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A Behavioural Pseudometric based on λ–Bisimilarity [1]

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Abstract

In order to describe approximate equivalence among processes, the notions of λ–bisimilarity and behavioural pseudometric have been introduced by Ying and van Breugel respectively. Van Breugel provides a distance function induced by λ–bisimilarity, and conjectures that his behavioural pseudometric coincides with this function. This paper is inspired by this conjecture. We give a negative answer for van Breugel's conjecture first. Moreover, we show that the distance function induced by λ–bisimilarity is a pseudometric on states, and provide a fixed point characterization of this pseudometric.

Keywords: Process algebra; λ–bisimilarity; behavioural pseudometric; fixed point characterization

1 Introduction

The notion of bisimilarity is one of the central concepts in process algebra [9,10]. Roughly speaking, two processes are said to be bisimilar if they can perform same actions to reach bisimilar states. Two bisimilar processes always are thought to be equivalent [9,10]. However, when we consider labelled transition systems whose states or actions contain quantitative data, the notion of bisimilarity seems not to be very suitable for describing the equivalence among processes. For instance, in real time systems, time delays play a key role and there is often a bit difference between time delays. It is too restrictive that time delays can be matched only when they are identical. To overcome this defect, a number of theories have

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been presented to describe approximate equivalence among processes (for example, [1,2,3,4,5,6,7,8,11,12,13]). In these papers, two different approaches have been adopted.

One approach is to introduce approximate equivalence relations over processes. For example, Giacalone et al. provide ε–bisimilarity for probabilistic transition systems [5]. In the same framework, Di Pierro et al. introduce a technique for defining approximate versions of various process equivalences [11]. Ying presents λ–bisimilarity [12,13] based on labelled transition system (LTS) and metric spaces over actions. Girard and Pappas introduce δ–approximate bisimulation in the framework of LTS with observations and metrics over observations [6,7,8].

Another approach is based on distance functions over processes (or, states). For instance, Giacalone et al. [5] and Desharnais et al. [3,4] define the pseudometrics on the states of probabilistic transition systems respectively. In the framework of LTS, van Breugel provides three avenues to define behavioural pseudometrics [2]. The behavioural pseudometric is a distance function between states, which is a quantitative analogue of bisimilarity. In the framework of quantitative transition system (QTS), de Alfaro et al. define the distance between traces and lift this trace distance to distances over states [1]. Furthermore, they provide the notion of branching distance, which generalizes the notion of bisimilarity.

Recently, the relationship between these two approaches has been considered in the literature. For example, a relationship between branching distance and δ–approximate bisimulation is established by Girard and Pappas [8]. Van Breugel also presents the following conjecture, which concerns the relationship between his notion of behavioural pseudometric and Ying's λ–bisimilarity [2]:

$$d(x, y) = \inf\{\lambda : s \sim_\lambda t\}$$

where d is the behavioural pseudometric provided by van Breugel in [2] and $s \sim_\lambda t$ means that s and t are λ–bisimilar [12,13].

This paper is motivated by the above conjecture. We first give a negative answer for this conjecture. Then we consider the distance function d_I defined by $d_I(s, t) = \inf\{\lambda : s \sim_\lambda t\}$ for each states s and t. We show that this distance function is a pseudometric. Furthermore, a fixed point characterization of this distance function is established.

The rest of this paper is organized as follows. In Section 2, we recall related definitions in the literature. In particular, we recall van Breugel's conjecture and provide a counterexample for this conjecture. Section 3 shows that the distance function d_I induced by λ–bisimilarity is a pseudometric. In Section 4, we obtain a fixed point characterization of this pseudometric. Section 5 finally compares our work with related work.

2 λ–bisimilarity and van Breugel's conjecture

This section will recall the notion of λ–bisimilarity [12,13] and van Breugel's conjecture [2]. The interested reader may refer to [2,12,13] for more information.

Definition 2.1 Let X be a nonempty set. The pair $(X,\, \rho)$ is a metric space if ρ is a mapping from $X \times X$ into $[0, \infty]$ such that for any $x,\, y,\, z \in X$,

(1) $\rho(x,\, y) = 0$ if and only if $x = y$,

(2) $\rho(x,\, y) = \rho(y,\, x)$,

(3) $\rho(x,\, z) \leq \rho(x,\, y) + \rho(y,\, z)$.

If (1) is weakened by

$(1)'$ $\rho(x,\, x) = 0$ for each $x \in X$,

then ρ is called a pseudometric.

Definition 2.2 ([2]) A metric LTS is a triple $(S,\, A,\, \rightarrow)$ consisting of a pseudometric space S of states, a metric space A of actions and a labelled transition relation $\rightarrow \subseteq S \times A \times S$.

As mentioned in introductory section, the notion of bisimilarity may not be very suitable for describing approximate equivalence. Ying presents the following notion to overcome this defect.

Definition 2.3 ([12]) Let $\sigma = (S,\, A,\, \{\xrightarrow{a} : a \in A\})$ be a metric LTS, ρ a metric on A, $R \subseteq S \times S$ and $\lambda \in [0, \infty]$. The relation R is a λ–bisimulation if and only if for each $(s,\, t) \in R$, for each $\theta > \lambda$ and for each $a \in A$,

(1) whenever $s \xrightarrow{a} s_1$ then there exist $b \in A$ and $t_1 \in S$ such that $\rho(a,\, b) < \theta$, $t \xrightarrow{b} t_1$ and $(s_1,\, t_1) \in R$; and

(2) whenever $t \xrightarrow{a} t_1$ then there exist $b \in A$ and $s_1 \in S$ such that $\rho(a,\, b) < \theta$, $s \xrightarrow{b} s_1$ and $(s_1,\, t_1) \in R$.

As usual, we say that s and t are λ–bisimilar, in symbols $s \sim_\lambda t$, if $(s, t) \in R$ for some λ–bisimulation R. In other words, λ–bisimilarity $\sim_\lambda$ is defined as $\sim_\lambda =_{def} \bigcup\{R : R \text{ is a } \lambda\text{–bisimulation}\}$.

Clearly, λ–bisimilarity does not always force matched actions to be identical and admits some difference between them. Such difference is depicted by metrics on actions. Moreover, it is easy to see that the usual notion of bisimilarity may be regarded as λ–bisimilarity with the discrete metric (i.e., $\rho(a,\, b) = \infty$ for all $a \neq b$ and $\rho(a, a) = 0$ for each $a \in A$).

Proposition 2.4 ([12]) *(1) For any $\lambda_1, \lambda_2 \in [0, \infty]$, if $\lambda_1 \leq \lambda_2$ then $\sim_{\lambda_1} \subseteq \sim_{\lambda_2}$.*

(2) For any $\lambda \in [0, \infty]$, $\sim_\lambda$ is a λ–bisimulation and it is reflexive and symmetric.

As mentioned in the introduction, van Breugel provides three different but equivalent characterizations of a behavioural pseudometric: a fixed point, a logical and a coalgebraic characterization [2]. In the same paper, he presents a conjecture concerning the relationship between his behavioural pseudometric and Ying's λ–bisimilarity. The work of this paper is inspired by this conjecture. In the following, we recall some related definitions in [2].

Definition 2.5 ([2]) The relation $\sqsubseteq$ on pseudometrics on S is defined by

$$d_1 \sqsubseteq d_2 \text{ if } d_1(s,t) \geq d_2(s,t) \text{ for all } s, t \in S.$$

Definition 2.6 ([2]) Let $(S,\ A,\ \rightarrow)$ be a metric LTS, and let d be a pseudometric on S and ρ be the metric on A. The function $\Delta(d)$: $S \times S \rightarrow [0,\ \infty]$ is defined by

$$\Delta(d)(s,t) = \max\left\{ \sup_{s \xrightarrow{a} s_1} \inf_{t \xrightarrow{b} t_1} \rho(a,b) + d(s_1,t_1),\ \sup_{t \xrightarrow{b} t_1} \inf_{s \xrightarrow{a} s_1} \rho(a,b) + d(s_1,t_1) \right\}$$

Here, $\inf \emptyset = \infty$ and $\sup \emptyset = 0$. The distance function $d_f : S \times S \rightarrow [0, \infty]$ is defined to be the greatest fixed point of Δ.

It should pointed out that defining a behavioural pseudometric as a greatest fixed point was first done by Desharnais et al. [4]. Van Breugel conjectures that the following equation holds [2]:

$$d_f(x,y) = \inf\{\lambda : s \sim_\lambda t\}.$$

However, this equation does not always hold. A counterexample is provided below.

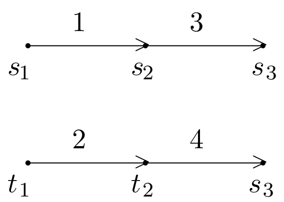

Figure 1

Example 2.7 Consider the processes s_1 and t_1, whose behaviours are depicted in Figure 1. Here, 1, 2, 3 and 4 denote actions and the distance between actions a and b is defined as $|a - b|$. We set

$$R = \{(s_1,t_1), (s_2,t_2), (s_3,s_3)\}.$$

It is easy to check that R is 1–bisimulation. Further, we get $s_1 \sim_1 t_1$. Thus, we obtain

$$\inf\{\lambda : s \sim_\lambda t\} \leq 1 \tag{2.7.1}$$

Next we will show $d_f(s_1,t_1) > 1$ by proving the following claim.

Claim. $d_f(s_2,t_2) = 1$ and $d_f(s_1,t_1) = 2$.

We compute $d_f(s_2,t_2)$ first. Since $s_2 \xrightarrow{3} s_3$, $t_2 \xrightarrow{4} s_3$ and s_2 and t_2 can not perform any other action, we get

$$\sup_{s_2 \xrightarrow{a} u} \inf_{t_2 \xrightarrow{b} v} \rho(a,b) + d_f(u,v) = \rho(3,4) + d_f(s_3,s_3). \tag{2.7.2}$$

On the other hand, since d_f is a pseudometric, we have $d_f(s_3,s_3) = 0$. Further, it follows from (2.7.2) and $\rho(3,4) = 1$ that

$$\sup_{s_2 \xrightarrow{a} u} \inf_{t_2 \xrightarrow{b} v} \rho(a,b) + d_f(u,v) = 1.$$

Similarly, we obtain $\sup_{t_2 \xrightarrow{a} u} \inf_{s_2 \xrightarrow{b} v} \rho(a,b) + d_f(u,v) = 1$. Thus, it follows from Definition 2.6 that

$$\Delta(d_f)(s_2,\ t_2) = 1.$$

Since d_f is the fixed point of Δ, we get $d_f(s_2, t_2) = 1$.

In the following, we will show $d_f(s_1, t_1) = 2$. Since $s_1 \xrightarrow{1} s_2$, $t_1 \xrightarrow{2} t_2$ and s_1 and t_1 can not perform any other action, it follows that

$$\sup_{s_1 \xrightarrow{a} u} \inf_{t_1 \xrightarrow{b} v} \rho(a,b) + d_f(u,v) = \rho(1,2) + d_f(s_2, t_2).$$

Further, since $d_f(s_2, t_2) = 1$ and $\rho(1,2) = 1$, we obtain

$$\sup_{s_1 \xrightarrow{a} u} \inf_{t_1 \xrightarrow{b} v} \rho(a,b) + d_f(u,v) = 2.$$

Similarly, we have $\sup_{t_1 \xrightarrow{a} u} \inf_{s_1 \xrightarrow{b} v} \rho(a,b) + d_f(u,v) = 2$. Hence, it follows from Definition 2.6 that

$$\Delta(d_f)(s_1, t_1) = 2.$$

So, $d_f(s_1, t_1) = 2$ follows from the fact that d_f is the fixed point of Δ.

Hence, by the above claim and (2.7.1), we get $d_f(s_1, t_1) \neq \inf\{\lambda : s_1 \sim_\lambda t_1\}$. □

This negative answer should be anticipated. In fact, λ–bisimilarity and van Breugel's behavioural pseudometric d_f characterize different aspects of approximate equivalence. Roughly speaking, the former describes the similarity of states in terms of the distances between the corresponding actions, while the latter captures the similarity of the behaviour of states based on the total of distances between the corresponding actions. More formally, if $s \sim_\lambda t$ and s can make a sequence of actions $a_1,\ a_2,\ \cdots a_n$, then for any $\varepsilon > 0$, t can make a sequence of actions b_1, $b_2,\ \ldots b_n$ such that $\rho(a_i, b_i) < \lambda + \varepsilon$ $(1 \leq i \leq n)$. In contrast, if $d_f(s,t) = \lambda$ and s can make a sequence of actions $a_1,\ a_2,\ \ldots a_n$, then t can make a sequence of actions $b_1,\ b_2,\ \ldots b_n$ such that $\Sigma_{1 \leq i \leq n} \rho(a_i, b_i) \leq \lambda$.

3 Behavioural λ–pseudometric

When we consider system approximations, one may argue that the notion of distance, rather than relation (for example, trace equivalence, simulation and bisimulation), is particularly useful in the quantitative setting [1]. In this section, we will illustrate that Ying's approach also provides a simple characterization for distances between states. To this end, the function d_I is introduced below. Similarly, in the framework of probabilistic transition systems, Giacalone et al. have provided their pseudometric in terms of ε–bisimilarity [5].

Definition 3.1 (behavioural λ–pseudometric) The function $d_I : S \times S \to [0, \infty]$ is defined as

$$d_I(s,t) = \inf\{\lambda : s \sim_\lambda t\}$$

In the following, d_I is said to be a behavioural λ–pseudometric.

Clearly, if $s \sim_\lambda t$ then $d_I(s,t) \leq \lambda$. Someone may raise the following conjecture: if $d_I(s,t) = \lambda$ then $s \sim_\lambda t$. The next counterexample illustrates that this does not always hold. A similar example has been considered in [12].

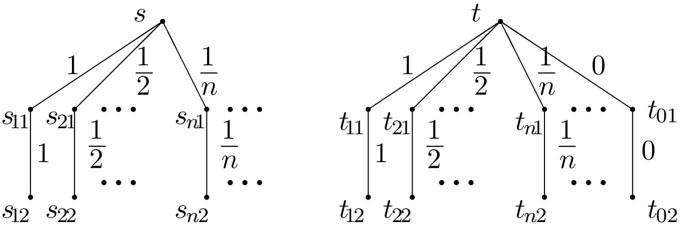

Figure 2

Example 3.2 Consider the processes s and t, whose behaviours are depicted in Figure 2. Here, $1/n$ and 0 denote actions for any positive integer n. The distance between actions a and b is defined as $|a-b|$.

Let $\lambda > 0$. We set

$$\begin{aligned} R_\lambda = \{(s,\ t)\} &\cup \{(s_{i1},\ t_{i1}) : i > 0\} \cup \{(s_{i1},\ t_{01}) : 1/i > \lambda\} \\ &\cup \{(s_{i2},\ t_{i2}) : i > 0\} \cup \{(s_{i2},\ t_{02}) : 1/i > \lambda\} \end{aligned}$$

It is easy to check that R_λ is a λ–bisimulation. Thus, $s \sim_\lambda t$. It follows that $d_I(s,t) \leq \lambda$. Since λ is arbitrary, we get $\{\lambda\colon\ s \sim_\lambda t\} = \{\lambda : \lambda > 0\}$. So, $d_I(s,t) = 0$.

On the other hand, it is clear that $s_{i1} \not\sim_0 t_{01}$ for each $i > 0$ [4]. So, there is not $u \in S$ such that

$$s \xrightarrow{a} u \text{ and } u \sim_0 t_{01} \text{ for some action } a.$$

Further, by $t \xrightarrow{0} t_{01}$, we get $s \not\sim_0 t$. Thus, $d_I(s,t) = 0$ and $s \not\sim_0 t$. □

In the rest of this section, we will show that d_I is a pseudometric over states. Before showing it, we first prove two preliminary results.

Lemma 3.3 *Let $s,t \in S$ and $k \in [0,\infty)$. Then*

(i) if $d_I(s,t) < \infty$ then for any $\varepsilon > 0$, $s \sim_{d_I(s,t)+\varepsilon} t$; and

(ii) if $d_I(s,t) < k$ and $s \xrightarrow{a} s_1$ then there exist $b \in A$ and $t_1 \in S$ such that $t \xrightarrow{b} t_1$, $\rho(a,b) < k$ and $d_I(s_1,t_1) < k$.

Proof. We show that (i) holds first. Assume that

$$s \not\sim_{d_I(s,t)+\varepsilon} t \quad \text{for some } \varepsilon > 0.$$

Then it follows from (1) in Proposition 2.4 that $s \not\sim_k t$ for any $k \leq d_I(s,t) + \varepsilon$. So we obtain

[4] Let $i > 0$ and $\theta = 1/2i$. Then $t_{01} \xrightarrow{0} t_{02}$ and s_{i1} can not perform an action a with $\rho(0,a) < 1/2i$. Thus by Definition 2.3, $s_{i1} \not\sim_0 t_{01}$.

$$d_I(s,t) = \inf\{\lambda : s \sim_\lambda t\} \geq d_I(s,t) + \varepsilon > d_I(s,t).$$

This is a contradiction.

Next we prove that (ii) holds. Let $d_I(s,t) < k$ and $s \xrightarrow{a} s_1$. We set $\varepsilon = (k - d_I(s,t))/2$. Clearly, $\varepsilon > 0$. Then

$$\inf\{\lambda : s \sim_\lambda t\} = d_I(s,t) < k - \varepsilon.$$

So there is $\lambda \in [0,\infty)$ such that

$$s \sim_\lambda t \text{ and } \lambda < k - \varepsilon.$$

By (2) from Proposition 2.4, it follows from $s \sim_\lambda t$ and $s \xrightarrow{a} s_1$ that there exist $b \in A$ and $t_1 \in S$ such that

$$t \xrightarrow{b} t_1,\ \rho(a,b) < \lambda + \varepsilon \textit{ and } s_1 \sim_\lambda t_1.$$

This together with $\lambda < k - \varepsilon$ implies $\rho(a,b) < k$ and $d_I(s_1,t_1) \leq \lambda < k$. □

Lemma 3.4 *Let $k \in [0,\infty)$. Then*

$$R_k =_{def} \{(s,t) : d_I(s,u) + d_I(u,t) < k \textit{ for some } u \in S\}$$

is a k–bisimulation.

Proof. Let $(s,t) \in R_k$ and $s \xrightarrow{a} s_1$. It is enough to show that there exist $b \in A$ and $t_1 \in S$ such that $\rho(a,b) < k,\ t \xrightarrow{b} t_1$ *and* $(s_1,t_1) \in R_k$

Since $(s,t) \in R_k$, we get $d_I(s,u) + d_I(u,t) < k$ for some $u \in S$. It follows that there exists $\varepsilon > 0$ such that

$$d_I(s,u) + d_I(u,t) + 4\varepsilon < k.$$

By Lemma 3.3, we have $s \sim_{d_I(s,u)+\varepsilon} u$. Then by (2) from Proposition 2.4, for some $b \in A$ and $u_1 \in S$, we obtain

$$u \xrightarrow{b} u_1,\ \rho(a,b) < d_I(s,u) + 2\varepsilon \text{ and } s_1 \sim_{d_I(s,u)+\varepsilon} u_1.$$

Similarly, by Lemma 3.3, we get $u \sim_{d_I(u,t)+\varepsilon} t$. Then by (2) from Proposition 2.4, it follows from $u \xrightarrow{b} u_1$ and $u \sim_{d_I(u,t)+\varepsilon} t$ that there exist $c \in A$ and $t_1 \in S$ such that

$$t \xrightarrow{c} t_1,\ \rho(b,c) < d_I(u,t) + 2\varepsilon \text{ and } u_1 \sim_{d_I(u,t)+\varepsilon} t_1.$$

Since ρ is a metric, we have

$$\rho(a,c) \leq \rho(a,b) + \rho(b,c) < d_I(s,u) + d_I(u,t) + 4\varepsilon < k.$$

Moreover, it follows from $s_1 \sim_{d_I(s,u)+\varepsilon} u_1$ and $u_1 \sim_{d_I(u,t)+\varepsilon} t_1$ that

$$d_I(s_1,u_1) + d_I(u_1,t_1) \leq d_I(s,u) + d_I(u,t) + 2\varepsilon < k.$$

So, $(s_1,\ t_1) \in R_k$. □

In the following, we will show that the function d_I is indeed a pseudometric on S. In other words, the notion of λ–bisimilarity also can induce a natural distance function between processes.

Theorem 3.5 *d_I is a pseudometric on S.*

Proof. Let $s, t, u \in S$. It follows from $s \sim_0 s$ that $d_I(s, s) = 0$. Moveover, by (2) from Proposition 2.4, $\sim_\lambda$ is symmetric for each $\lambda \in [0, \infty)$ and therefore $d_I(s, t) = d_I(t, s)$. It remains to show that $d_I(s, t) \leq d_I(s, u) + d_I(u, t)$. We consider the following two cases.

Case 1 $d_I(s, u) + d_I(u, t) = \infty$. Then $d_I(s, t) \leq d_I(s, u) + d_I(u, t)$ holds trivially.

Case 2 $d_I(s, u) + d_I(u, t) < \infty$. Let $\varepsilon > 0$. Then it follows from Lemma 3.4 that

$$s \sim_{d_I(s,u)+d_I(u,t)+\varepsilon} t.$$

Then

$$d_I(s, t) \leq d_I(s, u) + d_I(u, t) + \varepsilon.$$

Since ε is arbitrary, we get $d_I(s, t) \leq d_I(s, u) + d_I(u, t)$ [5]. □

4 The fixed point characterization of d_I

In [1,2,4], distance functions are defined in terms of fixed points (see Definition 2.6 in this paper, Definition 4 in [1] and Definition 3.5 in [4]). In the following, we will demonstrate that behavioural λ–pseudometric d_I also has a fixed point characterization. To this end, we introduce the operator F below. This operator has been mentioned in the conclusion of [2].

Definition 4.1 Let $(S, A, \rightarrow)$ be a metric LTS, and let d be a pseudometric on S and ρ be the metric on A. The function $F(d)$: $S \times S \rightarrow [0, \infty]$ is defined by

$$F(d)(s,t) = \max\left\{ \sup_{s \xrightarrow{a} s_1} \inf_{t \xrightarrow{b} t_1} \max\{\rho(a,b), d(s_1,t_1)\}, \sup_{t \xrightarrow{b} t_1} \inf_{s \xrightarrow{a} s_1} \max\{\rho(a,b), d(s_1,t_1)\} \right\}$$

This section aims to prove that d_I is the greatest fixed point of F. Before showing it, we prove a series of auxiliary results.

Lemma 4.2 *Let $k \in [0, \infty)$. Then*

$$R_k =_{def} \{(s, t) : d_I(s, t) < k \text{ or } F(d_I)(s, t) < k\}$$

is a k–bisimulation.

Proof. Let $(s, t) \in R_k$ and $s \xrightarrow{a} s_1$. It is enough to show that there exist $b \in A$ and $t_1 \in S$ such that $\rho(a, b) < k$, $t \xrightarrow{b} t_1$ and $(s_1, t_1) \in R_k$. Since $(s, t) \in R_k$, we get

$$\text{either } d_I(s, t) < k \text{ or } F(d_I)(s, t) < k.$$

[5] Otherwise, we set $\varepsilon = (d_I(s, t) - d_I(s, u) - d_I(u, t))/2$. Then $d_I(s, u) + d_I(u, t) + \varepsilon = (d_I(s, t) + d_I(s, u) + d_I(u, t))/2 < d_I(s, t)$, a contradiction. In the rest of this paper, similar arguments will be omitted.

By (ii) from Lemma 3.3, it is enough to consider the nontrivial case where $F(d_I)(s,t) < k$. From the definition of F, we have

$$\sup_{s \xrightarrow{a_1} u} \inf_{t \xrightarrow{b_1} w} \max\{\rho(a_1, b_1), d_I(u, w)\} < k.$$

Further, it follows from $s \xrightarrow{a} s_1$ that

$$\inf_{t \xrightarrow{c} v} \max\{\rho(a, c), d_I(s_1, v)\} < k. \tag{4.2.1}$$

Thus, since $\inf \emptyset = \infty$, we obtain $\{v : t \xrightarrow{c} v \text{ for some } c \in A\} \neq \emptyset$. Then there exist $b \in A$ and $t_1 \in S$ such that [6]

$$\rho(a, b) < k,\ t \xrightarrow{b} t_1 \text{ and } d_I(s_1, t_1) < k.$$

So, $(s_1, t_1) \in R_k$, as desired. □

Lemma 4.3 *Let $s, t \in S$ and $\lambda \in [0, \infty)$. If $s \sim_\lambda t$ then $F(d_I)(s,t) \leq \lambda$.*

Proof. Let $s \sim_\lambda t$, $s \xrightarrow{a} s_1$ and $\varepsilon > 0$. Then by (2) from Proposition 2.4, there exist $b \in A$ and $t_1 \in S$ such that

$$t \xrightarrow{b} t_1,\ \rho(a, b) < \lambda + \varepsilon \text{ and } s_1 \sim_\lambda t_1.$$

It follows that $\max\{\rho(a, b),\, d_I(s_1, t_1)\} < \lambda + \varepsilon$. Thus

$$\inf_{t \xrightarrow{c} v} \max\{\rho(a, c), d_I(s_1, v)\} < \lambda + \varepsilon.$$

Since a and s_1 are arbitrary, we get

$$\sup_{s \xrightarrow{a_1} u} \inf_{t \xrightarrow{b_1} w} \max\{\rho(a_1, b_1), d_I(u, w)\} \leq \lambda + \varepsilon.$$

Furthermore, since ε is arbitrary, we obtain

$$\sup_{s \xrightarrow{a_1} u} \inf_{t \xrightarrow{b_1} w} \max\{\rho(a_1, b_1), d_I(u, w)\} \leq \lambda.$$

Similarly, $\sup_{t \xrightarrow{b_1} t_1} \inf_{s \xrightarrow{a_1} s_1} \max\{\rho(a_1, b_1), d(s_1, t_1)\} \leq \lambda$. So, $F(d_I)(s,t) \leq \lambda$. □

Lemma 4.4 *Let d be any fixed point of F and $k \in [0, \infty)$. Then*

$$U_k =_{def} \{(s, t) : d(s, t) < k\}$$

is a k–bisimulation.

Proof. Let $(s, t) \in U_k$ and $s \xrightarrow{a} s_1$. It is enough to show that there exist $b \in A$ and $t_1 \in S$ such that $\rho(a, b) < k$, $t \xrightarrow{b} t_1$ and $(s_1, t_1) \in U_k$.

Since d is a fixed point of F, we have

$$F(d)(s, t) = d(s, t) < k.$$

By the definition of F, we get

[6] Otherwise, for each $b \in A$ and each $t_1 \in S$ such that $t \xrightarrow{b} t_1$, we have either $\rho(a, b) \geq k$ or $d_I(s_1, t_1) \geq k$. Then $\inf_{t \xrightarrow{b_1} w} \max\{\rho(a_1, b_1), d_I(u, w)\} \geq k$, which contradicts (4.2.1).

$$\sup_{s \xrightarrow{a_1} u} \inf_{t \xrightarrow{b_1} w} \max\{\rho(a_1, b_1), d(u, w)\} < k.$$

It follows from $s \xrightarrow{a} s_1$ that

$$\inf_{t \xrightarrow{c} v} \max\{\rho(a, c), d(s_1, v)\} < k.$$

Similar to Lemma 4.2, we have $\{v : t \xrightarrow{c} v \text{ for some } c \in A\} \neq \emptyset$. Then there exist $b \in A$ and $t_1 \in S$ such that

$$t \xrightarrow{b} t_1, \rho(a, b) < k \text{ and } d(s_1, t_1) < k.$$

It follows that $(s_1, t_1) \in U_k$. □

Now we arrive at the main result of this section, which provides a fixed point characterization of the behavioural λ–pseudometric d_I.

Theorem 4.5 *d_I is the greatest fixed point of F.*

Proof. We will demonstrate the following three claims in turn.

Claim 1. $F(d_I) \sqsubseteq d_I$.

Let $s, t \in S$. If $F(d_I)(s, t) = \infty$ then $F(d_I)(s, t) \geq d_I(s, t)$ holds trivially. In the following, we consider the other case where $F(d_I)(s, t) < \infty$. Let $\varepsilon > 0$. It follows from Lemma 4.2 that

$$s \sim_{F(d_I)(s,t)+\varepsilon} t.$$

Thus $F(d_I)(s, t) + \varepsilon \geq d_I(s, t)$. Further, since ε is arbitrary, we obtain

$$F(d_I)(s, t) \geq d_I(s, t).$$

So $F(d_I) \sqsubseteq d_I$, as desired.

Claim 2. $d_I \sqsubseteq F(d_I)$.

Let $s, t \in S$. $F(d_I)(s, t) \leq d_I(s, t)$ holds trivially if $d_I(s, t) = \infty$. In the following, we deal with the nontrivial case where $d_I(s, t) < \infty$. Let $\varepsilon > 0$. By Lemma 3.4, we get $s \sim_{d_I(s,t)+\varepsilon} t$. So it follows from Lemma 4.3 that

$$F(d_I)(s, t) \leq d_I(s, t) + \varepsilon.$$

Since ε is arbitrary, we get

$$F(d_I)(s, t) \leq d_I(s, t).$$

Then we have $d_I \sqsubseteq F(d_I)$, as desired.

Claim 3. Let d be any fixed point of F. Then $d \sqsubseteq d_I$.

Suppose d is a fixed point of F. Let $s, t \in S$. If $d(s, t) = \infty$ then $d(s, t) \geq d_I(s, t)$ holds trivially. Thus, we only need to consider the case where $d(s, t) < \infty$. Let $\varepsilon > 0$. By Lemma 4.4, we obtain

$$s \sim_{d(s,t)+\varepsilon} t.$$

So $d(s,t) + \varepsilon \geq d_I(s,t)$. Since ε is arbitrary, we get $d(s,t) \geq d_I(s,t)$. Thus $d \sqsubseteq d_I$. $\square$

By Theorem 3.5, the function d_I is indeed a distance function. Moreover, by Theorem 4.5, it also has a fixed point characterization. We provide an example below to illustrate the difference between d_I and d_f.

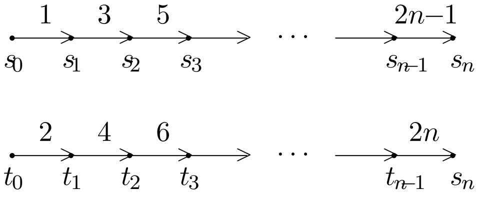

Figure 3

Example 4.6 Consider the processes s_0 and t_0, whose behaviours are depicted in Figure 3. Here, $n > 0$ and positive integer i $(1 \leq i \leq 2n)$ denotes an action. The distance between actions a and b is defined as $|a - b|$.

In the following, we will show $d_I(s_0, t_0) = 1$ and $d_f(s_0, t_0) = n$. We show $d_I(s_0, t_0) = 1$ first. Since d_I is a pseudometric, we have

$$d_I(s_n, s_n) = 0.$$

In order to compute $d_I(s_0, t_0)$, we demonstrate the following claim.

Claim. $d_I(s_{n-k}, t_{n-k}) = 1$ for each k such that $1 \leq k \leq n$.

We proceed by induction on k.

Let $k = 1$. Since $s_{n-1} \overset{2n-1}{\to} s_n$, $t_{n-1} \overset{2n}{\to} s_n$ and s_{n-1} and t_{n-1} can not perform any other action, we obtain

$$\sup_{s_{n-1} \overset{a}{\to} u} \ \inf_{t_{n-1} \overset{b}{\to} v} \max\{\rho(a,b), d_I(u,v)\} = \max\{\rho(2n-1, 2n), d_I(s_n, s_n)\}.$$

Further, since $\rho(2n-1, 2n) = 1$ and $d_I(s_n, s_n) = 0$, we get

$$\sup_{s_{n-1} \overset{a}{\to} u} \ \inf_{t_{n-1} \overset{b}{\to} v} \max\{\rho(a,b), d_I(u,v)\} = 1.$$

Similarly, we have $\sup_{t_{n-1} \overset{a}{\to} u} \ \inf_{s_{n-1} \overset{b}{\to} v} \max\{\rho(a,b), d_I(u,v)\} = 1$. So, by Definition 4.1, we obtain

$$F(d_I)(s_{n-1}, t_{n-1}) = 1.$$

Since d_I is a fixed point of F, we get

$$d_I(s_{n-1}, t_{n-1}) = F(d_I)(s_{n-1}, t_{n-1}) = 1.$$

Suppose $k = m + 1 \leq n$ and $d_I(s_{n-m}, t_{n-m}) = 1$. Let $l = n - m$. It is enough to show that $d_I(s_{l-1}, t_{l-1}) = 1$.

Since $s_{l-1} \stackrel{2l-1}{\rightarrow} s_l$, $t_{l-1} \stackrel{2l}{\rightarrow} t_l$ and s_{l-1} and t_{l-1} can not perform any other action, we have

$$\sup_{s_{l-1}\stackrel{a}{\rightarrow}u}\ \inf_{t_{l-1}\stackrel{b}{\rightarrow}v} \max\{\rho(a,b), d_I(u,v)\} = \max\{\rho(2l-1,2l), d_I(s_l,s_l)\}.$$

By induction, we obtain $d_I(s_l,t_l) = 1$. Further, since $\rho(2l-1,2l) = 1$, we get

$$\sup_{s_{l-1}\stackrel{a}{\rightarrow}u}\ \inf_{t_{l-1}\stackrel{b}{\rightarrow}v} \max\{\rho(a,b), d_I(u,v)\} = 1.$$

Similarly, we may show $\sup_{s_{l-1}\stackrel{a}{\rightarrow}u}\ \inf_{t_{l-1}\stackrel{b}{\rightarrow}v} \max\{\rho(a,b), d_I(u,v)\} = 1$. So, by Definition 4.1, we have

$$F(d_I)(s_{l-1},t_{l-1}) = 1.$$

Since d_I is a fixed point of F, we obtain $d_I(s_{l-1},t_{l-1}) = 1$, as desired.

Thus by the above claim, we get

$$d_I(s_0,t_0) = 1.$$

Similar to the proof of $d_I(s_0,t_0) = 1$, we can show [7]

$$d_f(s_0,t_0) = n.$$

Hence, $d_I(s_0,t_0) \neq d_f(s_0,t_0)$ if $n > 1$. □

The above example illustrates that the function d_I measures distances between states in terms of maximal distances between corresponding actions of states, while d_f measures distances between states using the total of distances of corresponding actions.

5 Comparison with related work

The work of Girard and Pappas [8] is relevant to ours. They introduce the notion of LTS with observations, which contains a set of observations and maps states to observations. In this framework, a labelled transition system may have more than one initial state. Based on this framework and metrics over observations, Girard and Pappas define the notion of δ–approximate bisimulation. Further, they introduce a *bisimulation metric* d over systems as follows: $d(T_1,T_2) = \inf\{\delta : T_1 \widetilde{=}_\delta T_2\}$, where $T_1 \widetilde{=}_\delta T_2$ means that there is a δ–approximate bisimulation relation R between systems T_1 and T_2 such that for each initial state s of a system, there exists an initial state t of another system with $(s,t) \in R$.

The difference between their bisimulation metric and our behavioural λ–pseudometric lies in the following. The behavioural λ–pseudometric is defined based on the actions performed by states, while the bisimulation metric is presented in terms of observations of states. Such a difference comes from the differ-

[7] It is enough to show $d_f(s_n,s_n) = 0$ and $d_f(s_{n-k},t_{n-k}) = k$ for each k such that $1 \leq k \leq n$. Their proofs are similar to the above. We leave them to the interested reader.

ence between λ–bisimilarity and δ–approximate bisimulation. Rough speaking, λ–bisimilarity admits some difference between corresponding actions of states, whereas δ–approximate bisimulation forces matched actions to be identical and admits some difference between observations of states. Thus, λ–bisimilarity is quite different from δ–approximate bisimulation, and this brings some drastic differences between proofs in this paper and ones in [8].

On the other hand, the difference between the bisimulation metric and the behavioural λ–pseudometric leads to different applications of these two notions. For example, the behavioural λ–pseudometric can be used to analyze approximate implementations of real time systems, while the bisimulation metric may be helpful to simplify safety verification of continuous and hybrid systems [8].

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