Available online at www.sciencedirect.com

SciVerse ScienceDirect

journal homepage: http://ees.elsevier.com/ejbas/default.asp

Full Length Article

Distribution of zeros of solutions of self-adjoint fourth order differential equations

S.H. Saker [a,*], Donal O'Regan [b]

[a] Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt
[b] School of Mathematics, Statistics and Applied Mathematics, National University of Ireland, Galway, Ireland

ARTICLE INFO

Article history:
Available online 1 October 2013

2000 Mathematics Subject Classification:
34K11
34C10

Keywords:
Opial and Wirtinger inequalities
Fourth-order differential equations
Bending of rods

ABSTRACT

In this paper, for self-adjoint fourth order differential equations, we establish some lower bounds on the distance between zeros of a nontrivial solution and also lower bounds on the distance between zeros of a solution and/or its derivatives. We also give new results related to boundary value problems which arise in the bending of rods. The main results will be proved by making use of some generalizations of Hardy, Opial and Wirtinger type inequalities.

1. Introduction

The oscillation and nonoscillation properties of the solutions of selfadjoint fourth order differential equations

$$(p(t)x''(t))'' + q(t)x(t) = 0, \tag{1.1}$$

and

$$(p(t)x''(t))'' - q(t)x(t) = 0, \tag{1.2}$$

were the subject of an extensive study in the fundamental paper of Lighton and Nehari [25] where the coefficients p and q are continuous positive functions. The investigation of the oscillatory behaviour of this type of equations originated with the vibrating rod problem of mathematical physics (see Ref. [38]). If the rod is clamped at its two endpoints $t = \alpha$ and $t = \beta$, it is well known that the deflection of the rod at time zero is an eigenfunction for the (1.2) with the boundary condition

$$x(\alpha) = x'(\alpha) = x(\beta) = x'(\beta) = 0. \tag{1.3}$$

Later these equations and their general forms have been studied extensively by other authors, we refer the reader to the papers [4,15,17–20,23,24,26,27,31,32,34–36] and the book [33] and the references cited therein. By a solution of (1.1) or (1.2) on the interval $J \subseteq I \equiv [\alpha_0, \infty)$, we mean a nontrivial real-valued function $x \in C^3(J)$, which has the property that $p(t)x''(t) \in C^2(J)$ and satisfies equation (1.1) or (1.2) on J. In this paper, we assume that (1.1) or (1.2) possesses such a nontrivial solution on I. The nontrivial solution x of (1.1) or (1.2) is said to be oscillate or to be oscillatory, if it has arbitrarily large zeros.

* Corresponding author.
E-mail addresses: shsaker@mans.edu.eg, active_5481@yahoo.com (S.H. Saker).
Peer review under responsibility of Mansoura University

Production and hosting by Elsevier

2314-808X/$ – see front matter
http://dx.doi.org/10.1016/j.ejbas.2013.09.001

An equation of the form (1.1) or (1.2) is said to be disconjugate on an interval I if no nontrivial solution has more than three zeros on I counting multiplicities. If (1.1) or (1.2) is not oscillatory (i.e., if all solutions have only finitely many zeros), then the equation is disconjugate on some interval $[\alpha_1,\infty)$ for $\alpha_1 \geq \alpha_0$ (see Ref. [25]). In general, an nth-order differential equation

$$x^{(n)}(t) + a_1(t)x^{(n-1)}(t) + \ldots + a_n(t)x(t) = 0, \tag{1.4}$$

is said to be $(k, n-k)$ disconjugate on an interval I if no nontrivial solution has a zero of order k followed by a zero of order $n-k$. This means that, for every pair of points $\alpha, \beta \in I$, $\alpha < \beta$, there does not exist a nontrivial solution of (1.4) which satisfies

$$\left.\begin{aligned} x^{(i)}(\alpha) &= 0, \quad i = 0,\ldots,k-1, \\ x^{(j)}(\beta) &= 0, \quad j = 0,\ldots,n-k-1. \end{aligned}\right\} \tag{1.5}$$

The least value of β such that there exists a nontrivial solution which satisfies (1.5), is called the $(k,n-k)$-conjugate point of α.

For equation (1.1), disconjugacy is equivalent to (3,1)-disconjugacy (which, since equation (1.1) is selfadjoint, is also equivalent to (1,3)-disconjugacy), and for equation (1.2), disconjugacy is equivalent to (2,2)-disconjugacy (see Ref. [25]). The equation (1.2) is said to be (2,2)-disconjugate on $[\alpha,\beta]$ if there is no nontrivial solution $x(t)$ and c, $d \in [\alpha,\beta]$, $c < d$ such that $x(c) = x'(c) = x(d) = x'(d) = 0$.

Our motivation in this paper comes from the old paper by C. de la Vallée Poussin [30] and the papers [10,12,14,28]. In Ref. [30] the author considered the linear nth-order differential equation (1.4) with real continuous coefficients a_j and asserts that the equation (1.4) is disconjugate on any interval sufficiently short with respect to the magnitude of the coefficients of the equation. More precisely, he proved that if $|a_j(t)| \leq b_j$ on $[\alpha,\beta]$ and the inequality

$$\sum_{j=1}^{n} \frac{b_j(\beta-\alpha)^j}{j} < 1, \tag{1.6}$$

holds, then (1.4) is disconjugate. In Ref. [37] it is shown that if x is a solution of the fourth order differential equation

$$x^{(4)}(t) + q(t)x(t) = 0, \tag{1.7}$$

which satisfies $x(\alpha) = x'(\alpha) = x(\beta) = x'(\beta) = 0$, then

$$(\beta-\alpha)^3 \geq 192 / \int_\alpha^\beta |q(t)|dt,$$

and if x satisfies $x(\alpha) = x(\beta) = x''(\alpha) = x''(\beta) = 0$, then

$$(\beta-\alpha)^2 \geq 4 / \int_\alpha^\beta |q(t)|dt.$$

In Ref. [10] the author proved that if x is a solution of (1.7) which satisfies $x(\alpha) = x(\beta) = x''(\alpha) = x''(\beta) = 0$, then

$$(\beta-\alpha)^3 \geq 16 / \int_\alpha^\beta |q(t)|dt.$$

In this paper, we obtain new lower bounds for the spacing $(\beta-\alpha)$ subject to the following boundary conditions:

$$\left.\begin{aligned} &x(\alpha) = x'(\alpha) = x''(\alpha) = x''(\beta) = 0, \\ \text{or } &x(\beta) = x'(\beta) = x''(\beta) = x''(\alpha) = 0, \end{aligned}\right\} \tag{1.8}$$

$$\left.\begin{aligned} &x(\alpha) = x'(\alpha) = x''(\alpha) = x(\beta) = 0, \\ \text{or } &x(\beta) = x'(\beta) = x''(\beta) = x(\alpha) = 0, \end{aligned}\right\} \tag{1.9}$$

and the boundary conditions

$$x(\alpha) = x''(\alpha) = x(\beta) = x''(\beta) = 0, \tag{1.10}$$

which correspond to a rod hinged or supported at both ends. We also consider the boundary conditions

$$x(\alpha) = x'(\alpha) = x''(\beta) = x'''(\beta) = 0, \tag{1.11}$$

which correspond to a rod clamped at $t = \alpha$ and free at $t = \beta$, and the boundary conditions

$$x(\beta) = x'(\beta) = x''(\alpha) = x'''(\alpha) = 0, \tag{1.12}$$

which correspond to a rod clamped at $t = \beta$ and free at $t = \alpha$.

The paper is organized as follows: In Section 2, we present some inequalities of Hardy, Opial and Wirtinger types. In Section 3, we prove several results for the equations (1.1) and (1.2) subject to the above boundary conditions. In particular, the results for the equation (1.1) will be proved in Section 3.1 subject to the boundary conditions (1.8)–(1.10). The results for the equation (1.2) will be proved in Section 3.2 subject to the boundary conditions (1.3), (1.8) and (1.11) when $p(t) < 0$. The case when (1.12) holds similar to the case when (1.11) holds and will be left to the interested reader. In Section 4, we give some illustrative examples.

2. Hardy, Opial and Wirtinger inequalities

In this section, we present the inequalities that we will need to prove the main results. For more details, we refer the reader to the books [2,21,22]. The Hardy inequality [21,22] of the differential form that we will need in this paper is given in the following theorem.

Theorem 2.1. [21,22]. If y is absolutely continuous on (α,β) with $y(\alpha) = 0$ or $y(\beta) = 0$, then the following inequality holds

$$\left(\int_\alpha^\beta q(t)|y(t)|^n dt\right)^{\frac{1}{n}} \leq C\left(\int_\alpha^\beta r(t)|y'(t)|^m dt\right)^{\frac{1}{m}}, \tag{2.1}$$

where q, r the weighted functions, are measurable positive functions in the interval (α,β) and m, n are real parameters satisfy $0 < n \leq \infty$ and $1 \leq m \leq \infty$ and the constant C satisfies

$$C \leq k(m,n)A(\alpha,\beta), \text{ for } 1 < m \leq n, \tag{2.2}$$

where $k(m,n) := n^{1/m}(m^*)^{1/m^*}$,

$$A(\alpha,\beta) := \sup_{\alpha<t<\beta}\left(\int_t^\beta q(t)dt\right)^{\frac{1}{n}}\left(\int_\alpha^t r^{1-m^*}(s)ds\right)^{1/m^*}, \text{ if } y(\alpha) = 0,$$

$$A(\alpha,\beta) := \sup_{\alpha<t<\beta}\left(\int_\alpha^t q(t)dt\right)^{\frac{1}{n}}\left(\int_t^\beta r^{1-m^*}(s)ds\right)^{1/m^*}, \text{ if } y(\beta) = 0,$$

and $m^* = m/(m-1)$.

Note that the inequality (2.1) can be considered when $y(\alpha) = y(\beta) = 0$. In this case, we see that (2.1) is satisfied with

$$A(\alpha,\beta) = \sup_{(c,d)\subset(\alpha,\beta)} \left(\int_c^d q(t)dt\right)^{\frac{1}{n}} \times \min\left\{\left(\int_\alpha^c r^{1-m^*}(s)ds\right)^{1/m^*}, \left(\int_d^\beta r^{1-m^*}(s)ds\right)^{1/m^*}\right\}. \tag{2.3}$$

The Opial inequalities that we will use in this paper are given in the following theorems.

Theorem 2.2. [2]. Assume that the functions ϑ and ϕ are non-negative and measurable on the interval (α,β), m, n are real numbers such that $\mu/m > 1$, and $0 \le k \le n-1$ $(n \ge 1)$ fixed. Let $x \in C^{(n-1)}[\alpha,\beta]$ be such that $x^{(n-1)}(t)$ is absolutely continuous on (α,β). If $x^{(i)}(\alpha) = 0$, for $k \le i \le n-1$ $(n \ge 1)$, then

$$\int_\alpha^\beta \phi(t)|x^{(k)}(t)|^l |x^{(n)}(t)|^m dt \le K_1(\alpha,\beta)\left[\int_\alpha^\beta \vartheta(t)|x^{(n)}(t)|^\mu dt\right]^{(l+m)/\mu}, \tag{2.4}$$

where

$$K_1(\alpha,\beta) := \frac{\left(\frac{m}{l+m}\right)^{\frac{m}{\mu}}}{((n-k-1)!)^l}\left[\int_\alpha^\beta \left(\phi^\mu(t)\vartheta^{-m}(t)\right)^{1/(\mu-m)} (P_{1,k}(t))^{l(\mu-1)/(\mu-m)} dt\right]^{\frac{\mu-m}{\mu}},$$

$$P_{1,k}(t) := \int_\alpha^t (t-s)^{(n-k-1)\mu/(\mu-1)} (\vartheta(s))^{-1/(\mu-1)} ds. \tag{2.5}$$

If we replace $x^{(i)}(\alpha) = 0$ by $x^{(i)}(\beta) = 0$, then (2.4) holds where K_1 is replaced by K_2 which is given by

$$K_2(\alpha,\beta) := \frac{\left(\frac{m}{l+m}\right)^{\frac{m}{\mu}}}{((n-k-1)!)^l} \times \left[\int_\alpha^\beta \left(\phi^\mu(t)\vartheta^{-m}(t)\right)^{1/(\mu-m)} (P_{2,k}(t))^{l(\mu-1)/(\mu-m)} dt\right]^{\frac{\mu-m}{\mu}}, \tag{2.6}$$

where

$$P_{2,k}(t) := \int_t^\beta (s-t)^{(n-k-1)\mu/(\mu-1)} (\vartheta(s))^{-1/(\mu-1)} ds.$$

Theorem 2.3. [3]. Let $p(t)$, $q(t)$ be non-negative measurable functions on (α,β) and $0 \le k \le n-1$ $(n \ge 1)$ fixed. If $x \in C^{n-1}[\alpha,\beta]$ is such that $x^{(i)}(\alpha) = 0$, $k \le i \le n-1$, $x^{(n-1)}$ is absolutely continuous on (α,β), then

$$\int_\alpha^\beta q(t)|x^{(k)}(t)||x^{(k+1)}(t)|dt \le C_\alpha \int_\alpha^\beta p(t)|x^{(n)}(t)|^2 dt, \tag{2.7}$$

where

$$C_\alpha := \frac{1}{2((n-k-1)!)^2} \max_{t\in[\alpha,\beta]} q(t) \int_\alpha^\beta \frac{(\beta-s)^{2(n-k-1)}}{p(s)} ds.$$

If $x \in C^{n-1}[\alpha,\beta]$ is such that $x^{(i)}(\beta) = 0$, $k \le i \le n-1$, $x^{(n-1)}$ is absolutely continuous on (α,β) then (2.7) holds with C_α is replaced by C_β where

$$C_\beta := \frac{1}{2((n-k-1)!)^2} \max_{t\in[\alpha,\beta]} q(t) \int_\alpha^\beta \frac{(s-\alpha)^{2(n-k-1)}}{p(s)} ds.$$

Theorem 2.4. [6]. If $y \in C^1[\alpha,\beta]$ with $y(\alpha) = 0$ (or $y(\beta) = 0$), then

$$\int_\alpha^\beta |y(t)|^\nu |y'(t)|^\eta dt \le N(\nu,\eta,s)(\beta-\alpha)^\nu \left(\int_\alpha^\beta |y'(t)|^s dt\right)^{\frac{\nu+\eta}{s}}, \tag{2.8}$$

where $\nu > 0$, $s > 1$, $0 \le \eta < s$,

$$N(\nu,\eta,s) := \frac{(s-\eta)\nu^\nu}{(s-1)(\nu+\eta)(I(\nu,\eta,s))^\nu}\sigma^{\nu+\eta-s}, \quad \sigma := \left\{\frac{\nu(s-1)+(s-\eta)}{(s-1)(\nu+\eta)}\right\}^{\frac{1}{s}}, \tag{2.9}$$

and

$$I(\nu,\eta,s) := \int_0^1 \left\{1 + \frac{s(\eta-1)}{s-\eta}t\right\}^{-(\nu+\eta+s\nu)/s\nu} [1+(\eta-1)t]t^{1/\nu-1} dt.$$

Note that the inequality (2.8) can be considered when $y(\alpha) = y(\beta) = 0$. Choose $c = (\alpha+\beta)/2$ and apply (2.8) to $[\alpha,c]$ and $[c,\beta]$ and then add we obtain

$$\int_\alpha^\beta |y(t)|^\nu |y'(t)|^\eta dt \le N(\nu,\eta,s)\left(\frac{\beta-\alpha}{2}\right)^\nu \left(\int_\alpha^\beta |y'(t)|^s dt\right)^{\frac{\nu+\eta}{s}}, \tag{2.10}$$

where $N(\nu,\eta,s)$ is defined as in (2.9). The inequality (2.8) can be considered when $\eta = s$ and $y(\alpha) = 0$ (or $y(\beta) = 0$). In this case the equation (2.8) becomes

$$\int_\alpha^\beta |y(t)|^\nu |y'(t)|^\eta dt \le L(\nu,\eta)(\beta-\alpha)^\nu \left(\int_\alpha^\beta |y'(t)|^\eta dt\right)^{\frac{\nu+\eta}{\eta}}, \tag{2.11}$$

where

$$L(\nu,\eta) := \frac{\eta\nu^\eta}{\nu+\eta}\left(\frac{\nu}{\nu+\eta}\right)^{\frac{\nu}{\eta}} \left(\frac{\Gamma\left(\frac{\eta+1}{\eta}+\frac{1}{\nu}\right)}{\Gamma\left(\frac{\eta+1}{\eta}\right)\Gamma\left(\frac{1}{\nu}\right)}\right)^\nu, \tag{2.12}$$

and Γ is the Gamma function.

In the following, we present a special case of the Wirtinger type inequality proved by Agarwal et al. in Ref. [1].

Theorem 2.5. [1]. For $I = [\alpha,\beta]$ and a positive function $\lambda \in C^1(I)$ with either $\lambda'(t) > 0$ or $\lambda'(t) < 0$ on I, we have

$$\int_\alpha^\beta \frac{\lambda^2(t)}{|\lambda'(t)|}|y'(t)|^2 dt \ge \frac{1}{4}\int_\alpha^\beta |\lambda'(t)||y(t)|^2 dt, \tag{2.13}$$

for any $y \in C^1(I)$ with $y(\alpha) = 0 = y(\beta)$.

If we put $y(t) = x''(t)$ with $x''(\alpha) = 0 = x''(\beta)$ and $Q(t) = \lambda'(t)$, then we have the following inequality which gives a relation between $x'''(t)$ and $x''(t)$ on the interval $[\alpha,\beta]$. For $I = [\alpha,\beta]$, then we have

$$\int_\alpha^\beta |p(t)||x'''(t)|^2 dt \ge \frac{1}{4}\int_\alpha^\beta |Q(t)||x''(t)|^2 dt, \tag{2.14}$$

for any $x \in C^3(I)$ with $x''(\alpha) = 0 = x''(\beta)$, where $p(t)$ and $Q(t)$ satisfy the equation

$$(p(t)(\lambda'(t)))' - 2Q(t)\lambda(t) = 0, \tag{2.15}$$

for any function $\lambda(t)$ satisfies $\lambda'(t) \neq 0$.

Remark 1. Note that the equation (2.15) holds if one chooses $p(t) = Q(t) = 1$, where in this case $\lambda(t) = \exp\sqrt{2}t$.

Also, the inequality (2.14) holds if $p(t) = Q(t)$. In this case the function $p(t)$ satisfies the differential equation

$$(p(t)(\lambda'(t)))' = 2p(t)\lambda(t), \tag{2.16}$$

for any function $\lambda(t)$ satisfies $\lambda(t) \neq 0$.

3. Main results

In this section, we will prove the main results. Throughout this paper in most of the results we will assume that $p(t)$ and $q(t)$ are positive function and in the case when $p(t) < 0$, we will indicate it. We will also assume throughout the paper that p is absolutely continuous on $[\alpha,\beta]$ and the appropriate integrals exist. Also, we assume throughout that there exists a differentiable function $Q(t)$ with $q(t) = Q'(t)$.

3.1. *The results for equation (1.1)*

For simplicity, we introduce the following notations:

$$\left.\begin{aligned}
\Phi_1(Q,p,P_{1,0}) &:= \frac{1}{2\sqrt{2}}\left[\int_\alpha^\beta \frac{Q^2(t)}{p(t)}P_{1,0}(t)dt\right]^{\frac{1}{2}}, \quad P_{1,0}(t) = \int_\alpha^t \frac{(t-s)^4}{p(s)}ds,\\
\Psi_1(Q,p,P_{1,1}) &:= \sqrt{\tfrac{1}{2}}\left[\int_\alpha^\beta \frac{Q^2(t)}{p(t)}P_{1,1}(t)dt\right]^{\frac{1}{2}}, \quad P_{1,1}(t) = \int_\alpha^t \frac{1}{p(s)}ds,\\
\Lambda_1(p',p,P_{1,2}) &:= \sqrt{\tfrac{1}{2}}\left[\int_\alpha^\beta \frac{(p'(t))^2}{p(t)}P_{1,2}(t)dt\right]^{\frac{1}{2}}, \quad P_{1,2}(t) = \int_\alpha^t \frac{1}{p(s)}ds,
\end{aligned}\right\} \tag{3.1}$$

and

$$\left.\begin{aligned}
\Phi_2(Q,p,P_{2,0}) &:= \frac{1}{2\sqrt{2}}\left[\int_\alpha^\beta \frac{Q^2(t)}{p(t)}P_{2,0}(t)dt\right]^{\frac{1}{2}}, \quad P_{2,0}(t) := \int_t^\beta \frac{(s-t)^4}{p(s)}ds,\\
\Psi_2(Q,p,P_{2,1}) &:= \sqrt{\tfrac{1}{2}}\left[\int_\alpha^\beta \frac{Q^2(t)}{p(t)}P_{2,1}(t)dt\right]^{\frac{1}{2}}, \quad P_{2,1}(t) = \int_t^\beta \frac{1}{p(s)}ds,\\
\Lambda_2(p',p,P_{2,2}) &:= \sqrt{\tfrac{1}{2}}\left[\int_\alpha^\beta \frac{(p'(t))^2}{p(t)}P_{2,2}(t)dt\right]^{\frac{1}{2}}, \quad P_{2,2}(t) := \int_t^\beta \frac{1}{p(s)}ds.
\end{aligned}\right\} \tag{3.2}$$

Theorem 3.1. Suppose that x is a nontrivial solution of (1.1). If $x^{(i)}(\alpha) = 0$, for $i = 0,1,2$ and $x''(\beta) = 0$, then

$$\Phi_1(Q,p,P_{1,0}) + 4\Psi_1(Q,p,P_{1,1}) + \Lambda_1(p',p,P_{1,2}) \geq 1. \tag{3.3}$$

If instead $x^{(i)}(\beta) = 0$, for $i = 0,1,2$ and $x''(\alpha) = 0$, then

$$\Phi_2(Q,p,P_{2,0}) + 4\Psi_2(Q,p,P_{2,1}) + \Lambda_2(p',p,P_{2,2}) \geq 1. \tag{3.4}$$

Proof. We prove (3.3). Multiplying (1.1) by $x''(t)$ and integrating by parts we get

$$\int_\alpha^\beta (p(t)x''(t))''x''(t)dt = (p(t)x''(t))'x''(t)|_\alpha^\beta - \int_\alpha^\beta (p(t)x''(t))'x'''(t)dt = -\int_\alpha^\beta q(t)x''(t)x(t)dt.$$

Using the assumptions $x''(\alpha) = x''(\beta) = 0$, we have

$$\int_\alpha^\beta (p(t)x''(t))'x'''(t)dt = \int_\alpha^\beta Q'(t)x''(t)x(t)dt = Q(t)x''(t)x(t)|_\alpha^\beta - \int_\alpha^\beta Q(t)x''(t)x'(t)dt - \int_\alpha^\beta Q(t)x'''(t)x(t)dt.$$

This implies after using the assumption $x''(\alpha) = x''(\beta) = 0$, that

$$\int_\alpha^\beta p(t)(x'''(t))^2dt = -\int_\alpha^\beta Q(t)x''(t)x'(t)dt - \int_\alpha^\beta Q(t)x'''(t)x(t)dt - \int_\alpha^\beta p'(t)x''(t)x'''(t)dt.$$

Hence

$$\int_\alpha^\beta p(t)|x'''(t)|^2dt \leq \int_\alpha^\beta |Q(t)||x'(t)||x''(t)|dt + \int_\alpha^\beta |Q(t)||x(t)||x'''(t)|dt + \int_\alpha^\beta |p'(t)||x''(t)||x'''(t)|dt. \tag{3.5}$$

Applying the inequality (2.4) on the integral

$$\int_\alpha^\beta |Q(t)||x(t)||x'''(t)|dt,$$

with $\phi(t) = |Q(t)|$, $\vartheta(t) = p(t)$, $m = 1$, $k = 0$, $l = 1$, $n = 3$ and $\mu = 2$, we get (note that $x^{(i)}(\alpha) = 0$, for $i = 0,1,2$) that

$$\int_\alpha^\beta |Q(t)x(t)x'''(t)|dt \leq \Phi_1(Q,p,P_{1,0})\left[\int_\alpha^\beta p(t)|x'''(t)|^2dt\right], \tag{3.6}$$

where $\Phi_1(Q, p, P_{1,0})$ is defined as in (3.1). Applying the inequality (2.4) again on the integral

$$\int_\alpha^\beta |Q(t)||x'(t)||x''(t)|dt,$$

with $\phi(t) = Q(t)$, $\vartheta(t) = p(t)$, $k = 1$, $n = 2$, $l = m = 1$ and $\mu = 2$, we see that

$$\int_\alpha^\beta |Q(t)||x'(t)||x''(t)|dt \leq \Psi_1(Q,p,P_{1,1})\left[\int_\alpha^\beta p(t)|x''(t)|^2dt\right], \tag{3.7}$$

where $\Psi_1(Q, p, P_{1,1})$ is defined as in (3.1). Applying the Wirtinger inequality (2.14) on the integral

$$\int_\alpha^\beta p(t)|x''(t)|^2 dt, \tag{3.8}$$

where $x''(\alpha) = 0 = x''(\beta)$, we see that

$$\int_\alpha^\beta p(t)|x''(t)|^2 dt \le 4\int_\alpha^\beta p(t)|x'''(t)|^2 dt, \tag{3.9}$$

where $p(t)$ satisfies the equation (2.16) for any positive function $\lambda(t)$. Substituting (3.9) into (3.7), we have

$$\int_\alpha^\beta |Q(t)||x'(t)||x''(t)|dt \le 4\Psi_1(Q,p,P_{1,1})\int_\alpha^\beta p(t)|x'''(t)|^2 dt. \tag{3.10}$$

Applying the inequality (2.4) again on the integral

$$\int_\alpha^\beta |p'(t)||x''(t)||x'''(t)|dt,$$

with $\phi(t) = |p'(t)|$, $\vartheta(t) = p(t)$, $k = 2$, $n = 3$, $l = m = 1$ and $\mu = 2$, we see that

$$\int_\alpha^\beta |p'(t)||x''(t)||x'''(t)|dt \le \Lambda_1(p',p,P_{1,2})\left[\int_\alpha^\beta p(t)|x'''(t)|^2 dt\right], \tag{3.11}$$

where $\Lambda_1(Q, p, P_{1,2})$ is defined as in (3.1). Substituting (3.6), (3.10) and (3.11) into (3.5) and cancelling the term $\int_\alpha^\beta p(t)|x'''(t)|^2 dt$, we have

$$\Phi_1(Q,p,P_{1,0}) + 4\Psi_1(Q,p,P_{1,1}) + \Lambda_1(p',p,P_{1,2}) \ge 1,$$

which is the desired inequality (3.3). The proof of (3.4) is similar by using integration by parts and the constants $\Phi_1(Q, p, P_{1,0})$, $\Psi_1(Q, p, P_{1,1})$ and $\Lambda_1(P, p, P_{1,2})$ will be replaced by $\Phi_2(Q, p, P_{2,0})$, $\Psi_2(Q, p, P_{2,1})$ and $\Lambda_2(P, p, P_{2,2})$ which are defined as in (3.2). The proof is complete.

Remark 2. Note that when $p(t)$ is a constant then the third term Λ_i for $i = 1,2$ will disappear from the results in Theorem 3.1.

In the following, we apply the inequality in Theorem 2.3 to obtain a new result by using the maximum value of $|Q|$. In this case $\Psi_1(Q, p, P_{1,1})$ and $\Psi_2(Q, p, P_{2,1})$ will be replaced by C_2 and C_2^* that we will determine below. As in the proof of Theorem 3.1, we suppose that the solution $x(t)$ of (1.1) satisfies $x'(\alpha) = x''(\alpha) = 0$. Then the application of the inequality (2.7) with $k = 1$ and $n = 3$ on the term $\int_\alpha^\beta |Q(t)||x'(t)||x''(t)|dt$, gives us

$$\int_\alpha^\beta |Q(t)||x'(t)||x''(t)|dt \le C_2\int_\alpha^\beta p(t)|x'''(t)|^2 dt, \tag{3.12}$$

where

$$C_2 := \frac{1}{2}\max_{t\in[\alpha,\beta]}|Q(t)|\int_\alpha^\beta \frac{(\beta-s)^2}{p(s)}ds.$$

If instead $x'(\beta) = x''(\beta) = 0$, then (3.12) holds where C_2 is replaced by

$$C_2^* := \frac{1}{2}\max_{t\in[\alpha,\beta]}|Q(t)|\int_\alpha^\beta \frac{(s-\alpha)^2}{p(s)}ds.$$

Using C_2 and C_2^* instead of $\Psi_1(Q, p, P_{1,1})$ and $\Psi_2(Q, p, P_{2,1})$ in the proof of Theorem 3.1, we obtain the following result.

Theorem 3.2. Suppose that x is a nontrivial solution of (1.1). If $x^{(i)}(\alpha) = 0$, for $i = 0,1,2$ and $x''(\beta) = 0$, then

$$\Phi_1(Q,p,P_{0,1}) + \frac{1}{2}\max_{t\in[\alpha,\beta]}|Q(t)|\int_\alpha^\beta \frac{(\beta-s)^2}{p(s)}ds + \Lambda_1(p',p,P_{1,2}) \ge 1.$$

If $x^{(i)}(\beta) = 0$, for $i = 0,1,2$ and $x''(\alpha) = 0$, then

$$\Phi_2(Q,p,P_{0,2}) + \frac{1}{2}\max_{t\in[\alpha,\beta]}|Q(t)|\int_\alpha^\beta \frac{(s-\alpha)^2}{p(s)}ds + \Lambda_2(p',p,P_{2,2}) \ge 1.$$

If the function $p(t)$ is non-increasing on $[\alpha,\beta]$, then we see that

$$\left.\begin{aligned} \int_\alpha^\beta \frac{(t-s)^2}{p(s)}ds &\le \frac{1}{p(\beta)}\int_\alpha^\beta (\beta-s)^2 ds \le \frac{(\beta-\alpha)^3}{3p(\beta)}, \\ \int_\alpha^\beta \frac{(s-t)^2}{p(s)}ds &\le \frac{1}{p(\beta)}\int_\alpha^\beta (s-\alpha)^2 ds \le \frac{(\beta-\alpha)^3}{3p(\beta)}. \end{aligned}\right\} \tag{3.13}$$

Substituting (3.13) into Theorem 3.2, we have the following result.

Theorem 3.3. Assume that $p(t)$ is a non-increasing function and x is a nontrivial solution of (1.1). If $x^{(i)}(\alpha) = 0$, for $i = 0,1,2$ and $x''(\beta) = 0$, then

$$\Phi_1(Q,p,P_{1,0}) + \frac{(\beta-\alpha)^3}{6p(\beta)}\max_{t\in[\alpha,\beta]}\left|\int_\alpha^t q(s)ds\right| + \Lambda_1(p',p,P_{1,2}) \ge 1.$$

If instead $x^{(i)}(\beta) = 0$, for $i = 0,1,2$ and $x''(\alpha) = 0$, then

$$\Phi_2(Q,p,P_{2,0}) + \frac{(\beta-\alpha)^3}{6p(\beta)}\max_{t\in[\alpha,\beta]}\left|\int_\alpha^t q(s)ds\right| + \Lambda_2(p',p,P_{2,2}) \ge 1.$$

In the proof of Theorem 3.1, we have applied the Wirtinger inequality (2.14) on the term (3.8). Applying the inequality (2.1) on the term (3.8) with $y(t) = x''(t)$ (where $x''(\alpha) = x''(\beta) = 0$), we see that

$$\int_\alpha^\beta p(t)|x''(t)|^2 dt \le \Delta^2\int_\alpha^\beta p(t)|x'''(t)|^2 dt, \tag{3.14}$$

where $\Delta^2 = 4A_*^2(\alpha,\beta)$, and

$$A_*(\alpha,\beta) = \sup_{(c,d)\subset(\alpha,\beta)}\left(\int_c^d p(t)dt\right)^{\frac{1}{n}} \times\min\left\{\left(\int_\alpha^c \frac{1}{p(s)}ds\right)^{1/2}, \left(\int_d^\beta \frac{1}{p(s)}ds\right)^{1/2}\right\}.$$

Now, we can use the inequality (3.14) in the proof of Theorem 3.1 to obtain new results but in this case the constant 4 in front of the coefficient Ψ will be replaced by Δ^2. The details will be left to the interested reader.

In the following, we will apply the Boyd inequality in Theorem 2.4. By applying the Schwarz inequality

$$\int_\alpha^\beta |f(t)g(t)|dt \le \left(\int_\alpha^\beta |f(t)|^2 dt\right)^{\frac{1}{2}} \times \left(\int_\alpha^\beta |g(t)|^2 dt\right)^{\frac{1}{2}}, \tag{3.15}$$

on the term

$$\int_\alpha^\beta |Q(t)||x'(t)||x''(t)|dt,$$

we see that

$$\int_\alpha^\beta |Q(t)||x'(t)||x''(t)|dt \le \left(\int_\alpha^\beta |Q(t)|^2 dt\right)^{\frac{1}{2}} \left(\int_\alpha^\beta |x'(t)|^2|x''(t)|^2 dt\right)^{\frac{1}{2}}. \tag{3.16}$$

Now, by applying the inequality (2.11) on the integral

$$\int_\alpha^\beta |x'(t)|^2|x''(t)|^2 dt,$$

with $\nu = \eta = 2$ and $y = x'$ (note that $x'(\alpha) = 0$), we see that

$$\int_\alpha^\beta |x'(t)|^2|x''(t)|^2 dt \le \frac{4(\beta-\alpha)^2}{\pi^2 p^2(\beta)}\left[\int_\alpha^\beta p(t)|x''(t)|^2 dt\right]^2, \tag{3.17}$$

where we assumed that $p(t)$ is a non-increasing function (note that the inequality (3.17) is also valid if $x'(\beta) = 0$). Substituting (3.17) into (3.16), we have

$$\int_\alpha^\beta |Q(t)||x'(t)||x''(t)|dt \le \frac{2}{\pi}\frac{(\beta-\alpha)}{p(\beta)}\left(\int_\alpha^\beta |Q(t)|^2 dt\right)^{\frac{1}{2}} \int_\alpha^\beta p(t)|x''(t)|^2 dt.$$

Again applying the Wirtinger inequality (2.14) on the integral

$$\int_\alpha^\beta p(t)|x''(t)|^2 dt,$$

where $x''(\alpha) = 0 = x''(\beta)$, we have

$$\int_\alpha^\beta p(t)|x''(t)|^2 dt \le 4\int_\alpha^\beta p(t)|x'''(t)|^2 dt,$$

where $p(t)$ satisfies the equation (2.16) for any positive function $\lambda(t)$. This implies that

$$\int_\alpha^\beta |Q(t)||x'(t)||x''(t)|dt \le \frac{8}{\pi}\frac{(\beta-\alpha)}{p(\beta)}\left(\int_\alpha^\beta |Q(t)|^2 dt\right)^{\frac{1}{2}} \int_\alpha^\beta p(t)|x'''(t)|^2 dt. \tag{3.18}$$

Using this inequality and proceeding as in the proof of Theorem 3.1, we obtain the following result.

Theorem 3.4. Assume that $p(t)$ is a non-increasing function and x is a nontrivial solution of (1.1). If $x^{(i)}(\alpha) = 0$, for $i = 0,1,2$ and $x''(\beta) = 0$, then

$$\Phi_1(Q,p,P_{1,0}) + \frac{8}{\pi}\frac{(\beta-\alpha)}{p(\beta)}\left(\int_\alpha^\beta |Q(t)|^2 dt\right)^{\frac{1}{2}} + \Lambda_1(p',p,P_{1,2}) \ge 1.$$

If $x^{(i)}(\beta) = 0$, for $i = 0,1,2$ and $x''(\alpha) = 0$, then

$$\Phi_2(Q,p,P_{2,0}) + \frac{8}{\pi}\frac{(\beta-\alpha)}{p(\beta)}\left(\int_\alpha^\beta |Q(t)|^2 dt\right)^{\frac{1}{2}} + \Lambda_2(p',p,P_{2,2}) \ge 1.$$

In the following, we apply the Opial inequality due to Bessack and Das [5] to obtain new results for (1.1) subject to the boundary conditions (1.10). This inequality is a generalization of the classical Opial inequality [29] and states that if y is absolutely continuous on $[a,b]$ with $y(a) = 0$, then the following inequality holds

$$\int_a^b B(t)|y(t)|^m|y'(t)|^n dt \le K_1(m,n)\int_a^b A(t)|y'(t)|^{m+n} dt, \tag{3.19}$$

where m, n are real numbers such that $mn > 0$ and $m+n > 1$, A and B are nonnegative, measurable functions on (a,b) such that $\int_a^b (A^{-1/(m+n-1)}(s)ds < \infty$, and

$$K_1(m,n) := \left(\frac{n}{n+m}\right)^{\frac{n}{n+m}}\left[\int_a^b \frac{B^{\frac{n+m}{m}}(t)}{A^{\frac{n}{m}}(t)}\left(\int_a^t \left(A^{\frac{-1}{(m+n-1)}}(s)ds\right)^{m+n-1} dt\right]^{\frac{m}{m+n}}. \tag{3.20}$$

If we replace $y(a) = 0$ by $y(b) = 0$, then (3.19) holds where $K_1(m,n)$ is replaced by

$$K_2(m,n) := \left(\frac{n}{n+m}\right)^{\frac{n}{n+m}}\left[\int_a^b \frac{B^{\frac{n+m}{m}}(t)}{A^{\frac{n}{m}}(t)}\left(\int_t^b \left(A^{\frac{-1}{(m+n-1)}}(s)ds\right)^{m+n-1} dt\right]^{\frac{m}{m+n}}. \tag{3.21}$$

Note that the inequality (3.19) can be considered when $y(a) = y(b)$ =10. In this case we will assume that there exists $\tau \in (a,b)$ such that

$$\int_\tau^b \left(A^{\frac{-1}{(m+n-1)}}(s)ds = \int_a^\tau \left(A^{\frac{-1}{(m+n-1)}}(s)ds. \tag{3.22}$$

In this case the inequality (3.19) holds with a new constant $K(m,n)$ which is given from the equation

$$K(m,n) = K_1(m,n) = K_2(m,n),$$

when (3.22) is satisfied. In the following, we assume that there exists $\tau \in (\alpha,\beta)$ such that

$$\int_\tau^\beta p^{-1}(s)ds = \int_\alpha^\tau p^{-1}(s)ds. \tag{3.23}$$

and assume that

$$K^*(p',p) = K_1(1,1) = K_2(1,1), \tag{3.24}$$

where

$$K_1(1,1) := \frac{1}{\sqrt{2}}\left[\int_\alpha^\beta \frac{|p'(t)|^2}{p(t)}\int_\alpha^t p^{-1}(s)dsdt\right]^{\frac{1}{2}},$$

$$K_2(1,1) := \frac{1}{\sqrt{2}}\left[\int_\alpha^\beta \frac{|p'(t)|^2}{p(t)}\int_t^\beta p^{-1}(s)dsdt\right]^{\frac{1}{2}}.$$

Theorem 3.5. Assume that $p(t)$ is a non-increasing function. Suppose that x is a nontrivial solution of (1.1). If $x(\alpha) = x''(\alpha) = x(\beta) = x''(\beta) = 0$, then

$$\frac{(\beta-\alpha)^{7/2}}{72p(\beta)}\left(\int_\alpha^\beta |q(t)|^2 dt\right)^{1/2} + K^*(p',p) \geq 1, \tag{3.25}$$

where $K^*(p',p)$ is defined as in (3.24).

Proof. Multiplying (1.1) by $x''(t)$ and proceeding as in the proof of Theorem 3.1 we get

$$\int_\alpha^\beta p(t)|x'''(t)|^2 dt \leq \int_\alpha^\beta |q(t)||x(t)||x''(t)|dt + \int_\alpha^\beta |p'(t)||x''(t)||x'''(t)|dt. \tag{3.26}$$

Applying the inequality the Schwarz inequality (3.15) on the integral

$$\int_\alpha^\beta |q(t)||x(t)||x''(t)|dt \leq \left(\int_\alpha^\beta |q(t)|^2 dt\right)^{1/2}\left(\int_\alpha^\beta |x(t)|^2|x''(t)|^2 dt\right)^{1/2}.$$

Applying the inequality ([11], Theorem 4.5]) (note that $x(\alpha) = x(\beta) = 0$)

$$\left(\int_\alpha^\beta |x(t)|^2|x''(t)|^2 dt\right)^{1/2} \leq \frac{(\beta-\alpha)^{3/2}}{12}\int_\alpha^\beta |x''(t)|^2 dt,$$

we get that

$$\left(\int_\alpha^\beta |q(t)|^2\right)^{1/2}\left(\int_\alpha^\beta |x(t)|^2|x''(t)|^2 dt\right)^{1/2} \leq \frac{(\beta-\alpha)^{3/2}}{12}\left(\int_\alpha^\beta |q(t)|^2 dt\right)^{1/2}\int_\alpha^\beta |x''(t)|^2 dt, \tag{3.27}$$

Applying the Wirtinger-inequality, see Brnetić and Pečarić [7],

$$\int_\alpha^\beta y^2(t)dt \leq \frac{(\beta-\alpha)^2}{6}\int_\alpha^\beta (y'(t))^2 dt, \tag{3.28}$$

for any $y \in C^1[\alpha,\beta]$ and $y(\alpha) = y(\beta) = 0$, with $y(t) = x''$ (note that $x''(\alpha) = 0 = x''(\beta)$) and the assumption that p is a non-increasing function, we have

$$\frac{(\beta-\alpha)^{3/2}}{12}\left(\int_\alpha^\beta |q(t)|^2 dt\right)^{1/2}\int_\alpha^\beta |x''(t)|^2 dt \leq \frac{(\beta-\alpha)^{7/2}}{72p(\beta)}\left(\int_\alpha^\beta |q(t)|^2 dt\right)^{1/2}\int_\alpha^\beta p(t)|x'''(t)|^2 dt. \tag{3.29}$$

This implies that

$$\int_\alpha^\beta |q(t)||x(t)||x''(t)|dt \leq \frac{(\beta-\alpha)^{7/2}}{72p(\beta)}\left(\int_\alpha^\beta |q(t)|^2 dt\right)^{1/2}\int_\alpha^\beta p(t)|x'''(t)|^2 dt. \tag{3.30}$$

Applying the Opial inequality (3.19) on the integral (note that $x''(\alpha) = 0 = x''(\beta)$)

$$\int_\alpha^\beta |p'(t)||x''(t)||x'''(t)|dt,$$

with $B(t) = |p'(t)|$ and $A(t) = p(t)$, and $y(t) = x''(t)$, we see that

$$\int_\alpha^\beta |p'(t)||x''(t)||x'''(t)|dt \leq K^*(p',p)\int_\alpha^\beta p(t)|x'''(t)|^2 dt, \tag{3.31}$$

where $K^*(p',p)$ is defined as in (3.24). Substituting (3.30) and (3.31) into (3.26) and cancelling the term $\int_\alpha^\beta p(t)|x'''(t)|^2$, we obtain the desired inequality (3.25). The proof is complete.

As a special case when $p(t) = 1$ in Theorem 3.5, we have the following result.

Corollary 3.1. Let x is a nontrivial solution of

$$x^{(4)}(t) + q(t)x(t) = 0, \quad t \in [\alpha,\beta],$$

which satisfies $x(\alpha) = x''(\alpha) = x(\beta) = x''(\beta) = 0$. Then

$$(\beta-\alpha)^7 \geq 5184 / \int_\alpha^\beta q^2(t)dt. \tag{3.32}$$

Remark 3. One can also obtain new results by multiplying (1.1) by $p(t)x''(t)$ and considering the case when $p(t) < 0$. In this case after integrating by parts, we have

$$\begin{aligned}\int_\alpha^\beta (p(t)x''(t))''p(t)x''(t)dt &= (p(t)x''(t))'p(t)x''(t)|_\alpha^\beta \\ &- \int_\alpha^\beta (p'(t)x''(t) + p(t)x'''(t))^2 dt \\ &= -\int_\alpha^\beta p(t)q(t)x''(t)x(t)dt.\end{aligned}$$

Using the assumption $x''(\alpha) = x''(\beta) = 0$, we have

$$\begin{aligned}\int_\alpha^\beta p^2(t)|x'''(t)|^2 dt = -2\int_\alpha^\beta p(t)p'(t)x''(t)x'''(t)dt \\ - \int_\alpha^\beta (p'(t))^2(x''(t))^2 dt \\ + \int_\alpha^\beta Q_1'(t)x(t)x''(t)dt,\end{aligned}$$

where $Q_1(t)$ is the antiderivative of $p(t)q(t)$. Integrating by parts the last term in the right hand side, we see that

$$\begin{aligned}\int_\alpha^\beta Q_1'(t)x''(t)x(t)dt = Q_1(t)x(t)x(t)|_\alpha^\beta - \int_\alpha^\beta Q_1(t)x'(t)x''(t)dt \\ - \int_\alpha^\beta Q_1(t)x(t)x'''(t)dt.\end{aligned}$$

Using the assumption $x''(\beta) = x''(\alpha) = 0$, we see that

$$\int_\alpha^\beta Q_1'(t)x(t)x''(t)dt = -\int_\alpha^\beta Q_1(t)x'(t)x''(t)dt - \int_\alpha^\beta Q_1(t)x(t)x'''(t)dt.$$

Hence we obtain

$$\int_\alpha^\beta p^2(t)|x'''(t)|^2dt \le \int_\alpha^\beta |Q_1(t)||x(t)||x'''(t)|dt$$
$$+\int_\alpha^\beta |Q_1(t)||x'(t)||x''(t)|dt$$
$$+2\int_\alpha^\beta |p(t)p'(t)||x''(t)||x'''(t)|dt$$
$$+\int_\alpha^\beta |p'(t)|^2|x''(t)|^2dt.$$

One can apply the inequalities in Section 2 to establish new results. This will be left to the interested reader.

3.2. *The results for equation (1.2)*

We begin with the boundary conditions $x(\alpha)=x'(\alpha)=x(\beta)=x'(\beta)=0$, which correspond to a rod clamped at each end.

Theorem 3.6. Suppose that x is a nontrivial solution of (1.2). If $x(\alpha)=x'(\alpha)=x(\beta)=x'(\beta)=0$, then

$$\max_{t\in[\alpha,\beta]}\left|\int_\alpha^t q(s)ds\right|\int_\alpha^\beta \frac{(\beta-s)^2}{p(s)}ds \ge 1. \tag{3.33}$$

Proof. Multiplying (1.2) by $x(t)$ and integrating by parts, we have

$$\int_\alpha^\beta (p(t)x'''(t))''x(t)dt = x(t)(p(t)x''(t))'\Big|_\alpha^\beta - \int_\alpha^\beta x'(t)(p(t)x''(t))'dt = \int_\alpha^\beta q(t)x^2dt. \tag{3.34}$$

Using the assumptions that $x(\alpha)=x(\beta)=0$ and $Q'(t)=q(t)$, we get that

$$\int_\alpha^\beta x'(t)(p(t)x''(t))'dt = -\int_\alpha^\beta Q'(t)x^2(t)dt. \tag{3.35}$$

Integrating by parts the right hand side, we see that

$$\int_\alpha^\beta Q'(t)x^2(t)dt = Q(t)x^2(t)\Big|_\alpha^\beta - 2\int_\alpha^\beta Q(t)x(t)x'(t)dt.$$

Using the assumption $x(\alpha)=x(\beta)=0$, we see that

$$\int_\alpha^\beta Q'(t)x^2(t)dt = -2\int_\alpha^\beta Q(t)x(t)x'(t)dt. \tag{3.36}$$

Integrating by parts the left hand side of (3.35), we see that

$$\int_\alpha^\beta x'(t)(p(t)x''(t))'dt = p(t)x'(t)x''(t)\Big|_\alpha^\beta - \int_\alpha^\beta p(t)(x''(t))^2dt. \tag{3.37}$$

Using the assumption $x'(\alpha)=x'(\beta)=0$, we have

$$\int_\alpha^\beta x'(t)(p(t)x''(t))'dt = -\int_\alpha^\beta p(t)(x''(t))^2dt. \tag{3.38}$$

Substituting (3.36) and (3.38) into (3.35), we have

$$\int_\alpha^\beta p(t)|x''(t)|^2dt \le 2\int_\alpha^\beta |Q(t)||x(t)||x'(t)|dt. \tag{3.39}$$

Applying the inequality (2.7) on the integral

$$\int_\alpha^\beta |Q(t)||x(t)||x'(t)|dt,$$

with $p(t)=Q(t)$, $k=0$ and $n=2$, we see that

$$\int_\alpha^\beta |Q(t)||x(t)||x'(t)|dt \le \left(\frac{1}{2}\max_{t\in[\alpha,\beta]}|Q(t)|\int_\alpha^\beta\frac{(\beta-s)^2}{p(s)}ds\right)\int_\alpha^\beta p(t)|x''(t)|^2dt, \tag{3.40}$$

where $x(\alpha)=x'(\alpha)=0$ (or $x(\beta)=x'(\beta)=0$). Substituting (3.40) into (3.39) and cancelling the term $\int_\alpha^\beta p(t)|x''(t)|^2dt$, we have

$$\max_{t\in[\alpha,\beta]}\left|\int_\alpha^t q(s)ds\right|\int_\alpha^\beta\frac{(\beta-s)^2}{p(s)}ds \ge 1,$$

which is the desired inequality (3.33). The proof is complete.

Note that when $p(t)$ is nonincreasing, we see that the inequalities in (3.13) are satisfied. Using these two inequalities in Theorem 3.6 give us the following result.

Corollary 3.2. Suppose that x is a nontrivial solution of (1.2), $p(t)$ is nonincreasing. If $x(\alpha)=x'(\alpha)=x(\beta)=x'(\beta)=0$, then

$$(\beta-\alpha)^3 \ge 3p(\beta)\Big/\max_{t\in[\alpha,\beta]}\left|\int_\alpha^t q(s)ds\right|. \tag{3.41}$$

Remark 4. Corollary 3.2 gives us a condition for (2,2)-disconjugacy of (1.2). In particular, if

$$\max_{t\in[\alpha,\beta]}\left|\int_\alpha^t q(s)ds\right| < \frac{3p(\beta)}{(\beta-\alpha)^3}, \tag{3.42}$$

then (1.2) is (2,2)-disconjugate in $[\alpha,\beta]$. This means that there is no nontrivial solution of (1.2) in $[\alpha,\beta]$ satisfies $x(\alpha)=x'(\alpha)=x(\beta)=x'(\beta)=0$.

Theorem 3.7. Assume that $p(t)$ is nonincreasing. If x is a nontrivial solution of (1.2) which satisfies $x(\alpha)=x'(\alpha)=x(\beta)=x'(\beta)=0$, then

$$\max_{t\in[\alpha,\beta]}|Q(t)| \ge \frac{96p(\beta)}{(\beta-\alpha)^3}. \tag{3.43}$$

Proof. Proceeding as in the proof of Theorem 3.7 to obtain

$$\int_\alpha^\beta p(t)|x''(t)|^2dt \le 2\max_{t\in[\alpha,\beta]}|Q(t)|\int_\alpha^\beta |x(t)||x'(t)|dt. \tag{3.44}$$

Applying the inequality (see ([8], Inequality (5.8)])) on the integral

$$\int_\alpha^\beta |x(t)||x'(t)|dt,$$

we see (note that $x(\alpha) = x'(\alpha) = 0 = x(\beta) = x'(\beta) = 0$) that

$$\int_\alpha^\beta |x(t)||x'(t)|dt \le \frac{(\beta-\alpha)^3}{192p(\beta)} \int_\alpha^\beta p(t)|x''(t)|^2 dt. \tag{3.45}$$

Substituting (3.45) into (3.44) and cancelling the term $\int_\alpha^\beta p(t)|x''(t)|^2 dt$, we have

$$\max_{t\in[\alpha,\beta]} |Q(t)| \frac{(\beta-\alpha)^3}{96p(\beta)} \ge 1,$$

which is the desired inequality (3.43). The proof is complete.

From Theorem 3.7 and Lemma 3.1, we have the following result.

Corollary 3.3. Assume that $p(t)$ is nonincreasing. If x is a nontrivial solution of (1.2) which satisfies $x(\alpha) = x'(\alpha) = x(\beta) = x'(\beta) = 0$, then

$$\int_\alpha^\beta |q(t)|dt \ge \frac{192p(\beta)}{(\beta-\alpha)^3}. \tag{3.46}$$

Remark 5. The contrapositive of the result in Corollary 3.4 yields a sufficient condition for (2,2)-disconjugacy of the equation (1.2).

In the following, we consider the boundary conditions $x(\alpha) = x'(\alpha) = x''(\beta) = x'''(\beta) = 0$ which correspond to a beam hinged or supported at both ends. The proof will be as in the proof of Theorem 3.6, by using these boundary conditions and gives us the following result.

Theorem 3.8. Suppose that x is a nontrivial solution of (1.2). If $x(\alpha) = x'(\alpha) = x''(\beta) = x'''(\beta) = 0$, then

$$\max_{t\in[\alpha,\beta]} \left| \int_\alpha^t q(s)ds \right| \int_\alpha^\beta \frac{(\beta-s)^2}{p(s)} ds \ge 1.$$

Corollary 3.4. Suppose that x is a nontrivial solution of (1.2), $p(t)$ is nonincreasing. If $x(\alpha) = x'(\alpha) = x''(\beta) = x'''(\beta) = 0$, then

$$(\beta-\alpha)^3 \ge 3p(\beta) / \max_{t\in[\alpha,\beta]} \left| \int_\alpha^t q(s)ds \right|. \tag{3.47}$$

From Corollary 3.4 and the arguments before Corollary 3.1, we have the following result.

Corollary 3.5. Assume that $p(t)$ is nonincreasing. If x is a nontrivial solution of (1.2) which satisfies $x(\alpha) = x'(\alpha) = x''(\beta) = x'''(\beta) = 0$, then

$$\int_\alpha^\beta |q(t)|dt \ge \frac{6p(\beta)}{(\beta-\alpha)^3}.$$

Next, in the following, we establish some results which allow us to consider the case when $p(t) < 0$. For simplicity, we denote

$$\left.\begin{aligned} K_1^*(|p'(t)|, P_{1,2}) &:= \tfrac{1}{\sqrt{2}}\left[\int_\alpha^\beta |p'(t)|^2 P_{1,2}(t)dt\right]^{\frac12}, \\ K_2^*(|p'(t)|, P_{2,2}) &:= \tfrac{1}{\sqrt{2}}\left[\int_\alpha^\beta |p'(t)|^2 P_{2,2}(t)dt\right]^{\frac12}, \end{aligned}\right\} \tag{3.48}$$

where

$$P_{1,1}(t) := \int_\alpha^t \frac{1}{p^2(s)}ds,\ P_{1,2}(t) := \int_\alpha^t \frac{1}{p^2(s)}ds,$$

$$P_{2,1}(t) := \int_t^\beta \frac{1}{p^2(s)}ds,\ P_{2,2}(t) := \int_t^\beta \frac{1}{p^2(s)}ds.$$

and $\Delta_1 = 4(A^*(\alpha,\beta))^2$ where

$$\begin{aligned} A^*(\alpha,\beta) = &\sup_{(c,d)\subset(\alpha,\beta)} \left(\int_c^d p^2(t)dt\right)^{\frac12} \\ &\times \min\left\{\left(\int_\alpha^c \frac{1}{p(s)}ds\right)^{1/2}, \left(\int_d^\beta \frac{1}{p(s)}ds\right)^{1/2}\right\}, \end{aligned} \tag{3.49}$$

and $\Delta_2 = 4(A^{**}(\alpha,\beta))^2$ where

$$\begin{aligned} A^{**}(\alpha,\beta) = &\sup_{(c,d)\subset(\alpha,\beta)} \left(\int_c^d |p'(t)|dt\right)^{\frac12} \\ &\times \min\left\{\left(\int_\alpha^c \frac{1}{p(s)}ds\right)^{1/2}, \left(\int_d^\beta \frac{1}{p(s)}ds\right)^{1/2}\right\}. \end{aligned} \tag{3.50}$$

Theorem 3.9. Suppose that x is a nontrivial solution of (1.2) and there exists a function $Q_1 \in C^1[\alpha,\beta]$ such that $Q_1' = pq$. If $x^{(i)}(\alpha) = 0$, for $i = 0,1,2$ and $x''(\beta) = 0$, then

$$\Phi_1(Q_1, p^2, P_{1,0}) + \Delta_1 \Psi_1(Q_1, p^2, P_{1,1}) + K_1^*(|p'(t)|, P_{1,2}) + \Delta_2 \ge 1. \tag{3.51}$$

If $x^{(i)}(\beta) = 0$, for $i = 0,1,2$ and $x''(\alpha) = 0$, then

$$\Phi_2(Q_1, p^2, P_{2,0}) + \Delta_1 \Psi_2(Q_1, p^2, P_{2,1}) + K_2^*(|p'(t)|, P_{2,2}) + \Delta_2 \ge 1. \tag{3.52}$$

Proof. We prove (3.51). Multiply (1.2) by $p(t)x''(t)$. In this case after integrating by parts, we have

$$\begin{aligned} \int_\alpha^\beta (p(t)x''(t))'' p(t)x''(t)dt &= (p(t)x''(t))' p(t)x''(t)\Big|_\alpha^\beta \\ &- \int_\alpha^\beta (p'(t)x''(t) + p(t)x'''(t))^2 dt \\ &= \int_\alpha^\beta p(t)q(t)x''(t)x(t)dt. \end{aligned}$$

Using the assumption $x''(\alpha) = x''(\beta) = 0$, we have

$$\int_{\alpha}^{\beta} p^2(t)|x'''(t)|^2 dt \le 2\int_{\alpha}^{\beta} |p(t)p'(t)||x''(t)||x'''(t)|dt + \int_{\alpha}^{\beta} (p'(t))^2(x''(t))^2 dt + \int_{\alpha}^{\beta} |Q_1'(t)||x(t)||x''(t)|dt.$$

Integrating by parts the last term in the right hand side, we see that

$$\int_{\alpha}^{\beta} |Q_1'(t)||x''(t)||x(t)|dt = |Q_1(t)||x''(t)||x(t)||_{\alpha}^{\beta} - \int_{\alpha}^{\beta} |Q_1(t)||x'(t)||x''(t)|dt - \int_{\alpha}^{\beta} |Q_1(t)||x(t)||x'''(t)|dt.$$

Using the assumption $x''(\beta) = x''(\alpha) = 0$, we see that

$$\int_{\alpha}^{\beta} |Q_1'(t)||x''(t)||x(t)|dt = -\int_{\alpha}^{\beta} |Q_1(t)||x'(t)||x''(t)|dt - \int_{\alpha}^{\beta} |Q_1(t)||x(t)||x'''(t)|dt.$$

Hence we obtain

$$\int_{\alpha}^{\beta} p^2(t)|x'''(t)|^2 dt \le \int_{\alpha}^{\beta} |Q_1(t)||x(t)||x'''(t)|dt + \int_{\alpha}^{\beta} |Q_1(t)||x'(t)||x''(t)|dt + 2\int_{\alpha}^{\beta} |p(t)p'(t)||x''(t)||x'''(t)|dt + \int_{\alpha}^{\beta} |p'(t)|^2|x''(t)|^2 dt. \tag{3.53}$$

Applying the inequality (2.4) on the integral

$$\int_{\alpha}^{\beta} |Q_1(t)||x(t)||x'''(t)|dt,$$

with $\phi(t) = |Q_1(t)|$, $\vartheta(t) = p^2(t)$, $m = 1$, $k = 0$, $l = 1$, $n = 3$ and $\mu = 2$, we get (note that $x^{(i)}(\alpha) = 0$, for $i = 0,1,2$) that

$$\int_{\alpha}^{\beta} |Q_1(t)||x(t)||x'''(t)|dt \le \Phi_1(Q_1, p^2, P_{1,0})\left[\int_{\alpha}^{\beta} p^2(t)|x'''(t)|^2 dt\right], \tag{3.54}$$

where $\Phi_1(Q_1, p^2, P_{1,0})$ is defined as in (3.1) and Q is replaced by Q_1. Applying the inequality (2.4) again on the integral

$$\int_{\alpha}^{\beta} |Q_1(t)||x'(t)||x''(t)|dt,$$

with $\phi(t) = Q_1(t)$, $\vartheta(t) = p^2(t)$, $k = 1$, $n = 2$, $l = m = 1$ and $\mu = 2$, we see that

$$\int_{\alpha}^{\beta} |Q_1(t)||x'(t)||x''(t)|dt \le \Psi_1(Q_1, p^2, P_{1,1})\int_{\alpha}^{\beta} p^2(t)|x''(t)|^2 dt, \tag{3.55}$$

where $\Psi_1(Q_1, p^2, P_{1,1})$ is defined as in (3.2) and Q is replaced by Q_1 and p is replaced by p^2. Applying the inequality (2.1) on the term (3.8) with $y(t) = x''(t)$ (where $x''(\alpha) = x''(\beta) = 0$), we see that

$$\int_{\alpha}^{\beta} p^2(t)|x''(t)|^2 dt \le \Delta_1 \int_{\alpha}^{\beta} p^2(t)|x'''(t)|^2 dt, \tag{3.56}$$

where $\Delta_1 = 4(A^*(\alpha,\beta))^2$ and $A^*(\alpha,\beta)$ is defined as in (3.49). Substituting (3.56) into (3.55), we have

$$\int_{\alpha}^{\beta} |Q_1(t)||x'(t)||x''(t)|dt \le \Delta_1\Psi_1(Q_1, p^2, P_{1,1})\int_{\alpha}^{\beta} p^2(t)|x'''(t)|^2 dt. \tag{3.57}$$

Applying the inequality (2.4) on the integral

$$\int_{\alpha}^{\beta} |p(t)p'(t)||x''(t)||x'''(t)|dt$$

with $\phi(t) = |.p(t)p'(t)|.$, $\vartheta(t) = p^2(t)$, $m = 1$, $k = 2$, $l = 1$, $n = 3$ and $\mu = 2$, we get (note that $x^{(i)}(\alpha) = 0$, for $i = 0,1,2$) that

$$\int_{\alpha}^{\beta} |p(t)p'(t)||x''(t)||x'''(t)|dt \le K_1^*(|p'(t)|, P_{1,2})\int_{\alpha}^{\beta} p^2(t)|x'''(t)|^2 dt, \tag{3.58}$$

where $K_1^*(|p'(t)|, P_{1,2})$ is defined as in (3.48). Applying the inequality (2.1) on the term $\int_{\alpha}^{\beta} |p'(t)|^2|x''(t)|^2 dt$ with $y(t) = x''(t)$ (where $x''(\alpha) = x''(\beta) = 0$), we see that

$$\int_{\alpha}^{\beta} |p'(t)|^2|x''(t)|^2 dt \le \Delta_2 \int_{\alpha}^{\beta} p^2(t)|x'''(t)|^2 dt, \tag{3.59}$$

where $\Delta_2 = 4(A^{**}(\alpha,\beta))^2$ and $A^{**}(\alpha,\beta)$ is defined as in (3.50). Substituting (3.54), (3.57) and (3.58) and (3.59) into (3.53) and cancelling the term $\int_{\alpha}^{\beta} p^2(t)|x'''(t)|^2 dt$, we have

$$\Phi_1(Q_1, p^2, P_{1,0}) + \Delta_1\Psi_1(Q_1, p^2, P_{1,1}) + 2K_1^*(|p'(t)|, P_{1,2}) + \Delta_2 \ge 1,$$

which is the desired inequality (3.51). The proof of (3.52) is similar to (3.51) by using the integration by parts and the constants

$$\Phi_1(Q_1, p^2, P_{1,0}),\ \Psi_1(Q_1, p^2, P_{1,1}),\ K_1^*(|p'(t)|, P_{1,2}),$$

are replaced by

$$\Phi_2(Q_1, p^2, P_{2,0}),\ \Psi_2(Q_1, p^2, P_{2,1}),\ K_2^*(|p'(t)|, P_{2,2}),$$

which are defined as in (3.1) and (3.48). The proof is complete.

4. Examples

The following examples illustrate the results.

Example 1. Consider the equation

$$x^{(4)}(t) + \lambda|\cos(\alpha t)|x(t) = 0, 0 \le t \le \pi, \tag{4.1}$$

where λ and α are positive constants. If x is a solution of (4.1) with $x(0) = x''(0) = x(\pi) = x''(\pi) = 0$, we see from Corollary 3.1 that

$$\frac{\lambda}{2}\int_0^{\pi}[1+\cos(2\alpha t)]dt = \frac{\lambda}{2}\left[\pi + \frac{1}{2\alpha}\sin(2\alpha\pi)\right] \geq \frac{5184}{\pi^7} = 1.7164.$$

provided that $\lambda \geq 1$. Then the condition (3.32) reads

$$\frac{\lambda\pi}{2} + \frac{\lambda}{4\alpha} \geq 1.7164. \tag{4.2}$$

for any $\lambda \geq 1$ and $\alpha > 0$.

Example 2. Consider the equation

$$x''''(t) - \frac{\lambda}{t^4}x(t) = 0, \ \alpha \leq t \leq \beta, \tag{4.3}$$

where λ is a positive constant and x is a solution of (4.3) which satisfies $x(0) = x'(0) = x(\beta) = x'(\beta) = 0$. Then the condition (3.46) implies that

$$\int_\alpha^\beta \frac{\lambda}{t^4}dt = \frac{\lambda(\beta^3 - \alpha^3)}{3\alpha^3\beta^3} > \frac{192}{(\beta-\alpha)^3},$$

which gives us that

$$\frac{(\beta-\alpha)^3(\beta^3-\alpha^3)}{\alpha^3\beta^3} > \frac{576}{\lambda}$$

This implies that (4.3) is disconjugate on $[\alpha,\beta]$ if

$$\lambda < \frac{576\alpha^3\beta^3}{(\beta-\alpha)^3(\beta^3-\alpha^3)}.$$

REFERENCES

[1] Agarwal RP, Bohner M, O'Regan D, Saker SH. Some Wirtinger-type inequalities on time scales and their applications. Pac J Math 2011;252:1–26.

[2] Agarwal RP, Pang PYH. Opial inequalities with applications in differential and difference equations. Dordrecht: Kluwer; 1995.

[3] Agarwal RP, Pang PYH. Sharp Opial-type inequalities involving higher order derivatives of two functions. Math Nachr 1995;174:5–20.

[4] Barrett JH. Disconjugacy of a self-adjoint differential equations of the fourth order. Pac J Math 1961;11:25–37.

[5] Bessack PR, Das KM. Extensions of Opial's inequality. Pac J Math 1968;26:215–32.

[6] Boyd D. Best constants in class of integral inequalities. Pac J Math 1969;30:367–83.

[7] Brnetić I, Pečarić J. Some new Opial-type inequalities. Math Inequal Appl 1998;3:385–90.

[8] Brown R, Hinton D. Lyapunov inequalities and their applications. In: Rassias T, editor. Survey on classical inequalities, edited by T. Dordrecht: Kluwer; 2000. p. 1–25.

[10] Cakmak D. Lyapunov-type integral inequalities for certain higher order differential equations. Appl Math Comp 2010;216:368–73.

[11] Clark S, Hinton D. Some disconjugacy criteria for differential equations with oscillatory coefficients. Math Nachr 2005;278:1476–89.

[12] Clark S, Hinton D. A Liapunov inequality for Hamiltonian systems. Math Inequal Appl 1998;1:201–9.

[14] Elias U. Necessary conditions and sufficient conditions for disfocality and disconjugacy of a differential equation. Pac J Math 1979;81:379–97.

[15] Erbe L. Hill-Wintner type comparison theorem for selfadjoint fourth order linear differential equations. Proc Am Math Soc 1980;80:417–22.

[17] Fišnarova S. Oscillatory properties of fourth order self-adjoint differential equations. Arch Math 2004;40:457–69.

[18] Howard H. Oscillation criteria for fourth-order linear differential equations. Trans Am Math Soc 1960;96:296–311.

[19] Jones GR, Rankin SM. Oscillation properties of a certain self-adjoint differential equations of the fourth order. Pac J Math 1976;63:179–84.

[20] Kreith K. Comparison theorems for a class of selfadjoint fourth order differential equations. Proc Am Math Soc 1977;67:67–73.

[21] Kufner A, Persson LE. Weighted inequalities of Hardy type. Singapore: World Scientific Publishing; 2003.

[22] Kufner A, Maligranda L, Persson LE. The Hardy inequalities: about its history and some related results. Pilsen; 2007.

[23] Kura T. Existence of oscillatory solutions for fourth order superlinear ordinary differential equations. Hiroshima Math J 1983;13:653–64.

[24] Kusano T, Manojlović J, Tanigawa T. Sharp oscillation criteria for a class of fourth order nonlinear differential equations. Rocky Mount J Math 2011;41:249–74.

[25] Lighton W, Nehari Z. On the oscillation of solutions of self-adjoint linear differential equations of the fourth order. Trans Am Math Soc 1958;89:325–77.

[26] Naito M, Wu F. On the existence of eventually positive solutions of fourth-order quasilinear differential equations. Nonlinear Anal 2004;57:253–63.

[27] Naito M, Wu F. A note on the existence and asymptotic behavior of nonoscillatory solutions of fourth order quasilinear differential equations. Acta Math Hung 2004;102:177–202.

[28] Nehari Z. Green's functions and disconjugacy. Arch Ration Mech Anal 1976;62:53–76.

[29] Opial Z. Sur uné inegalité. Ann Pol Math 1960;8:29–32.

[30] de la Valle Poussin C. Sur l'equation differentielle du second ordre. J Math Pur et Appl 1929;9:125–44.

[31] Ou CH, Wong JSW. Oscillation and nonoscillation theorems for superlinear Emden-Fowler equations of the fourth order. Ann Mat 2004;183:25–43.

[32] Saker SH, Agarwal RP, O'Regan D. New gaps between zeros of fourth-order differential equations via Opial inequalities. J Inequal Appl 2012;1:1–19.

[33] Swanson CA. Comparison and oscillation theory of linear differential equations. New York: Academic Press; 1968.

[34] Taylor Jr WE. On oscillatory and asymptotic behavior of solutions of a certain fourth order linear differential equations. Hiroshima Math J 1977;3:667–74.

[35] Tanigawa T. Oscillation and nonoscillation theorems for a class of fourth order quasilinear functional differential equations. Hiroshima Math J 2003;33:297–316.

[36] Thandapani E, Savitri R. Oscillation and nonoscillation of fourth order nonlinear neutral differential equations. Indian J Pure Appl Math 2001;32:1631–42.

[37] Yang X. On a Liapunov-type inequality for a certain higher-order differential equations. Appl Math Comp 2003;134:307–17.

[38] Zill DG, Wright WR, Cullen MR. Advanced engineering mathematics. 4th ed. Canada: Jones and Bartlett Publishers; 2011.