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Even-power of Cycles With Many Vertices are Type 1 Total Colorable [1]

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Abstract

A power of cycle graph C_n^k is the graph obtained from the chordless cycle C_n by adding an edge between any pair of vertices of distance at most k. Power of cycle graphs have been extensively studied in the literature, in particular with respect to coloring problems, and both vertex-coloring and edge-coloring problems have been solved in the class. The total-coloring problem, however, is still open for power of cycle graphs. A graph G is Type 1 if can be totally colored with $\Delta(G)+1$ colors and is Type 2 if can not be colored with $\Delta(G)+1$ and can be colored with $\Delta(G)+2$, with $\Delta(G)$ representing the maximum degree of a vertex in G. Although recent works from Campos and de Mello [4] and from Almeida et al. [1] point partial results for specific values of n and k, the total-coloring problem is far from being solved in the class. One remarkable conjecture from Campos and de Mello [4] states that C_n^k, with $2 \leq k < \lfloor n/2 \rfloor$, is Type 2, if and only if n is odd and $n < 3(k+1)$. In particular, the conjecture would imply that, for each $k \geq 2$, the number of Type 2 graphs is finite and every power of cycle graph C_n^k with $n \geq 3(k+1)$ would be Type 1. We address Campos and de Mello conjecture for power of cycle graphs and we prove a weaker version of the conjecture for even-power of cycle graphs: every C_n^k with even $k \geq 2$ and $n \geq 4k^2+2k$ is Type 1. Our proof is actually stronger and shows that, for each even $k < \lfloor n/2 \rfloor$, the number of Type 2 power of cycle graphs C_n^k is at most $2k^2-k$. Our results, therefore, constitute a strong evidence in favor of Campos and de Mello conjecture for power of cycle graphs.

Keywords: Graph theory, graph algorithms, total coloring, power of cycle graphs.

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k \ n	2k+1	2k+2	2k+3	2k+4	2k+5	2k+6	2k+7	...	odd n < 3(k+1)	r(2k+1)	r(2k+1) + k	...	even n	odd n ≥ 3(k+1)
2	Δ+1	Δ+1	Δ+2	Δ+1	Δ+1	Δ+1	Δ+1	...	C	Δ+1	Δ+1	...	Δ+1	Δ+1
3	Δ+1	Δ+1	Δ+2	Δ+1	Δ+2	Δ+1	Δ+1	...	C	Δ+1	Δ+1	...	Δ+1	Δ+1
4	Δ+1	Δ+1	Δ+2	Δ+1	Δ+2	Δ+1	Δ+1	...	C	Δ+1	Δ+1	...	Δ+1	Δ+1
5	Δ+1	Δ+1	Δ+2	TCC	Δ+2	TCC	O	...	C	Δ+1	TCC	...	TCC	O
6	Δ+1	Δ+1	Δ+2	TCC	Δ+2	TCC	O	...	C	Δ+1	TCC	...	TCC	O
...	...	...	...	...	...	...	...	...	...	...	...	...	...	...
k	Δ+1	Δ+1	Δ+2	TCC	Δ+2	TCC	O	...	C	Δ+1	TCC	...	TCC	O

Fig. 1. The state of the art of the two problems (TCC and total coloring) in the graphs C_n^k. The rows of the table represent the parameter k and the columns represent the parameter n as a function of k. The entry $\Delta + 1$ represents graphs that are Type 1; $\Delta + 2$ represents graphs that are Type 2; TCC represents that the TCC has been proved but the classification between Type 1 and Type 2 is unknown; C represents the graphs that are not Type 1 because of the Conformable condition. Label C was omitted in the entries that already have another label, but every entry with odd n and $2k + 1 < n < 3(k + 1)$ is also a C entry; and O represents the graphs where both problems are unknown. The row $k = 2$ is fully solved, due to [3]. The rows $k = 3$ and $k = 4$ are fully solved, due to [1]. The column $2k + 1$ consists of complete graphs, since $2k + 1$ is odd, the graphs are Type 1 [15]. The column $2k + 2$ is fully solved, due to [15]. The TCC was proved for the columns $2k + 3$ to $2k + 5$, due to [15]. The TCC was proved for even n, due to [4]. The TCC was proved for C_n^k, with $n = r(2k + 1) + k$ and $r > 0$, due to [3]. Note that this result also proves the TCC for some specific values of odd n. The graphs C_n^k, with $n = r(2k + 1)$ and $r > 0$, are Type 1, due to [3].

1 Introduction

The *Total Chromatic Number* $\chi_T(G)$ is the least number of colors needed to color the vertices and edges of a graph G such that no incident or adjacent elements (vertices and edges) receive the same color. The well known *Total Coloring Conjecture* (TCC), which was proposed independently by Behzad [2] and Vizing [14], states that for every simple graph G, $\chi_T(G) \leq \Delta(G) + 2$, with $\Delta(G)$ being the maximum degree of a vertex in $V(G)$. The TCC has been settled for restricted graph families, such as the complete r-partite graphs, split graphs, dually chordal graphs, and graphs with a high maximum degree. However, the TCC for general graphs is open for more than fifty years and remains open when restricted to chordal graphs and power of cycle graphs.

We call *total coloring* a function $\mathscr{C}: E(G) \cup V(G) \to [t]$, where $[t] = \{0, \ldots, t\}$, and adjacent elements must receive distinct colors. A color of a vertex v_i is denoted by $\mathscr{C}(v_i)$, similarly a color of an edge v_iv_j is denoted by $\mathscr{C}(v_iv_j)$. It is easy to see that no graph can be totally colored with less than $\Delta(G) + 1$ colors, which implies a lower bound for the total chromatic number $\chi_T(G) \geq \Delta(G) + 1$. If a graph G can be totally colored with $\Delta(G) + 1$ colors it is called *Type 1*, if it can not be colored with $\Delta(G) + 1$ colors but can be colored with $\Delta(G) + 2$ colors it is called *Type 2*. Sánchez-Arroyo [12] proved that the *total-coloring* problem of determining the total chromatic number is $\mathcal{NP}$-hard. It is important to note that there is no "implication relation" between the validity of the TCC in a class and the complexity of the total-coloring problem. For example, for bipartite graphs, the TCC is solved, but the total-coloring problem is $\mathcal{NP}$-hard [12]. On the other hand, for graphs with bounded tree-width, the total-coloring problem is polynomial but the TCC is not yet settled [8].

There exists a strong parallel between edge coloring and total coloring. The celebrated Vizing's theorem implies the edge-coloring classification problem into Class 1 (edge-colorable graphs with Δ colors) and Class 2 (edge-colorable graphs with $\Delta+1$ colors), while the TCC would imply the total-coloring classification of graphs into Type 1 and Type 2. Note that both edge-coloring and total-coloring problems are

$\mathcal{NP}$-complete. In particular, the total-coloring problem was first proved to be $\mathcal{NP}$-hard [12] by a reduction from edge coloring; this could suggest that total coloring would be a harder problem, nevertheless, we know classes where edge coloring is $\mathcal{NP}$-hard but total-coloring is polynomial [9]. Since both edge-coloring and total-coloring are $\mathcal{NP}$-complete problems, effort has been done in the sense of defining "easy" sufficient or necessary conditions for a graph to be Class 1 or Type 1, and we discuss two of such conditions next. The *overfull* condition for edge coloring states that the number of edges is greater than $\Delta \lfloor n/2 \rfloor$ and is a sufficient condition for a graph to be Class 2 — for a graph to be Class 1 it is necessary to be non-overfull. For example, any regular graph with odd n is overfull and therefore Class 2. The basic classes of cycle and complete graphs are fully classified with respect to edge coloring by using the overfull condition. The *conformable* condition for total coloring is a necessary condition for a graph to be Type 1, and requires the existence of a particular vertex coloring with $\Delta + 1$ colors with the property that "many" color classes have the same parity as the order of the graph — where the number of such color classes is related to the difference between the maximum degree and the average degree. For regular graphs, when that difference is zero, the conformable condition requires that every color class has the same parity as the order of the graph [5].

A power of cycle graph, denoted by C_n^k, is a graph where $V(C_n^k) = \{v_0, v_1, \ldots, v_{n-1}\}$, where $v_0, v_1, \ldots, v_{n-1}$ is a spanning cycle, and $E(C_n^k) = E^1 \cup \cdots \cup E^k$, where $E^i = \{v_j v_{(j+i)} \mid 0 \leq j \leq n-1\}$. In this work, when we refer to a vertex $v_i \in V(C_n^k)$ we mean $v_{i \bmod n}$. Power of cycle graphs are a well studied class of graphs with respect to several variations of coloring problems. The vertex-coloring problem can be solved in the class with a greedy algorithm [3] and the edge-coloring problem can be solved using the overfull condition [10].

By definition, the power of cycle graphs C_n^k generalize the cycle graphs $C_n = C_n^1$ and the complete graphs $K_n = C_n^k$, with $k \geq \lfloor n/2 \rfloor$. The basic class of complete graphs is fully classified with respect to total coloring by using the conformable condition: the Type 1 complete graphs K_n are precisely the conformable ones, i.e., the graphs K_n with $n \bmod 2 = 1$. However, the conformable condition does not characterize the Type 1 cycle graphs, as there are conformable cycle graphs that are Type 2, for example, graph C_7.

There are results towards the TCC and the total-coloring problem for power of cycle graphs C_n^k for specific values of n and k. Figure 1 summarizes these results, the first column $n = 2k+1$ represents the Type 1 complete graphs since n is odd. If n is odd and $2k+1 < n < 3(k+1)$, then C_n^k is not Type 1 [3]; If $k = 2$ and $n > 2k+1$, then C_n^k is Type 2 if $n = 7$ and Type 1 otherwise [3]; if $k = 3$ and $n > 2k+1$, then C_n^k is Type 2 if $n = 9$ or $n = 11$ and Type 1 otherwise; and if $k = 4$ and $n > 2k+1$, then C_n^k is Type 2 if $n = 11$ or $n = 13$ and Type 1 otherwise [1]. The last two results were presented at VI Latin American Workshop on Cliques in Graphs in 2014 and they are published in the conference book of abstracts (with no detailed proofs). If n is even and $n > 2k+1$ or $n = r(2k+1)+k$ with integer $r > 0$, then the TCC for such C_n^k graphs was proved [4]; If $n = r(2k+1)$ with integer

$r > 0$, then C_n^k is Type 1. This result is specially interesting because it is a corollary of the pullback algorithm by de Figueiredo, Meidanis, and de Mello [6] and of the Very Greedy Neighborhood Coloring ($VGNC$) algorithm by Golumbic [7].

Campos [3] showed that if a power of cycle graph C_n^k, with $2 < k < \lfloor n/2 \rfloor$ and odd n, is conformable, then $n \geq 3(k+1)$. In other words, for $2 < k < \lfloor n/2 \rfloor$, if n is odd and $n < 3(k+1)$, then the C_n^k is not Type 1. Campos and de Mello [4] proposed a remarkable conjecture that those would be precisely the Type 2 power of cycle graphs for that range of k:

Conjecture 1.1 ([4]) *Let $G = C_n^k$, with $2 \leq k < \lfloor n/2 \rfloor$. If $k > \frac{n}{3} - 1$ and n is odd, then $\chi_T(G) = \Delta + 2$. Otherwise, $\chi_T(G) = \Delta + 1$.*

We give a strong evidence for Conjecture 1.1: we prove that every even-power of cycle graph C_n^k, with $n \geq 4k^2 + 2k$ is Type 1. Moreover, we prove that, for each even $k < \lfloor n/2 \rfloor$, the number of Type 2 power of cycle graphs C_n^k is at most $2k^2 - k$, by showing a Type 1 total coloring for every other graph under the considered range. We also prove that for every fixed $k \geq 2$, if n is bigger than a function $f(k)$ we can decompose a graph C_n^k into two graphs $C_{n_1}^k$ and $C_{n_2}^k$, with $n = n_1 + n_2$, giving a decomposition-based polynomial-time algorithm that total-colors a power of cycle C_n^k by total-coloring the graphs $C_{n_1}^k$ and $C_{n_2}^k$ and applying the same reasoning recursively for $C_{n_1}^k$ and $C_{n_2}^k$ if $n_1 > f(k)$ or $n_2 > f(k)$.

The paper is organized as follows. In Section 2 we define operations and operators that are used in Section 3 to present a threshold for a even-power of cycle graph to be Type 1. In Section 4 we present a framework to decompose a power of cycle graph into a set of smaller power of cycle graphs. In Section 5 we present our conclusions and possible further directions and applications of the developed tools.

2 Compatibility and bottom-up composition

In this section, we define two operations used as part of the process of construct a Type 1 total coloring of a graph C_n^k, from two other Type 1 graphs $C_{n_1}^k$ and $C_{n_2}^k$, such that $n_1 + n_2 = n$.

A power of path graph, denoted by P_n^k, is a graph where $V(P_n^k) = \{v_0, v_1, \ldots, v_{n-1}\}$, where $v_0, v_1, \ldots, v_{n-1}$ is a spanning path, and $E(P_n^k) = E^1 \cup \cdots \cup E^k$, where $E^i = \{v_j v_{(j+i)} \mid 0 \leq j \leq n-1\}$. In power of path graphs v_{j+i} is not a modular operation.

We aim to decompose a power of cycle graph C_n^k into two power of path graphs $P_{n_1}^k$ and $P_{n_2}^k$, with $n_1 + n_2 = n$, and two sets of edges $N^-(w_x)N^+(w_{x-1})$ and $N^-(w_y)N^+(w_{y-1})$, with $w_x, w_y \in V(C_n^k)$, in such way that we can use a total coloring of the graphs $C_{n_1}^k$ and $C_{n_2}^k$ to totally color $P_{n_1}^k$ and $P_{n_2}^k$, respectively, using the colors of the edges of $E(C_{n_1}^k) \setminus E(P_{n_1}^k)$ and $E(C_{n_2}^k) \setminus E(P_{n_2}^k)$ to color the edges of $N^-(w_x)N^+(w_{x-1})$ and $N^-(w_y)N^+(w_{y-1})$ to obtain a valid total color of C_n^k. The operation of transfer total colorings is called *pullback* and it was defined by de Figueiredo, Meidanis, and de Mello [6]. The pullback from G to G' is a function $f : V(G) \to V(G')$, such that: (i) f is a homomorphism, i.e., if $pq \in E(G)$, then

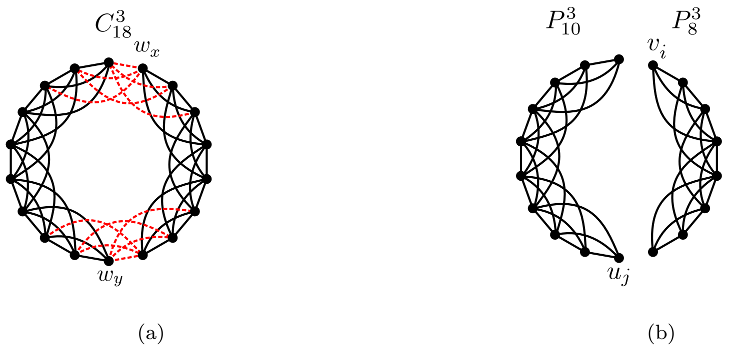


Fig. 2. The illustration of a composition process, note that the vertex $w_x \in C^k_n$ represent the vertex $v_i \in C^k_{n_1}$ and the vertex $w_y \in C^k_n$ represent the vertex $u_j \in C^k_{n_2}$. (a) A graph C^k_n, with $n = 18$ and $k = 3$. The red dashed edges represent the sets $N^-(w_x)N^+(w_{x-1})$ and $N^-(w_y)N^+(w_{y-1})$. (b) The vertex v_i represents the first vertex of the graph P^3_{10}, the same occurs with the vertex u_j which is the first vertex of P^3_8. The edges of the sets $N^-(w_x)N^+(w_{x-1})$ and $N^-(w_y)N^+(w_{y-1})$ was omitted to highlight the graphs P^3_8 and P^3_{10}, but the sets can be seen in 2a.

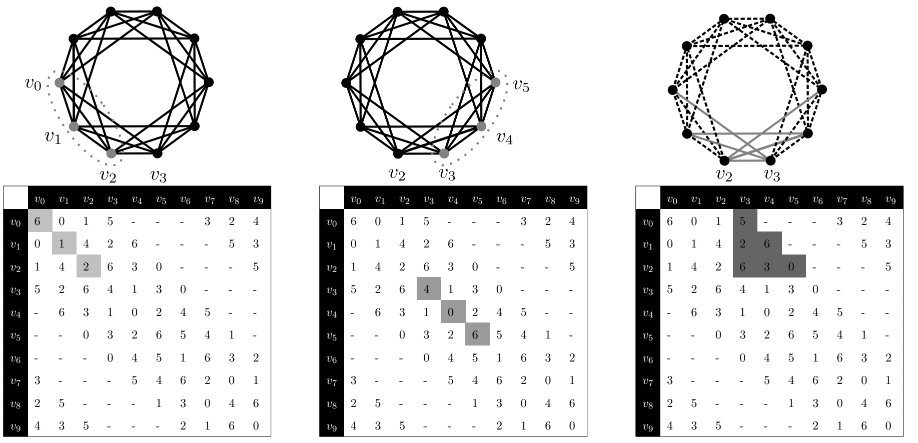
$f(p)f(q) \in E(G')$; (ii) f is injective when restricted to $N(p)$, with $N(p)$ representing the open neighborhood of p, for $p \in V(G)$. The pullback is a powerful tool that allows us to transfer colorings from one graph to other.

We define more formally the operations and the operators of this procedure in Definitions 2.1 and 2.2.

Definition 2.1 We say that a *k-power of path decomposition* of a power of cycle graph C^k_n is a decomposition into two power of path graphs $P^k_{n_1}$ and $P^k_{n_2}$, and two sets of edges $N^-(w_x)N^+(w_{x-1})$ and $N^-(w_y)N^+(w_{y-1})$, with the distance between $w_x, w_y \in C^k_n$ in the spanning cycle is greater than $2k$. w_x represents the vertex $v_i \in P^k_{n_1}$ which is the first vertex of the induced path of $P^k_{n_1}$ and w_y represents the vertex $u_j \in P^k_{n_2}$ which is the first vertex of the induced path of $P^k_{n_2}$. Note that every power of cycle graph C^k_n with $n \geq 4k+2$ has a k-power of path decomposition. Figure 2 illustrates a 3-power of path decomposition of C^3_{18}.

A semi-cut of vertices of a graph C^k_n, is a set of k consecutive vertices in $V(G)$, considering the cyclic order. We denote two special semi-cuts of vertices: $N^-(w_x) = \{w_q \mid x - k \leq q < x\}$ and $N^+(w_{x-1}) = \{v_q \mid x \leq q < x + k\}$. A semi-cut of edges of a graph C^k_n is a set $N^-(w_x)N^+(w_{x-1})$ which represents the edges that have one extreme in $N^-(w_x)$ and the other extreme in $N^+(w_{x-1})$, more formally $N^-(w_x)N^+(v_{x-1}) = \{v_q v_l \mid x - k < q \leq x,\ x < l \leq x + k \text{ e } l - q \leq k\}$. Figure 3 shows a graph C^3_{10} and the sets $N^-(v_3)$, $N^+(v_2)$, and $N^-(v_3)N^+(v_2)$. Each set is a shade highlighted in one table. The tables show total colorings, in such way that the color of an edge $v_i v_j$ is represented by the cell i, j of the matrix. The color of the vertex v_i is represented by the cell i, i.

Definition 2.2 We say that two graphs $C^k_{n_1}$ and $C^k_{n_2}$ are Type 1-*compatible* if each graph has a Type 1 total coloring $\mathscr{C}_1$ and $\mathscr{C}_2$ and pivot vertices v_i and u_j, respectively, such that the following holds: (a) $N^-(v_i)$ is compatible with $N^+(u_{j-1})$, meaning $\mathscr{C}_1(v_{i-r}) \neq \mathscr{C}_2(u_{j+s})$; (b) $N^+(v_{i-1})$ is compatible with $N^-(u_j)$, meaning $\mathscr{C}_1(v_{i+s}) \neq \mathscr{C}_2(u_{j-r})$, for every $r \in \{0, \dots, k-1\}$ and every $s \in \{1, \dots, k-r\}$;



(a) (b) (c)

Fig. 3. (a) The vertices of the set $N^{-}(v_3)$ are the vertices v_0, v_1, v_2. (b) The vertices of the set $N^{+}(v_2)$ are the vertices v_3, v_4, v_5. (c) The edges of the set $N^{-}(v_3)N^{+}(v_2)$ are represented for the shaded gray edges.

(c) $N^{-}(v_i)N^{+}(v_{i-1})$ is compatible with $N^{-}(u_j)N^{+}(u_{j-1})$, meaning $\mathscr{C}_1(v_{i-r}v_{i+s}) = \mathscr{C}_2(u_{j-r}u_{j+s})$, for every $r \in \{0, \dots, k-1\}$ and every $s \in \{1, \dots, k\}$. By definition, every Type 1 graph is Type 1-compatible with itself.

3 A threshold for power of cycle graphs to be Type 1

In this section, we prove the main results of this work. Theorem 3.1 shows an operation of composition between two graphs.

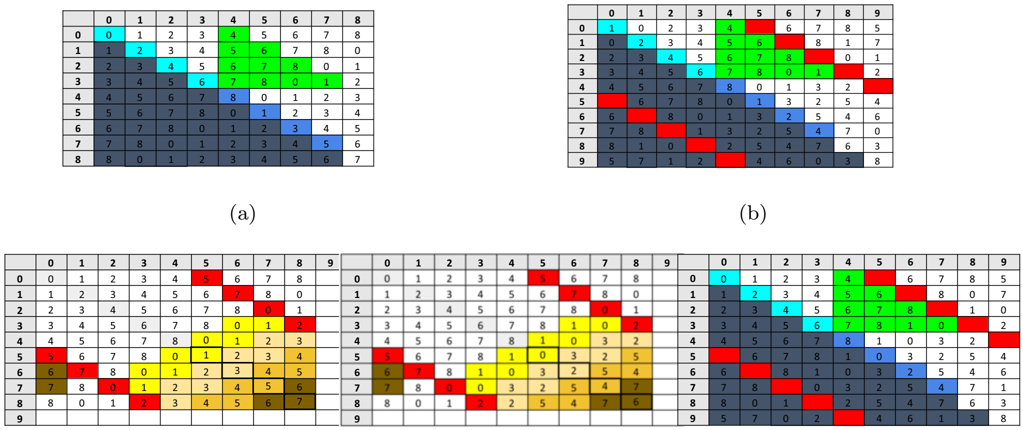
Theorem 3.1 *If two graphs $C^k_{n_1}$ and $C^k_{n_2}$ are Type 1-compatible, then the graph C^k_n, with $n = n_1 + n_2$, is Type 1-compatible with $C^k_{n_1}$ and $C^k_{n_2}$.*

Proof. Since $C^k_{n_1}$ and $C^k_{n_2}$ are Type 1 - compatible, there are two Type 1 total colorings $\mathscr{C}_1$ and $\mathscr{C}_2$ and vertices $v_i \in V(C^k_{n_1})$ and a vertex $u_j \in V(C^k_{n_2})$ that fulfill the restrictions of the Definition 2.2.

The graph $P^k_{n_1}$ is formed from the graph $C^k_{n_1}$ by removing the edges of $N^{-}(v_i)N^{+}(v_{i-1})$. And the graph $P^k_{n_2}$ the graph $C^k_{n_2}$ by removing the edges of $N^{-}(u_j)N^{+}(u_{j-1})$.

Since the graph C^k_n has a k-power of path decomposition, we can use a pullback of the total coloring of the graphs $C^k_{n_1}$ and $C^k_{n_2}$ to the power of path graphs $P^k_{n_1}$ and $P^k_{n_2}$. Then we only need to color the edges of the sets $N^{-}(w_x)N^{+}(w_{x-1})$ and $N^{-}(w_y)N^{+}(w_{y-1})$, but we can make the same pullback technique to the colors of $N^{-}(v_i)N^{+}(v_{i-1})$ and $N^{-}(u_j)N^{+}(u_{j-1})$. $\square$

In the next Theorem, we make a recoloring procedure from a famous *latin square* L to a valid total coloring represented by a matrix M, of order $2k+2$, in such way that this matrices represents Type 1-compatible total colorings of the graphs C^k_{2k+1} and C^k_{2k+2}, respectively. The procedure can be divided in three steps: First, from the conformable condition, we know that we have to change the colors of some vertices in such way that every color colors a even number of vertices. For this,



(a) (b)

(c) (d) (e)

Fig. 4. The red cells represent the edges that do not exist in the graph C^k_{2k+2}; the green cells represent the cells of the set $N^-(v_4)N^+(v_3)$; the light blue cells represent the set $N^-(v_4)$; the dark blue cells represent the set $N^+(v_3)$. (a) The latin square L represents a Type 1-total coloring of the graph C^k_{2k+1}. (b) The matrix M after all the recoloring procedure, representing a Type 1-total coloring of the graph C^k_{2k+2}, which is compatible with the coloring of the graph C^k_{2k+1}. (c) The matrix M at the start of the procedure. The highlighted cells represent the color of the vertices that starts the procedure. The different shades of the cells represent the colors that have to be swapped. (d) The matrix M after swapping colors. (e) The matrix M after adding the colors of the cells that are in M but not in L.

we have to swap some colors using the fact that the matrix M has cells that not represent valid edges in the graph C^k_{2k+2}. The second step is the inclusion of the new colors in M that represent the vertex and the edges that are in C^k_{2k+2} but not in C^k_{2k+1}. The third step made is to make the two total coloring conformable with other. Combining Theorem 3.2 with Theorem 3.1 we show that every power of cycle graph C^k_n, with even k, is Type 1 except for $2k^2 - k$ graphs.

Theorem 3.2 *The graphs C^k_{2k+1} and C^k_{2k+2}, with even k, are Type 1-compatible.*

Proof. The matrix M represents the total coloring of the graph C^k_{2k+2}. The order of M is $2k+2$ and it has the same properties of the a latin square L, which is used to total coloring complete graphs [15]. The diagonal cells of M represent the colors of the vertices and the off-diagonal cells represent the colors of the edges. Set each cell $L[i,j] = M[i,j] = (i+j) \bmod 2k+1$, with $0 \le i,j \le 2k$. Figure 4a represents the matrix L and Figure 4c represents the matrix M. Note that the cells $M[x, k+1+x]$, with $x \in \{0,\ldots,k\}$ represent edges that do not exist in the graph C^k_{2k+2} because of the distance between the vertices, so these cells can be ignored in our procedure.

Figure 4d has the vertices that start the following procedure highlighted and the cells that have colors swapped have the same shade in the figure. We make a recoloring procedure on the matrix M, this procedure can be made in three steps:

(i) For $s \in \{1,\ldots,k/2\}$ we change the color of the cells $M[k+s-r, k+s+r]$, to the color of the cells $M[k+s-r-1, k+s+r]$, with $r \in \{0,\ldots,k/2\}$, note that the cells $M[k+s-k/2+1-1, k+s+k/2+1] = M[k/2+s, k/2+s+k+1]$

represent the edges that do not exist in the graph C^k_{2k+2}, so this allows us to stop the procedure with no conflict of colors in M.

(ii) For $s = k/2 + 1$ we change the color of the cells $M[k + s - r, k + s + r]$, to the color of the cells $M[k + s - r - 1, k + s + r]$, with $r \in \{0, \dots, k/2 - 1\}$. Note that the cell $M[k + s - (k/2 - 1), k + k/2 + 1 + k/2 - 1] = M[k + 2, 2k]$ receive the original color of the cell $M[k + 1, 2k] = k + 1 = M[0, k + 1]$, which is a cell that represents an edge that do not exists in C^k_{2k+2}. So we end this case with no conflict of colors.

(iii) For the case where $s \in \{k/2 + 2, \dots, k\}$ we change the color of the cells $M[k + s - r, k + s + r]$, to the color of the cells $M[k + s - r - 1, k + s + r]$, with $r \in \{0, \dots, k/2 - 1\}$. Note that in this case we have to use module $2k + 1$. The cell $M[k+s-(k/2-1)-1, k+s+(k/2-1)] = M[k/2+s, (k+k/2+s-1) \bmod 2k + 1]$ represent the edges that do not exist in the graph C^k_{2k+2}, so this allows us to stop the procedure with no conflict of colors in M.

To color the row $2k+1$ of M we use the following procedure: $M[k+i+1, 2k+1] = L[i, k + 1]$, representing the colors of the edges which do not exist in the graph, and $M[i, 2k + 1] = L[i, k]$, representing the colors that were changed by procedure of recoloring, with $i \in \{0, \dots, k - 1\}$. And the cell $M[2k + 1, 2k + 1] = 2k$, the only color possible for the new vertex. This step is illustrated in Figure 4e.

To show that the matrix M is a valid total coloring for a graph C^k_{2k+2} we have to show that:

- The diagonal cells represents a proper vertex coloring: Since we use as base the colors of L, in which all the diagonal cells are distinct, we only change the colors of the cells $M[k + i + 1, k + i + 1]$ to the color $L[k + i, k + i + 1] = (2k + 2i + 1) \bmod 2k + 1 = 2i$, with $i \in \{0, \dots, k - 1\}$ and add the color $2k$ to the cell $M[2k + 1, 2k + 1]$. As the color $2i$ is used as a diagonal color in the cell $M[i][i]$ and in the cell $M[k + i + 1][k + i + 1]$, this can be used as a proper vertex coloring of the graph C^k_{2k+2}, since the only repetition of colors happen in vertices which have distance $k + 1$, therefore do not adjacent.
- Every row and column of M, excluding the cells which represents an edge that do not exists in C^k_{2k+2}, is a permutation of $2k + 1$ elements: Since we use L as base and by construction of the recoloring procedure we only stop the recoloring when the conflict was in one of the cells that represent the edges that do not exist in the graph C^k_{2k+2}, then M is a valid total coloring of C^k_{2k+2}.

Now, we have to swap the colors of all the cells which have colors $M[i, i]$ to the color $M[i, i + 1]$, with $i \in \{0, \dots, k/2 - 2\}$ and $k > 2$. Note that this swap do not change any property of the M.

Now M and L represents the Type 1 total colorings of C^k_{2k+2} and C^k_{2k+1}, respectively, in such way that the first vertex in the spanning cycle is represented by the first row (or column) in the respectively matrix, the second vertex is represented by the second row in the matrix and so on.

To show that C^k_{2k+2} and C^k_{2k+1} are compatible, we have to fulfill the conditions of the Definition 2.2. So lets call C^k_{2k+2} of $C^k_{n_1}$ and C^k_{2k+1} of $C^k_{n_2}$, the vertex $v_i \in C^k_{n_1}$, which $i = k-1$ and $u_j \in C^k_{n_2}$ with $j = k-1$.

To prove (a) of the Definition 2.2: Note that $N^-(v_{i-1})$ have the vertices v_p, with $p \in \{0,\dots,k-1\}$, and $\mathscr{C}_1(v_p) = 2p$. Now note that $N^+(u_{j-1})$ have the vertices u_q, with $q \in \{k,\dots,2k-1\}$, and $\mathscr{C}_2(u_q) = 2(q-k)$, if $q > k$ and if $q = k$, $\mathscr{C}_2(u_q) = 2k$, so the distance between the vertex which have the same color is $k+1$, clearly $N^-(v_{i-1})$ and $N^+(v_{j-1})$ are compatible.

To prove (b) of the Definition 2.2: Note that $N^+(v_{i-1})$ have the vertices v_p, with $p \in \{k,\dots,2k-1\}$, and $\mathscr{C}_2(v_p) = 2(p-k)$, if $p > k$ and if $p = k$, $\mathscr{C}_1(v_p) = 2k$. Note that $N^-(u_{j-1})$ have the vertices u_q, with $q \in \{0,\dots,k-1\}$, and $\mathscr{C}_1(u_q) = 2p+1$, if $q \leq k/2-1$ and $\mathscr{C}_1(u_q) = 2q$, otherwise.

To prove (c) of the Definition 2.2: Since we use the same latin square L to generate the total colorings and the only part of the recoloring procedure that change the color of a cell which represent an edge in $N^-(v_i)N^+(v_{i-1})$ is the Item (i) and we make the changes in L to change the exactly the colors that are affected by (i), the sets $N^-(v_i)N^+(v_{i-1})$ is compatible with $N^-(u_j)N^+(u_{j-1})$. Figure 4b show the matrix M representing a valid total coloring that is compatible with the coloring represented by the latin square L. □

Lemma 3.3 is used in the Theorem 3.4 to prove our main result.

Lemma 3.3 *If $z \geq a^2 - a$, $\exists x, y \geq 0$, such that $z = xa + y(a+1)$.*

Proof. The prove is made by induction in z. If $z = a^2 - a$, then $z = xa + 0(a+1)$, with $x = a-1$. Supose that for any $z_1 > a^2 - a$, $z_1 = x_1 a + y_1(a+1)$. For $z_2 = z_1 + 1$: if $x_1 > 0$, than $z_2 = (x_1 - 1)a + (y_1+1)(a+1)$. If $x_1 = 0$, then: $z_1 = 0a + y_1(a+1)$ and as $z_1 > a^2 - a$, $y_1 \geq a-1$. So we can write $z_1 = (a+1)(a-1) + (y_1 - (a-1))(a+1) = a^2 - 1 + (y_1 - (a-1))(a+1)$ and then $z_2 = x_2 a + y_2(a+1)$, with $x_2 = a$ and $y_2 = y_1 - (a-1)$. □

Note that the threshold given in the Lemma 3.3 is tight, as this is a classic linear diophantine equation, is known that if $z = a^2 - a - 1$, then $z = xa + y(a+1)$ has no valid integer solutions.

Theorem 3.4 *Every graph C^k_n with even $k \geq 2$ and $n \geq 4k^2 + 2k$ is Type 1.*

Proof. Following Lemma 3.3, $n = x(2k+1) + y(2k+2)$, then we only need to compose x times the graph C^k_{2k+1} and y times C^k_{2k+2}. □

Proposition 3.5 *For a fixed even k, with $2k+1 \leq n \leq 4k^2 + 2k$, $2k^2 - k$ graphs C^k_n are Type 1.*

Proof. The proof can be made by induction in s, which denote the number of graphs that will compose the target graph. For $s = 2$: We have the graphs C^k_{4k+2}, C^k_{4k+3}, and C^k_{4k+4}, which have a Type 1 colorings by doing compositions of the graphs C^k_{2k+1} and C^k_{2k+1}, C^k_{2k+1} and C^k_{2k+2}, and C^k_{2k+2} and C^k_{2k+2}, respectively. Note that $2k-2$ — the graphs C^k_n with $2k+3 \leq n \leq 4k+1$ — were not totally colored. For $s = 3$: We have the graphs C^k_{6k+3}, C^k_{6k+4}, C^k_{6k+5}, and C^k_{6k+6}, this

k \ n	2k+1	2k+2	...	4k+1	4k+2	4k+3	4k+4	4k+5	...	n ≥ 4k²+2k
3	Δ+1	Δ+1	...	Δ+1	Δ+1	Δ+1	Δ+1	Δ+1	...	Δ+1
4	Δ+1	Δ+1	...	Δ+1	Δ+1	Δ+1	Δ+1	Δ+1	...	Δ+1
5	Δ+1	Δ+1	...	O	Δ+1	Δ+1	Δ+1	O	...	Δ+1
6	Δ+1	Δ+1	...	O	Δ+1	Δ+1	Δ+1	O	...	Δ+1
7	Δ+1	Δ+1	...	O	Δ+1	Δ+1	Δ+1	O	...	Δ+1
8	Δ+1	Δ+1	...	O	Δ+1	Δ+1	Δ+1	O	...	Δ+1
9	Δ+1	Δ+1	...	O	TCC	O	TCC	O	...	-
10	Δ+1	Δ+1	...	O	Δ+1	Δ+1	Δ+1	O	...	Δ+1
...	...	...	...	...	...	...	...	...	...	-
even k	Δ+1	Δ+1	...	O	Δ+1	Δ+1	Δ+1	O	...	Δ+1
odd k	Δ+1	Δ+1	...	O	TCC	O	TCC	O	...	-

Fig. 5. The contributions of this work are highlighted. The previous results can be seen in Figure 1.

graphs can be obtained by collating the graphs C^k_{2k+1} and C^k_{2k+2} in the graphs obtained in the step $s = 2$. Note that $2k - 3$ graphs were not totally colored. For a given $3 < s < 2k$: We have the graphs $C^k_{(2s)k+s+l}$, with $l \in \{0, \dots, s\}$, obtained by collating the graphs C^k_{2k+1} and C^k_{2k+2} in the graphs obtained in the step $s-1$; Note that $2k - s$ graphs were not totally colored. And for any C^k_n, with $n \geq 4k^2 + 2k$ all the graphs are Type 1.

Note that for each $2 \leq s \leq 2k$, only $2k - s$ graphs were not totally colored, so only $2k^2 - k$ graphs were not colored. $\square$

The next result shows that for small values of odd k the technique presented in Theorem 3.2 is valid. Thus, the results presented in Theorem 3.4 hold for these graphs.

Theorem 3.6 *The graphs C^k_{2k+1} and C^k_{2k+2} are Type 1-compatible, for $k = 3$, $k = 5$ and $k = 7$.*

4 Top-down decomposition

In this section, we give a framework to decompose any power of cycle graph with many vertices into two power of cycle graphs with a limited number of the vertices. This result shows that we only need to consider a finite number of power of cycle graphs for a given fixed k.

Let $C^k_n[N[v]]$ be the graph induced in $G = C^k_n$ by the closed neighborhood of a vertex $v \in V(G)$ and let $f(k)$ be the number of distinct total colorings of $C^k_n[N[v]]$. Theorem 4.1 shows that every power of cycle with large order — in the sense that $n > f(k)$ — can be decomposed into smaller compatible graphs.

Theorem 4.1 *If $n > f(k)$, then C^k_n can be decomposed into two power of cycle graphs $C^k_{n_1}$ and $C^k_{n_2}$, such that $n = n_1 + n_2$ and $C^k_{n_1}, C^k_{n_2}$ are compatible.*

Proof. Suppose that G is t-total colorable and $n > f(k)$.

Since $n > f(k)$ then, by pigeonhole principle, there are at least one pair of vertices $w_x, w_y \in V(C^k_n)$ such that $C^k_n[N[w_x]]$ and $C^k_n[N[w_y]]$ have the same total coloring (preserving the cyclic order of the elements). Thus, we can decompose C^k_n into two power of cycle graphs $C^k_{n_1}$ and $C^k_{n_2}$, with $n_1 + n_2 = n$, using the reverse operation applied in Theorem 3.1, with w_x representing the vertex v_j and w_y representing the vertex u_j. $\square$

N. Trotignon and K. Vušković [13] defined that a decomposition is said to be *extremal* if at least one of the blocks is *simple*, i.e., it can not be decomposed into smaller blocks. If we apply same reasoning of Theorem 4.1 recursively, we can generate an extremal *total coloring tree decomposition*, such that: the root of the tree is the original power of cycle graph C_n^k, two sibling nodes are compatible graphs, the collage of these two nodes generate the father node, and every leaf node of the tree is a simple block - $C_{n_\ell}^k$ graph which $n_\ell \leq f(k)$. Note that the size of each node is smaller than the size of the original graph C_n^k and that the number of nodes of the decomposition tree is a linear function of n. In addition, the size of each leaf node is smaller than $f(k)$. Hence, for fixed k, we can devise a decomposition-based polynomial-time algorithm that total-color a power of cycle C_n^k by attributing to each leaf a total-coloring that is compatible with the coloring of its sibling node. An appropriate use of data structures and refined complexity analysis and would actually show that such algorithm is linear in n assuming fixed k.

5 Conclusion

In this work, we proved that if n is bigger than a given threshold then every graph C_n^k with even k is Type 1. Our result is actually stronger, as we prove the TCC for such graphs and give a linear time optimal algorithm to total color them. The result proves asymptotically the conjecture proposed by Campos and de Mello [4], reducing to a small number the graphs that are still unknown to satisfy the conjecture. Our results are illustrated in Figure 5, which combined with Figure 1 shows the contribution of our results to the state of the art on total coloring power of cycle graphs.

In addition to our main results for even k, for odd k we also have computational results for specific values of k, namely $k = 5$ and $k = 7$. These results point to a path to follow towards a more general technique which can also be applied for odd k.

All techniques presented in this work can be expanded: we can extend the set of graphs colored by the technique presented in Section 3 by simply increasing the size of the seed set used to generate the total colorings. As Definition 2.2 does not present any restriction on the number of colors, the same idea can be used to prove the TCC as well, by giving $\Delta(G) + 2$-compatible total colorings. The framework presented in Section 4 can be extended to totally color other classes of graphs, mostly the classes of graphs that have similar structural decomposition, for example circulant graphs.

The total coloring tree decomposition proposed in this paper for power of cycles is closed related to the notion of tree decomposition, as well as treewidth, defined by Robertson and Seymour [11]. Therefore, another interesting direction would be to extend our technique in order to improve the state of art of the total-coloring problem for classes of graphs with bounded treewidth.

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