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Computability of the Spectrum of Self-Adjoint Operators and the Computable Operational Calculus

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Abstract

Self-adjoint operators and their spectra play a crucial rôle in analysis and physics. Therefore, it is a natural question whether the spectrum of a self-adjoint operator and its eigenvalues can be computed from a description of the operator. We prove that given a "program" of the operator one can obtain positive information on the spectrum as a compact set in the sense that a dense subset of the spectrum can be enumerated and a bound on the set can be computed. This generalizes some non-uniform results obtained by Pour-El and Richards, which imply that the spectrum of any computable self-adjoint operator is a recursively enumerable compact set. Additionally, we show that the eigenvalues of self-adjoint operators can be computed in the sense that we can compute a list of indices such that those elements of the already computed dense subset of the spectrum, whose indices are not enumerated in this list, form the set of eigenvalues. Beside these main results we prove some computability results about the operational calculus, which we need in our proofs.

Keywords: Computable functional analysis, Hilbert spaces, self-adjoint operators, spectrum and eigenvalues, operational calculus, TTE.

1 Introduction

A linear operator $T : \mathcal{H} \to \mathcal{H}$ on a Hilbert space $\mathcal{H}$ over some field $\mathbb{F} \in \{\mathbb{C}, \mathbb{R}\}$ is called *self-adjoint*, if $T = T^*$ where T^* is the *adjoint operator* of T, which is the unique operator that satisfies $\langle Tx, y\rangle = \langle x, T^*y\rangle$. Any self-adjoint operator is *normal*, which means that $T^*T = TT^*$ holds. We will apply all these notions also to partial operators $T :\subseteq \mathcal{H} \to \mathcal{H}$, but in this case we additionally demand that $\mathrm{dom}(T)$ is dense in $\mathcal{H}$.

The *spectrum* $\sigma(T)$ of a bounded operator $T \in \mathcal{B}(\mathcal{H})$ is the set of all values $\lambda \in \mathbb{F}$ such that $\lambda - T$ has no inverse in the set $\mathcal{B}(\mathcal{H})$ of bounded linear operators

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$T : \mathcal{H} \to \mathcal{H}$. In particular, all *eigenvalues* of T, i.e. all $\lambda \in \mathbb{F}$ such that there exists a non-zero $x \in \mathcal{H}$ with $Tx = \lambda x$, are elements of the spectrum. The spectrum of a bounded linear operator is known to be a compact set and in absolute value it is bounded by $||T|| = \sup_{||x||=1} ||Tx||$, i.e. by the *operator norm* of the corresponding operator $T \in \mathcal{B}(\mathcal{H})$. For self-adjoint operators the spectrum is known to be real, i.e. $\sigma(T) \subseteq \mathbb{R}$ (see [10] for statements regarding linear operators on Hilbert spaces).

¿From the perspective of computable analysis a natural question is whether the spectrum and the eigenvalues of a self-adjoint operator can be computed in some natural sense. In general, there are at least two variants of such a result, which could be of interest, a uniform and a non-uniform one:

(i) (Uniform) the maps $T \mapsto \sigma(T)$ and/or $T \mapsto \sigma_p(T)$ are computable, [2]

(ii) (Non-uniform) $\sigma(T)$ and/or $\sigma_p(T)$ are computable for any computable T.

It is clear that any uniform result implies the corresponding non-uniform one (as computable maps map computable inputs to computable outputs). However, in general the uniform result is much stronger. What complicates things here is that the spectrum and also the operator might be computable in different senses. Regarding operators we will typically represent them in a way, which corresponds to programs (i.e. we will rather use the compact-open topology and not the operator norm topology). Regarding compact subsets, we will consider a variant of computability, which includes positive information on the compact set.

The main result of this paper, presented in Section 4, is that for self-adjoint operators in complex Hilbert spaces we obtain the following uniform and non-uniform results:

(i) (Uniform) the map $T \mapsto \sigma(T)$ is lower semi-computable (that is computable with respect to positive information on $\sigma(T)$),

(ii) (Uniform) the map $T \mapsto \sigma_p(T)$ is computable in the sense that we can enumerate a sequence in $\sigma(T)$ and a list of indices such that those elements of the sequence whose indices are not enumerated in the list form the set of eigenvalues of T,

(iii) (Non-uniform) $\sigma(T)$ is a recursively enumerable compact set for any computable T,

(iv) (Non-uniform) all eigenvalues are computable reals for any computable T.

The non-uniform version of this result also follows from the Second Main Theorem of Pour-El and Richards [9,8]. In [4] we have already proved a result corresponding to the first "uniform" item. But here we present another proof for this result, which is similar to the proof in [9] and gives us the possibility to obtain also some information about the eigenvalues. In [4] we also proved that the result for $\sigma(T)$ cannot be strengthened to recursive compactness, because any recursively enumerable compact set can be represented as the spectrum of some computable normal operator. This is in contrast to the finite-dimensional case where the spectrum map

[2] By $\sigma_p(T)$ we denote the point spectrum of T, i.e. the set of all eigenvalues of T.

$T \mapsto \sigma(T)$ is computable in a stronger sense [13,14].

For the proof of our main results we need a computable version of the operational calculus, which we derive in Section 3. In Section 5 we introduce some "prerequisites" that we need for the proof of the main result, namely triangle functions and the norms of the operational calculus. Most of the technical part of the proof is done in Section 6 where we define two auxiliary mappings, which have some interesting properties and do the main part of computing the spectrum. We finish the proof of our main result in Section 7.

In the following Section 2 we briefly introduce some required notions from computable analysis, the Turing machine based theory of computability and complexity, which is the approach that we will use throughout this paper (see [9,7,11] for comprehensive introductions).

2 Computable Hilbert Spaces

In this section we briefly introduce the required tools from computable analysis, which we will use in the following. For a more comprehensive introduction the reader is referred to [11] and the other cited references. We will not introduce notions from functional analysis here and the reader is referred to standard textbooks in this case. In the following we will discuss operators $T :\subseteq \mathcal{H} \to \mathcal{H}$ on Hilbert spaces $\mathcal{H}$ and we are in particular interested in computable Hilbert spaces, which we define below (the inclusion symbol "$\subseteq$" indicates that T might be partial). In general we assume that $\mathcal{H}$ is defined over the field $\mathbb{F}$, which might either be $\mathbb{R}$ or $\mathbb{C}$. Throughout the paper, we assume that $\mathcal{H} \neq \{0\}$.

Definition 2.1 A *computable Hilbert space* $(\mathcal{H}, \langle . \rangle, e)$ is a separable Hilbert space $(\mathcal{H}, \langle . \rangle)$ together with a fundamental sequence $e : \mathbb{N} \to \mathcal{H}$ (i.e. the closure of the linear span of range(e) is dense in $\mathcal{H}$) such that the induced normed space is a computable normed space.

The induced normed space is the normed space with the norm given by $||x|| := \sqrt{\langle x, x \rangle}$. A *computable normed space* is a normed space such that the metric d induced by $d(x, y) := ||x - y||$ together with the sequence $\alpha_e : \mathbb{N} \to \mathcal{H}$, defined by $\alpha_e\langle k, \langle n_0, ..., n_k\rangle\rangle := \sum_{i=0}^{k} \alpha_{\mathbb{F}}(n_i)e_i$, form a computable metric space such that the linear operations (vector space addition and scalar multiplication) become computable. Here $\alpha_{\mathbb{F}}$ is a standard numbering of $\mathbb{Q}_{\mathbb{F}}$ where $\mathbb{Q}_{\mathbb{F}} = \mathbb{Q}$ in case of $\mathbb{F} = \mathbb{R}$ and $\mathbb{Q}_{\mathbb{F}} = \mathbb{Q}[i]$ in case of $\mathbb{F} = \mathbb{C}$. We assume that there is some $n \in \mathbb{N}$ with $\alpha_{\mathbb{F}}(n) = 0$. Without loss of generality, we can even assume that $(e_n)_{n \in \mathbb{N}}$ is an orthonormal basis of $\mathcal{H}$. A *computable metric space* X is a separable metric space together with a sequence $\alpha : \mathbb{N} \to X$ such that range(α) is dense in X and $d \circ (\alpha \times \alpha)$ is a computable (double) sequence of reals.

If not mentioned otherwise, then we assume that all computable Hilbert spaces $\mathcal{H}$ are represented by their Cauchy representation $\delta_{\mathcal{H}}$ (of the induced computable metric space). The *Cauchy representation* $\delta :\subseteq \Sigma^\omega \to X$ of a computable metric space X is defined such that a sequence $p \in \Sigma^\omega$ represents a point $x \in X$, if it encodes a sequence $(\alpha(n_i))_{i\in\mathbb{N}}$ that rapidly converges to x, where rapid means that $d(\alpha(n_i), \alpha(n_j)) < 2^{-j}$ for all $i > j$. Here Σ^ω denotes the set of infinite sequences over some finite set Σ (the *alphabet*) and Σ^ω is endowed with the product topology with respect to the discrete topology on Σ. All computability statements with respect to Hilbert spaces are to be understood with respect to the Cauchy representation.

In general a *representation* of a set X is a surjective map $\delta :\subseteq \Sigma^\omega \to X$. Here the inclusion symbol "$\subseteq$" indicates that the corresponding map might be partial. Given representations $\delta :\subseteq \Sigma^\omega \to X$ and $\delta' :\subseteq \Sigma^\omega \to Y$, a map $f :\subseteq X \to Y$ is called (δ, δ')*–computable*, if there exists a computable map $F :\subseteq \Sigma^\omega \to \Sigma^\omega$ such that $\delta' F(p) = f\delta(p)$ for all $p \in \mathrm{dom}(f\delta)$. Analogously, one can define computability for multi-valued functions $f :\subseteq X \rightrightarrows Y$. In this case the equation above has to be replaced by the condition $\delta' F(p) \in f\delta(p)$. Here a function $F :\subseteq \Sigma^\omega \to \Sigma^\omega$ is called *computable* if there exists a Turing machine which computes F. Similarly, one can define the concept of *continuity* with respect to representations, where the computable function F is replaced by a continuous function.

Cauchy representations of computable metric spaces X are known to be *admissible* and for such representations continuity with respect to representations coincides with ordinary continuity. If X, Y are computable metric spaces, then we assume that the space $\mathcal{C}(X, Y)$ of continuous functions $f : X \to Y$ is represented by $[\delta_X \to \delta_Y]$, which is a canonical function space representation. This representation satisfies two characteristic properties, evaluation and type conversion, which can be performed computably (see [11] for details). If $Y = \mathbb{F}$, then we write for short $\mathcal{C}(X) = \mathcal{C}(X, \mathbb{F})$.

It is clear that the inner product of any computable Hilbert space is a computable map.

Proposition 2.2 *The inner product* $\langle.\rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{F}, (x, y) \mapsto \langle x, y\rangle$ *of any computable Hilbert space* $\mathcal{H}$ *is computable.*

Proof. This follows from the fact that the norm $||.|| : \mathcal{H} \to \mathbb{R}$ is a computable map for any computable normed space and the fact that the inner product satisfies the polar identities

$$\langle x, y\rangle = \frac{1}{4}(||x + y||^2 - ||x - y||^2)$$

in case of $\mathbb{F} = \mathbb{R}$ and

$$\langle x, y\rangle = \frac{1}{4}(||x + y||^2 - ||x - y||^2 + i||x + iy||^2 - i||x - iy||^2)$$

for $\mathbb{F} = \mathbb{C}$. □

By $\delta_\mathbb{F}$ and $\delta_\mathbb{R}$ we denote standard representations of $\mathbb{F}$ and $\mathbb{R}$, respectively. For the countable sets $\mathbb{N}$ and $\mathbb{Z}$ we use standard notations $\nu_\mathbb{N}$ and $\nu_\mathbb{Z}$, respectively.

3 The Computable Operational Calculus

In the proof of their Second Main Theorem Pour-El and Richards use some properties of the operational calculus to compute a dense sequence in the spectrum of a self-adjoint operator and to distinguish between elements of the continuous spectrum and eigenvalues [9]. Thus we need some computability results about the operational calculus to obtain a uniform version of the proof.

3.1 Continuous Functions with Compact Domain

For the main result in this section, we need a representation for the partial continuous functions from $\mathbb{R}$ to $\mathbb{F}$ (not only for those with a common fixed domain). As it is impossible to obtain such a representation for the set of all these functions, we restrict ourselves to the set of continuous functions with nonempty compact domain.

In the following we denote the set of all compact subsets of $\mathbb{R}$ by $\mathcal{K} := \{K \subseteq \mathbb{R} \mid K \text{ compact}\}$ and the set of all nonempty compact subsets by $\mathcal{K}^* := \mathcal{K} \setminus \{\emptyset\}$. Given a set $A \subseteq \mathbb{R}$, we define $\mathcal{C}(A) := \mathcal{C}(A, \mathbb{F})$. As we only use this notion $\mathcal{C}(A)$ in conjunction with a Hilbert space $\mathcal{H}$, it will always be clear what set we mean by $\mathcal{C}(A)$. Given $A \in \mathcal{K}^*$ we use the supremum norm $\|f\|_\infty := \max_{x \in A} |f(x)|$ on $\mathcal{C}(A)$.

Definition 3.1 We denote the set of all polynomials on $\mathbb{R}$ with rational coefficients by

$$\mathrm{Pn} := \left\{ f \in \mathcal{C}(\mathbb{R}) \mid \exists n \in \mathbb{N}\, \exists a_0, \ldots, a_n \in \mathbb{Q}_\mathbb{F}\, \forall t \in \mathbb{R} : f(t) = \sum_{i=0}^{n} a_i t^i \right\}.$$

Let ν_{Pn} be some standard notation of Pn. Given some nonempty compact set $A \subseteq \mathbb{R}$, we denote the set of all polynomials with rational coefficients and domain A by

$$\mathrm{Pn}_A := \{p\rfloor_A \mid p \in \mathrm{Pn}\} \quad \text{with} \quad A \in \mathcal{K}^*.$$

Let ν_{Pn_A} be the standard notation of Pn_A derived from ν_{Pn}:

$$\nu_{\mathrm{Pn}_A}(w) = p \;:\Longleftrightarrow\; \nu_{\mathrm{Pn}}(w) = q \text{ and } p = q\rfloor_A$$

for $w \in \Sigma^*$, $p \in \mathrm{Pn}_A$ and $q \in \mathrm{Pn}$.

For any nonempty compact subset $A \subseteq \mathbb{R}$ the set Pn_A is dense in the space $(\mathcal{C}(A), \|\,\|_\infty)$ (The Stone-Weierstrass theorem). Thus using the notation ν_{Pn_A}, we can define a Cauchy representation of $\mathcal{C}(A)$, i.e. the set of all continuous functions with common compact domain $A \in \mathcal{K}^*$ (see also [11]).

Definition 3.2 [Cauchy representation of $\mathcal{C}(A)$] For any nonempty compact set $A \subseteq \mathbb{R}$, we define the Cauchy representation δ^A_{Cp} of $\mathcal{C}(A)$ by

$$\begin{aligned} \delta^A_{\mathrm{Cp}}(p) = f : \Longleftrightarrow\; p = \iota(w_0)\iota(w_1)\ldots, \text{ such that} \\ \|\nu_{\mathrm{Pn}_A}(w_i) - \nu_{\mathrm{Pn}_A}(w_k)\|_\infty < 2^{-i} \text{ for } i < k \\ \text{and } f = \lim_{i\to\infty} \nu_{\mathrm{Pn}_A}(w_i). \end{aligned}$$

That is, a name of a function in $\mathcal{C}(A)$ consists of a rapidly converging sequence of polynomials with rational coefficients. Now we can define a representation for the set of all continuous functions with compact domain.

Definition 3.3 [Representation of $\mathcal{C}_{\mathcal{K}}(\subseteq \mathbb{R})$] We denote the set of all continuous functions $f :\subseteq \mathbb{R} \to \mathbb{F}$ with nonempty compact domain $\mathrm{dom}(f) \in \mathcal{K}^*$ by

$$\mathcal{C}_{\mathcal{K}}(\subseteq \mathbb{R}) := \bigcup_{A \in \mathcal{K}^*} \mathcal{C}(A)\,.$$

We define a representation δ_{Cp} of the set $\mathcal{C}_{\mathcal{K}}(\subseteq \mathbb{R})$ of continuous functions $f :\subseteq \mathbb{R} \to \mathbb{F}$ with compact domain by

$$\delta_{\mathrm{Cp}}\langle p, q\rangle = f :\Longleftrightarrow \kappa(q) = \mathrm{dom}(f) \neq \emptyset \text{ and } \delta_{\mathrm{Cp}}^{\kappa(q)}(p) = f\,.$$

That is, a name of a continuous function f with compact domain contains two types of information, a name q of the domain $\mathrm{dom}(f)$ of f as a compact subset of $\mathbb{R}$ and a name p of f as a continuous map with (fixed) compact domain as a rapidly converging sequence of polynomials with rational coefficients. Here by κ we denote the standard representation of $\mathcal{K}$ using positive and negative information (see [11]). If not mentioned otherwise, we will assume that $\mathcal{C}_{\mathcal{K}}(\subseteq \mathbb{R})$ is represented by δ_{Cp}.

Given a a total continuous function $f : \mathbb{R} \to \mathbb{F}$ and a nonempty compact subset $K \in \mathcal{K}^*$, we can compute the restriction $f\rfloor_K \in \mathcal{C}_{\mathcal{K}}(\subseteq \mathbb{R})$.

Proposition 3.4 *The mapping*

$$F :\subseteq \mathcal{C}(\mathbb{R}) \times \mathcal{K} \to \mathcal{C}_{\mathcal{K}}(\subseteq \mathbb{R}),\ (f, K) \mapsto f\rfloor_K$$

with $\mathrm{dom}(f) := \mathcal{C}(\mathbb{R}) \times \mathcal{K}^*$ *is* $([[\delta_{\mathbb{R}} \to \delta_{\mathbb{F}}], \kappa], \delta_{Cp})$*-computable.*

Proof. Given $f \in \mathcal{C}(\mathbb{R})$, $K \in \mathcal{K}^*$, we systematically search a polynomial $p_n \in \mathrm{Pn}$ such that $\|f\rfloor_K - p_n\rfloor_K\|_\infty < 2^{-n}$ with $n \in \mathbb{N}$. This can be done computably and uniformly in n (Effective Weierstraß Theorem, [11]). The sequence of all these polynomials forms a name of $f\rfloor_K \in \mathcal{C}_{\mathcal{K}}(\subseteq \mathbb{R})$. □

On the other hand, given a function $f \in \mathcal{C}_{\mathcal{K}}(\subseteq \mathbb{R})$ we can obtain a rapidly converging sequence of polynomials and the domain of f, as both information are encoded in a name of f.

Lemma 3.5 *The multi-valued mapping*

$$F_1 : \mathcal{C}_{\mathcal{K}}(\subseteq \mathbb{R}) \rightrightarrows \mathrm{Pn}^{\mathbb{N}}$$

with

$$(p_n)_{n\in\mathbb{N}} \in F_1(f) :\Longleftrightarrow \|p_i\rfloor_{\mathrm{dom}(f)} - p_k\rfloor_{\mathrm{dom}(f)}\|_\infty < 2^{-i} \text{ for } i < k \text{ and } f = \lim_{i\to\infty} p_i\rfloor_{\mathrm{dom}(f)}$$

is $\left(\delta_{Cp}, \nu_{\mathrm{Pn}}^{\mathbb{N}}\right)$*-computable, and the mapping*

$$F_2 : \mathcal{C}_{\mathcal{K}}(\subseteq \mathbb{R}) \rightrightarrows \mathcal{K},\ f \mapsto \mathrm{dom}(f)$$

is (δ_{Cp}, κ)*-computable.*

3.2 Computability of the Continuous Operational Calculus

In this section we use the operational calculus as it is defined in [12, VII.1.3/4], though we denote the operational calculus corresponding to an operator T by Φ_T

(and not by Φ) to point out the dependency between Φ_T and T. In particular, we use the derivation of the continuous operational calculus as the unique continuous extension of the mapping $(T, p) \mapsto p(T)$ defined for operators $T \in \mathcal{B}(\mathcal{H})$ and polynomials p. Here we briefly state some statements about the operational calculus

Proposition 3.6 (Continuous Operational Calculus [12]) *Let $\mathcal{H}$ be a complex Hilbert space and $T \in \mathcal{B}(\mathcal{H})$ be self-adjoint. Then there exists a unique mapping $\Phi_T : \mathcal{C}(\sigma(T)) \to \mathcal{B}(\mathcal{H})$ such that*

(i) $\Phi_T(\mathbf{t}) = T$, $\Phi_T(\mathbf{1}) = \mathrm{id}$,

(ii) Φ_T *is linear,*

(iii) $\Phi_T(f \cdot g) = \Phi_T(f) \circ \Phi_T(g)$,

(iv) $\Phi_T(\overline{f}) = \Phi_T(f)^*$,

(v) Φ_T *is continuous.*

Instead of $\Phi_T(f)$ we also write $f(T)$. Additionally, Φ_T has the following properties:

(i) $p(T) = \sum_{i=0}^{n} a_i T^i$ *for every polynomial p with* $p(t) = \sum_{i=0}^{n} a_i t^i$.

(ii) $\|f(T)\| = \|f\|_\infty (= \sup_{\lambda \in \sigma(T)} |f(\lambda)|)$.

(iii) $Tx = \lambda x$ *implies* $f(T)x = f(\lambda)x$.

(iv) $\sigma(f(T)) = f(\sigma(T)) (= \{f(\lambda) \mid \lambda \in \sigma(T)\})$.

The operational calculus can be extended such that it is defined for measurable functions f. We only need this extension in the way that we can transfer some results regarding characteristic functions and the spectrum to the here used continuous triangle functions: Given a sequence $(f_n)_{n\in\mathbb{N}}$ of measurable functions defined on $\sigma(T)$ that converges pointwise to a function f and is uniformly bounded, the sequence $(f_n(T))_{n\in\mathbb{N}}$ converges pointwise to $f(T)$.

Given a computable normed space X, $\delta_{\mathcal{B}(X)} := [\delta_X \to \delta_X] |^{\mathcal{B}(X)}$ is a representation of the space $\mathcal{B}(X)$ of all linear bounded operators $T : X \to X$. If not mentioned otherwise, we will assume that $\mathcal{B}(X)$ is represented by $\delta_{\mathcal{B}(X)}$.

First we prove that given an operator T and polynomial p we can compute the operator $p(T) = \Phi_T(p)$.

Lemma 3.7 *Let X be a computable normed space. Then the mapping*

$$P :\subseteq \mathcal{B}(X) \times \mathrm{Pn} \to \mathcal{B}(X),\ (T, p) \mapsto p(T)$$

is $\left(\left[\delta_{\mathcal{B}(X)}, \nu_{\mathrm{Pn}}\right], \delta_{\mathcal{B}(X)}\right)$-computable.
Here $p(T)$ is defined as $p(T) = \sum_{i=0}^{n} a_i T^i$ if $p(t) = \sum_{i=0}^{n} a_i t^i$ with $t \in \mathbb{R}$.

Proof. The mapping $(T, n) \mapsto T^n$ is $\left(\left[\delta_{\mathcal{B}(X)}, \nu_{\mathbb{N}}\right], \delta_{\mathcal{B}(X)}\right)$-computable, and $\mathcal{B}(X)$ together with the representation $\delta_{\mathcal{B}(X)}$ is a computable vector space. Thus we can compute $\sum_{i=0}^{n} a_i T^i = p(T)$. □

By the definitions of Φ_T (see [12, VII.1.3/4]) and P we have

$$\|\Phi_T(f)\| = \|f\|_\infty = \sup_{\lambda \in \sigma(T)} |f(\lambda)|$$

for self-adjoint $T \in \mathcal{B}(\mathcal{H})$ and functions $f \in \mathcal{C}(\sigma(T))$, and furthermore the (in)equalities

$$\|P(T,p)\| = \|\Phi_T(p\rfloor_{\sigma(T)})\| = \sup_{\lambda\in\sigma(T)} |p(\lambda)| \leq \sup_{\lambda\in K} |p(\lambda)| = \|p\rfloor_K\|_\infty$$

for self-adjoint $T \in \mathcal{B}(\mathcal{H})$, $p \in \mathrm{Pn}$ and nonempty $K \in \mathcal{K}$ such that $\sigma(T) \subseteq K$. Since P is linear in the second component it follows that

$$P(T,p_1) - p(T,p_2)\| = \|P(T,p_1 - p_2)\| \leq \|(p_1 - p_2)\rfloor_K\|_\infty = \|p_1\rfloor_K - p_2\rfloor_K\|_\infty$$

for self-adjoint $T \in \mathcal{B}(\mathcal{H})$, $p_1, p_2 \in \mathrm{Pn}$ and nonempty $K \in \mathcal{K}$ such that $\sigma(T) \subseteq K$.

Due to these estimations the following lemma holds.

Lemma 3.8 *Let $\mathcal{H}$ be a computable complex Hilbert space. By* $\mathrm{Lim} :\subseteq \mathcal{B}(\mathcal{H})^{\mathbb{N}} \to \mathcal{B}(\mathcal{H})$ *we denote the mapping defined by*

$$\mathrm{dom(Lim)} := \left\{(T_n)_{n\in\mathbb{N}} \in \mathcal{B}(\mathcal{H})^{\mathbb{N}} \mid \|T_i - T_k\| < 2^{-i} \textit{ for } i < k\right\}$$

and

$$\mathrm{Lim}\left((T_n)_{n\in\mathbb{N}}\right) = T :\Longleftrightarrow \lim_{n\to\infty} T_n = T$$

for $(T_n)_{n\in\mathbb{N}} \in \mathrm{dom(Lim)}$*. Let* $T \in \mathcal{B}(\mathcal{H})$ *be self-adjoint and* $f \in \mathcal{C}_{\mathcal{K}}(\subseteq \mathbb{R})$ *such that* $\sigma(T) \subseteq \mathrm{dom}(f)$*. Then we have*

$$(P(T,p_n))_{n\in\mathbb{N}} \in \mathrm{dom(Lim)}$$

and

$$\Phi_T(f\rfloor_{\sigma(T)}) = \mathrm{Lim}((P(T,p_n))_{n\in\mathbb{N}})$$

for all sequences $(p_n)_{n\in\mathbb{N}} \in F_1(f)$*.*[3]

Proof. Let $T \in \mathcal{B}(\mathcal{H})$ be self-adjoint, $f \in \mathcal{C}_{\mathcal{K}}(\subseteq \mathbb{R})$ such that $\sigma(T) \subseteq \mathrm{dom}(f)$ and $(p_n)_{n\in\mathbb{N}} \in F_1(f)$. Then by the definition of F_1 we obtain

$$\|p_i\rfloor_{\mathrm{dom}(f)} - p_k\rfloor_{\mathrm{dom}(f)}\|_\infty < 2^{-i}$$

for $i < k$. Due to the previous estimations it follows

$$\|P(T,p_i) - P(T,p_k)\| \leq \|p_i\rfloor_{\mathrm{dom}(f)} - p_k\rfloor_{\mathrm{dom}(f)}\|_\infty < 2^{-i}$$

for $i < k$. Thus $(P(T,p_n))_{n\in\mathbb{N}} \in \mathrm{dom(Lim)}$. By the continuity of Φ_T we obtain

$$\lim_{n\to\infty} P(T,p_n) = \lim_{n\to\infty} \Phi_T(p_n\rfloor_{\sigma(T)}) = \Phi_T\left(\lim_{n\to\infty} p_n\rfloor_{\sigma(T)}\right) = \Phi_T(f\rfloor_{\sigma(T)})$$

because of $\sigma(T) \subseteq \mathrm{dom}(f)$. Hence it follows

$$\Phi_T(f\rfloor_{\sigma(T)}) = \mathrm{Lim}((P(T,p_n))_{n\in\mathbb{N}})\,.$$

□

In particular, all sequences $(p_n)_{n\in\mathbb{N}}$ of polynomials that converge rapidly to $f \in \mathcal{C}(\mathbb{R})$ on a compact superset of $\sigma(T)$ result in the same limit $\Phi_T(f\rfloor_{\sigma(T)})$ of $((P(T,p_n))_{n\in\mathbb{N}})$.

[3] Here by F_1 we denote the mapping defined in Lemma 3.5.

Theorem 3.9 *Let $\mathcal{H}$ be a computable complex Hilbert space. Then the mapping*

$$\Phi :\subseteq \mathcal{B}(\mathcal{H}) \times \mathcal{C}_{\mathcal{K}}(\subseteq \mathbb{R}) \to \mathcal{B}(\mathcal{H}),\ (T, f) \mapsto \Phi_T(f\rfloor_{\sigma(T)})$$

with domain

$$\mathrm{dom}(\Phi) := \{(T, f) \in \mathcal{B}(\mathcal{H}) \times \mathcal{C}_{\mathcal{K}}(\subseteq \mathbb{R}) \mid T \textit{ is self-adjoint and } \sigma(T) \subseteq \mathrm{dom}(f)\}$$

is $\left(\left[\delta_{\mathcal{B}(\mathcal{H})}, \delta_{Cp}\right], \delta_{\mathcal{B}(\mathcal{H})}\right)$-computable.

Proof. Given $f \in \mathcal{C}_{\mathcal{K}}(\subseteq \mathbb{R})$ we can obtain a rapidly converging sequence of polynomials p_n with $\lim_{n\to\infty} p_n = f$ as such a sequence is encoded in the name of f (see Lemma 3.5). By Lemma 3.8 we have $\mathrm{Lim}((P(T, p_n))_{n\in\mathbb{N}}) = \Phi_T(f\rfloor_{\sigma(T)})$ for each such sequence $(p_n)_{n\in\mathbb{N}}$. Given a sequence $(p_n)_{n\in\mathbb{N}} \in \mathrm{Pn}^{\mathbb{N}}$ we can compute the sequence $(P(T, p_n))_{n\in\mathbb{N}}) \in \mathcal{B}(\mathcal{H})^{\mathbb{N}}$. $(\mathcal{B}(\mathcal{H}), \|\,\|, \delta_{\mathcal{B}(\mathcal{H})})$ is a general computable Banach space [2]. Hence $(T, (p_n)_{n\in\mathbb{N}}) \mapsto \mathrm{Lim}((P(T, p_n))_{n\in\mathbb{N}}))$ is $\left(\left[\delta_{\mathcal{B}(\mathcal{H})}, \nu^{\mathbb{N}}_{\mathrm{Pn}}\right], \delta_{\mathcal{B}(\mathcal{H})}\right)$-computable. As mentioned before, the result of $\mathrm{Lim}((P(T, p_n))_{n\in\mathbb{N}}))$ only depends on $T \in \mathcal{B}(\mathcal{H})$ and $f \in \mathcal{C}_{\mathcal{K}}(\subseteq \mathbb{R})$, but not on the particular sequence $(p_n)_{n\in\mathbb{N}} \in F_1(f)$ that the machine realizing F_1 computes from the given name of f. Thus we can compute Φ as a single-valued function although we have to use the multi-valued function F_1 during our computation. $\square$

Now, we obtain a computable version of the operational calculus as a simple corollary. We just have to restrict the domain of Φ.

Corollary 3.10 *Let $\mathcal{H}$ be a computable complex Hilbert space. Then the mapping*

$$\Phi_{org} :\subseteq \mathcal{B}(\mathcal{H}) \times \mathcal{C}_{\mathcal{K}}(\subseteq \mathbb{R}) \to \mathcal{B}(\mathcal{H}),\ (T, f) \mapsto \Phi_T(f\rfloor_{\sigma(T)})$$

with domain

$$\mathrm{dom}(\Phi_{org}) := \{(T, f) \in \mathcal{B}(\mathcal{H}) \times \mathcal{C}_{\mathcal{K}}(\subseteq \mathbb{R}) \mid T \textit{ is self-adjoint and } \sigma(T) = \mathrm{dom}(f)\}$$

is $\left(\left[\delta_{\mathcal{B}(\mathcal{H})}, \delta_{Cp}\right], \delta_{\mathcal{B}(\mathcal{H})}\right)$-computable.

In the next section, we need another computable version of the operational calculus, namely one that expects a function $f \in \mathcal{C}(\mathbb{R})$ instead of a function $f \in \mathcal{C}_{\mathcal{K}}(\subseteq \mathbb{R})$ (and of course also expects an operator $T \in \mathcal{B}(\mathcal{H})$).

Proposition 3.11 *Let $\mathcal{H}$ be a computable complex Hilbert space. Then the mapping*

$$\Phi_{\mathcal{C}(\mathbb{R})} :\subseteq \mathcal{B}(\mathcal{H}) \times \mathcal{C}(\mathbb{R}) \to \mathcal{B}(\mathcal{H}),\ (T, f) \mapsto \Phi_T(f\rfloor_{\sigma(T)})$$

with

$$\mathrm{dom}(\Phi_{\mathcal{C}(\mathbb{R})}) := \{T \in \mathcal{B}(\mathcal{H}) \mid T \textit{ is self-adjoint}\} \times \mathcal{C}(\mathbb{R})$$

is $\left(\left[\delta_{\mathcal{B}(\mathcal{H})}, [\delta_{\mathbb{R}} \to \delta_{\mathbb{F}}]\right], \delta_{\mathcal{B}(\mathcal{H})}\right)$-computable.

Proof. Given $T \in \mathcal{B}(\mathcal{H})$ we can compute an upper bound $M \in \mathbb{N}$ of $\|T\|$ [1,3]. Furthermore the mapping $M \mapsto [-M, M]$ is $(\nu_{\mathbb{N}}, \kappa)$-computable and $[-M, M] \neq \emptyset$. Since $\sigma(T) \subseteq [-\|T\|, \|T\|] \subseteq [-M, M]$ for self-adjoint $T \in \mathcal{B}(\mathcal{H})$, we can compute a compact nonempty superset of $\sigma(T)$. Then by Proposition 3.4 we can compute

$f\rfloor_{[-M,M]} \in \mathcal{C}_{\mathcal{K}}(\subseteq \mathbb{R})$. As $[-M, M]$ is superset of $\sigma(T)$ we have $(T, f\rfloor_{[-M,M]}) \in \mathrm{dom}(\Phi)$, and by Theorem 3.9 we can compute $\Phi_T(f\rfloor_{\sigma(T)})$. [4] □

Now we obtain the computability of the map $T \mapsto \Phi_T$ as a corollary of Proposition 3.11, if we apply *type conversion*.

Corollary 3.12 *Let $\mathcal{H}$ be a computable complex Hilbert space. Then the mapping*

$$\widetilde{\Phi} :\subseteq \mathcal{B}(\mathcal{H}) \to \mathcal{C}(\mathcal{C}(\mathbb{R}), \mathcal{B}(\mathcal{H})),\ T \mapsto \widetilde{\Phi}_T$$

with

$$\mathrm{dom}(\widetilde{\Phi}) := \{T \in \mathcal{B}(\mathcal{H}) \mid T \textit{ is self-adjoint}\}$$

and

$$\widetilde{\Phi}_T : \mathcal{C}(\mathbb{R}) \to \mathcal{B}(\mathcal{H}),\ f \mapsto \Phi_T(f\rfloor_{\sigma(T)})$$

is $\big(\delta_{\mathcal{B}(\mathcal{H})}, \big[[\delta_{\mathbb{R}} \to \delta_{\mathbb{F}}] \to \delta_{\mathcal{B}(\mathcal{H})}\big]\big)$*-computable.*

4 The Spectrum and Eigenvalues of Self-Adjoint Operators

In this section we present our main result, which is a uniform version of the first two items of the Second Main Theorem of Pour-El and Richards [9]. We also prove corollaries of this result. The proof of the main result is suspended to the following sections because we need some technical preparations for it.

In the following by En we denote the enumeration representation of $2^{\mathbb{N}}$, the set of all subsets of $\mathbb{N}$.

Theorem 4.1 *Let $\mathcal{H}$ be a computable complex Hilbert space. Then there exists a* $\big(\delta_{\mathcal{B}(\mathcal{H})}, \big[\delta_{\mathbb{R}}^{\mathbb{N}}, \mathrm{En}\big]\big)$*-computable multi-valued mapping*

$$\mathrm{H} :\subseteq \mathcal{B}(\mathcal{H}) \rightrightarrows \mathbb{R}^{\mathbb{N}} \times 2^{\mathbb{N}}$$

with domain

$$\mathrm{dom}(\mathrm{H}) := \{T \in \mathcal{B}(\mathcal{H}) \mid T \textit{ is self-adjoint}\}$$

such that

(i) $\overline{\{\lambda_n \mid n \in \mathbb{N}\}} = \sigma(T)$ *and*

(ii) $\sigma_p(T) = \{\lambda_n \mid n \in \mathbb{N} \setminus A\}$

for all $((\lambda_n)_{n\in\mathbb{N}}, A) \in \mathrm{H}(T)$.

In the proof we use some further properties of the operational calculus, which we briefly state here. By χ_A we denote the characteristic function of a set $A \subseteq \mathbb{R}$, i.e. $\chi_A(x) = 1$ if $x \in A$, and $\chi_A(x) = 0$ if $x \notin A$.

[4] Here again we have to use a multi-valued function, namely the mapping of T to some upper bound of $\|T\|$, during the computation of the single-valued function $\Phi_{\mathcal{C}(\mathbb{R})}$. This is possible as the result of $\Phi(T, f)$ only depends on T and the values of f on $\sigma(T)$, but not on the particular domain $\mathrm{dom}(f)$, which only has to be a superset of $\sigma(T)$.

Proposition 4.2 *Let $\mathcal{H}$ be a complex Hilbert space and $T \in \mathcal{B}(\mathcal{H})$ be self-adjoint.*

(i) *$\lambda \in \rho(T)$, iff there exists an open interval U containing λ such that $\Phi_T(\chi_U) = 0$.*

(ii) *$\lambda \in \sigma_p(T)$, iff $\Phi_T(\chi_{\{\lambda\}}) \neq 0$. In this case $\Phi_T(\chi_{\{\lambda\}})$ is the projection on the eigenspace corresponding to the eigenvalue λ.*

Using these properties in Section 6 we construct an algorithm that starts with "possible" eigenvectors [5] and tries to find the corresponding eigenvalues.

As a corollary we obtain the result that given a self-adjoint operator we can compute its spectrum as a compact set. By $\mathcal{K}(\mathbb{F})$ we denote the set of all compact subsets of $\mathbb{F}$, and equip it with the representation $\kappa_<$, which represents a compact set by positive information about it and a bound of it (see [11] for further information).

Corollary 4.3 *Let $\mathcal{H}$ be a computable complex Hilbert space. Then the spectrum map*

$$\sigma :\subseteq \mathcal{B}(\mathcal{H}) \to \mathcal{K}(\mathbb{F}),\, T \mapsto \sigma(T)$$

with domain

$$\mathrm{dom}(\sigma) := \{T \in \mathcal{B}(\mathcal{H}) \mid T \textit{ is self-adjoint}\}$$

is $(\delta_{\mathcal{B}(\mathcal{H})}, \kappa_<)$-computable.

Proof. Given $T \in \mathrm{dom}(\sigma)$ by Theorem 4.1 we can compute a dense sequence $(\lambda_n)_{n\in\mathbb{N}}$ in the spectrum of T and some upper bound of $\|T\|$ [1,3]. These information are equivalent to the information encoded by the representation $\kappa_<$. □

We already proved this result in [4], even for normal operators and real Hilbert spaces. But the result there only gave us information about the spectrum. The result presented here shows that we also can get some information about the eigenvalues of a self-adjoint operator in complex Hilbert spaces.

¿From the uniform results we can derive some non-uniform results, namely the first two points of the Second Main Theorem of Pour-El and Richards [9] and the fact that the spectrum of a computable self-adjoint operator in a computable complex Hilbert space is a recursively enumerable compact set.

Corollary 4.4 *Let $\mathcal{H}$ be a computable complex Hilbert space. Let $T \in \mathcal{B}(\mathcal{H})$ be $\delta_{\mathcal{B}(\mathcal{H})}$-computable and self-adjoint. Then there exist a $\delta_{\mathbb{R}}^{\mathbb{N}}$-computable sequence $(\lambda_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ and an* En*-computable subset $A \subseteq \mathbb{N}$ such that*

(i) $\overline{\{\lambda_n \mid n \in \mathbb{N}\}} = \sigma(T)$ *and*

(ii) $\sigma_p(T) = \{\lambda_n \mid n \in \mathbb{N} \setminus A\}$

Corollary 4.5 *Let $\mathcal{H}$ be a computable complex Hilbert space. Let $T \in \mathcal{B}(\mathcal{H})$ be $\delta_{\mathcal{B}(\mathcal{H})}$-computable and self-adjoint. Then the spectrum $\sigma(T)$ is $\kappa_<$-computable, thus a recursively enumerable compact subset of $\mathbb{F}$. Furthermore, every eigenvalue λ of T is a computable real.*

[5] more precisely, with a dense sequence in the unit sphere

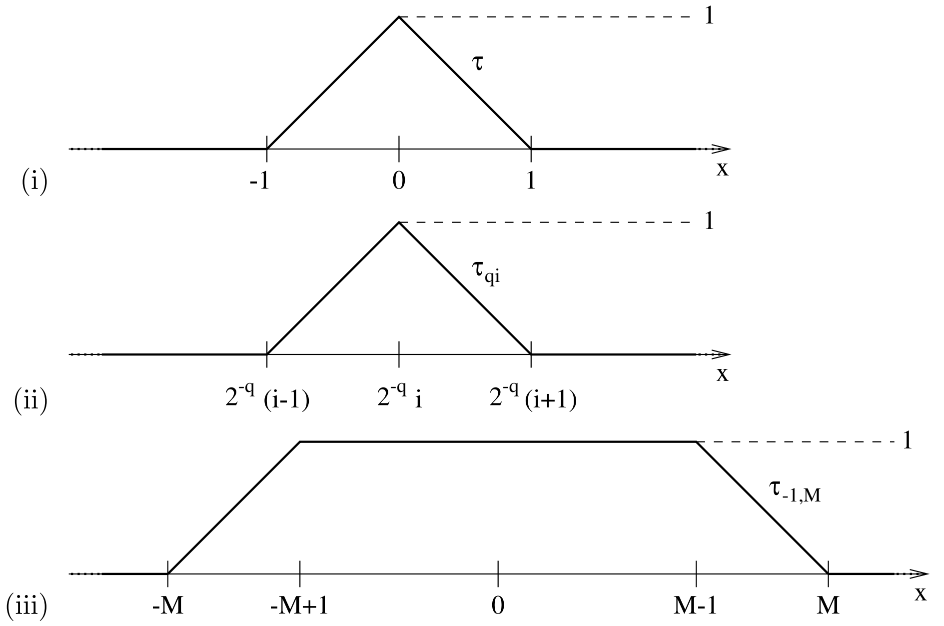

Fig. 1. Triangle functions: (i) τ, (ii) τ_{qi} with $q \in \mathbb{N}, i \in \mathbb{Z}$, (iii) $\tau_{-1,M}$ with $M \in \mathbb{N} \setminus \{0\}$.

In the next sections we present some technical results that we need for the proof of Theorem 4.1. The proof itself is presented in Section 7.

5 Prerequisites for the Proof

In this section we briefly introduce some functions that we use in the proof instead of the non-continuous characteristic functions and prove some (computability) results about them in conjunction with the operational calculus.

5.1 The Triangle Functions

In functional analysis the operational calculus is used in conjunction with characteristic functions to describe the spectrum of self-adjoint and normal operators. In computable analysis we need continuous functions, for which we have representations. Thus instead of characteristic functions, we will use some kind of *triangle functions* as a continuous substitute. Pour-El and Richards already used this kind of functions in the proof of the Second Main Theorem [9]. In contrast to their proof, we divide the intervals into three overlapping subintervals, and not into 15 subintervals like it is done in [9].

Definition 5.1 We define the following functions (see Figure 1).

(i) $\tau : \mathbb{R} \to \mathbb{R}$, $x \mapsto \max\{1 - |x|, 0\}$.

(ii) $\tau_{qi} : \mathbb{R} \to \mathbb{R}, x \mapsto \tau(2^q x - i)$ for $q \in \mathbb{N}$ and $i \in \mathbb{Z}$.

(iii) $\tau_{-1,M} : \mathbb{R} \to \mathbb{R}$, $x \mapsto \max\{\min\{M - |x|, 1\}, 0\}$ for $M \in \mathbb{N} \setminus \{0\}$.

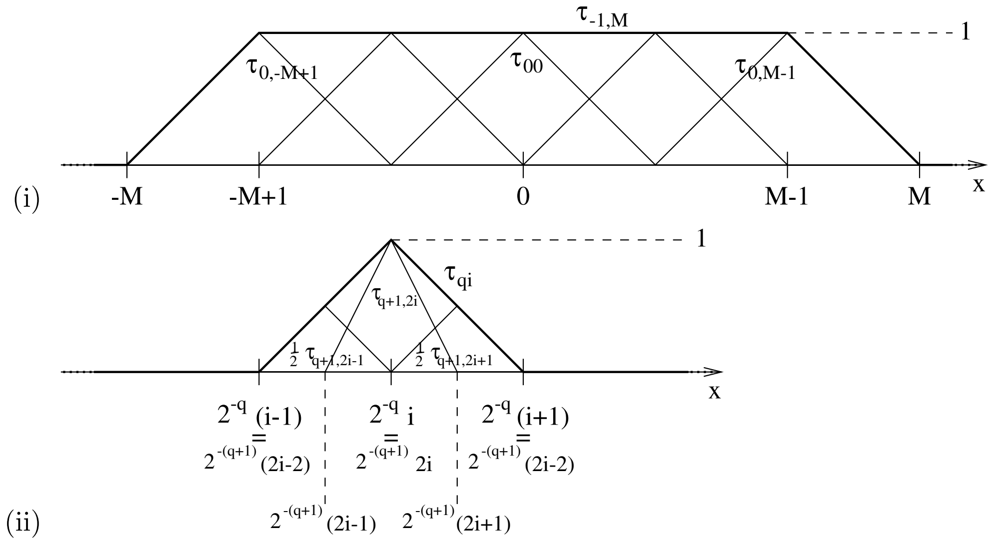

Fig. 2. Properties of the triangle functions: connection between (i) $\tau_{-1,M}$ and τ_{0i}, (ii) τ_{qi} and $\tau_{q+1,i}$.

We can derive the following properties from Definition 5.1 (see also Figure 2). In particular, the functions τ_{qi} are overlapping in such a way that for each stage q and each point $x \in \mathbb{R}$ we can find a function τ_{qi} that is greater than $\frac{1}{2}$ at x. Detailed proofs of these properties can be found in [6].

Lemma 5.2 (i) *Let $M \in \mathbb{N} \setminus \{0\}$. Then it holds*

$$\tau_{-1,M} = \sum_{i=-M+1}^{M-1} \tau_{0i}\,.$$

(ii) *Let $q \in \mathbb{N}$ and $i \in \mathbb{N}$. Then it holds*

$$\tau_{qi} = \frac{1}{2}\,\tau_{q+1,2i-1} + \tau_{q+1,2i} + \frac{1}{2}\,\tau_{q+1,2i+1}\,.$$

(iii) *Let $M \in \mathbb{N}$ and $x \in [-M, M]$. Then there exists some $i \in \mathbb{Z}$ such that*

$$-M \leq i \leq M \quad \text{and} \quad \tau_{0i} \geq \frac{1}{2}\,.$$

(iv) *Let $x \in \mathbb{R}$ and $q \in \mathbb{N}$. Then for all $i \in \mathbb{Z}$ it holds*

$$\tau_{qi}(x) \geq \frac{1}{4} \quad \Longrightarrow \quad \exists j \in \{2i-1, 2i, 2i+1\} : \tau_{q+1,j}(x) \geq \frac{1}{2}\,.$$

In contrast to the characteristic functions of intervals and points, the triangle functions τ_{qi} are continuous and even computable. Furthermore the mapping $(q, i) \mapsto \tau_{qi}$ is computable.

Lemma 5.3 *The mapping*

$$\mathcal{T} : \mathbb{N} \times \mathbb{Z} \to \mathcal{C}(\mathbb{R}, \mathbb{R}),\ (q, i) \mapsto \tau_{qi}$$

is $([\nu_{\mathbb{N}}, \nu_{\mathbb{Z}}], [\delta_{\mathbb{R}} \to \delta_{\mathbb{R}}])$-computable.
Particularly, the functions τ_{qi} with $q \in \mathbb{N}$ and $i \in \mathbb{Z}$ are $(\delta_{\mathbb{R}}, \delta_{\mathbb{R}})$-computable.

The functions $\tau_{-1,M}$ with $M \in \mathbb{N} \setminus \{0\}$ and the mapping $M \mapsto \tau_{-1,M}$ are also computable. But we do not need this fact in the following proofs.

5.2 The Norms of the Operational Calculus

In this section we prove some technical lemmas about the norms[6] $\left\|\Phi_T(\tau_{qi}\rfloor_{\sigma(T)})x\right\|$ and the mapping FN defined by

$$\mathrm{FN} :\subseteq \mathcal{B}(\mathcal{H}) \times \mathbb{N} \times \mathbb{Q} \times \mathcal{H} \to \mathbb{R},\ (T, q, i, x) \mapsto \left\|\Phi_T(\tau_{qi}\rfloor_{\sigma(T)})x\right\|$$

and

$$\mathrm{dom}(\mathrm{FN}) := \{T \in \mathcal{B}(\mathcal{H}) \mid T \text{ is self-adjoint}\} \times \mathbb{N} \times \mathbb{Z} \times S_{\mathcal{H}}\,.$$

Here by S_X we denote the unit sphere $S_X := S(0,1) := \{x \in X \mid \|x\| = 1\}$ of a normed space X. The above mentioned norms and the mapping FN become important in the proof of the main result of this paper.

Lemma 5.4 *Let $\mathcal{H}$ be a complex Hilbert space and $T \in \mathcal{B}(\mathcal{H})$ be a self-adjoint operator in $\mathcal{H}$.*

(i) *Let $M \in \mathbb{N}$ such that $\|T\| + 1 \le M$ and let $x \in S_{\mathcal{H}}$. Then*

$$\max\left\{\left\|\Phi_T(\tau_{0i}\rfloor_{\sigma(T)})x\right\| \mid i \in \mathbb{Z} \cap [-M+1, M-1]\right\} \ge \frac{1}{2M-1}\,.$$

(ii) *Let $q \in \mathbb{N}$, $i \in \mathbb{Z}$ and $x \in S_{\mathcal{H}}$. Then*

$$\max\left\{\left\|\Phi_T(\tau_{q+1,j}\rfloor_{\sigma(T)})x\right\| \mid j \in \{2i-1, 2i, 2i+1\}\right\} \ge \frac{1}{3}\left\|\Phi_T(\tau_{qi})\rfloor_{\sigma(T)})x\right\|\,.$$

Proof. The above estimations can be derived from the (in)equalities given in Lemma 5.2 and the properties of the continuous functional calculus.

(i) The function $\tau_{-1,M}$ is the sum of $2M-1$ functions τ_{0i} (Lemma 5.2), and the restriction to the domain $\sigma(T)$ is equal to the restriction of the function with constant value 1 to this domain. Therefore applying the properties of the operational calculus we obtain for $x \in S_{\mathcal{H}}$

$$\begin{aligned}1 = \|x\| = \|\operatorname{id}_{\mathcal{H}} x\| &= \left\|\sum_{i=-M+1}^{M-1} \Phi_T(\tau_{0i}\rfloor_{\sigma(T)})x\right\| \\ &\le (2M-1)\cdot\max\left\{\left\|\Phi_T(\tau_{0i}\rfloor_{\sigma(T)})x\right\| \mid i \in \mathbb{Z}\cap[-M+1, M-1]\right\}\,.\end{aligned}$$

(ii) Using the fact that the function τ_{qi} of stage q are a combination of three such function of stage $q+1$ and the properties of the operational calculus we obtain for $q \in \mathbb{N}$, $i \in \mathbb{Z}$ and $x \in S_{\mathcal{H}}$

$$\begin{aligned}&\left\|\Phi_T(\tau_{qi}\rfloor_{\sigma(T)})x\right\| \\ &= \left\|\frac{1}{2}\,\Phi_T(\tau_{q+1,2i-1}\rfloor_{\sigma(T)})x + \Phi_T(\tau_{q+1,2i}\rfloor_{\sigma(T)})x + \frac{1}{2}\,\Phi_T(\tau_{q+1,2i+1}\rfloor_{\sigma(T)})x\right\| \\ &\le 3\cdot\max\left\{\left\|\Phi_T(\tau_{q+1,j}\rfloor_{\sigma(T)})x\right\| \mid j \in \{2i-1, 2i, 2i+1\}\right\}\,.\end{aligned}$$

[6] In the following by *norms of the operational calculus* these norms $\left\|\Phi_T(\tau_{qi}\rfloor_{\sigma(T)})x\right\|$ are meant.

□

Now we obtain that given a self-adjoint operator $T \in \mathcal{B}(\mathcal{H})$, $M \in \mathbb{N}$ such that $\|T\| + 1 \leq M$ and $x \in S_{\mathcal{H}}$ there exists a sequence $(i_q)_{q\in\mathbb{N}}$ in $\mathbb{Z}$ with the following properties:

- The support of $\tau_{q+1,i_{q+1}}$ is a subset of the support of τ_{q,i_q}.
- The sequence $(2^{-q}i_q)_{q\in\mathbb{N}}$ is a rapidly converging sequence in $\mathbb{R}$.[7]
- It holds $\left\|\Phi_T(\tau_{qi_q}\rfloor_{\sigma(T)})x\right\| \geq \frac{1}{2M-1} \cdot \frac{1}{3^q}$. That is for each q the norm is greater than zero, but the difference between the norm and zero depends on q.

The limit λ of such a sequence $(2^{-q}i_q)_{q\in\mathbb{N}}$ is an element of the spectrum of T. If λ is an eigenvalue of T then the difference between zero and the norm is independent of q, if additionally the distance between x and the eigenspace corresponding to λ is small enough.

Lemma 5.5 *Let $\mathcal{H}$ be a complex Hilbert space and $T \in \mathcal{B}(\mathcal{H})$ be a self-adjoint operator on $\mathcal{H}$. Let $z \in S_{\mathcal{H}}$ be a normalized eigenvector corresponding to an eigenvalue λ of T.*

(i) *Let $M \in \mathbb{N}$ such that $\|T\| \leq M$ and $x \in S_{\mathcal{H}}$. Then there exists some integer $i \in [-M, M]$ such that*
$$\left\|\Phi_T(\tau_{0i}\rfloor_{\sigma(T)})x\right\| \geq \frac{1}{2} - \|x - z\| .$$

(ii) *Let $q \in \mathbb{N}$, $i \in \mathbb{Z}$ and $x \in S_{\mathcal{H}}$ such that*
$$\left\|\Phi_T(\tau_{qi}\rfloor_{\sigma(T)})x\right\| \geq \frac{1}{4} + \|x - z\| .$$
Then there exists some $j \in \{2i-1, 2i, 2i+1\}$ such that
$$\left\|\Phi_T(\tau_{q+1,j}\rfloor_{\sigma(T)})x\right\| \geq \frac{1}{2} - \|x - z\| .$$

Proof. As z is a normalized eigenvector of T we obtain
$$\left\|\Phi_T(\tau_{qi}\rfloor_{\sigma(T)})z\right\| = |\tau_{qi}(\lambda)|\,\|z\| = \tau_{qi}(\lambda) .$$

(i) By Lemma 5.2 there exists some $i \in \mathbb{Z}$ such that $-M \leq i \leq M$ and
$$\begin{aligned}\left\|\Phi_T(\tau_{0i}\rfloor_{\sigma(T)})x\right\| &\geq \left\|\Phi_T(\tau_{0i}\rfloor_{\sigma(T)})z\right\| - \|x-z\| = \tau_{0i}(\lambda) - \|x - z\| \\ &\geq \frac{1}{2} - \|x - z\| .\end{aligned}$$

(ii) If $\left\|\Phi_T(\tau_{qi}\rfloor_{\sigma(T)})x\right\| \geq \frac{1}{4} + \|x - z\|$ then it holds
$$\tau_{qi}(\lambda) = \left\|\Phi_T(\tau_{qi}\rfloor_{\sigma(T)})x\right\| \geq \left\|\Phi_T(\tau_{qi}\rfloor_{\sigma(T)})z\right\| - \|x - z\| \geq \frac{1}{4} .$$
By Lemma 5.2 there exists some $j \in \{2i-1, 2i, 2i+1\}$ such that
$$\begin{aligned}\left\|\Phi_T(\tau_{q+1,j}\rfloor_{\sigma(T)})x\right\| &\geq \left\|\Phi_T(\tau_{q+1,j}\rfloor_{\sigma(T)})z\right\| - \|x - z\| = \tau_{q+1,j}(\lambda) - \|x - z\| \\ &\geq \frac{1}{2} - \|x - z\| .\end{aligned}$$

[7] The elements $2^{-q}i_q$ are the centers of the support intervals of the functions $\tau q i_q$. The intervals are nested and their length converges rapidly to 0. Therefore $(2^{-q}i_q)_{q\in\mathbb{N}}$ converges.

□

Beside these pure mathematical properties of the norms $\left\|\Phi_T(\tau_{qi}\rfloor_{\sigma(T)})x\right\|$ the mapping FN is computable.

Lemma 5.6 *Let $\mathcal{H}$ be a computable complex Hilbert space. Then the mapping*

$$\mathrm{FN} :\subseteq \mathcal{B}(\mathcal{H}) \times \mathbb{N} \times \mathbb{Q} \times \mathcal{H} \to \mathbb{R},\ (T, q, i, x) \mapsto \left\|\Phi_T(\tau_{qi}\rfloor_{\sigma(T)})x\right\|$$

with

$$\mathrm{dom}(\mathrm{FN}) := \{T \in \mathcal{B}(\mathcal{H}) \mid T \textit{ is self-adjoint}\} \times \mathbb{N} \times \mathbb{Z} \times S_{\mathcal{H}}$$

is $\left(\left[\delta_{\mathcal{B}(\mathcal{H})}, \nu_{\mathbb{N}}, \nu_{\mathbb{Z}}, \delta_{\mathcal{H}}\right], \delta_{\mathbb{R}}\right)$*-computable.*

Proof. Since the mappings $\mathcal{T} : (q, i) \mapsto \tau_{qi}$ and $\Phi_{\mathcal{C}(\mathbb{R})} : (T, f) \mapsto \Phi_T(f\rfloor_{\sigma(T)})$ are computable, the composition $(T, q, i) \mapsto \Phi_T(\tau_{qi}\rfloor_{\sigma(T)})$ is $\left(\left[\delta_{\mathcal{B}(\mathcal{H})}, \nu_{\mathbb{N}}, \nu_{\mathbb{Z}}\right], \delta_{\mathcal{B}(\mathcal{H})}\right)$-computable. Along with the computability of the norm we obtain the computability of the mapping FN. □

6 The Technical Part of the Proof

In this section most of the technical part of the proof is done. We define two "technical" mappings, and prove their computability and some properties regarding the spectrum of a self-adjoint operator.

6.1 The mapping F

In this section we prove the computability of a mapping F that maps a tuple (T, M, a, x) on a sequence $(i_q)_{q\in\mathbb{N}}$. Here T is a self-adjoint operator, M is an upper bound of the norm of T, a is an upper bound of an error and x is a vector with norm 1. $(i_q)_{q\in\mathbb{N}}$ is a sequence of integers that converges[8] to a real. This sequence meets some particular properties regarding the spectrum of T and the vector x. The properties also depend on the parameters M and a. In the next section we study a similar function that no longer depends on M and a.

Given a sequence $(i_q)_{q\in\mathbb{N}} \in \mathbb{Z}^{\mathbb{N}}$ we define the intervals I_q, $q \in \mathbb{N}$, by

$$I_q := \left[2^{-q}(i_q - 1), 2^{-q}(i_q + 1)\right] .$$

Lemma 6.1 *Let $\mathcal{H}$ be a computable complex Hilbert space. Then there exists a* $\left(\left[\delta_{\mathcal{B}(\mathcal{H})}, \nu_{\mathbb{N}}, \nu_{\mathbb{Q}}, \delta_{\mathcal{H}}\right], \nu_{\mathbb{Z}}^{\mathbb{N}}\right)$*-computable multi-valued mapping*

$$\mathrm{F} :\subseteq \mathcal{B}(\mathcal{H}) \times \mathbb{N} \times \mathbb{Q} \times \mathcal{H} \rightrightarrows \mathbb{Z}^{\mathbb{N}}$$

with

$$\mathrm{dom}(\mathrm{F}) := \{(T, M, a, x) \in \mathcal{B}(\mathcal{H}) \times \mathbb{N} \times \mathbb{Q} \times \mathcal{H} \mid T \textit{ is self-adjoint}, \\ \|T\| + 1 \le M,\ 0 < a \le 1 \textit{ and } \|x\| = 1\}$$

[8] We explain later what is meant by *converges* in this context.

such that given $(T, M, a, x) \in \mathrm{dom(F)}$ *for each sequence* $(i_q)_{q\in\mathbb{N}} \in \mathrm{F}(T, M, a, x)$ *the following properties are fulfilled:*

(i) $(2^{-q} i_q)_{q\in\mathbb{N}}$ *is a rapidly converging Cauchy sequence in* $\mathbb{R}$.

(ii) $I_0 \subseteq [-M, M]$ *and* $\forall q \in \mathbb{N} : I_{q+1} \subseteq I_q$.

(iii) $\forall q \in \mathbb{N} : \left\|\Phi_T(\tau_{q,i_q}\rfloor_{\sigma(T)})x\right\| \geq \frac{1}{2M} \cdot \frac{1}{4^q}$.

(iv) *For* $q = 0$ *it holds*

$$\left\|\Phi_T(\tau_{0i_0}\rfloor_{\sigma(T)})x\right\| > \max\left\{\left\|\Phi_T(\tau_{0i}\rfloor_{\sigma(T)})x\right\| \mid -M+1 \leq i \leq M-1\right\} - a\,,$$

and for all $q \in \mathbb{N}$ *it holds*

$$\begin{aligned}&\left\|\Phi_T(\tau_{q+1,i_{q+1}}\rfloor_{\sigma(T)})x\right\| \\ &\qquad > \max\left\{\left\|\Phi_T(\tau_{q+1,j}\rfloor_{\sigma(T)})x\right\| \mid j \in \{2i_q - 1, 2i_q, 2i_q + 1\}\right\} - a\,.\end{aligned}$$

Proof. The proof is a slight modification of the proof of the *Second Main Theorem* of Pour-El and Richards [9]. Using the computable version of the operational calculus the computation in [9] can be done in a way that is uniform in the operator T. Since we divide the interval in three overlapping subintervals (and not in 15 subintervals) we have to use different error bounds than in [9].

In the following we describe a machine M_0 that computes a sequence $(i_q)_{q\in\mathbb{N}}$ from a self-adjoint operator T, an upper bound M of its norm, an upper bound a of an error and a vector x. Then we prove that the sequence computed by M_0 has the desired properties.

Given a tuple $(T, M, a, x) \in \mathrm{dom(F)}$ the machine M_0 works in stages $q = 0, 1, 2, \ldots$. In stage $q = 0$ M_0 computes $2M - 1$ values $r_{0i} := \mathrm{FN}(T, 0, i, x) = \left\|\Phi_T(\tau_{0i}\rfloor_{\sigma(T)})x\right\|$ for $i = -M+1, \ldots, M-1$ and determines an index i_0 of an approximate maximum r_{0i_0} with an error bound of $\frac{1}{2M-1} \cdot \frac{1}{2M} \cdot a$. That is the difference between the approximate maximum r_{0i_0} and the exact maximum $\max\{r_{0i} \mid i = -M+1, \ldots, M-1\}$ is less than the error bound: It holds $r_{0i_0} \geq \max\{r_{0i} \mid i = -M+1, \ldots, M-1\} - \frac{1}{2M-1} \cdot \frac{1}{2M} \cdot a$. Then M_0 continues with stage 1.

In stage $q \geq 1$ M_0 computes three values $r_{qi} := \mathrm{FN}(T, q, i, x) = \left\|\Phi_T(\tau_{qi}\rfloor_{\sigma(T)})x\right\|$ for $i = 2i_{q-1} - 1, 2i_{q-1}, 2i_{q-1} + 1$ and determines an index i_q of an approximate maximum r_{qi_q} with an error bound of $\frac{1}{3} \cdot \frac{1}{2M} \cdot \frac{1}{4^q} \cdot a$. Then M_0 continues with stage $q + 1$.

The result of M_0 is the sequence $(i_q)_{q\in\mathbb{N}}$ of the indices of the approximate maxima. The sequence has the following properties.

(i) For each $q \in \mathbb{N}$ we obtain $|2^{-q} i_q - 2^{-(q+1)} i_{q+1}| = 2^{-(q+1)} |2i_q - i_{q+1}| \leq 2^{-(q+1)}$. Thus $(2^{-q} i_q)_{q\in\mathbb{N}}$ is a rapidly converging sequence in $\mathbb{R}$.

(ii) For $q \in \mathbb{N}$ we obtain

$$\begin{aligned} y \in I_{q+1} &\iff 2^{-(q+1)}(i_{q+1} - 1) \leq y \leq 2^{-(q+1)}(i_{q+1} + 1) \\ &\implies 2^{-q}(i_q - 1) \leq y \leq 2^{-q}(i_q + 1) \\ &\iff y \in I_q \end{aligned}$$

and because of $\mathrm{length}(I_q) = 2^{-q} > 2^{-(q+1)} = \mathrm{length}(I_{q+1})$ it follows $I_{q+1} \subsetneq I_q$.

Additionally we obtain $I_0 = [i_0 - 1, i_0 + 1] \subseteq [-M, M]$.

(iii) For $q = 0$ we obtain

$$\begin{aligned}
&\left\|\Phi_T(\tau_{0i_0}\rfloor_{\sigma(T)})x\right\| \\
&\quad \geq \max\left\{\left\|\Phi_T(\tau_{0i}\rfloor_{\sigma(T)})x\right\| \mid -M+1 \leq i \leq M-1\right\} - \frac{1}{2M-1}\cdot\frac{1}{2M}\cdot a \\
&\quad \geq \frac{1}{2M}\cdot\frac{1}{4^0}
\end{aligned}$$

If we assume that we have shown the estimation for $q-1$, we obtain for $q \geq 1$

$$\begin{aligned}
&\left\|\Phi_T(\tau_{qi_q}\rfloor_{\sigma(T)})x\right\| \\
&\quad \geq \max\left\{\left\|\Phi_T(\tau_{qj}\rfloor_{\sigma(T)})x\right\| \mid j \in \{2i_{q-1}-1, 2i_{q-1}, 2i_{q-1}+1\}\right\} \\
&\qquad\qquad - \frac{1}{3}\cdot\frac{1}{2M}\cdot\frac{1}{4^q}\cdot a \\
&\quad \geq \frac{1}{3}\left\|\Phi_T(\tau_{q-1,i_{q-1}}\rfloor_{\sigma(T)})x\right\| - \frac{1}{3}\cdot\frac{1}{2M}\cdot\frac{1}{4^q}\cdot a \\
&\quad \geq \frac{1}{2M}\cdot\frac{1}{4^q}\,.
\end{aligned}$$

(iv) Because of the chosen error bounds by the computation of the approximate maximum it holds

$$\begin{aligned}
&\left\|\Phi_T(\tau_{0i_0}\rfloor_{\sigma(T)})x\right\| \\
&\quad \geq \max\left\{\left\|\Phi_T(\tau_{0i}\rfloor_{\sigma(T)})x\right\| \mid -M+1 \leq i \leq M-1\right\} - \frac{1}{2M-1}\cdot\frac{1}{2M}\cdot a \\
&\quad > \max\left\{\left\|\Phi_T(\tau_{0i}\rfloor_{\sigma(T)})x\right\| \mid -M+1 \leq i \leq M-1\right\} - a\,,
\end{aligned}$$

and for each $q \in \mathbb{N}$

$$\begin{aligned}
&\left\|\Phi_T(\tau_{qi_q}\rfloor_{\sigma(T)})x\right\| \\
&\quad \geq \max\left\{\left\|\Phi_T(\tau_{qj}\rfloor_{\sigma(T)})x\right\| \mid j \in \{2i_{q-1}-1, 2i_{q-1}, 2i_{q-1}+1\}\right\} \\
&\qquad\qquad - \frac{1}{3}\cdot\frac{1}{2M}\cdot\frac{1}{4^q}\cdot a \\
&\quad > \max\left\{\left\|\Phi_T(\tau_{qj}\rfloor_{\sigma(T)})x\right\| \mid j \in \{2i_{q-1}-1, 2i_{q-1}, 2i_{q-1}+1\}\right\} - a\,.
\end{aligned}$$

□

The limit of rapidly converging sequences of reals is computable. More precisely, the function $\mathrm{Lim} :\subseteq \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$, $(r_n)_{n\in\mathbb{N}} \mapsto \lim_{n\to\infty} r_n$ with $\mathrm{dom}(\mathrm{Lim}) := \{(r_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} \mid |r_i - r_k| < 2^{-i} \text{ for all } i < k\}$ is $\left(\delta_{\mathbb{R}}^{\mathbb{N}}, \delta_{\mathbb{R}}\right)$-computable. Hence, we can compute the real number that is the limit of the sequence $(2^{-q}i_q)_{q\in\mathbb{N}}$ corresponding to the sequence $(i_q)_{q\in\mathbb{N}}$ that we have computed in the proof of the Lemma above. We define a mapping

$$\mathrm{G} :\subseteq \mathbb{Z}^{\mathbb{N}} \to \mathbb{R},\ (i_q)_{q\in\mathbb{N}} \mapsto \lim_{q\to\infty} 2^{-q}i_q = \mathrm{Lim}((2^{-q}i_q)_{q\in\mathbb{N}})$$

with domain

$$\mathrm{dom}(\mathrm{G}) := \left\{(i_q)_{q\in\mathbb{N}} \in \mathbb{Z}^{\mathbb{N}} \mid (2^{-q}i_q)_{q\in\mathbb{N}} \in \mathrm{dom}(\mathrm{Lim})\right\}.$$

Because of $\mathrm{F}(T, M, a, x) \subseteq \mathrm{dom}(\mathrm{G})$ under the assumption $(T, M, a, x) \in \mathrm{dom}(\mathrm{F})$, we obtain $\mathrm{dom}(\mathrm{F}) = \mathrm{dom}(\mathrm{G} \circ \mathrm{F})$. In the following paragraphs we prove some properties of the elements of $\mathrm{G} \circ \mathrm{F}(T, M, a, x)$ regarding the spectrum of T.

Subset of the Spectrum

For each $(T, M, a, x) \in \mathrm{dom}(\mathrm{F})$ the set $\mathrm{G} \circ \mathrm{F}(T, M, a, x)$ is a subset of $\sigma(T)$.

Lemma 6.2 *Let $\mathcal{H}$ be a computable complex Hilbert space and F be defined as in Lemma 6.1. Let $(T, M, a, x) \in \mathrm{dom}(\mathrm{F}) = \mathrm{dom}(\mathrm{G} \circ \mathrm{F})$. Then*

$$\lim_{q \to \infty} 2^{-q} i_q \in \sigma(T)$$

holds for all $(i_q)_{q \in \mathbb{N}} \in \mathrm{F}(T, M, a, x)$, hence

$$\lambda \in \mathrm{G} \circ \mathrm{F}(T, M, a, x) \Longrightarrow \lambda \in \sigma(T)$$

and $\mathrm{G} \circ \mathrm{F}(T, M, a, x) \subseteq \sigma(T)$.

Proof. Since $(2^{-q} i_q)_{q \in \mathbb{N}}$ is a rapidly converging Cauchy sequence, its limit $\lambda := \lim_{q \to \infty} 2^{-q} i_q$ is well defined. By Lemma 6.1 we obtain

$$\left\| \Phi_T(\tau_{q i_q} \rfloor_{\sigma(T)}) x \right\| \geq \frac{1}{2M} \cdot \frac{1}{4^q}$$

for all $q \in \mathbb{N}$. Hence $\tau_{q i_q} \rfloor_{\sigma(T)} \neq \mathbf{0} \rfloor_{\sigma(T)}$ and $\mathrm{support}(\tau_{q i_q}) \cap \sigma(T) \neq \emptyset$ for all $q \in \mathbb{N}$. Because of $\mathrm{support}(\tau_{q i_q}) = I_q$ it follows $I_q \cap \sigma(T) \neq \emptyset$ for all $q \in \mathbb{N}$. Thus for each $q \in \mathbb{N}$ there exists an element $\mu_q \in I_q$ in the spectrum $\sigma(T)$ such that $|\mu_q - 2^{-q} i_q| \leq 2^{-q}$. As the intervals I_q are nested and their length converges rapidly to 0, the sequences $(i_q)_{q \in \mathbb{N}}$ and $(\mu_q)_{q \in \mathbb{N}}$ converge to the same limit λ. This limit is an element of $\sigma(T)$ because it is the limit of elements of the spectrum and the spectrum is closed. Therefore all elements of $\mathrm{G} \circ \mathrm{F}(T, M, a, x)$ are elements of $\sigma(T)$ if $(T, M, a, x) \in \mathrm{dom}(\mathrm{F})$. □

Dense in the Spectrum

If we compute $\mathrm{F}(T, M, a, x)$ for fixed T, M and a with $\|T\| + 1 \leq M$ and $0 < a \leq 1$ and for all $x \in A$ where A is a dense subset of the unit sphere $S_{\mathcal{H}}$, then we obtain a dense subset of the spectrum. It even suffices to pick one sequence $(i_q)_{i \in \mathbb{N}} \in \mathrm{F}(T, M, a, x)$ for each $x \in A$.

Lemma 6.3 *Let $\mathcal{H}$ be a computable complex Hilbert space and F be defined as in Lemma 6.1. Let $T \in \mathcal{B}(\mathcal{H})$ be self-adjoint, $M \in \mathbb{N}$ such that $\|T\| + 1 \leq M$ and $a \in \mathbb{Q}$ such that $0 < a \leq 1$. Let $A \subseteq S_X$ be dense in $S_{\mathcal{H}}$ and $\lambda \in \sigma(T)$. Then*

$$\forall \varepsilon > 0 \, \exists x \in A \, \forall (i_q)_{q \in \mathbb{N}} \in \mathrm{F}(T, M, a, x) : \left| \lambda - \lim_{q \to \infty} 2^{-q} i_q \right| < \varepsilon \,,$$

or equivalent

$$\forall \varepsilon > 0 \, \exists x \in A \, \forall \mu \in \mathrm{G} \circ \mathrm{F}(T, M, a, x) : |\lambda - \mu| < \varepsilon \,.$$

Proof. We have to prove that given a self-adjoint operator $T \in \mathcal{B}(\mathcal{H})$, an "upper bound" $M \geq \|T\| + 1$ of its norm, an error bound $a \in \mathbb{Q}$ with $0 < a \leq 1$ and an element $\lambda \in \sigma(T)$ of the spectrum there exists a unit vector $x \in S_{\mathcal{H}}$ such that all elements $\mu \in \mathrm{G} \circ \mathrm{F}(T, M, a, x)$ are "sufficiently near" to λ.

Since $\sigma(T) \subseteq \mathbb{R}$ for self-adjoint T, it holds $\lambda \in \mathbb{R}$. Given $\varepsilon > 0$ we choose $q_0 \in \mathbb{N}$ such that $2^{-q_0} < \frac{\varepsilon}{4}$. Since $\lambda \in \sigma(T)$ we obtain $\Phi_T(\chi_{(\lambda-\frac{\varepsilon}{2},\lambda+\frac{\varepsilon}{2})}) \neq 0$ so that the subspace

$$H_{\lambda,\frac{\varepsilon}{2}} := \mathrm{range}(\Phi_T(\chi_{(\lambda-\frac{\varepsilon}{2},\lambda+\frac{\varepsilon}{2})}))$$

is not the trivial subspace $\{0\}$. Thus there exists a vector $z \in H_{\lambda,\frac{\varepsilon}{2}} \cap S_{\mathcal{H}}$ and a corresponding $x \in A$ such that

$$\|x - z\| < \frac{1}{2M} \cdot \frac{1}{4^{q_0}}\,.$$

We obtain $\left\|\Phi_T(\tau_{q_0 i_{q_0}}\rfloor_{\sigma(T)})x\right\| > 0$, hence $\Phi_T(\tau_{q_0 i_{q_0}}\rfloor_{\sigma(T)}) \neq 0$ for all sequences $(i_q)_{q\in\mathbb{N}} \in \mathrm{F}(T, M, a, x)$. It follows $\tau_{q_0 i_{q_0}}\rfloor_{\sigma(T)} \cdot \chi_{(\lambda-\frac{\varepsilon}{2},\lambda+\frac{\varepsilon}{2})}) \neq 0$. Therefore, there exists some $\mu \in (\lambda - \frac{\varepsilon}{2}, \lambda + \frac{\varepsilon}{2})$ such that $\tau_{q_0 i_{q_0}}\rfloor_{\sigma(T)}(\mu) \neq 0$, hence $|\mu - \lambda| < \frac{\varepsilon}{2}$ and $|\mu - 2^{-q_0} i_{q_0}| < 2^{-q_0}$. It follows

$$\left|\lambda - \lim_{q\to\infty} 2^{-q} i_q\right| \leq |\lambda - \mu| + \left|\mu - 2^{-q_0} i_{q_0}\right| + \left|2^{-q_0} i_{q_0} - \lim_{q\to\infty} 2^{-q} i_q\right| < \varepsilon\,.$$

Along with the definition of G we obtain the desired result. □

The Continuous Spectrum

For each sequence $(i_q)_{q\in\mathbb{N}} \in \mathrm{F}(T, M, a, x)$ such that $\mathrm{G}((i_q)_{q\in\mathbb{N}})$ is an element of the continuous spectrum of T the norms $\left\|\Phi_T(\tau_{q i_q}\rfloor_{\sigma(T)})x\right\|$ fall below any bound. We use this property to identify the continuous spectrum within the spectrum.

Lemma 6.4 *Let $\mathcal{H}$ be a computable complex Hilbert space and* F *be defined as in Lemma 6.1. Let $T \in \mathcal{B}(\mathcal{H})$ be self-adjoint, $M \in \mathbb{N}$ such that $\|T\| + 1 \leq M$, $a \in \mathbb{Q}$ such that $0 < a \leq 1$, and $x \in S_{\mathcal{H}}$. Then*

$$\lim_{q\to\infty} 2^{-q} i_q \in \sigma_c(T) \Longrightarrow \left[\ \forall \varepsilon > 0\, \exists q_0 \in \mathbb{N}\, \forall q > q_0 : \left\|\Phi_T(\tau_{q i_q}\rfloor_{\sigma(T)})x\right\| < \varepsilon\ \right]$$

holds or equivalently

$$\mathrm{G}((i_q)_{q\in\mathbb{N}}) \in \sigma_c(T) \Longrightarrow \lim_{q\to\infty} \left\|\Phi_T(\tau_{q i_q}\rfloor_{\sigma(T)})x\right\| = 0$$

for every sequence $(i_q)_{q\in\mathbb{N}} \in \mathrm{F}(T, M, a, x)$.

Proof. Let $\lambda := \lim_{q\to\infty} 2^{-q} i_q$. The sequence $(\tau_{qi} \cdot (1 - \chi_{\{\lambda\}}))_{q\in\mathbb{N}}$ converges pointwise to zero and is uniformly bounded. Therefore $\lim_{q\to\infty} \Phi_T(\tau_{qi} \cdot (1-\chi_{\{\lambda\}})\rfloor_{\sigma(T)})z = 0$ for each $z \in \mathcal{H}$. If $\lambda \notin \sigma_p(T)$ then $\Phi_T(\chi_{\{\lambda\}}\rfloor_{\sigma(T)}) = 0$. It follows

$$\Phi_T(\tau_{qi} \cdot (1 - \chi_{\{\lambda\}})\rfloor_{\sigma(T)}) = \Phi_T(\tau_{qi}\rfloor_{\sigma(T)})$$

and furthermore

$$\lim_{q\to\infty} \Phi_T(\tau_{qi}\rfloor_{\sigma(T)})x = \lim_{q\to\infty} \Phi_T(\tau_{qi} \cdot (1 - \chi_{\{\lambda\}})\rfloor_{\sigma(T)})x = 0\,.$$

Since T is self-adjoint, thus normal, $\sigma_c(T) = \sigma(T) \setminus \sigma_p(T)$ holds. Therefore $\lambda \notin \sigma_p(T)$ is equivalent to $\lambda \in \sigma_c(T)$ because of $\lambda \in \sigma(T)$. Hence

$$\lim_{q\to\infty} \left\|\Phi_T(\tau_{qi}\rfloor_{\sigma(T)})x\right\| = 0$$

and the desired result follows. □

The Point Spectrum (Eigenvalues)

Given a self-adjoint operator $T \in \mathcal{B}(\mathcal{H})$, an upper bound $M \geq \|T\| + 1$, and an error bound $a \in \mathbb{Q}$ such that $0 < a \leq 1$, the set of all values $\mathrm{G} \circ \mathrm{F}(T, M, a, x)$ forms a dense subset of $\sigma(T)$ if x varies over some dense subset $A \subseteq S_{\mathcal{H}}$. This set even contains all eigenvalues of T if $a < \frac{1}{4}$. Furthermore in the case that $\mathrm{G}((i_q)_{q\in\mathbb{N}})$ is an eigenvalue of T the norms $\left\|\Phi_T(\tau_{qi_q}\rfloor_{\sigma(T)})x\right\|$ stay above $\frac{1}{4}$ for each $q \in \mathbb{N}$, if x is "sufficiently near" to the corresponding eigenspace.

Lemma 6.5 *Let $\mathcal{H}$ be a computable complex Hilbert space and* F *be defined as in Lemma 6.1. Let $T \in \mathcal{B}(\mathcal{H})$ be self-adjoint and $M \in \mathbb{N}$ such that $\|T\| + 1 \leq M$. Let $\lambda \in \sigma_p(T)$ and $z \in S_{\mathcal{H}}$ such that $Tz = \lambda z$. Let $a \in \mathbb{Q}$ such that $0 < a < \frac{1}{4}$ and $x \in S_{\mathcal{H}}$ such that $\|x - z\| \leq \frac{1}{8} - \frac{1}{2} \cdot a$. Then $\lim_{q\to\infty} 2^{-q} i_q = \lambda$ for all $(i_q)_{q\in\mathbb{N}} \in \mathrm{F}(T, M, a, x)$ or equivalent $\mathrm{G} \circ \mathrm{F}(T, M, a, x) = \{\lambda\}$. Furthermore*

$$\forall q \in \mathbb{N} : \left\|\Phi_T(\tau_{qi_q}\rfloor_{\sigma(T)})x\right\| > \frac{1}{2} - (\|x - z\| + a) \geq \frac{3}{8} - \frac{1}{2} \cdot a > \frac{1}{4}$$

for all $(i_q)_{q\in\mathbb{N}} \in \mathrm{F}(T, M, a, x)$.

Proof. Due to Lemma 5.5 and 6.1 it holds that given T, M, λ and z as above and $a \in \mathbb{Q}$ such that $0 < a \leq 1$ and $x \in S_{\mathcal{H}}$ we obtain

$$\left\|\Phi_T(\tau_{q_0 i_{q_0}}\rfloor_{\sigma(T)})x\right\| > \frac{1}{2} - (\|x - z\| + a)$$

as well as

$$\left\|\Phi_T(\tau_{qi_q}\rfloor_{\sigma(T)})x\right\| \geq \frac{1}{4} - \|x - z\|$$
$$\Longrightarrow \left\|\Phi_T(\tau_{q+1,i_{q+1}}\rfloor_{\sigma(T)})x\right\| > \frac{1}{2} - (\|x - z\| + a)$$

for all $(i_q)_{q\in\mathbb{N}} \in \mathrm{F}(T, M, a, x)$. Using the additional assumptions $a < \frac{1}{4}$ and $\|x-z\| \leq \frac{1}{8} - \frac{1}{2} \cdot a$ it follows

$$\left\|\Phi_T(\tau_{q_0 i_{q_0}}\rfloor_{\sigma(T)})x\right\| > \frac{1}{2} - (\|x - z\| + a) \geq \frac{3}{8} - \frac{1}{2} \cdot a > \frac{1}{4}$$

and

$$\left\|\Phi_T(\tau_{q+1,i_{q+1}}\rfloor_{\sigma(T)})x\right\| > \frac{1}{2} - (\|x - z\| + a) \geq \frac{3}{8} - \frac{1}{2} \cdot a > \frac{1}{4}$$

for those $q \in \mathbb{N}$ such that $\left\|\Phi_T(\tau_{qi_q}\rfloor_{\sigma(T)})x\right\| \geq \frac{3}{8} - \frac{1}{2} \cdot a$ because this implies $\left\|\Phi_T(\tau_{qi_q}\rfloor_{\sigma(T)})x\right\| \geq \frac{1}{4} + \|x - z\|$. Hence

$$\left\|\Phi_T(\tau_{qi_q}\rfloor_{\sigma(T)})x\right\| > \frac{1}{2} - (\|x - z\| + a) \geq \frac{3}{8} - \frac{1}{2} \cdot a > \frac{1}{4}$$

holds for all $q \in \mathbb{N}$. Furthermore for all $q \in \mathbb{N}$

$$\tau_{qi_q}(\lambda) \geq \left\|\Phi_T(\tau_{qi_q}\rfloor_{\sigma(T)})x\right\| - \|x - z\| \geq \frac{1}{4}$$

holds, and therefore $\lambda \in \mathrm{support}(\tau_{qi_q})$ for all $q \in \mathbb{N}$. It follows $\lim_{q\to\infty} 2^{-q} i_q = \lambda$. $\square$

If we compute the values $\mathrm{G} \circ \mathrm{F}(T, M, a, x)$ for all $x \in A$ where $A \subseteq S_{\mathcal{H}}$ is some dense subset of the unit sphere of $\mathcal{H}$, then for each eigenvalue λ of T we also compute this values for some vector x that is "sufficiently near" to the corresponding eigenspace. Therefore the whole point spectrum of T is "computed" during this procedure.

Corollary 6.6 *Let $\mathcal{H}$ be a computable complex Hilbert space and* F *be defined as in Lemma 6.1. Let $A \subseteq S_{\mathcal{H}}$ be dense in $S_{\mathcal{H}}$ and $a \in \mathbb{Q}$ such that $0 < a \leq \frac{1}{4}$. Let $T \in \mathcal{B}(\mathcal{H})$ be self-adjoint and $M \in \mathbb{N}$ such that $\|T\|+1 \leq M$. Let $\lambda \in \sigma_p(T)$ and $z \in S_{\mathcal{H}}$ such that $Tz = \lambda z$. Then there exists some $x \in A$ such that $\lim_{q\to\infty} 2^{-q} i_q = \lambda$ and $\forall q \in \mathbb{N} : \left\|\Phi_T(\tau_{qi_q}\rfloor_{\sigma(T)})x\right\| \geq \frac{3}{8} - \frac{1}{2}\cdot a > \frac{1}{4}$ for all $(i_q)_{q\in\mathbb{N}} \in \mathrm{F}(T, M, a, x)$.*

Proof. Since A is a dense subset of the unit sphere $S_{\mathcal{H}}$ there exists some $x \in A$ such that $\|x - z\| \leq \frac{1}{8} - \frac{1}{2}\cdot a$. Now we apply Lemma 6.5. □

6.2 The mapping H_a

In this section we define a mapping H_a that is similar to F but only with the self-adjoint operator T and the unit vector x as input. More precisely H_a is based on F in such a way that the upper bound M of T is computed from T and the error bound a is fixed for H_a. The additional parameters of F compared to H_a have been necessary to describe the properties of F (and $\mathrm{G}\circ\mathrm{F}$) properly.

Lemma 6.7 *Let $\mathcal{H}$ be a computable complex Hilbert space. Given $a \in \mathbb{Q}$ such that $0 < a < \frac{1}{4}$ there exists a $\left(\left[\delta_{\mathcal{B}(\mathcal{H})}, \delta_{\mathcal{H}}\right], \nu_{\mathbb{Z}}^{\mathbb{N}}\right)$-computable multi-valued mapping*

$$\mathrm{H}_a :\subseteq \mathcal{B}(\mathcal{H}) \times \mathcal{H} \rightrightarrows \mathbb{Z}^{\mathbb{N}}$$

with

$$\mathrm{dom}(\mathrm{H}_a) := \{T \in \mathcal{B}(\mathcal{H}) \mid T \textit{is self-adjoint}\} \times S_{\mathcal{H}}$$

such that $\mathrm{range}(\mathrm{H}_a) \subseteq \mathrm{dom}(\mathrm{G})$ *and the following properties are fulfilled for each $T \in \mathcal{B}(\mathcal{H})$ and $A \subseteq S_{\mathcal{H}}$.*

(i) *It holds that*

$$\mathrm{G}\circ\mathrm{H}_a[\{T\} \times S_{\mathcal{H}}] \subseteq \sigma(T)$$

or equivalent

$$\forall (T, x) \in \mathrm{dom}(\mathrm{H}_a) : \Big[\ \lambda \in \mathrm{G}\circ\mathrm{H}_a(T, x) \Longrightarrow \lambda \in \sigma(T)\Big] .$$

In particular given $x \in A$ all elements of $\mathrm{G}\circ\mathrm{H}_a(T, x)$ are elements of the spectrum of T.

(ii) *If A is dense in $S_{\mathcal{H}}$ then*

$$\forall \lambda \in \sigma(T)\, \forall \varepsilon > 0\, \exists x \in A\, \forall \mu \in \mathrm{G}\circ\mathrm{H}_a(T, x) : |\lambda - \mu| < \varepsilon .$$

If h_a is some choice function of H_a and A is dense in $S_{\mathcal{H}}$, then $\mathrm{G}\circ\mathrm{h}_a[\{T\}\times A]$ is dense in $\sigma(T)$.

(iii) *It holds that*

$$\forall x \in S_{\mathcal{H}}\, \forall (i_q)_{q\in\mathbb{N}} \in \mathrm{H}_a(T, x) \ : \\ \Big[\ \mathrm{G}((i_q)_{q\in\mathbb{N}}) \in \sigma_c(T) \Longrightarrow \lim_{q\to\infty} \left\|\Phi_T(\tau_{qi_q}\rfloor_{\sigma(T)})x\right\| = 0\ \Big]$$

or equivalent

$$\forall \varepsilon > 0\, \forall x \in S_{\mathcal{H}}\, \forall (i_q)_{q\in\mathbb{N}} \in \mathrm{H}_a(T,x)\ :$$
$$\Big[\ \mathrm{G}((i_q)_{q\in\mathbb{N}}) \in \sigma_c(T) \implies \Big[\ \exists q_0 \in \mathbb{N}\, \forall q > q_0\ :\ \left\|\Phi_T(\tau_{qi_q}\rfloor_{\sigma(T)})x\right\| < \varepsilon\ \Big]\Big].$$

Given $x \in A$ the norms of the operational calculus of any result of $\mathrm{H}_a(T,x)$ *become arbitrarily small, in particular smaller than $\frac{1}{4}$, if the image under* G *is in the continuous spectrum.*

(iv) *If A is dense in $S_{\mathcal{H}}$, it holds that*

$$\forall \lambda \in \sigma_p(T)\, \exists x \in A\ :$$
$$\Big[\ \mathrm{G} \circ \mathrm{H}_a(T,x) = \{\lambda\}\ \ \textit{and}\ \Big[\ \forall (i_q)q \in \mathbb{N} \in \mathrm{H}_a(T,x) \forall q \in \mathbb{N}\ :$$
$$\left\|\Phi_T(\tau_{qi_q}\rfloor_{\sigma(T)})x\right\| \geq \frac{3}{8} - \frac{1}{2}\cdot a > \frac{1}{4})\ \Big]\Big]$$

If h_a *is some choice function of* H_a *and A is dense in $S_{\mathcal{H}}$ then $\sigma_p(T) \subseteq$* $\mathrm{G} \circ \mathrm{h}_a[\{T\} \times A]$. *For each $\lambda \in \sigma_p(T)$ there even exists some $x \in A$ such that the norm of the operational calculus stays above $\frac{1}{4}$ for all results of* $\mathrm{H}_a(T,x)$ *and the image under* G *is λ for every element of* $\mathrm{H}_a(T,x)$.

Proof. Let F be defined as in Lemma 6.1. Given some operator $T \in \mathcal{B}(\mathcal{H})$ we can compute an upper bound of $\|T\|$ [1,3] and therefore also some $M \in \mathbb{N}$ such that $M \geq \|T\| + 1$. Now $(T, M, a, x) \in \mathrm{dom}(\mathrm{F})$ holds if $0 < a < \frac{1}{4}$ and $(T,x) \in \mathrm{dom}(\mathrm{H}_a)$. Hence we can compute $\mathrm{F}(T,M,a,x)$ as the result of $\mathrm{H}_a(T,x)$. Using the properties of F we obtain the following properties for the results $(i_q)_{q\in\mathbb{N}}$ of $\mathrm{H}_a(T,x)$.

(i) By Lemma 6.2 $G((i_q)_{q\in\mathbb{N}}) \in \sigma(T)$ holds.

(ii) Let $\lambda \in \sigma(T)$ and A be a dense subset of $S_{\mathcal{H}}$. By Lemma 6.3 for each $\varepsilon > 0$ there exists some $x \in A$ such that $|\lambda - \mathrm{G}((i_q)_{q\in\mathbb{N}})| < \varepsilon$ if $(i_q)_{q\in\mathbb{N}}$ is computed using this particular x.

(iii) If $\mathrm{G}((i_q)_{q\in\mathbb{N}}) \in \sigma_c(T)$, then by Lemma 6.4 $\lim_{q\to\infty} \left\|\Phi_T(\tau_{qi_q}\rfloor_{\sigma(T)})x\right\| = 0$ holds.

(iv) Let $\lambda \in \sigma_p(T)$ and A be a dense subset of $S_{\mathcal{H}}$. Let $z \in S_{\mathcal{H}}$ such that $Tz = \lambda z$. Then by Corollary 6.6 there exists some $x \in A$ such that $\mathrm{G}((i_q)_{q\in\mathbb{N}}) = \lambda$ and $\forall q \in \mathbb{N} : \left\|\Phi_T(\tau_{qi_q}\rfloor_{\sigma(T)})x\right\| \geq \frac{3}{8} - \frac{1}{2}\cdot a > \frac{1}{4}$ if $(i_q)_{q\in\mathbb{N}}$ is computed using this particular x.

Hence the claimed properties of H_a hold. □

The computable mapping H_a, $0 < a < \frac{1}{4}$, provides an opportunity to compute the spectrum and the eigenvalues of self-adjoint operators $T \in \mathcal{B}(\mathcal{H})$ using the "program", thus a $\delta_{\mathcal{B}(\mathcal{H})}$-name of T, in a uniform way.

- Given some operator T and a vector x we can compute a sequence $(i_q)_{q\in\mathbb{N}} \in \mathrm{H}_a(T,x)$ and furthermore a real number $\lambda = G((i_q)_{q\in\mathbb{N}}$.
- If we compute a sequence in $(\mathrm{G} \circ \mathrm{H}_a(T,x_n))_{n\in\mathbb{N}}$ using a sequence $(x_n)_{n\in\mathbb{N}}$ that is dense in $S_{\mathcal{H}}$ and pick *one* real number out of $\mathrm{H}_a(T,x_n)$ for each $n \in \mathbb{N}$, then we obtain a dense sequence in the spectrum $\sigma(T)$ that contains all eigenvalues of T.
- For the elements $(i_q)_{q\in\mathbb{N}} \in \mathrm{H}_a(T,x)$ of the sequence $(\mathrm{G} \circ \mathrm{H}_a(T,x_n))_{n\in\mathbb{N}}$ we can determine if $G((i_q)_{q\in\mathbb{N}})$ is an eigenvalue or not by having a look at the norms

$\left\|\Phi_T(\tau_{qi_q}\rfloor_{\sigma(T)})x_n\right\|$. In the case of an eigenvalue, the norm stays above $\frac{1}{4}$ provided x_n is "sufficiently near" to the corresponding eigenspace. In the other case the norm tends to zero, hence it becomes smaller than $\frac{1}{4}$ at some stage.

7 The Proof: Computability of the Spectrum and the Eigenvalues

Now we are prepared to prove the main result of this paper. The sequence $(i_q)_{q\in\mathbb{N}}$ computed by H_a corresponds to the sequence that Pour-El and Richards compute in the proof of the Second Main Theorem [9] for a single vector x. In this section we define a mapping based on H_a that applies the mapping H_a to a sequence $(x_n)_{n\in\mathbb{N}}$ of unit vectors such that we obtain a sequence $(\lambda_n)_{n\in\mathbb{N}}$ in the spectrum $\sigma(T)$ and a set $A \subseteq \mathbb{N}$ that contains the indices of the "non-eigenvalues" [9]. Hence given some self-adjoint operator T this mapping computes a sequence and a set, corresponding to the sequence and the set of part (i) and (ii) of the Second Main Theorem [9].

First we recall the main result stated in Section 4. Then we prove it.

Theorem 7.1 *Let $\mathcal{H}$ be a computable complex Hilbert space. Then there exists a $\left(\delta_{\mathcal{B}(\mathcal{H})}, \left[\delta_{\mathbb{R}}^{\mathbb{N}}, \mathrm{En}\right]\right)$-computable multi-valued mapping*

$$\mathrm{H} :\subseteq \mathcal{B}(\mathcal{H}) \rightrightarrows \mathbb{R}^{\mathbb{N}} \times 2^{\mathbb{N}}$$

with domain

$$\mathrm{dom}(\mathrm{H}) := \{T \in \mathcal{B}(\mathcal{H}) \mid T \textit{ is self-adjoint}\}$$

such that

(i) $\overline{\{\lambda_n \mid n \in \mathbb{N}\}} = \sigma(T)$ *and*

(ii) $\sigma_p(T) = \{\lambda_n \mid n \in \mathbb{N} \setminus A\}$

for all $((\lambda_n)_{n\in\mathbb{N}}, A) \in \mathrm{H}(T)$.

Proof. The unit sphere $S_{\mathcal{H}}$ is a recursive closed [10] subset of $\mathcal{H}$ as it has the computable distance function $\mathrm{dist}_{S_{\mathcal{H}}}(x) = |\|x\| - 1|$ [5, Corollary 3.14]. Hence there exists a $\delta_{\mathcal{H}}^{\mathbb{N}}$-computable sequence $(x_n)_{n\in\mathbb{N}}$ that is dense in $S_{\mathcal{H}}$ [5, Corollary 3.14]. We fix such a sequence for the following procedure and choose $a := \frac{1}{8}$. Using the computability of $\mathrm{H}_a = \mathrm{H}_{\frac{1}{8}}$ and $(x_n)_{n\in\mathbb{N}}$ we can compute a double sequence $((i_q^n)_{q\in\mathbb{N}})_{n\in\mathbb{N}} \in (\mathrm{H}_a(T, x_n))_{n\in\mathbb{N}}$ that has the properties described in the preceding sections.

Using this sequence we can compute a sequence $(\lambda_n)_{n\in\mathbb{N}} = (\mathrm{G}((i_q^n)_{q\in\mathbb{N}}))_{n\in\mathbb{N}}$ such that $\lambda_n = \mathrm{G}((i_q^n)_{q\in\mathbb{N}})$ for $n \in \mathbb{N}$. Furthermore we can compute the sequence $\left(\left\|\Phi_T(\tau_{qi_q^n}\rfloor_{\sigma(T)})x_n\right\|\right)_{q\in\mathbb{N}}$ for each $n \in \mathbb{N}$ and scan it whether at some stage the value becomes less than $\frac{1}{4}$. If this is the case we "add" n to the sequence that represents the set A. [11] As proved before all n such that λ_n is an eigenvalue and

[9] We explain later what exactly is meant by "non-eigenvalues".

[10] Closed and compact subsets of computable metric spaces are studied e.g. in [5].

[11] We can determine whether this is the case. But we cannot determine the other case.

x_n is "sufficiently near" to the eigenspace corresponding to λ_n are never listed in this sequence because for these n the norms are greater than $\frac{1}{4}$ at any stage. As for every eigenvalue λ of T there exists some vector x_n that is near enough to the corresponding eigenspace the set $\mathbb{N} \setminus A$ contains at least one n such that $\lambda_n = \lambda$. □

A more detailed proof can be found in [6].

8 Conclusions

In this paper we have studied the computability of the spectrum of self-adjoint operators in complex Hilbert spaces using the framework of TTE. We have proved a uniform version of the Second Main Theorem of Pour-El and Richards [9]. Roughly speaking, we have shown that the spectrum map of self-adjoint operators on computable complex Hilbert spaces is lower-semicomputable, and that we can identify the continuous spectrum within the spectrum in such a way that the remaining elements of a dense sequence in the spectrum form the set of eigenvalues of a given self-adjoint operator.

Additionally we have stated a computable version of the operational calculus, which we used in the proofs regarding the spectrum.

There remain several interesting questions regarding the computability of the spectrum of linear operators. In [9] Pour-El and Richards also gave a version of the Second Main Theorem for bounded normal operators and for unbounded closed self-adjoint operators. In [4] we stated our results not only for normal operators on complex Hilbert spaces, but also for self-adjoint operators on real Hilbert spaces as well as normal operators on complex Hilbert spaces. It is still an open question whether a uniform version of the results of Pour-El and Richards regarding normal and unbounded operators can be proved, i.e. if it is possible to obtain information about the eigenvalues from a program of the operator in these cases.

More detailed proofs of most of the results presented here can be found in [6].

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