Available online at www.sciencedirect.com

ScienceDirect

Electronic Notes in Theoretical Computer Science 345 (2019) 3–35

Electronic Notes in Theoretical Computer Science

www.elsevier.com/locate/entcs

Domain-complete and LCS-complete Spaces

Matthew de Brecht [a,1,2]

[a] *Graduate School of Human and Environmental Studies , Kyoto University, Kyoto, Japan*

Jean Goubault-Larrecq [b,3,4] Xiaodong Jia [b,3,5]
Zhenchao Lyu [b,3,6]

[b] *LSV, ENS Paris-Saclay, CNRS, Université Paris-Saclay, France*

Abstract

We study G_δ subspaces of continuous dcpos, which we call domain-complete spaces, and G_δ subspaces of locally compact sober spaces, which we call LCS-complete spaces. Those include all locally compact sober spaces—in particular, all continuous dcpos—, all topologically complete spaces in the sense of Čech, and all quasi-Polish spaces—in particular, all Polish spaces. We show that LCS-complete spaces are sober, Wilker, compactly Choquet-complete, completely Baire, and $\odot$-consonant—in particular, consonant; that the countably-based LCS-complete (resp., domain-complete) spaces are the quasi-Polish spaces exactly; and that the metrizable LCS-complete (resp., domain-complete) spaces are the completely metrizable spaces. We include two applications: on LCS-complete spaces, all continuous valuations extend to measures, and sublinear previsions form a space homeomorphic to the convex Hoare powerdomain of the space of continuous valuations.

Keywords: Topology, domain theory, quasi-Polish spaces, G_δ subsets, continuous valuations, measures

1 Motivation

Let us start with the following question: for which class of topological spaces X is it true that every (locally finite) continuous valuation on X extends to a measure on X, with its Borel σ-algebra? The question is well-studied, and Klaus Keimel and

---

[1] The first author was supported by JSPS Core-to-Core Program, A. Advanced Research Networks and by JSPS KAKENHI Grant Number 18K11166.

[2] Email: matthew@i.h.kyoto-u.ac.jp

[3] This research was partially supported by Labex DigiCosme (project ANR-11-LABEX-0045-DIGICOSME) operated by ANR as part of the program "Investissement d'Avenir" Idex Paris-Saclay (ANR-11-IDEX-0003-02).

[4] Email: goubault@ens-paris-saclay.fr

[5] Email: jia@lsv.fr

[6] Email: zhenchaolyu@gmail.com

https://doi.org/10.1016/j.entcs.2019.07.014

1571-0661/

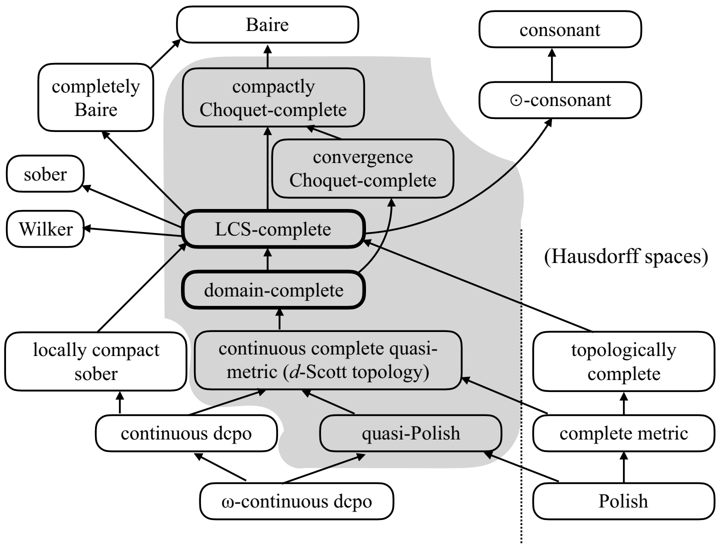

Fig. 1. Domain-complete and LCS-complete spaces in relation to other classes of spaces

Jimmie Lawson have rounded it up nicely in [24]. A result by Mauricio Alvarez-Manilla *et al.* [2] (see also Theorem 5.3 of the paper by Keimel and Lawson) states that every locally compact sober space fits.

Locally compact sober spaces are a pretty large class of spaces, including many non-Hausdorff spaces, and in particular all the continuous dcpos of domain theory. However, such a result will be of limited use to the ordinary measure theorist, who is used to working with Polish spaces, including such spaces as Baire space $\mathbb{N}^{\mathbb{N}}$, which is definitely not a locally compact space.

It is not too hard to extend the above theorem to the following larger class of spaces (and to drop the local finiteness assumption as well):

Theorem 1.1 *Let X be a (homeomorph of a) G_δ subset of a locally compact sober space Y. Every continuous valuation ν on X extends to a measure on X with its Borel σ-algebra.*

We defer the proof of that result to Section 18. The point is that we do have a measure extension theorem on a class of spaces that contains both the continuous dcpos of domain theory and the Polish spaces of topological measure theory. We will call such spaces *LCS-complete*, and we are aware that this is probably not an optimal name. *Topologically complete* would have been a better name, if it had not been taken already [5].

Another remarkable class of spaces is the class of *quasi-Polish* spaces, discovered and studied by the first author [7]. This one generalizes both ω-continuous dcpos and Polish spaces, and we will see in Section 5 that the class of LCS-complete spaces is a proper superclass. We will also see that there is no countably-based LCS-complete space that would fail to be quasi-Polish. Hence LCS-complete spaces can be seen as an extension of the notion of quasi-Polish spaces, and the extension is strict only for non-countably based spaces.

Generally, our purpose is to locate LCS-complete spaces, as well as the related *domain-complete* spaces inside the landscape formed by other classes of spaces. The result is summarized in Figure 1. The gray area is indicative of what happens with countably-based spaces: for such spaces, all the classes inside the the gray area coincide.

We proceed as follows. We recall some background in Section 2, and we give basic definitions in Section 3. The rest of the paper (apart from the final Section 18, which is independent of the others, and where we prove the promised Theorem 1.1), is the result of our findings on domain-complete and LCS-complete spaces, in no particular order. We show that continuous complete quasi-metric spaces, quasi-Polish spaces and topologically complete spaces are all LCS-complete in Sections 4–6. Then we show that all LCS-complete spaces are sober (Section 7), Wilker (Section 8), Choquet-complete and in fact a bit more (Section 9), Baire and even completely Baire (Section 10), consonant and even $\odot$-consonant (Section 12). In the process, we explore the Stone duals of domain-complete and LCS-complete spaces in Section 11. While the class of LCS-complete spaces is strictly larger than the class of domain-complete spaces, in Section 9, we also show that for countably-based spaces, LCS-complete, domain-complete, and quasi-Polish are synonymous. We give a first application in Section 13: when X is LCS-complete, the Scott and compact-open topologies on the space $\mathcal{L}X$ of lower semicontinuous maps from X to $\overline{\mathbb{R}}_+$ coincide; hence $\mathcal{L}X$ with the Scott topology is locally convex, allowing us to apply an isomorphism theorem [14, Theorem 4.11] beyond core-compact spaces, to the class of all LCS-complete spaces. In the sequel (Sections 14–16), we explore the properties of the categories of LCS-complete, resp. domain-complete spaces: countable products and arbitrary coproducts exist and are computed as in topological spaces, but those categories have neither equalizers nor coequalizers, and are not Cartesian-closed; we also characterize the exponentiable objects in the category of quasi-Polish spaces as the countably-based locally compact sober spaces. Section 17 is of independent interest, and characterizes the compact saturated subsets of LCS-complete spaces, in a manner reminiscent of a well-known theorem of Hausdorff on complete metric spaces. We prove Theorem 1.1 in Section 18, and we conclude in Section 19.

Acknowledgement

The second author thanks Szymon Dolecki for pointing him to [9, Proposition 7.3].

2 Preliminaries

We assume that the reader is familiar with domain theory [12,1], and with basic notions in non-Hausdorff topology [13].

We write **Top** for the category of topological spaces and continuous maps.

$\mathbb{R}_+$ denotes the set of non-negative real numbers, and $\overline{\mathbb{R}}_+$ is $\mathbb{R}_+$ plus an element ∞, larger than all others. We write $\leq$ for the underlying preordering of any preordered set, and for the specialization preordering of a topological space. The notation $\uparrow A$ denotes the upward closure of A, and $\downarrow A$ denotes its downward closure. When $A = \{y\}$, this is simply written $\uparrow y$, resp. $\downarrow y$. We write $\bigcup^{\uparrow}$ for directed unions, $\sup^{\uparrow}$ for directed suprema, and $\bigcap^{\downarrow}$ for filtered intersections.

Compactness does not imply separation, namely, a compact set is one such that one can extract a finite subcover from any open cover. A *saturated* subset is a subset that is the intersection of its open neighborhoods, equivalently that is an upwards-closed subset in the specialization preordering.

We write $\ll$ for the way-below relation on a poset Y, and $\twoheaduparrow y$ for the set of points $z \in Y$ such that $y \ll z$.

We write $int(A)$ for the interior of a subset A of a topological space X, and $\mathcal{O}\,X$ for its lattice of open subsets.

A space is *locally compact* if and only if every point has a base of compact saturated neighborhoods. It is *sober* if and only if every irreducible closed subset is the closure of a unique point. It is *well-filtered* if and only if given any filtered family $(Q_i)_{i \in I}$ of compact saturated subsets and every open subset U, if $\bigcap_{i \in I} Q_i \subseteq U$ then $Q_i \subseteq U$ for some $i \in I$. In a well-filtered space, the intersection $\bigcap_{i \in I} Q_i$ of any such filtered family is compact saturated. Sobriety implies well-filteredness, and the two properties are equivalent for locally compact spaces.

A space X is *core-compact* if and only if $\mathcal{O}\,X$ is a continuous lattice. Every locally compact space is core-compact, and in that case the way-below relation on open subsets is given by $U \Subset V$ if and only if $U \subseteq Q \subseteq V$ for some compact saturated set Q. Conversely, every core-compact sober space is locally compact.

3 Definition and basic properties

A G_δ subset of a topological space Y is the intersection of a countable family $(W_n)_{n \in \mathbb{N}}$ of open subsets of Y. Replacing W_n by $\bigcap_{i=0}^{n} W_i$ if needed, we may assume that the family is *descending*, namely that $W_0 \supseteq W_1 \supseteq \cdots \supseteq W_n \cdots$.

Definition 3.1 A *domain-complete space* is a (homeomorph of a) G_δ subset of a continuous dcpo, with the subspace topology from the Scott topology.

An *LCS-complete space* is a (homeomorph of a) G_δ subset of a locally compact sober space, with the subspace topology.

Remark 3.2 There is a pattern here. For a class $\mathcal{C}$ of topological spaces, one might call *$\mathcal{C}$-complete* any homeomorph of a G_δ subset of a space in $\mathcal{C}$. For example, if $\mathcal{C}$ is the class of stably (locally) compact spaces, we would obtain *SC-complete* (resp., *SLC-complete*) spaces. By an easy trick which we shall use in Lemma 14.1, SC-complete and SLC-complete are the same notion.

Proposition 3.3 *Every locally compact sober space is LCS-complete, in particular every quasi-continuous dcpo is LCS-complete. Every continuous dcpo is domain-complete. Every domain-complete space is LCS-complete.*

Proof. Every space is G_δ in itself. Every quasi-continuous dcpo is locally compact (being locally finitary compact [13, Exercise 5.2.31]) and sober [13, Exercise 8.2.15]. The last part follows from the fact that every continuous dcpo is locally compact and sober. □

We will see other examples of domain-complete spaces in Sections 4, 5, and 6.

Remark 3.4 Given any continuous dcpo (resp., locally compact sober space) Y, and any descending family $(W_n)_{n\in\mathbb{N}}$ of open subsets of Y, $X \stackrel{\text{def}}{=} \bigcap^{\downarrow}_{n\in\mathbb{N}} W_n$ is domain-complete (resp., LCS-complete). We can then define $\mu\colon Y \to \overline{\mathbb{R}}_+$ by $\mu(y) \stackrel{\text{def}}{=} \inf\{1/2^n \mid y \in W_n\}$. This is continuous from Y to $\overline{\mathbb{R}}^{op}_+$, i.e., $\overline{\mathbb{R}}_+$ with the Scott topology of the reverse ordering $\geq$. Indeed, $\mu^{-1}([0,a)) = W_n$ where n is the smallest natural number such that $1/2^n < a$. Then X is equal to the *kernel* $\ker\mu \stackrel{\text{def}}{=} \mu^{-1}(\{0\})$ of μ. Conversely, any space that is (homeomorphic to) the kernel of some continuous map $\mu\colon Y \to \overline{\mathbb{R}}^{op}_+$ from a continuous dcpo (resp., locally compact space) Y is equal to $\bigcap^{\downarrow}_{n\in\mathbb{N}} \mu^{-1}([0,1/2^n))$, hence is domain-complete (resp., LCS-complete). This should be compared with Keye Martin's notion of *measurement* [28], which is a map μ as above with the additional property that for every $x \in \ker\mu$, for every open neighborhood V of x in Y, there is an $\epsilon > 0$ such that $\downarrow x \cap \mu^{-1}([0,\epsilon)) \subseteq V$.

4 Continuous complete quasi-metric spaces

A *quasi-metric* on a set X is a map $d\colon X \times X \to \overline{\mathbb{R}}_+$ satisfying the laws: $d(x,x) = 0$; $d(x,y) = d(y,x) = 0$ implies $x = y$; and $d(x,z) \leq d(x,y) + d(y,z)$ (*triangular inequality*). The pair X, d is then called a *quasi-metric space*.

Given a quasi-metric space, one can form its poset $\mathbf{B}(X,d)$ of *formal balls*. Its elements are pairs (x,r) with $x \in X$ and $r \in \mathbb{R}_+$, and are ordered by $(x,r) \leq^{d^+} (y,s)$ if and only if $d(x,y) \leq r - s$. Instead of spelling out what a complete (a.k.a., *Yoneda-complete* quasi-metric space) is, we rely on the Kostanek-Waszkiewicz Theorem [25] (see also [13, Theorem 7.4.27]), which characterizes them in terms of $\mathbf{B}(X,d)$: X, d is *complete* if and only if $\mathbf{B}(X,d)$ is a dcpo.

We will also say that X, d is a *continuous complete* quasi-metric space if and only if $\mathbf{B}(X,d)$ is a continuous dcpo. This is again originally a theorem, not a definition [16, Theorem 3.7]. The original, more complex definition, is due to Mateusz Kostanek and Paweł Waszkiewicz.

There is a map $\eta\colon X \to \mathbf{B}(X,d)$ defined by $\eta(x) \stackrel{\text{def}}{=} (x,0)$. The coarsest topology that makes η continuous, once we have equipped $\mathbf{B}(X,d)$ with its Scott topology, is called the *d-Scott topology* on X [13, Definition 7.4.43]. This is our default topology on quasi-metric spaces, and turns η into a topological embedding.

Every poset X can be seen as a quasi-metric space with equipping it with the quasi-metric $d(x,y) \stackrel{\text{def}}{=} 0$ if $x \leq y$, ∞ otherwise. In that case, the d-Scott topology coincides with the Scott topology [25, Example 1.8], and if X is a continuous dcpo then X, d is continuous complete [25, Example 3.12].

The d-Scott topology coincides with the usual open ball topology when d is a metric (i.e., $d(x, y) = d(y, x)$ for all x, y) or when X, d is a so-called Smyth-complete quasi-metric space [13, Propositions 7.4.46, 7.4.47]. We will not say what Smyth-completeness is (see Section 7.2, ibid.), except that every Smyth-complete quasi-metric space is continuous complete, by the Romaguera-Valero theorem [33] (see also [13, Theorem 7.3.11]).

Theorem 4.1 *For every continuous complete quasi-metric space X, d, the space X with its d-Scott topology is domain-complete.*

Proof. Every standard quasi-metric space X, d embeds as a G_δ set into $\mathbf{B}(X, d)$ [16, Proposition 2.6], and every complete quasi-metric space is standard (Proposition 2.2, ibid.) Explicitly, X is homeomorphic to $\bigcap_{n \in \mathbb{N}} W_n$ where $W_n \stackrel{\text{def}}{=} \{(x, r) \in \mathbf{B}(X, d) \mid r < 1/2^n\}$ is Scott-open. Since X, d is continuous complete, $\mathbf{B}(X, d)$ is a continuous dcpo. □

When d is a metric, $\mathbf{B}(X, d)$ is a continuous poset [11, Corollary 10], with $(x, r) \ll (y, s)$ if and only if $d(x, y) < r - s$; also, $\mathbf{B}(X, d)$ is a dcpo if and only if X, d is complete in the usual, Cauchy sense [11, Theorem 6]. Hence every complete metric space is continuous complete in our sense.

Corollary 4.2 *Every complete metric space is domain-complete in its open ball topology.* □

5 Quasi-Polish spaces

Quasi-Polish spaces were introduced in [7], and can be defined in many equivalent ways. The original definition is: a quasi-Polish space is a separable Smyth-complete quasi-metric space X, d, seen as a topological space with the open ball topology. By *separable* we mean the existence of a countable dense subset in X with the open ball topology of d^{sym}, where d^{sym} is the symmetrized metric $d^{sym}(x, y) \stackrel{\text{def}}{=} \max(d(x, y), d(y, x))$. By a lemma due to Künzi [27], a quasi-metric space is separable if and only if its open ball topology is countably-based.

Since Smyth-completeness implies continuous completeness and also that the open ball and d-Scott topologies coincide [13, Theorem 7.4.47], Theorem 4.1 implies:

Proposition 5.1 *Every quasi-Polish space is domain-complete.*

Not every domain-complete space is quasi-Polish. In fact, the following remark implies that not every domain-complete space is even first-countable. We will see that all countably-based domain-complete spaces *are* quasi-Polish in Section 9.

Remark 5.2 Let us fix an uncountable set I, and let $X \stackrel{\text{def}}{=} Y \stackrel{\text{def}}{=} \mathbb{P}(I)$, with the Scott topology of inclusion. This is an algebraic, hence continuous dcpo, hence a domain-complete space. I is its top element. We claim that every collection of open neighborhoods of I whose intersection is $\{I\}$ must be uncountable. Imagine we had a countable collection $(V_n)_{n\in\mathbb{N}}$ of open neighborhoods of I whose intersection is $\{I\}$. For each $n \in \mathbb{N}$, I is in some basic open set $\uparrow A_n \stackrel{\text{def}}{=} \{B \in \mathbb{P}(I) \mid A_n \subseteq B\}$ (where each A_n is a finite subset of I) included in V_n. Then $\bigcap_{n\in\mathbb{N}} \uparrow A_n$ is still equal to $\{I\}$. However, $\bigcap_{n\in\mathbb{N}} \uparrow A_n = \uparrow A_\infty$, where A_∞ is the countable set $\bigcup_{n\in\mathbb{N}} A_n$, and must contain some (uncountably many) points other than I.

6 Topologically complete spaces

In 1937, Eduard Čech defined *topologically complete* spaces as those topological spaces that are G_δ subsets of some compact Hausdorff space, or equivalently those completely regular Hausdorff spaces that are G_δ subsets of their Stone-Čech compactification [5], and proved that a metrizable space is completely metrizable if and only if it is topologically complete.

The following is then clear:

Fact 6.1 *Every topologically complete space in Čech's sense is LCS-complete.*

7 Sobriety

A $\mathbf{\Pi}^0_2$ subset of a topological space Y is a space of the form $\{y \in Y \mid \forall n \in \mathbb{N}, y \in U_n \Rightarrow y \in V_n\}$, where U_n and V_n are open in Y. Every G_δ subset of Y is $\mathbf{\Pi}^0_2$ (take $U_n = Y$ for every n), and every closed subset of Y is $\mathbf{\Pi}^0_2$ (take U_n equal to the complement of that closed subset for every n, and V_n empty). More generally, we consider *Horn* subsets of Y, defined as sets of the form $\{y \in Y \mid \forall i \in I, y \in U_i \Rightarrow y \in V_i\}$, where U_i, V_i are (not necessarily countably many) open subsets of Y.

Proposition 7.1 *Every Horn subset X of a sober space Y is sober. In particular, every LCS-complete space is sober.*

Proof. We prove the first part. In the case of $\mathbf{\Pi}^0_2$ subsets, that was already proved in [8, Lemma 4.2]. Let $X \stackrel{\text{def}}{=} \{y \in Y \mid \forall i \in I, y \in U_i \Rightarrow y \in V_i\}$, with U_i and V_i open. $\mathbb{P}(I)$, with the inclusion ordering, is an algebraic dcpo, whose finite elements are the finite subsets of I. Let $f : Y \to \mathbb{P}(I)$ map y to $\{i \in I \mid y \in U_i\}$, and g map y to $\{i \in I \mid y \in U_i \cap V_i\}$. Both are continuous, since $f^{-1}(\uparrow\{i_1, \cdots, i_k\}) = \bigcap_{j=1}^k U_{i_j}$ and $g^{-1}(\uparrow\{i_1, \cdots, i_k\}) = \bigcap_{j=1}^k U_{i_j} \cap V_{i_j}$ are open. The equalizer of f and g is $\{y \in Y \mid f(y) = g(y)\} = \{y \in Y \mid \forall i \in I, y \in U_i \Leftrightarrow y \in U_i \cap V_i\} = X$, with the subspace topology. But every equalizer of continuous maps from a sober space to a T_0 topological space is sober [13, Lemma 8.4.12] (note that "T_0" is missing from the statement of that lemma, but T_0-ness is required). $\square$

8 The Wilker condition

A space X satisfies *Wilker's condition*, or is *Wilker*, if and only if every compact saturated set Q included in the union of two open subsets U_1 and U_2 of X is included in the union of two compact saturated sets $Q_1 \subseteq U_1$ and $Q_2 \subseteq U_2$. The notion is used by Keimel and Lawson [24, Theorem 6.5], and is due to Wilker [35, Theorem 3]. Theorem 8 of the latter states that every KT_4 space, namely every space in which every compact subspace is T_4, is Wilker. In particular, every Hausdorff space is Wilker.

We will need the following lemma several times in this paper. The proof of Theorem 8.2 is typical of several arguments in this paper.

Lemma 8.1 *Let X be a subspace of a topological space Y. For every subset E of X,*

(i) *E is compact in X if and only if E is compact in Y;*

(ii) *if X is upwards-closed in Y, then E is saturated in X if and only if E is saturated in Y;*

(iii) *if X is a G_δ subset of Y, then E is G_δ in X if and only if E is G_δ in Y.*

Proof. 1. Assume E compact in X. For every open cover $(\widehat{U}_i)_{i\in I}$ of E by open subsets of Y, $(\widehat{U}_i \cap X)_{i\in I}$ is an open cover of E by open subsets of X. Hence E has a subcover $(\widehat{U}_i \cap X)_{i\in J}$, with J finite, and $(\widehat{U}_i)_{i\in J}$ is a finite subcover of the original cover of E, showing that E is compact in Y.

Conversely, if E is compact in Y (and included in X), we consider an open cover $(U_i)_{i\in I}$ of E by open subsets of X. For each $i \in I$, we can write U_i as $\widehat{U}_i \cap X$ for some open subset $\widehat{U}_i$ of Y. Then $(\widehat{U}_i)_{i\in I}$ is an open cover of E in X. We extract a subcover $(\widehat{U}_i)_{i\in J}$ with J finite. Since E is included in X, E is included in $X \cap \bigcup_{i\in J} \widehat{U}_i = \bigcup_{i\in J} U_i$. This shows that E is compact in X.

2. This follows from the fact that the specialization preordering on X is the restriction of the specialization preordering on Y to X. If E is saturated in X, then for every $x \in E$ and every y above x in Y, then y is in X since X is saturated, and then in E by assumption. Therefore E is saturated in Y. Conversely, if E is saturated in Y, then for every $x \in E$ and every y above x in X, y is also above x in Y, hence in E since E is saturated in Y. This shows that E is saturated in X.

3. Since X is G_δ in Y, X is equal to $\bigcap_{n\in\mathbb{N}} W_n$ where each W_n is open in Y.

If E is a G_δ subset of X, say $E \stackrel{\text{def}}{=} \bigcap_{m\in\mathbb{N}} U_m$, where each U_m is open in X, we write U_m as $\widehat{U}_m \cap X$ for some open subset $\widehat{U}_m$ of Y. It follows that E is equal to $\bigcap_{m,n\in\mathbb{N}} (\widehat{U}_m \cap W_n)$. This is a countable intersection of open subsets of Y, hence a G_δ subset.

Conversely, if E is a G_δ subset of Y, say $E \stackrel{\text{def}}{=} \bigcap_{m\in\mathbb{N}} \widehat{U}_m$, then since E is included in X, E is also equal to $\bigcap_{m\in\mathbb{N}} (\widehat{U}_m \cap X)$, showing that E is G_δ in X. $\square$

Theorem 8.2 *Every LCS-complete space is Wilker.*

Proof. We start by showing that every locally compact space Y is Wilker, and in fact satisfies the following stronger property: $(*)$ for every compact saturated subset Q of Y, for all open subsets U_1 and U_2 such that $Q \subseteq U_1 \cup U_2$, there are two compact saturated subsets Q_1 and Q_2 such that $Q \subseteq int(Q_1) \cup int(Q_2)$, $Q_1 \subseteq U_1$, and $Q_2 \subseteq U_2$. For each $x \in Q$, if x is in U_1, then we pick a compact saturated neighborhood Q_x of x included in U_1, and if x is in $U_2 \setminus U_1$, then we pick a compact saturated neighborhood Q'_x of x included in U_2. From the open cover of Q consisting of the sets $int(Q_x)$ and $int(Q'_x)$, we extract a finite cover. Namely, there are a finite set E_1 of points of $Q \cap U_1$ and a finite set E_2 of points of $Q \setminus U_1$ (hence in $Q \cap U_2$) such that $Q \subseteq \bigcup_{x \in E_1} int(Q_x) \cup \bigcup_{x \in E_2} int(Q'_x)$. We let $Q_1 \stackrel{\text{def}}{=} \bigcup_{x \in E_1} Q_x$, $Q_2 \stackrel{\text{def}}{=} \bigcup_{x \in E_2} Q'_x$.

Let X be (homeomorphic to) the intersection $\bigcap^{\downarrow}_{n \in \mathbb{N}} W_n$ of a descending sequence of open subsets of a locally compact sober space Y. Let Q be compact saturated in X, and included in the union of two open subsets U_1 and U_2 of X. Let us write U_1 as $\widehat{U}_1 \cap X$ where $\widehat{U}_1$ is open in Y, and similarly U_2 as $\widehat{U}_2 \cap X$. By Lemma 8.1, Q is compact saturated in Y. By property $(*)$, there are two compact saturated subsets Q_{01} and Q_{02} of Y such that $Q \subseteq int(Q_{01}) \cup int(Q_{02})$, $Q_{01} \subseteq \widehat{U}_1 \cap W_0$, $Q_{02} \subseteq \widehat{U}_2 \cap W_0$. By $(*)$ again, there are two compact saturated subsets Q_{11} and Q_{12} of Y such that $Q \subseteq int(Q_{11}) \cup int(Q_{12})$, $Q_{11} \subseteq int(Q_{01}) \cap W_1$, $Q_{12} \subseteq int(Q_{02}) \cap W_1$. Continuing this way, we obtain two compact saturated subsets Q_{n1} and Q_{n2} for each $n \in \mathbb{N}$ such that $Q \subseteq int(Q_{n1}) \cup int(Q_{n2})$, $Q_{(n+1)1} \subseteq int(Q_{n1}) \cap W_{n+1}$, and $Q_{(n+1)2} \subseteq int(Q_{n2}) \cap W_{n+1}$. Let $Q_1 \stackrel{\text{def}}{=} \bigcap^{\downarrow}_{n \in \mathbb{N}} Q_{n1} = \bigcap^{\downarrow}_{n \in \mathbb{N}} int(Q_{n1})$. This is a filtered intersection of compact saturated sets in a well-filtered space, hence is compact saturated. Since $Q_{(n+1)1} \subseteq W_{n+1}$ for every $n \in \mathbb{N}$, Q_1 is included in X, hence is compact saturated in X by Lemma 8.1. Similarly, $Q_2 \stackrel{\text{def}}{=} \bigcap^{\downarrow}_{n \in \mathbb{N}} Q_{n2} = \bigcap^{\downarrow}_{n \in \mathbb{N}} int(Q_{n2})$ is compact saturated in X.

We note that Q is included in $Q_1 \cup Q_2$. Otherwise, there would be a point x in Q and outside both Q_1 and Q_2, hence outside $int(Q_{m1})$ for some $m \in \mathbb{N}$ and outside $int(Q_{n1})$ for some $n \in \mathbb{N}$, hence outside $int(Q_{k1}) \cup int(Q_{k2})$ with $k \stackrel{\text{def}}{=} \max(m, n)$. That is impossible since $Q \subseteq int(Q_{k1}) \cup int(Q_{k2})$.

Finally, Q_1 is included in U_1 because $Q_1 \subseteq Q_{01} \cap X \subseteq \widehat{U}_1 \cap W_0 \cap X = U_1$, and similarly Q_2 is included in U_2. ☐

Remark 8.3 The proof of Theorem 8.2 shows that Q_1 and Q_2 are even compact G_δ subsets of X, being obtained as $\bigcap^{\downarrow}_{n \in \mathbb{N}} int(Q_{n1})$, hence also as $\bigcap^{\downarrow}_{n \in \mathbb{N}} int(Q_{n1}) \cap X$ (resp., $\bigcap^{\downarrow}_{n \in \mathbb{N}} int(Q_{n2}) \cap X$). This suggests that there are many compact G_δ sets in every LCS-complete space. Note that not all compact saturated sets are G_δ in general LCS-complete spaces: even the upward closures $\uparrow x$ of single points may fail to be G_δ, as Remark 5.2 demonstrates.

Remark 8.4 Pursuing Remark 3.2, every SC-complete space X is not only LCS-complete, but also *coherent*: the intersection of two compact saturated sets Q_1, Q_2 is compact. Indeed, let X be G_δ in some stably compact space Y; by Lemma 8.1, items 1 and 2, Q_1 and Q_2 are compact saturated in Y, then $Q_1 \cap Q_2$ is compact saturated in Y and included in X, hence compact in X. This implies that there are LCS-complete, and even domain-complete spaces, that are not SC-complete: take any non-coherent dcpo, for example $\mathbb{Z}^- \cup \{a, b\}$, where $\mathbb{Z}^-$ is the set of negative integers with the usual ordering, and a and b are incomparable and below $\mathbb{Z}^-$.

9 Choquet completeness

The *strong Choquet game* on a topological space X is defined as follows. There are two players, α and β. Player β starts, by picking a point x_0 and an open neighborhood V_0 of x_0. Then α must produce a smaller open neighborhood U_0 of x_0, i.e., one such that $U_0 \subseteq V_0$. Player β must then produce a new point x_1 in U_0, and a new open neighborhood V_1 of x_1, included in U_0, and so on. An *α-history* is a sequence $x_0, V_0, U_0, x_1, V_1, U_1, x_2, V_2, \cdots, x_n, V_n$ where $V_0 \supseteq U_0 \supseteq V_1 \supseteq U_1 \supseteq V_2 \supseteq \ldots \supseteq V_n$ is a decreasing sequence of opens and $x_0 \in U_0$, $x_1 \in U_1$, $x_2 \in U_2$, …, $x_{n-1} \in U_{n-1}$, $x_n \in V_n$, $n \in \mathbb{N}$. A *strategy* for α is a map σ from α-histories to open subsets U_n with $x_n \in U_n \subseteq V_n$, and defines how α plays in reaction to β's moves. (For details, see Section 7.6.1 of [13].)

X is *Choquet-complete* if and only if α has a winning strategy, meaning that whatever β plays, α has a way of playing such that $\bigcap_{n\in\mathbb{N}} U_n$ ($= \bigcap_{n\in\mathbb{N}} V_n$) is non-empty. X is *convergence Choquet-complete* if and only if α can always win in such a way that $(U_n)_{n\in\mathbb{N}}$ is a base of open neighborhoods of some point. The latter notion is due to Dorais and Mummert [10]. We introduce yet another, related notion: a space is *compactly Choquet-complete* if and only if α can always win in such that way that $(U_n)_{n\in\mathbb{N}}$ is a base of open neighborhoods of some non-empty compact saturated set. We do not assume the strategies to be stationary, that is, the players have access to all the points x_n and all the open sets U_n, V_n played earlier.

The following generalizes [16, Theorem 4.3], which states that every continuous complete quasi-metric space is convergence Choquet-complete in its d-Scott topology.

Proposition 9.1 *Every domain-complete space is convergence Choquet-complete. Every LCS-complete space is compactly Choquet-complete.*

Proof. Let X be the intersection of a descending sequence $(W_n)_{n\in\mathbb{N}}$ of open subsets of Y. Given any open subset U of X, we write $\widehat{U}$ for some open subset of Y such that $\widehat{U} \cap X = U$ (for example, the largest one).

We first assume that Y is a continuous dcpo. The proof is a variant of [13, Exercise 7.6.6]. We define α's winning strategy so that U_n is of the form $\twoheaduparrow y_n \cap X$ for some $y_n \in Y$. Given the last pair (x_n, V_n) played by β, x_n is the supremum of a directed family of elements way-below x_n. One of them will be in $\widehat{V_n} \cap W_n$, and also in $\twoheaduparrow y_{n-1}$ if $n \geq 1$, because $x_n \in V_n \subseteq \widehat{V_n}$, $x_n \in X \subseteq W_n$, and (if $n \geq 1$) $x_n \in U_{n-1} = \twoheaduparrow y_{n-1} \cap X \subseteq \twoheaduparrow y_{n-1}$. Pick one such element y_n from $\widehat{V_n} \cap W_n \cap \twoheaduparrow y_{n-1}$ (if $n \geq 1$, otherwise from $\widehat{V_n} \cap W_n$), and let α play $U_n \stackrel{\text{def}}{=} \twoheaduparrow y_n \cap X$, as announced. Formally, the strategy σ that we are defining for α is $\sigma(x_0, V_0, U_0, x_1, V_1, U_1, x_2, V_2, \cdots, x_n, V_n) \stackrel{\text{def}}{=} \twoheaduparrow y(x_0, V_0, U_0, x_1, V_1, U_1, x_2, V_2, \cdots, x_n, V_n) \cap X$, where $y(x_0, V_0, U_0, x_1, V_1, U_1, x_2, V_2, \cdots, x_n, V_n)$ is defined by induction on n as a point in $\widehat{V_n} \cap W_n$, and also in $\twoheaduparrow y(x_0, V_0, U_0, x_1, V_1, U_1, x_2, V_2, \cdots, x_{n-1}, V_{n-1})$ if $n \geq 1$.

Given any play $x_0, V_0, U_0, x_1, V_1, U_1, x_2, V_2, \cdots, x_n, V_n, U_n, \cdots$ in the game, let

$x \stackrel{\text{def}}{=} \sup_{n\in\mathbb{N}} y_n$ (where $y_n \stackrel{\text{def}}{=} y(x_0, V_0, U_0, x_1, V_1, U_1, x_2, V_2, \cdots, x_n, V_n)$). This is a directed supremum, since $y_0 \ll y_1 \ll \cdots \ll y_n \ll \cdots \ll x$. Since $y_n \in W_n$ for every $n \in \mathbb{N}$, x is in $\bigcap_{n\in\mathbb{N}} W_n = X$. For every $n \in \mathbb{N}$, we have $y_n \ll x$, so x is in $U_n = \Uparrow y_n \cap X$. In order to show that $(U_n)_{n\in\mathbb{N}}$ is a base of open neighborhoods of x in X, let U be any open neighborhood of x in X. Since $x = \sup_{n\in\mathbb{N}} y_n$, some y_n is in $\widehat{U}$, so $U_n = \Uparrow y_n \cap X$ is included in $\uparrow y_n \cap X \subseteq \widehat{U} \cap X = U$.

The argument is similar when Y is a locally compact sober space instead. Instead of picking a point y_n in $\widehat{V}_n \cap W_n$ (and in $\Uparrow y_{n-1}$ if $n \geq 1$), α now picks a compact saturated subset Q_n whose interior contains x_n, and included in $\widehat{V}_n \cap W_n$ (and in $int(Q_{n-1})$ if $n \geq 1$), and defines U_n as $int(Q_n) \cap X$. This is possible because Y is locally compact. We let $Q \stackrel{\text{def}}{=} \bigcap_{n\in\mathbb{N}} Q_n$. This is a filtered intersection, since $Q_0 \supseteq Q_1 \supseteq \cdots \supseteq Q_n \supseteq \cdots$.

Because Y is sober hence well-filtered, Q is compact saturated in Y. It is also non-empty: if $Q = \bigcap_{n\in\mathbb{N}} Q_n$ were empty, namely, included in $\emptyset$, then Q_n would be included in $\emptyset$ by well-filteredness, which is impossible since $x_n \in int(Q_n)$. Also, since $Q_n \subseteq W_n$ for every $n \in \mathbb{N}$, Q is included in $\bigcap_{n\in\mathbb{N}} W_n = X$. By Lemma 8.1, item 3, Q is a compact saturated subset of X.

Since $Q \subseteq Q_{n+1} \subseteq \widehat{V}_{n+1}$ for every $n \in \mathbb{N}$, we have $Q = Q \cap X \subseteq \widehat{V}_{n+1} \cap X = V_{n+1} \subseteq U_n$. In order to show that $(U_n)_{n\in\mathbb{N}}$ forms a base of open neighborhoods of Q in X, let U be any open neighborhood of Q in X. Then $\widehat{U}$ contains $Q = \bigcap_{n\in\mathbb{N}} Q_n$, so by well-filteredness some Q_n is included in $\widehat{U}$. Now $U_n = int(Q_n) \cap X$ is included in $\widehat{U} \cap X = U$. □

In the case of LCS-complete spaces, notice that $Q = \bigcap_{n\in\mathbb{N}} U_n$ is not only compact, but also a G_δ subset of X. This again suggests that there are many compact G_δ sets in every LCS-complete space, as in Remark 8.3.

The following—at last—justifies the "complete" part in "LCS-complete".

Theorem 9.2 *The metrizable LCS-complete (resp., domain-complete) spaces are the completely metrizable spaces.*

Proof. One direction is Corollary 4.2. Conversely, an LCS-complete space is Choquet-complete (Proposition 9.1) and every metrizable Choquet-complete space is completely metrizable [13, Corollary 7.6.16]. □

Remark 9.3 There is an LCS-complete but not domain-complete space. The space $\{0,1\}^I$, where $\{0,1\}$ is given the discrete topology, is compact Hausdorff, hence trivially LCS-complete. We claim that it is not domain-complete if I is uncountable. In order to show that, we first show that: $(*)$ for every point $\boldsymbol{a}$ of $\{0,1\}^I$, every countable family of open neighborhoods $(V_n)_{n\in\mathbb{N}}$ of $\boldsymbol{a}$ must be such that $\bigcap_{n\in\mathbb{N}} V_n \neq \{\boldsymbol{a}\}$. We write a_i for the ith component of $\boldsymbol{a}$. For each subset J of I, let $V_J(\boldsymbol{a}) \stackrel{\text{def}}{=} \{\boldsymbol{b} \in \{0,1\}^I \mid \forall i \in J, a_i = b_i\}$; this is a basic open subset of the product topology if J is finite. Since $\boldsymbol{a} \in V_n$, there is a finite subset J_n of I such that $\boldsymbol{a} \in V_{J_n}(\boldsymbol{a}) \subseteq V_n$. Then $\bigcap_{n\in\mathbb{N}} V_n$ contains $\bigcap_{n\in\mathbb{N}} V_{J_n}(\boldsymbol{a}) = V_{\bigcup_{n\in\mathbb{N}} J_n}(\boldsymbol{a})$, which contains uncountably many elements other than $\boldsymbol{a}$. Having established $(*)$, it is clear that no point has a countable base of open neighborhoods. In particular, $\{0,1\}^I$ is not convergence Choquet-complete, hence not domain-complete.

The situation simplifies for countably-based spaces. We will use the notion of *supercompact* set: $Q \subseteq X$ is *supercompact* if and only if for every open cover $(U_i)_{i \in I}$ of Q, there is an index $i \in I$ such that $Q \subseteq U_i$. By [18, Fact 2.2], the supercompact subsets of a topological space X are exactly the sets $\uparrow x$, $x \in X$.

Proposition 9.4 *Every countably-based, compactly Choquet-complete space X is convergence Choquet-complete.*

Proof. Let σ be a strategy for α such that the open sets $(U_n)_{n \in \mathbb{N}}$ played by α form a base of open neighborhoods of some compact saturated set. Let also $(B_n)_{n \in \mathbb{N}}$ be a countable base of the topology, and let us write $B(x,n)$ for $\bigcap\{B_i \mid 0 \leq i \leq n, x \in B_i\}$. (In case there is no B_i containing x for any i, $0 \leq i \leq n$, this is the whole of X.) We define a new strategy σ' for α by using a game stealing argument: when β plays x_n and V_n, α simulates what he would have done if β had played x_n and $V_n \cap B(x_n, n)$ instead. Formally, we define $\sigma'(x_0, V_0, U_0, x_1, V_1, U_1, x_2, V_2, \cdots, x_n, V_n) \stackrel{\text{def}}{=} \sigma(x_0, V_0, U_0, x_1, V_1, U_1, x_2, V_2, \cdots, x_n, V_n \cap B(x_n, n))$. Let $(U'_n)_{n \in \mathbb{N}}$ denote the open sets played by α using σ': $U'_n = \sigma(x_0, V_0, U'_0, x_1, V_1, U'_1, x_2, V_2, \cdots, x_n, V_n \cap B(x_n, n))$. Since X is compactly Choquet-complete, $(U'_n)_{n \in \mathbb{N}}$ is a countable base of open neighborhoods of some non-empty compact saturated set Q. We claim that Q is of the form $\uparrow x$.

We only need to show that Q is supercompact. We start by assuming that Q is included in the union of two open sets U and V, and we will show that Q is included in one of them. We can write U and V as unions of basic open sets B_n, hence by compactness there are two finite sets I and J of natural numbers such that $Q \subseteq \bigcup_{i \in I \cup J} B_i$, $\bigcup_{i \in I} B_i \subseteq U$, and $\bigcup_{j \in J} B_j \subseteq V$. Since $(U'_n)_{n \in \mathbb{N}}$ is a base of open neighborhoods of Q, some U'_n is included in $\bigcup_{i \in I \cup J} B_i$. We choose n higher than every element of $I \cup J$. Since $x_n \in U'_n$, x_n is in $\bigcup_{i \in I \cup J} B_i$. If x_n is in B_i for some $i \in I$, then $B(x_n, n)$ is included in B_i, and then $Q \subseteq U'_n \subseteq B(x_n, n)$ (by the definition of σ') $\subseteq B_i \subseteq U$. Otherwise, by a similar argument $Q \subseteq V$.

It follows that if Q is included in the union of $n \geq 1$ open sets, then it is included in one of them. Given any open cover $(U_i)_{i \in I}$ of Q, there is a finite subcover $(U_i)_{i \in J}$ of Q. J is non-empty, since $Q \neq \emptyset$. Hence Q is included in U_i for some $i \in J$. This shows that Q is supercompact. Hence $Q = \uparrow x$ for some x. Since $(U'_n)_{n \in \mathbb{N}}$ is a countable base of open neighborhoods of Q, it is also one of x. □

Theorem 9.5 *The following are equivalent for a countably-based T_0 space X:*

(i) *X is domain-complete;*

(ii) *X is LCS-complete;*

(iii) *X is quasi-Polish;*

(iv) *X is compactly Choquet-complete;*

(v) *X is convergence Choquet-complete.*

Proof. (iii)⇒(i) is by Proposition 5.1, (i)⇒(ii) is by Proposition 3.3, (ii)⇒(iv) is by Proposition 9.1, (iv)⇒(v) is by Proposition 9.4. Finally, (v)⇒(iii) is the contents of Theorem 51 of [7], see also [6, Theorem 11.8]. □

Remark 9.6 $\mathbb{Q}$, with the usual metric topology, is not quasi-Polish. Theorem 9.5 implies that it is not LCS-complete either. One can show directly that it is not Choquet-complete, as follows. We fix an enumeration $(q_n)_{n\in\mathbb{N}}$ of $\mathbb{Q}$, and we call first element of a non-empty set A the element $q_k \in A$ with k least. At step 0, β plays $x_0 \stackrel{\text{def}}{=} q_0$, $V_0 \stackrel{\text{def}}{=} \mathbb{Q}$. At step $n+1$, β plays V_{n+1}, defined as U_n minus its first element, and lets x_{n+1} be the first element of V_{n+1}. This is possible since every non-empty open subset of $\mathbb{Q}$ is infinite. One checks easily that, whatever α plays, $\bigcap_{n\in\mathbb{N}} V_n$ is empty.

10 The Baire property

Every Choquet-complete space is *Baire* [13, Theorem 7.6.8], where a Baire space is a space in which every intersection of countably many dense open sets is dense.

Corollary 10.1 *Every LCS-complete space is Baire.*

This generalizes Isbell's result that every locally compact sober space is Baire [21].

We will refine this below. We need to observe the following folklore result.

Lemma 10.2 *Let Y be a continuous dcpo, and C be a closed subset of Y. The way-below relation on C is the restriction of the way-below relation $\ll$ on Y to C. C is a continuous dcpo, with the restriction of the ordering $\leq$ of Y to C.*

Proof. First, C is a dcpo under the restriction $\leq_C$ of $\leq$ to C, and directed suprema are computed as in Y.

Let $\ll_C$ denote the way-below relation on C. For all $x, y \in C$, if $x \ll y$ (in Y) then $x \ll_C y$: every directed family of elements of C whose supremum (in C, equivalently in Y) lies above y must contain an element above x.

It follows that C is a continuous dcpo: every element x of C is the supremum of the directed family of elements y that are way-below x in Y, and all those elements are in C and way-below x in C.

Conversely, we assume $x \ll_C y$, and we consider a directed family D in Y whose supremum lies above y. Every continuous dcpo is meet-continuous [12, Theorem III-2.11], meaning that if $y \leq \sup D$ for any directed family D in Y, then y is in the Scott-closure of $\downarrow D \cap \downarrow y$. (The theory of meet-continuous dcpos is due to Kou, Liu and Luo [26].) In the case at hand, $\downarrow D \cap \downarrow y$ is included in $\downarrow y$ hence in C. Since C is a continuous dcpo and $x \ll_C y$, the set $\twoheaduparrow_C x \stackrel{\text{def}}{=} \{z \in C \mid x \ll_C z\}$ is Scott-open in C, and contains y. Then $\twoheaduparrow_C x$ intersects the Scott-closure of $\downarrow D \cap \downarrow y$, and since it is open, it also intersects $\downarrow D \cap \downarrow y$ itself, say at z. Then $x \ll_C z \leq d$ for some $d \in D$, which implies $x \leq d$. Therefore $x \ll y$. □

Proposition 10.3 *Every G_δ subset, every closed subset of a domain-complete (resp., LCS-complete) space is domain-complete (resp., LCS-complete).*

Proof. Let X be the intersection of a descending sequence $(W_n)_{n\in\mathbb{N}}$ of open subsets of Y, where Y is a continuous dcpo (resp., a locally compact sober space).

Given any G_δ subset $A \stackrel{\text{def}}{=} \bigcap_{m\in\mathbb{N}} V_m$ of X, where each V_m is open in X, let $\widehat{V}_m$ be some open subset of Y such that $\widehat{V}_m \cap X = V_m$. Then A is also equal to the countable intersection $\bigcap_{m,n\in\mathbb{N}} \widehat{V}_m \cap W_n$, hence A is G_δ in Y.

Given any closed subset C of X, C is the intersection of X with some closed subset C' of Y. If Y is a continuous dcpo, then C' is also a continuous dcpo by Lemma 10.2. Then $C = \bigcap_{n\in\mathbb{N}}(C' \cap W_n)$, showing that C is a G_δ subset of C', hence is domain-complete. If Y is a locally compact sober space, C' is sober as a subspace (being the equalizer of the indicator function of its complement and of the constant 0 map), and is locally compact: for every $x \in C'$, for every open neighborhood $U \cap C'$ of x in C' (where U is open in Y), there is a compact saturated neighborhood Q of x in Y included in U; then $Q \cap C'$ is a compact saturated neighborhood of x in C' included in $U \cap C'$. Again $C = \bigcap_{n\in\mathbb{N}}(C' \cap W_n)$, showing that C is a G_δ subset of C', hence is LCS-complete. $\square$

A space is *completely Baire* if and only if all its closed subspaces are Baire. This is strictly stronger than the Baire property. Proposition 10.3 and Corollary 10.1 together entail the following, which generalizes the fact that every quasi-Polish space is completely Baire [8, beginning of Section 5].

Corollary 10.4 *Every LCS-complete space is completely Baire.*

11 Stone duality for domain-complete and LCS-complete spaces

There is an adjunction $\mathcal{O} \dashv \mathsf{pt}$ between **Top** and the category of locales **Loc**—the opposite of the category of frames **Frm**. (See [13, Section 8.1], for example.) The functor $\mathcal{O}\colon \mathbf{Top} \to \mathbf{Loc}$ maps every space X to $\mathcal{O}X$, and every continuous map f to the frame homomorphism $\mathcal{O}f\colon U \mapsto f^{-1}(U)$. Conversely, $\mathsf{pt}\colon \mathbf{Loc} \to \mathbf{Top}$ maps every frame L to its set of completely prime filters, with the topology whose open sets are $\mathcal{O}_u \stackrel{\text{def}}{=} \{x \in \mathsf{pt}\, L \mid u \in x\}$, for each $u \in L$. This adjunction restricts to an adjoint equivalence between the full subcategories of sober spaces and spatial locales, between the category of locally compact sober spaces and the opposite of the category of continuous distributive complete lattices by the Hofmann-Lawson theorem [19] (see also [13, Theorem 8.3.21]), and between the category of continuous dcpos and the opposite of the category of completely distributive complete lattices [13, Theorem 8.3.43].

Let us recall what quotient frames are, following [17, Section 3.4]. More generally, the book by Picado and Pultr [31] is a recommended reference on frames and locales. A *congruence preorder* on a frame L is a transitive relation $\preceq$ such that $u \leq v$ implies $u \preceq v$ for all $u, v \in L$, $\bigvee_{i\in I} u_i \preceq v$ whenever $u_i \preceq v$ for every

$i \in I$, and $u \preceq \bigwedge_{i=1}^{n} v_i$ whenever $u \preceq v_i$ for every i, $1 \leq i \leq n$. We can then form the *quotient frame* $L/{\preceq}$, whose elements are the equivalence classes of L modulo $\preceq \cap \succeq$. Given any binary relation R on L, there is a least congruence preorder $\preceq_R$ such that $u \preceq_R v$ for all $(u, v) \in R$. In particular, for every subset A of L, there is a least congruence preorder $\preceq_A$ such that $\top \preceq_A v$ for every $v \in A$, where $\top$ is the largest element of L. Using [31, Section 11.2] for instance, one can check that $L/{\preceq_A}$ can be equated with the subframe of L consisting of the A-*saturated* elements of L, namely those elements $u \in L$ such that $u = (a \Rightarrow u)$ for every $a \in A$, where $\Rightarrow$ is Heyting implication in L ($a \Rightarrow u \stackrel{\text{def}}{=} \bigvee\{b \mid a \wedge b \leq u\}$).

Theorem 11.1 *The adjunction $\mathcal{O} \dashv \mathsf{pt}$ restricts to an adjoint equivalence between the category of LCS-complete spaces (resp., domain-complete spaces) and the opposite of the category of quotient frames $L/{\preceq_A}$, where A is a countable subset of L and L is a continuous distributive (resp., completely distributive) continuous lattice.*

Proof. We use the following theorem, due to Heckmann [17, Theorem 3.13]: given any completely Baire space Y, and any countable relation $R \subseteq \mathcal{O}Y \times \mathcal{O}Y$, the quotient frame $\mathcal{O}Y/{\preceq_R}$ is spatial, and isomorphic to the frame of open sets of $\bigcap_{(U,V)\in R}(Y \setminus U) \cup V$.

For every domain-complete (resp., LCS-complete) space X, written as $\bigcap^{\downarrow}_{n\in\mathbb{N}} W_n$, where each W_n is open in the continuous dcpo (resp., locally sober space) Y, Y is itself LCS-complete (Proposition 3.3) hence completely Baire (Corollary 10.4). It follows from Heckmann's theorem that $\mathcal{O}X$ is isomorphic to $\mathcal{O}Y/{\preceq_A}$ where $A \stackrel{\text{def}}{=} \{W_n \mid n \in \mathbb{N}\}$. Therefore $\mathcal{O}X$ is a quotient frame of a continuous distributive (resp., completely distributive) continuous lattice by the countable set A.

By Proposition 7.1, every LCS-complete space is sober, so the unit $x \in X \mapsto \{U \in \mathcal{O}X \mid x \in U\} \in \mathsf{pt}\,\mathcal{O}X$ is a homeomorphism [13, Proposition 8.2.22, Fact 8.2.5].

In the other direction, let L be a completely distributive (resp., continuous distributive) continuous lattice. By the Hofmann-Lawson theorem, L is isomorphic to the open set lattice of some locally compact sober space Y. Without loss of generality, we assume that $L = \mathcal{O}Y$. As above, Y is LCS-complete hence completely Baire. By Heckmann's theorem, for every countable relation R on L, $L/{\preceq_R}$ is isomorphic to $\mathcal{O}X$ where $X \stackrel{\text{def}}{=} \bigcap_{(U,V)\in R}(Y \setminus U) \cup V$. In particular, for any countable subset $A \stackrel{\text{def}}{=} \{W_n \mid n \in \mathbb{N}\}$ of L, we can equate $L/{\preceq_A}$ with $\mathcal{O}X$ where $X \stackrel{\text{def}}{=} \bigcap_{n\in\mathbb{N}} W_n$. By construction, X is domain-complete (resp., LCS-complete). Finally, the counit $U \in L \mapsto \mathcal{O}_U$ is an isomorphism because L is spatial [13, Proposition 8.1.17]. $\square$

12 Consonance

For a subset Q of a topological space X, let $\blacksquare Q$ be the family of open neighborhoods of Q. A space is *consonant* if and only if, given any Scott-open family $\mathcal{U}$ of open sets, and given any $U \in \mathcal{U}$, there is a compact saturated set Q such that $U \in \blacksquare Q \subseteq \mathcal{U}$.

Equivalently, if and only if every Scott-open family of opens is a union of sets of the form $\blacksquare Q$, Q compact saturated.

In a locally compact space, every open subset U is the union of the interiors $int(Q)$ of compact saturated subsets Q of U, and that family is directed. It follows immediately that every locally compact space is consonant. Another class of consonant spaces is given by the regular Čech-complete spaces, following Dolecki, Greco and Lechicki [9, Theorem 4.1 and footnote 8].

Consonance is not preserved under the formation of G_δ subsets [9, Proposition 7.3]. Nonetheless, we have:

Proposition 12.1 *Every LCS-complete space is consonant.*

Proof. Let X be the intersection of a descending sequence $(W_n)_{n\in\mathbb{N}}$ of open subsets of a locally compact sober space Y. Let $\mathcal{U}$ be a Scott-open family of open subsets of X, and $U \in \mathcal{U}$.

By the definition of the subspace topology, there is an open subset $\widehat{U}$ of Y such that $\widehat{U} \cap X = U$. By local compactness, $\widehat{U} \cap W_0$ is the union of the directed family of the sets $int(Q)$, where Q ranges over the family $\mathcal{Q}_0$ of compact saturated subsets of $\widehat{U} \cap W_0$. We have $\bigcup^{\uparrow}_{Q\in\mathcal{Q}_0} int(Q) \cap X = \widehat{U} \cap W_0 \cap X = \widehat{U} \cap X = U$. Since U is in $\mathcal{U}$ and $\mathcal{U}$ is Scott-open, $int(Q) \cap X$ is in $\mathcal{U}$ for some $Q \in \mathcal{Q}_0$. Let Q_0 be this compact saturated set Q, $\widehat{U}_0 \stackrel{\text{def}}{=} int(Q_0)$, and $U_0 \stackrel{\text{def}}{=} \widehat{U}_0 \cap X$. Note that $U_0 \in \mathcal{U}$, $\widehat{U}_0 \subseteq Q_0 \subseteq \widehat{U} \cap W_0$.

We do the same thing with $\widehat{U}_0 \cap W_1$ instead of $\widehat{U} \cap W_0$. There is a compact saturated subset Q_1 of $\widehat{U}_0 \cap W_1$ such that $int(Q_1) \cap X$ is in $\mathcal{U}$. Then, letting $\widehat{U}_1 \stackrel{\text{def}}{=} int(Q_1)$, $U_1 \stackrel{\text{def}}{=} \widehat{U}_1 \cap X$, we obtain that $U_1 \in \mathcal{U}$, $\widehat{U}_1 \subseteq Q_1 \subseteq \widehat{U}_0 \cap W_1$.

Iterating this construction, we obtain for each $n \in \mathbb{N}$ a compact saturated subset Q_n and an open subset $\widehat{U}_n$ of Y, and an open subset U_n of X such that $U_n \in \mathcal{U}$ for each n, and $\widehat{U}_{n+1} \subseteq Q_{n+1} \subseteq \widehat{U}_n \cap W_n$.

Let $Q \stackrel{\text{def}}{=} \bigcap_{n\in\mathbb{N}} Q_n$. Since Y is sober hence well-filtered, Q is compact saturated in Y.

Since $Q \subseteq \bigcap_{n\in\mathbb{N}} W_n = X$, Q is a compact saturated subset of Y that is included in X, hence a compact subset of X by Lemma 8.1, item 3.

We have $Q \subseteq Q_0 \subseteq \widehat{U} \cap W_0 \subseteq \widehat{U}$, and $Q \subseteq X$, so $Q \subseteq \widehat{U} \cap X = U$. Therefore $U \in \blacksquare Q$.

For every $W \in \blacksquare Q$, write W as the intersection of some open subset $\widehat{W}$ of Y with X. Since $Q = \bigcap_{n\in\mathbb{N}} Q_n \subseteq \widehat{W}$, by well-filteredness some Q_n is included in $\widehat{W}$. Hence $\widehat{U}_n \subseteq Q_n \subseteq \widehat{W}$. Taking intersections with X, $U_n \subseteq W$. Since U_n is in $\mathcal{U}$, so is W. □

Remark 12.2 In the proof of Proposition 12.1, Q is a G_δ subset of X. Indeed, $\widehat{U}_{n+1} \subseteq Q_{n+1} \subseteq \widehat{U}_n \cap W_n$ for every $n \in \mathbb{N}$, hence $Q = \bigcap_{n\in\mathbb{N}} Q_n = \bigcap_{n\in\mathbb{N}} Q_n \cap X = \bigcap_{n\in\mathbb{N}} (\widehat{U}_n \cap X)$. Hence we can refine Proposition 12.1 to: in an LCS-complete space X, every Scott-open family $\mathcal{U}$ of open subsets of X is a union of sets $\blacksquare Q$, where the sets Q are compact G_δ, not just compact.

13 The space $\mathcal{L}X$ for X LCS-complete

The topological coproduct of two consonant spaces is not in general consonant [29, Example 6.12], whence the need for the following definition. Let $n \odot X$ denote the coproduct of n identical copies of X.

Definition 13.1 [$\odot$-consonant] A topological space X is called $\odot$-*consonant* if and only if, for every $n \in \mathbb{N}$, $n \odot X$ is consonant.

In particular, every $\odot$-consonant space is consonant.

Lemma 13.2 *Every LCS-complete space is $\odot$-consonant.*

Proof. Every topological coproduct of LCS-complete spaces is LCS-complete, as we will see in Proposition 15.1. For now, let us just write the LCS-complete space X as $\bigcap^{\downarrow}_{m\in\mathbb{N}} W_m$, where each W_m is open in the locally compact sober space Y. Then $n \odot Y$ is sober, because coproducts of sober spaces are sober [13, Lemma 8.4.2]. Since $\mathcal{O}(n \odot Y)$ is isomorphic to $(\mathcal{O}\,Y)^n$, it is a continuous lattice, so $n \odot Y$ is core-compact, hence locally compact. Then $n \odot X$ arises as the G_δ subset $\bigcap^{\downarrow}_{m\in\mathbb{N}} n \odot W_m$ of $n \odot Y$. Finally, by Proposition 12.1, $n \odot X$ is consonant. □

Let $[X \to Y]$ denote the space of all continuous maps from X to Y. A *step function* g from X to Y is a continuous map whose image is finite. For every y in the image $\mathrm{Im}\, g$ of g, there is an open neighborhood V_y of y such that $V_y \cap \mathrm{Im}\, g = \uparrow y \cap \mathrm{Im}\, g$, namely, that contains only the elements from $\mathrm{Im}\, g$ that are above y. This is because $\uparrow y$ is the filtered intersection of the family $(V_i)_{i\in I}$ of open neighborhoods of y, and $(V_i \cap \mathrm{Im}\, g)_{i\in I}$ is filtered and finite, hence reaches its infimum. Then $U_y \stackrel{\mathrm{def}}{=} g^{-1}(\uparrow y)$ is open because it is also equal to $g^{-1}(V_y)$. Moreover, g is the map that sends every element of $U_y \setminus \bigcup_{y'\in\mathrm{Im}\, g, y<y'} U_{y'}$ to y. When Y also has a least element $\bot$, g is then also the pointwise supremum $\sup_{y\in\mathrm{Im}\, g} U_y \searrow y$, where the elementary step function $U_y \searrow y$ maps every element of U_y to y and all other elements to $\bot$. (The sup is always defined in this case, whatever Y is provided it has a least element.) This generalizes the usual notion of step function. The following should be familiar to domain theorists—except that the step functions we build are not required to be way-below f.

A *bounded* family is a set of elements that has an upper bound.

Lemma 13.3 *Let X be a topological space, and Y be a continuous poset in which every finite bounded family of elements has a least upper bound, with its Scott topology. Every continuous map $f : X \to Y$ is the pointwise supremum of a directed family of step functions.*

Proof. In Y, the empty family has a least upper bound, meaning that Y has a least element $\bot$. The constant $\bot$ map is a step function below f. Given any two step functions g, h below f, let k map every $x \in X$ to the supremum of $g(x)$ and $h(x)$, which exists because the family $\{g(x), h(x)\}$ is bounded by $f(x)$. The image of k is clearly finite. We claim that k is continuous. For every open subset V of Y, let D_V be the set of pairs $(y_1, y_2) \in \mathrm{Im}\, g \times \mathrm{Im}\, h$ such that the supremum of y_1 and y_2 is in V. This is a finite set. Then $k^{-1}(V) = \bigcup_{(y_1,y_2)\in D_V} g^{-1}(\uparrow y_1) \cap h^{-1}(\uparrow y_2)$ is open. Since k is continuous and $\mathrm{Im}\, k$ is finite, k is a step function. This shows that the family $\mathcal{D}$ of step functions pointwise below f is directed.

For every $x \in X$, and every $y \ll f(x)$ in Y, the step function $f^{-1}(\twoheaduparrow y) \searrow y$ is in $\mathcal{D}$, and its value at x is y. Since the supremum of all the elements y way-below $f(x)$ is $f(x)$, $\sup\{g(x) \mid g \in \mathcal{D}\} = f(x)$. □

Given a finite subset B of Y, where Y is a poset in which every finite bounded family J of elements has a least upper bound $\sup J$, and given any $|B|$-tuple $(V_y)_{y\in B}$ of open subsets of X, the notation $\sup_{y\in B} V_y \searrow y$ defines a step function if and only if every subset $J \subseteq B$ such that $\bigcap_{y\in J} V_y \neq \emptyset$ is bounded: in that case $\sup_{y\in B} V_y \searrow y$ maps every point $x \in X$ to $\sup J$, where $J \stackrel{\text{def}}{=} \{y \in B \mid x \in V_y\}$; otherwise, we say that $\sup_{y\in B} V_y \searrow y$ is *undefined.*

Proposition 13.4 *Let X be a $\odot$-consonant space. Let Y be a continuous poset in which every finite bounded family of elements has a least upper bound, with its Scott topology. The compact-open topology on $[X \to Y]$ is equal to the Scott topology.*

Proof. The compact-open topology has subbasic open sets $[Q \subseteq V] \stackrel{\text{def}}{=} \{f \in [X \to Y] \mid Q \subseteq f^{-1}(V)\}$, where Q is compact saturated in X and V is open in Y. It is easy to see that $[Q \subseteq V]$ is Scott-open. In the converse direction, let $\mathcal{W}$ be a Scott-open subset of $[X \to Y]$, and $f \in \mathcal{W}$. Our task is to find an open neighborhood of f in the compact-open topology that is included in $\mathcal{W}$.

The function f is the pointwise supremum of a directed family of step functions, by Lemma 13.3, hence one of them, say g_0, is in $\mathcal{W}$. We can write g_0 as $g_0 \stackrel{\text{def}}{=} \sup_{y\in B} U_y \searrow y$, with B finite, and where each U_y is open.

Consider the maps $\sup_{y\in B} U_y \searrow z_y$, where $z_y \ll y$ for each $y \in B$. Those maps are defined: for every $J \subseteq B$ such that $\bigcap_{y\in J} U_y \neq \emptyset$, $\sup J$ exists and is an upper bound of $\{z_y \mid y \in J\}$. Explicitly, those maps $\sup_{y\in B} U_y \searrow z_y$ send each $x \in X$ to $\sup\{z_y \mid y \in J\}$, where $J \stackrel{\text{def}}{=} \{y \in B \mid x \in U_y\}$. Those maps form a directed family whose supremum is g_0, hence one of them, say $g \stackrel{\text{def}}{=} \sup_{y\in B} U_y \searrow z_y$, is in $\mathcal{W}$.

Let G be the set of subsets J of B such that $Z_J \stackrel{\text{def}}{=} \{z_y \mid y \in J\}$ is bounded. For each $J \in G$, Z_J has a least upper bound $\sup Z_J$, by assumption. Let $\mathcal{V}$ be the set of $|B|$-tuples $(V_y)_{y\in B}$ of open subsets of X such that $\sup_{y\in B} V_y \searrow z_y$ is undefined or in $\mathcal{W}$. Ordering those tuples by componentwise inclusion, we claim that $\mathcal{V}$ is Scott-open.

We first check that $\mathcal{V}$ is upwards-closed. Let $(V_y)_{y\in B}$ be an element of $\mathcal{V}$, and $(V'_y)_{y\in B}$ be a family of open sets such that $V_y \subseteq V'_y$ for every $y \in B$. If $\sup_{y\in B} V_y \searrow z_y$ is undefined, then there is a subset J of B, not in G, and such that $\bigcap_{y\in J} V_y \neq \emptyset$. Then $\bigcap_{y\in J} V'_y$ is non-empty as well, so $\sup_{y\in B} V'_y \searrow z_y$ is undefined, too. If $\sup_{y\in B} V_y \searrow z_y$ is defined, then either $\sup_{y\in B} V'_y \searrow z_y$ is undefined, or $\sup_{y\in B} V_y \searrow z_y \leq \sup_{y\in B} V'_y \searrow z_y$. In both cases, $(V'_y)_{y\in B}$ is in $\mathcal{V}$.

Next, let $(V_y)_{y\in B}$ be a $|B|$-tuple of open subsets of X, let I be some indexing set and assume that for every $y \in B$, $V_y = \bigcup^{\uparrow}_{i \in I} V_{yi}$, where each V_{yi} is open. If $\sup_{y \in B} V_y \searrow z_y$ is undefined, then there is a subset J of B, not in G, and such that $\bigcap_{y \in J} V_y \neq \emptyset$. We pick an element x from $\bigcap_{y \in J} V_y$. For each $y \in B$, there is an index $i \in I$ such that $x \in V_{yi}$, and we can take the same i for every $y \in B$ by directedness. Then $\sup_{y \in B} V_{yi} \searrow z_y$ is undefined, hence $(V_{yi})_{y \in B}$ is in $\mathcal{V}$. If instead $\sup_{y \in B} V_y \searrow z_y$ is defined, then every map $\sup_{y \in B} V_{yi} \searrow z_y$, $i \in I$, is defined, too. We claim that $\sup_{y \in B} V_y \searrow z_y = \sup^{\uparrow}_{i \in I}(\sup_{y \in B} V_{yi} \searrow z_y)$. We fix $x \in X$, and we let $J \stackrel{\text{def}}{=} \{y \in B \mid x \in V_y\}$. For every $y \in J$, x is in $V_y = \sup^{\uparrow}_{i \in I} V_{yi}$ so $x \in V_{yi}$ for some $i \in I$. By directedness, we can choose the same i for every $y \in J$. It follows that $(\sup_{y \in B} V_{yi} \searrow z_y)(x) = \sup Z_J = (\sup_{y \in B} V_y \searrow z_y)(x)$. This shows the claim. Now that we know that $\sup_{y \in B} V_y \searrow z_y = \sup^{\uparrow}_{i \in I}(\sup_{y \in B} V_{yi} \searrow z_y)$, and since that is in the Scott-open set $\mathcal{W}$, $\sup_{y \in B} V_{yi} \searrow z_y$ is in $\mathcal{W}$ for some $i \in I$, in particular $(V_{yi})_{y \in B}$ is in $\mathcal{V}$.

We know that $\mathcal{V}$ is Scott-open. Moreover, and recalling that $g = \sup_{y \in B} U_y \searrow z_y$ is in $\mathcal{W}$, the $|B|$-tuple $(U_y)_{y \in B}$ is in $\mathcal{V}$. We may equate $|B|$-tuples of open subsets with open subsets of $|B| \odot X$, and then the compact saturated subsets of $|B| \odot X$ are naturally equated with $|B|$-tuples of compact saturated subsets of X. Since X is $\odot$-consonant, there is a $|B|$-tuple of compact saturated subsets Q_y, $y \in B$, such that $Q_y \subseteq U_y$ for every $y \in B$ and such that every $|B|$-tuple $(V_y)_{y \in B}$ of open sets such that $Q_y \subseteq V_y$ for every $y \in B$ is in $\mathcal{V}$.

Let us consider the compact-open open subset $\mathcal{W}' \stackrel{\text{def}}{=} \bigcap_{y \in B}[Q_y \subseteq \Uparrow z_y]$. Since $Q_y \subseteq U_y$ for every $y \in B$, f is in $\mathcal{W}'$: for every $y \in B$, for every $x \in Q_y$, x is in U_y, so $f(x)$, which is larger than or equal to $g_0(x)$, hence to y, is in $\Uparrow z_y$. We claim that $\mathcal{W}'$ is included in $\mathcal{W}$. Let h be any element of $\mathcal{W}'$. For every $y \in B$, let $V_y \stackrel{\text{def}}{=} h^{-1}(\Uparrow z_y)$. Since $h \in [Q_y \subseteq \Uparrow z_y]$, $Q_y \subseteq V_y$, so $(V_y)_{y \in B}$ is in $\mathcal{V}$, meaning that $\sup_{y \in B} V_y \searrow z_y$ is undefined or in $\mathcal{W}$. But it cannot be undefined: for every $x \in X$, letting $J \stackrel{\text{def}}{=} \{y \in B \mid x \in V_y\}$, $h(x)$ is an upper bound of $\{z_y \mid y \in J\}$, by the definition of V_y. The same argument shows that $\sup_{y \in B} V_y \searrow z_y \leq h$. Since $\sup_{y \in B} V_y \searrow z_y$ is in $\mathcal{W}$ and $\mathcal{W}$ is upwards-closed, h is also in $\mathcal{W}$. □

Let $\mathcal{L}X$ denote the space of all continuous maps from X to $\overline{\mathbb{R}}_{+\sigma}$, the set of extended non-negative real numbers under the Scott topology. Those are usually known as the *lower semicontinuous* maps from X to $\overline{\mathbb{R}}_+$. $Y \stackrel{\text{def}}{=} \overline{\mathbb{R}}_{+\sigma}$ certainly satisfies the assumptions of Proposition 13.4. Hence:

Corollary 13.5 *Let X be a $\odot$-consonant space, for example an LCS-complete space. The compact-open topology on $\mathcal{L}X$ is equal to the Scott topology on $\mathcal{L}X$.* □

As an application, let us consider Theorem 4.11 of [14]. (We will give another application in Section 16.) This expresses a homeomorphism between two kinds of objects. The first one is the space $\mathbb{P}_{\mathtt{AP}}(X)$ of *sublinear previsions* on X, namely Scott-continuous sublinear maps F from $\mathcal{L}X$ to $\overline{\mathbb{R}}_{+\sigma}$, where sublinear means that $F(ah) = aF(h)$ and $F(h + h') \leq F(h) + F(h')$ for all $a \in \mathbb{R}_+$, $h, h' \in \mathcal{L}X$. $\mathbb{P}_{\mathtt{AP}}(X)$ is equipped with the *weak topology*, whose subbasic open sets are $[h > r] \stackrel{\text{def}}{=} \{F \in \mathbb{P}_{\mathtt{AP}}(X) \mid F(h) > r\}$, $h \in \mathcal{L}X$, $r \in \mathbb{R}_+$. The second one is $\mathcal{H}^{cvx}(\mathbf{V}_{\mathrm{w}}(X))$, where $\mathbf{V}_{\mathrm{w}}(X)$ is the space of continuous valuations on X [23,22] (more details in Section 18), or equivalently the space of *linear* previsions (defined as sublinear previsions, except that $F(h+h') = F(h) + F(h')$ replaces the inequality $F(h+h') \leq F(h) + F(h')$), $\mathcal{H}(Y)$ is the space of non-empty closed subsets of Y with the lower Vietoris topology, and $\mathcal{H}^{cvx}(Y)$ is the subspace of $\mathcal{H}(Y)$ consisting of its convex sets. The already cited Theorem 4.11 of [14] states that $\mathbb{P}_{\mathtt{AP}}(X)$ and $\mathcal{H}^{cvx}(\mathbf{V}_{\mathrm{w}}(X))$ are homeomorphic if $\mathcal{L}X$ is *locally convex* in its Scott topology, meaning that every element of $\mathcal{L}X$ has a base of convex open neighborhoods. The homeomorphism is given by $r_{\mathtt{AP}} : \mathcal{H}^{cvx}(\mathbf{V}_{\mathrm{w}}(X)) \mapsto \mathbb{P}_{\mathtt{AP}}(X)$, $r_{\mathtt{AP}}(C)(h) \stackrel{\text{def}}{=} \sup_{\nu \in C} \int_{x \in X} h(x) d\nu$, and $s_{\mathtt{AP}} : \mathbb{P}_{\mathtt{AP}}(X) \to \mathcal{H}^{cvx}(\mathbf{V}_{\mathrm{w}}(X))$, $s_{\mathtt{AP}}(F) \stackrel{\text{def}}{=} \{\nu \in \mathbf{V}_{\mathrm{w}}(X) \mid \forall h \in \mathcal{L}X, \int_{x \in X} h(x) d\nu \leq F(h)\}$. The primary case when those form a homeomorphism is when X is core-compact. We have a second class of spaces where that holds:

Lemma 13.6 *For every $\odot$-consonant space X, for example an LCS-complete space, $\mathcal{L}X$ is locally convex in its Scott topology.*

Proof. By Corollary 13.5, it suffices to show that it is locally convex in its compact-open topology, namely that every element of $\mathcal{L}X$ has a base of convex open neighborhoods. It is routine to show that every basic open $\bigcap_{i=1}^{n} [Q_i \subseteq (a_i, \infty]]$ is convex. □

Corollary 13.7 *For every LCS-complete space X, the maps $s_{\mathtt{AP}}$ and $r_{\mathtt{AP}}$ define a homeomorphism between $\mathbb{P}_{\mathtt{AP}}(X)$ and $\mathcal{H}^{cvx}(\mathbf{V}_w(X))$.* □

This holds in particular for all continuous complete quasi-metric spaces in their d-Scott topology, in particular for all complete metric spaces in their open ball topology.

14 Categorical limits

Lemma 14.1 *Every domain-complete space is a G_δ subset of a pointed continuous dcpo. Every LCS-complete space is a G_δ subset of a compact, locally compact and sober space.*

Proof. Let X be the intersection $\bigcap_{n \in \mathbb{N}} W_n$ of a descending family of open subsets of Y. We define the lifting $Y_\perp$ of Y as Y plus a fresh element $\perp$ below all others (when Y is a dcpo), or as Y plus a fresh element, with open sets those of Y plus $Y_\perp$ itself (if Y is a topological space). If Y is a continuous dcpo, then so is $Y_\perp$ (it is easy to see that x is way-below y in $Y_\perp$ if and only if it is in Y, or $x = \perp$), and $Y_\perp$ is pointed; if Y is locally compact then so is $Y_\perp$ [13, Exercise 4.8.6]; and if Y is sober then so is $Y_\perp$ [13, Exercise 8.2.9]; and $Y_\perp$ is compact. Every open subset of Y is open in $Y_\perp$. Therefore X is the G_δ subset $\bigcap_{n \in \mathbb{N}} W_n$ of $Y_\perp$. □

Proposition 14.2 *The topological product of a countable family of domain-complete (resp., LCS-complete) spaces is domain-complete (resp., LCS-complete).*

Proof. For each $i \in \mathbb{N}$, let X_i be the intersection $\bigcap_{n \in \mathbb{N}} W_{in}$ of a descending family of open subsets of a continuous dcpo Y_i. We may assume that Y_i is pointed, too, by Lemma 14.1. The product of pointed continuous dcpos is a continuous dcpo, and the Scott topology on the product is the product topology [13, Proposition 5.1.56]. Then the topological product $\prod_{i \in \mathbb{N}} X_i$ arises as the G_δ subset $\bigcap_{n \in \mathbb{N}} \left(\prod_{i=0}^{n} W_{i(n-i)} \times \prod_{i=n+1}^{+\infty} Y_i\right)$ of $\prod_{i \in \mathbb{N}} Y_i$.

We use a similar argument when each Y_i is locally compact and sober instead. By Lemma 14.1, we may assume that Y_i is compact. Every product of a family of compact, locally compact spaces is (compact and) locally compact [13, Proposition 4.8.10], and every product of sober spaces is sober [13, Theorem 8.4.8]. □

Proposition 14.3 *The categories of domain-complete, resp. LCS-complete spaces, do not have equalizers.*

Proof. Let $X \stackrel{\text{def}}{=} \mathbb{R}$, with its usual topology, and $Y \stackrel{\text{def}}{=} \mathbb{P}(\mathbb{R})$, with the Scott topology of inclusion. Those are domain-complete spaces. Define $f, g \colon X \to Y$ by $f(x) = (\mathbb{R} \setminus \{x\}) \cup \mathbb{Q}$ and $g(x) \stackrel{\text{def}}{=} \mathbb{R}$. Those are continuous maps: in the case of f, this is because $f^{-1}(\uparrow A)$, for every finite $A \subseteq \mathbb{R}$, is the complement of the finite set $A \setminus \mathbb{Q}$. The equalizer of f and g in **Top** is $\mathbb{Q}$, which is not LCS-complete (Remark 9.6). That is not enough to show that f and g do not have an equalizer in the category of LCS-complete (resp., domain-complete) spaces, hence we argue as follows.

Assume f and g have an equalizer $i \colon Z \to X$ in the category of LCS-complete spaces, resp. of domain-complete spaces. For every $z \in Z$, $f(i(z)) = g(i(z))$, so $i(z) \in \mathbb{Q}$. Since i is a (regular) mono, and the one-point space $\{*\}$ is domain-complete, i is injective: any two distinct points in Z define two distinct morphisms from $\{*\}$ to Z, whose compositions with i must be distinct. If there is a rational point q that is not in the image of i, then the inclusion map $j \colon \{q\} \to X$ is continuous, $\{q\}$ is domain complete, $f \circ j = g \circ j$ since q is rational, but j does not factor through i: contradiction. Hence the image of i is exactly $\mathbb{Q}$. This allows us to equate Z with $\mathbb{Q}$, with some topology, and i with the inclusion map. Since i is continuous, the topology on Z is finer than the usual topology on $\mathbb{Q}$—the subspace topology from $\mathbb{R}$.

We claim that the topology of Z is exactly the usual topology on $\mathbb{Q}$. Let C be a closed subset of Z, and let $cl(C)$ be its closure in $\mathbb{R}$. It suffices to show that $cl(C) \cap Z$ is included in, hence equal to C: this will show that C is closed in $\mathbb{Q}$ with its usual topology. Take any point x from $cl(C) \cap Z$. Since $\mathbb{R}$ is first-countable, there is a sequence $(x_n)_{n \in \mathbb{N}}$ of elements of C that converges to x. Let us consider $\mathbb{N}_\infty$, the one-point compactification of $\mathbb{N}$, where $\mathbb{N}$ is given the discrete topology. This is a compact Hausdorff space, hence it is trivially LCS-complete. It is also countably-based, hence domain-complete (and quasi-Polish) by Theorem 9.5. The map $j \colon \mathbb{N}_\infty \to X$ defined by $j(n) \stackrel{\text{def}}{=} x_n$, $j(\infty) \stackrel{\text{def}}{=} x$ is continuous, and $f \circ j = g \circ j$ since the image of j is included in $\mathbb{Q}$. By the universal property of equalizers, $j = i \circ h$ for some continuous map $h \colon \mathbb{N}_\infty \to Z$. We must have $h(n) = x_n$ and $h(\infty) = x$. Since ∞ is a limit of the numbers $n \in \mathbb{N}$ in $\mathbb{N}_\infty$, x must be a limit of $(x_n)_{n \in \mathbb{N}}$ in Z. The fact that C is closed in Z implies that x is in C, too, completing the argument.

Hence Z is $\mathbb{Q}$, and has the same topology. But this is impossible, since $\mathbb{Q}$ is not LCS-complete. □

Remark 14.4 In contrast, the category of quasi-Polish spaces has equalizers, and they are obtained as in **Top**. Indeed, for all continuous maps $f, g\colon X \to Y$ between two countably-based T_0 spaces X and Y, the coequalizer $[f = g] \stackrel{\text{def}}{=} \{x \in X \mid f(x) = g(x)\}$ in **Top** is a $\mathbf{\Pi}^0_2$ subspace of X [7, Corollary 10], and the $\mathbf{\Pi}^0_2$ subspaces of a quasi-Polish space are exactly its quasi-Polish subspaces [7, Corollary 23]. We note that those properties fail in domain-complete and LCS-complete spaces: the singleton subspace $\{I\}$ of $\mathbb{P}(I)$ (see Remark 5.2) is trivially quasi-Polish but not $\mathbf{\Pi}^0_2$ in $\mathbb{P}(I)$, because the $\mathbf{\Pi}^0_2$ subsets of $\mathbb{P}(I)$ that contain I, the top element, must be G_δ subsets, and we have seen that $\{I\}$ is not G_δ in $\mathbb{P}(I)$. The reason of the failure is deeper: as the following proposition shows, the $\mathbf{\Pi}^0_2$ subspaces of an LCS-complete space can fail to be LCS-complete.

Using a named coined by Heckmann [17], let us call *UCO subset* of a space X any union of a closed subset with an open subset. All UCO subsets are trivially $\mathbf{\Pi}^0_2$, and $\mathbf{\Pi}^0_2$ subsets are countable intersections of UCO subsets.

Proposition 14.5 *The UCO subsets of compact Hausdorff spaces are not in general compactly Choquet-complete. In particular, the UCO subsets of LCS-complete spaces are not in general LCS-complete.*

Proof. The second part follows from the first part by Fact 6.1 and Proposition 9.1.

Let $X \stackrel{\text{def}}{=} [0, 1]^I$, for some uncountable set I, and where $[0, 1]$ has the usual metric topology. This is compact Hausdorff. Let us fix a closed subset C of $[0, 1]$ with empty interior and containing 0 and at least one other point a (for example, $\{0, a\}$), and let U be its complement. Note that U is dense in $[0, 1]$. C^I is closed in X, since its complement is the open subset $\bigcup_{i\in I} \pi_i^{-1}(U)$, where $\pi_i\colon X \to [0, 1]$ is projection onto coordinate i. Let us define Y as the UCO set $\{\mathbf{0}\} \cup (X \setminus C^I)$, where $\mathbf{0}$ is the point whose coordinates are all 0. We claim that Y is not compactly Choquet-complete.

To this end, we assume it is, and we aim for a contradiction. In the strong Choquet game, let β play $x_n \stackrel{\text{def}}{=} \mathbf{0}$ at each round of the game. Let U_n, $n \in \mathbb{N}$, be the open sets played by α. By assumption, $\bigcap_{n\in\mathbb{N}} U_n$ is a compact subset Q of Y, hence also of X by Lemma 8.1. For each $n \in \mathbb{N}$, U_n is the intersection of Y with an open neighborhood of $\mathbf{0}$ in X, and that open neighborhood contains a basic open set $\bigcap_{i\in J_n} \pi_i^{-1}(V_{ni})$, where J_n is finite and V_{ni} is an open neighborhood of 0 in $[0, 1]$. In particular, U_n contains $\bigcap_{i\in J_n} \pi_i^{-1}(\{0\}) \cap Y$. It follows that $K = \bigcap_{n\in\mathbb{N}} U_n$ contains $\bigcap_{i\in J} \pi_i^{-1}(\{0\}) \cap Y$, where $J \stackrel{\text{def}}{=} \bigcup_{n\in\mathbb{N}} J_n$ is countable.

Then $I \setminus J$ is uncountable, hence non-empty. Let k be any element of $I \setminus J$. Since $\pi_k^{-1}(C)$ contains C^I, and Y contains $X \setminus C^I$, Y contains $\pi_k^{-1}(U)$, so K contains $\bigcap_{i\in J} \pi_i^{-1}(\{0\}) \cap \pi_k^{-1}(U)$. We claim that K must contain $\bigcap_{i\in J} \pi_i^{-1}(\{0\})$. For every element $\boldsymbol{x}$ in $\bigcap_{i\in J} \pi_i^{-1}(\{0\})$, let x_k be its kth coordinate, and for every $b \in [0, 1]$, write $\boldsymbol{x}[k := b]$ for the same element with coordinate k changed to b. Since U is dense in $[0, 1]$, x_k is the limit of a sequence $(b_n)_{n\in\mathbb{N}}$ of elements of U. Then $\boldsymbol{x}[k := b_n]$, $n \in \mathbb{N}$, form a sequence in K that converges to $\boldsymbol{x}$. Since K is compact in a Hausdorff space, hence closed, $\boldsymbol{x}$ is in K.

Since K is included in Y, Y contains $\bigcap_{i\in J} \pi_i^{-1}(\{0\})$, too. However $\mathbf{0}[k := a]$ is in the latter, but not in the former since it is different from $\mathbf{0}$ and in C^I. □

15 Colimits

Proposition 15.1 *The topological coproduct of an arbitrary family of domain-complete (resp., LCS-complete) spaces is domain-complete (resp., LCS-complete).*

Proof. For each $i \in I$, let X_i be the intersection $\bigcap_{n\in\mathbb{N}} W_{in}$ of a descending family of open subsets of a continuous dcpo Y_i. The coproduct of continuous dcpos is a continuous dcpo again, and the Scott topology is the coproduct topology [13, Proposition 5.1.59]. Then we can express the coproduct $\coprod_{i\in I} X_i$ as the G_δ subset $\bigcap_{n\in\mathbb{N}} \coprod_{i\in I} W_{in}$ of $\coprod_{i\in I} Y_i$.

When each Y_i is locally compact and sober, we use a similar argument, observing the following facts. First, the compact saturated subsets of each Y_i are compact saturated in $\coprod_{i\in I} Y_i$. It follows easily that $\coprod_{i\in I} Y_i$ is locally compact. The coproduct of arbitrarily many sober spaces is sober, too [13, Lemma 8.4.2]. □

In order to show that coequalizers fail to exist, we make the following observation.

Lemma 15.2 *Every countable compactly Choquet-complete space is first-countable, hence countably-based.*

Proof. Let X be countable and compactly Choquet-complete. Assume that X is not first-countable. There is a point x that has no countable base of open neighborhoods. For each $y \in X \setminus \uparrow x$, $X \setminus \downarrow y$ is an open neighborhood of x, and the intersection of those sets is $\uparrow x$. Since X is countable, we can therefore write $\uparrow x$ as the intersection of countably many open sets $(W_n)_{n\in\mathbb{N}}$. Note that this does *not* say that those open set form a base of open neighborhoods: we do not have a contradiction yet.

In the strong Choquet game, we let β play the same point $x_n \stackrel{\text{def}}{=} x$ at each step. Initially, $V_0 \stackrel{\text{def}}{=} W_0$, and at step $n+1$, β plays $V_{n+1} \stackrel{\text{def}}{=} U_n \cap W_{n+1}$, where U_n was the last open set played by α. Note that $\bigcap^{\downarrow}_{n\in\mathbb{N}} V_n \subseteq \bigcap^{\downarrow}_{n\in\mathbb{N}} W_n = \uparrow x$, while the converse inclusion is obvious. Since X is compactly Choquet-complete, $(V_n)_{n\in\mathbb{N}}$ is a base of open neighborhoods of some compact saturated set Q, and since $\bigcap^{\downarrow}_{n\in\mathbb{N}} V_n = \uparrow x$, $Q = \uparrow x$, and therefore $(V_n)_{n\in\mathbb{N}}$ is a base of open neighborhoods of x: contradiction.

Finally, every countable first-countable space is countably-based. □

Proposition 15.3 *The categories of domain-complete, resp. LCS-complete spaces, do not have coequalizers.*

Proof. Let $\mathbb{N}_\infty$ be the one-point compactification of $\mathbb{N}$, the latter with its discrete topology. Let us form the coproduct Y of countably many copies of $\mathbb{N}_\infty$. Its elements are (k, n) where $k \in \mathbb{N}$, $n \in \mathbb{N}_\infty$. The *sequential fan* is the quotient of Y by the equivalence relation that equates every (k, ∞), $k \in \mathbb{N}$. This is a known example of a countable space that is not countably-based. That can be realized as the coequalizer of $f, g\colon X \to Y$ in **Top**, where X is $\mathbb{N}$ with the discrete topology, $f(k) \stackrel{\text{def}}{=} (k, \infty)$, $g(k) \stackrel{\text{def}}{=} (0, \infty)$. Note that X and Y are domain-complete: X is trivially locally compact and sober (since Hausdorff), and countably-based, then use Theorem 9.5; for similar reasons, $\mathbb{N}_\infty$ is domain-complete, then use Proposition 15.1 to conclude that Y is, too.

Let us assume that f and g have a coequalizer $q\colon Y \to Z$ in the category of LCS-complete spaces, resp. of domain-complete spaces. There is no reason to believe that Z is the sequential fan, hence we have to work harder. There is no reason to believe that q is surjective either, since epis in concrete categories may fail to be surjective. However, q is indeed surjective, as we now show. This is done in several steps. Let $z \in Z$.

The closure of z is $\downarrow z$, so $\chi_{Z \setminus \downarrow z}\colon Z \to \mathbb{S}$ is continuous, where $\mathbb{S} \stackrel{\text{def}}{=} \{0 < 1\}$ is Sierpiński space—trivially a continuous dcpo, hence a domain-complete space. Let $\mathbf{1}$ be the constant map equal to $1 \in \mathbb{S}$. If $\downarrow z$ did not intersect the image of q, then $\chi_{Z \setminus \downarrow z} \circ q$ would be equal to $\mathbf{1} \circ q$, although $\chi_{Z \setminus \downarrow z} \neq \mathbf{1}$, and that is impossible since q is epi. Therefore $\downarrow z$ intersects the image of q.

Imagine that there were two distinct points $q(k_1, n_1)$, $q(k_2, n_2)$ in $\downarrow z$. In particular, (k_1, n_1) and (k_2, n_2) are distinct. Also, not both n_1 and n_2 are equal to ∞, since otherwise $q(k_1, n_1) = q(k_1, \infty) = q(f(k_1)) = q(g(k_1)) = q(g(k_2))$ (since g is a constant map) $= q(f(k_2)) = q(k_2, \infty) = q(k_2, n_2)$. Without loss of generality, let us say that $n_1 \neq \infty$. We consider the map $\chi_{\{(k_1, n_1)\}}\colon Y \to \{0, 1\}$, where $\{0, 1\}$ has the discrete topology (and is a continuous dcpo with the equality ordering, hence domain-complete). Observe that this is a continuous map, owing to the fact that $n_1 \neq \infty$. Since $\chi_{\{(k_1, n_1)\}} \circ f = \chi_{\{(k_1, n_1)\}} \circ g$ $(= 0)$, $\chi_{\{(k_1, n_1)\}} = h \circ q$ for some unique continuous map $h\colon Z \to \{0, 1\}$, by the definition of a coequalizer. Then $h(q(k_1, n_1)) = 1$, while $h(q(k_2, n_2)) = 0$, but since h is continuous it must be monotonic with respect to the underlying specialization orderings, so $q(k_1, n_1) \leq z$ implies $h(q(k_1, n_1)) = h(z)$, and similarly $h(q(k_2, n_2)) = h(z)$. This would imply $1 = h(z) = 0$, a contradiction. Hence there is exactly one point $z' \leq z$ in the image of q.

Consider the two maps $\chi_{Z \setminus \downarrow z'}, \chi_{Z \setminus \downarrow z}\colon Z \to \mathbb{S}$. For every $(k, n) \in Y$, if $\chi_{Z \setminus \downarrow z'}(q(k, n)) = 0$, then $q(k, n) \leq z' \leq z$, so $\chi_{Z \setminus \downarrow z}(q(k, n)) = 0$; conversely, if $\chi_{Z \setminus \downarrow z}(q(k, n)) = 0$, then $q(k, n)$ is below z and is therefore the unique point $z' \leq z$ in the image of q, so $\chi_{Z \setminus \downarrow z'}(q(k, n)) = \chi_{Z \setminus \downarrow z'}(z') = 0$. Hence we have two morphisms which yield the same map when composed with q. Since q is epi, they must be equal. It follows that $\downarrow z = \downarrow z'$, and since Z is T_0 (since sober, see Proposition 7.1), $z = z'$. Therefore z is in the image of q. This completes the proof that q is surjective.

Since q is surjective, and Y is countable, so is Z. By Lemma 15.2, Z is first-countable. Let $\omega \stackrel{\text{def}}{=} q(0, \infty)$. For every $k \in \mathbb{N}$, $q(k, \infty) = q(f(k)) = q(g(k)) = \omega$. Let $(B_k)_{k\in\mathbb{N}}$ be a countable base of open neighborhoods of ω in Z. For each $k \in \mathbb{N}$, since $(k, n)_{n\in\mathbb{N}}$ converges to (k, ∞) in Y, $(q(k, n))_{n\in\mathbb{N}}$ converges to ω, so $q(k, n)$ is in B_k for n large enough. Let us fix some $n_k \in \mathbb{N}$ such that $(k, n) \in q^{-1}(B_k)$ for every $n \geq n_k$. Let $h\colon Y \to \{0, 1\}$ map every point (k, n) to 0 if $n \leq n_k$, to 1 if $n > n_k$. This is continuous, $h \circ f = \mathbf{1} = h \circ g$, so $h = h' \circ q$ for some unique continuous map $h'\colon Z \to \{0, 1\}$. Since $h(0, \infty) = 1$, $h'(\omega) = 1$. By definition of a base, the open neighborhood $h'^{-1}(\{1\})$ of ω contains some B_k. Recall that (k, n_k) is in $q^{-1}(B_k)$, hence also in $q^{-1}(h'^{-1}(\{1\})) = h^{-1}(\{1\})$, so $h(k, n_k) = 1$. However, by definition of h, $h(k, n_k) = 0$. We reach a contradiction, so the coequalizer of f and g does not exist. □

Remark 15.4 The same proof shows that the category of quasi-Polish spaces does not have coequalizers.

16 The failure of Cartesian closure

Proposition 16.1 *In the category of domain-complete, resp. LCS-complete spaces, every exponentiable object is locally compact sober. The categories of domain-complete, resp. LCS-complete spaces, are not Cartesian-closed.*

Proof. Let X be an exponentiable object in any of those categories. By [13, Theorem 5.5.1], in any full subcategory of **Top** with finite products and containing $1 \stackrel{\text{def}}{=} \{*\}$ as an object, and up to a unique isomorphism, the exponential Y^X of two objects X, Y is the space $[X \to Y]$ of all continuous maps from X to Y, with some uniquely determined topology. We take $Y \stackrel{\text{def}}{=} \mathbb{S}$. Then $[X \to Y]$ can be equated with the lattice $\mathcal{O}X$ of open subsets of X. The application map from $[X \to Y] \times X$ to Y is continuous, and notice that product $\times$ here is just topological product (Proposition 15.1). It follows that the graph $(\in)$ of the membership relation on the topological product $X \times \mathcal{O}X$ is open. By [13, Exercise 5.2.7], this happens if and only if X is core-compact. Since X is also sober (Proposition 7.1), and sober core-compact spaces are locally compact [13, Theorem 8.3.10], X must be locally compact. Now take any non-locally compact LCS-complete space, for example Baire space $\mathbb{N}^{\mathbb{N}}$, which is Polish but not locally compact. □

Remark 16.2 The same proof shows that the category of quasi-Polish spaces is not Cartesian-closed. A similar proof, with $[0, 1]$ replacing $\mathbb{S}$, would show that the category of Polish spaces is not Cartesian-closed, using Arens' Theorem [3] (see also [13, Exercise 6.7.25]): the completely regular Hausdorff spaces that are exponentiable in the category of Hausdorff spaces are exactly the locally compact Hausdorff spaces.

We can be more precise on the subject of quasi-Polish spaces.

Theorem 16.3 *The exponentiable objects X in the category of quasi-Polish spaces are the locally compact quasi-Polish spaces, i.e., the countably-based locally compact sober spaces. For every quasi-Polish space Y, the exponential object is $[X \to Y]$ with the compact-open topology.*

Proof. We first note that every quasi-Polish space is sober and countably-based, and that conversely every countably-based locally compact sober is quasi-Polish [7, Theorem 44].

Assume X is locally compact quasi-Polish, and Y is quasi-Polish. The only thing we must show is that $[X \to Y]$, with the compact-open topology, is quasi-Polish. Indeed, the application map from $[X \to Y] \times X$ to Y will automatically be continuous, and the currification $z \mapsto (x \mapsto f(z,x))$ of every continuous map $f : Z \times X \to Y$ will be continuous from Z to $[X \to Y]$, because X is exponentiable in **Top** [13, Theorem 5.4.4] and the exponential object is $[X \to Y]$, with the compact-open topology, owing to the fact that X is locally compact [13, Exercise 5.4.8].

Up to homeomorphism Y is a $\mathbf{\Pi}^0_2$ subspace of $\mathbb{P}(\mathbb{N})$ [7, Corollary 24]. Hence write Y as $\{z \in \mathbb{P}(\mathbb{N}) \mid \forall n \in \mathbb{N}, z \in U_n \Rightarrow z \in V_n\}$, where U_n and V_n are open. As in the proof of Proposition 7.1, we define $f, g : \mathbb{P}(\mathbb{N}) \to \mathbb{P}(\mathbb{N})$ by $f(z) \stackrel{\text{def}}{=} \{n \in \mathbb{N} \mid z \in U_n\}$, $g(z) \stackrel{\text{def}}{=} \{n \in \mathbb{N} \mid z \in U_n \cap V_n\}$. The equalizer of f and g in **Top** is Y.

Since X is exponentiable, the exponentiation functor $\_^X$ on **Top** is well-defined and is right adjoint to the product functor $\_ \times X$ on **Top**. Since right adjoints preserve limits, in particular equalizers, Y^X is the equalizer of the maps $f^X, g^X : (\mathbb{P}(\mathbb{N}))^X \to (\mathbb{P}(\mathbb{N}))^X$. Since X is locally compact, we know that Y^X is $[X \to Y]$ with the compact-open topology (see [13, Exercise 5.4.11] for example).

Similarly, $(\mathbb{P}(\mathbb{N}))^X = [X \to \mathbb{P}(\mathbb{N})]$ with the compact-open topology. Recall that every quasi-Polish space is LCS-complete hence $\odot$-consonant (Lemma 13.2), and $\mathbb{P}(\mathbb{N})$ is an algebraic complete lattice. By Proposition 13.4, the compact-open topology on $[X \to \mathbb{P}(\mathbb{N})]$ is the Scott topology.

We now use [12, Proposition II-4.6], which says that if X is core-compact and L (here $\mathbb{P}(\mathbb{N})$) is an injective T_0 space (i.e., a continuous complete lattice by [12, Theorem II-3.8]), then $[X \to L]$ is a continuous complete lattice. We claim that $[X \to \mathbb{P}(\mathbb{N})]$ is countably-based. This follows froms [12, Corollary III-4.10], which says that when X is a T_0 core-compact space and L is a continuous lattice such that $w \stackrel{\text{def}}{=} \max(w(X), w(L))$ is infinite ($w(L)$ is the minimal cardinality of a basis of L, and is ω in our case; $w(X)$ is the *weight* of X, namely the minimal cardinality of a base of X, and is less than or equal to ω, by assumption), then $w([X \to L]) \leq w(\mathcal{O}[X \to L]) \leq w$. We have shown that $(\mathbb{P}(\mathbb{N}))^X = [X \to \mathbb{P}(\mathbb{N})]$ is a countably-based continuous dcpo, hence an ω-continuous dcpo by a result of Norberg [30, Proposition 3.1] (see also [13, Lemma 7.7.13]), hence a quasi-Polish space.

The equalizer (in **Top**) of two continuous maps between countably-based T_0 spaces is a $\mathbf{\Pi}^0_2$ subspace of the source space [7, Corollary 10]. Hence Y^X is $\mathbf{\Pi}^0_2$ in $(\mathbb{P}(\mathbb{N}))^X$. Since the $\mathbf{\Pi}^0_2$ subspaces of a quasi-Polish space are exactly its quasi-Polish subspaces [7, Corollary 23], $Y^X = [X \to Y]$ is quasi-Polish. ☐

17 Compact subsets of LCS-complete spaces

A well-known theorem due to Hausdorff states that, in a complete metric space, a subset is compact if and only if it is closed and precompact, where precompact means that for every $\epsilon > 0$, the subset can be covered by finitely many open balls of radius ϵ. An immediate consequence is as follows. Build a finite union A_0 of closed balls of radii at most 1. Then build a finite union A_1 of closed balls of radii at most 1/2 included in A_0, then a finite union A_2 of closed balls of radii at most 1/4 included in A_1, and so on. Then $\bigcap^{\downarrow}_{n\in\mathbb{N}} A_n$ is compact. (That argument is the key to showing that every bounded measure on a Polish space is tight, for example.) We show that a similar construction works in LCS-complete spaces.

In this section, we fix a presentation of an LCS-complete space X as $\ker \mu$ for some continuous map $\mu\colon Y \to \overline{\mathbb{R}}^{op}_{+}$, Y locally compact sober (see Remark 3.4). Replacing μ by $\frac{2}{\pi} \arctan \circ \mu$, we may assume that μ takes its values in $[0, 1]$.

For every non-empty compact saturated subset Q of Y, the image $\mu[Q]$ of Q by μ is compact in $[0, 1]^{op}$, hence has a largest element. Let us call that largest value the *radius* $r(Q)$ of Q. Note that this depends not just on Y, but also on μ. Note also that $r(\uparrow y) = \mu(y)$ for every $y \in Y$, and that $r(\bigcup_{i=1}^{n} Q_i) = \max\{r(Q_i) \mid 1 \leq i \leq n\}$.

Remark 17.1 . The name "radius" comes from the following observation. In the special case where $Y = \mathbf{B}(X, d)$ for some continuous complete quasi-metric space X, d, we may define $\mu(x, r) \stackrel{\text{def}}{=} r$, and in that case the radius of Q is $\max\{r \mid x \in X, (x, r) \in Q\}$.

Lemma 17.2 *Let X, Y, μ be as above. For every filtered family $(Q_i)_{i\in I}$ of non-empty compact saturated subsets of Y such that $\inf_{i\in I} r(Q_i) = 0$, $\bigcap^{\downarrow}_{i\in I} Q_i$ is a non-empty compact saturated subset of X.*

Proof. Since Y is sober hence well-filtered, $Q \stackrel{\text{def}}{=} \bigcap^{\downarrow}_{i\in I} Q_i$ is a non-empty compact saturated subset of Y. We show that Q is included in X by showing that, for every $y \in Q$, for every $\epsilon > 0$, $\mu(y) < \epsilon$. Indeed, since $\inf_{i\in I} r(Q_i) = 0$, we can find an index $i \in I$ such that $r(Q_i) < \epsilon$. Then $\mu(y) \leq r(Q_i)$, by definition of radii, and since $y \in Q_i$.

Hence Q is compact saturated in Y, and included in X, hence it is compact saturated in X, by Lemma 8.1, items 1 and 2. □

Lemma 17.3 *Let X, Y, μ be as above. For every non-empty compact saturated subset Q of X, for every open neighborhood U of Q in Y, for every $\epsilon > 0$, there is a non-empty compact saturated subset Q' of Y such that $Q \subseteq int(Q') \subseteq Q' \subseteq U$ and $r(Q') < \epsilon$.*

If Y is a continuous dcpo, we can even take Q' of the form $\uparrow A$ for some non-empty finite set $A = \{y_1, \cdots, y_n\}$, where $\mu(y_i) < \epsilon$ for every i.

Proof. $U \cap \mu^{-1}([0, \epsilon))$ is open, hence by local compactness it is the directed union of sets of the form $int(Q')$, where each Q' is compact saturated and included in

$U \cap \mu^{-1}([0, \epsilon))$. The open sets $int(Q')$ form a cover of Q, which is compact saturated in Y by Lemma 8.1, items 1 and 2, so some Q' as above is such that $Q \subseteq int(Q')$. By construction, $Q' \subseteq U$. Also, $r(Q') < \epsilon$ because $Q' \subseteq \mu^{-1}([0, \epsilon))$.

We prove the second part of the lemma in the more general case where Y is quasi-continuous. Then Y is locally finitary compact [13, Exercise 5.2.31], meaning that we can replay the above argument with Q' of the form $\uparrow A$ for A finite. □

Theorem 17.4 *Let X, Y, μ be as above. The non-empty compact saturated subsets of X are exactly the filtered intersections $\bigcap^{\downarrow}_{i \in I} Q_i$ of (interiors of) non-empty compact saturated subsets Q_i of Y such that $\inf_{i \in I} r(Q_i) = 0$. Moreover, we can choose that filtered intersection to be equal to $\bigcap^{\downarrow}_{i \in I} int(Q_i)$.*

When Y is a continuous dcpo, we can even take Q_i of the form $\uparrow A_i$, A_i finite.

Proof. One direction is Lemma 17.2. Conversely, let Q be compact saturated in X, and let $(Q_i)_{i \in I}$ be the family of compact saturated subsets of Y such that $Q \subseteq int(Q_i)$ (respectively, only those of the form $\uparrow A_i$ with A_i finite, if Y is a continuous dcpo). By Lemma 17.3 with $U \stackrel{\text{def}}{=} Y$, for every $\epsilon > 0$ there is an index $i \in I$ such that $r(Q_i) < \epsilon$, so $\inf_{i \in I} r(Q_i) = 0$. This also shows that the family is non-empty. For any two elements Q_i, Q_j of the family, we apply Lemma 17.3 with $U \stackrel{\text{def}}{=} int(Q_i) \cap int(Q_j)$ (and ϵ arbitrary), and we obtain an element Q_k such that $Q_k \subseteq int(Q_i) \cap int(Q_j)$. This shows that the family is filtered.

For every open neighborhood U of Q in Y, Lemma 17.3 (again) shows the existence of an index $i \in I$ such that $Q_i \subseteq U$. Therefore $Q = \bigcap^{\downarrow}_{i \in I} Q_i$. Finally, $Q \subseteq int(Q_i)$ for every $i \in I$, so $Q \subseteq \bigcap^{\downarrow}_{i \in I} int(Q_i) \subseteq \bigcap^{\downarrow}_{i \in I} Q_i = Q$, so all the terms involved are equal. □

In particular, if X, d is a continuous complete quasi-metric space, and taking $Y \stackrel{\text{def}}{=} \mathbf{B}(X, d)$ and $\mu(x, r) \stackrel{\text{def}}{=} r$, then the compact saturated subsets of X (in its d-Scott topology) are exactly the filtered intersections of sets $C_i \stackrel{\text{def}}{=} Q_i \cap X$. For each i, we can take Q_i of the form $\uparrow\{(x_1, r_1), \ldots, (x_n, r_n)\}$ where $r(Q_i) = \max\{r_1, \cdots, r_n\}$ is arbitrarily small. Then the sets C_i are easily seen to be finite unions of closed balls $B_{x_i, \le r_i}$ of arbitrarily small radius. That explains the connection with Hausdorff's theorem cited earlier. Note, however, that closed balls are in general not closed (except when X, d is metric), and need not be compact either.

18 Extensions of continuous valuations

Continuous valuations were introduced in [23,22]. As far as measure theory is concerned, we refer the reader to any standard reference, such as [4].

A *valuation* ν on a space X is a map from the lattice of open subsets $\mathcal{O}X$ of X to $\overline{\mathbb{R}}_+$ that is *strict* ($\nu(\emptyset) = 0$) and *modular* ($\nu(U) + \nu(V) = \nu(U \cup V) + \nu(U \cap V)$). A *continuous valuation* is additionally Scott-continuous. Every continuous valuation ν defines a linear prevision G by $G(h) \stackrel{\text{def}}{=} \int_{x \in X} h(x) d\nu$, and conversely any linear prevision defines a continuous valuation ν by $\nu(U) \stackrel{\text{def}}{=} G(\chi_U)$, where χ_U is the characteristic map of U.

Any pointwise directed supremum of continuous valuations is a continuous valuation again.

A continuous valuation ν is *locally finite* if and only if every point has an open neighborhood U such that $\nu(U) < \infty$. It is *bounded* if and only if $\nu(X) < \infty$. Let $\mathcal{A}(\mathcal{O}\,X)$ be the smallest Boolean algebra of subsets of X containing $\mathcal{O}\,X$. The elements of $\mathcal{A}(\mathcal{O}\,X)$ are the finite disjoint unions of *crescents*, where a crescent is a difference $U \setminus V$ of two open sets. The Smiley-Horn-Tarski theorem [34,20] states that every bounded valuation extends to a unique strict modular map from $\mathcal{A}(\mathcal{O}\,X)$ to $\mathbb{R}_+$.

Given any open set U, $\nu_{|U}$ is the continuous valuation defined by $\nu_{|U}(V) \stackrel{\text{def}}{=} \nu(U \cap V)$; that is bounded if and only if $\nu(U) < \infty$.

Let us write $\mathcal{B}(X)$ for the Borel σ-algebra of X. A measure on X is a σ-additive map from $\mathcal{B}(X)$ to $\overline{\mathbb{R}}_+$, or equivalently a strict, modular and ω-continuous map from $\mathcal{B}(X)$ to $\overline{\mathbb{R}}_+$. The latter makes it clear that the pointwise directed supremum of a family (even uncountable) of measures is a measure.

We will use the following standard fact, which we shall call *Kolmogorov's criterion*: given a bounded measure μ, and a descending sequence $(W_n)_{n\in\mathbb{N}}$ of Borel sets, $\mu(\bigcap^{\downarrow}_{n\in\mathbb{N}} W_n) = \inf_{n\in\mathbb{N}} \mu(W_n)$. We will also use the following: any two bounded measures that agree on $\mathcal{O}\,X$ agree on the whole of $\mathcal{B}(X)$.

If X is countably-based, or more generally if X is *hereditarily Lindelöf* (viz., every directed family of open subsets has a cofinal monotone sequence), every measure μ on X with its Borel σ-algebra restricts to a continuous valuation on the open sets. The following theorem shows that, conversely, every continuous valuation ν on an LCS-complete space extends to a measure μ. We recall that this holds for locally finite continuous valuations on locally compact sober spaces [2,24].

Lemma 18.1 *Let ν be a bounded valuation on a topological space X. If ν has an extension to a measure μ on $\mathcal{B}(X)$, then μ coincides with the* crescent outer measure *ν^* on $\mathcal{B}(X)$: $\nu^*(E) \stackrel{\text{def}}{=} \inf_{\mathcal{F}} \sum_{C\in\mathcal{F}} \nu(C)$, where $\mathcal{F}$ ranges over the countable families of crescents whose union contains E.*

Note that $\nu(C)$ makes sense by the Smiley-Horn-Tarski theorem.

Proof. For every open set U, taking $\mathcal{F} \stackrel{\text{def}}{=} \{U\}$, we obtain $\nu^*(U) \leq \nu(U) = \mu(U)$. Conversely, for every countable family $\mathcal{F}$ of crescents C whose union contains U, $\sum_{C\in\mathcal{F}} \nu(C) = \sum_{C\in\mathcal{F}} \mu(C) \geq \mu(\bigcup_{C\in\mathcal{F}} C) \geq \mu(U) = \nu(U)$, so $\nu^*(U) = \mu(U)$.

It is standard that ν^* defines a measure on the σ-algebra of *measurable sets*, where a subset A of X is called measurable if and only if for all subsets B of X, $\nu^*(B) = \nu^*(B \cap A) + \nu^*(B \setminus A)$ (see, e.g., [24, Theorem 3.2]). We claim that every open set U is measurable. Let us fix a subset B of X. For every crescent C, $C \cap U$ and $C \setminus U$ are crescents again. Hence, for every countable family $\mathcal{F} \stackrel{\text{def}}{=} (C_n)_{n\in\mathbb{N}}$ of crescents whose union contains B, $\sum_{C\in\mathcal{F}} \nu(C) = \sum_{n\in\mathbb{N}} \nu(C_n \cap U) + \nu(C_n \setminus U) \geq$

$\nu^*(B \cap U) + \nu^*(B \setminus U)$. Taking infima over $\mathcal{F}$, $\nu^*(B) \geq \nu^*(B \cap U) + \nu^*(B \setminus U)$. Conversely, for every countable family $\mathcal{F}$ of crescents whose union contains $B \cap U$, for every countable $\mathcal{F}'$ of crescents whose union contains $B \setminus U$, $\mathcal{F} \cup \mathcal{F}'$ is a countable family of crescents whose union contains B, so $\nu^*(B \cap U) + \nu^*(B \setminus U) \geq \nu^*(B)$, whence the equality. Since the measurable sets contain all the open sets, they also contain $\mathcal{B}(X)$.

Hence we have two measures on $\mathcal{B}(X)$, μ and ν^*, which coincide on the open sets. In particular, $\mu(X) = \nu^*(X) < \infty$, so they are bounded. It follows that μ and ν^* agree on the whole of $\mathcal{B}(X)$. □

Theorem 1.1 (recap). Let X be an LCS-complete space. Every continuous valuation ν on X extends to a measure on X with its Borel σ-algebra.

Proof. Let ν be a continuous valuation on X, and let X be written as $\bigcap^{\downarrow}_{n\in\mathbb{N}} W_n$, where each W_n is open in some locally compact sober space Y.

Let $(U_i)_{i\in I}$ be the family of open subsets of X of finite ν-measure. This is a directed family, since $\nu(U_i \cup U_j) \leq \nu(U_i) + \nu(U_j)$. We write U_∞ for $\bigcup^{\uparrow}_{i\in I} U_i$. If ν were locally finite, then U_∞ would be equal to X, but we do not assume so much.

For each $i \in I$, $\nu_{|U_i}$ is a bounded continuous valuation. Letting $e\colon X \to Y$ be the inclusion map, the image of $\nu_{|U_i}$ by e is another bounded continuous valuation, which we write as ν'_i: for every open subset V of Y, $\nu'_i(V) = \nu_{|U_i}(e^{-1}(V)) = \nu(V \cap U_i)$. Note that $i \sqsubseteq j$ implies $\nu'_i \leq \nu'_j$ (namely, $\nu'_i(V) \leq \nu'_j(V)$ for every V).

We claim that $i \sqsubseteq j$ implies that for every crescent C, $\nu'_i(C) \leq \nu'_j(C)$. In order to show that, let us write C as $U \setminus V$, where U and V are open in Y. Replacing V by $U \cap V$ if needed, we may assume $V \subseteq U$. For every $k \sqsubseteq j$, we have:

$$\begin{aligned}
\nu_{|U_j}(C \cap U_k) &= \nu_{|U_j}((U \cap U_k) \setminus (V \cap U_k)) \\
&= \nu_{|U_j}(U \cap U_k) - \nu_{|U_j}(V \cap U_k) \qquad \text{since } \nu_{|U_j} \text{ is additive on } \mathcal{A}(\mathcal{O}\,X) \\
&= \nu(U \cap U_k) - \nu(V \cap U_k) \qquad \text{since } U_k \subseteq U_j \\
&= \nu_{|U_k}(U) - \nu_{|U_k}(V) = \nu_{|U_k}(C). \qquad (1)
\end{aligned}$$

Taking $k \stackrel{\text{def}}{=} i$ in (1), $\nu_{|U_i}(C) = \nu_{|U_j}(C \cap U_i)$, which is less than or equal to $\nu_{|U_j}(C \cap U_j)$ (the difference is $\nu_{|U_j}(C \cap U_j \setminus U_i) \geq 0$), and the latter is equal to $\nu_{|U_j}(C)$ by (1) with $k \stackrel{\text{def}}{=} j$.

We have seen that ν'_i extends to a measure μ_i on Y. By Lemma 18.1, $\mu_i = \nu'^*_i$. Using the formula for the crescent outer measure, we obtain that if $i \sqsubseteq j$, then $\mu_i(E) \leq \mu_j(E)$ for every $E \in \mathcal{B}(Y)$.

Since X is G_δ hence Borel in Y, $\mathcal{B}(X)$ is included in $\mathcal{B}(Y)$. Hence μ_i also defines a measure on the smaller σ-algebra $\mathcal{B}(X)$. We still write it as μ_i, and we note that $i \sqsubseteq j$ implies that $\mu_i(E) \leq \mu_j(E)$ for every $E \in \mathcal{B}(X)$. Also, μ_i extends $\nu_{|U_i}$, as we now claim. Let U be any open subset of X. By definition of the subspace topology, U is the intersection of some open subset $\widehat{U}$ of Y with X. U is then equal to $\bigcap^{\downarrow}_{n\in\mathbb{N}} \widehat{U} \cap W_n$. Now $\mu_i(U) = \mu_i(\bigcap^{\downarrow}_{n\in\mathbb{N}} \widehat{U} \cap W_n) = \inf_{n\in\mathbb{N}} \mu_i(\widehat{U} \cap W_n)$ (Kolmogorov's criterion) $= \inf_{n\in\mathbb{N}} \nu'_i(\widehat{U} \cap W_n) = \inf_{n\in\mathbb{N}} \nu_{|U_i}(U)$ (since $\widehat{U} \cap W_n \cap U_i = U \cap U_i$) $= \nu_{|U_i}(U)$.

Any directed supremum of measures is a measure. Hence consider $\mu(E) \stackrel{\text{def}}{=} \sup^{\uparrow}_{i\in I} \mu_i(E)$. For every open subset U of X, $\mu(U) = \sup^{\uparrow}_{i\in I} \mu_i(U) = \sup^{\uparrow}_{i\in I} \nu_{|U_i}(U) = \sup^{\uparrow}_{i\in I} \nu(U \cap U_i) = \nu(U \cap U_\infty) = \nu_{|U_\infty}(U)$, so μ extends $\nu_{|U_\infty}$. Let ι be the indiscrete measure on $X \setminus U_\infty$, namely $\iota(E)$ is equal to ∞ if E intersects $X \setminus U_\infty$, to 0 if $E \subseteq U_\infty$. We check that the measure $\mu + \iota$ extends ν. For every open subset U of X, either $U \subseteq U_\infty$ and $\nu(U) = \nu_{|U_\infty}(U) = \mu(U) = \mu(U) + \iota(U)$, or U intersects $X \setminus U_\infty$, say at x. In the latter case, $\iota(U) = \infty$ so $\mu(U) + \iota(U) = \infty$, while $\nu(U) = \infty$ because, by definition, x has no open neighborhood of finite ν-measure.□

Remark 18.2 More generally, the proof of Theorem 1.1 would work on $\mathbf{\Pi}^0_2$ subsets of locally compact sober spaces. (That is a strict extension, by Proposition 14.5.) In that case, we write X as $\bigcap_{n\in\mathbb{N}} W_n$ where each W_n is the union of a closed and an open set. Replacing W_n by $\bigcap_{i=0}^{n} W_i$, we make sure that the sequence of sets W_n is descending, and W_n is still in $\mathcal{A}(\mathcal{O}\,Y)$. The rest of the proof is unchanged.

19 Conclusion

We have given two applications of the theory of LCS-complete spaces (Theorem 1.1, Corollary 13.7). We should mention a final application [15, Theorem 9.4], which will be published elsewhere: given a projective system $(p_{ij} \colon X_j \to X_i))_{i\sqsubseteq j\in I}$ of LCS-complete spaces such that I has a countable cofinal subset, given locally finite continuous valuations ν_i on X_i that are compatible in the sense that for all $i \sqsubseteq j$ in I, ν_i is the image valuation of ν_j by p_{ij}, there is a unique continuous valuation ν on the projective limit X of the projective system such that ν projects back to ν_i for every $i \in I$. This extends a famous theorem of Prohorov's [32], which appears as the subcase where each X_i is Polish and each ν_i is a measure.

One question that remains open, though, is: (i) Is the projective limit X of a projective system of LCS-complete spaces as above again LCS-complete?

That is only one of many remaining open questions: (ii) Is every sober compactly Choquet-complete space LCS-complete? (iii) Is every sober convergence Choquet-complete space domain-complete? (iv) Is every coherent LCS-complete space a G_δ subset of a stably (locally) compact space? (v) Is every $\mathbf{\Pi}^0_2$ subset of an domain-complete space again domain-complete? (A similar result fails for LCS-complete spaces, by Proposition 14.5.) (vi) Is every *countably correlated* space (i.e., every space homeomorphic to a $\mathbf{\Pi}^0_2$ subset of $\mathbb{P}(I)$ for some, possibly uncountable set I, see [6]) LCS-complete? (vii) Is every LCS-complete space countably correlated? $(viii)$ Are regular Čech-complete spaces LCS-complete, where Čech-complete is understood as in [13, Exercise 6.21]? (ix) Are all regular LCS-complete spaces Čech-complete?

Note added to the final version. Conjecture (v) was recently solved positively by the second author: every $\mathbf{\Pi}^0_2$ subset of a domain-complete space is domain-complete. As a consequence, (vi) is true as well; in fact, every countably correlated space is even domain-complete. This will be published elsewhere.

References

[1] Abramsky, S. and A. Jung, *Domain theory*, in: S. Abramsky, D. M. Gabbay and T. S. E. Maibaum, editors, *Handbook of Logic in Computer Science vol. III*, Oxford University Press, 1994 pp. 1–168.

[2] Alvarez-Manilla, M., A. Edalat and N. Saheb-Djahromi, *An extension result for continuous valuations*, Journal of the London Mathematical Society **61** (2000), pp. 629–640.

[3] Arens, R. F., *A topology for spaces of transformations*, Annals of Mathematics **47** (1946), pp. 480–495.

[4] Billingsley, P., "Probability and Measure," Wiley series in probability and mathematical statistics, John Wiley and Sons, 1995, 3rd edition.

[5] Čech, E., *On bicompact spaces*, Annals of Mathematics **38** (1937), pp. 823–844.

[6] Chen, R., *Notes on quasi-Polish spaces*, arXiv:1809.07440v1 [math.LO] (2018).

[7] de Brecht, M., *Quasi-Polish spaces*, Annals of Pure and Applied Logic **164** (2013), pp. 356–381.

[8] de Brecht, M., *A generalization of a theorem of Hurewicz for quasi-polish spaces*, Logical Methods in Computer Science **14(1:13)** (2018), pp. 1–18.

[9] Dolecki, S., G. H. Greco and A. Lechicki, *When do the upper Kuratowski topology (homeomorphically, Scott topology) and the co-compact topology coincide?*, Transactions of the American Mathematical Society **347** (1995), pp. 2869–2884.

[10] Dorais, F. G. and C. Mummert, *Stationary and convergent strategies in Choquet games*, Fundamenta Mathematicae **209** (2010), pp. 59–79.

[11] Edalat, A. and R. Heckmann, *A computational model for metric spaces*, Theoretical Computer Science **193** (1998), pp. 53–73.

[12] Gierz, G., K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove and D. S. Scott, "Continuous Lattices and Domains," Encyclopedia of Mathematics and its Applications **93**, Cambridge University Press, 2003.

[13] Goubault-Larrecq, J., "Non-Hausdorff Topology and Domain Theory—Selected Topics in Point-Set Topology," New Mathematical Monographs **22**, Cambridge University Press, 2013.

[14] Goubault-Larrecq, J., *Isomorphism theorems between models of mixed choice*, Mathematical Structures in Computer Science **27** (2017), pp. 1032–1067.
URL http://journals.cambridge.org/article_S0960129515000547

[15] Goubault-Larrecq, J., *Products and projective limits of continuous valuations on t_0 spaces*, arXiv:1803.05259v1 [math.PR] (2018).

[16] Goubault-Larrecq, J. and K. M. Ng, *A few notes on formal balls*, Logical Methods in Computer Science **13** (2017), pp. 1–34, special Issue of the Domains XII Workshop.
URL https://lmcs.episciences.org/4100

[17] Heckmann, R., *Spatiality of countably presentable locales (proved with the Baire category theorem)*, Mathematical Structures in Computer Science **25** (2015), pp. 1607–1625.

[18] Heckmann, R. and K. Keimel, *Quasicontinuous domains and the Smyth powerdomain*, Electronic Notes in Theoretical Computer Science **298** (2013), pp. 215–232.

[19] Hofmann, K.-H. and J. D. Lawson, *The spectral theory of distributive continuous lattices*, Transactions of the American Mathematical Society **246** (1978), pp. 285–310.

[20] Horn, A. and A. Tarski, *Measures in boolean algebras*, Transactions of the American Mathematical Society **64** (1948).

[21] Isbell, J. R., *Function spaces and adjoints*, Mathematica Scandinavica **36** (1975), pp. 317–339.

[22] Jones, C., "Probabilistic Non-Determinism," Ph.D. thesis, University of Edinburgh (1990), technical Report ECS-LFCS-90-105.

[23] Jones, C. and G. Plotkin, *A probabilistic powerdomain of evaluations*, in: *Proceedings of the 4th Annual Symposium on Logic in Computer Science* (1989), pp. 186–195.

[24] Keimel, K. and J. Lawson, *Measure extension theorems for T_0-spaces*, Topology and its Applications **149** (2005), pp. 57–83.

[25] Kostanek, M. and P. Waszkiewicz, *The formal ball model for $\mathcal{Q}$-categories*, Mathematical Structures in Computer Science **21** (2010), pp. 1–24.

[26] Kou, H., Y.-M. Liu and M.-K. Luo, *On meet-continuous dcpos*, in: G. Zhang, J. Lawson, Y. Liu and M. Luo, editors, *Domain Theory, Logic and Computation*, Semantic Structures in Computation **3**, Springer Netherlands, 2003 pp. 117–135.

[27] Künzi, H.-P. A., *An introduction to quasi-uniform spaces*, in: F. Mynard and E. Pearl, editors, *Beyond Topology*, Contemporary Mathematics **486**, American Mathematical Society, 2009 pp. 239–304.

[28] Martin, K., *The measurement process in domain theory*, in: *Proc. 27th International Conference on Automata, Languages and Programming (ICALP'00)* (2000), pp. 116–126.

[29] Nogura, T. and D. Shakhmatov, *When does the Fell topology on a hyperspace of closed sets coincide with the meet of the upper Kuratowski and the lower Vietoris topologies?*, Topology and its Applications **70** (1996), pp. 213–243.

[30] Norberg, T., *Existence theorems for measures on continuous posets, with applications to random set theory*, Mathematica Scandinavica **64** (1989), pp. 15–51.

[31] Picado, J. and A. Pultr, "Frames and Locales: Topology without points," Frontiers in Mathematics, Birkhaüser, 2011.

[32] Prohorov, Y. V., *Convergence of random processes and limit theorems in probability theory*, Theory of Probabilities and Applications **1** (1956), pp. 156–214.

[33] Romaguera, S. and O. Valero, *Domain theoretic characterisations of quasi-metric completeness in terms of formal balls*, Mathematical Structures in Computer Science **20** (2010), pp. 453–472. URL http://dx.doi.org/10.1017/S0960129510000010

[34] Smiley, M. F., *An extension of metric distributive lattices with an application to general analysis*, Transactions of the American Mathematical Society **56(3)** (1944), pp. 435–447.

[35] Wilker, P., *Adjoint product and hom functors in general topology*, Pacific Journal of Mathematics **34** (1970), pp. 269–283.