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Effective Randomness for Computable Probability Measures

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Abstract

Any notion of effective randomness that is defined with respect to arbitrary computable probability measures canonically induces an equivalence relation on such measures for which two measures are considered equivalent if their respective classes of random elements coincide. Elaborating on work of Bienvenu [1], we determine all the implications that hold between the equivalence relations induced by Martin-Löf randomness, computable randomness, Schnorr randomness, and weak randomness, and the equivalence and consistency relations of probability measures, except that we do not know whether two computable probability measures need to be equivalent in case their respective concepts of weakly randomness coincide.

Keywords: computable probability measures, Martin-Löf randomness, computable randomness, Schnorr randomness, weak randomness, equivalence of probability measures, consistency of probability measures.

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1 Introduction

Since the work of von Mises around 1920, several concepts of randomness for individual infinite sequences of zeros and ones have been proposed. The most important and most satisfactory concept known today is Martin-Löf randomness (introduced by Martin-Löf [5] in 1966), but other notions have received a lot of attention, too (for a detailed and comprehensive account, see the upcoming monograph of Downey and Hirschfledt [2]). On the one hand, there are various notions of stochasticity, which are defined in terms of selection rules. On the other hand, there are notions that can be defined in terms of betting strategies such as Martin-Löf randomness or computable randomness. The concepts of stochasticity and randomness are generally applied in connection with the uniform measure on Cantor spaces. Indeed, notions of stochasticity, which rely on the converge of frequencies as asserted in the law of large numbers, cannot be extended to arbitrary or even to arbitrary computable probability measures. In contrast to this, concepts of randomness defined in terms of betting strategies usually extend naturally to arbitrary computable probability measures. Accordingly, we will focus on standard randomness notions that can be defined in terms of martingales. These notions are Martin-Löf randomness, computable randomness (also called recursive randomness), Schnorr randomness, and weak randomness (also called Kurtz randomness).

Relations between the various notions of stochasticity and randomness have been extensively studied in the context of uniform measures. In the sequel, we pursue a different approach, as we compare randomness notions with respect to their behavior when the underlying probability measure is varied. In classical probability theory, two probability measures are said to be *equivalent* if they have the same nullsets, or, in other words, if they have the same sets of measure 1, which means that they are in some sense quite similar. Since defining a notion of randomness means choosing for each computable measure μ a particular set of μ-measure 1 and calling its elements *random*, it is natural to define an equivalence relation by saying that two measures are equivalent if they have the same random elements. In this paper,l we investigate into the question which relations hold between the equivalence relations that are obtained from the various randomness concepts. More precisely, we ask for which pairs of randomness concepts it is the case that equivalence with respect to the first concept implies equivalence with respect to the second.

2 Definitions and concepts

2.1 The Cantor space

In what follows, we will only deal with randomness in the Cantor space (although effective randomness can be extended to more general topological spaces, see for example [3]). The Cantor space, which we denote by 2^ω is the set of infinite binary sequences. It is canonically endowed with the product topology, a basis of which is given by the open sets of the form $u.2^\omega = \{\alpha \in 2^\omega : u \sqsubset \alpha\}$ with $u \in 2^*$ (2^* is the set of finite binary sequences, and $\sqsubset$ is the prefix relation, defined on $2^\omega \cup 2^*$). If $\alpha \in 2^* \cup 2^\omega$, $\alpha \upharpoonright n$ denotes the finite word consisting of the first n bits of α. If $\mathcal{X} \subseteq 2^\omega$, $\overline{\mathcal{X}}$ denotes the complement of $\mathcal{X}$ is 2^ω.

By Caratheodory's extension theorem, every function m defined on the subsets of 2^ω and taking its values in $[0, 1]$, such that $m(2^\omega) = 1$ and for all $u \in 2^*$ $m(u.2^\omega) = m(u0.2^\omega) + m(u1.2^\omega)$, induces a unique probability measure on 2^ω. Hence, from now on we can identify a probability measure with its restriction to the open sets of the form $u.2^\omega$, and we abbreviate $\mu(u.2^\omega)$ by $\mu(u)$. The canonical measure on 2^ω is the Lebesgue measure λ, defined by $\lambda(u) = 2^{-|u|}$ for all $u \in 2^*$. Moreover, since they are the only measures we will consider, we will abbreviate "probability measure on the Cantor space" by "measure".

Definition 2.1 A measure μ is computable if there exists a computable function $f : 2^* \times \mathbb{N} \to \mathbb{Q}$ such that for all (u, n), $|f(u, n) - \mu(u)| \leq 2^{-n}$.

The reason why we only consider computable measures is that the notions of randomness we consider, initially defined for the the uniform measure, can be extended in a very natural way to computable measures, whereas there is no such completely natural extension in the case of non-computable measures.

In classical probability theory, there are two main relations on probability measures:

Definition 2.2 Two probability measures μ and ν are EQUIVALENT (denoted by $\mu \sim \nu$) if they have the same nullsets.
Two probability measures μ and ν are CONSISTENT if there is no set which has measure μ-measure 1 and ν-measure 0.

2.2 Martingales

Following Ville [10], we now introduce the notion of martingale. It can be seen as describing the capital of a player who is trying to guess the bits of an infinite binary sequence, betting money (never more than his current capital) on their values, and is rewarded in a fair way. Of course, the fairness of the game depends on the underlying measure.

Definition 2.3 Let μ be a measure. A μ-MARTINGALE is a function $d : 2^* \to \mathbb{R}^+ \cup \{+\infty\}$ such that for all $u \in 2^*$: $d(u)\mu(u) = d(u0)\mu(u0) + d(u1)\mu(u1)$ (with the convention $+\infty.0 = 0$). A martingale d is said to be normed if $d(\varepsilon) = 1$ (ε being the empty word). It is said to be computable if there exists a computable function $f : 2^* \times \mathbb{N} \to \mathbb{Q}$ such that for all (u, n), $|f(u, n) - d(u)| \leq 2^{-n}$.

The following lemma shows that there exists an exact correspondence between measures and martingales:

Lemma 2.4 *For every measure (resp. computable measure) μ, the normed μ-martingales (resp. computable normed μ-martingales) are exactly the functions of the form $\frac{\xi}{\mu}$, where ξ is a measure (resp. computable measure).*

(The proof is straightforward).

The next theorem is a well-known result on martingales, which will be of great use in the sequel.

Theorem 2.5 (Ville [10]) *Let μ be a measure and d a μ-martingale. For all $k \in \mathbb{R}^+$, $\mu\{\alpha \in 2^\omega : sup_n\ d(\alpha \upharpoonright n) \geq k\} \leq 1/k$.*

2.3 Martin-Löf randomness

Definition 2.6 An open set $\mathcal{V}$ is said to be computably enumerable (c.e.) if there exists a computably enumerable $A \subset 2^*$ such that $\mathcal{V} = \bigcup_{u \in A} u.2^\omega$.

A collection $\{\mathcal{V}_n\}_{n\in\mathbb{N}}$ of c.e. open sets is said to be computable if there exists a computable function $(n, k) \in \mathbb{N}^2 \mapsto u_{n,k} \in 2^*$ such that for all $n \in \mathbb{N}$, $\mathcal{V}_n = \bigcup_{k\in\mathbb{N}} u_{n,k}.2^\omega$.

A μ-Martin-Löf test is a computable collection of c.e. open sets $\{\mathcal{V}_n\}_{n\in\mathbb{N}}$ such that for all n, $\mu(\mathcal{V}_n) \leq 2^{-n}$.

$\alpha \in 2^\omega$ is said to pass the μ-Martin-Löf test $\{\mathcal{V}_n\}_{n\in\mathbb{N}}$ if $\alpha \notin \bigcap_n \mathcal{V}_n$.

$\alpha \in 2^\omega$ is said to be μ-MARTIN-LÖF RANDOM (μ-ML random for short) if it passes all μ-Martin-Löf tests. We denote by μMLR the set of μ-ML-random infinite sequences.

Remark 2.7 The notion of ML randomness remains the same if we define a Martin-Löf test to be a computable collection of c.e. open sets $\{\mathcal{V}_n\}_n$ such that $\mu(\mathcal{V}_n)$ is bounded by some computable real-valued function $f(n)$ which is decreasing and tends to 0 as n tends to infinity.

Martin-Löf randomness can be expressed in the framework of martingales (see [2]) but we will not need this.

2.4 Computable randomness

C. P. Schnorr proposed in [8] and [9] two weaker, but in some sense more effective, alternative notions of effective randomess. They are now called respectively computable randomness and Schnorr randomness. Computable randomness is based on the so-called unpredictability paradigm: a sequence is random if no computable strategy/martingale succeeds on it.

Definition 2.8 Let μ be a computable probability measure. A sequence $\alpha \in 2^\omega$ is μ-COMPUTABLY RANDOM if there is no computable μ-martingale d such that $\sup_n\ d(\alpha \upharpoonright n) = +\infty$. We denote by μCR the set of μ-computably random sequences.

Remark 2.9 The notion of computable randomness remains the same if we replace the winning condition $\sup_n\ d(\alpha \upharpoonright n) = +\infty$ by $\lim_n\ d(\alpha \upharpoonright n) = +\infty$.

2.5 Schnorr randomness

Schnorr randomness is even weaker than computable randomness. A sequence α is declared to be not Schnorr random if there exists a computable strategy/martingale which not only succeeds on it, but succeeds at a reasonnable pace:

Definition 2.10 An order is a function $g : \mathbb{N} \to \mathbb{N}$ which is nondecreasing and unbounded. A sequence α is μ-SCHNORR RANDOM if there exists no computable μ-martingale d and computable order g such that $d(\alpha \upharpoonright n) \geq g(n)$ for infinitely many n. We denote by μSR the set of μ-Schnorr random sequences.

This definition can be rephrased as follows:

Proposition 2.11 *A sequence α is μ-Schnorr random iff there exists no computable μ-martingale d and computable function $f : \mathbb{N} \to \mathbb{N}$ such that $d(\alpha \upharpoonright f(n)) \geq n$ for infinitely many n.*

2.6 Weak randomness

Weak randomness, introduced by S. Kurtz [4] (and hence also known as Kurtz randomness) is in some sense the dual of Martin-Löf randomness. Instead of requiring a random sequence to avoid all the sets effectively of measure 0, we require it to belong to all the effective sets of measure 1:

Definition 2.12 A sequence α is μ-WEAKLY RANDOM if it belongs to all c.e. open set $\mathcal{U}$ of μ-measure 1. We denote by μWR the set of μ-weakly random sequences.

Weak randomness can be characterized using martingales, showing in particular that Schnorr randomness implies weak randomness.

Proposition 2.13 (Wang [11]) *A sequence $\alpha \in 2^\omega$ is μ-weakly random if there is no computable μ-martingale d and computable order g such that $d(\alpha \upharpoonright n) \geq g(n)$ for all n. We denote by μWR the set of μ-weakly random sequences.*

3 A classification of equivalence relations

The rest of the paper will be devoted to the proof of the following classification:

Theorem 3.1 *For all computable probability measures μ and ν, the following implications hold:*

$$
\begin{array}{ccccc}
\mu CR = \nu CR & & & & \\
\downarrow & & & & \\
\mu MLR = \nu MLR & & & & \mu SR = \nu SR \\
 & \searrow & & \swarrow & \\
 & & \mu \sim \nu & & \\
 & & \downarrow & & \\
 & & \mu WR = \nu WR & & \\
 & & \downarrow & & \\
 & & \mu, \nu \textit{ consistent} & &
\end{array}
$$

We will show that no other implication holds between these equivalence relations, except a possible equality between classical equivalence and WR-equivalence (although we conjecture this is not the case) which we leave as an open question.

Remark 3.2 It is interesting to note that the above implications between the different equivalence relations are not at all related to the implications between the underlying notions of randomness.

The first implication (if two computable measures have the same computably random elements, then they have the same Martin-Löf random elements) is proven in [6] (see also [1]). We will prove all the others.

Proposition 3.3 *Let μ and ν be two computable measures.*
(a) If $\mu MLR = \nu MLR$ then $\mu \sim \nu$
(b) If $\mu SR = \nu SR$ then $\mu \sim \nu$

To prove this, we need the following lemma:

Lemma 3.4 *Let $A = u_1, .., u_N$ be a finite prefix-free set of words (i.e. no one is a prefix of another). For all computable measure μ, there exists a normed μ-martingale d^μ_A, effectively computable from A, such that for all $1 \leq i \leq N$: $d^\mu_A(u_i) = \left(\sum_{i=1}^N \mu(u_i)\right)^{-1}$.*

Proof. For all $u \in 2^*$, let d^μ_u be the normed μ-martingale which tries to win as much as possible on u (and hence loses everything on all other words of length $|u|$). Formally:

$$d^\mu_u(w) = \begin{cases} \mu(w)^{-1} & \text{if } w \sqsubset u \\ \mu(u)^{-1} & \text{if } u \sqsubset w \\ 0 & \text{otherwise} \end{cases}$$

□

Then let $d^\mu_A = \left(\sum_{i=1}^N \mu(u_i)\right)^{-1} \sum_{i=1}^N \mu(u_i) d^\mu_{u_i}$. d^μ_A is a martingale as it is a weighted sum of martingales, and by construction is normed and satisfies the required property.

Proof. [of Proposition 3.3] We prove (a) and (b) at the same time. Suppose that μ and ν are not equivalent, i.e. for example there exists a set $\mathcal{X}$ such that $\mu(\mathcal{X}) = 0$ and $\nu(\mathcal{X}) > 0$. Let q be a rational number such that $\nu(\mathcal{X}) > q > 0$. By definition of a measure: $\mu(\mathcal{X}) = \inf \ \{\mu(\mathcal{W}) : \mathcal{W} \text{ open}\}$. Hence, for all $k \in \mathbb{N}$, there exists an open set $\mathcal{W} \supset \mathcal{X}$ such that $\mu(\mathcal{W}) < 2^{-2k}$ (and of course $\nu(\mathcal{W}) > q$). Since the $u.2^\omega$ are a base for the Cantor space topology, there exists a finite (prefix-free) set of words $w_1, ..., w_N$ such that $\bigcup_{i=1}^N w_i.2^\omega \subset \mathcal{W}$ and such that $\nu(\bigcup_{i=1}^N w_i.2^\omega) \geq q$ (and of course $\mu(\bigcup_{i=1}^N w_i.2^\omega) < 2^{-2k}$).

Hence, for a given k, one can effectively find a (prefix-free) finite set of words $A_k = u_1^k, ..., u_{N_k}^k$ such that, setting $\mathcal{V}_k = \bigcup_{i=1}^{N_k} u_i.2^\omega$, we have $\nu(\mathcal{V}_k) \geq q$ and $\mu(\mathcal{V}_k) \leq 2^{-2k}$ (it suffices to enumerate the finite sets of words until we find one which satisfies these properties, which will eventually happen by the above discussion). Then let $\mathcal{Z} = \bigcap_n \bigcup_{k>n} \mathcal{V}_k = \{\alpha : \alpha \in \mathcal{V}_k \text{ for infinitely many } k\}$. It is easy to see that $\nu(\mathcal{Z}) \geq q$ and hence that $\mathcal{Z}$ contains ν-Martin Löf random sequences. We now show that $\mathcal{Z} \cap \mu SR = \emptyset$. By the above lemma, for all k, there exists a normed μ-martingale $d_k = d^\mu_{A_k}$ such that for all u_i^k: $d_k(u_i^k) \geq 2^{2k}$. Next, let $d = \sum_{k\in\mathbb{N}} 2^{-k} d_k$. It is a normed μ-martingale as it is the weighted sum, with sum of weights equal to 1, of normed μ-martingales. It is computable since for all w, $\left|d(w) - \sum_{k=1}^m 2^{-k} d_k(w)\right| \leq 2^{-m}\ \mu(w)^{-1}$ (hence one can approximate $d(w)$ by taking m large enough, the error bound being computable in m and tending to 0 as m tends to infinity). Let f be the function defined by $f(k) = \max\{|u_i^k| : 1 \leq i \leq N_k\}$. For all $\alpha \in \mathcal{Z}$, there are infinitely many u_i^k which are prefixes of α. By construction of d, for all k and all $i \leq N_k$, we have $d(u_i^k) \geq 2^{-k} d_k(u_i^k) \geq 2^k$. Hence for infinitely many n: $d(\alpha \restriction f(n)) \geq 2^n$. This, by Proposition 2.11, asserts that α is not μ-Schnorr random. We have proven that $\mathcal{Z} \cap \nu MLR \neq \emptyset$ and $\mathcal{Z} \cap \mu SR = \emptyset$, which completes the proof. □

Proposition 3.5 *Let μ and ν be two computable measures. If $\mu \sim \nu$, then $\mu WR = \nu WR$.*

Proof. This is trivial. If a sequence α is, say, ν-weakly random and not μ-weakly random, then this means that there exists an effectively open set $\mathcal{U}$ of μ-measure 1 such that $\alpha \notin \mathcal{U}$. Since α is ν-weakly random, $\mathcal{U}$ must have ν-measure less than 1. Hence, $\mathcal{U}$ witnesses that μ and ν are not equivalent. □

Proposition 3.6 *Let μ and ν be two computable measures. If $\mu WR = \nu WR$ then μ and ν are consistent.*

Proof. Suppose μ and ν are not consistent, that is, there exists a set $\mathcal{X}$ such that $\nu(\mathcal{X}) = 1$ and $\mu(\mathcal{X}) = 0$. We argue, similarly to the proof of Proposition 3.3, that for all $k \in \mathbb{N}$, we can effectively find a finite prefix-free set of words $u_1^k, ..., u_{N_k}^k$ such that $\mu(\bigcup_{i=1}^{N_k} u_i.2^\omega) \leq 2^{-k-2}$ and $\nu(\bigcup_{i=1}^{N_k} u_i.2^\omega) \geq 1 - 2^{-k}$. Then, $\mathcal{U} = \bigcup_k \bigcup_{i=1}^{N_k} u_i^k.2^\omega$ is an effectively open set, of ν-measure 1 (and hence contains all the ν-weakly random sequences), and of μ-measure less than 1/2 (and hence does not contain all the μ-weakly random sequences, which form a set of μ-measure 1). □

We have proven all the implications of Theorem 3.1. We now begin the more delicate task of showing that there is no other valid implication between the equivalence relations we study (with the possible exception mentioned above).

In order to construct counter-examples, the following quantities will play a crucial role.

Definition 3.7 Let μ and ν be two computable measures and $k \in \mathbb{R}^+ \cup \{+\infty\}$. We set

$$\mathcal{L}^k_{\nu/\mu} = \{\alpha \in 2^\omega : \sup_{n\in\mathbb{N}} \frac{\nu(\alpha \restriction n)}{\mu(\alpha \restriction n)} \geq k\}$$

(with the following convention: if $\mu(\alpha \restriction n) = \nu(\alpha \restriction n) = 0$, $\frac{\nu(\alpha\restriction n)}{\mu(\alpha\restriction n)} = 1$, and if $\mu(\alpha \restriction n) = 0$ and $\nu(\alpha \restriction n) > 0$, $\frac{\nu(\alpha\restriction n)}{\mu(\alpha\restriction n)} = +\infty$)

¿From Lemma 2.4 and Theorem 2.5, we get:

Corollary 3.8 *For all computable measures μ, ν: $\mathcal{L}^\infty_{\nu/\mu} \cap \mu CR = \emptyset$. In particular, this implies that $\mathcal{L}^\infty_{\nu/\mu} \cap \mu MLR = \emptyset$. Moreover, for every $k \in \mathbb{R}^+$: $\mu(\mathcal{L}^k_{\nu/\mu}) \leq 1/k$ (and hence $\mu(\mathcal{L}^\infty_{\nu/\mu}) = 0$).*

Proof. $\frac{\nu}{\mu}$ is a μ-martingale (by Lemma 2.4), and $\mathcal{L}^\infty_{\nu/\mu}$ is exactly the set of sequences on which it succeeds. Thus, $\mathcal{L}^\infty_{\nu/\mu} \cap \mu CR = \emptyset$. The fact that $\mu(\mathcal{L}^k_{\nu/\mu}) \leq 1/k$ is an immediate consequence of Theorem 2.5. □

The next proposition shows how some of the equivalence relations we study are related to the $\mathcal{L}^\infty_{\mu/\nu}$.

Proposition 3.9 *For every couple (μ, ν) of computable measures:*
(a) $\mu \sim \nu$ iff $\mu(\mathcal{L}^\infty_{\mu/\nu}) = \nu(\mathcal{L}^\infty_{\nu/\mu}) = 0$,
(b) $\mu MLR = \nu MLR$ iff $\mathcal{L}^\infty_{\mu/\nu} \cap \mu MLR = \mathcal{L}^\infty_{\nu/\mu} \cap \nu MLR = \emptyset$,
(c) $\mu CR = \nu CR$ iff $\mathcal{L}^\infty_{\mu/\nu} \cap \mu CR = \mathcal{L}^\infty_{\nu/\mu} \cap \nu CR = \emptyset$.

Proof. For (a), (b) and (c), the "only if" direction is a direct consequence of Corollary 3.8. Let us now prove the "if" directions. We will use the following fact: for every open set $\mathcal{U} \subseteq 2^\omega$ and all measures μ and ν: $\mu(\mathcal{U} \cap \overline{\mathcal{L}^k_{\mu/\nu}}) \leq k\,\nu(\mathcal{U})$ (this is a trivial consequence of the definition of $\mathcal{L}^k_{\mu/\nu}$).

(a) Suppose $\mu(\mathcal{L}^\infty_{\mu/\nu}) = \nu(\mathcal{L}^\infty_{\nu/\mu}) = 0$, and let $\mathcal{X} \subseteq 2^\omega$ such that $\nu(\mathcal{X}) = 0$. Let $k \in \mathbb{N}$. Let $\mathcal{U}$ be an open set such that $\mathcal{X} \subseteq \mathcal{U}$ and $\nu(\mathcal{U}) \leq 1/k^2$. Then:

$$\begin{aligned} \mu(\mathcal{X}) &\leq \mu(\mathcal{U}) \\ &\leq \mu(\mathcal{U} \cap \mathcal{L}^k_{\mu/\nu}) + \mu(\mathcal{U} \cap \overline{\mathcal{L}^k_{\mu/\nu}}) \\ &\leq \mu(\mathcal{L}^k_{\mu/\nu}) + k\ \nu(\mathcal{U}) \\ &\leq \mu(\mathcal{L}^k_{\mu/\nu}) + 1/k \end{aligned}$$

This being true for all k, and since by assumption $\mu(\mathcal{L}^\infty_{\mu/\nu}) = 0$ (which is equivalent to $\lim_{k\to+\infty} \mu(\mathcal{L}^k_{\mu/\nu}) = 0$), this proves that $\mu(\mathcal{X}) = 0$.

(b) Suppose $\mathcal{L}^\infty_{\mu/\nu} \cap \mu MLR = \mathcal{L}^\infty_{\nu/\mu} \cap \nu MLR = \emptyset$. Let $\alpha \notin \nu MLR$. If $\alpha \in \mathcal{L}^\infty_{\mu/\nu}$, then by assumption $\alpha \notin \mu MLR$, and we are done. If not, then there exists $k \in \mathbb{R}^+$ s.t. $\alpha \notin \mathcal{L}^k_{\mu/\nu}$. Let $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ be a ν-Martin-Löf test s.t. $\alpha \in \bigcap_n \mathcal{U}_n$. Let $\{u_{n,i} : (n,i) \in \mathbb{N}^2\}$ be an effective enumeration of finite strings such that for all n, $\mathcal{U}_n$ is the disjoint union of the $\{u_{n,i}2^\omega : i \in \mathbb{N}\}$. Define for all n $\mathcal{V}_n = \bigcup_i \{u_{n,i}2^\omega : i \in \mathbb{N},\ \mu(u_{n,i}) < k\ \nu(u_{n,i})\}$. It is then clear that $\{\mathcal{V}_n\}_{n\in\mathbb{N}}$ is a computable family of c.e. open sets such that for all n, $\mu(\mathcal{V}_n) \leq k\ \nu(\mathcal{U}_n)$ (hence $\{\mathcal{V}_n\}_{n\in\mathbb{N}}$ is a μ-Martin-Löf test) and $\alpha \in \bigcap_n \mathcal{V}_n$. Thus, $\alpha \notin \mu MLR$.

(c) Suppose $\mathcal{L}^\infty_{\mu/\nu} \cap \mu CR = \mathcal{L}^\infty_{\nu/\mu} \cap \nu CR = \emptyset$. Let $\alpha \notin \nu CR$. By Lemma 2.4, there exists a computable measure ξ such that $\sup_n \frac{\xi(\alpha\restriction n)}{\nu(\alpha\restriction n)} = +\infty$. If $\sup_n \frac{\xi(\alpha\restriction n)}{\mu(\alpha\restriction n)} = +\infty$, then $\alpha \notin \mu CR$, and we are done. Otherwise, there exists a constant k such that for all n, $\frac{\xi(\alpha\restriction n)}{\mu(\alpha\restriction n)} \leq k$. Hence, $\sup_n \frac{\mu(\alpha\restriction n)}{\nu(\alpha\restriction n)} \geq \sup_n \frac{\mu(\alpha\restriction n)}{\xi(\alpha\restriction n)}\frac{\xi(\alpha\restriction n)}{\nu(\alpha\restriction n)} \geq \sup_n \frac{1}{k}\frac{\xi(\alpha\restriction n)}{\nu(\alpha\restriction n)} = +\infty$. Thus $\alpha \in \mathcal{L}^\infty_{\mu/\nu}$ which by assumption implies $\alpha \notin \mu CR$. □

Proposition 3.10 *Let $\alpha \in \Delta^0_2 \cap \lambda SR$. There exists a computable measure μ such that $\mathcal{L}^\infty_{\mu/\lambda} = \emptyset$, $\mathcal{L}^\infty_{\lambda/\mu} = \{\alpha\}$ and $\alpha \notin \mu SR$*

Proof. We will in fact construct a computable λ-martingale d such that: $\lim_n d(\alpha \restriction n) = 0$ and if $\beta \neq \alpha$, $d(\beta \restriction n)$ will be eventually constant. We will then argue that the computable measure μ defined by $\mu(u) = \lambda(u)d(u)$ for all u is as desired. The construction is done by stages. At stage s, $d(u)$ will be defined for all words u such that $|u| \leq 3^s$.

Since α is Δ^0_2, it is the pointwise limite of a sequence of words $\{w_s\}_{s\in\mathbb{N}}$. We can moreover assume that $\lim_{s\to+\infty} |w_s| = +\infty$, that $|w_s| \leq 3^s$ for all s, and that w_s is a prefix of α for infinitely many s.

Let $E = \{u1^{2|u|} : u \in 2^*\}$. Notice that every λ-Schnorr random sequence has only finitely many prefixes in E. Hence, up to changing a finite number of its bits, we can assume that α has no prefix in E. Let us now proceed to the construction of d.

*S*tage $s = 0$. Set $d(0) = d(1) = 1/2$.

*S*tage $s + 1$. We define $d(u)$ for every u of length 3^{s+1}:

- if u is not an extension of w_s, set $d(u) = d(u \upharpoonright 3^s)$
- if u is an extension of w_s and is not in E, set $d(u) = \frac{d(u \upharpoonright 3^s)}{s+1}$
- if u is an extension of w_s and is in E, set $d(u)$ in such a way that the average value of $\{d(v) : v \upharpoonright 3^s = u \upharpoonright 3^s\}$ is $d(u \upharpoonright 3^s)$

Then, for the words u such that $3^s < |u| < 3^{s+1}$, set inductively (in decreasing order of length) $d(u) = \frac{d(u0)+d(u1)}{2}$.

The martingale d is as desired: since α has no prefix in E, it follows that for infinitely many s, w_s is a prefix of α and $\alpha \upharpoonright 3^s$ is not in E. For all such s, by definition of d: $d(\alpha \upharpoonright 3^s) \leq 1/s$. On the other hand, if $\beta \neq \alpha$, there exists s_0 such that if $s > s_0$, w_s is not a prefix of β, and thus, for all $n > 3^{s_0}$, $d(\beta \upharpoonright n) = d(\beta \upharpoonright 3^{s_0})$.

Let us now consider $\mu = \lambda\ d$. By the above discussion, $\mathcal{L}^\infty_{\mu/\lambda} = \emptyset$, $\mathcal{L}^\infty_{\lambda/\mu} = \{\alpha\}$. Moreover, for infinitely many s, $\frac{\lambda(\alpha \upharpoonright 3^s)}{\mu(\alpha \upharpoonright 3^s)} \geq s$. Hence, by Lemma 2.4 and Proposition 2.11, $\alpha \notin \mu SR$.

□

Before we can apply the above proposition to the construction of counter examples, we need the following important theorem.

Theorem 3.11 (Nies, Stephan and Terwijn [7]) *Let $\alpha \in 2^\omega$. The following are equivalent:*
(i) α has high Turing degree,
(ii) There exists β in the Turing degree of α such that $\beta \in \lambda CR \setminus \lambda MLR$,
(iii) There exists β in the Turing degree of α such that $\beta \in \lambda SR \setminus \lambda CR$.

We are now ready to prove:

Proposition 3.12 *(a) There exists a computable measure μ such that $\lambda \sim \mu$ and nonetheless $\lambda MLR \neq \mu MLR$, $\lambda CR \neq \mu CR$, $\lambda SR \neq \mu SR$.*
(b) There exists a computable measure μ such that $\lambda MLR = \mu MLR$ and $\lambda CR \neq \mu CR$.
(c) There exists a computable measure μ such that $\lambda CR = \mu CR$ and $\lambda SR \neq \mu SR$.

Proof. (a) Let μ be, by Proposition 3.10, a computable measure such that $\mathcal{L}^\infty_{\mu/\lambda} = \emptyset$, $\mathcal{L}^\infty_{\lambda/\mu} = \{\Omega\}$ and $\Omega \notin \mu SR$. Here Ω denotes Chaitin's omega number, which is known to be left-r.e.(i.e. the set $\{q \in \mathbb{Q} : q < \Omega\}$ is recursively enumerable, which in particular implies that Ω is Δ^0_2). Since $\lambda(\{\Omega\}) = 0$, by Proposition 3.9, we have $\lambda \sim \mu$. Moreover, since $\Omega \in \lambda MLR \subset \lambda CR \subset \lambda SR$, and $\Omega \notin \mu SR$, it follows that $\lambda MLR \neq \mu MLR$, $\lambda CR \neq \mu CR$, $\lambda SR \neq \mu SR$.

(b) By Theorem 3.11, let α be a Δ^0_2 sequence such that $\alpha \in \lambda CR \setminus \lambda MLR$. By Proposition 3.10, there exists a computable measure μ such that $\mathcal{L}^\infty_{\mu/\lambda} = \emptyset$, $\mathcal{L}^\infty_{\lambda/\mu} = \{\alpha\}$ and $\alpha \notin \mu SR$. By Proposition 3.9, we have $\lambda MLR = \mu MLR$ (since $\alpha \notin \lambda MLR$) and $\lambda CR \neq \mu CR$ (since $\alpha \in \lambda CR \setminus \mu CR$).

(c) By Theorem 3.11, let α be a Δ^0_2 sequence such that $\alpha \in \lambda SR \setminus \lambda CR$. By Proposition 3.10, there exists a computable measure μ such that $\mathcal{L}^\infty_{\mu/\lambda} = \emptyset$, $\mathcal{L}^\infty_{\lambda/\mu} = \{\alpha\}$ and $\alpha \notin \mu SR$. By Proposition 3.9, we have $\lambda CR = \mu CR$ (since $\alpha \notin \lambda CR$) and $\lambda SR \neq \mu SR$ (since $\alpha \in \lambda SR \setminus \mu SR$).

$\square$

Proposition 3.13 *There exists a computable measure μ such that $\lambda SR = \mu SR$, $\lambda CR \neq \mu CR$ and $\lambda MLR \neq \mu MLR$*

Once again, we need a preliminary lemma.

Lemma 3.14 *Let μ and ν be two computable measures and $\alpha \in 2^\omega$. If $\alpha \in \nu SR \setminus \mu SR$, then there exists a computable order g such that $\frac{\nu(\alpha\restriction n)}{\mu(\alpha\restriction n)} \geq g(n)$ infinitely often.*

Proof. Let $\alpha \in \nu SR \setminus \mu SR$. By Lemma 2.4, there exists a computable measure ξ and a computable order g such that $\frac{\xi(\alpha\restriction n)}{\mu(\alpha\restriction n)} \geq g(n)$ for infinitely many n. Since $\alpha \in \nu SR$ and since $\sqrt{g}$ is a computable order, for almost all n, $\frac{\xi(\alpha\restriction n)}{\nu(\alpha\restriction n)} \leq \sqrt{g(n)}$. Hence, for infinitely many n: $\frac{\nu(\alpha\restriction n)}{\mu(\alpha\restriction n)} = \frac{\xi(\alpha\restriction n)}{\mu(\alpha\restriction n)} \frac{\nu(\alpha\restriction n)}{\xi(\alpha\restriction n)} \geq \frac{g(n)}{\sqrt{g(n)}} = \sqrt{g(n)}$. $\square$

Proof. [of Proposition 3.13] We will construct, in a very similar way as for Proposition 3.10 a computable measure μ such that $\mathcal{L}^\infty_{\mu/\lambda} = \emptyset$, $\mathcal{L}^\infty_{\lambda/\mu} = \{\Omega\}$, but this time we want Ω to be μ-Schnorr random. By the above lemma, it will be sufficient to ensure that $\frac{\lambda(\Omega\restriction n)}{\mu(\Omega\restriction n)}$ tends to infinity slower than any computable order. Hence, we will again construct a λ-martingale d such that $\lim_n d(\Omega \restriction n) = 0$ and if $\beta \neq \Omega$, $d(\beta \restriction n)$, ensuring that $d(\Omega \restriction n)$ decreases very slowly.

Ω being left-r.e., let $\{w_s\}_{s\in\mathbb{N}}$ be a sequence of words such that Ω is the pointwise limit of this sequence, $\lim_{s\to+\infty} |w_s| = +\infty$, $|w_s| \leq 3^s$ for all s, w_s is a prefix of α for infinitely many s, and if w_s is a prefix of Ω, $w_s \sqsubset w_t$ for all $t > s$. We also assume (up to modifying Ω for a finite number of bits) that Ω has no prefix in E (defined in the proof of Proposition 3.10).

Let $F : \mathbb{N} \to \mathbb{N}$ such that $F(0) = 0$, and for all $s > 0$, if $w_s \sqsubset \Omega$, $F(s+1) = |w_s|$, and otherwise $F(s+1) = F(s)$. By definition of the w_s, for all s, the initial segment of w_s which coincides with Ω has at least length $F(s)$. F tends to infinity slower than any computable order: suppose this is not the case, i.e. there exists a computable order g such that $g(s) \leq F(s)$ for infinitely many s. Then, for infinitely many s, looking at w_s, it is possible to guess the first $g(s)$ bits of Ω. From this remark, it is routine to construct a λ-martingale which asserts that Ω is not Schnorr random, a contradiction.

We construct a λ-martingale d such that for all s, $d(\Omega \upharpoonright 3^s) = F(s)^{-1}$, and if $\beta \neq \Omega$, $d(\beta \upharpoonright n)$ is eventually constant:

*S*tage $s = 0$. Set $d(0) = d(1) = 1/2$.

*S*tage $s + 1$. We define $d(u)$ for every u of length 3^{s+1}:

- if u is not an extension of w_s, set $d(u) = d(u \upharpoonright 3^s)$
- if u is an extension of w_s and is not in E, set $d(u) = 1/|w_s|$
- if u is an extension of w_s and is in E, set $d(u)$ in such a way that the average value of $\{d(v) : v \upharpoonright 3^s = u \upharpoonright 3^s\}$ is $d(u \upharpoonright 3^s)$

Then, for the words u such that $3^s < |u| < 3^{s+1}$, set inductively (in decreasing order of length) $d(u) = \frac{d(u0)+d(u1)}{2}$.

Finally, set $\mu = d\,\lambda$. It remains to show that μ is as desired. First, we have $\mathcal{L}^{\infty}_{\mu/\lambda} = \emptyset$. We also see that $\mathcal{L}^{\infty}_{\lambda/\mu} = \{\Omega\}$. Indeed, for all s, $d(\Omega \upharpoonright 3^s) = F(s)^{-1}$, and hence $\frac{\lambda(\Omega\upharpoonright 3^s)}{\mu(\Omega\upharpoonright 3^s)} = F(s)$, which by definition of F implies $\sup_n \frac{\lambda(\Omega\upharpoonright n)}{\mu(\Omega\upharpoonright n)} = +\infty$. Moreover, if $n < 3^s$, by construction of d, $d(\Omega \upharpoonright n) \geq d(\Omega \upharpoonright 3^s)$ and hence $\frac{\lambda(\Omega\upharpoonright n)}{\mu(\Omega\upharpoonright n)} \leq F(\log_3(n))$ for all n. And of course, $F(\log_3(n)) = o(g(n))$ for all computable order g. Finally, if $\beta \neq \Omega$, as we saw in the proof of Proposition 3.10, $\sup_n \frac{\lambda(\beta\upharpoonright n)}{\mu(\beta\upharpoonright n)} < +\infty$.

To complete the proof, notice that by Proposition 3.9, we have $\lambda MLR \neq \mu MLR$ and $\mu CR \neq \mu CR$ (since $\Omega \in \lambda MLR$ and $\Omega \notin \mu CR$). However, we have $\lambda SR = \mu SR$. Indeed by the previous lemma, $\mu SR \setminus \lambda SR = \emptyset$ since $\mathcal{L}^{\infty}_{\mu/\lambda} = \emptyset$, and $\lambda SR \setminus \mu SR = \emptyset$ since $\mathcal{L}^{\infty}_{\lambda/\mu} = \{\Omega\}$ and $\frac{\lambda(\Omega\upharpoonright n)}{\mu(\Omega\upharpoonright n)} = o(g(n))$ for every computable order g. $\square$

Proposition 3.15 *There exist a computable measure μ such that μ and λ are consistent and $\lambda WR \neq \mu WR$*

Proof. Let δ be the measure such that $\delta(\{0^\omega\}) = 1$ (which is clearly computable). Set $\mu = \delta/2 + \lambda/2$. λ and μ are consistent: let $\mathcal{X} \subseteq 2^\omega$. If $0^\omega \in \mathcal{X}$, then $\mu(\mathcal{X}) = 1/2 + \lambda(\mathcal{X})$ and if $0^\omega \notin \mathcal{X}$, $\mu(\mathcal{X}) = \lambda(\mathcal{X})$. In both cases, it is impossible to have $\mu(\mathcal{X}) = 0$ and $\lambda(\mathcal{X}) = 1$, or vice-versa. On the other hand, $0^\omega \in \mu WR$ and $0^\omega \notin \lambda WR$. $\square$

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