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Hull and Geodetic Numbers for Some Classes of Oriented Graphs[1]

Julio Araujo and Pedro Arraes[2]

Department of Mathematics
Universidade Federal do Ceará
Fortaleza - CE, Brazil

Abstract

An oriented graph D is an orientation of a simple graph, i.e. a directed graph whose underlying graph is simple. A directed path from u to v with minimum number of arcs in D is an (u,v)-geodesic, for every $u,v \in V(D)$. A set $S \subseteq V(D)$ is (geodesically) convex if, for every $u,v \in S$, all the vertices in each (u,v)-geodesic and in each (v,u)-geodesic are in S. For every $S \subseteq V(D)$ the (convex) hull of S is the smallest convex set containing S and it is denoted by $[S]$. A hull set of D is a set $S \subseteq V(D)$ whose hull is $V(D)$. The cardinality of a minimum hull set is the hull number of D and it is denoted by $\overrightarrow{\mathrm{hn}}(D)$. A geodetic set of D is a set $S \subseteq V(D)$ such that each vertex of D lies in an (u,v)-geodesic, for some $u,v \in S$. The cardinality of a minimum geodetic set is the geodetic number of D and it is denoted by $\overrightarrow{\mathrm{gn}}(D)$.
In this work, we first present an upper bound for the hull number of oriented split graphs. Then, we turn our attention to the computational complexity of determining such parameters. We first show that computing $\overrightarrow{\mathrm{hn}}(D)$ is NP-hard for partial cubes, a subclass of bipartite graphs, and that computing $\overrightarrow{\mathrm{gn}}(D)$ is also NP-hard for directed acyclic graphs (DAG). Finally, we present a positive result by showing how to compute such parameters in polynomial time when the input graph is an oriented cactus.

Keywords: convexity; oriented graphs; hull number; geodetic number; computational complexity.

1 Introduction

For basic notions on graph theory and computational complexity, the reader is referred to [4,13]. All graphs in this work are simple and finite, unless explicitly stated otherwise.

A directed graph $D = (V, A)$ whose underlying graph is simple is an *oriented graph*. Given an oriented graph D, a (directed) (u,v)-*path* P is a subgraph of D such that $V(P) = \{u = u_0, u_1, \ldots, u_k = v\}$ and $A(P) = \{(u_{i-1}, u_i) \mid i \in \{1, \ldots, k\}\}$.

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[2] E-mails: julio@mat.ufc.br and pedro.arraes@alu.ufc.br

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We can also denote it by $(u, u_1, \ldots, u_{k-1}, v)$; to represent a path in a non-oriented graph G we remove the parentheses. When w is a vertex of P different than u and v we say that it is an *internal* vertex of P. The set of internal vertices of P we call the *interior* of P. An (u, v)-path that uses the least number of arcs possible is called an (u, v)-*geodesic*. We denote its length by $d_D(u, v)$ which represents de distance in D from u to v. In the sequel, whenever D is clear in the context, we only use $d(u, v)$.

Given a vertex $v \in V(D)$ we define $N^-(v) := \{u \in V(D) \mid (u, v) \in A(D)\}$ and $N^+(v) := \{u \in V(D) \mid (v, u) \in A(D)\}$. Moreover we respectively define the *indegree* and the *outdegree* of v by $d^-(v) = |N^-(v)|$ and $d^+(v) = |N^+(v)|$.

For two oriented graphs D_1, D_2 such that D_1 is a subgraph of D_2 we denote this fact by $D_1 \subseteq D_2$. Given D an oriented graph and $C \subseteq D$ such that its underlying graph is a cycle, we say that C is simply a *cycle*, the fact that it is oriented is already implied by being a subgraph of D. However, when C is such that $V(C) = \{v_1, \ldots, v_n\}$ and $A(C) = \{(v_1, v_2), \ldots, (v_{n-1}, v_n), (v_n, v_1)\}$ we say that it is a *directed* cycle.

The *interval function* $I \colon \mathcal{P}(V(D)) \to \mathcal{P}(V(D))$ satisfies that, for each vertex set $S \subseteq V(D)$ with at least two elements, $I[S]$ is composed by all the vertices in an (u, v)-geodesic, for every $u, v \in S$; when S is unitary we have $I[S] = S$. For every positive integer n we recursively define $I^0[S] = S$ and $I^n[S] := I[I^{n-1}[S]]$, for every $n \geq 1$. A subset $S \subseteq V(G)$ is *convex* if $I[S] = S$; if $S^{\complement} = V(D) \setminus S$ is convex we say that S is *co-convex*. The *convex hull* of S is the smallest convex set which contains S and is denoted by $[S]$. There are two interesting properties for this set. One is that it is the intersection of all convex sets containing S. A noteworthy consequence of this fact is that if S does not intersect a given co-convex set, then its convex hull also does not intersect it. The other is that it is obtained when we iterate the interval function on S until we reach a convex set $I^k[S](= [S])$. Assuming that $V(D)$ is finite, the convex hull for every subset of $V(D)$ is well-defined.

If the convex hull of S is $V(D)$ we say that S is a *hull set* of D. When S is a hull set of minimum cardinality, we define the *hull number* of D as $\overrightarrow{\mathrm{hn}}(D) = |S|$ [5]. Similarly, if $I[S] = V(D)$ we say that S is a *geodetic set* of D. When S is a geodetic set of minimum cardinality, we say that the *geodetic number* of D is $\overrightarrow{\mathrm{gn}}(D) = |S|$ [7]. Notice that a geodetic set is also a hull set, so every assertion we make for all hull sets of a given oriented graph D is also valid for the geodetic sets.

Now that we have presented the main parameters of our research, we define a very important type of vertex. It was first introduced in [11] for the undirected case, however we use the definitions given in [5]. A vertex $v \in V(D)$ is called *extreme* if it is of one of the three types below:

(i) Transmitter (source): $d^-(v) = 0, d^+(v) > 0$;

(ii) Receiver (sink): $d^-(v) > 0, d^+(v) = 0$;

(iii) Transitive: $d^-(v) > 0$, $d^+(v) > 0$ and $(u, w) \in A(D)$, for every $u \in N^-(v)$ and every $w \in N^+(v)$.

For undirected graphs we have the *simplicial* vertices, which are the ones with a clique for neighborhood. What is so interesting about the extreme vertices is that they must be in every hull set of the oriented graph [5]. There is a similar result for the simplicials [8].

Although the first papers related to convexity in graphs study directed graphs [20,17,10], most of the papers we can find in the literature about graph convexities deal with undirected graphs. For instance, the hull and geodetic numbers with respect to undirected graphs [11,14] were first studied in the literature around a decade before their corresponding directed versions [5,7].

It is important to emphasize that as D is an orientation of a simple graph, then it cannot have both arcs (u, v) and (v, u), for distinct $u, v \in V(D)$. Thus, the parameters $\overrightarrow{\text{hn}}(D)$ and $\overrightarrow{\text{gn}}(D)$ are not equivalent to their undirected versions. For instance, the hull and geodetic numbers of a path P on $2k$ edges are both equal to two in the undirected version, while if D is an orientation of P where each vertex is either a source or a sink, then $\overrightarrow{\text{hn}}(D) = \overrightarrow{\text{gn}}(D) = 2k + 1$.

With respect to the directed case, most results in the literature provide bounds on the maximum and minimum values of $\overrightarrow{\text{hn}}(D(G))$ and $\overrightarrow{\text{gn}}(D(G))$ among all possible orientations $D(G)$ of a given undirected simple graph G [5,7,12]. It is important to emphasize the results on the parameter $\text{hn}^+(G)$, the upper orientable hull number of a graph G, since these are the only ones related to the upper bound we present. Such parameter is defined in [5], as the maximum value of $\overrightarrow{\text{hn}}(D)$ among all possible orientations D of a simple graph G. In the same article, the authors prove that for a non-oriented graph G, $\text{hn}^+(G) = n(G)$ if and only if there is an orientation D of G such that every vertex is extreme. They also compare this parameter with others, such as the lower orientable hull number ($\text{hn}^-(G)$) and the lower and upper orientable geodetic numbers ($\text{gn}^-(G)$ and $\text{gn}^+(G)$ respectively), defined analogously.

There are also few results about some related parameters: the forcing hull and geodetic numbers [19,6], the pre-hull number [18] and the Steiner number [15] are a few examples.

In this work, we first present a general tight upper bound for the hull number of an oriented split graph, in Section 2. Note that such bound is also an upper bound to $\text{hn}^+(G)$, whenever G is a split graph.

Then, we consider as input an oriented graph D and we study the computational complexity of determining $\overrightarrow{\text{hn}}(D)$ and $\overrightarrow{\text{gn}}(D)$, when the underlying graph of D belongs to some particular graph class. Up to our best knowledge, this is the first work to consider such questions.

It is known that determining the hull number of an undirected partial cube is NP-hard [1]. In Section 3, we prove that such result can be used to prove that determining whether $\overrightarrow{\text{hn}}(D) \leq k$, when D is an orientated partial cube, is NP-complete. Although the proof requires a careful analysis, the idea is quite simple: by replacing each edge of a partial cube G with a directed C_4, we obtain an oriented graph $D(G)$ whose underlying graph is a partial cube, and whose hull number $\overrightarrow{\text{hn}}(D(G))$ is the same as $\text{hn}(G)$. It is important to recall that partial cubes are bipartite graphs. In the same section, we also prove that determining $\overrightarrow{\text{gn}}(D) \leq k$ is also an NP-complete problem, even if D is a directed acyclic graph (DAG) whose underlying graph is bipartite.

Finally, we prove in Section 4 that $\overrightarrow{\mathrm{hn}}(D)$ and $\overrightarrow{\mathrm{gn}}(D)$ can be computed in polynomial time if D is a cactus, i.e. a graph whose blocks are either edges or induced cycles.

In Section 5, we present avenues for further research.

2 Upper bound on the hull number of an oriented split graph

The hull number problem for (non-oriented) split graphs has already been studied in [8]. Given a split graph $G = (S \cup C, E)$ with S a maximal stable set and C a clique, the authors prove that either the set of simplicial vertices of G is a minimum hull set or $\mathrm{hn}(G) = |S| + 1$. Moreover, the hull number of G can be decided in polynomial time.

Notice that in a split graph the vertices of S are simplicials. Thus, they belong to any hull set in the non-oriented case. However, in an orientation of G, maybe none of the vertices of S is extreme.

The extreme vertices of S must be in every hull set of D (an orientation of G), leaving the non-extreme of S to analyse. Given a non-extreme vertex $v \in S$, by definition there must be $u, w \in C$ such that $(u, v), (v, w), (w, u) \in A(D)$, which means that $v \in I[\{u, w\}]$. Using that argument for every non-extreme vertex of S gives us $S \setminus \mathrm{Ext}(D) \subseteq I[C]$, where $\mathrm{Ext}(D)$ denotes the set of extreme vertices of an oriented graph D. We then focus on the hull number problem for tournaments.

Let us first observe that extreme vertices in a tournament do not need to be considered when applying the interval function, as no (u, v)-geodesic with non-empty interior ends in an extreme vertex.

Lemma 2.1 *Let D be a tournament and $u, v \in V(D)$ two distinct vertices such that $u \in \mathrm{Ext}(D)$. Then $I[\{u, v\}] = \{u, v\}$.*

If D is a tournament, this means that for every $S' \subseteq V(D)$, we have $I[S'] = (S' \cap \mathrm{Ext}(D)) \cup I[S' \setminus \mathrm{Ext}(D)]$. By iterating this argument, we obtain the following result:

Corollary 2.2 *If D is a tournament, then S' is a hull set of D if, and only if, $\mathrm{Ext}(D) \subseteq S'$ and $[S' \setminus \mathrm{Ext}(D)] = V(D) \setminus \mathrm{Ext}(D)$.*

Let us analyse the non-extreme vertices of D. Given a non-extreme vertex $v \in V(D)$ there must be $u, w \in V(D)$ such that $(u, v), (v, w), (w, u) \in A(D)$, which implies that u, w are also non-extreme vertices. This means that each non-extreme vertex of D lies in a directed $C_3 \subseteq D$ and that v lies in a (u, w)-geodesic.

Thus, for each non-extreme vertex v that does not belong to a minimum hull set S' of D, at most two other vertices must belong to S' to ensure that $v \in [S']$. Now we can state the main result for tournaments.

Proposition 2.3 *If D is a tournament, then $\overrightarrow{hn}(D) \leq |\operatorname{Ext}(D)| + \frac{2}{3}|V(D) \setminus \operatorname{Ext}(D)|$ and this bound is tight.*

Next we show a couple of definitions that enable us to present a tournament which achieves the above bound.

Take D, D' oriented graphs. The *lexicographic product* of D by D', denoted by $D \circ D'$, is the oriented graph which satisfies $V(D \circ D') = \{(u, v) \mid u \in V(D) \text{ and } v \in V(D')\}$ and $((u_1, v_1), (u_2, v_2)) \in A(D \circ D')$ if, and only if, either $(u_1, u_2) \in A(D)$ or $u_1 = u_2$ and $(v_1, v_2) \in A(D')$. In other words, for each vertex $v \in V(D)$ we take a copy of D', namely D'_v, and if $(u, v) \in A(D)$, then we add the arcs (u', v'), for each $u' \in V(D'_u)$ and each $v' \in V(D'_v)$. A *transitive orientation* of a graph G is an oriented graph D obtained from G such that every vertex is extreme. In this case, a transitive orientation of a complete graph is a transitive tournament.

Proposition 2.4 *If D' is a directed C_3, D_k is a transitive tournament with k vertices and $D = D' \circ D_k$, then $\overrightarrow{hn}(D) = \frac{2}{3}|V(D)|$ for every $k \geq 1$.*

Now we return to oriented split graphs. Since in Proposition 2.3 we use only paths of length at most two, we can still find a subset of vertices C' in the clique C such that $[C' \cup \operatorname{Ext}(D[C])] \supseteq C$ and $|C'| \leq \frac{2}{3}(|C \setminus \operatorname{Ext}(D[C])|)$. We have seen before that $S \setminus \operatorname{Ext}(D) \subseteq I[C]$, from where we may deduce the following:

Corollary 2.5 *Let $D = (S \cup C, A)$ be an oriented split graph such that S is maximal and $|C| \geq 2$. Then, $\overrightarrow{hn}(D) \leq |\operatorname{Ext}(D) \cap S| + |\operatorname{Ext}(D[C])| + \frac{2}{3}|C \setminus \operatorname{Ext}(D[C])|$.*

3 NP-completeness on oriented bipartite graphs

In this section, we first study the hull number of a subclass of bipartite graphs, then we argue that computing the geodetic number is also NP-hard for bipartite graphs.

3.1 Hull number

In the undirected case, it was proven that determining the hull number of a graph is an NP-hard problem, even for bipartite graphs [2]. Recall that if arcs in both senses were allowed, then this result would imply the NP-hardness on the oriented case. Here, we first prove that replacing each edge by an oriented C_4 has roughly the same effect in the class of bipartite graphs.

Given a bipartite graph G, we create an oriented bipartite graph $G_{\overrightarrow{C4}}$ with the following procedure. Given that $V(G) = \{v_1, \ldots, v_n\}$, for each pair of adjacent vertices v_i and v_j we add the vertices $v_{i,j}, v_{j,i}$ to $V(G_{\overrightarrow{C4}})$ and the arcs $(v_i, v_{i,j}), (v_{i,j}, v_j), (v_j, v_{j,i}), (v_{j,i}, v_i)$ to $A(G_{\overrightarrow{C4}})$. So we have that $V(G_{\overrightarrow{C4}}) = V(G) \cup \{v_{i,j}, v_{j,i} \mid v_iv_j \in E(G)\}$ and $A(G_{\overrightarrow{C4}}) = \{(v_i, v_{i,j}), (v_{i,j}, v_j), (v_j, v_{j,i}), (v_{j,i}, v_i) \mid v_iv_j \in E(G)\}$. Thus, we replaced each edge with a directed C_4. Therefore, if there was a path $P = (v_i =)v_{i_0}, v_{i_1}, \ldots, v_{i_k}(= v_j)$ in G, in $G_{\overrightarrow{C4}}$ we would have the oriented paths $(v_i = v_{i_0}, v_{i_0,i_1}, v_{i_1}, \ldots, v_{i_{k-1},i_k}, v_{i_k} = v_j)$ and $(v_j = v_{i_k}, v_{i_k,i_{k-1}}, v_{i_{k-1}}, \ldots, v_{i_1,i_0}, v_{i_0} = v_i)$. Thus, one can observe that any hull set of G can be used to obtain a hull set of

$G_{\overrightarrow{C4}}$. We were also able to prove that, since G is bipartite, any minimum hull set of $G_{\overrightarrow{C4}}$ can be used to obtain a hull set of G with the same cardinality.

Theorem 3.1 *If G is a bipartite graph, then $hn(G) = \overrightarrow{hn}(G_{\overrightarrow{C4}})$.*

Thus, one can combine the result in [2] with Theorem 3.1 to deduce that, given an oriented bipartite graph D and a positive integer k, deciding whether $\overrightarrow{\mathrm{gn}}(D) \leq k$ is an NP-complete problem. However, one can observe that such reduction can also be applied to partial cubes, a subclass of bipartite graphs, which we define in the sequel.

The *hypercube graph of dimension n Q_n* is the (non-oriented) graph such that $V(Q_n) = \{(v^1, \ldots, v^n) \mid v^i \in \{0, 1\}\ \forall i \in \{1, \ldots, n\}\}$ and $E(Q_n) = \{uv \mid \exists! i \in \{1, \ldots, n\} : u^i \neq v^i\}$. In other words, the vertices are n-tuples of 0's and 1's and two vertices are adjacent if the differ in only one entry. A graph G is a *partial cube* if there are a natural number n and an injective mapping $\varphi : V(G) \to V(Q_n)$ such that $d_G(u, v) = d_{Q_n}(\varphi(u), \varphi(v))$ for every $u, v \in V(G)$.

Computing the hull number of partial cubes is also an NP-hard problem [1]. Thus, we only have to show that our reduction would return an oriented partial cube.

Proposition 3.2 *If G is a partial cube, then $G_{\overrightarrow{C4}}$ is an oriented partial cube.*

The idea for the proof is that, when we take $G_{\overrightarrow{C4}}$ for a partial cube $G \subseteq Q_n$ with n minimum, we have $G_{\overrightarrow{C4}} \subseteq Q_{2n}$. For each vertex $u = (u^1, \ldots, u^n) \in V(G)$ we take $u_* = (u_*^1, \ldots, u_*^{2n}) \in G_{\overrightarrow{C4}}$ such that $u_*^{2i-1} = u_*^{2i} = u^i$ for every $i \in \{1, \ldots, n\}$. Moreover, for each edge $uv \in E(G)$ we know that u_* and v_* differ in exactly two consecutive entries: $u_*^{2i-1} = u_*^{2i} \neq v_*^{2i-1} = v_*^{2i}$. We thus add two vertices such that the other $2n - 2$ entries for them both are the same as in u_* and in v_*, and the other two are respectively u^{2i-1}, v^{2i} and v^{2i-1}, u^{2i}, giving us the desired C_4.

Corollary 3.3 *Given an oriented partial cube D and a positive integer k, it is* NP*-complete to decide whether $\overrightarrow{hn}(D) \leq k$.*

3.2 Geodetic number

Our goal in this section is to prove that the following problem is NP-complete:

GEODETIC NUMBER

Instance: Oriented graph D and a positive integer k

Question: $\overrightarrow{\mathrm{gn}}(D) \leq k$?

Theorem 3.4 GEODETIC NUMBER *is an* NP*-complete problem, even if the input oriented graph D has no directed cycle and its underlying graph is bipartite.*

Proof. Given a subset of vertices $S \subseteq V(D)$, one can compute (u, v)-geodesics, for every $u, v \in V(D)$, and decide whether S is a geodetic set in polynomial time similarly to the undirected case [3]. Consequently, the problem is in NP.

We reduce the well-known Set Cover [16] problem to Geodetic number:

Set Cover

Instance: $U = \{1, 2, \ldots, n\}$, $\mathcal{F} \subseteq \mathcal{P}(U)$ such that $\bigcup \mathcal{F} = U$ and a positive integer k

Question: Does there exist $\mathcal{F}' \subseteq \mathcal{F}$ such that $\bigcup \mathcal{F}' = U$ and $|\mathcal{F}'| \leq k$?

Let $(U = \{1, 2, \ldots, n\}, \mathcal{F} = \{F_1, \ldots, F_m\}, k)$ be an input to Set Cover. We shall construct an oriented graph D such that $(U, \mathcal{F}, k)$ is an YES-instance if, and only if, $\overrightarrow{\text{gn}}(D) \leq k + 3$.

The vertex set of D consists of two subsets of vertices X and Y and three more vertices u, v and w. In X there is one vertex u_i corresponding to $F_i \in \mathcal{F}$, for every $i \in \{1, \ldots, m\}$. In Y there is a vertex v_j corresponding to each element in U, for every $j \in \{1, \ldots, n\}$.

In the arc set of D there is the arc $(u_i, v_j) \in A(D)$ whenever $j \in F_i$, for every $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$. Moreover, $A(D)$ has the arcs $(u, u_i), (u_i, w)$ for every $i \in \{1, \ldots, m\}$, the arcs (v_j, v) for every $j \in \{1, \ldots, n\}$, and finally the arc (u, v).

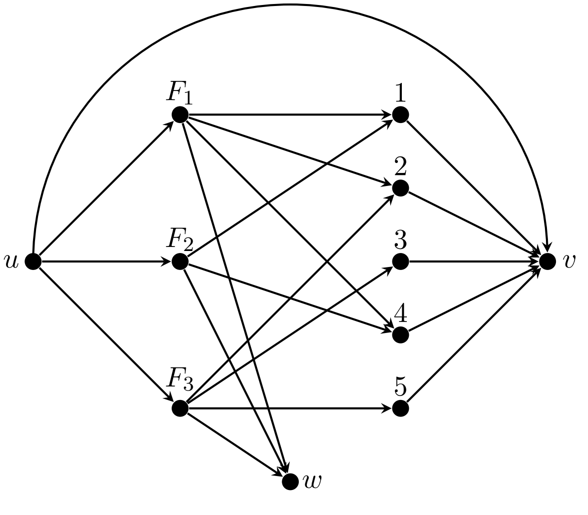

Fig. 1. Given the set $\{1, 2, 3, 4, 5\}$ and the family $\{\{1, 2, 3\}, \{1, 4\}, \{2, 3, 5\}\}$, the procedure described above returns the oriented graph of this figure.

By construction, D is clearly a DAG whose underlying graph is bipartite (with partition $V(D) = (X \cup \{v\}) \cup (Y \cup \{u, w\})$). Notice that u is a source and that v and w are sinks, thus they belong to any geodetic set. Besides, $(u, w) \notin A(D)$ and for every $u_i \in X$ we have the path (u, u_i, w). Since $(u, v) \in A(D)$ we have $I[u, v, w] = X \cup \{u, v, w\}$.

Let $\mathcal{F}' = \{F_i \mid i \in I\} \subseteq \mathcal{F}$, for some $I \subseteq \{1, \dots, m\}$ such that $\bigcup_{i \in I} F_i = U$ and $|I| \leq k$. We then take $X' = \{u_i \mid i \in I\} \cup \{u, v, w\}$, which has cardinality at most $k+3$. Thus, for every $v_j \in Y$ there is $u_i \in X'$ such that $(u_i, v_j) \in A(D)$, from where we have the geodesic (u_i, v_j, v). Therefore, one can observe that X' is a geodetic set of D.

On the other hand, let S be a geodetic set of D with at most $k + 3$ vertices. Thus, $\{u, v, w\} \subseteq S$ and at most k vertices of S belong to $X \cup Y$. If some $v_j \in S$, one can observe that by replacing v_j with a vertex u_i such that $(u_i, v_j) \in A(D)$, we obtain another geodetic set S' such that $|S'| \leq k + 3$. Thus, without loss of generality, we assume that $S \setminus \{u, v, w\} \subseteq X$. Let $I = \{i \in \{1, \dots, m\} \mid u_i \in S\}$. One can also observe that the family $\mathcal{F}' = \{F_i \in F \mid i \in I\}$ satisfies $\bigcup \mathcal{F}' = U$ and $|\mathcal{F}'| \leq k$. □

4 Polynomial-time algorithm for cacti

The hull and geodetic numbers of an undirected tree T are equal to the number of leaves of T. Notice that the leaves of a tree are simplicial vertices. Moreover, any node belongs to an uv-path for some distinct leaves $u, v \in V(G)$, and that path is a geodesic because it is unique. That means that the set of simplicial vertices of a tree is a minimum hull and geodetic set. A similar statement is true for the oriented case.

Proposition 4.1 *Let D be an oriented tree. Then,* $\mathrm{Ext}(D)$ *is both a minimum hull set and a minimum geodetic set of D. Consequently, it is unique and* $\overrightarrow{hn}(D) = \overrightarrow{gn}(D) = |\mathrm{Ext}(D)|$.

Thus, one can also observe that the hull and geodetic numbers of oriented trees, a subclass of oriented bipartite graphs, can be both computed in linear time.

This result led us to work on the cacti graphs. A graph is called a *cactus* if each block is either an edge or a cycle. Consequently, every cycle is an induced cycle and two cycles intersect in at most one vertex.

For cacti, the extreme vertices may not suffice in order to obtain a hull or a geodetic set. It is necessary to include some non-extreme vertices of some particular cycles. We introduce below a few important notions in order to define such cycles.

In the remainder of this section, let D be an oriented cactus graph. When we refer to a cycle subgraph of D it does not mean that it has a specific orientation, only that its underlying graph is a cycle. Let $C \subseteq D$ be a cycle and $u \in V(C)$ be a cut-vertex of D. If there is an arc $(u, v) \in A(D) \setminus A(C)$, we say that u is a *transmitter* cut-vertex of C and use the initials TCV. Analogously, if $(v, u) \in A(D) \setminus A(C)$ we say that u is a *receiver* cut-vertex and use the initials RCV. Notice that we can have a cut-vertex which is both an RCV and a TCV.

A cycle $C \subseteq D$ is called a *leaf cycle* if it has only one cut-vertex. We say that a *trap cycle* is a directed cycle C such that its cut-vertices are either all transmitters or all receivers. If C is a trap cycle with a TCV we say that it is a *transmitter* trap cycle. Similarly, a trap cycle with an RCV is a *receiver* trap cycle. At last, we say that a cycle C is *unsatisfactory* if one of the following holds:

Type 1: C is a trap cycle;

Type 2: C is a directed leaf cycle that is not a trap cycle;

Type 3: there are only two vertices in $\mathrm{Ext}(C)$, say u_1 and u_2, such that the two (u_1, u_2)-paths in C have different lengths and the longest one does no have internal cut vertices.

When none of these happens, we say that the cycle is *satisfactory.*

Lemma 4.2 *Let D be an oriented cactus graph and C an unsatisfactory cycle of D. $V(C)$ contains a co-convex set S and if C is of type:*

- *1, $S = V(C)$;*
- *2, $S = V(C) \setminus \{w\}$ where w is the cut-vertex of C;*
- *3 with $\mathrm{Ext}(C) = \{u, v\}$, S is composed by the internal vertices of the longest (u, v)-path.*

Moreover, there is no intersection between any two of these co-convex sets.

Recall that any hull set must intersect any co-convex set. Thus, any hull set must have at least one vertex of each unsatisfactory cycle. Let us now give some intuition on which vertices must belong to a hull set of an oriented cactus.

Given a vertex v, we consider a maximal directed path P containing v, such that P exits each cycle as soon as possible. There are three ways to end P: it finishes at either an extreme vertex or a trap cycle or an unsatisfactory cycle of type 2. In the last two cases, there is not a way out of the cycle, either because all its other cut-vertices are also receivers or there is only one, and we necessarily run through all the vertices of the cycle. In every case we can end the path in a vertex that will be in any hull set. This is roughly how we prove the following result.

Lemma 4.3 *Let D be an oriented cactus and $u \in V(D)$. For every $v \in N^+(u)$ ($v \in N^-(u)$) there is a maximal path $P = (v_1, \ldots, v_q)$ with $u = v_1$ and $v = v_2$ ($u = v_q$ and $v = v_{q-1}$) such that v_q (v_1) either is extreme or belongs to an unsatisfactory cycle C of type either 1 or 2. Moreover, in the latest two cases all the vertices of C are in $V(P)$.*

With that we can obtain a path between two vertices in any hull set having v in its interior, but it does not guarantee the existence of a geodesic. However, for some vertices if there was a geodesic not containing v then that would create a block that is not allowed in a cactus graph. Next we present some restrictions for these paths.

Lemma 4.4 *Let D be an oriented cactus graph with $C \subseteq D$ a cycle, $u, v \in V(D)$ and P an (u, v)-path.*

(i) *If there is a $w \in V(P)$ that does not belong to any cycle of D then every (u, v)-path contains w.*

(ii) *Let $(w_{1_P}, \ldots, w_{q_P}) = P' \subseteq P \cap C$ be maximum with $q_P \geq 2$. Then:*
   (a) every (u, v)-path goes through C;

(b) if $u \notin V(C)$ then w_{1_P} is the same vertex for every path P; and
(c) if $v \notin V(C)$ then w_{q_P} is the same vertex for every path P.

By combining the arguments provided by Lemmas 4.2, 4.3 and 4.4, one may deduce how to obtain a minimum hull set of an oriented cactus.

Theorem 4.5 *Let D be an oriented cactus graph. Then there exists a minimum hull set S of D composed by the extreme vertices of D and by exactly one non-extreme vertex of each unsatisfactory cycle. Moreover, $I[S]$ contains all vertices that are not in a satisfactory cycle and $I^2[S] = V(D)$.*

The proof argues which vertex must be chosen in each unsatisfactory cycle. All such vertices can be found in linear-time. Thus, $\overrightarrow{\mathrm{hn}}(D)$ can be found in linear-time, for every oriented cactus D.

The above theorem motivated us to also work on the geodetic number for these oriented graphs. Since a geodetic set is also a hull set, every geodetic set must have a non-extreme vertex of each unsatisfactory cycle. Besides, seen as only some satisfactory cycles may have vertices that are not obtained in the first iteration of the interval function, we studied such cycles in order to obtain a minimum geodetic set. As a result, we define the *falsely satisfactory* cycles, FSC for short. These can be of two types:

Type 1: $\mathrm{Ext}(C) = \{u_1, u_2\}$, u_1 is a source and u_2 is a sink. The (u_1, u_2)-paths have distinct lengths, P being the longest. P has length at least three and one of its internal vertices is a cut-vertex in D. Besides, all the following statements hold:
- (1) If there is an RCV v_1 internal to P, the (u_1, v_1)-path must have length at least two;
- (2) If there is a TCV v_2 internal to P, the (v_2, u_2)-path must have length at least two;
- (3) If there are both an RCV v_1 and a TCV v_2 internal to P, (1) and (2) hold and we must have $P = (u_1, \ldots, v_2, \ldots, v_1, \ldots, u_2)$. Moreover, the (v_2, v_1)-path must also have length at least two.

Type 2: The cycle C is directed and there are distinct RCV v_1 and TCV v_2 in C such that:
- (1) $d_C(v_2, v_1) \geq 1$; and
- (2) all the other cut-vertices are internal to $P = (w_0, w_1, \ldots, w_{k-1}, w_k)$, where $w_0 = v_1$ and $w_k = v_2$. Besides, if w_i is an RCV and w_j is a TCV then $i \leq j$ for every $i, j \in \{0, 1, \ldots, k\}$.

Otherwise, we say that the cycle is *truly satisfactory* and use TSC to simplify.

Lemma 4.6 *Let D be an oriented cactus graph and $C \subseteq D$ be a satisfactory cycle. Then:*

- *if C is truly satisfactory and S is a minimum hull set of D then $I[S] \supseteq V(C)$;*
- *if C is falsely satisfactory and $\mathcal{S} = N^+(V(C)) \cup N^-(V(C))$ then $I[\mathcal{S}] \not\supseteq V(C)$. Moreover the vertices not in $I[S]$ are the following ones:*

⋆ *if C is of type 1, the internal vertices of the (w_1, w_2)-path where $w_1 \in \{u_1, v_2\}$ and $w_2 \in \{u_2, v_1\}$ are as close as possible;*
⋆ *if C is of type 2, the internal vertices of the (v_2, v_1)-path.*
And also none of these vertices is a cut-vertex.

Since, for each FSC, the vertices not in $I[S]$ are not connected to the rest of the graph, we conclude that every geodetic set must have at least one non-extreme vertex of each FSC.

Theorem 4.7 *Let D be an oriented cactus graph. There is a geodetic set composed by all the extreme vertices, one non-extreme of each unsatisfactory cycle and one of each FSC. Moreover, this geodetic set is minimum.*

Once more, such minimum geodetic set can be found in linear time by just analyzing the cycles and determining which ones are (truly/falsely) satisfactory and unsatisfactory.

5 Further research

We first proved that the hull number of an oriented split graph $D = (S \cup C, A)$ is roughly $\frac{2}{3}(|C|)$ plus the number of its extreme vertices and $|\operatorname{Ext}(G[C])|$. A natural question is whether a similar bound holds for the geodetic number of an oriented split graph or, at least, a tournament.

Here we also proved that, given an oriented graph D, determining $\overrightarrow{\mathrm{hn}}(D)$ and $\overrightarrow{\mathrm{gn}}(D)$ are NP-hard problems even if the underlying graph of D is bipartite. Equivalent results were known in the literature [1,9] for the undirected case. We believe that the same is true concerning the class of chordal graphs. Determining $\mathrm{hn}(G)$ and $\mathrm{gn}(G)$ are NP-hard problems even if G is chordal [3,9]. A first open problem would be:

Problem 5.1 *Is it true that, given an oriented graph D and a positive integer k, then determining whether $\overrightarrow{hn}(D) \leq k$ or whether $\overrightarrow{gn}(D) \leq k$ are* NP*-complete problems, even if the underlying graph of D is chordal?*

In fact, even determining such parameters for tournaments seems a hard task.

Another natural problem is to find some graph class $\mathcal{G}$ for which determining $\overrightarrow{\mathrm{hn}}(D)$ is an NP-hard problem, while determining $\mathrm{hn}(G)$ can be solved in polynomial time, for some simple graph $G \in \mathcal{G}$ and some orientation D of G. The same should also be studied for the geodetic number.

Finally, bounds and complexity results for other graph classes (e.g. planar graphs, graphs with bounded treewidth, graphs with few P_4's, etc.) are also widely open.

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