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A Characterization of Constructive Dimension

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Abstract

In the context of Kolmogorov's algorithmic approach to the foundations of probability, Martin-Löf defined the concept of an individual random sequence using the concept of a constructive measure 1 set. Alternate characterizations use constructive martingales and measures of impossibility. We prove a direct conversion of a constructive martingale into a measure of impossibility and vice versa, such that their success sets, for a suitably defined class of computable probability measures, are equal. The direct conversion is then generalized to give a new characterization of constructive dimensions, in particular, the constructive Hausdorff dimension and the constructive packing dimension.

Keywords: Algorithmic Randomness, Measures of Impossibility, s-Gale, Martingale, Constructive Dimension

1 Introduction

One of the prime successes of the algorithmic approach to the foundations of probability theory envisioned by Kolmogorov is Martin-Löf's definition of an individual random sequence [11] using a constructive measure theory. The measure-theoretic approach to the definition of random sequences identifies a property of "typical" sets. A random sequence is one that belongs to every reasonable majority of sequences [4]. The notion of a reasonable majority is formulated as an effective version of measure 1. Each measure 1 set has a complement set of measure 0. It is hence sufficient to define the concept of the effective measure zero set.

Let P be a computable probability measure defined on the Cantor Space (defined in section 3). For finite strings x, we consider cylinders C_x, the set of all infinite sequences with x as a prefix. A set S of sequences from the sample space of all

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sequences has P-measure zero if, for each $\varepsilon > 0$, there is a sequence of cylinders $C_{x_0}, C_{x_1}, \ldots, C_{x_i}, \ldots$ of cylinder sets such that

$$S \subseteq \cup_i C_{x_i} \text{ and } P(\cup_i C_{x_i}) < \varepsilon.$$

A set of sequences S has effective P-measure zero if there is a computable function $h(i, \varepsilon)$ such that $h(i, \varepsilon) = C_{x_i}$ for each i.

Martin-Löf proved a universality property – that for any computable measure P, there is a unique largest effective P-measure zero set. The elements in the complement of this set are the set of P-random sequences.

Another tool in the study of effective randomness is the concept of martingales. Introduced by Ville in the 1930s [17] (being implicit in the early work of Lévy [5], [6]), they were applied in theoretical computer science by Schnorr in the early 1970s [12], [13], [14] in his investigation of Martin-Löf randomness, and by Lutz [7], [8], [9] in the development of resource-bounded measure. A martingale is a betting strategy, which, for a probability measure P defined on the Cantor Space, obeys the conditions,

$$\begin{aligned} d(\lambda) &\leq 1 \\ d(w)P(w) &= d(w0)P(w0) + d(w1)P(w1). \end{aligned} \tag{1}$$

Intuitively, it can be seen as betting strategy on an infinite sequence, where, for each prefix w of the infinite sequence, the amount $d(w)$ is the capital that is in hand after betting. A martingale can be seen as a fair betting condition where the expected value after every bet is the same as the expected value before the bet is made. It is said to succeed on a sequence ω if

$$\limsup_{n\to\infty} d(\omega[0 \ldots n-1]) = \infty.$$

The success set of a martingale, $S^\infty[d]$, is the set of all individual sequences on which it succeeds. In this work, we consider constructive martingales. There is a universal martingale d' which is constructive and for every ω, if there is a constructive martingale d which succeeds on ω, then d' succeeds on ω. The theory of martingales and their applications to the field of resource-bounded measure, complexity theory, and resource-bounded dimension has proved to be remarkably fruitful. In this work, we wish to establish connections between martingales and a third approach of defining randomness, *viz.*, that of a measure of impossibility.

This third approach to define a random sequence is to characterize a degree of disagreement between any sequence ω and the probability P. Following [18], a *measure of impossibility* is a function $p(\omega)$ which describes the quantitative level to which ω is impossible with respect to the probability measure P. A measure of impossibility is defined to be a lower semicomputable function p, which is integrable with respect to P. It can be seen that if ω is P-random, then $p(\omega) < \infty$. There is an optimal measure of impossibility $\tilde{p}$ such that a sequence ω is random if and only if $\tilde{p}(\omega) = \infty$ [18], [19]. This concept is a central one in V'yugin's proof of an effective version of the Ergodic Theorem [19].

The relation between martingales and Martin-Löf's definition of randomness was studied by Schnorr [12], [13], [14]. The proof that the notion of randomness defined by Martin-Löf corresponds to that of the ones defined via the measures of impossibility is due to Vovk and V'yugin [18] and V'yugin [19]. We establish a direct correspondence between the notions of constructive martingales and measures of impossibility.

We then apply this construction to come up with an analogous new definition of constructive dimension in terms of a generalized version of the notion of a measure of impossibility.

The main difference between a proof based on martingales and one using a measure of impossibility is that a martingale is defined on the basis cylinders, and a measure of impossibility is a pointwise notion. Measure of impossibility seems to be an easier tool in dealing with theorems in which we have to reason about the convergence of general random variables defined on the points in the sample space. However, we show that at the constructive level, these tools are equivalent. Since there are universal objects available in both the cases, there exists an indirect conversion between the two such that the success set of a martingale can be converted into that of a measure of impossibility and conversely; this work contributes a direct constructive conversion of one into another. The theory of algorithmic randomness has been remarkably fruitful to date. (For a survey of the field, see [2].) Martingales have proved to have greater apparent utility in some cases than Martin-Löf tests in studying randomness, and measures of impossibility have been of use in establishing a remarkable result in the study of algorithmic randomness. We hope that the explicit transformation of this work will improve the understanding, and perhaps the utility of measure of impossibility. Moreover, in the absence of universal objects which happens at computable and other levels a conversion of this nature may be useful.

2 Preliminaries

We consider the binary alphabet $\Sigma = \{0, 1\}$. The empty string is denoted by λ. The set of finite strings from the alphabet is denoted as Σ^* and the set of infinite sequences, as $\mathbf{C}$, the Cantor Space. For finite strings y, x, and infinite sequences ω, we denote x to be a prefix of y or of ω as $x \sqsubseteq y$, or as $x \sqsubseteq \omega$, respectively. We adopt the convention, for all sequences and strings x, for all $0 \leq n \leq |x|$, the n-length prefix of x is denoted $x[0 \ldots n-1]$ (this is always finite), and for $n > |x|$, we have $x[0 \ldots n-1] = x$ by notation.

Following the common notation, $\mathbb{N}$ represents the set of natural numbers, $\mathbb{Q}$ the set of rational numbers and $\mathbb{Z}$ the set of integers. Denote $\overline{\mathbb{R}}$ for $\mathbb{R} \cup \{-\infty, \infty\}$, $\mathbb{R}^+$ for the non-negative reals, and $\overline{\mathbb{R}}^+$ for $\mathbb{R}^+ \cup \{\infty\}$.

We define the notion of lower semicomputability for the natural topology on the product space $\mathbf{C} \times \mathbb{Q}$ or $\Sigma^* \times \mathbb{Q}$. The natural topology on $\mathbb{Q}$ or Σ^* is discrete (i.e., the topology made of the set of all subsets of $\mathbb{Q}$ or of Σ^*). The natural topology on $\mathbf{C}$ is generated by the cylinders $C_x = \{\omega \mid x \sqsubseteq \omega\}$, where $x \in \Sigma^*$. A function

$f : \Sigma^* \cup \mathbf{C} \to \overline{\mathbb{R}}$ is called lower semicomputable if its *graph* $S = \{(\omega, q) \mid \omega \in \Sigma^* \cup \mathbf{C}$ and $q \in \mathbb{Q}, q < f(\omega)\}$ is a union of a computably enumerable sequence of intervals in the natural topology on $\mathbb{Q} \times \Sigma^*$. The function f is lower semicomputable, if for any rational number q and any finite string w, the assertion $q < f(\omega)$ is true can be verified in a computable manner.

We prove an equivalent notion of lower semicomputability:

Lemma 2.1 *The following hold.*

i. A function $f : \mathbf{C} \to \overline{\mathbb{R}}$ is lower semicomputable if and only if there exists a computable function $\hat{f} : \Sigma^* \times 0^{\mathbb{N}} \to \mathbb{Q} \cup \{-\infty, \infty\}$ such that the following hold: For all $\omega \in \mathbf{C}$,

(a) *Monotonicity: For all $m, n \in \mathbb{N}$, $\hat{f}(\omega[0 \ldots m-1], 0^n) \leq \hat{f}(\omega[0 \ldots m-1], 0^{n+1}) \leq f(\omega)$, and $\hat{f}(\omega[0 \ldots m-1], 0^n) \leq \hat{f}(\omega[0 \ldots m], 0^n) \leq f(\omega)$.*

(b) *Convergence: We have*

$$\lim_{n \to \infty} \hat{f}(\omega[0 \ldots n-1], 0^n) = f(\omega).$$

ii. $f : \Sigma^* \to \overline{\mathbb{R}}$ is lower semicomputable if and only if there exists a function $\hat{f} : \Sigma^* \times 0^{\mathbb{N}} \to \mathbb{Q} \cup \{-\infty, \infty\}$ such that the following hold: For all $x \in \Sigma^*$,

a'. Monotonicity: For all $n \in \mathbb{N}$, $\hat{f}(x, 0^n) \leq \hat{f}(x, 0^{n+1}) \leq f(x)$.

b'. Convergence: $\lim_{n \to \infty} \hat{f}(x, 0^n) = f(x)$.

Proof. The characterization for the case when f is defined on the domain Σ^* is standard in the literature (see [10]), and we prove the formulation for the case when the domain is $\mathbf{C}$.

For the case when $f : \mathbf{C} \to \overline{\mathbb{R}}$ is lower semicomputable, first assume that the set $S = \{(\omega, q) \mid \omega \in \mathbf{C}, q \in \mathbb{Q}, f(\omega) > q\}$ is the union of a computable enumeration $S : 0^{\mathbb{N}} \to \Sigma^* \times (\mathbb{Q} \cup \{-\infty\})$ of cylinders in the natural topology on $\mathbf{C} \times \mathbb{Q}$. The projection functions $\pi_1 : \Sigma^* \times \mathbb{Q} \to \Sigma^*$ and $\pi_2 : \Sigma^* \times \mathbb{Q} \to \mathbb{Q}$ are defined as $\pi_1(w, q) = w$ and $\pi_2(w, q) = q$. We design a witness function $\hat{f} : \Sigma^* \times 0^{\mathbb{N}} \to \mathbb{Q} \cup \{-\infty\}$ in the following algorithm.

Algorithm 1 *procedure $\hat{f}(w, 0^n)$*

Set $\leftarrow \{-\infty\}$.
$i \leftarrow 0$.
while $(i \leq n)$
  if $(\pi_1(S(i)) \sqsubseteq w)$
    Set $\leftarrow$ *Set* $\cup\ \pi_2(S(i))$
  end if
  $i \leftarrow i + 1$.
end while
return max(*Set*)
end procedure

The monotonicity condition is satisfied, because in the algorithm, for every n, the sets have the following relationships.

$$\{\pi_2(S(i)) \mid \pi_1(S(i)) \sqsubseteq w, 0 \le i \le n\} \subseteq \{\pi_2(S(i)) \mid \pi_1(S(i)) \sqsubseteq w, 0 \le i \le n+1\},$$

and, for strings w' and w, if $w' \sqsubseteq w$, then

$$\{\pi_2(S(i)) \mid \pi_1(S(i)) \sqsubseteq w', 0 \le i \le n\} \subseteq \{\pi_2(S(i)) \mid \pi_1(S(i)) \sqsubseteq w, 0 \le i \le n\}.$$

For the convergence, it is obvious that

$$\lim_{n\to\infty} f(\omega[0 \dots n-1], 0^n) \tag{2}$$

exists, since it is a monotone bounded sequence in a compact space. To see that the limit is $f(\omega)$, assume that the limit (2) is $r < f(\omega)$. Then there exists an r' such that $r < r' < f(\omega)$ such that no pair (w, r') where w is some prefix of ω occurs in the enumeration of S. This is a contradiction. Hence the condition is satisfied, and limit (2) is $f(\omega)$.

Conversely, let $f : \mathbf{C} \to \overline{\mathbb{R}}$ be a lower semicomputable function with witness $\hat{f} : \Sigma^* \times 0^{\mathbb{N}} \to \mathbb{Q}$ satisfying lower semicomputability conditions. We prove that the set $S = \{(\omega, r) : r \in \mathbb{Q}, r < f(\omega)\}$ is the union of a computable enumeration of cylinders in $\mathbf{C} \times \mathbb{Q}$.

We show that for every $r \in \mathbb{Q}$, $r < f(\omega)$, there is a prefix w of ω, such that (w, r) is accepted by an algorithm. This is routine to see, since, we can dovetail the execution of $\hat{f}$ on $\Sigma^* \times 0^{\mathbb{N}}$. If $r < f(\omega)$, there is an $r' > r$ such that $\hat{f}$ produces r' on some prefix w and some 0^m, and then we can accept (w, r). □

A function f is called upper semicomputable if $-f$ is lower semicomputable. A function f is called computable if it is both lower and upper semicomputable. This may be seen to be equivalent to the following definition in the case of functions defined over Σ^*.

Definition 2.2 A function $f : \Sigma^* \to \mathbb{R}$ is said to be computable if there exists a function $\hat{f} : \Sigma^* \times 0^{\mathbb{N}} \to \mathbb{Q}$ such that for every $n \in \mathbb{N}$ and every $x \in \Sigma^*$,

$$|\hat{f}(x, 0^n) - f(x)| \le 2^{-n}.$$

Note 1 *It is easy to show that if $\hat{f}$ is a witness function to the computability of f, then for all n, $\hat{f}(x, 0^n) - 2.2^{-n}$ is a lower semicomputation, and $\hat{f}(x, 0^n) + 2.2^{-n}$ is an upper semi-computation of f.*

3 Effective Randomness

Let $(\Omega, \mathcal{F}, P)$ be the probability space, where Ω is the sample space, $\mathcal{F}$ is the Borel σ-algebra (members of $\mathcal{F}$ are the *events*), and $P : \mathcal{F} \to [0, 1]$ is the probability. We will be concerned with the sample space $\Omega = \mathbf{C}$, the set of all infinite binary sequences. $\mathcal{F}$ is the σ-algebra generated by the cylinders $C_x = \{\omega \mid \omega \in \mathbf{C}, x \sqsubseteq \omega\}$.

Definition 3.1 A probability measure P defined on the Cantor Space of infinite binary sequences is characterized by the following:

(i) $P(\lambda) = 1$.

(ii) For every string w, $0 \leq P(w) = P(w0) + P(w1)$.

A probability measure $P : 2^{\Omega} \to [0,1]$ is called computable if the probability measure $P : \Sigma^* \to [0,1]$ is a computable function. The notatation $P(w)$ for a string $w \in \Sigma^*$ stands for $P(C_w)$, the probability of the cylinder C_w.

To define the notion of Martin-Löf random sequences, we introduce two concepts – that of a measure of impossibility, and that of martingales.

Definition 3.2 A function $p : \Omega \to \overline{\mathbb{R}}^{+}$ is called a *measure of impossibility* with respect to the probability space $(\Omega, \mathcal{F}, P)$ if the following hold:

P1. p is lower semicomputable.

P2. $E_P\, p \leq 1$, where $E_P\, f$ is the expectation of the function f with respect to probability measure P.

A *measure of impossibility* p of ω with respect to the computable probability distribution P denotes whether ω is random with respect to the given probability distribution or not. In particular, we can see that $p(\omega) < \infty$ if ω is random with respect to the probability distribution P [18].

Definition 3.3 Let $\omega \in \mathbf{C}$. Then ω is said to be P-impossible if $p(\omega) = \infty$. A set $X \subseteq \mathbf{C}$ is said to be P-impossible as witnessed by p, if

$$X \subseteq \{\omega : \omega \text{ is } P\text{-impossible}\}.$$

V'yugin and Vovk [18] proved that there is an optimal mesure of impossibility $\tilde{p} : \mathbf{C} \to \overline{\mathbb{R}}^{+}$ such that a sequence ω is P-random if and only if $\tilde{p}(\omega) < \infty$. The set of Martin-Löf random sequences with respect to P is exactly the complement of the set of all $\omega \in \mathbf{C}$ that are P-impossible.

We also consider martingales.

Definition 3.4 A P-martingale $d : \Sigma^* \to \overline{\mathbb{R}}^{+}$, is a function which obeys the properties,

M1. $d(\lambda) \leq 1$.

M2. For all strings w, the following holds:

$$d(w)P(w) = d(w0)P(w0) + d(w1)P(w1).$$

A martingale is said to "succeed" on sequence ω if

$$\limsup_{n \to \infty} d(\omega[0 \ldots n-1]) = \infty.$$

A martingale is said to "strongly succeed" on sequence ω if

$$\liminf_{n \to \infty} d(\omega[0 \ldots n-1]) = \infty.$$

The success set of a martingale d, denoted $S^\infty[d]$, is defined to be the set of binary sequences on which d succeeds. The strong success set of a martingale d, denoted $S^\infty_{\mathrm{str}}[d]$, is the set of binary sequences on which d strongly succeeds.

A constructive martingale is a lower semicomputable martingale.

We show, the concept of a measure of impossibility and that of a constructive supermartingale are equivalent, in that every measure of impossibility p corresponds to a super-martingale which wins on an ω if and only if $p(\omega) = \infty$. Since there is a universal supermartingale which succeeds on the set of non-random sequences, and there is a universal measure of impossibility which attains ∞ on the set of non-random sequences, it is indirectly known that there is a conversion between the success criteria of martingales and that of measures of impossibility. The new result here is a direct conversion of a supermartingale into a measure of impossibility and vice versa, such that the success sets of both are the same (under some assumptions on the probability measure).

4 Converting a Martingale into a Measure of Impossibility

Let P be a computable probability measure. We wish to convert a lower semicomputable P-martingale which succeeds on a constructive P-measure-zero set, to a *measure of impossibility* $p : \mathbf{C} \to \overline{\mathbb{R}}^+$ with respect to P, such that $S^\infty[d]$ is P-impossible as witnessed by p. We show that if $d : \Sigma^* \to \mathbb{R}^+$ is a lower semicomputable P-martingale, then there exists a *measure of impossibility* $p : \Omega \to \overline{\mathbb{R}}^+$ such that $\forall \omega \in \Omega\ \limsup_{n\to\infty} d(\omega[0 \dots n-1]) = \infty$ only if $p(\omega) = \infty$, and both are less than ∞ otherwise.

We proceed in stages.

It is well-known that a sequence ω is non-random if and only if there is a martingale d which is such that $\liminf_n d(\omega[0 \dots n-1]) = \infty$. The following well-known lemma is stated here because we use the construction to prove results about the measure of impossibility.

Lemma 4.1 (Folk) *If $d : \Sigma^* \to \overline{\mathbb{R}}^+$ is a constructive martingale such that $\omega \in S^\infty[d]$, then there is a constructive martingale $d' : \Sigma^* \to \overline{\mathbb{R}}^+$ such that $\omega \in S^\infty_{str}[d']$. Moreover, $d' = \mathtt{bc} + \mathtt{sa}$ where $\mathtt{bc}, \mathtt{sa} : \Sigma^* \to \overline{\mathbb{R}}^+$ are such that $\mathtt{bc}$ is non-negative and $\mathtt{sa}$ monotone increasing and non-negative.*

Proof (Sketch) For every $w \in \Sigma^*$, let $d'(w) = \mathtt{bc}(w) + \mathtt{sa}(w)$ where $\mathtt{bc}(w)$ ("betting capital") and $\mathtt{sa}(w)$ ("savings account") are defined as follows.

For $d'(\lambda)$, we set $\mathtt{bc}(\lambda) = d'(\lambda)$ and $\mathtt{sa}(\lambda) = 0$.

Let $d(w1) = \alpha d(w)$, and $d(w0) = (\alpha + \frac{(1-\alpha)P(w)}{P(w0)})$, where $\alpha \in [0, 1/P(w1)]$. Then d' bets

$$\mathtt{bc}(w)\alpha$$

on $w1$, and

$$\mathtt{bc}(w)(\alpha + \frac{(1-\alpha)P(w)}{P(w0)})$$

on $w0$. Then, for $b \in \{0,1\}$, $\mathtt{bc}(wb)$ is decremented by 1 if it exceeds 2, and $\mathtt{sa}(wb)$ is incremented by 1. Otherwise, $sa(w0) = sa(w1) = sa(w)$. This is taken as the new betting capital and savings account on wb. It is routine to verify that d' is a constructive martingale if d is.

Note that $\mathtt{sa}$ is a monotone increasing function in the length of the string, such that $\mathtt{sa}(w) \le d'(w)$.

Let $\omega \in \mathbf{C}$. Then $\limsup_{n\to\infty} d(\omega[0\ldots n-1]) = \infty$ implies that for every integer m, there is an $n > m$ such that $\frac{d(\omega[0\ldots n-1])}{d(\omega[0\ldots m-1])} > 2$. This implies that $\mathtt{sa}(\omega[0\ldots n-1])$ increments by 1 for infinitely many n. Thus, $\liminf_{n\to\infty} d'(\omega[0\ldots n-1]) = \infty$, so ω is in the strong success set of d'. □

Let $d = \mathtt{bc}+\mathtt{sa}$ be a P-martingale as defined above. The measure of impossibility p is defined as follows.

$$p(\omega) = \lim_n \mathtt{sa}(\omega[0\ldots n-1]). \tag{3}$$

We prove that p is a measure of impossibility which attains ∞ on all sequences on which d strongly succeeds.

Lemma 4.2 *p defined in (3) is a measure of impossibility.*

Proof. Let $\mathtt{d}$ be lower computable with $\hat{d}$ as witness, and let $\hat{\mathtt{sa}}$ be the witness of the lower semicomputability of $\mathtt{sa}$. Then $\hat{\mathtt{sa}}$ witnesses that p is lower semicomputable.

Moreover,

$$\begin{aligned} E_P\, p &= \int p(\omega)dP \\ &\le \liminf_{n\to\infty} \sum_{w'\in\{0,1\}^n} \int \mathtt{sa}(w')dP \\ &= \lim_{n\to\infty} \sum_{w'\in\{0,1\}^n} \int \mathtt{sa}(w')dP, \end{aligned}$$

by Fatou's Lemma. But, since $\mathtt{sa}(w') \le d(w')$ for all w', it follows by Kraft's inequality that $\sum_{w'\in\{0,1\}^n} \int \mathtt{sa}(w')dP \le 1$ for all n. Hence it follows that $\int p(\omega)dP \le 1$. Therefore p defines a *measure of impossibility.* □

It is clear that, since $\liminf_{n\to\infty} d'(\omega[0\ldots n-1]) = \infty$ implies that $\sup_n \mathtt{sa}(\omega[0\ldots n-1]) = \infty$, we have $p(\omega) = \infty$. Thus p satisfies the conditions of being a measure of impossibility which attains ∞ on ω.

5 Converting a Measure of Impossibility into a Martingale

We assume $P : \mathcal{F} \to [0,1]$ is a computable probability measure. If $p : \Omega \to \overline{\mathbb{R}}^+$ is a *measure of impossibility*, we prove: there exists a constructive martingale $d : \Sigma^* \to \mathbb{R}^+$ such that d succeeds on every ω on which p assumes ∞– *i.e.* $\{\omega : \limsup_{n\to\infty} d(\omega[0\ldots n-1]) = \infty\} \supseteq \{\omega : p(\omega) = \infty\}$, with equality if P is a measure which assigns positive probability to every cylinder. In the following discussion, P is a probability measure, p is a measure of impossibility, d is a P-martingale, each of the types described in this paragraph.

We make the following restrictions: We ensure that P is not just a computable probability measure, but also *very strongly* positive: if $\hat{P}$ testifies to the fact that P is computable, then there exists a program $f : \Sigma^* \times 0^{\mathbb{N}} \to \mathbb{Q}$ such that for every cylinder C_x, the probability of the cylinder $P(C_x) > 0$ if and only if for all positive integers n, we have $\hat{P}(x, 0^n) > f(x, 0^n)$.

We define the P-martingale d.

For the empty string, $d(\lambda) = E_P[p]$. For all strings w, and $b \in \{0,1\}$,

$$d(wb) = \begin{cases} E_P[p \mid C_{wb}] & \text{if } P(C_{wb}) > 0 \\ 2 \cdot d(w) & \text{otherwise.} \end{cases} \tag{4}$$

Lemma 5.1 *If p is a measure of impossibility with respect to a very strongly positive, computable probability measure P, then d defined in (4), is a lower semicomputable P-martingale.*

Proof. We show that d is a P-martingale, and d is lower semicomputable,

d is a P-martingale:

We have $d(\lambda) = \frac{\int_\Omega p(\omega)dP}{P(\Omega)} \leq 1$.

For a string w, if all of C_w, C_{w0} and C_{w1} have non-zero probability, then the stipulation (M2) is satisfied by linearity of expectation. If, $P(C_w) \neq 0$, but one of $P(C_{w0})$, $P(C_{w1})$ is zero, without loss of generality, say $P(C_{w0}) = 0$, then

$$\begin{aligned} d(w0)P(C_{w0}) + d(w1)P(C_{w1}) &= 2d(w) \times 0 + E[p|C_{w1}]P[C_{w1}] \\ &= E[p|C_{w1}]P(C_w) = E[p|C_w]P(C_w) \\ &= d(w)P(C_w). \end{aligned}$$

d is lower semicomputable:

Consider the following program: Algorithm for $\hat{d} : \Sigma^* \times \mathbb{N} \to \mathbb{Q}$:

1. Input $xb \in \Sigma^*(b \in \Sigma)$ and $n \in \mathbb{N}$.
2. If $f(xb, 0^n) > \hat{P}(xb, 0^n)$, then $\hat{d}(xb, 0^n) = 2\,\hat{d}(x, 0^n)$.
3. Else,

$$\hat{d}(xb, 0^n) = \frac{\sum_{w\in\{0,1\}^n} \max_{y \sqsubseteq xbw}\{\hat{p}(y, 0^n)\} \times (\hat{P}(xbw, 0^{2n+1}) - 2 \cdot 2^{-2n-1})}{P(xb, 0^{2n+1}) + 2 \cdot 2^{-2n-1}}.$$

To show that $\hat{d}(xb, \cdot)$ is a lower semicomputation of $d(xb)$, we proceed as follows. We prove that the numerator in line 3 converges to the appropriate limit. From this, it follows that the output of the program converges to the value of d for the given string xb from below in a lower semicomputable way.

Define the following:

$$\forall xb \in \Sigma^*, m \in \mathbb{N}\ f_m^{xb0^\infty} = \sum_{w \in \{0,1\}^m} \max_{y \sqsubseteq xbw} \{\hat{p}(y, 0^m)\} \left[\hat{P}(xbw, 0^{2m+1}) - 2 \cdot 2^{-2m-1}\right]$$
$$\forall xb \in \Sigma^*, m \in \mathbb{N}\ S_m^{xb0^\infty} = \sum_{w \in \{0,1\}^m} \max_{y \sqsubseteq xbw} \{\hat{p}(y, 0^m)\} P(xbw)$$

The following claims suffice to prove that $\hat{d} : \mathbb{N} \times \Sigma^* \to \mathbb{Q}$ is a lower semicomputation of d.

Lemma 5.2 $\forall m \in \mathbb{N}\ f_m^{xb0^\infty} \leq S_m^{xb0^\infty} \leq \int_{C_{xb}} p(\omega)\, dP.$

Proof (Sketch) Lower semicomputability of P with witness $\hat{P}(\cdot, 0^m) - 2.2^{-m}$ for all $m \in \mathbb{N}$ implies the first inequality. The second is by the lower semicomputability of $\hat{p}$ with respect to p, and the fact that each $S_m^{x0^\infty}$ is the integral of a step function defined on Ω, $\hat{p} < p$ (everywhere), and by the definition of the Lebesgue integral. □

Now, we show that the sum converges as $n \to \infty$ to the required integral:

Lemma 5.3 *The series $f_m^{xb0^\infty}$ converges uniformly to the same limit as of the sum series $S_m^{xb0^\infty}$ as $m \to \infty$.*

Proof. By the computability witness $\hat{P}$ of P, we have, for any xbw, $m \in \mathbb{N}$,

$$P(xbw) - \hat{P}(xbw, 0^{2(m+1)}) < \frac{1}{2^{(2m+1)}},$$

whereby

$$\begin{aligned} |f_m^{xb0^\infty} - S_m^{xb0^\infty}| &< \frac{1}{2^{2m+1}} \times 2^m \\ &= \frac{1}{2^{m+1}} \end{aligned}$$

□

The fact that $S_m^{xb0^\infty} \to \int_{C_{xb}} p(\omega) dP$ as $m \to \infty$, follows due to the fact that $\hat{p}$ is a lower semicomputation of p. Property (1) of lower semicomputability ensures that p dominates the step function summed in $S_m^{xb0^\infty}$. The convergence property of lower semicomputability ensures that the the function $S_m^{xb0^\infty}$ converges to the integral $\int pdP$.

These claims suffice to establish the condition that $\hat{d}$ has to satisfy to be a lower semicomputation of d.

□

Lemma 5.4 *For any $\omega \in \mathbf{C}$, $p(\omega) = \infty$ implies* $\limsup_{n\to\infty} d(\omega[0 \to n-1]) = \infty$.

Proof. First, if $P(C_{\omega[0\ldots n-1]}) = 0$ for some n, this is routine to prove.

So, assume for all n, $P(C_{\omega[0\ldots n-1]}) > 0$, so that for all n, $d(\omega[0\ldots n-1]) = E[p|C_{\omega[0\ldots n-1]}]$. We show that d succeeds on ω. It is enough to show that for every rational q, there is some $x \sqsubseteq \omega$ such that $d(x) > q$.

Let S be the graph of p, described by the union of a computable enumeration $\mathcal{C}$ of cylinders in $\mathbf{C} \times \mathbb{Q}$. If $p(\omega) = \infty$, then there is an $(x, q) \in \mathcal{C}$ for every $q \in \mathbb{Q}$ such that x is some prefix of ω. If this is so, then for every $\sigma \in C_x$, $p(\sigma) > q$, whence $E[p|C_x] > q$, which proves the result. □

6 A New Characterization of Constructive Dimension

We limit ourselves to computable, and strongly positive probability measures P which assign positive probability to every cylinder C_w. For $s \in [0, \infty)$, we introduce the notion of a set being s-improbable with respect to a measure of impossibility.

Definition 6.1 Let $X \subseteq \mathbf{C}$. We say that X is s-improbable with respect to a P-measure of impossibility $p : \mathbf{C} \to \overline{\mathbb{R}}^+$ if for every infinite binary sequence $\omega \in X$, we have

$$\limsup_{n\to\infty} \frac{\int_{C_{\omega[0\ldots n-1]}} p(\omega)dP}{P^s(C_{\omega[0\ldots n-1]})} = \infty. \tag{5}$$

X is strongly s-improbable if

$$\liminf_{n\to\infty} \frac{\int_{C_{\omega[0\ldots n-1]}} p(\omega)dP}{P^s(C_{\omega[0\ldots n-1]})} = \infty. \tag{6}$$

The concept of s-improbability generalizes the concept of improbability.

Lemma 6.2 *Let $p : \mathbf{C} \to \overline{\mathbb{R}}^+$ be a P-measure of impossibility.*

(i) *For any $\omega \in \mathbf{C}$, if $p(\omega) = \infty$, then* $\limsup_{n\to\infty} \frac{\int_{C_{\omega[0\ldots n-1]}} p(\omega)dP}{P(C_{\omega[0\ldots n-1]})} = \infty$.

(ii) *For any $\omega \in \mathbf{C}$, if* $\limsup_{n\to\infty} \frac{\int_{C_{\omega[0\ldots n-1]}} p(\omega)dP}{P(C_{\omega[0\ldots n-1]})} = \infty$, *then there exists a P-measure of impossibility $p' : \mathbf{C} \to \overline{\mathbb{R}}^+$ such that $p'(\omega) = \infty$.*

Proof. 1.) Let $\omega \in \mathbf{C}$ be such that $p(\omega) = \infty$. Since p is lower semicomputable and $P(C_w) > 0$ for all strings w, the function $d : \Sigma^* \to \mathbb{R}^+$ defined by $d(w) = E_P[p|C_w]$ is a martingale, such that $\omega \in S^\infty[d]$. Thus, $\limsup_{n\to\infty} \frac{\int_{C_{\omega[0\ldots n-1]}} p(\omega)dP}{P(C_{\omega[0\ldots n-1]})} = \infty$.

2.) Since $\int p(\omega)dP \le 1$ and $P(C_w) > 0$ for all w, we take the equivalent characterization that $\sup_n \frac{\int_{C_{\omega[0\ldots n-1]}} p(\omega)dP}{P(C_{\omega[0\ldots n-1]})} = \infty$. Since p is lowersemicomputable and P is computable, $f' : \mathbf{C} \to \overline{\mathbb{R}}^+$ defined by $f'(\omega) = \sup_n \frac{\int_{C_{\omega[0\ldots n-1]}} p(\omega)dP}{P(C_{\omega[0\ldots n-1]})} = \infty$ is a measure of impossibility. Hence the optimal measure $\tilde{p}$ attains infinity on any ω where $f'(\omega) = \infty$. □

We now review the notion of a lower semicomputable s-P-gale, which, following Lutz [10], we use to give a definition of constructive Hausdorff (or constructive Billingsley) dimension.

Definition 6.3 (Lutz [10]) Let $s \in [0, \infty)$. An s-P-gale $d : \Sigma^* \to \mathbb{R}^+$ is a function that satisfies the condition for all $w \in \Sigma^*$,

$$d(w)P^s(w) = [d(w0)P^s(w0) + d(w1)P^s(w1)] \tag{7}$$

Definition 6.4 (Lutz [10]) Let d be an s-P-gale, where $s \in [0, \infty)$.

- We say that d succeeds on $\omega \in \mathbf{C}$ if

$$\limsup_{n\to\infty} d(\omega[0\ldots n-1]) = \infty.$$

- The success set of d is

$$S^\infty[d] = \{\omega \in \mathbf{C} \mid d \text{ succeeds on } \omega\}.$$

- We say that d strongly succeeds on $\omega \in \mathbf{C}$ if

$$\liminf_{n\to\infty} d(\omega[0\ldots n-1]) = \infty.$$

- The strong success set of d is

$$S^\infty_{str}[d] = \{\omega \in \mathbf{C} \mid d \text{ strongly succeeds on } \omega\}.$$

The notion of Billingsley dimension is outlined in Hausdorff's work on Hausdorff dimension[3]. The notion of constructive Hausdorff dimension is a constructive analogue of Hausdorff dimension where the covers of a set is defined to be constructive. We use the equivalent definition given in Lutz [10], which uses constructive s-P-gales.

Remark 6.5 [Lutz [10]] For every $s \in [0, \infty)$, the function $d : \Sigma^* \to \overline{\mathbb{R}}^+$ is a P-s-gale if and only if the function $d' : \Sigma^* \to \overline{\mathbb{R}}^+$ defined by $d'(w) = P^{(s-1)}(w)d(w)$ is a martingale.

Definition 6.6 (Lutz [10]) The constructive Hausdorff dimension of a set $X \subseteq \mathbf{C}$ is

$\dim_H(X) = \inf\{s \in [0,\infty) \mid$ There is a constructive s-P-gale for which $X \subseteq S^\infty[d].\}$

Analogously, the notion of constructive packing dimension is defined as the constructive analogue of the classical packing dimension [16], [15]. We use the following equivalent notion defined using strong success of s-P-gales.

Definition 6.7 (Athreya et al. [1]) The constructive packing dimension of a set $X \subseteq \mathbf{C}$ is

$\mathrm{Dim}_H(X) = \inf\{s \in [0,\infty) \mid$ There is a constructive s-P-gale for which $X \subseteq S^\infty_{\mathrm{str}}[d].\}$

Notation. For $X \subseteq \mathbf{C}$, let $\mathcal{G}(X)$ be the set of all $s \in [0,\infty)$ such that there is an s-P-gale d for which $X \subseteq S^\infty[d]$ and $\mathcal{P}(X)$ be the set of all $s \in [0,\infty)$ such that X is s-improbable with respect to some P-measure of impossibility. Similarly, let $\mathcal{G}_{\mathrm{str}}(X)$ be the set of all $s \in [0,\infty)$ such that there is an s-P-gale d for which $X \subseteq S^\infty_{\mathrm{str}}[d]$ and $\mathcal{P}_{\mathrm{str}}(X)$ be the set of all $s \in [0,\infty)$ such that X is strongly s-improbable with respect to some P-measure of impossibility.

Lemma 6.8 *Let $X \subseteq \mathbf{C}$ and $s \in [0,\infty)$. Then $s \in \mathcal{G}(X)$ if and only if $s \in \mathcal{P}(X)$, and $s \in \mathcal{G}_{str}(X)$ if and only if $s \in \mathcal{P}_{str}(X)$.*

Proof. First, assume $s \in \mathcal{G}(X)$. Then there is an s-P-gale d such that d succeeds on every $\omega \in X$. As in the case of constructive martingales, we can form another s-P-gale d' consisting of bc and sa such that the following hold.

(i) For all strings w, $d'(w) = \mathtt{bc}(w) + \mathtt{sa}(w)$.

(ii) $\mathtt{bc}(\lambda) = 1$.

(iii) $\mathtt{sa}(\lambda) = 0$.

If d bets $\alpha d(w)$ on $w1$, and $(\frac{(1-\alpha)P^s(C_w)}{P^s(C_{w0})} + \alpha)d(w)$ on $w0$, then d' bets the same ratio $\alpha\mathtt{bc}(w)$ on $w1$, and $(\frac{(1-\alpha)P^s(C_w)}{P^s(C_{w0})} + \alpha)\mathtt{bc}(w)$ on $w0$. If for $b \in \{0,1\}$ the capital on wb is greater than 2, say $2+c$, then $\mathtt{bc}(wb)$ is set to 1, and the new value of the savings account is set to $\mathtt{sa}(w) + 1 + c$. It is routine to verify that d' is a s-P-gale.

Unlike the martingale case, we cannot say that $\mathtt{sa}(w)$ is monotone increasing in the length of w. However, we can use the remark previously noted, to construct a martingale $\tilde{d}$, which for every w is defined as $\mathtt{bc}(w)P^{s-1}(C_w) + \mathtt{sa}(w)P^{s-1}(C_w)$. We can see that $\mathtt{sa}(w)P^{s-1}(w)$ is monotone increasing with the length of w.

We define $\overline{p} : \Sigma^* \to \mathbb{R}^+$ by

$$\overline{p}(w) = \mathtt{sa}(w)P^{s-1}(C_w),$$

for finite strings w.

Now, we define $p : \mathbf{C} \to \overline{\mathbb{R}}^+$ by

$$p(\omega) = \lim_{n\to\infty} \overline{p}(\omega[0 \ldots n-1]).$$

If d is lower semicomputable, it is routine to see that p is a measure of impossibility.

Also,

$$\frac{\int_{C_w} p(\omega)dP}{P^s(C_w)} = \mathtt{sa}(w)$$

for all strings w. Since $\mathtt{bc}(w)$ is always finite, we have that

$$\limsup_{n\to\infty} d(\omega[0\ldots n-1]) = \infty \equiv \limsup_{n\to\infty} \mathtt{sa}(\omega[0\ldots n-1]) = \infty,$$

and

$$\liminf_{n\to\infty} d(\omega[0\ldots n-1]) = \infty \equiv \liminf_{n\to\infty} \mathtt{sa}(\omega[0\ldots n-1]) = \infty.$$

Hence

$$\limsup_{n\to\infty} d(\omega[0\ldots n-1]) = \infty \implies \limsup_{n\to\infty} \frac{\int_{C_{\omega[0\ldots n-1]}} p(\omega)dP}{P(C_{\omega[0\ldots n-1]})} = \infty$$

and

$$\liminf_{n\to\infty} d(\omega[0\ldots n-1]) = \infty \implies \liminf_{n\to\infty} \frac{\int_{C_{\omega[0\ldots n-1]}} p(\omega)dP}{P(C_{\omega[0\ldots n-1]})} = \infty$$

Thus $s \in \mathcal{P}_{\mathrm{str}}(X)$ if $s \in \mathcal{G}_{\mathrm{str}}(X)$, and $s \in \mathcal{P}(X)$ if $s \in \mathcal{G}(X)$.

Conversely, let $s \in \mathcal{P}(X)$. Then there exists a measure of impossibility p such that X is s-improbable with respect to p. We define a martingale $d : \Sigma^* \to \mathbb{R}^+$ as follows. For finite strings w,

$$d(w) = E_P[p|C_w]P^{s-1}(C_w).$$

It is routine to see that d is a lowersemicomputable s-P-gale. Moreover,

$$\limsup_{n\to\infty} \frac{\int_{C_{\omega[0\ldots n-1]}} pdP}{P^s(C_{\omega[0\ldots n-1]})} = \infty \implies \limsup_{n\to\infty} d(\omega[0\ldots n-1]) = \infty.$$

Thus, we can see that $s \in \mathcal{G}(X)$. Similarly, we can establish that if $s \in \mathcal{P}_{\mathrm{str}}(X)$, then $s \in \mathcal{G}_{\mathrm{str}}(X)$. □

Using this, we can characterize effective Hausdorff and packing dimensions in the following way.

Corollary 6.9 *(Alternate Characterization of Constructive Dimension) For any set $X \subseteq \mathbf{C}$, the constructive Hausdorff dimension of $X = \inf \mathcal{P}(X)$, and the constructive packing dimension of $X = \inf \mathcal{P}_{str}(X)$.*

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References

[1] Athreya, K. B., J. M. Hitchcock, J. H. Lutz and E. Mayordomo, *Effective strong dimension, algorithmic information, and computational complexity*, in: *Proceedings of the 21st International Symposium on Theoretical Aspects of Computer Science* (2004), pp. 632–643.

[2] Downey, R. and D. Hirschfeldt, "Algorithmic randomness and complexity," Book Draft, 2006.

[3] Hausdorff, F., *Dimension und äußeres Mass*, Mathematische Annalen **79** (1919), pp. 157–179.

[4] Kolmogorov, A. N. and V. A. Uspenksy, *Algorithms and randomness*, Theory of Probability and Its Applications **32** (1987), pp. 389–412.

[5] Lévy, P., *Propriétés asymptotiques des sommes de variables indépendantes ou enchainées*, Journal des mathématiques pures et appliquées. Series 9. **14** (1935), pp. 347–402.

[6] Lévy, P., "Théorie de l'Addition des Variables Aleatoires," Gauthier-Villars, 1937 (second edition 1954).

[7] Lutz, J. H., *Almost everywhere high non-uniform complexity*, Journal of Computer and System Sciences **44** (1992), pp. 220–258.

[8] Lutz, J. H., *The quantitative structure of exponential time*, in: L. A. Hemaspaandra and A. L. Selman, editors, *Complexity Theory Retrospective II*, Springer-Verlag, 1997 pp. 225–254.

[9] Lutz, J. H., *Resource-bounded measure*, in: *Proceedings of the 13th IEEE Conference on Computational Complexity* (1998), pp. 236–248.

[10] Lutz, J. H., *Dimensions of individual strings and sequences*, Information and Computation **187** (2003), pp. 49–79.

[11] Martin-Löf, P., *The definition of random sequences*, Information and Control **9** (1966), pp. 602–619.

[12] Schnorr, C. P., *Klassifikation der Zufallsgesetze nach Komplexität und Ordnung*, Z. Wahrscheinlichkeitstheorie verw. Geb. **16** (1970), pp. 1–21.

[13] Schnorr, C. P., *A unified approach to the definition of random sequences*, Mathematical Systems Theory **5** (1971), pp. 246–258.

[14] Schnorr, C. P., "Zufälligkeit und Wahrscheinlichkeit," Springer-Verlag, Berlin, 1971.

[15] Sullivan, D., *Entropy, Hausdorff measures old and new, and limit sets of geometrically finite Kleinian groups*, Acta Mathematica **153** (1984), pp. 259–277.

[16] Tricot, C., *Two definitions of fractional dimension*, Mathematical Proceedings of the Cambridge Philosophical Society **91** (1982), pp. 57–74.

[17] Ville, J., "Étude Critique de la Notion de Collectif," Gauthier-Villars, Paris, 1939.

[18] Vovk, V. G. and V. V. V'yugin, *On the empirical validity of the bayesian method*, Journal of Royal Statistical Society B **55** (1993), pp. 253–266.

[19] V'yugin, V. V., *Ergodic theorems for individual random sequences*, Theoretical Computer Science **207** (1998), pp. 343–361.