}_{t=1}^s)$. Suppose that G is connected. If $r(\pi) = 1$, then $UB(G, \pi) = c^*(G) = nc_1$. If $g_{q(\pi)}^\pi$ is even, or*

$r(\pi) = 2$, *then*

$$UB(G,\pi) = \sum_{t=1}^{q(\pi)-1} (g^{\pi}_{t-1} - g^{\pi}_{t})c_t + (g^{\pi}_{q(\pi)-1} - 2(g^{\pi}_{q(\pi)} - 1))c_{q(\pi)} + 2(g^{\pi}_{q(\pi)} - 1)c_{r(\pi)}.$$

Proposition 4.4 *Let G be the two striped connected w.u. graph presentable as $G(n; a_1, a_2; c_1, c_2)$. If $c_1 = c_2$, then, $c^*(G) = nc_1$.*

Proof. We observe that $nc_1 \le c^*(G) \le nc_2$, as any Hamiltonian cycle contains n edges of cost at least c_1, and at most c_2. As $c_1 = c_2$, the claim follows. □

A lower bound for SCTSP, and some sufficient conditions for reaching such bound appear independently in [13], and in [9]. We present here the lower bound (*Theorem 4.5*), and two of these sufficient conditions in the two stripe case (*Proposition 4.6*).

Theorem 4.5 *Let G be the w.u. graph presentable as $G(n; \pi; \{c_t\}_{t=1}^{s})$. Suppose that G is connected. Then,*

$$c^*(G) \;\ge\; LB(G,\pi) \;=\; \sum_{t=1}^{r(\pi)-1} (g^{\pi}_{t-1} - g^{\pi}_{t})c_t + g^{\pi}_{r(\pi)-1}c_{r(\pi)}.$$

As $g^{\pi}_{t} = 1$, for any $r(\pi) \le t \le s$ (by *Definition 4.2*), and *Theorem 3.3* holds, it follows that $LB(G,\pi) = SHP(G) + c_{r(\pi)}$. As the cost of the shortest Hamiltonian path of G, and the $r(\pi)$-th cost of G do not depend on the considered presentation, it follows that also $LB(G,\pi)$ does not depend on the considered presentation. This is the reason why we will denote such lower bound simply by $LB(G)$.

Proposition 4.6 *Let G be the two striped connected w.u. graph presentable as $G(n; a_1, a_2; c_1, c_2)$. Suppose that $r(\pi) = 2$. If $g^{\pi}_{1} = 2$, or there exists an integer y_1 such that $0 \le y_1 \le g^{\pi}_{1}$, and $(2y_1 - g^{\pi}_{1})a_1 + g^{\pi}_{1}a_2 \equiv_n 0$ holds, then, $c^*(G) = LB(G) = (n - g_1)c_1 + g_1c_2$.*

Example 4.7 Let G be the w.u. graph presentable as $G(20; 8, 5; 1, 2)$, and let $\pi = (8, 5)$. As $g^{\pi}_{1} = 4$, $g^{\pi}_{2} = 1$, and the equation $(2y_1 - 4)8 + 20 \equiv_{20} 0$ has solution $y_1 = 2 \le 4 = g^{\pi}_{1}$, it follows that $c^*(G) = 36$. A minimal Hamiltonian cycle is depicted in *Figure 2*: the bold black ones are $(+a_1)$-arcs, the thin black ones are $(-a_1)$-arcs, the red ones are edges of stripe a_2. The integer $y_1 = 2$ denotes how many times an edge of stripe a_2 is followed by a $(-a_1)$-arc.

5 The two stripe case

Symmetric Circulant Traveling Salesman Problem for any two striped circulant w.u. graph not belonging to the set

$$\mathcal{G} = \{G(n; a_1, a_2; c_1, c_2) \;:\; c_1 < c_2,\, \gcd(n, a_1) \ge 3,\, \gcd(n, a_1, a_2) = 1\}$$

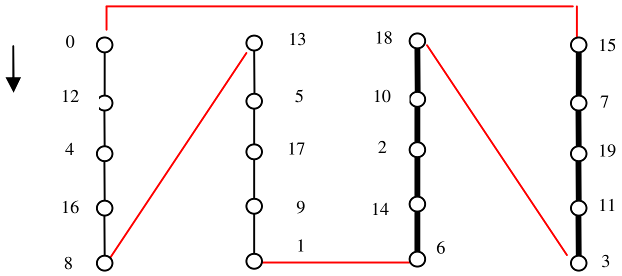

Fig. 2. A minimal Hamiltonian cycle for $G(20; 8, 1; 1, 2)$

has been already solved. In particular, the not connected case is solved by *Proposition 4.1*, the case $c_1 = c_2$ is solved by *Proposition 4.4*, the case $\gcd(n, a_1) = 1$ is solved by *Theorem 4.3*, and, finally, the case $\gcd(n, a_1) = 2$ is solved by *Theorem 4.6*.

In order to end the analysis of the two stripe case, we may consider only w.u. graphs in $\mathcal{G}$. Any such w.u. graph, has just a presentation, say it π, as $c_1 < c_2$. This is the reason why we will use the notation $G = G(n; a_1, a_2; c_1, c_2)$, instead of G is presentable as $G(n; a_1, a_2; c_1, c_2)$. Moreover, we will omit any superscript from $g_1 = \gcd(n, a_1)$, and $g_2 = \gcd(n, a_1, a_2)$, and denote $UB(G, \pi)$, simply by $UB(G)$. Finally, let us note that always $r(\pi) = 2$.

Let $G = G(n; a_1, a_2; c_1, c_2)$ be a w.u. graph in $\mathcal{G}$. For any path P in G, we denote by α_P (respectively, β_P) the number of $(+a_2)$-arcs (respectively, $(-a_2)$-arcs) contained in P.

Note that $(\alpha_P + \beta_P)$ denotes the number of edges of stripe a_2 in P, as $a_2 \neq n/2$. Indeed, if n is even, $a_2 = n/2$, and $g_2 = 1$, then $g_1 \leq 2$, as g_1 divides n, and $g_2 = \gcd(g_1, a_2)$. As G in $\mathcal{G}$, and, then, $g_1 \geq 3$, it can not happen that an arc is at the same time a $(+a_2)$-arc, and a $(-a_2)$-arc.

Theorem 5.1 *Let $G(n; a_1, a_2; c_1, c_2)$ be a w.u. graph in $\mathcal{G}$. There exists a minimal Hamiltonian cycle C such that $(\alpha_C - \beta_C) \in \{0, g_1\}$.*

Proof. Let $C' = [u_0, u_1, \ldots, u_n]$ be a minimal Hamiltonian cycle. Let us observe that $0 = (u_n - u_0) = \sum_k (u_k - u_{k-1})$, and that each summand belongs to the set $\{a_1,\ a_2,\ n - a_1,\ n - a_2\}$. In particular, a_2 is summed $\alpha_{C'}$ times, and $(n - a_2)$ is summed $\beta_{C'}$ times. Since g_1 divides n, and a_1, it follows that $(\alpha_{C'} - \beta_{C'})a_2 \equiv_{g_1} 0$. As $g_2 = \gcd(g_1, a_2) = 1$, then a_2 is invertible in $\mathbb{Z}_{g_1}$, and $(\alpha_{C'} - \beta_{C'}) \equiv_{g_1} 0$ holds.

On the other hand, it follows by *Theorem 4.3*, and *Theorem 4.5* that

$$(n - g_1)c_1 + g_1 c_2 \ \leq\ c(C') \ \leq\ (n - 2(g_1 - 1))c_1 + 2(g_1 - 1)c_2.$$

As $c_1 < c_2$, the number of edges of stripe a_2 in C', that is, $(\alpha_{C'} + \beta_{C'})$, verifies $g_1 \leq (\alpha_{C'} + \beta_{C'}) \leq 2(g_1 - 1)$. Hence, $|\alpha_{C'} - \beta_{C'}| \ \leq\ 2(g_1 - 1)$. It follows from $(\alpha_{C'} - \beta_{C'}) \equiv_{g_1} 0$ that $(\alpha_{C'} - \beta_{C'}) \ \in \{-g_1, 0, g_1\}$.

If $(\alpha_{C'} - \beta_{C'}) \in \{0, g_1\}$, the claim follows for $C = C'$. Otherwise, it follows for

$C = -C'$, as $\alpha_C = \beta_{C'}$, $\beta_C = \alpha_{C'}$, and then $(\alpha_C - \beta_C) = g_1$. □

Theorem 5.2 *Let $G = G(n; a_1, a_2; c_1, c_2)$ be a w.u. graph in $\mathcal{G}$. If, for some minimal cycle C, $(\alpha_C - \beta_C) = 0$, then, $c^*(G) = (n - 2(g_1 - 1))c_1 + 2(g_1 - 1)c_2$.*

Proof. Let $C = [u_0, u_1, \ldots, u_n]$ be a minimal Hamiltonian cycle such that $(\alpha_C - \beta_C) = 0$. Any of the $(\alpha_C + \beta_C) = 2\alpha_C$ edges of stripe a_2 in C costs c_2. Any other edge in C costs c_1. Hence, $c^*(G) = c(C) = (n - 2\alpha_C)c_1 + (2\alpha_C)c_2$. The claim follows if we show that $\alpha_C = g_1 - 1$.

Theorem 4.3 implies that $c^*(G) \leq (n - 2(g_1 - 1))c_1 + 2(g_1 - 1)c_2$. Hence, we have that $\alpha_C \leq g_1 - 1$. Here we prove the converse.

Without loss of generality we may assume that $(u_1 - u_0) \equiv_n \pm a_1$. For any $h = 1, \ldots, n$, let P_h be the path $[u_0, \ldots, u_h]$, let $M(h) = \max\{\alpha_{P_h}, \beta_{P_h}\}$, and let $C(h) = |\{\langle u_k \rangle_{g_1} : 1 \leq k \leq h\}|$. Clearly, $P_n = C$, $M(n) = \alpha_C$, and $C(n) = g_1$. The last relation holds, as C is a Hamiltonian cycle.

We claim that, for any $h = 1, \ldots, n$, $(C(h) - 1) \leq M(h)$.

As $(u_1 - u_0) \equiv_n \pm a_1$, it follows that $u_0 \equiv_{g_1} u_1$. Hence, $C(1) = 1$, and $M(1) = 0$. The claim thus holds for $h = 1$.

Assume, now, that $(C(h') - 1) \leq M(h')$, for some $h' < n$.

If $C(h' + 1) = C(h')$, then $(C(h' + 1) - 1) \leq M(h' + 1)$ holds, since $M(h)$ is a non decreasing function.

If $C(h' + 1) = C(h') + 1$, then $u_{h'+1} \equiv_{g_1} (u_{h'} \pm a_2)$. Suppose $\alpha_{P_{h'}} = \beta_{P_{h'}}$. Then, $M(h' + 1) = M(h') + 1$, and the claim holds also for $(h' + 1)$.

Suppose, now, $\alpha_{P_{h'}} > \beta_{P_{h'}}$. For any $j = 1, \ldots, h'$, let $\delta(j) = \alpha_{P_j} - \beta_{P_j}$. Let us note that

$$u_j - u_0 \equiv_{g_1} \sum_{k=1}^{j} (u_k - u_{k-1}) \equiv_{g_1} \alpha_{P_j} \cdot a_2 + \beta_{P_j} \cdot (-a_2) \equiv_{g_1} \delta(j) a_2,$$

and that $|\delta(j) - \delta(j - 1)| \leq 1$, if $j > 1$.

Hence, there exists $j' < h'$ such that $u_{j'} - u_0 \equiv_{g_1} (\delta(h') - 1)a_2$, as $\delta(1) = 0$, and $u_{h'} - u_0 \equiv_{g_1} \delta(h')a_2$. In particular, $u_{j'} \equiv_{g_1} (u_{h'} - a_2)$. Since $C(h' + 1) = C(h') + 1$, then $u_{h'+1} \not\equiv_{g_1} u_{j'}$. Hence, $u_{h'+1} \equiv_{g_1} (u_{h'} + a_2)$, and $[u_{h'}, u_{h'+1}]$ is a $(+a_2)$-arc. So, $\alpha_{P_{h'+1}} = \alpha_{P_{h'}} + 1 > \alpha_{P_{h'}} > \beta_{P_{h'}} = \beta_{P_{h'+1}}$. As $C(h' + 1) = C(h') + 1$, and $(C(h') - 1) \leq M(h')$ hold, it follows that

$$C(h' + 1) - 1 \;\leq\; M(h') + 1 \;=\; \alpha_{P_{h'}} + 1 \;=\; \alpha_{P_{h'+1}} \;=\; M(h' + 1).$$

The case $\alpha_{P_{h'}} < \beta_{P_{h'}}$ is similar to the latter one. The claim is thus proved. For $h = n$, we obtain that $\alpha_C = M(n) \geq C(n) - 1 = g_1 - 1$. The lemma is thus proved. □

Theorem 5.3 *Let $G = G(n; a_1, a_2; c_1, c_2)$ be a w.u. graph in $\mathcal{G}$, and let $A_G = \{y \in \mathbb{Z} : 0 \leq y < n/g_1, (2y - g_1)a_1 + g_1 a_2 \equiv_n 0\}$. The following statements hold.*

(i) If A_G is empty, then $c^*(G) = UB(G) = (n - 2(g_1 - 1))c_1 + 2(g_1 - 1)c_2$.

(ii) If A_G is not empty, let y_1, and y_2 be, respectively, the minimum, and the maximum of A_G, and let $m = \min\{y_1 - g_1, n/g_1 - y_2\}$.
If $m \le 0$, then $c^*(G) = LB(G) = (n - g_1)c_1 + g_1 c_2$. Otherwise, there exists a Hamiltonian cycle for G of cost $(n - g_1 - 2m)c_1 + (g_1 + 2m)c_2$.

Proof. Suppose, first, that A_G is empty. We are in the case (i). If we show that no Hamiltonian cycles C for G such that $(\alpha_C - \beta_C) = g_1$ exist, the claim follows by *Theorem 5.1*, and by *Theorem 5.2*.

Suppose, ad absurdum, that there exists $C = [u_0, u_1, \dots, u_n]$, Hamiltonian cycle for G, such that $(\alpha_C - \beta_C) = g_1$. Let

$$\gamma_C = |\{k \in \{1, \dots, n\} \,:\, (u_k - u_{k-1}) \equiv_n a_1\}|$$
$$\delta_C = |\{k \in \{1, \dots, n\} \,:\, (u_k - u_{k-1}) \equiv_n -a_1,\, (u_k - u_{k-1}) \not\equiv_n a_1\}|$$

Note that, also in the case $a_1 = n/2$, any edge of stripe a_1 in C is considered once. By definition of α_C, and β_C, we have that $n = \alpha_C + \beta_C + \gamma_C + \delta_C$. As $(\alpha_C - \beta_C) = g_1$, it follows that $(\gamma_C - \delta_C) = -2(\beta_C + \delta_C) + (n - g_1)$. Let $y_0 = \langle -(\beta_C + \delta_C)\rangle_{n/g_1}$. Note that $(\gamma_C - \delta_C)a_1 \equiv_n (2y_0 - g_1)a_1$, as $g_1 = \gcd(n, a_1)$, and note that

$$0 \equiv_n \sum_{k=1}^{n}(u_k - u_{k-1}) \equiv_n (\gamma_C - \delta_C)a_1 + (\alpha_C - \beta_C)a_2.$$

Hence, $0 \le y_0 < n/g_1$, and $(2y_0 - g_1)a_1 + g_1 a_2 \equiv_n 0$, that is $y_0 \in A_G$, contradicting the hypothesis. Case (i) is thus proved.

Suppose, now, that A_G is not empty. We are in the case (ii).

If $m \le 0$, then $m = (y_1 - g_1)$, as $y_2 \in A_G$ implies $(n/g_1 - y_2) > 0$. Hence, $y_1 \in \mathbb{Z}$ verifies $0 \le y_1 \le g_1$, as $m \le 0$, and $(2y_1 - g_1)a_1 + g_1 a_2 \equiv_n 0$, as $y_1 \in A_G$. The claim on $c^*(G)$, then, follows by *Theorem 4.6*.

Otherwise, $m = \min\{y_1 - g_1, n/g_1 - y_2\}$ is a positive integer less than $n/2g_1$. Indeed, $2m \le (y_1 - g_1) + (n/g_1 - y_2) \le n/g_1 - g_1 < n/g_1$. Let us denote by Δ_λ, for any $\lambda \in \mathbb{Z}_{g_1}$, the set $\{v \in \mathbb{Z}_n \,:\, v \equiv_{g_1} \lambda a_2\}$, and by n' the integer n/g_1. $\Delta_0, \Delta_1, \dots, \Delta_{g_1-1}$ forms a partition of $\mathbb{Z}_n$, the node set of G, and the equivalence $n'a_1 \equiv_n 0$ holds. Finally, $v \in \mathbb{Z}$ denotes the node $\langle v\rangle_n$ of G.

For any $\varepsilon \in \{+1, -1\}$, P_m^ε is the path

$$\begin{aligned}&[0, \varepsilon(n'-1)a_1, \dots, \varepsilon(2m+1)a_1, \varepsilon(2m+1)a_1 + a_2, \varepsilon 2ma_1 + a_2, \varepsilon 2ma_1,\\ &\varepsilon(2(m-1)+1)a_1, \dots, \varepsilon 3a_1, \varepsilon 3a_1 + a_2, \varepsilon 2a_1 + a_2, \varepsilon 2a_1, \varepsilon a_1, \varepsilon a_1 + a_2,\\ &a_2, \varepsilon(n'-1)a_1 + a_2, \dots, \varepsilon(2m+2)a_1 + a_2, \varepsilon(2m+2)a_1 + 2a_2].\end{aligned}$$

P_m^ε is an elementary path passing through any node in Δ_0, and Δ_1. Moreover, $c(P_m^\varepsilon) = (2n/g_1 - 2m)c_1 + (2 + 2m)c_2$.

For any $\lambda \in \mathbb{Z}_{g_1}$, and for any $\varepsilon \in \{+1, -1\}$, Q_λ^ε is the path

$$\begin{aligned}&[\varepsilon(2m+\lambda)a_1 + \lambda a_2, \varepsilon(2m+\lambda-1)a_1 + \lambda a_2, \dots, \lambda a_2, \varepsilon(n'-1)a_1 + \lambda a_2, \dots,\\ &\varepsilon(2m+\lambda+1)a_1 + \lambda a_2, \varepsilon(2m+\lambda+1)a_1 + (\lambda+1)a_2].\end{aligned}$$

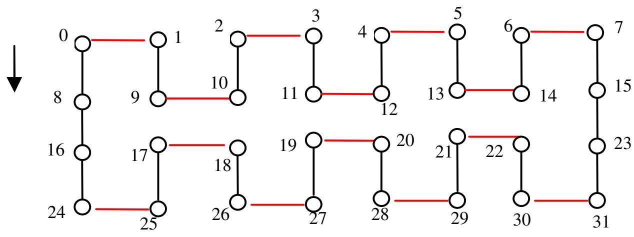

Fig. 3. A minimal Hamiltonian cycle for $G(32; 8, 1; 1, 2)$

$Q^{\varepsilon}_{\lambda}$ is an elementary path passing through any node in Δ_{λ}. Its cost verifies $c(Q^{\varepsilon}_{\lambda}) = (n/g_1 - 1)c_1 + c_2$. Finally, note that the ending point of P^{ε}_m coincides with the starting point of Q^{ε}_2, and that the ending point of $Q^{\varepsilon}_{\lambda}$ coincides with the starting point of $Q^{\varepsilon}_{\lambda+1}$, for any $\lambda < g_1 - 1$.

Let C^{ε}_m be the path $P^{\varepsilon}_m \cdot Q^{\varepsilon}_2 \cdot \ldots \cdot Q^{\varepsilon}_{g_1-1}$, for any $\varepsilon \in \{+1, -1\}$. C^{ε}_m starts from 0, and passes through any node in G. Its cost verifies

$$c(C^{\varepsilon}_m) = c(P^{\varepsilon}_m) + (g_1 - 2)c(Q^{\varepsilon}_m) = (n - g_1 - 2m)c_1 + (g_1 + 2m)c_2.$$

If $m = y_1 - g_1$, C^{+1}_m is a Hamiltonian cycle for G, as it ends in

$$v \equiv_n (2m + g_1)a_1 + g_1 a_2 \equiv_n (2y_1 - g_1)a_1 + g_1 a_2 \equiv_n 0.$$

If $m = n/g_1 - y_2$, C^{-1}_m is a Hamiltonian cycle for G, as it ends in

$$v \equiv_n (2m + g_1)a_1 - g_1 a_2 \equiv_n (2y_2 - g_1)a_1 + g_1 a_2 \equiv_n 0.$$

In both cases, it follows that there exists a Hamiltonian cycle for G of cost $(n - g_1 - 2m)c_1 + (g_1 + 2m)c_2$. □

Example 5.4 *Let G_1 be the w.u. graph $G(32; 8, 1; 1, 2)$. We have that $g_1 = 8$, $n/g_1 = 4$, and $g_2 = 1$. Since $(2y - 8)8 + 8 \equiv_{32} 0$ has no integer solutions, A_{G_1} is empty. Hence, $c^*(G_1) = UB(G_1) = 46$. A minimal cycle of cost 46 is depicted in* Figure 3.

Example 5.5 *Let G_2 be the w.u. graph $G(243; 18, 1; 1, 2)$. We have that $g_1 = 9$, $n/g_1 = 27$, and $g_2 = 1$. $A_{G_2} = \{25\}$, as $y_1 = 25$ is the unique integer solutions in $[0, 26]$ of the equation $(2y - 9)18 + 9 \equiv_{243} 0$. Hence, $m = n/g_1 - y_1 = 2$, and C^{-1}_2 is a Hamiltonian cycle for G_2 (see* Figure 4*).*

In general, we may observe that, given $G = G(n; a_1, a_2; c_1, c_2)$ in $\mathcal{G}$, the following statements hold:

- A_G is empty, if n/g_1 is even, and g_1 or a_2 are even.
- A_G contains just an element, if n/g_1 is odd.
- A_G contains two elements, if n/g_1 is even, g_1 is odd, a_2 is odd.

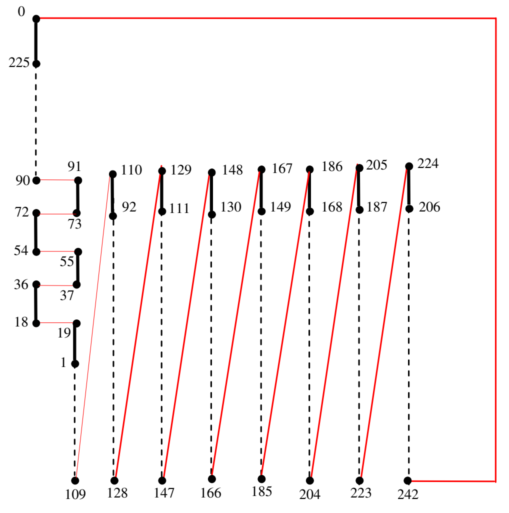

Fig. 4. A non trivial cycle for $G(243; 18, 1; 1, 2)$

Finally, let us consider the w.u. graph $G = G(45; 20, 9; 1, 2)$. It is easy to verify that $g_1 = 5$, $A_G = \{7\}$, and, so, $m = 2$. *Theorem 5.3* assures the existence of a Hamiltonian cycle of cost 54. Such cycle is not a minimal one, as $UB(G) = 53$ by *Theorem 4.3*.

Hence, if A_G is not empty, and $m > 0$, the Hamiltonian cycle found in *Theorem 5.3* is not necessarily minimal. Anyway, we conjecture that it happens whenever its cost is less than $UB(G)$.

6 Conclusions

Although a solution of SCTSP has not been found, we think that SCTSP is polynomial time solvable at least in the case in which any two stripes have different costs. Actually, if we understand how a Hamiltonian cycle, or, more generally, a Hamiltonian path for a circulant w.u. graphs with s stripes can be transferred to circulant w.u. graphs with more stripes, we could be very close to the solution of SCTSP.

To this aim we have analyzed SCTSP on the w.u. graphs with 2 stripes. *Theorem 5.3*, in particular, gives an algebraic characterization of those w.u. graphs having the cost of its minimal Hamiltonian cycle equal either to the upper bound, or to the lower bound. Moreover, it proposes a new method for constructing Hamiltonian cycles in the remaining cases. We conjecture that such cycles are also minimal, but it is not yet proved it. Finally, we are planning to run some heuristics for SCTSP in order to evaluate the soundness of this conjecture.

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