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Amplitude-frequency Relationship for the Relativistic Oscillator

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Abstract

The Hamiltonian approach and the variational approach are utilized to treat the relativistic harmonic oscillator for the amplitude-frequency relationship. The nice reliability is shown by the result comparison with that from open literature. The simplicity and efficiency of the methods are also disclosed for different range of the initial amplitude during looking for the amplitude-frequency relationship for the nonlinear relativistic harmonic oscillator.

Keywords: Relativistic harmonic oscillator; Amplitude-frequency relationship; Hamiltonian approach; Variational approach

1. The Oscillator

Consider the nonlinear relativistic harmonic oscillator [1-3]

$$\frac{d^2u}{dt^2}+\frac{u}{\sqrt{1+u^2}}=0 \tag{1}$$

with initial conditions

$$u(0)=A \text{ and } \frac{du(0)}{dt}=0, \tag{2}$$

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where A is the initial amplitude. This is an example of a conservative nonlinear oscillatory system in which the restoring force has an irrational form. The relativistic oscillator is important in physics because it is usually used as the basis for analyzing more complicated motion.

2. Application of the Hamiltonian approach

The Hamiltonian approach has been successfully used to study the nonlinear vibrating equations [5, 6] since it was discovered by Ji-huan He [4]. Here, the corresponding Hamiltonian can be easily obtained, which reads

$$H(u) = \frac{1}{2}\left(\frac{du}{dt}\right)^2 + \sqrt{1+u^2} \tag{3}$$

According to [4], integrating Eq. (3), there is

$$\overline{H}(u) = \int_0^{T/4} \left\{\frac{1}{2}\left(\frac{du}{dt}\right)^2 + \sqrt{1+u^2}\right\} dt, \tag{4}$$

where T is the period of the nonlinear oscillator.

Assume the solution of the oscillator in the form

$$u(t) = A\cos\omega t, \tag{5}$$

where A and ω are respectively the amplitude and frequency of the oscillator.
Substituting Eq. (5) to (4) yields

$$\overline{H}(u) = \int_0^{T/4} \left\{\frac{1}{2}A^2\omega^2\sin^2\omega t + \sqrt{1+A^2\cos^2\omega t}\right\} dt$$

$$= \int_0^{\pi/2} \left\{\frac{1}{2}A^2\omega\sin^2 t + \frac{1}{\omega}\sqrt{1+A^2\cos^2 t}\right\} dt \cdot \tag{6}$$

Go on with the approach

$$\frac{\partial}{\partial A}\left(\frac{\partial \overline{\mathrm{H}}}{\partial(1/\omega)}\right) = -\frac{\pi}{4}A\omega^2 + \frac{\partial}{\partial A}\left(\int_0^{\pi/2}\sqrt{1+A^2\cos^2 t}\,dt\right) = 0 \cdot \tag{7}$$

From Eq. (7), there has

$$\omega^2 = \frac{4\int_0^{\pi/2}\frac{\cos^2 t}{\sqrt{1+A^2\cos^2 t}}dt}{\pi} \cdot \tag{8}$$

This is the same as that obtained by the max-min approach in Ref. [2] (see Eq. (7) and Eq. (10) in Ref. [2]).

3. Application of the Variational approach

Variation is one of the two basic ways to describe a physical problem [7, 9]. Next, we apply the variational method [7, 9] to the nonlinear relativistic harmonic oscillator. The functional formulation can be constructed, which reads

$$J(u) = \int_0^{T/4} \left\{ -\frac{1}{2}\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)^2 + \sqrt{1+u^2} \right\} dt, \tag{9}$$

where T is the period of the nonlinear oscillator.

Assume the solution of the oscillator is the same as Eq. (5), i.e.

$$u(t) = A\cos\omega t,$$

where A and ω are the amplitude and frequency of the oscillator respectively.

Taking the Eq. (5) into (9), we have

$$J(A) = \int_0^{T/4} \left\{ -\frac{1}{2} A^2\omega^2 \sin^2 t + \sqrt{1+A^2\cos^2 \omega t} \right\} dt$$

$$= \int_0^{\pi/2} \left\{ -\frac{1}{2} A^2\omega \sin^2 t + \frac{1}{\omega}\sqrt{1+A^2\cos^2 t} \right\} dt \tag{10}$$

By He's variational method [7, 9], there becomes

$$\frac{\mathrm{d}J}{\mathrm{d}A} = \int_0^{\pi/2} -A\omega \sin^2 t dt + \frac{1}{\omega}\int_0^{\pi/2} \frac{A\cos^2 t}{\sqrt{1+A^2\cos^2 t}} dt = 0. \tag{11}$$

Then the result is

$$\omega^2 = \frac{4\int_0^{\pi/2} \frac{\cos^2 t}{\sqrt{1+A^2\cos^2 t}} dt}{\pi} \tag{12}$$

This agrees well with that obtained by the Hamiltonian approach, Eq. (8); equals that obtained by the max–min approach in Ref. [2] (see Eq. (7) and Eq. (10) in Ref. [2]).

4. Discussions

Rewriting with the elliptic integral, Eqs. (8) and (12) become

$$\omega^2 = -\frac{4\left(K\left(\sqrt{-A^2}\right) - E\left(\sqrt{-A^2}\right)\right)}{A^2\pi}, \tag{13}$$

where $K(m) = \int_0^{\pi/2} \frac{1}{\sqrt{1-m\sin^2 t}} dt$ and $E(m) = \int_0^{\pi/2} \sqrt{1-m\sin^2 t} dt$.

The exact period [8] T_{ex} for the relativistic oscillator is

$$T_{ex} = \left[4\sqrt{4+A^2} E\left(\frac{A^2}{4+A^2}\right) - \frac{8}{\sqrt{4+A^2}} K\left(\frac{A^2}{4+A^2}\right) \right]^{-1}, \tag{14}$$

where $K(m)=\int_0^{\pi/2}\frac{1}{\sqrt{1-m\sin^2 t}}dt$ and $E(m)=\int_0^{\pi/2}\sqrt{1-m\sin^2 t}dt$.

So

$$\omega_{ex}=2\pi\left[4\sqrt{4+A^2}E\left(\frac{A^2}{4+A^2}\right)-\frac{8}{\sqrt{4+A^2}}K\left(\frac{A^2}{4+A^2}\right)\right]. \quad (15)$$

Though ω in Eq. (13) and Eq. (15) is different in form, we know the approximation is good compared with the exact one after symbolic computation with distinct initial amplitude. Maybe because the methods are based on energy, the resulted approximation amplitude-frequency relationship is valid for different range of the initial amplitude, no matter the amplitude is large or small.

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