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Robustness of f- and g-generated Fuzzy (Co)Implications: The Yager's (Co)Implication Case Study

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Abstract

This paper studies the robustness of intuitionistic fuzzy implications in fuzzy reasoning based on Atanassov's intuitionistic fuzzy logic. Starting with an evaluation of the sensitivity in representable fuzzy negations, we apply the results in the Yager's classes of fuzzy implications called the f- and g-generated fuzzy implications. The paper formally states that the robustness preserves the projection functions in such class and also discusses their corresponding dual operators.

Keywords: Robustness analysis, Intuitionistic fuzzy logic, Yager's implications, f- and g-generated implications.

1 Introduction

Since Yager's classes of fuzzy implications called the f- and g-generated implications [24] have been used in common sense reasoning, there is a practical need for intuitionistic fuzzy versions of these operations, i.e., an operation $I_f(x, y)$ that uses the membership degrees $\mu_A(u) = a$ and $\mu_B(u) = b$ of two intuitionistic fuzzy sets A and B to estimate the uncertainty degree of confidence in the statement

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$A \rightarrow_f B$. These operations are also extensions of the corresponding crisp operation: $I_f(0,1){=}I_f(0,0){=}1$, $I_f(1,0){=}0$ and $I_f(1,0) = 1$.

The concepts of maximum and average perturbations of fuzzy sets [25], estimating the maximum and average perturbation parameters for various methods of fuzzy reasoning is relevant for systems based on fuzzy logic (FL) and as a consequence for intuitionistic fuzzy logic (IFL).

1.1 Main related works

In [15], Li et al. study properties of some measures of robustness (or sensitivity) of fuzzy connectives and implication operators and discuss their relationships with perturbation properties of fuzzy sets. Many other works have discussed the robustness analysis also including the δ-sensitive approach, see e.g. [14], [15], [16], [17] and [18].

This paper extends the δ-sensitivity study of some intuitionistic fuzzy connectives (IFCs) according with results previously presented in [15], based on Atanassov's Intuitionistic Fuzzy Logic (A-IFL), as presented in [1].

In [24], some properties of the Yager's classes of fuzzy implications including the h-generated implications are discussed, describing their relationships amongst themselves and with the well established strong and residual implication classes [9].

Additionally, in [19], the Bandler-Kohout subproduct relational inference system with the fuzzy implication interpreted as the Yager's classes of implications are reported, studying many of the desirable properties as interpolativity, continuity, robustness and computational efficiency, expanding the choice of operations available to practitioners.

A semantic behaviour of a fuzzy rule model is proposed in [13] as a pair of fuzzy implication and modus ponens generating function used for inference. Such methodology is applied to Yager's models which are obtained from Yager implication function. By Yager's implicative implication, it is shown to be midway between the usual residual and strong implications generated from the product t-norm. In fact, Yager's implication belongs to a more general family of implications that can also be generated from the t-norms.

Such analysis can improve the study of the stability of systems based on intuitionistic fuzzy rules. The notion of δ-sensitivity of fuzzy connectives in the fuzzy intuitionistic approach, which is characterized by the non-complementary relationship between the membership and non-membership functions, as proposed in [2], is considered in this work.

1.2 Main contribution of the paper

Following preliminary studies introduced in [21], this paper considers the robustness analysis defined on δ-sensitivity of the Atanassov intuitionistic fuzzy approach of the Yager's implication classes, the f- and g-generated implications [24], focusing on their pointwise components obtained by the projections related to membership and non-membership functions.

Extending previous work in [20], [21] and [27], the paper provides an interpretation of IFCs based on δ-sensitivity which is closely related to truth and non-truth in conditional fuzzy rules. Thus, not only the robustness of representable intuitionistic fuzzy negations are proposed but also the analyse of their perturbations based on δ-sensitive of the Yager's implication classes and their dual construction, the f- and g-generated coimplications.

As the main result, the robustness of intuitionistic fuzzy f- and g-generated (co)implications can be expressed by the robustness of their arguments, by corresponding fuzzy f- and g-generated (co)implications. These results are summarized in commutative diagrams showing that the δ-sensitivity operator commutes with the N_S-dual operator, by considering both approaches of FL and IFL.

1.3 Outline of the paper

The paper is organized as follows.

Firstly, the preliminaries describe the basic concepts of FCs and IFCs.

General results of robustness of FCs are stated in Sections 3 including the study of δ-sensitivity of fuzzy negations and fuzzy implications, mainly related to f- and g-generated implications.

In Section 4, we consider $\tilde{\delta} = (\delta_1, \delta_2) \in U^2$, in the δ-sensitivity of an intuitionistic operator f_I at point $\tilde{\mathbf{x}} \in \tilde{U}^n$ in terms of its left-projection $(l_{\tilde{U}^n}(\tilde{\mathbf{x}}))$ and right-projection $(r_{\tilde{U}^n}(\tilde{\mathbf{x}}))$, which are related to the δ-sensitive of the membership and non-membership degrees of an element $x \in \chi$ associated with the IFS $f_I(\tilde{U}^n)$. Thus, the study of δ-sensitivity of intuitionistic fuzzy negations and intuitionistic fuzzy implications, mainly related to f- and g-generated implications are considered.

Final remarks are reported in the conclusion.

2 Preliminaries

By recalling some basic concepts of FL and IFL, we firstly report notions of FL as conceived by Zadeh [26] concerning negations and (co)implications [10]. Relevant papers studied different classes of fuzzy implications, see [5,11,12] and [24].

Let $U = [0, 1]$ be the unit interval of real numbers. Recall that a function $N : U \to U$ is a **fuzzy negation** if it satisfies, for all$\in U$ the properties:

N1: $N(0)=1$ and $N(1)=0$; **N2**: If $x \geq y$ then $N(x) \leq N(y)$.

A fuzzy negation satisfying the involutive property:

N3: $N(N(x)) = x, \forall x \in U$; is called a **strong fuzzy negation** (SFN), e.g. the standard negation $N_S(x) = 1 - x$.

When $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in U^n$ and N is a fuzzy negation, the following notation is considered: $N(\mathbf{x}) = (N(x_1), N(x_2), \ldots, N(x_n))$.

Let N be a negation. The N**-dual function** of $f : U^n \to U$ is given by:

$$f_N(\mathbf{x}) = N(f(N(\mathbf{x}))), \ \forall \mathbf{x} \in U^n. \tag{1}$$

An **implicator operator** $I : U^2 \to U$ extends the classical implication:

I0: $I(1,1){=}I(0,1){=}I(0,0){=}1$, $I(1,0){=}0$.

Definition 2.1 [10] In the sense of J. Fodor and M. Roubens, when $x, y, z \in U^2$, a fuzzy implication $I : U^2 \to U$ is an implicator also verifying:

I1: $I(x,y) \geq I(z,y)$ if $x \leq z$ (first place antitonicity);

I2: $I(x,y) \leq I(x,z)$ if $y \leq z$ (second place isotonicity);

I3: $I(0,y) = 1$ (dominance of falsity);

I4: $I(x,1) = 1$ (boundary condition);

Analogously, a coimplicator $J : U^2 \to U$ verifies the conditions:

***J*0**: $J(0,0){=}J(1,0){=}J(1,1){=}0, J(0,1){=}1$.

It is immediate that a fuzzy coimplication is a coimplicator analogously defined as a fuzzy implication, replacing **I3** and **I4** in Definition 2.1 by **J3** :$J(x,0) = 0$ and **J4** :$J(1,y) = 0$, respectively.

Among many classes of fuzzy (co)implication functions (see, e.g., [6] and [7]), the class of **axiomatic representation** of fuzzy implications, named A-implications, is described in [23] in terms of non-commutativity property related to t-norms. The A-implications are based on a subset of the axioms listed in [10]. In this paper, we focus on Yager's implication.

In [24], Yager proposed two new classes of fuzzy implications, called f-generated implications and g-generated implications, which can not be fulfilled in the above presented classes.

By [24, Sect.3], let $f : [0,1] \to [0,\infty]$ be an f-generator, which means, a strictly decreasing and continuous function such that $f(0) = 1$ and its pseudo-inverse $f^{(-1)} : [0,\infty] \to [0,1]$ is defined by: $f^{(-1)}(x) = f^{-1}(x), \text{if } x \leq f(0);$ and $0, \text{otherwise}$. When $0{\cdot}\infty{=}0$, an f-generated fuzzy implication $I_f{:}U^2{\to}U$ is given by

$$I_f(x,y) = f^{(-1)}\left(x \cdot f(y)\right).$$

Moreover, let $g : [0,1] \to [0,\infty]$ be a g-generator, which means, a strictly increasing and continuous function such that $g(0) = 0$ and its pseudo-inverse $g^{(-1)} : [0,\infty] \to [0,1]$ is defined by: $g^{(-1)}(x) = g^{-1}(x), \text{if } x \leq g(1);$ and $1, \text{otherwise}$. When $\frac{1}{0}{=}\infty$ and $\infty \cdot 0 = \infty$, a g-generated fuzzy implication $I_g{:}U^2{\to}U$ is given by

$$I_g(x,y) = g^{(-1)}\left(\frac{1}{x} \cdot g(y)\right).$$

Proposition 2.2 *[24] The binary function $I_Y, (J_Y){:}U^2{\to}U$ given by*

$$I_{Y_f}(x,y) = 1, \ \text{if } x = y = 0; \text{and } I_{Y_f}(x,y) = y^x, \text{otherwise}. \tag{2}$$

$$J_{Y_f}(x,y) = 0, \ \text{if } x = y = 1; \text{and } J_{Y_f}(x,y) = 1 - (1-y)^{1-x}, \text{otherwise}. \tag{3}$$

is an f-generated fuzzy (co)implication called Yager(co)implication. Additionally, the function $I_g, (J_g){:}U^2{\to}U$ given by

$$I_{Y_g}(x,y) = 1, \text{ if } x = y = 0; \text{ and } I_{Y_g}(x,y) = 1 - (1-y)^{\frac{1}{x}}, \text{ otherwise.} \quad (4)$$

$$J_{Y_g}(x,y) = 0, \text{ if } x = y = 1; \text{ and } J_{Y_g}(x,y) = y^{\frac{1}{1-x}}, \text{ otherwise.} \quad (5)$$

is an g-generated fuzzy (co)implication.

Example 2.3 [4, Examples 3 and 4] Eqs. (2) and Eq. (4) (Eqs. (3) and Eq. (5)) define fuzzy (co)implications, according to Definition 2.1.

Corollary 2.4 *Both functions (I_{Y_g}, J_{Y_g}) and (I_{Y_f}, J_{Y_f}) define pairs of N_S-dual fuzzy (co)implications.*

Proof. Straightforward. □

2.1 Intuitionistic fuzzy connectives

This section briefly study intuitionistic fuzzy connectives. For further references, see [1,2,3,8] and [9].

According to [1], an intuitionistic fuzzy set (IFS) A_I in a non-empty, universe χ, is expressed as $A_I = \{(x, \mu_A(x), \nu_A(x)) : x \in \chi, \mu_A(x) + \nu_A(x)) \leq 1\}$. Thus, an intuitionistic fuzzy truth value of an element x in an IFS A_I is related to the ordered pair $(\mu_A(x), \nu_A(x))$. Moreover, an IFS A_I generalizes a FS $A = \{(x, \mu_A(x)) : x \in \chi\}$, since $\nu_A(x)$, which means that the non-membership degree of an element x, is less than or equal to the complement of its membership degree $\mu_A(x)$, and therefore $\nu_A(x)$ is not necessarily equal to its complement $1 - \mu_A(x)$.

Let $\tilde{U} = \{(x_1, x_2) \in U^2 | x_1 \leq N_S(x_2)\}$ be the set of all intuitionistic fuzzy values and $l_{\tilde{U}}, r_{\tilde{U}} : \tilde{U} \to U$ be the projection functions on $\tilde{U}$, which are given by $l_{\tilde{U}}(\tilde{x}) = l_{\tilde{U}}(x_1, x_2) = x_1$ and $r_{\tilde{U}}(\tilde{x}) = r_{\tilde{U}}(x_1, x_2) = x_2$, respectively.

Thus, for all $\tilde{\mathbf{x}} = (\tilde{x}_1, \ldots, \tilde{x}_n) \in \tilde{U}^n$, such that $\tilde{x}_i = (x_{i1}, x_{i2})$ and $x_{i1} \leq N_S(x_{i2})$ when $1 \leq i \leq n$, considering $l_{\tilde{U}^n}, r_{\tilde{U}^n} : \tilde{U}^n \to U^n$ as the projections given by:

$$l_{\tilde{U}^n}(\tilde{\mathbf{x}}) = (l_{\tilde{U}}(\tilde{x}_1), l_{\tilde{U}}(\tilde{x}_2), \ldots, l_{\tilde{U}}(\tilde{x}_n)) = (x_{11}, x_{21}, \ldots x_{n1}); \quad (6)$$

$$r_{\tilde{U}^n}(\tilde{\mathbf{x}}) = (r_{\tilde{U}}(\tilde{x}_1), r_{\tilde{U}}(\tilde{x}_2), \ldots r_{\tilde{U}}(\tilde{x}_n)) = (x_{12}, x_{22}, \ldots x_{n2}). \quad (7)$$

By [2], for $\tilde{x}, \tilde{y} \in \tilde{U}$, the order relation $\leq_{\tilde{U}}$ is given as

$$\tilde{x} \leq_{\tilde{U}} \tilde{y} \Leftrightarrow x_1 \leq y_1 \text{ and } x_2 \geq y_2, such that \tilde{0} = (0,1) \leq_{\tilde{U}} \tilde{x} \text{ and } \tilde{1} = (1,0) \geq_{\tilde{U}} \tilde{x} \quad (8)$$

Moreover, the folowing expression is known:

$$\tilde{x} \preceq_{\tilde{U}} \tilde{y} \Leftrightarrow x_1 \leq y_1 \text{ and } x_2 \leq y_2, \quad (9)$$

An *intuitionistic fuzzy negation* (IFN shortly) $N_I : \tilde{U} \to \tilde{U}$ satisfies, for all $\tilde{x}, \tilde{y} \in \tilde{U}$, the following properties:

$\mathbf{N_I1}$: $N_I(\tilde{0}) = N_I(0,1) = \tilde{1}$ and $N_I(\tilde{1}) = N_I(1,0) = \tilde{0}$;

$\mathbf{N_I2}$: If $\tilde{x} \geq \tilde{y}$ then $N_I(\tilde{x}) \leq N_I(\tilde{y})$.

Additionally, N_I is a **strong intuitionistic fuzzy negation** (SIFN) verifying the condition:

$\mathbf{N_I3}$: $N_I(N_I(\tilde{x})) = \tilde{x}, \forall \tilde{x} \in \tilde{U}$.

Consider N_I as IFN in $\tilde{U}$ and $\tilde{f} : \tilde{U}^n \to \tilde{U}$. For all $\tilde{\mathbf{x}} = (\tilde{x}_1, \ldots, \tilde{x}_n) \in \tilde{U}^n$, the N_I**-dual intuitionistic function of** $\tilde{f}$, denoted by $\tilde{f}_{N_I} : \tilde{U}^n \to \tilde{U}$, is given by:

$$\tilde{f}_{N_I}(\tilde{\mathbf{x}}) = N_I(\tilde{f}(N_I(\tilde{x}_1), \ldots, N_I(\tilde{x}_n))). \tag{10}$$

When $\tilde{N}_I$ is a SIFN, $\tilde{f}$ is a self-dual intuitionistic function. Additionally, by [3, Theorem 1] [8], a SIFN $N_I : \tilde{U} \to \tilde{U}$ is a SIFN iff there exists a SFN $N : U \to U$ such that:

$$N_I(\tilde{x}) = (N(N_S(x_2)), N_S(N(x_1))), \tag{11}$$

Additionally, if $N = N_S$, Eq. 11 can be reduced to $N_I(\tilde{x}) = (x_2, x_1)$.

According with [7, Definition 3], an Atanassov intuitionistic fuzzy implication $I_I : \widetilde{U}^2 \to \widetilde{U}$ is an intuitionistic fuzzy implicator such that, the analogous conditions from $\mathbf{I}_I\mathbf{1}$ to $\mathbf{I}_I\mathbf{4}$ in Definition 2.1 are verified with the additional property:

$\mathbf{I}_I\mathbf{5}$**:** If $\tilde{x} = (x_1, x_2)$ such that $x_1 + x_2 = 1$ it holds that $N_S(x_1 + x_2) = 0$

Thus, recovering Definition 2.1 of a fuzzy implication in the sense of J. Fodor and M. Roubens' work [10], an intuitionistic fuzzy implication also reproduces fuzzy (co)implications if, for all $\tilde{x} = (x_1, x_2)$, $\tilde{y} = (y_1, y_2) \in \widetilde{U}$ we have $x_1 = N_S(x_2)$ and $y_1 = N_S(y_2)$.

According to [2], another way of defining an operator I_I is to consider boundary conditions in $\mathbf{I}_I0$ and properties $\mathbf{I}_I1$ and $\mathbf{I}_I2$. In [7], Bustince et al. constructed fuzzy implications for intuitionistic fuzzy logic, in the sense of [7, Definition 3], based on aggregation operators and SFNs.

Considering the above results, the functions $I_I(J_I) : \widetilde{U}^2 \to \widetilde{U}$ are **representable fuzzy (co)implications** based on SFN $N_S : \widetilde{U} \to \widetilde{U}$ if there exist fuzzy (co)implications $I_a, I_b(J_a, J_b) : U^2 \to U$ such that, for all $\tilde{x}, \tilde{y} \in \tilde{U}$, the following holds:

$$I_I(\tilde{x}, \tilde{y}) = (I_a(N_S(x_2), y_1), N_S(I_b(x_1, N_S(y_2)))); \tag{12}$$

$$J_I(\tilde{x}, \tilde{y}) = (J_a(N_S(x_2), y_1), N_S(J_b(x_1, N_S(y_2)))). \tag{13}$$

Proposition 2.5 *Let* $I_f, I_g, (J_f, J_g) : U^2 \to U$ *be* f*- (*g*-)generated fuzzy (co)implications defined in Proposition 2.2. The functions* $I_{Y_{f\,I}}, I_{Y_{g\,I}}(J_{Y_{f\,I}}, J_{Y_{g\,I}}) : \widetilde{U}^2 \to \widetilde{U}$ *are representable fuzzy (co)implications expressed as:*

$$I_{Y_{f\,I}}(\tilde{x}, \tilde{y}) = \left(y_1^{1-x_2}, 1 - (1 - y_2)^{x_1}\right); J_{Y_{f\,I}}(\tilde{x}, \tilde{y}) = \left(1 - (1 - y_1)^{x_2}, y_1), y_2^{1-x_1}\right) \tag{14}$$

$$I_{Y_{g\,I}}(\tilde{x}, \tilde{y}) = \left(1 - (1 - y_1)^{\frac{1}{1-x_2}}, y_2^{\frac{1}{x_1}}\right); J_{Y_{g\,I}}(\tilde{x}, \tilde{y}) = \left(y_1^{\frac{1}{x_2}}, 1 - (1 - y_2)^{\frac{1}{1-x_1}}\right). \tag{15}$$

Proof. Eq.(14) follows from Eq.(12) by taking $I_a = I_b = I_{Y_f}$ and $J_a = J_b = J_{Y_f}$. And, Eq.(15) follows from Eq.(13) by taking $I_a = I_b = I_{Y_g}$ and $J_a = J_b = J_{Y_g}$. □

3 Pointwise sensitivity of fuzzy connectives

Based on [15] and [20], the study of a δ-sensitivity of $n-$order function f at point $\mathbf{x}$ on the domain U is considered, in the context of robustness of fuzzy logic, mainly related to the class of (S,N)-implications.

Definition 3.1 [15, Definition 1] Let $f : U^n \to U$ be an $n-$order function, $\delta \in U$ and $\mathbf{x} = (x_1, x_2, \dots x_n)$, $\mathbf{y} = (y_1, y_2, \dots y_n) \in U^n$. The **$\delta$-sensitivity of f at point x**, denoted by $\Delta_f(\mathbf{x}, \delta)$, is given by

$$\Delta_f(\mathbf{x}, \delta) = \sup\{|f(\mathbf{x}) - f(\mathbf{y})| : \mathbf{y} \in U^n \text{ and } \bigvee(\mathbf{x}, \mathbf{y}) \leq \delta\} \tag{16}$$

wherever $\bigvee(\mathbf{x}, \mathbf{y}) = \max\{|x_i - y_i| : i = 1, \dots, n\}$. Additionally, **the maximum δ sensitivity of f**, denoted as $\Delta_f(\delta)$, is defined as follows:

$$\Delta_f(\delta) = \bigvee_{\mathbf{x} \in U^n} \Delta_f(\mathbf{x}, \delta). \tag{17}$$

Proposition 3.2 *[20, Theorem 1] If $N = N_S$ and f_N is the N-dual function of f then **the sensitivity of f_N at point x** is given by*

$$\Delta_{f_N}(\mathbf{x}, \delta) = \Delta_f(N(\mathbf{x}), \delta). \tag{18}$$

3.1 δ sensitivity of $f-$ and $g-$generated fuzzy (co)implications

Now, we investigate the δ-sensitivity in FCs, in terms of Definition 3.1 based on results previously presented in [15]. In order to provide an easier notation, when $f : U^2 \to U$ and $\mathbf{x} = (x, y) \in U^2$, consider the following notations:

$$f\lfloor \mathbf{x} \rceil \equiv f((x - \delta) \vee 0, (y + \delta) \wedge 1);\ f\lceil \mathbf{x} \rfloor \equiv f((x + \delta) \wedge 1, (y - \delta) \vee 0).$$

Proposition 3.3 *[15, Theorem 1] Consider $f : U^2 \to U$, $\delta \in U$ and $\mathbf{x} = (x, y) \in U^2$. If f verifies both properties, first place antitonicity (**I1**) and second place isotonicity (**I2**), then:*

$$\Delta_f(\mathbf{x}, \delta) = (f(\mathbf{x}) - f\lceil \mathbf{x} \rfloor) \vee (f\lfloor \mathbf{x} \rceil - f(\mathbf{x})). \tag{19}$$

Proposition 3.4 *The δ-sensitivity of the functions $I_{Y_f}, J_{Y_f}, I_{Y_g}, J_{Y_g}$ defined in Proposition 2.2 by Eqs. (2)-(5) is given by Eq. (19).*

Proof. Straightforward Proposition 3.3 and Corollary 2.3. □

3.2 Maximum sensitivity of $f-$ and $g-$generated fuzzy (co)implications

In the following, we consider the maximum δ sensitivity of the f-and g-generated fuzzy implications I_{Y_f} and I_{Y_g}, showing that they coincide with their N_S-dual constructions, (J_{Y_f}) and (J_{Y_f}), by considering the endpoints of unitary interval in U^2.

Remark 3.5 Based on Eqs. (19) and (18), also including results in Proposition18, we obtain the following:

$\Delta_{I_{Y_f}}((0,0), \delta) = (1 - 0) \vee (1 - 1) = 1 = \Delta_{J_{Y_f}}((1,1), \delta)$;

$\Delta_{I_{Y_f}}((0,1), \delta) = (1 - (1 - \delta)^{\delta}) \vee (1 - 1) = 1 - (1 - \delta)^{\delta} = \Delta_{J_{Y_f}}((1,0), \delta)$;

$\Delta_{I_{Y_f}}((1,1), \delta) = (1 - (1 - \delta)) \vee (1 - 1) = \delta = \Delta_{J_Y}((0,0), \delta)$;

$\Delta_{I_{Y_f}}((1,0), \delta) = (0 - 0) \vee (\delta 1 - \delta - 0) = \delta^{(1-\delta)} = \Delta_{J_{Y_f}}((0,1), \delta)$.

Remark 3.6 Based on Eqs. (19) and (18), also including results in Proposition18, we obtain the following:
$\Delta_{I_{Y_g}}((0,0),\delta) = 1^{\frac{1}{\delta}} \vee (-1-\delta)^{\infty} = 1 = \Delta_{J_{Y_g}}((1,1),\delta)$;
$\Delta_{I_{Y_g}}((0,1),\delta) = \delta^{\frac{1}{\delta}} \vee 0 = \delta^{\frac{1}{\delta}} = \Delta_{J_{Y_g}}((1,0),\delta)$;
$\Delta_{I_{Y_g}}((1,1),\delta) = \delta \vee 0 = \delta = \Delta_{J_{Y_g}}((0,0),\delta)$;
$\Delta_{I_{Y_g}}((1,0),\delta) = 0 \vee 1-(1-\delta)^{1-\delta} = 1-(1-\delta)^{\frac{1}{1-\delta}} = \Delta_{J_{Y_g}}((0,1),\delta)$.

Proposition 3.7 *The maximum sensitivity of the f- and g-generated fuzzy (co)implications $I_{Y_f}, J_{Y_f}, I_{Y_g}, J_{Y_g}$, as defined in Proposition 2.2 by Eqs. (2)-(5), is given as follows:*

$$\Delta_{I_{Y_f}}(\delta) = 1 = \Delta_{I_{Y_g}}(\delta) \ \text{ and } \ \Delta_{J_{Y_g}}(\delta) = 1 = \Delta_{J_{Y_f}}((\delta); \tag{20}$$

$$\Delta_{I_{Y_g}}(\delta) = 1 = \Delta_{I_{Y_g}}(\delta) \ \text{ and } \ \Delta_{J_{Y_g}}(\delta) = 1 = \Delta_{J_{Y_g}}(\delta). \tag{21}$$

Proof. Straightforward from Remarks 3.5 and 3.6. □

Based on Proposition 3.7, the maximum δ sensitivity of the f- and g-generated fuzzy (co)implications I_{Y_f} and I_{Y_g} is related to the endpoints $(0,0)$ and $(1,1)$, respectively. Such results are closely related to their corresponding definition.

Moreover, based on Remarks 3.5 and 3.6, one can observe that I_{Y_g} is more robust than I_{Y_f} at point $(0,1)$, since $\Delta_{I_{Y_f}}((0,1),\delta) > \Delta_{I_{Y_g}}((0,1),\delta)$. In the converse, I_{Y_f} is more robust than I_{Y_g} at point $(1,0)$, since $\Delta_{I_{Y_f}}((1,0),\delta) < \Delta_{I_{Y_g}}((1,0),\delta)$.

4 Robustness of intuitionistic fuzzy connectives

In this section, we consider $Y_I \in \{Y_{f_I}, Y_{g_I}\}$ which means, Y_I denotes either $f-$ or $g-$generated intuitionistic fuzzy implication.

In order to provide a formal definition of robustness which can be applied to n-order intuitionistic fuzzy f- and g-generated implication operators, consider definition of the δ-sensitivity of an $n-$order fuzzy negation $f_I : \tilde{U}^n \to \tilde{U}$ at point $\tilde{\mathbf{x}} = (\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2, \ldots, \tilde{\mathbf{x}}_n) \in \tilde{U}^n$.

Thus, when $\tilde{\delta} = (\delta_1, \delta_2) \in U^2$, the δ-sensitivity of an intuitionistic operator f_I at point $\tilde{\mathbf{x}} \in \tilde{U}^n$ is defined in terms of its left-projection $(l_{\tilde{U}^n}(\tilde{\mathbf{x}}))$ and right-projection $(r_{\tilde{U}^n}(\tilde{\mathbf{x}}))$, which are related to the δ-sensitive of the membership and non-membership degrees of an element $x \in \chi$ associated with the IFS $f_I(\tilde{U}^n)$.

Definition 4.1 For $\tilde{\mathbf{y}} \in \tilde{U}^n$, the **$\delta$-sensitivity of f_I at point $\tilde{\mathbf{x}}$** is defined by

$$\Delta_{f_I}(\tilde{\mathbf{x}},\tilde{\delta}) = \sup\{|f_I(\tilde{\mathbf{x}}) - f_I(\tilde{\mathbf{y}})| : \bigvee(l_{\tilde{U}^n}(\tilde{\mathbf{x}}), l_{\tilde{U}^n}(\tilde{\mathbf{y}})) \le \delta_1 \text{ and } \bigwedge(r_{\tilde{U}^n}(\tilde{\mathbf{x}}), r_{\tilde{U}^n}(\tilde{\mathbf{y}})) \le \delta_2\},$$

when $\bigvee(\mathbf{x},\mathbf{y}) = \max\{|x_{i1} - y_{i1}| : i = 1,\ldots,n\}$, $\bigwedge(\mathbf{x},\mathbf{y}) = \min\{|x_{i2} - y_{i2}| : i = 1,\ldots,n\}$.

4.1 Preserving the robustness of representable negations

The next proposition states that the pointwise sensitivity is preserved by the projection functions applied to an intuitionistic fuzzy negation (IFN) which is representable in the same sense of [3] and [8].

Proposition 4.2 *[27] Let $N_I : \tilde{U}^n \to \tilde{U}$ be a SIFN as defined by Eq.(11). When $\tilde{\delta} = (\delta_1, \delta_2) \in U^2$ and $\tilde{\mathbf{x}} \in \tilde{U}^n$, the **$\delta$-sensitivity of** N_I **at point** $\tilde{\mathbf{x}}$, is given by*

$$\Delta_{N_I}(\tilde{\mathbf{x}}, \tilde{\delta}) = (\Delta_{N \circ N_S}(r_{\tilde{U}^n}(\tilde{\mathbf{x}}), \delta_2), \Delta_{N_S \circ N}(l_{\tilde{U}^n}(\tilde{\mathbf{x}}), \delta_1)). \tag{22}$$

Corollary 4.3 *[27] When $\tilde{\delta} = (\delta_1, \delta_2) \in U^2$, $N_I = N_{SI}$ and $\tilde{\mathbf{x}} \in \tilde{U}^n$, the **$\delta$-sensitivity of** N_I **at point** $\tilde{\mathbf{x}}$, can also be expressed as*

$$l_{\tilde{U}}(\Delta_{N_I}(\tilde{\mathbf{x}}, \tilde{\delta}))=\Delta_{l_{\tilde{U}} \circ N_I}(r_{\tilde{U}^n}(\tilde{\mathbf{x}}), r_{\tilde{U}}(\tilde{\delta})); \tag{23}$$

$$r_{\tilde{U}}(\Delta_{N_I}(\tilde{\mathbf{x}}, \tilde{\delta}))=\Delta_{r_{\tilde{U}} \circ N_I}(l_{\tilde{U}^n}(\tilde{\mathbf{x}}), l_{\tilde{U}}(\tilde{\delta})). \tag{24}$$

In particular, we have that $\Delta_{N_{SI}}(\tilde{\mathbf{x}}, \tilde{\delta}) = (\delta_2, \delta_1)$.

The diagram below summarizes the main results of Proposition 4.2 and Corollary 4.3: the robustness of representable IFNs can be expressed by robustness of their arguments:

$$\begin{tikzcd}
(\tilde{\mathbf{x}}, \delta) \arrow[r, "Eq.\ (22)"] \arrow[d, "Eqs.(6)(7)"] & \Delta_{N_{SI}}(\tilde{\mathbf{x}}, \delta) \arrow[d, "Eq.(6)(7)"'] \\
(r_{\tilde{U}^2}(\tilde{\mathbf{x}}, \delta), l_{\tilde{U}^2}(\tilde{\mathbf{x}}, \delta)) \arrow[r, "Eqs.\ (23),\ (24)"] & (\Delta_{r_{\tilde{U}} \circ N_I}(\tilde{\mathbf{x}}, \delta), \Delta_{r_{\tilde{U}} \circ N_I}(\tilde{\mathbf{x}}, \delta))
\end{tikzcd}$$

Fig. 1. Robustness operator on the class of representable IFNs.

4.2 Robustness of intuitionistic fuzzy functions and N_{SI}-dual constructions

From the δ-sensitivity of $f_I : \tilde{U}^n \to \tilde{U}$ at point $\tilde{\mathbf{x}}$ one can obtain the δ-sensitivity of corresponding dual construction, as described in the following proposition:

Proposition 4.4 *Let $f_I : \tilde{U}^n \to \tilde{U}$ be a representable fuzzy (co)implication defined by Eq.(12) (Eq.(13)) and $\Delta_{f_I}(\tilde{\mathbf{x}}, \delta)$ be the δ-sensitivity of f_I at point $\tilde{\mathbf{x}}$. When $\tilde{\delta} = (\delta_1, \delta_2) \in U^2$, $N_I = N_{SI}$ and $f_{I_{N_I}}$ is the N_I-dual function of f_I, **the δ-sensitivity of** $f_{I_{N_I}}$ **at point** $\tilde{\mathbf{x}}$ is given by*

$$\Delta_{(f_I)_{N_{SI}}}(\tilde{\mathbf{x}}, \tilde{\delta}) = \left(\Delta_{f_I}(N_{SI}(\tilde{\mathbf{x}}), \tilde{\delta})\right). \tag{25}$$

Proof. For all $\tilde{\mathbf{x}}, \tilde{\mathbf{y}} \in \tilde{U}^n$, it holds that:

$$\begin{aligned}
\Delta_{(f_I)_{N_S}}(\tilde{\mathbf{x}}, \tilde{\delta}) &= (\Delta_{f_N}(N_S(x_{12}), x_{21}), \delta_1), \Delta_{N_S \circ f_N}(x_{11}, N_S(x_{22})), \delta_2)) \text{ by Eqs. (6),(7)} \\
&= (\Delta_f((x_{12}, N_S(x_{21})), \delta_1), \Delta_{N_S \circ f}((N_S(x_{11}), x_{22}), \delta_2)) \text{ by Eq.(18)} \\
&= \left(\Delta_{f_I}(l_{\tilde{U}}(N_I(\tilde{\mathbf{x}}), \tilde{\delta}), (\Delta_{f_I}(r_{\tilde{U}}(N_I(\tilde{\mathbf{x}}), \tilde{\delta}))\right) \text{ by Eqs.(23),(24)} \\
&= (\Delta_{f_I}(N_{SI}(\tilde{\mathbf{x}}), N_{SI}(\delta))) \text{ by Eqs.(6),(7)}
\end{aligned}$$

Therefore Eq.(25) holds. □

4.3 δ-sensitivity of $f-$ and $g-$generated intuitionistic fuzzy (co)implications

In this section, we study the robustness of the Atanassov intuitionistic fuzzy approach related to the $f-$ and $g-$generated intuitionistic fuzzy (co)implications $I_{Y_{f\,I}}$ $(J_{Y_{g\,I}})$ at point $\tilde{\mathbf{x}} \in \tilde{U}^2$. For that, when $f_I : \tilde{U}^2 \to \tilde{U}$, $\tilde{\delta} = (\delta_1, \delta_2) \in \tilde{U}$ and $\tilde{\mathbf{x}} = (\tilde{x}, \tilde{y}) \in \tilde{U}^2$, we follow the notations below:

$$f_I\lfloor\tilde{\mathbf{x}}\rceil \equiv f_I((\tilde{x} - \tilde{\delta}) \vee \tilde{0}, (\tilde{y} + \tilde{\delta}) \wedge \tilde{1});\ f\lceil\tilde{\mathbf{x}}\rfloor \equiv f_I((\tilde{x} + \tilde{\delta}) \wedge \tilde{1}, (\tilde{y} - \tilde{\delta}) \vee \tilde{0}).$$

Proposition 4.5 *Consider $f_I : \tilde{U}^2 \to \tilde{U}$, $\tilde{\delta} = (\delta_1, \delta_2) \in \tilde{U}$ and $\tilde{\mathbf{x}} \in \tilde{U}^2$. If f_I verifies both properties, first place antitonicity and second place isotonicity, then:*

$$\Delta_{f_I}(\tilde{\mathbf{x}}, \tilde{\delta}) = (f_I(\tilde{\mathbf{x}}) - f\lceil\tilde{\mathbf{x}}\rfloor) \vee (f\lfloor\tilde{\mathbf{x}}\rceil - f(\tilde{\mathbf{x}})) \tag{26}$$

Proof. Straightforward Proposition 3.3. □

Proposition 4.6 *Let $I_{Y_{f\,I}}(J_{Y_{f\,I}}), I_{Y_{g\,I}}(J_{Y_{g\,I}}) : \tilde{U}^2 \to \tilde{U}$ be a representable f- and g-generated (co)implications as given by Eqs.(14) and (15). If $\tilde{\delta} = (\delta_1, \delta_2) \in U^2$ and $\tilde{\mathbf{x}} = (\mathbf{x}_1, \mathbf{x}_2) \in \tilde{U}^2$ the δ-sensitivity of both $I_{Y_{f\,I}}(J_{Y_{f\,I}}), I_{Y_{g\,I}}(J_{Y_{g\,I}})$ at point $\tilde{\mathbf{x}}$ is defined by Eq.(26).*

Proof. Straightforward, since they verify both Properties $\mathbf{I1}_I$ and $\mathbf{I2}_I$, by Proposition2.5. □

Proposition 4.7 *Let $I_{Y_{f\,I}}(J_{Y_{f\,I}}), I_{Y_{g\,I}}(J_{Y_{g\,I}}) : \tilde{U}^2 \to \tilde{U}$ be a representable f- and g-generated (co)implications as given by Eqs.(14) and (15). If $\tilde{\delta} = (\delta_1, \delta_2) \in U^2$ and $\tilde{\mathbf{x}} = (\mathbf{x}_1, \mathbf{x}_2) \in \tilde{U}^2$ the δ-**sensitivity of both** $I_{Y_{f\,I}}$ **and** $I_{Y_{g\,I}}$ **at point** $\tilde{\mathbf{x}}$ can be expressed as follows:*

$$\Delta_{I_{Y_{f\,I}}}(\tilde{\mathbf{x}}, \tilde{\delta}) = \Big(\Delta_{I_{Y_f}}(l_{\tilde{U}}(N_S(\mathbf{x}_1), \mathbf{x}_2), \delta_1), \Delta_{I_{Y_f}}(r_{\tilde{U}}(N_S(\mathbf{x}_1), \mathbf{x}_2)), \delta_2)\Big)\,; \tag{27}$$

$$\Delta_{I_{Y_{g\,I}}}(\tilde{\mathbf{x}}, \tilde{\delta}) = \Big(\Delta_{I_{Y_g}}(l_{\tilde{U}}(N_S(\mathbf{x}_1), \mathbf{x}_2), \delta_1), \Delta_{I_{Y_g}}(r_{\tilde{U}}(N_S(\mathbf{x}_1), \mathbf{x}_2)), \delta_2)\Big)\,. \tag{28}$$

Analogously, the δ-sensitivity of $J_{Y_{f\,I}}$ and $J_{Y_{g\,I}}$ at point $\tilde{\mathbf{x}}$ is defined by

$$\Delta_{J_{Y_{f\,I}}}(\tilde{\mathbf{x}}, \tilde{\delta}) = \Big(\Delta_{J_{Y_f}}(l_{\tilde{U}}(\mathbf{x}_1, N_S(\mathbf{x}_2)), \delta_1), \Delta_{J_{Y_f}}(r_{\tilde{U}}(\mathbf{x}_1, N_S(\mathbf{x}_2))), \delta_2)\Big)\,; \tag{29}$$

$$\Delta_{J_{Y_{g\,I}}}(\tilde{\mathbf{x}}, \tilde{\delta}) = \Big(\Delta_{J_{Y_g}}(l_{\tilde{U}}(\mathbf{x}_1, N_S(\mathbf{x}_2)), \delta_1), \Delta_{J_{Y_g}}(r_{\tilde{U}}(\mathbf{x}_1, N_S(\mathbf{x}_2))), \delta_2)\Big)\,. \tag{30}$$

Proof. Let $I_{Y_{f\,I}}$ be an intuitionistic fuzzy Yager's implication which is representable by the Yager fuzzy implication I_{Y_f} and the standard negation N_S, as defined by Eq.(12), then:

$$
\begin{aligned}
&\Delta_{I_Y}(\tilde{\mathbf{x}},\tilde{\delta}) = \\
&=\sup\{|I_Y(\tilde{\mathbf{x}})-I_Y(\tilde{\mathbf{y}})| : \tilde{\mathbf{y}}\in\tilde{U}^2, \bigvee(l_{\tilde{U}^2}(\tilde{\mathbf{x}}), l_{\tilde{U}^2}(\tilde{\mathbf{y}}))\le\delta_1 \text{ and } \bigwedge(r_{\tilde{U}^2}(\tilde{\mathbf{x}}), r_{\tilde{U}^2}(\tilde{\mathbf{y}}))\le\delta_2\} \\
&=\sup\{|I_Y((x_{11},x_{12}),(x_{21},x_{22}))-I_Y((y_{11},y_{12}),(y_{21},y_{22}))| : \tilde{\mathbf{y}}\in\tilde{U}^2 \text{ and} \\
&\qquad\qquad \bigvee(l_{\tilde{U}^2}(\tilde{\mathbf{x}}), l_{\tilde{U}^2}(\tilde{\mathbf{y}}))\le\delta_1 \text{ and } \bigwedge(r_{\tilde{U}^2}(\tilde{\mathbf{x}}), r_{\tilde{U}^2}(\tilde{\mathbf{y}}))\le\delta_2\} \\
&=\sup\{|(I_Y(N_S(x_{12}),x_{21}), N_S(I_Y(x_{11},N_S(x_{22}))-(I_Y(N_S(y_{12}),y_{21}), N_S(I_Y(y_{11},N_S(y_{22}))| : \\
&\quad \tilde{\mathbf{y}}\in\tilde{U}^2 \text{ and } \bigvee(l_{\tilde{U}^n}(\tilde{\mathbf{x}}), l_{\tilde{U}^2}(\tilde{\mathbf{y}}))\le\delta_1 \text{ and } \bigwedge(r_{\tilde{U}^2}(\tilde{\mathbf{x}}), r_{\tilde{U}^2}(\tilde{\mathbf{y}}))\le\delta_2\} \;\; \textit{by Eq.}(\ 12) \\
&=\sup\{|l_{\tilde{U}^2}(I_Y(\mathbf{x}_1,\mathbf{x}_2))-l_{\tilde{U}}(I_Y(\mathbf{y}_1,\mathbf{y}_2))| : \tilde{\mathbf{y}}\in\tilde{U}^2 \text{ and } \bigvee(l_{\tilde{U}^2}(\tilde{\mathbf{x}}), l_{\tilde{U}^2}(\tilde{\mathbf{y}}))\le\delta_1\}, \\
&\quad \sup\{|r_{\tilde{U}^2}(I_Y(\mathbf{x}_1,\mathbf{x}_2))-r_{\tilde{U}}(I_Y(\mathbf{y}_1,\mathbf{y}_2))| : \tilde{\mathbf{y}}\in\tilde{U}^2 \text{ and } \bigwedge(r_{\tilde{U}^2}(\tilde{\mathbf{x}}), l_{\tilde{U}^2}(\tilde{\mathbf{y}}))\le\delta_2\}) \\
&\;=(\Delta_{I_Y}(l_{\tilde{U}}(N_S(\mathbf{x}_1),\mathbf{x}_2),\delta_1), \Delta_{I_Y}(r_{\tilde{U}}(N_S(\mathbf{x}_1),\mathbf{x}_2),\delta_2)) \;\; \textit{by Eq.}(\ 16).
\end{aligned}
$$

Therefore, for all $\tilde{\mathbf{x}},\tilde{\mathbf{y}}\in\tilde{U}^2$, by Eqs. (6) and (7), it follows that $l_{\tilde{U}^2}(\Delta_{I_Y}(\tilde{\mathbf{x}},\tilde{\delta})) = \Delta_{I_Y}(l_{\tilde{U}}(N_S(\mathbf{x}_1),\mathbf{x}_2),\delta_1)$; and $r_{\tilde{U}^2}(\Delta_{I_Y}(\tilde{\mathbf{x}},\delta)) = \Delta_{I_Y}(r_{\tilde{U}}(N_S(\mathbf{x}_1),\mathbf{x}_2)),\delta_2)$. In analogous manner, Eq. (28) and corresponding dual constructions can be proved. □

The diagram below summarizes the main results of Propositions 4.6 and 4.7 related to an f-generated intuitionistic fuzzy (co)implications: the robustness of intuitionistic fuzzy f-generated (co)implication can be expressed by the robustness of their arguments, by corresponding fuzzy f-generated (co)implications:

$$
\begin{tikzcd}
(\tilde{\mathbf{x}},\tilde{\delta}) \arrow[r, "Eq.\ (26)"] \arrow[d, "Eqs.(6)(7)"] & \Delta_{I_{Y_I}}(\tilde{\mathbf{x}},\tilde{\delta}) \arrow[d, "Eqs.(6)(7)"'] \\
(r_{\tilde{U}^2}(\tilde{\mathbf{x}},\tilde{\delta}), l_{\tilde{U}^2}(\tilde{\mathbf{x}},\tilde{\delta})) \arrow[r, "Eqs.\ (27)(29)"] & (\Delta_{l_{\tilde{U}}\circ I_{Y_I}}(\tilde{\mathbf{x}},\tilde{\delta}), \Delta_{r_{\tilde{U}}\circ I_{Y_I}}(\tilde{\mathbf{x}},\tilde{\delta}))
\end{tikzcd}
$$

Fig. 2. Robustness operator on the class of representable IFNs.

The following theorem extends the results in [15]:

Theorem 4.8 *Consider $\tilde{\delta}\in\tilde{U}$ e $\tilde{\mathbf{x}}\in\tilde{U}^2$. It follows that:*

(i) $\Delta_{I_{Y\,I}}(\tilde{\mathbf{x}},\tilde{\delta}) = \Delta_{J_{Y\,I}}(N_S(\tilde{\mathbf{x}}),\tilde{\delta})$ *when I_Y (J_Y) is an f-generated fuzzy (co)implication;*

(ii) $\Delta_{I_{Y\,I}}(\tilde{\mathbf{x}},\tilde{\delta}) = \Delta_{J_{Y\,I}}(N_S(\tilde{\mathbf{x}}),\tilde{\delta})$ *when I_Y (J_Y) is an g-generated fuzzy (co)implication.*

Proof. Straightforward Proposition 4.4. □

Table4.3 summarizes the δ-sensitivity of the Atanassov intuitionistic approach of the Yager's (co)implication, in the endpoints of $\tilde{U}$.

In the following, we discuss the examples in the first line. Other cases in Table4.3 can be analogously extended:

$\tilde{\mathbf{x}}$	$\Delta_{I_{Y_f}}(\tilde{\mathbf{x}},\tilde{\delta})$	$\Delta_{J_{Y_{f_I}}}(\tilde{\mathbf{x}},\tilde{\delta})$
$(\tilde{0},\tilde{0})$	$(1,1)$	(δ_1,δ_2)
$(\tilde{0},\tilde{1})$	$\left(1-(1-\delta_1)^{\delta_1},1-(1-\delta_2)^{\delta_2}\right)$	$\left(\delta_1^{(1-\delta_1)},\delta_2^{(1-\delta_2)}\right)$
$(\tilde{1},\tilde{0})$	$\left(\delta_1^{(1-\delta_1)},{\delta_2}^{(1-\delta_2)}\right)$	$\left(1-(1-\delta_1)^{\delta_1},1-(1-\delta_2)^{\delta_2}\right)$
$(\tilde{1},\tilde{1})$	(δ_1,δ_2)	$(1,1)$

Table 1
Sensitivity analysis for the intuitionistic approach of an f-generated (co)implication

$$\begin{aligned}\Delta_{I_{Y_{f_I}}}((\tilde{0},\tilde{0}),\tilde{\delta}) &= (\Delta_{I_{Y_f}}((0,0),\delta_1),\Delta_{J_{Y_f}}((1,1),\delta_2)\\ &=((I_{Y_f}(0,0)-I_{Y_f}\lceil 0,0\rfloor)\vee(I_{Y_f}\lfloor 0,0\rceil-I_{Y_f}(0,0)),\\ &\quad(J_{Y_f}(1,1)-J_{Y_f}\lceil 1,1\rfloor)\vee(J_{Y_f}\lfloor 1,1\rceil-J_{Y_f}(1,1)))\\ &=((1-0)\vee(1-1),(0-1)\vee(1-0))=(1,1).\end{aligned}$$

$$\begin{aligned}\Delta_{J_{Y_{f_I}}}((\tilde{0},\tilde{0}),\tilde{\delta}) &= (\Delta_{J_{Y_f}}((0,0),\delta_1),\Delta_{I_{Y_f}}((1,1),\delta_2)\\ &=((J_{Y_f}(0,0)-J_{Y_f}\lceil 0,0\rfloor)\vee(J_{Y_f}\lfloor 0,0\rceil-J_{Y_f}(0,0)),\\ &\quad(I_{Y_f}(1,1)-I_{Y_f}\lceil 1,1\rfloor)\vee(I_{Y_f}\lfloor 1,1\rceil-I_{Y_f}(1,1)))\\ &=(\delta_1,\delta_2).\end{aligned}$$

Analogously, Table4.3 summarizes the δ-sensitivity of the g-generated (co)implication, in the endpoints of $\tilde{U}$.

$\tilde{\mathbf{x}}$	$\Delta_{I_{Y_{g_I}}}(\tilde{\mathbf{x}},\tilde{\delta})$	$\Delta_{J_{Y_{g_I}}}(\tilde{\mathbf{x}},\tilde{\delta})$
$(\tilde{0},\tilde{0})$	$(1,1)$	(δ_1,δ_2)
$(\tilde{0},\tilde{1})$	$\left((1-\delta_1)^{\frac{1}{1-\delta_1}},(1-\delta_2)^{\frac{1}{1-\delta_2}}\right)$	$\left(1-(1-\delta_1)^{\frac{1}{1-\delta_1}},1-(1-\delta_2)^{\frac{1}{1-\delta_2}}\right)$
$(\tilde{1},\tilde{0})$	$\left(1-(1-\delta_1)^{\frac{1}{1-\delta_1}},1-(1-\delta_2)^{\frac{1}{1-\delta_2}}\right)$	$\left((1-\delta_1)^{\frac{1}{1-\delta_1}},(1-\delta_2)^{\frac{1}{1-\delta_2}}\right)$
$(\tilde{1},\tilde{1})$	(δ_1,δ_2)	$(1,1)$

Table 2
Sensitivity analysis for the intuitionistic approach of an fuzzy g-generated (co)implication

Proposition 4.9 *The maximum sensitivity of the intuitionistic fuzzy f- and g-generated (co)implications $I_{Y_{f_I}}, J_{Y_{f_I}}, I, J_{Y_g}$, as defined in Proposition 4.7 by Eqs. (27)-(30), is given as follows:*

$$\Delta_{I_{Y_{f_I}}}(\tilde{\delta})=(1,1)=\Delta_{I_{Y_{g_I}}}(\tilde{\delta}) \text{ and } \Delta_{J_{Y_{g_I}}}(\tilde{\delta})=(1,1)=\Delta_{J_{Y_{f_I}}}(\tilde{\delta}); \tag{31}$$

$$\Delta_{I_{Y_{g_I}}}(\tilde{\delta})=(1,1)=\Delta_{I_{Y_{g_I}}}(\tilde{\delta}) \text{ and } \Delta_{J_{Y_{g_I}}}(\tilde{\delta})=(1,1)=\Delta_{J_{Y_{g_I}}}(\tilde{\delta}). \tag{32}$$

Proof. Straightforward from Proposition 4.4 and the results reported in Tables 4.3 and 4.3. □

Corollary 4.10 *Consider f- and g-generated (co)implications $I_{Y_{f\,I}}, J_{Y_{f\,I}}, {}_I, J_{Y_g}$ as defined in Proposition 4.7 by Eqs. (27)-(30). Then the following holds:*

(i) $I_{Y_{g\,I}}$ is at least as robust as $I_{Y_{f\,I}}$ at point $\tilde{\mathbf{x}} \in \{(\tilde{0}, \tilde{0}), (\tilde{1}, \tilde{1})\}$;

(ii) $J_{Y_{g\,I}}$ is at least as robust as $J_{Y_{f\,I}}$ at point $\tilde{\mathbf{x}} \in \{(\tilde{0}, \tilde{0}), (\tilde{1}, \tilde{1})\}$;

(iii) $J_{Y_{f\,I}}$ is more robust than $I_{Y_{f\,I}}$ at point $(\tilde{0}, \tilde{1})$; and

(iv) $I_{Y_{f\,I}}$ is more robust than $J_{Y_{f\,I}}$ at point $(\tilde{1}, \tilde{0})$.

Proof. Straightforward. □

5 Conclusion

Estimating the sensitivity to small changes is related to reducing sensitivity in the corresponding pointwise components of such fuzzy connectives. Thus, in this paper, by taking the class of strong fuzzy negation (standard negation), the paper formally states that the sensitivity of an n-order intuitionistic fuzzy connective at a point $\mathbf{x} \in U^n$ preserves its projections related to the sensitivity of its fuzzy approach at the same point, when representable fuzzy negations are considered.

The main contribution is concerned with the study of robustness on Atanassov intuitionistic fuzzy approach related to the f- and g-generated (co)implication. Some additional studies, considering δ-sensitivity of A-implications and their corresponding dual construction should be carried out.

Ongoing work, focussing on the sensitivity of fuzzy inference dependent on intuitionistic fuzzy rules based on intuitionistic fuzzy connectives, including the extension of the robustness studies of R-(co)implications, will also be investigated.

To sum up, future research aims to contribute with fundamental theoretical results for applications dealing with main results in the robustness analysis considering their principal operators, e.g. erosion, dilation, closing, opening operators used in the mathematical morphology.

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