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On Caterpillars of Game Chromatic Number 4[1]

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Abstract

The coloring game is played by Alice and Bob on a finite graph G. They take turns properly coloring the vertices with t colors. If at any point there is an uncolored vertex without available color, then Bob wins. The game chromatic number of G is the smallest number t such that Alice has a winning strategy. It is known that for forests this number is at most 4. We find exact values for the game chromatic number of an infinite subclass of forests (composed by caterpillars), in order to contribute to the open problem of characterizing forests with different game chromatic numbers. Moreover, we show two sufficient conditions and two necessary conditions for any tree and caterpillar to have game chromatic number 4, respectively.

Keywords: Graph Theory, Colorings, Coloring game, Combinatorial games, Game chromatic number, Caterpillar

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1 Introduction

The *coloring game* is a combinatorial game conceived by Steven Brams with the aim of finding an alternative proof for the Four Color Theorem, firstly published in 1981 by Martin Gardner [6], still in the context of *map-coloring game*. In 1991, the game was reinvented by Bodlaender [1], who studied it in the context of graphs. Since then, the game has been much studied for different graph classes in order to obtain better upper and lower bounds to $\chi_g(G)$ [2,3,7,8,10,11,12]. The first application of a variation of this game to a non-game graph packing problem was presented in 2009 by Kierstead and Kostochka [9].

Given t colors, Alice and Bob take turns properly coloring an uncolored vertex. The goal of Alice is to color the input graph with t colors, and Bob does his best to prevent it. Alice wins if at the end the graph is completely properly colored with t colors; otherwise, Bob wins. Bob wins if after some point, we cannot color a new vertex in a proper way with t colors. The *game chromatic number* $\chi_g^a(G)$ (or simply $\chi_g(G)$) of G is the smallest number t of colors that ensures that Alice wins when she starts the game. We find convenient to allow Bob to start the game as well, and we introduce the auxiliary parameter $\chi_g^b(G)$ that is the smallest number t of colors that ensures that Alice wins when Bob starts the game, which is a powerful tool in our proofs.

Observe that $\chi(G) \leq \chi_g(G) \leq \Delta(G) + 1$ (for details about the relationship between these graph invariants, see [11]), where $\chi(G)$ denotes the chromatic number and $\Delta(G)$ the maximum degree. So, we have that a complete graph K_n has $\chi_g(K_n) = n$ and an independent set S_n has $\chi_g(S_n) = 1$. Analyzing the game chromatic number of paths P_n, with n vertices, we can quickly check that $\chi_g(P_1) = 1$ and $\chi_g(P_2) = \chi_g(P_3) = 2$. For $n \geq 4$, we have that $\chi_g(P_n) = 3$ because, regardless of where Alice colors at her first turn, Bob can always give a different color to a vertex at distance 2 of the vertex that Alice colored, forcing the third color. Using the same idea, we have that a cycle C_n, with $n \geq 3$, has $\chi_g(C_n) = 3$. A star $K_{1,p}$, with $p \geq 1$, has $\chi_g(K_{1,p}) = 2$, and moreover, the stars are the only connected graphs satisfying $\chi_g(G) = 2$ [3].

Bodlaender [1] showed an example of a tree with game chromatic number at least 4 and proved that every tree has game chromatic number at most 5. In 1993, Faigle et al. [4] improved this bound by proving that every forest F has game chromatic number at most 4. Extending Faigle et al.'s algorithm that proves that $\chi_g(F) \leq 4$, we prove that $\chi_g^b(F) \leq 4$. Despite the vast literature in this area, only in 2015, Dunn et al. [3] considered the distinction between forests with different game chromatic numbers, by investigating special cases. They characterized forests with game chromatic number 2, and they investigated the distinction between forests with game chromatic number 3 and 4, using properties that depend on a certain set of vertices to be colored during the game. Actually, they do not provide any kind of characterization of these forests, suggesting as an open problem, due to the difficulty concerning this subject. In our work, we contribute to this study by analyzing a special tree called the *caterpillar*.

A caterpillar $H = cat(k_1, k_2, \ldots, k_s)$ is a tree obtained from a central path $v_1, v_2, v_3, ..., v_s$ (called *spine*) by joining k_i leaf vertices to v_i, for each $i = 1, \ldots, s$. If $k_i \geq 1$, then we say that the vertex v_i has k_i adjacent *leg leaves*.

The motivation to focus on caterpillars relies on the fact that Bodlaender [1] proved the existence of a tree with $\chi_g(T) \geq 4$ by considering the caterpillar $H_d = cat(0, 2, 2, 2, 2, 0)$ depicted in Figure 1. Actually, Dunn et al. [3] proved that the caterpillar H_d is the smallest tree such that $\chi_g(T) = 4$, and is the unique tree with fourteen vertices and game chromatic number 4.

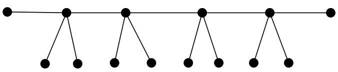

Fig. 1. The caterpillar H_d satisfies $\chi_g(H_d) = 4$.

Faigle et al. [4] proved for trees that $\chi_g(T) \leq 4$ and stated that there is no loss of generality when we think of forests, that is, the proof that $\chi_g(T) \leq 4$ can be extended to obtain $\chi_g(F) \leq 4$.

Dunn et al. [3] aimed to determine if the maximum degree was a relevant characteristic for classifying the game chromatic number 3, and they conjectured that the maximum degree alone is not relevant, and we agree with this conjecture. In [5], we contributed to this problem by characterizing two infinite subclasses of caterpillars:

- a caterpillar H with maximum degree 3 that is not a star has $\chi_g(H) = 3$ (Theorem 2.1 [5]);
- a caterpillar H without vertex of degree 2 has $\chi_g(H) = 4$ if, and only if, H has at least four vertices of degree at least 4 (Theorem 2.7 [5]).

Motivated by this work, in Section 3, we analyze the infinite subclass of caterpillars without vertex of degree 3, determining its game chromatic number: a caterpillar H without vertex of degree 3 has $\chi_g(H) = 4$ if, and only if, H has at least one induced subgraph of family Q (that we present further in Section 3). This result is presented in 3.5. But first, in Section 2, we show two sufficient conditions and two necessary conditions for any caterpillar to have $\chi_g(H) = 4$.

Since Faigle et al. [4] and Dunn et al. [3] worked with forests, the natural question that we can ask is: in a forest F with r tree connected components T_i, $1 \leq i \leq r$, is it true that $\chi_g(F) = max\{\chi_g(T_i)|\ i = 1, ..., r\}$? And $\chi_g(F) \leq max\{\chi_g(T_i)|\ i = 1, ..., r\}$? The answer is no, and we have presented counterexamples in [5].

In Section 4, we determine $\chi_g(F)$, for a forest, by knowing $\chi_g^a(T_i)$, $\chi_g^b(T_i)$ and the parity of $|V(T_i)|$ for each tree component T_i of F.

2 Caterpillar

We recall that, by Faigle et al. [4], every caterpillar H, which is not a star, has $3 \leq \chi_g(H) \leq 4$. So, we are interested in characterizing caterpillars with $\chi_g(H)$ equal to 3 or 4. We refer to Figure 2 for the four caterpillars H_1, H_2, H_3 and H_4

that we shall use throughout the paper.

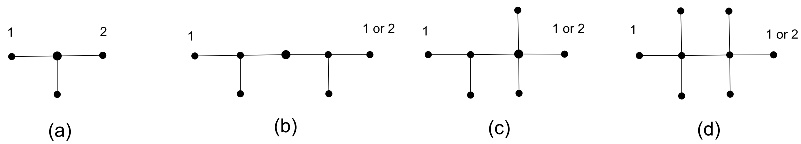

Fig. 2. The partially colored caterpillars (a) H_1 (b) H_2 (c) H_3 (d) H_4.

First, we introduce two auxiliary parameters. Let Z be the colored subset of vertices of $V(G)$, i.e., for all $v \in Z$, $c(v) \neq \emptyset$, where $c(v)$ is the color of vertex v. We denote by $\chi_g(G,Z)$ the smallest number t of colors that ensures that Alice wins in the partially colored graph G.

Lemma 2.1 *[5] Let H_1 be the caterpillar (claw) $cat(0,1,0)$, H_2 be the caterpillar $cat(0,1,0,1,0)$, H_3 be the caterpillar $cat(0,1,2,0)$, and H_4 be the caterpillar $cat(0,2,2,0)$. Considering the partially colored cases, as in Figure 2, we have that:*

(i) $\chi_g^a(H_1,Z)=3$ *and* $\chi_g^b(H_1,Z)=4$, *with* $Z=\{v_1,v_3|c(v_1)\neq c(v_3)\}$;

(ii) $\chi_g^a(H_2,Z)=3$ *and* $\chi_g^b(H_2,Z)=4$, *with* $Z=\{v_1,v_5\}$;

(iii) $\chi_g^a(H_3,Z)=3$ *and* $\chi_g^b(H_3,Z)=4$, *with* $Z=\{v_1,v_4\}$;

(iv) $\chi_g^a(H_4,Z)=3$ *and* $\chi_g^b(H_4,Z)=4$, *with* $Z=\{v_1,v_4|c(v_1)\neq c(v_4)\}$, *but* $\chi_g^a(H_4,Z')=\chi_g^b(H_4,Z')=4$, *where* $Z'=\{v_1,v_4|c(v_1)=c(v_4)\}$.

In Theorems 2.2 and 2.3, we will show sufficient conditions for $\chi_g(T)=4$ for any tree T.

Theorem 2.2 *If a tree T has an induced subtree T', such that $\chi_g^a(T')=\chi_g^b(T')=4$, then $\chi_g^a(T)=\chi_g^b(T)=4$.*

Proof. If Bob starts, then he colors in $V(T')$ and $\chi_g^b(T)=4$. If Alice starts, then she can color in $V(T')$ or not. If she colors in $V(T')$, then $\chi_g^a(T)=4$. If she avoids to color in $V(T')$, then Bob does it and, $\chi_g^a(T)=4$. $\square$

Theorem 2.3 *If a tree T has two induced subtrees T' and T'', such that $\chi_g^b(T')=\chi_g^b(T'')=4$, then $\chi_g^a(T)=\chi_g^b(T)=4$.*

Proof. If Alice starts the game in T, then she can color any vertex. Bob colors in $V(T')$ or $V(T'')$, choosing a subtree that is not colored. Thus, $\chi_g^a(T)=4$. If Bob starts, then he colors in $V(T')$ and $\chi_g^b(T)=4$. $\square$

In Theorems 2.11 and 2.13, we will show necessary conditions for $\chi_g(H)=4$ for any caterpillar H, but first we need auxiliary lemmas.

Lemma 2.4 *Let H be a caterpillar $cat(k_1,\ldots,k_s)$ such that $k_1=k_s=0$, $k_{s-1}=1$, and $k_i\leq 1$, for all $i=2,...,s-2$. We have that $\chi_g^a(H,Z),\chi_g^b(H,Z)\leq 3$, with $Z=\{v_1,v_s,\lambda_{s-1}|c(v_1)\neq c(v_s),c(v_s)=c(\lambda_{s-1})\}$, where λ_{s-1} is the leg leaf adjacent to v_{s-1}.*

Proof. If Alice starts, then she colors v_{s-1}, if s is odd; or v_{s-2}, if s is even, and by Claim 1 of Lemma 2.1 [5], $\chi_g^a(H,Z) \leq 3$. If Bob starts, then the proof follows by induction in s. □

Lemma 2.5 *Let H be a caterpillar $cat(k_1,\ldots,k_s)$ with s odd, $k_1 = k_{s-1} = k_s = 0$, and $\Delta(H) \leq 3$. We have that H has $\chi_g^b(H,Z) \leq 3$, with $Z = \{v_1, v_s\}$.*

Proof. This proof follows by induction in s.

When $s = 3$, we have $cat(0,0,0)$ with v_1 and v_3 colored and, $\chi_g^b(H,Z) \leq 3$.

Assume that the result is true for any caterpillar H with $3 \leq s \leq z$, for z odd. Now, we analyze the caterpillar H' with $s = z+2$.

If Bob colors a spine vertex v_i, then the result follows by Claim 1 of Lemma 2.1 [5] and induction hypothesis. If Bob colors a leg leaf adjacent to v_i, where i is odd, then Alice colors v_{i-1}, and the result follows by Claim 1 of Lemma 2.1 [5] and Lemma 2.4; i is even, then Alice colors v_i, and the result follows by Claim 1 of Lemma 2.1 [5]. □

Lemma 2.6 *Let H be a caterpillar $cat(k_1,\ldots,k_s)$ such that $k_1 = k_s = 0$, $k_2 \geq 2$, $k_{s-1} = 1$, and $k_i \leq 1$, for all $i = 3,\ldots,s-2$. We have that $\chi_g^a(H,Z), \chi_g^b(H,Z) \leq 3$, with $Z = \{v_1, v_s, \lambda_{s-1} | c(v_1) \neq c(v_s), c(v_s) = c(\lambda_{s-1})\}$, where λ_{s-1} is the leg leaf adjacent to v_{s-1}.*

Proof. First, when Alice starts, she colors v_2 with the same color of v_s, if s is odd, or she colors v_3 with the same color of v_1, if s is even. So, the induced subcaterpillar $cat(k_2,\ldots,k_s)$ or $cat(k_3,\ldots,k_s)$, respectively, is a caterpillar with maximum degree 3 and has both extremes colored (since $c(v_s) = c(\lambda_{s-1})$, we can ignore λ_{s-1}) and, by Claim 1 of Lemma 2.1 [5], $\chi_g^a(H,Z) \leq 3$.

Now, we prove the case when Bob starts by induction in s.

If $s = 4$, then $\chi_g^b(H,Z) \leq 3$. Assume that $\chi_g^b(H,Z) \leq 3$, for $4 \leq s \leq z$. When $s = z+1$, Bob can start coloring a spine vertex or a leg leaf. If Bob colors a leg leaf adjacent to v_i, then Alice colors v_{i+1}, and the result follows by induction hypothesis and Claim 1 of Lemma 2.1 [5]. If Bob colors v_i and v_{i-1} has an adjacent leg leaf, then Alice colors it, which is analogous to the previous case. If Bob colors v_i and v_{i-1} does not have an adjacent leg leaf ($d(v_{i-1}) = 2$), then Alice colors v_{i+1}, and the result follows by induction hypothesis and Lemma 2.4. □

Lemma 2.7 *Let H be a caterpillar $cat(k_1,\ldots,k_s)$, such that $k_1 = k_s = 0$, $k_2 \geq 2$ and $k_i \leq 1$, for all $i = 3,\ldots,s-1$. We have that $\chi_g^a(H,Z), \chi_g^b(H,Z) \leq 3$, with $Z = \{v_1\}$.*

Proof. If Alice starts, then she colors v_2, and the result follows by Claim 2 of Lemma 2.1 [5].

If Bob starts, then he can color a spine vertex v_i or a leg leaf. If he colors v_i, then Alice colors v_2 (for i is odd) or v_3 with the same color of v_1 (for i is even) and, by Claim 1 and 2 of Lemma 2.1 [5], the result follows. If Bob colors a leg leaf adjacent to v_i, then Alice colors v_{i+1}, and the result follows by Claim 2 of Lemma 2.1 [5] and Lemma 2.6. □

Lemma 2.8 *Let H be a caterpillar $cat(k_1, \ldots, k_s)$ such that $k_1 = k_s = 0$, $k_{s-1} = 1$, there exists $j \in \{3, \ldots, s-2\}$ with $k_j \geq 2$, and $k_i \leq 1$, for all $i = 2, ..., j-1, j+1, \ldots, s-2$. We have that $\chi_g^a(H, Z), \chi_g^b(H, Z) \leq 3$, with $Z = \{v_1, v_s, \lambda_{s-1} | c(v_1) \neq c(v_s), c(v_s) = c(\lambda_{s-1})\}$, where λ_{s-1} is the leg leaf adjacent to v_{s-1}.*

Proof. If j is odd, then Alice colors v_{j-1}, and the result follows by Claim 1 of Lemma 2.1 [5] and Lemma 2.6. Else, Alice colors v_j, and the result follows by Claim 1 of Lemma 2.1 [5].

When Bob starts, the proof follows by induction similarly to Lemma 2.6 (induction basis $s = 5$). □

Lemma 2.9 *Let H be a caterpillar $cat(k_1, \ldots, k_s)$ such that $k_1 = k_s = 0$, there exists $j \in \{3, \ldots, s-2\}$ with $k_j \geq 2$ and $k_i \leq 1$, for all $i = 2, \ldots, j-1, j+1, \ldots, s-1$. We have that $\chi_g^a(H, Z) \leq 3$ and $\chi_g^b(H, Z) \leq 4$, with $Z = \{v_1, v_s | c(v_1) \neq c(v_s)\}$.*

Proof. If there is a leg leaf adjacent to v_2 or v_{s-1}, then Alice colors it with the same color of v_1 or v_s, respectively, and the result follows by Lemma 2.8. Else, Alice colors v_j, and the result follows by Lemma 2.5.

The proof that $\chi_g^b(H, Z) \leq 4$ is presented in Lemma 2.5 [5]. □

Lemma 2.10 *Let H be a caterpillar $cat(k_1, \ldots, k_s)$, such that $k_1 = k_s = 0$, there exists $j \in \{2, \ldots, s-1\}$ with $k_j \geq 2$ i and $k_i \leq 1$, for all $i = 2, \ldots, j-1, j+1, \ldots, s-1$. We have that $\chi_g^a(H, Z), \chi_g^b(H, Z) \leq 3$, with $Z = \{v_1\}$.*

Proof. If j is even, then Alice colors v_j, and the result follows by Claim 1 and 2 of Lemma 2.1 [5]. If j is odd, then Alice colors v_{j-1}, and the result follows by Claim 1 of Lemma 2.1 [5] and Lemma 2.7.

When Bob starts, he can color a spine vertex v_i or a leg leaf. If he colors: *(i)* a spine vertex v_i, for $i > j$, then the result follows by Lemma 2.9 and Claim 2 of Lemma 2.1 [5]; *(ii)* a spine vertex v_i, for $i = j$, the result follows by Claim 1 and 2 of Lemma 2.1 [5]; *(iii)* a leg leaf adjacent to v_i, for $i \geq j$, then Alice colors v_{i+1}, and the result follows by Lemma 2.6 and Claim 2 of Lemma 2.1 [5]; *(iv)* a spine vertex v_i or a leg leaf adjacent to v_i, for $i < j$, we use induction in s to prove the result as follows.

If $s = 3$, then Bob can color v_2 or the leaf adjacent to it. In both cases, $\chi_g^b(H, Z) \leq 3$.

Assume that $\chi_g^b(H, Z) \leq 3$ for $s \leq z$. When $s = z + 1$, Bob can color a spine vertex or a leg leaf. If he colors a spine vertex v_i, the result follows by Claim 1 or 2 of Lemma 2.1 [5] and by induction hypothesis. If he colors a leg leaf adjacent to v_i, then Alice colors v_{i+1} with the same color, and the result follows by Lemma 2.6 and by induction hypothesis. □

Theorem 2.11 *If a caterpillar H has $\chi_g(H) = 4$, then H has at least four vertices of degree at least 4.*

Proof. If H has just one vertex of degree at least 4, v_i, then Alice colors v_i, and by Claim 2 of Theorem 2.1 [5], we have that $\chi_g(H) = 3$.

If H has exactly two vertices of degree at least 4, v_i and v_j, then Alice colors v_i, and by Theorem 2.1 [5] and Lemma 2.10, we have that $\chi_g(H) = 3$.

If H has exactly three vertices of degree at least 4, v_i, v_j and v_k, with $i < j < k$, then Alice colors v_j, and by Lemma 2.10, we have that $\chi_g(H) = 3$. □

Lemma 2.12 *Let $H' = cat(k_1, \ldots, k_p)$ and $H'' = cat(k_q, \ldots, k_s)$ be caterpillars such that $p < q$, $k_p = k_q = 0$, $\chi_g^a(H') = \chi_g^a(H'') = \chi_g^b(H'') = 3$, and $\chi_g^b(H') \geq 3$. Let H be a caterpillar constructed by linking vertex v_p of H' to vertex v_q of H'' by a path, that is $H = cat(k_1, \ldots, k_p, \ldots, k_q, \ldots, k_s)$, for $p \neq q$ and $k_i = 0$, for all $i \in \{p, \ldots, q\}$. We have that $\chi_g^a(H) = 3$.*

Proof. Alice colors a vertex in H'. If Bob colors in $V(H')$ or $V(H'')$, then Alice colors in the same subcaterpillar. If Bob colors a vertex v_i, $i \in \{p, \ldots, q\}$, then Alice continues coloring vertices in H' or H'' until all them are colored, because even if just Bob colors the vertices in the subpath $v_p, \ldots, v_q$, it does not increases the game chromatic number of H. □

Theorem 2.13 *If H is a minimal caterpillar with respect to $\chi_g^a(H) = 4$, then H does not have consecutive vertices of degree 2, unless H has two edge disjoint induced subcaterpillars H' and H'' that are minimal with respect to $\chi_g^b(H') = 4$ and $\chi_g^b(H'') = 4$.*

Proof. Suppose that there are at least two consecutive vertices of degree 2 in $V(H)$, and at most one induced subcaterpillar that is minimal with respect to game chromatic number 4 when Bob starts. Let $H = cat(k_1, ..., k_p, ..., k_q, ..., k_s)$ such that $k_i = 0$, $p \leq i \leq q$ and $p \neq q$. Let $H' = cat(k_1, ..., k_p)$ and $H'' = cat(k_q, ..., k_s)$ be two induced subcaterpillars of H. Since H is minimal with respect to $\chi_g^a(H) = 4$, we have that $\chi_g^a(H'), \chi_g^a(H'') \leq 3$. And since there does not exist two edge disjoint induced subcaterpillars that are minimal with respect to game chromatic number 4 when Bob starts, we have that only one of $\chi_g^b(H')$ and $\chi_g^b(H'')$ is 4. So, by Lemma 2.12, $\chi_g^a(H) = 3$.

If there are two edge disjoint induced subcaterpillars H' and H'' that are minimal with respect to $\chi_g^b(H') = 4$ and $\chi_g^b(H'') = 4$, by Theorem 2.3, we have that $\chi_g^a(H) = 4$, and H could be minimal with respect to $\chi_g^a(H) = 4$. □

3 Caterpillar without vertex of degree 3

We define the Family Q: infinite subclass of caterpillars without vertices of degree 3 which has game chromatic number 4. In Theorem 3.5 we show that having a caterpillar of Family Q as a subgraph is a necessary and sufficient condition for a caterpillar without vertex of degree 3 to have game chromatic number equals 4.

Lemma 3.1 *If H is a caterpillar $cat(k_1, ..., k_s)$ without vertex of degree 3 and H has no consecutive vertices of degree at least 4, then $\chi_g^a(H), \chi_g^b(H) \leq 3$.*

Proof. First, Alice colors a spine vertex of degree at least 4. Bob colors a vertex $u \in V(H)$. If there is an uncolored vertex of degree at least 4, then Alice colors the closest vertex of degree at least 4 to the vertex u. This process continues until there is no uncolored vertex of degree at least 4. So, in this moment there are just uncolored vertices of degree at most 2. To color each leaf it is necessary just a different color of its adjacent vertex in the spine, so, three colors are sufficient. By Claim 1 of Theorem 2.1 [5], to color the induced partially colored subpaths, three colors are sufficient too. □

To characterize the caterpillars without vertex of degree 3 and $\chi_g(H) = 4$, by Lemma 3.1, it remains to consider caterpillars that have an induced $H_4 = cat(0, 2, 2, 0)$ (as in Lemma 2.1). We define $H_t = cat(0, 2, 2, 2, 0)$ and $H_\alpha = cat(0, 2, 0, 2, 0, ..., 0, 2, 0)$ with α vertices of degree 2. So, H_α has $2\alpha - 1$ vertices, for $\alpha \geq 2$. Now, we connect these caterpillars by identifying at least two vertices. Please refer to Figure 3.

- H_{33}: composed by two copies of H_t; we identify v_5 of a H_t with v_1 of another H_t;
- $H_{[\alpha]}$: composed by H_α and two copies of H_4; we identify v_4 of a H_4 with v_1 of a H_α, and $v_{2\alpha-1}$ of H_α with v_1 of H_4;
- $H_{[\alpha][\beta]}$: composed by H_t, H_α and three copies of H_4; we identify v_4 of a H_4 with v_1 of a H_α, and $v_{2\alpha-1}$ of H_4 with v_1 of H_4, and v_4 of the second H_4 with v_1 of H_β, and finally $v_{2\beta-1}$ of H_β with v_1 of a third H_4;
- $H_{[\alpha]3[\beta]}$: composed by H_t, H_α, H_β and two copies of H_4; we identify v_4 of a H_4 with v_1 of a H_α, and $v_{2\alpha-1}$ of H_α with v_1 of H_t, and v_5 of the H_t with v_1 of H_β, and finally $v_{2\beta-1}$ of H_β with v_1 of a third H_4.

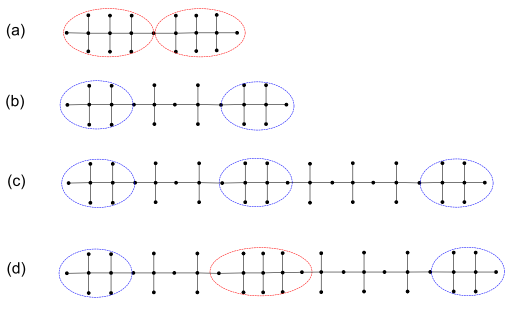

Fig. 3. The caterpillars: (a) H_{33} (b) $H_{[3]}$ (c) $H_{[3][4]}$ (d) $H_{[3]3[4]}$.

Lemma 3.2 *If a caterpillar H has an induced subcaterpillar H_d (Figure 1), then $\chi_g^a(H) = \chi_g^b(H) = 4$.*

Proof. We know that $\chi_g^a(H_d) = 4$ by Theorem 18 of [3]. Let us prove that $\chi_g^b(H_d) = 4$. Bob colors $v_1 \in V(H_d)$ with color 1. To prevent the hypothesis of Lemma 2.1(iv), Alice can color a spine vertex or a leg leaf. If she colors v_2 with color 2 (or v_3 with color 1), then Bob colors v_5 with color 2 (or v_6 with color 1, resp.) and, by

Lemma 2.1.(iv), $\chi_g^a(H) \geq 4$. If Alice colors a leg leaf adjacent to v_2 (or v_3), then Bob colors the other leg leaf adjacent to v_2 (or v_3, resp.) with a different color, forcing Alice to color v_2 (v_3), and by Lemma 2.1(iv), $\chi_g^b(H) = 4$.

Thus, $\chi_g^a(H) = \chi_g^b(H) = 4$, by Theorem 2.2. □

Lemma 3.3 *If H is a caterpillar without vertex of degree 3, $\chi_g^a(H) = 3$, H has no induced subcaterpillar with game chromatic number 4, and H is minimal with respect to $\chi_g^b(H) = 4$, then H does not have consecutive vertices of degree 2.*

Proof. Suppose that there are at least two consecutive vertices of degree 2 in $V(H)$. Let $H = cat(k_1, ..., k_p, ..., k_q, ..., k_s)$ such that $k_i = 0$, $p \leq i \leq q$ and $p \neq q$. Let $H' = cat(k_1, ..., k_p)$ and $H'' = cat(k_q, ..., k_s)$ be two induced subcaterpillars of H. Since H is minimal with respect to $\chi_g^b(H) = 4$, we have that $\chi_g^b(H'), \chi_g^b(H'') \leq 3$; and since there are no induced subcaterpillar with game chromatic number 4, we have that $\chi_g^a(H'), \chi_g^b(H'') \leq 3$. By Lemma 2.12, $\chi_g^a(H) = 3$, which is a contradiction. □

Lemma 3.4 *H is a caterpillar without vertex of degree 3, $\chi_g^a(H) = 3$, H has no induced subcaterpillar with game chromatic number 4, and H is minimal with respect to $\chi_g^b(H) = 4$ if, and only if, $H = H_{[\alpha]}$, for $l \geq 1$.*

Proof. By Lemma 3.1, H need at least two consecutive vertices of degree at least 4 to obtain $\chi_g(H) = 4$. By Lemma 3.2, we know that $\chi_g^b(H_d) = 4$, but $\chi_g^a(H_d) = 4$ by Theorem 18 of [3]. So, we search caterpillars with two or three consecutive vertices of degree at least 4.

Assume that H has two consecutive vertices of degree at least 4. Let us construct this caterpillar: $d(v_1) = 0$, $d(v_2), d(v_3) \geq 4$ and, necessarily, $d(v_4) = 2$. By Lemma 3.3, $d(v_5) \geq 4$. If $d(v_6) \geq 4$, then we have a minimal caterpillar with respect to $\chi_g^b(H) = 4$ and $\chi_g^a(H) = 3$, that is $H_{[1]} = cat(0, 2, 2, 0, 2, 2, 0)$. If $d(v_6) = 2$, then we analyze v_7. By Lemma 3.3, $d(v_7) \geq 4$. We can construct a sequence of $d_{4+2j} = 2$ and $d_{5+2j} \geq 4$, for $i \geq 1$ and it is not sufficient to get $\chi_g^b(H) = 4$. We finish this sequence with two consecutive vertices of degree at least 4, getting exactly $H_{[\alpha]}$.

Assume that H has three consecutive vertices of degree at least 4. So, we have that $d(v_1) = 0$, $d(v_2), d(v_3), d(v_4) \geq 4$ and $d(v_5) = 2$. By Lemma 3.3, $d(v_6) \geq 4$. If $d(v_7) \geq 4$, then H has an induced subcaterpillar $H_{[1]}$ that has $\chi_g^b(H_{[1]}) = 4$. So, $d(v_7) = 2$. By Lemma 3.3, $d(v_8) \geq 4$, and we can construct a sequence of $d_{5+2j} = 2$ and $d_{6+2j} \geq 4$, for $i \geq 1$ and it is not sufficient to get $\chi_g^b(H) = 4$. We finish this sequence with two adjacent vertices of degree at least 4, getting a caterpillar H that has $H_{[\alpha]}$ as an induced subcaterpillar. So, H is not minimal with respect to $\chi_g^b(H) = 4$.

Conversely, we have that $H_{[\alpha]}$ does not have vertex of degree 3. If Alice starts, then she colors v_3 and $\chi_g^a(H_{[l]}) = 3$. If Bob starts, then he colors v_4 and $\chi_g^b(H_{[\alpha]}) = 4$. □

We denote by $H_{[\alpha]} \cup H_{[\beta]}$ the caterpillar that has two edge disjoint induced subcaterpillars $H_{[\alpha]}$ and $H_{[\beta]}$, for $\alpha, \beta \geq 1$.

Let *family Q* be the set of caterpillars H_d, H_{33}, $H_{[\alpha]} \cup H_{[\beta]}$, $H_{[\alpha][\beta]}$ and $H_{[\alpha]3[\beta]}$.

Theorem 3.5 *A caterpillar H without vertex of degree 3 has $\chi_g(H) = 4$ if, and only if, H has a caterpillar of family Q as an induced subcaterpillar.*

Proof. First, if caterpillar H without vertex of degree 3 has $\chi_g(H) = 4$, then H has at least four vertices of degree at least 4, by Theorem 2.11. Moreover, H does not have consecutive vertices of degree 2 or there exists two minimal induced subcaterpillars with respect to game chromatic number 4 when Bob starts the game, by Theorem 2.13.

Let us analyze H without consecutive vertices of degree 2. According to Lemma 3.4, since H has no vertex of degree 3, the unique caterpillar that has game chromatic number 4 when Bob starts the game is $H_{[\alpha]}$. So, the caterpillar that has no vertex of degree 3 and two induced subcaterpillars with game chromatic number 4 when Bob starts the game is $H_{[\alpha]} \cup H_{[\beta]}$, for $\alpha, \beta \geq 1$.

Analyzing the case in which H does not have consecutive vertices of degree 2, we have the following. If H has four consecutive vertices of degree at least 4, then we have $H = H_d$, that is minimal with respect to $\chi_g(H_d) = 4$. By Lemma 3.1, H should have consecutive vertices of degree at least 4 to satisfy $\chi_g(H) = 4$. So, we have to study the case when H has two or three consecutive vertices of degree at least 4. We assume that $d(v_1) = 0$.

Assume that H has two consecutive vertices of degree at least 4. So, we have two vertices of degree at least 4 followed by a vertex of degree 2 (because H has no vertex of degree 3), that is followed by a vertex of degree at least 4 (because H has no consecutive vertices of degree 2). We can have an arbitrary sequence of vertices of degree 2 and at least 4, but this caterpillar has game chromatic number 3, because Alice colors v_3. So, this sequence need to finish for obtain $\chi_g(H) = 4$, and we do this avoiding other pair of consecutive vertices of degree at least 4. At this point, we constructed $H_{[\alpha]}$, for $\alpha \geq 1$, but $\chi_g(H_{[\alpha]}) = 3$, which made us continue with a sequence of vertices of degree 2 and at least 4, getting a game chromatic number 3 yet, by Alice coloring vertex v_5. So, we finish this sequence, and we do this avoiding another pair of consecutive vertices of degree at least 4, constructing $H_{[\alpha][\beta]}$, that is minimal with respect to $\chi_g(H_{[\alpha][\beta]}) = 4$.

Now, assume that H has three consecutive vertices of degree at least 4. So, we have three vertices of degree at least 4 followed by a vertex of degree 2 (because H has no vertex of degree 3), that is followed by a vertex of degree at least 4 (because H has no consecutive vertices of degree 2). We can have an arbitrary sequence of vertices of degree 2 and at least 4, but this caterpillar has game chromatic number 3, because Alice colors v_3. So, we finish this sequence for obtain $\chi_g(H) = 4$, and we do this avoiding other three consecutive vertices of degree at least 4 (to distinguish from the last case). At this point, the game chromatic number is 3, which made us continue with a sequence of vertices of degree 2 and at least 4, getting a game chromatic number 3 yet, by Alice coloring vertex v_6. So, we finish this sequence, and we do this avoiding another pair of consecutive vertices of degree at least 4. This caterpillar has $H_{[\alpha]3[\beta]}$ as an induced subcaterpillar, which is minimal with respect to $\chi_g(H_{[l]3[r]}) = 4$.

Assuming that H has at most three consecutive vertices of degree at least 4 is also possible that we have these three vertices of degree at least 4 followed by a vertex of degree 2 (because H has no vertex of degree 3), that is followed by three vertices of degree at least 4. So, we construct H_{33}, that is minimal with respect to $\chi_g(H_{33}) = 4$.

Conversely, by Theorem 2.2, we need to prove that each subclass of caterpillar of Family Q has game chromatic number 4 when both Alice and Bob starts the game: H_d was proved in Lemma 3.2; $H_{[\alpha]} \cup H_{[\beta]}$ by Lemma 2.1(iv) and Theorem 2.3; H_{33}, $H_{[\alpha][\beta]}$ and $H_{[\alpha][\beta]}$ by Lemma 2.1(iv). □

4 Forest

First, we observe that $\chi_g^b(F) \leq 4$, for F forest, using the same idea of Faigle et al.'s algorithm to prove that $\chi_g^a(F) \leq 4$. Since, stars are the only connected graphs with game chromatic number equals 2, we have that $(\chi_g^a(T), \chi_g^b(T)) \subseteq \{(1,1),(2,2),(3,2),(3,3),(3,4),(4,3),(4,4)\}$. Finally, we analyze the game chromatic number of a forest.

Theorem 4.1 *Let F be a forest composed by r trees T_1, ..., T_r. Assume that $\chi_g^a(T_1) \leq \chi_g^a(T_2) \leq ... \leq \chi_g^a(T_r)$, and, if there exist two trees with the same game chromatic number, then T_i and T_j are ordered in a way that $\chi_g^b(T_i) \leq \chi_g^b(T_j)$, for $i < j$. We have that:*

(i) *If $\chi_g^b(T_r) > \chi_g^a(T_r), \chi_g^b(T_{r-1})$, then $\chi_g(F) = \chi_g^a(T_r)$;*

(ii) *If $\chi_g^b(T_r) = \chi_g^b(T_{r-1}) > \chi_g^a(T_r)$, then $\chi_g(F) = \chi_g^b(T_r)$;*

(iii) *If $\chi_g^a(T_r) = \chi_g^b(T_r)$, then $\chi_g(F) = \chi_g^a(T_r) = \chi_g^b(T_r)$;*

(iv) *If $\chi_g^b(T_r) < \chi_g^a(T_r)$ and $\sum_{i=1}^{r-1} |V(T_i)|$ is even, then $\chi_g(F) = \chi_g^a(T_r)$;*

(v) *If $\chi_g^b(T_r) < \chi_g^a(T_r)$ and $\sum_{i=1}^{r-1} |V(T_i)|$ is odd, then $\chi_g(F) = max\left\{\chi_g^a(F\backslash T_r), \chi_g^b(T_r)\right\}$.*

Proof.

(i) Alice plays in T_r to avoid that Bob starts in T_r in his first turn. So, $\chi_g(F) = max\left\{\chi_g^a(T_r), \chi_g^b(T_{r-1})\right\} = \chi_g^a(T_r)$.

(ii) Independent of where Alice plays, Bob can start playing in T_{r-1} or T_r. So, $\chi_g(F) = \chi_g^b(T_r)$.

(iii) Since $\chi_g^a(T_r)$ and $\chi_g^b(T_r)$ are maximum values of game chromatic numbers in F, independent of who starts in T_r, $\chi_g(F) = \chi_g^a(T_r) = \chi_g^b(T_r)$.

(iv) Since $\chi_g^a(T_r)$ is the maximum value of game chromatic numbers in F, Alice tries not to play in T_r, but as $\sum_{i=1}^{r-1} |V(T_i)|$ is even, Bob can force Alice to start in T_r. So, $\chi_g(F) = \chi_g^a(T_r)$.

(v) Since $\chi_g^a(T_r)$ is the maximum value of game chromatic numbers in F, Alice tries not to play in T_r, and as $\sum_{i=1}^{r-1} |V(T_i)|$ is odd, Alice can force Bob to start in T_r. So, $\chi_g(F) = max\left\{\chi_g^a(F \backslash T_r), \chi_g^b(T_r)\right\}$.

$\square$

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