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Subclasses of Circular-Arc Bigraphs: Helly, Normal and Proper

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Abstract

The class of circular-arc graphs, as well as its Helly, normal, and proper subclasses, has been extensively studied in the literature. Circular-arc bigraphs, a bipartite variation on circular-arc graphs, remains a relatively new field, with only a few studies on the class and its proper and unit subclasses existing. In this paper, we introduce a Helly subclass for circular-arc bigraphs, based on the concept of bipartite-Helly families, and provide a polynomial-time recognition algorithm for it. We also introduce an alternative proper circular-arc bigraph subclass to the one in the literature, as well as two different normal circular-arc bigraph subclasses based on the definition of normal circular-arc graphs. We present containment relations between the different proper and normal classes.

Keywords: Biclique, Circular-arc, recognition, Helly, Normal, Proper

1 Introduction

A circular-arc graph is the intersection graph of a family of arcs on a circle. The class has been extensively studied since the second half of the 20th century [9, 18], and characterizations and linear-time recognition algorithms for the class [14] as well as many subclasses [13] have been discovered, such as the Helly [12, 17] and proper subclasses [3, 7], as well as intersection classes such as proper Helly [10] and normal Helly [11].

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Circular-arc bigraphs arise as a bipartite variation of circular-arc graphs. A bipartite graph is said to be a circular-arc bigraph if there exists a one-to-one correspondence between its vertices and a family of arcs on a circle such that, for every pair of vertices in different parts, they are neighbors precisely if their corrsesponding arcs intersect. The class is not very widely studied, although matrix-based characterizations of the class and its unit and proper subclasses exist [1], as well as a polynomial-time recognition algorithm for the proper subclass [2]. Currently no mention of a Helly or normal subclass of circular-arc bigraphs exists in the literature.

In this paper, we define the Helly subclass of circular-arc bigraphs, and provide a polynomial-time recognition algorithm of the class over graphs without isolated vertices. We also present an alternative proper circular-arc bigraph class, alongside two different definitions for normal circular-arc bigraphs, and demonstrate containments between the classes.

2 Definitions and Notation

In this paper, we denote bipartite graphs as triples (V, W, E) with V, W being the graph's partite sets and E being its edge set. If $G = (V, W, E)$ is a bipartite graph, we say that V and W are *opposite* partite sets to each other. For any graph G, we denote by G^* the graph resulting from adding, to G, an isolated vertex. For any integer $n > 0$, C_n denotes the cycle on n vertices.

A graph G is *complete* if, for every $v, w \in V(G)$, $vw \in E(G)$. A *clique* of a graph G is a maximal subset $K \subseteq V(G)$ such that the subgraph induced by K is a complete graph. Similarly, a bipartite graph $H = (V, W, E)$ is *bipartite-complete* if, for every $v \in V, w \in W$, $vw \in E(H)$. A *biclique* of a graph G is a maximal subset $K \subseteq V(G)$ which induces a bipartite-complete graph. To simplify notation, we refer to the set of bicliques of G as $b(G)$, and the set of bicliques of G that contain a vertex $v \in V(G)$ as $b_G(v)$. The graph subscript is omitted when the referred graph is clear from context.

For any graph G, define the *square* of G as $G^2 = (V(G), E^2(G))$, where $E^2(G) = \{vw | vw \in E(G) \vee \exists x \in V(G) : vx, xw \in E(G)\}$. If two vertices $v, w \in V(G)$ are such that $N(v) = N(w)$, then v and w are called *twins*. Vertices of equal open neighborhoods are commonly called *false twins* in the literature, with vertices of equal closed neighborhood being called twins, but since our study is on bipartite graphs, we use *twins* to refer to graphs of equal open neighborhood. The *twin-free version* of graph G is the graph that results from removing, from every set of twins, every vertex but one, and then repeating the process until no twins remain.

To simplify notation, we treat the circular arc intersection models of graphs as *circular-arc models* as defined by [17]. We also introduce our analogous definition of *bi-circular-arc models*.

A *circular-arc model* is a pair $(C, \mathbb{A})$ such that C is a circle, and $\mathbb{A}$ is a family of arcs over C. The *corresponding graph* of model $(C, \mathbb{A})$ is the intersection graph of $\mathbb{A}$.

A graph G is a circular-arc graph if and only if it is the corresponding graph of a circular-arc model. If $(C, \mathbb{A})$ is a circular-arc model for which G is the corresponding graph, we say that G *admits* model $(C, \mathbb{A})$. Note that there exists a one-to-one correspondence between $V(G)$ and $\mathbb{A}$ such that, for every pair of vertices $v, w \in V(G)$, $vw \in E(G)$ if and only if the arcs corresponding to the vertices intersect.

A *bi-circular-arc model* is a triple $(C, \mathbb{I}, \mathbb{E})$ such that C is a circle, and $\mathbb{I}, \mathbb{E}$ are arcs over C. The corresponding graph of a bi-circular-arc model is built by creating a vertex v_A for each arc $A \in \mathbb{I} \cup \mathbb{E}$ and, for every pair of arcs $I \in \mathbb{I}, E \in \mathbb{E}$, an edge $v_I v_E$ is added if and only if $I \cap E \neq \emptyset$. A bipartite graph G is a *circular-arc bigraph* if and only if it is the corresponding graph of a bi-circular-arc model.

If G is the corresponding graph of a model $(C, \mathbb{I}, \mathbb{E})$, we say that G *admits* the model. Note that families $\mathbb{I}$ and $\mathbb{E}$ are interchangeable, that is, models $(C, \mathbb{I}, \mathbb{E})$ and $(C, \mathbb{E}, \mathbb{I})$ can be considered equal.

To simplify notation, we refer to the arc corresponding to a vertex v in a circular-arc (bi-circular-arc) model as the *v-arc*, or $a(v)$ and the vertex corresponding to an arc A as the *A-vertex*, or $v(A)$.

If $(C, \mathbb{I}, \mathbb{E})$ is a bi-circular-arc model, we say that $\mathbb{I}$ and $\mathbb{E}$ are *opposite families* to each other. Furthermore, if $\mathbb{I}' \subseteq \mathbb{I}$ and $\mathbb{E}' \subseteq \mathbb{E}$, we say $(C, \mathbb{I}', \mathbb{E}')$ is a *submodel* of $(C, \mathbb{I}, \mathbb{E})$.

When graphically representing a bi-circular-arc model, we represent C as a dotted circle, with family $\mathbb{I}$ represented as arcs inside the circle, and family $\mathbb{E}$ represented as arcs outside the circle, as shown in Figure 3.

In this paper, we consider all arcs to be open unless otherwise stated. If A is an arc on a circle, we use $s(A)$ and $t(A)$ to denote its counter-clockwise and clockwise endpoints, respectively. To simplify notation, we call counter-clockwise endpoints s-endpoints, and clockwise endpoints t-endpoints. For any arc A in circle C, we define the *complement* of A as an arc $\bar{A} = C - A$. Denote the length of arc A as $|A|$.

If p, q are two points in a circle C, we define (p, q) as an open arc such that $s((p, q)) = p, t((p, q)) = q$. We say that a sequence $(p_1, ..., p_n)$ of points in circle C is a clockwise (counter-clockwise) order if, for every $0 < i < n$, the open arc (p_i, p_{i+1}) (the open arc (p_{i+1}, p_i)) does not contain any point in the sequence.

For any indexed set or sequence, index summation is considered cyclic. For example, in a set $\{s_1, ..., s_n\}$, we consider $s_{1-1} = s_n$ and $s_{n+1} = s_1$.

Define, for two points $p, q \in C$, the *distance* between p and q as $d(p, q) = min\{|(p, q)|, |(q, p)|\}$. For $0 \leq c < |C|$, define, for any point p, the points $p - c$ and $p + c$ to be such that $|(p - c, p)| = |(p, p + c)| = c$. That is, the point $p - c$ $(p + c)$ is at a counter-clockwise (clockwise) offset of length c from p.

A *rotation* of a sequence $(p_1, ..., p_n)$ is a permutation of the form $(p_i, ..., p_n, p_1, ..., p_{i-1})$ for $1 \leq i \leq n$. Note that the identity permutation is a rotation. A set $S \subset \{p_1, ..., p_n\}$ is said to be *circularly consecutive* in $(p_1, ..., p_n)$ if there exists a rotation of the sequence in which S is consecutive.

A set of arcs $\{A_1, ..., A_n\}$ on a circle C is said to *cover the circle* if $A_1 \cup ... \cup A_n = C$. In this paper, assume that, in every circular-arc (bi-circular-arc) model over a circle C, there is no individual arc A such that $A = C$.

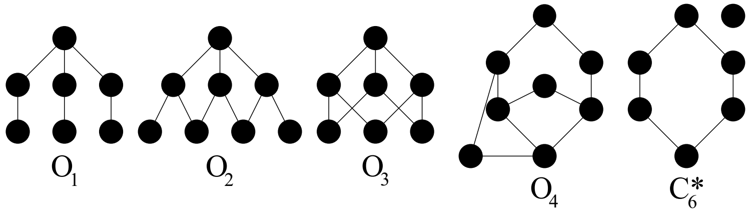

Figure 1. Forbidden induced subgraphs for the Helly circular-arc bigraph class.

A family $\mathbb{F}$ is said to be *intersecting* if, for every pair $E, F \in \mathbb{F}$, we have $E \cap F \neq \emptyset$. Similarly, we say that a pair of families $\mathbb{E}, \mathbb{F}$ is *bipartite-intersecting* if, for every $E \in \mathbb{E}, F \in \mathbb{F}$, $E \cap F \neq \emptyset$.

3 Helly circular-arc bigraphs and interval bigraphs

Many clique related computational problems have well known biclique variations, such as the NP-complete maximum edge biclique problem [16]. These hard problems call for the study of biclique structures on several bipartite graph subclasses, motivating the introduction of the Helly class we present.

The class we introduce is analogous to the *Helly circular-arc graph* class, based on the concept of *Helly families*. A family $\mathbb{F}$ is said to be *Helly* if, for every intersecting subfamily $\mathbb{F}' \subseteq \mathbb{F}$, $\bigcap_{F \in \mathbb{F}'} F \neq \emptyset$.

Helly circular-arc graphs are then defined as the class of graphs that admit a model $(C, \mathbb{A})$ in which $\mathbb{A}$ is a Helly family [12]. Equivalently, a graph G is a Helly circular-arc graph if and only if it admits a circular-arc model such that, for every clique $K \subseteq V(G)$, there exists a point $p \in C$ such that, for every $v \in K$, $a(v)$ contains p. A model $(C, \mathbb{A})$ in which $\mathbb{A}$ is a Helly family is said to be a *Helly circular-arc model*.

To define an analogous Helly subclass for circular-arc bigraphs, we apply the concept of *bipartite-Helly* families, a variation of the Helly property for bipartite sets introduced in [5].

Definition 3.1 [5] A pair of families $\mathbb{A}, \mathbb{B}$ is said to be *bipartite-Helly* if, for every bipartite-intersecting pair of subfamilies $\mathbb{A}' \subseteq \mathbb{A}, \mathbb{B}' \subseteq \mathbb{B}$, $\bigcap_{A \in \mathbb{A}' \cup \mathbb{B}'} A \neq \emptyset$.

We then define Helly circular-arc bigraphs as the graphs that admit a bi-circular-arc model $(C, \mathbb{I}, \mathbb{E})$ in which the pair $\mathbb{I}, \mathbb{E}$ is bipartite-Helly. We call a model with that property a *Helly bi-circular-arc model*. Equivalently, a bipartite graph G is a Helly circular-arc bigraph if and only if it admits a bi-circular-arc model $(C, \mathbb{I}, \mathbb{E})$ in which, for every biclique $K \subset V(G)$, there exists a point $p \in C$ such that $p \in a(v)$ for every $v \in K$.

It is easy to verify that, if the twin-free version of a graph G is a Helly circular-arc bigraph, then so is G. Figure 1 contains forbidden induced subgraphs we discovered for the class and use throughout this section.

An important subclass of circular-arc bigraphs, for which we introduce a similar

Helly subclass, is the class of interval bigraphs, which has been extensively studied and characterized [8, 15] alongside its proper subclass [2]. We define a *bi-interval model* to be a pair of families $(\mathbb{E}, \mathbb{F})$ of intervals on the real line. The *corresponding graph* of a bi-interval model $(\mathbb{E}, \mathbb{F})$ is constructed by creating a vertex for each element of $\mathbb{E} \cup \mathbb{F}$ and, for every pair of intervals $E \in \mathbb{E}, F \in \mathbb{F}$, an edge between the corresponding vertices of E and F is added if and only if $E \cap F \neq \emptyset$. A bipartite graph is an interval bigraph if and only if it is the corresponding graph of a bi-interval model.

We then define Helly interval bigraphs as graphs that admit a bi-interval model $(\mathbb{E}, \mathbb{F})$ that verifies the bipartite-Helly property. It is easy to verify that a bipartite graph is a Helly interval bigraph if and only if its square is an interval graph. We introduce a similar result for Helly circular-arc bigraphs in Lemma 3.2.

Lemma 3.2 *Let $G = (V, W, E)$ be a bipartite graph. If G^2 is a Helly circular-arc graph, G is a Helly circular-arc bigraph.*

Proof Let $(C, \mathbb{A})$ be a Helly circular-arc model of G^2. Let $\mathbb{I}, \mathbb{E} \subset \mathbb{A}$ be such that $\mathbb{I}$ contains the arcs corresponding to V in G, and $\mathbb{E}$ contains the arcs corresponding to W. The bi-circular-arc model $(C, \mathbb{I}, \mathbb{E})$ is a Helly model of G. $\square$

Lemma 3.3 is used in multiple proofs throughout this section.

Lemma 3.3 *A bipartite graph G without isolated vertices is a Helly circular-arc bigraph if and only if, given a circle C, it is possible to attribute to every biclique K of G a point $p_K \in C$ such that, for all $v \in V(G)$, the points attributed to $b(v)$ are consecutive. We call said points the* biclique points *of each biclique.*

Proof If we have a Helly bi-circular-arc model of a bipartite graph G, we can easily derive a set of biclique points by picking, for each biclique, one point that is contained in all the arcs corresponding to its vertices.

Conversely, let S be a set of biclique points for $G = (V, W, E)$ on circle C such that, for every $v \in V \cup W$, the points attributed to the members of $b(v)$ are consecutive. Define $\epsilon = \frac{1}{10} min\{d(a, b)|a, b \in S\}$. Construct a bi-circular-arc model by making, for each vertex v, a v-arc A_v with the following process: let A be the shortest closed arc that contains every point of $b(v)$ and no point of $b(G) - b(v)$ (with $s(A)$ and $t(A)$ being biclique points), then $A_v = (s(A) - \epsilon, t(A) + \epsilon)$. It is easy to verify that this process creates a Helly model of G. $\square$

3.1 Recognition algorithm for Helly circular-arc bigraphs

The recognition algorithm we present treats two separate cases:

- When the input graph is C_6-free, the problem is reduced to the recognition of Helly circular-arc graphs.
- When the input graph contains an induced C_6, the algorithm searches for forbidden induced subgraphs.

Treatment of the first case depends on Theorem 3.7, which depends on Lemmas 3.4, 3.5 and Corollary 3.6.

Lemma 3.4 *[6] If G is a hereditary open neighborhood Helly graph, every clique $K \in G^2$ is a biclique in G.*

Lemma 3.5 *[4] A graph is hereditary open neighborhood Helly if and only if it does not contain C_6 nor C_3 as induced subgraphs.*

Corollary 3.6 *A C_6-free bipartite graph G is such that, for every clique $K \subseteq G^2$, K is a biclique in G.*

Theorem 3.7 *A C_6-free bipartite graph $G = (V, W, E)$ with no isolated vertices is a Helly circular-arc bigraph if and only if G^2 is a Helly circular-arc graph.*

Proof ($\Leftarrow$) Follows from Lemma 3.2.

($\Rightarrow$) Construct a Helly bi-circular-arc model $(C, \mathbb{I}, \mathbb{E})$ based on a set of biclique points S as seen in the proof of Lemma 3.3. Note that arcs $A, B \in \mathbb{I} \cup \mathbb{E}$ intersect if and only if $v(A), v(B)$ are contained in a common biclique of G, and that, for every biclique $K \subset V \cup W$, there is a point $p \in C$ contained in all arcs corresponding to vertices of K. Therefore, by Corollary 3.6, $(C, \mathbb{I} \cup \mathbb{E})$ is a Helly circular-arc model of G^2. □

Theorem 3.7 implies that, for the C_6-free case, it is possible to verify if graph G is a Helly circular-arc bigraph by first constructing G^2, and then verifying if G is a Helly circular-arc graph.

To treat the cases where a graph has an induced C_6, we prove that if a bipartite graph containing an induced C_6 has none of the graphs in Figure 1 as induced subgraphs, then it is a Helly circular-arc bigraph. Since a graph is Helly if and only if its twin-free version is, we focus on the twin-free case.

Lemma 3.8 *Let G be a twin-free bipartite graph that contains an induced C_6 in the cycle $C = (c_1, ..., c_6)$. If G is a Helly circular-arc bigraph, then for every vertex $v \in V(G) - C$, v is neighbor to exactly one vertex of C.*

Proof If v is neighbor to no vertices of C, then G contains ${C_6}^*$ as an induced subgraph. If v is neighbor to more than three vertices of C, then G is not bipartite. If v is neighbor to exactly three vertices of C (of the same partite set), then G contains an induced O_3 from Figure 1.

Suppose, now, that v has two neighbors in C. Without loss of generality, let $N(v) \cap C = \{c_1, c_3\}$. Since $N(v) \neq N(c_2)$, there must exist a vertex w that is neighbor to v or to c_2, but not both.

Suppose that w is neighbor to c_2. If w is not neighbor to any other vertex in C, then G contains an induced ${C_6}^*$. If w is neighbor to some other vertex of C (either c_4 or c_6), then G contains an induced O_4.

Now suppose that w is neighbor to v. If w is neighbor to only one vertex of C, G contains an induced O_4, and if it is neighbor to both c_4 and c_6, then G contains an induced O_3. □

Lemma 3.9 *Let G be a twin-free Helly circular-arc bigraph that contains an induced C_6 in the cycle $C = (c_1, ..., c_6)$. For every $1 \leq i \leq 6$, if $v, w, x \in V(G) - C$ are such that $N(v) \cap C = \{c_i\}$, $N(w) \cap C = \{c_{i+1}\}$ and $N(x) \cap C = \{c_{i+2}\}$, then $vw \notin E(G)$ or $wx \notin E(G)$.*

Proof If $vw, wx \in E(G)$, an O_2 is induced with $c_{i+1}, c_i, w, c_{i+2}, c_{i-1}, v, x, c_{i+3}$. □

Lemma 3.10 *Let G be a twin-free Helly circular-arc bigraph that contains an induced C_6 in the cycle $C = (c_1, ..., c_6)$. If $v_1, v_2, w_1, w_2 \in V(G)$ are such that $N(v_1) \cap C = N(v_2) \cap C = \{c_1\}$, $N(w_1) \cap C = N(w_2) \cap C = \{c_2\}$, $v_1w_1 \in E(G)$ and $v_2w_2 \in E(G)$, then $v_1w_2 \in E(G)$ or $v_2w_1 \in E(G)$.*

Proof If $v_1w_2, v_2w_1 \notin E(G)$, an O_1 is induced by $c_1, c_6, v_1, v_2, c_5, w_1, w_2$. □

Lemma 3.11 *Let G be a twin-free Helly circular-arc bigraph that contains an induced C_6 in cycle $C = (c_1, ..., c_6)$. If v, w are such that $N(v) \cap C = c_1$ and $N(w) \cap C = c_4$, then $vw \notin E(G)$.*

Proof If $vw \in E(G)$, an O_1 is induced with $c_1, c_2, v, c_6, c_3, w, c_5$. □

Lemma 3.9 implies that, in a twin-free Helly circular-arc bigraph G with an induced C_6 in cycle $C = (c_1, ..., c_6)$, any vertex $v \in V(G) - C$ that is a neighbor of c_i can be neighbor to vertices in $N(c_{i+1})$ or $N(c_{i-1})$, but not both. Lemma 3.10 implies that, if two vertices v, w are neighbors to c_i, and both of them have neighbors in $N(c_{i+1}) - C$ (or $N(c_{i-1}) - C$), then their neighborhoods are comparable. Furthermore, Lemma 3.11 implies that v cannot be neighbor to any vertex from $N(c_{i+3})$.

Note that the three Lemmas, together with Lemma 3.8, imply that any twin-free Helly circular-arc bigraph that contains an induced C_6 is an induced subgraph of the graph in Lemma 3.12, presented in the sequence.

Lemma 3.12 *Let G be a bipartite graph such that $V(G)$ is the union of the following subsets, for $n_1, ..., n_6 \geq 0$:*

- $C = \{c_1, c_2, c_3, c_4, c_5, c_6\}$.
- $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$.
- $W_i = \{w_{i,1}, ..., w_{i,n_i}\}$ *for all* $1 \leq i \leq 6$.
- $X_i = \{x_{i,1}, ..., x_{i,n_i}\}$ *for all* $1 \leq i \leq 6$.

And let the neighborhoods of the vertices in $V(G)$ be the following, for all $1 \leq i \leq 6$:

- $N(c_i) = \{c_{i-1}, c_{i+1}, v_i\} \cup W_i \cup X_{i-1}$.
- $N(v_i) = \{c_i\}$.
- $N(w_{i,j}) = \{c_i\} \cup \{x_{i,k} \in X_i | k \leq j\}$, *for all* $1 \leq j \leq n_i$.
- $N(x_{i,j}) = \{c_{i+1}\} \cup \{w_{i,k} \in W_i | k \geq j\}$, *for all* $1 \leq j \leq n_i$.

Then G is a Helly circular-arc bigraph. See Figure 2 for induced subgraphs showcasing the structure of G.

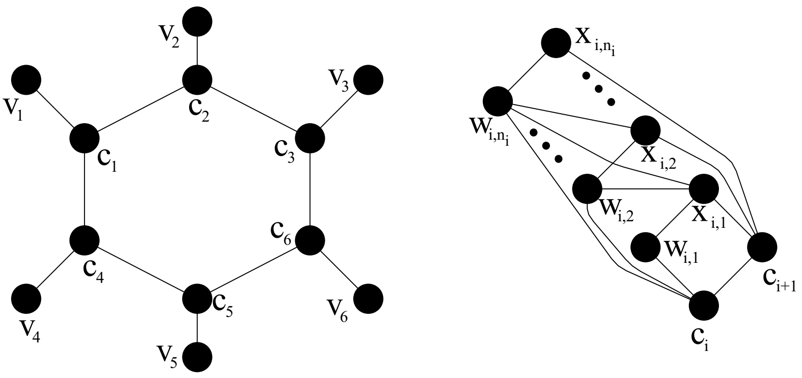

Figure 2. The subgraph of G from Lemma 3.12 induced by $C \cup V$ (left), and the subgraph induced by $\{c_i, c_{i+1}\} \cup W_i \cup X_i$ (right).

Proof The graph's bicliques are the following:

- $A_i = \{c_{i-1}, c_i, c_{i+1}, v_i\} \cup W_i \cup X_{i-1}$ for all $1 \leq i \leq 6$.
- $B_{i,j} = \{c_i, c_{i+1}\} \cup \{w_{i,k} | k \geq j\} \cup \{x_{i,k} | k \leq j\}$ for all $1 \leq i \leq 6$, and $1 \leq j \leq n_i$.

It is easy to verify that every set of the form A_i or $B_{i,j}$ is a biclique, and that every biclique of G is either of the form A_i or $B_{i,j}$ for some i, j.

To prove that G is a Helly circular-arc bigraph, we apply Lemma 3.3. Let $S = \{a_1, ..., a_6\} \cup \{b_{i,j} | 1 \leq i \leq 6, 1 \leq j \leq n_i\}$ be a set of points around circle C such that the following sequence is a clockwise order:

$$(a_1, b_{1,1}, ..., b_{1,n_1}, a_2, b_{2,1}, ..., b_{2,n_2}, a_3, b_{3,1}, ..., b_{3,n_3}, a_4, b_{4,1}, ..., b_{4,n_4},$$
$$a_5, b_{5,1}, ..., b_{5,n_5}, a_6, b_{6,1}, ..., b_{6,n_6}).$$

Now, consider the family of bicliques for each vertex:

- For every $1 \leq i \leq 6$, $b(c_i) = \{A_{i-1}, A_i, A_{i+1}\} \cup \{B_{i,j} | 1 \leq j \leq n_i\} \cup \{B_{i-1,j} | 1 \leq j \leq n_{i-1}\}$.
- For every $1 \leq i \leq 6$, $b(v_i) = \{A_i\}$.
- For every $1 \leq i \leq 6$, and $1 \leq j \leq n_i$, $b(w_{i,j}) = \{A_i\} \cup \{B_{i,k} | 1 \leq k \leq j\}$.
- For every $1 \leq i \leq 6$, and $1 \leq j \leq n_i$, $b(x_{i,j}) = \{A_{i+1}\} \cup \{B_{i,k} | j \leq k \leq n_i\}$.

Note that, if we attribute point a_i to biclique A_i for every $1 \leq i \leq 6$, and point $b_{i,j}$ to biclique $B_{i,j}$ for every $1 \leq i \leq 6, 1 \leq j \leq n_i$, the points attributed to the bicliques in $b(v)$ for every $v \in V(G)$ will be consecutive. Therefore, G is a Helly circular-arc bigraph by Lemma 3.3. □

Lemma 3.12 allows us to conclude Theorem 3.13, the final theorem we need for the algorithm.

Theorem 3.13 *Let G be a bipartite graph with no isolated vertices that has an induced C_6. Then G is a Helly circular-arc bigraph if and only if it does not contain $O_1, O_2, O_3, O_4, C_6{}^*$ as an induced subgraph.*

Algorithm 3.1: Recognition(G)

if G^2 is a Helly circular-arc graph **then**
  return Yes
if G does not contain an induced C_6 **then**
  return No
if G does not contain $O_1, O_2, O_3, O_4, C_6{}^*$ as an induced subgraph **then**
  return Yes
return No

Proof Let G^- be the twin-free version of G, and let the vertices of the induced C_6 form the cycle $C = (c_1, ..., c_6)$. Furthermore, assume G contains no induced $O_1, O_2, O_3, O_4, C_6{}^*$.

According to the proof of Lemma 3.8, G^- is such that, for every vertex $v \in V(G^-) - C$, $|N(v) \cap C| = 1$. Similarly, by the proof of Lemma 3.9, if there are three vertices $v, w, x \in V(G^-) - C$ such that $N(v) \cap C = \{c_{i-1}\}$, $N(w) \cap C = \{c_i\}$, $N(x) \cap C = \{c_{i+1}\}$, then w cannot be neighbor to both v and x. Furthermore, if $N(v) \cap C = \{c_i\}$ and $N(w) \cap C = \{c_j\}$ with $j \neq i+1, i-1$, then $vw \notin E(G^-)$ by the proof of Lemma 3.11.

Let $V_i = \{v \in V(G^-) - C | N(v) \cap C = \{c_i\}\}$ and $V_{i,j} = \{v \in V_i | N_{G^-}(v) - C \subseteq V_j\}$, for $1 \leq i, j \leq 6$. We know that, if $k \neq j$, then $V_{i,j} \cap V_{i,k} = \emptyset$ (Lemma 3.9). By the proof of Lemma 3.10, we know that, if $v, w \in V_{i,j}$, then $N(v) \subset N(w)$ or $N(w) \subset N(v)$.

Consider the relation between two sets of the form $V_{i,i+1}$ and $V_{i+1,i}$. We use w.l.o.g. the sets $V_{1,2}$ and $V_{2,1}$. Let $V_{1,2} = \{v_1, ..., v_{n_1}\}$ for some $n_1 \geq 0$ such that, for every $1 \leq i < n_1$, $N(v_i) \subset N(v_{i+1})$. Consider v_i and v_{i+1} for some value of i, and let $w \in N(v_{i+1}) - N(v_i)$. Notice that $N(w) = \{c_2\} \cup \{v_j \in V_{1,2} | j > i\}$, which implies, for every $1 \leq i < n_1$, $|N_{G^-}(v_{i+1}) - N_{G^-}(v_i)| = 1$ and, since $|N(v_1) \cap V_{2,1}| = 1$, $|V_{1,2}| = |V_{2,1}|$. Define, for every $1 \leq k \leq 6$, $n_k = |V_{k,k+1}|$. Note that, for these values of $n_1, n_2, n_3, n_4, n_5, n_6$, G^- is an induced subgraph of the graph from Lemma 3.12, implying G^- is a Helly circular-arc bigraph and, therefore, so is G. □

Theorem 3.13 demonstrates that Algorithm 3.1 is a correct recognition algorithm for Helly circular-arc bigraphs without isolated vertices. Note that every step in the algorithm can be computed in polynomial time. Let $G = (V, E)$ be a bipartite graph such that $|V| = n, |E| = m$:

- Calculating the square G^2 of graph G can be done in $O(n^2)$.
- Verifying whether G^2 is a Helly circular-arc graph can be done in linear time [10].
- Searching for any fixed induced subgraph of order k in G can be done in $O(n^k)$.

As previously mentioned, Algorithm 3.1 is only correct for graphs without isolated vertices, as it may fail otherwise. The reason for that is some C_6-free Helly circular-arc bigraphs with isolated vertices may be Helly even when their squares are not Helly circular-arc graphs, as is the case with the graph in Figure 3.

A C_6-free bipartite graph G with isolated vertices is Helly if and only if the

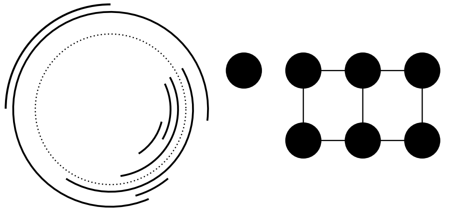

Figure 3. A Helly circular-arc bigraph that is not recognized by Algorithm 3.1 due to its isolated vertex. The bi-circular model presented verifies the Helly property. The graph's square contains a $C_4{}^*$, which is not a circular-arc graph.

graph G' resulting from the removal of all isolated vertices from G admits a Helly model $(C, \mathbb{I}, \mathbb{E})$ such that there exists an arc in $A \subset C$, $A \notin \mathbb{I} \cup \mathbb{E}$, that intersects no arcs of $\mathbb{I}$. Recognizing the graphs that verify this property is an open problem.

4 Proper and Normal circular-arc bigraphs

Normal circular-arc graphs are defined as graphs that admit a circular-arc model $(C, \mathbb{A})$ such that no two arcs $A, B \in \mathbb{A}$ cover the circle. Based on the concept of prohibiting pairs of arcs in a model from covering the circle, we propose two normal circular-arc bigraph subclasses.

Definition 4.1 A bipartite graph G is a *type-1 normal circular-arc bigraph* if it admits a bi-circular-arc model $(C, \mathbb{I}, \mathbb{E})$ such that there are no two arcs $A, B \in \mathbb{I}$ that cover the circle, and no two arcs $A, B \in \mathbb{E}$ that cover the circle. A model with those properties is called a *type-1 normal bi-circular-arc model.*

A bipartite graph G is a *type-2 normal circular-arc bigraph* if it admits a bi-circular-arc model $(C, \mathbb{I}, \mathbb{E})$ such that, for all $A \in \mathbb{I}, B \in \mathbb{E}$, $A \cup B \subsetneq C$. Such a model is called a *type-2 normal bi-circular-arc model.*

A family $\mathbb{F}$ of arcs is said to be *proper* if there are no two elements $E, F \in \mathbb{F}$ such that $E \subset F$. Proper circular-arc graphs are defined as graphs that admit a circular-arc model $(C, \mathbb{A})$ such that $\mathbb{A}$ is a proper family. The following definition of *proper circular-arc bigraphs* is based on this definition.

Definition 4.2 [1] A bipartite graph G is a *proper circular-arc bigraph* if it admits a bi-circular-arc model $(C, \mathbb{I}, \mathbb{E})$ such that $\mathbb{I}$ and $\mathbb{E}$ are proper families.

We refer to the class of proper circular-arc bigraphs from Definition 4.2 as *type-1 proper circular-arc bigraphs*, and the models that verify the property as *type-1 proper bi-circular-arc models.*

Also based on the concept of prohibiting proper containments in a model, we introduce the following class.

Definition 4.3 A bipartite graph G is a *type-2 proper circular-arc bigraph* if it admits a bi-circular-arc model $(C, \mathbb{I}, \mathbb{E})$ such that no two arcs $I \in \mathbb{I}, E \in \mathbb{E}$ are such that $I \subset E$ or $E \subset I$. A model with those properties is called a *type-2 proper bi-circular-arc model.*

The class of type-1 proper circular-arc bigraphs has known characterizations [1] and a polynomial-time recognition algorithm [2], but the recognition of the classes introduced in this section is an open problem. In the sequence, we present known inclusions.

Lemma 4.4 *[2] Every type-1 proper circular-arc bigraph admits a bi-circular-arc model that is both type-1 proper and type-2 normal.*

Lemma 4.4 is implied by Theorem 2 of [2], and allows us to conclude the following inclusion.

Corollary 4.5 *Type-1 proper circular-arc bigraphs are a proper subclass of type-2 normal circular-arc bigraphs.*

The next lemma proves a similar inclusion we discovered between the classes we introduced.

Lemma 4.6 *Every type-2 proper circular-arc bigraph admits a bi-circular-arc model that is both type-2 proper and type-1 normal.*

Proof Let $(C, \mathbb{I}, \mathbb{E})$ be a type-2 proper bi-circular-arc model of graph G. Suppose there are two arcs $A, B \in \mathbb{I}$ such that $A \cup B = C$. Let $S = \{s(A)|A \in \mathbb{I} \cup \mathbb{E}\} \cup \{t(A)|A \in \mathbb{I} \cup \mathbb{E}\}$, define $\epsilon = \frac{1}{10}min\{d(a,b)|a,b \in S\}$.

Since the model is type-2 proper, every arc in $\mathbb{E}$ intersects both A and B, meaning it is possible to replace arc A with arc $(t(B)+\epsilon, s(B)-\epsilon)$ without changing the corresponding graph of the model, and without the model losing the type-2 proper property.

If the same process is applied to every pair of arcs of the same family that cover the circle, redefining ϵ after every replacement, the resulting model will be type-1 normal when no pairs remain. □

Corollary 4.7 *Type-2 proper circular-arc bigraphs are a proper subclass of type-1 normal circular-arc bigraphs.*

It is currently unknown whether the class of type-1 proper (type-2 proper) circular-arc bigraphs is a proper subclass of type-1 normal (type-2 normal) circular-arc bigraphs. It is also unknown whether type-1 proper circular-arc bigraphs are a proper subclass of type-2 proper circular-arc bigraphs.

5 Conclusion

In this paper, the class of Helly circular-arc bigraphs was introduced, and a polynomial-time recognition algorithm was presented for graphs without isolated vertices. The recognition problem for graphs that contain isolated vertices remains open.

Two circular-arc bigraph classes based on normal circular-arc graphs were introduced and called type-1 and type-2 normal circular-arc bigraphs. Furthermore, as an alternative to the class of proper circular-arc bigraphs presented in [1], which we dubbed “type-1”, the class of type-2 proper circular-arc bigraphs was introduced. We proved that type-2 proper circular-arc bigraphs are a proper subclass of type-1 normal circular-arc bigraphs.

Several open problems remain, such as the recognition of the type-2 proper class and both type-1 and type-2 normal classes, and determining whether the type-1 proper class is contained in the type-2 proper class. Future research involves studying those problems, as well as intersection classes based on the ones studied in [10, 11].

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