Coalgebraic Trace Semantics for Combined Possibilitistic and Probabilistic Systems

Bart Jacobs

Institute for Computing and Information Sciences,
Radboud University of Nijmegen
P.O. Box 9010, 6500 GL Nijmegen, The Netherlands
Email: `bart@cs.ru.nl`   *URL:* `http://www.cs.ru.nl/~bart`

Abstract

Non-deterministic (also known as possibilistic) and probabilistic state based systems (or automata) have been studied for quite some time. Separately, they are reasonably well-understood. The combination however is difficult, both for conceptual and technical reasons. Here we study the combination from a coalgebraic perspective and identify a monad $\mathcal{CM}$ that captures the combination—following work of Varacca. We use this monad to apply the coalgebraic framework for (finite) trace semantics in this setting. It yields a smooth, but not entirely trivial, description of traces.

Keywords: coalgebra, probabilistic nondeterminism, probabilistic nondeterminism

1 Introduction

The combination of non-determinism and probability is an important but difficult topic of research, which has received much attention. There is a wide variety of possible combinations. We shall not try to give an overview or a historical account and refer to [3] for such an overview and a classification, in coalgebraic terms.

Within this coalgebraic setting an abstract description of trace semantics has emerged [10,9] that exploits finality within a Kleisli category of a monad. This works well for non-determinism—via the powerset monad $\mathcal{P}$—and also for probability—for the distribution monad $\mathcal{D}$, but so far only when $\mathcal{P}$ and $\mathcal{D}$ are considered separately. The combination of $\mathcal{P}$ and $\mathcal{D}$ has defied integration attempts. The technical reason is that there is no distributive law $\mathcal{DP} \Rightarrow \mathcal{PD}$, see *e.g.* [21], describing a (counter)argument due to Plotkin.

Varacca in his thesis [20] (see also [21]) proposes two solutions, namely to either replace the distribution monad by a new monad of "indexed valuations" (for which there is indeed a distributive law with powerset) or to use one monad of convex subsets (which acts on a different category) for the whole combination. Here we

shall follow the latter approach. What we contribute is first of all a reformulation of this second approach in terms of semimodules [7]. In algebra, a module (see *e.g.* [15]) is like a vector space, but with a ring of scalars, instead of a field. A semimodule is even weaker, and has only a semiring of scalars. Such a semiring is in fact a combination of two monoids, with one distributing over the other. There are natural examples of semirings in this setting, namely the sets of non-negative natural, rational, or real numbers, possibly extended with infinity ∞.

Our first step is to describe the (more or less standard) construction of free semimodules over sets, via a multiset functor that counts elements with values in a semiring. These multisets can be described as formal sums $\sum_i a_i x_i$ with multiplicity a_i for element x_i. We do not impose the requirement $\sum_i a_i = 1$, which is typical of probability distributions. The more general formulation of multisets not only gives a nicer mathematical theory (with free semimodules) but also allows more general interpretations of the a_i than probabilities, for instance involving cost or time or resource consumption.

In a next step the notion of convex subset can be defined naturally over a semimodule, namely as a subset that is closed under linear combinations (with scalars adding up to 1). Our first aim is to reformulate the setting of Varacca in terms of free constructions of semimodules. In doing so we slightly extend his work, by formulating it with a semiring as parameter, and with non-finitely generated convex subsets. The latter are needed since a trace is generally not a finite (or finitely generated) set.

Our second contribution is to show that the monad $\mathcal{CM}$ that is obtained from the free construction of semimodules over (affine) complete lattices is indeed suitable for coalgebraic trace semantics. This is shown in two steps, namely by verifying that $\mathcal{CM}$ satisfies almost all the technical conditions of [10] for trace semantics—in particular that its Kleisli category is enriched over directed complete partial orders—and by calculating traces in a concrete example, following this coalgebraic approach. There is actually one condition from [10] that is not satisfied, namely the presence of a bottom element in Kleisli homsets. We do however have a zero element, which is enough, after some manipulation. For expository reasons we will start with the example and subsequently develop the required mathematics.

This paper makes a modest step itself, but hopefully forms the starting point for an integration of research lines in the area of possibilistic and probabilistic systems. We conjecture, for instance, that the approach to traces based on schedulers (see *e.g.* [17,21,5]) gives the same outcome as the coalgebraic approach that is developed here. This will be elaborated in a next version of this paper.

2 Example

We shall consider a concrete state-based system with combined possibilistic and probabilistic behaviour in order to illustrate the calculation of traces of states. This is meant as a sketch of what this paper achieves. Later sections will elaborate the underlying technical details. Hence, possibly, not everything is clear at this stage.

In particular, some notions and notations (like for convex closure) will be used that are explained later on. Hopefully, the intuition of what is happening is helpful.

Our example system has state space $X = \{p, q, r\}$ and set of labels $A = \{a, b, c, d, e\}$ with the following picture, in which the symbol ✓ is used to indicate termination.

(1)
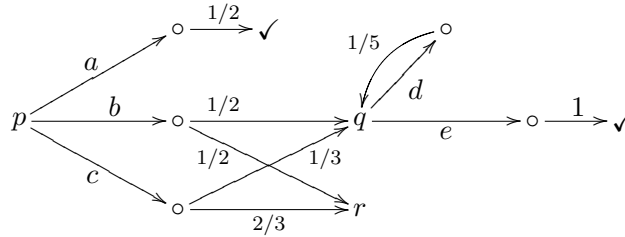
There are two kinds of arrows in this picture. The arrows ending in circles ∘ describe non-deterministic (labeled) transitions. Their targets are not states, but distributions (actually multisets) of states: they have outgoing arrows to states, with probabilities as labels, indicating how likely that transition is.

This system may be described as a coalgebra of the form $\gamma \colon X \to \mathcal{CM}(A + A \times X)$, namely as:

$$
\begin{aligned}
\gamma(p) &= \overline{\{0, \tfrac{1}{2}a, \tfrac{1}{2}\langle b, q\rangle + \tfrac{1}{2}\langle b, r\rangle, \tfrac{1}{3}\langle c, q\rangle + \tfrac{2}{3}\langle c, r\rangle\}} \\
\gamma(q) &= \overline{\{0, \tfrac{1}{5}\langle d, q\rangle, 1e\}} \\
\gamma(r) &= \{0\}.
\end{aligned}
$$

At this stage we only describe $\mathcal{CM}$ informally as containing convex subsets of distributions. The overlining describes convex closure. Hopefully the match between these equations and the picture is sufficiently convincing. The zero elements are included for technical reasons, but are not written in the picture. They could be written as arrows $x \xrightarrow{\ell} \circ$ for every state x and label ℓ, but doing so does not make things clearer.

A crucial point is that $\mathcal{CM}$ is a "monad", so that we can use what is called "Kleisli" composition. This allows us to compose the coalgebra γ with itself, and obtain iterates $\gamma_n \colon X \to \mathcal{CM}(A^{\leq n})$, where $A^{\leq n}$ is the set of sequences of elements from A with length at most n. The first step is given by $\gamma_0(x) = \{0\}$—where 0 is

the "null" distribution—and the subsequent ones by:

$$\gamma_{n+1}(p)$$
$$= \bigcup \left\{ \{0\}, \{\tfrac{1}{2}a\}, \left\{ \sum_{\sigma \in A^+} \tfrac{1}{2}\psi(\sigma)(b\sigma) \Big| \psi \in \gamma_n(q) \right\}, \left\{ \sum_{\sigma \in A^+} \tfrac{1}{3}\psi(\sigma)(c\sigma) \Big| \psi \in \gamma_n(q) \right\} \right\}$$

$$\gamma_{n+1}(q)$$
$$= \bigcup \left\{ \{0\}, \{1e\}, \left\{ \sum_{\sigma \in A^+} \tfrac{1}{5}\psi(\sigma)(d\sigma) \Big| \psi \in \gamma_n(q) \right\} \right\}$$

$$\gamma_{n+1}(r)$$
$$= \{0\}.$$

These formulas will be justified later on. For now we shall compute some these sets. To start with:

$$\gamma_1(p) = \overline{\bigcup \left\{ \{0\}, \{\tfrac{1}{2}a\} \right\}} = \overline{\{0, \tfrac{1}{2}a\}} = \{0, \tfrac{1}{2}a\}$$
$$\gamma_1(q) = \overline{\bigcup \left\{ \{0\}, \{1e\} \right\}} = \{0, 1e\}.$$

In a next step we get:

$$\gamma_2(p) = \overline{\bigcup \left\{ \{0\}, \{\tfrac{1}{2}a\}, \{\tfrac{1}{2}be\}, \{\tfrac{1}{3}ce\} \right\}} = \overline{\{0, \tfrac{1}{2}a, \tfrac{1}{2}be, \tfrac{1}{3}ce\}}$$
$$\gamma_2(q) = \overline{\bigcup \left\{ \{0\}, \{1e\}, \{\tfrac{1}{5}de\} \right\}} = \overline{\{0, 1e, \tfrac{1}{5}de\}}.$$

The multisets appearing here, like $\tfrac{1}{2}be$ in $\gamma_2(p)$ correspond to a 2-step path, from p to $\checkmark$, with multiplication of probabilities that occur on the way.

We make one more step:

$$\gamma_3(p) = \overline{\bigcup \left\{ \{0\}, \{\tfrac{1}{2}a\}, \{\tfrac{1}{2}be, \tfrac{1}{10}bde\}, \{\tfrac{1}{3}ce, \tfrac{1}{15}cde\} \right\}} = \overline{\{0, \tfrac{1}{2}a, \tfrac{1}{2}be, \tfrac{1}{3}ce, \tfrac{1}{10}bde, \tfrac{1}{15}cde\}}$$
$$\gamma_3(q) = \overline{\bigcup \left\{ \{0\}, \{1e\}, \{\tfrac{1}{5}de, \tfrac{1}{25}dde\} \right\}} = \overline{\{0, 1e, \tfrac{1}{5}de, \tfrac{1}{25}dde\}}.$$

By continuing in this way we get the trace as supremum:

$$\mathrm{tr}(p) = \overline{\left\{0, \tfrac{1}{2}a\right\} \cup \left\{\tfrac{1}{2}\tfrac{1}{5^n}bd^ne \,\Big|\, n \in \mathbb{N}\right\} \cup \left\{\tfrac{1}{3}\tfrac{1}{5^n}cd^ne \,\Big|\, n \in \mathbb{N}\right\}}$$
$$\mathrm{tr}(q) = \overline{\left\{0\right\} \cup \left\{\tfrac{1}{5^n}d^ne \,\Big|\, n \in \mathbb{N}\right\}}.$$

Such trace descriptions will be justified in the remainder of this paper.

3   Monoids, semirings and semimodules

We start with an abstract description to arrive at the notion of a semimodule in a category. One can also use the more concrete description, given by operations and

equations as in (2) below.

Standard "universes" in this paper are the category **Sets** of sets and functions and the category **ACL** of "affine" complete lattices (posets with joins of all non-empty subsets) and non-empty join preserving functions between them (see [12]). An affine complete lattice is thus different from an ordinary complete lattice because it need not have a bottom element $\perp$—as join of the empty subset. The category **Sets** has finite products $(1, \times)$ in the usual way; **ACL** has a monoidal structure $(I, \otimes)$, where a homomorphism $X \otimes Y \to Z$ corresponds to a function $X \times Y \to Z$ that preserves non-empty joins in both arguments separately (is "bilinear"). This follows work of Kock on tensors in categories of Eilenberg-Moore algebras, see [12] again for a concise description.

Let $\mathbb{C}$ be an arbitrary category with a symmetric monoidal structure $(I, \otimes)$—which may informally be understood as products without projections or diagonals. In such a setting one can define the notion of commutative monoid. It consists of a "carrier" object $M \in \mathbb{C}$ with two maps $I \xrightarrow{0} M \xleftarrow{+} M \otimes M$ making obvious diagrams commute, expressing that $(0, +)$ satisfy the standard requirements for commutative monoids. These structures may be organised in a category $cMon(\mathbb{C})$ in which homomorphisms are maps in $\mathbb{C}$ between the carriers that commute appropriately with the monoid structures.

In this way one obtains for instance the category $cMon(\mathbf{Sets})$ of "ordinary" commutative monoids or $cMon(\mathbf{ACL})$ of (commutative, unital and "affine") quantales [16]. In the latter case the carrier is an affine complete lattice and addition preserves non-empty joins, in both arguments.

Given a monoid $M \in cMon(\mathbb{C})$ there is a notion of "M-action". It consists of an object $X \in \mathbb{C}$ with a map $\sigma \colon M \otimes X \to X$ satisfying:

A homomorphism $(X, \sigma) \to (Y, \tau)$ of actions is a map $f \colon X \to Y$ in $\mathbb{C}$ with $f \circ \sigma = \tau \circ \mathrm{id} \otimes f$. This yields a category $Act_M(\mathbb{C})$, with forgetful functor $Act_M(\mathbb{C}) \to \mathbb{C}$, see [14, Ch. VII.4]. It has a left adjoint, given by $X \mapsto M \otimes X$.

Often these categories $cMon(\mathbb{C})$ also have a monoidal structure $(I, \otimes)$ themselves. In that case one can consider the category $cMon(cMon(\mathbb{C}))$ of "double" monoids. These are commonly called *semirings*. They are objects $S \in \mathbb{C}$ for which one has an additive structure $(0, +)$ and a multiplicative structure $(1, \cdot)$ where multiplication is a homomorphism wrt. the additive structure, in both arguments. This amounts to the familiar distributivity laws:

$$(x + y) \cdot z = x \cdot z + y \cdot z \quad \text{and} \quad 0 \cdot z = 0.$$

Notice that in this setting a semiring has a multiplicative unit 1 and is commutative, both additivily and multiplicatively. We shall abbreviate $cMon(cMon(\mathbb{C}))$ as $SRng(\mathbb{C})$, assuming that appropriate tensors exist.

For a semiring S in a category $\mathbb{C}$ we can perform the above action construction wrt. the category $cMon(\mathbb{C})$ of commutative monoids in $\mathbb{C}$. This yields a category $Act_S(cMon(\mathbb{C}))$ which we shall write as $SMod_S(\mathbb{C})$. It is the category of *semimodules* in $\mathbb{C}$, see *e.g.* [7]. An object of $SMod_S(\mathbb{C})$ is a commutative monoid M with an action $S \otimes M \to M$, which we shall typically write as $\bullet$. In usual notation the following equations hold.

$$
\begin{array}{lll}
& 1 \bullet x = x & (a+b) \bullet x = a \bullet x + b \bullet x \\
(2) & (a \cdot b) \bullet x = a \bullet (b \bullet x) & a \bullet 0 = 0 \\
& 0 \bullet x = 0 & a \bullet (x+y) = a \bullet x + a \bullet y.
\end{array}
$$

We shall be especially interested in the categories $SMod_S(\mathbf{Sets})$ and $SMod_S(\mathbf{ACL})$, for semirings S like $\mathbb{N} \cup \{\infty\}$ or $[0, \infty] = \{a \in \mathbb{R} \mid a \geq 0\} \cup \{\infty\}$ of extended non-negative (natural and real) numbers. Notice that these two semirings are complete lattices, with the semiring operations $+$ and $\cdot$ preserving joins. The unit interval $[0, 1]$ of real numbers is a semiring (in complete lattices) with $(0, \max)$ as additive and $(1, \cdot)$ as multiplicative structure. This is a "semifield", in which the non-zero elements form a multiplicative group, see [7].

4 Free semimodules

For a semiring $S \in SRng(\mathbf{Sets})$ we shall write $\mathcal{M}_S \colon \mathbf{Sets} \to \mathbf{Sets}$ for the finite "multiset" functor that counts in S. It is defined as:

$$
\mathcal{M}_S(X) = \{\varphi \colon X \to S \mid \text{supp}(\varphi) = \{x \in X \mid \varphi(x) \neq 0\} \text{ is finite}\}.
$$

For a function $f \colon X \to Y$, a "multiset" $\varphi \in \mathcal{M}_S(X)$, and an element $y \in Y$, we write:

$$
\mathcal{M}_S(f)(\varphi)(y) = \sum_{x \in f^{-1}(y)} \varphi(x) = \sum_{x \in f^{-1}(y) \cap \text{supp}(\varphi)} \varphi(x).
$$

This makes $\mathcal{M}_S$ a functor.

These sets $\mathcal{M}_S(X)$ form commutative monoids via pointwise operations. Elements $\varphi \in \mathcal{M}_S(X)$ will often be written as formal sum $\sum_x \varphi(x) x$ or as $\sum_i a_i x_i$ if $\text{supp}(\varphi) = \{x_1, \ldots, x_n\}$ and $\varphi(x_i) = a_i$. The element $a_i \in S$ describes the "multiplicity" of the element x_i in the finite "multiset" φ. These monoids $\mathcal{M}_S(X)$ also carry an S-action, namely:

$$
a \bullet \varphi = \lambda x.\, a \cdot \varphi(x).
$$

It is not hard to see that this makes $\mathcal{M}_S(X)$ a semimodule. In fact, it is the free one on the set X.

Proposition 4.1 *The $\mathcal{M}_S(-)$ construction yields free semimodules: it forms a left adjoint to the forgetful functor $SMod_S(\mathbf{Sets}) \to \mathbf{Sets}$. In fact, $SMod_S(\mathbf{Sets})$ is the category of (Eilenberg-Moore) algebras of the induced monad $\mathcal{M}_S: \mathbf{Sets} \to \mathbf{Sets}$.*

Proof For a function $f: X \to M$, where M is a semimodule over S, one obtains a unique extension $\widehat{f}: \mathcal{M}_S(X) \to M$ by $\widehat{f}(\varphi) = \sum_x \varphi(x) \bullet f(x)$. Then $\widehat{f} \circ \eta = f$, where $\eta(x) = 1x$. This $\widehat{f}$ is the unique semimodule homomorphism with this property because each multiset $\varphi \in \mathcal{M}_S(X)$ can be written as finite sum $\varphi = \sum_x \varphi(x) \bullet \eta(x)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The following diagram is an adaptation of [21]. It describes the structure of what follows in the remainder of this section.

(3)
$$
\begin{array}{ccc}
 & \mathcal{C} & \\
SMod_S(\mathbf{Sets}) \xleftarrow{\quad\perp\quad} & & SMod_S(\mathbf{ACL}) \\
 & \mathcal{M}_S \searrow \quad \dashv \quad \nearrow & \\
 & \mathbf{Sets} &
\end{array}
$$

The straight arrows are forgetful functors, and the bent ones are their left adjoints. The upper adjoint $\mathcal{C}$ involves "convex" subsets in a semimodule. This notion is introduced first.

For a semimodule $M \in SMod(\mathbf{Sets})$ and an arbitrary $U \subseteq M$ one defines the convex closure $\overline{U} \subseteq M$ of U as:

$$\overline{U} = \{a_1 \bullet x_1 + \cdots + a_n \bullet x_n \mid x_i \in U, a_i \in S, \sum_i a_i = 1\}.$$

It is not hard to see that $U \subseteq \overline{U}$, $\overline{\overline{U}} = \overline{U}$ and $U \subseteq V \Rightarrow \overline{U} \subseteq \overline{V}$—making $\overline{\cdot}$ indeed a closure operation.

One calls the subset U convex if $\overline{U} = U$. Now we put:

$$\mathcal{C}(M) = \{U \subseteq M \mid U \text{ is non-empty and convex}\}.$$

It is essential that $\mathcal{C}(M)$ contains *non-empty* subsets, and not all subsets, for instance in the proof of Lemma 4.2 below—to show $0 \bullet U = 0$—and in order to get $\mathcal{C}M(0) = 1$ later on in this paper. A consequence of using non-empty subsets is that we have no bottom element, and thus an affine lattice.

For a map $f: M \to N$ in $SMod_S(\mathbf{Sets})$ we obtain $\mathcal{C}(f): \mathcal{C}(M) \to \mathcal{C}(N)$ simply as image:

$$\mathcal{C}(f)(U) = \{f(x) \mid x \in U\}.$$

It is easy to see that this image is indeed convex. The set $\mathcal{C}(M)$, ordered by inclusion, is an *affine* complete lattice, with joins over non-empty index sets I given by:

$$\bigvee_{i \in I} U_i = \overline{\bigcup_{i \in I} U_i}.$$

Next we define monoid operations on subsets of M.

(4) $\qquad\qquad 0 = \{0\} \qquad \text{and} \qquad U + V = \{x + y \mid x \in U, y \in V\}.$

where $U, V \subseteq M$ are arbitrary subsets. It is not hard to see that $\overline{U} + \overline{V} = \overline{U + V}$, making $+$ a well-defined operation on $\mathcal{C}(M)$. The direction $(\supseteq)$ is obvious and for $(\subseteq)$ it suffices to prove $\overline{U} + V \subseteq \overline{U + V}$. This is done as follows. Assume $x + y \in \overline{U} + V$, say $x = \sum_j a_j \bullet x_j$ with $x_j \in U$ and $\sum_j a_j = 1$. Then $y = 1 \bullet y = (\sum_j a_j) \bullet y = \sum_j a_j \bullet y$ so that $x + y = \sum_j a_j \bullet (x_j + y)$, where $x_j + y \in U + V$. Then $x + y \in \overline{U + V}$.

These $0, +$ make $\mathcal{C}(M)$ a commutative monoid. There is also an action, given as:

(5)
$$a \bullet U = \{a \bullet x \mid x \in U\}.$$

We have $\overline{a \bullet U} = a \bullet \overline{U}$, since $\sum_j a_j \bullet (a \bullet x_j) = \sum_j (a \cdot a_j) \bullet x_j = a \bullet (\sum_j a_j \bullet x_j)$. Hence also the action on $\mathcal{C}(M)$ is well-defined.

The singleton map $\{-\}\colon M \to \mathcal{C}(M)$ is clearly a map of semimodules.

The essence of the next series of results can be traced back to [19,21]. For completeness and convenience we include many aspects of the proofs.

Lemma 4.2 *Taking convex subsets yields a functor* $\mathcal{C}\colon SMod_S(\mathbf{Sets}) \to SMod_S(\mathbf{ACL})$ *when* $S \in SRng(\mathbf{ACL})$ *is a semifield which is "zerosumfree", i.e. satisfies* $a + b = 0 \Rightarrow a = b = 0$.

From now on we shall assume that $S \in SRng(\mathbf{ACL})$ is such a zerosumfree semifield.

Proof Clearly $0, +$ from (4) form a commutative monoid on $\mathcal{C}(M)$ and $\bullet$ from (5) an action. We have to check that the action preserves the monoid structure:

$$
\begin{aligned}
a \bullet \{0\} &= \{a \bullet 0\} \\
&= \{0\} \\
a \bullet (U + V) &= \{a \bullet (x + y) \mid x \in U, y \in V\} \\
&= \{a \bullet x + a \bullet y \mid x \in U, y \in V\} \\
&= a \bullet U + a \bullet V \\
0 \bullet U &= \{0 \bullet x \mid x \in U\} \\
&= \{0\} \qquad \text{since } U \text{ is non-empty} \\
(a + b) \bullet U &= \{(a + b) \bullet x \mid x \in U\} \\
&= \{a \bullet x + b \bullet x \mid x \in U\} \\
&\overset{(*)}{=} \{a \bullet x + b \bullet y \mid x, y \in U\} \\
&= a \bullet U + b \bullet U.
\end{aligned}
$$

The marked equation $\overset{(*)}{=}$ requires some care. The direction $(\subseteq)$ is obvious, but $(\supseteq)$ requires convexity of U and division in S. Suppose we have $x, y \in U$. We may assume $a + b \neq 0$, because otherwise $a + b = 0$ yields $a = b = 0$ so that the equation obviously holds. Take $z = \frac{a}{a+b} \bullet x + \frac{b}{a+b} \bullet y$, which is in U because U is convex,

and also:

$$
\begin{aligned}
a \bullet z + b \bullet z &= \tfrac{a^2}{a+b} \bullet x + \tfrac{ab}{a+b} \bullet y + \tfrac{ba}{a+b} \bullet x + \tfrac{b^2}{a+b} \bullet y \\
&= \tfrac{a^2+ab}{a+b} \bullet x + \tfrac{b^2+ab}{a+b} \bullet y \\
&= a \bullet x + b \bullet y.
\end{aligned}
$$

Next we need to prove that joins are preserved.

$$
\begin{aligned}
\left(\bigvee_i U_i\right) + V &= \left(\overline{\bigcup_i U_i}\right) + V \\
&= \left(\overline{\bigcup_i U_i}\right) + \overline{V} \\
&= \overline{\left(\bigcup_i U_i\right) + V} \qquad \text{as shown after (4)} \\
&= \overline{\bigcup_i (U_i + V)} \\
&= \bigvee_i (U_i + V) \\
a \bullet \left(\bigvee_i U_i\right) &= a \bullet \left(\overline{\bigcup_i U_i}\right) \\
&= \overline{a \bullet \left(\bigcup_i U_i\right)} \qquad \text{see after (5)} \\
&= \overline{\bigcup_i (a \bullet U_i)} \\
&= \bigvee_i (a \bullet U_i).
\end{aligned}
$$

Finally we need to check that if f is a map of semimodules, then so is $\mathcal{C}(f)$. This is easy. Additionally, $\mathcal{C}(f)$ must preserve joins. This follows from the fact that $\overline{\cdot}$ commutes with images: $\mathcal{C}(f)(\overline{U}) = \overline{\mathcal{C}(f)(U)}$. $\qquad \square$

The following lemma is typical for semimodules over lattices: it combines the sum, action and join. It is a mild generalisation of [21, Prop. 5.5].

Lemma 4.3 *In a semimodule $M \in SMod_S(\mathbf{ACL})$ one has:*

$$
\sum_{i \leq n} (a_i \bullet x_i) \leq \left(\sum_{i \leq n} a_i\right) \bullet \left(\bigvee_{i \leq n} x_i\right).
$$

Proof By induction on n. The case $n = 0$ involves summation over 1 and is obvious. Further:

$$
\begin{aligned}
&\left(\textstyle\sum_{i \leq n+1} a_i\right) \bullet \left(\bigvee_{i \leq n+1} x_i\right) \\
&= (b + a_{n+1}) \bullet (y \vee x_{n+1}) \quad \text{where } b = \textstyle\sum_{i \leq n} a_i \text{ and } y = \bigvee_{i \leq n} x_i \\
&= b \bullet (y \vee x_{n+1}) + a_{n+1} \bullet (y \vee x_{n+1}) \\
&= (b \bullet y \vee b \bullet x_{n+1}) + (a_{n+1} \bullet y \vee a_{n+1} \bullet x_{n+1}) \\
&= (b \bullet y + a_{n+1} \bullet y) \vee (b \bullet y + a_{n+1} \bullet x_{n+1}) \vee \\
&\qquad (b \bullet x_{n+1} + a_{n+1} \bullet y) \vee (b \bullet x_{n+1} + a_{n+1} \bullet x_{n+1}) \\
&= (b + a_{n+1}) \bullet y \vee (b \bullet y + a_{n+1} \bullet x_{n+1}) \vee \\
&\qquad (b \bullet x_{n+1} + a_{n+1} \bullet y) \vee (b + a_{n+1}) \bullet x_{n+1} \\
&= (b + a_{n+1}) \bullet (y \vee x_{n+1}) \vee (b \bullet y + a_{n+1} \bullet x_{n+1}) \vee (b \bullet x_{n+1} + a_{n+1} \bullet y).
\end{aligned}
$$

Now we are almost done:

$$
\begin{aligned}
&\textstyle\sum_{i\leq n+1}(a_i \bullet x_i) \\
&= \left(\textstyle\sum_{i\leq n}(a_i \bullet x_i)\right) + a_{n+1} \bullet x_{n+1} \\
&\leq \left(\textstyle\sum_{i\leq n} a_i\right) \bullet \left(\bigvee_{i\leq n} x_i\right) + a_{n+1} \bullet x_{n+1} \quad \text{by induction hypothesis} \\
&= b \bullet y + a_{n+1} \bullet x_{n+1} \quad \text{with } b, y \text{ as before} \\
&\leq (b + a_{n+1}) \bullet (y \vee x_{n+1}) \vee (b \bullet y + a_{n+1} \bullet x_{n+1}) \vee (b \bullet x_{n+1} + a_{n+1} \bullet y) \\
&\qquad \text{since in general } u \leq v \vee u \vee w \\
&= \left(\textstyle\sum_{i\leq n+1} a_i\right) \bullet \left(\bigvee_{i\leq n+1} x_i\right) \quad \text{as shown above.} \qquad \square
\end{aligned}
$$

Proposition 4.4 *The functor* $\mathcal{C}: SMod_S(\mathbf{Sets}) \to SMod_S(\mathbf{ACL})$ *is left adjoint to the forgetful functor.*

Proof For $M \in SMod_S(\mathbf{Sets})$ and $N \in SMod_S(\mathbf{ACL})$ the extension of a module morphism $f: M \to N$ to $\widehat{f}: \mathcal{C}(M) \to N$ is given by $\widehat{f}(U) = \bigvee \{f(x) \mid x \in U\}$. Obviously, $\widehat{f} \circ \{-\} = f$. In order to prove that $\widehat{f}$ is a homomorphism we first need that for arbitrary $U \subseteq M$

(6) $$\widehat{f}(\overline{U}) = \widehat{f}(U).$$

The direction $(\geq)$ is obvious, and for $(\leq)$ we need to show that $f(y) \leq \widehat{f}(U)$ for $y \in \overline{U}$. So let $y = \sum_i a_i \bullet y_i$ with $y_i \in U$. Then:

$$
\begin{aligned}
f(y) = f(\textstyle\sum_i a_i \bullet y_i) = \textstyle\sum_i a_i \bullet f(y_i) &\leq \left(\textstyle\sum_i a_i\right) \bullet \left(\bigvee_i f(y_i)\right) \quad \text{by Lemma 4.3} \\
&= 1 \bullet \left(\bigvee_i f(y_i)\right) \\
&= \bigvee_i f(y_i) \\
&\leq \bigvee \{f(x) \mid x \in U\} = \widehat{f}(U).
\end{aligned}
$$

Then, for non-empty joins:

$$
\begin{aligned}
\widehat{f}(\bigvee_i U_i) &= \widehat{f}(\overline{\bigcup_i U_i}) \\
&= \widehat{f}(\textstyle\bigcup_i U_i) \quad \text{by (6)} \\
&= \bigvee \{f(x) \mid x \in \textstyle\bigcup_i U_i\} \\
&= \bigvee \textstyle\bigcup_i \{f(x) \mid x \in U_i\} \\
&= \bigvee_i \bigvee \{f(x) \mid x \in U_i\} \\
&= \bigvee_i \widehat{f}(U_i).
\end{aligned}
$$

Uniqueness of $\widehat{f}$ follows from the fact that each $U \in \mathcal{C}(M)$ can be written as non-empty join $U = \overline{U} = \overline{\bigcup_{x \in U}\{x\}} = \bigvee_{x \in U}\{x\}$. $\qquad \square$

This adjunction is related to a more elementary one between sets and affine

complete lattices, as in:

$$
\begin{array}{ccc}
SMod_S(\mathbf{Sets}) & \xleftarrow{\quad\mathcal{C}\quad} & SMod_S(\mathbf{ACL}) \\
\mathcal{U}\downarrow & \xrightarrow{\mathcal{P}^+} & \downarrow\mathcal{U} \\
\mathbf{Sets} & \xleftarrow{\quad V \quad} & \mathbf{ACL}
\end{array}
$$

where the category **ACL** is the category of algebras of this non-empty powerset monad $\mathcal{P}^+$, see [12]. There is a map of adjunctions from the upper adjunction to the lower one, given by natural transformations $\mathcal{U}\mathcal{C} \Rightarrow \mathcal{P}^+\mathcal{U}$ and $\mathcal{U}V \Rightarrow V\mathcal{U}$. The first one is given by inclusions $\mathcal{C}(M) \subseteq \mathcal{P}^+(M)$ and the second one by identities.

As a result there is a map of monads between the induced monads, on $SMod_S(\mathbf{Sets})$ and on **Sets**, as described on the left in the above diagram.

5 The monad for both nondeterminism and probability

In this section we combine Propositions 4.1 and 4.4, about diagram (3), to obtain a monad $\mathcal{CM}$ on **Sets** that combines both possibilistic and probabilistic aspects. Recall that we often leave the (zerosumfree) semifield S over which we work implicit.

Proposition 5.1 *By composition of adjoints, the functor $\mathcal{CM} = \mathcal{C} \circ \mathcal{M}$ yields free semimodules in the situation:*

$$
\begin{array}{c}
SMod_S(\mathbf{ACL}) \\
\mathcal{CM}\left(\dashv\;\right\downarrow \\
\mathbf{Sets}
\end{array}
$$

We shall write $\mathcal{CM}\colon\mathbf{Sets} \to \mathbf{Sets}$ for the induced monad. An element $U \in \mathcal{CM}(X)$ is then a non-empty convex set of multisets of elements from X.

*Given a semimodule $M \in SMod_S(\mathbf{ACL})$ and a set X, the associated extension of a function $f\colon X \to M$ in **Sets** to a map $\widehat{f}\colon\mathcal{CM}(X) \to M$ in $SMod_S(\mathbf{ACL})$ is given by:*

$$
\widehat{f}(U) = \bigvee\nolimits_{\varphi \in U} \sum\nolimits_{x \in \mathrm{supp}(\varphi)} \varphi(x) \bullet f(x). \qquad\qquad \square
$$

The unit $\eta\colon X \to \mathcal{CM}(X)$ and multiplication $\mu\colon\mathcal{CM}^2(X) \to \mathcal{CM}(X)$ of the induced monad $\mathcal{CM}\colon\mathbf{Sets} \to \mathbf{Sets}$ are:

$$
\eta(x) = \{1x\}
$$
$$
\mu(P) = \bigvee\nolimits_{\Phi \in P} \sum\nolimits_{U \in \mathrm{supp}(\Phi)} \Phi(U) \bullet U \;=\; \bigcup\nolimits_{\Phi \in P} \sum\nolimits_{U} \{\Phi(U) \bullet \varphi \mid \varphi \in U\}.
$$

It is not hard to see that there is a map of monads $\mathcal{CM} \Rightarrow \mathcal{P}$, given by $U \mapsto \bigcup_{\varphi \in U} \mathrm{supp}(\varphi)$.

A standard construction for a monad T on **Sets** is the associated strength operation $\mathrm{st}\colon A \times T(X) \to T(A \times X)$, given by $\mathrm{st}(a, u) = T(\lambda x.\, \langle a, x \rangle)(u)$. This strength

map commutes appropriately with the monad's unit and multiplication. There is an associated map $\text{st}': T(X) \times A \to T(X \times A)$, obtained by twisting (twice). The monad T is called commutative if the two resulting maps $T(X) \times T(Y) \rightrightarrows T(X \times Y)$ are the same.

Lemma 5.2 *The monad* $\mathcal{CM}: \textbf{Sets} \to \textbf{Sets}$ *has strength map* $\text{st}: A \times \mathcal{CM}(X) \to \mathcal{CM}(A \times X)$ *given by:*

$$
\begin{aligned}
\text{st}(a, U) &= \mathcal{CM}(\lambda x. \langle a, x \rangle)(U) \\
&= \{ \mathcal{M}(\lambda x. \langle a, x \rangle)(\varphi) \mid \varphi \in U \} \\
&= \{ \text{st}_{\mathcal{M}}(a, \varphi) \mid \varphi \in U \} \\
&= \{ \textstyle\sum_x \varphi(x) \langle a, x \rangle \mid \varphi \in U \}.
\end{aligned}
$$

This monad is commutative, with associated "double strength" map $\text{dst}: \mathcal{CM}(X) \times \mathcal{CM}(Y) \to \mathcal{CM}(X \times Y)$ *given by:*

$$
\text{dst}(U, V) = \overline{\{ \varphi \cdot \psi \mid \varphi \in U, \psi \in U \}},
$$

where $\varphi \cdot \psi \in \mathcal{M}(X \times Y)$ *is defined by multiplication:* $(\varphi \cdot \psi)(x, y) = \varphi(x) \cdot \psi(y).$

Proof By straightforward calculation.   □

Remark 5.3 Actions on complete lattices have been used before, for instance in [1]. There, the context is completely different. The starting point are quantales, which are monoids in the category of complete lattices. The free quantale on a set A, for instance, is the lattice $\mathcal{P}(A^\star)$ of languages over A. What is observed (and exploited) in [1] is that a non-deterministic A-labelled transition system $X \to \mathcal{P}(A \times X)$ is the same as an action (or module) $\mathcal{P}(A^\star) \otimes \mathcal{P}(X) \to \mathcal{P}(X)$, via the following correspondences.

$$
\cfrac{\cfrac{\cfrac{X \longrightarrow \mathcal{P}(A \times X) \cong (\mathcal{P}(X))^A}{A \longrightarrow \mathcal{P}(X \times X) \cong (\mathcal{P}(X) \multimap \mathcal{P}(X))}}{\mathcal{P}(A^\star) \longrightarrow (\mathcal{P}(X) \multimap \mathcal{P}(X))}}{\mathcal{P}(A^\star) \otimes \mathcal{P}(X) \longrightarrow \mathcal{P}(X)}
$$

Here we have written $\otimes$ for the tensor of complete lattices and $\multimap$ for the associated function space of linear maps. The middle correspondence arises by freeness, because $\mathcal{P}(X) \multimap \mathcal{P}(X)$ is both a complete lattice and a monoid (via composition). Such actions are used in [1] to capture various kinds of process equivalences, for labelled transition systems.

   This setting is quite different from ours, not only because we deal with different transition systems—with monad $\mathcal{CM}$ instead of $\mathcal{P}$—but also because we consider actions wrt. a semiring like $[0, \infty]$, *i.e.* a "double" monoid, in **ACL** and not just a "single" monoid $\mathcal{P}(A^\star)$.

The terminology may lead to confusion: the actions of a *monoid* used in [1] are called modules, like in [13], whereas a (semi)module for us is an action of a *semiring* (following [7] and standard use of the term 'module' in algebra, see *e.g.* [15]).

6 The Kleisli category

Now that we have seen the monad $\mathcal{CM}$ we can investigate its Kleisli category $\mathcal{Kl}(\mathcal{CM})$ whose morphisms capture computations $X \to \mathcal{CM}(Y)$ mapping elements of X to a (convex) subset of multisets (or distributions) on Y. We shall be especially interested in the order enrichment of this category, to make sure that it satisfies the requirements needed for "coalgebraic trace semantics", as formulated in [10, Thm. 3.3].

We start with composition in $\mathcal{Kl}(\mathcal{CM})$—also known as Kleisli composition. It involves the extension operation $\widehat{\cdot}$ from Proposition 5.1 (or multiplication μ) in the following way. For $f \colon X \to \mathcal{CM}(Y)$ and $g \colon Y \to \mathcal{CM}(Z)$ we have their composite $g \circ f \colon X \to \mathcal{CM}(Z)$ given as:

$$(7) \qquad \begin{aligned} (g \circ f)(x) &= \widehat{g}(f(x)) \\ &= \bigvee_{\varphi \in f(x)} \sum_{y \in \mathrm{supp}(\varphi)} \varphi(y) \bullet g(y) \\ &= \bigcup_{\varphi \in f(x)} \sum_{y \in \mathrm{supp}(\varphi)} \{\varphi(y) \bullet \psi \mid \psi \in g(y)\}. \end{aligned}$$

Each homset $\mathcal{Kl}(\mathcal{CM})(X, Y)$ of functions $f \colon X \to \mathcal{CM}(Y)$ is ordered pointwise: $f \sqsubseteq g$ iff $\forall x \in X. \, f(x) \subseteq g(x)$. This forms an affine complete lattice, with pointwise joins. In order to obtain an enriched category we need to check that Kleisli composition preserves these joins. Here it turns out that we need to restrict to directed joins $\bigvee^{\uparrow}$, because of the property that a function in two arguments preserves directed joins in each argument separately if and only if it preserves directed joins. We shall apply this in the form $(\bigvee_{i \in I}^{\uparrow} x_i) + (\bigvee_{i \in I}^{\uparrow} y_i) = \bigvee_{i \in I}^{\uparrow}(x_i + y_i)$. Recall that a directed set is by definition non-empty, so that a directed join is a special form of non-empty join.

Kleisli composition preserves non-empty joins in the second component, and directed joins in the first one:

$$\begin{aligned} \Big(g \circ (\textstyle\bigvee_i f_i)\Big)(x) &= \widehat{g}(\textstyle\bigvee_i f_i(x)) \\ &= \textstyle\bigvee_i \widehat{g}(f_i(x)) \quad \text{since } \widehat{g} \text{ preserves joins} \\ &= \Big(\textstyle\bigvee_i(g \circ f_i)\Big)(x). \\ \Big((\textstyle\bigvee_i^{\uparrow} g_i) \circ f\Big)(x) &= \textstyle\bigvee_{\varphi \in f(x)} \sum_y \varphi(y) \bullet (\textstyle\bigvee_i^{\uparrow} g_i(y)) \\ &= \textstyle\bigvee_{\varphi \in f(x)} \sum_y (\textstyle\bigvee_i^{\uparrow} \varphi(y) \bullet g_i(y)) \\ &= \textstyle\bigvee_{\varphi \in f(x)} \textstyle\bigvee_i^{\uparrow} \sum_y \varphi(y) \bullet g_i(y) \quad \text{because the join is directed} \\ &= \textstyle\bigvee_i^{\uparrow} \textstyle\bigvee_{\varphi \in f(x)} \sum_y \varphi(y) \bullet g_i(y) \\ &= \textstyle\bigvee_i^{\uparrow} \widehat{g_i}(f(x)) \\ &= \Big(\textstyle\bigvee_i^{\uparrow}(g_i \circ f)\Big)(x). \end{aligned}$$

As a result, the Kleisli category $\mathcal{K}\ell(\mathcal{CM})$ is enriched over the category directed complete partial orders.

Each Kleisli homset has a special zero element $\mathbf{0}_{Y,Z} = \lambda y \in Y. \{0\} \colon Y \to \mathcal{CM}(Z)$. Composition is strict wrt. this zero in both arguments.

$$
\begin{aligned}
(\mathbf{0}_{Y,Z} \circ f)(x) &= \bigvee\nolimits_{\varphi \in f(x)} \textstyle\sum_y \varphi(y) \bullet \{0\} \\
&= \bigvee\nolimits_{\varphi \in f(x)} \{0\} \\
&= \{0\} \\
&= \mathbf{0}_{X,Z}(x) \\
(g \circ \mathbf{0}_{X,Y})(x) &= \bigvee\nolimits_{\varphi \in \{0\}} \textstyle\sum_y \varphi(y) \bullet g(y) \\
&= \{0\} \\
&= \mathbf{0}_{X,Z}(x).
\end{aligned}
$$

As shown in [10, Lemma 3.5] the first of these equations ("left strictness" $\mathbf{0} \circ f = \mathbf{0}$) means that the initial (empty) set 0 is both initial and final in $\mathcal{K}\ell(\mathcal{CM})$, because $\mathcal{CM}(0) = 1$. We shall use this fact later.

We summarise what we have found in this section.

Proposition 6.1 *The Kleisli category $\mathcal{K}\ell(\mathcal{CM})$ of the monad $\mathcal{CM}$ from Proposition 5.1 is enriched over the category of "pointed" directed complete orders.* $\qquad\square$

Our setting differs from [10] in the sense that our point $\mathbf{0}$ in homsets need not be a bottom element.

7 The transition type functor

The category $SMod(\mathbf{Sets})$ of semimodules is algebraic over $\mathbf{Sets}$, via the monad $\mathcal{M}$. Hence it is cocomplete, see for instance [2, §3.4, Theorem 1 and §9.3, Proposition 4] or [4, Volume 2, §4.3]. Finite colimits are special. For instance, the coproduct of two semimodules $M, N \in SMod(\mathbf{Sets})$ is the product $M \times N$: it is a "biproduct". Similarly, the copower $A \cdot M$, for a (finite) set A, is given by the function space M^A. The coprojections $\kappa_a \colon M \to M^A$ are given by $\kappa_a(x)(b) =$ if $a = b$ then x else 0. Since elements of this copower $A \cdot M$ are of the form $\kappa_a(x)$ for $a \in A$ and $x \in M$ we shall also write a copower $A \cdot M$ informally in set-theoretic notation as $A \times M$ with tuples $\langle a, x \rangle = \kappa_a(x)$.

The generic trace theory from [10] works for coalgebras of the form $X \to T(F(X))$ where T is a suitable monad and F is a "transition type" functor. Here we shall use $F = A + (A \times -)$, for a fixed set A. Its initial algebra is of course the

set A^+ of non-empty (finite) sequences of elements of A. Then we can write:

$$
\begin{aligned}
\mathcal{CM}(F(X)) &= \mathcal{C}(\mathcal{M}(A + A \times X)) \\
&\cong \mathcal{C}(\mathcal{M}(A \times (1 + X))) \\
(8) \qquad &\cong \mathcal{C}(\mathcal{M}(A \cdot (1 + X))) && \text{where } \cdot \text{ is copower in } \mathbf{Sets} \\
&\cong \mathcal{C}(A \cdot \mathcal{M}(1 + X)) && \text{since } \mathcal{M} \text{ preserves colimits, as left adjoint} \\
&= \mathcal{C}(A \times \mathcal{M}(1 + X)) && \text{using the above convention.}
\end{aligned}
$$

Coalgebras $X \to \mathcal{CM}(F(X))$ thus correspond to "Segala-style" systems [17], with first a possibilistic choice (via $\mathcal{C}$) followed by a probabilistic one (via $\mathcal{M}$). This last formulation $\mathcal{C}(A \times \mathcal{M}(1 + X))$ is useful in pictures of systems, like in Section 2.

Because the monad $\mathcal{CM}$ is commutative (see Lemma 5.2) and the functor F is "shapely" (built out of coproducts and (finite) products), there is by [10, Lemma 2.3] a distributive law $\lambda \colon F\mathcal{CM} \Rightarrow \mathcal{CM}F$ with components:

$$
(9) \qquad A + A \times \mathcal{CM}(X) \xrightarrow{\quad \lambda_X \quad} \mathcal{CM}(A + A \times X)
$$

given by:

$$
\lambda_X = [\mathcal{CM}(\kappa_1) \circ \eta, \mathcal{CM}(\kappa_2) \circ \mathrm{st}],
$$

where $\mathrm{st} \colon A \times \mathcal{CM}(X) \to \mathcal{CM}(A \times X)$ is the strength operator. Thus:

$$
\lambda(\kappa_1 a) = \{1(\kappa_1 a)\} \quad \text{and} \quad \lambda(\kappa_2 \langle a, U \rangle) = \{\textstyle\sum_x \varphi(x)\kappa_2 \langle a, x \rangle \mid \varphi \in U\}.
$$

As a result there is a "lifting" to a functor $\overline{F} \colon \mathcal{K\ell}(\mathcal{CM}) \to \mathcal{K\ell}(\mathcal{CM})$ given by:

$$
\begin{aligned}
X &\longmapsto FX \\
\left(X \xrightarrow{f} \mathcal{CM}(Y)\right) &\longmapsto \left(FX \xrightarrow{Ff} F\mathcal{CM}(Y) \xrightarrow{\lambda_Y} \mathcal{CM}(FY)\right)
\end{aligned}
$$

More concretely, we have $\overline{F}(f) \colon A + A \times X \to \mathcal{CM}(A + A \times Y)$ given by:

$$
\overline{F}(f)(\kappa_1 a) = \{1(\kappa_1 a)\} \quad \text{and} \quad \overline{F}(f)(\kappa_2 \langle a, x \rangle) = \{\textstyle\sum_y \varphi(y)\kappa_2 \langle a, y \rangle \mid \varphi \in f(x)\}.
$$

It is obvious that $\overline{F}$ is locally monotone, *i.e.* satisfies $f \sqsubseteq g \Rightarrow \overline{F}(f) \sqsubseteq \overline{F}(g)$. In fact, it is also locally continuous.

At this stage we have almost established sufficiently many properties about the monad $\mathcal{CM}$ and the functor F to apply the main result [10, Thm. 3.3] for trace semantics, stating that the initial algebra $F(A^+) \xrightarrow{\cong} A^+$ yields a final coalgebra $A^+ \xrightarrow{\cong} \overline{F}(A^+)$ in the Kleisli category $\mathcal{K\ell}(\mathcal{CM})$. This trace semantics, for a coalgebra $\gamma \colon X \to \mathcal{CM}(F(X))$, is constructed via an ascending sequence of Kleisli maps $\gamma_n \colon X \to \mathcal{CM}(F^n(0))$, for $n \in \mathbb{N}$.

$$
(10) \qquad
\begin{aligned}
\gamma_0 &= \mathbf{0} \colon X \longrightarrow 1 = \mathcal{CM}(0) = \mathcal{CM}(F^0(0)) \\
\gamma_{n+1} &= \overline{F}(\gamma_n) \circ \gamma \colon X \longrightarrow F(X) \longrightarrow F^{n+1}(0) \quad \text{in } \mathcal{K\ell}(\mathcal{CM}).
\end{aligned}
$$

From now on we shall assume that our coalgebra γ satisfies $0 \in \gamma(x)$ for each state x. This can always be enforced by adding 0's, if needed. It means that after each non-deterministic transition the system/coalgebra can choose to do nothing. Adding such 0's does not have influence on the trace behaviour. But adding 0's means that the following two systems become the same.

With the assumption $0 \in \gamma(x)$ we get $\gamma_0(x) \subseteq \gamma_1(x)$, and more generally $\gamma_n \sqsubseteq \gamma_{n+1}$ so that we have an ascending sequence.

The initial algebra A^+ is standardly constructed as colimit of the ω-chain $F^n(0) = A + A^2 + \cdots + A^n = A^{\leq n}$. In order to be precise we shall write the (colimit) co-projections as $\kappa_n \colon F^n(0) \to A^+$. The trace map $\mathrm{tr} \colon X \to \mathcal{CM}(A^+)$ is then defined as directed join in the Kleisli homset:

$$(11) \qquad \mathrm{tr} = \bigvee_n^\uparrow \mathcal{CM}(\kappa_n) \circ \gamma_n = \lambda x \in X. \bigvee_n^\uparrow \mathcal{CM}(\kappa_n)(\gamma_n(x)).$$

The following result says that trace semantics for combined possibilistic and probabilistic systems can be obtained via finality in a Kleisli category.

Theorem 7.1 *This map* $\mathrm{tr} \colon X \to \mathcal{CM}(A^+)$ *forms the unique coalgebra homomorphism to the final coalgebra* A^+ *in the Kleisli category* $\mathcal{Kl}(\mathcal{CM})$, *as in:*

$$
\begin{array}{ccc}
\overline{F}X & \xrightarrow{\ \overline{F}(\mathrm{tr})\ } & \overline{F}(A^+) \\
{\scriptstyle \gamma}\big\uparrow & & \big\uparrow{\scriptstyle \cong} \\
X & \xrightarrow{\ \ \mathrm{tr}\ \ } & A^+
\end{array}
$$

(where we assume $0 \in \gamma(x)$, for all $x \in X$).

Very little in this result actually depends on the particular shape of the transition type functor $F = A + A \times (-)$. But at this stage we are not interested in full generality.

The proof of the trace theorem in [10] proceeds via the Smyth-Plotkin coincidence of limits and colimits [18]. Here it does not work because we do not have bottom elements (but zero elements) in the Kleisli homsets of the monad $\mathcal{CM}$. The proof that is given below—and continued in the appendix—proceeds along the lines of [9].

Proof For clarity let's write $J \colon \mathbf{Sets} \to \mathcal{Kl}(\mathcal{CM})$ for the standard functor, given by $J(X) = X$ and $J(f) = \eta \circ f$ and $\odot$ for composition in the Kleisli category. We need to show that tr is the unique map satisfying $f = J(\alpha) \odot \overline{F}(f) \odot \gamma$, where $\alpha \colon F(A^+) \xrightarrow{\ \cong\ } A^+$ is the initial algebra. By construction as colimit, it satisfies $\alpha \circ F(\kappa_n) = \kappa_{n+1}$.

We first compute:

$$J(\alpha) \odot \overline{F}(\mathcal{CM}(\kappa_n) \circ \gamma_n) \odot \gamma = \mu \circ \mathcal{CM}(\eta \circ \alpha) \circ \overline{F}(\mathcal{CM}(\kappa_n) \circ \gamma_n) \odot \gamma$$
$$= \mathcal{CM}(\alpha) \circ \mu \circ \mathcal{CM}(\lambda \circ F(\mathcal{CM}(\kappa_n) \circ \gamma_n)) \circ \gamma$$
(12)
$$= \mathcal{CM}(\alpha) \circ \mu \circ \mathcal{CM}^2(F\kappa_n) \circ \mathcal{CM}(\lambda \circ F(\gamma_n)) \circ \gamma$$
$$= \mathcal{CM}(\alpha \circ F\kappa_n) \circ \overline{F}(\gamma_n) \odot \gamma$$
$$= \mathcal{CM}(\kappa_{n+1}) \circ \gamma_{n+1}.$$

Thus, using that $\overline{F}$ is locally continuous,

$$\varphi \in \Big(J(\alpha) \odot \overline{F}(\mathrm{tr}) \odot \gamma \Big)(x) \Longleftrightarrow \varphi \in \Big(J(\alpha) \odot \overline{F}(\bigvee_n^{\uparrow} \mathcal{CM}(\kappa_n) \circ \gamma_n) \odot \gamma \Big)(x)$$
$$\Longleftrightarrow \varphi \in \bigcup_n \Big(J(\alpha) \odot \overline{F}(\mathcal{CM}(\kappa_n) \circ \gamma_n) \odot \gamma \Big)(x)$$
$$\Longleftrightarrow \varphi \in \bigcup_n \Big(\mathcal{CM}(\kappa_{n+1}) \circ \gamma_{n+1} \Big)(x)$$
$$\Longleftrightarrow \varphi \in \mathrm{tr}(x) - \{0\}.$$

Since $0 \in \gamma(x)$ and thus $0 \in (J(\alpha) \odot \overline{F}(\mathrm{tr}) \odot \gamma)(x)$ we obtain that the restriction '$-\{0\}$' can be removed from the last line, and thus that the diagram in the theorem commutes.

In order to prove uniqueness, assume we have a coalgebra homomorphism $f \colon X \to \mathcal{CM}(A^+)$. Then $f = J(\alpha) \odot \overline{F}(f) \odot \gamma$. We need to prove $f = \mathrm{tr}$. The direction ($\sqsupseteq$) is easy: since $0 \in \gamma(x)$ we have $0 \in (J(\alpha) \odot \overline{F}(f) \odot \gamma)(x) = f(x)$, so that $\mathcal{CM}(\kappa_0) \circ \gamma_0 \sqsubseteq f$. This forms the basis for induction:

$$\mathcal{CM}(\kappa_{n+1}) \circ \gamma_{n+1} = J(\alpha) \odot \overline{F}(\mathcal{CM}(\kappa_n) \circ \gamma_n) \odot \gamma \quad \text{by (12)}$$
$$\sqsubseteq J(\alpha) \odot \overline{F}(f) \odot \gamma \qquad\qquad \text{by induction}$$
$$= f.$$

Hence $\mathrm{tr} = \bigvee_n^{\uparrow} \mathcal{CM}(\kappa_n) \circ \gamma_n \sqsubseteq f$.

The proof of the reverse direction is non-trivial, and postponed to the appendix.

$\square$

8   Example, revisited

Now that we have a sufficiently strong theoretical basis we shall reconsider the example from Section 2. First of all, the system as pictured in (1) may be described as a coalgebra of the form $\gamma \colon X \to \mathcal{C}(A \times \mathcal{M}(1 + X))$, where $1 = \{\checkmark\}$.

$$\gamma(p) = \overline{\{\langle a, \tfrac{1}{2}\checkmark \rangle, \langle b, \tfrac{1}{2}q + \tfrac{1}{2}r \rangle, \langle c, \tfrac{1}{3}q + \tfrac{2}{3}r \rangle\}} \cup \{\langle \ell, 0 \rangle \mid \ell \in A\}$$
$$\gamma(q) = \overline{\{\langle d, \tfrac{1}{5}q \rangle, \langle e, 1\checkmark \rangle\}} \cup \{\langle \ell, 0 \rangle \mid \ell \in A\}$$
$$\gamma(r) = \{\langle \ell, 0 \rangle \mid \ell \in A\}.$$

This representation closely follows the picture, except for the zero-steps $\{\langle \ell, 0 \rangle \mid \ell \in$

A} which are not written in (1). The convex combination captures non-determinism. For instance, for the semiring $[0, \infty]$, the above set $\gamma(q)$ may be described explicitly as all convex combinations:

$$\alpha\langle d, \tfrac{1}{5}q\rangle + (1 - \alpha)\langle e, 1\checkmark\rangle, \qquad \text{for } \alpha \in [0, 1].$$

The parameter α captures that no choice is made explicitly. Hence non-determinism is represented as an unknown distribution. By combining these "non-deterministic" parameters with the actual "probabilistic" ones iteratively, one obtains traces.

Using the isomorphisms in (8) we can also write the system as a coalgebra $\gamma\colon X \to \mathcal{CM}(F(X)) = \mathcal{CM}(A + A \times X)$. In doing so we shall omit coprojections κ_i and simply write $\ell \in A + A \times X$ for $\kappa_1\ell$ and $\langle \ell, x\rangle \in A + A \times X$ for $\kappa_2\langle \ell, x\rangle$, assuming that no confusion arises. We then have:

$$\gamma(p) = \overline{\{0, \tfrac{1}{2}\langle a, \checkmark\rangle, \tfrac{1}{2}\langle b, q\rangle + \tfrac{1}{2}\langle b, r\rangle, \tfrac{1}{3}\langle c, q\rangle + \tfrac{2}{3}\langle c, r\rangle\}}$$
$$\gamma(q) = \overline{\{0, \tfrac{1}{5}\langle d, q\rangle, 1\langle e, \checkmark\rangle\}}$$
$$\gamma(r) = \{0\}.$$

as described in Section 2. If we elaborate the formula for γ_{n+1} from (10) we get:

$$\gamma_{n+1}(x) =$$
$$\bigcup \left\{ \left\{ \sum_{\ell \in A} \varphi(\langle\ell, \checkmark\rangle)\ell \right\} + \sum_{\ell \in A, y \in X} \left\{ \sum_{\sigma \in A^+} \varphi(\langle\ell, y\rangle)\psi(\sigma)(\ell\sigma) \,\Big|\, \psi \in \gamma_n(y) \right\} \,\bigg|\, \varphi \in \gamma_n(x) \right\}$$

where $\ell \in A^+$ is a singleton sequence and $\ell\sigma \in A^+$ is the sequence $\mathsf{cons}(\ell, \sigma)$.

It is not hard to see that $\gamma_n(r) = \{0\}$ for all $n \in \mathbb{N}$. For $x = p, q$ we have:

$$\gamma_{n+1}(p)$$
$$= \bigcup \left\{ \{0\}, \{\tfrac{1}{2}a\}, \left\{ \sum_{\sigma \in A^+} \tfrac{1}{2}\psi(\sigma)(b\sigma) \,\Big|\, \psi \in \gamma_n(q) \right\} + \left\{ \sum_{\sigma \in A^+} \tfrac{1}{2}\psi(\sigma)(b\sigma) \,\Big|\, \psi \in \gamma_n(r) \right\}, \right.$$
$$\left. \left\{ \sum_{\sigma \in A^+} \tfrac{1}{3}\psi(\sigma)(c\sigma) \,\Big|\, \psi \in \gamma_n(q) \right\} + \left\{ \sum_{\sigma \in A^+} \tfrac{2}{3}\psi(\sigma)(c\sigma) \,\Big|\, \psi \in \gamma_n(r) \right\} \right\}$$
$$= \bigcup \left\{ \{0\}, \{\tfrac{1}{2}a\}, \left\{ \sum_{\sigma \in A^+} \tfrac{1}{2}\psi(\sigma)(b\sigma) \,\Big|\, \psi \in \gamma_n(q) \right\}, \left\{ \sum_{\sigma \in A^+} \tfrac{1}{3}\psi(\sigma)(c\sigma) \,\Big|\, \psi \in \gamma_n(q) \right\} \right\}$$
$$\gamma_{n+1}(q)$$
$$= \bigcup \left\{ \{0\}, \{1e\}, \left\{ \sum_{\sigma \in A^+} \tfrac{1}{5}\psi(\sigma)(d\sigma) \,\Big|\, \psi \in \gamma_n(q) \right\} \right\}$$

These formulas can then be used to calculated traces, as already illustrated in Section 2.

9 Conclusion and further work

Now that the combination of possibilistic and probabilistic computation fits within the coalgebraic framework, many follow-up questions arise. We mention a few.

- What is the appropriate coalgebraic modal logic (see *e.g.* [6] for a recent reference) for the functor $\mathcal{CM}$? One expects modal operators $\Box_r$, for $r \in [0,1] \cap \mathbb{Q}$, acting on a subset $P \subseteq X$ of the state space of a coalgebra $\gamma: X \to \mathcal{CM}(X)$ as:

$$\Box_r(P) = \{x \in X \mid \forall \varphi \in \gamma(x). \ \textstyle\sum_{y \in P} \varphi(y) \geq r\}.$$

- What about simulations [8] in this setting?
- Is this coalgebraic trace semantics really the same as scheduler semantics?
- Is this trace semantics compositional wrt. standard process combinators like parallel composition, see [11] and also [5])?
- Now that the Smyth-Plotkin setting of [10] turns out to be too restrictive for the $\mathcal{CM}$-coalgebras used here—because it assumes bottom elements—the question arises: what is the most general setting for trace semantics?

Acknowledgement

References

[1] S. Abramsky and S.J. Vickers. Quantales, observational logic and process semantics. *Math. Struct. in Comp. Sci.*, 3:161–227, 1993.

[2] M. Barr and Ch. Wells. *Toposes, Triples and Theories*. Springer, Berlin, 1985. Revised and corrected version available from URL: `www.cwru.edu/artsci/math/wells/pub/ttt.html`.

[3] F. Bartels, A. Sokolova, and E. de Vink. A hierarchy of probabilistic system types. *Theor. Comp. Sci.*, 327(1-2):3–22, 2004.

[4] F. Borceux. *Handbook of Categorical Algebra*, volume 50, 51 and 52 of *Encyclopedia of Mathematics*. Cambridge Univ. Press, 1994.

[5] L. Cheung. *Reconciling Nondeterministic and Probabilistic Choices*. PhD thesis, Univ. Nijmegen, 2006.

[6] C. Cïrstea and D. Pattinson. Modular proof systems for coalgebraic logics. *Theor. Comp. Sci.*, 388:83–108, 2007.

[7] J. S. Golan. *Semirings and their Applications*. Kluwer Academic Publishers, 1999.

[8] I. Hasuo. Generic forward and backward simulations. In C. Baier and H. Hermanns, editors, *International Conference on Concurrency Theory (CONCUR 2006)*, number 4137 in Lect. Notes Comp. Sci., pages 406–420. Springer, Berlin, 2006.

[9] I. Hasuo and B. Jacobs. Context-free languages via coalgebraic trace semantics. In J.L. Fiadeiro, N. Harman, M. Roggenbach, and J. Rutten, editors, *Algebra and Coalgebra in Computer Science (CALCO'05)*, number 3629 in Lect. Notes Comp. Sci., pages 213–231. Springer, Berlin, 2005.

[10] I. Hasuo, B. Jacobs, and A. Sokolova. Generic trace theory. *Logical Methods in Comp. Sci.*, 3(4:11), 2007.

[11] I. Hasuo, B. Jacobs, and A. Sokolova. The microcosm principle and concurrency in coalgebra. In *Foundations of Software Science and Computation Structures*, LNCS. Springer, Berlin, 2008.

[12] B. Jacobs. Semantics of weakening and contraction. *Ann. Pure & Appl. Logic*, 69(1):73–106, 1994.

[13] A. Joyal and M. Tierney. An extension of the Galois theory of Grothendieck. *Memoirs of the AMS*, 51(309-4), 1984.

[14] S. Mac Lane. *Categories for the Working Mathematician*. Springer, Berlin, 1971.

[15] S. Lang. *Algebra*. Addison-Wesley, 2nd rev. edition, 1984.

[16] K.I. Rosenthal. *Quantales and their applications*. Number 234 in Pitman Research Notes in Math. Longman Scientific & Technical, 1990.

[17] R. Segala. A compositional trace-based semantics for probabilistic automata. In I. Lee and S.A. Molka, editors, *Concur'95: Concurrency Theory*, number 962 in Lect. Notes Comp. Sci., pages 234–248. Springer, Berlin, 1995.

[18] M.B. Smyth and G.D. Plotkin. The category theoretic solution of recursive domain equations. *SIAM Journ. Comput.*, 11:761–783, 1982.

[19] R. Tix, K. Keimel, and G. Plotkin. *Semantic Domains for Combining Probability and Non-Determinism*. Number 129 in Elect. Notes in Theor. Comp. Sci. Elsevier, Amsterdam, 2005.

[20] D. Varacca. *Probability, Nondeterminism and Concurrency: Two Denotational Models for Probabilistic Computation*. PhD thesis, Univ. Aarhus, 2003. BRICS Dissertation Series, DS-03-14.

[21] D. Varacca and G. Winskel. Distributing probability over non-determinism. *Math. Struct. in Comp. Sci.*, 16:87–113, 2006.

A   Appendix

We shall write an injection between sets as $X \rightarrowtail Y$ and use **Inj** as the subcategory **Inj** $\hookrightarrow$ **Sets** of sets and injective functions between them. This restriction will be used in the next few lemmas.

Lemma A.1 *There is a functor $\mathcal{M}^\circ\colon \mathbf{Inj}^{op} \to SMod_S(\mathbf{Sets})$ which is $\mathcal{M}$ on objects and on a morphism $m\colon X \rightarrowtail Y$ given as follows. For a multiset $\psi \in \mathcal{M}(Y)$,*

$$\mathcal{M}^\circ(m)(\psi) = \psi \circ m\colon X \longrightarrow Y \longrightarrow S.$$

Then: $\mathcal{M}^\circ(m) \circ \mathcal{M}(m) = id$.
   And if $\mathrm{supp}(\psi) \subseteq Im(m)$, then also $\mathcal{M}(m)(\mathcal{M}^\circ(m)(\psi)) = \psi$.

Proof Notice that the support of $\mathcal{M}^\circ(\psi)$ is finite because m is an injection. The mapping $\mathcal{M}^\circ$ is obviously functorial, and $\mathcal{M}^\circ(m)$ preserves the semimodule structure. For $\varphi \in \mathcal{M}(X)$ we have:

$$\begin{aligned}
\mathcal{M}^\circ(m)(\mathcal{M}(m)(\varphi)) &= \lambda x.\, \mathcal{M}(m)(\varphi)(m(x)) \\
&= \lambda x.\, \textstyle\sum_{x' \in m^{-1}(m(x))} \varphi(x') \\
&= \lambda x.\, \varphi(x) \\
&= \varphi.
\end{aligned}$$

Now assume $\text{supp}(\psi) \subseteq Im(m)$. Then:

$$
\begin{aligned}
\mathcal{M}(m)(\mathcal{M}^\circ(m)(\psi))(y) &= \sum_{x \in m^{-1}(y)} \mathcal{M}^\circ(m)(\psi)(x) \\
&= \begin{cases} \psi(m(x)) & \text{if there is a (unique) } x \text{ with } m(x) = y \\ 0 & \text{otherwise} \end{cases} \\
&= \psi(y). \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square
\end{aligned}
$$

Lemma A.2 *There is also a functor* $\mathcal{CM}^\circ \colon \mathbf{Inj}^{op} \to SMod_S(\mathbf{ACL})$ *which is* $\mathcal{CM}$ *on objects and on a morphism* $m \colon X \rightarrowtail Y$ *and multiset* $V \in \mathcal{CM}(Y)$,

$$
\mathcal{CM}^\circ(V) = \{ \mathcal{M}^\circ(m)(\psi) \mid \psi \in V \}.
$$

Then: $\mathcal{CM}^\circ(m) \circ \mathcal{CM}(m) = id$.

Proof We only check the last equation:

$$
\begin{aligned}
\mathcal{CM}^\circ(m)(\mathcal{CM}(m)(U)) &= \{ \mathcal{M}^\circ(m)(\psi) \mid \psi \in \mathcal{CM}(m)(U) \} \\
&= \{ \mathcal{M}^\circ(m)(\mathcal{M}(m)(\varphi)) \mid \varphi \in U \} \\
&= \{ \varphi \mid \varphi \in U \} \qquad \text{by the previous result} \\
&= U. \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square
\end{aligned}
$$

Next we describe how $\mathcal{CM}^\circ$ interacts with the Kleisli category. For clarity we shall (again) write $\odot$ for Kleisli composition, as described in (7).

Lemma A.3 *For an injection* m,

(i) $\mathcal{CM}^\circ(m) \circ (g \odot f) = (\mathcal{CM}^\circ(m) \circ g) \odot f$.

(ii) $\mathcal{CM}^\circ(Fm) \circ \overline{F}(f) = \overline{F}(\mathcal{CM}^\circ(m) \circ f)$.

Proof For the first point we use that $\mathcal{CM}^\circ(m)$ is a map in $SMod_S(\mathbf{ACL})$, in:

$$
\begin{aligned}
\left(\mathcal{CM}^\circ(m) \circ (g \odot f) \right)(x) &= \mathcal{CM}^\circ(m) \left(\bigvee_{\varphi \in f(x)} \sum_{y \in \text{supp}(\varphi)} \varphi(y) \bullet g(y) \right) \\
&= \bigvee_{\varphi \in f(x)} \mathcal{CM}^\circ(m) \left(\sum_{y \in \text{supp}(\varphi)} \varphi(y) \bullet g(y) \right) \\
&= \bigvee_{\varphi \in f(x)} \sum_{y \in \text{supp}(\varphi)} \mathcal{CM}^\circ(m) \left(\varphi(y) \bullet g(y) \right) \\
&= \bigvee_{\varphi \in f(x)} \sum_{y \in \text{supp}(\varphi)} \varphi(y) \bullet \mathcal{CM}^\circ(m)(g(y)) \\
&= \bigvee_{\varphi \in f(x)} \sum_{y \in \text{supp}(\varphi)} \varphi(y) \bullet (\mathcal{CM}^\circ(m) \circ g)(y) \\
&= (\mathcal{CM}^\circ(m) \circ g) \odot f.
\end{aligned}
$$

For the second point we calculate:

$$
\begin{aligned}
\mathcal{CM}^\circ(Fm) \circ \overline{F}(f) &= \mathcal{CM}^\circ(Fm) \circ \lambda \circ F(f) \qquad \text{with } \lambda \text{ from (9)} \\
&\overset{(*)}{=} \lambda \circ F(\mathcal{CM}^\circ(m)) \circ F(f) \\
&= \overline{F}(\mathcal{CM}^\circ(m) \circ f)
\end{aligned}
$$

For the marked equation $\stackrel{(*)}{=}$ we have to check that the distributive law $\lambda\colon F\mathcal{CM} \Rightarrow \mathcal{CM}F$ from (9) is also a natural transformation $\lambda\colon F\mathcal{CM}^\circ \Rightarrow \mathcal{CM}^\circ F$, *i.e.* that for $m\colon X \rightarrowtail Y$ one has:

$$\lambda_X \circ (\mathrm{id} + \mathrm{id} \times \mathcal{CM}^\circ(m)) = \mathcal{CM}^\circ(\mathrm{id} + \mathrm{id} \times m) \circ \lambda_Y.$$

This follows by an easy calculation. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Now we can fill in the missing step in the proof of Theorem 7.1, namely to show that $f \sqsubseteq \mathrm{tr}$ for a coalgebra homomorphism $f\colon X \to \mathcal{CM}(A^+)$.

Assume therefore $\varphi \in f(x)$, where $\varphi \in \mathcal{M}(A^+)$ is a finite multiset of sequences. By finiteness there is an $n \in \mathbb{N}$ such that φ is a multiset over sequences of length at most n, *i.e.* $\varphi \in \mathcal{M}(A^{\leq n}) = \mathcal{M}(F^n 0)$.

More precisely, we have found an $n \in \mathbb{N}$ such that $\mathrm{supp}(\varphi) \subseteq Im(\kappa_n)$, so that we have $\varphi = \mathcal{M}(\kappa_n)(\psi)$ where $\psi = \mathcal{M}^\circ(\kappa_n)(\varphi) \in \mathcal{CM}^\circ(\kappa_n)(f(x))$ by Proposition A.1. Now it suffices to prove:

$$\text{(A.1)} \qquad\qquad \mathcal{CM}^\circ(\kappa_n) \circ f = \gamma_n\colon X \longrightarrow \mathcal{CM}(F^n(0))$$

because then we are done: we have $\psi \in \mathcal{CM}^\circ(\kappa_n)(f(x)) = \gamma_n(x)$ and thus $\varphi = \mathcal{M}(\kappa_n)(\psi) \in \mathcal{CM}(\kappa_n)(\gamma_n(x)) \subseteq \mathrm{tr}(x)$.

We prove (A.1) by induction. The case $n = 0$ is easy because both sides are maps to the terminal object $\mathcal{CM}(F^0(0)) = \mathcal{CM}(0) = 1$. The induction step goes much like earlier in the proof, but this time with $\mathcal{CM}^\circ$ instead of $\mathcal{CM}$, and using Lemma A.3.

$$
\begin{aligned}
\mathcal{CM}^\circ(\kappa_{n+1}) \circ f \;&=\; \mathcal{CM}^\circ(\kappa_n) \circ J(\alpha) \odot \overline{F}(f) \odot \gamma \\
&=\; \mathcal{CM}^\circ(\kappa_{n+1}) \circ \mathcal{CM}(\alpha) \circ \overline{F}(f) \odot \gamma \\
&=\; \mathcal{CM}^\circ(\kappa_{n+1}) \circ \mathcal{CM}^\circ(\alpha^{-1}) \circ \overline{F}(f) \odot \gamma \\
&=\; \mathcal{CM}^\circ(\alpha^{-1} \circ \kappa_{n+1}) \circ \overline{F}(f) \odot \gamma \\
&=\; \mathcal{CM}^\circ(F\kappa_n) \circ \overline{F}(f) \odot \gamma \\
&=\; (\mathcal{CM}^\circ(F\kappa_n) \circ \overline{F}(f)) \odot \gamma \\
&=\; \overline{F}(\mathcal{CM}^\circ(\kappa_n) \circ f) \odot \gamma \\
&\stackrel{\text{(IH)}}{=}\; \overline{F}(\gamma_n) \odot \gamma \\
&=\; \gamma_{n+1}.
\end{aligned}
$$