Available online at www.sciencedirect.com
ScienceDirect

Electronic Notes in Theoretical Computer Science

Electronic Notes in Theoretical Computer Science 173 (2007) 67–84

www.elsevier.com/locate/entcs

Probabilistic Completion of Nondeterministic Models

Guy Beaulieu[1]

Department of Mathematics and Statistics
University of Ottawa
Ottawa, Canada

Abstract

This work continues ongoing research in combining theories of nondeterminism and probabilistic choice. First, we adapt the above choice theories to allow for uncountably indexed nondeterministic operators, and countably indexed probabilistic operators. Classically, models for mixed choice were obtained by enhancing arbitrary models for probabilistic choice with appropriately distributive nondeterministic operations. In this paper, we focus on the dual approach: constructing mixed choice models by completing nondeterministic models with suitably behaved probabilistic operations. We introduce a functorial construction, called *convex completion*, which freely computes set-theoretical and posetal mixed choice models from the appropriate semilattices. The completion construction relies upon a new closure operation on convex sets, dependant on the given semilattice. Finally, we show that building a free mixed choice model is equivalent to applying the convex completion functor to its corresponding free nondeterministic model.

Keywords: combining categorical theories, mixed choice theory, functorial factorization, convex completion w.r.t a semilattice

1 Introduction

This paper addresses the problem of how to add appropriately behaved probabilistic operators to models of nondeterminism, which is the focus of the author's upcoming PhD thesis, [2]. While there is a large literature on probabilistic theories [5,7,8,9,15] and also nondeterministic theories [14,16,18] in isolation, theories with interacting (mixed) choice operators are a more recent phenomenon. Among many developments in the area of mixed choice [4,10,12,13,19,20,21], the works that have most influenced our viewpoints are Mislove,Ouaknine, and Worrell [13], and Keimel, Plotkin, and Tix [10], along with the methods on combining Lawvere theories introduced in Power [17], and in Hyland, Plotkin and Power [6].

We consider mixed choice theories which admit probabilistic operators of countable arity and nondeterministic operators of possibly uncountable arity. Separately,

[1] Email: gbeaua181@rogers.com

1571-0661
doi:10.1016/j.entcs.2007.02.028

each choice theory is associated to a monad. In our case, the probabilistic choice theory is associated to the distributions monad and the nondeterministic choice theory is associated to the non-empty powerset monad. These will be described later on in the paper. The use of uncountably indexed nondeterministic operators in our framework, is motivated by two important factors. First, the probabilistic completion of nondeterministic models with uncountable nondeterministic operators has a simpler presentation and a greater degree of generality than its countable counterpart. Second, let $\boxplus$ denote nondeterministic choice and for $\lambda \in [0,1]$, let $A \oplus_\lambda B$ denote the probabilistic operator which evaluates to A with probability λ and evaluates to B with probability $1-\lambda$. In our framework, the nondeterministic choice between terms A and B can be interpreted as the probabilistic choice of indeterminate weight of the constituents. This viewpoint is the same as the one proposed by Mislove [12] and also Tix [19] in their approach to creating a mixed choice theory. However, our use of uncountable nondeterministic operators is a new feature allowing us to express the above interpretation of nondeterminism formally as $A \boxplus B = \bigboxplus_{\lambda \in [0,1]} (A \oplus_\lambda B)$. This equation is derivable from our axioms defining a mixed choice theory.

Works by Power [17], and Hyland, Plotkin and Power [6] develop machinery for merging enriched Lawvere theories to combine computational effects, such as combining nondeterminism and probabilistic choice. In particular, Power [17] defines (among other things) a subset of enriched countable Lawvere theories which can be merged under a distributive tensor. The distributive tensor between the Lawvere theories $\mathbf{L}$ and $\mathbf{L}'$, denoted $\mathbf{L} \triangleright \mathbf{L}'$, is the theory which admits all the operations and axioms from $\mathbf{L}$ and $\mathbf{L}'$ but also imposes distributivity axioms which make every operation in $\mathbf{L}$ distribute over every operation in $\mathbf{L}'$. This describes exactly our intended structure for mixed choice.

We proceed by constructing such a distributive tensor, however on a more general combination of Lawvere theories. In our case the theory of nondeterminism has *uncountable* operators and does not appear to fit in Power's framework. In the end our axioms for mixed choice agree with the works of Mislove, Ouaknine and Worrell [13] and Keimel, Plotkin and Tix [10].

In the above literature on combining nondeterministic and probabilistic choice, the approach to constructing models of mixed choice in a category $\mathbf{C}$, is always the same:

(i) First, freely construct probabilistic models over the category $\mathbf{C}$. Thus for set-theoretical models one would use the *distributions functor*, $\mathcal{D}$, and for domain-theoretical models one would use the *extended probabilistic powerdomain*, $\mathcal{V}$.

(ii) Then, freely complete the probabilistic models obtained in (i) with suitable distributive nondeterministic operations, in such a way as to obtain a model for mixed choice. For set-theoretical models the *convex powerset functor*, $\mathcal{P}^*_{cvx}$ is used and for domain-theoretical models there are three types of *convex powerdomains*: lower, upper and biconvex.

The above algorithm applied over the category **Set** corresponds to composing the

$$
\begin{tikzcd}
 & & \mathbf{Mod}(\mathbf{T_{MC}},\mathbf{Set}) \arrow[dll, "\mathcal{U}_{\mathcal{P}^*_{cvx}}"'] \arrow[drr, "\mathcal{U}_{\mathcal{C}_{cvx}}"] \arrow[dd, bend right, "\mathcal{U}_{\mathcal{G}_{cvx}}"'] & & \\
\mathbf{Mod}(\mathbf{L_P},\mathbf{Set}) \arrow[urr, bend left, "\mathcal{P}^*_{cvx}"] \arrow[drr, "\mathcal{U}_{\mathcal{D}}"] & & & & \mathbf{Mod}(\mathbf{T_{ND}},\mathbf{Set}) \arrow[ull, bend right, "\mathcal{C}_{cvx}"'] \arrow[dll, "\mathcal{U}_{\mathcal{P}^*}"'] \\
 & & \mathbf{Set} \arrow[uu, bend right, "\mathcal{G}_{cvx}"'] \arrow[ull, bend left, "\mathcal{D}"] \arrow[urr, bend right, "\mathcal{P}^*"'] & &
\end{tikzcd}
$$

Fig. 1. Mixed Choice Factored Through Probabilistic and Nondeterministic Choice

functors along the left-hand path in the commuting diagram in Figure 1.

Through general categorical results from Barr and Wells [1], we observed that models of mixed choice could also be obtained by completing arbitrary models of nondeterminism with appropriately distributive probabilistic operators. Thus, allowing for an alternative algorithm to constructing models for mixed choice.

(i') First, freely construct nondeterministic models over the category $\mathbf{C}$. Thus for set-theoretical models one would use the *non-empty powerset functor, $\mathcal{P}^*$*.

(ii') Then, freely complete the nondeterministic models obtained in (i') with suitable distributive probabilistic operations, in such a way as to obtain a model for mixed choice. For set-theoretical models we will define the *convex completions functor, $\mathcal{C}_{cvx}$*, to do so.

This algorithm corresponds to composing the functors along the right-hand path in the commutative diagram in Figure 1. This new approach allows for a wider range of mixed choice models to be studied: those that would arise from arbitrary semilattices, hence arbitrary nondeterministic models with a possible non-standard definition of nondeterminism. This paper will focus on giving a concrete definition of the functors on the right-hand path. More specifically, how to explicitly compute the *convex completions functor*, $\mathcal{C}_{cvx} : \mathbf{Mod}(\mathbf{T_{ND}}, \mathbf{Set}) \to \mathbf{Mod}(\mathbf{T_{MC}}, \mathbf{Set})$, (in the "northeast quadrant" of Figure 1) which assigns a model of mixed choice to an arbitrary nondeterministic model.

In the first section, we provide a quick review of the various categorical notions used, with a short list of important notation. In the second section we give a quick overview on combining the choice theories. The third section contains our main result on completing arbitrary nondeterministic models with appropriately distributive probabilistic operators. We introduce the theory necessary to construct the convex completions functor and give a full account of its adjunction structure. The final section presents how our functors can be lifted to find posetal models.

2 Categorical Preliminaries

The following are various categorical definitions and results used below. We follow the treatments in [1,3,11].

Definition 2.1 [Monads] A *monad*, $\mathbb{T}$, on a category $\mathbf{C}$ is given by $\mathbb{T} = (\mathcal{T}, \eta, \mu)$ where $\mathcal{T} : \mathbf{C} \to \mathbf{C}$ is a functor and $\eta : 1_{\mathbf{C}} \Rightarrow \mathcal{T}$, $\mu : \mathcal{T}\mathcal{T} \Rightarrow \mathcal{T}$ are natural transformations satisfying the commutativity conditions: $\mu \circ \eta_T = 1_T = \mu \circ T\eta$, and $\mu \circ \mu_T = \mu \circ T\mu$.

Example 2.2 [Non-empty Powerset Monad] The monad $\mathbb{P}^* = (\mathcal{P}^*, \eta, \mu)$, called the *non-empty powerset monad*, is defined such that for $X \in \mathbf{Set}$,
(a) $\mathcal{P}^*$ is the *non-empty powerset functor*, i.e. $\mathcal{P}^*(X) = \{Y \mid \emptyset \neq Y \subseteq X\}$;
(b) η_X is the *singleton map*, for $x \in X, \eta_X(x) = \{x\}$;
(c) μ_X is the *big union map*, for any $\mathcal{Y} \in \mathcal{P}^*\mathcal{P}^*(\mathcal{X})$,
$\mu_X(\mathcal{Y}) = \bigcup_{\mathcal{Y} \in \mathcal{Y}} \mathcal{Y}$.

Definition 2.3 [$\mathbb{T}$-Algebras] Let $\mathbb{T} = (\mathcal{T}, \eta, \mu)$ be a monad on a category $\mathbf{C}$. A $\mathbb{T}$-*algebra* is a pair $(C, \mathcal{T}(C) \xrightarrow{\zeta} C)$ with C and ζ in $\mathbf{C}$ such that $\zeta \circ \eta_C = 1_C$ and $\zeta \circ \mathcal{T}\zeta = \zeta \circ \mu_C$. If $(D, \mathcal{T}(D) \xrightarrow{\xi} D)$ is another $\mathbb{T}$-algebra, a morphism $f : (C, \zeta) \to (D, \xi)$ of $\mathbb{T}$-algebras is a morphism $f : C \to D$ in $\mathbf{C}$ satisfying $f \circ \zeta = \xi \circ \mathcal{T}f$.

Proposition 2.4 *Let $\mathbb{T} = (\mathcal{T}, \eta, \mu)$ be a monad on a category $\mathbf{C}$. The $\mathbb{T}$-algebras and their morphisms constitute a category, written $\mathbf{C}^{\mathbb{T}}$, called the* "Eilenberg-Moore" category for $\mathbb{T}$ over $\mathbf{C}$.

Example 2.5 The Eilenberg-Moore category for $\mathbb{P}^*$ is equivalent to $\mathbf{SLat}$: the category of $\vee$-semilattices and $\vee$-preserving maps.

The following definition for *categorical theories* is a generalization of the definition of algebraic theory presented in Borceux [3, p.130], where we allow for arbitrary powers of the generating object.

Definition 2.6 [Categorical Theories]
(a) A category $\mathbf{T}$, is a *categorical theory*, if its objects are given by possibly uncountable (cartesian) products, T^W, on some generating object T.
(b) A *countable Lawvere theory* $\mathbf{L}$, is a categorical theory whose objects consist of countable products of the generator.
(c) Let $\mathbf{T}$ be a categorical theory and $\mathbf{C}$ a category with the same (possibly uncountable) product structure as $\mathbf{T}$. *Models of* $\mathbf{T}$ *in* $\mathbf{C}$ and morphisms between them can be described under two equivalent forms:
 (i) As product preserving functors $\mathcal{F} : \mathbf{T} \to \mathbf{Set}$, with morphisms being natural transformations between such functors.
 (ii) As universal algebras: objects $C \in \mathbf{C}$ and a set of W-ary operations on C, $\left\{\hat{f} : C^W \to C^{W'} \mid f : T^W \to T^{W'} \in \mathbf{T}\right\}$ which satisfy the same equations as the corresponding morphisms in $\mathbf{T}$. Morphisms between such algebras are maps in C which preserve every operation in the set of operations on C.

We denote by $\mathbf{Mod}(\mathbf{T}, \mathbf{C})$, the category of models of $\mathbf{T}$ in $\mathbf{C}$.

Proposition 2.7 *The following categories coincide;*
(a) the model category of a countable Lawvere theory $\mathbf{L}$ *over* $\mathbf{Set}$, $\mathbf{Mod}(\mathbf{L}, \mathbf{Set})$*;*
(b) the category of $\mathbb{T}$*-algebras, where* $\mathbb{T}$ *is a monad of countable rank on* $\mathbf{Set}$.

2.1 Notation

(i) **Categorical Theories**
  (a) $\mathbf{T_{ND}}$ - nondeterministic choice theory.
  (b) $\mathbf{L_{ND}}$ - countable Lawvere theory of nondeterministic choice.
  (c) $\mathbf{L_{P}}$ - countable Lawvere theory of probabilistic choice.
  (d) $\mathbf{T_{MC}}$ - mixed choice theory.
  (e) $\mathbf{L_{MC}}$ - countable Lawvere theory of mixed choice.

(ii) **Monads**
  (a) $\mathbb{P}^* : \mathbf{Set} \to \mathbf{Set}$ - non-empty powerset monad.
  (b) $\mathbb{D} : \mathbf{Set} \to \mathbf{Set}$ - distributions monad.
  (c) $\mathbb{G}_{cvx} : \mathbf{Set} \to \mathbf{Set}$ - geometrically convex powerset monad.
  (d) $\mathbb{C}_{cvx} : \mathbf{Set} \to \mathbf{Set}$ - convex completion monad.

(iii) **Functors**
  (a) $\mathcal{P}^* : \mathbf{Set} \to \mathbf{Mod}(\mathbf{T_{ND}}, \mathbf{Set})$ - non-empty powerset functor.
  (b) $\mathcal{D} : \mathbf{Set} \to \mathbf{Mod}(\mathbf{L_P}, \mathbf{Set})$ - distributions functor.
  (c) $\mathcal{G}_{cvx} : \mathbf{Set} \to \mathbf{Mod}(\mathbf{T_{MC}}, \mathbf{Set})$ - geometrically convex powerset functor.
  (d) $\mathcal{C}_{cvx} : \mathbf{Mod}(\mathbf{T_{ND}}, \mathbf{Set}) \to \mathbf{Mod}(\mathbf{T_{MC}}, \mathbf{Set})$ - convex completion functor.

(iv) **Operators**
  (a) $\boxplus$ - binary nondeterministic choice.
  (b) $\oplus_\lambda$ - binary probabilistic choice weighted by λ.
  (c) $\underset{W}{\boxplus}$ - nondeterministic choice indexed by W.
  (d) $\bigoplus_{(\lambda_i)_I}$ - probabilistic choice weighted by the sequence $(\lambda_i)_I$.

3 Nondeterministic, Probabilistic & Mixed Choice

In this section we recall the definitions of each choice theory and their associated monads. This includes the definitions of all the functors present in Figure 1, except for the convex completion functor, $\mathcal{C}_{cvx} : \mathbf{Mod}(\mathbf{T_{ND}}, \mathbf{Set}) \to \mathbf{Mod}(\mathbf{T_{MC}}, \mathbf{Set})$, which we present in full details in the following section.

3.1 The Theory of Nondeterminism and the Non-empty Powerset Monad

Definition 3.1 [Nondeterministic Choice Theory] The *nondeterministic choice theory*, $\mathbf{T_{ND}}$, is the categorical theory given as follows:

(a) **Objects**: Arbitrary products of some fixed generating object N. So objects are denoted by N^W for some indexing set W.

(b) **Morphisms**: In addition to the required morphisms for the product structure, there exists *nondeterminism operators (ND-ops)* of possibly uncountable arity. Given a non-empty indexing set W, the ND-op of arity W is denoted as the morphism $\underset{W}{\boxplus} : N^W \to N$.

The nondeterminism operators must satisfy the following axioms:

$$\text{(N-Ax1)} \quad \boxplus_{w\in W} A_w = \boxplus_{u\in U} \left(\boxplus_{v\in V_u} A_v\right), \text{ with } \{V_u \mid u \in U\} \text{ a partition of W.}$$

$$\text{(N-Ax2)} \quad \boxplus_{w\in W} A = A.$$

Remark 3.2 [Axioms for Finite ND-ops] Consider the binary ND-op, denoted by $A_1 \boxplus A_2$, then every finitely indexed ND-op is generated by the binary ND-op. As a consequence of (N-Ax1) and (N-Ax2) we can derive the usual semilattice axioms:

$$\begin{array}{rc} \text{(N-Assoc)} & (A_1 \boxplus A_2) \boxplus A_3 = A_1 \boxplus (A_2 \boxplus A_3) \\ \text{(N-Com)} & A_1 \boxplus A_2 = A_2 \boxplus A_1 \\ \text{(N-Idem)} & A \boxplus A = A \end{array}$$

Definition 3.3 [Models of Nondeterminism] Let $\mathbf{C}$ be a category with arbitrary products. A *model of* $\mathbf{T_{ND}}$ *in* $\mathbf{C}$, denoted $(C, \{\boxplus\})$, consists of an object $C \in \mathbf{C}$ and for each non-empty indexing set W (W possibly uncountable) a nondeterministic operator $\boxplus_W : C^W \to C$ satisfying (N-Ax1) and (N-Ax2).

Definition 3.4 [Countable Lawvere Theory of Nondeterministic Choice] The countable Lawvere theory of nondeterministic choice, denoted $\mathbf{L_{ND}}$, is the full subcategory of $\mathbf{T_{ND}}$ generated by countable products of N.

Proposition 3.5 *The category of models of the nondeterministic theory* $\mathbf{T_{ND}}$ *over* $\mathbf{Set}$*,* $\mathbf{Mod}(\mathbf{T_{ND}}, \mathbf{Set})$*, is equivalent to the category of* $\mathbb{P}^*$*-algebras,* $\mathbb{P}^*$-$\mathbf{Alg}$*, where* $\mathbb{P}^*$ *is the non-empty powerset monad as defined in Example 2.2.*

3.2 The Theory of Probabilistic Choice and the Distributions Functor

Definition 3.6 [Probabilistic Weighting] A *probabilistic weighting*, $(\lambda_i)_I$, is a countable sequence of elements in the interval $(0, 1]$ such that $\sum_{i\in I} \lambda_i = 1$. We will denote the set of all probabilistic weightings by *Prob*. Whenever the indexing set is clear from the context we shall denote write (λ_i) in lieu of $(\lambda_i)_I$.

Definition 3.7 [Countable Lawvere Theory of Probabilistic Choice] The *countable Lawvere theory of probabilistic choice*, $\mathbf{L_P}$, is given as follows:

(a) **Objects**: Countable products of some fixed generating object P. So objects are denoted by P^I for some countable indexing set I.

(b) **Morphisms**: In addition to the required morphisms for the product structure, there are *probability operators (P-ops)* indexed by probabilistic weightings $(\lambda_i)_I$. The P-op weighted by $(\lambda_i)_I \in Prob$, is denoted by the morphism $\bigoplus_{(\lambda_i)_I} : P^I \to P$.

The probabilistic operators must satisfy the following axioms:

(P-Ax1) $\bigoplus_{(\lambda_i)_I} A_i = \bigoplus_{(\rho_j)_J} (\bigoplus_{(\nu_k)_{K_j}} A_k)$, with $\{K_j \mid j \in J\}$ a partition of I,
$\rho_j = \sum_{k \in K_j} \lambda_k$ and $\nu_k = \frac{\lambda_k}{\rho_j}$.

(P-Ax2) $\bigoplus_{(\lambda_i)_I} A = A$.

Remark 3.8 [Axioms for Finite P-ops] Any P-op indexed by a finite probabilistic weighting can be generated by binary P-ops. We shall denote the binary P-op indexed by $(\lambda, 1-\lambda) \in \mathit{Prob}$ by $A_1 \oplus_\lambda A_2$ and define $\oplus_1$, $\oplus_0 : P^2 \to P$ to be the left and right projections on P^2 respectively. The axioms we derive for finite P-ops correspond to the usual axioms for probabilistic choice in the literature.

(P-Assoc) $(A_1 \oplus_\lambda A_2) \oplus_\rho A_3 = A_1 \oplus_{\lambda\rho} (A_2 \oplus_{\frac{(1-\lambda)\rho}{1-\lambda\rho}} A_3)$, where $\lambda\rho \neq 1$

(P-Com) $A_1 \oplus_\lambda A_2 = A_2 \oplus_{1-\lambda} A_1$,

(P-Idem) $A \oplus_\lambda A = A$

Definition 3.9 [Models of Probabilistic Choice] Let $\mathbf{C}$ be a category with countable products. A *model of* $\mathbf{L_P}$ *in* $\mathbf{C}$, $(C, \{\bigoplus\})$, is given by an object $C \in \mathbf{C}$ and for each probabilistic weighting $(\lambda_i)_I \in \mathit{Prob}$ a probabilistic operator $\bigoplus_{(\lambda_i)_I} : C^I \to C$ satisfying (P-Ax1) and (P-Ax2).

Definition 3.10 [Distributions Functor] The *distributions functor*, $\mathcal{D} : \mathbf{Set} \to \mathbf{Mod}(\mathbf{L_P}, \mathbf{Set})$, is given by:

(a) **On objects**: For $X \in \mathbf{Set}$, $\mathcal{D}(X) = (\mathcal{D}(X), \{\bigoplus\})$, such that
  (i) $\mathcal{D}(X) = \{d : X \to [0,1] \mid (d(x))_{supp(d)} \in \mathit{Prob}\}$,
  where $supp(d) = \{x \in X \mid d(x) \neq 0\}$ is the *support* of d, and
  (ii) For $(\lambda_i)_I \in \mathit{Prob}$ and $(d_i)_I \in \mathcal{D}(X)^I$, $\bigoplus_{(\lambda_i)_I} d_i = \sum_{i \in I} \lambda_i d_i$.

(b) **On morphisms**: For $f : X \to X' \in \mathbf{Set}$, $d \in \mathcal{D}(X)$ and $x' \in X'$, then $(\mathcal{D}f(d))(x') = \sum_{\{x \in X \mid f(x) = x'\}} d(x)$.

Proposition 3.11 (Distributions Monad) *The distributions functor* $\mathcal{D} : \mathbf{Set} \to \mathbf{Mod}(\mathbf{L_P}, \mathbf{Set})$ *is left adjoint to the forgetful functor* $\mathcal{U_D} : \mathbf{Mod}(\mathbf{L_P}, \mathbf{Set}) \to \mathbf{Set}$. *Thus it forms a monad* $\mathbb{D} = (\mathcal{U_D} \circ \mathcal{D}, \eta, \mu)$ *over* $\mathbf{Set}$, *with adjunction structure:*

(a) **The unit**(η): *for* $X \in \mathbf{Set}$ *and* $x \in X$, $\eta_X(x) = \delta_x$, *where* $\delta_x : X \to [0,1]$ *is the* Dirac distribution *given by* $\delta_x(x') = \begin{cases} 1 \text{ if } x' = x \\ 0 \text{ otherwise} \end{cases}$

(b) **The counit**(ε): *for* $(P, \{\bigoplus\}) \in \mathbf{Mod}(\mathbf{L_P}, \mathbf{Set})$ *and* $d \in \mathcal{D}(P)$, $\varepsilon_{(P,\{\oplus\})}(d) = \bigoplus_{(d(p))_{supp(d)}} p$.

Proposition 3.12 *The category of models of the probabilistic choice theory* $\mathbf{L_P}$ *over* $\mathbf{Set}$, $\mathbf{Mod}(\mathbf{L_P}, \mathbf{Set})$, *is equivalent to the category of* $\mathbb{D}$*-algebras,* $\mathbb{D}$-$\mathbf{Alg}$.

3.3 The Theory of Mixed Choice and the Geometrically Convex Powerset Functor

Definition 3.13 [Mixed Choice Theory] The *mixed choice theory*, $\mathbf{T_{MC}}$, is given as follows:

(a) **Objects**: Arbitrary products of some fixed generating object M. So objects are denoted by M^W for some indexing set W.

(b) **Morphisms**: In addition to the required morphism for the product structure we include ND-ops, $\underset{W}{\boxplus} : M^W \to M$ and P-ops, $\bigoplus_{(\lambda_i)_I} : M^I \to M$.

The ND-ops must satisfy (N-Ax1), (N-Ax2) and the P-ops must satisfy (P-Ax1), (P-Ax2). Furthermore the following distributivity axioms must hold:

$$\text{(Dist)} \quad \bigoplus_{(\lambda_i)_I} (\underset{w \in W_i}{\boxplus} A_w) = \underset{\boldsymbol{w} \in \prod_I W_i}{\boxplus} (\bigoplus_{(\lambda_i)_I} A_{\pi_i(\boldsymbol{w})})$$

Remark 3.14 [Axioms for Finite ND-ops and P-ops] For the binary ND-op and P-ops we can derive the following axiom from (Dist):

$$\text{(FinDist)} \qquad (A_1 \boxplus A_2) \oplus_\lambda A_3 = (A_1 \oplus_\lambda A_3) \boxplus (A_2 \oplus_\lambda A_3)$$

Thus, (FinDist) together with the previously derived axioms (N-Assoc), (N-com), (N-Idem), (P-Assoc), (P-Com), (P-Idem), form the usual axiom set for mixed choice found in the literature.

Definition 3.15 [Models of Mixed Choice] Let $\mathbf{C}$ be a category with arbitrary products. A *model of* $\mathbf{T_{MC}}$ *in* $\mathbf{C}$, $(C, \{\boxplus\}, \{\bigoplus\})$, is given by an object $C \in \mathbf{C}$, for each probabilistic weighting $(\lambda_i)_I \in \mathit{Prob}$ a P-op $\bigoplus_{(\lambda_i)_I} : C^I \to C$ and for each non-empty indexing set W a ND-op $\underset{W}{\boxplus} : C^W \to C$ subject to the axioms (N-Ax1), (N-Ax2), (P-Ax1), (P-Ax2) and (Dist).

General Axioms		*Derived Axioms for Finite Operations*	
(N-Ax1)	$\boxplus_W A_w = \boxplus_U (\boxplus_{V_u} A_v)$,	(N-Assoc)	$(A_1 \boxplus A_2) \boxplus A_3 = A_1 \boxplus (A_2 \boxplus A_3)$
(N-Ax2)	$\boxplus_W A = A$,	(N-Com)	$A_1 \boxplus A_2 = A_2 \boxplus A_1$
(P-Ax1)	$\oplus_{(\lambda_i)_I} A_i = \oplus_{(\rho_j)_J} (\oplus_{(\nu_k)_{K_j}} A_k)$,	(N-Idem)	$A \boxplus A = A$
(P-Ax2)	$\oplus_{(\lambda_i)_I} A = A$,	(P-Assoc)	$(A_1 \oplus_\lambda A_2) \oplus_\rho A_3 =$ $A_1 \oplus_{\lambda\rho} (A_2 \oplus_{\frac{(1-\lambda)\rho}{1-\lambda\rho}} A_3)$,
(Dist)	$\oplus_{(\lambda_i)_I} (\boxplus_{W_i} A_w) =$ $\boxplus_{\prod_I W_i} (\oplus_{(\lambda_i)_I} A_{\pi_i(\boldsymbol{w})})$,	(P-Com)	$A_1 \oplus_\lambda A_2 = A_2 \oplus_{1-\lambda} A_1$,
		(P-Idem)	$A \oplus_\lambda A = A$
		(FinDist)	$(A_1 \boxplus A_2) \oplus_\lambda A_3 =$ $(A_1 \oplus_\lambda A_3) \boxplus (A_2 \oplus_\lambda A_3)$

Fig. 2. Axioms for $\mathbf{T_{MC}}$ at a Glance

Definition 3.16 [Countable Lawvere Mixed Choice Theory] The countable Lawvere mixed choice theory, $\mathbf{L_{MC}}$, is given by $\mathbf{L_P} \triangleright \mathbf{L_{ND}}$, the distributive tensor between $\mathbf{L_P}$ and $\mathbf{L_{ND}}$. Equivalently, it is the full subcategory of $\mathbf{T_{MC}}$ generated by countable products of M.

Definition 3.17 [Convex Subsets] Let X be a set equipped with P-ops, $\bigoplus_{(\lambda_i)} : X^I \to X$. A subset Y of X is *convex* if for every $(\lambda_i)_I \in \mathit{Prob}$ and $(y_i)_I \in Y^I$, $\bigoplus_{(\lambda_i)} y_i \in Y$.

Definition 3.18 [Geometrically Convex Powerset Functor] The *geometrically convex powerset functor*, $\mathcal{G}_{cvx} : \mathbf{Set} \to \mathbf{Mod}(\mathbf{T_{MC}}, \mathbf{Set})$, is given by:

(a) **On objects**: For $X \in \mathbf{Set}$, $\mathcal{G}_{cvx}(X) = (\mathcal{G}_{cvx}(X), \{\boxplus\}, \{\bigoplus\})$, such that
   (i) $\mathcal{G}_{cvx}(X) = \{Y \,|\, Y \text{ is a convex subset of } \mathcal{D}(X)\}$,
   (ii) For $(\lambda_i)_I \in \mathit{Prob}$ and $(Y_i)_I \in \mathcal{G}_{cvx}(X)^I$, $\bigoplus_{(\lambda_i)_I} Y_i = \left\{\sum_{i\in I} \lambda_i y_i \,|\, y_i \in Y_i\right\}$,
   (iii) For a non-empty family $(Y_w)_W \in \mathcal{G}_{cvx}(X)^W$, $\displaystyle\boxplus_{w\in W} Y_w = \left\{\sum_{i\in I} \rho_i y_i \,|\, I \subseteq W, y_i \in Y_i, (\rho_i)_I \in \mathit{Prob}\right\}$.

(b) **On morphisms**: For $f : X \to X' \in \mathbf{Set}$ and $Y \in \mathcal{G}_{cvx}(X)$, then $\mathcal{G}_{cvx} f(Y) = \{\mathcal{D}f(y) \,|\, y \in Y\}$.

Proposition 3.19 *The geometrically convex powerset functor* $\mathcal{G}_{cvx} : \mathbf{Set} \to \mathbf{Mod}(\mathbf{T_{MC}}, \mathbf{Set})$ *is left adjoint to the forgetful functor* $\mathcal{U}_{\mathcal{G}_{cvx}} : \mathbf{Mod}(\mathbf{T_{MC}}, \mathbf{Set}) \to \mathbf{Set}$*. Thus it forms a monad* $\mathbb{G}_{cvx} = (\mathcal{U}_{\mathcal{G}_{cvx}} \circ \mathcal{G}_{cvx}, \eta, \mu)$ *over* $\mathbf{Set}$*. We include the adjunction structure:*

(a) **The unit**(η)*: for* $X \in \mathbf{Set}$ *and* $x \in X$, $\eta_X(x) = \{\delta_x\}$.

(b) **The counit**(ε)*: for* $(M, \{\boxplus\}, \{\bigoplus\}) \in \mathbf{Mod}(\mathbf{T_{MC}}, \mathbf{Set})$ *and* $Y \in \mathcal{G}_{cvx}(M)$,
$$\varepsilon_{(P,\{\boxplus\},\{\oplus\})}(Y) = \boxplus_{y\in Y} \Big(\bigoplus_{(y(m))_{supp(y)}} m\Big).$$

Proposition 3.20 *The category of models of the mixed choice theory* $\mathbf{T_{MC}}$ *in* $\mathbf{Set}$, $\mathbf{Mod}(\mathbf{T_{MC}}, \mathbf{Set})$*, is equivalent to the category of* $\mathbb{G}_{cvx}$*-algebras,* $\mathbb{G}_{cvx}$-$\mathbf{Alg}$.

4 Mixed Choice Models from Nondeterministic Models

In this section we construct the *convex completion functor*, $\mathcal{C}_{cvx} : \mathbf{Mod}(\mathbf{T_{ND}}, \mathbf{Set}) \to \mathbf{Mod}(\mathbf{T_{MC}}, \mathbf{Set})$, the functor which freely constructs models of $\mathbf{T_{MC}}$ over $\mathbf{Set}$ from semilattices, i.e. models of $\mathbf{T_{ND}}$ over $\mathbf{Set}$. First we consider the following special case of a theorem of Barr and Wells [1, p. 133].

Theorem 4.1 *Consider the diagram below satisfying the following conditions,*

$$\begin{tikzcd}
\mathbf{Mod}(\mathbf{T_{MC}},\mathbf{Set}) \arrow[dd, bend right, "\mathcal{U}_{\mathcal{G}_{cvx}}"'] \arrow[dr, "\mathcal{U}_{\mathcal{C}_{cvx}}"] & \\
& \mathbf{Mod}(\mathbf{T_{ND}},\mathbf{Set}) \arrow[ul, dotted, bend right, "\mathcal{C}_{cvx}"'] \arrow[dl, bend left, "\mathcal{U}_{\mathcal{P}^*}"] \\
\mathbf{Set} \arrow[uu, bend right, "\mathcal{G}_{cvx}"'] \arrow[ur, bend right, "\mathcal{P}^*"'] &
\end{tikzcd}$$

(a) $\mathcal{G}_{cvx}$ *is left adjoint to* $\mathcal{U}_{\mathcal{G}_{cvx}}$,
(b) $\mathcal{P}^*$ *is left adjoint to* $\mathcal{U}_{\mathcal{P}^*}$,
(c) $\mathcal{U}_{\mathcal{P}^*} \circ \mathcal{U}_{\mathcal{C}_{cvx}}$ *is naturally isomorphic to* $\mathcal{U}_{\mathcal{G}_{cvx}}$,
(d) $\mathcal{U}_{\mathcal{P}^*}$ *is of descent type, and*
(e) $\mathbf{Mod}(\mathbf{T_{MC}}, \mathbf{Set})$ *has coequalizers.*

Then $\mathcal{U}_{\mathcal{C}_{cvx}}$ *has a left adjoint* $\mathcal{C}_{cvx}$ *for which* $\mathcal{C}_{cvx} \circ \mathcal{P}^* \cong \mathcal{G}_{cvx}$.

Although Theorem 4.1 states the existence of the left adjoint $\mathcal{C}_{cvx}$, it only gives a partially constructive proof. The proof clearly states that the image of a free object $(\mathcal{P}^{*}(X), \cup) \in \mathbf{Mod}(\mathbf{T_{ND}}, \mathbf{Set})$ under $\mathcal{C}_{cvx}$ is isomorphic to $\mathcal{G}_{cvx}(X) \in \mathbf{Mod}(\mathbf{T_{MC}}, \mathbf{Set})$. However, the image of a non-free object is dependant on computing coequalizers in $\mathbf{Mod}(\mathbf{T_{MC}}, \mathbf{Set})$. Hence, to define $\mathcal{C}_{cvx}$ one must determine how to calculate coequalizers in $\mathbf{Mod}(\mathbf{T_{MC}}, \mathbf{Set})$.

In this section we shall give a concrete definition for the left adjoint of $\mathcal{U}_{\mathcal{C}_{cvx}} : \mathbf{Mod}(\mathbf{T_{MC}}, \mathbf{Set}) \to \mathbf{Mod}(\mathbf{T_{ND}}, \mathbf{Set})$, which we call the *convex completion functor*, $\mathcal{C}_{cvx} : \mathbf{Mod}(\mathbf{T_{ND}}, \mathbf{Set}) \to \mathbf{Mod}(\mathbf{T_{MC}}, \mathbf{Set})$. In the first subsection, we introduce the concepts needed to define $\mathcal{C}_{cvx}$ and then give its full definition and structure in the next subsection.

4.1 Completion of Convex Sets w.r.t. a Semilattice

In this section we define a closure operation on elements of $\mathcal{G}_{cvx}(S)$, called *completion w.r.t.* $(S, \vee)$, where $(S, \vee)$ is an arbitrary semilattice. These completions are very important in defining the left adjoint of $\mathcal{U}_{\mathcal{C}_{cvx}} : \mathbf{Mod}(\mathbf{T_{MC}}, \mathbf{Set}) \to \mathbf{Mod}(\mathbf{T_{ND}}, \mathbf{Set})$ due to the following correspondence.

> Two convex sets $C_1, C_2 \in \mathcal{G}_{cvx}(S)$ are related by the congruence relation generated by the coequalizers from the proof of Theorem 4.1 if and only if their completions w.r.t $(S, \vee)$ are equal.

The congruence relation generated in the proof of Theorem 4.1 is given by the smallest congruence relation over $\mathcal{G}_{cvx}(S)$ containing the set $\{(\mathcal{D}(T), \mathcal{D}(\{\vee T\})) \,|\, T \in \mathcal{P}^{*}(S)\}$. One can see that the elements of $\mathcal{G}_{cvx}(S)$ of the form $\mathcal{D}(T)$, with $T \in \mathcal{P}^{*}(S)$, play an important role in determining if two elements are congruent. They will also be an important aspect in defining the completion operation.

Definition 4.2 [Faces] Given a semilattice $(S, \vee)$,

(a) An element $Y \in \mathcal{G}_{cvx}(S)$ is a *face of* $\mathcal{G}_{cvx}(S)$, if $Y = \mathcal{D}(T)$, where $T \in \mathcal{P}^{*}(S)$. Note that if $(S, \vee)$ is a finite semilattice, then the faces correspond exactly to the faces of the polytope $\mathcal{D}(S)$.

(b) An element $F \in \mathcal{G}_{cvx}(S)$ is a *facial polytope of* $\mathcal{G}_{cvx}(S)$, if F is an element of *Poly*, the convex subset of $\mathcal{G}_{cvx}(S)$ generated by the faces of $\mathcal{G}_{cvx}(S)$, i.e $F \in \mathit{Poly} = \left\{ \bigoplus_{(\lambda_i)_I} \mathcal{D}(T_i) \,|\, (\lambda_i)_I \in \mathit{Prob}, T_i \subseteq S \right\}$.

Example 4.3 Consider the semilattices S_1 and S_2 from Figure 3. Since the underlying sets of S_1 and S_2 are equal, each semilattice will generate the same set of faces and facial polytopes. We give some examples and non-examples of elements of *Poly* for S_1 and S_2 in Figure 4.

For a semilattice $(S, \vee)$, consider a face $\mathcal{D}(T)$, $T \in \mathcal{P}^{*}(S)$. Recall that the congruence relation discussed in the proof of Theorem 4.1 was generated by the set $\{(\mathcal{D}(T), \mathcal{D}(\{\vee T\})) \,|\, T \in \mathcal{P}^{*}(S)\}$. Therefore, $\mathcal{D}(T)$ is congruent to $\mathcal{D}(\{\vee T\})$. Moreover there is a largest face which is congruent to $\mathcal{D}(\{\vee T\})$, namely $\mathcal{D}(\downarrow\{\vee T\})$, where

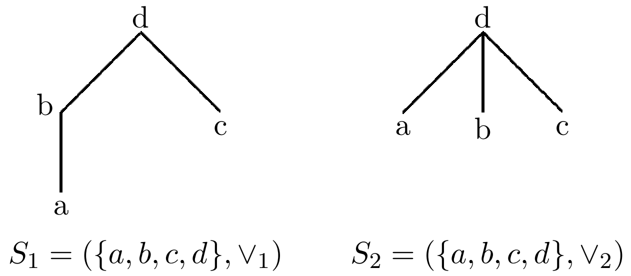


Fig. 3. Semilattices S_1 and S_2

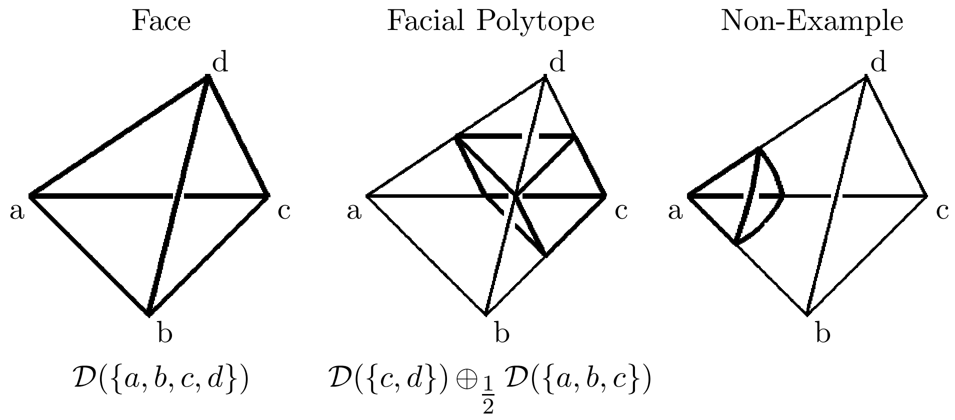


Fig. 4. Examples of Faces and Facial Polytopes in S_1 and S_2

$\downarrow\{\vee T\}$ is the face generated by the *downset* of $\{\vee T\}$, $\downarrow\{\vee T\} = \{s \in S \,|\, s \leq \vee T\}$, where for $s, t \in S$, $s \leq t$ if and only if $s \vee t = t$. Next, we define a *facial polytope expansion* construction on facial polytopes based on the above observations.

Definition 4.4 [Facial Polytope Expansion, $(F^{\downarrow})$] Given a semilattice $(S, \vee)$, let $F = \bigoplus_{(\lambda_i)_I} \mathcal{D}(T_i) \in \mathit{Poly}$, where $(\lambda_i)_I \in \mathit{Prob}$ and $T_i \in \mathcal{P}^*(S)$. The *facial polytope expansion of* F, denoted $F^{\downarrow}$, is given by $F^{\downarrow} = \bigoplus_{(\lambda_i)_I} \mathcal{D}(\downarrow\{\vee T_i\})$.

Example 4.5 Consider the semilattices S_1 and S_2 from Figure 3. We list below the faces of $\mathcal{G}_{cvx}(S_1)$ and $\mathcal{G}_{cvx}(S_2)$ on which their facial polytope expansion disagree.

Face	Expansion in S_1	Expansion in S_2
$\mathcal{D}(\{b\})$	$\mathcal{D}(\{a,b\})$	$\mathcal{D}(\{b\})$
$\mathcal{D}(\{a,b\})$	$\mathcal{D}(\{a,b\})$	$\mathcal{D}(\{a,b,c,d\})$

Given a convex set $C \in \mathcal{G}_{cvx}(S)$, if it contains a face, say $\mathcal{D}(T)$, then $C = C \boxplus \mathcal{D}(T)$. By our previous observations, $\mathcal{D}(T)$ is congruent to $\mathcal{D}(\downarrow\{\vee T\})$, hence C is congruent to $C \boxplus \mathcal{D}(\downarrow\{\vee T\})$. A similar reasoning using facial polytopes implies that if $F \in \mathit{Poly}$ and $F \subseteq C$ then C is congruent to $C \boxplus F^{\downarrow}$. Since $F \subseteq F^{\downarrow}$, we have that $C \boxplus F^{\downarrow}$ is a possibly larger representative than C in their common congruence class. Since we are searching for the largest representative in a given congruence class, we focus on convex sets which are invariant under all facial expansions, i.e. for any facial polytope F such that $F \subseteq C$ then $C = C \boxplus F^{\downarrow}$. We shall call such a convex set *complete*. Thus, a convex set $C \in \mathcal{G}_{cvx}(S)$ is *complete* if for any convex subset contained in C generated by a set of facial polytopes $D \subseteq Poly$, then the convex subset generated by their facial expansions $\{F^{\downarrow} \mid F \in D\}$, must also be contained in C.

Definition 4.6 [Complete] Given a semilattice $(S, \vee)$ and $C \in \mathcal{G}_{cvx}(S)$. We say C is *complete under* $(S, \vee)$ if for every $F \in Poly$ such that $F \subseteq C$ then $F^{\downarrow} \subseteq C$. We shall denote by Cmp, the set of all complete convex sets in $\mathcal{G}_{cvx}(S)$.

Example 4.7 Consider our semilattices S_1 and S_2 from Example 4.3. We present two convex sets in $\mathcal{G}_{cvx}(S_1) = \mathcal{G}_{cvx}(S_2)$ which are complete in one semilattice but not the other.

(a) The face $\mathcal{D}(\{b\})$ is complete in $\mathcal{G}_{cvx}(S_2)$. However, it is not complete in $\mathcal{G}_{cvx}(S_1)$, since $\mathcal{D}(\{b\})^{\downarrow} = \mathcal{D}(\{a, b\}) \not\subseteq \mathcal{D}(\{b\})$.

(b) The face $\mathcal{D}(\{a, b\})$ is complete in $\mathcal{G}_{cvx}(S_1)$. However, it is not complete in $\mathcal{G}_{cvx}(S_2)$, since $\mathcal{D}(\{a, b\})^{\downarrow} = \mathcal{D}(\{a, b, c, d\}) \not\subseteq \mathcal{D}(\{a, b\})$.

Proposition 4.8 *Let W be a possibly uncountable, non-empty indexing set. Consider the family $(C_w)_W$, where $C_w \in \mathcal{G}_{cvx}(S)$. If every C_w is complete, then $\bigcap_W C_w$ is also complete.*

The above proposition states that for any $C \in \mathcal{G}_{cvx}(S)$ there exists a smallest complete convex set, $C^{\Downarrow}$ in $\mathcal{G}_{cvx}(S)$ which contains C. It is given by the intersection of all complete convex sets $D \in Cmp$ such that $C \subseteq D$, i.e. $C^{\Downarrow} = \bigcap_{\{D \in Cmp \mid C \subseteq D\}} D$.

Definition 4.9 [Completion, $(C^{\Downarrow})$] Given a semilattice $(S, \vee)$ and $C \in \mathcal{G}_{cvx}(S)$. The *completion of C under* $(S, \vee)$, denoted by $C^{\Downarrow}$, is the smallest complete convex set in $\mathcal{G}_{cvx}(S)$ containing C.

Example 4.10 Consider the semilattices from Example 4.3.

(a) The completion of $\mathcal{D}(\{b\})$ in S_1 is given by $(\mathcal{D}(\{b\}))^{\Downarrow} = \mathcal{D}(\{a, b\})$.

(b) The completion of $\mathcal{D}(\{a, b\})$ in S_2 is given by $(\mathcal{D}(\{a, b\}))^{\Downarrow} = \mathcal{D}(\{a, b, c, d\})$.

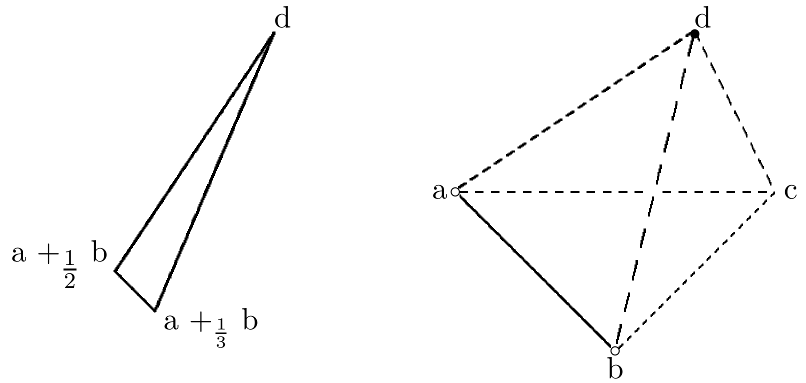
(c) It is not always the case that the completion of a finitely generated convex set results in a finitely generated convex set. For example consider the convex hull of $\{a +_{\frac{1}{3}} c, a +_{\frac{1}{2}} c, b\}$ in $\mathcal{G}_{cvx}(S_2)$, denote it by $hull(\{a +_{\frac{1}{3}} c, a +_{\frac{1}{2}} c, b\})$. The completion of $hull(\{a +_{\frac{1}{3}} c, a +_{\frac{1}{2}} c, b\})$ is the convex set obtained by removing $\mathcal{D}(\{a, d\})$, $\mathcal{D}(\{b, c\})$, $\mathcal{D}(\{b, d\})$ and $\mathcal{D}(\{c, d\})$ from $\mathcal{D}(S_2)$ and then adding $\{d\}$, as can be seen in Figure 5 .

Proposition 4.11 *Given a semilattice $(S, \vee)$, completion w.r.t $(S, \vee)$ is a closure operator on $\mathcal{G}_{cvx}(S)$. In other words, completion w.r.t. $(S, \vee)$, $(\cdot)^{\Downarrow} : \mathcal{G}_{cvx}(S) \to \mathcal{G}_{cvx}(S)$, satisfies the following properties: let $C, D \in \mathcal{G}_{cvx}(S)$*

(a) $C \subseteq C^{\Downarrow}$; $((\cdot)^{\Downarrow}$ is extensive),

(b) $(C^{\Downarrow})^{\Downarrow} = C^{\Downarrow}$; $((\cdot)^{\Downarrow}$ is idempotent),

(c) if $C \subseteq D$, *then* $C^{\Downarrow} \subseteq D^{\Downarrow}$; $((\cdot)^{\Downarrow}$ is monotone).



$hull(\{a +_{\frac{1}{3}} c, a +_{\frac{1}{2}} c, b\})$ $\qquad$ $(hull(\{a +_{\frac{1}{3}} c, a +_{\frac{1}{2}} c, b\}))^{\Downarrow}$

Fig. 5. Completion of $hull(\{a +_{\frac{1}{3}} c, a +_{\frac{1}{2}} c, b\})$

Given a convex set $C \in \mathcal{G}_{cvx}(S)$, for any non-empty family $(F_w)_W \in Poly^W$ such that $F_w \subseteq C$, then $C \subseteq \mathop{\boxplus}\limits_{w\in W} (C \boxplus F_w{}^{\downarrow}) \subseteq C^{\Downarrow}$. Moreover, we can make a similar argument with $\mathop{\boxplus}\limits_{w\in W} (C \boxplus F_w{}^{\downarrow})$ and a non-empty family $(F_u)_U \in Poly^U$ such that $F_u \subseteq \mathop{\boxplus}\limits_{w\in W} (C \boxplus F_w{}^{\downarrow})$. We shall call an element obtained by the above construction a *partial completion on* C *under* $(S, \vee)$. Below, we give a recursive definition for the set of all partial completions on C under $(S, \vee)$ and use it to compute the completion of C under $(S, \vee)$.

Definition 4.12 [Partial Completions] Given a semilattice $(S, \vee)$ and $C \in \mathcal{G}_{cvx}(S)$. The set of partial completions on C under $(S, \vee)$, $Parcmp(C)$, is defined by:

(i) $C \in Parcmp(C)$,

(ii) for $Y \in Parcmp(C)$ and $F \in Poly$ such that $F \subseteq Y$, then $Y \boxplus F^{\downarrow} \in Parcmp(C)$,

(iii) for a non-empty family $(Y_w)_W \in Parcmp(C)^W$, then $\mathop{\boxplus}\limits_{w\in W} Y_w \in Parcmp(C)$.

Theorem 4.13 *Let* $C \in \mathcal{G}_{cvx}(S)$*, then* $C^{\Downarrow} = \mathop{\boxplus}\limits_{Y \in Parcmp(C)} Y$.

4.2 Probabilistic Completion of Nondeterministic Models

Definition 4.14 [Convex Completion Functor] The *convex completion functor*, $\mathcal{C}_{cvx} : \mathbf{Mod}(\mathbf{T_{ND}}, \mathbf{Set}) \to \mathbf{Mod}(\mathbf{T_{MC}}, \mathbf{Set})$ is defined as follows:

(a) **On objects**: For $(S, \vee) \in \mathbf{Mod}(\mathbf{T_{ND}}, \mathbf{Set})$, $\mathcal{C}_{cvx}((S, \vee)) = (\mathcal{C}_{cvx}(S), \{\boxplus\}, \{\bigoplus\})$, such that

 (i) $\mathcal{C}_{cvx}(S) = \{C^{\Downarrow} \mid C \in \mathcal{G}_{cvx}(S)\}$,

 (ii) for $(\lambda_i)_I \in Prob$ and $(C_i)_I \in \mathcal{G}_{cvx}(S)^I$, $\bigoplus_{(\lambda_i)_I} C_i{}^{\Downarrow} = (\bigoplus_{(\lambda_i)_I} C_i{}^{\Downarrow})^{\Downarrow}$,

 (iii) for a non-empty family $(C_w)_W \in \mathcal{G}_{cvx}(S)^W$, $\mathop{\boxplus}\limits_{w\in W} C_w{}^{\Downarrow} = (\mathop{\boxplus}\limits_{w\in W} C_w{}^{\Downarrow})^{\Downarrow}$.

(b) **On morphisms**: For $f : (S, \vee) \to (S', \vee') \in \mathbf{Mod}(\mathbf{T_{ND}}, \mathbf{Set})$ and $C \in \mathcal{G}_{cvx}(S)$, $\mathcal{C}_{cvx} f(C^{\Downarrow}) = (\mathcal{G}_{cvx} f(C^{\Downarrow}))^{\Downarrow}$.

Theorem 4.15 *The convex completion functor* $\mathcal{C}_{cvx} : \mathbf{Mod}(\mathbf{T_{ND}}, \mathbf{Set}) \to \mathbf{Mod}(\mathbf{T_{MC}}, \mathbf{Set})$ *is left adjoint to the forgetful functor* $\mathcal{U}_{\mathcal{C}_{cvx}} : \mathbf{Mod}(\mathbf{T_{MC}}, \mathbf{Set}) \to \mathbf{Mod}(\mathbf{T_{ND}}, \mathbf{Set})$. *Thus it forms a monad* $\mathbb{C}_{cvx} = (\mathcal{U}_{\mathcal{C}_{cvx}} \circ \mathcal{C}_{cvx}, \eta, \mu)$ *over* $\mathbf{Mod}(\mathbf{T_{ND}}, \mathbf{Set})$. *We include the adjunction structure:*

(a) **The unit**(η)*: for* $(S, \vee) \in \mathbf{Mod}(\mathbf{T_{ND}}, \mathbf{Set})$ *and* $s \in S$, $\eta_{(S,\vee)}(s) = \{\delta_s\}^{\Downarrow}$.

(b) **The counit**(ε)*: for* $(M, \{\boxplus\}, \{\bigoplus\}) \in \mathbf{Mod}(\mathbf{T_{MC}}, \mathbf{Set})$ *and* $C \in \mathcal{G}_{cvx}(S)$,
$$\varepsilon_{(M,\{\boxplus\},\{\oplus\})}(C^{\Downarrow}) = \boxplus_{c \in C} \big(\bigoplus\nolimits_{(c(m))_{supp(c)}} m\big).$$

Theorem 4.16 *We can factorize the geometrically convex powerset functor,* $\mathcal{G}_{cvx}$*, through the non-empty powerset functor* $\mathcal{P}^*$ *by using the convex completion functor* $\mathcal{C}_{cvx}$*, i.e.* $\mathcal{G}_{cvx} \cong \mathcal{C}_{cvx} \circ \mathcal{P}^*$.

Proof. [Sketch] We define two inverse natural transformations:

(a) $\phi : (\mathcal{C}_{cvx} \circ \mathcal{P}^*) \Rightarrow \mathcal{G}_{cvx}$, where for an $X \in \mathbf{Set}$, the mixed choice morphism $\phi_X : (\mathcal{C}_{cvx} \circ \mathcal{P}^*)(X) \to \mathcal{G}_{cvx}(X)$ on $C^{\Downarrow} \in (\mathcal{C}_{cvx} \circ \mathcal{P}^*)(X)$, is given by
$$\phi_X(C^{\Downarrow}) = \boxplus_{c \in C^{\Downarrow}} \big(\bigoplus\nolimits_{(c(A))_{supp(c)}} \big(\boxplus_{a \in A} \mathcal{D}(\{a\})\big)\big).$$

(b) $\psi : \mathcal{G}_{cvx} \Rightarrow (\mathcal{C}_{cvx} \circ \mathcal{P}^*)$, where for an $X \in \mathbf{Set}$, the mixed choice morphism $\psi_X : \mathcal{G}_{cvx}(X) \to (\mathcal{C}_{cvx} \circ \mathcal{P}^*)(X)$ on $C \in \mathcal{G}_{cvx}(X)$ is given by
$$\psi_X(C) = \Big(\boxplus_{c \in C} \big(\bigoplus\nolimits_{(c(a))_{supp(c)}} \mathcal{D}(\{\{a\}\})\big)\Big)^{\Downarrow}.$$

To show that the above natural transformations are inverse we need the following technical lemmas.

Lemma 4.17 *For any* $X \in \mathbf{Set}$ *and* $C \in \mathcal{G}_{cvx}(X)$ *the convex set* $\boxplus_{c \in C} \big(\bigoplus_{(c(a))_{supp(c)}} \mathcal{D}(\{\{a\}\})\big)$ *is complete.*

Lemma 4.18 *For any* $X \in \mathbf{Set}$ *and* $C^{\Downarrow} \in (\mathcal{C}_{cvx} \circ \mathcal{P}^*)(X)$, $C^{\Downarrow} = \Big(\boxplus_{c \in C^{\Downarrow}} \big(\bigoplus_{(c(A))_{supp(c)}} \mathcal{D}(\{\{a\} \mid a \in A\})\big)\Big)^{\Downarrow}$.

Let $X \in \mathbf{Set}$ and $C \in \mathcal{G}_{cvx}(X)$, we compute $\phi_X \circ \psi_X(C)$.
$$\begin{aligned}
\phi_X \circ \psi_X(C) &= \phi_X\Big(\Big(\boxplus_{c \in C} \big(\bigoplus\nolimits_{(c(a))_{supp(c)}} \mathcal{D}(\{\{a\}\})\big)\Big)^{\Downarrow}\Big) \\
&= \phi_X\Big(\boxplus_{c \in C} \big(\bigoplus\nolimits_{(c(a))_{supp(c)}} \mathcal{D}(\{\{a\}\})\big)\Big) \qquad \text{by Lemma 4.17} \\
&= \boxplus_{c \in C} \big(\bigoplus\nolimits_{(c(a))_{supp(c)}} \mathcal{D}(\{a\})\big) \\
&= C
\end{aligned}$$

Let $X \in \mathbf{Set}$ and $C^{\Downarrow} \in (\mathcal{C}_{cvx} \circ \mathcal{P}^*)(X)$, we compute $\psi_X \circ \phi_X(C^{\Downarrow})$.

$$\begin{aligned}\psi_X \circ \phi_X(C^{\Downarrow}) &= \psi_X(\boxplus_{c\in C^{\Downarrow}} (\bigoplus\nolimits_{(c(A))_{supp(c)}} (\boxplus_{a\in A} \mathcal{D}(\{a\})))) \\ &= (\boxplus_{c\in C^{\Downarrow}} (\bigoplus\nolimits_{(c(A))_{supp(c)}} (\boxplus_{a\in A} \mathcal{D}(\{\{a\}\}))))^{\Downarrow} \\ &= (\boxplus_{c\in C^{\Downarrow}} (\bigoplus\nolimits_{(c(A))_{supp(c)}} \mathcal{D}(\{\{a\} \mid a \in A\})))^{\Downarrow} \\ &= C^{\Downarrow} \qquad \text{by Lemma 4.18}\end{aligned}$$

□

5 Lifting to Posets

In this section we lift the above construction from the category **Set** to the category **Poset**. That is, we wish to consider our initial diagram in Figure 1 with **Set** replaced by **Poset**. In the case of domains, Keimel, Plotkin and Tix [10], develop three convex powerdomains (lower, upper and biconvex) in order to construct mixed choice models from probabilistic models. Thus by taking their composition with the probabilistic powerdomain [7,8], we obtain three possible powerdomains for constructing mixed choice models. Once again this approach uses the classical algorithm for constructing mixed choice models and is represented by composing functors along the left-hand path in our posetal version of Figure 1.

Below we construct posetal mixed choice models, based on our alternate algorithm proposed in the Introduction. Hence, we shall define the functors on the right-hand path of the posetal version of Figure 1. We begin by recalling the known powerdomain structures over **Poset**, the probabilistic powerdomain and the three powerdomain constructions which model nondeterminism: the Plotkin powerdomain, the Hoare (Lower) Powerdomain and the Smyth (Upper) Powerdomain. Finally, we present for each type of powerdomain for nondeterminism the corresponding posetal completion functor in order to build posetal models for mixed choice.

5.1 Probabilistic and Nondeterministic Powerdomains

We begin by presenting the posetal probabilistic powerdomain.

Definition 5.1 [Probabilistic Powerdomain over **Poset**] The *probabilistic powerdomain* over **Poset**, $\mathcal{D} : \mathbf{Poset} \to \mathbf{Mod}(\mathbf{L_P}, \mathbf{Poset})$ is defined as follows:

(a) **On objects**: $(X, \sqsubseteq)$, $(X, \sqsubseteq) \mapsto ((\mathcal{D}(X), \preceq_{\sqsubseteq}), \{\bigoplus\})$, where $\preceq_{\sqsubseteq}$ is *the distributions order generated by* $\sqsubseteq$. For $d, d' \in \mathcal{D}(X)$, $d \preceq_{\sqsubseteq} d'$ if and only if for all $Y \subseteq X$, $d(\uparrow_{\sqsubseteq} Y) \leq d'(\uparrow_{\sqsubseteq} Y)$, where $d(Y) = \sum_{y\in Y} d(y)$.

(b) **On morphisms**: For $f : (X, \sqsubseteq) \to (X', \sqsubseteq') \in \mathbf{Poset}$, $d \in \mathcal{D}(X)$ and $x' \in X'$, $\mathcal{D}f(x') = \sum_{\{x\in X \mid f(x)=x'\}} d(x)$.

Next we present the posetal definition of the three powerdomains for nondeterminism. As previously stated, each of the following powerdomains gives rise to models of nondeterminism with associative, commutative and idempotent nondeterministic operators. However, in the cases of the Hoare and Smyth powerdomains, two extra axioms concerning the ordering are assumed, $A \sqsubseteq A \boxplus B$ and $A \boxplus B \sqsubseteq A$, respectively.

Definition 5.2 [Plotkin Powerdomain over **Poset**] The *Plotkin Powerdomain* over **Poset**, $\mathcal{P}^* : \mathbf{Poset} \to \mathbf{Mod}(\mathbf{T_{ND}}, \mathbf{Poset})$ is defined as follows:

(a) **On objects**: $(X, \sqsubseteq)$, $(X, \sqsubseteq) \mapsto ((\mathcal{P}^*(X), \downarrow\uparrow_{\sqsubseteq}), \cup)$, where $\downarrow\uparrow_{\sqsubseteq}$ is the *Egli-Milner order generated by* $\sqsubseteq$. For $A, B \in \mathcal{P}^*(X)$, $A \downarrow\uparrow_{\sqsubseteq} B$ if and only if $\downarrow_{\sqsubseteq} A \subseteq \downarrow_{\sqsubseteq} B$ and $\uparrow_{\sqsubseteq} B \subseteq \uparrow_{\sqsubseteq} A$.

(b) **On morphisms**: For $f : (X, \sqsubseteq) \to (X', \sqsubseteq') \in \mathbf{Poset}$ and $Y \in \mathcal{P}^*(X)$, $\mathcal{P}^* f(Y) = \{f(y) \mid y \in Y\}$.

Definition 5.3 [Hoare (Lower) Powerdomain over **Poset**] The *Hoare Powerdomain* over **Poset**, ${\mathcal{P}^*}_H : \mathbf{Poset} \to \mathbf{Mod}(\mathbf{T_{ND}}, \mathbf{Poset})$ is defined as follows:

(a) **On objects**: $(X, \sqsubseteq)$, $(X, \sqsubseteq) \mapsto ((\mathcal{P}^*(X), \sqsubseteq_H), \cup)$, where $\sqsubseteq_H$ is the *Hoare ordering generated by* $\sqsubseteq$. For $A, B \in \mathcal{P}^*(X)$, $A \sqsubseteq_H B$ if and only if $A \subseteq \downarrow_{\sqsubseteq} B$.

(b) **On morphisms**: For $f : (X, \sqsubseteq) \to (X', \sqsubseteq') \in \mathbf{Poset}$ and $Y \in \mathcal{P}^*(X)$, ${\mathcal{P}^*}_H f(Y) = \{f(y) \mid y \in Y\}$.

Definition 5.4 [Smyth (Upper) Powerdomain over **Poset**] The *Smyth Powerdomain* over **Poset**, ${\mathcal{P}^*}_S : \mathbf{Poset} \to \mathbf{Mod}(\mathbf{T_{ND}}, \mathbf{Poset})$ is defined as follows:

(a) **On objects**: $(X, \sqsubseteq)$, $(X, \sqsubseteq) \mapsto ((\mathcal{P}^*(X), \sqsubseteq_S), \cup)$, where $\sqsubseteq_S$ is the *Smyth ordering generated by* $\sqsubseteq$. For $A, B \in \mathcal{P}^*(X)$, $A \sqsubseteq_S B$ if and only if $B \subseteq \uparrow_{\sqsubseteq} A$.

(b) **On morphisms**: For $f : (X, \sqsubseteq) \to (X', \sqsubseteq') \in \mathbf{Poset}$ and $Y \in \mathcal{P}^*(X)$, ${\mathcal{P}^*}_S f(Y) = \{f(y) \mid y \in Y\}$.

5.2 Upper Convex, Lower Convex and Biconvex Completion

For each of the possible powerdomains capturing nondeterminism, we associate a posetal completion functor.

Definition 5.5 [Biconvex Completion Functor over $\mathbf{Mod}(\mathbf{T_{ND}}, \mathbf{Poset})$] The *biconvex completion functor* over posetal models of nondeterminism, $\mathcal{C}_{cvx} : \mathbf{Mod}(\mathbf{T_{ND}}, \mathbf{Poset}) \to \mathbf{Mod}(\mathbf{T_{MC}}, \mathbf{Poset})$ is defined as follows:

(a) **On objects**: $((S, \sqsubseteq), \vee)$, $((S, \sqsubseteq), \vee) \mapsto ((\mathcal{C}_{cvx}(S), \downarrow\uparrow_{\preceq}), \{\boxplus\}, \{\bigoplus\})$. Where $\downarrow\uparrow_{\preceq}$ is the Egli-Milner ordering generated by $\preceq_{\sqsubseteq}$, the distributions ordering generated by $\sqsubseteq$.

(b) **On morphisms**: $f : ((S, \sqsubseteq), \vee) \to ((S', \sqsubseteq'), \vee') \in \mathbf{Mod}(\mathbf{T_{ND}}, \mathbf{Poset})$ and $C \in \mathcal{G}_{cvx}(S)$, $\mathcal{C}_{cvx} f(C^{\Downarrow}) = \mathcal{G}_{cvx} f(C)^{\Downarrow}$.

Definition 5.6 [Lower Convex Completion Functor over $\mathbf{Mod}(\mathbf{T_{ND}}, \mathbf{Poset})$] The *lower convex completion functor* over posetal models of nondeterminism, $\mathcal{C}^H_{cvx} : \mathbf{Mod}(\mathbf{T_{ND}}, \mathbf{Poset}) \to \mathbf{Mod}(\mathbf{T_{MC}}, \mathbf{Poset})$ is defined as follows:

(a) **On objects**: $((S, \sqsubseteq), \vee)$, $((S, \sqsubseteq), \vee) \mapsto ((\mathcal{C}_{cvx}(S), \preceq_L), \{\boxplus\}, \{\bigoplus\})$. Where $\preceq_L$ is the Hoare ordering generated by $\preceq_{\sqsubseteq}$, the distributions ordering generated by $\sqsubseteq$.

(b) **On morphisms**: $f : ((S, \sqsubseteq), \vee) \to ((S', \sqsubseteq'), \vee') \in \mathbf{Mod}(\mathbf{T_{ND}}, \mathbf{Poset})$ and $C \in \mathcal{G}_{cvx}(S)$, $\mathcal{C}^H_{cvx} f(C^{\Downarrow}) = \mathcal{G}_{cvx} f(C)^{\Downarrow}$.

Definition 5.7 [Upper Convex Completion Functor over $\mathbf{Mod}(\mathbf{T_{ND}}, \mathbf{Poset})$] The *upper convex completion functor* over posetal models of nondeterminism, $\mathcal{C}^S_{cvx} : \mathbf{Mod}(\mathbf{T_{ND}}, \mathbf{Poset}) \to \mathbf{Mod}(\mathbf{T_{MC}}, \mathbf{Poset})$ is defined as follows:

(a) **On objects**: $((S, \sqsubseteq), \vee)$, $((S, \sqsubseteq), \vee) \mapsto ((\mathcal{C}_{cvx}(S), \preceq_U), \{\boxplus\}, \{\bigoplus\})$. Where $\preceq_U$ is the Smyth ordering generated by $\preceq_{\sqsubseteq}$, the distributions ordering generated by $\sqsubseteq$.

(b) **On morphisms**: $f : ((S, \sqsubseteq), \vee) \to ((S', \sqsubseteq'), \vee') \in \mathbf{Mod}(\mathbf{T_{ND}}, \mathbf{Poset})$ and $C \in \mathcal{G}_{cvx}(S)$, $\mathcal{C}^S_{cvx} f(C^{\Downarrow}) = \mathcal{G}_{cvx} f(C)^{\Downarrow}$.

Theorem 5.8 *The compositions of the above powerdomain functors with their associated convex completion functor, for the biconvex case $\mathcal{C}_{cvx} \circ \mathcal{P}^*$, the lower convex case $\mathcal{C}^L_{cvx} \circ \mathcal{P}^*{}_H$ and the upper convex case $\mathcal{C}^U_{cvx} \circ \mathcal{P}^*{}_S$ are equivalent to their counterparts obtained by following the left-hand path in the posetal Figure 1.*

6 Further Directions

As we have seen, the classical way of constructing mixed choice models on a category $\mathbf{C}$ was done by calculating $\mathbf{L_P}$ models in $\mathbf{C}$ then extending them by adding appropriately distributive nondeterministic operators. This allowed us not only to construct the free mixed choice models from objects in $\mathbf{C}$ but also to construct mixed choice models from $\mathbf{L_P}$ models over $\mathbf{C}$. By focusing on the dual approach, we are able to calculate $\mathbf{T_{MC}}$ models from $\mathbf{T_{ND}}$ models in $\mathbf{C}$. Thus given any non-standard model of nondeterministic choice in $\mathbf{C}$, we can extend it to mixed choice.

It would be interesting to apply the convex completions functor to existing non-standard models of nondeterminism, perhaps associated to process calculi which admit a non-standard definition of nondeterminism. This would give mixed choice extensions of such models which could not previously be determined by the available machinery.

One important aspect of combining mixed choice was properly capturing the interaction between the nondeterministic operators and probabilistic operators, given by considering the distributive tensor between their respective Lawvere theories. We would be interested in generalizing our work to encompass, not only the combining of nondeterminism and probability, but for any distributive combination of theories.

7 Acknowledgements

I would like to acknowledge, first and foremost, my supervisor Philip J. Scott. His advice, understanding and patience throughout the development of my PhD thesis, on which this paper is based, have been a real asset to me. Also, I would like to thank Michael Mislove and James Worrell for the many interesting discussions we had on this topic during my visit to Tulane. Finally, I'd like to thank all the anonymous referees for their insightful comments and useful suggestions.

References

[1] M. Barr and C. Wells. Toposes, Triples and Theories. Springer-Verlag, 1985.

[2] G. Beaulieu. PhD thesis in progress, University of Ottawa.

[3] F. Borceux. Handbook of Categorical Algebra. Cambridge University Press, 1994.

[4] J.I. den Hartog. Probabilistic Extensions of Semantical Models. PhD thesis, Vrije Universiteit Amsterdam, 2002

[5] R. Heckmann. Probabilistic Domains. *CAAP '94: Proceedings of the 19th International Colloquium on Trees in Algebra and Programming*, Lecture Notes in Computer Science, volume 787, pages 142-156, Sprigner-Verlag 1994.

[6] M. Hyland, G. Plotkin and J. Power. Combining effects: sum and tensor. *Theoretical Computer Science*, volume 357(1) , pages 70-99, Elsevier 2006.

[7] C. Jones. Probabilistic Nondeterminism, PhD Thesis, University of Edinburgh, 1990.

[8] C. Jones and G. Plotkin. A probabilistic powerdomain of evaluations. In *Proceedings of the Fourth Annual Symposium on Logic in Computer Science*, pages 186-195, IEEE Computer Society Press, 1989.

[9] A. Jung and R. Tix. The Troublesome Probabilistic Powerdomain. *Proceedings of Third Workshop on Computation and Approximation*, Electronic Notes in Theoretical Computer Science, volume 13, 1998.

[10] K. Keimel, P. Plotkin and R. Tix. Semantic Domains for Combining Probability and Non-Determinism. *Electronic Notes in Theoretical Computer Science*, volume 129, pages 1-104, 2005.

[11] J. Lambek and P. J. Scott. Introduction to Higher Order Categorical Logic. Cambridge University Press, 1986.

[12] M. Mislove. Nondeterminism and probabilistic choice: obeying the laws. In *Proceedings 11th CONCUR*, Lecture Notes in Computer Science, volume 1877, pages 350-364, 2000.

[13] M. Mislove, J. Ouaknine, J. Worrell. Axioms for Probability and Nondeterminism. *Proceedings of EXPRESS 03*, Electronic Notes in Computer Science, volume 96, 2004.

[14] G. Plotkin. A powerdomain construction. *SIAM Journal on Computing*, volume 5, pages 452-487, 1976.

[15] G. Plotkin. Probabilistic powerdomains. In *Proceedings CAAP*, pages 271-287, 1982.

[16] G. Plotkin. A powerdomain for countable non-determinism. In M. Nielson and E. M. Schmidt, editors, *Automata, Language and programming*, Lecture Notes in Computer Science, volume 140, pages 412-428, Springer-Verlag 1982.

[17] J. Power. Discrete Lawvere Theories. *Proceedings of CALCO 2005*, Lecture Notes in Computer Science, volume 3629, pages 348-363, Springer 2005.

[18] M. B. Smyth. Powerdomains. *Journal of Computer and Systems Sciences*, volume 16, pages 23-36, 1978.

[19] R. Tix. Continuous D-cones: Convexity and Powerdomain Constructions. PhD thesis, Technische Universität Darmstadt, 1999.

[20] D. Varacca. The Powerdomain of Indexed Valuations. In *Proceedings of 17th LICS*, IEEE Computer Society Press, 2002.

[21] D. Varacca and G. Winskel. Distributing probability over non-determinism. *Mathematical Structures in Computer Science*, volume 16, pages 87-113, 2006.