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# Directed Homology

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**Abstract**

We introduce a new notion of directed homology for semicubical sets. We show that it respects directed homotopy and is functorial, and that it appears to enjoy some good algebraic properties. Our work has applications to higher-dimensional automata.

*Keywords:* Directed homology, directed topology, directed homotopy, cubical sets, $\omega$-categories, higher-dimensional automata

## 1 Introduction

One can gain valuable insights in concurrency theory by exploring the geometry of concurrent systems. This point of view has been promoted for some time, and it appears that it is gaining territory. In this paper, we are introducing a notion of *directed homology* for semicubical sets, which should hopefully have various applications as a strong invariant of higher-dimensional automata [11], systems of weakly synchronizing PV processes [1], and other related formalisms for concurrent systems.

One of the characteristic features of algebraic topology is the interplay of homotopy and homology as invariants of topological spaces. For the *directed* topological spaces [2,5] used for the geometric modeling of concurrent systems, one has good notions of directed *homotopy* [6,12], but the concept of directed *homology* has hitherto been lacking.

In his recent papers [7,8], M. Grandis is working with a notion of directed combinatorial homology of cubical sets. What we define in the present paper differs considerably from his notion, and the relationship between the two is

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yet to be explored.

*Exposition*

We set out by introducing semicubical sets and relating them to the formalism of higher-dimensional automata. Then we define the central notion of this paper, the *chain $\omega$-category* of a semicubical set. Our exploration benefits considerably from the $\omega$-categorical viewpoint made possible by this.

We define three (graded) equivalence relations on semicubical sets; (directed) homotopy, weak homotopy, and homology, and we show some example calculations. All three equivalences give rise to graded quotients, which themselves are multiple categories. We end the paper by exploring some ways of extracting information from these quotients.

## 2 Cubical Sets and Their Morphisms

A *semicubical set* is a graded set $X = \{X_n\}_{n \in \mathbb{N}}$ together with mappings (*face maps*) $\delta_i^\alpha : X_n \to X_{n-1}$ ($i = 1, \dots, n$, $\alpha = 0, 1$) [1] satisfying the *semicubical axiom*

$$\delta_i^\alpha \delta_j^\beta = \delta_{j-1}^\beta \delta_i^\alpha \quad (i < j) \tag{1}$$

A *cubical set* is a semicubical set together with mappings (*degeneracies*) $\varepsilon_i : X_n \to X_{n+1}$ ($i = 1, \dots, n+1$), such that

$$\varepsilon_i \varepsilon_j = \varepsilon_{j+1} \varepsilon_i \quad (i \le j) \qquad\qquad \delta_i^\alpha \varepsilon_j = \begin{cases} \varepsilon_{j-1} \delta_i^\alpha & (i < j) \\ \varepsilon_j \delta_{i-1}^\alpha & (i > j) \\ \mathrm{id} & (i = j) \end{cases}$$

Cubical sets were introduced by Serre in [13]; they can be enriched with various other mappings; *connections*, *compositions*, and *reflections*, see [9] for an overview. The standard example of a cubical set is the singular cubical complex of a topological space [10]: If $X$ is a topological space, let $S_n X = \mathsf{Top}(I^n, X)$, the set of all continuous maps $I^n \to X$, where $I$ is the unit interval. If the faces and degeneracies are given by

$$\begin{aligned} \delta_i^\alpha f(t_1, \dots, t_{n-1}) &= f(t_1, \dots, t_{i-1}, \alpha, t_i, \dots, t_{n-1}) \\ \varepsilon_i f(t_1, \dots, t_n) &= f(t_1, \dots, \hat{t}_i, \dots, t_n) \end{aligned}$$

then $SX = \{S_n X\}$ is a cubical set.

[1] We always use $\alpha$ to mean one of 0 or 1, or $+$ or $-$. Also, the set $\mathbb{N}$ of natural numbers includes 0.

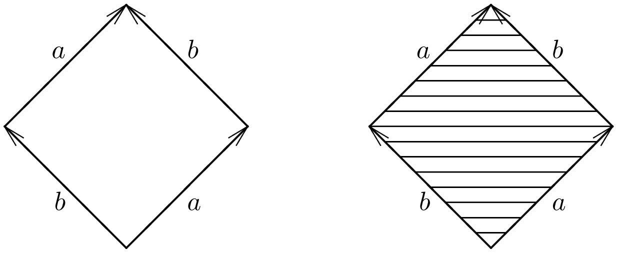


Fig. 1. Choice vs. concurrency.

### 2.1 Higher-Dimensional Automata

Higher-dimensional automata were introduced by Pratt [11], and their relation to cubical sets was established in [3,4]. In our notation, a higher-dimensional automaton over an alphabet $\Sigma$ is a cubical set $X = \{X_n\}$ together with a specified initial state $I \in X_0$ and a labeling mapping $\ell : X_1 \to \Sigma$ satisfying the condition that $\ell(\delta_1^0 x) = \ell(\delta_1^1 x)$ and $\ell(\delta_2^0 x) = \ell(\delta_2^1 x)$ for all $x \in X_2$.

Higher-dimensional automata are a generalization of finite automata that allow for the specification of *true concurrency*. As an example, consider figure 1, picturing two simple higher-dimensional automata over the alphabet $\{a, b\}$. In the left automaton, the interior of the rectangle is empty, specifying that there is a choice between executing $a.b$ or $b.a$. In the right automaton, there is a 2-cube connecting $a.b$ and $b.a$, with the semantics that $a$ and $b$ can be executed *simultaneously*. That is, the left automaton expresses a choice between two sequential behaviours, the right expresses that $a$ and $b$ are *truly concurrent*.

For a thorough treatment of higher-dimensional automata as a model of concurrent systems we refer to [4].

### 2.2 An Example

We introduce here a simple example of a semicubical set, which we shall refer to occasionally later. It consists of five 2-cubes glued together to form a hollow 3-cube without bottom face, a "turned-over open box." Figure 2 shows an image; for clarification we list the face maps from $X_2$ to $X_1$, the others should be obvious from the figure:

$$
\begin{array}{llll}
\delta_1^0 f_1 = e_5 & \delta_1^1 f_1 = e_6 & \delta_2^0 f_1 = e_1 & \delta_2^1 f_1 = e_9 \\
\delta_1^0 f_2 = e_6 & \delta_1^1 f_2 = e_7 & \delta_2^0 f_2 = e_2 & \delta_2^1 f_2 = e_{10} \\
\delta_1^0 f_3 = e_8 & \delta_1^1 f_3 = e_7 & \delta_2^0 f_3 = e_3 & \delta_2^1 f_3 = e_{11} \\
\delta_1^0 f_4 = e_5 & \delta_1^1 f_4 = e_8 & \delta_2^0 f_4 = e_4 & \delta_2^1 f_4 = e_{12} \\
\delta_1^0 f_5 = e_{12} & \delta_1^1 f_5 = e_{10} & \delta_2^0 f_5 = e_9 & \delta_2^1 f_5 = e_{11}
\end{array}
$$

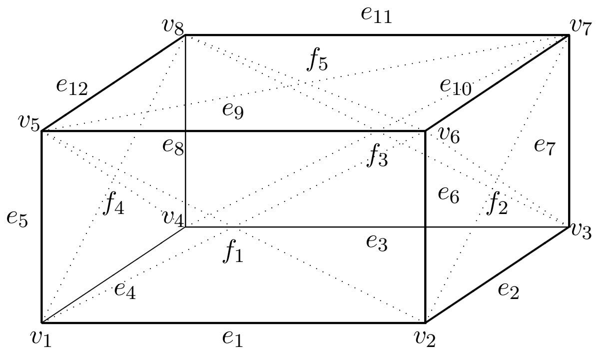


Fig. 2. A turned-over open box.

## 3 The Chain $\omega$-Category

We define an analogue of the chain complex of a cubical set. Our "directed chain complexes" are, in fact, strict globular $\omega$-categories.

Let $X = \{X_n\}$ be a semicubical set. For $x \in X_n$, define $\mathrm{d}^- x$, $\mathrm{d}^+ x$ by

$$\mathrm{d}^- x = \sum_{k=1}^{n} \delta_k^{(k+1) \bmod 2} x \qquad \mathrm{d}^+ x = \sum_{k=1}^{n} \delta_k^{k \bmod 2} x$$

This gives mappings $\mathrm{d}^-, \mathrm{d}^+ : X_n \to \mathbb{N} \cdot X_{n-1}$, where $\mathbb{N} \cdot X_{n-1}$ denotes the free abelian monoid on $X_{n-1}$. Extend these boundary mappings to be defined on $\mathbb{N} \cdot X_n$ by

$$\mathrm{d}^\alpha\Big(\sum \alpha_j x_j\Big) = \sum \alpha_j \, \mathrm{d}^\alpha x_j$$

Our mappings satisfy a weak version of the globular equality $\mathrm{d}^\alpha \mathrm{d}^- = \mathrm{d}^\alpha \mathrm{d}^+$; the proof is by direct calculation.

**Lemma 3.1** $\mathrm{d}^+\mathrm{d}^+ + \mathrm{d}^-\mathrm{d}^- = \mathrm{d}^+\mathrm{d}^- + \mathrm{d}^-\mathrm{d}^+$.

Our basic objects of study will not be (formal sums of) cubes, but rather formal sums of cubes *with specified lower boundaries*. So instead of studying, say, a "pure" element $x \in \mathbb{N} \cdot X_2$, we will consider "enriched versions" of $x$, which are 5-tuples $(x, \check{x}_1, \hat{x}_1, \check{x}_0, \hat{x}_0)$, where $\check{x}_1$ is to be thought of as a specified lower 1-boundary of $x$, $\hat{x}_1$ as a specified upper 1-boundary, etc.

Formally, we define our sets of *filtered* cubes $C_n X$ as follows:

$$\begin{aligned}
&C_0 X = \mathbb{N} \cdot X_0 \\
&C_1 X = \{(x_1, \check{x}_0, \hat{x}_0) \subseteq \mathbb{N} \cdot X_1 \times (\mathbb{N} \cdot X_0)^2 \mid \check{x}_0 + \mathrm{d}^+ x_1 = \hat{x}_0 + \mathrm{d}^- x_1\} \\
&C_n X = \{(x_n, \check{x}_{n-1}, \hat{x}_{n-1}, \dots, \check{x}_0, \hat{x}_0) \in \mathbb{N} \cdot X_n \times (\mathbb{N} \cdot X_{n-1})^2 \times \cdots \times (\mathbb{N} \cdot X_0)^2 \mid \\
&\qquad \check{x}_{n-1} + \mathrm{d}^+ x_n = \hat{x}_{n-1} + \mathrm{d}^- x_n, \forall i = 0, \dots, n-2 : \check{x}_i + \mathrm{d}^+ \check{x}_{i+1} = \hat{x}_i + \mathrm{d}^- \check{x}_{i+1}\}
\end{aligned}$$

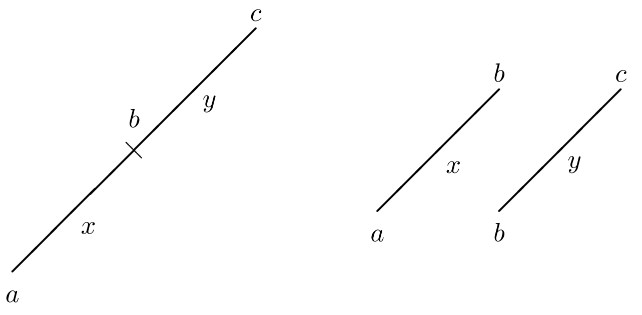


Fig. 3. The filtered cubes $(x+y,a,c)$ and $(x+y,a+b,b+c)$.

The intuition behind enriching (formal sums of) $n$-cubes with specified lower boundaries is that the same sum of cubes can have many different interpretations. This is inspired by a suggestion of Marco Grandis; here is a simple example:

Let $x, y \in X_1$, with $\delta_0^0 x = a$, $\delta_0^1 x = b$, $\delta_0^0 y = b$, $\delta_0^1 y = c$ (cf. figure 3). Then $x+y$ can be "used" for connecting $a$ to $c$, or for connecting $a+b$ to $b+c$. The former interpretation is expressed by the filtered cube $(x+y,a,c)$, the latter by $(x+y,a+b,b+c)$.

It is not difficult to show that the last condition in the definition of $C_nX$ is equivalent to demanding that $\check{x}_i + \mathrm{d}^+\hat{x}_{i+1} = \hat{x}_i + \mathrm{d}^-\hat{x}_{i+1}$, i.e. with the second and last "checks" replaced by "hats."

Now define mappings $\mathrm{d}^-, \mathrm{d}^+ : C_nX \to C_{n-1}X$, $\mathrm{e} : C_nX \to C_{n+1}X$ by

$$\begin{aligned}
\mathrm{d}^-(x_n, \check{x}_{n-1}, \hat{x}_{n-1}, \dots, \hat{x}_0) &= (\check{x}_{n-1}, \check{x}_{n-2}, \hat{x}_{n-2}, \dots, \hat{x}_0)\\
\mathrm{d}^+(x_n, \check{x}_{n-1}, \hat{x}_{n-1}, \dots, \hat{x}_0) &= (\hat{x}_{n-1}, \check{x}_{n-2}, \hat{x}_{n-2}, \dots, \hat{x}_0)\\
\mathrm{e}(x_n, \dots, \hat{x}_0) &= (0, x_n, x_n, \dots, \hat{x}_0)
\end{aligned}$$

then these satisfy $\mathrm{d}^\alpha \mathrm{d}^- = \mathrm{d}^\alpha \mathrm{d}^+$ and $\mathrm{d}^\alpha \mathrm{e} = \mathrm{id}$, that is, the graded set $CX = \{C_nX\}$ together with these mappings has a structure of *reflexive globular set.*

For $m < n \in \mathbb{N}$ let

$$C_nX \times_m C_nX = \{(x,y) \in C_nX \times C_nX \mid (\mathrm{d}^+)^{n-m} x = (\mathrm{d}^-)^{n-m} y\}$$

that is, $(x,y) = ((x_n, \dots, \check{x}_m, \hat{x}_m, \dots, \hat{x}_0), (y_n, \dots, \check{y}_m, \hat{y}_m, \dots, \hat{y}_0)) \in C_nX \times_m C_nX$ if and only if $\hat{x}_m = \check{y}_m$, and $\check{x}_i = \check{y}_i$ and $\hat{x}_i = \hat{y}_i$ for all $i = 0, \dots, m-1$. Define operations $\circ_m : C_nX \times_m C_nX \to C_nX$ by

$$\begin{aligned}
&(x_n, \dots, \hat{x}_0) \circ_m (y_n, \dots, \hat{y}_0) =\\
&\qquad (x_n + y_n, \dots, \check{x}_{m+1} + \check{y}_{m+1}, \hat{x}_{m+1} + \hat{y}_{m+1}, \check{x}_m, \hat{y}_m, \check{x}_{m-1}, \hat{x}_{m-1}, \dots, \hat{x}_0)
\end{aligned}$$

Note that the operations $\circ_m$ are *not commutative*: Given $x, y \in C_nX$, only one of $x \circ_m y$, $y \circ_m x$ might be defined, or they both may be defined, but have different values.

**Proposition 3.2** *$CX$ with operations $\circ_m$ and mappings $\mathrm{d}^-$, $\mathrm{d}^+$, $\mathrm{e}$ is a strict globular $\omega$-category, that is, $\mathrm{d}^\alpha \mathrm{d}^- = \mathrm{d}^\alpha \mathrm{d}^+$ and $\mathrm{d}^\alpha \mathrm{e} = \mathrm{id}$, and if $m < n \in \mathbb{N}$, $(x, y) \in C_n X \times_m C_n X$, then $\mathrm{e}x \circ_m \mathrm{e}y = \mathrm{e}(x \circ_m y)$ and*

$$\mathrm{d}^-(x \circ_m y) = \begin{cases} \mathrm{d}^- x \\ \mathrm{d}^- x \circ_m \mathrm{d}^- y \end{cases} \qquad \mathrm{d}^+(x \circ_m y) = \begin{cases} \mathrm{d}^+ y & \text{if } m = n-1 \\ \mathrm{d}^+ x \circ_m \mathrm{d}^+ y & \text{if } m < n-1 \end{cases}$$

*For any $z \in C_n X$,*

$$\mathrm{e}^{n-m}((\mathrm{d}^-)^{n-m} z) \circ_m z = z \circ_m \mathrm{e}^{n-m}((\mathrm{d}^+)^{n-m} z) = z$$

*and if also $(y, z) \in C_n X \times_m C_n X$, then*

$$(x \circ_m y) \circ_m z = x \circ_m (y \circ_m z)$$

*Also, if $(x', y') \in C_n X \times_m C_n X$ such that $(x, x'), (y, y') \in C_n X \times_p C_n X$ for some $p < n$, then*

$$(x \circ_m y) \circ_p (x' \circ_m y') = (x \circ_p x') \circ_m (y \circ_p y')$$

The proof is straight-forward.

In addition to the operations $\circ_m$, $0 \le m < n$, as defined above, we also have an operation $\circ_{-1}$ defined for all pairs $(x, y) \in C_n X \times C_n X \to C_n X$ and given by $(x_n, \dots, \hat{x}_0) \circ_{-1} (y_n, \dots, \hat{y}_0) = (x_n + y_n, \dots, \hat{x}_0 + \hat{y}_0)$. With this operation, $CX$ is a monoidal $\omega$-category.

We can turn the object mapping $\mathsf{SCub} \to \omega\mathsf{Cat}$ defined above into a functor the following way: Let $f : X \to Y$ be a morphism of semicubical sets, i.e. fulfilling $\delta_i^\alpha f = f \delta_i^\alpha$. Extend $f$ to a function $\mathbb{N} \cdot X_n \to \mathbb{N} \cdot Y_n$ by $f(\sum_i \alpha_i x_i) = \sum_i \alpha_i f(x_i)$, and define $\tilde{f} : CX \to CY$ by $\tilde{f}(x_n, \dots, \hat{x}_0) = (f x_n, \dots, f \hat{x}_0)$. Then $f \check{x}_i + \mathrm{d}^+ f \check{x}_{i+1} = f \check{x}_i + f\, \mathrm{d}^+ \check{x}_{i+1} = f(\check{x}_i + \mathrm{d}^+ \check{x}_{i+1}) = f(\hat{x}_i + \mathrm{d}^- \check{x}_{i+1}) = f \hat{x}_i + \mathrm{d}^- f \check{x}_{i+1}$, so $\tilde{f}$ is in fact a mapping $CX \to CY$. Also, $\tilde{f} \mathrm{d}^\alpha = \mathrm{d}^\alpha \tilde{f}$, $\tilde{f} \mathrm{e} = \mathrm{e} \tilde{f}$, and $\tilde{f}(x \circ_m y) = \tilde{f} x \circ_m \tilde{f} y$, so $\tilde{f}$ is a morphism of $\omega$-categories.

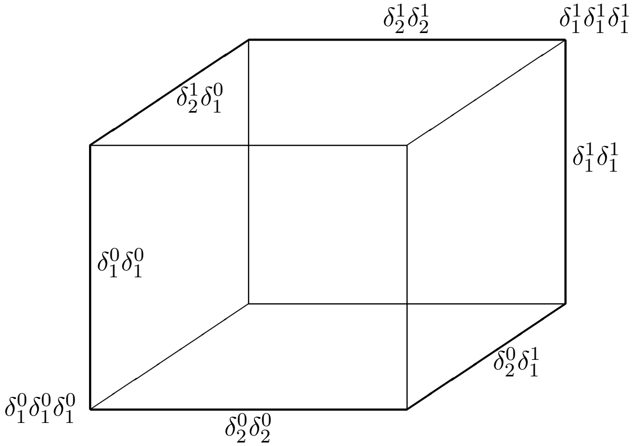


Fig. 4. The minimal 3-cube with specified 1- and 0-boundaries.

### 3.1 Minimal Representatives

Given an $n$-cube $x \in X_n$, define its *minimal representative* $\complement x = (x, \check{x}_{n-1}, \dots, \hat{x}_0) \in C_n X$ by

$$\check{x}_k = \sum_{i_1=1}^{k+1} \sum_{i_2=1}^{i_1} \cdots \sum_{i_{n-k}=1}^{i_{n-k+1}} \delta_{i_1}^{i_1+n-k} \cdots \delta_{i_{n-k}}^{i_{n-k}+n-k} x$$

$$\hat{x}_k = \sum_{i_1=1}^{k+1} \sum_{i_2=1}^{i_1} \cdots \sum_{i_{n-k}=1}^{i_{n-k+1}} \delta_{i_1}^{i_1+n-k+1} \cdots \delta_{i_{n-k}}^{i_{n-k}+n-k+1} x$$

(where the superscripts are to be understood modulo 2). The minimal representative owes its name to the following proposition, whose proof is a tedious but routine application of the semicubical identity (1):

**Proposition 3.3** *Given $y = (y_n, \dots, \hat{y}_0) \in C_n X$ such that $y_n \in X_n$, then there exists $z \in C_{n-1} X$ such that $y = \complement y_n \circ_{-1} z$.*

**Example 3.4** For a 3-cube $x \in X_3$,

$$\complement x = (x, \delta_1^0 x + \delta_2^1 x + \delta_3^0 x, \delta_1^1 x + \delta_2^0 x + \delta_3^1 x,$$
$$\delta_1^0\delta_1^0 x + \delta_2^1\delta_1^0 x + \delta_2^1\delta_2^1 x, \delta_1^1\delta_1^1 x + \delta_2^0\delta_1^1 x + \delta_2^0\delta_2^0 x, \delta_1^0\delta_1^0\delta_1^0 x, \delta_1^1\delta_1^1\delta_1^1 x)$$

Figure 4 displays the 3-cube in standard orientation (i.e. $\delta_1^0$ to the left, $\delta_2^0$ in front, and $\delta_3^0$ as the bottom face), where we have labeled the 1- and 0-boundaries of its minimal representative.

## 4 Weak Homotopy in $\omega$-Categories

We shall define three (graded) equivalences on the chain $\omega$-category of a semicubical set; (directed) homotopy, weak homotopy, and homology. We shall see that homotopy implies weak homotopy, which in turn implies homology, and that homotopy is essentially the same as the *combinatorial dihomotopy* relation of [2]. Weak homotopy applies to general $\omega$-categories, so we start with that one.

Note that there are some differences between our notion of homotopy and what one would call "standard" homotopy, in that what we are studying is a relation not between cubes, but rather between *formal sums* of cubes.

Given an $\omega$-category $C = \{C_n\}$, with boundary mappings $\mathrm{d}^\alpha$, identity mappings e, and compositions $\circ_n$, let $\sim_n \subseteq C_n \times C_n$ be the equivalence relation generated by the $n+1$-cells. That is, $\sim_n$ is the transitive, symmetric closure of the elementary relation $R_n$ defined by "$x R_n y$ if and only if there exists $A \in C_{n+1}$ such that $x = \mathrm{d}^- A$, $y = \mathrm{d}^+ A$."

**Lemma 4.1** *Assume $x \sim_n y \in C_n$. Then $\mathrm{d}^\alpha x = \mathrm{d}^\alpha y$, and if $x' \sim_n y' \in C_n$ are such that $(x, x') \in C_n \times_m C_n$ for some $m < n$, then also $(y, y') \in C_n \times_m C_n$, and $x \circ_m x' \sim_n y \circ_m y'$.*

Note that taking the symmetric closure of $R_n$ amounts to formally inverting $n+1$-cells. Let $\hat{C}_{n+1} = C_{n+1} \cup C_{n+1}^{\mathrm{op}}$ denote $C_{n+1}$ with formal inverses added for all cells; then $x \sim_n y$ if and only if there exist $A_1, \dots, A_k \in \hat{C}_{n+1}$ such that $\mathrm{d}^- A_1 = x$, $\mathrm{d}^+ A_i = \mathrm{d}^- A_{i+1}$ for all $i = 1, \dots, k-1$, and $\mathrm{d}^+ A_k = y$.

**Proof.** Let $A_1, \dots, A_k \in \hat{C}_{n+1}$ be a sequence of $n+1$-cells connecting $x$ and $y$. Then

$$\mathrm{d}^\alpha x = \mathrm{d}^\alpha \mathrm{d}^- A_1 = \mathrm{d}^\alpha \mathrm{d}^+ A_1 = \mathrm{d}^\alpha \mathrm{d}^- A_2 = \cdots = \mathrm{d}^\alpha \mathrm{d}^+ A_k = \mathrm{d}^\alpha y \tag{2}$$

Let $A'_1, \dots, A'_\ell \in \hat{C}_{n+1}$ be a sequence of $n+1$-cells connecting $x'$ and $y'$, then similarly $\mathrm{d}^\alpha x' = \mathrm{d}^\alpha \mathrm{d}^- A'_i = \mathrm{d}^\alpha \mathrm{d}^+ A'_i = \mathrm{d}^\alpha y'$. Hence $(\mathrm{d}^+)^{n-m} y = (\mathrm{d}^+)^{n-m} x = (\mathrm{d}^-)^{n-m} x' = (\mathrm{d}^-)^{n-m} y'$, and therefore $(y, y') \in C_n \times_m C_n$. Define $B_1, \dots, B_{k+\ell} \in \hat{C}_{n+1}$ by

$$B_i = \begin{cases} A_i \circ_m \mathrm{e}x' & \text{for } i = 1, \dots, k \\ \mathrm{e}y \circ_m A'_{i-k} & \text{for } i = k+1, \dots, k+\ell \end{cases}$$

We claim that $B_1, \dots, B_{k+\ell}$ is a sequence of $n+1$-cells connecting $x \circ_m x'$ to $y \circ_m y'$.

First we need to check that all $(A_i, \mathrm{e}x'), (\mathrm{e}y, A'_i) \in C_{n+1} \times_m C_{n+1}$, however as $n+1-m \geq 2$, we can use (2) to show that

$$(\mathrm{d}^+)^{n+1-m} A_i = (\mathrm{d}^+)^{n-m} \mathrm{d}^+ A_i = (\mathrm{d}^+)^{n-m} x = (\mathrm{d}^-)^{n-m} x' = (\mathrm{d}^-)^{n+1-m} \mathrm{e} x'$$

and similarly for showing that $(\mathrm{d}^-)^{n+1-m} A'_i = (\mathrm{d}^+)^{n+1-m} \mathrm{e} y$.

Now as $n+1-m \geq 2$, we have $\mathrm{d}^- B_1 = \mathrm{d}^- A_1 \circ_m \mathrm{d}^- \mathrm{e} x' = x \circ_m x'$ and $\mathrm{d}^+ B_{k+\ell} = \mathrm{d}^+ \mathrm{e} y \circ_m \mathrm{d}^+ A'_\ell = y \circ_m y'$. The last condition, $\mathrm{d}^+ B_i = \mathrm{d}^- B_{i+1}$ for all $i = 1, \dots, k+\ell-1$, is easily seen to be true for $i = 1, \dots, k-1$ and $i = k+1, \dots, k+\ell-1$. For $i = k$, $\mathrm{d}^+ B_k = \mathrm{d}^+ A_k \circ_m \mathrm{d}^+ \mathrm{e} x' = y \circ_m x'$ and $\mathrm{d}^- B_{k+1} = \mathrm{d}^- \mathrm{e} y \circ_m \mathrm{d}^- A'_1 = y \circ_m x'$. □

Let $D_n = C_n/\sim_n$, and define $\mathrm{d}^\alpha[x] = \mathrm{d}^\alpha x$, $[x] \circ_m [y] = [x \circ_m y]$. The degeneracies $\mathrm{e} : C_{n-1} \to C_n$ can be composed with the quotient mappings $C_n \to D_n$, yielding new degeneracies $\mathrm{e} : C_{n-1} \to D_n$. For each $n \in \mathbb{N}$ we define the *weak homotopy quotient* in dimension $n$ of $C$ by

$$\tilde{\pi}_n C = \left\{ D_n \underset{\mathrm{e}}{\overset{\mathrm{d}^\alpha}{\rightleftarrows}} C_{n-1} \underset{\mathrm{e}}{\overset{\mathrm{d}^\alpha}{\rightleftarrows}} \cdots \underset{\mathrm{e}}{\overset{\mathrm{d}^\alpha}{\rightleftarrows}} C_0 \right\}$$

which, with operations $\circ_m$ as above, is seen to be an $n$-category for all $n \in \mathbb{N}$.

We can turn the described object mappings $\tilde{\pi}_n : \omega\mathsf{Cat} \to n\mathsf{Cat}$ into functors as follows: Let $f = \{f_n\} : C \to D$ be a morphism of $\omega$-categories, i.e. fulfilling $\mathrm{d}^\alpha f = f \mathrm{d}^\alpha$, $\mathrm{e} f = f \mathrm{e}$, and $f(x \circ_m y) = fx *_m fy$. Then it can be shown that $x \sim_n y \in C_n$ implies $f_n x \sim_n f_n y \in D_n$, hence $f_n$ induces a mapping $f_n^\sharp : C_n/\sim_n \to D_n/\sim_n$.

We then have $\mathrm{d}^\alpha f_n^\sharp = f_{n-1} \mathrm{d}^\alpha$ and $f_n^\sharp([x] \circ_m [y]) = f_n^\sharp[x] *_m f_n^\sharp[y]$, hence $f_n^\sharp$ can be assembled with the $f_{n-1}, \dots, f_0$ to yield an $n$-morphism $f_\sharp : \tilde{\pi}_n C \to \tilde{\pi}_n D$.

## 5 Directed Homotopy of Semicubical Sets

If $X$ is a semicubical set, weak homotopy as above defines equivalence relations $\sim_n$ on the $C_n X$, where $CX = \{C_n X\}$ is the chain $\omega$-category associated with $X$. We can obtain finer relations by restricting the generating relations to single cubes instead of formal sums of these:

Let $R_n \subseteq C_n X \times C_n X$ be the relation defined by "$x R_n y$ if and only if there exist $A \in X_{n+1}$, $z \in C_n X$ such that $\mathrm{d}^-(\complement A \circ_{-1} z) = x$, $\mathrm{d}^+(\complement A \circ_{-1} z) = y$." Note that this amounts to saying that $x = (x_n, \dots, \hat{x}_0)$, $y = (y_n, \dots, \hat{y}_0)$ are such that $\mathrm{d}^\alpha x = \mathrm{d}^\alpha y$ and $(A, x_n, y_n, \dots, \hat{x}_0) \in C_{n+1} X$.

We define directed homotopy $\approx_n \subseteq C_n X \times C_n X$ to be the equivalence relation generated by the $R_n$, and we prove below that in dimension 1, our directed homotopy is essentially the same as the *combinatorial dihomotopy* relation of [2]. Note that $x \approx_n y$ implies $x \sim_n y$, hence we can form *directed* *homotopy quotients* ($n$-categories)

$$\pi_n C = \left\{ C_n/{\approx_n} \overset{\mathrm{d}^\alpha}{\underset{\mathrm{e}}{\rightleftarrows}} C_{n-1} \overset{\mathrm{d}^\alpha}{\underset{\mathrm{e}}{\rightleftarrows}} \cdots \overset{\mathrm{d}^\alpha}{\underset{\mathrm{e}}{\rightleftarrows}} C_0 \right\}$$

In order to state the next proposition, we need some definitions from [2]: A *dipath* in a cubical set $X = \{X_n\}$ is a sequence $x = (x_1, \dots, x_k) \subseteq X_1$ of 1-cubes such that for all $i = 1, \dots, k-1$, $\delta_1^1 x_i = \delta_1^0 x_{i+1}$. If $y = (y_1, \dots, y_k)$ is another such dipath (of the same length), then $x$ and $y$ are said to be *elementarily dihomotopic* if there exist $j \in \{1, \dots, k-1\}$ and $A \in X_2$ such that $x_i = y_i$ for all $i \neq j, j+1$, $\mathrm{d}^- A = x_j + x_{j+1}$, and $\mathrm{d}^+ A = y_j + y_{j+1}$. The relation of *combinatorial dihomotopy* is defined to be the reflexive, symmetric, and transitive closure of the elementary-dihomotopy relation.

**Proposition 5.1** *Let $x = (x_1, \dots, x_k)$, $y = (y_1, \dots, y_k)$ be dipaths in $X_1$, and write $\mathsf{C}x = \mathsf{C}x_1 \circ_0 \cdots \circ_0 \mathsf{C}x_k$, $\mathsf{C}y = \mathsf{C}y_1 \circ_0 \cdots \circ_0 \mathsf{C}y_k$, then $x$ and $y$ are combinatorially dihomotopic if and only if $\mathsf{C}x \approx_1 \mathsf{C}y$.*

**Proof.** It it enough to prove this for the generating relations. So let $A \in X_2$ be an elementary dihomotopy from $x$ to $y$. Note that

$$\mathsf{C}x = (x_1 + \cdots + x_n, \delta_1^0 x_1, \delta_1^1 x_n) \qquad \mathsf{C}y = (y_1 + \cdots + y_n, \delta_1^0 y_1, \delta_1^1 y_n)$$

and that we by the elementary dihomotopy know that $\delta_1^0 x_1 = \delta_1^0 y_1$, $\delta_1^1 x_n = \delta_1^1 y_n$. Also, as $A$ is an elementary dihomotopy, we have $\mathrm{d}^- A + y_1 + \cdots + y_n = \mathrm{d}^+ A + x_1 + \cdots + x_n$, hence the filtered cube

$$(A, x_1 + \cdots + x_n, y_1 + \cdots + y_n, \delta_1^0 x_1, \delta_1^1 x_n) \in C_2 X$$

provides a directed homotopy from $\mathsf{C}x$ to $\mathsf{C}y$.

For the other direction, assume that $\mathsf{C}x \approx_1 \mathsf{C}y$, that is, there exists $A \in X_2$ such that $(A, x_1 + \cdots + x_n, y_1 + \cdots + y_n, \delta_1^0 x_1, \delta_1^1 x_n) \in C_2 X$. Then

$$\delta_1^0 A + \delta_2^1 A + y_1 + \cdots + y_n = \delta_1^1 A + \delta_2^0 A + x_1 + \cdots + x_n$$

hence after cancellation we have indices $i, j, k, \ell$ such that $\delta_1^0 A = x_i$, $\delta_2^1 A = x_j$, $\delta_2^0 A = y_k$, $\delta_1^1 A = y_\ell$, cf. figure 5.

As this implies that $\delta_1^0 x_i = \delta_1^0 y_k$, $\delta_1^1 x_i = \delta_1^0 x_j$, $\delta_1^1 y_k = \delta_1^0 y_\ell$, and $\delta_1^1 x_j = \delta_1^1 y_\ell$, the $x_1, \dots, x_n$, $y_1, \dots, y_n$ can be rearranged in such a way that $i = k$ and $j = \ell = i+1$, hence $A$ provides an elementary dihomotopy from $x$ to $y$. $\square$

The fundamental category $\vec{\Pi}_1 X$ of a semicubical set $X$, cf. [12], is the category with object set $X_0$ and morphisms dihomotopy classes of dipaths,

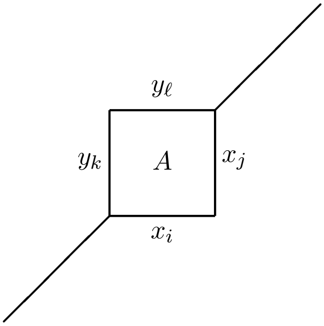


Fig. 5. An elementary dihomotopy.

i.e.

$$\vec{\Pi}_1 X(a,b) = \{[(x_1,\dots,x_k)] \mid (x_1,\dots,x_k) \text{ is a dipath from } a \text{ to } b\}$$

The following is an easy consequence of the preceding proposition:

**Proposition 5.2** $\vec{\Pi}_1 X$ *is isomorphic to the full subcategory of* $\pi_1 X$ *induced by the inclusion* $X_0 \subseteq C_0 X = \mathbb{N} \cdot X_0$.

### *5.1 Example*

Continuing the example from section 2.2, we see that the dipath $e_1 + e_2 + e_7$ is homotopic to the dipath $e_3 + e_4 + e_7$, by the following elementary relations:

$$\begin{aligned}
e_1 + e_2 + e_7 &\approx_1 e_1 + e_6 + e_{10} && \text{by} \quad \mathrm{d}^- f_2 = e_6 + e_{10}, \quad \mathrm{d}^+ f_2 = e_2 + e_7 \\
&\approx_1 e_5 + e_9 + e_{10} && \text{by} \quad \mathrm{d}^- f_1 = e_5 + e_9, \quad \mathrm{d}^+ f_1 = e_1 + e_6 \\
&\approx_1 e_5 + e_{12} + e_{11} && \text{by} \quad \mathrm{d}^- f_5 = e_{11} + e_{12}, \quad \mathrm{d}^+ f_5 = e_9 + e_{10} \\
&\approx_1 e_4 + e_8 + e_{11} && \text{by} \quad \mathrm{d}^- f_4 = e_5 + e_{12}, \quad \mathrm{d}^+ f_4 = e_4 + e_8 \\
&\approx_1 e_4 + e_3 + e_7 && \text{by} \quad \mathrm{d}^- f_3 = e_8 + e_{11}, \quad \mathrm{d}^+ f_3 = e_3 + e_7
\end{aligned}$$

Note that in the relation $e_1 + e_2 + e_7 \approx_1 e_3 + e_4 + e_7$, the edge $e_7$ cannot be canceled: There is no directed homotopy between $e_1 + e_2$ and $e_3 + e_4$.

## 6 Directed Homology of Semicubical Sets

Directed homology provides yet other equivalences on the chain $\omega$-category of a semicubical set $X$, which are coarser than weak homotopy and have better algebraic structure:

Let $\mathbb{Z} \cdot X_n$ denote the free abelian group on $X_n$, define boundary mappings $\mathrm{d}^\alpha : \mathbb{Z} \cdot X_n \to \mathbb{Z} \cdot X_{n-1}$ by $\mathrm{d}^\alpha(\sum \alpha_j x_j) = \sum \alpha_j \, \mathrm{d}^\alpha x_j$, and introduce sets

$\bar{C}_n X \supseteq C_n X$ by

$$\bar{C}_n X = \{(x_n, \check{x}_{n-1}, \dots, \hat{x}_0) \in \mathbb{Z} \cdot X_n \times \prod_{i=1}^n (\mathbb{N} \cdot X_{n-i})^2 \mid \check{x}_{n-1} + \mathrm{d}^+ x_n = \hat{x}_{n-1} + \mathrm{d}^- x_n, \forall i = 0, \dots, n-2 : \check{x}_i + \mathrm{d}^+ \check{x}_{i+1} = \hat{x}_i + \mathrm{d}^- \check{x}_{i+1}\}$$

So for an element $x = (x_n, \dots, \hat{x}_0) \in \bar{C}_n X$, the "cube itself" $x_n$ can have negative components, but all its boundaries are positive.

We define boundary mappings $\mathrm{d}^\alpha$ and operations $\circ_m$ on the $\bar{C}_n$ by the same formulas as in section 3. It is then easily seen that the graded set $\{\bar{C}_n X, C_{n-1} X, \dots, C_0 X\}$ with these mappings and operations has a structure of $n$-category.

Given $x, y \in C_n X$, say that $x \simeq_n y$ if there exists $A \in \bar{C}_{n+1} X$ such that $\mathrm{d}^- A = x$, $\mathrm{d}^+ A = y$. This defines equivalence relations $\simeq_n \subseteq C_n X \times C_n X$ which we shall refer to as *directed homology*.

**Proposition 6.1** *Given $x, y \in C_n X$; if $x \sim_n y$, then $x \simeq_n y$.*

**Proof.** Let $A_1, \dots, A_k \in C_{n+1} X \cup (C_{n+1} X)^{\mathrm{op}}$ be a sequence of $n+1$-cells connecting $x$ to $y$. The elements of $(C_{n+1} X)^{\mathrm{op}}$ are sequences $(a_{n+1}, \check{a}_n, \dots, \hat{a}_0) \in \mathbb{N} \cdot X_n \times \prod_{i=1}^n (\mathbb{N} \cdot X_{n-i})^2$ satisfying $\check{a}_n + \mathrm{d}^- a_{n+1} = \hat{a}_n + \mathrm{d}^+ a_{n+1}$ and $a_i + \mathrm{d}^+ \check{a}_{i+1} = \hat{a}_i + \mathrm{d}^- \check{a}_{i+1}$, so the set $(C_{n+1} X)^{\mathrm{op}}$ is in one-to-one correspondence with

$$C'_{n+1} X = \{(a_{n+1}, \check{a}_n, \dots, \hat{a}_0) \in (-\mathbb{N} \cdot X_{n+1}) \times \prod_{i=0}^n (\mathbb{N} \cdot X_{n-i})^2 \mid \check{a}_n + \mathrm{d}^+ a_{n+1} = \hat{a}_n + \mathrm{d}^- a_{n+1}, \forall i = 0, \dots, n-1 : \check{a}_i + \mathrm{d}^+ \check{a}_{i+1} = \hat{a}_i + \mathrm{d}^- \check{a}_{i+1}\}$$

Now $C'_{n+1} X \subseteq \bar{C}_{n+1} X$, so $(C_{n+1} X)^{\mathrm{op}}$ can be included in $\bar{C}_{n+1} X$, and we can think of the $A_i$ as elements of $\bar{C}_{n+1}$. Write $A_i = (A_i^{n+1}, \dots, \hat{A}_i^0)$ and define

$$A = A_1 \circ_n \cdots \circ_n A_k = (\textstyle\sum_{j=1}^k A_j^{n+1}, \check{A}_1^n, \hat{A}_k^n, \check{x}_{n-1}, \dots, \hat{x}_0)$$

then $A \in \bar{C}_{n+1} X$, $\mathrm{d}^- A = \mathrm{d}^- A_1 = x$, and $\mathrm{d}^+ A = \mathrm{d}^+ A_k = y$. □

**Lemma 6.2** *Assume $x \simeq_n y \in C_n X$. Then $\mathrm{d}^\alpha x = \mathrm{d}^\alpha y$, and if $x' \simeq_n y' \in C_n X$ are such that $(x, x') \in C_n X \times_m C_n X$ for some $m < n$, then also $(y, y') \in C_n X \times_m C_n X$, and $x \circ_m x' \simeq_n y \circ_m y'$.*

The proof is similar to the one of lemma 4.1.

So we can again define mappings $\mathrm{d}^\alpha : C_n X/{\simeq_n} \to C_{n-1} X$ and operations $\circ_m$ on $C_n X/{\simeq_n}$, and we assemble these to introduce *directed homology quotients*

$$H_n X = \left\{ C_n X/{\simeq_n} \overset{\mathrm{d}^\alpha}{\underset{\mathrm{e}}{\rightleftarrows}} C_{n-1} X \overset{\mathrm{d}^\alpha}{\underset{\mathrm{e}}{\rightleftarrows}} \cdots \overset{\mathrm{d}^\alpha}{\underset{\mathrm{e}}{\rightleftarrows}} C_0 X \right\}$$

which, with operations $\circ_m$ as above, are seen to be $n$-categories for all $n \in \mathbb{N}$.

Functoriality of this construction is obtained the same way as for directed homotopy; if $f : X \to Y$ is a morphism of semicubical sets, and $\tilde{f} : CX \to CY$ is the induced morphism of $\omega$-categories, then $x \simeq_n y \in C_n X$ implies $\tilde{f}_n x \simeq_n \tilde{f}_n y \in C_n Y$, hence we have an induced mapping $\tilde{f}_n^* : C_n X/{\simeq_n} \to C_n Y/{\simeq_n}$, which can be assembled with the other functions $\tilde{f}_{n-1}, \dots, \tilde{f}_0$ to yield a mapping $f_* : H_n X \to H_n Y$.

### 6.1 Example

For the example introduced in section 2.2, we have a directed homology between (the minimal representatives of) $e_1 + e_2 + e_7$ and $e_3 + e_4 + e_7$, mediated by the 2-cell

$$(f_1 + f_2 - f_3 - f_4 + f_5, e_3 + e_4 + e_7, e_1 + e_2 + e_7, v_1, v_7)$$

However the edge $e_7$ can be canceled, arriving at a new 2-cell

$$(f_1 + f_2 - f_3 - f_4 + f_5, e_3 + e_4, e_1 + e_2, v_1, v_3)$$

and hence also $e_1 + e_2$ and $e_3 + e_4$ are homologous. By looking at certain "fibres over 0-cells" we hope to obtain restricted dihomology relations which would disallow such cancellation (note that the end point $v_7$ has been replaced by $v_3$ in the second 2-cell). This is subject to further research.

### 6.2 Properties

The notion of directed homotopy introduced in section 5 seems to have some unusual properties, one of them being that the "Hurewicz mappings" $\pi_n X \to H_n X$ are *surjective*: By proposition 6.1, the homology quotient mappings $C_n X \to C_n X/{\simeq_n}$ pass to the *homotopy* quotient $C_n X/{\approx_n}$, hence the identity induces mappings $h_n : \pi_n X \to H_n X$, and these are indeed surjective.

Also, as the $H_n X$ are $n$-categories, we can extract various kinds of information from them by restricting our attention to certain fibres. One example of such restriction are the sets

$$H_1 X_a^b = \{x \in H_1 X \mid \mathrm{d}^- x = a, \mathrm{d}^+ x = b\}$$

for $a, b \in X_0$. The operation $\circ_0$ then provides a mapping $H_1 X_a^b \times H_1 X_b^c \to H_1 X_a^c$; moreover, if $a \simeq_0 a'$ and $b \simeq_0 b'$, then $H_1 X_a^b$ and $H_1 X_{a'}^{b'}$ are isomorphic.

Restriction to fibres can also be applied *before* taking homology quotients; an interesting example are the categories

$$C_2 X_a^b = \{x \in C_2 X \mid \mathrm{d}^- \mathrm{d}^- x = a, \mathrm{d}^+ \mathrm{d}^+ x = b\}$$

again for $a, b \in X_0$. This is what should lead to the "restricted dihomology relations" hinted at in section 6.1; note that, with the notation of the running example, *within* $C_2 X_{v_1}^{v_3}$ there is *no* equivalence of $e_1 + e_2$ with $e_3 + e_4$.

## 7 Future Work

It appears that our chain $\omega$-categories and dihomology quotients have just enough algebraic structure to make possible a meaningful notion of the quotient chain $\omega$-category induced by a semicubical subset, and to fit this quotient into an exact sequence. This in turn should make possible some Mayer-Vietoris like arguments, which should open up for actual computations of directed homology quotients. We plan to do this in a sequel paper.

With a look to applications, we note that higher-dimensional automata have symmetries (*reflections*, cf. [9]) in dimensions $\geq 2$, which essentially mean that for any $x \in X_n$, $n \geq 2$, there exists $x' \in X_n$ such that $\delta_1^\alpha x = \delta_i^{1-\alpha} x'$ for all $i = 1, \dots, n$. This suggests that the sets

$$H_n X_a^a = \{x \in H_n X \mid \mathrm{d}^- x = \mathrm{d}^+ x = a\}$$

for $n \geq 2$ and $a \in X_{n-1}$ should capture much of the information in the $H_n X$, and there should also be a strong relationship to "usual" homology.

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