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# $s_2$-Quasialgebraic Posets

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**Abstract**

In this paper, the concept of $s_2$-quasialgebraic posets is introduced. The main results are: (1) A poset is an $s_2$-quasialgebraic iff the $\sigma_2$-topology is a hypercontinuous and algebraic lattice; (2) A poset is $s_2$-algebraic iff it is meet $s_2$-continuous and $s_2$-quasialgebraic.

*Keywords:* $s_2$-Algebraic poset, meet $s_2$-continuous poset, $s_2$-quasialgebraic poset, $\sigma_2$-topology

## 1 Introduction

The theory of continuous domains, due to its strong background in computer science, general topology and topological algebra has been extensively studied by people from various areas (see [1,8]). Quasicontinuous domains is an very interesting topic in domain theory (see [6,7,10,11,14,15]). In [19], the concept of $s_2$-quasicontinuous posets is introduced as a common generalization of both $s_2$-continuous posets [2] and

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quasicontinuous domains by making use of the cut operator instead of joins. The notion of $s_2$-quasicontinuity admits to generalize most important characterizations of quasicontinuity from dcpos to arbitrary posets and has the advantage that not even the existence of directed joins has to be required.

Algebraic and quasialgebraic domains also proved to be very useful in various fields of order theory, topology and computer science (see [1,4,8,12,13,14,16,17,21]). In [4,20], the concept of $s_2$-algebraic posets is introduced as a common generalization of both $s_2$-continuous posets and algebraic domains. In this paper, we introduce new concept of $s_2$-quasialgebraic posets as a common generalization of both $s_2$-algebraic posets and $s_2$-quasicontinuous posets. It is proved that a poset is $s_2$-algebraic iff it is meet $s_2$-continuous and $s_2$-quasialgebraic. We also give the characterization that a poset is an $s_2$-quasialgebraic iff the $\sigma_2$-topology is a hypercontinuous and algebraic lattice. In last section, we investigate some dual categories on posets.

## 2 Preliminaries

For a poset $P$, let $P^{(<\omega)} = \{F \subseteq P : F \text{ is finite}\}$. For all $x \in P$, $A \subseteq P$, let $\uparrow x = \{y \in P : x \leq y\}$ and $\uparrow A = \bigcup_{a\in A} \uparrow a$; $\downarrow x$ and $\downarrow A$ are defined dually. $A^{\uparrow}$ and $A^{\downarrow}$ denote the sets of all upper and lower bounds of $A$, respectively. Let $A^{\delta} = (A^{\uparrow})^{\downarrow}$ and $\delta(P) = \{A^{\delta} : A \subseteq P\}$.

For a poset $P$, the topology generated by the collection of sets $P \backslash \downarrow x$ (as a subbase) is called the *upper topology* and denoted by $\upsilon(P)$; the *lower topology* $\omega(P)$ on $P$ is defined dually. A subset $U$ of $P$ is called *Scott open* provided that $U = \uparrow U$ and $D \cap U \neq \emptyset$ for all directed sets $D \subseteq P$ with $\bigvee D \in U$ whenever $\bigvee D$ exists. The topology formed by all the Scott open sets of $P$ is called the *Scott topology* on $P$, written as $\sigma(P)$. The topology $\lambda(P) = \sigma(P) \vee \omega(P)$ is called the *Lawson topology* on $P$.

We order the collection of nonempty subsets of a poset $P$ by $G \leq H$ if $\uparrow H \subseteq \uparrow G$ (this is only a preorder, not an order, since it is typically not antisymmetric). We say that a nonempty family of sets is *directed* if given $F_1$, $F_2$ in the family, there exists $F$ in the family such that $F_1, F_2 \leq F$, i.e., $F \subseteq \uparrow F_1 \cap \uparrow F_2$. For non-empty subsets $F$ and $G$ of a dcpo $L$, we say $F$ *approximates* $G$ if whenever a directed subset $D$ satisfies $\bigvee D \in \uparrow G$, then $d \in \uparrow F$ for some $d \in D$. A dcpo $L$ is called a *quasicontinuous domain* if for all $x \in L$, $\uparrow x$ is the directed (with respect to reverse inclusion) intersection of sets of the form $\uparrow F$, where $F$ approximates $\{x\}$ and $F$ is finite. If in addition, it is possible to choose the finite sets $F$ such that $F$ approximates $F$ for each $F$, then $L$ is called a *quasialgebraic domain*.

**Lemma 2.1** ([5]) *Let $P$ be a poset.*

(1) *The maps $(-)^{\uparrow} : (2^P)^{op} \to 2^P$, $A \mapsto A^{\uparrow}$ and $(-)^{\downarrow} : 2^P \to (2^P)^{op}$, $A \mapsto A^{\downarrow}$ are order preserving.*

(2) *$((-)^{\uparrow}, (-)^{\downarrow})$ is a Galois connection between $(2^P)^{op}$ and $2^P$, that is, for all $A$, $B \subseteq P$, $B^{\uparrow} \supseteq A \Leftrightarrow B \subseteq A^{\downarrow}$. Thus both $\delta : 2^P \to 2^P$, $A \mapsto A^{\delta} = (A^{\uparrow})^{\downarrow}$ and $\delta^* : 2^P \to 2^P$, $A \mapsto (A^{\downarrow})^{\uparrow}$ are closure operators.*

(3) *Let* $L = \delta(P)$. *For all* $\{A_i{}^\delta : i \in I\} \subseteq L$, $\bigwedge_L \{A_i{}^\delta : i \in I\} = \bigcap \{A_i{}^\delta : i \in I\}$, $\bigvee_L \{A_i{}^\delta : i \in I\} = (\bigcup \{A_i{}^\delta : i \in I\})^\delta = (\bigcup\limits_{i \in I} A_i)^\delta$.

**Definition 2.2** ([4,20]) Let $P$ be a poset.

(1) Given any two elements $x$ and $y$ in $P$, we say that $x$ *approximates* $y$, written $x \ll y$, if for all directed sets $D \subseteq P$ with $y \in D^\delta$ , there exists $d \in D$ with $x \leq d$. The set $\{y \in P : y \ll x\}$ will be denoted $\Downarrow x$ and $\{y \in P : x \ll y\}$ denoted $\Uparrow x$. Let $K(P) = \{x \in P : x \ll x\}$.

(2) $P$ is called $s_2$*-continuous* if for all $x \in P$, $x \in (\Downarrow x)^\delta$ and $\Downarrow x$ is directed.

(3) $P$ is called $s_2$*-algebraic* if for all $x \in P$, $x \in (\downarrow x \cap K(P))^\delta$ and $\downarrow x \cap K(P)$ is directed.

**Definition 2.3** ([3,9]) A poset $P$ is called *meet* $s_2$*-continuous* if for any $x \in P$ and any directed set $D$ with $x \in D^\delta$, then $x$ is in the $\sigma_2$-closure of $\downarrow x \cap \downarrow D$.

**Corollary 2.4** ([19]) *If $F$ is a finite set in a meet $s_2$-continuous poset, then we have*

$$int_{\sigma_2(P)} \uparrow F \subseteq \bigcup \{\Uparrow x : x \in F\}.$$

**Definition 2.5** ([7]) For a complete lattice $L$, define a relation $\prec$ on $L$ by $x \prec y \Leftrightarrow y \in \mathrm{int}_{\upsilon(L)} \uparrow x$. $L$ is called *hypercontinuous* if $x = \bigvee \{u \in L : u \prec x\}$ for all $x \in L$.

**Definition 2.6** ([2,3]) Let $P$ be a poset. A subset $U \subseteq P$ is called $\sigma_2$*-open* if for all directed sets $D \subseteq P$, $D^\delta \cap U \neq \emptyset$ implies $D \cap U \neq \emptyset$.

The collection of all $\sigma_2$-open subsets of $P$ forms a topology, it will be called $\sigma_2$-*topology* of $P$ and will be denoted by $\sigma_2(P)$. The topology $\lambda_2(P) = \sigma_2(P) \vee \omega(P)$ is called the $\lambda_2$*-topology* on $P$. Obviously, $\upsilon(P) \subseteq \sigma_2(P) \subseteq \sigma(P)$.

**Remark 2.7** (1) For dcpos, the $\sigma_2$-topology is the same as the Scott topology and the $\lambda_2$-topology the same as the Lawson topology.

(2) $U$ is $\sigma_2$-open iff $U = \uparrow U$ and $U \ll U$. Hence $x \in U$ entails $U \ll x$.

**Definition 2.8** ([19]) Let $P$ be a poset.

(1) For all $G$, $H \subseteq P$, we say that $G$ is *way below* $H$ or $G$ *approximates* $H$ and write $G \ll H$ if for all directed sets $D \subseteq P$, $\uparrow H \cap D^\delta \neq \emptyset$ implies $\uparrow G \cap D \neq \emptyset$. We write $G \ll x$ for $G \ll \{x\}$ and $y \ll H$ for $\{y\} \ll H$. The set $\{x \in P : F \ll x\}$ will be denoted $\Uparrow F$ and $\{x \in P : x \ll F\}$ denoted $\Downarrow F$. Let $w(x) = \{F \subseteq P : F$ is finite and $F \ll x\}$ and $k(x) = \{F \subseteq P : F$ is finite and $F \ll F \leq x\}$.

(2) $P$ is called an $s_2$*-quasicontinuous poset* if for each $x \in P$, $\uparrow x = \bigcap \{\uparrow F : F \in w(x)\}$ and $w(x)$ is directed.

**Proposition 2.9** ([19]) *Let $\mathcal{F}$ be a directed family of nonempty finite sets in a poset. If $G \ll x$ and $\bigcap\limits_{F \in \mathcal{F}} \uparrow F \subseteq \uparrow x$, then $F \subseteq \uparrow G$ for some $F \in \mathcal{F}$.*

## 3 $s_2$-Quasialgebraic posets

In this section, the concept of $s_2$-quasialgebraic posets is introduced and some topological characterizations of $s_2$-quasialgebraic posets are given. Particularly, we show that a poset $P$ is $s_2$-algebraic if and only if it is both meet $s_2$-continuous and $s_2$-quasialgebraic.

**Definition 3.1** A poset $P$ is called an $s_2$-*quasialgebraic poset* if for each $x \in P$, $\uparrow x = \bigcap\{\uparrow F : F \in k(x)\}$ and $k(x)$ is directed.

**Remark 3.2** For dcpos the preceding definition of $s_2$-quasialgebraic posets is equivalent to the standard one ([8, Definition III-3.23]).

**Proposition 3.3** *For a poset $P$, the following conditions are equivalent:*

(1) *$P$ is an $s_2$-quasialgebraic poset;*

(2) *$P$ is an $s_2$-quasicontinuous poset, and $F \ll x$ iff there exists a finite $G \ll G$ with $x \in \uparrow G \subseteq \uparrow F$.*

**Proof.** (1) $\Rightarrow$ (2): Obviously, $P$ is an $s_2$-quasicontinuous poset. If $F \ll x$, then there exists a finite $G \ll G$ with $x \in \uparrow G \subseteq \uparrow F$ by Proposition 2.9. Conversely, if there exists a finite $G \ll G$ with $x \in \uparrow G \subseteq \uparrow F$, then $F \leq G \ll G \leq x$. Thus $F \ll x$.

(2) $\Rightarrow$ (1): Let $x \in P$ and $F_1$, $F_2 \in k(x)$. Then $F_1 \ll x$ and $F_2 \ll x$. Since $w(x)$ is directed, there exists $F \in w(x)$ such that $F \subseteq \uparrow F_1 \cap \uparrow F_2$. By (2), there exists a finite $G \ll G$ with $x \in \uparrow G \subseteq \uparrow F$. So $k(x)$ is directed.

Obviously, $\uparrow x \subseteq \bigcap\{\uparrow F : F \in k(x)\}$. If $y \notin \uparrow x$, then there exists $H \in w(x)$ such that $y \notin \uparrow H$ since $P$ is $s_2$-quasicontinuous. By (2), there exists a finite $E \ll E$ with $x \in \uparrow E \subseteq \uparrow H$. Thus $E \in k(x)$ and $y \notin \uparrow E$. Therefore, $P$ is $s_2$-quasialgebraic. $\square$

Now we give the topological characterizations of $s_2$-quasialgebraic posets.

**Proposition 3.4** *For a poset $P$, the following conditions are equivalent:*

(1) *$P$ is an $s_2$-quasialgebraic poset;*

(2) *For each $(x, U) \in P \times \sigma_2(P)$ with $x \in U$, there exists $F \in P^{(<\omega)}$ such that $x \in int_{\sigma_2(P)}\uparrow F = \uparrow F \subseteq U$;*

(3) *$(\sigma_2(P), \subseteq)$ is a hypercontinuous and algebraic lattice;*

(4) *For each compact set $K$ in $(P, \sigma_2(P))$ and $U \in \sigma_2(P)$, if $K \subseteq U$, then there exists $F \in P^{(<\omega)}$ such that $K \subseteq int_{\sigma_2(P)}\uparrow F = \uparrow F \subseteq U$;*

(5) *For each compact set $K$ in $(P, \sigma_2(P))$, the family $\{F \subseteq P : F$ is finite and $K \subseteq int_{\sigma_2(P)}\uparrow F = \uparrow F\}$ is directed and $\uparrow K = \bigcap\{\uparrow F\colon F$ is finite and $K \subseteq int_{\sigma_2(P)}\uparrow F = \uparrow F\}$.*

**Proof.** (1) $\Rightarrow$ (2): Let $(x, U) \in P \times \sigma_2(P)$ with $x \in U$. Then $U \ll x$. By Proposition 2.9, there exists $F \in k(x)$ such that $\uparrow F \subseteq U$. Since $F \ll F$ implies $int_{\sigma_2(P)}\uparrow F = \uparrow F$, $x \in int_{\sigma_2(P)}\uparrow F = \uparrow F \subseteq U$.

(2) $\Leftrightarrow$ (3): This follows from Lemma 2.2 of [18].

(2) ⇒ (4): Suppose that $K$ is a compact set in $(P, \sigma_2(P))$ and $K \subseteq U \in \sigma_2(P)$. For each $x \in K$, by (2), there is $F_x \in P^{(<\omega)}$ such that $x \in int_{\sigma_2(P)}\uparrow F_x = \uparrow F_x \subseteq U$. Hence $K \subseteq \bigcup_{x\in K} int_{\sigma_2(P)}\uparrow F_x = \bigcup_{x\in K} \uparrow F_x \subseteq U$. By the compactness of $K$, there exists a finite set $\{x_1, x_2, \ldots, x_n\} \subseteq K$ such that $K \subseteq \bigcup_{i=1}^{n} int_{\sigma_2(P)}\uparrow F_{x_i}$. Let $F = \bigcup_{i=1}^{n} F_{x_i}$. Then $F \in P^{(<\omega)}$ and $K \subseteq int_{\sigma_2(P)}\uparrow F = \uparrow F \subseteq U$.

(4) ⇒ (5): Suppose $F_1$, $F_2 \in \{F \subseteq P : F$ is finite and $K \subseteq int_{\sigma_2(P)}\uparrow F = \uparrow F\}$. Then $K \subseteq int_{\sigma_2(P)}\uparrow F_1 \cap int_{\sigma_2(P)}\uparrow F_2 \in \sigma_2(P)$. By (4), there is $F_3 \in P^{(<\omega)}$ such that $K \subseteq int_{\sigma_2(P)}\uparrow F_3 = \uparrow F_3 \subseteq int_{\sigma_2(P)}\uparrow F_1 \cap int_{\sigma_2(P)}\uparrow F_2$; and hence $F_3 \in \{F \subseteq P : F$ is finite and $K \subseteq int_{\sigma_2(P)}\uparrow F = \uparrow F\}$ and $\uparrow F_3 \subseteq \uparrow F_1 \cap \uparrow F_2$. Therefore, $\{F \subseteq P : F$ is finite and $K \subseteq int_{\sigma_2(P)}\uparrow F = \uparrow F\}$ is directed. Clearly, $\uparrow K \subseteq \bigcap\{\uparrow F\colon F$ is finite and $K \subseteq int_{\sigma_2(P)}\uparrow F = \uparrow F\}$. If $z \notin \uparrow K$, then $K \subseteq P\backslash\downarrow z \in \sigma_2(P)$. By (4), there is $G \in P^{(<\omega)}$ with $K \subseteq int_{\sigma_2(P)}\uparrow G = \uparrow G \subseteq P\backslash\downarrow z$. It follows that $G \in \{F\colon F$ is finite and $K \subseteq int_{\sigma_2(P)}\uparrow F = \uparrow F\}$ and $z \notin \uparrow G$. Therefore, $\uparrow K = \bigcap\{\uparrow F\colon F$ is finite and $K \subseteq int_{\sigma_2(P)}\uparrow F = \uparrow F\}$.

(5) ⇒ (1): Since the set $\{x\}$ is compact for each $x \in P$. □

**Corollary 3.5** *For a poset $P$, the following conditions are equivalent:*

(1) *$P$ is an $s_2$-quasialgebraic poset;*

(2) *$P$ is an $s_2$-quasicontinuous poset, and the $\sigma_2$-topology has a base consisting of compact-open sets;*

(3) *$P$ is an $s_2$-quasicontinuous poset, and $(\sigma_2(P), \subseteq)$ is an algebraic lattice.*

Observe that an $s_2$-quasialgebraic poset is generally not actually an $s_2$-algebraic poset ($s_2$-algebraic posets are a special kind of $s_2$-quasialgebraic posets in which the collection of finite sets $F \ll F$ is replaced by a collection of singleton subsets). However, $s_2$-quasialgebraic posets have many properties similar to those of $s_2$-algebraic posets and quasialgebraic domains.

**Example 3.6** Let $P = \{a\} \cup \{a_n : n \in \mathbb{N}\}$. The partial order on $P$ is defined by setting $a_n < a_{n+1}$ for all $n \in \mathbb{N}$, and $a_1 < a$. Then $P$ is an $s_2$-quasialgebraic poset which is not $s_2$-algebraic.

**Theorem 3.7** *Let $P$ be a poset. The following conditions are equivalent:*

(1) *$P$ is an $s_2$-algebraic poset;*

(2) *$P$ is a meet $s_2$-continuous and $s_2$-quasialgebraic poset;*

(3) *$P$ is a meet $s_2$-continuous poset, $\downarrow x \cap K(P)$ is directed for all $x \in P$, and whenever $x \not\leq y$ in $P$, then there are $U \in \sigma_2(P)$ and $V \in \omega(P)$ such that $x \in U$, $y \in V$, $U \cap V = \emptyset$ and $U \cup V = P$.*

**Proof.** (1) ⇒ (2): Obviously.

(2) ⇒ (3): Firstly, we assume that $x \not\leq y$ in $P$. Then $x \in P\backslash\downarrow y \in \sigma_2(P)$. By Proposition 2.9, there exists $F \in P^{(<\omega)}$ such that $x \in int_{\sigma_2(P)}\uparrow F = \uparrow F \subseteq P\backslash\downarrow y$. Let $U = int_{\sigma_2(P)}\uparrow F$ and $V = P\backslash\uparrow F$. Then $x \in U \in \sigma_2(P)$, $y \in V \in \omega(P)$,

$U \cap V = \emptyset$ and $U \cup V = P$.

Then we show that $\downarrow x \cap K(P)$ is directed for all $x \in P$. On the one hand, let $u, v \in \downarrow x \cap K(P)$. Then $x \in \uparrow u \cap \uparrow v \in \sigma_2(P)$. By Proposition 2.9, there exists $F \in P^{(<\omega)}$ such that $x \in int_{\sigma_2(P)}\uparrow F = \uparrow F \subseteq \uparrow u \cap \uparrow v$. Since $F$ is finite, $\uparrow F = \uparrow \mathrm{Min(F)}$ where $\mathrm{Min(F)}$ is the set of all minimal elements in $F$. By Corollary 2.4, $x \in \uparrow \mathrm{Min(F)} = \uparrow \mathrm{F} = \mathrm{int}_{\sigma_2(\mathrm{P})}\uparrow \mathrm{F} = \mathrm{int}_{\sigma_2(\mathrm{P})}\uparrow \mathrm{Min(F)} \subseteq \bigcup\{\Uparrow \mathrm{t} : \mathrm{t} \in \mathrm{Min(F)}\}$. So there exists $t \in \mathrm{Min(F)}$ with $t \ll x$. Since $\uparrow \mathrm{Min(F)} \subseteq \bigcup\{\Uparrow \mathrm{t} : \mathrm{t} \in \mathrm{Min(F)}\}$, there exists $s \in \mathrm{Min(F)}$ with $s \ll t$, hence $s \leq t$. So $s = t$ since $s, t \in \mathrm{Min(F)}$. Thus $t \in \downarrow x \cap K(P)$ and $t \in \uparrow u \cap \uparrow v$. On the other hand, since $P$ is an $s_2$-quasialgebraic poset, there exists $G \in P^{(<\omega)}$ such that $x \in int_{\sigma_2(P)}\uparrow G = \uparrow G \subseteq P$. Similarly, we can show that there is a $y \in \mathrm{Min(G)}$ with $y \in \downarrow x \cap K(P)$. Thus $\downarrow x \cap K(P) \neq \emptyset$. Therefore, $\downarrow x \cap K(P)$ is directed.

(3) $\Rightarrow$ (1): For all $x \in P$, we show $x = \bigvee(\downarrow x \cap K(P))$. Clearly, $x$ is an upper bound of $\downarrow x \cap K(P)$. Let $y$ be any upper bound of $\downarrow x \cap K(P)$ and assume $x \not\leq y$. By (3), there are $U \in \sigma_2(P)$ and $V \in \omega(P)$ such that $x \in U$, $y \in V$, $U \cap V = \emptyset$ and $U \cup V = P$. We may assume that $V$ is a basic $\omega$-open set, i.e., there exists $F \in P^{(<\omega)}$ such that $V = P \backslash \uparrow F$. From $U \cap V = \emptyset$ and $U \cup V = P$, $U = \uparrow F$ follows. So $x \in int_{\sigma_2(P)}\uparrow F = \uparrow F$. It is similar to the proof of (2) $\Rightarrow$ (3), there exists $t \in \downarrow x \cap K(P)$ such that $t \in \uparrow F$. Since $y$ is an upper bound of $\downarrow x \cap K(P)$, $y \in \uparrow F$, a contradiction to $y \in V = P \backslash \uparrow F$. □

**Corollary 3.8** *Let $P$ be a dcpo. The following conditions are equivalent:*

(1) *$P$ is an algebraic domain;*

(2) *$P$ is a meet continuous and quasialgebraic domain;*

(3) *$P$ is a meet continuous domain, $\downarrow x \cap K(P)$ is directed for all $x \in P$, and whenever $x \not\leq y$ in $P$, then there are $U \in \sigma(P)$ and $V \in \omega(P)$ such that $x \in U$, $y \in V$, $U \cap V = \emptyset$ and $U \cup V = P$.*

Let $(X, \tau, \leq)$ be a partially ordered topological space. $(X, \tau, \leq)$ is called *totally order-disconnected* if whenever $x \not\leq y$ there is a clopen upper set $U$ such that $x \in U$ and $y \notin U$. A compact order-disconnected space is called a *Priestley space* ([16]).

**Corollary 3.9** *Let $P$ be a complete lattice. The following conditions are equivalent:*

(1) *$P$ is an algebraic lattice;*

(2) *$P$ is a meet continuous and quasialgebraic lattice;*

(3) *$P$ is a meet continuous lattice, and $(P, \lambda(P), \leq)$ is a Priestley space.*

Since the lattice of open sets of any topological space is a Heyting algebra, we have the following.

**Corollary 3.10** *For a topological space $X$, let $\mathcal{O}(X)$ denote the lattice of open sets of $X$. Then the following conditions are equivalent:*

(1) *$(\mathcal{O}(X), \subseteq)$ is an algebraic lattice;*

(2) *$(\mathcal{O}(X), \subseteq)$ is a quasialgebraic lattice;*

(3) *The Lawson topology of* $(\mathcal{O}(X), \subseteq)$ *is a Priestley space.*

# 4 Dual categories on posets

A function $f : P \to Q$ between posets is $\sigma_2$-*continuous* iff it is continuous with respect to the $\sigma_2$-topologies, that is, $f^{-1}(U) \in \sigma_2(P)$ for all $U \in \sigma_2(Q)$. In [4], Erné proved that $f : P \to Q$ is $\sigma_2$-continuous iff $f(D^\delta) \subseteq f(D)^\delta$ for all directed subsets $D$ of $P$.

**Definition 4.1** Let $P$, $Q$ be two posets. $Q$ is called a $\sigma_2$-*continuous retract* of $P$ if there exist $\sigma_2$-continuous functions $f : P \to Q$ and $g : Q \to P$ such that $f \circ j = g_Q$.

**Theorem 4.2** *Let $P$, $Q$ be two posets. If $P$ is $s_2$-quasicontinuous and $Q$ is a $\sigma_2$-continuous retract of $P$, then $Q$ is $s_2$-quasicontinuous.*

**Proof.** Since $Q$ is a $\sigma_2$-continuous retract of $P$, there exist $\sigma_2$-continuous functions $f : P \to Q$ and $g : Q \to P$ such that $f \circ j = g_Q$. For every $x \in Q$ and finite set $F \ll g(x)$, we claim that $f(F) \ll x$. Indeed, let $D$ be a directed set of $Q$ with $x \in D^\delta$; then $g(x) \in g(D^\delta) \subseteq g(D)^\delta$ and we obtain an element $d \in D$ such that $g(d) \in \uparrow F$ because $F \ll g(x)$. So we get $d = f(g(d)) \in f(\uparrow F) \subseteq \uparrow f(F)$, and the claim is true. Since $P$ is $s_2$-quasicontinuous and $f$ is order preserving, the family $\{f(F) : F \text{ is finite and } F \ll g(x)\}$ is directed.

Given $x$, $y \in Q$ with $x \not\leq y$, then $x = f(g(x)) \in Q \backslash \downarrow y$ and we get $g(x) \in f^{-1}(Q \backslash \downarrow y) \in \sigma_2(P)$. Since $P$ is $s_2$-quasicontinuous, there exists finite $F \ll g(x)$ such that $F \subseteq f^{-1}(Q \backslash \downarrow y)$. This means that $f(F) \subseteq Q \backslash \downarrow y$. By the claim above we know that $f(F) \ll x$. So for every $x \in Q$, we have $\uparrow x = \bigcap \{\uparrow f(F) : F \text{ is finite and } F \ll g(x)\}$. Thus $Q$ is $s_2$-quasicontinuous. □

**Definition 4.3** ([8]) Let $P$ and $Q$ be two posets. We shall say that a pair $(g, d)$ of functions $f : P \to Q$ and $g : Q \to P$ is a *Galois connection*, if both $f$ and $g$ are order preserving, and for all $(x, y) \in P \times Q$, $f(x) \geq y$ iff $x \geq g(y)$. In an adjunction $(f, g)$, the function $f$ is called the *upper adjoint* and $g$ the *lower adjoint*.

**Proposition 4.4** *Let $d : T \to S$ be the upper adjoint of a monotone map $g : S \to T$ between posets. Then $d(A^\delta) \subseteq d(A)^\delta$ for all $A \subseteq T$. In particular, $d$ is $\sigma_2$-continuous.*

**Proof.** Let $A \subseteq T$. If there exists $x \in A^\delta$ with $d(x) \notin d(A)^\delta$, then there is $y \in S$ with $d(A) \subseteq \downarrow y$ such that $d(x) \not\leq y$. Thus $x \not\leq g(y)$, hence $A \not\subseteq \downarrow g(y)$. Since $d(A) \subseteq \downarrow y$, $d(a) \leq y$ for all $a \in A$. So $a \leq g(y)$ for all $a \in A$, which implies $A \subseteq \downarrow g(y)$, a contradiction to $A \not\subseteq \downarrow g(y)$. □

**Proposition 4.5** *Let $P$ and $Q$ be posets and $f : P \to Q$ the upper adjoint of $g : Q \to P$. Consider the following conditions:*

(1) *$f$ is $\sigma_2$-continuous;*

(2) *for all $U \in \sigma_2(Q)$, $\uparrow g(U) \in \sigma_2(P)$;*

(3) *$g$ preserves $\ll$, that is, if $F \ll G$ in $Q$, then $g(F) \ll g(G)$ in $P$.*

*Then* (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3). *If $Q$ is an $s_2$-quasicontinuous poset, then* (3) $\Rightarrow$ (1).

**Proof.** (1) $\Rightarrow$ (2): Let $U \in \sigma_2(Q)$. In order to show that $\uparrow g(U) \in \sigma_2(P)$, we take a directed set $D \subseteq P$ with $D^\delta \cap \uparrow g(U) \neq \emptyset$ and we show that $D \cap \uparrow g(U) \neq \emptyset$. Since $D^\delta \cap \uparrow g(U) \neq \emptyset$, there exists $x \in D^\delta$ and $y \in U$ such that $g(y) \leq x$. So $y \leq f(x)$. By (1), $f(x) \in f(D^\delta) \subseteq f(D)^\delta$. Since $f$ is order preserving, $f(D)$ is directed and $y \in f(D)^\delta$. Thus $U \cap f(D) \neq \emptyset$, that is, there is a $d \in D$ with $f(d) \in U$; hence $gf(d) \in g(U)$. But $gf(d) \leq d$, so $d \in \uparrow g(U)$. Thus $D \cap \uparrow g(U) \neq \emptyset$.

(2) $\Rightarrow$ (1): Let $D \subseteq P$ be a directed set with $x \in D^\delta$. If $f(x) \notin f(D)^\delta$, then there exists $y \in Q$ such that $f(D) \subseteq \downarrow y$ and $f(x) \not\leq y$. So $f(x) \in Q \backslash \downarrow y \in \sigma_2(Q)$. Let $U = Q \backslash \downarrow y$. Since $gf(x) \in g(U)$ and $gf(x) \leq x$, By (2), $x \in \uparrow g(U) \in \sigma_2(P)$. Thus $D \cap \uparrow g(U) \neq \emptyset$, that is, there exists $d \in D$ and $z \in U$ such that $g(z) \leq d$; hence $z \leq f(d)$. Thus $f(d) \in U = Q \backslash \downarrow y$, a contradiction to $f(D) \subseteq \downarrow y$. So $f(D^\delta) \subseteq f(D)^\delta$.

(1) $\Rightarrow$ (3): Suppose $F \ll G$ in $Q$ and let $D \subseteq P$ be directed with $\uparrow g(G) \cap D^\delta \neq \emptyset$. So there is a $y \in G$ with $g(y) \in D^\delta$. By hypothesis $y \leq fg(y) \in f(D^\delta) \subseteq f(D)^\delta$. Since $f$ is order preserving, $f(D)$ is directed and $y \in f(D)^\delta$, that is, $\uparrow G \cap f(D)^\delta \neq \emptyset$. Thus $\uparrow F \cap f(D) \neq \emptyset$, that is, there exist $d \in D$ and $x \in F$ such that $x \leq f(d)$; hence $g(x) \leq d$. Thus $g(F) \ll g(G)$.

(3) $\Rightarrow$ (1): Suppose $Q$ is an $s_2$-quasicontinuous poset. For all $U \in \sigma_2(Q)$, we show that $f^{-1}(U) \in \sigma_2(P)$. Let $D \subseteq P$ be directed with $D^\delta \cap f^{-1}(U) \neq \emptyset$. Then there exists $x \in D^\delta$ with $f(x) \in U$. Since $Q$ is an $s_2$-quasicontinuous poset, there exists a finite $F \ll f(x)$ with $F \subseteq U$. By (3), $g(F) \ll gf(x) \leq x$; hence $g(F) \ll x$. So $\uparrow g(F) \cap D \neq \emptyset$, that is, there exist $d \in D$ and $z \in F$ such that $g(z) \leq d$. Since $z \leq fg(z)$, $fg(z) \in U$ and $g(z) \in f^{-1}(U)$. So $D \cap f^{-1}(U) \neq \emptyset$ since $f$ is order preserving. Thus $f^{-1}(U) \in \sigma_2(P)$. $\square$

**Definition 4.6** We introduce the following categories:

(1) $\mathbf{PO_G}$ has as objects posets and as morphisms $\sigma_2$-continuous maps $g$ that have a lower adjoint.

(2) $\mathbf{PO_D}$ has as objects posets and as morphisms maps $d$ that have an upper adjoint and the property that for each $\sigma_2$-open $U$ in the domain of $d$ the set $\uparrow d(U)$ is $\sigma_2$-open in the range. (Note that such maps are $\sigma_2$-continuous.)

(3) $\mathbf{QC_G}$ has as objects $s_2$-quasicontinuous posets and as morphisms $\sigma_2$-continuous maps that have a lower adjoint.

(4) $\mathbf{QC_D}$ has as objects $s_2$-quasicontinuous posets and as morphisms maps that have an upper adjoint and preserve the way-below relation $\ll$.

By [8, Lemma IV-1.2] and Proposition 4.5, we have the following

**Theorem 4.7** *The following pairs of categories are dual under the adjoint functors $D$ and $G$:*

(1) $\mathbf{PO_G}$ *and* $\mathbf{PO_D}$.

(2) $\mathbf{QC_G}$ *and* $\mathbf{QC_D}$.

# References


[1] Abramsky, S., and A. Jung, *Domain theory*, in: S. Abramsky, D.M. Gabbay, T.S.E. Maibaum (Eds.), Handbook of Logic in Computer Science, vol. **3**, Oxford University Press, Oxford, 1994, pp. 1–168.

[2] Erné, M., *Scott convergence and Scott topology on partially ordered sets II.* In: B. Banaschewski and R.-E. Hoffman, eds., Continuous Lattices, Bremen 1979, Lecture Notes in Math. **871**, Springer-Verlag, Berlin-Heidelberg-New York, 1981, pp. 61–96.

[3] Erné, M., *Infinite distributive laws versus local connectedness and compactness properties*, Topology and its Applications **156** (2009), 2054–2069.

[4] Erné, M., *Categories of locally hypercompact spaces and quasicontonuous posets*, Applied categorical structure **26** (2018), 823–854.

[5] Erné, M., *The Dedekind-MacNeille completion as a reflector*, Order **8**(2) (1991), 159–173.

[6] Gierz, G., J. D. Lawson and A. R. Stralka, *Quasicontinuous posets*, Houston J. Math. **9** (1983), 191–208.

[7] Gierz, G., and J. D. Lawson, *Generalized continuous and hypercontinuous lattices*, Rocky Mt. J. Math. **11** (1981), 271–296.

[8] Gierz, G. et al., *Continuous Lattices and Domains*, Cambridge University Press, Cambridge, 2003.

[9] Huang, M.Q., Q. G. Li and J.B. Li, *Generalized Continuous Posets and a New Cartesian Closed Category*, Applied Categorical Structures **17** (2009), 29–42.

[10] Heckmann, R., *An upper power domain construction in terms of strongly compact sets*, Lecture Notes in Comp. Science **598** (1992), 272–293.

[11] Kou, H., Y.M. Liu and M.K. Luo, *On meet-continuous dcpos*, in: Guoqiang Zhang, J.D. Lawson, Yingming Liu, Maokang Luo (Eds.), Domain Theory, Logic and Computation, Kluwer Academic Publishers, 2003, pp. 137–149.

[12] Mislove, M., *Topology, domain theory and theoretical computer science*, Topology and its Applications **89** (1998), 3–59.

[13] Mislove, M., *Algebraic posets, algebraic CPO's and Models of Concurrency*, in Topology and Category Theory in Computer Science, G.M. Reed, A.W. Roscoe, and R.F. Wachter (eds), Oxford University Press (1991), 75–111.

[14] Mao, X.X., L.S. Xu, *Quasicontinuity of Posets via Scott Topology and Sobrification*, Order **23** (2006), 359–369.

[15] Venugopalan, P., *Quasicontinuous posets*, Semigroup Forum **41** (1990), 193-200.

[16] Venugopalan, P., *Priestley spaces*, Proceedings of the American Mathematics Society **109**(3) (1990), 605–610.

[17] Yokoyama, T., *A Poset with Spectral Scott Topology is a Quasialgebraic Domain*, Order **26** (2009), 331–335.

[18] Xu, X.Q., and J.B.Yang, *Topological representations of distributive hypercontinuous lattices*, Chin. Ann. Math. **30B**(2) (2009), 199–206.

[19] Zhang, W.F., and X.Q. Xu, *$s_2$-Quasicontinuous posets*, Theoretical Computer Science **574** (2015), 78–85.

[20] Zhang, W.F., and X.Q. Xu, *$s_2$-Algebraic posets*, Fuzzy Systems and Mathematics **29** (2015), 71–74 (in Chinese).

[21] Zhao, B., and X.R. Liang, *Some properties on quasialgebraic Domain*, Journal of Shaanxi Normal University(Natural Science Edition) **33** (2005), 1–5 (in Chinese).