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Paraconsistent Arithmetic with a Local Consistency Operator and Global Selfreference

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Abstract

The Provability Logic and Proof-Theory of the system of Paraconsistent Arithmetic **PRACI** are presented. **PRACI** is based on the paraconsistent predicate calculus **CI** corresponding to the **C**-system **Ci** introduced by Carnielli et al. [7]. **PRACI** can support an infinity of contradictions $B \wedge \neg B$ without trivializing, but reject identifications between different numbers such as $0 = 1$. In **PRACI** a new propositional connective $^\circ(.)$ is added, so that $^\circ A$ can be read as "A is consistent". We obtain a system with a *local selfreference*, based on the *local consistency assertions* $^\circ A$, and a *global selfeference*, based statements involving $\mathrm{Pr}_{\mathbf{PRACI}}(.)$. The *fundamental relation* $\mathrm{Pr}_{\mathbf{T}}(\#^\circ B) \to \neg\, \mathrm{Pr}_{\mathbf{T}}(\#B)$ betweeen local and global consistency is investigated. It states that in a paraconsistent setting, the provability of the non-trivialty of Arithmetic could be reduced to that of some suitable local consistency assertions, so that we can speak of a possible weakened Hilbert's program.

Keywords: Paraconsistent Arithmetic, Provability Logic, Local and Global Selfreference, Weakened Hilbert's program.

1 Introduction

In this paper we explore Provability Logic and Proof-Theory of the system of Paraconsistent Arithmetic **PRACI**, based on the paraconsistent predicate calculus **CI**, which extends the propositional **C**-system **Ci** introduced by Carnielli et al. [7]. As already pointed out in [1] for the presentation of the theory **PCA**, we propose systems of Paraconsistent Arithmetic which are essentially new w.r.t. most of paraconsistent arithmetical theories existing in the preceding relevant literature. In our thinking paraconsistent reasoning, intuitionistic reasoning and classical reasoning can be compared as different methods to investigate the *same* elementary mathematical objects: that is, we introduce a paraconsitent reasoning about *standard*

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numbers. Thus, the theories **PCA** and **PRACI** can support an infinity of relevant contradictions $B \wedge \neg B$ without trivializing, but reject identifications between different numbers, such as $0 = 1$ and so on. An interesting feature of the **C**-system based arithmetical theories proposed here, is that a new propositional connective $^\circ(.)$ is added, expressing a notion of consistency referred to a particular formula, so that $^\circ A$ can be read as "A is consistent". **PRACI** exhaustively formalizes the properties of local consitency assertions. Thus, by further introducing the global (i.e. referred to a system) canonical provability predicate $\mathrm{Pr}_{\mathbf{PRACI}}(.)$, we obtain a system where two kinds of selreference are possible: a *local selfreference*, based on the *local consistency assertions* $^\circ A$, and a *global selfeference*, based on the provability logic statements involving $\mathrm{Pr}_{\mathbf{PRACI}}(.)$. In the paper the free cut-elimination and paraconsistency properties of **PRACI** are presented (Sections 3 and 4), the peculiar features of **PRACI**-Provability Logic are illustrated (Section 5), the intensional meaning of the paraconsistent negation is discussed (Section 5), and the *fundamental relation* betweeen local and global consistency is investigated (Section 6). That is, starting from the result $<< \vdash \mathrm{Pr}_{\mathbf{CI}}(\#^\circ B) \rightarrow \neg\,\mathrm{Pr}_{\mathbf{CI}}(\#B)$ is **PRACI**-provable for each sentence B which has not the form $^\circ F >>$ (Theorem 6.5), a kind of weakened Hilbert's program can be suggested. Indeed, some weaker versions of the previous implication, referred to **PRACI**, i.e. including the predicate $\mathrm{Pr}_{\mathbf{PRACI}}(.)$ and not symply $\mathrm{Pr}_{\mathbf{CI}}(.)$, are **PRACI**-provable. Therefore, in a paraconsistent setting, the provability of the non-trivialty (which corresponds to classical consistency) of Arithmetic could be reduced to that of some suitable local consistency assertions. Thus, very elementary and constructive extensions of **PRACI**, with the same induction rule and withoud adding mathematical information, could prove the non triviality of **PRACI**.

2 The paraconsistent sequent predicate calculus CI

In Benassi-Gentilini [1] the sequent version **BC** of the system **bC** is presented. **bC** is the basic system of the hierarchy of paraconsistent **C**-systems, introduced by Carnielli, Marcos and other authors [7,8,9,10,11,12], whose language is the extension of the classical one through a monadic propositional connective $^\circ(.)$,which plays an essential role in the introduction of the paraconsistent negation. As we shall see, both $^\circ(.)$ and the propositional negation $\neg$ result as intensional logical operators. The intended meaning of $^\circ B$ is "B is consistent " that is " $<<B$ and not $B>>$ does not hold". Thus, $^\circ B$ is a kind of formal translation of a metatheoretic statement, as for the provability predicate $\mathrm{Pr}_{\mathbf{T}}(.)$ happens. We call the formulas of the form $^\circ B$ *local consistency assertions*. The **C**-system **Ci** is the **bC**-extension that explicitly expresses the *local consistency* properties: indeed, **bC** cannot prove theorems of the form $^\circ B$, while **Ci** has a relevant class of such theorems. $^\circ B$ is neither **bC**- nor **Ci**-equivalent to $\neg(B \wedge \neg B)$. We present here the sequent version **CI** of **Ci**. We recall that (see [5,23,14]) a sequent S is an expression of the form $X \vdash Y$ where X and Y are finite (possibly empty) sets of formulas. X is called the *antecedent* of S, Y the *succedent* of S. We will use the symbols $X, Y, \Lambda, \Gamma, \ldots$ as meta-expressions for sets of formulas, $A, B, C, D, \ldots$ for formulas. The intended meaning of a sequent $A_1, A_2, \ldots, A_n \vdash B_1, B_2, \ldots, B_m$ is $\wedge_i A_i \longrightarrow \vee_j B_j$ and such equivalence holds both in a classical and in a paraconsistent setting. Given a rule $\dfrac{S_1 ... S_n}{S}$, the sequents $S_1, ..., S_n$ are the *premises* of the rule, the sequent S is the *conclusion* of the rule. The proofs are trees, whose leaves are *axioms*, and whose branches are formed by sequent rules. The writing Λ, Γ stands for $\Lambda \cup \Gamma$.

We say that a sequent formulated system or theory *trivializes* (or *is trivial*) if and only if it proves each sequent of the form $\vdash A$. A **BC**- or **CI**-based system is trivial if and only if it proves the empty sequent $\vdash$. A system **T** is *negation consistent* if **T** cannot prove any formula of the form $B \wedge \neg B$. We recall that a formula A is in *prenex form* if A is $\mathsf{Q}_1 .... \mathsf{Q}_n B$ where each $\mathsf{Q}_j \in \{\forall x_j, \exists x_j\}_{j=1,\ldots,n}$ and B is quantifier-free; in general, any formula F is not **CI**-equivalent to a formula D in prenex form, since the interdefinability of quantifiers does not hold in the paraconsistent setting presented here. In writing formulas we adopt the convention that $\vee, \wedge, \neg$ link more than $\rightarrow$, and that $\rightarrow$ links more than $\leftrightarrow$. The sequent system **BC** is given by (see also [1,12]):

BC$-$*Axioms*: $A \vdash A$

BC$-$*Positive propositional logical rules:*

$$\frac{B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta} \wedge -L \quad \frac{B, \Gamma \vdash \Delta}{B \wedge A, \Gamma \vdash \Delta} \wedge -L \quad \frac{\Gamma \vdash \Delta, A \quad \Lambda \vdash X, B}{\Gamma, \Lambda \vdash \Delta, X, A \wedge B} \wedge -R$$

$$\frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A \vee B} \vee -R \quad \frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, B \vee A} \vee -R \quad \frac{A, \Gamma \vdash \Delta \quad B, \Lambda \vdash X}{A \vee B, \Gamma, \Lambda \vdash \Delta, X} \vee -L$$

$$\frac{A, \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \rightarrow B} \longrightarrow -R \qquad \frac{\Gamma \vdash \Delta, A B, \Lambda \vdash X}{A \rightarrow B, \Gamma, \Lambda \vdash \Delta, X} \longrightarrow -L$$

BC$-$*Negation rules:*

$$\frac{A, \Gamma \vdash \Delta}{\neg\neg A, \Gamma \vdash \Delta} \neg - L1 \qquad \frac{{}^\circ A, \Gamma \vdash \Delta, A}{{}^\circ A, \neg A, \Gamma \vdash \Delta} \neg - L3$$

$$\frac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg A} \neg - R$$

We call the formula ${}^\circ A$ in the rule $\neg - L3$ *constraint formula* of the rule.

BC$-$*Quantifier rules*:

$$\frac{[t/x]\, A, \Gamma \vdash \Delta}{\forall x A, \Gamma \vdash \Delta} \forall - L \qquad \frac{\Gamma \vdash \Delta, [b/x]\, A}{\Gamma \vdash \Delta, \forall x A} \forall - R$$

$$\frac{[b/x]\, A, \Gamma \vdash \Delta}{\exists x A, \Gamma \vdash \Delta} \exists - L \qquad \frac{\Gamma \vdash \Delta, [t/x]\, A}{\Gamma \vdash \Delta, \exists x A} \exists - R$$

where t is an arbitrary term and b is a free variable which does not occur in Γ, Δ. Moreover, t may be not fully quantified while b must be uniformly replaced by x (see [23]).

BC$-$*Structural rules:*

Weakening rules:

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A} W - R \qquad \frac{\Gamma \vdash \Delta}{A, \Gamma \vdash \Delta} W - L; \text{ Cut rule: } \frac{\Gamma \vdash \Delta, A A, \Lambda \vdash X}{\Gamma, \Lambda \vdash \Delta, X} Cut$$

The system **CI** is given by adding to **BC** the following *proper* **CI**-*rules*:

$$\frac{\Gamma \vdash \Delta, {}^\circ A}{\neg {}^\circ A, \Gamma \vdash \Delta} \neg - L4 \quad \frac{A \wedge \neg A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, {}^\circ A} \quad R\,\mathrm{Ci}$$

In the rule $R\,\mathrm{Ci}$ the formula $A \wedge \neg A$ in the premise antecedent is the $R\,\mathrm{Ci}-$*auxiliary formula,* the formula ${}^\circ A$ in the conclusion succedent is the $R\,\mathrm{Ci}-$*principal formula.* Note that system **CI** only has a rule introducing the connective ${}^\circ(.)$. Moreover, the classical predicate calculus **LK** [19,23] can be obtained from **BC** by replacing the pair $\neg - L3$, $\neg - L1$, with the rule: $\dfrac{\Gamma \vdash \Delta, A}{\neg A, \Gamma \vdash \Delta} \neg - L2.$

Theorem 2.1 *The system* **CI** *admits of cut-elimination.*

For the proof see [13].

It must be stressed that **CI** cannot define the connective ${}^\circ(.)$. In particular, the following sequents are **CI**-provable: $\vdash {}^\circ B \to \neg(B \wedge \neg B)$, $\vdash \neg {}^\circ B \leftrightarrow B \wedge \neg B$. But $\vdash \neg(B \wedge \neg B) \to {}^\circ B$ *is not* **CI**-provable.

3 Paraconsistent Arithmetic with a *local consistency operator*

The system of **CI**-*based Paraconsistent Recursive Arithmetic* **PRACI** is so defined. The language of **PRACI** is that of Primitive Recursive Arithmetic **PRA**, plus the monadic propositional connective ${}^\circ(.)$ of the **C**-systems. For the proof theory of classical Arithmetic we refer to [4,6,17,19,23].

We choose a version of **PRA** with the only predicate $= (.,.)$, the individual constant 0, symbols for numerals, and a function letter for each primitive recursive function. We assume as identified the writings $= (t, s)$ and $t = s$. All the primitive recursive predicates R different from $= (.,.)$ are expressed by their characteristic function X_R, and $X_R(t_1, ..., t_n) = 1$ means that $R(t_1, ..., t_n)$ holds, $X_R(t_1, ..., t_n) = 0$ means that $R(t_1, ..., t_n)$ does not hold. Moreover, we establish that each proper axiom set we shall present in this work is closed under term substitution.

PRACI is given by the system **CI** plus the set **AxPRACI** of proper axioms and the rule **Ind**.

AxPRACI is the following axiom set:

1) Arithmetical axioms defining primitive recursive function (in the following sequents all the explicitly indicated variables x_i, y_j, are free):

1j)Definitions of the basic recursive functions (*zero function, successor function, projection function*):

$\vdash Z_k\,(x_1, ..., x_k) = 0$ for each $k \geq 1$ (Z_k *zero* function);

$S\,(x) = 0 \vdash$ (S *successor* function);

$S(x) = S(y) \vdash x = y$

$\vdash P_i^k\,(x_1, ..., x_k) = x_i$ for each $k \geq 1$,$i \leq k$ (P_i^k *projection* function);

1jj) Composition schema:

$\vdash f\,(x_1, ..., x_m) = h\,(g_1\,(x_1, ..., x_m)\,, ..., g_m\,(x_1, ..., x_m))$

where $g_1, ..., g_m$ are n-ary function letters and h is a m-ary function letter.

1jjj) Recursion schema :

$\vdash f(x_1, ..., x_n, 0) = g(x_1, ..., x_n)$

$\vdash f(x_1, ..., x_n, S(y)) = h(x_1, ..., x_n, y, f(x_1, ..., x_n, y))$

where g is a n-ary function letter and h is a $n+1$-ary function letter.

2)Equality axioms:

$\vdash x = x$

$x_1 = y_1, ..., x_n = y_n \vdash f(x_1, ..., x_n) = f(y_1, ..., y_n)$

$x_1 = y_1, x_2 = y_2, x_1 = x_2 \vdash y_1 = y_2$

Convention:

for each natural number m we write $\overline{m}$ for the term $S...S(0)$, with m occurences of S on the left. The terms of the form $\overline{m}$ are called *numerals*. We will briefly write $1, 2, ...,$for numerals, each time no confusion arises.

The induction rule **Ind** of **PRACI** has the following form:

$$\frac{F(x), X \vdash Y, F(S(x))}{F(0), X \vdash Y, F(t)} \quad \textbf{Ind}$$

where $F(x)$ is an atomic formula; the free variable x, called the eigenvariable of the rule, does not occur in X, Y, t ; t is an arbitrary term which we say *introduced by* **Ind**; $F(0), F(t)$ are the *principal formulas* of **Ind**; $F(x)$, $F(S(x))$ are the *auxiliary formulas* of **Ind**.

Note that the usual definitions by recursion of sum and product are included in the schemata of **AxPRACI** Moreover, the axiom $S(x) = 0 \vdash$ gives rise to a set of *bottom particles* (see [9]) for **PRACI**, so that any formula of the form $S(t) = 0$ trivializes **PRACI**. We note that the deduction apparatus of classical Primitive Recursive Arithmetic **PRA** is given by the classical predicate calculus **LK** plus **AxPRACI** and the rule **Ind**. *Classical full Arithmetic* **PA** is the extension of **PRA** by induction rules admitting arbitrary induction formulas, and analogously is defined the **CI**-*based Full Paraconsistent Arithmetic* **PACI**. Note that the **PRACI**-language strictly includes the **PRA**-language due to the occurrence of the connective $\circ(.)$. The system of *Paraconsistent Recursive Arithmetic* **PCA** introduced in [1] is the subsystem of **PRACI** obtained by restricting **CI** to **BC**. We recall that a term t of the **PRACI**-language is called a *ground term* if no variables occur in it. We write Δ_0 for the class of primitive recursive relations, Σ_1 for the class of formulas $\exists x A(x)$ with $A \in \Delta_0, \Pi_1$ for the class of formulas $\neg A$ with $A \in \Sigma_1$ (see also [6,22]). Note however that, e.g., the class Π_1 is not the same as in the classical case since the translation into a prenex form fails in **PRACI**. We will write **N** for the standard model of **PRA** and $\mathbb{N}$ for the natural number set. As usual, at the metatheoretic level of the discourse, we assume the consistency of classical Arithmetic.

In order to study arithmetical theories **T** of the form **W**+ **AxT** + **Ind**, **AxT** including **AxPRACI**, where $\mathbf{W} \in \{\mathbf{LK}, \mathbf{BC}, \mathbf{CI}\}$, it is important to establish whether the proofs in **T** admits of the elimination of some class of cut inferences. In [1] we have already considered the case with $\mathbf{W} \equiv \mathbf{BC}$. According to the canonical exposition of Buss [5], p 43, we employ the notion of *free cut*:

Definition 3.1 Let P be a proof in $\mathbf{W}+\mathbf{AxT}+\mathbf{Ind}, \mathbf{W} \in \{\mathbf{LK}, \mathbf{BC}, \mathbf{CI}\}$. We say that a formula occurrence B in P is *anchored* if B is the direct descendant either of a formula occurring in an initial sequent belonging to **AxT** or of a principal formula of an induction rule. A cut inference in P is called a *free cut* if either : i) the cut formula is not atomic and both the cut formula occurrences in the premises are not anchored, or ii) at least one cut formula occurrence in the premises is not anchored and is introduced in P by weakenings or logical axioms only. A cut inference which is not free is said to be *anchored*.

In Buss [5], pp. 43-47, the following result is proven:

Theorem 3.2 *(Classical free-cut elimination theorem) Let* $\mathbf{T} \equiv \mathbf{LK}+\mathbf{AxT}+\mathbf{Ind}$, *be a theory of arithmetic with* **AxT** *closed under term substitution. Then* **T** *admits of free-cut elimination.*

The following is a classical consequence of the theorem mentioned above:

Proposition 3.3 *Cut-elimination of non atomic cuts holds in* **PRA**.

The extension of the free-cut elimination theorem to the system **PRACI** holds:

Lemma 3.4 *Let* $\mathbf{U} \equiv \mathbf{CI}+\mathbf{AxV}$ *with* **AxV** *closed under term substitution. Then* **U** *admits of free-cut elimination.*

Theorem 3.5 *(Free-cut elimination theorem) Let* $\mathbf{T} \equiv \mathbf{CI}+\mathbf{AxT}+\mathbf{Ind}$ *be a theory of arithmetic with* **AxT** *closed under term substitution. Then* **T** *admits of the elimination of free cuts.*

For the sake of brevity we cannot present the proofs here.

Corollary 3.6 *Cut-elimination of non atomic cuts holds in* **PRACI**.

Definition 3.7 Let $\mathbf{T} \equiv \mathbf{U}+\mathbf{AxT}$, $\mathbf{U} \in \{\mathbf{PRA}, \mathbf{PCA}, \mathbf{PRACI}\}$, **AxT** possibly empty axiom set. Then a proof P in **T** is called *normal* if free cuts do not occur in P.

Lemma 3.8 *Let* $X \vdash Y$ *be a sequent such that* X *and* Y *are not both empty and include at most atomic formulas. Then* $X \vdash Y$ *is* **PRACI**-*provable if and only if it is* **PRA**-*provable.*

Proof. If $X \vdash Y$ is **PRACI**-provable, then, it admits of a normal **PRACI**-proof Q in which only atomic formulas occur. Therefore, Q is a **PRA**-proof too. On the other hand, if $X \vdash Y$ is **PRA**-provable it admits of a normal **PRA**-proof P in which only atomic formulas occur, and P is also a **PRACI**-proof. □

Let's conclude the section by checking the non triviality and the negation consistency of **PRACI**:

Definition 3.9 We call *reduced translation* of a **PRACI**-formula B into **PRA** the formula B^* obtained from B by replacing each occurrence of a B-subformula of the form $^\circ F$ with $\neg(F \wedge \neg F)$. We call reduced **PRA**-translation of a **PRACI**-tree Q

the tree Q^* in the conservative extension **PRACI** $+ \neg - L1$ obtained from Q by replacing each formula with its reduced translation.

Proposition 3.10 **PRACI** *is non trivial and negation consistent.*

Proof. From a **PRACI**-proof tree Q of the empty sequent we would obtain, by the reduced **PRA**-translation, a proof tree Q* of the empty sequent in **PRA** $+ \neg - L1$: indeed each $R\,$Ci instance is translated into a $\neg - R$ rule. But this is absurd under our assumptions. □

We remark that, through the reduced translation, we also obtain a kind of interpetation of **PRACI** into the standard model **N**. However *this is not a proper semantics for* **PRACI**, which, for example, must falsify an infinity of formulas $\neg(B \wedge \neg B)$. **PRACI**-semantics could be defined as an extension of the semantics for **Ci** presented in [7], but a different approach is proposed in [13].

4 Paraconsistency properties of PRACI

No contradiction in the *classical* language is either proved or rejected by **BC** or **CI**. However, in the arithmetical systems **PCA** and **PRACI** we have a more complex situation since, as already discussed in [1], **PCA** rejects the identifications between different numbers. Since this is guaranteed by the axiom $S(x) = 0 \vdash$, **PRACI** too rejects the identification between different numbers. We call *numerical absurdity* any atomic formula $s = t$ where s and t are ground terms which are respectively **PRACI**-provably equal to m and n, with m, n different numerals. Thus **PRACI** is trivialized by each sequent $\vdash s = t$, where $s = t$ is any numerical absurdity, and furthermore by a lot of classical contradictions involving numerical absurdities:

Proposition 4.1 **PRACI** *is trivialized by sequents of the form* $\vdash (m = n) \wedge \neg(m = n)$, *where* $m = n$ *is a numerical absurdity and, moreover, by an infinity of sequents of the form* $\vdash (A \wedge \neg A)$, *where only the classical connectives at most occur in* A.

Theorem 4.2 *Let* $\vdash \neg(A \wedge \neg A)$ *be a* **PRACI**-*provable sequent such that in* A *classical connectives at most occur, through a normal proof* Q. *Then* $\neg(A \wedge \neg A)$ *must be a descendant in* Q *of a sequent* $s = t \vdash$ *such that* $s = t$ *has at least one instance which is a numerical absurdity, and atomic formulas having numerical absurdities as instances necessarily occur in* A.

The proofs are similar to that of the analogous theses for **PCA** (see [1]). Of course, **PRACI** maintains many relevant paraconsistency properties already owned by **PCA**:

Theorem 4.3 *Sequents of the form* $\vdash (m = m) \wedge \neg(m = m)$, m *any numeral, do not trivialize* **PRACI**.

Proposition 4.4 *Sequents of the form* $\vdash \neg(t = t \rightarrow t = t), t$ *any term, do not trivialize* **PRACI**.

Theorem 4.5 *Sequents of the form* $\vdash \Pr_{\mathbf{V}}(\#A \wedge \neg A) \wedge \neg \Pr_{\mathbf{V}}(\#A \wedge \neg A)$ *, where* **V** *is any non trivial recursively axiomatized system extending* **PRACI***, do not trivialize* **PRACI**.

Proposition 4.6 *There is a denumerable infinity of formulas A such that* **PRACI** *does not prove any sequent of the form* $\vdash \neg(A \wedge \neg A)$.

For the sake of brevity we do not present the proofs of the theses above. The reader can see the similar proofs of the corresponding results for **PCA** in [1]. However, as expected, the paraconsistency properties of **PRACI** cannot be the same as for **PCA**, for example:

Proposition 4.7 *Each sequent of the forms* $\vdash\ {}^{\circ}F \wedge \neg{}^{\circ}F$ *or* ${}^{\circ\circ}F \vdash$ *trivializes* **PRACI** *and does not trivialize* **PCA**.

Proof. Consider the following **PRACI**-proof: $\dfrac{\dfrac{{}^{\circ}F \vdash {}^{\circ}F}{{}^{\circ}F, \neg{}^{\circ}F \vdash}}{{}^{\circ}F \wedge \neg{}^{\circ}F \vdash}$. As to the second part of the thesis, it follows from the properties of normal proofs in **PCA**. □

5 Global Selfreference and Provability Logic of PRACI

In [1] we have already illustrated the formalization of metatheory inside **PCA**. As to the basic notions and properties the extension to **PRACI** is straightforward.The writing $\#E$ stands for the gödel-number of any expression E of the **PCA**-language. Therefore, we can define a binary primitive recursive predicate $Prov_{\mathbf{PRACI}}(.,.)$ such that $Prov_{\mathbf{PRACI}}(m,n)$ holds iff m is the gödel-number of a **PRACI**- proof of the sentence with gödel-number n. We recall that in our language each primitive recursive predicate R is expressed by the characteristic function X_R, so that we employ the fuction $X_{\mathrm{Prov}-\mathbf{PRACI}}$, and $Prov_{\mathbf{PRACI}}(m,n)$ corresponds to $X_{\mathrm{Prov}-\mathbf{PRACI}}(m,n) = 1$ in the **PRACI**-language. However, we will briefly write $Prov_{\mathbf{PRACI}}(m,n)$ for $X_{\mathrm{Prov}-\mathbf{PRACI}}(m,n) = 1$, and so on. The formula $\exists y X_{\mathrm{Prov}-\mathbf{PRACI}}(y, \#B) = 1$ means "the sentence B is **PRACI**-provable" and we also write it as $\Pr_{\mathbf{PRACI}}(\#B)$. [2] In general if K is a recursive relation between terms $t_1, ..., t_n$ we formally express it by $X_K(t_1, ..., t_n) = 1$, X_K characteristic function, and its recursive complementary relation by $X_K(r_1, ..., r_n) = 0$: *note that we do not establish in* **PRACI** *any link between* $X_K(r_1, ..., r_n) = 0$ *and the formula* $\neg(X_K(r_1, ..., r_n) = 1)$, since, as happens for **PCA**, the *logical* **PRACI**-negation is not the boolean negation. Conversely, in the classical **PRA** the logical negation is boolean. $\Pr_{\mathbf{PRACI}}(\cdot)$ is the non recursive Σ_1-provability predicate of **PRACI**. We recall that a slightly different canonical predicate $\Pr_{\mathbf{PRACI}}[.]$ is definable, such that if B is any open formula with free variables $x, y, ...$, then $\Pr_{\mathbf{PRACI}}[\#B]$ has the same $x, y, ...$as free variables and means "the formula B is **PCA**-provable"; $\Pr_{\mathbf{PRACI}}[\#B]$ coincides with $\Pr_{\mathbf{PRACI}}(\#B)$ if B is closed. We briefly write $\Pr_{\mathbf{PRACI}}(\#B)$ even if B is open, with the convention that $\Pr_{\mathbf{PRACI}}(\#B)$ has the same free variables

[2] As to the classical Provability Logic see [2,3,21]. For a possible conjectural approach see [18]. For the links with modal logic see [14,15,16,20,22].

as B. We extend in a straightforward way the godelization to sequents, and so $\mathrm{Pr}_{\mathbf{PRACI}}(\#X \vdash Y)$ means "sequent $X \vdash Y$ is **PRACI**-provable" and it is evident, since we are working with sequent formulated systems, that $\mathrm{Pr}_{\mathbf{PRACI}}(\#B)$ has the same meaning as $\mathrm{Pr}_{\mathbf{PRACI}}(\# \vdash B)$. For each recursively axiomatized theory $\mathbf{T}$ of the form $\mathbf{PRACI} + \mathbf{AxT}$, $\mathbf{AxT}$ proper axiom set, we can analogously define a provability predicate $\mathrm{Pr}_{\mathbf{T}}(.)$ for $\mathbf{T}$.

As to the **PRACI**-representation of the non triviality of any paraconsistent rec. axiom. system $\mathbf{T}$ extending **PRACI**, we remark that the situation is different with respect to the **PRA**-representation of the consistentcy of any classical rec. axiom. system $\mathbf{U}$ extending **PRA**. In the classical case we have an infinity of different formulas stating the consistency of $\mathbf{U}$, that are all **PRA**-equivalent to $\neg\,\mathrm{Pr}_{\mathbf{U}}(\# \vdash)$, which we also write $Con(\mathbf{U})$. In the paraconsistent case, if S and F are two different non **PRACI**-provable sequents or formulas, then both $\neg\,\mathrm{Pr}_{\mathbf{T}}(\#S)$ and $\neg\,\mathrm{Pr}_{\mathbf{T}}(\#F)$ express the non-triviality of $\mathbf{T}$, *but they are not* **PRACI**-*equivalent*, and this holds even if S and F are **PRACI**-equivalent. Moreover, observe that $\neg\,\mathrm{Pr}_{\mathbf{T}}(\#F)$ is a Π_1-formula in our paraconsistent sense, i.e. it cannot be equivalently written in the form $\forall x A(x)$. We call *global selfrefence sentences of* **PRACI** those including also subformulas of the form $\mathrm{Pr}_{\mathbf{T}}(\#S)$, with $\mathbf{T}$ any **PRACI**-extension.

The deep difference between paraconsitent and classical selfreference is announced by the following *non-transparency* results:

Proposition 5.1 *Let $B(z)$ be any formula in which the free variable z occurs; then the sequent $r = t,\, B(r) \vdash B(t)$ is in general not* **PRACI**-*provable, so that* **PRACI** *is not transparent.*

The proof is similar to that for **PCA** (see [1]). An elementary example is the **PRACI**-unprovability of sequents of the form $r = t, \neg(f(r) = k) \vdash \neg(f(t) = k)$, with r, t, k suitable terms, f any unary function letter.

Also the local consistency connective $^\circ(.)$ has a non transparent behaviour, that is:

Proposition 5.2 *Sequents of the form $x = y, {}^\circ A(x) \vdash {}^\circ A(y)$, such that x, y are different free variables and $A(z)$ is any atomic formula with z free variable, are in general not* **PRACI**-*provable.*

Proof. Suppose ad absurdum that $x = y, {}^\circ(Z(x) = 0) \vdash {}^\circ(Z(y) = 0)$ is the root of a normal **PRACI**-proof Q, where Z is the zero-function. By free-cut elimination property, by recalling that all non-logical axioms of **PRACI** are sequents of atomic formulas, and that the **PRACI**-induction acts on atomic formulas only, each ancestor in Q of $^\circ(Z(x) = 0)$ in the antecedent must be introduced by weakenings only, and never is the constraint formula of a $\neg - L3$ rule . If we delete such wakenings, and also delete weakenings introducing atomic formulas, in the most general case we get a normal proof Q' of $x = y \vdash {}^\circ(Z(y) = 0)$. We assume that in Q and Q' all the atomic cuts occur above any propositional rule, so that the premise of the Q'-root is, in the most general case, a $R\,\mathrm{Ci}-$premise of the form $x = y, Z(y) = 0 \wedge \neg Z(y) = 0$ $\vdash$. Then, by the free-cut elimination, we have that $x = y, Z(y) = 0 \vdash$ must be provable. Since $\vdash Z(y)=0$ is a **PRACI**-axiom, we get $x=y\vdash$ as a theorem, which is absurd. □

The definitive arguments to conclude that $\neg$ and $^\circ(.)$ are in **PRACI** intensional connectives will be given in the sequel through provability logic statements. First, we mention what standard provabilty logic statements are preserved in **PRACI**:

Proposition 5.3 *If* **PRACI** *proves* $\vdash B$*, then* **PRACI** *proves* $\vdash \mathrm{Pr}_{\mathbf{PRACI}}(\#B)$*.*
Moreover, the following sequents are **PRACI** *-provable: (D1)* $\mathrm{Pr}_{\mathbf{PRACI}}(\#A\to B) \vdash \mathrm{Pr}_{\mathbf{PRACI}}(\#A) \to \mathrm{Pr}_{\mathbf{PRACI}}(\#B)$; *(D2)* $\mathrm{Pr}_{\mathbf{PRACI}}(\#A\to B) \wedge \mathrm{Pr}_{\mathbf{PRACI}}(\#A) \vdash \mathrm{Pr}_{\mathbf{PRACI}}(\#B)$; *(D3)* $\vdash \mathrm{Pr}_{\mathbf{PRACI}}(\#A\wedge B) \leftrightarrow \mathrm{Pr}_{\mathbf{PRACI}}(\#A) \wedge \mathrm{Pr}_{\mathbf{PRACI}}(\#B)$; *(D4)*$\mathrm{Pr}_{\mathbf{PRACI}}(\#B) \vdash \mathrm{Pr}_{\mathbf{PRACI}}(\#\,\mathrm{Pr}_{\mathbf{PRACI}}(\#B))$.

Furthermore, it must be remarked that the proof-strength of **PRACI** is relevant, for example:

Theorem 5.4 **PRACI** *proves the consistency (non triviality) of each consistent (non trivial) finitely axiomatized subsystem of* **PRA** *(of* **PRACI***).*

Proof. Let **V** be any finitely axiomatized subsystem as mentioned in the thesis. By fundamental results of classical proof-theory (see [19]) we have that **PRA** proves $\vdash \neg\,\mathrm{Pr}_{\mathbf{V}}(\#\vdash)$ and then $\mathrm{Pr}_{\mathbf{V}}(\#\vdash)\vdash$. By the predicate calculus **LK**, the atomic sequent $Prov_{\mathbf{V}}(a,\#\vdash)\vdash$, a free variable, is **PRA**-provable. Then, by Lemma 3.8, it is **PRACI**-provable too. □

Corollary 5.5 *Let* **V** *be any finitely axiomatized subsystem of* **PRA** *or* **PRACI***. Then* **PRACI** *proves* $\vdash {}^\circ\,\mathrm{Pr}_{\mathbf{V}}(\#\vdash)$.

Notwithstanding such meaningful proof power, the main tool of classical Provability Logic fails in **PRACI**. We recall that Gödel's Diagonal Lemma for **PRA** (see e.g. [21,22]) states that:

Lemma 5.6 *(Classical Diagonal Lemma) Let $A(u)$ be a* **PRA***-formula in which the free variable u occurs. Then there is a formula B such that $A(\#B)\leftrightarrow B$ is* **PRA***-provable. Moreover, $A(\#B)$ and B have exactly the same free variables.*

We have that:

Theorem 5.7 *Classical Diagonal Lemma does not hold for* **PRACI***.*

Theorem 5.8 *(Paraconsistent diagonal lemma for* **PRACI***) Let $A(u)$ be a* **PRACI***-formula in which the free variable u occurs. Then there are a formula B and a finite set $\{^\circ D_s\}_{s=1,\ldots,d}$ of local consistency assertions such that:*

i)the sequent $\{(\forall x_j)^\circ D_s\}_{s=1,\ldots,d}\vdash A(\#B)\leftrightarrow B$ is **PRACI***-provable;*

ii) $A(\#B)$ and B have exactly the same free-variable set V and the free variables occurring in $\{(\forall x_j)^\circ D_s\}_{s=1,\ldots,d}$ belong to V;

iii) each formula D_s can be obtained by term renaming from a proper subformula G_s of $A(u)$ such that G_s has not the form $^\circ F$ and $\neg G_s$ too is an $A(u)$-subformula.

The classical form of diagonal lemma can be re-obtained in **PRACI** for a significant particular case, due to the **PRACI**- provability of some classes of local consistency assertions:

Proposition 5.9 *Let $A(u)$ be a* **PRACI***-formula in which the free variable u occurs, such that each negated $A(u)-$subformula has the form $\neg^\circ G$. Then the thesis of the Diagonal Lemma holds in* **PRACI** *for A in the classical way.*

The lack of diagonal lemma in its standard form shows that the proofs of Gödel's theorem cannot be the same as in the classical case. On the other hand, the antecedent of Paraconsistent Diagonal Lemma shows the relevant role of local consistency assertions in providing new formulations of important principles of classical selfreference.

We know that in the classical case $A \vdash \mathrm{Pr}_{\mathbf{PRA}}(\#A)$ is **PRA**-provable for each A which is a Σ_1-formula (see [2,22]). In the paracosistent case such property in general does not hold:

Proposition 5.10 *The sequent $B \vdash \mathrm{Pr}_{\mathbf{PRACI}}(\#B)$ is in general not* **PRACI***-provable for any arbitrary quantifier free formula B; in particular, if A is any quantifier free non negated formula such that $\vdash \neg A$ is not* **PRACI***-provable, then $\neg A \vdash \mathrm{Pr}_{\mathbf{PRACI}}(\#\neg A)$ is not* **PRACI***-provable.*

Proposition 5.11 *If $^\circ B$ is not a* **PRACI***−theorem, then $^\circ B \vdash \mathrm{Pr}_{\mathbf{PRACI}}(\#^\circ B)$ is not* **PRACI***-provable.*

Proof. Assume ad absurdum that a normal **PRACI**-proof Q of $^\circ B \vdash \mathrm{Pr}_{\mathbf{PRACI}}(\#^\circ B)$ exists. By free-cut elimination, $^\circ B$ in the antecedent can be introduced in Q by weakenings only, and no ancestor of it can be the constraint formula of a $\neg - L3$ rule. If we delete such weakenings we have that $\vdash \mathrm{Pr}_{\mathbf{PRACI}}(\#^\circ B)$ is provable. Then, since **PRACI** is not trivial, a normal proof P of $\vdash \mathrm{Pr}_{\mathbf{PRACI}}(\#^\circ B)$ exists, such that the root succedent is introduced in P also by a $\exists$-R inference having the premise of the form: $\vdash X_{\mathrm{Prov}-\mathbf{PRACI}}(t_1, \#^\circ B) = 1, ..., X_{\mathrm{Prov}-\mathbf{PRACI}}(t_m, \#^\circ B) = 1$, with $t_1, ..., t_m$, closed terms, $m \geq 1$. But, by hypotheses, each formula in such premise is a false ground recursive relation, and the normal proof of such premise is a **PRA**-proofs too. This is absurd, since **PRA** does not prove any disjunction of false ground recursive relations. □

Therefore, the **PRACI**-logical negation $\neg$ anf the local consistency connective $^\circ(.)$ are intensional, and, moreover, cannot be expressed through Σ_1-formulas. Thus, for example, $\neg 1 = 1 \vdash \mathrm{Pr}_{\mathbf{PRACI}}(\#\neg 1 = 1)$ and $^\circ(1 = 1) \vdash \mathrm{Pr}_{\mathbf{PRACI}}(\#^\circ(1 = 1))$ are not **PRACI**-provable. Observe however that for the boolean negation of a recursive relation the property is preserved , that is $X_=(x, y) = 0 \vdash \mathrm{Pr}_{\mathbf{PRACI}}(\#X_=(x, y) = 0)$, x, y, free variables, is **PRACI**-provable.

We wish to emphasize the fact that the sentences expressing the non triviality of a **PRACI**-based system **T** are in general not **PRACI**-equivalent. In particular:

Proposition 5.12 *Let B any non* **T***-provable sentence of a paraconsistent system* $\mathbf{T} \equiv \mathbf{PRACI} + \mathbf{AxT}$ *whose non-triviality is not* **PRA***-provable. Then neither*

$\neg\operatorname{Pr}_{\mathbf{T}}(\# \vdash) \vdash \neg\operatorname{Pr}_{\mathbf{T}}(\#B)$ *nor* $\neg\operatorname{Pr}_{\mathbf{T}}(\#B) \vdash \neg\operatorname{Pr}_{\mathbf{T}}(\# \vdash)$ *are* **PRACI**-*provable.*

Proof. First, we establish that, if S is any non **T**-provable sequent, then $\operatorname{Pr}_{\mathbf{T}}(\#S) \vdash$ is not **PRACI**-provable. Indeed, if so it is we would have, by predicate calculus **CI**, $Prov_{\mathbf{T}}(a, \#S) \vdash$ in **PRACI** and then, by Lemma 3.8, $Prov_{\mathbf{T}}(a, \#S) \vdash$ should be **PRA**-provable, against the hypotheses. Thus, since $\vdash \neg\operatorname{Pr}_{\mathbf{T}}(\#S)$ cannot be **PRACI**-provable, the thesis follows from free-cut elimination. □

Remark that the non-triviality statements mentioned above do not play equivalent roles w.r.t. classical **PRA**. For example, $\neg\operatorname{Pr}_{\mathbf{PRACI}}(\# \vdash) \leftrightarrow \neg\operatorname{Pr}_{\mathbf{PRA}}(\# \vdash)$ is **PRA**-provable, but $\neg\operatorname{Pr}_{\mathbf{PRACI}}(\#\neg(1 = 1 \wedge \neg 1 = 1)) \rightarrow \neg\operatorname{Pr}_{\mathbf{PRA}}(\#\neg(1 = 1 \wedge \neg 1 = 1))$ is false in the standard model **N**.

6 Relations between local and global consistency in PRACI-Provability Logic

Proposition 6.1 *The schema* $^\circ A \rightarrow A$ *makes* **PRACI** *trivial.*

Proof. Consider any formula B which has not the form $^\circ F$ an such that $B \vdash$ is **PRACI**-provable. Then, by $\wedge - R$ we get $B \wedge \neg B \vdash$ from which, by $R\,\mathrm{Ci}$, we have $\vdash {}^\circ B$. Thus, from the schema-instance $^\circ B \rightarrow B$ we get $\vdash B$, that with $B \vdash$ produces the empty sequent. □

Proposition 6.2 *Let* B *any sentence such that* $B \vdash \operatorname{Pr}_{\mathbf{PRACI}}(\#B)$ *is* **PRACI**-*provable. Then the foollwing sequents are* **PRACI**-*provable: i)* $\vdash \operatorname{Pr}_{\mathbf{PRACI}}(\#B) \vee {}^\circ B$ *and ii)* $\vdash \neg{}^\circ B \rightarrow \operatorname{Pr}_{\mathbf{PRACI}}(\#B)$.

Proof. Both theses follow from the application of the proper **PRACI** rules $R\,\mathrm{Ci}$ and $\neg - L4$. □

Note that the propositions above do not hold for **PCA**.

Lemma 6.3 *Assume that the formula* B *is not a negated formula and has not the form* $^\circ F$*. Then, if* **CI** *proves* $\vdash {}^\circ B$*, the formula* B *is not* **CI**-*provable.*

Proof. By the cut elimination for **CI**-proofs, the proof Q of $\vdash {}^\circ B$ must have a $R\,\mathrm{Ci}$ as end-rule, with premise $B \wedge \neg B \vdash$. By cut elimination, this implies that either $B \vdash$ or $\neg B \vdash$ is **CI**-provable. The latter case would be absurd, by the hypotheses on B and by cut elimination. Then, $B \vdash$ is **CI**-provable. Therefore, if $\vdash B$ would be **CI**-provable we obtain the empty sequent, against the cut-elimination for **CI**.□

If $cut - el(\mathbf{CI})$ is the formula representing in the **PRA**-language the cut-elimiantion property of **CI**, we have:

Lemma 6.4 *The following sequents are both* **PRA**- *and* **PRACI**-*provable: i)* $\vdash cut - el(\mathbf{CI})$*; ii)* $\vdash \neg\operatorname{Pr}_{\mathbf{CI}}(\# \vdash)$.

Note that for **CI** the non triviality is a straightforward consequence of cut-elimination. Conversely, it is obvious that the stated free-cut elimination for **PRA** and **PRACI** does not imply their consistency or non-triviality.

Theorem 6.5 *(Fundamental relation between local and global consistency) The proof of Lemma 6.3 can be translated into* **PRA** *and* **PRACI** *so that :*

$$\vdash \mathrm{Pr}_{\mathbf{CI}}(\#^\circ B) \to \neg\, \mathrm{Pr}_{\mathbf{CI}}(\#B)$$

is **PRACI**-*provable for each sentence B which has not the form $^\circ F$.*

Proof. The starting point is the **PRA**-provability of $\vdash \mathrm{Pr}_{\mathbf{CI}}(\#^\circ B) \to \neg\, \mathrm{Pr}_{\mathbf{CI}}(\#B)$, due to the standard representation power of **PRA** (see [21,22]). Then, it can be shown that the **PRA**-proof must also be a **PRACI**-proof. □

The *fundamental relation* connects the provability of a local consistency assertion with the global non triviality of the system. An important fact is that the fundamental relation is true for **PRACI** too:

Lemma 6.6 *Assume that the formula B has not the form $^\circ F$, and that* **PRACI** *does not prove the empty sequent. Then, if* **PRACI** *proves $\vdash {}^\circ B$, the formula B is not* **PRACI** *-provable.*

Proof. Without any loss of generality we can suppose that in the normal proof Q of $\vdash {}^\circ B$ all atomic cuts occur above each logical rule. Then, recalling that $^\circ B$ cannot be introduced in Q by Induction rules, we proceed as in the proof of Lemma 6.3, by employing free-cut elimination. □

The main goal is to select a minimal constructive **PRACI**-extension that proves the fundamental relation for **PRACI**.

Let $free{-}cut{-}el(\mathbf{PRACI})$ be the formula representing the free-cut elimination property for **PRACI**. Then we have the following preliminary fact:

Proposition 6.7 *Assume that the formula B has not the form $^\circ F$, and that* **PRACI** *does not prove the empty sequent. Then the proof of Lemma 6.6 can be translated into* **PRA** *and* **PRACI** *so that :*

$$free-cut-el(\mathbf{PRACI}) \wedge {}^\circ\, \mathrm{Pr}_{\mathbf{PRACI}}(\# \vdash) \wedge \neg\, \mathrm{Pr}_{\mathbf{PRACI}}(\# \vdash) \wedge \mathrm{Pr}_{\mathbf{PRACI}}(\#^\circ B) \vdash \neg\, \mathrm{Pr}_{\mathbf{PRACI}}(\#B)$$

is **PRACI**-*provable.*

A *weakened Hilbert's program* could be so declared: to find a very weak and constructive **PRACI**-extension **W** such that the *fundamental relation for* **PRACI**, i.e. the sequent $\mathrm{Pr}_{\mathbf{PRACI}}(\#^\circ B) \vdash \neg\, \mathrm{Pr}_{\mathbf{PRACI}}(\#B)$, is **W**-provable. Thus, the problem of proving the non triviality of **PRACI** (i.e a *global* selfreference statement) could be reduced to the provability of suitable *local* consistency assertions−that is, **PRACI** *almost* would establish its own non triviality. In essence, the conjecture is the following: the **CI**-based paraconsistent arithmetical systems have, w.r.t non-triviality, more constructive and efficient proof capabilities than that owned by classical atithmetical systems w.r.t. consistency.

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