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Electronic Notes in Theoretical Computer Science 345 (2019) 281–292

Electronic Notes in Theoretical Computer Science

www.elsevier.com/locate/entcs

The Duality Theory of General $\mathcal{Z}$-continuous Posets

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Abstract

In this paper, we research further into $\mathcal{Z}$-predistributive and $\mathcal{Z}$-precontinuous posets introduced by Erné. We focus on duality theorems based on the application of Galois connections whenever $\mathcal{Z}$ is a closed subset selection. For example, there is a duality between the categories $\mathcal{Z}$-$\mathbf{PD}_G$ and $\mathcal{Z}$-$\mathbf{PD}_D$ of all $\mathcal{Z}$-predistributive posets with weakly $\mathcal{Z}^{\triangle}$-continuous maps which have a lower adjoint, and maps preserve $\mathcal{Z}$-below relation that have an upper adjoint, respectively, as morphisms. We introduce the concept of $\mathcal{Z}_0$-approximating auxiliary relation, and have made a slight improvement on $\mathcal{Z}$-precontinuity, so that there is a generalization of the classical equivalence between domains and auxiliary relations.

Keywords: poset, $\mathcal{Z}$-predistributive, $\mathcal{Z}$-precontinuous, $\mathcal{Z}$-closed, Galois connection, auxiliary relation.

1 Introduction

The "way-below" relation is an essential ingredient in continuous posets and domains [1,10,11] and plays a central role in the applications of computer sciences. Continuous poset is based on the axiom of approximation, where the classical "way-below" relation is associated with all directed subsets which have supremum, but not for arbitrary subsets. In [16], Wright, J. B., Wagner, E. G. and Thatcher, J. W. introduced the concept of subset systems $\mathcal{Z}$, got rid of the restriction to directed subsets and replaced by "$\mathcal{Z}$-sets". After this, a theory of $\mathcal{Z}$-continuous posets was developed by Bandelt and Erné [3,4], Novak [13], Venugopalan [15]. The theory has been pursued by others, such as [2,8,9,19] and so on. $\mathcal{Z}$-continuity inherits the basic idea to use variants of the "way-below" relation, associated with $\mathcal{Z}$-sets which have a least upper bound.

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[3] This work is supported by National Natural Science Foundation of China (No. 11771134).

https://doi.org/10.1016/j.entcs.2019.07.030
1571-0661/© 2019 Published by Elsevier B.V.

In [8], Erné used $\mathcal{Z}$ to denote a subset selection, which assigns every poset P to a certain collection $\mathcal{Z}P$ of subsets and is more extensive than subset system. Some types of posets with "$\mathcal{Z}$-approximation" from below were put forward, for instance, $\mathcal{Z}$-predistributive and $\mathcal{Z}$-precontinuous posets. The "$\mathcal{Z}$-approximation" involves the cut operator $\triangle$ (others may use δ) of subsets instead of the existence of supremum. Erné characterized these posets by certain homomorphism properties and adjunctions. In recent years, there are other results about $\mathcal{Z}$-precontinuity, see [12,14,17,18], but none discussed the dual category on these posets. The purpose of our paper is to discuss that.

In Section 3, we give some statements of $\mathcal{Z}$-predistributive and $\mathcal{Z}$-precontinuous posets. Galois connections play an important role in the framework of category theory. Let $\mathcal{Z}$ be a closed subset selection. A duality is built up between categories $\mathcal{Z}$-$\mathbf{PD}_G$ and $\mathcal{Z}$-$\mathbf{PD}_D$ of all $\mathcal{Z}$-predistributive posets with $\mathcal{Z}^{\triangle}$-morphisms and $\mathcal{Z}^{\triangle}$-comorphisms, as morphisms respectively, in particular the full subcategories $\mathcal{Z}$-$\mathbf{PC}_G$ and $\mathcal{Z}$-$\mathbf{PC}_D$ of all $\mathcal{Z}$-precontinuous posets. We also show that the image of a $\mathcal{Z}$-precontinuous poset under a $\mathcal{Z}^{\triangle}$-morphism is $\mathcal{Z}$-precontinuous. We characterize $\mathcal{Z}$-precontinuity with appropriate auxiliary relations in Section 4.

2 Preliminaries

Let us recall some basic definitions. For each poset P and $A \subseteq P$, we denote ${\downarrow}A := \{y \in P \mid (\exists x \in A)\, y \le x\}$ and ${\downarrow}x := {\downarrow}\{x\}$, ${\downarrow}A$ is said to be the *lower set generated by* A. $A^u := \{x \in P \mid (\forall y \in A)\, y \le x\}$ is called the *upper bound set* of A. The least element of A^u if it exists is called the *supremum* of A and is denoted by $\bigvee A$; $A^\ell := \{x \in P \mid (\forall y \in A)\, y \ge x\}$ is the *lower bound set* of A. The *cut operator* $\triangle$ is written by $\triangle A := A^{u\ell}$. A *subset selection* $\mathcal{Z}$ denotes a function which assigns to each poset P a set $\mathcal{Z}P$ of subsets of P, and $\mathcal{Z}$ is called a *subset system* if

(i) there exists a poset P such that $\mathcal{Z}P$ contains some nonempty set;

(ii) if $f : P \to Q$ is a monotone map from P into a poset Q, then $f(Z) \in \mathcal{Z}Q$ for all $Z \in \mathcal{Z}P$.

By (ii), for any subset $B \subseteq P$, $Z \in \mathcal{Z}(B)$ implies $Z \in \mathcal{Z}(P)$. The frequently used examples of subset selections are:

- $\mathcal{A}$ where $\mathcal{A}P$ is the collection of all lower sets;
- $\mathcal{B}$ where $\mathcal{B}P$ is the collection of all nonempty upper bounded subsets;
- $\mathcal{C}$ where $\mathcal{C}P$ is the collection of all nonempty chains;
- $\mathcal{D}$ where $\mathcal{D}P$ is the collection of all directed subsets;
- $\mathcal{E}$ where $\mathcal{E}P$ is the collection of all one-element subsets;
- $\mathcal{F}$ where $\mathcal{F}P$ is the collection of all finite subsets;
- $\mathcal{P}$ where $\mathcal{P}P$ is the collection of all subsets.

Among these, all except $\mathcal{A}$ are subset systems. But note that ${\downarrow}f(Z) \in \mathcal{A}Q$ for all $Z \in \mathcal{A}P$.

For any subset selection $\mathcal{Z}$, we denote by $\mathcal{Z}^\wedge P = \{\downarrow Z : Z \in \mathcal{Z}P \bigcup \mathcal{E}P\}$, the collection of all $\mathcal{Z}$*-ideals*. However, for $\mathcal{A}$ and subset system $\mathcal{Z}$, $\mathcal{Z}^\wedge P$ is just the set $\{\downarrow Z : Z \in \mathcal{Z}P\}$. A subset selection $\mathcal{Z}$ such that $Y \in \mathcal{Z}(\mathcal{Z}^\wedge P)$ implies $\bigcup Y \in \mathcal{Z}^\wedge P$ for all posets P is called *union-complete*. We denote

$$\mathcal{Z}^\triangle P := \{Y \in \mathcal{A}P \mid Z \in \mathcal{Z}P \text{ and } Z \subseteq Y \text{ implies } \triangle Z \subseteq Y\},$$

this is called $\triangle$-*ideal completion* of poset P. Obviously, $\downarrow x \in \mathcal{Z}^\triangle P$ for $x \in P$ and $\triangle X \in \mathcal{Z}^\triangle P$ for $X \subseteq P$. The closure X^- of any subset X is defined by $X^- := \bigcap\{Y \in \mathcal{Z}^\triangle P \mid X \subseteq Y\}$. Let $\sigma_{\mathcal{Z}}(P) = \{P \backslash Y \mid Y \in \mathcal{Z}^\triangle P\}$, which generalizes the classical Scott topology. It is easy to show that $U \in \sigma_{\mathcal{Z}}(P)$ iff $U = \uparrow U$ and for all $Z \in \mathcal{Z}^\wedge P$, $\triangle Z \cap U \neq \emptyset$ implies $Z \cap U \neq \emptyset$.

The function f is called *weakly* $\mathcal{Z}^\triangle$*-continuous* if $f^{-1}(\downarrow x) \in \mathcal{Z}^\triangle P$ for all $x \in Q$; f is said to be $\mathcal{Z}$*-closed* if for every $Z \in \mathcal{Z}^\wedge P$ implies $\downarrow f(Z) \in \mathcal{Z}^\wedge Q$. The subset selection $\mathcal{Z}$ is called *closed* if every monotone map is $\mathcal{Z}$-closed. Some tedious manipulation yields that all the subset systems are closed, including subset selection $\mathcal{A}$.

3 Duality of $\mathcal{Z}$-predistributive posets

In this paper, unless otherwise stated, $\mathcal{Z}$ denotes a subset selection. We will consider variants of $\mathcal{Z}$-continuity: $\mathcal{Z}$-predistributive and $\mathcal{Z}$-precontinuous posets, some properties will be given. In order to make connections between categories, we need to define suitable morphisms, that is, $\mathcal{Z}^\triangle$-morphisms and $\mathcal{Z}^\triangle$-comorphisms which involve the Galois connections for any closed subset selection.

Now, we firstly recall the "way-below" relation on posets with respect to $\mathcal{Z}$-sets. The $\mathcal{Z}$*-below ideal* of an element x in a poset P is the set $\Downarrow^{\mathcal{Z}} x = \bigcap\{Z \in \mathcal{Z}^\wedge P \mid x \in \triangle Z\}$. For $x, y \in P$, we write $y \ll^{\mathcal{Z}} x$ if $Z \in \mathcal{Z}^\wedge P$ and $x \in \triangle Z$ imply $y \in Z$, the relation $\ll^{\mathcal{Z}}$ is called $\mathcal{Z}$*-below relation*. Denote the set $\{v \in P \mid x \ll^{\mathcal{Z}} v\}$ by $\Uparrow^{\mathcal{Z}} x$, and for $A \subseteq P$, $\Downarrow^{\mathcal{Z}} A = \{u \in P \mid (\exists y \in A)\ u \ll^{\mathcal{Z}} y\}$, $\Uparrow^{\mathcal{Z}} A = \{v \in P \mid (\exists y \in A)\ y \ll^{\mathcal{Z}} v\}$. The properties of the relation $\ll^{\mathcal{Z}}$ are as follows.

Proposition 3.1 *For any poset P, the following statements hold for $x, y, u, v \in P$:*

(1) $x \ll^{\mathcal{Z}} y$ implies $x \leq y$;

(2) $u \leq x \ll^{\mathcal{Z}} y \leq v$ implies $u \ll^{\mathcal{Z}} v$;

(3) $0 \ll^{\mathcal{Z}} x$ whenever P has bottom element 0 and $x \neq 0$.

Remark 3.2 The empty set may confuse us. If $\emptyset \in \mathcal{Z}^\wedge P$ whenever P has bottom element 0, then $0 \in \triangle\emptyset$. Thus 0 is impossible to have $\mathcal{Z}$-below relationship with itself. Otherwise, $0 \ll^{\mathcal{Z}} x$ for all $x \in P$.

Proposition 3.3 *Let P be a poset. Suppose that there exists a $\mathcal{Z}$-set $Z \subseteq \Downarrow^{\mathcal{Z}} x$ with $x = \bigvee Z$. Then $\Downarrow^{\mathcal{Z}} x \in \mathcal{Z}^\wedge P$ and $x = \bigvee \Downarrow^{\mathcal{Z}} x$.*

Definition 3.4 [8] Let P be a poset.

(i) P is called $\mathcal{Z}$*-predistributive* if $x = \bigvee \Downarrow^{\mathcal{Z}} x$ for each $x \in P$;

(ii) P is called *$\mathcal{Z}$-precontinuous* if it is $\mathcal{Z}$-predistributive and $\Downarrow^{\mathcal{Z}} x \in \mathcal{Z}^{\wedge}P$ for each $x \in P$.

Actually, $\mathcal{Z}$-predistributive posets were called completely $\mathcal{Z}$-distributive in [7]. If $\mathcal{Z}$ is a subset system, then $\mathcal{Z}$-precontinuous is the Z_δ-continuous poset essentially (see [18]); for a subset system which requires the existence of a non-singleton $\mathcal{Z}$-set, $\mathcal{Z}$-predistributive is the weak s_Z-continuous, and $\mathcal{Z}$-precontinuous is s_Z-continuous in the sense of [14]. $\mathcal{A}$-precontinuous is just the completely precontinuous in [17], $\mathcal{D}$-precontinuous is s_2-continuous in [5].

In a continuous poset, the classical "way-below" relation satisfies the *interpolation property*, that is, $x \ll y$ implies $x \ll u \ll y$. For any union-complete, lower fine subset system $\mathcal{Z}$, the interpolation property holds in $\mathcal{Z}$-precontinuous poset in the sense of [18]. We have a similar property for subset selections.

Proposition 3.5 *Let $\mathcal{Z}$ be a union-complete and closed subset selection. Then the $\mathcal{Z}$-below relation of a $\mathcal{Z}$-precontinuous poset P satisfies the interpolation property.*

Proof. Take $x \ll^{\mathcal{Z}} y$ in P. Since P is a $\mathcal{Z}$-precontinuous poset, we have that $y = \bigvee \Downarrow^{\mathcal{Z}} y$ and $\Downarrow^{\mathcal{Z}} y \in \mathcal{Z}^{\wedge}P$. Note that $\triangle(\Downarrow^{\mathcal{Z}} y) = \triangle(\bigcup_{a \ll^{\mathcal{Z}} y} \Downarrow^{\mathcal{Z}} a)$ and the function $c \mapsto \Downarrow^{\mathcal{Z}} c : P \to \mathcal{Z}^{\wedge}P$ is monotone. Then $\downarrow\{\Downarrow^{\mathcal{Z}} a : a \ll^{\mathcal{Z}} y\}$ is a $\mathcal{Z}$-ideal of $\mathcal{Z}^{\wedge}P$ by $\mathcal{Z}$ being closed. It suffices that there exists $\mathcal{U} \in \mathcal{Z}(\mathcal{Z}^{\wedge}P)$ such that $\downarrow\{\Downarrow^{\mathcal{Z}} a : a \ll^{\mathcal{Z}} y\} = \downarrow\mathcal{U}$ in $\mathcal{Z}^{\wedge}P$. Thus $\bigcup\mathcal{U} \in \mathcal{Z}^{\wedge}P$ by union-completeness and $\triangle(\bigcup_{a \ll^{\mathcal{Z}} y} \Downarrow^{\mathcal{Z}} a) = \triangle(\bigcup\mathcal{U})$. Hence, $x \in \Downarrow^{\mathcal{Z}} a$ for some $a \ll^{\mathcal{Z}} y$. $\square$

Next, we use the relationship between elements to describe the variants of continuity.

Theorem 3.6 *Let P be a poset. Then the following conditions are equivalent:*

(1) P is $\mathcal{Z}$-predistributive;

(2) $A \subseteq \triangle(\Downarrow^{\mathcal{Z}} A)$ for all $A \in \mathcal{Z}^{\wedge}P$;

(3) there is a $u \in P$ such that $u \ll^{\mathcal{Z}} x$ with $u \nleq y$ whenever $x \nleq y$ for $x, y \in P$;

(4) $P \setminus \downarrow y = \bigcup\{\Uparrow^{\mathcal{Z}} x \mid x \in P \setminus \downarrow y\}$ for each $y \in P$;

(5) $P \setminus \triangle A = \Uparrow^{\mathcal{Z}}(P \setminus A)$ for each $A \in \mathcal{Z}^{\wedge}P$.

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) and (5) $\Rightarrow$ (4) $\Rightarrow$ (3), are straightforward.

(3) $\Rightarrow$ (5): For every $y \in \Uparrow^{\mathcal{Z}} x$ where $x \in P \setminus A$. Assume that $y \in \triangle A$. Then $x \in A$ by the $\mathcal{Z}$-below relation, which is contradictory. On the other hand, $P \setminus \triangle A \subseteq \Uparrow^{\mathcal{Z}}(P \setminus A)$ if the upper bound set of A is empty; otherwise, for any $y \in P \setminus \triangle A$, then we have $t \in A^u$ such that $y \nleq t$. According to (3), there exists $x \in \Downarrow^{\mathcal{Z}} y$ such that $x \nleq t$, that is, x belongs to $P \setminus \downarrow t$ which contains in the complement of A. It follows that y is a member of $\Uparrow^{\mathcal{Z}}(P \setminus A)$. $\square$

Galois connection is one of the most efficient tools in dealing with complete lattices. Moreover, Galois connections can be used as morphisms to define the functors between categories. It is a natural point in our study to use them to discuss duality theorems on the variants of $\mathcal{Z}$-continuity.

Definition 3.7 [10] Let P and Q be two posets. We say that a pair (g, d) of functions $g : P \to Q$ and $d : Q \to P$ is a *Galois connection* or an *adjunction* between P and Q provided that

(i) both g and d are monotone, and

(ii) the relations $g(s) \geq t$ and $s \geq d(t)$ are equivalent for all pairs of elements $(s, t) \in P \times Q$.

In an adjunction (g, d), the function g is called the *upper adjoint* and d the *lower adjoint*.

Lemma 3.8 *Let (g, d) be a Galois connection between posets P and Q. Then $d(\triangle A) \subseteq \triangle d(A)$ for any subset A of Q.*

Proof. Let A be a subset of poset Q. It is formed in case $\triangle A$ is an empty subset. For each $x \in \triangle A$, if $d(A)^u = \emptyset$, then $\triangle d(A) = P$; otherwise, note that $v \in d(A)^u$ is equivalent to $g(v) \in A^u$, then $x \leq g(v)$, that is, $d(x) \leq v$. Therefore $d(\triangle A) \subseteq \triangle d(A)$ holds in all situations. □

The following proposition is needed for later proof, slightly different from in [12, Proposition 2] and [6, Proposition 1.8].

Proposition 3.9 *Let f be a map between posets P and Q. Consider the following conditions:*

(1) f is a weakly $\mathcal{Z}^{\triangle}$-continuous map;

(2) for every $Z \in \mathcal{Z}P$, $f(\triangle Z) \subseteq \triangle f(Z)$.

Then (1) implies (2) for any subset selection $\mathcal{Z}$; if $\mathcal{Z}$ is a subset system, then (1) $\Leftrightarrow$ (2). If f is monotone, then conditions are equivalent for all subset selections.

Remark 3.10 For an arbitrary subset selection $\mathcal{Z}$, the condition of monotonicity of f is essential when (2) implies (1) in the above proposition. See the following example as $\mathcal{Z} = \mathcal{A}$. Let P be the set $\{a, b, \top\}$ with $a, b \leq \top$ and Q be the chain $\mathbf{2}$. Consider the function f which sends a, b to 1 and $\top$ to 0, simple verification shows that $f(\triangle Z) \subseteq \triangle f(Z)$ for every lower set Z of P. However, a weakly $\mathcal{A}^{\triangle}$-continuous map is monotone but f is not.

From Lemma 3.8 and Proposition 3.9, a lower adjoint d of map g between posets is always weakly $\mathcal{Z}^{\triangle}$-continuous.

There is a well-known duality on posets. The categories $\mathbf{POSET}_G$ and $\mathbf{POSET}_D$ have the class of all posets with the order preserving maps g which have a lower adjoint d and the order preserving maps d having an upper adjoint g as morphisms, respectively. We know that the categories $\mathbf{POSET}_G$ and $\mathbf{POSET}_D$ are dual via functors D and G, where for any poset P we write simply $D(P) = P$ and $G(P) = P$; for every morphism $g : P \to Q$ of $\mathbf{POSET}_G$, $D(g) : Q \to P$ is the lower adjoint of g; for each morphism $d : Q \to P$ of $\mathbf{POSET}_D$, $G(d) : P \to Q$

is the upper adjoint of d. Then our following task is to investigate other duality theories in the context of subset selections. First, we see how the functors D and G translate certain preservation properties of morphisms.

Proposition 3.11 *Let $\mathcal{Z}$ be a closed subset selection. If (g, d) is a Galois connection between posets P and Q. Then the following statements are equivalent:*

(1) g is weakly $\mathcal{Z}^{\triangle}$-continuous;

(2) if $U \in \sigma_{\mathcal{Z}}(Q)$, then $\uparrow d(U) \in \sigma_{\mathcal{Z}}(P)$.

These conditions imply

(3) d preserves $\mathcal{Z}$-below relation $\ll^{\mathcal{Z}}$, that is, $x \ll^{\mathcal{Z}} y$ in Q implies $d(x) \ll^{\mathcal{Z}} d(y)$ in P.

and if Q is $\mathcal{Z}$-predistributive, we have all three conditions are equivalent.

Proof. (1) $\Rightarrow$ (2): Let U be an element of $\sigma_{\mathcal{Z}}(Q)$. We take a $A \in \mathcal{Z}^{\wedge}P$ with $\triangle A \cap \uparrow d(U) \neq \emptyset$ and should show that $A \cap \uparrow d(U) \neq \emptyset$. Then there exists $u \in U$ such that $d(u) \in \triangle A$ by $\triangle A \cap \uparrow d(U) \neq \emptyset$. Without loss of generality, let $A = \downarrow Z$ with $Z \in \mathcal{Z}P$. It is easy to see that $u \in \triangle g(Z) = \triangle(\downarrow g(A))$. Since $\mathcal{Z}$ is closed and $U \cap \triangle(\downarrow g(A)) \neq \emptyset$, there exists $t \in A$ such that $g(t) \in U$. We obtain that $t \in \uparrow d(U)$, so $\uparrow d(U) \in \sigma_{\mathcal{Z}}(P)$.

(2) $\Rightarrow$ (1): Assume that $v \in \triangle Z$ such that $g(v) \in g(\triangle Z) \backslash \triangle g(Z)$ for $Z \in \mathcal{Z}P$. This means that we have an element $x \in g(Z)^u$ such that $g(v) \nleq x$. Let $U = Q \setminus \downarrow x$. Then we have $U \in \sigma_{\mathcal{Z}}(Q)$ and $g(v) \in U$. By hypothesis (2) we know that $\uparrow d(U) \in \sigma_{\mathcal{Z}}(P)$. So, $v \in \uparrow d(U)$, we have a $t \in Z$ with $t \in \uparrow d(U)$, that is, $d(u) \leq t$ for some $u \in U$. It follows that $u \leq g(t) \leq x$, a contradiction.

(1) $\Rightarrow$ (3): Suppose that $x \ll^{\mathcal{Z}} y$ in Q and $A \in \mathcal{Z}^{\wedge}P$ with $d(y) \in \triangle A$. It suffices to show that if $A = \downarrow Z$ for some $Z \in \mathcal{Z}P$, then $d(x) \in A$. By Proposition 3.9, $g(d(y)) \in \triangle g(Z) = \triangle(\downarrow g(Z))$. Recall that $\downarrow g(Z) = \downarrow g(\downarrow Z)$ since g is monotone, thus $y \in \triangle(\downarrow g(A))$ due to $y \leq g(d(y))$. Because $\mathcal{Z}$ is a closed subset selection, $\downarrow g(A)$ is a $\mathcal{Z}$-ideal, we have $x \in \downarrow g(A)$. Hence $d(x) \in d(\downarrow g(A)) \subseteq A$. Therefore, $d(x) \ll^{\mathcal{Z}} d(y)$ in P.

It remains (3) $\Rightarrow$ (1). Suppose that Q is a $\mathcal{Z}$-predistributive poset. We claim $g(\triangle Z) \subseteq \triangle g(Z)$ for each $Z \in \mathcal{Z}P$. Assume there is an element $x \in g(\triangle Z) \backslash \triangle g(Z)$. Then we have an upper bound y of set $g(Z)$ such that $x \nleq y$ in Q. Thus there exists u such that $u \in \twoheaddownarrow^{\mathcal{Z}} x$ but $u \nleq y$ in Q by Theorem 3.6. It follows that $d(u) \ll^{\mathcal{Z}} d(x)$ by hypothesis (3). Naturally $d(u) \in \downarrow Z$ since $d(x) \in d(g(\triangle Z)) \subseteq \triangle(\downarrow Z)$, as a result, $u \in g(\downarrow Z) \subseteq \downarrow g(Z)$. Hence $u \leq y$ and this is the desired contradiction. $\square$

For simplicity of presentation, we assume that subset selection $\mathcal{Z}$ is closed throughout the rest of this section.

Definition 3.12 Let S, T be two posets.

(i) A map $g : S \to T$ is said to be a $\mathcal{Z}^{\triangle}$*-morphism* if g is weakly $\mathcal{Z}^{\triangle}$-continuous and has a lower adjoint.

(ii) A map $d : T \to S$ is called a $\mathcal{Z}^{\triangle}$*-comorphism* if d preserves $\mathcal{Z}$-below relation and has an upper adjoint.

(iii) A map $f : S \to T$ is called a *quasiopen* if $U \in \sigma_{\mathcal{Z}}(S)$ implies $\uparrow f(U) \in \sigma_{\mathcal{Z}}(T)$.

Corollary 3.13 *Let $g : P \to Q$ be a map between posets which has a lower adjoint d. If g is a $\mathcal{Z}^{\triangle}$-morphism, then d is a $\mathcal{Z}^{\triangle}$-comorphism. If Q is $\mathcal{Z}$-predistributive and d preserves $\mathcal{Z}$-below relation, then g is a $\mathcal{Z}^{\triangle}$-morphism.*

We introduce the subcategories of $\mathbf{POSET}_G$ and $\mathbf{POSET}_D$, in order to reformulate Proposition 3.11 in terms of duality.

Definition 3.14 We define the following categories.

(i) $\mathcal{Z}$-$\mathbf{POS}_G$ is the category of posets with $\mathcal{Z}^{\triangle}$-morphisms.

(ii) $\mathcal{Z}$-$\mathbf{POS}_D$ is the category of posets and maps d as morphisms which are quasiopen and have an upper adjoint.

(iii) $\mathcal{Z}$-$\mathbf{PD}_G$ and $\mathcal{Z}$-$\mathbf{PD}_D$ have the same objects of all $\mathcal{Z}$-predistributive posets; the morphisms of $\mathcal{Z}$-$\mathbf{PD}_G$ are $\mathcal{Z}^{\triangle}$-morphisms and the morphisms of $\mathcal{Z}$-$\mathbf{PD}_D$ are $\mathcal{Z}^{\triangle}$-comorphisms.

(iv) $\mathcal{Z}$-$\mathbf{PC}_G$ is the full subcategory of $\mathcal{Z}$-$\mathbf{PD}_G$ consisting of all $\mathcal{Z}$-precontinuous posets.

(v) $\mathcal{Z}$-$\mathbf{PC}_D$ is the full subcategory of $\mathcal{Z}$-$\mathbf{PD}_D$ consisting of all $\mathcal{Z}$-precontinuous posets.

Theorem 3.15 *The following categories are dual under the functors D and G given through the Galois connection of functions:*

(1) $\mathcal{Z}$-$\mathbf{POS}_G$ and $\mathcal{Z}$-$\mathbf{POS}_D$;

(2) $\mathcal{Z}$-$\mathbf{PD}_G$ and $\mathcal{Z}$-$\mathbf{PD}_D$;

(3) $\mathcal{Z}$-$\mathbf{PC}_G$ and $\mathcal{Z}$-$\mathbf{PC}_D$.

We explore the constructions of new $\mathcal{Z}$-precontinuous posets. The following example gives us some constructions of posets, and the Table 1 will show whether the $\mathcal{Z}$-precontinuity is preserved under these structures for a frequent subset selection $\mathcal{Z}$.

Example 3.16 Let P and Q be two posets. We have the following five kinds of "disjoint" sums: (1) (*Disjoint sum*) $P \sqcup Q$, the disjoint union of P and Q (with the obvious partial ordering: elements $x \in P$ and $y \in Q$ are incomparable); (2) (*Coalesced sum*) $P \oplus Q$, the disjoint sum $P \sqcup Q$ with the bottom elements identified, if they have them; (3) (*Separated sum*) $P + Q = (P \sqcup Q)_0$, that is, the disjoint sum with a new bottom element adjoined; (4) $P +_1 Q = (P \oplus Q)^1$, the coalesced sum with a new top element adjoined; (5) $P +_2 Q$, the coalesced sum with the top elements identified if they have them.

Suppose that P and Q are $\mathcal{Z}$-precontinuous posets where $\mathcal{Z}$ is the subset selection $\mathcal{B}$, $\mathcal{C}$ or $\mathcal{D}$ respectively. Then it is easy to check that the sum corresponding to the first three is still $\mathcal{Z}$-precontinuous. Pick P is the chain $\mathbf{3}$ and Q is a 4-element lattice. It is clear that $\mathcal{A}$, $\mathcal{F}$ and $\mathcal{P}$-precontinuity may be destroyed under all these constructions. If P and Q are poset $\mathbb{N}$, then $P +_1 Q$ and $P +_2 Q$ are not $\mathcal{B}$, $\mathcal{C}$ or $\mathcal{D}$-precontinuous posets. As shown in Table 1.

$\mathcal{Z}$ \ sums	(1)	(2)	(3)	(4)	(5)
$\mathcal{A}$	no	no	no	no	no
$\mathcal{B}$	yes	yes	yes	no	no
$\mathcal{C}$	yes	yes	yes	no	no
$\mathcal{D}$	yes	yes	yes	no	no
$\mathcal{F}$	no	no	no	no	no
$\mathcal{P}$	no	no	no	no	no

Table 1
The $\mathcal{Z}$-precontinuity of the sums

Lemma 3.17 *Let (g, d) be an adjunction between posets P and Q. If g is surjective, then $\hat{g} : \mathcal{Z}^\wedge P \to \mathcal{Z}^\wedge Q$ defined by $\hat{g}(A) = \downarrow g(A)$ is surjective. Moreover, $(\hat{g}, \hat{d})$ is an adjunction between $\mathcal{Z}^\wedge P$ and $\mathcal{Z}^\wedge Q$, where $\hat{d}$ is similarly defined.*

Proof. $\hat{g}$ is well-defined by $\mathcal{Z}$ being closed. For all subsets A of P, $g(\downarrow A) \subseteq \downarrow g(A)$ by the monotonicity of g. Since $x \in \downarrow g(A)$ implies $d(x) \in \downarrow A$, we have $g(\downarrow A) = \downarrow g(A)$. Recall that $gd = \mathrm{id}_Q$ (see [10, Proposition O-3.7]). Then $\hat{g}(\downarrow d(B)) = \downarrow g(d(B)) = B$ for $B \in \mathcal{Z}^\wedge Q$. As a result, $\hat{g}$ maps $\mathcal{Z}$-ideals of poset P onto $\mathcal{Z}$-ideals of Q. It is easy to show that $(\hat{g}, \hat{d})$ is an adjunction. □

Theorem 3.18 *Let $g : P \to Q$ be a $\mathcal{Z}^\triangle$-morphism from $\mathcal{Z}$-precontinuous poset P onto poset Q. Then Q is $\mathcal{Z}$-precontinuous. In particular, the image of a $\mathcal{Z}$-precontinuous poset under a $\mathcal{Z}^\triangle$-morphism is $\mathcal{Z}$-precontinuous.*

Proof. Let d be the lower adjoint of g. First, we claim that $u \ll^{\mathcal{Z}} d(x)$ in P implies $g(u) \ll^{\mathcal{Z}} x$ in Q. If $x \in \triangle(\downarrow B) = \triangle B$ for $B \in \mathcal{Z}Q$, then $d(x) \in d(\triangle B) \subseteq \triangle d(B)$ by Lemma 3.8. Thus $u \in \downarrow d(B)$, this means that $g(u) \in g(\downarrow d(B)) = \downarrow B$. We obtain $g(\Downarrow^{\mathcal{Z}} d(x)) \subseteq \Downarrow^{\mathcal{Z}} x$, furthermore, $g(\Downarrow^{\mathcal{Z}} d(x)) = \Downarrow^{\mathcal{Z}} x$ in Q by Proposition 3.11. Since P is $\mathcal{Z}$-precontinuous, $\Downarrow^{\mathcal{Z}} d(x)$ is a $\mathcal{Z}$-ideal of P. It follows that $\Downarrow^{\mathcal{Z}} x \in \mathcal{Z}^\wedge Q$ due to Lemma 3.17. Hence, we have

$$x = g(d(x)) \in g(\triangle[\Downarrow^{\mathcal{Z}} d(x)]) \subseteq \triangle g(\Downarrow^{\mathcal{Z}} d(x)) = \triangle(\Downarrow^{\mathcal{Z}} x),$$

i.e., $x = \bigvee \Downarrow^{\mathcal{Z}} x$. So far, we reach the conclusion that Q is a $\mathcal{Z}$-precontinuous poset. □

4 $\mathcal{Z}_0$-approximating auxiliary relation

In this section, we take a closer look at the $\mathcal{Z}$-below relation and auxiliary relations. We use appropriate auxiliary relations to characterize the improved $\mathcal{Z}$-precontinuity.

Definition 4.1 [10] We say that a binary relation $\prec$ on a poset P is an auxiliary relation, or an auxiliary order, if it satisfies the following conditions for all u, x, y, v:

(i) $x \prec y$ implies $x \leq y$;

(ii) $u \leq x \prec y \leq v$ implies $u \prec v$;

(iii) if a bottom element 0 exists, then $0 \prec x$.

The set of all auxiliary relations on P is denoted by $\mathrm{Aux}(P)$.

Based on Remark 3.2, for any subset selection $\mathcal{Z}$, we denote the *truncated selection* $\mathcal{Z}_0$ by $\mathcal{Z}_0 P = \mathcal{Z}P \backslash \{\emptyset\}$; $\mathcal{Z}_0$*-ideals* by $\mathcal{Z}_0{}^{\wedge}P = \{\downarrow Z : Z \in \mathcal{Z}_0 P \bigcup \mathcal{E}P\}$. For $x, y \in P$, we write $x \ll^{\mathcal{Z}_0} y$ if $Z \in \mathcal{Z}_0{}^{\wedge}P$ and $y \in \triangle Z$ imply $x \in Z$, the relation $\ll^{\mathcal{Z}_0}$ is called $\mathcal{Z}_0$*-below relation* of poset P. Similarly, we may define $\mathcal{Z}_0$-*predistributive* and $\mathcal{Z}_0$*-precontinuous* posets. $\ll^{\mathcal{Z}}$ and $\ll^{\mathcal{Z}_0}$ are equal whenever $\emptyset \notin \mathcal{Z}P$. Obviously, the $\mathcal{Z}_0$-below relation is an auxiliary relation.

We may introduce some definitions that help us to characterize $\mathcal{Z}_0$-precontinuous posets with auxiliary relations.

Definition 4.2 An auxiliary relation $\prec$ on a poset P is said to be $\mathcal{Z}_0$*-approximating* iff the set $\downarrow_{\prec} x = \{y \in P : y \prec x\}$ is a $\mathcal{Z}_0$-ideal and $x = \bigvee \downarrow_{\prec} x$ for all $x \in P$. The set of all $\mathcal{Z}_0$-approximating auxiliary relations is written $\mathrm{App}_{\mathcal{Z}_0}(P)$.

Definition 4.3 A poset P is called $\mathcal{Z}_0$*-meet-precontinuous* if $\downarrow x \cap Y^{-} \subseteq (\downarrow x \cap Y)^{-}$ for all $x \in P$ and $Y \in \mathcal{Z}_0{}^{\wedge}P$.

As what we have anticipated, every $\mathcal{Z}_0$-precontinuous poset is $\mathcal{Z}_0$-meet-precontinuous (see [12,14]). Let $\mathrm{Low}(P)$ denote the set of all lower sets of P. We know the assignment

$$\prec \mapsto s_{\prec} = (x \mapsto \{y : y \prec x\})$$

is an isomorphism from $\mathrm{Aux}(P)$ onto monotone functions $s : P \to \mathrm{Low}P$, whose inverse associates to each monotone function s the relation $\prec_s$ given by $x \prec_s y$ iff $x \in s(y)$ in [10]. If P is a semilattice, then we consider for each $Z \in \mathcal{Z}_0{}^{\wedge}P$ the monotone function $m_Z : P \to \mathrm{Low}P$ given by

$$m_Z(x) = \begin{cases} \downarrow x \cap Z = x \wedge Z, & \text{if } x \in \triangle Z, \\ \downarrow x, & \text{otherwise.} \end{cases}$$

Let P be a semilattice. The unary meet operation $\wedge_x : P \to P : y \mapsto x \wedge y$ is monotone for $x, y \in P$, we say that $\wedge_x$ is $\mathcal{Z}_0$*-closed* if for all $Z \in \mathcal{Z}_0{}^{\wedge}P$ implies $\wedge_x(Z) \in \mathcal{Z}_0{}^{\wedge}P$. The truncated selection $\mathcal{Z}_0$ is called $\wedge$*-closed* if each unary meet operation on all semilattices is $\mathcal{Z}_0$-closed. There is no doubt that each closed subset selection $\mathcal{Z}_0$ is $\wedge$-closed. Let $\mathcal{Z}_0$ be the all nonempty Frink ideals. Then $\mathcal{Z}_0$ is a $\wedge$-closed subset selection but not closed.

Proposition 4.4 *For any $\wedge$-closed subset selection $\mathcal{Z}_0$, a semilattice P is $\mathcal{Z}_0$-meet-precontinuous iff the unary meet operations $\wedge_x : P \to P : y \mapsto x \wedge y$ are weakly $\mathcal{Z}_0{}^{\triangle}$-continuous.*

Proof. For all $Z \in \mathcal{Z}_0 P$, $\triangle Z$ is equal to Z^- by [12, Lemma 1]. We have $\wedge_x(\triangle Z) = x \wedge Z^- = {\downarrow}x \cap ({\downarrow}Z)^-$ for a semilattice. Let P be a $\mathcal{Z}_0$-meet-precontinuous. Then ${\downarrow}x \cap ({\downarrow}Z)^- \subseteq ({\downarrow}x \cap {\downarrow}Z)^- \subseteq \triangle(x \wedge Z)$. Thus $\wedge_x$ is weakly ${\mathcal{Z}_0}^{\triangle}$-continuous. Conversely, we just need to prove that ${\downarrow}x \cap ({\downarrow}Z)^- \subseteq ({\downarrow}x \cap {\downarrow}Z)^-$ for all $x \in P$ and $Z \in \mathcal{Z}_0 P$. Indeed,

$$
{\downarrow}x \cap ({\downarrow}Z)^- = x \wedge (\triangle Z) \subseteq \triangle(x \wedge Z) = [{\downarrow}(x \wedge Z)]^- = ({\downarrow}x \cap {\downarrow}Z)^-,
$$

the second inequality holds as the map $\wedge_x$ is weakly ${\mathcal{Z}_0}^{\triangle}$-continuous, third equation because $\mathcal{Z}_0$ is $\wedge$-closed. This completes the proof. □

Lemma 4.5 *Let $\mathcal{Z}$ be a subset selection which truncated selection $\mathcal{Z}_0$ is $\wedge$-closed. Then for a $\mathcal{Z}_0$-meet-precontinuous semilattice P, all relations $\prec_{m_Z}$ for $Z \in {\mathcal{Z}_0}^{\wedge} P$ are $\mathcal{Z}_0$-approximating.*

Proof. Let $x \in P$. If $x \in \triangle Z$, then $\{y \in P : y \prec_{m_Z} x\} = x \wedge Z$. $x \wedge Z$ is a $\mathcal{Z}_0$-ideal since $\mathcal{Z}_0$ is $\wedge$-closed. It follows that $\triangle(x \wedge Z) = (x \wedge Z)^-$. Thus $\triangle(x \wedge Z) = {\downarrow}x \cap Z^- = {\downarrow}x \cap \triangle Z = {\downarrow}x$ by $\mathcal{Z}_0$-meet-precontinuity. If $x \notin \triangle Z$, then $\{y \in P : y \prec_{m_Z} x\} = {\downarrow}x$. We have $x = \bigvee\{y \in P : y \prec_{m_Z} x\}$ in all cases. □

Proposition 4.6 *Let P be a poset and $\mathcal{Z}_0$ a $\wedge$-closed subset selection. Then the $\mathcal{Z}_0$-below relation $\ll^{\mathcal{Z}_0}$ is contained in all $\mathcal{Z}_0$-approximating auxiliary relations, and is equal to their intersection if P is a $\mathcal{Z}_0$-meet-precontinuous semilattice.*

Proof. It is straightforward that $\ll^{\mathcal{Z}_0}$ is contained in all $\mathcal{Z}_0$-approximating auxiliary relations. If P is a $\mathcal{Z}_0$-meet-precontinuous semilattice, then by Lemma 4.5 we obtain

$$
\begin{aligned}
\bigcap\{s_{\prec}(x) \mid \prec \in \mathrm{App}_{\mathcal{Z}_0}(P)\} &\subseteq \bigcap\{m_Z(x) \mid Z \in {\mathcal{Z}_0}^{\wedge} P\} \\
&= \bigcap_{x \in \triangle Z} ({\downarrow}x \cap Z) \cap \bigcap_{x \notin \triangle Z} {\downarrow}x \\
&= {\downarrow}x \cap \bigcap_{x \in \triangle Z} Z \\
&= \mathop{\downarrow}\limits^{\mathcal{Z}_0} x,
\end{aligned}
$$

thus $\ll^{\mathcal{Z}_0}$ includes the intersection of all $\mathcal{Z}_0$-approximating auxiliary relations. □

For a poset, $\ll^{\mathcal{Z}_0}$ is itself not a $\mathcal{Z}_0$-approximating relation necessarily. But we may now derive the following theorem.

Theorem 4.7 *Let P be a poset and $\mathcal{Z}_0$ a $\wedge$-closed subset selection. Consider the following conditions:*

(1) P is $\mathcal{Z}_0$-precontinuous;

(2) $\ll^{\mathcal{Z}_0}$ is the smallest $\mathcal{Z}_0$-approximating auxiliary relation on P;

(3) there is a smallest $\mathcal{Z}_0$-approximating auxiliary relation on P.

Then (1) ⇔ (2) ⇒ (3), and if P is a $\mathcal{Z}_0$-meet-precontinuous semilattice, then conditions are equivalent.

Proof. (1) $\Leftrightarrow$ (2): Observe that P is a $\mathcal{Z}_0$-precontinuous poset iff $\ll^{\mathcal{Z}_0}$ is a $\mathcal{Z}_0$-approximating auxiliary relation. Then the equivalence follows from Proposition 4.6.

(2) $\Rightarrow$ (3) is clear.

Let P be a $\mathcal{Z}_0$-meet-precontinuous semilattice. Then $\mathcal{Z}_0$-below relation is equal to the intersection of all $\mathcal{Z}_0$-approximating auxiliary relations by Proposition 4.6. Thus, $\ll^{\mathcal{Z}_0}$ has to be the smallest $\mathcal{Z}_0$-approximating auxiliary relation. Therefore we have (3) $\Rightarrow$ (1). $\square$

5 Conclusion

The collection of all directed subsets is a crucial subset selection in domain theory. The present paper has further exhibited some results of general $\mathcal{Z}$-continuity which enriches the $\mathcal{Z}$-theory, a generalization of domain theory. In closed subset selections, we investigated the duality theory for $\mathcal{Z}$-predistributive ($\mathcal{Z}$-precontinuous) posets. Finally, we used $\mathcal{Z}_0$-approximating auxiliary relations to characterize $\mathcal{Z}_0$-precontinuous posets. Naturally, it may take into consideration whether these results are suitable for more subset selections.

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