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On Approximating Grey Model DGM(2,1)

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Abstract

Based on the solution structure of white differential equation of DGM(2,1) model, we deduce the new 2-order grey derivative expression. In the specific application, we use new 2-order grey derivative expression instead of old expression to estimate parameters. Through accumulating example, we can see that the optimized DGM(2,1) model has higher simulation precision obviously.

Keywords: DGM(2,1); Optimization; 2-order grey derivative

1. Introduction

The grey system theory which founded by professor J.L. Deng is a kind of theory which analyze the system on uncertainty in less data, incomplete information and devoid of experience[1]. As the elementary model, GM(1,1) model has applied extensively in the past twenty seven years, and obtain some satisfactory effect. But in the process of application, many scholars also discover the defects of GM(1,1) model. In order to make up the defect of GM(1,1) model, let grey model has more extensive application., scholars have done a lot of jobs. These jobs can be divided into two sorts generally, one sort is to optimize GM(1,1) model straightly by changing background value, grey derivative, the accumulation method of parameters, the optimized GM(1,1) model can be applied extensive[4,5,6,7]. Another sort is to construct some new grey models, such as GM(2,1)

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model, DGM(2,1) model, Verhulst model, and so on. DGM(2,1) model is a kind of new grey model which is constructed by grey derivative and second-order grey derivative[2], [3]. DGM(2,1) model can make up some defect of GM(1,1) model, but it still has its own defects. In this article, we deduce the new 2-order grey derivative expression based on the solution structure of white differential equation, and estimate the initial value in time responsive according to the principle of minimum of error sum of squares. so we construct optimized GM(2,1) model by optimizing 2-order grey derivative in order to make white differential equation and grey differential equation be more consistent. Through accumulating example, we see that the optimized DGM(2,1) model has higher simulation precision obviously.

2. Basic definitions and properties

Definition 1[2]:

$$\alpha^{(1)}x^{(0)}(k)+ax^{(0)}(k)=b \tag{1}$$

is called direct DGM(2,1) model.

$$\frac{d^2x^{(1)}}{dt^2}+a\frac{dx^{(1)}}{dt}=b \tag{2}$$

is called white differential equation of DGM(2,1) model.

Theorem 1[2]: If $B=\begin{bmatrix}-x^{(0)}(2) & 1\\ -x^{(0)}(3) & 1\\ \cdots & \cdots\\ -x^{(0)}(n) & 1\end{bmatrix}, Y=\begin{bmatrix}\alpha^{(1)}(2)\\ \alpha^{(1)}(3)\\ \cdots\\ \alpha^{(1)}(n)\end{bmatrix}=\begin{bmatrix}x^{(0)}(2)-x^{(0)}(1)\\ x^{(0)}(3)-x^{(0)}(2)\\ \cdots\\ x^{(0)}(n)-x^{(0)}(n-1)\end{bmatrix}$, then

$$\hat{a}=[a,b]^T=\left(B^TB\right)^{-1}B^TY. \tag{3}$$

Theorem 2[2]: The time response sequence of white differential equation is

$$\hat{x}^{(1)}(k+1)=\left(\frac{b}{a^2}-\frac{x^{(0)}(1)}{a}\right)e^{-ak}+\frac{b}{a}(k+1)+\left(x^{(0)}(1)-\frac{b}{a}\right)\frac{1+a}{a} \tag{4}$$

$$\hat{x}^{(0)}(k+1)=\hat{x}^{(1)}(k+1)-\hat{x}^{(1)}(k)\quad=\left(\frac{b}{a^2}-\frac{x^{(0)}(1)}{a}\right)\left(1-e^{a}\right)e^{-ak}+\frac{b}{a} \tag{5}$$

3. Optimization of DGM(2,1)

Paragraph [8] shows that the connotation expression of DGM(2,1) model is non-homogeneous exponential function, and we can see from theorem 2 mentioned above that the restored value deduced from the solution of white differential equation is also non-homogeneous exponential type. So we assume

$$x^{(1)}(t)=B_1e^{A(t-1)}+B_2t+B_3 \tag{6}$$

then

$$x^{(0)}(k+1)=B_1\left(1-e^{-A}\right)e^{Ak}+B_2. \tag{7}$$

In order to improve the simulation and prediction precision, we should make white differential equation and grey differential equation be more matching, so we should get a new expression of $\beta^{(1)}x^{(0)}(k)$ instead of

$$\alpha^{(1)}x^{(0)}(k)=x^{(0)}(k)-x^{(0)}(k-1) \tag{8}$$

as 2-order grey derivative.

Let $x^{(1)}(t)=B_1e^{A(t-1)}+B_2t+B_3$ (9)

Then

$$\frac{dx^{(1)}}{dt}=AB_1e^{A(t-1)}+B_2 \tag{10}$$

$$\Rightarrow \frac{d^2x^{(1)}}{dt^2}=A^2B_1e^{A(t-1)}=A\frac{dx^{(1)}}{dt}-AB_2 \tag{11}$$

$$\Rightarrow x^{(1)}(k-i)=B_1e^{A[k-(i+1)]}+B_2(k-i)+B_3 \qquad (i=0,1,2,3) \tag{12}$$

$$\Rightarrow x^{(0)}(k-i)=B_1e^{A[k-(i+2)]}\left(e^A-1\right)+B_2 \qquad (i=0,1,2) \tag{13}$$

$$\Rightarrow x^{(0)}(k)-x^{(0)}(k-1)=B_1e^{A(k-3)}\left(e^A-1\right)^2 \tag{14}$$

$$\Rightarrow x^{(0)}(k-1)-x^{(0)}(k-2)=B_1e^{A(k-4)}\left(e^A-1\right)^2 \tag{15}$$

$$\Rightarrow \frac{x^{(0)}(k)-x^{(0)}(k-1)}{x^{(0)}(k-1)-x^{(0)}(k-2)}=e^A \tag{16}$$

$$\Rightarrow A=\ln\frac{x^{(0)}(k)-x^{(0)}(k-1)}{x^{(0)}(k-1)-x^{(0)}(k-2)} \tag{17}$$

take（17）into（14）,then

$$B_1=\frac{\left(x^{(0)}(k)-x^{(0)}(k-1)\right)\left(x^{(0)}(k-1)-x^{(0)}(k-2)\right)^{k-1}}{\left(x^{(0)}(k)-2x^{(0)}(k-1)+x^{(0)}(k-2)\right)^2\left(x^{(0)}(k)-x^{(0)}(k-1)\right)^{k-3}} \tag{18}$$

take（17）、（18）into（13）, then

$$B_2=\frac{x^{(0)}(k-2)x^{(0)}(k)-\left[x^{(0)}(k-1)\right]^2}{x^{(0)}(k)-2x^{(0)}(k-1)+x^{(0)}(k-2)} \tag{19}$$

When $\left[x^{(1)}(t)\right]'=x^{(0)}(k)$, the new 2-order grey derivative expression is

$$\beta^{(1)}x^{(0)}(k)=\frac{d^2x^{(1)}}{dt^2}=Ax^{(0)}(k)-AB_2$$

$$=\frac{\left[x^{(0)}(k)-x^{(0)}(k-1)\right]^2}{x^{(0)}(k)-2x^{(0)}(k-1)+x^{(0)}(k-2)}\ln\frac{x^{(0)}(k)-x^{(0)}(k-1)}{x^{(0)}(k-1)-x^{(0)}(k-2)} \tag{20}$$

Theorem 3: Let original data series be $X^{(0)}=\left(x^{(0)}(1),x^{(0)}(2),\cdots,x^{(0)}(n)\right)$, 1-AGO sequence $X^{(1)}$ be $X^{(1)}=\left(x^{(1)}(1),x^{(1)}(2),\cdots,x^{(1)}(n)\right)$, if $\hat{a}=[a,b]^T$ is parameter series,

$$\beta^{(1)}x^{(0)}(k)=\frac{\left[x^{(0)}(k)-x^{(0)}(k-1)\right]^2}{x^{(0)}(k)-2x^{(0)}(k-1)+x^{(0)}(k-2)}\ln\frac{x^{(0)}(k)-x^{(0)}(k-1)}{x^{(0)}(k-1)-x^{(0)}(k-2)},$$

$$B=\begin{bmatrix}-x^{(0)}(3) & 1\\ -x^{(0)}(4) & 1\\ \cdots & \cdots\\ -x^{(0)}(n) & 1\end{bmatrix},\quad Y=\begin{bmatrix}\beta^{(1)}x^{(0)}(3)\\ \beta^{(1)}x^{(0)}(4)\\ \cdots\\ \beta^{(1)}x^{(0)}(n)\end{bmatrix}$$, then $DGM(2,1)$ model

$$\beta^{(1)}x^{(0)}(k)+ax^{(0)}(k)=b \tag{21}$$

meet

$$\hat{a}=[a,b]^T=\left(B^TB\right)^{-1}B^TY\,. \tag{22}$$

Theorem 4: Let $D=\dfrac{x^{(0)}(k)-x^{(0)}(k-1)}{x^{(0)}(k-1)-x^{(0)}(k-2)}$, when $D\to 1$, $\beta^{(1)}x^{(0)}(k)\to x^{(0)}(k)-x^{(0)}(k-1)$.

Proof: According to theorem 3,

$$\begin{aligned}\beta^{(1)}x^{(0)}(k)&=\frac{\left[x^{(0)}(k)-x^{(0)}(k-1)\right]^2}{x^{(0)}(k)-2x^{(0)}(k-1)+x^{(0)}(k-2)}\ln\frac{x^{(0)}(k)-x^{(0)}(k-1)}{x^{(0)}(k-1)-x^{(0)}(k-2)}\\&=\frac{\left[x^{(0)}(k)-x^{(0)}(k-1)\right]}{\left[x^{(0)}(k)-x^{(0)}(k-1)\right]-\left[x^{(0)}(k-1)-x^{(0)}(k-2)\right]}\ln\frac{x^{(0)}(k)-x^{(0)}(k-1)}{x^{(0)}(k-1)-x^{(0)}(k-2)}\\&=\frac{\left[x^{(0)}(k)-x^{(0)}(k-1)\right]}{1-\dfrac{x^{(0)}(k-1)-x^{(0)}(k-2)}{x^{(0)}(k)-x^{(0)}(k-1)}}\ln\frac{x^{(0)}(k)-x^{(0)}(k-1)}{x^{(0)}(k-1)-x^{(0)}(k-2)}\end{aligned} \tag{23}$$

then

$$\beta^{(1)}x^{(0)}(k)=\frac{\left[x^{(0)}(k)-x^{(0)}(k-1)\right]}{1-\dfrac{1}{D}}\ln D \tag{24}$$

So $\lim\limits_{D\to 1}\beta^{(1)}x^{(0)}(k)=\left[x^{(0)}(k)-x^{(0)}(k-1)\right]\lim\limits_{D\to 1}\dfrac{\ln D}{1-\dfrac{1}{D}}=\left[x^{(0)}(k)-x^{(0)}(k-1)\right]\lim\limits_{D\to 1}\dfrac{\dfrac{1}{D}}{\dfrac{1}{D^2}}$

$$=\left[x^{(0)}(k)-x^{(0)}(k-1)\right]\lim_{D\to 1}D=x^{(0)}(k)-x^{(0)}(k-1) \tag{25}$$

especially,

when $x^{(0)}(k)-x^{(0)}(k-1)=x^{(0)}(k-1)-x^{(0)}(k-2)$, $\beta^{(1)}x^{(0)}(k)=x^{(0)}(k)-x^{(0)}(k-1)$

Theorem 5: The time response sequence of white differential equation of $DGM(2,1)$ model is,

$$\hat{x}^{(1)}(t)=\left(\frac{b}{a^2}-\frac{\gamma}{a}\right)e^{-a(t-1)}+\frac{b}{a}t+\left(\gamma-\frac{b}{a}\right)\frac{1+a}{a} \tag{26}$$

The time response sequence of $DGM(2,1)$ model is,

$$\hat{x}^{(1)}(k+1)=\left(\frac{b}{a^2}-\frac{\gamma}{a}\right)e^{-ak}+\frac{b}{a}(k+1)+\left(\gamma-\frac{b}{a}\right)\frac{1+a}{a} \tag{27}$$

$$\gamma=\frac{\sum_{i=1}^{k}\left[\frac{b}{a^2}(1-e^a)e^{-2a(i-1)}+\frac{b}{a}e^{-a(i-1)}-x^{(0)}(i)e^{-a(i-1)}\right]}{\sum_{i=1}^{k}\frac{1-e^a}{a}e^{-2a(i-1)}} \tag{28}$$

Proof：According to (27) we can get $\hat{x}^{(0)}(k)=\hat{x}^{(1)}(k)-\hat{x}^{(1)}(k-1)=\left(\frac{b}{a^2}-\frac{\gamma}{a}\right)(1-e^a)e^{-a(k-1)}+\frac{b}{a}$

Let $f(\gamma)=\sum_{i=1}^{k}\left[x^{(0)}(i)-\hat{x}^{(0)}(i)\right]^2=\sum_{i=1}^{k}\left[x^{(0)}(i)-\left(\frac{b}{a^2}-\frac{\gamma}{a}\right)(1-e^a)e^{-a(i-1)}-\frac{b}{a}\right]^2$

Then $f'(\gamma)$

$$=2\frac{1-e^a}{a}\left[\sum_{i=1}^{k}x^{(0)}(i)e^{-a(i-1)}-\frac{b}{a^2}(1-e^a)\sum_{i=1}^{k}e^{-2a(i-1)}+\gamma\frac{1-e^a}{a}\sum_{i=1}^{k}e^{-2a(i-1)}-\frac{b}{a}\sum_{i=1}^{k}e^{-a(i-1)}\right]$$

Let $f'(\gamma)=0$，then

$$\gamma=\frac{\sum_{i=1}^{k}\left[\frac{b}{a^2}(1-e^a)e^{-2a(i-1)}+\frac{b}{a}e^{-a(i-1)}-x^{(0)}(i)e^{-a(i-1)}\right]}{\sum_{i=1}^{k}\frac{1-e^a}{a}e^{-2a(i-1)}}$$

According to the theorems mentioned above, we can generalize the modelling steps of the optimization of DGM(2,1)

1） Accumulating the series of parameters $\hat{a}=[a,b]^T$ according to theorem 3.

2） Accumulating γ in the time response sequence according to solution (28).

3） Taking the solution of step 2 into (27) , then accumulate $\hat{x}^{(1)}(k+1)$.
Using $\hat{x}^{(0)}(k+1)=\hat{x}^{(1)}(k+1)-\hat{x}^{(1)}(k)$ to get restored value.

4. Example

Table 1 The electricity consumption of Shanxi Province from 1979 to 1984 （10^9 kw/h）

The electricity consumption of Shanxi Province from 1979 to 1984

Year	1979	1980	1981	1982	1983	1984
Consumption of electricity	1.11	1.19	1.27	1.36	1.46	1.58

$x^{(0)}=(1.11,1.19,1.27,1.36,1.46,1.58)$，establish $GM(1,1)$ model（model 1），original $DGM(2,1)$ model（model 2）,optimized $DGM(2,1)$ model（model 3）,the time response sequence is

Model 1: $\hat{x}^{(1)}(k) = 16.0667e^{0.07094(k-1)} - 14.966$

Model 2: $\hat{x}^{(1)}(k) = 6.0198e^{0.1052(k-1)} + 0.4781k - 5.3879$

Model 3: $\hat{x}^{(1)}(k) = 2.4304e^{0.1601(k-1)} + 0.7739k - 2.0403$

Table 2 Comparison of the simulation precision

Number	Model 1		Model 2		Model 3	
	Simulation value	Relative error (%)	Simulation value	Relative error (%)	Simulation value	Relative error (%)
2	1.1718	-1.5294	1.1459	-3.7059	1.1959	0.4957
3	1.2681	-0.1496	1.22	-3.937	1.2691	0.0708
4	1.3612	0.0882	1.3022	-4.25	1.3552	0.352
5	1.4613	0.0891	1.3937	-4.5411	1.4561	0.267
6	1.5687	-0.7152	1.4953	-5.3607	1.5745	0.348
Average relative error (%)	0.5143		4.3589		0.3067	

From table 2, we can see that the optimized DGM(2,1) model(Model 3) has higher simulation precision obviously.

5. Conclusion

This article deduce the new 2-order grey derivative expression based on the solution structure of white differential equation of DGM(2,1) model, and proof that the old expression of 2-order grey derivative is the special situation of new 2-order grey derivative expression, so the new expression is more suitable. Through accumulating example, we can see that the optimized DGM(2,1) model has higher simulation precision obviously.

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