On the Convergence of Fourier Series of Computable Lebesgue Integrable Functions

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Abstract

This paper studies how well computable functions can be approximated by their Fourier series. To this end, we equip the space of L^p-computable functions (computable Lebesgue integrable functions) with a size notion, by introducing L^p-computable Baire categories. We show that L^p-computable Baire categories satisfy the following three basic properties. Singleton sets $\{f\}$ (where f is L^p-computable) are meager, suitable infinite unions of meager sets are meager, and the whole space of L^p-computable functions is not meager. We give an alternative characterization of meager sets via Banach Mazur games. We study the convergence of Fourier series for L^p-computable functions and show that whereas for every $p > 1$, the Fourier series of every L^p-computable function f converges to f in the L^p norm, the set of L^1-computable functions whose Fourier series does not diverge almost everywhere is meager.

Keywords: Computable Functions, Fourier Series.

1 Introduction

Fourier series are trigonometric polynomials that are useful for approximating arbitrary periodic functions. Areas of applications include electrical engineering, acoustics, optics, signal and image processing, and data compression. The goal of this paper is to study how well computable functions can be approximated by their Fourier series. Our main result shows that almost all computable Lebesgue integrable functions cannot be approximated by their Fourier series.

Our work is based in the setting of computable Lebesgue integrable functions (see [6,7]), a natural extension of the standard bit-computable (BC) model (see [6]), where a function f is said computable if a TM given a good approximation for x, can compute a good approximation for $f(x)$. One of the limitation of the BC model, is the fact that every bit-computable function is continuous. Therefore

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even simple step functions – a tool extensively used in functional analysis – are not computable in the BC model. A natural extension of the BC model known as L^p-computability, consists of a computable version of L^p (see [6,7]), the space of p-power Lebesgue integrable functions (where L^p is the class of functions f such that $|f|^p$ is Lebesgue integrable). In functional analysis, the spaces L^p form an important class of examples of Banach spaces. The notion of L^p-computable functions can be seen as an extension of the BC-model, which informally corresponds to the computable version of the class of continuous functions.

In order to be able to prove quantitative results (i.e. of the form: *almost every* function f satisfies ...), we equip the space of L^p computable functions with a size notion, by introducing L^p-computable Baire categories. Classically (see [5]), Baire categories are a topological size notion that allow to characterize the size of subsets of L^p, and that satisfy the following three basic properties. Singleton sets $\{f\}$ ($f \in L^p$) are meager (i.e. small), countable unions of meager sets are meager, and the whole space L^p is not meager. Baire categories can be used to show quantitative results instead of existential ones, i.e. show that most functions have some property P instead of proving the mere existence of one such function. Unfortunately, classical Baire categories cannot be used directly as a size notion on the set of computable functions, because this set is countable hence meager. What is needed is a computable version of Baire categories. We introduce such a notion on the space of L^p-computable functions, and show that our notion satisfies the three basic properties. Classically, meager sets can be characterized via Banach Mazur games (infinite two players games) (see [5]). We show that a similar characterization of meagerness holds for our L^p-computable Baire categories.

Our work extends previous notions of Baire categories introduced in the more restricted setting of bit-computability [4,1].

We then investigate the convergence of Fourier series for L^p-computable functions. We use our Baire category notion to understand how well L^1-computable functions can be approximated by their Fourier series. It is well know that for $p > 1$, the Fourier series of any function in L^p converges to f in the L^p-norm, and this is true for L^p-computable functions also. But for $p = 1$, things change dramatically: In the early 20's, Kolmogorov [3] constructed a function $f \in L^1$ whose Fourier series diverges almost everywhere. Unfortunately, Kolmogorov's result gives no information in the setting of L^1-computable functions.

We show that the analogue is true in the computable case, and that it is a *typical* property of L^1-computable functions, i.e. a majority of L^1-computable functions are very "complicated" functions, in the sense that most of them cannot be approximated by their Fourier series. More precisely we show that the set of L^1-computable functions whose Fourier series does not diverge a.e. is L^p-meager, i.e. negligible. As corollaries, our result implies the existence of an L^1-computable function whose Fourier series diverges almost everywhere, as well as Kolmogorov's result [3].

Due to lack of space, all proofs will appear in the full version of this paper.

2 Preliminaries

We take as known the elements of Lebesgue's theory of measure and integration, see for example [8,2]. If not mentioned, all functions we consider are over $[0, 2\pi]$. $\mathbb{R}^{[0,2\pi]}$ denotes $\{f|\, f : [0, 2\pi] \to \mathbb{R}\}$. L is the set of functions $f(x)$ integrable in the sense of Lebesgue over $[0, 2\pi]$. A measure zero set is called a *null set*: null sets are irrelevant in the theory of integration. We say $f = g$ *almost everywhere* (a.e.) if f and g differ only on a null set. For $p \geq 1$, we say f is in L^p if f is measurable and $|f|^p$ is in L. L^1 is L. The *p-norm* is

$$\|f\|_p = \left(\int_0^{2\pi} |f|^p \, dx \right)^{1/p}$$

L^p is a metric space, with the distance given by

$$d_p(f, g) = \|f - g\|_p.$$

We will not distinguish two functions that are equal a.e., and use the notation f for representing the class of functions equal a.e. to f.

3 The Class of L^p-computable Functions

A function f is said L^p-computable if a Turing machine can compute a step function approximating f in the p-norm. More precisely,

Definition 3.1 [see [6,7]] Let $p \in \mathbb{Z}^+$. A (class of) function $f : [0, 2\pi] \to \mathbb{R}$ is in L^p_{rec} (also said L^p-computable) if there exists a TM that on input any $m \in \mathbb{N}$, outputs a set of rationals $\{(a_i, b_i)|\, 1 \leq i \leq t_m\}$ such that the step function $\psi|_{[a_i, a_{i+1})} = b_i \quad (1 \leq i \leq t_m - 1)$ satisfies $\|\psi - f\|_p < 2^{-m}$.

The above step function ψ is called a rational step function with endpoints $\{(a_i, b_i)|\, 1 \leq i \leq t_m\}$. For families of functions the definition is similar: a family of functions $\{f_n\}_{n \geq 0}$ is in L^p_{rec} (or is uniformly L^p-computable) if there is a TM that on input any two integers m, n, outputs integers corresponding to a step function ψ such that $\|\psi - f_n\|_p < 2^{-m}$.

More generally we call a family of objects $\{O_n\}_{n \geq 0}$ (rational numbers, functions, etc.) to be uniformly computable if there is a single TM that given n computes O_n.

4 Baire Categories on L^p_{rec}

Baire categories yield a topological size notion for function spaces. Intuitively a class of functions is meager if it is full of "holes". Let us give a precise definition.

Definition 4.1 Given any rational step function ψ and any rational $\epsilon > 0$, we define the open ball centered in ψ and of radius ϵ by

$$B(\psi, \epsilon) = \{f : [0, 2\pi] \to \mathbb{R}|\, \|f - \psi\|_p < \epsilon\} .$$

We call such a set a rational ball. We denote by $\mathcal{B}$ the set of all rational balls.

A constructor (also called a strategy) is a function $\alpha : \mathcal{B} \to \mathcal{B}$ such that for any $B \in \mathcal{B}$, $\alpha(B) \subseteq B$. A computable constructor is a constructor $\alpha : \mathcal{B} \to \mathcal{B}$ such that there is a TM which on input any $B \in \mathcal{B}$ outputs $\alpha(B)$. We shall also consider indexed computable constructors, i.e. of the form $\alpha : \mathcal{B} \times \mathbb{N} \to \mathcal{B}$ such that there is a TM M such that $M(B, i) = \alpha(B, i)$.

Constructors are used to testify that a meager set is full of "holes".

Definition 4.2 A set $X \subseteq \mathcal{P}(\mathbb{R}^{[0,2\pi]})$ is L^p_{rec}-meager if $X = \cup_{i \geq 0} X_i$ and there exists a computable indexed constructor α such that for any $i \in \mathbb{N}$ and for any rational ball B

$$\alpha(B, i) \cap X_i = \emptyset.$$

The α in the above definition is said to avoid X or to testify the meagerness of X.

Meagerness is preserved for a special case of unions, called L^p_{rec}-unions. Here is a definition.

Definition 4.3 $X = \cup_{i \geq 0} X_i \subseteq \mathcal{P}(\mathbb{R}^{[0,2\pi]})$ is an L^p_{rec}-union of L^p_{rec}-meager sets if there exists a computable indexed constructor $\alpha : \mathcal{B} \times \mathbb{N} \times \mathbb{N} \to \mathcal{B}$ such that for any $i \in \mathbb{N}$, $\alpha(\cdot, \cdot, i)$ testifies X_i's meagerness.

The following result states that L^p-computable Baire categories indeed yield a size notion on L^p_{rec}, i.e. the whole space is not meager, any point in the space is meager, and meagerness is preserved by suitable unions.

Theorem 4.4 (i) *If $f \in L^p_{rec}$ then $\{f\}$ is L^p_{rec}-meager.*

(ii) *L^p_{rec} is not L^p_{rec}-meager.*

(iii) *If $X = \cup_{i \geq 0} X_i$ is a L^p_{rec}-union of L^p_{rec}-meager sets, then X is L^p_{rec}-meager.*

4.1   The Banach-Mazur Game Characterization

In the classical theory of Baire categories (see [5]), and also in the BC-model [1], there is an alternative characterization of meagerness by Banach-Mazur games. Informally speaking, a Banach-Mazur game is a game between two strategies α and β, where the game begins with some rational ball B. Then $\beta \circ \alpha$ is applied successively on B. Such a game yields a unique function called the result of the game between α and β. A strategy β wins the game against a class of functions X if it can force the result of the game starting with any α and B to be a function not in X. It is a classical result that the existence of a winning strategy against X is equivalent to the meagerness of X. In the following section, we show that this alternative characterization also holds for L^p-computable Baire categories.

Given two indexed constructors α, β where β is a shrinking strategy (i.e. for every rational ball $B \in \mathcal{B}$, the radius of $\beta(B)$ is less than half the radius of B), the Banach-Mazur game between α, β proceeds in infinitely many rounds, starting with a rational ball B, where α and β are applied successively; i.e. $R_0 = \beta(\alpha(B, 0), 0)$,

and round $i \in \mathbb{Z}^+$ corresponds to the rational ball $R_i = \beta(\alpha(R_{i-1}, i), i)$. The result of the game between α, β with initial ball B, denoted $R(\alpha, \beta, B)$ is the unique L^p function f, such that $f \in R_i$ for all $i \in \mathbb{N}$.

Given a shrinking indexed constructor β, and a set $X \subseteq \mathbb{R}^{[0,2\pi]}$, β is said to win the Banach-Mazur game against X, if for any indexed constructor α, and any rational ball $B \in \mathcal{B}$, $R(\alpha, \beta, B) \notin X$.

The following result states that if both strategies are L^p-computable, then the resulting function also is.

Theorem 4.5 *Let α, β be two computable indexed constructors, with β shrinking, and let $B \in \mathcal{B}$. Then $R(\alpha, \beta, B) \in L^p_{rec}$.*

The following result states that the classical characterization of meagerness by Banach-Mazur games also holds for L^p-computable Baire categories.

Theorem 4.6 *Let $X \subseteq \mathbb{R}^{[0,2\pi]}$. X is L^p_{rec}-meager iff there exists a shrinking L^p_{rec}-computable constructor β that wins the Banach-Mazur game against X.*

5　Convergence of Fourier Series for Functions in L^p_{rec}

In this section we investigate how well can Fourier series approximate L^p-computable functions. Fourier series are trigonometric series that are broadly used for the approximation of arbitrary periodic functions in many different areas including electrical engineering, signal and image processing, and data compression.

It is a classical result [8,2] that for any $p > 1$, the Fourier series of any function in L^p-converges to f in the L^p norm, therefore this also holds for L^p-computable functions.

For L^1, the situation is different: Kolmogorov [3] constructed a function $f \in L^1$ whose Fourier series diverges almost everywhere. Using some of his techniques we show a stronger, *typical* result for the computable case, namely that the class of L^1-computable functions captures some very "complicated" functions, so that most of them cannot be approximated by their Fourier series; more precisely we show that the set of L^1-computable functions whose Fourier series does not diverge a.e. is L^1_{rec}-meager, i.e. negligible. As corollaries, our result implies the existence of a function in L^1_{rec} whose Fourier series diverges almost everywhere, as well as Kolmogorov's result [3].

First let us give some notation. The Fourier series of any function $f \in \mathbb{R}^{[0,2\pi]}$ is given by

$$S[f] = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \cos nt \, dt, \quad b_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \sin nt \, dt \quad (n = 1, 2, \ldots).$$

$S_l(f, x)$ denotes the partial lth sum of the Fourier series of f evaluated at x.

The following is the main result of this section.

Theorem 5.1 *The set $\{f \in \mathbb{R}^{[0,2\pi]}|\ S[f]\ does\ not\ diverge\ a.e.\}$ is L^1_{rec}-meager.*

Corollary 5.2 *There exists $f \in L^1_{rec}$ whose Fourier series diverges almost everywhere.*

Corollary 5.3 (Kolomogorov [3]) *There exists $f \in L^1$ whose Fourier series diverges almost everywhere.*

Acknowledgement

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