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Electronic Notes in Theoretical Computer Science 308 (2014) 3–23

Electronic Notes in Theoretical Computer Science

www.elsevier.com/locate/entcs

On Continuous Nondeterminism and State Minimality

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Abstract

This paper is devoted to the study of nondeterministic closure automata, that is, nondeterministic finite automata (nfas) equipped with a strict closure operator on the set of states and continuous transition structure. We prove that for each regular language L there is a unique minimal nondeterministic closure automaton whose underlying nfa accepts L. Here minimality means no proper sub or quotient automata exist, just as it does in the case of minimal dfas. Moreover, in the important case where the closure operator of this machine is topological, its underlying nfa is shown to be state-minimal. The basis of these results is an equivalence between the categories of finite semilattices and finite strict closure spaces.

Keywords: Canonical Nondeterministic Automata, State Minimality, Closure Spaces, Semilattices

1 Introduction

Why are state-minimal deterministic finite automata (dfas) easy to construct, whilst no efficient minimization procedure for nondeterministic finite automata (nfas) is known? Let us start with the observation that minimal dfas are built inside the category Set_f of finite sets and functions and are characterized by having no proper subautomata (reachability) and no proper quotient automata (simplicity). Nfas can be regarded as dfas interpreted in the category Rel_f of finite sets and relations, and so one might hope to build minimal nfas in the same way as minimal dfas, but now in Rel_f. However, there is a significant difference: Set_f is both finitely complete and cocomplete, yet Rel_f does not have coequalizers, i.e., canonical quotients. The lack of such canonical constructions provides evidence for the lack of *canonical* state-minimal nfas.

http://dx.doi.org/10.1016/j.entcs.2014.10.002

This suggests the following strategy: form the cocompletion of Rel_f obtained by freely adding canonical quotients, which turns out to be the category JSL_f of finite join-semilattices (see Appendix), and build minimal automata in this larger category. Every nfa may be viewed as a dfa in JSL_f via the usual subset construction. In order to obtain more efficient presentations of nfas, avoiding the full power set of states, we make use of a categorical equivalence between JSL_f and the category Cl_f of finite strict closure spaces [2]. The objects of the latter are finite sets Z equipped with a strict closure operator (i.e., an extensive, monotone and idempotent map $\mathbf{cl}_Z : \mathcal{P}Z \to \mathcal{P}Z$ preserving the empty set), and the morphisms are continuous relations, see Definitions 2.9 and 2.10 below. For example, every finite topological space induces a finite strict closure space; these closures are called *topological*.

Just as nfas may be viewed as deterministic automata interpreted in Rel_f, *nondeterministic closure automata* (ncas) are deterministic automata interpreted in Cl_f: an nca is an nfa with a strict closure operator on its set of states, continuous transition relations, an open set of final states and a closed set of initial states. Since the category Cl_f has the same relevant properties as Set_f, we derive for each regular language $L \subseteq \Sigma^*$ the existence of a unique minimal nca $\mathcal{N}(L)$ whose underlying nfa (forgetting the closure operator) accepts L. It is minimal in the sense that it has no proper subautomata (reachability) and no proper quotient automata (simplicity), and can be constructed in a way very much analogous to Brzozowski's classical construction of the minimal dfa [6]: starting with any nca $\mathcal{N}$ accepting L, one has

$$\mathcal{N}(L) = \mathsf{reach} \circ \mathsf{rev} \circ \mathsf{reach} \circ \mathsf{rev}(\mathcal{N})$$

where reach and rev are continuous versions of the reachable subset construction and the reversal operation for nfas, respectively.

The states of $\mathcal{N}(L)$ are the *prime derivatives* of L, i.e., those non-empty derivatives $w^{-1}L = \{v \in \Sigma^* : wv \in L\}$ of L that do not arise as a union of other derivatives. The underlying nfa of $\mathcal{N}(L)$ accepts L, thus it is natural to ask when this nfa is *state-minimal*. Our main result is:

If the closure of $\mathcal{N}(L)$ is topological then the underlying nfa is state-minimal.

In other words, we identify a natural class of regular languages for which *canonical* state-minimal nondeterministic acceptors exist.

Related Work. Our paper is inspired by the work of Denis, Lemay and Terlutte [7] who define a canonical nondeterministic acceptor for each regular language L. In fact, the underlying nfa of our minimal nca $\mathcal{N}(L)$ is precisely their 'canonical residual finite state automaton', and our Brzozowski construction of $\mathcal{N}(L)$ in Section 3 generalizes their construction in [7, Theorem 5.2]. The main conceptual difference is that the latter works on the level of nfas, while our construction takes the continuous structure of nondeterministic closure automata into account. We hope to convince the reader that ncas provide the proper setting in which to study these canonical nfas and their construction.

We have introduced nondeterministic closure automata in [2] where we demonstrated that ncas – as well as related machines like the átomaton of Brzozowski and Tamm [5] – are instances of a uniform coalgebraic construction. There we also gave various simple criteria for nondeterministic state minimality. The present paper can be understood as an in-depth study of ncas, extending the results from [2] and [7] in two ways: firstly, we provide a richer and more conceptual way of constructing the minimal nca $\mathcal{N}(L)$ (compared to [7]) by working with closures. Secondly, we prove that the underlying nfa of $\mathcal{N}(L)$ is state-minimal provided that $\mathcal{N}(L)$ has topological closure, thereby generalizing a much weaker criterion from [2].

2 From Deterministic JSL-Automata to Nondeterministic Closure Automata

In this section we consider deterministic automata interpreted in the category of join-semilattices, and explain how they induce nondeterministic closure automata. We shall assume familiarity with basic concepts from category theory (categories, functors, duality and equivalence).

Notation 2.1 *Throughout this paper we fix an alphabet Σ. The composition of relations $R \subseteq A \times B$ and $S \subseteq B \times C$ is $S \circ R = \{(a, c) : \exists b \in B.(a, b) \in R \wedge (b, c) \in S\}$. Moreover $R[A'] = \{b \in B : \exists a \in A'.(a, b) \in R\}$ denotes the R-image of a subset $A' \subseteq A$, and we write $R[a]$ instead of $R[\{a\}]$.*

Let us first recall deterministic and nondeterministic finite automata and provide them with suitable morphisms.

Definition 2.2 (1) A *nondeterministic finite automaton* (*nfa*) $N = (Z, R_a, F)$ consists of a finite set Z of states, transition relations $R_a \subseteq Z \times Z$ for every $a \in \Sigma$, and a set $F \subseteq Z$ of final states. A *pointed* nfa (N, I) is additionally equipped with a set of initial states $I \subseteq Z$.

(2) Nfas form a category Nfa whose morphisms $\mathcal{B} : (Z, R_a, F) \to (Z,', R'_a, F')$ are relations $\mathcal{B} \subseteq Z \times Z'$ that preserve and reflect transitions (i.e. $R'_a \circ \mathcal{B} = \mathcal{B} \circ R_a$) and moreover $z \in F$ iff $\mathcal{B}[z] \cap F' \neq \emptyset$. Likewise we have the category Nfa_* of pointed nfas, whose morphisms $\mathcal{B} : (N, I) \to (N', I')$ are additionally required to satisfy $\mathcal{B}[I] = I'$.

Remark 2.3 Our choice of Nfa-morphisms $\mathcal{B}$ is sound: for each $z \in Z$ the pointed nfas $(Z, R_a, F, \{z\})$ and $(Z', R'_a, F', \mathcal{B}[z])$ accept the same language.

Definition 2.4 (1) A *deterministic finite automaton* (*dfa*) is an nfa $D = (Q, \delta_a, F)$ whose transition relations $\delta_a : Q \to Q$ are functions. A *pointed dfa* (D, q_0) is a dfa equipped with a single initial state $q_0 \in Q$. Morphisms of (pointed) dfas are precisely the Nfa- (resp. Nfa_*-)morphisms that are functions.

(2) A *deterministic automaton* (*da*) is defined in analogy to to (1), except that the set of states is not required to be finite.

We are mainly interested in d(f)as carrying a semilattice structure.

Notation 2.5 Let JSL denote the category of (join-)semilattices with a bottom element $\bot$, whose morphisms are $\bot$-preserving semilattice homomorphisms. JSL_f is the full subcategory of finite semilattices.

One can view the final states of a da (Q, δ_a, F) as a predicate $f : Q \to \{0, 1\}$ with $F = f^{-1}(\{1\})$. This suggests the following definition: a JSL-*da* is a da whose state set Q carries a semilattice structure, such that the transitions $\delta_a : Q \to Q$ and the final state predicate $f : Q \to \mathbf{2}$ are semilattice morphisms. Here $\mathbf{2}$ denotes the 2-chain $0 < 1$. Note that to give a morphism $f : Q \to \mathbf{2}$ means precisely to give a prime filter $F \subseteq Q$, i.e., $\bot_Q \notin F$ and $q \vee_Q q' \in F$ iff $q \in F$ or $q' \in F$. Indeed, given f, the set $F = f^{-1}(\{1\})$ is a prime filter, and conversely every prime filter of Q arises in this way. Therefore:

Definition 2.6 *(a) A* JSL-da *is a triple $\mathcal{D} = (Q, \delta_a, F)$ where Q is a semilattice of states, the a-transitions $\delta_a : Q \to Q$ are semilattice homomorphisms for $a \in \Sigma$, and the final states $F \subseteq Q$ form a prime filter. Given another* JSL-*da $\mathcal{D}' = (Q', \delta'_a, F')$, a* morphism *$h : \mathcal{D} \to \mathcal{D}'$ is a semilattice homomorphism $h : Q \to Q'$ such that $\delta'_a \circ h = h \circ \delta_a$ and $q \in F$ iff $h(q) \in F'$. We denote by* Jda *the category of all* JSL-*das, and by* Jdfa *the full subcategory of* JSL-dfas*, that is,* JSL-*das with a finite set of states.*

(b) A pointed JSL-da *$(\mathcal{D}, q_0)$ is a* JSL-*da $\mathcal{D} = (Q, \delta_a, F)$ with an initial state $q_0 \in Q$. Morphisms of pointed* JSL-*das must additionally preserve the initial state.* Jda_* *denotes the category of pointed* JSL-*das, and* Jdfa_* *the full subcategory of pointed* JSL-*dfas.*

(c) The language accepted by a pointed JSL-*da $(\mathcal{D}, q_0)$ is the language accepted by its underlying da, that is, the set*

$$\mathcal{L}_{\mathcal{D}}(q_0) = \{w \in \Sigma^* : \delta_w(q_0) \in F\}$$

where $\delta_w : Q \to Q$ is the usual inductive extension of the transition function given by $\delta_\varepsilon = \mathsf{id}_Q$ and $\delta_{wa} = \delta_a \circ \delta_w$.

Example 2.7 (1) Take any nfa $N = (Z, R_a, F)$. The usual determinization via the subset construction is a JSL-dfa. Indeed, the states $\mathcal{P}Z$ form a semilattice w.r.t. union, the transitions preserve binary unions and the empty set, and the final states $\{A \subseteq Z : A \cap F \neq \emptyset\}$ form a prime filter.

(2) Let $\mathcal{P}\Sigma^*$ be the semilattice (w.r.t. union) of all languages over Σ. It carries the structure of a JSL-da whose transitions are given by $L \to a^{-1}L$ $(a \in \Sigma)$ and whose final states F are precisely the languages containing the empty word. This automaton $\mathcal{D}_{\mathcal{P}\Sigma^*} = (\mathcal{P}\Sigma^*, a^{-1}(-), F)$ is easily seen to be the *final* JSL-da: every JSL-da has a unique Jda-morphism into $\mathcal{D}_{\mathcal{P}\Sigma^*}$, namely the function mapping each state to the language it accepts.

(3) Let $\mathcal{D}_{\mathsf{Reg}(\Sigma)}$ be the subautomaton of $\mathcal{D}_{\mathcal{P}\Sigma^*}$ whose states are the regular languages over Σ. It can be characterized up to isomorphism by the property that every JSL-dfa has a unique Jda-morphism into $\mathcal{D}_{\mathsf{Reg}(\Sigma)}$.

Remark 2.8 Readers familiar with the theory of coalgebras will immediately notice that JSL-das correspond to coalgebras for the functor $T = \mathbf{2} \times \mathsf{Id}^\Sigma : \mathsf{JSL} \to \mathsf{JSL}$. The examples (2) and (3) above then state precisely that $\mathcal{D}_{\mathcal{P}\Sigma^*}$ is the final T-coalgebra and $\mathcal{D}_{\mathsf{Reg}(\Sigma)}$ is the final locally finite T-coalgebra, see [9].

In [2] it was proved that the category JSL_f of finite semilattices is equivalent to the category of finite strict closure spaces. We recall the necessary concepts.

Definition 2.9 *A* closure operator *(shortly, a* closure*) on a set Z is a monotone, idempotent and extensive function $\mathbf{cl}_Z : \mathcal{P}Z \to \mathcal{P}Z$, that is,*

$$\frac{A \subseteq B}{\mathbf{cl}_Z(A) \subseteq \mathbf{cl}_Z(B)}, \qquad \mathbf{cl}_Z(A) \supseteq A, \qquad \mathbf{cl}_Z \circ \mathbf{cl}_Z = \mathbf{cl}_Z, \qquad \textit{for all } A, B \subseteq Z.$$

A closure space *$(Z, \mathbf{cl}_Z)$ is a set with a closure defined on it. It is* finite *if Z is finite and* strict *if $\mathbf{cl}_Z(\emptyset) = \emptyset$. A subset $A \subseteq Z$ is* closed *if $\mathbf{cl}_Z(A) = A$ and* open *if its complement $\bar{A} \subseteq Z$ is closed. We write $\mathbf{cl}_Z(z)$ for $\mathbf{cl}_Z(\{z\})$.*

Definition 2.10 *Let Z_1 and Z_2 be finite strict closure spaces. Then a relation $\mathcal{B} \subseteq Z_1 \times Z_2$ is said to be* continuous *if:*

(i) *For each $z \in Z_1$, the image $\mathcal{B}[z] \subseteq Z_2$ is closed in Z_2.*

(ii) *$\mathcal{B}[\mathbf{cl}_{Z_1}(A)] \subseteq \mathbf{cl}_{Z_2}(\mathcal{B}[A])$ for all subsets $A \subseteq Z_1$.*

Given two continuous relations $\mathcal{B}_1 : Z_1 \to Z_2$ and $\mathcal{B}_2 : Z_2 \to Z_3$ we define their continuous composition *as follows:*

$$\mathcal{B}_2 \bullet \mathcal{B}_1 = \{(z_1, z_3) \in Z_1 \times Z_3 : z_3 \in \mathbf{cl}_{Z_3}(\mathcal{B}_2 \circ \mathcal{B}_1[z_1])\} : Z_1 \to Z_3.$$

That is, one forms the usual relational composition and takes the closure in Z_3. The continuous identity *is defined by $\mathsf{id}_Z = \{(z, z') \in Z \times Z : z' \in \mathbf{cl}_Z(z)\}$.*

Definition 2.11 Let Cl_f denote the category of finite strict closure spaces and continuous relations, with continuous composition and identities.

Remark 2.12 Strict closure spaces can be regarded as generalized topological spaces. Indeed, every topological space Z induces a strict closure space $(Z, \mathbf{cl}_Z)$ where $\mathbf{cl}_Z$ is the usual topological closure operator. It preserves finite unions, i.e.,

$$\mathbf{cl}_Z(A \cup B) = \mathbf{cl}_Z(A) \cup \mathbf{cl}_Z(B) \quad \text{for all } A, B \subseteq Z. \tag{$*$}$$

Conversely, every strict closure space Z satisfying $(*)$ arises from a topological space. Moreover, if $\mathcal{B} : Z_1 \to Z_2$ is a function between topological spaces and Z_2 is a T_1 space, then $\mathcal{B}$ is continuous in the sense of topology iff it is continuous in the sense of Definition 2.10.

Definition 2.13 A closure $\mathbf{cl}_Z$ satisfying $(*)$ is called *topological*.

Definition 2.14 *Let Q be a finite semilattice, so that it is a lattice with meet*

$$q \wedge_Q q' = \bigvee_Q \{r : r \leq_Q q \textit{ and } r \leq_Q q'\}$$

and top element

$$\top_Q = \bigvee_Q \{q : q \in Q\}.$$

Then $q \in Q$ is join-irreducible *(resp.* meet-irreducible*) if (i) $q \neq \bot_Q$ (resp. $q \neq \top_Q$) and (ii) whenever $q = r \vee_Q r'$ (resp. $q = r \wedge_Q r'$) then $q = r$ or $q = r'$. Let $J(Q)$, $M(Q) \subseteq Q$ be the sets of join-irreducible and meet-irreducible elements of Q.*

Lemma 2.15 (see [2]) *The categories of finite semilattices and finite strict closure spaces are equivalent. Indeed, the following functor $G : \mathsf{JSL}_f \to \mathsf{Cl}_f$ is an equivalence:*

$$GQ = (J(Q), \mathbf{cl}_Q) \quad \textit{where } \mathbf{cl}_Q(A) = \{j \in J(Q) : j \leq_Q \bigvee_Q A\}$$

$$Gf = \{(j, j') \in J(Q) \times J(Q') : j' \leq_Q f(j)\} : GQ \to GQ'$$

for any semilattice homomorphism $f : Q \to Q'$.

Remark 2.16 The associated equivalence $C : \mathsf{Cl}_f \to \mathsf{JSL}_f$ maps a finite strict closure space Z to the semilattice CZ of all closed subsets of Z w.r.t. inclusion order. Its bottom is $\emptyset$ and it has joins $A \vee_{CZ} B = \mathbf{cl}_Z(A \cup B)$ (the meet being intersection). A continuous relation $\mathcal{B} : Z_1 \to Z_2$ is mapped to

$$C\mathcal{B} : CZ_1 \to CZ_2, \quad C\mathcal{B}(A) = \mathbf{cl}_{Z_2}(\mathcal{B}[A]).$$

The equivalence G lifts to an equivalence of automata, assigning to each JSL-dfa a corresponding 'nondeterministic closure automaton' in Cl_f.

Definition 2.17 *(a) A* nondeterministic closure automaton *(*nca*) is a triple $\mathcal{N} = (Z, R_a, F)$ where Z is a finite strict closure space (of states), the transition relations $R_a \subseteq Z \times Z$ are continuous for $a \in \Sigma$, and the final states $F \subseteq Z$ form an open set. Given another nca $\mathcal{N}' = (Z', R'_a, F')$, a* morphism *$\mathcal{B} : \mathcal{N} \to \mathcal{N}'$ is a continuous relation $\mathcal{B} : Z \to Z'$ such that $R'_a \bullet \mathcal{B} = \mathcal{B} \bullet R_a$ for each $a \in \Sigma$, and $z \in F$ iff $\mathcal{B}[z] \cap F' \neq \emptyset$. The category of ncas (and the above morphisms with continuous composition) is denoted* Nca.

(b) A pointed nca *$(\mathcal{N}, I)$ is an nca $\mathcal{N} = (Z, R_a, F)$ equipped with a* closed *subset $I \subseteq Z$ of initial states. Morphisms $\mathcal{B}$ between pointed ncas must additionally satisfy $\mathbf{cl}_{Z'}(\mathcal{B}[I]) = I'$. The category of pointed ncas is denoted* Nca_*.

(c) The language accepted by a pointed nca $(\mathcal{N}, I)$ is the set $\mathcal{L}_{\mathcal{N}}(I) \subseteq \Sigma^*$ of words w such that some state in I has some w-path to a final state.

Remark 2.18 (1) Every nfa $N = (Z, R_a, F)$ may be viewed as an nca where Z is discrete, i.e., it has the identity closure $\mathbf{cl}_Z = \mathsf{id}_{\mathcal{P}Z}$. This nca is well-defined because (i) every relation between discrete closure spaces is continuous and (ii) every subset of a discrete closure space is both open and closed. Therefore we have full embeddings $\mathsf{Nfa} \hookrightarrow \mathsf{Nca}$ and $\mathsf{Nfa}_* \hookrightarrow \mathsf{Nca}_*$.

(2) Every (pointed) nca has an *underlying (pointed) nfa* where we forget the closure. In contrast to the previous statement, this does not define functors $\mathsf{Nca} \to \mathsf{Nfa}$

and $\mathsf{Nca}_* \to \mathsf{Nfa}_*$ because composition of Nca-morphisms is not the relational composition. Note that the language $\mathcal{L}_{\mathcal{N}}(I)$ accepted by a pointed nca $(\mathcal{N}, I)$ is the language accepted by its underlying pointed nfa.

Lemma 2.19 (see [2]) *The categories of (pointed)* JSL*-dfas and (pointed) ncas are equivalent. Indeed, the equivalence* $G : \mathsf{JSL}_f \to \mathsf{Cl}_f$ *described above lifts to equivalences:*

$$\mathbb{G} : \mathsf{Jdfa} \to \mathsf{Nca} \qquad \textit{with} \qquad \mathbb{G}(Q, \delta_a, F) = (GQ, G\delta_a, F')$$

$$\mathbb{G}_* : \mathsf{Jdfa}_* \to \mathsf{Nca}_* \qquad \textit{with} \qquad \mathbb{G}_*(Q, \delta_a, F, q_0) = (GQ, G\delta_a, F', I_{q_0})$$

where $F' = J(Q) \cap F$ *and* $I_{q_0} = \{q \in J(Q) : q \leq_Q q_0\}$*. On morphisms we have* $\mathbb{G}f = Gf$ *and* $\mathbb{G}_* f = Gf$*.*

Proof. [Sketch] This follows from Lemma 2.15. Briefly, $G : \mathsf{JSL}_f \to \mathsf{Cl}_f$ defines an equivalence and one can apply it to the carrier Q and each homomorphism δ_a of a JSL-dfa. Furthermore, the final states arise as a morphism $Q \to \mathbf{2}$ and initial states as a morphism $\mathbf{2} \to Q$, so one may again apply G. The resulting structure is the equivalent (pointed) nca. □

Hence JSL-dfas and ncas are essentially the same structures, although the latter have the significant advantage of having fewer states. For example, if a JSL-dfa has free carrier, then the corresponding nca is exponentially smaller. The languages accepted by JSL-dfas and ncas are by definition just the languages accepted by their underlying dfas and nfas, respectively, and are preserved by the equivalence:

Lemma 2.20 *A pointed* JSL*-dfa accepts the same language as its equivalent pointed nca, i.e.,* $\mathcal{L}_{\mathcal{D}}(q) = \mathcal{L}_{\mathcal{N}}(I)$ *where* $(\mathcal{N}, I) = \mathbb{G}_*(\mathcal{D}, q)$*.*

Proof. Let $\mathcal{D} = (Q, \delta_a, F)$ and recall $I = J(Q) \cap {\downarrow_Q} q$, where ${\downarrow_Q} q = \{q' \in Q : q' \leq q\}$. Then $w \in \mathcal{L}_{\mathcal{D}}(q)$ iff $\delta_w(q) \in F$ by definition. Equivalently $w \in \mathcal{L}_{\mathcal{D}}(q)$ iff $f \circ \delta_w \circ i = \mathsf{id}_{\mathbf{2}}$ where the JSL_f-morphism $i : \mathbf{2} \to Q$ is defined $i(1) = q$ (and, necessarily, $i(0) = \bot_Q$) and the JSL_f-morphism $f : Q \to \mathbf{2}$ is defined $f(q) = 1$ iff $q \in F$, recalling that F is a prime filter.

Now $f \circ \delta_w \circ i = \mathsf{id}_{\mathbf{2}}$ iff $Gf \bullet G\delta_w \bullet Gi = G\mathsf{id}_{\mathbf{2}} = \mathsf{id}_{\{1\}}$ (where $\{1\}$ has identity closure) because G is faithful, being an equivalence. Observe that $Gi[1] = J(Q) \cap {\downarrow_Q} q = I$ and also $Gf \subseteq J(Q) \times \{1\}$ is such that $Gf[j] = \{1\}$ iff $j \in F$. We now show

$$Gf \bullet G\delta_w \bullet Gi = \mathsf{id}_{\{1\}} \tag{1}$$

iff there exists $j \in I$ such that $G\delta_w[j] \cap F \neq \emptyset$. The latter is equivalent to saying that $\mathbb{G}\mathcal{D}$ accepts w via the initial states I.

Assuming acceptance implies (1) because the relation $G\delta_w \bullet Gi$ contains $G\delta_w \circ Gi$. Conversely, assuming (1), first observe that:

$$\begin{aligned} G\delta_w \bullet Gi[1] &= \mathbf{cl}_Q(G\delta_w \circ Gi[1]) \\ &= \mathbf{cl}_Q(\textstyle\bigcup_{j\in I}\{j_0 \in J(Q) : j_0 \leq_Q \delta_w(j)\}) \\ &= \{j' \in J(Q) : j' \leq \textstyle\bigvee_Q \{\delta_w(j) : j \in I\}\}. \end{aligned}$$

Then (1) implies that $G\delta_w \bullet Gi[1]$ has non-empty intersection with F, i.e., there exists $j' \leq_Q \bigvee_Q\{\delta_w(j) : j \in I\}$ such that $j' \in F \cap J(Q)$, so in particular $j' \neq \bot_Q$. Since F is upwards closed, there is some non-zero $\delta_w(j) \in F$ (where $j \in I$) and hence also some non-zero join-irreducible beneath $\delta_w(j)$ lies in F. This implies $G\delta_w[j] \cap F$ is non-empty. □

3 Reversal, Reachability and Minimality

The purpose of the present section is to prove that every regular language L has an associated *minimal* pointed nca $\mathcal{N}(L)$ accepting L, which is unique up to isomorphism. We also present a construction of this minimal pointed nca, which is analogous to Brzozowski's classical construction of the minimal pointed dfa [6] (see also [4] for a (co-)algebraic view). Recall that the latter takes any pointed nfa (N, I) accepting L and constructs L's minimal dfa as follows:

$$\mathsf{reach} \circ \mathsf{rev} \circ \mathsf{reach} \circ \mathsf{rev}(N, I) \tag{2}$$

Here rev reverses transitions and also swaps the final and initial states,

$$(N, I) = (Z, R_a, F, I) \quad \Longrightarrow \quad \mathsf{rev}(N, I) = (Z, \breve{R}_a, I, F),$$

where $\breve{R}_a$ denotes the converse of the relation R_a. Furthermore, reach performs the reachable subset construction, i.e., it forms the subset dfa and takes its reachable part. In this section we introduce these two operations for pointed ncas. We then prove that the minimal pointed nca $\mathcal{N}(L)$ arises in exactly the same way as (2), only now taking any pointed nca accepting L as input. In particular, any pointed nfa will do.

The above nfa operation rev extends to a self-duality of the category Nfa_* of pointed nfas, defined on objects as above and on morphisms by $\mathcal{B} \mapsto \breve{\mathcal{B}}$. To see that it works on the final/initial states, let $\mathfrak{F} = \{(z, *) : z \in F\} \subseteq Z \times \mathbf{1}$ and $\mathfrak{I} = \{(*, z) : z \in I\} \subseteq \mathbf{1} \times Z$ where $\mathbf{1} = \{*\}$. Then we can rewrite our conditions on Nfa_*-morphisms (see Definition 2.2) as $\mathfrak{F}' \circ \mathcal{B} = \mathfrak{F}$ and $\mathcal{B} \circ \mathfrak{I} = \mathfrak{I}'$, which clearly dualize under converse. Therefore in order to generalize rev to pointed ncas, we describe a suitable self-duality of Nca_*. It is based on the well-known self-duality of JSL_f:

Lemma 3.1 *The following functor $D : \mathsf{JSL}_f^{op} \to \mathsf{JSL}_f$ is an equivalence: on objects let $DQ = Q^{op}$ (which has carrier Q, bottom $\top_Q$ and join $\wedge_Q$) and on morphisms*

$f : Q_1 \to Q_2$ define $Df^{op} : Q_2^{op} \to Q_1^{op}$ by

$$Df^{op}(q_2) = \bigvee_{Q_1} \{q_1 \in Q_1 : f(q_1) \leq_{Q_2} q_2\}.$$

Proof. [Sketch] The self-duality of JSL_f follows from the adjoint functor theorem for posets. Finite join-semilattices are finite posets with all joins (= colimits) and join-semilattice morphisms are monotone maps that preserve all joins. Consequently, each $f : Q_1 \to Q_2$ has a right adjoint $g : Q_2^{op} \to Q_1^{op}$ where the order is reversed because right adjoints preserve all meets. The uniqueness of adjoints implies that this is an equivalence. Its explication yields the above action on the morphisms. □

Since JSL_f is equivalent to Cl_f (see Lemma 2.15 and Remark 2.16), it follows that Cl_f is also self-dual, with dual equivalence

$$\mathbf{D} = (\,\mathsf{Cl}_f^{op} \xrightarrow{C^{op}} \mathsf{JSL}_f^{op} \xrightarrow{D} \mathsf{JSL}_f \xrightarrow{G} \mathsf{Cl}_f\,).$$

We now describe this self-duality explicitly. Recall that CZ denotes the semilattice of closed subsets of a finite strict closure space Z, and that $M(CZ)$ is the set of meet-irreducibles of CZ.

Proposition 3.2 *For any finite strict closure space Z we have*

$$\mathbf{D}Z = M(CZ) \qquad \mathbf{cl}_{\mathbf{D}Z}(X) = \{A \in M(CZ) : \bigcap X \subseteq A\},$$

and for any $\mathcal{B} : Z_1 \to Z_2$, the continuous relation $\mathbf{D}\mathcal{B}^{op} : M(CZ_2) \to M(CZ_1)$ consists of all those $(A_2, A_1) \in M(CZ_2) \times M(CZ_1)$ such that:

$$\mathcal{B}[A] \subseteq A_2 \quad \Longrightarrow \quad A \subseteq A_1 \qquad \textit{for every } A \in J(CZ_1).$$

Proof. We have $\mathbf{D}Z = G((CZ)^{op}) = (M(CZ), \mathbf{cl})$ where $M(CZ) = J((CZ)^{op}) \subseteq Q$ are the meet-irreducibles and:

$$\begin{aligned}\mathbf{cl}(S) &= \{j \in J((CZ)^{op}) : j \leq_{(CZ)^{op}} \textstyle\bigvee_{(CZ)^{op}} S\}\\ &= \{m \in M(CZ) : \textstyle\bigwedge_{CZ} S \leq_{CZ} m\}\\ &= \{m \in M(CZ) : \textstyle\bigcap S \subseteq m\}\end{aligned}$$

using the fact that the meet in CZ is intersection. Likewise every JSL_f-morphism $f : Q \to Q'$ has a dual morphism $Df^{op} : Q'^{op} \to Q^{op}$ defined $Df^{op}(q') = \bigvee_Q f^{-1}(\downarrow_{Q'} q')$. Therefore every continuous relation $\mathcal{B} : Z \to Z'$ has a *dual* continuous relation $\mathbf{D}\mathcal{B}^{op} : \mathbf{D}Z' \to \mathbf{D}Z$ defined as follows: let $f = C\mathcal{B}$ be the JSL_f-morphism corresponding to $\mathcal{B}$ and then apply the equivalence $G : \mathsf{JSL}_f \to \mathsf{Cl}_f$ to the homomorphism

Df^{op}. Then

$$\begin{aligned}\mathbf{D}\mathcal{B}^{op} &= \{(j',j) \in J(Q'^{op}) \times J(Q^{op}) : j \leq_{Q^{op}} Df^{op}(j')\}\\ &= \{(m',m) \in M(Q') \times M(Q) : \textstyle\bigvee_Q f^{-1}(\downarrow_{Q'} m') \leq_Q m\}\\ &= \{(m',m) : \forall j \in J(Q).(f(j) \leq_{Q'} m' \Rightarrow j \leq_Q m)\}\\ &= \{(m',m) : \forall j \in J(Q).(\mathbf{cl}_{Z_2}(\mathcal{B}[j]) \leq_{Q'} m' \Rightarrow j \leq_Q m)\}\\ &= \{(m',m) : \forall j \in J(Q).(\mathcal{B}[j] \leq_{Q'} m' \Rightarrow j \leq_Q m)\}\end{aligned}$$

In the last step we use that $B[j]$ is closed since $\mathcal{B}$ is continuous. □

Given any pointed nca (Z, R_a, F, I), the previous duality naturally leads to a pointed nca with states $M(CZ)$ by applying $\mathbf{D}$ to (a) $R_a : Z \to Z$, (b) F considered as a continuous relation $F : Z \to \{1\}$ and (c) I considered as a continuous relation $I : \{1\} \to Z$. This is the *reversal* of pointed ncas: by applying $\mathbf{D}$ to F we get the subset $I^d \subseteq M(CZ)$ of all $A \in M(CZ)$ containing $Z \setminus F$. By applying $\mathbf{D}$ to I we get the subset $F^d \subseteq M(CZ)$ of all $A \in M(CZ)$ with $I \not\subseteq A$.

Definition 3.3 *The* reversal *of a pointed nca is defined by*

$$\mathsf{rev}(Z, R_a, F, I) = (M(CZ), \mathbf{D}R_a^{op}, F^d, I^d).$$

Example 3.4 Take any pointed nfa and view it as a pointed nca with identity closure. Then $CZ = \mathcal{P}Z$ has meet-irreducibles $Z \setminus \{z\}$ for $z \in Z$ and the reversal is the classical nfa reversal, modulo the bijection $z \mapsto Z \setminus \{z\}$.

Theorem 3.5 *The category* Nca_* *is self-dual: the object map* rev *extends to an equivalence* $\mathsf{rev} : \mathsf{Nca}_*^{op} \to \mathsf{Nca}_*$. *It assigns to every morphism* $\mathcal{B} : (Z, R_a, I, F) \to (Z', R'_a, I', F')$ *the morphism* $\mathsf{rev}(\mathcal{B}) \subseteq M(CZ') \times M(CZ)$ *of all pairs* (A', A) *such that*

$$\mathcal{B}[X] \subseteq A' \quad \Longrightarrow \quad X \subseteq A \qquad \text{for every } X \in J(CZ).$$

Proof. In Definition 3.3 we defined the object part of the dual equivalence $\mathsf{rev} : \mathsf{Nca}_*^{op} \to \mathsf{Nca}$. We now prove that it actually defines a functor.

(i) The action on continuous relations R_a and Nca_*-morphisms $\mathcal{B}$ is the action of $\mathbf{D}$, see Proposition 3.2.

(ii) The initial states form a closed set $I \in CZ$, or equivalently a join-semilattice morphism $i : \mathbf{2} \to CZ$ such that $i(1) = I$. Dually we have the join-semilattice morphism $i' = Di^{op} : (CZ)^{op} \to \mathbf{2}^{op}$ defined by

$$i'(x) = \bigvee_{\mathbf{2}} i^{-1}(\downarrow_{CZ} x) = \begin{cases} \top_{\mathbf{2}^{op}} = 0 & \text{if } i(1) \not\leq_{CZ} x \\ 1 & \text{otherwise} \end{cases}$$

Therefore the new final states are:

$$\begin{aligned} F^d &= J((CZ)^{op}) \cap i'^{-1}(\top_{\mathbf{2}^{op}}) \\ &= \{m \in M(CZ) : I \not\leq_{CZ} m\} \\ &= \{m \in M(CZ) : I \not\subseteq m\}. \end{aligned}$$

(iii) The final states F form an open set in $(Z, \mathbf{cl}_Z)$, or equivalently a join-semilattice morphism $f : CZ \to \mathbf{2}$ such that $Z \setminus F = \bigvee_{CZ} f^{-1}(\{0\})$. The dual join-semilattice morphism $f' = Df^{op} : \mathbf{2}^{op} \to (CZ)^{op}$ is defined by $f'(b) = \bigvee_{CZ} f^{-1}(\downarrow_{\mathbf{2}} b)$. Consequently:

$$\begin{aligned} I^d &= \{j \in J((CZ)^{op}) : j \leq_{(CZ)^{op}} f'(\top_{\mathbf{2}^{op}})\} \\ &= \{m \in M(CZ) : \textstyle\bigvee_{CZ} f^{-1}(\{0\}) \leq_{CZ} m\} \\ &= \{m \in M(CZ) : Z \setminus F \subseteq m\}. \end{aligned}$$

□

Proposition 3.6 *If a pointed nca accepts L then its reverse pointed nca accepts the reversed language* $\mathsf{rev}(L)$.

Proof. A pointed JSL-dfa $(\mathcal{D}, q) = (Q, \delta_a, F, q)$ accepts a word w iff $f \circ \delta_w \circ i = \mathsf{id}_{\mathbf{2}}$, where $f : Q \to \mathbf{2}$ represents the final states, δ_w is the inductive extension of the δ_a's and $i : \mathbf{2} \to Q$ represents $q \in Q$. Now Jdfa_* is self-dual (because Nca_* is) with dual pointed JSL-dfa $(Q^{op}, D\delta_a^{op}, \uparrow_{Q^{op}} q, \bigvee_Q (Z \setminus F))$. The dual of the equality above is $Di^{op} \circ D\delta_{w^r}^{op} \circ Df^{op} = \mathsf{id}_{\mathbf{2}^{op}}$ and furthermore Di corresponds to the *final* states in the dual machine, just as Df corresponds to the *initial* states. Consequently $(\mathcal{D}, q)$ accepts w iff its dual machine accepts the reversed word w^r. This property is inherited by Nca_* because the equivalence $\mathbb{G}_* : \mathsf{Jdfa}_* \to \mathsf{Nca}_*$ preserves languages, see Lemma 2.20. □

Next we extend the operation reach from nfas to pointed ncas. A pointed nfa is *reachable* if each state is reached from some initial state by transitions. Equivalently, the pointed nfa has no proper sub nfas. Here 'sub nfa' refers to the category Nfa_* i.e. N' is a sub nfa of N if there is a morphism $\mathcal{B} : N' \to N$ where $\mathcal{B}$ is an injective function. Implicitly one uses the (onto relation, injective function) factorization system in Rel_f and lifts it to Nfa_*. Similar remarks apply to sub dfas: (i) they arise as injective dfa morphisms via the (surjection, injection) factorization system of Set_f, (ii) a pointed dfa is reachable iff it has no proper sub dfas.

In order to define *reachable pointed ncas*, we first need the appropriate concept of *sub nca*. To this end, we take the (epi, mono) = (surjection, injection) factorization system of JSL_f and apply the equivalence of JSL_f and Cl_f to obtain a corresponding factorization system in Cl_f.

Lemma 3.7 *Every continuous relation $\mathcal{B} : Z_1 \to Z_2$ has an essentially unique (epi,* *mono)-factorization in* Cl_f*. Moreover,* $\mathcal{B}$ *is monic (resp. epic) iff the function*

$$C\mathcal{B} : CZ_1 \to CZ_2, \quad C\mathcal{B}(S) = \mathbf{cl}_{Z_2}(\mathcal{B}[S]),$$

is injective (resp. surjective).

Proof. The functor $C : \mathsf{Cl}_f \to \mathsf{JSL}_f$ preserves and reflects monos and epis, being an equivalence. □

Definition 3.8 *(a) A pointed nca* $(\mathcal{N}', I')$ *is a* sub nca *of* $(\mathcal{N}, I)$ *if there exists an* Nca_**-monomorphism* $m : (\mathcal{N}', I') \rightarrowtail (\mathcal{N}, I)$*.*

(b) A pointed nca $(\mathcal{N}', I')$ *is a* quotient nca *of* $(\mathcal{N}, I)$ *if there exists an* Nca_**-epimorphism* $e : (\mathcal{N}, I) \twoheadrightarrow (\mathcal{N}', I')$*.*

(c) A pointed nca is called reachable *if it has no proper sub ncas,* simple *if it has no proper quotient ncas, and* minimal *if it is both reachable and simple.*

Proposition 3.9 *Any sub or quotient nca of* $(\mathcal{N}, I)$ *accepts the same language* $\mathcal{L}_{\mathcal{N}}(I)$*.*

Proof. Viewed as their equivalent pointed JSL-dfa, we have injective or surjective deterministic automata morphisms which preserve the initial state. These certainly preserve the language and by Lemma 2.20 the respective pointed ncas accept the same languages. □

To obtain a more concrete characterization of reachability and simplicity, we shall restrict to ncas whose closure is normalized in the following sense:

Lemma 3.10 *Every finite strict closure space* Y *is isomorphic to another finite strict closure space* Z *such that:*

(i) *Z is separable, that is, $z \neq z' \in Z$ implies $\mathbf{cl}_Z(z) \neq \mathbf{cl}_Z(z')$.*

(ii) *$S \in CZ$ is join-irreducible in CZ iff $S = \mathbf{cl}_Z(\{z\})$ for some $z \in Z$.*

Proof.

(i) Recall from Lemma 2.15 and Remark 2.16 the equivalence $G : \mathsf{JSL}_f \to \mathsf{Cl}_f$ and its associated equivalence $C : \mathsf{Cl}_f \to \mathsf{JSL}_f$. Then Y is isomorphic to the closure space $Z = GCY$ whose carrier $J(CY)$ is the set of join-irreducibles in CY and whose closure is defined by $\mathbf{cl}_{GCY}(S) = \{j \in J(CY) : j \subseteq \mathbf{cl}_Y(\bigcup S)\}$ for any $S \subseteq J(CY)$. This closure space GCY is separable: given distinct join-irreducibles $j \neq j' \in J(CY)$ then wlog $j \not\leq_{CY} j'$, hence $j \not\subseteq j' = \mathbf{cl}_Y(j')$ and $j \notin \mathbf{cl}_{GCY}(\{j'\})$. But clearly $j' \in \mathbf{cl}_{GCY}(\{j'\})$.

(ii) In *any* closure space Z', every join-irreducible in CZ' is the closure of some singleton set. For if $S = \mathbf{cl}_{Z'}(S)$ is join-irreducible and $S = S_1 \cup S_2$ then $S = \mathbf{cl}_{Z'}(S_1) \vee_{CZ'} \mathbf{cl}_{Z'}(S_2)$ and hence wlog $S = \mathbf{cl}_{Z'}(S_1)$. Continuing we get either $S = \mathbf{cl}_{Z'}(\emptyset) = \emptyset$ (a contradiction) or S is the closure of a singleton set. For the particular closure space $GCY = (J(CY), \mathbf{cl}_{CY})$ we also have the converse, i.e., the closure of a singleton subset of $J(CY)$ is join-irreducible in $C(GCY)$. This follows because the closed sets of GCY take the form $J(CY) \cap {\downarrow_{CY}} S$ for some $S \in CY$ and in particular the closure of a singleton $\{j\}$, $j \in J(CY)$, consists of all join-irreducibles smaller than or equal to j. It follows that if $\mathbf{cl}_{GCY}(\{j\}) = K_1 \vee_{C(GCY)} K_2$ then some K_i contains j by join-irreducibility of $\{j\}$, wlog $j \in K_1$. Then $\mathbf{cl}_{GCY}(\{j\}) \subseteq \mathbf{cl}_{GCY}(K_1) = K_1$, and the converse is clear.

□

Definition 3.11 A (pointed) nca is *normalized* if its closure satisfies the conditions of Lemma 3.10.

Corollary 3.12 *Every nca is isomorphic to a normalized nca.*

Proposition 3.13 *A normalized pointed nca (Z, R_a, F, I) is reachable iff for every $z \in Z$ there exists a word $w_z \in \Sigma^*$ such that:*

(i) There is a w_z-path from some initial state to z in the underlying nfa.

(ii) Every w_z-path from every initial state terminates at an element of $\mathbf{cl}_Z(z)$.

Proof. Suppose $(\mathcal{N}, I)$ is a reachable pointed nca. Then by the equivalence of $\mathbb{G}_* : \mathsf{Jdfa}_* \to \mathsf{Nca}_*$, we have a corresponding pointed JSL-dfa $(\mathcal{D}, I) = (CZ, CR_a, F', I)$. Its final states $F' \subseteq CZ$ are defined $F' = \{A \in CZ : A \cap F \neq \emptyset\}$. Note $I \in CZ$ is now a *single state*. By the equivalence of Nca_* and Jdfa_* we know that $(\mathcal{D}, I)$ has no proper subobjects i.e. every injective Jdfa_*-morphism into $(\mathcal{D}, I)$ is bijective. Viewing $\mathcal{D}$ as its underlying dfa, one can list those states reachable from the state $I \in CZ$ via transitions and then construct the join-subsemilattice of CZ generated by this set. This defines a pointed sub JSL-dfa, using the fact that the transition functions $CR_a : CZ \to CZ$ are JSL_f-morphisms, hence an injective Jdfa_*-morphism which is necessarily bijective. It follows that *every* $A \in CZ$ arises as a join of elements reachable from the single state I via transitions. In particular the join-irreducible elements must be reachable from I via transitions, since they form the minimal generating set. So take any element $z \in Z$ and consider its closure $A = \mathbf{cl}_Z(z)$, this being an element of $\mathcal{D}$'s carrier. By our assumption that the closure has been normalized, A is join-irreducible in CZ. Therefore there exists some $w_z \in \Sigma^*$ such that $CR_{w_z}(I) = A$ i.e. we have a deterministic w_z-path in $\mathcal{D}$ from the initial state I to the state A. Then $A = \mathbf{cl}_Z(R_{w_z}[I])$ and since A is join-irreducible and $\mathbf{cl}_Z$ is separable we deduce $z \in R_{w_z}[I]$, i.e., the first condition holds. The second condition follows because $z' \in R_{w_z}[z_0]$ implies $z' \in CR_{w_z}(I) = A = \mathbf{cl}_Z(z)$.

Conversely, suppose the two conditions hold for some pointed nca (Z, R_a, F, I). Consider its equivalent pointed JSL-dfa with carrier CZ. The conditions imply that every join-irreducible in CZ is reachable from the single state I. We can form a sub JSL-dfa by closing under the transitions and then forming the generated subalgebra. Since every join-irreducible is reachable, this gives the original JSL-dfa. Furthermore this is the smallest sub JSL-dfa, which implies the respective pointed nca is reachable. □

Remark 3.14 The above condition (ii) may be felt surprising. However, recall that reachability for pointed ncas was defined by complete analogy with nfas: no proper sub nca exists. For pointed dfas (viewed as ncas with identity closure) this is the usual notion of reachability, since there is exactly one w_z-path from the unique initial state. However the same cannot be said for pointed nfas.

Proposition 3.15 *Viewed as a pointed nca with identity closure, a pointed nfa is reachable iff its reachable subset construction (that is, the reachable part of the subset dfa) contains all singleton sets.*

Proof. Let (Z, R_a, F, I) be a pointed nfa. If every singleton subset lies in its reachable subset construction, then every $z \in Z$ has some $w_z \in \Sigma^*$ such that the unique path from the single state I terminates at z. Since the path is unique and $z \in \mathbf{cl}_Z(z) = \{z\}$, it follows that we have a reachable pointed nca by Proposition 3.13. Conversely, suppose this pointed nfa defines a reachable pointed nca with identity closure. Then by Proposition 3.13 each $z \in Z$ has some w_z such that $I \xrightarrow{w_z} \{z\}$ because no other state lies in the closure of $\{z\}$. □

We now provide further properties of reachable pointed ncas.

Proposition 3.16 *Suppose one has a normalized reachable pointed nca accepting L, then:*

(i) *Its underlying pointed nfa is reachable.*

(ii) *Its individual states accept derivatives of L.*

(iii) *Varying the (closed) set of initial states, the languages it accepts are precisely the unions of L's derivatives.*

Proof. The first statement follows immediately from Proposition 3.13 via the first condition (i). The second statement follows because for each $z \in Z$, its closure is reachable from I in the equivalent pointed JSL-dfa. Thus this closed set accepts a derivative of L and this language is preserved by the equivalence (see Lemma 2.20). Finally, for (iii) observe that the JSL-dfa equivalent to $\mathcal{N}$ accepts precisely the unions of derivatives of L when varying the initial state: indeed, the states reachable via transitions accept precisely the derivatives of L, and all other states arise as a join of such states. Since languages are preserved by the equivalence, (iii) follows. □

Every pointed nfa has a smallest sub nfa, which is necessarily reachable. That is, one simply discards all those states not reachable from the initial states by transitions. From the categorical standpoint this means that the intersection of all pointed sub nfas exists. Similarly, for any pointed nca, the intersection of all pointed sub ncas exists i.e. we can always construct a unique reachable sub nca.

Definition 3.17 *The* reachable part $\mathsf{reach}(\mathcal{N}, I)$ *of a pointed nca $(\mathcal{N}, I)$ is the unique reachable pointed sub nca, i.e. the intersection of all pointed sub ncas.*

Notation 3.18 *Given any nca with carrier Z, any word $w \in \Sigma^*$ and any subset $I \subseteq Z$, we write $z \xrightarrow{w} z'$ whenever there is a w-path from z to z' in the underlying nfa. Then $w \cdot I \in CZ$ denotes the closure of $\{z' \in Z : \exists z \in I. z \xrightarrow{w} z'\}$.*

Proposition 3.19 $\mathsf{reach}(Z, R_a, F, I)$ *is isomorphic to* (Z', R'_a, F', I') *where:*

(i) $Z' \subseteq CZ$ *is the set of* $w \cdot I$*'s not arising as joins of other* $v \cdot I$*'s in* CZ*;*

(ii) $R'_a = \{(u \cdot I, v \cdot I) \in Z' \times Z' : v \cdot I \subseteq ua \cdot I\}$ *for each* $a \in \Sigma$*;*

(iii) $F' = \{A \in Z' : F \cap A \neq \emptyset\}$*;*

(iv) $I' = \{A \in Z' : A \subseteq I\}$*.*

Proof. [Sketch] This follows by considering equivalent pointed JSL-dfa i.e. we close under the deterministic transitions from I, form the generated subalgebra, and then convert this JSL-dfa back into a pointed nca. □

Remark 3.20 Applying this to an nfa (i.e. a pointed nca with identity closure), one finds that $\mathsf{reach}(\mathcal{N}, I)$ is never larger than the reachable subset construction $\{w \cdot I : w \in \Sigma^*\}$.

Next we characterize *simple* pointed ncas. Recall that a dfa is simple iff distinct states accept distinct languages. Analogously:

Proposition 3.21 *A pointed nca* $(\mathcal{N}, I)$ *with carrier* Z *is simple iff distinct closed subsets accept distinct languages i.e. if* $A \neq B \in CZ$ *then* $\mathcal{L}_{\mathcal{N}}(A) \neq \mathcal{L}_{\mathcal{N}}(B)$*.*

Proof. Let $\mathcal{D}$ be the JSL-dfa equivalent to N. It is simple since $\mathcal{N}$ is, so the unique map into the final JSL-da $\mathcal{D}_{\mathcal{P}\Sigma^*}$ (see Example 2.7) is injective. This means that distinct states of $\mathcal{D}$ accept distinct languages, hence the statement follows from Lemma 2.20. □

By Theorem 3.5 we know Nca_* is self-dual. Moreover *reachable* and *simple* are dual concepts, see Definition 3.8. Then if $(\mathcal{N}, I)$ is any pointed nca accepting L, it follows that

$$\mathsf{sim}(\mathcal{N}, I) \stackrel{\mathsf{def}}{=} \mathsf{rev} \circ \mathsf{reach} \circ \mathsf{rev}(\mathcal{N}, I)$$

is a simple pointed nca accepting L i.e. the *simplification* of $(\mathcal{N}, I)$. Categorically, it is the cointersection of all quotient ncas.

Next consider $\mathsf{reach} \circ \mathsf{sim}(\mathcal{N}, I)$ which is certainly reachable. Importantly it is also simple, using Proposition 3.21 and the fact $\mathsf{reach}(\mathcal{N}, I)$ is a sub nca of $(\mathcal{N}, I)$. Then by definition it is a *minimal* pointed nca accepting L. In fact:

Proposition 3.22 *For every regular languages* L*, there is up to isomorphism only one minimal pointed nca accepting* L*.*

Proof. This follows from a more general result in [1, Lemma 3.22] since the minimal nca is an instance of a *well-pointed coalgebra*. Let us briefly sketch the argument. Suppose one has two minimal pointed ncas accepting L. Equivalently one has two minimal pointed finite JSL-dfas accepting L, where minimal now means reachable and simple in Jdfa_*. Each one has a unique Jda-morphism to the deterministic JSL-automaton $\mathcal{D}_{\mathsf{Reg}(\Sigma)}$ of regular languages (see Example 2.7(c)), assigning to each state the language it accepts. These morphisms factorize into a surjective morphism followed by an inclusion i.e. their image defines a sub dfa of $\mathcal{D}_{\mathsf{Reg}(\Sigma)}$. Since they are both simple, they are each isomorphic to their respective image. Since they are both reachable, by Proposition 3.16 the carrier of this image is precisely the set of unions of derivatives of L. Therefore they are isomorphic to each other. $\square$

Notation 3.23 *Let Q_L denote the semilattice of all unions of derivatives of L.*

Proposition 3.24 *The following pointed nca $\mathcal{N}(L)$ for is minimal nca for L:*

(i) States $Z_L = J(Q_L)$ i.e. the non-empty derivatives of L that are not unions of others derivatives, endowed with the closure:

$$\mathbf{cl}_L(A) = \{K \in Z_L : K \subseteq \bigcup A\},$$

(ii) transitions $R_a = \{(K, K') \in Z_L \times Z_L : K' \subseteq a^{-1}K\}$ for each $a \in \Sigma$,

(iii) as final states those $K \in Z_L$ containing ε,

(iv) as initial states those $K \in Z_L$ which are subsets of L.

Proof. It is easy to see that the minimal pointed JSL-dfa accepting L is the finite subautomaton of $\mathcal{D}_{\mathsf{Reg}(\Sigma)}$ (see Example 2.7(c)) generated by L. Hence is has states Q_L, transitions $K \to a^{-1}K$ ($K \in Q_L$), initial state L, and the final states are precisely those languages in Q_L containing ε. Now apply the equivalence of Jdfa_* and Nca_*. $\square$

Then the main result in this section follows:

Theorem 3.25 *For any pointed nca $(\mathcal{N}, I)$ accepting L, the pointed nca:*

$$\mathsf{reach} \circ \mathsf{rev} \circ \mathsf{reach} \circ \mathsf{rev}(\mathcal{N}, I)$$

is a minimal pointed nca accepting L, and is hence isomorphic to $\mathcal{N}(L)$.

Proof. By Proposition 3.6, the dual of a pointed nca accepts the reversed language. Since reach preserves the accepted language, the above pointed nca accepts L. so by Proposition 3.22 it suffices to show it is a minimal pointed nca. It is clearly reachable. Moreover, $\mathsf{rev} \circ \mathsf{reach} \circ \mathsf{rev}(\mathcal{N}, I)$ is the dual of a reachable pointed nca and hence is simple. Finally, reach preserves simplicity (as previously explained), so we are done. $\square$

Finally, since Nca_* is self-dual and reachability and simplicity are dual:

Proposition 3.26 *For each regular language L, the reverse of the minimal pointed nca $\mathcal{N}(L)$ is isomorphic to the minimal pointed nca for $\mathsf{rev}(L)$, shortly,*

$$\mathsf{rev}(\mathcal{N}(L)) \cong \mathcal{N}(\mathsf{rev}(L)).$$

Proof. By Proposition 3.6 we know the dual of $\mathcal{N}(L)$ accepts $\mathsf{rev}(L)$. Furthermore minimality is a *self-dual* property because 'no proper subobjects' and 'no proper quotients' dualize. Hence, the dual of $\mathcal{N}(L)$ is minimal and is isomorphic to $\mathcal{N}(\mathsf{rev}(L))$ by Proposition 3.22. $\square$

4 State-minimal Nondeterministic Automata

Each regular language L is accepted by the underlying nfa of its minimal pointed nca $\mathcal{N}(L)$. Although this nfa is never larger than the minimal dfa, it generally need not be a state-minimal *nondeterministic* acceptor [7]. In this section we present a natural sufficient condition for state minimality. Recall that a strict closure $\mathbf{cl}_Z$ is *topological* if it is induced by a topology on Z, i.e., it satisfies the equation

$$\mathbf{cl}_Z(A \cup B) = \mathbf{cl}_Z(A) \cup \mathbf{cl}_Z(B) \quad \text{for all } A, B \subseteq Z.$$

Definition 4.1 *An nca is* topological *if its closure is topological.*

Lemma 4.2 *$\mathcal{N}(L)$ is topological iff the lattice $Q_L \cong CZ_L$ is distributive.*

Proof. If $\mathbf{cl}_{Z_L}$ is topological then $CZ_L \subseteq \mathcal{P}Z_L$ is closed under union and intersection, hence it is a distributive lattice. Conversely suppose that $Q_L \cong CZ_L$ is distributive, and let $A = \{K_1, \dots, K_m\}$ and $B = \{L_1, \dots, L_n\}$ be closed subsets of Z_L. We need to show that $A \cup B$ is closed, so suppose that $K \in \mathbf{cl}_{Z_L}(A \cup B)$, that is, $K \subseteq K_1 \cup \dots \cup K_m \cup L_1 \cup \dots \cup L_n$ by the definition of $\mathbf{cl}_{Z_L}$. Note that for any join-irreducible element x of a finite distributive lattice, $x \le y \vee z$ implies $x \le y$ or $x \le z$ (because $x \le y \vee z$ implies $x = x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$, hence $x = x \wedge y$ or $x = x \wedge z$ since x is join-irreducible). We conclude that $K \subseteq K_1 \cup \dots \cup K_m$ or $L \subseteq L_1 \cup \dots \cup L_n$, so $K \in \mathbf{cl}_{Z_L}(A) = A$ or $K \in \mathbf{cl}_{Z_L}(B) = B$. Thus $\mathbf{cl}_{Z_L}(A \cup B) \subseteq A \cup B$. □

Example 4.3 (1) If $\mathcal{N}(L)$ is an nfa (i.e., has identity closure) then it is topological. For example, this is the case whenever $d_L = 2^{n_L}$ where d_L (resp. n_L) is the number of states of a state-minimal dfa (resp. nfa) accepting L, see [2].

(2) If $\mathcal{N}(L)$ is topological then so is $\mathcal{N}(\mathsf{rev}(L))$. Indeed, recall that they are duals. Then since $\mathcal{N}(L)$ is topological iff the closed sets CZ_L form a distributive lattice, its order-dual is also distributive and is isomorphic to $CZ_{\mathsf{rev}(L)}$.

(3) If $\mathcal{N}(L)$ is topological and $f : \Delta^* \to \Sigma^*$ is a surjective monoid morphism then $\mathcal{N}(f^{-1}(L))$ is also topological. Here one uses the fact that $f^{-1} : \mathcal{P}\Sigma^* \to \mathcal{P}\Delta^*$ is an injective boolean morphism, providing an isomorphism between the semilattices Q_L and $Q_{f^{-1}(L)}$.

(4) If L is *intersection-closed* (that is, each binary intersection of derivatives of L arises as a union of derivatives of L), then $\mathcal{N}(L)$ is topological by Lemma 4.2. Examples of intersection-closed languages include:
   (a) the languages Σ^* and $\{w\}$ for $w \in \Sigma^*$,
   (b) the tail languages $(a+b)^*b(a+b)^{n-1}$ $(n \ge 1)$,
   (c) linear codes, i.e., linear subspaces of the vector space $\mathbb{Z}_2^n$ (viewed as languages over the binary alphabet);
   (d) the languages $L_f = \{w \in 2^n : f(w) = 1\} \subseteq 2^*$ where $f : 2^n \to 2$ is either the parity function, the majority function or any $\mathbb{R}$-weighted threshold function, i.e.,

$$f(b_1, \dots b_n) = 1 \quad \text{iff} \quad \sum k_i b_i \ge t$$

for some real-valued constants $k_1, \dots, k_n$ and t.

(5) The language $L = a(a+b) + b(b+c)$ is not intersection-closed, yet $\mathcal{N}(L)$ is topological. Indeed, the derivatives of L are $\emptyset$, L, $a+b$, $b+c$ and ε, and $(a+b)\cap(b+c) = b$ is not a union of derivatives. However, the lattice Q_L is isomorphic to the lattice of all subsets of a four-element set, and hence distributive.

In [2] is was proved that $\mathcal{N}(L)$ is state-minimal provided that L is intersection-closed. The following theorem is a generalization of this result, as witnessed by Example 4.3(4),(5) above.

Theorem 4.4 *Let L be a regular language. If the minimal pointed nca $\mathcal{N}(L)$ is topological then its underlying nfa is state-minimal.*

Proof. (1) By Lemma 2.19, the categories of pointed JSL-dfas and pointed nondeterministic closure automata are equivalent. In fact, $\mathbb{G}_*$ has the associated equivalence $\mathbb{C}_* : \mathsf{Nca}_* \to \mathsf{Jdfa}_*$ defined by

$$\mathbb{C}_*(Z, R_a, F, I) = (CZ, CR_a, F', I) \quad \text{and} \quad \mathbb{C}_*\mathcal{B} = C\mathcal{B},$$

where $F' = \{A \in CZ : A \cap F \neq \emptyset\}$. Here we use the equivalence $C : \mathsf{Cl}_f \to \mathsf{JSL}_f$, see Remark 2.16. Observe that $I \in CZ$ is now a *single state* in a JSL-dfa.

(2) Given any pointed nfa $(N, I) = (Z, R_a, F, I)$ accepting L. Viewing (N, I) as a pointed nca $(\mathcal{N}, I)$ with identity closure and applying the above equivalence yields the JSL-dfa $(\mathcal{D}, I)$ where $\mathcal{D} = (\mathcal{P}Z, CR_a, F')$ is the subset construction. Let

$$Q = \{\mathcal{L}_\mathcal{D}(q) : q \in \mathcal{P}Z\}$$

be the set of languages accepted by the individual states in $\mathcal{D}$. Since these are precisely those languages accepted by the nfa N (varying the set of initial states), we deduce that Q is closed under finite unions and derivatives, so Q is a semilattice of regular languages under $\emptyset$ and $\cup$. It forms the carrier of a JSL-dfa $\mathcal{D}' = (Q, \delta_a, \hat{F})$ with transitions $K \xrightarrow{a} a^{-1}K$ and final states $\hat{F} := \{K \in Q : \varepsilon \in K\}$. Furthermore, $\mathcal{L}_\mathcal{D}$ defines a surjective Jdfa_*-morphism

$$\mathcal{L}_\mathcal{D} : (\mathcal{D}, I) \twoheadrightarrow (\mathcal{D}', L),$$

where $L \in Q$ because (N, I) accepts L. Now every surjective morphism between finite semilattices has the property that the domain has at least as many join-irreducibles as the codomain (since the join-irreducibles form the minimal generating set). Hence we have shown that

$$|J(Q)| \leq |J(\mathcal{P}Z)| = |Z|.$$

(3) Next apply the above construction to the underlying nfa of $\mathcal{N}(L)$. We obtain a JSL-dfa $\mathcal{D}_L = (Q_L, a^{-1}(-), \tilde{F})$ where Q_L is the semilattice of languages which this nfa can accept, varying the set of initial states. In fact:

(a) Q_L is precisely the set of unions of derivatives of L, see Proposition 3.16 and use that $\mathcal{N}(L)$ is reachable and (isomorphic to) a normalized nca by Corollary 3.12.

(b) Applying $\mathbb{G}_*$ to $(\mathcal{D}_L, L)$ one obtains the minimal pointed nca $\mathcal{N}(L)$. In particular, Z_L has states $J(Q_L)$ – see Lemma 2.19 and Proposition 3.24.

(c) Therefore $CZ_L = CGQ_L \cong Q_L$, since G and C define an equivalence.

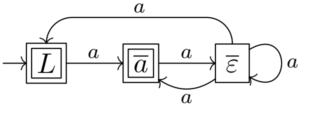
It follows from (a) that we have an injective Jdfa_*-morphism:

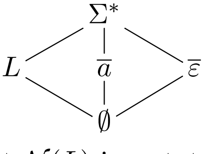
$$\iota : (\mathcal{D}_L, L) \hookrightarrow (\mathcal{D}', L).$$

Indeed $Q_L \subseteq Q$ because $L \in Q$ and the latter is closed under unions and derivatives. This inclusion certainly preserves unions. The transition structure and final states are inherited, so ι is well-defined.

(4) Now assume that $\mathcal{N}(L)$ is topological, hence $Q_L \cong CZ_L$ is distributive by Lemma 4.2. In view of (2), it remains to prove that $|J(Q_L)| \leq |J(Q)|$, for which we establish an auxiliary statement: given any finite distributive lattice D which is a sub semilattice of a finite semilattice S, we prove that $|J(D)| \leq |J(S)|$. Let $|J(S)| = n$, so we have a surjective join-semilattice homomorphism $\mathcal{P}n \twoheadrightarrow S$. By the self-duality of JSL_f we have an embedding of S^{op} into $(\mathcal{P}n)^{op} \cong \mathcal{P}n$. Thus any maximal chain in S (hence also in D) has at most n edges. Since the number of join-irreducibles in a finite distributive lattice equals the number of edges of any maximal chain [8, Corollary II.112], it follows that $|J(D)| \leq n = |J(S)|$. Now Q_L is a distributive sub-semilattice of Q by (3), so applying this result to $D = Q_L$ and $S = Q$ proves the theorem. □

Example 4.5 The converse of Theorem 4.4 is generally false. Let $L = \overline{aa}$ denote the complement of the singleton $\{aa\}$. The underlying nfa of $\mathcal{N}(L)$ and the lattice Q_L are depicted below:





Clearly Q_L is non-distributive but $\mathcal{N}(L)$ is a state-minimal nfa accepting L.

5 Conclusions and Future Work

It has been known since the early days of automata theory that nondeterministic finite automata suffer from two unpleasant phenomena, as opposed to their deterministic counterparts: the lack of canonical machines, and the lack of state-minimization. In this paper, we have demonstrated that both problems disappear when one augments nfas with a closure structure. Based on the equivalence between JSL-dfas and nondeterministic closure automata, we derived the "right" notion of minimality, which allowed us to establish the existence of a unique minimal nca for each regular language along with a Brzozowski-style construction method. Furthermore, by restricting to ncas with topological closure, we identified a very natural class of *canonical* state-minimal nfas.

One open question that we leave for future work is to what extent our main result (Theorem 4.4) can be reversed, that is, under which conditions the state-minimality of $\mathcal{N}(L)$ implies topologicity.

Another issue we aim to address in the future are the complexity-related implications of our results. Although the general state minimization problem for nfas is known to be PSPACE-complete, a good implementation of our operators reach and rev could lead to efficient state minimization procedures for the class of topological automata, and possibly even larger classes of automata.

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A Appendix

We prove the claim made in the Introduction that the category JSL_f of finite semilattices arises as a free cocompletion of Rel_f. More precisely, JSL_f is both

(a) a free cocompletion of Rel_f under reflexive coequalizers, and

(b) a conservative cocompletion of Rel_f under finite colimits.

Let us us first recall the general concepts.

Definition A.1 Let $\mathcal{A}$ be a category. A *free cocompletion under reflexive coequalizers* of $\mathcal{A}$ is a full embedding $E : \mathcal{A} \hookrightarrow \mathcal{A}'$ such that

(i) $\mathcal{A}'$ has reflexive coequalizers (i.e., coequalizers of pairs of retractions $f, g : X \to Y$ having a common coretraction $d : Y \to X$, $fd = \mathsf{id}_Y = gd$).

(ii) Every functor $F : \mathcal{A} \to \mathcal{B}$, where $\mathcal{B}$ has reflexive coequalizers, has an extension $F' : \mathcal{A}' \to \mathcal{B}$ (i.e., $F'E \cong F$) preserving reflexive coequalizers, which is unique up to natural isomorphism.

Definition A.2 Let $\mathcal{A}$ be a category with finite coproducts. A *conservative finite cocompletion* of $\mathcal{A}$ is a full embedding $E : \mathcal{A} \hookrightarrow \mathcal{A}'$ such that

(i*) E preserves finite coproducts and $\mathcal{A}'$ is finitely cocomplete.

(ii*) Every finite-coproduct preserving functor $F : \mathcal{A} \to \mathcal{B}$, where $\mathcal{B}$ is finitely cocomplete, has an extension to a finite-colimit preserving functor $F' : \mathcal{A}' \to \mathcal{B}$ (i.e. $F'E \cong F$) which is unique up to natural isomorphism.

Example A.3 Let $\mathcal{V}$ be a finitary variety, and let $\mathcal{V}_{fp}$ and $\mathcal{V}_{fgf}$ be the full subcategories of all finite presentable and finitely generated free algebras, respectively. Then $\mathcal{V}_{fp}$ is a free cocompletion of $\mathcal{V}_{fgf}$ under reflexive coequalizers, as well as a conservative finite cocompletion of $\mathcal{V}_{fgf}$, see [3, Theorem 7.3 and Theorem 17.11].

Corollary A.4 JSL_f *is a free cocompletion of* Rel_f *under reflexive coequalizers, as well as a conservative finite cocompletion of* Rel_f.

Proof. Apply the above example to the variety $\mathcal{V} = \mathsf{JSL}$. Here $\mathcal{V}_{fp} = \mathsf{JSL}_f$. Since finitely generated free semilattices are power sets $\mathcal{P}n$ $(n < \omega)$, it is easy to see that $\mathcal{V}_{fgf}$ is equivalent to Rel_f. Indeed, the functor $\mathcal{P} : \mathsf{Rel}_f \to \mathsf{JSL}_{fgf}$ assigning to each finite set n its power set and to each relation $h : n \to m$ the semilattice morphism

$$\mathcal{P}h : \mathcal{P}n \to \mathcal{P}m, \quad A \mapsto h[A],$$

is an equivalence of categories. □