Hardness and Efficiency on Minimizing Maximum Distances for Graphs With Few P_4's and (k, ℓ)-graphs [1]

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Abstract

A tree t-spanner of a graph G is a spanning subtree T in which the distance between any two adjacent vertices of G is at most t. The smallest t for which G has a tree t-spanner is called tree stretch index. The problem of determining the tree stretch index (MSST) has been studied by: establishing lower and upper bounds, based, for instance, on the girth value and on the minimum diameter spanning tree problem, respectively; and presenting some classes for which t is a tight value. In 1995, the computational complexity of MSST was settled to be NP-hard for $t \geq 4$, polynomial-time solvable for $t = 2$. However, deciding if $t = 3$ still remains an open problem. In this work, we show that graphs with few P_4's are 3-admissible, generalizing our previous results obtained on cographs. Considering (k, ℓ)-graphs, which are those graphs whose vertex set that can be partitioned into k independent sets and ℓ cliques, we partially classify the P vs NP-complete dichotomy for such a decision version. Although we prove that MSST for $(2, 1)$-graphs is NP-hard, and knowing, beforehand, that determining the stretch index for chordal graphs is NP-hard as well, we present exact tree stretch indexes for $(2, 1)$-chordal graphs. We also solve MSST for power cycle graphs, an interesting class under two different perspectives: they are families of $(0, \ell)$-graphs, class for which we prove it is NP-hard to determine the stretch index when ℓ is a linear function on the size of the graph; and their stretch indexes are, at the same time, far from the natural lower bound given by the girth, and tight with respect to the diameter spanning tree upper bound.

Keywords: Stretch index, graphs with few P_4's, (k, ℓ)-graphs, $(2, 1)$-chordal graphs, P vs NP-c dichotomy.

1   Introduction

The problem of looking for a spanning tree with constraints on the vertices' distances is a combinatorial challenge with many applications and approaches [2,12]. A *tree t-spanner* of a graph G is a spanning subtree T of G in which the distance between every pair of vertices is at most t times their distance in G or, equivalently, it is the spanning subtree T in which the distance between two adjacent vertices of G is at most t (*cf.* [7]). If a graph has a tree t-spanner, then it is called a *tree t-spanner admissible graph* (or simply *t-admissible*). The parameter t of a tree t-spanner is called the *tree stretch factor*, denoted by $\sigma(T)$, and the smallest t for which a graph G is t-admissible is the *tree stretch index* of G, denoted by $\sigma_T(G)$. Note that the problem of determining the *tree stretch index* of G, called the *minimum stretch spanning tree problem* (MSST), is one of the interesting min-max problems, which are studied not only in graphs, but in several other combinatorial problems, in such a way that bounds, algorithms and computational complexity studies are widely developed [1,11]. From now on, when we refer to MSST, we are dealing with the decision version of such a problem. Moreover, since disconnected graphs do not have spanning trees and trees are the unique 1-admissible graphs, we only consider connected and graphs distinct of trees.

A lower bound for the stretch index can be obtained considering the girth $g(G)$, i.e., the length of a minimum cycle of a graph G. If G is t-admissible, then $t \geq g(G)-1$ which is a tight value for some classes, for instance complete graphs, cycle graphs, wheel graphs, or complete r-partite graphs, for $r \geq 2$. However, establishing lower bounds is not a simple task, even dealing with MSST restricted to graph classes, after all the great majority of results on it concerns graphs admissibility (*cf.* [7]). On the other hand, the *minimum diameter spanning tree* yields an upper bound for the stretch index. In this polynomial-time solvable problem, the solution parameter, say $D_T(G)$, corresponds to the diameter of the minimum diameter spanning tree of G [13].

Theorem 1.1 *[7,13] Given $g(G)$ the girth of G, we have that $g(G)-1 \leq \sigma_T(G) \leq D_T(G)$.*

An intriguing aspect comes when we want to determine if a graph is t-admissible. In terms of the computational complexity, this task is still the greatest breakthrough we aim to solve, since deciding if $\sigma_T(G) \geq 4$ is NP-complete, whereas 2-admissible graphs are polynomial-time recognizable [7], and determining 3-admissible graphs is still an open problem. There are also some classes for which this problem was settled to be NP-complete, as bipartite and chordal with bounded diameter graphs [5,6], or classes for which the stretch index was proved to be bounded by specific values, as split and cographs (*cf.* [16]).

Still in the computational complexity approach, the characterization for tree 2-spanner admissible graphs [7], stated in Theorem 1.2, deals with triconnected components of a connected graph, defined as any maximal subgraph that does not contain two vertices whose removal disconnects the graph. Although it is known that complete graphs with n vertices, $n \geq 2$, are $(n-1)$-connected, i.e, the removal of

$n-2$ vertices does not disconnect the graph [3], the authors consider that edges and triangles are triconnected components. A *nonseparable* graph is a graph without a *cut vertex*, i.e., a vertex whose removal disconnects the graph. A *star* with $n+1$ vertices is the complete bipartite graph $K_{1,n}$. A *v-centered star* is a star centered on v.

Theorem 1.2 *[7] A nonseparable graph G has a tree 2-spanner if, and only if, G contains a spanning tree T such that for each triconnected component H of G, $T \cap H$ is a spanning star of H.*

Recently, we characterized the stretch index for cographs, the P_4-free graphs [8]. Hence, a natural question is to determine such a parameter for graphs with few P_4's. In this work, we present exact values for P_4-sparse and P_4-tidy graphs (Sec. 2). A graph is (k, ℓ) if its vertex set can be partitioned into k independent sets and ℓ cliques [4]. We are interested in classifying the MSST complexity for such a class. Table 1 summarizes the state of art, so far, on determining the stretch indexes for (k, ℓ)-graphs.

k \ ℓ	0	1	2	...
0	−	P	?	..
1	−	P [8]	?	..
2	NP-c [5]	?	?	..
3	?	?	?	..
⋮	⋮	⋮	⋮	..

Table 1
P *vs* NP-c dichotomy before this paper on deciding the stretch index for (k,ℓ)-graphs.

k \ ℓ	0	1	2	3	...	$f(n)$	...
0	−	P	P	?	...	NP-c	...
1	−	P [8]	?	?	...	NP-c	...
2	NP-c [5]	NP-c	NP-c	NP-c	NP-c	NP-c	...
3	NP-c	NP-c	NP-c	NP-c	NP-c	NP-c	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

Table 2
P *vs* NP-c dichotomy after this paper on deciding the stretch index for (k, ℓ)-graphs. $f(n)$ is a linear function on graphs with n vertices. Gray cells present this paper's results.

Note that the decision problem was treated just for a few values of k and ℓ. In spite of that, the problem of determining whether a graph is t-admissible was considered for $(0, 2)$ and for $(1, 1)$-graphs, which are 3-admissible (*cf.* [16]). Table 2 presents our contribution for MSST (k, ℓ)-graphs. Although we set most of them as NP-complete problems (Sec. 3.1), we present subclasses for which MSST is polynomial-time solvable (Sec. 3.2), as $(2, 1)$-chordal (even knowing that for chordal graphs, MSST is NP-complete [6]). We also determine the stretch index for p-power cycles graphs, a $(0, \ell)$-graph family, generalizing results obtained in [8]. Another relevance of considering p-power cycles graphs is: their stretch indexes are, at the same time, far from the lower bound given by the girth, and tight with respect to the diameter spanning tree upper bound of Theorem 1.1.

2   Stretch index for graphs with few P_4's

Given a graph $G = (V, E)$, $d_G(x, y)$ denotes the distance between x and y in G and $d_G(v)$, the degree of v in G. A *pendant vertex* is a vertex of degree 1. We say that a *non-edge* of a spanning tree T is an edge of $G \setminus T$. We say that a subgraph H is *seen by a vertex* v if $N_H(v) \neq \emptyset$, and in this case, v *sees* H. Moreover, a vertex v

is H-*universal* if $V(H) \subseteq N(v)$. A p-path is a path formed by p edges. A *chordal graph* is a C_k-free, for any $k \geq 4$. A *cograph* is a P_4-free graph, i.e. a graph that does not have a 3-path. A pair of vertices v, w is *true* (resp. *false*) *twins* if $N[v] = N[w]$ (resp. $N(v) = N(w)$). See [3] for other graph theory terminologies. Recently [8], we determined the tree stretch index for cographs, by observing structural properties on their cotrees. Based on that, now we consider cographs' superclasses. We are particularly interested in graphs with few P_4's, and in this section we deal with P_4-sparse graphs and P_4-tidy graphs. A graph is P_4-sparse if for each set of 5 vertices, there is at most one induced P_4. A graph G is P_4-*tidy* if, for each induced P_4 P of G, there is at most one vertex $v \in V(G) \setminus V(P)$ such that $V(P) \cup \{v\}$ induces at most two P_4's in G. Graphs with few P_4's can be constructed by a finite number of operations, as union and join. Given graphs $G_i = (V_i, E_i)$, $i = 1, \ldots, p$, we formally define the union and the join operations, respectively, as follows: $G_1 \textcircled{0} \cdots \textcircled{0} G_p = (V_1 \cup \cdots \cup V_p, E_1 \cup \cdots \cup E_p)$; $G_1 \textcircled{1} \cdots \textcircled{1} G_p = (V_1 \cup \cdots \cup V_p, E_1 \cup \cdots \cup E_p \cup \{xy \mid x \in V_i, y \in V_j, \ i \neq j, \ 1 \leq i, j \leq p\})$.

Stretch index *vs* join operation

Now, we deal with the join operation of any two graphs. Particularly, we prove that any graph obtained by the join of two graphs, $G = G_1 \textcircled{1} G_2$, is a 3-admissible graph.

Lemma 2.1 *Given two graphs G_1 and G_2 and $G = G_1 \textcircled{1} G_2$, then $\sigma_T(G) \leq 3$.*

Proof. Consider v_1 a vertex of G_1. Since $N_G(v_1) = N_{G_1}(v_1) \cup V(G_2)$, construct a v_1-centered star whose leaves are all vertices of $V(G_2)$. Next, choose a leaf arbitrarily, say v_2, and make v_2 adjacent to $V(G_1) \setminus \{v_1\}$. Let T be the resulting graph. We claim that T is a tree 3-spanner of G. First, observe that any pair of adjacent vertices u, u' of G_1 (resp. w, w' of G_2) is a non-edge of T. In this case, $d_T(u, u') = 2$ (resp. $d_T(w, w') = 2$), by the path uv_2u' (or wv_1w'). The other non-edges of T are *transversal*, i.e., they have one extremity in G_1 and the other in G_2. Let uw be a transversal non-edge. Thus, $d_T(u, w) = 3$, by the path uv_2v_1w.   □

Lemma 2.1 implies that complete bipartite graphs $K_{p,q}$ are 3-admissible, since they are the join of two $(1,0)$-graphs. Moreover, observe that in such a case, $\sigma_T(K_{p,q}) = 3$, since the only non-edges of T are transversal edges, and, regardless the spanning tree we consider, the smallest path connecting two non-adjacent vertices of a bipartite graph has size at least 3. However, there are graphs obtained by the join of two graphs whose stretch index is 2, for instance, any graph with a universal vertex. Interestingly, in [8] we proved that for cographs the stretch index is 2 if, and only if, it has a universal vertex.

Stretch index *vs* spider operation

A graph G is a *spider* if its vertex set can be partitioned into $\mathcal{S}, \mathcal{K}$ and $\mathcal{R}$ such that (i) $\mathcal{K}$ is a clique, $\mathcal{S}$ is an independent set and $|\mathcal{S}| = |\mathcal{K}| \geq 2$; (ii) each vertex of $\mathcal{R}$ is adjacent to all vertices of $\mathcal{K}$ (a join operation) and is non-adjacent to any

vertex of $\mathcal{S}$; (iii) There is a bijection $f : \mathcal{S} \mapsto \mathcal{K}$ such that, for all $x \in \mathcal{S}$, either $N(x) = \{f(x)\}$, called a *thin spider*, or $N(x) = \mathcal{K} - \{f(x)\}$, called a *thick spider*.

Jamison and Olariu [14] constructively characterized P_4-sparse graphs. A graph G is P_4-*sparse* if, and only if, for each one of its induced subgraphs H, exactly one of the following conditions is satisfied: (i) H is disconnected; (ii) $\overline{H}$ is disconnected; (iii) H is isomorphic to a spider. Note that items (i) and (ii) suggest the union and the join operations applied in a cograph construction. In order to construct a P_4-sparse graph, an operation concerning item (iii) is defined in the following. Let $G_1 = (V_1, \emptyset)$ and $G_2 = (V_2, E_2)$ be two disjoint graphs, where $V_2 = \{v\} \cup \mathcal{K} \cup \mathcal{R}$ and such that: (a) $|\mathcal{K}| = |V_1| + 1 \geq 2$; (b) $\mathcal{K}$ is a clique; (c) $x \in \mathcal{R}$ is adjacent to each vertex $x' \in \mathcal{K}$ and x is not adjacent to v; (d) there exists a vertex $v' \in \mathcal{K}$ such that $N_{G_2}(v) = \{v'\}$ or $N_{G_2}(v) = \mathcal{K} \setminus \{v'\}$. Choose a bijective function $f : V_1 \mapsto \mathcal{K} \setminus \{v'\}$ and define the operation ② as follows: G_1 ② $G_2 = (V_1 \cup V_2, E_2 \cup E')$, where $E' = \{xf(x) \mid x \in V_1\}$, if $N_{G_2}(v) = \{v'\}$, or $E' = \{xz \mid x \in V_1, z \in \mathcal{K} \setminus \{v'\}\}$, if $N_{G_2}(v) = \mathcal{K} \setminus \{v'\}$.

A graph is a spider if, and only if, it can be obtained by the two proper induced subgraphs generated by ②. Moreover, a spider is P_4-sparse if, and only if, the subgraph induced by $\mathcal{R}$ is P_4-sparse [14]. In this way, a graph G is P_4-sparse if, and only if, G can be obtained from trivial graphs, by applying, in any order, operations ⓪, ① and ② a finite number of times.

As a consequence, each P_4-sparse graph has an associated tree, called *PS-tree*. Essentially, in a PS-tree, leaves are the vertices of the graph, each internal node is labeled by $0, 1$ or 2 (accordingly to the operation applied to the associated subtree). See [14] for construction details.

Lemma 2.2 *Let G be a spider graph. If G is a thin spider, then $\sigma_T(G) = 2$. Otherwise, $\sigma_T(G) = 3$.*

Proof. If G is a thin spider, then each vertex in $\mathcal{K}$ is a $\mathcal{K} \cup \mathcal{R}$-universal vertex. Thus there is a spanning star of $\mathcal{K} \cup \mathcal{R}$ centered on any vertex of $\mathcal{K}$. Moreover, since each vertex of $\mathcal{S}$ is pendant in G, the edge incident to any pendant vertex is forced in T, and then $\sigma_T(G) = 2$. Now, assume $|\mathcal{K}| \neq 2$, otherwise G is also a thin spider, which was already considered in the previous case. If G is a thick spider with $G[\mathcal{K} \cup \mathcal{S}]$ not isomorphic to the Hajós graph, i.e., the graph $H(\{a, b, c, d, e, f\}, \{ab, ad, bd, bc, ce, ce, de, de, ef\}))$, then G is a triconnected component, and if $\sigma_T(G) = 2$, there is a spanning star T of G [7], a contradiction, because there is not a universal vertex in G. If G is isomorphic to the Hajós graph, then G is a split graph with $\sigma_T(G) = 3$ [8]. Else, if $G[\mathcal{K} \cup \mathcal{S}]$ is isomorphic to the Hajós graph, then $\mathcal{K} \cup \mathcal{R}$ is a triconnected component, and if $\sigma_T(G) = 2$, then there is a spanning tree T of G such that $T \cap (\mathcal{K} \cup \mathcal{R})$ is a star. Except from the case where $v \in \mathcal{R}$ is $\mathcal{R}$-universal, such a star must be centered on a $\mathcal{K}$ vertex. Note that it remains to place in T the vertices of $\mathcal{S}$, which have degree 2. Clearly, one vertex of $\mathcal{S}$ is not adjacent to the center and is adjacent in G to two leaves of the star, what yields $\sigma_T(G) = 3$. If $v \in \mathcal{R}$ is $\mathcal{R}$-universal and the star $T \cap (\mathcal{K} \cup \mathcal{R})$ is v-centered, then no vertex of $\mathcal{S}$ is adjacent to v. Consequently, $\sigma_T(G) = 3$. □

Lemmas 2.1 and 2.2 implies Lemma 2.3.

Lemma 2.3 *Let G be a P_4-sparse graph, then $\sigma_T(G) \leq 3$.*

An octahedral graph, O_k, is the $(2k-2)$-regular graph, i.e. the graph obtained by the removal of a perfect matching from a K_{2k}. Recently, we proved that $\sigma_T(O_k) = 3$, for $k > 2$ [8] (O_2 is isomorphic to a C_4, and the result follows trivially). Note that, observing the PS-tree, a connected P_4-sparse graph G that is not a spider and does not contain a universal vertex, has a generalized octahedral graph as an induced subgraph. More specifically, if G has a 1-labeled root with k subtrees, there is a generalized octahedral O_k that contains two vertices of $\mathcal{S}$, for each 2-labeled root's subtree, if they exist, and two non-adjacent vertices for each 0-labeled root's subtree (I). Based on that, we obtain the stretch index for P_4-sparse graphs. If G is not a tree and has a universal vertex then $\sigma_T(G) = 2$, as well if G is a thin spider. Next, we determine the stretch index for the other kinds of P_4-sparse graphs.

Lemma 2.4 *If G is a connected P_4-sparse graph which is not a thin spider and without a universal vertex, then $\sigma_T(G) = 3$.*

Proof. Considering the PS-tree of a connected P_4-sparse graph G without a universal vertex, there are only two possible cases:

(i) root's label is 2. In this case, G is a spider, and by Lemma 2.2 we know when G is 2-admissible;

(ii) root's label is 1. In this case, such a root has at least two children which are not leaves, otherwise we would have a universal vertex.

    (a) all root's children are labeled by 0. Thus in each root's subtree there is a pair of non-adjacent vertices whose the lowest common ancestor is the root, labeled by 0, of such a subtree. Therefore, there is a generalized octahedral graph, O_k, as an induced subgraph of G, and for every vertex $v \in V(G) \setminus V(O_k)$, $G[\{v\} \cup V(O_k)]$ does not have a universal vertex.

    (b) there is at least a root's child labeled by 2. In this case, G has at least one proper subgraph that induces a spider. If all spiders of G are thin, then there is a generalized octahedral graph as an induced subgraph of G, and for every vertex $v \in V(G) \setminus V(O_k)$, $G[\{v\} \cup V(O_k)]$ does not have a universal vertex. If there is a thick spider in G, say A, then there is a vertex $v \in V(A)$ which is universal with respect to all subgraphs of G that induce a generalized octahedral. Moreover, note that a subgraph of G induced by $V(O_k)$, $k \geq 2$, and all universal vertices with respect to O_k is a triconnected subgraph of G.

        We claim that even when there is, in G, a universal vertex with respect to the generalized octahedrals of G, $\sigma_T(G) = 3$. Let O_k be a generalized octahedral of G as described in (I), A be a thick spider of G, and $u \in V(A)$ an O_k-universal vertex. Suppose, by contradiction, that $\sigma_T(G) = 2$. In this case, there is a spanning tree T of G such that, for each triconnected component H of G, $T \cap H$ is a star [7]. Since an O_k and the O_k-universal vertices belong to a triconnected component H, and $\sigma_T(O_k) = 3$, each

vertex of O_k must be a leaf of the star $T \cap H$. Thus, the center of such a star must be an O_k-universal vertex, say u. Note that $u \in V(A)$ and at least two of these leaves belong to $V(A)$, say a, b. In particular, they belong to $\mathcal{S}$, by O_k's construction. Observe that there is exactly one vertex of A that is not adjacent to u, say x, and x has at least 2 neighbors in A. We have two possibilities: (i) all neighbors of u in G are neighbors of u in T, and in this case $d_T(x, y) \geq 3$, where $y \in N_A(x)$; (ii) there is at least one neighbor of u, say w, which is not in $N_T(u)$, but is adjacent to a or b in G. Hence, $d_T(w, i) \geq 3$, for $i \in \{a, b\}$.

Since the PS-tree's root is 1-labeled, and thus no 1-labeled descendent is allowed, it suffice to analyze cases (a) and (b) to finish the proof.

□

Theorem 2.5 *A P_4-sparse graph G is 2-admissible if, and only if, either G has a universal vertex; or G is a thin spider.*

Proof. Clearly, if G has a universal vertex or if G is a thin spider, then $\sigma_T(G) = 2$. For the converse, suppose G is not a thin spider and does not have a universal vertex. So, its PS-tree's root has label 2 (in this case G is a thick spider) or 1. Hence, by Lemmas 2.2 and 2.4, respectively, $\sigma_T(G) = 3$. □

A natural generalization of P_4-sparse graphs are the P_4-tidy graphs. A graph H is an *almost-spider* graph if H can be constructed from a spider graph $G = (\mathcal{S}, \mathcal{K}, \mathcal{R})$ by adding a vertex v' which is either a false twin of v or a true twin of v, such that $v \in \mathcal{S} \cup \mathcal{K}$ [15]. Hence, we call H a *$\mathcal{P}$-false-almost-spider* and *$\mathcal{P}$-true-almost-spider*, respectively, where $\mathcal{P}$ is the set to which v belongs, i.e, $\mathcal{P} \in \{\mathcal{S}, \mathcal{K}\}$. In the same way, if G is a thin (or thick) spider, then H is a *true* or *false-almost-thin (or thick)-spider*. A P_4-tidy graph G can be constructed by the following way: i) $G_1 \ⓞ\ G_2$, for G_1 and G_2 being P_4-tidy graphs; ii) $G_1 \ⓛ\ G_2$, for G_1 and G_2 being P_4-tidy graphs; iii) G is a spider; iv) G is an almost spider; v) G is P_5, C_5, $\overline{P_5}$, or K_1. Since PS-trees represent P_4-sparse graphs, we can develop in a similar way a tree representation of a P_4-tidy graph [15].

Lemma 2.6 *Let G be an almost-spider graph, then $\sigma_T(G) \leq 3$.*

By i) - v) and results above, if G is a P_4-tidy graph, then G is 3-admissible, excepted if G is a C_5.

Lemma 2.7 *Let G be an almost-spider graph. G is 2-admissible if, and only if, G is a $\mathcal{S}$-almost-thin-spider (false or true), or is a $\mathcal{K}$-true-almost-thin-spider.*

Similarly to Theorem 2.5, we are able to characterize the P_4-tidy graphs that are 2-admissible.

Theorem 2.8 *A P_4-tidy graph G is 2-admissible if only only if either: G has a universal vertex; or G is a thin spider; or G is a $\mathcal{S}$-almost-thin-spider (false or true); or G is a $\mathcal{K}$-true-almost-thin-spider.*

3   Stretch index for (k, ℓ)-graphs

As illustrated in Table 1, the MSST has not been treated for (k, ℓ)-graphs, considering $k + \ell \geq 3$. Although $(0, 2)$-graphs are known to be 3-admissible (*cf.* [7]), there is no characterization on the 2-admissibility of $(0, 2)$-graphs. (k, ℓ)-graphs fit on the framework of MSST interesting classes, since $(0, 2)$-graphs are 3-admissible whereas for $(2, 0)$-graphs the MSST is known to be NP-complete for $t \geq 5$ [5]. In Section 3.1, we present NP-complete cases considering (k, ℓ)-graphs, and in Section 3.2, we present (k, ℓ)-graphs' subclasses for which we are able to determine the stretch index in polynomial time.

3.1   Difficult cases

Let G be a graph. A *unicorn graph* is a graph obtained from G by the addition of a pendant vertex to an arbitrary vertex of G. If G belongs to a class $\mathcal{C}$, then a unicorn graph obtained from G is called a *unicorn-$\mathcal{C}$ graph*. Since the edge incident to any pendant vertex is forced in any spanning tree, we immediately have that given H a unicorn graph obtained from a graph G, $\sigma_T(H) = t$ if, and only if, $\sigma_T(G) = t$. Therefore, MSST is NP-complete even for unicorn bipartite graphs.

Note that a unicorn-(k, ℓ)-graph has a (k, ℓ)-partition in which the added pendant vertex is assigned to either one of the k independent sets of G or to one of the ℓ cliques of G (in this case, this clique is a K_1). Next we present two constructions in order to prove that MSST for $(k, 0)$-graphs, $k \geq 3$, and for $(k, \ell + 1)$-graphs, $k \geq 3, \ell \geq 0$, are NP-complete for $t \geq 5$.

Construction 1 *Given $H = (V_H, E_H)$ a unicorn bipartite graph, and consider v the added pendant vertex of H. Hence, we construct a graph $W = (V_W, E_W)$ from H as follows: i) $V_W = V_H \cup V(K_{k-1})$; ii) $E_W = E_H \cup \{vy \mid y \in V(K_{k-1})\} \cup E(K_{k-1})$.*

Let H be a graph and W a graph obtained from H by Construction 1. Hence, it is straightforward to check that, given $t \geq 2$, $\sigma_T(W) = t$ if, and only if, $\sigma_T(H) = t$, and so, we have what follows.

Lemma 3.1 *MSST for $(k, 0)$-graphs, for $k \geq 3$, is NP-complete for $t \geq 5$.*

Construction 2 *Let $H = (V_H, E_H)$ be a unicorn-(k, ℓ)-graph, and consider v the added pendant vertex of H. We construct a particular instance $W = (V_W, E_W)$ as follows: i) $V_W = V_H \cup V(K_{k+1})$; ii) $E_W = E_H \cup \{vy \mid y \in V(K_{k+1})\} \cup E(K_{k+1})$.*

Lemma 3.2 *If MSST for (k, ℓ)-graphs is NP-complete, for k, ℓ fixed integers, $k, \ell \geq 0$, then MSST for $(k, \ell + 1)$-graphs is NP-complete for $t \geq 5$.*

Proof. Given a unicorn-(k, ℓ)-graph H and the graph W obtained by Construction 2, it remains to prove that a unicorn-(k, ℓ)-graph H has stretch index $t > 2$ if, and only if, W is a $(k, \ell + 1)$-graph with stretch index $t > 2$. Clearly, W has a $(k, \ell + 1)$-partition. Moreover, by Construction 2, v is the unique vertex of H adjacent to the added clique K_{k+1}. Thus, there is no path between vertices of V_H that contains a vertex of K_{k+1}. Thus $\sigma_T(W) = t$. For the converse, suppose W is a

$(k, \ell + 1)$-graph. By Construction 2, we assure, by the pigeonhole principle, that at least one of the $l + 1$ cliques of W is composed only by added vertices (if v is in such a clique, it can be replaced in one of the k independent sets of W without any loss). Thus, after the K_{k+1} removal of W, the resulting graph H' is a unicorn-(k, ℓ)-graph and $\sigma_T(H') = t$, since we can guarantee that there are two vertices of $V'_H \setminus \{v\}$ whose distance in T is t, otherwise, we could improve $\sigma_T(W)$. $\qquad \square$

Lemmas 3.1 and 3.2 immediately imply Theorem 3.3.

Theorem 3.3 *MSST for (k, ℓ)-graphs, for $k \geq 2$ and $\ell \geq 0$, is* NP*-complete.*

Next we prove that MSST for $(0, \ell)$-graphs is NP-complete by presenting a polynomial-time reduction from the NP-complete problem MSST for $(2, 0)$-graphs. Since any graph with n vertices is a $(0, n)$-graph, hence MSST for $(0, n)$-graphs is already known to be NP-complete, for $t \geq 4$ [7]. However, it is an interesting question to decide what is the smallest ℓ for which MSST is NP-complete. In the following, we deal with this task.

Construction 3 *Given $G = (V_G, E_G)$ a bipartite graph, an integer d, we construct a graph $Q = (V_Q, E_Q)$ from G and $|E_G|$ copies of complete graphs K_d^i with vertices $u_1^i, u_2^i, \ldots, u_d^i$, for $i \in E_G$, and $|V_G|$ copies of complete graphs K_{d-1}^i with vertices $v_1^i, v_2^i, \ldots, v_{d-1}^i$, for $i \in V_G$, as follows:*

- $V_Q = V_G \bigcup\limits_{i \in E_G} V(K_d^i) \bigcup\limits_{i \in V_G} V(K_{d-1}^i);$

- $E_Q = E_G \cup \{au_1^i, bu_d^i | ab \in E_G\} \cup \{av_j^a | a \in V(G), j = 1, 2, \ldots, (d-1)\} \bigcup\limits_{i \in E_G} E(K_d^i) \bigcup\limits_{i \in V_G} E(K_{d-1}^i).$

Fact 3.4 *Let $Q = (V_Q, E_Q)$ be the graph constructed from $G = (V_G, E_G)$ and the integer d by Construction 3. We have that $|V_Q| = d(|V_G| + |E_G|)$, and Q is a $(0, |V_G| + |E_G|)$-graph.*

Lemma 3.5 *Let Q be the graph obtained from G by Construction 3 and the integer d. Then G is a $(2, 0)$-graph with $\sigma_T(G) = t$ if, and only if, Q is a $(0, \frac{|V_Q|}{d})$-graph with $\sigma_T(Q) = t + 2$.*

Since a *size* of a graph $G = (V, E)$ is $|V| + |E|$, we relate $(2, 0)$-graphs of size ℓ and $(0, \ell)$-graphs.

Theorem 3.6 *If MSST for $(2, 0)$-graphs of size ℓ is* NP*-complete, then MSST for $(0, \ell)$-graphs is* NP*-complete.*

An *urchin-G graph* is the graph obtained from a graph G by adding one pendant vertex for each vertex of G. Clearly, an urchin-G graph of a $(0, \ell)$-graph is a $(1, \ell)$-graph. Since an urchin-G graph H has $\sigma_T(H) = \sigma_T(G)$, we state Theorem 3.7.

Theorem 3.7 *If MSST for $(0, \ell)$ is* NP*-complete, then MSST for $(1, \ell)$-graphs is* NP*-complete.*

3.2 Easy cases

Although for several values of k and ℓ, MSST is NP-complete, we can determine the stretch index for some (k, ℓ)-graphs in polynomial-time. $(0, 2)$-Graphs are 3-admissible, *cf.* [16], and, in this work we characterize 2-admissible $(0, 2)$-graphs. Let $H = G[V_H]$, where V_H is the set of vertices incident to each transversal edge of G, i.e., edges with one extreme in K^1 and the other in K^2.

Lemma 3.8 *Let $G = (K^1 \cup K^2, E)$ be a $(0, 2)$-graph. G is 2-admissible if, and only if, either G has a universal vertex, G has a cut-vertex or H is a strict 2-connected graph that has not an induced C_4.*

As a consequence of Lemma 3.8, we have that a $(0, 2)$-graph G obtained by a union of $H = K_p$ and $Q = K_q$, for $p, q \geq 4$, and connecting them by an induced C_4 composed by two vertices of H and two vertices of Q, has $\sigma_T(G) = 3$.

$(2, 1)$-Chordal graphs

As proved in Section 3.1, MSST for $(2, 1)$-graphs is NP-complete, as well for chordal graphs [6]. However, for $(2, 1)$-chordal graphs (class studied in some other contexts [9,10]), the MSST is solved in polynomial-time. Note that a $(2, 0)$-chordal graph does not have cycles, so, a chordal graph is a $(2, 0)$-graph if, and only if, it is a forest. Thus, a $(2, 1)$-chordal graph is partitioned into a forest $\mathcal{F}$ and a clique $\mathcal{K}$. Given a $(\mathcal{F}, \mathcal{K})$-partition of an arbitrary graph G, edges with an extreme in $\mathcal{F}$ and the other in $\mathcal{K}$ are called *transversal* edges. Two transversal edges incident to a same tree in $\mathcal{F}$ create a cycle in G. If the two edges are incident to the same vertex of $\mathcal{K}$, then we have a *type 1* cycle, otherwise, a *type 2* cycle. In particular, if G is chordal, in a type 1 cycle the vertex in $\mathcal{K}$ must be adjacent to each other vertex of the cycle. In a type 2 cycle we have 2 possibilities: at least one of the $\mathcal{K}$-vertices are completely adjacent to the cycle vertices; or their neighborhoods cover the cycle with some intersection as described in [9]. In this work we consider that vertices v in $T_i \in \mathcal{F}$, for $i \in \{1, \ldots, |\mathcal{F}|\}$, are *type 1 vertices* if, and only if, $|\bigcup_{v \in T_i} N_{\mathcal{K}}(v)| = 1$. Otherwise, they are *type 2 vertices*.

Lemma 3.9 $(2, 1)$-*Chordal graphs are 4-admissible.*

Next we characterize 2-admissible $(2, 1)$-chordal graphs, with a $(\mathcal{F}, \mathcal{K})$-partition such that $\mathcal{K}$ is maximal. Note that if a given graph G is $(2, 1)$-chordal, but the clique is not maximal, there exists at least one and at most two universal vertices with respect to $\mathcal{K}$ in $T^i \in \mathcal{F}$ that can be placed in $\mathcal{K}$ without any loss. So, from now on, we only consider $(\mathcal{F}, \mathcal{K})$-partitions for $(2, 1)$-chordal graphs such that $\mathcal{K}$ is maximal.

A *bi-star* B is obtained by identifying one leaf of a star with the center of another star. A *v, w-centered bi-star* is a bi-star with centers v, w. We denote the leaf sets of v and w in B by $L(v)$ and $L(w)$, respectively.

Lemma 3.10 *Let G be a $(2, 1)$-chordal graph. If $\mathcal{K} \cap T$ is distinct of a star, then $\sigma_T(G) \geq 3$.*

In order to determine the stretch index of a $(2,1)$-chordal graph G, we first pre-process G, obtaining a graph G' called *cleaned $(2,1)$-chordal graph*, as follows: we remove $t \in V(\mathcal{F})$ such that $N_{\mathcal{K}}(t) = \emptyset$, and all type 1 vertices of $\mathcal{F}$. Next, considering each type 2 cycle C_{k_1,k_2}, $k_1, k_2 \in \mathcal{K}$, and a corresponding tree $T \in \mathcal{F}$ such that k_1, k_2 are not universal in C_{k_1,k_2}, we mark vertices $f \in C_{k_1,k_2} \cap \mathcal{F}$ such that $|N_{\mathcal{K}}(f)| = 1$ and are not adjacent to vertices in $N_{C_{k_1,k_2} \cap T}(k_1) \cap N_{C_{k_1,k_2} \cap T}(k_2)$. If a marked vertex belongs to another cycle and is not marked in such a case, then the mark is removed and the vertex cannot be marked anymore. In the end, all marked vertices are removed. Clearly, if the $(2,1)$-chordal graph G has $\sigma_T(G) = t$, then its cleaned $(2,1)$-chordal graph G' has $\sigma_T(G') = t$.

Lemma 3.11 *Let $G = (\mathcal{F}, \mathcal{K}, E)$ be a $(2,1)$-chordal graph, and $G' = (\mathcal{F}', \mathcal{K}, E')$ be its cleaned $(2,1)$-chordal graph. $\sigma_T(G) = 2$ if, and only if, one of the following conditions holds: (i) all cycles in G (not formed only by clique vertices) are type 1; (ii) there is a $\mathcal{F}'$-universal vertex in $\mathcal{K}$.*

Lemma 3.12 *Let $G = (\mathcal{F}, \mathcal{K}, E)$ be a $(2,1)$-chordal graph. If $\sigma_T(G) = 3$, then there is tree 3-spanning T such that $T \cap \mathcal{K}$ is a bi-star.*

Given a v,w-centered bi-star and pair of $\mathcal{K}$-vertices (x, z), an edge $tt' \in E(T_i)$ is called a (x, z)-*stress edge* if x sees only one extreme and z sees both t, t'.

Lemma 3.13 *Let $G = (\mathcal{F}, \mathcal{K}, E)$ be a non-2-admissible $(2,1)$-chordal graph and $G' = (\mathcal{F}', \mathcal{K}, E')$ be its cleaned $(2,1)$-chordal graph. $\sigma_T(G) = 3$ if, and only if, $\mathcal{K} \cap T$ is a v, w-centered bi-star such that, for each tree T_i in $\mathcal{F}', i = 1, \ldots, |\mathcal{F}'|$, exactly one of the following conditions holds:*

(i) $N_{T_i}(v) \cup N_{T_i}(w) = V(T_i)$;

(ii) $N_{T_i}(v) = N_{T_i}(w) = N_{T_i}(L(w)) = \emptyset$ *(resp. $N_{T_i}(L(v)) = \emptyset$) and there is $f \in L(v)$ (resp. $l \in L(w)$) which is T_i-universal.*

(iii) *Items 3.1 and 3.2 must hold simultaneously:*

    3.1. for each pair (e_1, e_2) of (f, x) and (x, f)-stress edges (resp. (l, x) and (x, l)-stress edges), $f \in L(v), x \in (\{w\} \cup L(v))$ (resp. $l \in L(w), x \in (\{v\} \cup L(w))$) there is a vertex $t \in N_{T_i}(x) \cap N_{T_i}(f)$ (resp. $N_{T_i}(x) \cap N_{T_i}(l)$) which belongs to the path between e_1 and e_2 in T_i and such that $vt \in E(G')$ (resp. $wt \in E(G')$).

    3.2. for each (f, l) or (l, f)-stress edges, say $e_1 = tt'$, at least one center see t or t', $f \in L(v)$ and $l \in L(w)$.

Lemmas 3.9, 3.11 and 3.13 give us the needed conditions to determine the stretch index of $(2,1)$-chordal graphs in polynomial time. More specifically, Lemma 3.11's conditions can be verified by looking for type 1 cycles or a $\mathcal{F}'$-universal vertex. Moreover, in order to recognize a 3-admissible $(2,1)$-chordal graph, we try to construct a v, w-centered bi-star that satisfies at least one of the Lemma 3.13's conditions. Thus, we consider each pair $v, w \in \mathcal{K}$ as bi-star center candidates and we determine $L(v)$ and $L(w)$ observing $N_{\mathcal{F}'}(v), N_{\mathcal{F}'}(w)$ and strongly following Lemma 3.13's conditions. If it is not possible to build such a bi-star for some pair v, w, another pair

is checked, and if all answers are negative, we conclude that $\sigma_T(G) = 4$. Note that Lemma 3.13's conditions restrict $L(v)$ and $L(w)$, and we do not build all possible bi-stars.

Theorem 3.14 *MSST for $(2,1)$-chordal graphs is polynomial-time solvable.*

Cycle power graphs

A *p-cycle-power graph*, C_n^p, is obtained from a C_n by adding edges between all vertices with distance at most p in C_n. Since $g(C_n^p) = 3$, then $\sigma_T(C_n^p) \geq 2$. A p-cycle-power graph can be partitioned into $\lceil \frac{n}{p+1} \rceil$ cliques. Therefore, any p-cycle-power graph is a $(0,\ell)$-graph, for $\lceil \frac{n}{p+1} \rceil \leq \ell \leq n$. In [8], we showed an optimum tree $\lfloor \frac{n}{2} \rfloor$-spanner for $p = 2$. Next, we show the index stretch for the general p-cycle power graphs. A *turn around path* between u_i and u_p is a path $u_i u_{j^{(1)}} u_{j^{(2)}} \dots u_{j^{(s)}} u_p$ such that $(i, j^{(1)}, j^{(2)}, \dots, j^{(s)}, p)$ is a monotonic circular sequence. A non turn around path is called a *zig-zag path*.

Lemma 3.15 *For any spanning tree T of $G = C_n^p$, there is at least an external non-edge $u_i u_{i+1}$, such that the path between u_i and u_{i+1} in T is a turn around path.*

As consequence of Lemma 3.15, we have that the spanner tree diameter value for a power cycle graph is equal to its stretch index (i.e. $D_T(C_n^p) = \sigma_T(C_n^p)$). Theorem 3.16 presents $\sigma_T(C_n^p)$ by showing upper bound that is equal to the length of a turn around path with respect to an external non-edge.

Theorem 3.16 *For any C_n^p, if $p \geq \lfloor \frac{n}{2} \rfloor$, then $\sigma_T(C_n^p) = 2$. Otherwise, if $p < \lfloor \frac{n}{2} \rfloor$, then: i) if $n \equiv x \pmod{p}$, for $x \in \{0,1\}$, then $\sigma_T(C_n^p) = \lfloor \frac{n}{p} \rfloor$; ii) if $n \not\equiv x \pmod{p}$, for $x \in \{0,1\}$, then $\sigma_T(C_n^p) = \lceil \frac{n}{p} \rceil$.*

4 Conclusions

In this work, we present tree stretch indexes for graphs with few P_4's and (k,ℓ)-graphs, graph classes that generalize cographs, bipartite graphs, and split graphs, for which the computational complexity of MSST was already settled before [5,8]. Following the strategies proposed in this work, we intend to continue obtaining optimum tree t-spanners for graphs constructed by vertex/edges operations. There are still some open problems concerning the results presented in here, for instance: determining a fixed value of ℓ for which MSST for $(0,\ell)$-graphs is NP-complete, if such a value exists; and finishing the fully classification of MSST for (k,ℓ)-graphs' P *vs* NP-complete dichotomy.

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