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A Hybridization of Irreflexive Modal Logics

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Abstract

This paper discusses a bimodal hybrid language with a sub-modality (called the irreflexive modality) associated with the intersection of the accessibility relation R and the inequality $\neq$. First, we provide the Hilbert-style axiomatizations (with and without the COV-rule) for logics of our language, and prove the Kripke completeness and the finite frame property for them. Second, with respect to the frame expressive power, we compare our language containing the irreflexive modality with the hybrid languages $\mathcal{H}$ and $\mathcal{H}(\mathrm{E})$. Finally, we establish the Goldblatt-Thomason-style characterization for our language.

Keywords: Modal and Hybrid Logics, Kripke Completeness, Finite Frame Property, (Modality for) Irreflexivity, Gabbay-style rule, Goldblatt-Thomason-style Characterization.

1 Introduction and Motivation

As is well known, the standard modal propositional language $\mathcal{M}$ with $\Diamond$ cannot define all the natural assumptions about the accessibility relation R on the set W of states, e.g., irreflexivity and antisymmetry. It has known that we can prove the completeness with respect to the frames with undefinable properties by adding Gabbay-style non-orthodox rules without changing the language [10,11]. In order to overcome the lack of expressive power of $\mathcal{M}$, on the other hand, extensions with various tools have been proposed, e.g., the difference operator D [8], nominals i [4], the global modality E [12], and the satisfaction operator $@_i$ [6].

For the same reason, the author of the present paper [16,18,19] proposed a new modest extension $\mathcal{M}(\blacklozenge)$ of $\mathcal{M}$, which consisted in adding an operator $\blacklozenge$ associated with the intersection of the accessibility relation R and the inequality $\neq$ (written $(R\cap\neq)$). E.g., in tense logic, $\blacklozenge\varphi$ means that φ holds in some future instant *different from now*. He [16,18] has already proved that various normal modal logics (some

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of which are kinds of Lemmon-Scott's Axioms $\Diamond^m\Box^n p \to \Box^j\Diamond^k p$ $(m, n, j, k \in \omega)$) in his extended language enjoy Kripke completeness (for some logics containing **K**, even in the predicate extension [17]) and the finite model property. He [19] has also shown that $\Diamond p \to \blacklozenge p$ defines irreflexivity and $\blacklozenge\blacklozenge p \to \blacklozenge p$ defines the conjunction of antisymmetry and transitivity, though antisymmetry is not independently definable. The definability of these properties of frames witnesses the strength of his extension. In addition, the author and Sato recently have provided the Van Benthem-style and Goldblatt-Thomason-style Characterizations for his extension [20].

We can point out that $\blacklozenge$ has a close connection with topological semantics for modal logics. $\Diamond$ has been topologically interpreted in at least two ways: as the closure operator or as the derivative operator. '$\Diamond$ as derivative' has been studied by e.g., [9]. By the recent work [2] (especially, Lemmas 1.62 and 1.64), we can understand that '$\Diamond$ as derivative' has the same meaning as our $\blacklozenge$ in the relational semantics. Thus, we can claim that the author's studies about $\blacklozenge$ contain, as a special case, a relational-semantical study of '$\Diamond$ as derivative'. In fact, when $\Diamond$ satisfies the axioms for **S4**, $\blacklozenge$, associated with $(R \cap \neq)$, satisfies those for **wK4** [2, sec.3.1.1], which is the basic axiom system for '$\Diamond$ as derivative' (Note that, if $\Diamond$ satisfies the reflexivity axiom **T**: $p \to \Diamond p$ at least, $\Diamond p$ is equivalent to $\blacklozenge p \vee p$).

Topological semantics for hybrid logic have been studied by, e.g., [21,23]. Most of these works deal with $\Diamond$ as the closure operator. There has been no study that tried to interpret $\Diamond$ as the derivative operator in the framework of hybrid logics. This paper is a preliminary step toward this. In this paper, we hybridize our $\mathcal{M}(\blacklozenge)$ and intend to give a relational basis for topological hybrid logics with '$\Diamond$ as derivative'.

We add a new sort of variables, *nominals*, which will be formulas used to denote points in the domains in the Kripke frames ($\mathcal{H}(\blacklozenge)$ denotes this hybridization of $\mathcal{M}(\blacklozenge)$). Nominals also allow us to define irreflexivity. Therefore, the expressive powers of nominals and the irreflexive modality $\blacklozenge$ overlap. We reveal the difference between the irreflexive modality and nominals with respect to the expressive powers and meta-logical properties.

Hybridization of our extension is also inspired by [13] (cf. [1, sec.2.2]). They showed that intersection of accessibility relations can be defined using nominals. For $\blacklozenge$, this means that the axiom scheme $\blacklozenge i \leftrightarrow \Diamond i \wedge \mathrm{D}i$ holds, where i is a nominal and D is the difference operator associated with the inequality $\neq$. In addition, they also showed that complement of the accessibility relation can be defined by nominals. Since the axiom scheme $i \leftrightarrow \neg\mathrm{D}i$ holds, we obtain the following scheme: $\blacklozenge i \leftrightarrow \Diamond i \wedge \neg i$.

We use such a scheme to provide our Hilbert-style axiomatization by using the traditional notion of possibility forms [14] (and the COV-rule [13]). We also show that hybridization of $\mathcal{M}(\blacklozenge)$ preserves Kripke completeness and the finite frame property (Section 3). Second, with respect to the frame expressive power, we compare our languages containing the irreflexive modality with the hybrid languages $\mathcal{H}$ and $\mathcal{H}(\mathrm{E})$. Finally, we give the Goldblatt-Thomason-style characterization for $\mathcal{H}(\blacklozenge)$ (Section 4).

2 Basic Notions

2.1 Syntax and Semantics

The *irreflexive hybrid language* $\mathcal{H}(\blacklozenge)$ is obtained by the basic modal language $\mathcal{M}$ containing $\Diamond$ with an infinite set of nominals $\mathsf{Nom} = \{ i, j, k, \dots \}$ and the irreflexive modality $\blacklozenge$. The set of irreflexive hybrid formulas is defined as

$$\varphi ::= \top \mid p \mid i \mid \neg \varphi \mid \varphi \wedge \psi \mid \Diamond\varphi \mid \blacklozenge\varphi,$$

where $p \in \mathsf{Prop}$, the set of all proposition letters, and i is a nominal. It is usually assumed that the set of nominals Nom as well as Prop is countable.

We define the following languages and their formulas similarly: The *hybrid language* $\mathcal{H}$ without $\blacklozenge$, the *irreflexive modal language* $\mathcal{M}(\blacklozenge)$ without Nom, and their extension with the *global modality* E (written, e.g., $\mathcal{H}(\blacklozenge, \mathrm{E})$). In addition to the usual abbreviations for material implication $\rightarrow$, disjunction $\vee$, logical equivalence $\leftrightarrow$, the falsum $\bot$, we use the following: $\Box\varphi := \neg\Diamond\neg\varphi$, $\blacksquare\varphi := \neg\blacklozenge\neg\varphi$. We use φ, ψ, $\theta, \dots$ to denote formulas and Γ, Δ, $\dots$ to sets of formulas. A formula φ is called *pure* if it contains no proposition letters (nominals are allowed).

A *bimodal frame* is a triple $\mathfrak{F} = \langle W, R, S \rangle$ of a non-empty set W, called a *domain*, and two binary relations R, S on W. A *bimodal model* is a pair $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$ of a bimodal frame $\mathfrak{F} = \langle W, R, S \rangle$ and a mapping $V : \mathsf{Prop} \cup \mathsf{Nom} \rightarrow \mathcal{P}(W)$ *with* $|V(i)| = 1$ for any $i \in \mathsf{Nom}$. A *unimodal frame* and a *unimodal model* are defined similarly. $|\mathfrak{M}|$ (or $|\mathfrak{F}|$) means the domain of a model $\mathfrak{M}$ (or, a frame $\mathfrak{F}$, respectively). For any binary relation R on W, $R[w]$ denotes $\{ w' \mid wRw' \}$.

For a bimodal model $\mathfrak{M} = \langle W, R, S, V \rangle$ (or unimodal model $\mathfrak{M} = \langle W, R, V \rangle$), $w \in W$ and a formula φ of $\mathcal{H}(\blacklozenge)$ (or $\mathcal{H}$, respectively), the satisfaction relation $\mathfrak{M}, w \Vdash \varphi$ is defined as usual. We assume that $\Diamond$ is associated with R, i.e., $\mathfrak{M}, w \Vdash \Diamond\varphi \iff [wRw'$ and $\mathfrak{M}, w' \Vdash \varphi]$ for some w', and that $\blacklozenge$ is associated with S. For E, the satisfaction is defined as follows: $\mathfrak{M}, w \Vdash \mathrm{E}\varphi \iff \mathfrak{M}, w' \Vdash \varphi$ for some w'.

A formula φ is *valid in a model* $\mathfrak{M}$ (written $\mathfrak{M} \Vdash \varphi$) if $\mathfrak{M}, w \Vdash \varphi$ for any w in $\mathfrak{M}$. φ is *valid in a frame* $\mathfrak{F}$ (written $\mathfrak{F} \Vdash \varphi$) if $\langle \mathfrak{F}, V \rangle \Vdash \varphi$ for any valuation $V : \mathsf{Prop} \cup \mathsf{Nom} \rightarrow \mathcal{P}(|\mathfrak{F}|)$. φ is *satisfiable in a model* $\mathfrak{M}$ *(or a frame* $\mathfrak{F}$*)* if $\mathfrak{M} \nVdash \neg\varphi$ (or $\mathfrak{F} \nVdash \neg\varphi$, respectively). φ is *valid in a class* F *of frames* (written $\mathsf{F} \Vdash \varphi$) if it is valid in every $\mathfrak{F} \in \mathsf{F}$. For a set of formulas, these notions are defined similarly. A set Γ of formulas *defines a class* F *of frames* if, for all frames $\mathfrak{F}$, $\mathfrak{F} \Vdash \Gamma \iff \mathfrak{F} \in \mathsf{F}$. A class F of frames is *modally definable* if there is some set of formulas that defines F.

A bimodal frame $\langle W, R, S \rangle$ satisfying $S = (R\cap \neq)$ is called an $\mathcal{H}(\blacklozenge)$*-frame*, where $w(R\cap \neq)w'$ means that wRw' and $w \neq w'$. The notion of $\mathcal{H}(\blacklozenge)$*-model* is defined similarly. Thus, for any $\mathcal{H}(\blacklozenge)$-model $\mathfrak{M}$, $\mathfrak{M}, w \Vdash \blacklozenge\varphi \iff [w(R\cap \neq)w'$ implies $\mathfrak{M}, w' \Vdash \varphi]$ for some w'. Observe that an $\mathcal{H}(\blacklozenge)$-frame $\langle W, R, (R\cap \neq) \rangle$ (or -model) is determined by the unimodal frame $\langle W, R \rangle$ (or model, respectively). Therefore, we often regard $\langle W, R \rangle$ as $\mathcal{H}(\blacklozenge)$-frame $\langle W, R, (R\cap \neq) \rangle$.

2.2 Unimodal and Bimodal p-morphisms

Definition 2.1 Let $\mathfrak{F} = \langle W, R, S \rangle$ and $\mathfrak{F}' = \langle W', R', S' \rangle$ be bimodal frames. A mapping $f : W \to W'$ is a *bimodal p-morphism from* $\mathfrak{F}$ *to* $\mathfrak{F}'$ (written $f : \mathfrak{F} \to \mathfrak{F}'$) if it satisfies the following:

- (R-forth) If wRv, then $f(w)R'f(v)$,
- (R-back) If $f(w)R'v'$, then wRv and $f(v) = v'$ for some $v \in W$,

and (S-forth) and (S-back) defined similarly.

For bimodal models $\mathfrak{M}$ and $\mathfrak{M}'$, $f : |\mathfrak{M}| \to |\mathfrak{M}'|$ is *a bimodal p-morphism from* $\mathfrak{M}$ *to* $\mathfrak{M}'$ (written $f : \mathfrak{M} \to \mathfrak{M}'$) if $f : \mathfrak{F} \to \mathfrak{F}'$ with $w \in V(a) \iff f(w) \in V'(a)$ for each $a \in \mathsf{Prop} \cup \mathsf{Nom}$ and each $w \in |\mathfrak{M}|$.

If there is a bimodal p-morphism f from $\mathfrak{M}$ to $\mathfrak{M}'$ such that f is surjective as a mapping between domains, $\mathfrak{M}'$ is called *a p-morphic image of* $\mathfrak{M}$ (written $\mathfrak{M} \twoheadrightarrow \mathfrak{M}'$).

Given two unimodal frames (or models) unimodal p-morphism between frames (or models, respectively), is defined by using the clauses (R-forth) and (R-back).

It is known that the following holds (see, e.g., Blackburn's thesis [3, Lemma 3.2.2]. He stated the tense logical analogue, though the generalization to $\mathcal{H}(\blacklozenge)$ (or $\mathcal{H}$) is obvious).

Fact 2.2 *Let* $\mathfrak{M}$ *and* $\mathfrak{M}'$ *be bimodal (or unimodal) models with* $f : \mathfrak{M} \to \mathfrak{M}'$*. Then, for each formula* φ *of* $\mathcal{H}(\blacklozenge)$ *(or* $\mathcal{H}$*, respectively), and each* w *in* $\mathfrak{M}$*,* $\mathfrak{M}, w \Vdash \varphi \iff \mathfrak{M}', f(w) \Vdash \varphi$*.*

Note that a surjective p-morphism between frames does not preserve validity for the modal language with nominals [3, sec 3.2.3].

The following is an application of Fact 2.2, i.e., a *unimodal* p-morphism *between models* preserves the satisfaction:

Proposition 2.3 *There exists no formula* $\varphi(q)$ *of* $\mathcal{H}$ *containing* q *such that* $\mathfrak{M}, w \Vdash \blacklozenge\psi \iff \mathfrak{M}, w \Vdash \varphi(\psi)$*, for any* ψ*, any model* $\mathfrak{M}$ *and any* $w \in |\mathfrak{M}|$*. In other words,* $\blacklozenge$ *is not definable in* $\mathcal{H}$ *in the level of models.*

Proof. Suppose for contradiction that $\blacklozenge$ is definable in $\mathcal{H}$ in the level of models. Then we define $\blacklozenge\neg i$ by $\mathcal{H}$-formulas $\varphi(\neg i)$. Consider two models and a unimodal p-morphism: $\mathfrak{M}_1 = \langle \{a, b, c\}, \{\langle a, b\rangle, \langle a, c\rangle\} \cup \{b, c\} \times \{b, c\}, V_1 \rangle$ where $V_1(i) = \{a\}$ for any $i \in \mathsf{Nom}$, $\mathfrak{M}_2 = \langle \{0, 1\}, \{\langle 0, 1\rangle, \langle 1, 1\rangle\}, V_2 \rangle$ where $V_2(i) = \{0\}$ for any $i \in \mathsf{Nom}$, and $f(a) = 0$ and $f(b) = f(c) = 1$. Then, $\mathfrak{M}_1, b \Vdash \varphi(\neg i)$, but $\mathfrak{M}_2, f(b) \nVdash \varphi(\neg i)$ hence $\mathfrak{M}_1, b \nVdash \varphi(\neg i)$ by Fact 2.2, a contradiction. $\square$

Remark that $f : |\mathfrak{M}_1| \to |\mathfrak{M}_2|$ in the proof is not a *bimodal* p-morphism between $\mathcal{H}(\blacklozenge)$-models. Consider the following model and mapping $g : |\mathfrak{M}_3| \to |\mathfrak{M}_1|$:

$$\mathfrak{M}_3 = \langle \omega \cup \{x\}, \{\langle m, m\rangle, \langle m, m+1\rangle, \langle x, m\rangle \mid m \in \omega\}, V_3 \rangle,$$

where $V_3(i) = \{x\}$ for any $i \in \mathsf{Nom}$, and $g(x) = a$, $g(2m) = b$ and $g(2m+1) = c$ for any $m \in \omega$. Then, g is a bimodal p-morphism between $\mathcal{H}(\blacklozenge)$-models.

Next, we introduce the notion of realizations. This is a generalization of bulldozing in hybrid logics (see, e.g., [3, ch.5]) and the tricks as Koymans [15, Theorem 4.3.3] and de Rijke [8, Theorem 3.2].

Definition 2.4 Let $\mathfrak{M}$ be a bimodal model, $\mathfrak{M}'$ an $\mathcal{H}(\blacklozenge)$-model. If a bimodal p-morphism $f : \mathfrak{M}' \to \mathfrak{M}$ is surjective as a mapping between domains, we call f a $\blacklozenge$-*realizer* and $\mathfrak{M}'$ a $\blacklozenge$-*realization* of $\mathfrak{M}$.

Proposition 2.5 *Suppose that a bimodal model* $\mathfrak{M} = \langle W, R, S, V \rangle$ *satisfies* $(R \cap \neq) \subset S \subset R$ *and that, for any* $i \in \mathsf{Nom}$, wSw *fails where* $\{ w \} = V(i)$. *Then,* $\mathfrak{M}$ *has a* $\blacklozenge$*-realization.*

Proof. Let $C = \{ w \in |\mathfrak{M}| \mid wSw \}$, $W^- = W \setminus C$, $2 = \{0, 1\}$, and $W' = W^- \cup (C \times 2)$, where we may assume $W^- \cap (C \times 2) = \emptyset$. $f : W' \to W$ is defined as follows: $f(x) := x$ if $x \in W^-$; w if $x = \langle w, i \rangle$ for some $i \in 2$ and some $w \in C$. Write $\mathfrak{M}' = \langle W', R', S', V' \rangle$, where $R' = \{ \langle x, y \rangle \mid f(x) R f(y) \}$, $S' = (R' \cap \neq)$, and $V'(a) = f^{-1}[V(a)] = \{ x \mid f(x) \in V(a) \}$ for any $a \in \mathsf{Prop} \cup \mathsf{Nom}$. Note that V' is a valuation since $|V(i)| \geq 1$ by the surjectiveness of f and $|V(i)| \leq 1$ by the injectiveness of $f \upharpoonright W^-$. It suffices to prove that f is a bimodal p-morphism from $\langle W', R', S' \rangle$ to $\langle W, R, S \rangle$. Since f is surjective by definition and $xR'y$ is equivalent to $f(x) R f(y)$ for any $x, y \in W'$, we have (R-forth) and (R-back). It is easy to see that (S-forth) and (S-back) hold. $\Box$

3 Kripke Completeness and Finite Frame Property

We axiomatize the irreflexive hybrid logic of $\mathcal{H}(\blacklozenge)$ using Goldblatt's notion of a *possibility form* [14] .

Definition 3.1 Fix an arbitrary symbol \$ not occurring in $\mathcal{H}(\blacklozenge)$. A *possibility form* (PF) of \$ (written: $m(\$)$) are defined inductively as follows: (1) \$ is a PF of \$. (2) If m is a PF of \$ and φ is a formulas of $\mathcal{H}(\blacklozenge)$, then $\varphi \wedge m$, $\Diamond m$ and $\blacklozenge m$ are PFs of \$.

Given $m(\$)$ and φ of $\mathcal{H}(\blacklozenge)$, we use $m(\varphi)$ to denote the formula obtained by replacing the unique occurrence of \$ in m by φ.

Definition 3.2 Axiomatization of $\mathbf{K}_{\mathcal{H}(\blacklozenge)}$ is given as follows:

(Taut) φ, for all classical tautologies φ;

(□1) $\Box(p \to q) \to (\Box p \to \Box q)$;

(■1) $\blacksquare(p \to q) \to (\blacksquare p \to \blacksquare q)$;

(■2) $i \to \neg \blacklozenge i$;

(M1) $p \wedge \blacksquare p \to \Box p$;

(M2) $\Box p \to \blacksquare p$;

(N1) $m_1(i \wedge p) \to \neg m_2(i \wedge \neg p)$ for any nominal i and any PFs m_1 and m_2;

(MP) from $\varphi \to \psi$ and φ, we may infer ψ;

(□-rule) from φ, we may infer $\Box\varphi$;

(Sub) from φ, we may infer $\varphi\sigma$, where σ is a substitution that uniformly replaces proposition letters by formulas and nominals by nominals.

Axiomatization of $\mathbf{K}^{+}_{\mathcal{H}(\blacklozenge)}$ is defined as axiomatization of $\mathbf{K}_{\mathcal{H}(\blacklozenge)}$ plus the following additional rule [13]:

(COV) From $\neg m(i)$, we may infer $\neg m(\top)$, for any PF m and any nominal i which does not appear in m.

A $\mathbf{K}_{\mathcal{H}(\blacklozenge)}$-logic is any set of formulas containing all the axioms of $\mathbf{K}_{\mathcal{H}(\blacklozenge)}$ and closed under all its rules. For every Λ of $\mathcal{H}(\blacklozenge)$, $\mathbf{K}_{\mathcal{H}(\blacklozenge)}\Lambda$ denotes the smallest logic containing Λ. For a $\mathbf{K}_{\mathcal{H}(\blacklozenge)}$-logic Λ, a set Γ and a formula φ of $\mathcal{H}(\blacklozenge)$, the deducibility relation $\Gamma \vdash_{\Lambda} \varphi$ is defined as usual [5, Definition 4.4]. The notion of $\mathbf{K}^{+}_{\mathcal{H}(\blacklozenge)}$-logic is defined similarly to $\mathbf{K}_{\mathcal{H}(\blacklozenge)}$-logic, except for closing in addition under the (COV)-rule. The notion of $\mathbf{K}^{+}_{\mathcal{H}(\blacklozenge)}\Lambda$ and the deducibility relation are defined similarly. For $\vdash_{\Lambda}$, we usually drop the subscript if it is clear from the context.

From ($\blacksquare$2), (M1) and (M2), we deduce that $\blacklozenge i \leftrightarrow \Diamond i \wedge \neg i$. Remark that $p \to \neg \blacklozenge p$ is not valid on any $\mathcal{H}(\blacklozenge)$-frames, though ($\blacksquare$2) is valid.

Fact 3.3 *The following are theorems and a derived rule of* $\mathbf{K}_{\mathcal{H}(\blacklozenge)}$, *therefore, of* $\mathbf{K}^{+}_{\mathcal{H}(\blacklozenge)}$: (i) $m_1(i \wedge \varphi) \wedge \neg m_1(\neg(i \wedge \psi)) \to m_1(i \wedge \varphi \wedge \psi)$; (ii) $m_1(i \wedge \varphi) \wedge m_2(i \wedge \psi) \to m_1(i \wedge \varphi \wedge \psi)$; (iii) $\vdash \varphi \implies \vdash \neg m(\neg \varphi)$, *where* m_1, m_2 *and* m *are PFs and* $i \in \mathsf{Nom}$.

Let K be the class of all $\mathcal{H}(\blacklozenge)$-frames.

Proposition 3.4 *For any* φ *of* $\mathcal{H}(\blacklozenge)$, $\vdash_{\mathbf{K}_{\mathcal{H}(\blacklozenge)}} \varphi \implies \mathsf{K} \Vdash \varphi$ *and* $\vdash_{\mathbf{K}^{+}_{\mathcal{H}(\blacklozenge)}} \varphi \implies \mathsf{K} \Vdash \varphi$.

Proof. By induction on $\vdash_{\mathbf{K}_{\mathcal{H}(\blacklozenge)}} \varphi$. Note that (M1) defines $(R\cap \neq) \subset S$, (M2) defines $S \subset R$ and ($\blacksquare$2) defines the irreflexivity of S. With respect to (N1) and (COV), adding $\blacklozenge$ does not change the usual argument for soundness of hybrid logic (see, e.g., [12, Proposition 5.1] for (COV)). □

3.1 Kripke Completeness without (COV)

Let Γ be a set of $\mathcal{H}(\blacklozenge)$. We say that Γ is *maximal* if, for any φ of $\mathcal{H}(\blacklozenge)$, $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$. For any $\mathbf{K}_{\mathcal{H}(\blacklozenge)}$-logic (or, $\mathbf{K}^{+}_{\mathcal{H}(\blacklozenge)}$-logic, respectively) Λ, we say that Γ is Λ-*consistent* if $\Gamma \nvdash_{\Lambda} \bot$ and Γ is Λ-*maximal consistent* if Γ is maximal and consistent with respect to Λ.

Definition 3.5 The canonical *bimodal* frame $\mathfrak{F}^{\mathbf{K}_{\mathcal{H}(\blacklozenge)}}$ is defined as follows:

- $W = \{\, w \mid w \text{ is a } \mathbf{K}_{\mathcal{H}(\blacklozenge)}\text{-maximal consistent set} \,\}$
- $wRw' \iff [\Box\varphi \in w \implies \varphi \in w']$ for any φ of $\mathcal{H}(\blacklozenge)$.
- $wSw' \iff [\blacksquare\varphi \in w \implies \varphi \in w']$ for any φ of $\mathcal{H}(\blacklozenge)$.

$Nom(w)$ denotes $\{\, i \in \mathsf{Nom} \mid i \in w \,\}$ for any w in $\mathfrak{F}^{\mathbf{K}_{\mathcal{H}(\blacklozenge)}}$.

Definition 3.6 Let $\mathfrak{F} = \langle W, R, S \rangle$ and $\mathfrak{F}' = \langle W', R', S' \rangle$ be two bimodal frames. $\mathfrak{F}'$ is *(bimodally) generated subframe* of $\mathfrak{F}$ (written $\mathfrak{F}' \rightarrowtail \mathfrak{F}$) if

- (i) $W' \subset W$,
- (ii) $R' = R \cap (W')^2$,
- (iii) $w \in W'$ implies $R[w] \subset W'$ and
- (ii′) and (iii′) about S, S' defined similar to (ii) and (iii).

For a subset $X \subset W$, the *subframe generated by* X (notation: $\mathfrak{F}_X$) is the smallest generated subframe of $\mathfrak{F}$ whose domain contains X. The *point-generated frame by* w (notation: $\mathfrak{F}_w$) is $\mathfrak{F}_{\{w\}}$ where w is called the *root* of the frame.

(N1) (especially, Fact 3.3 (ii)) forces that in every point-generated subframe, every nominal is satisfied in at most one point as follows:

Proposition 3.7 *Let w be in $\mathfrak{F}^{\mathbf{K}_{\mathcal{H}(\blacklozenge)}}$ and v, v' in $\mathfrak{F}_w^{\mathbf{K}_{\mathcal{H}(\blacklozenge)}}$. Then, $v \neq v'$ implies $Nom(v) \cap Nom(v') = \emptyset$.*

Lemma 3.8 *Let $\mathfrak{F}_w^{\mathbf{K}_{\mathcal{H}(\blacklozenge)}} = \langle W', R', S' \rangle$ be a point-generated subframe of $\mathfrak{F}^{\mathbf{K}_{\mathcal{H}(\blacklozenge)}}$. Then, (i) $(R' \cap \neq) \subset S'$, (ii) $S' \subset R'$ and (iii) for any v in W' with $Nom(v) \neq \emptyset$, $vS'v$ fails.*

Proof. Clearly, (ii) holds. It is easy to prove that (iii) by ($\blacksquare$2) (see, e.g., [3, Lemma 5.1.1]). We prove (i) as follows: Assume that xRx' and $x \neq x'$. $B \in x$ and $\neg B \in x'$ for some B of $\mathcal{H}(\blacklozenge)$ since x and x' are maximal consistent. Suppose for contradiction that not xSx'. Then, $\blacksquare C \in x$ and $\neg C \in x'$ for some C of $\mathcal{H}(\blacklozenge)$. We have $\neg B \wedge \neg C \in x'$ whence $B \vee C \notin x'$. Since xRx', $\Box(B \vee C) \notin x$. By $B, \blacksquare C \in x$, we have $B \vee C, \blacksquare(B \vee C) \in x$ and so $(B \vee C) \wedge \blacksquare(B \vee C) \in x$, which implies $\Box(B \vee C)$, a contradiction. □

Lemma 3.9 *Let $\mathfrak{F}_w^{\mathbf{K}_{\mathcal{H}(\blacklozenge)}} = \langle W', R', S' \rangle$ be a point-generated subframe of $\mathfrak{F}^{\mathbf{K}_{\mathcal{H}(\blacklozenge)}}$. Fix $\star \notin W$, the domain of $\mathfrak{F}^{\mathbf{K}_{\mathcal{H}(\blacklozenge)}}$. Define $\mathfrak{M}^\star = \langle W^\star, R^\star, S^\star, V^\star \rangle$ as follows: $W^\star = W \cup \{\star\}$; $R^\star = R'$; $S^\star = S'$;*

$$V^\star(a) = \begin{cases} \{ v \in W' \mid a \in v \} & \text{if } a \in \mathsf{Prop} \cup \bigcup_{x \in W'} Nom(x) \\ \{\star\} & \text{if } a \in \mathsf{Nom} \setminus \bigcup_{x \in W'} Nom(x) \end{cases}$$

Then, (i) *$(R^\star \cap \neq) \subset S^\star$ and $S^\star \subset R^\star$;* (ii) *for any v in $W^\star$ with $\{v\} = V(i)$ for some $i \in \mathsf{Nom}$, $vS^\star v$ fails;* (i) *for any $w \in W'$ and for any φ of $\mathcal{H}(\blacklozenge)$, $\mathfrak{M}^\star, w \Vdash \varphi \iff \varphi \in w$.*

Proof. (i) and (ii) are obvious by Lemma 3.8 and the construction. We can prove (iii) by induction on φ. Note that $V^\star$ is a valuation, i.e., $|V^\star(i)| = 1$ for any $i \in \mathsf{Nom}$, by Lemma 3.7 and the construction. □

Theorem 3.10 *Let φ be a formula of $\mathcal{H}(\blacklozenge)$. $\mathsf{K} \Vdash \varphi \implies \vdash_{\mathbf{K}_{\mathcal{H}(\blacklozenge)}} \varphi$.*

Proof. Suppose that $\neg\varphi$ is $\mathbf{K}_{\mathcal{H}(\blacklozenge)}$-consistent. $\neg\varphi \in w$ for some $\mathbf{K}_{\mathcal{H}(\blacklozenge)}$-maximal consistent set w. Take the point-generated subframe $\mathfrak{F}_w^{\mathbf{K}_{\mathcal{H}(\blacklozenge)}}$ of $\mathfrak{F}^{\mathbf{K}_{\mathcal{H}(\blacklozenge)}}$. Note that

$\mathfrak{M}^\star$ is a bimodal model. We need to duplicate the $S^\star$-reflexive points in $\mathfrak{M}^\star$. By Lemma 3.9 (i), (ii) and Proposition 2.5, there exists a $\blacklozenge$-realization $\mathfrak{N}$ such that $f : \mathfrak{N} \twoheadrightarrow \mathfrak{M}^\star$ for some f. However, by Lemma 3.9 (iii), $\mathfrak{M}^\star, w \Vdash \neg\varphi$. Thus, $\mathfrak{N}, x \Vdash \neg\varphi$ for some x in $\mathfrak{N}$ with $f(x) = w$. Thus, we conclude that $\mathsf{K} \nVdash \varphi$. $\Box$

3.2 Kripke Completeness with (COV)

Let φ be a pure formula of $\mathcal{H}(\blacklozenge)$. It is well known that φ defines the first-order property [3, p.92]. $\mathfrak{M} = \langle W, R, V \rangle$ is called *named* if, for all $w \in W$, there is some $i \in \mathsf{Nom}$ such that $V(i) = \{ w \}$. The following fact is quite useful for the general completeness theorem for pure axioms [5, Lemma 7.22].

Fact 3.11 *Let $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$ be a named model and φ a pure formula. Suppose that for all substitution σ, $\mathfrak{M} \Vdash \varphi\sigma$. Then, $\mathfrak{F} \Vdash \varphi$.*

We expand the language $\mathcal{H}(\blacklozenge)$ with the set J of new nominals as there are formulas in the language. $\mathcal{H}(\blacklozenge)[J]$ denotes the expanded language. Let Λ be a $\mathbf{K}^+_{\mathcal{H}(\blacklozenge)}$-logic.

Definition 3.12 Γ of $\mathcal{H}(\blacklozenge)[J]$ is *a named Λ-maximal consistent set* if Γ is a Λ-maximal consistent set and Γ satisfies the following: for any PF m in $\mathcal{H}(\blacklozenge)[J]$, $m(\top) \in \Gamma \implies [m(j) \in \Gamma$ for some $j \in J$ which does not appear in $m]$.

Lemma 3.13 *Every Λ-consistent set Γ of $\mathcal{H}(\blacklozenge)$ can be extended to a named Λ-maximal consistent set Γ^+ of $\mathcal{H}(\blacklozenge)[J]$.*

Note that adding the new nominals J preserves consistency. For the proof, see, e.g., [22, Lemma 5.3.9].

Definition 3.14 Let Γ be a named Λ-maximal consistent set. Define Nom' as $\{ j \in \mathsf{Nom} \cup J \mid m(j) \in \Gamma$ for some PF $m \}$. We define the equivalence relation $\sim$ on Nom' as follows: $m(j \wedge k) \in \Gamma$ for some PF m. The quotient $\mathsf{Nom}'/\sim$ is defined as usual and $[j] \in \mathsf{Nom}'/\sim$ denotes the equivalence class of j. $\mathfrak{F}^\Lambda_\Gamma = \langle W, R, S \rangle$ is defined as follows: $W = \mathsf{Nom}'/\sim$; $[j]R[k] \iff m(j \wedge \Diamond k) \in \Gamma$ for some PF m; $[j]S[k] \iff m(j \wedge \blacklozenge k) \in \Gamma$ for some PF m.

Since $\vdash_{\mathbf{K}_{\mathcal{H}(\blacklozenge)}} \blacklozenge i \leftrightarrow \Diamond i \wedge \neg i$, we can easily prove the following:

Lemma 3.15 *In $\mathfrak{F}^\Lambda_\Gamma$, $(R \cap \neq) = S$ holds.*

Lemma 3.16 *Fix $\star \notin W$. Define $\mathfrak{M}^\star = \langle W^\star, R^\star, S^\star, V^\star \rangle$ as follows: $W^\star = W \cup \{ \star \}$; $R^\star = R$; $S^\star = S$;*

$$V^\star(a) = \begin{cases} \{ [j] \in W \mid m(a \wedge j) \in \Gamma \text{ for some PF } m \} & \textit{if } a \in \mathsf{Prop} \cup \mathsf{Nom}' \\ \{ \star \} & \textit{if } a \in (\mathsf{Nom} \cup J) \setminus \mathsf{Nom}' \end{cases}$$

Then, (i) *$V^\star$ is a valuation from $\mathsf{Nom} \cup J$ to $\mathcal{P}(W^\star)$.* (ii) *$(R^\star \cap \neq) = S^\star$;* (iii) *for any $[j] \in W$ and any φ of $\mathcal{H}(\blacklozenge)[J]$, $\mathfrak{M}^\star, [j] \Vdash \varphi \iff [m(j \wedge \varphi) \in \Gamma$ for some PF $m]$.*

(i) is obvious by the construction and Fact 3.3 (ii). Clearly (ii) holds by Lemma 3.15. We can prove (iii) by induction on φ. Observe that $\mathfrak{M}^\star$ is a named model.

Theorem 3.17 *Let λ be a pure formula and φ a formula of $\mathcal{H}(\blacklozenge)$. Then, $\mathsf{K}_\lambda \Vdash \varphi \Longrightarrow \vdash_{\mathbf{K}^+_{\mathcal{H}(\blacklozenge)}\{\lambda\}} \varphi$, where K_λ is the class of frame defined by λ* [3] *.*

Proof. Suppose that $\neg\varphi$ is $\mathbf{K}^+_{\mathcal{H}(\blacklozenge)}\{\lambda\}$-consistent. $\neg\varphi \in \Gamma$ holds for some named $\mathbf{K}^+_{\mathcal{H}(\blacklozenge)}\{\lambda\}$-maximal consistent set Γ. Since $\top \in \Gamma$ and Γ is named, $j \in \Gamma$ for some $j \in J$. Thus, $j \wedge \neg\varphi \in \Gamma$. By Lemma 3.16 (iii), $\mathfrak{M}^\star, [j] \Vdash \neg\varphi$ hence $\mathfrak{F}^\star = \langle W^\star, R^\star, S^\star \rangle \nVdash \varphi$ (Note that $\mathfrak{M}^\star$ is a $\mathcal{H}(\blacklozenge)$-model). For any substitution σ, $\vdash_{\mathbf{K}^+_{\mathcal{H}(\blacklozenge)}\{\lambda\}} \lambda\sigma$, which implies $\mathfrak{M}^\star \Vdash \lambda$ and so $\mathfrak{F}^\star \Vdash \lambda$ by Fact 3.11. Since λ defines K_λ, $\mathfrak{F}^\star \in \mathsf{K}_\lambda$, which implies $\mathsf{K}_\lambda \nVdash \varphi$. □

3.3 Finite Frame Property and Decidability

Let $\mathsf{K}_{\text{finite}}$ be the class of all finite $\mathcal{H}(\blacklozenge)$-frames. In this subsection, we prove that $\mathbf{K}_{\mathcal{H}(\blacklozenge)}$ and $\mathbf{K}^+_{\mathcal{H}(\blacklozenge)}$ are complete with respect to $\mathsf{K}_{\text{finite}}$, i.e., we prove the finite frame property of $\mathbf{K}_{\mathcal{H}(\blacklozenge)}$ and $\mathbf{K}^+_{\mathcal{H}(\blacklozenge)}$.

Definition 3.18 Let $\mathfrak{M} = \langle W, R, S, V \rangle$ be a bimodal model and Σ a subformula-closed set of formulas. $w \sim_\Sigma v \iff [\mathfrak{M}, w \Vdash \varphi \iff \mathfrak{M}, v \Vdash \varphi]$ for every $\varphi \in \Sigma$. The quotient $W/\sim_\Sigma$ is defined as usual and $[w] \in W/\sim_\Sigma$ denotes the equivalence class of w. A bimodal model $\mathfrak{M}^f_\Sigma = \langle W^f, R^f, S^f, V^f \rangle$ is called *a bimodal finest filtration of $\mathfrak{M}$ through Σ* if $W^f = W/\sim_\Sigma$, $[w]R^f[v] \iff \exists w' \in [w]\, \exists v' \in [v]\, w'Rv'$, $[w]S^f[v] \iff \exists w' \in [w]\, \exists v' \in [v]\, w'Sv'$ and $V^f : \mathsf{Prop} \cup \mathsf{Nom} \to \mathcal{P}(W^f)$ is a function satisfying,

$$V^f(a) = \begin{cases} \{\, [w] \in W^f \mid w \in V(a) \,\} & \text{if } a \in \mathsf{Prop} \cup (\mathsf{Nom} \cap \Sigma) \\ \{\, [v] \,\} & \text{if } a \in \mathsf{Nom} \setminus \Sigma \end{cases}$$

where $[v]$ is an arbitrarily fixed element of W^f since $W^f \neq \emptyset$.

Note that V^f is a valuation (see [3, Lemma 2.4.2]). The following fact holds (see, e.g., [3, Theorem 2.4.1]) since a finest filtration is a special case of the usual filtration.

Fact 3.19 *Suppose that Σ is subformula closed. For any $\varphi \in \Sigma$ and any w in $\mathfrak{M}$, $\mathfrak{M}, w \Vdash \varphi \iff \mathfrak{M}^f_\Sigma, [w] \Vdash \varphi$.*

Lemma 3.20 *(i) $(R \cap \neq) \subset S$ implies $(R^f \cap \neq) \subset S^f$, (ii) $S \subset R$ implies $S^f \subset R^f$.*

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[3] One of the referees of the present paper comments on (the preliminary version of) Theorem 3.17 as follows: it seems to follow directly from the existing general completeness results for pure axioms since $\blacklozenge i \leftrightarrow \Diamond i \wedge \neg i$ is pure. If we remove ($\blacksquare$2), (M1), (M2) from the axiomatization of $\mathbf{K}^+_{\mathcal{H}(\blacklozenge)}$ and add $\blacklozenge i \leftrightarrow \Diamond i \wedge \neg i$ and ($\blacksquare$-rule) to the axiomatization, then we can immediately get the desired results. Note, however, that in the definition of $\mathbf{K}^+_{\mathcal{H}(\blacklozenge)}$-logic, we need not require ($\blacksquare$-rule), which is derivable from (M2) and ($\Box$-rule).

Proof. Clearly, (ii) holds. To prove (i), suppose $[w](R^f\cap \neq)[w']$, i.e., vRv' for some $v \in [w]$ and some $v' \in [w']$. Since $[w] \neq [w']$, $v \neq v'$ whence $v(R\cap \neq)v'$. By $(R\cap \neq) \subset S$, vSv' whence $[w]S^f[w']$. □

Definition 3.21 A valuation V on $\mathfrak{F}$ is S-*irreflexivity respecting* if $w \in V(i)$ is S-irreflexive for all $i \in \mathsf{Nom}$. A bimodal model $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$ is S-*irreflexivity respecting* if V is S-irreflexivity respecting.

Lemma 3.22 *Let $\mathfrak{M} = \langle W, R, S, V \rangle$ be an S-irreflexivity respecting bimodal model and Σ a subformula closed finite set. Then, there exists finite Σ' such that $\Sigma \subset \Sigma'$ and a bimodal finest filtration $\mathfrak{M}^f_{\Sigma'}$ is S-irreflexivity respecting.*

Proof. Define Σ_1 as follows: $\Sigma_1 = \Sigma$ (if $\Sigma \cap \mathsf{Nom} \neq \emptyset$); $\Sigma_1 = \Sigma \cup \{ j \}$ where j is the fixed element of Nom (if $\Sigma \cap \mathsf{Nom} = \emptyset$). Fix $j_0 \in \Sigma_1 \cap \mathsf{Nom}$. Take the bimodal finest filtration $\mathfrak{M}^f_{\Sigma_1}$ with $V^f(k) = V^f(j_0)$ for all nominals $k \notin \Sigma_1$.

Assume that $i \in \mathsf{Nom} \cap \Sigma_1$. $\mathfrak{M}, w \Vdash i$ holds for some w. Since $\mathfrak{M}$ is an S-irreflexivity respecting model, wSw fails. It follows from $i \in \Sigma_1$ that $\mathfrak{M}^f_{\Sigma_1}, [w] \Vdash i$ by Fact 3.19. Note that $[w] = V(i)$. Since $\mathfrak{M}^f_{\Sigma'}$ is the finest filtration and wSw fails, $[w]S^f[w]$ fails. Assume that $i \in \mathsf{Nom} \setminus \Sigma_1$. Let $V^f(i) = \{ [v] \}$. Since $V^f(i) = V^f(j_0)$, S^f-irreflexivity of $[v]$ holds. Thus, $\mathfrak{M}^f_{\Sigma_1}$ is S-irreflexivity respecting. □

Theorem 3.23 *Let φ be a formula of $\mathcal{H}(\blacklozenge)$.* $\mathsf{K}_{\text{finite}} \Vdash \varphi \implies \vdash_{\mathbf{K}^+_{\mathcal{H}(\blacklozenge)}} \varphi$.

Proof. Suppose that $\neg\varphi$ is $\mathbf{K}_{\mathcal{H}(\blacklozenge)}$-consistent. Construct the $\mathfrak{M}^\star$ as in the proof of Theorem 3.17. Let Σ be the set of all subformulas of $\neg\varphi$. By Lemmas 3.15, 3.16 and 3.22, there exists finite Σ' such that $\Sigma \subset \Sigma'$ and $\mathfrak{M}^{\star f}_{\Sigma'}$ is finite and S-irreflexivity respecting, and satisfies $(R^f\cap \neq) \subset S^f$ and $S^f \subset R^f$. Apply Proposition 2.5 as in the proof of Theorem 3.10. Note that the construction in the proof of Proposition 2.5 does not change the finite cardinality. □

Theorem 3.24 *Let φ be a formula of $\mathcal{H}(\blacklozenge)$.* $\mathsf{K}_{\text{finite}} \Vdash \varphi \implies \vdash_{\mathbf{K}_{\mathcal{H}(\blacklozenge)}} \varphi$.

Proof. Suppose that $\neg\varphi$ is $\mathbf{K}_{\mathcal{H}(\blacklozenge)}$-consistent. Construct the $\mathfrak{M}^\star$ as in the proof of Theorem 3.10. Apply the similar argument as in the proof of Theorem 3.23. □

Corollary 3.25 $\mathbf{K}_{\mathcal{H}(\blacklozenge)}$ *and* $\mathbf{K}^+_{\mathcal{H}(\blacklozenge)}$ *are decidable.* [4]

4 Frame Definablity

4.1 Relations between nominals and the irreflexive modality

Recall that Γ defines a class F of frames if, for all frames $\mathfrak{F}$, $\mathfrak{F} \Vdash \Gamma \iff \mathfrak{F} \in \mathsf{F}$.

Fact 4.1 *In $\mathcal{H}(\blacklozenge)$, the following define the corresponding properties of R. (i) $\Diamond p \to \blacklozenge p$: irreflexivity, (ii) $\Diamond\Diamond p \to \blacklozenge p$: strict partial ordering, (iii) $\blacklozenge\blacklozenge p \to \blacklozenge p$: antisymmetry and transitivity, (iv) $(p \to \Diamond p) \wedge (\blacklozenge\blacklozenge p \to \blacklozenge p)$: partial ordering.*

[4] Complexity of our decidability remains open. One of the referees suggests that it may be in PSPACE.

As for (iii), note that S-transitivity is equivalent in the $\mathcal{H}(\blacklozenge)$-frames to the conjunction of R-transitivity and R-antisymmetry. When adding $\blacklozenge$ to standard modal language without nominals, we cannot define antisymmetry [20] [5] [6]. We can, on the other hand, define it in $\mathcal{H}$ by $(i \wedge \Diamond(j \wedge \Diamond i)) \to j$ [22, p.45]. Thus, nominals increase the frame-expressive power of the irreflexive modality.

In this subsection, we deal with the following question:

(Q) Does the irreflexive modality $\blacklozenge$ increase the frame-expressive power of hybrid language?

Some uses of $\blacklozenge$ are inessential. For example, $\blacklozenge(i \wedge \varphi)$ and $\varphi \vee \blacklozenge\varphi$ can equally well be written (even in the level of pointed models) without using $\blacklozenge$, as $\Diamond(i \wedge \varphi) \wedge \neg i$ and $\varphi \vee \Diamond\varphi$, respectively. In the level of frames, the following proposition demonstrates another inessential use of $\blacklozenge$.

Proposition 4.2 *(1) $\blacklozenge\top$ (of $\mathcal{H}(\blacklozenge)$) defines $\forall x \exists y [xRy$ and $x \neq y]$. (2) $i \to \Diamond\neg i$ (of $\mathcal{H}$) also defines the same property.*

As for (1), observe that $\blacklozenge\top$ defines the seriality of S. For (2), the standard translation for hybrid languages, e.g., [5, Exercises 7.3.1] tells us the correspondence.

We have not obtained the full answer to (Q) yet. We, however, will give two partial answers to this. First answer is that $\blacklozenge$ increases the expressivity of $\mathcal{H}(\mathrm{E})$, where $\mathcal{H}(\mathrm{E})$ means $\mathcal{H}$ with the global modality E, whose accessibility relation is the total relation on W. Second answer is that, with respect to the *pure formulas*, $\blacklozenge$ increases the definability of $\mathcal{H}$.

4.1.1 A Comparison in the Setting of Adding the Global Modality

Here, we consider the extended languages $\mathcal{H}(\blacklozenge, \mathrm{E})$ and $\mathcal{H}(\mathrm{E})$. Recall that we define $\mathfrak{M}, w \Vdash \mathrm{E}\varphi \iff \mathfrak{M}, w' \Vdash \varphi$ for some w'. We can easily prove the following:

Proposition 4.3 $\mathrm{E}\blacklozenge\top$ *defines $\exists x \exists y [xRy$ and $x \neq y]$.*

$\exists x \exists y [xRy$ and $x \neq y]$, however, is undefinable in $\mathcal{H}(\mathrm{E})$ [7]. To show this, we introduce the frame construction, *ultrafilter morphic images* [22, Definition 4.2.5].

Definition 4.4 (1) Given a binary relation R on a set W, we define a unary operation m_R on $\mathcal{P}(W)$: $m_R(X) = \{ w \in W \mid R[w] \cap X \neq \emptyset \}$. (2) The *bimodal ultrafilter extension* $\mathfrak{ue}\,\mathfrak{F}$ of $\mathfrak{F} = \langle W, R, S \rangle$ is the frame $\langle W^{\mathfrak{ue}}, R^{\mathfrak{ue}}, S^{\mathfrak{ue}} \rangle$, where $W^{\mathfrak{ue}}$ is the set of (principal and non-principal) ultrafilters over W, $uR^{\mathfrak{ue}}u' \iff$ for any $X \subset W$, $X \in u'$ implies $m_R(X) \in u$, and $S^{\mathfrak{ue}}$ defined similarly by m_S. *The unimodal ultrafilter extension* of $\mathfrak{F} = \langle W, R \rangle$ is $\langle W^{\mathfrak{ue}}, R^{\mathfrak{ue}} \rangle$.

Definition 4.5 Let $\mathfrak{F}$ and $\mathfrak{G}$ be (unimodal) frames. $\mathfrak{G}$ is a *bimodal ultrafilter mor-*

[5] Consider $\mathfrak{F} = \langle \{a, b\}, \{a, b\}^2 \rangle$ and $\mathfrak{F}' = \langle \omega, \{ \langle m, m \rangle, \langle m, m+1 \rangle \mid m \in \omega \} \rangle$. Note that $\mathfrak{F}'$ is antisymmetric and $\mathfrak{F}$ is not. Define f by $f(2m) = a$, $f(2m+1) = b$ for any $m \in \omega$. Then, we can prove that f is a surjective bimodal p-morphism (in the sense of standard modal language) $f : \mathfrak{F}' \to \mathfrak{F}$, which violates antisymmetry.

[6] Even if we add the global modality E in addition to $\blacklozenge$, we cannot define it. This is because the surjective bimodal p-morphism with respect to R and $(R \cap \neq)$ preserve frame-validity of the irreflexive modal language with the global modality.

[7] Contrary to $\mathrm{E}\blacklozenge\top$, $\mathrm{E}(i \to \Diamond\neg i)$ does not define $\exists x \exists y [xRy$ and $x \neq y]$. $\mathrm{E}(i \to \Diamond\neg i)$ defines $\forall x \exists y . x \neq y$.

phic image of $\mathfrak{F}$ if there is a surjective bimodal p-morphism $f : \mathfrak{F} \to \mathfrak{ue}\,\mathfrak{G}$ such that $|f^{-1}[\{u\}]| = 1$ for all principal ultrafilters u in $\mathfrak{ue}\,\mathfrak{G}$. *Unimodal ultrafilter morphic images* are defined similarly.

It is known that validity of $\mathcal{H}(\mathrm{E})$-formulas is preserved under taking unimodal ultrafilter morphic images [22, Proposition 4.2.6].

Proposition 4.6 *$\exists x \exists y\,[xRy$ and $x \neq y]$ is undefinable in $\mathcal{H}(\mathrm{E})$.*

We prove this by the example used in the proof of [22, Proposition 4.2.7].

Proof. Take any frame $\mathfrak{F} = \langle W, \mathrm{id}_W \rangle$ with $|W| \geq \omega$, where id_W is the identity relation on W. Consider the ultrafilter extension $\langle W^{\mathfrak{ue}}, (\mathrm{id}_W)^{\mathfrak{ue}} \rangle$ of $\mathfrak{F}$. Note that $W^{\mathfrak{ue}}$ contains non-principal ultrafilters since $|W| \geq \omega$. It is easy to prove that $(\mathrm{id}_W)^{\mathfrak{ue}} = \mathrm{id}_{W^{\mathfrak{ue}}}$.

Next, we 'bulldoze' the non-principal ultrafilters in $W^{\mathfrak{ue}}$ as follows: Let P the set of all principal ultrafilters and NP the set of all non-principal ultrafilters. Define $\mathfrak{G} = \langle W', R' \rangle$ where $W' = P \cup (NP \times \{0,1\})$ and $R' = \mathrm{id}_{W'} \cup \{\langle\langle u,0\rangle,\langle u,1\rangle\rangle,\langle\langle u,1\rangle,\langle u,0\rangle\rangle \mid u \in NP\}$. Consider $f : W' \to W^{\mathfrak{ue}}$ as $u \in P \mapsto u$ and $\langle u,i\rangle \in NP \times \{0,1\} \mapsto u$. Then, we can prove that $\mathfrak{F}$ is an ultrafilter morphic image of $\mathfrak{G}$. $\mathfrak{G}$ satisfies $\exists x \exists y\,[xRy$ and $x \neq y]$ but $\mathfrak{F}$ does not. Thus, by the validity preservation under ultrafilter morphic images [22, Proposition 4.2.6], we can conclude that $\exists x \exists y\,[xRy$ and $x \neq y]$ is undefinable in $\mathcal{H}(\mathrm{E})$. □

Thus, $\mathcal{H}(\blacklozenge, \mathrm{E})$ is more expressive than $\mathcal{H}(\mathrm{E})$ with respect to the frame definability. We can also conclude that $\mathcal{M}(\blacklozenge, \mathrm{E})$, $\mathcal{M}(\blacklozenge)$ with E, is the different extension from $\mathcal{H}(\mathrm{E})$ (i.e., by the Gargov and Goranko Translation [12], $\mathcal{M}(\mathrm{D})$, the unimodal language with the *difference operator*, associated with the inequality $\neq$).

4.1.2 A Comparison with respect to the Pure Formulas

Recall that a formula φ of $\mathcal{H}(\blacklozenge)$ ($\mathcal{H}$ or $\mathcal{H}(\mathrm{E})$) is called *pure* if it contains no proposition letters.

The following proposition holds clearly:

Proposition 4.7 *(1) $\blacksquare\bot$ defines $\forall x \forall y\,[xRy \Rightarrow x = y]$. (2) $\mathrm{E}\blacksquare\bot$ (of $\mathcal{H}(\blacklozenge, \mathrm{E})$) defines $\exists x \forall y\,[xRy \Rightarrow x = y]$. (3) $\Diamond\blacksquare\bot$ defines $\forall x \exists y\,[xRy$ and $\forall z\,[yRz \Rightarrow y = z]]$.*

In $\mathcal{H}(\blacklozenge)$ (even in $\mathcal{M}(\blacklozenge)$), we can define the atomicity by $\Diamond\blacksquare\bot$ without the assumption of transitivity.

We will prove that the properties in (2), (3) of Proposition 4.7 are not definable by pure formulas of $\mathcal{H}(\mathrm{E})$. To prove this, we introduce the following new frame construction:

Definition 4.8 Let $\mathfrak{F}$ and $\mathfrak{G}$ be (unimodal) frames. $\mathfrak{G}$ is an *unimodal ultrafilter morphic domain* of $\mathfrak{F}$ if there is a surjective unimodal p-morphism $f : \mathfrak{ue}\,\mathfrak{G} \to \mathfrak{F}$ such that $f \restriction P$ is the identity function on P, where P is the set of all principal ultrafilters u in $\mathfrak{ue}\,\mathfrak{G}$.

Ultrafilter morphic domains are different from ultrafilter morphic images in the direction of surjective unimodal p-morphisms.

It is easy to show the following Lemma. Observe, however, that the surjectiveness of f is needed for the case where φ is $\mathrm{E}\psi$.

Lemma 4.9 *Let $\mathfrak{M}$ and $\mathfrak{N}$ be (unimodal) models. Let $f : \mathfrak{M} \to \mathfrak{N}$ be a surjective unimodal p-morphism. Then, for any formula φ in $\mathcal{H}(\mathrm{E})$ and any $w \in |\mathfrak{M}|$, $\mathfrak{M}, w \Vdash \varphi \iff \mathfrak{N}, f(v) \Vdash \varphi$.*

Proposition 4.10 *The validity of pure formulas in $\mathcal{H}(\mathrm{E})$ is preserved under taking ultrafilter morphic domains.*

Proof. Let $f : \mathfrak{ue}\,\mathfrak{G} \to \mathfrak{F}$ be a surjective unimodal p-morphism such that f is injective with respect to the set of all principal ultrafilters. Let φ be a pure formula in $\mathcal{H}(\mathrm{E})$. We prove the contraposition. Suppose that $\mathfrak{G} \nVdash \varphi$, which implies $\langle \mathfrak{G}, V \rangle, w \nVdash \varphi$ for some V and $w \in |\mathfrak{G}|$. From this, we can prove that $\langle \mathfrak{ue}\,\mathfrak{G}, V^{\mathfrak{ue}} \rangle, u_w \nVdash \varphi$ where $u_w = \{ X \in \mathcal{P}(|\mathfrak{G}|) \mid w \in X \}$ (see, e.g., the second paragraph in the proof of [22, Proposition 4.2.6]). Observe that this inference holds even for *any* formulas in $\mathcal{H}(\mathrm{E})$.

Then, we define a valuation V' (for Nom) on $\mathfrak{F}$ as follows: $V'(i) = f[V^{\mathfrak{ue}}(i)]$. This is a valuation since, for any $i \in \mathsf{Nom}$, the element of $V^{\mathfrak{ue}}(i)$ is the principal ultrafilter u_v where $V(i) = \{ v \}$ and so $|V'(i)| = |f[V^{\mathfrak{ue}}(i)]| = 1$ by the injectiveness of f with respect to the set of all principal ultrafilters. Clearly, for any u in $\mathfrak{ue}\,\mathfrak{G}$ and any $i \in \mathsf{Nom}$, $u \in V^{\mathfrak{ue}}(i) \iff f(u) \in V'(i)$. Since $\langle \mathfrak{ue}\,\mathfrak{G}, V^{\mathfrak{ue}} \rangle, u_w \nVdash \varphi$ and φ is pure, we deduce, by Lemma 4.9, that $\langle \mathfrak{F}, V' \rangle, f(u_w) \nVdash \varphi$ hence $\mathfrak{F} \nVdash \varphi$. □

Proposition 4.11 *(1) $\exists x \forall y [xRy \Rightarrow x = y]$ and (2) $\forall x \exists y [xRy$ and $\forall z [yRz \Rightarrow y = z]]$ are not definable by pure formulas in $\mathcal{H}(\mathrm{E})$, i.e., in $\mathcal{H}$.*

Proof. Consider the frame $\mathfrak{N} = \langle \mathbb{N}, < \rangle$, the natural numbers in their usual ordering. Take the ultrafilter extension $\mathfrak{ue}\,\mathfrak{N} = \langle \mathbb{N}^{\mathfrak{ue}}, <^{\mathfrak{ue}} \rangle$ of $\mathfrak{N}$. We can divided $\mathbb{N}^{\mathfrak{ue}}$ into two disjoint parts, P, the set of all principal ultrafilters, and NP, the set of all non-principal ultrafilters. As pointed out in [5, Example 2.58], this frame consists of an isomorphic copy P of natural numbers, followed by an uncountable cluster containing NP. Observe that for any pair u, u' of ultrafilters, if u' is non-principal, then $u <^{\mathfrak{ue}} u'$ [5, Example 2.58].

Construct $\mathfrak{G} = \langle W', R' \rangle$ as follows: $W' = P \cup \{ * \}$, $R' = \{ \langle u, u' \rangle \in P^2 \mid u <^{\mathfrak{ue}} u' \} \cup P \times \{ * \} \cup \{ \langle *, * \rangle \}$. Define $f : \mathbb{N}^{\mathfrak{ue}} \to W'$ as $u \in P \mapsto u$ and $u \in NP \mapsto *$. Then, it is easy to see that f is a surjective unimodal p-morphism and f is an identity function with respect to P. Thus, $\mathfrak{N}$ is an ultrafilter morphic domain of $\mathfrak{G}$.

$\mathfrak{G}$ satisfies (1), (2) but $\mathfrak{F}$ does not. Thus, by Proposition 4.10, we conclude that (1), (2) are undefinable in $\mathcal{H}(\mathrm{E})$. □

Therefore, the set of pure formulas of $\mathcal{H}(\blacklozenge)$ is more expressive than that of $\mathcal{H}$ with respect to the frame definability.

4.2 Goldblatt-Thomason-style Characterization

In this subsection, we use some notions from first-order model theory, e.g., submodel, elementary embedding, ω-saturatedness. The reader unfamiliar with them can refer to, e.g., [7]. In addition, we drop the usual assumption that Nom and Prop of $\mathcal{H}(\blacklozenge)$ have countable members.

The notion of *unimodal* generated subframes are defined similar to bimodal generated subframes without using the clauses related to the second accessibility relation S.

Proposition 4.12 *$\mathcal{H}(\blacklozenge)$- (or $\mathcal{H}$-) definable classes are closed under unimodal generated subframes.*

Proof. See [22, Proposition 4.2.1] or [3, Theorem 3.2.1]. Adding $\blacklozenge$ does not change the argument. $\square$

Definition 4.13 $\mathfrak{F}$ is an *hybrid amalgamation* of $\{\,\mathfrak{G}_k \,|\, k \in K\,\}$ if for any (unimodal) point-generated subframe $\mathfrak{F}_w$ of $\mathfrak{F}$ there exists $k \in K$ such that $\mathfrak{F}_w$ is a proper generated subframe of $\mathfrak{G}_k$, i.e., $\mathfrak{F}_w$ is a (unimodal) generated subframe of $\mathfrak{G}_k$ and $\mathfrak{F}_w \neq \mathfrak{G}_k$.

Proposition 4.14 *$\mathcal{H}(\blacklozenge)$- (or $\mathcal{H}$-) definable classes are closed under hybrid amalgamations.*

Proof. See [22, Proposition 4.2.2]. Adding $\blacklozenge$ does not change the argument. $\square$

With respect to ultrafilter morphic images, we can prove the following similarly to the unimodal case [22, Proposition 4.2.6]:

Proposition 4.15 *Validity (on $\mathcal{H}(\blacklozenge)$-frames) of $\mathcal{H}(\blacklozenge)$-formulas is preserved under taking bimodal ultrafilter morphic images.*

For $\mathcal{H}(\blacklozenge)$, we can give the following characterization:

Theorem 4.16 *An elementary frame class F is definable in $\mathcal{H}(\blacklozenge)$ iff F is closed under taking (i) bimodal ultrafilter morphic images, (ii) unimodally generated subframes, and (iii) hybrid amalgamations.*

Proof. We will prove that the right-to-left-direction. Let $\mathrm{Th}(\mathsf{F}) = \{\,\varphi \,|\, \mathsf{F} \Vdash \varphi\,\}$. It suffices to prove that, for any $\mathcal{H}(\blacklozenge)$-frame $\mathfrak{F}$, $\mathfrak{F} \Vdash \mathrm{Th}(\mathsf{F}) \implies \mathfrak{F} \in \mathsf{F}$.

Suppose that $\mathfrak{F} \Vdash \mathrm{Th}(\mathsf{F})$. Let us assume that $\mathfrak{F}$ is unimodally point generated (Otherwise, the proof that $\mathfrak{F} \in \mathsf{F}$ is similar to the proof of [22, Theorem 4.3.4], where we need the closure condition (iii). Note that Lemmas needed there hold even in $\mathcal{H}(\blacklozenge)$). Let w be the root of $\mathfrak{F}$.

We can suppose that $\mathsf{Prop} \cup \mathsf{Nom}$ contains a proposition letter p_X and nominal i_x for each $X \subset |\,\mathfrak{F}\,|$ and each $x \in |\,\mathfrak{F}\,|$, respectively. Let $\mathfrak{M} = \langle\,\mathfrak{F}, V_0\,\rangle$, where V_0 is a natural valuation with $V_0(p_X) = X$ and $V_0(i_x) = \{\,x\,\}$.

Let Δ be the set consisting of the following, for all $X, Y \subset |\mathfrak{F}|$ and $x \in |\,\mathfrak{F}\,|$,

$$p_{|\mathfrak{F}|\setminus X} \leftrightarrow \neg\, p_X;\; p_{X \cap Y} \leftrightarrow p_{X \wedge Y};\; i_x \leftrightarrow p_{\{\,x\,\}};\; p_{m_R(X)} \leftrightarrow \lozenge p_X;\; p_{m_S(X)} \leftrightarrow \blacklozenge p_X.$$

Let $\Delta_{\mathfrak{F}}$ be the following set:

$$\{\, i_w \,\} \cup \{\, \Box_1 \cdots \Box_m \varphi \mid \varphi \in \Delta \text{ and } m \in \omega \text{ and } \Box_i \in \{\, \Box, \blacksquare \,\} \text{ for any } i \;\}.$$

It is easy to see that $\Delta_{\mathfrak{F}}$ is satisfiable on $\mathfrak{F}$ at w under the natural valuations. Then we can prove the following claim (for the proof, see, e.g., [22, p.59, Claim 1]): $\Delta_{\mathfrak{F}}$ is satisfiable in $\mathfrak{G}$ for some $\mathfrak{G} \in \mathsf{F}$.

By the claim, we may infer that $\langle\, \mathfrak{G}, V \,\rangle, v \Vdash \Delta$ for some valuation V and some v in $\mathfrak{G}$ for some $\mathfrak{G} \in \mathsf{F}$. By the construction, all nominals of the set $\{\, i_x \mid x \in |\,\mathfrak{F}\,|\,\}$ denote a point in $\langle\, \mathfrak{G}, V \,\rangle$ that is reachable from v. Thus, we can think of V as a valuation for the frame $\mathfrak{G}_v$. In this way, we can consider the point-generated 'submodel' $\langle\, \mathfrak{G}_v, V \,\rangle$ of $\langle\, \mathfrak{G}, V \,\rangle$ (For the definition of *submodel*, see, e.g., [5, Definition 2.5]). Then we can prove that $\langle\, \mathfrak{G}_v, V \,\rangle, x \Vdash \Delta$ for any $x \in |\mathfrak{G}_v|$ and $\langle\, \mathfrak{G}_v, V \,\rangle, v \Vdash p_X$ for all $X \subset |\,\mathfrak{F}\,|$ with $w \in X$.

Let $\langle\, \mathfrak{G}_v^*, V^* \,\rangle$ be an ω-saturated elementary extension of $\langle\, \mathfrak{G}_v, V \,\rangle$. It follows that $\langle\, \mathfrak{G}_v^*, V^* \,\rangle, x \Vdash \Delta$ for any $x \in |\mathfrak{G}_v^*|$ and $\langle\, \mathfrak{G}_v^*, V^* \,\rangle, v^* \Vdash p_X$ for all $X \subset |\,\mathfrak{F}_w\,|$ with $w \in X$ where v^* is the corresponding element to v, since the satisfaction relation is elementary.

Claim 4.17 *$\mathfrak{F}$ is a bimodal ultrafilter morphic image of $\mathfrak{G}_v^*$.*

(Proof of Claim 4.17) For any s in $\mathfrak{G}_v^*$, $\{\, X \subset |\,\mathfrak{F}\,| \mid \langle\, \mathfrak{G}_v^*, V^* \,\rangle, s \Vdash p_X \,\}$ is an ultrafilter. This defines the mapping f from $|\,\mathfrak{G}_v^*\,|$ to $|\,\mathfrak{ue}\,\mathfrak{F}\,|$. We can prove that f is a surjective bimodal p-morphism and satisfies the condition of ultrafilter morphic image (For the detailed proof of these, see [22, Claim 2 in the proof of Theorem 4.3.4]). **(QED of Claim 4.17)**

Thus we can conclude that $\mathfrak{F} \in \mathsf{F}$ by $\mathfrak{G} \in \mathsf{F}$, by the elementariness of F and the closure conditions (i) and (ii). $\Box$

Thus, we can capture the precise frame expressivity of $\mathcal{H}(\blacklozenge)$ in terms of frame constructions.

5 Concluding Remarks

One of the referees of the present paper posed two interesting further directions. Both of them are concerned with the semantical relation (with respect to the frame definability) between the irreflexive modality (especially, $\mathcal{M}(\blacklozenge)$) and nominals (especially, $\mathcal{H}$). First one is a syntactical investigation of $\mathcal{H}$ to simulate (some parts of) $\mathcal{M}(\blacklozenge)$. For example, as the referee mentions, can we simulate $\mathcal{M}(\blacklozenge)$ by restricting nominals to occur only in specific conditions, e.g., in the scope of at most one modal operator? Second one is to generalize $\mathcal{M}(\blacklozenge)$ to the PDL-style language using program composition, union, and *intersection-with-inequality* (but not with the star) and to specify the scope for the idea of the irreflexive modality. These two directions will make our understanding of the irreflexive modality and nominals deeper.

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References

[1] Areces, C. and B. ten Cate, *Hybrid logics*, in: P. Blackburn, F. Wolter and J. van Benthem, editors, *Handbook of Modal Logics*, 2005 (In preparation).
URL http://www.loria.fr/~areces/content/papers/files/hml-arecestencate.pdf

[2] Bezhanishvili, G. and J. v. Benthem, *Modal logics of space*, Prepublication (PP) Series PP-2006-08, ILLC Publications (2006).
URL http://www.illc.uva.nl/Publications/ResearchReports/PP-2006-08.text.pdf

[3] Blackburn, P., "Nominal Tense Logic and Other Sorted Intensional Frameworks," Ph.D. thesis, Centre for Cognitive Science, University of Edinburgh (1990).

[4] Blackburn, P., *Nominal tense logic*, Notre Dame Journal of Formal Logic **34** (1993), pp. 56–83.

[5] Blackburn, P., M. de Rijke and Y. Venema, "Modal Logic," Cambridge Tracts in Theoretical Computer Science, Cambridge University Press, Cambridge, 2001.

[6] Blackburn, P. and J. Seligman, *Hybrid language*, Journal of Logic, Language and Information **4** (1995), pp. 251–72.

[7] Chang, C. C. and H. J. Keisler, "Model Theory," North-Holland Publishing Company, Amsterdam, 1990, 3 edition.

[8] de Rijke, M., *The modal logic of inequality.*, Journal of Symbolic Logic **57** (1992), pp. 56–84.

[9] Esakia, L., *Weak transitivity - restitution*, Study in Logic **8** (2001), pp. 244–54.

[10] Gabbay, D. M., *An irreflexivity lemma with applications to axiomatizations of conditions on tense frames*, in: U. Monnich, editor, *Aspects of Philosophical Logic*, Reidel, Dordrecht, 1981 pp. 91–117.

[11] Gabbay, D. M., I. M. Hodkinson and M. Reynolds, "Temporal Logic: Mathematical Foundations and Computational Aspects," Oxford Logic Guides **1**, Oxford University Press, 1994.

[12] Gargov, G. and V. Goranko, *Modal logic with names*, Journal of Philosophical Logic **22** (1993), pp. 607–36.

[13] Gargov, G., S. Passy and T. Tinchev, *Modal environment for boolean speculations (preliminary report)*, in: D. Skordev, editor, *Mathematical Logic and its Applications. Proceedings of the Summer School and Conference dedicated to the 80th Anniversary of Kurt Gödel*, Plenum Press, Druzhba, 1987 pp. 253–263.

[14] Goldblatt, R., "Axiomatizing the logic of computer programming," Lecture Notes in Computer Science **130**, Springer-Verlag, 1982.

[15] Koymans, R., "Specifying Message Passing and Time-Critical Systems with Temporal Logic," Springer-Verlag, Berlin, 1992.

[16] Sano, K., *Bimodal logics with irreflexive modality*, in: *Proceedings of the 38th MLG meeting at Gamagori*, 2004, pp. 50–2.
URL http://www.seto.nanzan-u.ac.jp/~sasaki/2004mlg/50-52.pdf

[17] Sano, K., *Predicate bimodal logics with irreflexive modality*, in: *Proceedings of the 39th MLG meeting at Gamagori*, 2005, pp. 2–5.
URL http://www.seto.nanzan-u.ac.jp/~sasaki/2005mlg/2--5.pdf

[18] Sano, K., *Using the irreflexive modality* (2006), (In preparation).

[19] Sano, K. and Y. Nakayama, *Bimodal logics with irreflexive modality*, Journal of the Japan Association for Philosophy of Science (in Japanese) (To appear).

[20] Sano, K. and K. Sato, *Semantical characterization for irreflexive and generalized modal languages*, accepted for Notre Dame Journal of Formal Logic.

[21] Sustretov, D., *Hybrid definability in topological spaces*, Technical Report PP-2005-14, ILLC Technical Report, University of Amsterdam (2005).
URL http://www.illc.uva.nl/Publications/ResearchReports/PP-2005-14.text.pdf

[22] ten Cate, B., "Model theory for extended modal languages," Ph.D. thesis, University of Amsterdam, Institute for Logic, Language and Computation (2005).

[23] ten Cate, B. and T. Litak, *Topological perspective on the hybrid proof rules*, in: *Proceeding of International workshop on hybrid logic 2006*, 2006, pp. 73–87.