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Electronic Notes in Theoretical Computer Science

Electronic Notes in Theoretical Computer Science 346 (2019) 725–734

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On the Complexity of Gap-[2]-vertex-labellings of Subcubic Bipartite Graphs[1]

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Abstract

A gap-$[k]$-vertex-labelling of a simple graph $G = (V, E)$ is a pair (π, c_π) in which $\pi : V(G) \to \{1, 2, \ldots, k\}$ is an assignment of labels to the vertices of G and $c_\pi : V(G) \to \{0, 1, \ldots, k\}$ is a proper vertex-colouring of G such that, for every $v \in V(G)$ of degree at least two, $c_\pi(v)$ is induced by the largest difference, i.e. the largest gap, between the labels of its neighbours (cases where $d(v) = 1$ and $d(v) = 0$ are treated separately). Introduced in 2013 by A. Dehghan et al. [3], they show that deciding whether a bipartite graph admits a gap-[2]-vertex-labelling is NP-complete and question the computational complexity of deciding whether cubic bipartite graphs admit such a labelling. In this work, we advance the study of the computational complexity for this class, proving that this problem remains NP-complete even when restricted to subcubic bipartite graphs.

Keywords: Gap-labellings, proper labellings, graph labellings.

1 Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. The *elements* of G are its vertices and its edges. The degree and the neighbourhood of a vertex $v \in V(G)$ are respectively denoted by $d(v)$ and $N(v)$, and the maximum degree of G is denoted by $\Delta(G)$. A *proper colouring* of G is an assignment $c : V(G) \to \mathcal{C}$, where $\mathcal{C}$ denotes a set of colours, such that for every edge $uv \in E(G)$, $c(u) \neq c(v)$. A *proper vertex-labelling* is a pair (π, c_π) where $\pi : V(G) \to \{1, 2, \ldots, k\}$ is an assignment of labels to the vertices of G and c_π is a proper colouring of G, *induced*

[1] This work was supported by grants CNPq Universal 425340/2016-3, FAPESP Temático 2015/11937-9 and CNPq 308689/2017-8.

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https://doi.org/10.1016/j.entcs.2019.08.063
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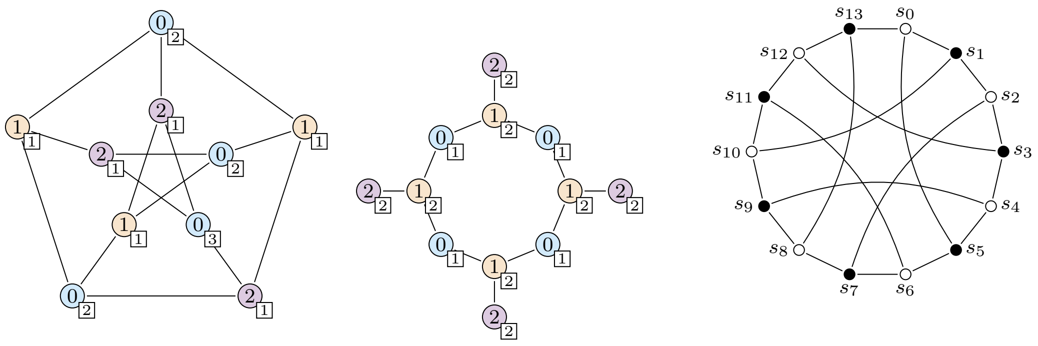

(a) (b) (c)

Fig. 1. In (a), a gap-[3]-vertex-labelling of the Petersen Graph; in (b), a gap-[2]-vertex-labelling of a subcubic bipartite graph inducing a 3-colouring; and, in (c), the Heawood Graph, which does not admit a gap-[2]-vertex-labelling. For every vertex, the number inside the box to its lower right corner corresponds to its assigned label

by π through some mathematical function over the set of labelled elements. In this work, we discuss proper vertex-labellings induced by gaps of labels.

Most authors trace the origins of proper graph labellings to 1967, when A. Rosa [7] introduced β-valuations. Since then, many different labellings have been studied. Each one varies the elements to which the labels are assigned and how the colouring is induced. In fact, there are several interesting surveys on the topic [4,6,10]. Concerning gap-labellings, they were first introduced as a vertex-distinguishing edge-labelling in 2012 by M. Tahraoui et al. [8], as a generalization of set and multiset labellings [1]. The vertex-labelling variant we consider in this text is defined as follows.

A *gap-*$[k]$*-vertex-labelling* of a graph G is a proper vertex-labelling of G over the set $\{1, 2, \ldots, k\}$ such that $c_\pi : V(G) \to \{0, 1, \ldots, k\}$ is a proper colouring of G in which, for every vertex $v \in V(G)$: (i) $c_\pi(v) = 1$ if $d(v) = 0$; (ii) $c_\pi(v) = \pi(u)$, $u \in N(v)$, if $d(v) = 1$; and (iii) $c_\pi(v) = \max_{u \in N(v)}\{\pi(u)\} - \min_{u \in N(v)}\{\pi(u)\}$ otherwise. Figures 1(a) and 1(b) exemplify gap-$[k]$-vertex-labellings using $k = 3$ and $k = 2$ labels, respectively. Observe in the latter that a gap-$[k]$-vertex-labelling may induce a proper $(k+1)$-colouring if there exists degree-one vertices in the graph.

Gap-$[k]$-vertex-labellings were introduced by A. Dehghan et al. [3] in 2013. In their article, the authors investigated both the computational complexity of decision problems associated with the gap-$[k]$-vertex-labelling of some families of graphs, as well as the least k for which some graphs admit such a labelling. They also show that deciding whether a given graph admits a gap-$[k]$-vertex-labelling is NP-complete for $k \geq 3$.

For the case $k = 2$, Dehghan et al. show that the problem remains NP-complete for 3-colourable and for bipartite graphs, but if the graph is bipartite and planar, then the problem can be solved in polynomial time. They also show that every r-regular bipartite graph, with $r \geq 4$, admits a gap-[2]-vertex-labelling, and question whether this is also true for cubic bipartite graphs. The dichotomy surrounding bipartite graphs was further investigated by A. Dehghan in 2016 [2]. The author showed that deciding whether a planar bipartite graph G admits a gap-$[k]$-vertex-labelling such that c_π is a proper 2-colouring of G is also an NP-complete problem. Recently, C. A. Weffort-Santos [9] investigated the gap-[k]-vertex-labelling of some classical families of graphs; of particular interest to this work, we cite cycles and unicyclic graphs. For these classes, the author characterized which graphs admit gap-[2]-vertex-labellings. These results imply that the gap-[2]-vertex-labelling decision problem can be solved in linear time for these particular instances.

Although the literature presents "positive" polynomial-time solvability results for r-regular bipartite graphs for $r = 2$ and $r \geq 4$, the computational complexity for gap-[2]-vertex-labellings of 3-regular (i.e. *cubic*) bipartite graphs, originally proposed by Dehghan et al. [3] in 2013, remains unknown. In this work, we advance the computational complexity analysis on bipartite graphs, showing that the problem remains NP-complete when restricted to the family of subcubic bipartite graphs. Our main results are presented in Theorem 1.1 and Corollary 1.2 and are proved in the next section. In order to simplify the notation, we denote by G2vl the problem of deciding whether a graph admits a gap-[2]-vertex-labelling.

Theorem 1.1 *Restricted to subcubic bipartite graphs,* G2vl *is* NP*-complete.* □

Corollary 1.2 G2vl *remains* NP*-complete when restricted to the family of subcubic bipartite graphs with minimum degree 2.* □

2 Proof of Theorem 1.1

Given a graph $G = (V, E)$, the Monochromatic Triangle (MT) problem questions if there exists a partition of E into two disjoint sets E_1, E_2 such that neither $G_1 = (V, E_1)$ nor $G_2 = (V, E_2)$ contains a triangle. This problem was proved to be NP-complete by Burr in 1976 (although this result was only published by Garey & Johnson in 1979 [5]). It can also be stated as an edge-colouring problem, where the question is whether G admits a {red, blue}-edge-colouring such that every triangle in G, i.e. a subgraph of G isomorphic to complete graph K_3, has at least one red edge and one blue edge. For an instance G of Monochromatic Triangle, we denote by $\mathcal{T} = (t_1, t_2, \ldots, t_p)$ an (arbitrary but fixed) ordering of all the p triangles in G. Also, we abuse notation to say that $t_i = \{e_x, e_y, e_z\}$ is the triangle of G induced by edges e_x, e_y and e_z of $E(G)$.

We prove Theorem 1.1 by reducing Monochromatic Triangle to G2vl in polynomial time. Equivalently, given an instance G of Monochromatic Triangle, we construct a subcubic bipartite graph G' such that G admits a 2-edge-colouring with no monochromatic triangles if and only if G' admits a gap-[2]-vertex-labelling. The reduction is accomplished with the aid of two gadgets: a *triangle gadget* and a *negation gadget*.

The first, denoted by $G^{\triangle}$, is an auxiliary simple bipartite graph with 19 vertices, 20 edges and is described as follows. For a triangle $t_i = \{e_x, e_y, e_z\}$, $G_i^{\triangle}$ has: a vertex u that represents $t_i \in G$ in G'; two adjacent vertices v_j and w_j for each edge e_j in t_i; and a path P_{12}, with $V(P_{12}) = \{q_0, \ldots, q_{11}\}$. We also link u to every v_j and add edges $w_x q_0$, $w_y q_4$ and $w_z q_8$. Fig. 2(a) illustrates the triangle gadget for a triangle

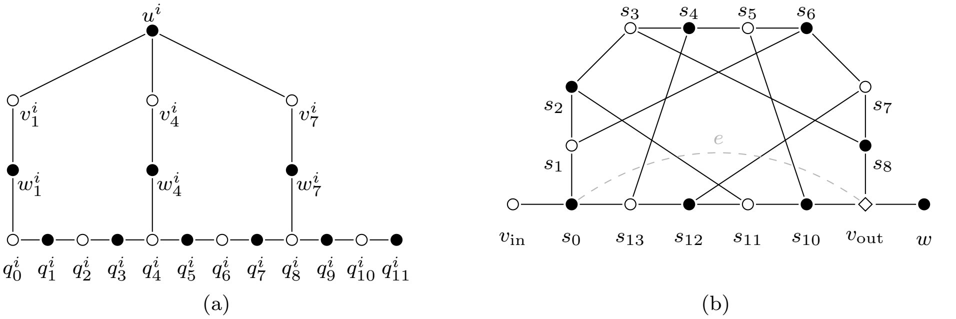

Fig. 2. In (a), triangle gadget $G_i^{\triangle}$ for triangle $t_i = \{e_1, e_4, e_7\}$; and in (b), the negation gadget $G^{\neg}$.

$t_3 = \{e_1, e_4, e_7\}$. Note in the image that we index all of the gadget's vertices with a superscript i, as well as the name $G^{\triangle}$, with a matching subscript. This is done so as to bind the vertices of triangle gadget $G_i^{\triangle}$ to the corresponding triangle t_i of G. We remark that $G^{\triangle}$ is bipartite since it does not contain any odd cycles. Furthermore, since no vertex has degree greater than 3, $G^{\triangle}$ is subcubic.

The negation gadget $G_{i,j}^{\neg}$ is used to connect vertices v_x^i and w_x^j, belonging to triangle gadgets $G_i^{\triangle}$ and $G_j^{\triangle}$ (not necessarily distinct). It is an auxiliary simple bipartite graph obtained by removing an edge e from the *Heawood Graph*, depicted in Fig. 1(c), and linking two new vertices, v_{in} and w, to the ends of e. Let H be the Heawood Graph, with $V(H) = \{s_0, \ldots, s_{13}\}$ and $e = s_0 s_9$. In the proof, we also refer to vertex s_9 as v_{out}. The construction of $G^{\neg}$ yields a subcubic bipartite graph with 16 vertices and 22 edges, which is illustrated in Fig. 2(b). Vertices v_x^i and w_x^j are identified with v_{in} and w from $G_{i,j}^{\neg}$, respectively. Note that, upon performing this operation, v_x^i and w_x^j in the corresponding triangle gadgets have degree 3.

We are now ready to show the reduction, which will be exemplified for graph G presented in Fig. 3(a). For every $t_i \in \mathcal{T}$, $t_i = \{e_x, e_y, e_z\}$, add a new triangle gadget $G_i^{\triangle}$ to G'. For every $1 \leq i \leq p-1$, add edge $q_{11}^i q_0^{i+1}$ to G'. Also, add a copy of cycle C_6, $V(C_6) = \{c_0, \ldots, c_5\}$, and edge $c_0 q_0^1$.

Next, let p_x denote the number of triangles to which an edge $e_x \in E(G)$ belongs to; note that $p_x \leq p$. Then, each e_x has p_x different triangle gadgets $G^{\triangle}$ in G', each of which has its own pair of vertices v_x and w_x. Now, let t_j^x denote the j-th triangle that contains e_x in $\mathcal{T}$. For instance, consider edge e_9 from graph G in Fig. 3(a), which belongs to $p_9 = 2$ triangles: t_3 and t_4. Then, t_1^9 is the first triangle in $\mathcal{T}$ containing e_9, namely t_3; analogously, t_2^9 is the second triangle containing e_9, i.e. t_4.

In order to complete our reduction, we refer to vertices v_x^i and w_x^i belonging to triangle t_l^x as $v_{x,l}$ and $w_{x,l}$, respectively. For example, $v_{5,3}$ is vertex v_5^i in $G_i^{\triangle}$ corresponding to the third triangle in $\mathcal{T}$ to contain edge e_5, namely t_3. Then, connect vertices $v_{x,j}$ and $w_{x,j+1}$ cyclically using negation gadget $G_{x,(j,j+1)}^{\neg}$, following the indices of triangles t_j^x in $\mathcal{T}$. This procedure — and the resulting graph G' — are illustrated in Fig. 3(b). We draw the reader's attention to edge e_5, which belongs to three triangles t_1, t_2 and t_3. Thus, vertices $v_{5,1}, w_{5,2}$ are connected with the negation gadget $G^{\neg}_{5,(1,2)}$, vertices $v_{5,2}$ and $w_{5,3}$, with $G^{\neg}_{5,(2,3)}$ and $v_{5,3}$ and $w_{5,1}$, with $G^{\neg}_{5,(3,1)}$.

Note that if an edge e_x belongs to a single triangle t_i in G, then v^i_x and w^i_x are in the same triangle gadget $G^{\triangle}_i$, and are connected with negation gadget $G^{\neg}_{x,(i,i)}$ as exemplified by edges $e_1, e_2, e_3, e_6, e_7, e_8$ and e_9 in the image. This completes the construction of graph G', a subcubic bipartite instance of G2vl. We remark that the reduction is accomplished in polynomial time on the size of the input graph G since $p = \mathcal{O}(n^3)$.

The structures used in the construction of G' have important properties that are essential to the reduction and, consequently, to the proof of our main result. We list these properties below and refer the reader to Weffort-Santos' Master's Thesis for their detailed proofs [9].

Proposition 2.1 *Let $G^{\triangle}_i$ be a triangle gadget. If $G^{\triangle}_i$ admits a gap-[2]-vertex-labelling (π, c_π) with $c_\pi(u^i) = 1$, then $\pi(q^i_0) = \pi(q^i_4) = \pi(q^i_8)$.*

Proof. Let $G^{\triangle}_i$ be a triangle gadget, and suppose $G^{\triangle}_i$ admits a gap-[2]-vertex-labelling such that $c_\pi(u^i) = 1$. Since the gadget is bipartite and every vertex v in $V(G^{\triangle}_i) \setminus \{q^i_{11}\}$ has $d(v) \geq 2$, the only possible colours induced in v by π are 0 and 1. Now, every vertex $q^i_l \in V(G^{\triangle}_i)$, l odd, is in the same part as vertex u in any bipartition of $G^{\triangle}_i$. Thus, for every vertex q^i_l, $l \leq 9$, $c_\pi(q^i_l) = c_\pi(u^i) = 1$, as illustrated in Fig. 4.

Let $a \in \{1, 2\}$ be the label assigned to q^i_0. Then, since $c_\pi(q^i_1) = 1$ and $N(q^i_1) = \{q^i_0, q^i_2\}$, we conclude that $\{\pi(q^i_0), \pi(q^i_2)\} = \{1, 2\}$ and, consequently, that $\pi(q^i_2) = b$, $b \in \{1, 2\} \setminus \{a\}$. Analogously, consider vertex q^i_3 and, observing that $c_\pi(q^i_3) = 1$ and $N(q^i_3) = \{q^i_2, q^i_4\}$, we conclude that $\pi(q^i_4) = a = \pi(q^i_0)$. By a similar reasoning, we conclude that the same holds for vertex q^i_8, and the result follows. □

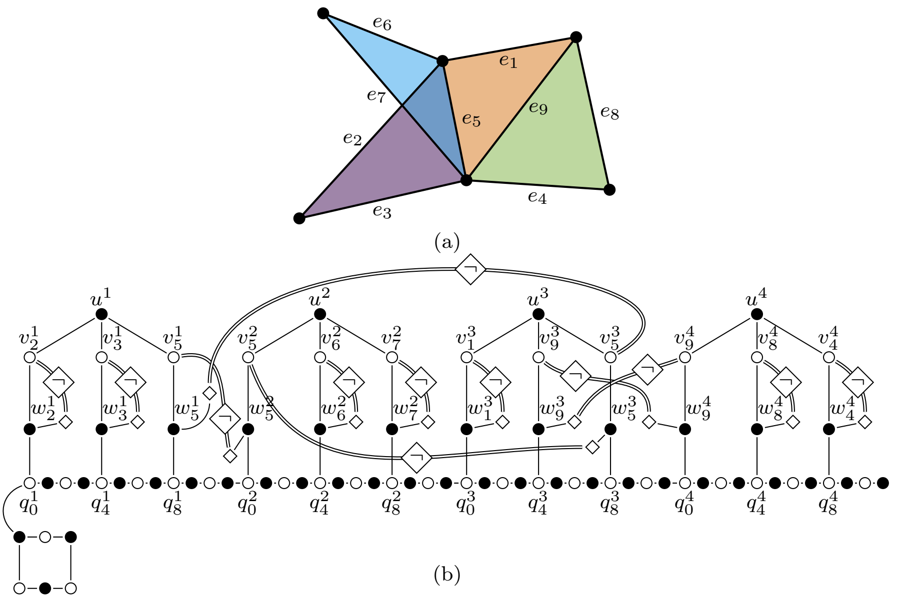

Fig. 3. (a) Instance G of Monochromatic Triangle; (b) Graph G' obtained by the reduction. Graph $G^{\neg}_{i,j}$ is represented by symbol $\neg$ in doubled lines; the diamond-shaped vertex is v_{out}.

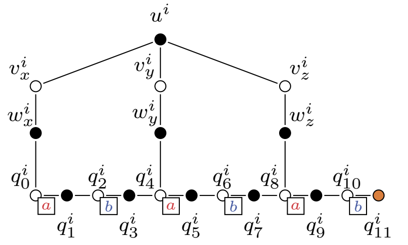

Fig. 4. A (partial) labelling of $G_i^{\triangle}$ when $c_\pi(u^i) = 1$. Vertices with induced colour 1 are filled in black, and vertices with colour 0, in white. Vertex q_{11}^i is coloured in orange, implying $c_\pi(q_{11}) \in \{1, 2\}$.

Proposition 2.2 *Let G' be a subcubic bipartite graph with $G_{i,j}^{\neg} \subseteq G'$ and $d(v_{in}) = d(w_x^j) = 3$. If G' admits a gap-[2]-vertex-labelling (π, c_π) such that $c_\pi(v_{in}) = 0$, then $\pi(v_{in}) \neq \pi(v_{out})$, for $v_{in} = v_x^i$, $v_{out} \in V(G_{i,j}^{\neg})$.*

Proof (Sketch) Let $G', G_{i,j}^{\neg}$ be as stated in the hypothesis. Suppose G' admits a gap-[2]-vertex-labelling such that $c_\pi(v_{\text{in}}) = 0$. Note that $c_\pi(v) \in \{0, 1\}$ for every $v \in V(G_{i,j}^{\neg})$ since $d_{G'}(v) = 3$. Also, since s_0 is adjacent to v_{in}, we conclude that $c_\pi(s_0) = 1$. Therefore, $c_\pi(s_l) = (l+1) \bmod 2$ for every s_l in the gadget. This implies that $\pi(s_i) = \pi(s_j)$ for every i, j even. The idea of the proof is to verify all possible label assignments to odd-indexed vertices s_l that induce colouring c_π.

Let $\{a, b\} = \{1, 2\}$. We begin by considering the label assignments to some neighbours of s_0, supposing $a = \pi(s_1) \neq \pi(s_{13})$ as illustrated in Fig. 5; note that this labelling induces $c_\pi(s_0) = 1$ without considering the label assigned to v_{in}. Now, suppose $\pi(s_3) = b$. Since $s_3, s_{13} \in N(s_4)$ and $\pi(s_3) = \pi(s_{13}) = b$, we conclude that $\pi(s_5) = a$. Then, since $s_1, s_5 \in N(s_6)$, $\pi(s_7) = b$. Analogously, we have $s_7, s_{13} \in N(s_{12})$ which implies $\pi(s_{11}) = a$. The only remaining odd-indexed vertex to be labelled is $s_9 = v_{\text{out}}$. However, if $\pi(v_{\text{out}}) = a$, then $\pi(v_{\text{out}}) = \pi(s_5) = \pi(s_{11})$ which induces $c_\pi(s_{10}) = 0$, contradicting the hypothesis. This allows us to conclude that our last assumption, i.e., $\pi(s_3) = b \neq \pi(s_1)$, is incorrect.

By attempting $\pi(s_3) = \pi(s_1)$ and following the same reasoning, we arrive at a similar contradiction, forcing us to backtrack to our first assumption and conclude that $\pi(s_1) = \pi(s_{13})$ in any such gap-[2]-vertex-labelling of G'. Finally, we suppose $\pi(v_{\text{in}}) = \pi(v_{\text{out}})$ and, with the knowledge that $\pi(s_1) = \pi(s_{13})$, we need only verify the two possible label assignments for vertex s_3 and how they affect the induced colouring. Since both labellings arrive at a contradiction, the result follows. □

Proposition 2.3 *Let e_x be an edge of G that belongs to $p_x \geq 2$ triangles and t_j^x the j-th triangle in $\mathcal{T}$ that contains e_x. If G' admits a gap-[2]-vertex-labelling (π, c_π), then, for every pair of triangle gadgets $G_{x,j}^{\triangle}$, $G_{x,j+1}^{\triangle}$ representing consecutive triangles t_j^x, t_{j+1}^x, $\pi(v_{x,j}) = \pi(v_{x,j+1})$.*

Proof (Sketch) Let G, G', (π, c_π) and e_x be as stated in the hypothesis. Now, consider the p_x triangle gadgets, $G_{x,j}^{\triangle}$, which have vertices $v_{x,j}$ corresponding to edges $e_x \in t_j$. Then, every vertex $v_{x,j}$ is connected to $w_{x,j+1}$ through the use of a negation gadget, and vertex $w_{x,j}$, to $v_{x,j-1}$. Also, recall that every vertex $w_{x,j}$ is adjacent to vertices v_{out}, $v_{x,j}$ and some vertex $q_l^{x,j}$, with $l \equiv 0 \pmod 4$; furthermore, every $q_l^{x,j}$ is assigned the same label $a \in \{1, 2\}$ by Proposition 2.1.

Suppose that there exist vertices $v_{x,j}$ and $v_{x,j+1}$ for which $\{\pi(v_{x,j}), \pi(v_{x,j+1})\} = \{a, b\}$. Adjust notation so that $\pi(v_{x,j}) = b$. By Proposition 2.2, $b = \pi(v_{x,j}) = \pi(v_{\text{in}}) \neq \pi(v_{\text{out}})$ for v_{out} connecting $v_{x,j}$ and $w_{x,j+1}$. Therefore, every vertex in $N(w_{x,j+1})$ has been assigned the same label a, which induces $c_\pi(w_{x,j+1}) = 0$. This is a contradiction since $w_{x,j+1}$ is adjacent to $v_{x,j+1}$, whose induced colour is 0 by hypothesis. We conclude that in a gap-[2]-vertex-labelling of G', every $v_{x,j}$ corresponding to an edge $e_x \in E(G)$ has received the same label, and the result follows. □

Proposition 2.4 *Let $C_6 \subseteq G'$ as constructed in the reduction. If G' admits a gap-[2]-vertex-labelling (π, c_π), then $\pi(c_3) = a$ and $\pi(c_1) = \pi(c_5) = b$, $\{a, b\} = \{1, 2\}$.*

Proof. Let G' as stated in the hypothesis and suppose G' admits a gap-[2]-vertex-labelling (π, c_π). We prove the result by contradiction. Suppose $c_\pi(c_3) = 1$. This implies that $\pi(c_4) \neq \pi(c_2)$. By the symmetry of the cycle, we assume, without loss of generality, that $\pi(c_2) = 1$. This implies that $\pi(c_0) = 2$ since $c_\pi(c_1) = 1$. Analogously, considering $N(c_5)$, we conclude that $\pi(c_4) = 1$, a contradiction. Therefore, $c_\pi(c_3) = 0$, which implies that vertices c_0, c_2 and c_4 all have induced colour 1. By letting $\pi(c_3) = a$, for $a \in \{1, 2\}$, we conclude that $\pi(c_1) = \pi(c_5) = b$,

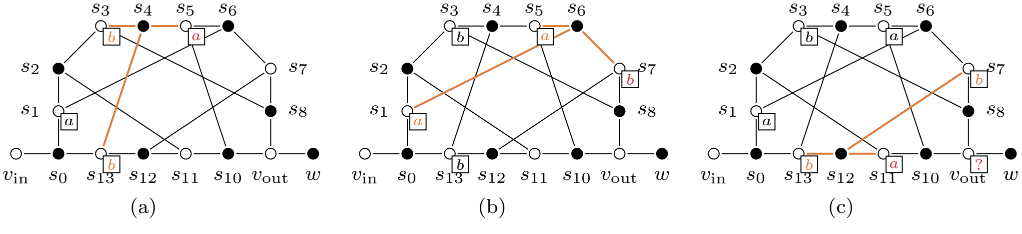

Fig. 5. Case $\pi(s_1) \neq \pi(s_{13})$. In (a), (b), and (c), vertex s_3 has been assigned label b, which determines the assignment of labels to vertices s_5, s_7 and s_{11}, respectively. In all figures, edges and labels highlighted in orange are those that force labels, which are highlighted in red. Also, we denote black vertices as those with induced colour 1 and white vertices, those with colour 0.

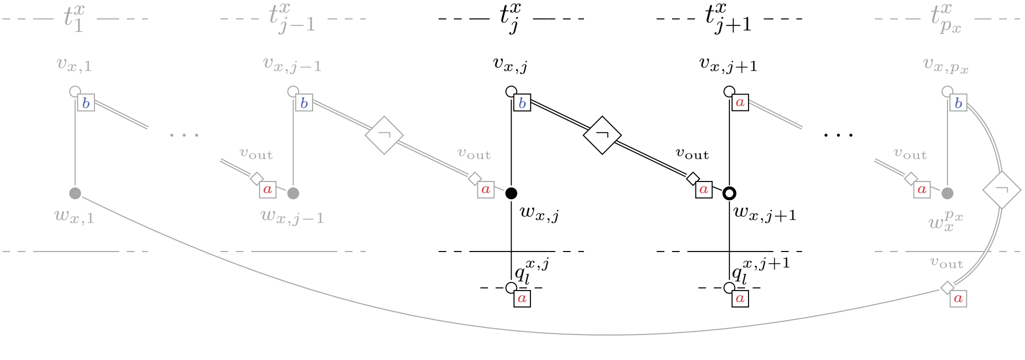

Fig. 6. Two vertices $v_{x,j}$ and $v_{x,j+1}$ labelled with b and a, respectively. The contradiction is reached when observing vertex $w_{x,j+1}$ which, in this labelling, would have induced colour 0.

$b \in \{1, 2\} \setminus \{a\}$. Both cases are illustrated in Fig. 7. □

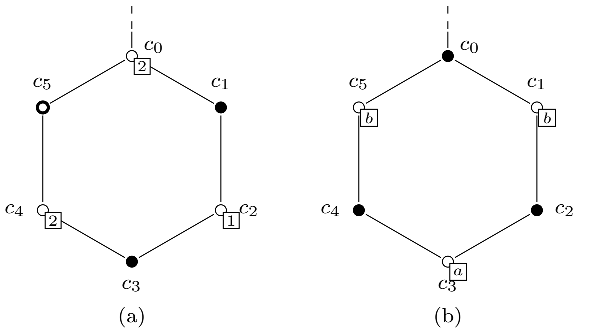

Fig. 7. In (a), c_3 with induced colour 1; and, in (b), colour 0. Black vertices have induced colour 1 and white vertices, colour 0.

In order to prove that Monochromatic Triangle $\preceq_p$ G2vl for subcubic bipartite graphs, we prove the following claims.

Claim 2.5 *If G admits an edge-colouring $c : E(G) \to \{red, blue\}$ such that no triangle is monochromatic, then G' admits a gap-*[2]*-vertex-labelling.*

Proof (Sketch) Let G, G' as stated and suppose G admits a {red, blue}-edge-colouring c without monochromatic triangles. Also, let $\{A, B\}$ be the bipartition of G' such that vertices $v_x^i \in A$. Figures 2 and 3 exemplify vertices in parts A and B as coloured in white and black, respectively.

Define a labelling $\pi : V(G) \to \{1, 2\}$ of G as follows. First, assign label 1 to every vertex in part B. Now, consider the vertices in part A. For every vertex $v_x^i \in G_i^{\triangle}$ (corresponding to edge $e_x \in E(G)$), assign $\pi(v_x) = 1$ if $c(e_x) =$ red, and $\pi(v_x) = 2$, otherwise. For negation gadgets $G_{i,j}^{\neg}$, assign labels: (a, b, b, b, a, a, a) to vertices $(s_1, s_3, \ldots, s_{13})$, with $b = \pi(v_x^i)$ and $a \in \{1, 2\} \setminus \{b\}$. For vertices $q_l^i \in P_{12} \subseteq G_i^{\triangle}$, let $\pi(q_l^i) = 1$ if $l \equiv 0 \pmod 4$ and $\pi(q_l^i) = 2$ if $l \equiv 2 \pmod 4$. Finally, assign $\pi(c_1) = \pi(c_5) = 2$ and $\pi(c_3) = 1$. Define colouring c_π as usual.

In order to prove that (π, c_π) is a gap-[2]-vertex-labelling of G', it suffices to show that c_π is a proper vertex-colouring of G'. First, since every vertex in part B received the same label and G' is connected, every vertex $v \in A$ has induced colour $c_\pi(v) = 0$. Thus, it remains to consider the induced colours of vertices in B. It is possible to inspect the induced colourings of cycle C_6, paths $P_{12} \subseteq G_i^{\triangle}$ and negation gadgets $G_{i,j}^{\neg}$ to observe that colours 0 and 1 alternate in the vertices in these structures (with the exception of vertex q_{11}^p in $G_p^{\triangle}$, whose colour is $c_\pi(q_{11}^p) \in \{1, 2\}$). Furthermore, by properties 2.2 and 2.3, every vertex $w_x^i \in G_i^{\triangle}$ has (at least) two neighbours, namely v_i^x and $v_{\text{out}} \in V(G^{\neg})$, with labels $\pi(v_i^x) \neq \pi(v_{\text{out}})$. Therefore, $c_\pi(w_x^i) = 1$.

To complete the proof, note that, since every triangle in G is not monochromatic, the edges of $t_i = \{e_x, e_y, e_z\}$ are coloured such that $\{c(e_x), c(e_y), c(e_z)\} = \{\text{red}, \text{blue}\}$. This implies that $\{\pi(v_x^i), \pi(v_y^i), \pi(v_z^i)\} = \{1, 2\}$ in every triangle gadget $G_i^{\triangle} \in G'$ which, in turn, induces $c_\pi(u^i) = 1$. We conclude that c_π is a proper colouring of G, and the result follows. $\square$

Claim 2.6 *If G' admits a gap-$[2]$-vertex-labelling, then G admits an edge-colouring $c : E(G) \to \{red, blue\}$ such that no triangle is monochromatic.*

Proof (Sketch) Let $G', (\pi, c_\pi)$ as stated in the hypothesis and let $\{A, B\}$ be a bipartition of $V(G')$ with $v_x^i \in A$. By Proposition 2.4, cycle $C_6 \subseteq G'$ has a unique gap-[2]-vertex-labelling which, in turn, defines the induced colour of every vertex in G' with degree at least two: vertices in A have induced colour 0 and vertices in $B \setminus \{q_{11}^p\}$, colour 1.

Now, since $u^i \in G_i^{\triangle}$ belongs to B, we know that $c_\pi(u^i) = 1$. Moreover, since $N(u^i) = \{v_x^i, v_y^i, v_z^i\}$, we have $\{\pi(v_x^i), \pi(v_y^i), \pi(v_z^i)\} = \{1, 2\}$. Then, define a {red, blue}-edge-colouring of G as follows. For every edge e_x, assign $c(e_x) = \text{red}$ if $\pi(v_x) = 1$ and $c(e_x) = \text{blue}$, otherwise. By Proposition 2.3, every v_x^i receives the same label and, therefore, no edge e_x is assigned two colours. Thus, we have $\{c(e_x), c(e_y), c(e_z)\} = \{\text{red}, \text{blue}\}$ in every triangle $t_i = \{e_x, e_y, e_z\}$, and every t_i has at least one edge coloured with red and one with blue. $\square$

By combining claims 2.5 and 2.6, we conclude that MT $\preceq_p$ G2VL (restricted to subcubic graphs), completing the proof of Theorem 1.1.

Now, consider the following modification to the previously described reduction. Remove vertices q_9^p, q_{10}^p and q_{11}^p from $G_p^{\triangle}$, i.e. the last three vertices in path P_{12} in the last triangle gadget of G' according to $\mathcal{T}$. Call the resulting graph G''. Note that this graph is subcubic having $\delta(G'') = 2$. However, this modification to the constructed graph does not alter any of the properties of G', used to prove Theorem 1.1. Therefore, Corollary 1.2, below, holds.

Corollary 1.2 G2VL *remains* NP*-complete when restricted to the family of subcubic bipartite graphs with minimum degree 2.* $\square$

3 Concluding remarks and open problems

The gap-$[k]$-vertex-labelling problem is fairly recent in the field of graph labellings. It is currently being studied under both the computational complexity and algorithmic points of view. In the former, the problem has been proven to be NP-complete for graphs in general when $k \geq 3$. The latter approach has shown that, for certain families of graphs (e.g. cycles, trees, crowns, wheels, unicyclic graphs and some families of snarks), it is possible to decide if they admit such a labelling in linear time. Moreover, there are known results on *optimal* gap-$[k]$-vertex-labellings for these families, i.e. labellings using the least number k of labels so as to induce a proper colouring of the graph. There have also been discoveries of classes of graphs that do not admit any gap-$[k]$-vertex-labelling for any $k \in \mathbb{N}$, such as complete graphs of order $n \geq 4$ and some powers of paths and cycles.

In particular, for $k = 2$, the family of bipartite graphs has proven to be quite challenging. It is known that deciding whether a bipartite graph admits a gap-[2]-vertex-labelling is NP-complete in general, but can be solved in polynomial time for some subfamilies, such as planar bipartite and r-regular graphs with $r \geq 4$. Our main result states that the problem remains NP-complete even when restricted to the families of subcubic bipartite graphs. Furthermore, if the minimum degree of these graphs is two, the problem remains NP-complete. As a consequence, the existence (or lack thereof) of degree-one vertices does not seem to facilitate deciding whether subcubic bipartite graphs admit gap-[2]-vertex-labellings. This result contrasts with known results for gap-vertex-labelable graphs since, in many cases, the presence of degree-one vertices helps the gap-[k]-vertex-labelling of certain graphs, e.g., every bipartite crown admits a gap-[2]-vertex-labelling whereas this is not the case for every even-length cycle.

In conclusion, we raise the following questions. Why do vertices of degree one have (apparently) no impact on the hardness of G2VL? Furthermore, although the problem remains NP-complete for subcubic bipartite graphs, can the structural properties of cubic graphs be used to decide if and when these graphs admit gap-[2]-vertex-labellings?

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