

# A Note on Finitely Derived Information Systems

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## Abstract

The notion of information system initially introduced by Scott provides an efficient approach to represent various kinds of domains. In this note, a new type of information systems named finitely derived information systems is introduced. For this notion, the requirement for the consistency predicate used in Scott's information systems is simplified, and the reflexive and transitive rules for the entailment relation are preserved while the finitely derived rule is introduced. A comprehensive investigation is made on the interrelation between finitely derived information systems and algebraic domains. It turns out that their corresponding categories are equivalent, which indicates that the proposed notion of finitely derived information system provides a concrete approach to representing algebraic domains.

**Keywords:** Finitely derived information system; Algebraic domain; Equivalence of categories

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# 1 Introduction

In Domain theory, an interesting topic is to represent various kinds of domain categories by other relatively concrete structures [2]. It can be traced back to the notions of information system and approximable mapping which were initially developed by Scott [7]. Then Larsen and Winskel [5] proved that the category of Scott's information systems is exactly equivalent to that of Scott Domains (i.e., bounded-complete algebraic domains) with Scott continuous functions being morphisms. Afterwards, Hoofman [3] generalized Scott's information systems to the continuous case and obtained the representation of bounded-complete continuous domains. Recently, Spren et al. [8] proposed a generalized version of continuous information systems which realizes the representation of continuous domains. In addition, other variations of information systems have been introduced in order to characterize various other kinds of domains [9,10,11].

Theoretically, in Scott's original work, an information system is a triple  $(A, Con, \vdash)$  where  $A$  is a set,  $Con$  is a set of subsets of  $A$  and  $\vdash$  is a binary relation between  $Con$  and  $A$ . As advised by Scott,  $A$  can be understood as a set of data objects,  $Con$  as consistent combinations of data objects and  $\vdash$  as an entailment relation which records the dependencies between consistent combinations of data objects and individual ones. As for  $Con$ , the following axioms are required: (a) it must be closed under subsets; (b) it contains all singletons; and (c) adjunction of an entailed object to a consistent combination preserves consistency. The entailment  $\vdash$  is required to be reflexive and transitive, that is, (i) for any  $X \in Con$ , if  $a \in X$ , then  $X \vdash a$  (reflexivity); (ii) for any  $X, Y \in Con$  and  $a \in A$ , if  $X \vdash Y \vdash a$ , then  $X \vdash a$  (transitivity).

If we consider  $A$  as a set of items (for instance, product items in the supermarket domain), then Scott's information systems may provide a mathematical infrastructure for association rule mining in which  $Con$  represents the antecedents of transactions over  $A$  and  $\vdash$  represents the atomic association rules which indicates the consequents of these antecedents. Nonetheless, we have to be vigilant on the axioms of Scott's information systems in that some of them may be too restrict from the practical viewpoint and, on the other hand, new axioms may be required to reflect some exclusive features of association rules. For instance, the restrictions that  $Con$  is closed under subsets and it contains all singletons are obviously unreasonable from the viewpoint of association rule mining. Moreover, we consider that the reflexive and transitive rules for  $\vdash$  are necessary to our consideration, but they are not sufficient to appropriately reflect some features of association rule systems. Therefore, we need to adapt Scott's information systems in order to provide a more rational model for association rule mining.

In [4], we proposed the notion of  $\mathcal{F}$ -context where  $\mathcal{F}$  is a finite-subsets-family structure on the attribute set of a formal context. We discussed the implication rule systems induced from consistent  $\mathcal{F}$ -contexts. The results demonstrate that the concept hierarchy of a consistent  $\mathcal{F}$ -context is just in correspondence with a special kind of subset family inherent in the induced implication rules system. Based on this result, we introduced the notion of formal implication rule system which

can be viewed as the axiomatization of the induced implication rule systems of consistent  $\mathcal{F}$ -contexts. By regarding the notion of consistent  $\mathcal{F}$ -context as a bridge, we demonstrated that the notion of formal implication rule system provides an approach to reconstructing algebraic domains.

The present note is based on [4]. First we rename the formal implication rule systems by finitely derived information systems (see Definition 3.1) in order to be consistent with the notions in Domain theory. Obviously, this type of information systems is still in Scott-style. However, comparing with Scott's information systems, our proposed notion has the following features: (i) *Con* is still a set of finite subsets of  $A$  but the requirements used in Scott's information systems are abandoned; (ii) the reflexive and transitive rules for  $\vdash$  are preserved but a new rule named finitely derived rule is introduced (Note: This is why the name "finitely derived information system" is chosen in this paper). Moreover, with the help of the notion of approximable mapping, we settle finitely derived information systems into a category. It turns out that this category is exactly equivalent to that of algebraic domains.

The rest of this paper is structured as follows: In Section 2, we recall the necessary definitions and results about equivalence of categories and algebraic domains. In Section 3, we introduce the notions of finitely derived information systems and approximable mappings which compose of a category. In Section 4, we study the order-theoretic properties of the family of information states with respect to set inclusion. It is shown that every poset generated in this way is an algebraic domain. We also discuss the one-to-one correspondence between approximable mappings and Scott continuous functions. In Section 5, we prove the equivalence between the category of finitely derived information systems and that of algebraic domains. In Section 6, we reach the conclusion.

## 2 Preliminaries

### 2.1 Equivalence of categories

We first recall the notion of the equivalence of categories. For more basic notions in Category Theory, we recommend the reader the book [6].

**Definition 2.1** Let  $\mathbf{C}$  and  $\mathbf{D}$  be categories. A functor  $\mathfrak{F} : \mathbf{C} \rightarrow \mathbf{D}$  is a pair of functions  $\mathfrak{F}_o : \mathbf{C}_o \rightarrow \mathbf{D}_o$  and  $\mathfrak{F}_a : \mathbf{C}_a \rightarrow \mathbf{D}_a$  which satisfy:

- (F1) if  $f : A \rightarrow B$  is a morphism in  $\mathbf{C}$ , then  $\mathfrak{F}_a(f) : \mathfrak{F}_o(A) \rightarrow \mathfrak{F}_o(B)$  is a morphism in  $\mathbf{D}$ ;
- (F2) for any object  $A$  of  $\mathbf{C}$ ,  $\mathfrak{F}_a(\text{id}_A) = \text{id}_{\mathfrak{F}_o(A)}$ ;
- (F3) if  $g \circ f$  is defined in  $\mathbf{C}$ , then  $\mathfrak{F}_a(g) \circ \mathfrak{F}_a(f)$  is defined in  $\mathbf{D}$  and  $\mathfrak{F}_a(g \circ f) = \mathfrak{F}_a(g) \circ \mathfrak{F}_a(f)$ .

The notion of equivalence of categories is used to demonstrate strong similarities or essential identicalness between mathematical structures which may appear unrelated at a superficial or intuitive level.

**Definition 2.2** A functor  $\mathfrak{F} : \mathbf{C} \rightarrow \mathbf{D}$  is called an *equivalence of categories* if there are

- (E1) a functor  $\mathfrak{G} : \mathbf{D} \rightarrow \mathbf{C}$ ;
- (E2) a family of isomorphisms  $\{\mu_C : C \rightarrow \mathfrak{G}_o(\mathfrak{F}_o(C)) \mid C \in \mathbf{C}_o\}$  with the property that for every morphism  $f : C \rightarrow C'$  of  $\mathbf{C}$ ,  $\mathfrak{G}_a(\mathfrak{F}_a(f)) \circ \mu_C = \mu_{C'} \circ f$ ;
- (E3) a family of isomorphisms  $\{\nu_D : D \rightarrow \mathfrak{F}_o(\mathfrak{G}_o(D)) \mid D \in \mathbf{D}_o\}$  with the property that for every morphism  $g : D \rightarrow D'$  of  $\mathbf{D}$ ,  $\mathfrak{F}_a(\mathfrak{G}_a(g)) \circ \nu_D = \nu_{D'} \circ g$ .

If  $\mathfrak{F} : \mathbf{C} \rightarrow \mathbf{D}$  is an equivalence of categories, the associated functor  $\mathfrak{G}$  is usually called a *pseudo-inverse* of  $\mathfrak{F}$ . In this case,  $\mathbf{C}$  and  $\mathbf{D}$  are said to be categorically equivalent.

## 2.2 Algebraic lattices and algebraic domains

We recommend the book [1] for the basic notions in Lattice Theory. In the sequel, we recall some notions about algebraic domains. Most of them are collected from [2].

Let  $(L, \leq)$  be a poset. A subset  $A \subseteq L$  is said to be *join-dense* in  $L$  if any element of  $L$  is the join of a subset of  $A$ . A non-empty subset  $D \subseteq L$  is said to be *directed* if for any elements  $x, y \in D$ , there exists  $z \in D$  such that  $x \leq z$  and  $y \leq z$ . If every directed subset  $D \subseteq L$  has the least upper bound  $\bigvee D$ , then  $L$  is called a *dcpo*. For any  $x, y \in L$ ,  $x$  is said to be *way below*  $y$ , written as  $x \ll y$ , if for every directed set  $D \subseteq L$  for which  $\bigvee D$  exists,  $y \leq \bigvee D$  always implies the existence of some  $d \in D$  such that  $x \leq d$ . If an element  $x \in L$  satisfies  $x \ll x$ , then it is said to be *compact* in  $(L, \leq)$ . Throughout this paper, we always use  $K(L)$  to denote the set of all compact elements of  $L$ . For any  $x \in L$ , the notation  $\downarrow x$  is used to denote the set  $\downarrow x \cap K(L)$ , i.e., the compact elements less than or equal to  $x$ . A subset  $B \subseteq L$  is called a *basis* of  $L$  if for every  $x \in L$ , the subset  $\downarrow x \cap B$  is directed and  $x = \bigvee(\downarrow x \cap B)$ . A dcpo  $(L, \leq)$  is called an *algebraic domain* if  $K(L)$  is a basis of  $L$ .

**Definition 2.3** Let  $(L_1, \leq_1)$  and  $(L_2, \leq_2)$  be dcpo's. A map  $\varphi : L_1 \rightarrow L_2$  is said to be *Scott continuous* if it preserves the least upper bounds of directed subsets, i.e., if  $D \subseteq L_1$  is directed, then  $\varphi(\bigvee D) = \bigvee \varphi(D)$ .

Throughout this paper, we always use the notation  $\subseteq_f$  to describe the finite subset relation between sets. Whenever both  $\mathcal{A}$  and  $\mathcal{B}$  are families of subsets of sets, by saying that a map  $\varphi : \mathcal{A} \rightarrow \mathcal{B}$  is Scott continuous we always mean that  $\varphi$  is Scott continuous with respect to the order of set inclusion. In this paper, we use **ALD** to denote the category of algebraic domains as objects and Scott continuous maps as morphisms.

## 3 The category of finitely derived information systems

In this section, we first introduce the notion of finitely derived information system. Then we put this kind of information systems into a category with appropriate approximable mappings being morphisms.

**Definition 3.1** A finitely derived information system  $\underline{A}$  is a triple  $(\text{Dom}_A, \text{Con}_A, \vdash_A)$  where  $\text{Dom}_A$  is a non-empty set,  $\text{Con}_A$  is a non-empty family of non-empty finite subsets of  $\text{Dom}_A$ , and  $\vdash_A$  is a binary relation from  $\text{Con}_A$  to  $\text{Dom}_A$ , i.e.,  $\vdash_A \subseteq \text{Con}_A \times \text{Dom}_A$ , which satisfies:

- (R1)  $(\forall F \in \text{Con}_A, a \in \text{Dom}_A) a \in F \Rightarrow F \vdash_A a$ ;
  - (R2)  $(\forall F_1, F_2 \in \text{Con}_A, a \in \text{Dom}_A) F_2 \vdash_A F_1 \vdash_A a \Rightarrow F_2 \vdash_A a$ ;
  - (R3)  $(\forall F \in \text{Con}_A, B \subseteq_f \text{Dom}_A) B \neq \emptyset \ \& \ F \vdash_A B \Rightarrow (\exists F' \in \text{Con}_A) F \vdash_A F' \supseteq B$ ,
- where  $F_2 \vdash_A F_1$  means  $F_2 \vdash_A a$  for any  $a \in F_1$ .

For convenience, given a finitely derived information system  $\underline{A}$  and  $F \in \text{Con}_A$ , we use  $\widetilde{F}$  to denote the subset  $\{a \in \text{Dom}_A \mid F \vdash_A a\}$ .

**Proposition 3.2** Let  $\underline{A}$  be a finitely derived information system. Then for any  $F, F_1, F_2 \in \text{Con}_A$ ,

- (1)  $F \subseteq \widetilde{F}$ ;
- (2)  $F_1 \subseteq F_2 \Rightarrow \widetilde{F_1} \subseteq \widetilde{F_2}$ ;
- (3)  $F_1 \subseteq \widetilde{F_2} \Leftrightarrow \widetilde{F_1} \subseteq \widetilde{F_2}$ .

**Definition 3.3** An approximable mapping between finitely derived information systems  $\underline{A}$  and  $\underline{B}$  is a binary relation  $\Theta \subseteq \text{Con}_A \times \text{Con}_B$  which satisfies that for any  $F, F_1, F_2 \in \text{Con}_A$  and  $G_1, G_2 \in \text{Con}_B$ ,

- (T1)  $F_1 \subseteq \widetilde{F_2} \ \& \ (F_1, G_1) \in \Theta \ \& \ G_2 \subseteq \widetilde{G_1} \Rightarrow (F_2, G_2) \in \Theta$ ;
- (T2)  $(F, G_1) \in \Theta \ \& \ (F, G_2) \in \Theta \Rightarrow (\exists G_3 \in \text{Con}_B) G_1 \cup G_2 \subseteq \widetilde{G_3} \ \& \ (F, G_3) \in \Theta$ .

Next, we present some basic properties of approximable mappings between finitely derived information systems which will be used in the subsequent sections.

**Proposition 3.4** Let  $\Theta$  be an approximable mapping between finitely derived information systems  $\underline{A}$  and  $\underline{B}$ . Then for any  $F \in \text{Con}_A$  and  $G \in \text{Con}_B$ , the following are equivalent:

- (1)  $(F, G) \in \Theta$ .
- (2) There exists  $F' \in \text{Con}_A$  such that  $F' \subseteq \widetilde{F}$  and  $(F', G) \in \Theta$ .
- (3) There exists  $G' \in \text{Con}_B$  such that  $(F, G') \in \Theta$  and  $G \subseteq \widetilde{G'}$ .

Given a finitely derived information system  $\underline{A}$ , a relation  $\text{id}_{\underline{A}} \subseteq \text{Con}_A \times \text{Con}_A$  is defined by

$$(F, F') \in \text{id}_{\underline{A}} \Leftrightarrow F' \subseteq \widetilde{F} \ (\Leftrightarrow F \vdash_A F').$$

Let  $\Theta : \underline{A} \rightarrow \underline{B}$  and  $\Upsilon : \underline{B} \rightarrow \underline{C}$  be approximable mappings. Define a relation  $\Upsilon \circ \Theta \subseteq \text{Con}_A \times \text{Con}_C$  by

$$(F, H) \in \Upsilon \circ \Theta \Leftrightarrow (\exists G \in \text{Con}_B) (F, G) \in \Theta \ \& \ (G, H) \in \Upsilon.$$

**Theorem 3.5** The finitely derived information systems being objects with approximable mappings being arrows form a category **FIS**.

**Proof.** It is routine by checking the clauses in the definition of a category.  $\square$

## 4 Correspondence between finitely derived information systems and algebraic domains

In this section, we study the relationship between finitely derived information systems and algebraic domains.

### 4.1 From finitely derived information systems to algebraic domains

We first introduce the notion of information state of finitely derived information systems. We also investigate the order-theoretic properties of the poset of information states with respect to set inclusion. As is shown, every such poset is an algebraic domain. Moreover, we study the correspondence between approximable mappings (between the given finitely derived information systems) and Scott continuous functions (between the induced algebraic domains).

**Definition 4.1** Let  $\underline{A}$  be a finitely derived information system. A non-empty subset  $D \subseteq \text{Dom}_A$  is said to be an information state of  $\underline{A}$  if

- (S1)  $(\forall F \in \text{Con}_A, a \in \text{Dom}_A) F \subseteq D \ \& \ F \vdash_A a \Rightarrow a \in D$ ;
- (S2)  $(\forall B \subseteq_f \text{Dom}_A) B \neq \emptyset \ \& \ B \subseteq D \Rightarrow (\exists F \in \text{Con}_A) F \subseteq D \ \& \ F \vdash_A B$ .

Intuitively, the condition (S1) means that  $D$  is closed under entailment and (S2) means that  $D$  is finitely derived in the sense that every combination of finite data objects of  $D$  can be derived from a consistent combinations of data objects of  $D$ .

In the sequel, we use  $\mathfrak{I}(\underline{A})$  to denote the set of all information states of  $\underline{A}$ .

**Proposition 4.2** Let  $\underline{A}$  be a finitely derived information system. Then for any  $F \in \text{Con}_A$ ,  $\tilde{F}$  is an information state of  $\underline{A}$ .

**Proof.** We only need to check that  $\tilde{F}$  satisfies (S1) and (S2). □

**Proposition 4.3** Let  $D$  be an information state of  $\underline{A}$ . Then

- (1) For any  $F \in \text{Con}_A$ , if  $F \subseteq D$ , then  $\tilde{F} \subseteq D$ ;
- (2) The family  $\{\tilde{F} \mid F \in \text{Con}_A \ \& \ F \subseteq D\}$  is directed;
- (3)  $D = \bigcup \{\tilde{F} \mid F \in \text{Con}_A \ \& \ F \subseteq D\}$ .

**Proposition 4.4** Let  $\{D_i\}_{i \in I}$  be a directed family of information states of  $\underline{A}$ . Then  $\bigcup_{i \in I} D_i$  is also an information state of  $\underline{A}$ .

**Proof.** We only need to check that  $\bigcup_{i \in I} D_i$  satisfies (S1) and (S2). □

**Theorem 4.5** Let  $\underline{A}$  be a finitely derived information system. Then  $(\mathfrak{I}(\underline{A}), \subseteq)$  is an algebraic domain.

**Proof.** We have proved in Proposition 4.4 that  $(\mathfrak{I}(\underline{A}), \subseteq)$  is a dcpo. Then we can check that  $D \subseteq \text{Dom}_A$  is compact in  $(\mathfrak{I}(\underline{A}), \subseteq)$  if and only if  $D = \tilde{F}$  for some  $F \in \text{Con}_A$ . Finally, by Proposition 4.3, we can see that the family  $\{\tilde{F} \mid F \in \text{Con}_A\}$  precisely forms a basis of  $(\mathfrak{I}(\underline{A}), \subseteq)$ . □

The following result shows that there is a one-to-one correspondence between approximable mappings (between the given finitely derived information systems) and Scott continuous functions (between the induced algebraic domains).

**Theorem 4.6** *Let  $\underline{A}$  and  $\underline{B}$  be finitely derived information systems. Then*

- (1) *For any approximable mapping  $\Theta : \underline{A} \rightarrow \underline{B}$ , the function  $\varphi_\Theta : \mathfrak{I}(\underline{A}) \rightarrow \mathfrak{I}(\underline{B})$  defined by*

$$\varphi_\Theta(D) = \bigcup \{ \tilde{G} \mid (\exists F \in \text{Con}_A, G \in \text{Con}_B) F \subseteq D \ \& \ (F, G) \in \Theta \}$$

*is Scott continuous;*

- (2) *For any Scott continuous function  $\varphi : \mathfrak{I}(\underline{A}) \rightarrow \mathfrak{I}(\underline{B})$ , the relation  $\Theta_\varphi \subseteq \text{Con}_A \times \text{Con}_B$  defined by*

$$(F, G) \in \Theta_\varphi \Leftrightarrow G \subseteq \varphi(\tilde{F})$$

*is an approximable mapping;*

- (3) *Moreover,  $\Theta_{\varphi_\Theta} = \Theta$  and  $\varphi_{\Theta_\varphi} = \varphi$ .*

**Proof.** It is routine to check that  $\varphi_\Theta$  is Scott continuous and  $\Theta_\varphi$  is an approximable mapping.

For  $\Theta_{\varphi_\Theta} = \Theta$ , given any  $F \in \text{Con}_A$  and  $G \in \text{Con}_B$ ,

$$(F, G) \in \Theta_{\varphi_\Theta} \Leftrightarrow G \subseteq \varphi_\Theta(\tilde{F})$$

$$\Leftrightarrow (\exists F' \in \text{Con}_A, G' \in \text{Con}_B) F' \subseteq \tilde{F} \ \& \ (F', G') \in \Theta \ \& \ G \subseteq \tilde{G'}$$

$$\Leftrightarrow (F, G) \in \Theta$$

For  $\varphi_{\Theta_\varphi} = \varphi$ , given any information state  $D$  of  $\underline{A}$ ,

$$\begin{aligned} \varphi_{\Theta_\varphi}(D) &= \bigcup \{ \tilde{G} \mid (\exists F \in \text{Con}_A, G \in \text{Con}_B) F \subseteq D \ \& \ (F, G) \in \Theta_\varphi \} \\ &= \bigcup \{ \tilde{G} \mid (\exists F \in \text{Con}_A, G \in \text{Con}_B) F \subseteq D \ \& \ G \subseteq \varphi(\tilde{F}) \} \\ &= \bigcup \{ \tilde{G} \mid G \in \text{Con}_B \ \& \ G \subseteq \varphi(D) \} \\ &= \varphi(D) \end{aligned}$$

□

#### 4.2 The finitely derived information systems induced from algebraic domains

In this section, we study the finitely derived information systems which can be generated from given algebraic domains. We also study the correspondence between Scott continuous functions (between the given algebraic domains) and approximable mappings (between the induced finitely derived information systems).

**Theorem 4.7** *Let  $\mathbf{L} = (L, \leq)$  be an algebraic domain. Then  $\underline{A}_L = (\text{Dom}_L, \text{Con}_L, \vdash_L)$  is a finitely derived information system, where*

- $\text{Dom}_L = K(L)$ ;
- $\text{Con}_L$  consists of the finite subsets of  $K(L)$  for which every  $F \in \text{Con}_L$  has its greatest element  $k_F$  with respect to  $\leq$ ;
- $\vdash_L \subseteq \text{Con}_L \times \text{Dom}_L$  for which  $F \vdash_L k$  if and only if  $k \leq k_F$ .

**Proof.** Straightforward by checking clauses in Definition 3.1.  $\square$

**Proposition 4.8** *Let  $\underline{A}_L$  be the induced finitely derived information system of  $(L, \leq)$ . Then for any  $F, F_1, F_2 \in \text{Con}_L$ ,*

$$F_1 \subseteq \widetilde{F_2} \Leftrightarrow k_{F_1} \leq k_{F_2} \Leftrightarrow \widetilde{F_1} \subseteq \widetilde{F_2}.$$

**Proposition 4.9** *Let  $(L, \leq)$  be an algebraic domain and  $D \subseteq K(L)$ . Then  $D$  is an information state of  $\underline{A}_L$  if and only if  $D$  is an ideal of  $(K(L), \leq)$ .*

**Proof.** Straightforward.  $\square$

The following result shows that there is a one-to-one correspondence between Scott continuous functions (between the given algebraic domains) and approximable mappings (between the induced finitely derived information systems). In the other word, the notion of approximable mapping provides an approach to represent Scott continuous functions.

**Theorem 4.10** *Let  $(L_1, \leq_1)$  and  $(L_2, \leq_2)$  be algebraic domains. Then:*

- (1) *For any Scott continuous function  $\psi : L_1 \rightarrow L_2$ , the relation  $\Pi_\psi \subseteq \text{Con}_{L_1} \times \text{Con}_{L_2}$  defined by*

$$(F, G) \in \Pi_\psi \Leftrightarrow k_G \leq_2 \psi(k_F)$$

*is an approximable mapping;*

- (2) *For any approximable mapping  $\Pi$  from  $\underline{A}_{L_1}$  to  $\underline{A}_{L_2}$ , the function  $\psi_\Pi : L_1 \rightarrow L_2$  defined by*

$$\psi_\Pi(x) = \bigvee \{k_G \mid (\exists F \in \text{Con}_{L_1}, G \in \text{Con}_{L_2}) (F, G) \in \Pi \ \& \ k_F \leq_1 x\}$$

*is a Scott continuous function;*

- (3) *Moreover,  $\Pi_{\psi_\Pi} = \Pi$  and  $\psi_{\Pi_\psi} = \psi$ .*

**Proof.** It is routine to check that  $\Pi_\psi$  is an approximable mapping and  $\psi_\Pi$  is a Scott continuous function.

For  $\Pi_{\psi_\Pi} = \Pi$ , given any  $F \in \text{Con}_{L_1}$  and  $G \in \text{Con}_{L_2}$ ,

$$\begin{aligned} (F, G) \in \Pi_{\psi_\Pi} &\Leftrightarrow k_G \leq_2 \psi_\Pi(k_F) \\ &\Leftrightarrow (\exists F' \in \text{Con}_{L_1}, G' \in \text{Con}_{L_2}) (F', G') \in \Pi \ \& \ k_{F'} \leq_1 k_F \ \& \ k_G \leq_2 k_{G'} \\ &\Leftrightarrow (\exists F' \in \text{Con}_{L_1}, G' \in \text{Con}_{L_2}) (F', G') \in \Pi \ \& \ F' \subseteq \widetilde{F} \ \& \ G \subseteq \widetilde{G'} \\ &\Leftrightarrow (F, G) \in \Pi \end{aligned}$$

For  $\psi_{\Pi_\psi} = \psi$ , given any  $x \in L_1$ ,



$$\begin{aligned}
\psi_{\Pi_\psi}(x) &= \bigvee \{k_G \mid (\exists F \in \text{Con}_{L_1}, G \in \text{Con}_{L_2}) (F, G) \in \Pi_\psi \ \& \ k_F \leq_1 x\} \\
&= \bigvee \{k_G \mid (\exists F \in \text{Con}_{L_1}, G \in \text{Con}_{L_2}) \ k_G \leq_2 \psi(k_F) \ \& \ k_F \leq_1 x\} \\
&= \bigvee \{\psi(k_F) \mid F \in \text{Con}_{L_1} \ \& \ k_F \leq_1 x\} \\
&= \psi(\bigvee \{k_F \mid F \in \text{Con}_{L_1} \ \& \ k_F \leq_1 x\}) \\
&= \psi(\bigvee \downarrow x) \\
&= \psi(x).
\end{aligned}$$

□

## 5 The equivalence between FIS and ALD

In this section, we study the relationship between the category of finitely derived information system and that of algebraic domains. In this end, we establish two functors between them in the following.

### 5.1 Functors between FIS and ALD

From **FIS** to **ALD**, define functions  $\mathfrak{F}_o : \mathbf{FIS}_o \rightarrow \mathbf{ALD}_o$  and  $\mathfrak{F}_a : \mathbf{FIS}_a \rightarrow \mathbf{ALD}_a$  as follows: For any finitely derived information system  $\underline{A}$ ,

$$\mathfrak{F}_o(\underline{A}) = (\mathfrak{I}(\underline{A}), \subseteq)$$

where  $(\mathfrak{I}(\underline{A}), \subseteq)$  is the induced algebraic domain of  $\underline{A}$  in the sense of Theorem 4.5. Let  $\underline{A}$  and  $\underline{B}$  be finitely derived information systems. For any approximable mapping  $\Theta$  from  $\underline{A}$  to  $\underline{B}$ ,

$$\mathfrak{F}_a(\Theta) = \varphi_\Theta$$

where  $\varphi_\Theta$  is the induced Scott continuous function of  $\Theta$  in the sense of Theorem 4.6(1).

It is routine to check that  $\mathfrak{F} = (\mathfrak{F}_o, \mathfrak{F}_a)$  is a functor from **FIS** to **ALD**.

Conversely, from **ALD** to **FIS**, define functions  $\mathfrak{G}_o : \mathbf{ALD}_o \rightarrow \mathbf{FIS}_o$  and  $\mathfrak{G}_a : \mathbf{ALD}_a \rightarrow \mathbf{FIS}_a$  as follows: For any algebraic domain  $\mathbf{L} = (L, \leq)$ ,

$$\mathfrak{G}_o(\mathbf{L}) = \underline{A}_L$$

where  $\underline{A}_L$  is the induced finitely derived information system of  $\mathbf{L}$  in the sense of Theorem 4.7. Let  $\mathbf{L}_1$  and  $\mathbf{L}_2$  be algebraic domains. For any Scott continuous function  $\psi$  from  $\mathbf{L}_1$  to  $\mathbf{L}_2$ ,

$$\mathfrak{G}_a(\psi) = \Pi_\psi$$

where  $\Pi_\psi$  is the induced approximable mapping of  $\psi$  in the sense of Theorem 4.10(1).

It is routine to check that  $\mathfrak{G} = (\mathfrak{G}_o, \mathfrak{G}_a)$  is a functor from **ALD** to **FIS**.

## 5.2 The equivalence between **FIS** and **ALD**

Based on the above two functors, starting from a finitely derived information system  $\underline{A}$ , we can get a new finitely derived information system  $\mathfrak{G}_o(\mathfrak{F}_o(\underline{A})) = (\text{Dom}_{\mathfrak{I}(\underline{A})}, \text{Con}_{\mathfrak{I}(\underline{A})}, \vdash_{\mathfrak{I}(\underline{A})})$ . Define a relation  $\Theta_{\underline{A}} \subseteq \text{Con}_A \times \text{Con}_{\mathfrak{I}(\underline{A})}$  by

$$(F, \mathcal{A}) \in \Theta_{\underline{A}} \Leftrightarrow k_{\mathcal{A}} \subseteq \tilde{F} \quad (\Leftrightarrow \mathcal{A} \subseteq \Downarrow \tilde{F})$$

where  $k_{\mathcal{A}}$  is the greatest element of  $\mathcal{A}$  in  $(\mathfrak{I}(\underline{A}), \subseteq)$  and  $\Downarrow \tilde{F}$  is the compact elements below  $\tilde{F}$  in  $(\mathfrak{I}(\underline{A}), \subseteq)$  (Note: We can easily check  $\Downarrow \tilde{F} = \{\tilde{F}' \mid F' \in \text{Con}_A \ \& \ F' \subseteq \tilde{F}\}$ ).

Moreover, define a relation  $\Pi_{\underline{A}} \subseteq \text{Con}_{\mathfrak{I}(\underline{A})} \times \text{Con}_A$  by

$$(\mathcal{A}, F) \in \Pi_{\underline{A}} \Leftrightarrow \tilde{F} \subseteq k_{\mathcal{A}}.$$

It is routine to check that  $\Theta_{\underline{A}}$  is an approximable mapping from  $\underline{A}$  to  $\mathfrak{G}_o(\mathfrak{F}_o(\underline{A}))$  and  $\Pi_{\underline{A}}$  is an approximable mapping from  $\mathfrak{G}_o(\mathfrak{F}_o(\underline{A}))$  to  $\underline{A}$ . Moreover, the following result shows that  $\underline{A}$  and  $\mathfrak{G}_o(\mathfrak{F}_o(\underline{A}))$  are isomorphic in category **FIS**.

**Proposition 5.1** *Let  $\underline{A}$  be a finitely derived information system. Then  $\Pi_{\underline{A}}$  and  $\Theta_{\underline{A}}$  are inverse with each other in **FIS**.*

**Proof.** We can check that  $(F, F') \in \Pi_{\underline{A}} \circ \Theta_{\underline{A}}$  if and only if  $(F, F') \in \text{id}_{\underline{A}}$  for any  $F, F' \in \text{Con}_A$ . This implies  $\Pi_{\underline{A}} \circ \Theta_{\underline{A}} = \text{id}_{\underline{A}}$ .

Moreover, we have  $(\mathcal{A}, \mathcal{A}') \in \Theta_{\underline{A}} \circ \Pi_{\underline{A}}$  if and only if  $(\mathcal{A}, \mathcal{A}') \in \text{id}_{\mathfrak{G}_o(\mathfrak{F}_o(\underline{A}))}$  for any  $\mathcal{A}, \mathcal{A}' \in \text{Con}_{\mathfrak{I}(\underline{A})}$ . This implies  $\Theta_{\underline{A}} \circ \Pi_{\underline{A}} = \text{id}_{\mathfrak{G}_o(\mathfrak{F}_o(\underline{A}))}$ .  $\square$

Given an algebraic domain  $\mathbf{L}$ , we can get a new algebraic domain  $\mathfrak{F}_o(\mathfrak{G}_o(\mathbf{L})) = (\mathfrak{I}(\underline{A}_L), \subseteq)$ . Define a function  $\varphi_L : L \rightarrow \mathfrak{I}(\underline{A}_L)$  by

$$\varphi_L(x) = \Downarrow x.$$

Define another function  $\psi_L : \mathfrak{I}(\underline{A}_L) \rightarrow L$  by

$$\psi_L(D) = \bigvee D.$$

By Proposition 4.9, both  $\psi_L$  and  $\varphi_L$  are well-defined.

It is routine to check that  $\varphi_L$  is a Scott continuous function from  $\mathbf{L}$  to  $\mathfrak{F}_o(\mathfrak{G}_o(\mathbf{L}))$  and  $\psi_L$  is a Scott continuous function from  $\mathfrak{F}_o(\mathfrak{G}_o(\mathbf{L}))$  to  $\mathbf{L}$ . The following result shows that  $\mathbf{L}$  is isomorphic to  $\mathfrak{F}_o(\mathfrak{G}_o(\mathbf{L}))$  in category **ALD**.

**Proposition 5.2** *Let  $\mathbf{L}$  be an algebraic domain. Then  $\varphi_L$  and  $\psi_L$  are inverse with each other.*

**Proof.** For any  $x \in L$ , we have  $\psi_L(\varphi_L(x)) = \bigvee \Downarrow x = x$ . This implies  $\psi_L \circ \varphi_L = \text{id}_L$ . For any information state  $D$  of  $\underline{A}_L$ , we have  $\varphi_L(\psi_L(D)) = \Downarrow (\bigvee D) = D$ . This implies  $\varphi_L \circ \psi_L = \text{id}_{\mathfrak{F}_o(\mathfrak{G}_o(\mathbf{L}))}$ .  $\square$

Finally, we can obtain the categorical equivalence between **FIS** and **ALD**, which indicates that the notion of finitely derived information system provides a concrete representation of algebraic domains.

**Theorem 5.3**  $\mathfrak{F}$  is a functor from **FIS** to **ALD** with  $\mathfrak{G}$  being the pseudo-inverse. Therefore, **FIS** is categorical equivalent to **ALD**.

**Proof.** Given any approximable mapping  $\Theta : \underline{A} \rightarrow \underline{B}$ , we can check that  $(F, \mathcal{B}) \in \Theta_{\underline{B}} \circ \Theta$  if and only if  $(F, \mathcal{B}) \in \mathfrak{G}_a(\mathfrak{F}_a(\Theta)) \circ \Theta_{\underline{A}}$  for any  $F \in \text{Con}_A$  and  $\mathcal{B} \in \text{Con}_{\mathfrak{J}(\underline{B})}$ . This implies  $\mathfrak{G}_a(\mathfrak{F}_a(\Theta)) \circ \Theta_{\underline{A}} = \Theta_{\underline{B}} \circ \Theta$ .

Given any Scott continuous function  $\varphi$  from  $\mathbf{L}_1$  to  $\mathbf{L}_2$ , we can check that  $\mathfrak{F}_a(\mathfrak{G}_a(\varphi)) \circ \varphi_{L_1}(x) = \varphi_{L_2}(\varphi(x))$  for any  $x \in L_1$ . This implies  $\mathfrak{F}_a(\mathfrak{G}_a(\varphi)) \circ \varphi_{L_1} = \varphi_{L_2} \circ \varphi$ .  $\square$

## 6 Conclusions

In this note, we introduced a new kind of information systems named finitely derived information systems. This notion can be seen as a revision of the original Scott's information systems. Towards the consistency predicate, we only required that it is only a set of non-empty finite subsets of the a set of tokens. Towards entailment relation, we preserved the reflexive and transitive rules which have been used in the Scott's information systems and introduced a new rule named finitely derived. We investigated systematically the interrelation between the category of finitely derived information systems with approximable mappings as morphisms and that of algebraic domains with Scott continuous functions as morphisms. By establishing two functors between these two categories, we proved that they are categorical equivalent. This result demonstrated that our proposed notion of finitely derived information system indeed provides a concrete representation of algebraic domains.

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