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Identifying Codes in the Complementary Prism of Cycles

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Abstract

We show that an identifying code of minimum order in the complementary prism of a cycle of order n has order $7n/9 + \Theta(1)$. Furthermore, we observe that the clique-width of the complementary prism of a graph of clique-width k is at most $4k$, and discuss some algorithmic consequences.

Keywords: identifying code, complementary prism, dominating set, cycle

1 Introduction

We consider finite, simple, and undirected graphs, and use standard notation and terminology.

For a positive integer d, a graph G, and a vertex u of G, let $N_G^{\leq d}[u]$ be the set of vertices of G at distance at most d from u. Note that the closed neighborhood $N_G[u]$ of u in G coincides with $N_G^{\leq 1}[u]$. A set C of vertices of a graph G is a *d-identifying code* in G for a positive integer d [18] if the sets $N_G^{\leq d}[u] \cap C$ are non-empty and distinct for all vertices u of G. A 1-identifying code is known simply as an *identifying code*. Let $\text{ic}(G)$ denote the minimum order of an identifying code in

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G. Note that a graph has an identifying code if and only if no two vertices have the same closed neighborhood.

It is algorithmically hard [1,5] to determine identifying codes of minimum order even for planar graphs of arbitrarily large girth. Exact values, density results, as well as good upper and lower bounds have been studied in detail for many special graphs; in particular for graphs that arise by product operations using simple factors such as grids [2,3,4,7,8,11,10,12,13,17,20]. In the present paper we study identifying codes in the complementary prism of cycles. The related notion of locating-domination was studied for such graphs in [16].

Complementary prisms were introduced by Haynes et al. [15] as a variation of the well-known *prism* of a graph [14]. For a graph G with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G)$, the *complementary prism of* G is the graph denoted by $G\bar{G}$ with vertex set $V(G\bar{G}) = \{v_1, \ldots, v_n\} \cup \{\bar{v}_1, \ldots, \bar{v}_n\}$ and edge set

$$E(G\bar{G}) = E(G) \cup \{\bar{v}_i\bar{v}_j : 1 \leq i < j \leq n \text{ and } v_iv_j \notin E(G)\} \cup \{v_1\bar{v}_1, \ldots, v_n\bar{v}_n\}.$$

In other words, the complementary prism $G\bar{G}$ of G arises from the disjoint union of the graph G and its complement $\bar{G}$ by adding the edges of a perfect matching joining corresponding vertices of G and $\bar{G}$. For every vertex u of G, we will consistently denote the corresponding vertex of $\bar{G}$ by $\bar{u}$, that is, $V(G\bar{G}) = V(G) \cup V(\bar{G})$ where $V(\bar{G}) = \{\bar{v}_1, \ldots, \bar{v}_n\}$. For a positive integer k, let $[k]$ denote the set of positive integers at most k. For an integer n at least 3, let C_n denote the cycle of order n.

In Section 2 we determine the minimum order of an identifying code in $C_n\bar{C}_n$ up to a small constant. Note that for $n \geq 6$ and $d \geq 2$, the graph $C_n\bar{C}_n$ contains distinct vertices u and v with $N^{\leq d}_{C_n\bar{C}_n}[u] = N^{\leq d}_{C_n\bar{C}_n}[v]$, which implies that there is no d-identifying code in $C_n\bar{C}_n$ for such values.

Before we proceed to Section 2, we make some more general algorithmic observations. In [1] Auger describes an involved linear time dynamic programming algorithm that determines an identifying code of minimum order for a given tree. In [6] Charon et al. present a similar algorithm for oriented trees, and explicitly mention that it is an open issue whether, for any fixed d at least 2, it is possible to determine a d-identifying code of minimum order for a given tree in polynomial time. In fact, the existence of such efficient algorithms follows immediately from general results [9] concerning graph of bounded clique-width, such as trees, which have clique-width at most 3. For a positive integer d, and two vertices u and v of a graph G, we have $v \in N^{\leq d}_G[u]$ if and only if

$$\begin{aligned}&\exists v_0, v_1, \ldots, v_d \in V(G) : (u = v_0) \wedge (v = v_d)\\ &\qquad \wedge\Big((v_0v_1 \in E(G)) \vee (v_0 = v_1)\Big) \wedge \cdots \wedge \Big((v_{d-1}v_d \in E(G)) \vee (v_{d-1} = v_d)\Big).\end{aligned}$$

Furthermore, a set C of vertices of G is a d-identifying code in G if and only if

$$\begin{aligned}&\Big(\forall u \in V(G) : \exists v \in C : v \in N^{\leq d}_G[u]\Big) \wedge\\ &\Big(\forall x, y \in V(G) : (x \neq y) \Rightarrow\\ &\qquad \exists z \in C : \ \Big((z \in N^{\leq d}_G[x]) \wedge (z \notin N^{\leq d}_G[y])\Big) \vee \Big((z \notin N^{\leq d}_G[x]) \wedge (z \in N^{\leq d}_G[y])\Big)\Big).\end{aligned}$$

These observations imply that the optimization problem to determine a d-identifying code of minimum order is expressible in the $LinEMSOL(\tau_1)$ logic [9]. Therefore, if cw is some constant, and $\mathcal{G}$ is a class of graphs such that every graph G in $\mathcal{G}$ has clique-width at most cw, and a clique-width expression for G using at most cw distinct labels can be determined in polynomial time, then d-identifying codes of minimum order can be determined in polynomial time for the graphs in $\mathcal{G}$ (cf. Theorem 4 in [9]). For the class of trees, this immediately implies the existence of linear time algorithms that determine a d-identifying code of minimum order for any fixed d. These algorithmic consequences extend to complementary prisms by the following result.

Proposition 1.1 *If G is a graph of clique-width cw, then $G\bar{G}$ has clique-width at most $4cw$.*

Proof: Let G be a graph of clique-width cw. In [19] it is shown that there is a rooted binary tree T whose leaves are the vertices of G such that, for every vertex s of T, the set V_s of vertices of G that are descendants of s in T partitions into at most cw equivalence classes with respect to the equivalence relation $\sim$, where $u \sim v$ for $u, v \in V_s$ if and only if $N_G[u] \setminus V_s = N_G[v] \setminus V_s$. Replacing in T every leaf u with parent x by three vertices u, $\bar{u}$, and y, and adding the arcs (x, y), (y, u), and $(y, \bar{u})$, we obtain a rooted binary tree T' whose leaves are the vertices of $G\bar{G}$. By the definition of $G\bar{G}$, we obtain that for every vertex s' of T', the set V'_s of vertices of $G\bar{G}$ that are descendants of s' in T' partitions into at most $2cw$ equivalence classes with respect to the equivalence relation $\sim'$, where $u \sim' v$ for $u, v \in V'_s$ if and only if $N_{G\bar{G}}[u] \setminus V'_s = N_{G\bar{G}}[v] \setminus V'_s$. Again by [19], this implies that the clique-width of $G\bar{G}$ is at most $4cw$. $\square$

2 Minimum identifying code in $C_n\bar{C}_n$

Throughout this section, let $C_n : v_1v_2 \ldots v_nv_1$ be a cycle of order n at least 3, and let $G = C_n\bar{C}_n$. We identify indices of vertices of G modulo n. For a subset C of $V(C_n)$, let $x(C)$ denote the characteristic vector of C, that is, $x(C) = (x_1, \ldots, x_n) \in \{0, 1\}^n$ where $x_i = 1$ if and only if $v_i \in C$ for $i \in [n]$. Similarly, for a subset $\bar{C}$ of $V(\bar{C}_n)$, let $x(\bar{C}) = (\bar{x}_1, \ldots, \bar{x}_n) \in \{0, 1\}^n$ where $\bar{x}_i = 1$ if and only if $\bar{v}_i \in \bar{C}$ for $i \in [n]$.

Lemma 2.1 *For an integer n at least 9, let $G = C_n\bar{C}_n$. Let $C \subseteq V(C_n)$ and $\bar{C} \subseteq V(\bar{C}_n)$. Let $x(C) = (x_1, \ldots, x_n)$ and $x(\bar{C}) = (\bar{x}_1, \ldots, \bar{x}_n)$.*

If $C \cup \bar{C}$ is an identifying code in G, then the following conditions hold for every

$i, j \in [n]$ with $(j-i) \bmod n \notin \{0,2\}$ (cf. Figure 1):

$$\begin{aligned}
C(i) &: x_{i-1} + x_i + \bar{x}_i + x_{i+1} &&\geq 1,\\
C(i,i+1) &: x_{i-1} + \bar{x}_i + \bar{x}_{i+1} + x_{i+2} &&\geq 1,\\
C(i,i+2) &: x_{i-1} + x_i + \bar{x}_i + x_{i+2} + \bar{x}_{i+2} + x_{i+3} &&\geq 1,\\
\bar{C}(i,j) &: \bar{x}_{i-1} + x_i + \bar{x}_{i+1} + \bar{x}_{j-1} + x_j + \bar{x}_{j+1} &&\geq 1, \textit{ and}\\
\bar{C}(i,i+2) &: \bar{x}_{i-1} + x_i + x_{i+2} + \bar{x}_{i+3} &&\geq 1.
\end{aligned}$$

Furthermore, if $|\bar{C}| \geq 4$, then $C \cup \bar{C}$ is an identifying code in G if and only if these conditions hold.

Proof. Note that for distinct vertices u and v of G, we have $N_G[u] \cap (C \cup \bar{C}) \neq N_G[v] \cap (C \cup \bar{C})$ if and only if $C \cup \bar{C}$ intersects $(N_G[u] \setminus N_G[v]) \cup (N_G[v] \setminus N_G[u])$. Therefore, for $i \in [n]$, we have that $C(i)$ is equivalent to $N_G(v_i) \cap (C \cup \bar{C}) \neq \emptyset$, $C(i,i+1)$ is equivalent to $N_G(v_i) \cap (C \cup \bar{C}) \neq N_G(v_{i+1}) \cap (C \cup \bar{C})$, $C(i,i+2)$ is equivalent to $N_G(v_i) \cap (C \cup \bar{C}) \neq N_G(v_{i+2}) \cap (C \cup \bar{C})$, and $\bar{C}(i,i+2)$ is equivalent to $N_G(\bar{v}_i) \cap (C \cup \bar{C}) \neq N_G(\bar{v}_{i+2}) \cap (C \cup \bar{C})$. For $i, j \in [n]$ with $(j-i) \bmod n \geq 3$, we have that $C(i)$ and $C(j)$ together are equivalent to $N_G(v_i) \cap (C \cup \bar{C}) \neq N_G(v_j) \cap (C \cup \bar{C})$. For $i, j \in [n]$ with $(j-i) \bmod n \notin \{0,2\}$, we have that $\bar{C}(i,j)$ is equivalent to $N_G(\bar{v}_i) \cap (C \cup \bar{C}) \neq N_G(\bar{v}_j) \cap (C \cup \bar{C})$. Hence, all these conditions are necessary. Note that $|\bar{C}| \geq 4$ implies $N_G(v_i) \cap (C \cup \bar{C}) \neq N_G(\bar{v}_j) \cap (C \cup \bar{C}) \neq \emptyset$ for every $i, j \in [n]$, in which case, the given conditions are also sufficient. □

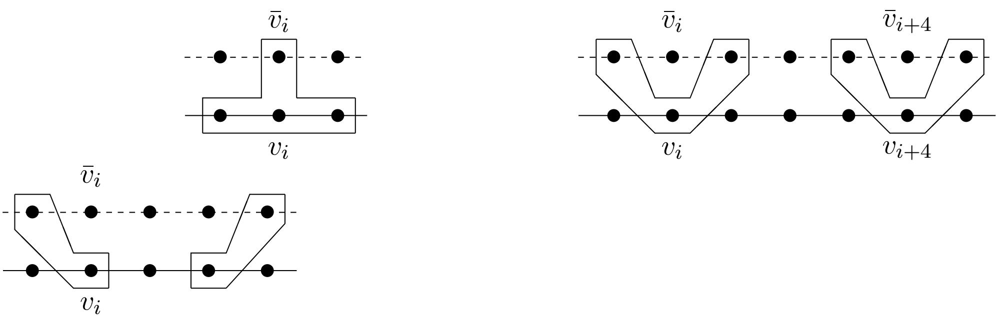

Fig. 1. Condition $C(i)$ implies that at least one of the four vertices indicated in the left figure belongs to $C \cup \bar{C}$. Condition $\bar{C}(i,i+4)$ implies that at least one of the six vertices indicated in the middle figure belongs to $C \cup \bar{C}$. Condition $\bar{C}(i,i+2)$ implies that at least one of the four vertices indicated in the right figure belongs to $C \cup \bar{C}$. Note that for $\bar{C}_n$, instead of indicating the edges, we indicate the non-edges by dashed lines.

Lemma 2.2 *For an integer n at least 9, let $G = C_n\bar{C}_n$. Let $C \subseteq V(C_n)$ and $\bar{C} \subseteq V(\bar{C}_n)$. Let $x(C) = (x_1, \ldots, x_n)$ and $x(\bar{C}) = (\bar{x}_1, \ldots, \bar{x}_n)$.*

If $k = \left\lfloor \frac{n}{9} \right\rfloor$, *and for* $i \in [n]$ *(cf. Figure 2),*

$$x_i = \begin{cases} 1, \ i \bmod 9 \in \{1,2,3\} \textit{ and } i \le 9k, \\ 1, \ i \ge 9k+1, \ \textit{ and} \\ 0, \ \textit{otherwise} \end{cases}$$

and

$$\bar{x}_i = \begin{cases} 1, \ i \bmod 9 \in \{5,6,7,8\} \textit{ and } i \le 9k, \ \textit{ and} \\ 0, \ \textit{otherwise,} \end{cases}$$

then $C \cup \bar{C}$ *is an identifying code in* G*. In particular,* $\mathrm{ic}(G) \le \frac{7}{9}n + \frac{16}{9}$.

Proof. Lemma 2.1 easily implies that $C \cup \bar{C}$ is an identifying code in G. Furthermore, $|C \cup \bar{C}| = 7k + (n - 9k) = n - 2k = n - 2\left\lfloor \frac{n}{9} \right\rfloor \le n - 2\frac{(n-8)}{9} = \frac{7}{9}n + \frac{16}{9}$. □

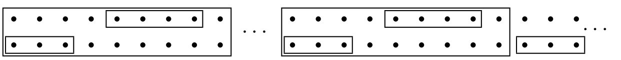

Fig. 2. Some identifying code for $C_n\bar{C}_n$.

Lemma 2.3 *For an integer* n *at least* 9*, let* $G = C_n\bar{C}_n$*. Let* $C \subseteq V(C_n)$ *and* $\bar{C} \subseteq V(\bar{C}_n)$ *be such that* $C \cup \bar{C}$ *is an identifying code in* G*. Let* $x(C) = (x_1, \dots, x_n)$ *and* $x(\bar{C}) = (\bar{x}_1, \dots, \bar{x}_n)$*. Let* $\bar{I} = \{i \in [n] : \bar{x}_{i-1} + x_i + \bar{x}_{i+1} = 0\}$.

If $|\bar{C}| \ge 6$ *and* $i \in [n]$ *is such that* $i-5, i, i+1, i+6 \notin \bar{I}$ *and* $\binom{\bar{x}_i \ \bar{x}_{i+1}}{x_i \ x_{i+1}} = \binom{0\ 0}{0\ 0}$*, then*

(i) either there are subsets $C' \subseteq V(C_n)$ *and* $\bar{C}' \subseteq V(\bar{C}_n)$ *such that* $C' \cup \bar{C}'$ *is an identifying code in* G *with* $|C' \cup \bar{C}'| \le |C \cup \bar{C}|$*, and*

$$\left|\left\{ j \in [n] : \begin{pmatrix} \bar{x}'_j \\ x'_j \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}\right| < \left|\left\{ j \in [n] : \begin{pmatrix} \bar{x}_j \\ x_j \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}\right|$$

where $x(C') = (x'_1, \dots, x'_n)$ *and* $x(\bar{C}') = (\bar{x}'_1, \dots, \bar{x}'_n)$*,*

(ii) or $\left\{ \binom{\bar{x}_{i-1} \ \bar{x}_i \ \dots \ \bar{x}_{i+7}}{x_{i-1} \ x_i \ \dots \ x_{i+7}}, \binom{\bar{x}_{i-6} \ \bar{x}_{i-5} \ \dots \ \bar{x}_{i+2}}{x_{i-6} \ x_{i-5} \ \dots \ x_{i+2}} \right\}$ *contains* $\binom{1\ 0\ 0\ 1\ 0\ 1\ 0\ 0\ 1}{1\ 0\ 0\ 1\ 0\ 1\ 0\ 0\ 1}$.

Proof. We use the conditions from Lemma 2.1. Let $i \in [n]$ be such that $i-5, i, i+1, i+6 \notin \bar{I}$ and $\binom{\bar{x}_i \ \bar{x}_{i+1}}{x_i \ x_{i+1}} = \binom{0\ 0}{0\ 0}$. $C(i)$ and $C(i+1)$ imply $x_{i-1} = x_{i+2} = 1$. Since $i, i+1 \notin \bar{I}$, we have $\bar{x}_{i-1} = \bar{x}_{i+2} = 1$.

Let $C' = C \cup \{v_i, v_{i+1}\}$ and $\bar{C}' = \bar{C} \setminus \{\bar{v}_{i-1}, \bar{v}_{i+2}\}$. Let $x(C') = (x'_1, \dots, x'_n)$ and $x(\bar{C}') = (\bar{x}'_1, \dots, \bar{x}'_n)$. If $C' \cup \bar{C}'$ is an identifying code in G, then (i) holds. Hence, we may assume that $C' \cup \bar{C}'$ is not an identifying code in G. Since $|\bar{C}'| \ge 4$, some condition from Lemma 2.1 is violated by $C' \cup \bar{C}'$. Since $(C \cup \bar{C}) \setminus (C' \cup \bar{C}') = \{\bar{v}_{i-1}, \bar{v}_{i+2}\}$, a violated condition must involve $\bar{x}'_{i-1}$ or $\bar{x}'_{i+2}$. By symmetry, we may assume that $\bar{x}'_{i+2}$ is involved in a violated condition. The conditions that involve $\bar{x}'_{i+2}$ are $C(i+2)$, $C(i+1, i+2)$, $C(i+2, i+3)$, $C(i, i+2)$, $C(i+2, i+4)$, $\bar{C}(i+1, j)$,

$\bar{C}(i+3,j)$, $\bar{C}(i-1,i+1)$, and $\bar{C}(i+3,i+5)$ for $j \in [n]$ with $(j-i) \bmod n \notin \{0,2\}$, where we replace x_j with x'_j and $\bar{x}_j$ with $\bar{x}'_j$ for all $j \in [n]$. Since $x'_i = 1$, the conditions $C(i+1,i+2)$ and $C(i,i+2)$ are not violated. Since $x'_{i+1} = 1$, the conditions $C(i+2,i+3)$, $\bar{C}(i+1,j)$, and $\bar{C}(i-1,i+1)$ are not violated. Since $x'_{i+2} = 1$, the conditions $C(i+2)$ and $C(i+2,i+4)$ are not violated. If $\bar{C}(i+3,j)$ is violated, then $x_{i+3} = x'_{i+3} = 0$ and $\bar{x}_{i+4} = \bar{x}'_{i+4} = 0$. If $\bar{C}(i+3,i+5)$ is violated, then $x_{i+3} = x_{i+5} = 0$ and $\bar{x}_{i+6} = 0$. Therefore, by symmetry, we may assume that either $x_{i+3} = \bar{x}_{i+4} = 0$, or $x_{i+3} = x_{i+5} = \bar{x}_{i+6} = 0$, and $\bar{x}_{i+4} = 1$. In the first case, $\bar{C}(i+1,i+3)$ is violated. Hence, we may assume $x_{i+3} = x_{i+5} = \bar{x}_{i+6} = 0$, and $\bar{x}_{i+4} = 1$. $C(i+1,i+3)$ implies that $\bar{x}_{i+3} = 1$ or $x_{i+4} = 1$. Let $C'' = C \setminus \{v_{i+2}\}$ and $\bar{C}'' = \bar{C} \cup \{\bar{v}_{i+1}\}$. If $C'' \cup \bar{C}''$ is an identifying code in G, then (i) holds. Let $x(C'') = (x''_1, \ldots, x''_n)$. Hence, we may assume that $C'' \cup \bar{C}''$ is not an identifying code in G. Since $|\bar{C}''| \geq 4$, some condition from Lemma 2.1 is violated by $C'' \cup \bar{C}''$. Since $(C \cup \bar{C}) \setminus (C'' \cup \bar{C}'') = \{v_{i+2}\}$, a violated condition must involve x''_{i+2}. Arguing as above, Lemma 2.1 implies that $\bar{x}_{i+3} = \bar{x}_{i+5} = x_{i+6} = 0$. As noted above, $\bar{x}_{i+3} = 0$ implies $x_{i+4} = 1$. Now $C(i+6)$ and $i+6 \notin \bar{I}$ imply $x_{i+7} = \bar{x}_{i+7} = 1$, that is, (ii) holds, which completes the proof. $\square$

Lemma 2.4 *If n is an integer at least 9, then* $\mathrm{ic}(C_n\bar{C}_n) \geq \frac{7}{9}n - 12$.

Proof. We prove the statement by induction on n, and use the conditions from Lemma 2.1. Clearly, we may assume that $n > \left\lfloor \frac{9 \cdot 12}{7} \right\rfloor = 15$. Let $G = C_n\bar{C}_n$, and let $C \subseteq V(C_n)$ and $\bar{C} \subseteq V(\bar{C}_n)$ be such that

- $C \cup \bar{C}$ is an identifying code in G with $\mathrm{ic}(G) = |C \cup \bar{C}|$, and
- subject to the previous condition, $\left|\left\{ i \in [n] : \binom{\bar{x}_i}{x_i} = \binom{0}{0} \right\}\right|$ is as small as possible.

Let $\bar{I} = \{i \in [n] : \bar{x}_{i-1} + x_i + \bar{x}_{i+1} = 0\}$. By $\bar{C}(i,j)$ and $\bar{C}(i,i+2)$, we may assume that $\bar{I} \subseteq [1]$.

If $|\bar{C}| \leq 5$, then there are at least $n - 1 - 2|\bar{C}|$ indices i with $i \notin \bar{I}$ and $\bar{x}_{i-1} = \bar{x}_{i+1} = 0$, which implies $x_i = 1$. Therefore, $|C| + |\bar{C}| \geq (n - 1 - 2|\bar{C}|) + |\bar{C}| = n - 1 - |\bar{C}| \geq n - 6 > \frac{7}{9}n - 12$. Hence, we may assume that $|\bar{C}| \geq 6$.

Claim 2.5 *There is no integer i with $7 \leq i \leq n-6$ such that* $\begin{pmatrix} \bar{x}_i & \bar{x}_{i+1} \\ x_i & x_{i+1} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Proof of Claim 2.5: If there is some integer i with $7 \leq i \leq n-6$ such that $\begin{pmatrix} \bar{x}_i & \bar{x}_{i+1} \\ x_i & x_{i+1} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, then $i-5, i, i+1, i+6 \notin \bar{I}$. Now Lemma 2.3 and the choice of $C \cup \bar{C}$ imply $\begin{pmatrix} \bar{x}_{i-1} & \bar{x}_i & \ldots & \bar{x}_{i+7} \\ x_{i-1} & x_i & \ldots & x_{i+7} \end{pmatrix} = \begin{pmatrix} 1&0&0&1&0&1&0&0&1 \\ 1&0&0&1&0&1&0&0&1 \end{pmatrix}$ or $\begin{pmatrix} \bar{x}_{i-6} & \bar{x}_{i-5} & \ldots & \bar{x}_{i+2} \\ x_{i-6} & x_{i-5} & \ldots & x_{i+2} \end{pmatrix} = \begin{pmatrix} 1&0&0&1&0&1&0&0&1 \\ 1&0&0&1&0&1&0&0&1 \end{pmatrix}$. By symmetry, we may assume that the former case occurs. Let $C' \subseteq V(C_{n-5})$ and $\bar{C}' \subseteq V(\bar{C}_{n-5})$ be such that

$$\begin{pmatrix} \bar{x}'_1 & \ldots & \bar{x}'_{n-5} \\ x'_1 & \ldots & x'_{n-5} \end{pmatrix} = \begin{pmatrix} \bar{x}_1 & \ldots & \bar{x}_{i+2} & \bar{x}_{i+8} & \ldots & \bar{x}_n \\ x_1 & \ldots & x_{i+2} & x_{i+8} & \ldots & x_n \end{pmatrix}$$

for $x(C') = (x'_1, \ldots, x'_{n-5})$ and $x(\bar{C}') = (\bar{x}'_1, \ldots, \bar{x}'_{n-5})$. Since $|\bar{C}| \geq 6$, we have $|\bar{C}'| \geq 4$. Considering the conditions from Lemma 2.1 easily implies that $C' \cup \bar{C}'$ is an identifying code in $C_{n-5}\bar{C}_{n-5}$. By induction, we obtain

$|C \cup \bar{C}| = |C' \cup \bar{C}'| + 4 \geq \frac{7}{9}(n-5) - 12 + 4 > \frac{7}{9}n - 12$. $\square$

Let $i_1 < i_2 < \ldots < i_k$ be the increasing sequence of integers i with $8 \leq i \leq n-6$ such that $\binom{\bar{x}_i}{x_i} = \binom{0}{0}$. Note that, by Claim 2.5, for $j \in [k]$, we have $\binom{\bar{x}_{i_j-1}}{x_{i_j-1}}, \binom{\bar{x}_{i_j+1}}{x_{i_j+1}} \neq \binom{0}{0}$.

For $j \in [k-1]$, let $I_j = \{i \in [n] : i_j \leq i \leq i_{j+1} - 1\}$. Note that $|I_j| \geq 2$ for $j \in [k-1]$.

For $I \subseteq [n]$, let $V(I) = \bigcup_{i \in I}\{v_i, \bar{v}_i\}$.

Claim 2.6 *If $k \geq 2$, then there are integers $\ell, j_1, j_2, \ldots, j_\ell$ with $\ell \geq 2$ and $1 = j_1 < j_2 < \ldots < j_\ell = k$ such that*

$$\left|(C \cup \bar{C}) \cap V(I_{j_1} \cup \cdots \cup I_{j_2-1})\right| \geq \frac{7}{9}\left|I_{j_1} \cup \cdots \cup I_{j_2-1}\right| - \frac{5}{9}, \tag{1}$$

$$\left|(C \cup \bar{C}) \cap V(I_{j_r} \cup \cdots \cup I_{j_{r+1}-1})\right| \geq \frac{7}{9}\left|I_{j_r} \cup \cdots \cup I_{j_{r+1}-1}\right|, \text{ for } r \in [\ell-2] \setminus [1], \tag{2}$$

$$\text{and } \left|(C \cup \bar{C}) \cap V(I_{j_{\ell-1}} \cup \cdots \cup I_{j_\ell-1})\right| \geq \frac{7}{9}\left|I_{j_{\ell-1}} \cup \cdots \cup I_{j_\ell-1}\right| - \frac{5}{9}. \tag{3}$$

Proof of Claim 2.6: If for some $j \in [k-1]$, there is some $i \in I_j$ with $\binom{\bar{x}_i}{x_i} = \binom{1}{1}$, then I_j is *dirty*; otherwise I_j is *clean*. Note that, if I_j is dirty, then

$$\left|(C \cup \bar{C}) \cap V(I_j)\right| \geq |I_j|, \tag{4}$$

and, if I_j is clean, then, since $|I_j| \geq 2$,

$$\left|(C \cup \bar{C}) \cap V(I_j)\right| = |I_j| - 1 \geq \frac{7}{9}|I_j| - \frac{5}{9}. \tag{5}$$

Let $j_1 = 1$.

Recall that $\binom{\bar{x}_{i_1-1}}{x_{i_1-1}} \neq \binom{0}{0}$. If $\binom{\bar{x}_{i_1-1}}{x_{i_1-1}} \neq \binom{1}{1}$, then the definition of j_2 follows the pattern of the definition of j_{r+1} for $r \geq 2$ described above, that is, in this case, (2) will be satisfied also for $r = 1$, which is a stronger inequality. If $\binom{\bar{x}_{i_1-1}}{x_{i_1-1}} = \binom{1}{1}$, then let j_2 be maximum such that $j_1 < j_2 \leq k$ and I_j is dirty for $j \in \{j_1, j_1+1, \ldots, j_2-2\}$. Note that, if I_{j_1} is clean, then $j_2 = j_1 + 1$. By (4) and (5), we obtain that (1) holds. If $j_2 = k$, then set $\ell = 2$, and terminate the definition of the sequence $j_1, \ldots, j_\ell$. Note that (3) coincides with (1) in this case. If $j_2 < k$, then, by the choice of j_2, we have that I_{j_2-1} is clean, which implies that $\binom{\bar{x}_{i_{j_2}-1}}{x_{i_{j_2}-1}} \neq \binom{1}{1}$.

Therefore, we may now assume that for some non-negative integer r, the indices $1 = j_1 < \cdots < j_r < k$ have already been defined in such a way that the corresponding conditions are satisfied, and that $\binom{\bar{x}_{i_{j_r}-1}}{x_{i_{j_r}-1}} \notin \left\{\binom{0}{0}, \binom{1}{1}\right\}$. We will define j_{r+1} with $j_r < j_{r+1} \leq k$ such that the corresponding condition is satisfied. We consider different cases. In each case, we consider potential choices j'_{r+1} and possibly j''_{r+1} for j_{r+1}. As before, if one of j_{r+1}, j'_{r+1}, or j''_{r+1} equals k, then set $\ell = r+1$, and terminate the definition of the sequence $j_1, \ldots, j_\ell$. In such a case, (4) and (5) will imply (3).

Let $t = i_{j_r}$.

Case 1 *I_{j_r} is clean and* $\begin{pmatrix} \bar{x}_t & \bar{x}_{t+1} \\ x_t & x_{t+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

$C(t)$ implies $x_{t-1} = 1$, and hence, $\bar{x}_{t-1} = 0$. Since $t+1 \notin \bar{I}$, we have $\bar{x}_{t+2} = 1$. Since I_{j_r} is clean, $x_{t+2} = 0$. $\bar{C}(t, t+2)$ implies $\bar{x}_{t+3} = 1$. Since I_{j_r} is clean, $x_{t+3} = 0$. $\bar{C}(t+1, t+3)$ implies $\bar{x}_{t+4} = 1$. Since I_{j_r} is clean, $x_{t+4} = 0$. This implies $i_{j_r+1} - i_{j_r} \geq 5$. Since $\left|\left(C \cup \bar{C}\right) \cap V(I_{j_r})\right| = |I_{j_r}| - 1 \geq \frac{4}{5}|I_{j_r}| > \frac{7}{9}|I_{j_r}|$, setting $j_{r+1} = j_r + 1$, we obtain condition (2) for r.

Case 2 *I_{j_r} is clean and* $\binom{\bar{x}_t \ \bar{x}_{t+1}}{x_t \ x_{t+1}} = \binom{0\ 0}{0\ 1}$.

Since $t \notin \bar{I}$, we have $\bar{x}_{t-1} = 1$, and hence, $x_{t-1} = 0$. $C(t, t+1)$ implies $x_{t+2} = 1$. Since I_{j_r} is clean, $\bar{x}_{t+2} = 0$. $C(t+1, t+2)$ implies $x_{t+3} = 1$. Since I_{j_r} is clean, $\bar{x}_{t+3} = 0$. This implies $i_{j_r+1} - i_{j_r} \geq 4$. If $i_{j_r+1} - i_{j_r} \geq 5$, then setting $j_{r+1} = j_r + 1$, we obtain condition (2) for r as in Case 1. Hence, we may assume that $i_{j_r+1} - i_{j_r} = 4$.

If I_{j_r+1} is clean, then $t + 4 \notin \bar{I}$ implies $\bar{x}_{t+5} = 1$, and hence, $x_{t+5} = 0$. Now, analogous arguments as in Case 1 imply $x_{t+6} = x_{t+7} = x_{t+8} = 0$ and $\bar{x}_{t+6} = \bar{x}_{t+7} = \bar{x}_{t+8} = 1$. Hence, $i_{j_r+2} - i_{j_r} \geq 9$, and

$$\left|\left(C \cup \bar{C}\right) \cap V(I_{j_r} \cup I_{j_r+1})\right| = |I_{j_r} \cup I_{j_r+1}| - 2 \geq \frac{7}{9}|I_{j_r} \cup I_{j_r+1}|,$$

that is, setting $j_{r+1} = j_r + 2$, we obtain condition (2) for r. Hence, we may assume that I_{j_r+1} is dirty.

Let j'_{r+1} be maximum such that $j_r < j'_{r+1} \leq k$ and I_j is dirty for $j \in \{j_r + 1, j_r + 2, \ldots, j'_{r+1} - 2\}$. Clearly, $j'_{r+1} = k$ or $I_{j'_{r+1}-1}$ is clean.

If $j'_{r+1} = k$, then set $\ell = r + 1$ and $j_{r+1} = k$. Note that, if $I_{\ell-1}$ is dirty, then $|I_{j_r} \cup \cdots \cup I_{\ell-1}| \geq 6$ and

$$\left|\left(C \cup \bar{C}\right) \cap V(I_{j_r} \cup \cdots \cup I_{j_{\ell}-1})\right| = |I_{j_r} \cup \cdots \cup I_{j_{\ell}-1}| - 1 \geq \frac{7}{9}\left|I_{j_r} \cup \cdots \cup I_{j_{\ell}-1}\right|,$$

and, if $I_{j_{\ell}-1}$ is clean, then $|I_{j_r} \cup \cdots \cup I_{j_{\ell}-1}| \geq 8$ and

$$\left|\left(C \cup \bar{C}\right) \cap V(I_{j_r} \cup \cdots \cup I_{j_{\ell}-1})\right| = |I_{j_r} \cup \cdots \cup I_{j_{\ell}-1}| - 2 \geq \frac{7}{9}\left|I_{j_r} \cup \cdots \cup I_{j_{\ell}-1}\right| - \frac{5}{9},$$

that is, in both cases (3) holds. Hence, we may assume that $j'_{r+1} < k$ and $I_{j'_{r+1}-1}$ is clean.

If $i_{j'_{r+1}-1} - i_{j_r+1} \geq 3$, then set $j_{r+1} = j'_{r+1}$. Since $\left|I_{j_r} \cup \cdots \cup I_{j_{r+1}-1}\right| \geq 9$ and

$$\left|\left(C \cup \bar{C}\right) \cap V(I_{j_r} \cup \cdots \cup I_{j_{r+1}-1})\right| = \left|I_{j_r} \cup \cdots \cup I_{j_{r+1}-1}\right| - 2 \geq \frac{7}{9}\left|I_{j_r} \cup \cdots \cup I_{j_{r+1}-1}\right|,$$

(2) holds for r. Hence, we may assume that $i_{j'_{r+1}-1} - i_{j_r+1} = 2$, which implies that I_{j_r+1} has exactly two elements, and $j'_{r+1} = j_r + 3$, that is, I_{j_r+2} is clean.

Let $s = i_{j'_{r+1}-1}$. Note that $s = t + 6$. If $\binom{\bar{x}_s \ \bar{x}_{s+1}}{x_s \ x_{s+1}} = \binom{0\ 1}{0\ 0}$, then, $s + 1 \notin \bar{I}$ implies $\bar{x}_{s+2} = 1$, which implies that $|I_{i_r+2}| \geq 3$. Again, setting $j_{r+1} = j'_{r+1}$ yields $\left|I_{j_r} \cup \cdots \cup I_{j_{r+1}-1}\right| \geq 9$, and (2) for r follows as above. Hence, we may assume that $\binom{\bar{x}_s \ \bar{x}_{s+1}}{x_s \ x_{s+1}} = \binom{0\ 0}{0\ 1}$, that is,

$$\begin{pmatrix} \bar{x}_t & \ldots & \bar{x}_{t+7} \\ x_t & \ldots & x_{t+7} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

Now, $\bar{C}(t + 4, t + 6)$ does not hold, which is a contradiction, and completes the second case.

For the remaining cases, we may assume that I_{j_r} is dirty. Let j'_{r+1} be maximum such that $j_r < j'_{r+1} \le k$ and I_j is dirty for $j \in \{j_r, j_r+1, \dots, j'_{r+1}-2\}$. Clearly, $j'_{r+1} = k$ or $I_{j'_{r+1}-1}$ is clean.

If $j'_{r+1} = k$, then set $\ell = r+1$ and $j_{r+1} = k$. Note that, if $I_{\ell-1}$ is dirty, then

$$\left|\left(C \cup \bar{C}\right) \cap V(I_{j_r} \cup \dots \cup I_{j_\ell - 1})\right| = |I_{j_r} \cup \dots \cup I_{j_\ell - 1}| \ge \frac{7}{9}\,|I_{j_r} \cup \dots \cup I_{j_\ell - 1}|,$$

and, if $I_{j_\ell - 1}$ is clean, then $|I_{j_r} \cup \dots \cup I_{j_\ell - 1}| \ge 4$ and

$$\left|\left(C \cup \bar{C}\right) \cap V(I_{j_r} \cup \dots \cup I_{j_\ell - 1})\right| = |I_{j_r} \cup \dots \cup I_{j_\ell - 1}| - 1 \ge \frac{7}{9}\,|I_{j_r} \cup \dots \cup I_{j_\ell - 1}| - \frac{5}{9},$$

that is, in both cases (3) holds. Hence, we may assume that $j'_{r+1} < k$ and $I_{j'_{r+1}-1}$ is clean.

The remaining two cases have some similarities with Cases 1 and 2.

Let $t = i_{j'_{r+1}-1}$.

Case 3 I_{j_r} *is dirty and* $\binom{\bar{x}_t\ \bar{x}_{t+1}}{x_t\ x_{t+1}} = \binom{0\ 1}{0\ 0}$.
Since $t \notin \bar{I}$, we have $\bar{x}_{t+2} = 1$. Since $I_{j'_{r+1}-1}$ is clean, $x_{t+2} = 0$. This implies $i_{j'_{r+1}} - i_{j_r} \ge 5$. Setting $j_{r+1} = j'_{r+1}$, condition (2) for r follows as in Case 1.

Case 4 I_{j_r} *is dirty and* $\binom{\bar{x}_t\ \bar{x}_{t+1}}{x_t\ x_{t+1}} = \binom{0\ 0}{0\ 1}$.
If $i_{j'_{r+1}} - i_{j_r} \ge 5$, then setting $j_{r+1} = j'_{r+1}$ satisfies (2) for r as in Case 3. Hence, we may assume that $i_{j'_{r+1}} - i_{j_r} = 4$, which implies $j'_{r+1} = j_r + 2$ and $|I_{j_r}| = |I_{j_r+1}| = 2$.

If I_{j_r+2} is clean, then $t+2 \notin \bar{I}$ implies $\bar{x}_{t+3} = 1$, and hence $x_{t+3} = 0$. Now similar arguments as in Case 1 imply $x_{t+4} = x_{t+5} = x_{t+6} = 0$ and $\bar{x}_{t+4} = \bar{x}_{t+5} = \bar{x}_{t+6} = 1$. Therefore, $i_{j_r+3} - i_{j_r} \ge 9$, and setting $j_{r+1} = i_{j_r+3}$ satisfies (2) for r as above. Note that if $i_{j_r+3} - i_{j_r} = 9$, then $\binom{\bar{x}_t\ \dots\ \bar{x}_{t+8}}{x_t\ \dots\ x_{t+8}}$ corresponds to the pattern used in the proof of Lemma 2.2. Hence, we may assume that I_{j_r+2} is dirty.

Let j''_{r+1} be maximum such that $j_r + 2 < j''_{r+1} \le k$ and I_j is dirty for $j \in \{j_r+2, j_r+3, \dots, j''_{r+1}-2\}$. Clearly, $j''_{r+1} = k$ or $I_{j''_{r+1}-1}$ is clean. If $j''_{r+1} = k$, then setting $\ell = r+1$ and $j_{r+1} = k$, and arguing similarly as in Case 2 yields (3). Hence, we may assume $j''_{r+1} < k$ and $I_{j''_{r+1}-1}$ is clean.

If $i_{j''_{r+1}} - i_{j_r} \ge 9$, then setting $j_{r+1} = j''_{r+1}$ yields (2) for r as above. Hence, we may assume that $i_{j''_{r+1}} - i_{j_r} = 8$, which implies that $j''_{r+1} = j_r + 4$ and $|I_{j_r+2}| = |I_{j_r+3}| = 2$. This implies

$$\begin{pmatrix}\bar{x}_{t-2} & \dots & \bar{x}_{t+6}\\ x_{t-2} & \dots & x_{t+6}\end{pmatrix} \in \left\{ \begin{pmatrix}0\,1\,0\,0\,0\,1\,0\,1\,0\\ 0\,1\,0\,1\,0\,1\,0\,0\,0\end{pmatrix}, \begin{pmatrix}0\,1\,0\,0\,0\,1\,0\,0\,0\\ 0\,1\,0\,1\,0\,1\,0\,1\,0\end{pmatrix} \right\}.$$

Now the first options leads to the contradiction $t+5 \in \bar{I}$, and the second option leads to the contradiction that $\bar{C}(t+2, t+4)$ does not hold.

This completes the proof of Claim 2.6. □

If $k \leq 1$, then $n > 15$ implies

$$\left|(C \cup \bar{C})\right| \geq \left|(C \cup \bar{C}) \cap V([n-6] \setminus [7])\right| \geq n - 6 - 7 - 1 = n - 14 > \frac{7}{9}n - 12.$$

Hence, we may assume that $k \geq 2$.

Since i_1 is the smallest integer $i \geq 8$ with $\binom{\bar{x}_i}{x_i} = \binom{0}{0}$, we have $\left|(C \cup \bar{C}) \cap V([i_1 - 1])\right| \geq i_1 - 1 - 7$. Since i_k is the largest integer $i \leq n-6$ with $\binom{\bar{x}_i}{x_i} = \binom{0}{0}$, we have $\left|(C \cup \bar{C}) \cap V([n] \setminus [i_k - 1])\right| \geq n - 6 - i_k$. By Claim 2.6, we obtain

$$\begin{aligned}
&\left|(C \cup \bar{C}) \cap V([i_k - 1] \setminus [i_1 - 1])\right| \\
&= \left|(C \cup \bar{C}) \cap V(I_1 \cup \cdots \cup I_{k-1})\right| \\
&= \left|(C \cup \bar{C}) \cap V(I_{j_1} \cup \cdots \cup I_{j_2 - 1})\right| + \sum_{r=2}^{\ell-2} \left|(C \cup \bar{C}) \cap V(I_{j_r} \cup \cdots \cup I_{j_{r+1}-1})\right| \\
&\quad + \left|(C \cup \bar{C}) \cap V(I_{j_{\ell-1}} \cup \cdots \cup I_{j_\ell - 1})\right| \\
&\geq \left(\frac{7}{9}\left|I_{j_1} \cup \cdots \cup I_{j_2-1}\right| - \frac{5}{9}\right) + \sum_{r=2}^{\ell-2}\left(\frac{7}{9}\left|I_{j_r} \cup \cdots \cup I_{j_{r+1}-1}\right|\right) \\
&\quad + \left(\frac{7}{9}\left|I_{j_{\ell-1}} \cup \cdots \cup I_{j_\ell - 1}\right| - \frac{5}{9}\right) \\
&= \frac{7}{9}\left|I_1 \cup \cdots \cup I_{k-1}\right| - \frac{10}{9} \\
&= \frac{7}{9}(i_k - i_1) - \frac{10}{9} \\
&= \frac{7}{9}\left((i_k - i_1) - \frac{10}{7}\right).
\end{aligned}$$

Altogether, this implies

$$\begin{aligned}
|C \cup \bar{C}| &\geq (i_1 - 1 - 7) + \left(\frac{7}{9}(i_k - i_1) - \frac{10}{9}\right) + (n - 6 - i_k) \\
&\geq \frac{7}{9}\left((i_1 - 1 - 7) + \left((i_k - i_1) - \frac{10}{7}\right) + (n - 6 - i_k)\right) \\
&= \frac{7}{9}n - \frac{108}{9} \\
&= \frac{7}{9}n - 12,
\end{aligned}$$

which completes the proof. □

We proceed to our main result.

Theorem 2.7 $\mathrm{ic}(C_n\bar{C}_n) = \frac{7}{9}n + \Theta(1)$ *for* $n \geq 3$.

Proof. This follows immediately from Lemma 2.2 and Lemma 2.4. □

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