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Fragments of Monadic Second-Order Logics Over Word Structures

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Abstract

In this paper, we explore the expressive power of fragments of monadic second-order logic enhanced with some generalized quantifiers of comparison of cardinality over finite word structures.
The full monadic second-order fragment of the logics that we study correspond to the famous linear hierarchy, see [10], and their existential fragments characterize some sequential recognizers. We prove that the first-order closure of the existential fragments of these logics is strictly beyond the existential fragments.

Keywords: Descriptive complexity, Monadic Second-Order Logic, Generalized Quantifiers.

1 Introduction

In the early sixties, Büchi, Elgot and Trakhtenbrot [3,7,22] proved that a "finite words" language is recognized by a finite automaton if, and only if, it is the class of "finite word structures" satisfying a MSO sentence. Since then, monadic second-order logic (MSO for short) have been intensively explored as one of the cornerstones of logic in computer science, see the expository paper [21] for historical background and state of art. Analogous results have been proved for infinite words, finite trees, infinite trees, and traces, see [21]. In all these cases, monadic second-order logic have the same expressive power than its existential fragment ($\exists MSO$ for short).

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In the finite graphs topic, R. Fagin proved that the existential fragment of MSO is strictly less expressive than MSO. The famous problem of directed graphs connectivity is one of the problems (queries in databases terminology) that separates these classes, see [13,5].

In his thesis [11], the author improved a result of T. Schwentick by proving that either if we allow generalized quantifiers of cardinality comparison (in some restricted form), over ordered graphs, graph connectivity is still not expressible in $\exists MSO$.

Ajtai and *al*, and Matz, in [1,18], have introduced and studied some fragments of MSO that contains $\exists MSO$. One of these fragments is the first-order closure of $\exists MSO$. It is the set of prenex formulas whose monadic second-order variables are existentially quantified and alternated by quantified first-order (universally, or existentially) variables. This class is denoted $FO(\exists MSO)$. In [1], it has been proved that $\exists MSO \subsetneq FO(\exists MSO) \subsetneq MSO$ over finite graphs. There were also affirmed that $FO(\exists MSO)$ is a "natural" extension of $\exists MSO$.

In this paper, we prove analogous results in the aim of finite word structures where we allow generalized quantifiers of cardinality comparison of quantified sets. These logics are monadic second-order logic enhanced by partial order constraints over the cardinality of quantified sets. We will use for this purpose Logical characterizations of some classes of languages and some complexity results.

This paper is organized as follows: In the next section, we introduce the logics we explore and recall the descriptive complexity results needed in the sequel. In section 3, we study the fragments of MSO augmented with some partial order over quantified subsets. In section 4, we study the expressive power of fragments of monadic second-order logic enhanced by the famous Rescher and Härtig quantifiers over finite word structures. In the conclusion of the paper we give some remarks over the results and some directions to explore.

2 Preliminary definitions and results

Let's first define how to identify words with logical structures, see for example [21,13].

Definition 2.1 We associate with each word $w = w_0 \dots w_{n-1}$ over the alphabet Σ, the *word structure* S_w, namely the relational structure $S_w = ([n], <, (P_a)_{a\in\Sigma})$, where $[n] = \{0, \dots, n-1\}$, $<$ is the linear order on $[n]$, and P_a is the unary predicate collecting the positions of w labeled a:

$$P_a = \{i \in [n] | w_i = a\}.$$

In the case of binary word structures, i.e. $\Sigma = \{0, 1\}$, we need a single predicate collecting the positions labeled 1.

$$P = \{i \in [n] | w_i = 1\}.$$

In the rest of this section, we define the enhanced monadic second-order logics we study in the next sections. We suppose the reader familiar with monadic second-order logic and its existential fragment, see [5,13,21] for detailed definitions. We after recall some descriptive complexity results that we need in the separation results.

Definition 2.2 Let $MSO(\leq_g, =_g)$ be the monadic-second order logic over word structures where atomic formulas are of the one of the forms $x = y, x < y, P(x), U(x), U =_g V$ and $U \leq_g V$, for some individual variables or constants x, y and set variables U, V.

The semantic of this logic is the natural one for monadic second-order logic, and the interpretations of $X =_g Y$ and $X \leq_g Y$ are *partial orders between subsets of the universe* $[n]$, as introduced in [19]:

$$([n], \cdots) \models X \leq_g Y \text{ iff for all } m < n,\ |X \cap [m]| \leq |Y \cap [m]|,$$

and

$$([n], \cdots) \models X =_g Y \text{ iff } X \leq_g Y \text{ and } |X| = |Y|.$$

Example 2.3 Let $\Sigma = \{(,)\}$. A word $w = u_0 \cdots u_{n-1}$ over Σ is a sequence of well balanced parentheses (a word of the two symbols Dyck language, see [12]) if, and only if:

$$S_w \models \{i < n | u_i = `)'\} =_g \{i < n | u_i = `('\}.$$

Let $MSO(Q_r)$ (resp. $MSO(Q_h)$) be monadic second-order logic where atomic formulas are of the form $x = y, x < y, U(x), P(x)$ and $Q_r(U, V)$ (resp. $Q_h(U, V)$), for individual variables or constants x, y and set variables U, V. The semantics of these logics are the same as monadic second-order logic where:

- Q_r is interpreted as the *Rescher quantifier* Q_r, or the majority of cardinality quantifier, defined by:

$$Q_r(X, Y) \equiv |X| < |Y|.$$

- Q_h is the *Härtig quantifier* Q_h, also called equicardinality quantifier, defined by:

$$Q_h(X, Y) \equiv |X| = |Y|.$$

Petri nets were introduced in the aim to study concurrency. This model is also studied as a model of sequential computing. It is this last point of view that interests us in this paper. Naturally, this sequential model has been compared to classical models of the Chomsky hierarchy, we will denote this class PNL. It has been proved that:

$$Reg \subsetneq PNL \subsetneq CS$$

where Reg (resp CS) denotes the class of regular languages (resp context sensitive). For a detailed introduction, and motivations that led to the study of the sequential behavior of Petri nets, see for example, chapter 6 of [20], the article [19] or [11].

Result 2.4 (Parigot and Pelz [19]) *Let L be a language over an alphabet Σ. The following are equivalent:*

(i) *L is a Petri net language;*

(ii) *L is defined by a sentence of $MSO(=_g, \leq_g)$ of the form $\exists \overline{X} \phi(\overline{X})$, where $\phi(\overline{X})$ is $MSO(=_g, \leq_g)$-first-order formula over $\Sigma \cup \overline{X}$;*

(iii) *L is defined by a sentence of $MSO(=_g, \leq_g)$ of the form $\exists \overline{X} \phi(\overline{X})$, where $\phi(\overline{X})$ is a positive combination[2] of formulas of the form $X =_g Y$ and first-order formulas in which $=_g$ and $\leq_g$ do not occur.*

In order to define non regular languages, we have to use a strictly more powerful logic than MSO. In the other hand, if we add a quantified binary predicate expresses all context free and some NP-complete languages, following a recent result of Eiter, Gottlob and Gurevich [6]. In [6], the authors proved that a prefix class of second-order logic, either expresses only regular languages, or defines some NP-complete problem. Furthermore, they proved that NP-hardness is present in formulas of the form $\exists R\ \phi$, for some binary predicate R and a first-order formula ϕ of the appropriate prefix. Lautemann, Schwentick and Thérien [16] chose a semantic approach in order to characterize the class of context free languages. They confined the binary second-order predicate to be a matching, i.e. an order preserving, non-crossing relation.

Definition 2.5 *A binary relation M over a word structure is called a* matching *if it satisfies the following conditions :*

(i) $\forall ij[(i, j) \in M \Rightarrow i < j]$.

[2] This means that we only use $\wedge$ and $\vee$ in the construction of formulas.

(ii) $\forall ij[(i,j) \in M \Rightarrow \forall k \neq i, j((i,k), (k,i), (j,k), \textit{ and } (k,j) \text{ are not in } M)]$.

(iii) $\forall ijkl[(i,j),(k,l) \in M \Rightarrow (i < k < j \rightarrow i < l < j)]$.

Let Match *denote the class of matchings on word structures.*

Let S be any word structure, $S \models \exists^{Match} M\, \phi$ means : there exists a relation $M \in Match$ such that $(S, M) \models \phi$.

Example 2.6 Suppose that the positions in P are opening parentheses. The formula:

$$\exists^{Match} M \forall x \forall y \exists z((M(x,z) \vee M(z,x)) \wedge (M(x,y) \rightarrow (P(x) \wedge \neg P(y))))$$

defines the two letters Dyck language.

Result 2.7 *A finite word language is context free if, and only if, it is the class of models of a formula of the form $\exists^{Match} M\ \phi$, where ϕ is a first-order formula using M.*

By combining this result with a result of Book and Greibach [2] which states: *"A language is in $NTime[n]$ if, and only if, it is the projection of a finite intersection of context free languages"*, Lautemann, Schwentick et Schweikardt [15] cited the following result:

Result 2.8 *Over binary word structures:*

$$NTime[n] = \exists^{Match} M_1 \dots M_k \exists \overline{R}(\phi_1 \wedge \dots \wedge \phi_k) = \exists^{Match} M_1 M_2 M_3 \exists \overline{R}(\phi_1 \wedge \phi_2 \wedge \phi_3)$$

where the M_i's are restricted to be matchings and the only binary relations in ϕ_i are M_i and $<$.

Proof. (of $\supseteq$) Let L be a language defined by a formula of the form:

$$\exists^{Match} M_1 \cdots M_k \exists \overline{R}(\phi_1 \wedge \cdots \wedge \phi_k)$$

with the condition that the only binary relations in ϕ_i are M_i and $<$, and all the R_j's are unary variables. We first guess the unary relations $\bar{R}$ in non-deterministic linear time. We assume these sets as part of the augmented signature. *We treat the R_i's as new letters of the alphabet, with minor modifications in order to have that one position have at most one label.*

We finally evaluate the formulas $\exists M_i \phi_i$, which describe context free languages by result 2.7, in non-deterministic linear time. The other direction is close to the one exhibited in [15] for a slightly modified logic. □

Definition 2.9 *We define the* first-order closure of $\exists MSO(=_g, \leq_g)$ *(resp. of $\exists MSO(Q_h, Q_r)$), and we note: $FO(\exists MSO(=_g, \leq_g))$ (resp. $FO(\exists MSO(Q_h, Q_r))$,* *as the set of prenex formulas of* $MSO(=_g, \leq_g)$ *(resp. of* $MSO(Q_h, Q_r)$ *) in which we authorize alternations between first-order quantifiers and existential monadic second-order ones. These are formulas of the form:*

$$\exists X \forall x \exists Y \cdots \theta$$

where upper case letters stand for set variables, lower case one for individual variables and θ *is a quantifier free formula.*

3 $FO(\exists MSO(=_g, \leq_g))$ versus $\exists MSO(=_g, \leq_g)$

In this section, we prove that $FO(\exists MSO(=_g, \leq_g))$ is strictly more expressive than $\exists MSO(=_g, \leq_g)$ over finite word structures. To this aim, we prove that $FO(\exists MSO(=_g, \leq_g))$ defines all context free languages, while it is known, see [20], that there are context free languages that are not Petri net languages (which corresponds to $\exists MSO(=_g, \leq_g)$ by [19]).

Theorem 3.1 *Over finite word structures:*

$$\exists MSO(=_g, \leq_g) \subsetneq FO(\exists MSO(=_g, \leq_g)).$$

In order to prove this result, let's prove first that the context free languages are definable in $FO(\exists MSO(=_g, \leq_g))$.

Theorem 3.2 *Over finite word structures:*

$$CFL \subseteq FO(\exists MSO(=_g, \leq_g)).$$

Proof. By a result of [16], characterizing CFL by sentences of $\exists Match\, FO$, it suffices to prove that the formulas in $\exists Match\, FO$ are expressible in $FO(\exists MSO(=_g, \leq_g))$. Let L a context free language definable by the formula:

$$\Phi_L \equiv \exists^{Match} M \phi(M),$$

The formula α_M :

$$\exists X_1 X_2 (X_1 =_g X_2 \wedge \forall x \neg (X_1(x) \wedge X_2(x))),$$

of $FO(\exists MSO(=_g, \leq_g))$, ensure us of the existence of two disjoint subsets of positions that could be interpreted as, the set of opening parentheses:

$$X_2 \equiv \{x | \exists y M(x, y)\},$$

and the set of closing parentheses:

$$X_1 \equiv \{x | \exists y M(y, x)\},$$

of the matching M. We after express in $FO(\exists MSO(=_g, \leq_g))$, the fact that two positions , x and y, are linked by the matching M, represented by the sets X_1 et X_2. This is made by the formula $\mu(x, y)$:

$$\exists Y_1 Y_2 \forall z[(Y_1(z) \leftrightarrow (X_1(z) \wedge x < z < y)) \\ \wedge (Y_2(z) \leftrightarrow (X_2(z) \wedge x < z < y)) \wedge (X_2(x) \wedge X_1(y) \wedge Y_1 =_g Y_2)].$$

In fact, this formula mimic an elementary algorithm of finding the closing parenthese of each opening one. In order to have uniquely existential second-order quantifiers, the occurences of the atomic formulas of the form $M(x, y)$ must be all positive. For this aim, we express $\neg M(x, y)$ by a positive formula in M.

$$\neg M(x, y) \equiv \neg X_2(x) \vee \neg X_1(y) \vee \exists z(z \neq y \wedge M(x, z)).$$

Our translation will be as follows:

We begin by replacing the negative occurrences of $M(x, y)$ in the initial formula by formulas where M occur only positively as given above.

After, we replace $\exists M$ by α_M, and the occurences of $M(x, y)$ by $\mu(x, y)$. We can easily check that this produces a logically equivalent $FO(\exists MSO(=_g, \leq_g))$-formula. □

As a direct consequence of this result and the one of Book and Greibach [2,15], see also result 2.8:

Corollary 3.3 *Over binary finite word structures:*

$$NTime[n] \subseteq FO(\exists MSO(=_g, \leq_g)).$$

Proof. (of corollary 3.3) In result 2.8, it is proved that:

$$NTime[n] = \exists^{Match} M_1, M_2, M_3 \exists \bar{R}(\phi_1 \wedge \phi_2 \wedge \phi_3),$$

where, for $i = 1,\ 2,\ 3$, $\phi_i \in FO[M_i, \bar{R}]$, and the R_i are unary. By the previous theorem, and the closure of $FO(\exists MSO(=_g, \leq_g))$ by conjunction (which expresses intersection) and by existential monadic second-order quantification (which expresses projection), we obtain the required claim. □

Proof. (of theorem 3.1) It is known that $CFL \not\subset PNL$, see for example [20]. This result is equivalent to $\exists Match\ FO \not\subset \exists MSO(=_g, \leq_g)$, by results of [16] and [19]. By theorem 3.2

$$CFL = \exists Match\ FO \subseteq FO(\exists MSO(=_g, \leq_g)).$$

We conclude:

$$\exists MSO(=_g, \leq_g) \subsetneq FO(\exists MSO(=_g, \leq_g)).$$

□

4 Fragments of $MSO(Q_h, Q_r)$ over word structures

In this section, we first investigate the expressive power of $\exists MSO(Q_h, Q_r)$. We give an upper bound for this class which is $NTime[n]$, we after give it a lower bound which is the union of the class of bounded context free languages and the regular ones.

After that, we prove that this class is strictly included in its first-order closure.

Theorem 4.1 *Over binary finite word structures:*

$$\exists MSO(Q_h, Q_r) \subseteq NTime[n].$$

Lemma 4.2 *Each formula of $\exists MSO(Q_h, Q_r)$ is equivalent to a one in which Q_h and Q_r occur only positively.*

Proof. We replace, in each formula, the occurrences of $\neg Q_h(X, Y)$ by

$$Q_r(X, Y) \vee Q_r(Y, X)$$

and we replace the occurrences of $\neg Q_r(X, Y)$ by

$$Q_h(X, Y) \vee Q_r(Y, X).$$

After this procedure, we obtain the desired result. □

Proof. (of theorem 4.1) We begin by replacing each formula of $\exists MSO(Q_h, Q_r)$ by an equivalent one in $\exists MSO(Q_h, Q_r)$ in which Q_h and Q_r appear only positively, as in the previous lemma.

Next, we replace the occurrences of atomic formulas of the form $Q_h(Z, T)$ by the sentence: $\exists M \forall xy[(M(x, y) \vee M(y, x)) \rightarrow ((Z(x) \wedge \neg T(x) \wedge T(y) \wedge \neg Z(y)) \vee ((T(x) \wedge \neg Z(x) \wedge Z(y) \wedge \neg T(y)))] \wedge \forall x((Z(x) \wedge \neg T(x)) \rightarrow \exists y(M(x, y) \vee M(y, x))) \wedge \forall x((T(x) \wedge \neg Z(x)) \rightarrow \exists y(M(x, y) \vee M(y, x)))$. This is possible because we always can find a bijection whose graph edges do not cross (as in the construction of parentheses). We associate with each element satisfying $Z(x) \wedge \neg T(x)$ the least element such that:

> $T(y) \wedge \neg Z(y)$ and there is as many elements satisfying Z and T between these positions and *vice versa*.

We after replace the occurrences of atomic formulas of the form $Q_r(Z, T)$ by the sentence: $\exists M \forall xy[(M(x, y) \vee M(y, x)) \rightarrow ((Z(x) \wedge \neg T(x) \wedge T(y) \wedge \neg Z(y)) \vee ((T(x) \wedge \neg Z(x) \wedge Z(y) \wedge \neg T(y)))] \wedge \forall x((Z(x) \wedge \neg T(x)) \rightarrow \exists y(M(x, y) \vee M(y, x))) \wedge \exists x((T(x) \wedge \neg Z(x)) \wedge \forall y \neg (M(x, y) \vee M(y, x)))$. Who corresponds to a non crossing bijection between a proper subset of $T \wedge \neg Z$ and $Z \wedge \neg T$. Because,

M appear only in a single translation, we can, via an appropriate renaming, put the $\exists M_i$'s in the beginning of the prenex formula. This formula is so in the form given in Result 2.8. We conclude that $\exists MSO(Q_h, Q_r) \subseteq NTime[n]$. □

Theorem 4.3 *Over binary finite word structures:*

$$\exists MSO(Q_h, Q_r) \not\subseteq CFL.$$

Proof. Let $L = \{a^n b^n c^n | n \in \mathbb{N}\}$ be the non context free language over $\Sigma = \{a, b, c\}$, see [12]. The $\exists MSO(Q_h, Q_r)$-formula:

$$\begin{gathered}\exists XYZ\exists xy\forall t[(X(t) \leftrightarrow t \leq x) \wedge (X(t) \leftrightarrow P_a(t)) \wedge \\ (Y(t) \leftrightarrow x < t \leq y) \wedge (Y(t) \leftrightarrow P_b(t)) \wedge \\ (Z(t) \leftrightarrow y < t) \wedge (Z(t) \leftrightarrow P_c(t)) \wedge Q_h(X, Y) \wedge Q_h(Y, Z)].\end{gathered}$$

defines L. We conclude that CFL does not contain $\exists MSO(Q_h, Q_r)$. □

Definition 4.4 *The class of bounded context free languages, denoted $BCFL$, introduced by Ginsburg in 1966 (see [20] pages 181-182) is the least class such that:*

(i) *Finite languages are in $BCFL$;*

(ii) *If L_1 and L_2 are in $BCFL$, then so are $L_1 L_2$ and $L_1 \cup L_2$;*

(iii) *If L is in $BCFL$, and u, v are finite words of Σ^*, then:*

$$\{u^i L v^i | i \geq 0\},$$

is in $BCFL$.

Remark: The class $BCFL$ does not contain all regular languages because Σ^* is not in $BCFL$ if Σ contains at least two letters. The class $BCFL$ is not included in the class of regular languages because $L = \{a^n b^n | n \in \mathbb{N}\}$, which is not regular, is in $BCFL$. □

Theorem 4.5 *Over finite word structures, $\exists MSO(Q_h, Q_r)$ contains regular languages and $BCFL$.*

Proof. Over finite word structures, the class of regular languages correspond to $\exists MSO$, see [3,21], so obviously, $Reg \subseteq \exists MSO(Q_h, Q_r)$. Let us show that $BCFL \subseteq \exists MSO(Q_h, Q_r)$ by structural induction over the construction of $BCFL$.

Basis. Let ϕ_w be the first-order sentence which is satisfied by the word $w = w_1 \cdots w_n$. So finite languages are definable by a finite disjunction of such formulas.

Closure operations. For union, it is easy to check whether the disjunction of sentences in $\exists MSO(Q_h, Q_r)$ is a sentence in $\exists MSO(Q_h, Q_r)$. For concatenation, it is a simple relativization of variables. Let X be a set (a unary predicate). We define $S_X(i,j)$, the successor relation relative to X by:

$$i < j \wedge X(i) \wedge X(j) \wedge \forall k(i < k < j \rightarrow \neg X(k)).$$

Let $Max_X(i)$ the predicate stating that i is the greatest element satisfying X, and let $Min_X(i)$ the predicate stating that i is the least element satisfying X. Let $\Phi_L \in \exists MSO(Q_h, Q_r)$ the formula defining L. We will mark the elements forming the last letter of u by a predicate X, and those forming the first letter of v by Y. The formula defining $\{u^i L v^i | i \in \mathbb{N}\}$ expresses that there are such sets X, Y and positions x, y such that $Max_X(x)$ and $Min_Y(y)$, and if $S_X(i,j)$ then $\phi_u(i+1,j)$, and if $S_Y(i,j)$ then $\phi_v(i,j-1)$.

$$\begin{gathered}\exists XY \exists xy[(x \leq y) \wedge Max_X(x) \wedge Min_Y(y) \wedge \\ \forall ij(S_X(i,j) \rightarrow \phi_u(i+1,j)) \wedge \phi_u(min, Min_X) \\ \wedge \forall ij(S_Y(i,j) \rightarrow \phi_v(i,j-1)) \wedge \phi_v(Max_Y, max)]\end{gathered}$$

$Q_h(X,Y)$ states that the word is of the form $u^i m v^i$ with $m \in \Sigma^*$. We finally relativize the formula Φ_L to the integer interval $[x+1, y-1]$. □

Definition 4.6 *Let's define the logic $\exists MSO^+(Q_h, Q_r)$. The formulas of this logic are the same as those of $\exists MSO(Q_h, Q_r)$ with the authorization to have atomic formulas of the form:*

$$Q_h(\phi(x), \psi(x)) \text{ and } Q_r(\phi(x), \psi(x)),$$

for first order formulas ϕ and ψ, over the signature augmented by the monadic predicates, having a single free variable and parameters.

Theorem 4.7 *Over finite word structures:*

$$NTime[n] \subseteq \exists MSO^+(Q_h, Q_r).$$

Proof. The formula :

$$Q_h(0 \leq i \leq x, y \leq i \leq z),$$

defines $z = x + y$. In this formula, x, y and z are parameters, they are so free in this atomic formula. We then have $\exists MSO(+) \subseteq \exists MSO^+(Q_h, Q_r)$. Using a result of Lynch in [17], we obtain:

$$NTime[n] \subseteq \exists MSO(+) \subseteq \exists MSO^+(Q_h, Q_r).$$

□

Here is a state of art of the results involved and proved in this section:

$$\exists MSO(Q_h, Q_r) \subseteq NTime[n] \subseteq NLIN \subseteq \exists MSO(+) \subseteq \exists MSO^+(Q_h, Q_r)$$

The inclusion $NLIN \subseteq MSO(+)$ is a result of Grandjean and Olive [9].

Theorem 4.8 *Over finite word structures:*

$$\exists MSO^+(Q_h, Q_r) \subseteq FO(\exists MSO(Q_h, Q_r))$$

Proof. The first step, will be to replace the formulas of $\exists MSO^+(Q_h, Q_r)$ by formulas in which Q_h and Q_r appear only positively, because in the proof of lemma 4.2 we can replace $\exists MSO(Q_h, Q_r)$ by $\exists MSO^+(Q_h, Q_r)$ without changing the proof. After we replace the occurrences of $Q_h(\phi, \psi)$ by

$$\exists X \exists Y \forall x((X(x) \leftrightarrow \phi(x)) \wedge (Y(x) \leftrightarrow \psi(x)) \wedge Q_h(X, Y))$$

for new (not previously used) variables X, Y and x. We do the same for $Q_r(\phi, \psi)$ by

$$\exists X \exists Y \forall x((X(x) \leftrightarrow \phi(x)) \wedge (Y(x) \leftrightarrow \psi(x)) \wedge Q_r(X, Y))$$

Because these variables do not occur in the other subformulas, we can put the quantifiers $\exists X \exists Y \forall x$ in front of the first-order formula and we obtain a formulas in $FO(\exists MSO(Q_h, Q_r))$. For example, let the formula

$$\forall x \forall y \exists z(\neg Q_h(0 \leq i \leq x, y \leq i \leq z))$$

that states that for any x and y, there is a z which is different from $x + y$. We begin by replacing the negative occurrence of Q_h and obtain:

$$\forall x \forall y \exists z(Q_r(0 \leq i \leq x, y \leq i \leq z) \vee Q_r(y \leq i \leq z, 0 \leq i \leq x))$$

By the second procedure, we have:

$$\begin{aligned}\forall x \forall y \exists z \exists X \exists Y \forall i((X(i) \leftrightarrow 0 \leq i \leq x) \wedge (Y(i) \leftrightarrow y \leq i \leq z)\\ \wedge(Q_r(X, Y) \vee Q_r(Y, X)))\end{aligned}$$

$\square$

It is also proved in [14] that:

Result 4.9 $\exists MSO(Q_h, Q_r)$ *does not express Petri Net languages.*

We then conclude that

$$\exists MSO(Q_h, Q_r) \subsetneq FO(\exists MSO(Q_h, Q_r))$$

5 Conclusion

In [1], the authors proved that in the presence of a binary predicate in the signature:

$$\exists MSO \subsetneq FO(\exists MSO) \subsetneq MSO$$

In a recent paper [10], the author proved that over binary word structures:

$$MSO(+) = MSO(=_g, \leq_g) = MSO(Q_r) = MSO(Q_h) = MSO(Maj) = LinH$$

It is an easy exercise to prove that :

$$FO(\exists MSO(+)) = FO(\exists MSO(=_g, \leq_g)) = FO(\exists MSO(Q_r, Q_h))$$

We then have that the existential fragments and all alternation classes of these logics correspond to "natural" complexity classes, which legitimate their study as "natural" classes. Does the classes $FO(\exists MSO)$ also correspond to some machine model? By the definition of these fragments, in all cases:

$$\exists MSO \subseteq FO(\exists MSO) \subseteq MSO$$

It will be interesting for further works, to investigate in which cases these inclusions are strict. Some consequences over collapse of complexity classes can be given.

In his PhD [18], Oliver Matz studied a finer hierarchy than MSO and compared it to the classical MSO hierarchy. We can try to set analogous results in the context of this work.

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