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Technical Report: Computation on the Extended Complex Plane and Conformal Mapping of Multiply-connected Domains

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Abstract

We introduce a system for computation on the extended complex plane based on the Type-Two Effectivity approach to computable analysis. Included are computations on meromorphic functions, open sets, and closed sets. Applications to Möbius transformations, boundaries of multiply connected domains, and conformal mapping of multiply connected domains are considered.

Keywords: Computable analysis, conformal mapping, Type-Two Effectivity

1 Introduction

The Riemann mapping theorem states that all simply connected domains with more than one boundary point are conformally equivalent. This theorem made possible the study of conformal mappings of such simply connected domains onto one and the same *canonical domain*, namely, the unit disk.

There has been considerable interest in obtaining constructive proofs of the Riemann mapping theorem. Paul Koebe's [9] approach to the simplification of the first complete proof of the theorem given by Carathéodory is the cornerstone of the subsequent constructive proofs given by Cheng [2], Bishop and Bridges [1], and of the computable proof given by Hertling [7].

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The study of conformal mappings of multiply connected domains onto canonical domains immediately encounters two major obstacles. The first is the fact that continuous mappings preserve connectivity and thus the possible canonical domain must be multiply connected. The second and the most important one is that two multiply connected domains of the same order of connectivity need not be conformally equivalent. For example, there is no conformal mapping between the annuli $r < |z| < 1$ and $R < |z| < 1$ whenever $r \neq R$ (for a simple proof of this fact, see [12, p. 333]).

Every doubly connected domain is conformally equivalent to an annulus $\mu < |z| < 1$, where μ is a unique real number, $0 < \mu < 1$, called the *modulus* of the doubly connected domain. It can be shown that the conformal type of an n-connected domain ($n > 2$) is determined by $3n - 6$ real numbers also called the *moduli* of the domain (for a derivation see [12, p. 354, Exercise 13]).

For connectivity $n > 2$ there are various canonical domain types: the parallel slit domain, the circular slit domain, the radial slit domain, the circle with concentric circular slits, the circular ring with concentric circular slits, and the circular domain. Two circular domains of the same order of connectivity are conformally equivalent if and only if there is a Möbius transfomation mapping one of them onto the other [14, p. 426, Theorem IX.36].

In a paper preceding his proof of the Riemann mapping theorem Koebe [8] gave a very short outline of his method for construction of a conformal mapping of a multiply connected domain onto a circular domain. A detailed convergence proof of Koebe's method was given by Gaier [3] almost fifty years later, and a more elaborate version of Gaier's proof can be found in Henrici [6, Theorem 17.7a]. Our aim is to present a computable proof of Koebe's theorem based on Gaier's approach.

This leads us to consider computation on the extended complex plane (*i.e.* the Riemann sphere) which is the mathematically most convenient setting for the study of conformal mapping. We develop a model of computation on the extended complex plane based on the Type-Two Effectivity (TTE) approach to computable analysis [15]. That is, we produce admissible representations of the extended complex plane, its open and closed sets, and the meromorphic functions. We then show that the fundamental operations such as the field operations, computations of zeros and poles, Möbius transformations, *etc.* are computable. In the last two sections, we take up the problem of determining the precise amount of information necessary to construct a conformal map between two non-degenerate multiply connected domains.

Unless otherwise mentioned, all definitions and notations for computable analysis are as in [15].

Proofs of many claims are omitted, but will appear in a future work.

2 Background from complex analysis

Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. We identify $\mathbb{C}$ with $\mathbb{R}^2$.

Definition 2.1 (Domain terminology)

(i) A *domain* is an open connected subset of $\hat{\mathbb{C}}$.

(ii) A domain is *n-connected* if its complement has exactly n connected components.

(iii) A domain is *degenerate* if a component of its complement has fewer than two points.

(iv) A domain is *circular* if it is the result of removing one or more disjoint closed disks from $\hat{\mathbb{C}}$.

Each circular domain whose complement has at least two connected components is associated with two constants, μ and δ. These are defined as follows. Let $D_r(z)$ denote the open disk with center z and radius r.

Definition 2.2 Let C be an n-connected circular domain with $n \geq 2$, and let $\overline{D_{r_1}(z_1)}, \ldots, \overline{D_{r_n}(z_n)}$ be the components of $\hat{\mathbb{C}} - C$. Define μ_C to be the reciprocal of

$$\min\{r > 1 \ : \ \exists 1 \leq j, k \leq n \ (j \neq k \ \wedge \ \overline{D_{rr_j}(z_j)} \cap \overline{D_{rr_k}(z_k)} \neq \emptyset)\}.$$

Definition 2.3 Let C be an n-connected circular domain with $n \geq 2$, and let $\overline{D_{r_1}(z_1)}, \ldots, \overline{D_{r_n}(z_n)}$ be the components of $\hat{\mathbb{C}} - C$. Denote the boundary of $D_{r_j}(z_j)$ by Γ_j. Let Γ_j^k denote the circle obtained by reflecting Γ_k into Γ_j. We define δ_C to be

$$\min\{d(\Gamma_j, \Gamma_j^k) \ : \ 1 \leq j, k \leq n \ \wedge \ j \neq k\}.$$

Let $\mathcal{D}_n(\hat{\mathbb{C}})$ be the set of all n-connected domains.

The Riemann Mapping Theorem states (among other things) that every proper, open, simply connected subset of the plane is conformally equivalent to the unit disk $\mathbb{D}$. Such a set is a non-degenerate 1-connected domain. There is an extension of this result to n-connected subsets of the extended plane.

Theorem 2.4 *If D is a non-degenerate n-connected domain, and if $z_0 \in D$, then there is a circular domain C and a conformal mapping f from D onto C such that $f(z_0) = \infty$.*

Unfortunately, when $n > 1$, it is not the case that all non-degenerate n-connected domains are conformally equivalent to the *same* circular region. However, we can get a weaker uniqueness result if we impose a restriction on the form of the conformal map.

Theorem 2.5 *If D is a non-degenerate n-connected domain that contains ∞ but not 0, then there is a unique circular region C_D such that there is a conformal map f_D of D onto C_D such that $f_D(z) = z + O(z^{-1})$. Furthermore, this map is unique.*

Theorem 2.5 implies Theorem 2.4. Theorem 2.5 is proven in [6]. In Section 10, we will investigate the computable content of Theorem 2.5.

We will use winding numbers to compute interiors and exteriors of continuously differentiable simple closed curves. These are defined as follows.

Definition 2.6 When γ is a rectifiable simple closed curve in $\mathbb{C}$ and $z \in \hat{\mathbb{C}} - \mathrm{ran}(\gamma)$, let

$$\eta(\gamma, z) = \frac{1}{2\pi i} \int_{\gamma} \frac{d\zeta}{\zeta - z}.$$

$\eta(\gamma, z)$ is called the *winding number of* γ *around* z.

If γ is a simple closed curve, then let $Int(\gamma)$ denote the interior of γ, and let $Ext(\gamma)$ denote the exterior of γ. A proof of the following can be found in [4].

Proposition 2.7 *Let γ be a rectifiable, simple closed curve.*

(i) *For all $z \notin \mathrm{ran}(\gamma)$, $\eta(\gamma, z)$ is an integer.*

(ii) $Int(\gamma) = \{z \in \hat{\mathbb{C}} \ : \ \eta(\gamma, z) \neq 0\}$.

(iii) $Ext(\gamma) = \{z \in \hat{\mathbb{C}} \ : \ \eta(\gamma, z) = 0\}$.

For computability purposes, we will approximate continuously differentiable curves with rational polygonal curves. These are defined as follows.

Definition 2.8 (i) A curve $\gamma : [a, b] \to \mathbb{C}$ is *polygonal* if it is continuous and there exist $t_0, \ldots, t_k$ such that $a = t_0 < \ldots < t_k = b$ and γ is linear on each of $[t_0, t_1], \ldots, [t_{k-1}, t_k]$.

(ii) If, in addition, the coördinates of $\gamma(t_j)$ are rational for each j, then we will call γ a *rational polygonal curve.*

The following lemma says that it is possible to simultaneously approximate a continuously differentiable simple closed curve and its derivative by a rational polygonal curve with arbitrary precision.

Lemma 2.9 *Suppose $\gamma : [a, b] \to \mathbb{C}$ is a simple closed curve with continuous derivative. Then, for every $\epsilon > 0$, there is a rational, polygonal, simple closed curve $\gamma_0 : [a, b] \to \mathbb{C}$ such that*

$$\max\{|\gamma(t) - \gamma_0(t)| \ : \ t \in [a, b]\} < \epsilon$$
$$\sup\{|\gamma'(t) - \gamma_0'(t)| \ : \ t \in \mathrm{dom}(\gamma_0')\} < \epsilon$$

Informally speaking, the following lemma says it is possible to surround the connected components of the complement of an unbounded, non-degenerate n-connected domain with rational polygonal curves that are well-separated.

Lemma 2.10 *Suppose D is a non-degenerate n-connected domain and $\infty \in D$. Let $C_1, \ldots, C_n$ be the components of the complement of D. Then, there exist rational polygonal curves $\gamma_1, \ldots, \gamma_n$ such that the following hold.*

(i) $\mathrm{ran}(\gamma_j) \subseteq D$.

(ii) *If $j \neq k$, then* $\mathrm{ran}(\gamma_j) \cap \mathrm{ran}(\gamma_k) = \emptyset$.

(iii) $C_j \subseteq Int(\gamma_j) - \bigcup_{k \neq j} \overline{Int(\gamma_k)}$.

Using Lemma 2.10, we can prove the following, which will be used later to recognize the connected components of the complement of an unbounded, non-degenerate n-connected domain. Let I^2 be the notation in Definition 4.1.2 of [15].

Lemma 2.11 *Suppose D is a non-degenerate n-connected domain and $\infty \in D$. Then, there exist words $w_1, \ldots, w_n$ and rational, polygonal, simple closed curves $\gamma_1, \ldots, \gamma_n$ such that the following.*

(i) $\mathrm{ran}(\gamma_j) \subseteq D$.

(ii) *If $j \neq k$, then* $\mathrm{ran}(\gamma_j) \cap \mathrm{ran}(\gamma_k) = \emptyset$.

(iii) $\overline{I^2(w_j)} \subseteq Int(\gamma_j) - \cup_{k\neq j}\overline{Int(\gamma_k)}$.

(iv) $I^2(w_j) \cap \partial D \neq \emptyset$.

Furthermore, if $w_1, \ldots, w_n, \gamma_1, \ldots, \gamma_n$ are such that (i) - (iv) hold, then it is possible to label the connected components of $\hat{\mathbb{C}} - D$, $C_1, \ldots, C_n$, so that $C_j \subseteq Int(\gamma_j) - \cup_{k\neq j}\overline{Int(\gamma_k)}$.

3 Computable analysis

3.1 Spaces to be considered and their default naming systems

Suppose X is a topological space. We let $\mathcal{O}(X)$ be the set of open subsets of X. We let $\mathcal{C}(X)$ be the set of closed subsets of X.

The following table indicates some of our default naming systems. Definitions can be found in [15].

Space	Default naming system
$\mathbb{N}$	$\nu_{\mathbb{N}}$
$\mathbb{Q}$	$\nu_{\mathbb{Q}}$
Σ^*	Id_{Σ^*}
$\mathbb{R}$	ρ
$\mathbb{C}$	ρ^2
$\mathcal{O}(\mathbb{C})$	$\theta^2_<$

We introduce an admissible naming system for $\hat{\mathbb{C}}$ by first building a computable topology on $\hat{\mathbb{C}}$. [4]

Definition 3.1 (Computable topology on the extended plane)

(i) Let

$$\sigma_{\hat{\mathbb{C}}} = \{I^2(w) \mid w \in dom(I^2)\} \cup \{\hat{\mathbb{C}} - \overline{I^2(w)} \mid w \in dom(I^2) \wedge 0 \in I^2(w)\}.$$

[4] This is not the only way to set up computation on $\hat{\mathbb{C}}$. There is also the paper [13] in which computation on these spaces is defined by embedding $\hat{\mathbb{C}}$ into $\mathbb{R}^3$.

Thus, $\sigma_{\hat{\mathbb{C}}}$ is a countable basis for the standard topology on $\hat{\mathbb{C}}$.

(ii) For all $w \in \mathrm{dom}(I^2)$, let $\nu_{\hat{\mathbb{C}}}(\langle 0, w\rangle) = I^2(w)$. If $0 \in I^2(w)$, let $\nu_{\hat{\mathbb{C}}}(\langle 1, w\rangle) = \hat{\mathbb{C}} - \overline{I^2(w)}$.

(iii) Let $\mathcal{S}_{\hat{\mathbb{C}}} = (\hat{\mathbb{C}}, \sigma_{\hat{\mathbb{C}}}, \nu_{\hat{\mathbb{C}}})$.

It follows that $\mathcal{S}_{\hat{\mathbb{C}}}$ is a computable topological space. Let $\delta_{\hat{\mathbb{C}}} = \delta_{\mathcal{S}_{\hat{\mathbb{C}}}}$. $\delta_{\hat{\mathbb{C}}}$ is our default naming system for $\hat{\mathbb{C}}$. Informally speaking, a $\delta_{\hat{\mathbb{C}}}$-name of a $z \in \hat{\mathbb{C}}$ is an exhaustive list of all basic neighborhoods to which z belongs. If $z = \infty$, then this list will only contain neighborhoods of the form $\mathbb{C} - \overline{I^2(w)}$. But, if $z \neq \infty$, then this list will contains neighborhoods of ∞ as well as finite neighborhoods. These observations lead to the following.

Proposition 3.2 *There is no computable $F :\subseteq \Sigma^\omega \to \{0, 1\}$ such that for all $p \in \mathrm{dom}(\delta_{\hat{\mathbb{C}}})$,*

$$F(p) = \begin{cases} 1 \ \text{if } \delta_{\hat{\mathbb{C}}}(p) = \infty \\ 0 \quad \text{otherwise} \end{cases}$$

We now turn to $\mathcal{O}(\hat{\mathbb{C}})$.

Definition 3.3 (Computable topology on open subsets of the extended plane)

(i) For all $w \in \mathrm{dom}(\nu_{\hat{\mathbb{C}}})$, let

$$\nu_{\mathcal{O}(\hat{\mathbb{C}})}(w) = \{U \in \mathcal{O}(\hat{\mathbb{C}}) \mid \overline{\nu_{\hat{\mathbb{C}}}(w)} \subseteq U\}.$$

Then, let $\sigma_{\mathcal{O}(\hat{\mathbb{C}})} = \mathrm{ran}(\nu_{\mathcal{O}(\hat{\mathbb{C}})})$.

(ii) Let $\mathcal{S}_{\mathcal{O}(\hat{\mathbb{C}})} = (\mathcal{O}(\hat{\mathbb{C}}), \sigma_{\mathcal{O}(\hat{\mathbb{C}})}, \nu_{\mathcal{O}(\hat{\mathbb{C}})})$.

Clearly, $\mathcal{S}_{\mathcal{O}(\hat{\mathbb{C}})}$ is a computable topological space. Let $\delta_{\mathcal{O}(\hat{\mathbb{C}})} = \delta_{\mathcal{S}_{\mathcal{O}(\hat{\mathbb{C}})}}$. $\delta_{\mathcal{O}(\hat{\mathbb{C}})}$ is our default representation of $\mathcal{O}(\hat{\mathbb{C}})$. Informally speaking, a $\delta_{\mathcal{O}(\hat{\mathbb{C}})}$-name of an open set is an exhaustive list of all basic neighborhoods whose closures are contained in that set. If this set is bounded, then this list will only contain bounded neighborhoods. Otherwise, it will contain both bounded and unbounded neighborhoods. These observations lead to the following.

Proposition 3.4 *There is no computable $F :\subseteq \Sigma^\omega \to \{0, 1\}$ such that for all $p \in \mathrm{dom}(\delta_{\mathcal{O}(\hat{\mathbb{C}})})$,*

$$F(p) = \begin{cases} 1 \ \text{if } \infty \in \delta_{\mathcal{O}(\hat{\mathbb{C}})}(p) \\ 0 \quad \text{otherwise.} \end{cases}$$

We now turn to closed sets.

Definition 3.5 (Computable topology on closed subsets of the extended plane)

(i) For all $w \in \mathrm{dom}(\nu_{\hat{\mathbb{C}}})$, let

$$\nu_{\mathcal{C}(\hat{\mathbb{C}})}(w) = \{X \in \mathcal{C}(\hat{\mathbb{C}}) \mid \nu_{\hat{\mathbb{C}}}(w) \cap X \neq \emptyset\}$$

Then, let $\sigma_{\mathcal{C}(\hat{\mathbb{C}})} = \mathrm{ran}(\nu_{\mathcal{C}(\hat{\mathbb{C}})})$.

(ii) Let $\mathcal{S}_{\mathcal{C}(\hat{\mathbb{C}})} = (\mathcal{C}(\hat{\mathbb{C}}), \sigma_{\mathcal{C}(\hat{\mathbb{C}})}, \nu_{\mathcal{C}(\hat{\mathbb{C}})})$.

Hence, $\mathcal{S}_{\mathcal{C}(\hat{\mathbb{C}})}$ is a computable topological space. Let $\delta_{\mathcal{C}(\hat{\mathbb{C}})} = \delta_{\mathcal{S}_{\mathcal{C}(\hat{\mathbb{C}})}}$. $\delta_{\mathcal{C}(\hat{\mathbb{C}})}$ is our default representation of $\mathcal{C}(\hat{\mathbb{C}})$. Informally speaking, a $\delta_{\mathcal{C}(\hat{\mathbb{C}})}$-name of a closed set lists all basic neighborhoods that intersect the set.

We now consider meromorphic functions. Let $\mathfrak{M}(\hat{\mathbb{C}})$ be the set of all meromorphic $f :\subseteq \hat{\mathbb{C}} \to \hat{\mathbb{C}}$.

Definition 3.6 For all $f \in \mathfrak{M}(\hat{\mathbb{C}})$ and all $p \in \Sigma^\omega$, let $\delta_{\mathfrak{M}(\hat{\mathbb{C}})}(p) = f$ if there exist q, r such that $p = \langle r, q\rangle$, $\eta_q^{\omega\omega}$ $(\delta_{\hat{\mathbb{C}}}, \delta_{\hat{\mathbb{C}}})$-realizes f, and $\mathrm{dom}(f) = \delta_{\mathcal{O}(\hat{\mathbb{C}})}(r)$.

$\delta_{\mathfrak{M}(\hat{\mathbb{C}})}$ is our default naming system for $\mathfrak{M}(\hat{\mathbb{C}})$. Informally speaking, a $\delta_{\mathfrak{M}(\hat{\mathbb{C}})}$-name of a meromorphic function contains a name of the domain of the function as well as an oracle Turing machine that computes the function. A thorough treatment of naming systems for function spaces is [5].

We have now defined our default naming systems for the spaces we will consider. We now describe the default naming systems for the derived spaces.

(i) **(Function spaces)** When δ_X, δ_Y are the default naming systems for X, Y respectively, then $[\delta_X \to \delta_Y]$ is the default naming system for the set of all (δ_X, δ_Y)-continuous functions with domain X. When, in addition, $A \subseteq X$, $[\delta_X \to \delta_Y]_A$ is the default naming system for the set of (δ_X, δ_Y)-continuous functions with domain A.

(ii) **(Products)** When $\delta_{X_1}, \ldots, \delta_{X_n}$ are the default naming systems for $X_1, \ldots, X_n$ respectively, then $[\delta_{X_1}, \ldots, \delta_{X_n}]$ is the default naming system for $X_1 \times \ldots \times X_n$. In addition, $[\delta_X]^n$ is the default naming system for X^n when δ_X is the default naming system for X and $n \leq \omega$.

When only the default naming systems are being used, we will suppress their mention. For example, we just say that a $\delta_{\hat{\mathbb{C}}}$-computable point in $\hat{\mathbb{C}}$ is computable. This convention will eliminate much notation from our discussions. To eliminate even more, let:

$$\begin{aligned}
z_p &= \delta_{\hat{\mathbb{C}}}(p)\\
D_p &= \delta_{\mathcal{O}(\hat{\mathbb{C}})}(p)\\
E_p &= \delta_{\mathcal{C}(\hat{\mathbb{C}})}(p)\\
f_p &= \delta_{\mathfrak{M}(\hat{\mathbb{C}})}(p)
\end{aligned}$$

3.2 Some lemmas for proving computability

Since $\eta(\gamma, z)$ is integer-valued, we immediately obtain the following by using the fact that integration is a computable operator (see, for example, Theorem 6.4.1.2 of [15]).

Proposition 3.7 *$\gamma \mapsto Int(\gamma)$ and $\gamma \mapsto Ext(\gamma)$ are computable.*

We now give some tools for proving computability without using type two machines.

Definition 3.8 A predicate $R \subseteq \Sigma^\omega \times \Sigma^*$ is *computable* if there is a type two machine M such that when $\iota(w)p$ is written on the input tape, M halts with output 1 if $R(p, w)$ holds and halts with output 0 if $R(p, w)$ does not hold. A predicate $S \subseteq (\Sigma^\omega)^n \times (\Sigma^*)^m$ is *computable* if there is a computable predicate $R \subseteq \Sigma^\omega \times \Sigma^*$ such that

$$S(p_1, \ldots, p_n, w_1, \ldots, w_m) \Leftrightarrow R(\langle p_1, \ldots, p_n\rangle, \langle w_1, \ldots, w_m\rangle).$$

Lemma 3.9 *Let $\mathcal{S}_1 = (M_1, \tau_1, \nu_1)$ be a computable topological space. Suppose* $\mathrm{dom}(\nu_1)$ *is computable. Let δ be a representation of M_0. Let $f :\subseteq M_0 \to M_1$. Then, the following are equivalent.*

(i) *f is $(\delta, \delta_{\mathcal{S}_1})$-computable.*

(ii) *There is a computable predicate $R \subseteq \Sigma^\omega \times \Sigma^* \times \Sigma^*$ such that for all $p \in \delta^{-1}(\mathrm{dom}(f))$ and all $w \in \mathrm{dom}(\nu_1)$*

$$f(\delta(p)) \in \nu_1(w) \Leftrightarrow \exists y R(p, w, y).$$

4 Operations on points

We claim that the operations of addition, multiplication, and division are computable on $\hat{\mathbb{C}}$. We first review how these operations are defined on $\hat{\mathbb{C}}$ and what their domains are.

Definition 4.1 The operations of addition and multiplication are extended to $\hat{\mathbb{C}}$ by the equations

$$z + \infty = \infty + z = \infty,\ z \in \hat{\mathbb{C}}$$
$$z \times \infty = \infty \times z = \infty,\ z \in \hat{\mathbb{C}} - \{0\}$$

Division is extended to $\hat{\mathbb{C}}$ by the equations

$$z/\infty = 0,\ z \in \mathbb{C}$$
$$\infty/z = \infty,\ z \in \mathbb{C}$$
$$z/0 = \infty,\ z \in \hat{\mathbb{C}} - \{0\}$$

Hence

$$\mathrm{dom}(+) = \hat{\mathbb{C}} \times \hat{\mathbb{C}}$$
$$\mathrm{dom}(\times) = \hat{\mathbb{C}} \times \hat{\mathbb{C}} - \{(0, \infty), (\infty, 0)\}$$
$$\mathrm{dom}(/) = \hat{\mathbb{C}} \times \hat{\mathbb{C}} - \{(\infty, \infty), (0, 0)\}$$

The following can be proven fairly easily using Lemma 3.9, the definition of $\delta_{\hat{\mathbb{C}}}$, and Definition 4.1.

Proposition 4.2 *Addition is computable on* $\hat{\mathbb{C}} \times \hat{\mathbb{C}} - \{(\infty, \infty)\}$. *Addition is not continuous at* (∞, ∞). *Hence, addition is not computable on* $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$.

Proposition 4.3 *Multiplication of points in* $\hat{\mathbb{C}}$ *is computable.*

Proposition 4.4 *Division of points in* $\hat{\mathbb{C}}$ *is computable.*

5 Uniform computability of Möbius transformations

Given distinct $z_1, z_2, z_3 \in \hat{\mathbb{C}}$, there is a unique bilinear transformation that maps z_1, z_2, z_3 to $1, 0, \infty$ respectively. Denote this transformation by $T_{(z_1,z_2,z_3)}$. Details may be found in [4]. We claim the following.

Theorem 5.1 *The map* $(z_1, z_2, z_3) \mapsto T_{(z_1,z_2,z_3)}$, *where* (z_1, z_2, z_3) *ranges over all pairwise distinct triples in* $\hat{\mathbb{C}} \times \hat{\mathbb{C}} \times \hat{\mathbb{C}}$, *is computable.*

6 Operations on meromorphic functions

We begin by extending some of the results in [10] and [11] on zeros to meromorphic functions.

Lemma 6.1 *The following maps on* $\mathfrak{M}(\hat{\mathbb{C}})$ *are computable.*

(i) $f \mapsto (\hat{\mathbb{C}} - f^{-1}[\{0\}]) \cap \mathrm{dom}(f)$.

(ii) $f \mapsto (\hat{\mathbb{C}} - f^{-1}[\{\infty\}]) \cap \mathrm{dom}(f)$.

Theorem 6.2 *The following maps are computable on* $\mathfrak{M}(\hat{\mathbb{C}})$.

(i) $f \ni f \not\equiv 0 \mapsto f^{-1}[\{0\}]$.

(ii) $f \ni f \not\equiv \infty \mapsto f^{-1}[\{\infty\}]$.

In order to prove that $f \mapsto f^{-1}$ is computable, we must first prove some things about operations on open sets.

7 Operations on closed and open sets

Theorem 7.1 (i) *The map* $(f, U) \mapsto f[U]$ *is computable on pairs such that* $f \in \mathfrak{M}(\hat{\mathbb{C}})$ *is not constant and* U *is an open subset of* $\mathrm{dom}(f)$.

(ii) *The map* $(f, C) \mapsto f[C]$ *is computable on pairs such that* $f \in \mathfrak{M}(\hat{\mathbb{C}})$ *and* C *is a closed set that is contained in* $\mathrm{dom}(f)$.

We can now show that computing inverses of injective meromorphic functions is possible.

Theorem 7.2 $f \mapsto f^{-1}$ *is computable on the set of injective meromorphic functions.*

8 Decomposing boundaries

For all non-degenerate $D \in \mathcal{D}_n(\hat{\mathbb{C}})$, let $Comp_n(D)$ consist of all $(C_1, \ldots, C_n)$ such that $C_1, \ldots, C_n$ are the connected components of the complement of D. Then, for all such D, let:

$$Comp_n^{>}(D) = \{(\hat{\mathbb{C}} - C_1, \ldots, \hat{\mathbb{C}} - C_n) \mid (C_1, \ldots, C_n) \in Comp_n(D)\}$$
$$Comp_n^{\partial}(D) = \{(\partial C_1, \ldots, \partial C_n) \mid (C_1, \ldots, C_n) \in Comp_n(D)\}$$

The proof of the following is based on Lemmas 2.10 and 2.11.

Theorem 8.1 *$(D, \partial D) \mapsto Comp_n^{>}(D)$ and $(D, \partial D) \mapsto Comp_n^{\partial}(D, \partial D)$ are computable multifunctions.*

9 Constants associated with a circular domain

We claim that the constants associated with a circular domain can be computed from the domain and its boundary. These constants will be used in Section 10.1.

Proposition 9.1 *$(C, \partial C) \mapsto \mu_C$ is computable (where C ranges over the circular regions in $\mathcal{D}_n(\hat{\mathbb{C}})$ for fixed n).*

Proposition 9.2 *$(C, \partial C) \mapsto \delta_C$ is computable (where C ranges over the circular regions in $\mathcal{D}_n(\hat{\mathbb{C}})$ for fixed n).*

10 Conformal mapping of n-connected domains

10.1 Koebe's algorithm

The following can be proven using Hertling's result on the Riemann Mapping Theorem [7] and the results of the previous sections.

Theorem 10.1 *When D ranges over all non-degenerate domains in $\mathcal{D}_1(\hat{\mathbb{C}})$ that contain ∞ but not 0, $(D, \partial D) \mapsto (C_D, \partial C_D, f_D)$ is computable.*

We will need the following for the proof of Theorem 10.3.

Lemma 10.2 *From a $\delta_{\mathfrak{M}(\hat{\mathbb{C}})}$-name of a map of the form $f|_{\mathbb{C}}$ such that $f \in \mathfrak{M}(\hat{\mathbb{C}})$ and f has no poles except at ∞, it is possible to compute a name of f.*

Theorem 10.3 *When D ranges over non-degenerate domains in $\mathcal{D}_n(\hat{\mathbb{C}})$ that contain ∞ but not 0, $(D, \partial D, C_D, \partial C_D) \mapsto f_D$ is computable.*

Proof. Given $(D, \partial D, C_D, \partial C_D)$, we define sequences $\{D_k\}_{k=0}^{\infty}$, $\{C_{k,1}\}_{k=0}^{\infty}$, $\ldots$, $\{C_{k,n}\}_{k=0}^{\infty}$, and $\{f_k\}_{k=0}^{\infty}$ by simultaneous recursion as follows.

To begin, let $C_1, \ldots, C_n$ be the connected components of $\hat{\mathbb{C}} - D$. We then let $D_0 = D$. Let $C_{0,j} = C_j$ for $j = 1, \ldots, n$. Let $f_0 = Id$. By Theorem 8.1, we can compute names of $\partial C_{0,j}$ and $\hat{\mathbb{C}} - \mathbb{C}_{0,j}$ from names of $D, \partial D$.

Let $k \in \mathbb{N}$, and suppose $f_k, D_k, C_{k,1}, \ldots, C_{k,n}$ have been defined. Assume we have also computed names of f_k, D_k, and the complements and boundaries of

$C_{k,1}, \dots, C_{k,n}$. Let $k' \in \{1, \dots, n\}$ be equivalent to k modulo n. Let f_{k+1} be the conformal map of $\hat{\mathbb{C}} - C_{k,k'}$ onto a circular domain C such that $f_{k+1}(z) = z + O(z^{-1})$. Since $\partial(\hat{\mathbb{C}} - C_{k,k'}) = \partial C_{k,k'}$, by Theorem 10.1, we can compute a name of f_{k+1}.

Now, let $D_{k+1} = f_{k+1}[D_k]$. Let $C_{k+1,j} = f_{k+1}[C_{k,j}]$ when $j \neq k'$. Let $C_{k+1,k'} = \hat{\mathbb{C}} - C$. Note that when $j \neq k'$, $C_{k,j} \subseteq \mathrm{dom}(f_{k+1})$ and so we can compute a name of $\partial C_{k+1,j}$. By Theorem 10.1, we can compute a name of $\partial C_{k+1,k'}$. At the same time, we note that $\hat{\mathbb{C}} - C_{k+1,j} = C_{k+1,k'} \cup f_{k+1}[\hat{\mathbb{C}} - C_{k,j}]$ when $j \neq k'$. Hence, we can compute names of the complements of $C_{k+1,1}, \dots, C_{k+1,n}$.

It follows that the sequence $\{f_k\}_{k=0}^{\infty}$ can be computed from D and ∂D. (See, for example, Theorem 2.1.14 of [15].) We now let:

$$\begin{aligned} g_0 &= f_1 \\ g_{k+1} &= f_{k+2} \circ g_k \end{aligned}$$

The following is proven in [6].

Lemma 10.4 *The sequence $\{g_k\}_{k=0}^{\infty}$ converges pointwise to a conformal map g of D onto C_D such that $g(z) = z + O(z^{-1})$.*

So $f_D = g$.

Choose ρ so that $\rho > \max\{|z| \ : \ z \in \hat{\mathbb{C}} - C_D\}$. Since this maximum occurs on ∂C_D, ρ can be computed from the given information.

Abbreviate C_D with C. Let $\mu = \mu_C$ and $\delta = \delta_C$. Let

$$\gamma = \frac{2\rho^2}{\pi\delta}\left[\frac{2[\pi\mu^{-1}]^2}{\ln \mu^{-1}} + 1\right].$$

The following is proven in [6].

Lemma 10.5 *For all $z \in D - \{\infty\}$ and all $j \in \mathbb{N}$, $|g_j(z) - g(z)| \leq \gamma\mu^{4\lfloor j/n \rfloor}$.*

It now follows from Theorem 6.2.2.2 of [15] that we can compute a $[\rho^2 \to \rho^2]$-name of the restriction of g to $D \cap \mathbb{C}$. Then, by Lemma 10.2 we can compute a name of g. This completes the proof.

□

From the proof of Theorem 10.3, we can extract a proof of the following non-uniform result.

Theorem 10.6 *Let $D \in \mathcal{D}_n(\hat{\mathbb{C}})$ be non-degenerate. If $D, \partial D$ are computable, then C_D, f_D are computable.*

Proof. Choose rational R, r such that $R > \gamma$ and $\mu < r < 1$. □

10.2 Necessity of parameters

Suppose we have a (possibly non-computable) function $x \mapsto f(x)$. A *sufficient parameter* for this function is a function g such that $(x, g(x)) \mapsto f(x)$ is computable. If $x \mapsto g(x)$ is computable, then we call g *superfluous* (and it follows that f is computable). If g is sufficient and $(x, f(x)) \mapsto g(x)$ is computable, then we call the parameter g *exact*. The parameter g is called *necessary* if it is both exact and non-superfluous.

This terminology can be extended to situations where we add several parameters. For example, suppose we add two parameters g_1, g_2 so that $(x, g_1(x), g_2(x)) \mapsto f(x)$ is computable. We say that g_1 is *exact relative to* g_2 if $(x, f(x), g_2(x)) \mapsto g_1(x)$ is computable. We then say that g_1 is *superfluous relative to* g_2 if $(x, g_2(x)) \mapsto g_1(x)$ is computable. We say that g_1 is *necessary relative to* g_2 if it is exact and non-superfluous relative to g_2. We similarly define these terms for g_2 relative to g_1. We then extend this terminology to the addition of $n > 2$ parameters in the obvious way.

We now claim that with respect to the map $D \mapsto f_D$, the parameter ∂D is exact with respect to the additional parameters $(C_D, \partial C_D)$. The proof is similar to the proof of Theorem 4.7 of [7].

Theorem 10.7 *The map* $(D, f_D, C_D, \partial C_D) \mapsto \partial D$*, where* D *ranges over non-degenerate domains in* $\mathcal{D}_n(\hat{\mathbb{C}})$ *that contain* ∞ *but not* 0 *is computable.*

We can similarly show that ∂C_D is exact with respect to $(\partial D, C_D)$. In fact, we can show that $(\partial D, f_D) \mapsto \partial C_D$ is computable. We can also show that C_D is exact with respect to $(\partial D, \partial C_D)$. In fact, it just follows from Theorem 7.1 that $f_D \mapsto C_D$ is computable. We do not know if these parameters are necessary.

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