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# On the Computability of Blochs Constant

Robert Rettinger<sup>1</sup>

*FernUniversität Hagen  
LG Komplexität und Algorithmen  
Universitätsstrasse 1  
D-58095 Hagen*

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## Abstract

We prove the computability of Blochs constant by presenting the first algorithm for this constant.

*Keywords:* computability, algorithm, Blochs constant.

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## 1 Introduction

It is well known that any holomorphic (i.e complex analytic) function is open. A quantitative version of this fact and with many respects surprising, is Blochs theorem, stating that for any  $r > 0$  and any holomorphic function  $f$  defined on a disc  $\mathbb{D}_r(z_0) = \{z \in \mathbb{C} \mid |z - z_0| < r\}$  with  $f'(z_0) \neq 0$  there exists a schlicht disc of radius  $|f'(z_0)| \cdot r \cdot c$  in the image  $f(\mathbb{D}_r(z_0))$ , where the constant  $c$  does not depend on  $f$ ! Obviously  $c$  is bounded from above, thus its supremum, the so called Bloch constant  $\beta$ , exists. The best upper bound known,

$$\beta \leq \frac{\Gamma(1/3)\Gamma(11/12)}{\Gamma(1/4)(1 + \sqrt{3})^{1/2}}$$

found by Ahlfors and Grunsky [2], is at the same time conjectured to be the exact value of  $\beta$ . However the best lower bound known so far (quite recently found by Chen and Gauthier [3]) is

$$\frac{\sqrt{3}}{4} + 2 \cdot 10^{-4} < \beta$$

We will give an algorithm to compute Blochs constant to any precision. Furthermore our algorithm can be adopted to other constants of the same type, e.g. Landaus constant.

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<sup>1</sup> Email: [robert.rettinger@fernuni-hagen.de](mailto:robert.rettinger@fernuni-hagen.de)

Next we will introduce some basic notions and notations mainly on the computability part of this paper. As mentioned above let  $\mathbb{D}_\varepsilon(z_0)$  denote the open disc of radius  $\varepsilon$  with center  $z_0$ . To simplify notation we use  $\mathbb{D}_\varepsilon := \mathbb{D}_\varepsilon(0)$  and  $\mathbb{D} = \mathbb{D}_1$ . A normed holomorphic function on a domain  $D$ ,  $0 \in D$ , is a holomorphic function  $f$  with  $f(0) = 0$  and  $f'(0) = 1$ . The space of normed functions on  $D$  is denoted by  $\mathcal{N}(D)$ . Given  $f \in \mathcal{N}(D)$  let  $\beta(D, f, z_0)$  denote the supremum  $r > 0$  so that the disc  $\mathbb{D}_r(z_0)$  is covered by  $f(D)$  and  $f^{-1}$  exists on  $\mathbb{D}_r(z_0)$ , i.e. there exists a function  $f^{-1} : \mathbb{D}_r(z_0) \rightarrow D$  so that  $f \circ f^{-1}$  is the identity function. These discs  $\mathbb{D}_r(z_0)$  are called schlicht (with respect to  $D$  and  $f$ ). Notice that in this case  $f^{-1}$  is uniquely determined and again holomorphic. Let  $\beta(D, f)$  be the supremum of  $\beta(D, f, z)$  for all  $z \in f(D)$ . Finally let  $\beta(D)$  be the infimum of  $\beta(D, f)$  for all  $f \in \mathcal{N}(D)$ . Then, by definition, we have  $\beta = \beta(\mathbb{D})$ .

We will frequently use the notation  $z = z' \pm \varepsilon$  instead of  $z \in (z' - \varepsilon, z' + \varepsilon)$ . Furthermore we will use the following simple lemma freely (which can be proven by the transformation  $f(z) \mapsto \varepsilon f(\frac{1}{\varepsilon}z)$ ):

**Lemma 1.1** *Let  $\varepsilon > 0$  and a domain  $D$  with  $\mathbb{D}_\varepsilon \subseteq D \subseteq \mathbb{D}$  be given. Then*

$$\varepsilon\beta \leq \beta(D) \leq \beta$$

To introduce computability on  $\mathbb{C}$  and  $\mathbb{R}$  (and  $\mathcal{N}(D)$ ) we will shortly repeat some notions of discrete and type-2-computability theory. Let  $\Sigma$  be some finite set (alphabet). Then partial computable functions  $f : \subseteq \Sigma^* \rightarrow \Sigma^*$  on the free monoid  $\Sigma^*$  can be defined by Turing-machines. Computability can be extended to  $\Sigma^\infty$ , the set of infinite sequences of elements of  $\Sigma$ , as follows: A function  $f : \subseteq \Sigma^\infty \rightarrow \Sigma^\infty$  is computable iff there exists a computable function  $g : \subseteq \Sigma^* \times \mathbb{N} \rightarrow \Sigma^*$  so that for all  $u, w$  with  $f(u) = w$  and all  $n \in \mathbb{N}$  there exists exactly one  $m \in \mathbb{N}$  so that  $g(u|_m, n) = w|_n$  is defined, where  $w|_n$  denotes the initial word of  $w$  of length  $n$ . Here and henceforth we assume that natural numbers are given by their binary representation.

To define computability on other spaces  $M$  we can use representations, i.e. surjections  $\nu : \subseteq \Sigma^\infty \rightarrow M$ . A word  $w$  with  $\nu(w) = m$  is also called ( $\nu$ -)name of  $m$ . A function  $f$  between represented spaces  $(M_1, \nu_1)$  and  $(M_2, \nu_2)$  is computable if we can realize  $f$  by a computable  $g$  in the sense that  $\nu_2 \circ g$  equals  $f \circ \nu_1$  wherever the latter is defined. Products, even for sequences of representations or names, and representations of integers and rational numbers can be defined straightforwardly, for example by using computable pairing functions  $\mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ . (Standard) representations of  $\mathbb{R}$  and thus  $\mathbb{C}$  (where we identify  $\mathbb{C}$  by  $\mathbb{R} \times \mathbb{R}$ ) can be introduced as follows: A name of a real  $x$  is a sequence  $(q_i)_i$  of rational numbers so that  $|q_i - x| \leq 2^{-i}$  for all  $i$ . Finally we will need a representation of holomorphic functions, defined on some domain containing 0. Using standard representations of  $\mathbb{C}$  we identify a sequence  $(a_0, a_1, a_2, \dots)$  with the series (and thus holomorphic function)  $f(z) = \sum_{i=0}^{\infty} a_i z^i$ . For more details see [5].

Beside computable functions we will also need computable multifunctions  $f : \subseteq M \rightrightarrows N$ , where  $m \in M$  can have many different values. Given representations of  $M$  and  $N$  we call such a multifunction computable if there exists a computable

realization as in the definition of functions (here the image of the realization can take any name of any value of the corresponding image of  $f$ ).

## 2 Test Functions

The main idea of our algorithm is to compute for several normed functions the corresponding  $\beta$ -values. By definition, we have to take the infimum for all normed functions, which cannot be done in finitely many steps. Thus we will restrict the space of functions we consider in our algorithm. In this section we will give such a smaller class of functions, which we will call test functions. In the next section we will adopt this class to our purposes, so that for each precision only finitely many functions have to be considered. Finally in Section 4 we will show how to compute the  $\beta$ -values for these finitely many functions.

Central to the definition of test functions is the following lemma.

**Lemma 2.1** *Let  $\varepsilon > 0$  be given. Then for any  $f \in \mathcal{N}(\mathbb{D})$  with  $\beta(\mathbb{D}, f) = \beta \pm \varepsilon$*

$$(1) \quad |f(z)| \leq c_0 \int_0^z \frac{1}{1-|t|^2} dt$$

for  $c_0 = \frac{4}{\sqrt{3}}(\beta + \varepsilon)$  and all  $z \in \mathbb{D}$ .

**Proof** According to a theorem of Ahlfors, any holomorphic function  $f$  in  $\mathbb{D}$  fulfills the following property: If  $|f'(z)| \cdot (1 - |z|^2) \geq M$  for some  $z \in \mathbb{D}$  then the image  $f(\mathbb{D})$  contains schlicht discs of radius  $\frac{\sqrt{3}}{4}M$ . Taking  $M = c_0 = \frac{4}{\sqrt{3}} \cdot (\beta + \varepsilon)$  and taking integrals, we see, that only functions for which (1) above holds can fulfill  $\beta(\mathbb{D}, f) \leq \beta + \varepsilon$ .  $\square$

To simplify things we fix a concrete value for  $c_0$  and call it  $c_1$ , say  $c_1 = 0.48 \cdot \frac{4}{\sqrt{3}}$ . Furthermore let  $\rho(z) = \int_0^z \frac{1}{1-|t|^2} dt$ . Then the set of test functions is defined to be the set

$$\mathcal{T} = \{f \in \mathcal{N}(\mathbb{D}) | \forall z \in \mathbb{D}. |f(z)| \leq c_1 \cdot \rho(z)\}$$

Thus the above lemma implies

**Corollary 2.2** *For each  $\varepsilon > 0$  there exist  $f \in \mathcal{T}$  so that*

$$|\beta(\mathbb{D}, f) - \beta| < \varepsilon$$

Finally we give a simple property on the Taylor series of test functions, which we will use in the next section.

**Lemma 2.3** *Let  $f \in \mathcal{T}$  and  $\varepsilon$ ,  $0 < \varepsilon < 1$  be given. Then for the Taylor series  $f(z) = \sum_{i=0}^{\infty} a_i z^i$  we have*

$$|a_i| \leq \frac{c_1 \cdot \rho(\varepsilon)}{\varepsilon^i}$$

**Proof** By Lemma 2.1 we have  $|f(z)| \leq c_1 \cdot \rho(\varepsilon)$  for all  $z$  with  $|z| = \varepsilon$ . In addition, by Cauchy formula, we have

$$|a_i| = \left| \frac{f^{(i)}(0)}{i!} \right| = \frac{1}{2\pi} \left| \int_{|z|=\varepsilon} \frac{f(t)}{(-t)^{i+1}} dt \right| \leq \frac{2\pi\varepsilon}{2\pi} \sup_{|z|=\varepsilon} \frac{|f(z)|}{\varepsilon^{i+1}}$$

□

### 3 The Main Theorem

Starting with Lemma 2.3 of the last section we define sets  $A_\varepsilon$  and  $B_\varepsilon$  of sequences  $(a_0, a_1, \dots)$  of complex coefficients, which we will identify with their functions  $f(z) = \sum_{i=0}^{\infty} a_i z^i$ . Let  $\varepsilon, \text{varepsilonpsilon}' > 0$  be a fixed constants. Then

$$A_\varepsilon := \{(0, 1, a_2, \dots) | a_i \in \mathbb{C}, |a_i| \leq \frac{c_1 \cdot \rho(\varepsilon^{1/4})}{\varepsilon^{i/4}}\}$$

and

$$B_\varepsilon := \{(0, 1, a_2, \dots) | a_i \in \mathbb{C}, |a_i| < \frac{(c_1 + \varepsilon') \cdot \rho(\varepsilon^{1/4})}{\varepsilon^{i/4}}\}$$

The reason to introduce  $B_\varepsilon$  is, that we can enumerate (in a computable sense!) the prefixes of the elements in  $B_\varepsilon$  but we cannot do this for  $A_\varepsilon$  as the "smaller or equal" condition in the definition cannot be decided. (Actually, we could use  $A_\varepsilon$  and restrict the class of prefixes accordingly.)

First we have to prove that the elements of  $B_\varepsilon$  do indeed define bounded, holomorphic functions on  $\mathbb{D}_\varepsilon$ :

**Lemma 3.1** *For each  $(a_i)_i \in B_\varepsilon$  we have*

- (i)  $\sum_i a_i z^i$  converges absolutely on  $\mathbb{D}_{\sqrt{\varepsilon}}$ ,
- (ii)  $|\sum_i a_i z^i| \leq \sqrt{\varepsilon} + (c_1 + \varepsilon') \cdot \rho(\varepsilon^{1/4}) / (1 - \varepsilon^{1/4})$  for all  $z \in \mathbb{D}_{\sqrt{\varepsilon}}$ .

**Proof** The following inequality proves 1. and 2.

$$\begin{aligned} |\sum_i a_i z^i| &\leq \sqrt{\varepsilon} + \sum_{i>1} \frac{(c_1 + \varepsilon') \cdot \rho(\varepsilon^{1/4})}{\varepsilon^{i/4}} \sqrt{\varepsilon}^i \\ &\leq \sqrt{\varepsilon} + (c_1 + \varepsilon') \cdot \rho(\varepsilon^{1/4}) \sum_{i>1} \varepsilon^{i/4} \end{aligned}$$

for all  $z \in \mathbb{D}_{\sqrt{\varepsilon}}$ .

□

Notice that the bound in 2. does actually not depend on the element of  $B_\varepsilon$ . Thus we can uniformly compute  $f$  on  $\mathbb{D}_\varepsilon$  (and even its closure) as shown e.g. in [4]. Using Cauchy inequality we see that also  $f'$  and  $f''$  can be computed uniformly on  $\mathbb{D}_\varepsilon$ , i.e. given  $f$  (as a sequence of coefficients in  $B_\varepsilon$ ) together with some  $z \in \mathbb{D}_\varepsilon$  we can compute  $f(z)$ ,  $f'(z)$  and  $f''(z)$ .

Next let us introduce a topological structure on  $B_\varepsilon$ : For  $\delta > 0$  define the neighborhoods  $U_\delta(f)$  by  $U_\delta(f) = \{g \in B_\varepsilon \mid |g(z) - f(z)| < \delta \text{ for all } z \in \mathbb{D}_\varepsilon\}$ . We will call the induced topological space again  $B_\varepsilon$  and its restriction  $A_\varepsilon$ . The decisive properties of  $B_\varepsilon$  will be given by the following lemma, where the first part follows from Lemma 2.3. We omit a detailed proof.

**Lemma 3.2** *Let  $\varepsilon$  with  $0 < \varepsilon < 1$  be given. Then*

- (i)  $\mathcal{T} \subseteq A_\varepsilon \subseteq B_\varepsilon$  and
- (ii)  $A_\varepsilon$  is compact.

The second property allows us to approximate  $\beta$  with fixed precision in finitely many steps. The main point is to prove, that we can find, for any  $f \in B_\varepsilon$ , an approximation  $\gamma$  of  $\beta(\mathbb{D}_\varepsilon, f)$  so that this approximation is actually an approximation for a whole neighborhood. Afterwards we can use standard compactness arguments (see below). The validity of the first step is guaranteed by the following lemma, which we will prove in the next section.

**Lemma 3.3** *There exist computable multifunctions  $F : \subseteq \mathbb{R} \times B_\varepsilon \rightrightarrows \mathbb{R} \times \mathbb{R}$  so that given any  $\varepsilon > 0$ ,  $0 < \varepsilon < 1$ ,  $f \in B_\varepsilon$  and  $(\gamma, \delta) \in F(\varepsilon, f)$  we have*

- (i)  $\delta > 0$ ,  $\gamma > 0$  and
- (ii)  $|\gamma - \beta(\mathbb{D}_\varepsilon, g)| < 1 - \varepsilon$  for all  $g \in U_\delta(f)$ .

The proof of our main theorem is now a simple realization of the above discussion:

**Theorem 3.4** *Blochs constant is computable.*

**Proof** We have to show that  $\beta$  can be approximated to arbitrary precision. Therefore let the precision  $d > 0$  be given. Then define  $\varepsilon := 1 - d/2$  and proceed as follows:

We will need a variable  $\beta'$  holding the currently best approximation to  $\beta$ . Furthermore we keep a list  $L = U_0, U_1, \dots$  of open subsets of  $B_\varepsilon$ . At the beginning let  $\beta'$  be some known upper bound on  $\beta$  and  $L$  be an empty list.

Next enumerate the prefixes  $\alpha_0, \alpha_1, \dots$  of all elements in  $B_\varepsilon$  and add in step  $i$  the neighborhood  $U_\delta(\alpha_i)$  given by the multifunction  $F$  of the above lemma, applied to  $\alpha_i$ , to  $L$ . At the same time update the value  $\beta'$  to the minimum of  $\beta'$  and the value  $\gamma$  given by  $F(\varepsilon, \alpha_i)$ . Finally check after each step, whether the neighborhoods encountered in  $L$  already cover  $A_\varepsilon$ . If so, the current value of  $\beta'$  is an accurate approximation of  $\beta$ . Otherwise proceed with the next step. Notice that the minimum above need only taken up to precision  $d/2$  to achieve the overall precision  $d$ .

There is a small technicality in applying the function  $F$ : As we enumerate only prefixes of the coefficients, the procedure described in the next section may need more information than this prefix provides. This can be either resolved by cancelling the step and proceeding with the next prefix in the enumeration, or assuming all digits not given by the prefix to be 0.

□

## 4 Proof of Lemma 6

Let  $\varepsilon$  with  $0 < \varepsilon < 1$  and  $f \in B_\varepsilon$  be given. W.l.o.g. we can assume that  $\varepsilon > 3/4$ . We proceed in 4 steps:

**Step 1:** Find all zeroes  $z_0, z_1, \dots, z_n$  of  $f'$  inside  $\mathbb{D}_\varepsilon$  and discs  $\mathbb{D}_{\varepsilon_i}(z_i)$  with

$$|f'(z)| < (1 - \varepsilon) \cdot (\sqrt{3} - 1)/4^2$$

for all  $z \in \mathbb{D}_{\varepsilon_i}(z_i)$ .

Notice that we are actually only interested in the discs  $\mathbb{D}_{\varepsilon_i}(z_i)$  (and  $\mathbb{D}_{\varepsilon_i/2}(z_i)$ ), because points of this discs are mapped to values near the boundary of schlicht discs. There are several ways to find the zeroes and discs. A quite simple one is by using the theorem of Rouché and polynomial approximations:  $f$  is given as the approximation of polynomials. To find the zeroes we can first find a strictly positiv lower bound of  $|f|$  on  $\{z \mid |z| = \varepsilon\}$  and approximate  $f$  by a polynomial so that the error is smaller than this lower bound. (By using small disturbances of  $\varepsilon$  such a strictly positiv lower bound can always be found.) Rouché's theorem then gives the number of zeroes we have to search for. By dividing  $\mathbb{D}_\varepsilon$  by circles we can repeatedly apply this method until we got all zeroes accurately enough. This can be improved by using a fast algorithm to find the roots of the polynomials approximating  $f$  and testing (again by Rouché's theorem) whether these zeroes are accurate approximations of the zeroes of  $f$  itself.

**Step 2:** Compute a lower bound  $b$  of and an upper bound  $B$  on  $|f'|$  on

$$D_{1/2} = \mathbb{D}_\varepsilon \setminus \bigcup_i \mathbb{D}_{\varepsilon_i/2}(z_i).$$

Then fix some  $\varepsilon' < \inf\{\varepsilon_i/3, b/\sup_{z \in \mathbb{D}_\varepsilon} |f''(z)|, (1 - \varepsilon)/4\}$  and let

$$D = \mathbb{D}_\varepsilon \setminus \bigcup_i \mathbb{D}_{\varepsilon_i}(z_i),$$

$$D_{2/3} = \mathbb{D}_{\varepsilon - \varepsilon'} \setminus \bigcup_i \mathbb{D}_{2\varepsilon_i/3}(z_i).$$

After step 2 we know that for any  $z \in D_{2/3}$  the discs  $\mathbb{D}_{\varepsilon'}(z)$  are mapped one to one onto  $f(\mathbb{D}_{\varepsilon'}(z))$ . Furthermore, by Bloch's theorem there exists discs of radius  $b \cdot \varepsilon' \cdot \sqrt{3}/4$  inside  $f(\mathbb{D}_{\varepsilon'}(z))$ .

**Step 3:** In this step we will approximate  $\beta(\mathbb{D}_\varepsilon, f)$ . Let

$$\varepsilon'' = \min\{b \cdot \varepsilon' \cdot \sqrt{3}/4, \varepsilon'/2\}$$

and

$$G := \left(\frac{\varepsilon''}{4} \cdot \mathbb{Z} + i\frac{\varepsilon''}{4} \cdot \mathbb{Z}\right) \cap f(\mathbb{D}_\varepsilon).$$

For each pair  $(z, f(z)) \in D_{2/3} \times G$  and each  $k \in \mathbb{N}$ ,  $k > 0$  we compute a value

$g_{(z,f(z))}^k \in \{0, 1\}$  so that

$$(2) \quad g_{(z,f(z))}^k = \begin{cases} 1 & \text{if } \beta(\mathbb{D}_\varepsilon, f, f(z)) > (k+1) \cdot \varepsilon''/4 \\ 0 & \text{if } \beta(\mathbb{D}_\varepsilon, f, f(z)) < (k-3) \cdot \varepsilon''/4 - (1-\varepsilon)/4 \end{cases}$$

Let  $D_{(z,f(z))}^k = \{y \in f(\mathbb{D}_\varepsilon) \mid |y - f(z)| < k \cdot \varepsilon''/4\}$  and  $G_{(z,f(z))}^k = D_{(z,f(z))}^k \cap G$ . We proceed in 2 steps, where we start with  $g_{(z,f(z))}^k = 1$ . Furthermore we will compute the values involved only up to precision  $\varepsilon''/(16B)$ :

- (A) Fix for each  $y \in G_{(z,f(z))}^k$  a shortest path  $\pi_{f(z),y}$  from  $f(z)$  to  $y$  inside  $D_{(z,f(z))}^k$ . Divide  $\pi_{f(z),y}$  by equidistant points  $x_0, \dots, x_t$  of  $\pi_{f(z),y}$ , so that  $|x_i - x_{i-1}| < \varepsilon''/4$ .
- (B) Compute for  $s = 0, 1, \dots, t$  the pre-images  $X = f^{-1}(\mathbb{D}_{3\varepsilon''/8}(x_s))$  one after the other and check whether  $X \subseteq D_{2/3}$ , i.e. if  $X \cap (\mathbb{C} \setminus \mathbb{D}_{\varepsilon-\varepsilon'}) \neq \emptyset$  or  $X \cap \mathbb{D}_{2\varepsilon_i/3}(z_i) \neq \emptyset$  we set  $g_{(z,f(z))}^k = 0$ .

Now we can approximate  $\beta(\mathbb{D}_\varepsilon, f)$  by

$$\max_{(z,f(z)) \in D_{2/3} \times G} \{k \cdot \varepsilon''/4 \mid g_{(z,f(z))}^k = 1\}.$$

The correctness of the approximation in the last step follows easily from (2), as

$$\begin{aligned} & |\beta(\mathbb{D}_\varepsilon, f) - \sup_{(z,f(z)) \in D \times G} \{k \cdot \varepsilon''/4 \mid g_{(z,f(z))}^k = 1\}| \\ & \leq \varepsilon''/4 + \sup_{(z,f(z)) \in D \times G} |\beta(\mathbb{D}_\varepsilon, f, f(z)) - \max\{k \cdot \varepsilon''/4 \mid g_{(z,f(z))}^k = 1\}| \\ & \leq \varepsilon''/4 + 4\varepsilon''/4 + (1-\varepsilon)/4 \\ & \leq (1-\varepsilon)/2. \end{aligned}$$

Thus to verify step 3 we have to prove (2) and the computability of the values  $g_{(z,f(z))}^k$ . The latter follows from the fact, that any decisions and computations are made only up to precision  $\varepsilon''/(8B)$ .

To prove (2), let first  $(z, f(z)) \in D_{2/3} \times G$  and  $k \in \mathbb{N}$  be given so that  $g_{(z,f(z))}^k = 1$ . We will show that then the disc  $\mathbb{D}_{k\varepsilon''/4}(f(z))$  is schlicht: Take any path  $\pi$  inside  $\mathbb{D}_{k\varepsilon''/4}(f(z))$  which starts at  $f(z)$ . By construction we have  $f^{-1}(\pi) \subset D$ , because  $\mathbb{D}_{k\varepsilon''/4}(f(z))$  is covered by the discs tested in step (A) (even if we take the reduced precision into account). That means that  $f^{-1}$  can be continued from  $f(z)$  along  $\pi$ . As  $\mathbb{D}_{k\varepsilon''/4}(f(z))$  is simply connected,  $f^{-1}$  thus exists on  $\mathbb{D}_{k\varepsilon''/4}(f(z))$ , i.e.  $\mathbb{D}_{k\varepsilon''/4}(f(z))$  is indeed schlicht.

Next let  $(z, f(z)) \in \mathbb{D}_\varepsilon \times G$  be given so that there exists a schlicht disc of radius  $\gamma > (1-\varepsilon)/4 - \varepsilon''$  with center  $f(z)$  inside  $f(\mathbb{D}_\varepsilon)$ . Let  $D' = \mathbb{D}_{\gamma-(1-\varepsilon)/4}(f(z))$  and  $D'' = f^{-1}(D')$ . Notice, that  $D'' \subset D$ , i.e.  $\mathbb{D}_{\varepsilon_i}(z_i) \cap D'' = \emptyset$ , because otherwise there exists  $z' \in D'' \cap \mathbb{D}_{\varepsilon_i}(z_i)$  for some  $i$  and we have  $|((f^{-1})'(f(z')))| > 4^2/(1-\varepsilon)(\sqrt{3}-1)$ . This implies by Blochs theorem that there exist discs of radius  $\frac{1-\varepsilon}{4} \cdot \frac{\sqrt{3}}{4}$ .  $|((f^{-1})'(f(z')))| > 1$  which is not possible. But now (2) immediately follows, as the tests in the above computation will for any path in  $\mathbb{D}_{\gamma-(1-\varepsilon)/4-\varepsilon''/2}(f(z))$  answer, that  $f^{-1}$  exists.

**Step 4:** Finally choose  $\delta$  small enough so that in all previous steps any values of  $f$

are needed only up to precision  $\delta$ .

## 5 Discussion

In this paper we have given the first algorithm to approximate Blochs constant to arbitrary precision. We can furthermore adopt this algorithm to other constants of this type as Landaus constant. Beside the pure calculation of the values of the constants, our algorithm probably allows to get more information about the holomorphic functions bounding  $\beta$ . Thus we hope, that an implementation of our algorithm is a step forward to prove the long standing conjecture of Ahlfors and Grunsky.

Concerning the complexity of our algorithm, we hope to achieve an efficient algorithm by recycling data of previously steps and restricting adaptively the set of test functions. However, an a priori proof of efficiency (i.e. polynomial time bound), seems to be unachievable from our current knowledge.

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## References

- [1] Ahlfors, L. *Complex Analysis*, McGraw-Hill, 1981.
- [2] Ahlfors, L., and H. Grunsky *Über die Blochsche Konstante*, Math.Z. 42, pp. 671 - 673, 1937.
- [3] Chen, H., and P.M. Gauthier *On Bloch's constant*, J. Anal.Math. 69, pp. 275 - 291, 1996.
- [4] Müller, N. *Polynomial-Time Computation of Taylor Series*, Proc. 22 JAIIO - PANEL '93, Part 2, Buenos Aires, 1993, pp 259-281.
- [5] Weihrauch, K. *Computable Analysis*, Springer-Verlag Berlin/Heidelberg, 2000.