

Monoid-labeled transition systems

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Abstract

Given a \vee -complete (semi)lattice \mathcal{L} , we consider \mathcal{L} -labeled transition systems as coalgebras of a functor $\mathcal{L}^{(-)}$, associating with a set X the set \mathcal{L}^X of all \mathcal{L} -fuzzy subsets. We describe simulations and bisimulations of \mathcal{L} -coalgebras to show that $\mathcal{L}^{(-)}$ weakly preserves nonempty kernel pairs iff it weakly preserves nonempty pullbacks iff \mathcal{L} is join infinitely distributive (JID).

Exchanging \mathcal{L} for a commutative monoid \mathcal{M} , we consider the functor $\mathcal{M}_\omega^{(-)}$ which associates with a set X all finite multisets containing elements of X with multiplicities $m \in M$. The corresponding functor weakly preserves nonempty pullbacks along injectives iff 0 is the only invertible element of \mathcal{M} , and it preserves nonempty kernel pairs iff \mathcal{M} is *refinable*, in the sense that two sum representations of the same value, $r_1 + \dots + r_m = c_1 + \dots + c_n$, have a common refinement matrix $(m_{i,j})$ whose k -th row sums to r_k and whose l -th column sums to c_l for any $1 \leq k \leq m$ and $1 \leq l \leq n$.

Key words: Coalgebra, transition system, fuzzy transition, multiset, weak pullback preservation, bisimulation, refinable monoid, distributive lattice.

1 Introduction

It is well known that transition systems can be described as coalgebras of the covariant powerset functor \mathcal{P} . The general theory of coalgebras automatically supplies the fundamental notions of *homomorphism* and *bisimulation*. The fact that the functor \mathcal{P} is well behaved, i.e. that it preserves (generalized) weak pullbacks, guarantees a collection of useful properties. In particular, the relational product of bisimulations is a bisimulation, the largest bisimulation is an equivalence relation and kernels of homomorphisms are always bisimulations. The subclass of image-finite transition systems corresponds to coalgebras of the finite powerset functor \mathcal{P}_ω . This functor, in addition, is bounded, so a terminal \mathcal{P}_ω -coalgebra exists, providing semantics and a proof principle (coinduction) for image-finite transition systems.

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Introducing labels does not complicate the situation. Given a set L of labels, an L -labeled transition system on a state set S is a ternary relation $T \subseteq S \times L \times S$. Again, the equivalent coalgebraic view as a map $\alpha : S \rightarrow \mathcal{P}(S)^L$, i.e. as a $\mathcal{P}(-)^L$ -coalgebra, better captures the dynamic aspects of labeled transition systems. What has been said above for \mathcal{P} -, resp. \mathcal{P}_ω -coalgebras carries over to $\mathcal{P}(-)^L$ -, resp. $\mathcal{P}_\omega(-)^L$ -coalgebras. In fact an L -labeled transition system is nothing but a collection of (plain) transition systems, one for each label.

The situation becomes more interesting, when the labels carry some algebraic structure. This structure is relevant, when arc labels denote, for instance, flow capacities or durations.

In general, we shall assume the labels to define a *commutative monoid*, i.e. a commutative, associative operation with a neutral element 0. This allows us a coalgebraic interpretation of labeled transition systems as a map from states to graded sets of successor states. This view will be seen to provide an interesting intertwining of coalgebraic structure with algebraic properties of the monoid.

Another motivation for this work is the study of *Set*-functors. In previous work ([Gum98],[GSb],[GS00]) we have studied the coalgebraic significance of various preservation properties of *Set*-endofunctors. Here we construct such functors, that are parameterized by a commutative monoid, so we can custom build functors by selecting commutative monoids with appropriate algebraic properties.

In the first sections, we start with an arbitrary \vee -complete semilattice \mathcal{L} , and we consider \mathcal{L} -multisets. A multiset S can be thought of as a set, each of whose elements s is contained in S with some multiplicity (probability, certainty) $l \in \mathcal{L}$. There are plenty of natural choices for \mathcal{L} , such as $\{0, 1\}$, $\mathbb{N} \cup \{\infty\}$, or the real interval $[0, 1]$, giving rise to the standard notions of (plain old) set, bag (multiset), or fuzzy set.

The semilattice \mathcal{L} gives rise to a functor $\mathcal{L}^{(-)}$ which generalizes the powerset functor \mathcal{P} for $\mathcal{L} = \{0, 1\}$. Thus $\mathcal{L}^{(-)}$ -coalgebras generalize transition systems to multiset transition systems and fuzzy transition systems.

We show that $\mathcal{L}^{(-)}$ always preserves nonempty pullbacks along injective maps, and that it preserves arbitrary nonempty weak pullbacks just in case \mathcal{L} is *join infinitely distributive*, that is, finite meets distribute over infinite joins.

In semilattices, the possibility of defining infinite sums is closely tied to the idempotent law. In arbitrary commutative monoids, we can form infinite sums only in cases where all but finitely many summands are zero. This leads to a slightly different functor, $\mathcal{M}_\omega^{(-)}$, for an arbitrary commutative monoid \mathcal{M} . We study it in the second half of this paper. This functor preserves nonempty pullbacks along injective maps iff the monoid is *positive* and it preserves nonempty weak pullbacks of arbitrary maps iff additionally the monoid is *refinable* in a sense to be defined later.

2 Basic notions

Let $R \subseteq A \times B$ and $S \subseteq B \times C$ be binary relations. We denote by $(R; S)$ their *composition* or *relational product*:

$$(R; S) := \{(a, c) \mid \exists b \in B. (a, b) \in R, (b, c) \in S\}.$$

With R^- we denote the converse of R , i.e. $R^- := \{(b, a) \mid (a, b) \in R\}$. We use “ aRb ” as a shorthand for “ $(a, b) \in R$ ”.

If $\varphi : A \rightarrow B$ is a map then we denote by $G(\varphi)$ its graph, that is

$$G(\varphi) := \{(a, \varphi(a)) \mid a \in A\}.$$

With these definitions we get: $G(\psi \circ \varphi) = G(\varphi); G(\psi)$.

2.1 F -coalgebras and homomorphisms

Let $F : \mathbf{Set} \rightarrow \mathbf{Set}$ be a functor from the category of sets to itself. A *coalgebra of type F* is a pair $\mathcal{A} = (A, \alpha)$, consisting of a set A and a map $\alpha : A \rightarrow F(A)$. A is called the *carrier set* and α is called the *structure map* of \mathcal{A} .

If $\mathcal{A} = (A, \alpha)$ and $\mathcal{B} = (B, \beta)$ are F -coalgebras, then a map $\varphi : A \rightarrow B$ is called a *homomorphism*, if

$$\beta \circ \varphi = F(\varphi) \circ \alpha,$$

that is, if the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ \alpha \downarrow & & \downarrow \beta \\ F(A) & \xrightarrow{F(\varphi)} & F(B) \end{array}$$

F -coalgebras and their homomorphisms form a category \mathbf{Set}_F . It is well known that all colimits in \mathbf{Set}_F exist, and they are formed just as in \mathbf{Set} . In particular, the sum $\sum_{i \in I} \mathcal{A}_i$ of a family of F -coalgebras $\mathcal{A}_i = (A_i, \alpha_i)$ has as carrier the disjoint union $\biguplus_{i \in I} A_i$ and the coalgebra structure is the unique map $\alpha : \biguplus_{i \in I} A_i \rightarrow F(\biguplus_{i \in I} A_i)$ with

$$\alpha \circ \iota_{A_i} = F(\iota_{A_i}) \circ \alpha_i$$

for all $i \in I$, where each ι_{A_i} is the canonical embedding of A_i into the disjoint union $\biguplus_{i \in I} A_i$.

2.2 Subcoalgebras

A subset $U \subseteq A$ is called a *subcoalgebra* of $\mathcal{A} = (A, \alpha)$, provided there exists a coalgebra structure $\nu : U \rightarrow F(U)$ so that the inclusion map $\subseteq_U^A : U \rightarrow A$ is a homomorphism from $\mathcal{U} = (U, \nu)$ to \mathcal{A} .

One reads off the definition of homomorphism, that U is a subcoalgebra of $\mathcal{A} = (A, \alpha)$ iff

$$\forall u \in U. \exists v \in F(U). \alpha(u) = F(\subseteq_U^A)(v).$$

$U = \emptyset$ is always a subcoalgebra. If $U \neq \emptyset$, then the inclusion map \subseteq_U^A has a left inverse. Consequently, $F(\subseteq_U^A)$ has a left inverse too, in particular it is injective. Thus, if a structure map ν as above exists, it is unique. For that reason we use the term “subcoalgebra” interchangeably for the coalgebra \mathcal{U} as well as for its carrier set U .

2.3 Bisimulations

A binary relation $R \subseteq A \times B$ is called a bisimulation between \mathcal{A} and \mathcal{B} if it is possible to define a coalgebra structure $\delta : R \rightarrow F(R)$, so that the projection maps $\pi_1 : R \rightarrow A$ and $\pi_2 : R \rightarrow B$ are homomorphisms. This can be expressed by the following commutative diagram:

$$\begin{array}{ccccc} A & \xleftarrow{\pi_1} & R & \xrightarrow{\pi_2} & B \\ \alpha \downarrow & & \downarrow \delta & & \downarrow \beta \\ F(A) & \xleftarrow{F(\pi_1)} & F(R) & \xrightarrow{F(\pi_2)} & F(B) \end{array}$$

If R is a bisimulation between \mathcal{A} and \mathcal{B} , then its converse R^- is a bisimulation between \mathcal{B} and \mathcal{A} . The union of a family of bisimulations is a bisimulation again, so there is always a largest bisimulation $\sim_{\mathcal{A}, \mathcal{B}}$ between coalgebras \mathcal{A} and \mathcal{B} .

If $\mathcal{A} = \mathcal{B}$, then we shall call R a bisimulation *on* \mathcal{A} . The diagonal relation $\Delta_A = \{(a, a) \mid a \in A\}$ is always a bisimulation, hence the largest bisimulation on \mathcal{A} , denoted by $\sim_{\mathcal{A}}$, is always reflexive and symmetric. In general, though, it need not be transitive.

A map $\varphi : A \rightarrow B$ is a homomorphism between coalgebras \mathcal{A} and \mathcal{B} , iff its graph $G(\varphi)$ is a bisimulation. If \mathcal{R} is a coalgebra, and $\varphi : \mathcal{R} \rightarrow \mathcal{A}$ and $\psi : \mathcal{R} \rightarrow \mathcal{B}$ are homomorphisms, then $\{(\varphi(r), \psi(r)) \mid r \in R\} = (G(\varphi)^- ; G(\psi))$ is a bisimulation between \mathcal{A} and \mathcal{B} . As a consequence, a bisimulation R between \mathcal{A} and \mathcal{B} gives rise to a bisimulation $\tilde{R} := \{(\iota_A(a), \iota_B(b)) \mid (a, b) \in R\}$ on their sum $\mathcal{A} + \mathcal{B}$.

3 The functor $\mathcal{L}^{(-)}$

Let \mathcal{L} be a complete \vee -semilattice. \mathcal{L} can be made into a complete lattice by defining for any subset S :

$$\bigwedge S = \bigvee \{l \in L \mid \forall s \in S. l \leq s\}.$$

For any set A , let \mathcal{L}^A denote the set of all maps $\sigma : A \rightarrow \mathcal{L}$. Each such σ can be thought of as the characteristic function of an \mathcal{L} -multiset:

$$x \in_l \sigma \iff \sigma(x) = l.$$

Given a map $f : A \rightarrow B$, we define a map $\mathcal{L}^f : \mathcal{L}^A \rightarrow \mathcal{L}^B$ by

$$\mathcal{L}^f(\sigma)(b) := \bigvee \{\sigma(a) \mid f(a) = b\},$$

then it is easy to check:

Lemma 3.1 $\mathcal{L}^{(-)}$ is a (covariant) *Set*-endofunctor.

Proof. Given sets A, B , and C and maps $f : A \rightarrow B, g : B \rightarrow C$, we calculate for arbitrary $\sigma : A \rightarrow \mathcal{L}, a \in A$, and $c \in C$:

$$\begin{aligned} \mathcal{L}^{id_A}(\sigma)(a) &= \bigvee \{\sigma(x) \mid id_A(x) = a\} \\ &= \sigma(a) \\ &= id_{\mathcal{L}^A}(\sigma)(a). \end{aligned}$$

$$\begin{aligned} (\mathcal{L}^g \circ \mathcal{L}^f)(\sigma)(c) &= \bigvee \{\mathcal{L}^f(\sigma)(b) \mid g(b) = c\} \\ &= \bigvee \{\bigvee \{\sigma(a) \mid f(a) = b\} \mid g(b) = c\} \\ &= \bigvee \{\sigma(a) \mid (g \circ f)(a) = c\} \\ &= \mathcal{L}^{g \circ f}(\sigma)(c). \end{aligned}$$

Obviously, when \mathcal{L} is the two-element lattice $\{0, 1\}$, this functor is naturally isomorphic to the powerset functor \mathcal{P} via $\eta : \mathcal{L}^{(-)} \rightarrow \mathcal{P}$, defined for any set X by

$$\eta_X(\sigma) := \{x \in X \mid \sigma(x) = 1\}.$$

It is also straightforward to check that each \bigvee -preserving map $\varphi : \mathcal{L} \rightarrow \mathcal{L}'$ induces a natural transformation between $\mathcal{L}^{(-)}$ and $\mathcal{L}'^{(-)}$.

3.1 \mathcal{L} -Coalgebras.

According to the general definition of coalgebras, an $\mathcal{L}^{(-)}$ -coalgebra is a pair (A, α) consisting of a set A and a map $\alpha : A \rightarrow \mathcal{L}^A$. Given two $\mathcal{L}^{(-)}$ -coalgebras (A, α) and (B, β) , a map $\varphi : A \rightarrow B$ is a homomorphism if $\mathcal{L}^\varphi \circ \alpha = \beta \circ \varphi$, that is, if for all $a \in A$ and $b \in B$ we have

$$\beta(\varphi(a))(b) = \mathcal{L}^\varphi(\alpha(a))(b) = \bigvee \{\alpha(a)(a') \mid a' \in A, \varphi(a') = b\}.$$

In the sequel we shall speak of \mathcal{L} -coalgebras rather than of $\mathcal{L}^{(-)}$ -coalgebras. It is convenient to introduce an “arrow notation” familiar from transition

systems as a shorthand. We write

$$a \xrightarrow{l} a' \quad \text{iff} \quad \alpha(a)(a') = l.$$

Obviously, the \mathcal{L} -coalgebra structure $\alpha : A \rightarrow \mathcal{L}^A$ may also be interpreted as an \mathcal{L} -graded relation $\hat{\alpha} : A \times A \rightarrow \mathcal{L}$ by setting $\hat{\alpha}(a, a') := \alpha(a)(a')$.

When \mathcal{L} is the two-element lattice then an \mathcal{L} -coalgebra is just a Kripke-frame. In the general case, \mathcal{L} provides a set of measures for indicating how “strong” a pair (a, a') should be in the \mathcal{L} -graded relation $\hat{\alpha}$, or, alternatively, how confident we might be that (a, a') is in $\hat{\alpha}$. If \mathcal{L} is the unit interval, then $\hat{\alpha}$ is just a *fuzzy relation*.

3.2 Subcoalgebras

The functor $\mathcal{L}^{(-)}$ is not *standard* in the sense defined in [Mos99], that is

$$\mathcal{L}^{(\subseteq_U^A)} \neq \subseteq_{\mathcal{L}^U}^{\mathcal{L}^A}.$$

Rather, we always have:

$$\mathcal{L}^{(\subseteq_U^A)}(\tau)(a) = \begin{cases} \tau(a), & \text{if } a \in U \\ 0, & \text{otherwise.} \end{cases}$$

From that one obtains straightforwardly:

Lemma 3.2 *$U \subseteq A$ is a subcoalgebra of $\mathcal{A} = (A, \alpha)$ iff*

$$\forall u \in U, a \in A. \forall m \neq 0. u \xrightarrow{m} a \implies a \in U.$$

Proof.

$$\begin{aligned} U \text{ is a subcoalgebra of } \mathcal{A} &= (A, \alpha) \\ \iff \forall u \in U. \exists \tau : U \rightarrow \mathcal{L}. \alpha(u) &= \mathcal{L}^{(\subseteq_U^A)}(\tau) \\ \iff \forall u \in U. \forall a \in (A - U). \alpha(u)(a) &= 0 \\ \iff \forall u \in U, a \in A \forall m \neq 0. u \xrightarrow{m} a &\implies a \in U. \end{aligned}$$

For coalgebras of arbitrary type F it is known that subcoalgebras are always closed under finite intersections (see [GSa]). For the functor $F = \mathcal{L}^{(-)}$, we even get from the above:

Corollary 3.3 *The intersection of an arbitrary family $(U_i)_{i \in I}$ of subcoalgebras of an \mathcal{L} -coalgebra \mathcal{A} is again a subcoalgebra of \mathcal{A} .*

3.3 \mathcal{L} -Simulations

Definition 3.4 We define a *simulation* between \mathcal{A} and \mathcal{B} as a relation $S \subseteq A \times B$ where for all $(a, b) \in S$ and all $a' \in A$

$$a \xrightarrow{m} a' \implies \bigvee \{l \mid \exists y \in B. b \xrightarrow{l} y, a' S y\} \geq m.$$

Thus, if a is simulated by b and if a' is an m -successor of a , then there must be sufficiently many l_i -successors b_i of b , each one simulating a' and together yielding $\bigvee_{i \in I} l_i \geq m$.

We write $\mathcal{A} \xrightarrow{S} \mathcal{B}$, if S is a simulation between \mathcal{A} and \mathcal{B} . Then we conclude directly from the definition:

Lemma 3.5 *If $\mathcal{A} \xrightarrow{S} \mathcal{B}$ and $\mathcal{B} \xrightarrow{T} \mathcal{C}$, then $\mathcal{A} \xrightarrow{S;T} \mathcal{C}$. Thus the class of all \mathcal{L} -coalgebras with simulations as arrows forms a category.*

3.4 \mathcal{L} -Homomorphisms

Homomorphisms turn out to be maps between coalgebras which strengthen the graded relation and whose converse is a simulation, that is:

Lemma 3.6 *A map $\varphi : A \rightarrow B$ between coalgebras $\mathcal{A} = (A, \alpha)$ and $\mathcal{B} = (B, \beta)$ is a homomorphism if and only if the following two conditions are satisfied for all $a, a' \in A$, all $b' \in B$ and for all $m \in \mathcal{L}$:*

$$a \xrightarrow{m} a' \implies \varphi(a) \xrightarrow{m'} \varphi(a') \text{ for some } m' \geq m \quad (1)$$

$$\varphi(a) \xrightarrow{m} b' \implies m \leq \bigvee \{l \mid \exists x \in A. a \xrightarrow{l} x, \varphi(x) = b'\}. \quad (2)$$

Proof. The homomorphism condition requires that for every $a \in A$ and every $b' \in B$ we have the equation

$$\beta(\varphi(a))(b') = \mathcal{L}^\varphi(\alpha(a))(b').$$

Translating this, using the arrow-notation, we get

$$\hat{\beta}(\varphi(a), b') = \bigvee \{l \mid \exists x \in A. a \xrightarrow{l} x, \varphi(x) = b'\}.$$

The two homomorphism conditions are equivalent to the inequalities which are obtained by replacing “=” above by “ \geq ” and “ \leq ”. From the first homomorphism condition we obtain $\hat{\beta}(\varphi(a), b') \geq l$ whenever $a \xrightarrow{l} x$ and $\varphi(x) = b'$, thus $\beta(\varphi(a))(b') \geq \bigvee \{l \mid \exists x \in A. a \xrightarrow{l} x, \varphi(x) = b'\}$. From this inequality, in turn, we get the first homomorphism condition back by considering only $x = a'$ in the right hand side, to obtain $\hat{\beta}(\varphi(a), \varphi(a')) \geq l$ whenever $a \xrightarrow{l} a'$. The second homomorphism condition is obviously equivalent with the inequality obtained from replacing “=” with “ \leq ”.

Using the description of homomorphisms and of subcoalgebras, it is now straightforward to check:

Corollary 3.7 *If $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ is a homomorphism and $V \leq B$ is a subcoalgebra of \mathcal{B} , then $\varphi^{-}[V]$ is a subcoalgebra of \mathcal{A} .*

Observe that the first, resp. second, homomorphism condition just states that $G(\varphi)$, resp. $G(\varphi)^{-}$, is a simulation, thus we get:

Corollary 3.8 *A map $\varphi : A \rightarrow B$ is a homomorphism between \mathcal{L} -coalgebras \mathcal{A} and \mathcal{B} if and only if both $G(\varphi)$ and $G(\varphi)^{-}$ are simulations.*

3.5 \mathcal{L} -Bisimulations

We have already remarked that a map between coalgebras \mathcal{A} and \mathcal{B} is a homomorphism iff its graph is a bisimulation. However, a relation R is not necessarily a bisimulation even if R and R^{-} are simulations. To see the connections we have to characterize bisimulations:

Proposition 3.9 *A relation R between coalgebras $\mathcal{A} = (A, \alpha)$ and $\mathcal{B} = (B, \beta)$ is a bisimulation if and only if for all $(a, b) \in R$ and all $a' \in A$, $b' \in B$ we have:*

$$a \xrightarrow{m} a' \implies \bigvee \{m \wedge l \mid \exists y \in B. b \xrightarrow{l} y, a' R y\} = m, \quad \text{and} \quad (3)$$

$$b \xrightarrow{m} b' \implies \bigvee \{m \wedge l \mid \exists x \in A. a \xrightarrow{l} x, x R b'\} = m. \quad (4)$$

Proof. Let R be a bisimulation, and $(a, b) \in R$, then there is some $\tau := \delta(a, b) : R \rightarrow \mathcal{L}$ with

$$\alpha(a) = \mathcal{L}^{\pi_1}(\tau), \quad \text{and} \quad (5)$$

$$\beta(b) = \mathcal{L}^{\pi_2}(\tau). \quad (6)$$

For $a' \in A$ we have therefore by (5)

$$\begin{aligned} \hat{\alpha}(a, a') &= \mathcal{L}^{\pi_1}(\tau)(a') \\ &= \bigvee \{\tau(u) \mid \pi_1(u) = a'\} \\ &= \bigvee \{\tau(a', y) \mid a' R y\}. \end{aligned}$$

By (6) we have for every y with $a' R y$:

$$\begin{aligned} \hat{\beta}(b, y) &= \mathcal{L}^{\pi_2}(\tau)(y) \\ &= \bigvee \{\tau(v) \mid \pi_2(v) = y\} \\ &= \bigvee \{\tau(x, y) \mid x R y\} \\ &\geq \tau(a', y). \end{aligned}$$

Combining the above, we get

$$\begin{aligned}
\hat{\alpha}(a, a') &= \bigvee \{ \tau(a', y) \mid a' R y \} \\
&= \bigvee \{ \hat{\alpha}(a, a') \wedge \tau(a', y) \mid a' R y \} \\
&\leq \bigvee \{ \hat{\alpha}(a, a') \wedge \hat{\beta}(b, y) \mid a' R y \} \\
&\leq \hat{\alpha}(a, a').
\end{aligned}$$

Hence, $\hat{\alpha}(a, a') = \bigvee \{ \hat{\alpha}(a, a') \wedge \hat{\beta}(b, y) \mid a' R y \}$, which is the first bisimulation condition. The second one is obtained symmetrically.

To show the converse, let the bisimulation conditions (3) and (4) be satisfied. We need to show that R is a bisimulation. Define a structure map $\delta : R \rightarrow \mathcal{L}^R$ by

$$\delta(a, b)(x, y) := \hat{\alpha}(a, x) \wedge \hat{\beta}(b, y).$$

For an arbitrary $a' \in A$ we have

$$\begin{aligned}
\mathcal{L}^{\pi_1}(\delta(a, b))(a') &= \bigvee \{ \delta(a, b)(u) \mid \pi_1(u) = a' \} \\
&= \bigvee \{ \delta(a, b)(a', y) \mid a' R y \} \\
&= \hat{\alpha}(a, a') \\
&= \alpha(a)(a').
\end{aligned}$$

Hence $\mathcal{L}^{\pi_1}(\delta(a, b)) = \alpha(a) = (\alpha \circ \pi_1)(a, b)$, thus $\mathcal{L}^{\pi_1} \circ \delta = \alpha \circ \pi_1$, and, analogously, $\mathcal{L}^{\pi_2} \circ \delta = \beta \circ \pi_2$. Therefore, R is a bisimulation.

Notice that lemma 3.6 could be inferred from the above, since for general coalgebras, homomorphisms are precisely those maps whose graph is a bisimulation. We also get the following corollary:

Corollary 3.10 *If R is a bisimulation then both R and R^- are simulations.*

4 The role of distributivity

In the applications of coalgebras one often has type functors which satisfy an important technical property: They preserve weak pullbacks. This is to say that a (weak) pullback diagram is transformed into a weak pullback diagram. Many results in coalgebra theory (e.g. [Rut00]) hinge on this property. In [GSa] it was shown that a functor F preserves nonempty weak pullbacks if and only if bisimulations between F -coalgebras are closed under composition.

Now consider corollary 3.10. If its converse is true then by lemma 3.5 bisimulations will be closed under composition, thus $\mathcal{L}^{(-)}$ will preserve weak pullbacks. In this section we shall show that this and related properties hinge on a certain distributivity condition on the lattice \mathcal{L} .

Definition 4.1 [[Grä98]] A lattice is called *join infinite distributive* (in short JID), if it satisfies the law

$$x \wedge \bigvee \{ x_i \mid i \in I \} = \bigvee \{ x \wedge x_i \mid i \in I \}.$$

If \mathcal{L} is a complete semilattice, we shall say that \mathcal{L} is JID iff this is the case for the lattice induced on \mathcal{L} .

Whenever \mathcal{L} is JID, the bisimilarity conditions for \mathcal{L} -coalgebras simplify to:

Lemma 4.2 *If \mathcal{L} is JID then a relation $R \subseteq A \times B$ is a bisimulation between coalgebras \mathcal{A} and \mathcal{B} if and only if*

$$a \xrightarrow{m} a' \implies \bigvee \{l \mid b \xrightarrow{l} y, a' R y\} \geq m, \quad \text{and} \quad (7)$$

$$b \xrightarrow{m} b' \implies \bigvee \{l \mid a \xrightarrow{l} x, x R b'\} \geq m. \quad (8)$$

The following theorem, amongst other things, provides a converse to this lemma and to corollary 3.10:

Theorem 4.3 *Let \mathcal{L} be a \bigvee -semilattice, then the following are equivalent:*

- (i) \mathcal{L} is JID.
- (ii) $R \subseteq A \times B$ is a bisimulation $\iff R$ and R^- are simulations.
- (iii) Bisimulations are closed under compositions.
- (iv) $\mathcal{L}^{(-)}$ preserves nonempty weak pullbacks.
- (v) $\mathcal{L}^{(-)}$ weakly preserves nonempty kernel pairs.
- (vi) The largest bisimulation $\sim_{\mathcal{A}}$ on every \mathcal{L} -coalgebra is transitive.

Proof. (i) \rightarrow (ii) follows from lemma 4.2. (ii) \rightarrow (iii) is a consequence of lemma 3.5. (iii) \rightarrow (iv) is shown for arbitrary functors in [GS00]. (iv) \rightarrow (v) \rightarrow (vi) can be found in [GS00], so we may concentrate on proving (vi) \rightarrow (i).

Let a family $(l_i)_{i \in I}$ and a further element m of \mathcal{L} be given, it suffices to show

$$m \wedge \bigvee_{i \in I} l_i \leq \bigvee_{i \in I} (m \wedge l_i),$$

since the reverse inclusion holds in any lattice. On the sets

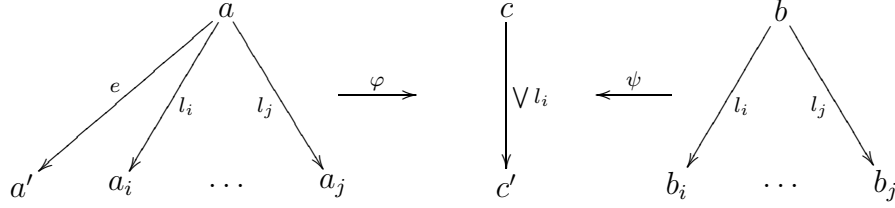
$$\begin{aligned} A &:= \{a_i \mid i \in I\} \cup \{a, a'\} \\ B &:= \{b_i \mid i \in I\} \cup \{b\} \\ C &:= \{c, c'\} \end{aligned}$$

define coalgebra structures α , β , and γ given in arrow-notation as follows:

$$\begin{aligned} a &\xrightarrow{e} a', \text{ where } e := m \wedge \bigvee l_i \\ a &\xrightarrow{l_i} a_i, \text{ for all } i \in I \\ b &\xrightarrow{l_i} b_i, \text{ for all } i \in I \\ c &\xrightarrow{\bigvee l_i} c'. \end{aligned}$$

Using lemma 3.6 it is easy to see that $\varphi : A \rightarrow C$, defined as $\varphi(a) = c$ and $\varphi(a') = \varphi(a_i) = c'$ for all $i \in I$, and $\psi : B \rightarrow C$ given by $\psi(b) = c$ and

$\psi(b_i) = c'$ for all $i \in I$ are homomorphisms. Consequently, $G(\varphi)$ and $G(\psi)^-$ are bisimulations.



Consider now the sum of the three coalgebras $\mathcal{A} + \mathcal{B} + \mathcal{C}$. We still have that $G(\varphi)$ and $G(\psi)^-$ are bisimulations, in particular, they are contained in the largest bisimulation $\sim = \sim_{\mathcal{A} + \mathcal{B} + \mathcal{C}}$. Thus $a \sim c$ and $c \sim b$. By assumption, now $a \sim b$.

Proposition 3.9 tells us that there is a subcollection $(b_j)_{j \in J \subseteq I}$ with

$$e \leq \bigvee_{j \in J} \{e \wedge l_j \mid b \xrightarrow{l_j} b_j\} \leq \bigvee_{i \in I} (e \wedge l_i).$$

Hence

$$m \wedge \bigvee_{i \in I} l_i = e \leq \bigvee_{i \in I} (e \wedge l_i) = \bigvee_{i \in I} (m \wedge l_i),$$

finishing the proof.

The implication $(i) \rightarrow (iv)$ is due to S. Pfeiffer [Pfe99]. A finite lattice is distributive iff it does not contain one of the characteristic five-element nondistributive sublattices M_3 or N_5 , see ([Grä98]). By separately excluding these cases, she also obtained the converse $(iv) \rightarrow (i)$ in case that \mathcal{L} is finite and distributive.

Observe that nonempty weak pullbacks along injective maps, resp. nonempty pullbacks of an arbitrary collection of injective maps, are always preserved, without assuming JID. In [GS00], these conditions have been shown to be equivalent to homomorphic preimages of subcoalgebras, resp. arbitrary intersection of subcoalgebras, being subcoalgebras again. Thus, these results follow from corollaries 3.7 and 3.3.

One might wonder, whether in general, simulations could not have been defined by just one clause of 3.9 so that condition (ii) would automatically be satisfied for arbitrary \bigvee -semilattices \mathcal{L} . Notice, however, that with such a definition we would not have been able to show that simulations are closed under composition.

5 Labeling with a commutative monoid

In the definition of the functor $\mathcal{L}^{(-)}$ it is essential that \mathcal{L} has arbitrary suprema, i.e. that \mathcal{L} is \bigvee -complete. When trying to replace \mathcal{L} by an arbitrary commutative monoid $\mathcal{M} = (M, +, 0)$, we do not have infinite sums available anymore,

unless when almost all summands are 0. Hence, we must redefine the functor by only considering maps $\sigma : X \rightarrow M$ with *finite support*:

Definition 5.1 Let $\mathcal{M} = (M, +, 0)$ be a commutative monoid. Whenever X is a set and all but finitely many elements of a family $(g(x))_{x \in X}$ are 0, we denote its sum by $\sum(g(x) \mid x \in X)$.

Given any set X and a map $\sigma : X \rightarrow M$, we call

$$\text{supp}(\sigma) := \{x \in X \mid \sigma(x) \neq 0\}$$

the *support* of σ . Let

$$\mathcal{M}_\omega^X := \{\sigma : X \rightarrow M \mid |\text{supp}(\sigma)| < \omega\}$$

be the set of all maps from X to M with finite support, and for any map $f : X \rightarrow Y$ let \mathcal{M}_ω^f be the map defined on any $\sigma \in \mathcal{M}_\omega^X$ by:

$$\forall y \in Y. \mathcal{M}_\omega^f(\sigma)(y) := \sum(\sigma(x) \mid x \in X, f(x) = y).$$

One easily checks that $\mathcal{M}_\omega^f(\sigma)$ is a map from Y to M with finite support, so using associativity and commutativity of $+$, one verifies as before:

Lemma 5.2 $\mathcal{M}_\omega^{(-)}$ is a *Set-endofunctor*.

When \mathcal{M} is the two-element Boolean algebra $(\{0, 1\}, \vee, 0)$ then $\mathcal{M}_\omega^{(-)}$ is just the *finite powerset functor* $\mathcal{P}_\omega(-)$.

Coalgebras of type $\mathcal{M}_\omega^{(-)}$, in the sequel called \mathcal{M}_ω -coalgebras, may again be viewed as graphs with arcs labeled by elements of M , so we continue using the arrow-notation as in the case of \mathcal{L} -coalgebras. In particular, if (A, α) is an \mathcal{M}_ω -coalgebra, $a, a' \in A$ and $m \in M$, we can choose between the equivalent notations

$$\alpha(a)(a') = m, \quad \text{or} \quad \hat{\alpha}(a, a') = m, \quad \text{or} \quad a \xrightarrow{m} a'.$$

For \mathcal{M}_ω -coalgebras, the basic coalgebraic constructions can be easily described:

Lemma 5.3 Let $\mathcal{A} = (A, \alpha)$ and $\mathcal{B} = (B, \beta)$ be \mathcal{M}_ω -coalgebras, then

(i) $U \subseteq A$ is a subcoalgebra of \mathcal{A} , iff for all $u \in U$, and all $a \in A$:

$$u \xrightarrow{m} a, m \neq 0 \implies a \in U.$$

(ii) $\varphi : A \rightarrow B$ is a homomorphism iff for all $a \in A, b' \in B$:

$$\varphi(a) \xrightarrow{m} b' \iff m = \sum(m' \mid \exists a' \in A. a \xrightarrow{m'} a', \varphi(a') = b').$$

Corollary 5.4 The intersection of an arbitrary family $(\mathcal{U})_{i \in I}$ of subcoalgebras of an \mathcal{M}_ω -coalgebra \mathcal{A} is again a subcoalgebra of \mathcal{A} .

If $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ is a homomorphism and $U \subseteq B$ a subcoalgebra of \mathcal{B} , then $\varphi^{-}[U]$ need not be a subcoalgebra of \mathcal{A} . This stands in contrast to the situation for lattice labeled coalgebras (corollary 3.7). Thus, the functor $\mathcal{M}_\omega^{(-)}$ does in general not preserve nonempty pullbacks along injective maps ([GS00]). In the next section, we shall study algebraic conditions on the monoid \mathcal{M} which are responsible for such properties of the functor.

We conclude this section with a characterization of bisimulations $R \subseteq A \times B$ between \mathcal{M}_ω -coalgebras \mathcal{A} and \mathcal{B} . For this we consider the elements of \mathcal{M}_ω^A as vectors with $|A|$ many components and the elements of \mathcal{M}_ω^B as $|A| \times |B|$ -matrices with entries from M . From the definition of bisimulation in section 2.3, we obtain:

Lemma 5.5 *Let $\mathcal{A} = (A, \alpha)$ and $\mathcal{B} = (B, \beta)$ be \mathcal{M}_ω -coalgebras. A relation $R \subseteq A \times B$ is a bisimulation iff for every $(a, b) \in R$ there exists an $|A| \times |B|$ -matrix $(m_{x,y})$ with entries from M such that:*

- *all but finitely many $m_{x,y}$ are 0,*
- *$m_{x,y} \neq 0$ implies $(x, y) \in R$,*
- *$\alpha(a)$ is the vector of all row-sums of $(m_{x,y})$, i.e.*

$$\forall x \in A. \hat{\alpha}(a, x) = \sum (m_{x,y} \mid y \in B),$$

- *$\beta(b)$ is the vector of all column-sums of $(m_{x,y})$, i.e.*

$$\forall y \in B. \hat{\beta}(b, y) = \sum (m_{x,y} \mid x \in A).$$

5.1 Positive monoids.

Any commutative semigroup can be turned into a commutative monoid by simply adjoining a new element 0. The obtained monoid is rather special though, it can be internally characterized by the fact that no nonzero element is invertible:

Definition 5.6 A monoid element $m \in M$ is called *invertible* if there exists some $m^- \in M$ with $m + m^- = 0$. A monoid $\mathcal{M} = (M, +, 0)$ is called *positive* if 0 is the only invertible element.

If a commutative monoid is *not* positive, we can obtain one by factoring out the invertible elements, or by deleting all invertible ones, except for 0, for it is easy to check:

Lemma 5.7 *The invertible elements of a commutative monoid form a group $I(\mathcal{M})$. Factoring \mathcal{M} by the induced congruence relation yields the largest positive factor of \mathcal{M} . At the same time, any commutative monoid is the union of the subgroup $I(\mathcal{M})$ of invertible elements with a positive submonoid \mathcal{M}_+ .*

Example 5.8 The following monoids are positive:

- (i) $(\mathbb{N}, +, 0)$
- (ii) $(\mathbb{N} \setminus \{0\}, \cdot, 1)$
- (iii) $(L, \vee, 0)$ for any (semi)lattice with 0.

5.2 Refinable monoids.

We shall need to consider a further monoid condition which we shall call *refinable*. For this, let us consider an $m \times n$ -matrix $(a_{i,j})$ of monoid elements. Consider their row-sums $r_i = \sum_{1 \leq j \leq n} a_{i,j}$, and their column sums $c_j = \sum_{1 \leq i \leq m} a_{i,j}$, then by associativity and commutativity one obviously has $r_1 + \dots + r_m = c_1 + \dots + c_n$. Refinability is just the inverse condition, that is:

Definition 5.9 Given $m, n \in \mathbb{N}$, a monoid \mathcal{M} is called (m, n) -*refinable*, if for any $r_1, \dots, r_m, c_1, \dots, c_n \in M$ with $r_1 + \dots + r_m = c_1 + \dots + c_n$ one can find an $m \times n$ -matrix $(a_{i,j})$ of elements of M , whose row sums are r_1, \dots, r_m and whose column sums are c_1, \dots, c_n .

$$\begin{array}{ccc|c} a_{1,1} & \cdots & a_{1,n} & r_1 \\ \vdots & \cdots & \vdots & \cdots \\ a_{m,1} & \cdots & a_{m,n} & r_m \\ \hline c_1 & \cdots & c_n & \end{array}$$

Obviously, when $m = 1$ or $n = 1$, the condition is vacuous. When $m > 1$ then \mathcal{M} is $(m, 0)$ -refinable iff it is positive. For the remaining cases we prove:

Proposition 5.10 *For any $m, n > 1$ we have: A commutative monoid is (m, n) -refinable, iff it is $(2, 2)$ -refinable.*

Proof. In an $(m, n+1)$ -refinement of $r_1 + \dots + r_m = c_1 + \dots + c_n + 0$, we can add corresponding elements from the last two rows to obtain an (m, n) -refinement of $r_1 + \dots + r_m = c_1 + \dots + c_n$, so one direction is clear.

The other direction is proved by an easy induction over the number of columns, followed by a similar induction over the number of rows. As a hint, we show how to get from $(2, 2)$ to $(3, 2)$:

Given $r_1 + r_2 + r_3 = c_1 + c_2$, use the $(2, 2)$ refinement property to find a 2×2 -matrix $(a_{i,j})$ with column sums c_1, c_2 and row sums $r_1, (r_2 + r_3)$. Now $a_{2,1} + a_{2,2} = r_2 + r_3$, so there is another 2×2 matrix with row sums r_2, r_3 and column sums $a_{2,1}, a_{2,2}$:

$$\begin{array}{cc|c} a_{1,1} & a_{1,2} & r_1 \\ a_{2,1} & a_{2,2} & r_2 + r_3 \\ \hline c_1 & c_2 & \end{array} \quad \text{and} \quad \begin{array}{cc|c} b_{1,1} & b_{1,2} & r_2 \\ b_{2,1} & b_{2,2} & r_3 \\ \hline a_{2,1} & a_{2,2} & \end{array}$$

obviously now, the following matrix solves the original problem:

$$\begin{array}{cc|c} a_{1,1} & a_{1,2} & r_1 \\ b_{1,1} & b_{1,2} & r_2 \\ b_{2,1} & b_{2,2} & r_3 \\ \hline c_1 & c_2 & \end{array}$$

As a consequence of this proposition, we may simply call a commutative monoid *refinable* if it is $(2, 2)$ -refinable. Refinability does not imply positivity, since nontrivial abelian groups, for instance, are refinable, but not positive.

For the next section, we shall need the following observation, referring to infinite matrices:

Lemma 5.11 *Suppose that \mathcal{M} is refinable. Given X, Y nonempty sets, $\sigma \in \mathcal{M}_\omega^X$ and $\tau \in \mathcal{M}_\omega^Y$ with $\sum(\sigma(x) \mid x \in X) = \sum(\tau(y) \mid y \in Y)$. There exists an $|X| \times |Y|$ -matrix $(m_{x,y})$ with row sums $\sum(m_{x,y} \mid y \in Y) = \sigma(x)$ and column sums $\sum(m_{x,y} \mid x \in X) = \tau(y)$, where all but finitely many $m_{x,y}$ are 0.*

Proposition 5.10 makes it easy to check that the first two instances of example 5.8 are refinable. In fact, any commutative monoid which is cancellative and which satisfies

$$\forall c, r \in M. \exists x \in M. c = x + r \text{ or } r = x + c$$

is easily seen to be refinable. This also covers the case of $(\mathbb{N}, +, 0)$.

In the case of $(\mathbb{N} \setminus \{0\}, \cdot, 1)$, refinability is a consequence of the fact that every element has a unique prime factor decomposition.

Refinability does not carry over to submonoids. Consider, for instance, the submonoid $(\mathbb{N} \setminus \{0, 2\}, \cdot, 1)$ of the previous example, and the refinement problem $5 \cdot 6 = 3 \cdot 10$. Any refinement would require the prime factor 2, which is unavailable.

In the case of a lattice \mathcal{L} , we obtain a familiar property:

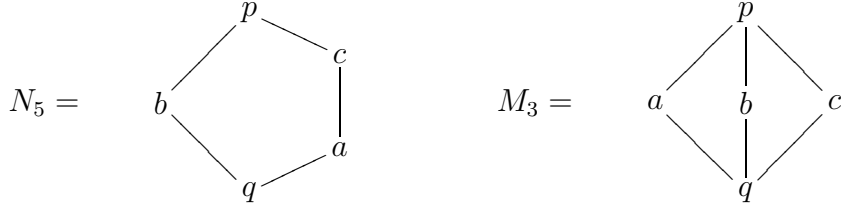
Lemma 5.12 *If \mathcal{L} is a lattice with smallest element 0, then $(L, \vee, 0)$ is refinable if and only if \mathcal{L} is distributive.*

Proof. Given a distributive lattice \mathcal{L} and $a, b, c, d \in L$ with $a \vee b = c \vee d$, then we have a refinement

$$\begin{array}{cc|c} a \wedge c & a \wedge d & a \\ b \wedge c & b \wedge d & b \\ \hline c & d & \end{array}$$

Conversely, if \mathcal{L} is not distributive, then one of the following lattices, known

as N_5 , resp. M_3 , must be a sublattice of \mathcal{L} (see e.g. [Grä98]):



In both cases, $a \vee b = b \vee c$. Suppose we had a refinement

x	y	a
z	u	b
b	c	

with $x, y, u, v \in L$. From the table, it follows that $u \leq b$ and $u \leq c$, so $u \leq b \wedge c = q \leq a$. Also, $y \leq a$, hence $u \vee y = c \leq a$. But $c \not\leq a$, both in M_3 and in N_5 .

5.3 Weak pullback preservation

We now study conditions under which the functor $\mathcal{M}_\omega^{(-)}$ weakly preserves nonempty kernel pairs, pullbacks along injectives, or arbitrary pullbacks.

A functor F is said to (weakly) preserve pullbacks along injective maps, provided for any $f : X \rightarrow Z$ and $g : Y \rightarrow Z$ with g injective, a (weak) pullback of f with g is transformed by F into a weak pullback of $F(f)$ with $F(g)$. In [GS00], we have shown that a $\mathcal{S}et$ -endofunctor F weakly preserves nonempty pullbacks along injective maps if and only the preimage $\varphi^{-}[V]$ of any F -subcoalgebra $\mathcal{V} \leq \mathcal{B}$ under a homomorphism $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ is again a subcoalgebra of \mathcal{A} .

For \mathcal{L} -coalgebras, this condition is always satisfied according to corollary 3.7. We shall see, however, that this is not necessarily the case for \mathcal{M}_ω -coalgebras. In fact we shall algebraically characterize those monoids \mathcal{M} for which $\mathcal{M}_\omega^{(-)}$ preserves weak pullbacks along injective maps.

Finally, we consider preservation of arbitrary weak pullbacks.

Theorem 5.13 *Let $\mathcal{M} = (M, +, 0)$ be a commutative monoid.*

- (i) $\mathcal{M}_\omega^{(-)}$ (weakly) preserves nonempty pullbacks along injective maps iff \mathcal{M} is positive.
- (ii) $\mathcal{M}_\omega^{(-)}$ weakly preserves nonempty kernel pairs iff \mathcal{M} is refinable.
- (iii) $\mathcal{M}_\omega^{(-)}$ weakly preserves nonempty pullbacks iff \mathcal{M} is positive and refinable.

Proof. (i): Assume that \mathcal{M} is positive, and let $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ be a homomorphism of \mathcal{M} -coalgebras and \mathcal{V} a subcoalgebra of \mathcal{B} . We need to show that

$\varphi^{-1}[V]$ is a subcoalgebra of \mathcal{A} . Given $a \in \varphi^{-1}[V]$, $a' \notin \varphi^{-1}[V]$ and $a \xrightarrow{m} a'$, we need to show that $m = 0$. We know that $\varphi(a) \in V$ and $\varphi(a') \notin V$, so by part (i) of lemma 5.3 we conclude $\varphi(a) \xrightarrow{0} \varphi(a')$. Part (ii) of the same lemma then yields $\sum (n \mid a \xrightarrow{n} x, \varphi(x) = \varphi(a')) = 0$. Positivity forces each summand to be 0, in particular $m = 0$.

To prove the converse, let $m_1, m_2 \in M$ be given with $m_1 + m_2 = 0$. Consider the coalgebra \mathcal{A} , given by a point p and two transitions to points q_1 and q_2 , labeled with m_1 and m_2 . Let \mathcal{B} consist of two points r and s with no transitions (all transitions labeled with 0). We get a homomorphism φ with $\varphi(p) = r$ and $\varphi(q_1) = \varphi(q_2) = s$. Now $\{r\}$ is a subcoalgebra of \mathcal{B} , and the assumption forces $\varphi^{-1}\{r\} = \{p\}$ to be a subcoalgebra of \mathcal{A} , but this implies $m_1 = m_2 = 0$.

$$\mathcal{A} = \begin{array}{c} p \\ \swarrow m_1 \quad \searrow m_2 \\ q_1 \quad q_2 \end{array} \xrightarrow{\varphi} \begin{array}{c} r \\ \downarrow 0 \\ s \end{array} = \mathcal{B}$$

A slight modification of this construction also gives us the backward direction of (ii): Assume that F weakly preserves nonempty kernel pairs, then kernels of homomorphisms are bisimulations. Suppose $m_1 + m_2 = s_1 + s_2$ in \mathcal{M} . We take a copy \mathcal{A}' of \mathcal{A} as above, but we label the arcs of \mathcal{A}' with s_1 and s_2 . If \mathcal{B}' is obtained from \mathcal{B} by changing the edge label to $m_1 + m_2 = s_1 + s_2$, there is an obvious homomorphism $\varphi : \mathcal{A} + \mathcal{A}' \rightarrow \mathcal{B}'$. Its kernel must be a bisimulation, so lemma 5.5, provides us with a refinement of $m_1 + m_2 = s_1 + s_2$.

We combine the 'if'-directions of (ii) and (iii): Assume that \mathcal{M} is refinable. Given homomorphisms $\varphi : \mathcal{A} \rightarrow \mathcal{C}$, and $\psi : \mathcal{B} \rightarrow \mathcal{C}$, we need to show that

$$pb(\varphi, \psi) := \{(a, b) \mid \varphi(a) = \psi(b)\}$$

is a bisimulation between \mathcal{A} and \mathcal{B} .

Let $(a, b) \in pb(\varphi, \psi)$ be given. We shall define an $|A| \times |B|$ -matrix $(m_{x,y})$ with entries from M , satisfying the conditions of lemma 5.5.

For any $c \in \varphi[A] \cap \psi[B]$, put $X := \varphi^{-1}(\{c\})$ and $r_x := \hat{\alpha}(a, x)$, i.e. $a \xrightarrow{r_x} x$, for any $x \in X$. Similarly, $Y := \psi^{-1}(\{c\})$ and $c_y = \hat{\beta}(b, y)$ for every $y \in Y$. With lemma 5.11 we obtain an $|X| \times |Y|$ matrix $(m_{x,y}^c)$ with row sums $(r_x)_{x \in X}$ and column sums $(c_y)_{y \in Y}$. Observe that we can achieve that

- for all but finitely many c is $(m_{x,y}^c)$ the 0-matrix
- all but finitely many entries in each $(m_{x,y}^c)$ are 0.

The final $|A| \times |B|$ -matrix $(m_{x,y})$ is obtained by putting all $(m_{x,y}^c)$ together

and filling up with zeroes:

$$\begin{array}{c|c|c}
 \dots & 0 & \dots \\
 \hline
 0 & (m_{x,y}^c) & 0 \\
 \hline
 \dots & 0 & \dots
 \end{array}
 \quad
 m_{x,y} := \begin{cases} m_{x,y}^c & \text{if } \varphi(x) = c = \psi(y), \\ 0 & \text{otherwise.} \end{cases}$$

By construction, $m_{x,y} \neq 0$ implies $(x, y) \in pb(\varphi, \psi)$. Moreover, all but finitely many entries of $(m_{x,y})$ are zero. Suppose now that $a \xrightarrow{r} a'$. We need to show that the a' -th row sum is r , i.e. $\sum(m_{a',b} \mid b \in B) = r$.

Let $c := \varphi(a')$. If $\psi^{-1}(\{c\}) \neq \emptyset$ then $\sum(m_{a',b} \mid b \in B) = \sum(m_{a',b}^c \mid b \in B) = r$. If $\psi^{-1}(\{c\}) = \emptyset$ (this case cannot happen in (ii)), we shall invoke positivity to show $r = 0$. Specifically, for s with $\varphi(a) \xrightarrow{s} c$ we have

$$\sum(m \mid a \xrightarrow{m} a', \varphi(a') = c) = s = \sum(m \mid b \xrightarrow{m} b', \psi(b') = c) = 0.$$

Hence $r + u = 0$ for some $u \in M$, whence $r = 0$.

A more elegant way to see (iii) is to directly conclude it from (i) and (ii), by invoking the following lemma from the second author's thesis:

Lemma 5.14 [Sch01] *A Set-endofunctor weakly preserves pullbacks iff it weakly preserves kernel pairs and pullbacks along injective maps.*

6 Discussion

One motivation for this study was to provide a repository of examples of *Set*-endofunctors with particular combinations of preservation properties. This we achieve by parameterizing a certain class of functors with algebraic structures and translating the functorial properties into corresponding algebraic laws. For instance, theorem 5.13 can be used to obtain an example of a functor weakly preserving nonempty kernel pairs, but not weakly preserving nonempty pullbacks: Simply choose for \mathcal{M} any nontrivial abelian group.

Of course, \mathcal{L} -coalgebras and $\mathcal{M}_\omega^{(-)}$ -coalgebras as \mathcal{L} -, resp. \mathcal{M} -labeled transition systems are of independent interest. \mathcal{L} -valued sets and relations are considered by Goguen in [Gog67]. In the book [FS90], Freyd and Scedrov consider the following operations on “ \mathcal{L} -valued relations” $R : A \times B \rightarrow \mathcal{L}$ and $S : B \times C \rightarrow \mathcal{L}$:

$$(R \circ S)(a, c) := \bigvee \{R(a, b) \wedge S(b, c) \mid b \in B\}.$$

When $L = \{0, 1\}$, this agrees with the familiar composition of relations. The

authors remark that this operation is associative iff \mathcal{L} is join infinitely distributive (JID), also called a *locale* in [Bor94].

L. Moss, in [Mos99], considers the following subfunctor of $\mathbb{R}_+^{(-)}$:

$$Q(X) := \{f : X \rightarrow \mathbb{R} \mid \text{supp}(f) \text{ finite, } \sum_{x \in X} f(x) = 1\}.$$

Coalgebras of this functor are stochastic transition systems ([Mos99], [dVR99]). Moss invokes the “Row/Column-theorem” for \mathbb{R}_+ , which is to say that \mathbb{R} is (m, n) -refinable for each $m, n > 1$, to show that Q weakly preserves pullbacks. (He attributes the proof of the Row/Column theorem to Saley Aliyari).

We have borrowed the term *refinable* from a classical line of algebraic investigation, asking for the existence of unique product decompositions of finite algebras. If $\mathcal{A}_1 \times \dots \times \mathcal{A}_m \cong \mathcal{B}_1 \times \dots \times \mathcal{B}_n$ are two representations of the same finite algebra as a product of indecomposables, one would like to conclude $m = n$ and $B_i \cong A_{\tau(i)}$, for some permutation τ .

It is easy to come up with examples of finite algebras that do not have unique decompositions. In such cases it may still be possible to prove a *refinement property*: Given that $\mathcal{A}_1 \times \dots \times \mathcal{A}_m \cong \mathcal{B}_1 \times \dots \times \mathcal{B}_n$, then each factor can be further decomposed into a product of smaller algebras $Q_{i,j}$, until one has the same collection of factors $Q_{i,j}$ on the left and on the right side.

J.D.H. Smith has reminded us of a result of B. Jónsson and A. Tarski [JT47] which states that a class of algebras, amongst whose operations are a binary operation $+$ and a constant 0 , which is neutral with respect to $+$ and idempotent for all fundamental operations, has the (m, n) -refinement property. This means that the class of all finite Jónsson-Tarski algebras, with the monoid structure given by the direct product (\times) and with $\{0\}$ as neutral element, is a refinable monoid.

Jónsson and Tarski needed the operations $+$ and 0 to represent direct products as “inner products”. Without some such assumptions, refinement is not possible in general, for refinement implies cancellability: $\mathcal{A} \times \mathcal{B} \cong \mathcal{A} \times \mathcal{C} \implies \mathcal{B} \cong \mathcal{C}$. When \mathcal{A} has a 1-element subalgebra, cancellability holds, according to L. Lovász ([Lov67]), but otherwise, one needs to replace “isomorphism” by the weaker notion of “isotopy”, see [Gum77]. A refinement theorem up to isotopy for algebras in congruence modular varieties has been proved in [GH79].

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