Available online at www.sciencedirect.com

ScienceDirect

Electronic Notes in Theoretical Computer Science 202 (2008) 137–170

Electronic Notes in Theoretical Computer Science

www.elsevier.com/locate/entcs

Notions of Probabilistic Computability on Represented Spaces

Volker Bosserhoff [1],[2]

Institut für Theoretische Informatik und Mathematik
Universität der Bundeswehr
Munich, Germany

Abstract

We define and compare several probabilistically weakened notions of computability for mappings from represented spaces (that are equipped with a measure or outer measure) into effective metric spaces. We thereby generalize definitions by Ko [9] and Parker [11,12], and furthermore introduce the new notion of *computability in the mean*. Some results employ a notion of *computable measure* that originates in definitions by Weihrauch [19] and Schröder [14]. In the spirit of the well-known Representation Theorem, we establish dependencies between the weakened computability notions and classical properties of mappings. We finally present some positive results on the computability of vector-valued integration on metric spaces, and discuss certain measurability issues arising in connection with our definitions.

Keywords: computable analysis, computable measure theory

1 Introduction

1.1 Motivation

The considerations in this article are inspired by real-world situations like the following: An agent (i.e. a person, a machine or a combination of such) has the task to perform a measurement ξ of a (physical) magnitude. Then a 2^{-k}-approximation to the value $f(\xi)$ shall be computed, where $k \in \mathbb{N}$ is a given precision parameter and $f : X \to Y$ is a given function that maps the state space X of the magnitude into a metric space (Y, d). *When it comes to computations, a realistic model of the abilities of the agent is a Turing machine*; so the results of the measurement must be available in machine readable form, i.e. encoded as a string over some finite alphabet Σ. The space X will typically not be countable, so the value ξ must be encoded as an infinite string. We assume that there is a surjective partial mapping

[1] The work was supported by DFG grant HE 2489/4-1.

[2] Email: volker.bosserhoff@unibw.de

doi:10.1016/j.entcs.2008.03.013

$\delta :\subseteq \Sigma^\omega \to X$, a so-called *representation* of X, and that the measuring device puts out a δ-name $p \in \mathrm{dom}(\delta)$ of ξ, i.e. $\delta(p) = \xi$.[3] *We do not model the details of this process, so we can make no assumptions about what particular δ-name of ξ will finally be extracted from the measurement.* The δ-name is progressively written onto the input tape of a Turing machine.[4] The codomain Y of f is typically not countable either, but we assume that Y has a countable dense subset A, and that there is a partial mapping $\alpha :\subseteq \Sigma^* \to A$, a so called *notation* of A. The question is: Is there a TM that takes a δ-name p of some measured ξ as well as a precision parameter k as inputs and halts (after a finite number of steps) with a word w on its output tape such that $d(f(\xi), \alpha(w)) \leq 2^{-k}$?

There are functions f for which there exists no Turing machine that could perform the above task. This is the case, for example, if there is a name $p \in \mathrm{dom}(\delta)$ and a precision parameter $k \in \mathbb{N}$ such that no prefix of p already determines $f(\delta(p))$ up to precision 2^{-k}. But even for functions, for which such a discontinuity does not occur, there is possibly no Turing machine for the above task, simply because there are "too many functions" and "too few Turing machines"; by now, however, no one has given an example of a function of the latter kind, that comes up naturally in an application.

Now, additionally assume that there is a σ-algebra $\mathcal{S}$ and a probability measure P such that $(X, \mathcal{S}, P)$ is a probability space, and that the observed magnitude is distributed according to P. The presence of a probability distribution allows us to weaken the demands on the Turing machine above in several meaningful ways; in particular, we might only ask for a TM that

(I) behaves correctly on P-almost every value of ξ, or

(II) behaves correctly, except on a set whose probability is at most 2^{-k}, or

(III) produces an approximation whose *expected* error is at most 2^{-k}.

In the following, it will be our aim to develop the foundations of a representation-based computability theory for these three settings. Although probability measures are most interesting for applications, we will also consider more general measures and outer measures whenever meaningful.

The general theory of Turing machine computability via representations is developed in the textbook of Weihrauch [20]; the present work is formulated to fit into this framework. We will recall some basic notions from computable analysis below, but refer to [20] for some more technical definitions.

We assume that the reader has a basic background on measure theory and descriptive set theory. All facts we use can be found in any introductory textbook; we occasionally refer to [7,8].

[3] In respect of the requirement of producing a δ-name from the outcome of the measurement, we imagine that the measurment is performed in two stages: First, an analogue "snapshot" of the magnitude is taken, which (ideally) completely resembles ξ. Then, a δ-name is progressively extracted from the analogue snapshot. This two-stage model is necessary, because the magnitude might change over time, and so we cannot extract the δ-name directly.

[4] Each character of the name is extracted from the snapshot before or just when the TM queries the corresponding tape cell for the first time.

1.2 Overview of the present work

In Section 2, we recall some definitions and results about continuity and computability via representations. We recall the definition of an effective topological space, and we also define what it means for it to be computably regular and computably compact. We introduce several (multi-)representations of Borel measures. We then give a useful result on computable measures on computable metric spaces. We finally recall some less common notions from measure theory.

Section 3 contains precise definitions of the three weakened concepts of computability corresponding to items (I), (II) and (III) above; by considering mixed settings, we arrive at a total of five concepts. Each of these computability concepts is accompanied by a weakened continuity concept; the focus is on working out relations between these weakened forms of continuity and classical properties of the representations, spaces, measures and mappings. We then study the pairwise relations between the five concepts: We either give a strong counter-example showing that one concept does not imply the other, show that one concept always implies the other, or show that one concept implies the other under mild additional assumptions.

Section 4 contains some positive results on the computability of (Pettis) integration of mappings from metric spaces into normed spaces.

The definitions in Section 3 depend on a certain "local error function" (Definition 3.2). In the final section, we investigate the measurability this function.

1.3 Related work

The book of Ko [9] deals with computability and complexity of real functions in a way that is consistent with [20]. For functions $f : [0,1] \to \mathbb{R}$ and the Lebesgue measure λ, a weakened notion of computability, that corresponds to item (II) above, is defined and studied in Chapter 5 of that book. Building on Ko's definitions, probabilistic computability notions for characteristic functions of subsets of $\mathbb{R}^n$ have been studied by Parker [11,12]; Parker's definitions correspond to concepts (I) and (II). The works of Ko and Parker can be said to have taken a "top-down" approach by restricting themselves to Euclidean spaces; we are attempting to go "bottom-up" and consider very general definitions.

One of the (multi-)representations of Borel-measures to be introduced below is a modification of a definition of Schröder [14], which itself generalizes a definition of Weihrauch [19]. The reader might also find the related articles [10] and [15] of interest. We would also like to mention the work of Gács [4], whose definition of a computable probability measure seems to be equivalent to Schröder's for the special case of metric spaces.

[23] introduces *computable measure spaces*; this notion is further studied in e.g. [21,22]. The focus of those works, however, is on representations (and the induced computability) of measurable sets and measurable functions, while we are interested in computability on points in a represented space that is also equipped with a measure.

Furthermore, measure and integration have been treated from the viewpoints of constructive mathematics [1], domain theory [2,3], and digital topology [18]. It is beyond the scope of this article to work out the relations between these approaches.

One motivation for the present work was to establish weakened computability notions that correspond to weakened notions of *solvability* (more precisely the "probabilistic setting" and the "average-case setting") studied in information-based complexity [16]. IBC is mainly concerned with numerical problems on function space and uses an algebraic (aka "real number"-) model of computation. We hope that our definitions and results will be useful for studying numerical problems in the Turing machine model.

2 Preliminaries

2.1 Computable analysis via representations

Let X be a nonempty set and $W \in \{\Sigma^*, \Sigma^\omega\}$. A *naming system* for X is a surjective partial mapping $\delta :\subseteq W \to X$. If $W = \Sigma^*$, a naming system is called a *notation*; if $W = \Sigma^\omega$, a naming system is called a *representation*. If X_1 and X_2 are sets with naming systems $\delta_1 :\subseteq W_1 \to X_1$, $\delta_2 \subseteq: W_2 \to X_2$, and f is a mapping $X_1 \to X_2$ (or a multi-valued mapping $X_1 \rightrightarrows X_2$), then a mapping $h :\subseteq W_1 \to W_2$ is called a (δ_1, δ_2)-*realization* for f, if for every $p \in \mathrm{dom}(\delta)$, one has $h(p) \in \mathrm{dom}(\delta')$ and $(\delta' \circ h)(p) = (f \circ \delta)(p)$ (or $(\delta' \circ h)(p) \in (f \circ \delta)(p)$, respectively, in the multi-valued case). f is called (δ_1, δ_2)-continuous (-computable), if there exists a continuous (computable) (δ_1, δ_2)-realization for f.

A naming system δ of some set X is said to be continuously (computably) *reducible* to another naming system δ' of X, if the identity on X is (δ, δ')-continuous (-computable); we write $\delta \leq_t \delta'$ ($\delta \leq \delta'$).

Below, we will frequently use the notations $\nu_\mathbb{N}$ and $\nu_\mathbb{Q}$, the representations ρ, ρ_C, $\rho_<$, $\rho_>$, and $\overline{\rho}_<$, and the wrapping function ι just as defined in [20]. We will also use standard devices to construct new naming systems from given ones; these are described in [20, Section 3.3]. We additionally use the convention: If X is a set with a naming system δ, then put

$$[\delta]^{<\omega} := [\delta]^0 \vee [\delta]^1 \vee [\delta]^2 \vee \cdots .$$

If X is a set with a representation δ, we shall write

$$W(\delta, w) := \delta(w\Sigma^\omega \cap \mathrm{dom}(\delta))$$

for every $w \in \Sigma^*$. We denote by $\sigma(\delta^{-1})$ the smallest σ-algebra on X which contains all sets $W(\delta, w)$, $w \in \Sigma^*$.

Suppose X_1, X_2 are topological and have naming systems δ_1, δ_2. An important question is concerned with the relation between (δ_1, δ_2)-continuity and classical continuity of a mapping $f : X_1 \to X_2$. A key result is the Kreitz-Weihrauch Representation Theorem [20, Theorem 3.2.11] which has later been generalized by

Schröder [13]: A representation of a topological space is called *admissible*, if it is continuous and every continuous representation of the same space is continuously reducible to it. If both δ_1 and δ_2 are admissible, then the (δ_1, δ_2)-continuous mappings are exactly the sequentially continuous mappings. Note that in most applications X_1 is countably based, and then sequential continuity is equivalent to topological continuity.

We finally note that any topological space that allows a continuous representation is hereditarily Lindelöf, i.e. every open covering of any subspace contains a countable covering.

2.2 Computable topological spaces

Below, we will frequently work with the notion of an *effective/computable topological space* and its *standard representation*. Effective topological spaces shall be defined as in [20, Definition 3.2.1]. For computable topological spaces, we use a sightly weaker definition than found there [5]:

Definition 2.1 An effective topological space (X, β, ϑ) is a computable topological space if $\mathrm{dom}(\vartheta)$ is r.e.

Definition 2.2 Let (X, β, ϑ) be an effective topological space. In a canonical way, one can define

- a notation $\vartheta^\cap$ of the set $\beta^\cap$ of all finite intersections of elements of β plus the empty set.
- a notation ϑ_{alg} of the algebra $\mathcal{A}(\beta)$ generated by β.

A representation $\vartheta_<$ of the hyperspace of open subsets $\mathcal{O}(X)$ of X shall then be defined by

$$\vartheta_<(p) = \bigcup_i U_i \;\; :\Longleftrightarrow \;\; [\vartheta^\cap]^\omega(p) = (U_i)_i.$$

Lemma 2.3 *Let (X, β, ϑ) be an effective topological space. Then the following mappings are computable w.r.t. the canonical representations given in Definition 2.2: Finite intersection on $\beta^\cap$; complementation, finite union and finite intersection on $\mathcal{A}(\beta)$; finite and countable union and finite intersection on $\mathcal{O}(X)$; the embeddings $\beta \hookrightarrow \beta^\cap$, $\beta^\cap \hookrightarrow \mathcal{A}(\beta)$, $\beta^\cap \hookrightarrow \mathcal{O}(X)$.* □

Lemma 2.4 *Let (X, β, ϑ) be a computable topological space with standard representation δ. Put $D := \{w \in \Sigma^* \; : \; \iota(v) \triangleleft w \Rightarrow v \in \mathrm{dom}(\vartheta)\}$. Then D is r.e., and for every $w \in D$ one has $W(\delta, w) = \bigcap_{\iota(v) \triangleleft w} \vartheta(v)$. The mapping $D \to \beta^\cap$, $w \mapsto W(\delta, w)$, is $(\mathrm{id}_{\Sigma^*}|^D, \vartheta^\cap)$-computable.* □

Computably regular T_0-spaces have been defined in [5]. We modify the definition to comprise exactly what we will need below:

[5] The advantage of this becomes clear when one compares Lemma 2.7(ii) to [20, Theorem 8.1.4.2].

Definition 2.5 An effective topological space (X, β, ϑ) is *computably regular* if the multi-valued mapping defined by the graph

$$\big\{(V,(V_n,U_n)_n) \in \beta \times (\beta^{\cap} \times \mathcal{O}(X))^{\omega} \;:\; V = \bigcup_n V_n \text{ and } (\forall\, n)\ [X \setminus V \subseteq U_n \text{ and } V_n \cap U_n = \emptyset]\big\}$$

is $(\vartheta, [\vartheta^{\cap}, \vartheta_<]^{\omega})$-computable.

Definition 2.6 Let (X, β, ϑ) be an effective topological space. Put

$$C := \{(U_n)_n \in (\beta^{\cap})^{\omega} \;:\; \bigcup_n U_n = X\}.$$

(X, β, ϑ) is *computably compact* if the multi-valued mapping defined by the graph

$$\{((U_n)_n, m) \in C \times \mathbb{N} \;:\; \bigcup_{n \leq m} U_n = X\}$$

is $([\vartheta^{\cap}]^{\omega}|^{C}, \nu_{\mathbb{N}})$-computable.

2.3 Computable metric spaces

Some motivation for the notion of an *effective/computable metric space* has already been given in the introduction. A formal definition is given in [20, Definition 8.1.2], where also the *Cauchy representation* is defined. (Notwithstanding [20], we write effective metric spaces as three-tuples (X, d, α), i.e. we omit the dense subset, which is understood to be range(α).)

For any metric space (X, d) define

$$(\forall\, x_0 \in X,\, \epsilon > 0)\ B(x_0, \epsilon) := \{x \in X \;:\; d(x_0, x) < \epsilon\}$$

and

$$(\forall\, x_0 \in X,\, \epsilon \geq 0)\ \overline{B}(x_0, \epsilon) := \{x \in X \;:\; d(x_0, x) \leq \epsilon\}$$

The following is well-known:

Lemma 2.7 *Let (X, d, α) be an effective metric space. An effective topological space (X, β, ϑ) can be defined by putting*

$$\beta := \{B(a, r) \;:\; a \in \text{range}(\alpha),\ r \in \mathbb{Q} \cap (0, \infty)\} \textit{ and}$$
$$\vartheta\langle u, v\rangle := B(\alpha(u), \nu_{\mathbb{Q}}(v)).$$

(i) If δ is the corresponding standard representation, then $\delta_X \equiv_t \delta$.

(ii) If (X, d, α) is computable, then (X, β, ϑ) is computable and computably regular. Furthermore $\delta_X \equiv \delta$.

□

If the codomain of a multi-valued mapping is a metric space, we will refer to the multi-valued mapping as an *operation*.

2.4 Computable measures

In this subsection, we assume that (X, β, ϑ) is an effective topological space.

Schröder [14] (generalizing a definition of Weihrauch [19]) defines a representation of the Borel probability measures on an admissibly represented topological space. His definition can be extended to non-probability measures in a straight-forward manner, but then is no longer a representation but merely a multi-representation.[6] On the other hand, we are only interested in measures on effective topological spaces. In this case, the mentioned multi-representation is easily seen to be equivalent to the following:

Definition 2.8 A multi-representation $\vartheta_{\mathcal{M}<}$ of the Borel measures on X is given by

$$\nu \in \vartheta_{\mathcal{M}<}(p) \;:\Longleftrightarrow\; [\vartheta_< \to \overline{\rho}_<](p) = \nu|_{\mathcal{O}(X)}.$$

Remark 2.9 One can deduce from Caratheodory's Extension Theorem (see [7, Theorem 2.5]) and the Lindelöf property of X that the restriction of $\vartheta_{\mathcal{M}<}$ to locally finite measures is single-valued. □

For finite measures, we consider the following two representations:

Definition 2.10 Representations of the finite Borel measures on X are given by

(i)

$$\vartheta^0_{\mathcal{M}<}\langle p, q\rangle = \nu \;:\Longleftrightarrow\; \vartheta_{\mathcal{M}<}(p) = \nu \text{ and } \rho_>(q) = \nu(X).$$

(ii)

$$\vartheta_{\mathcal{M}=}(p) = \nu \;:\Longleftrightarrow\; [\vartheta_{\mathrm{alg}} \to \rho](p) = \nu|_{\mathcal{A}(\beta)}.$$

It is easy to see that $\vartheta_{\mathcal{M}=} \le \vartheta^0_{\mathcal{M}<}$. Although there are $\vartheta^0_{\mathcal{M}<}$-computable measures that are not $\vartheta_{\mathcal{M}=}$-computable[7], one has the following useful result for metric spaces:

Theorem 2.11 *Let (X, d, α) be a computable metric space, and let (X, β, ϑ) be the computable topological space derived from it as in Lemma 2.7. Suppose that ν is a $\vartheta^0_{\mathcal{M}<}$-computable finite Borel measure on X. Then there is a computable topological space (X, β', ϑ') such that*

(i) ν is $\vartheta'_{\mathcal{M}=}$-computable.

(ii) $\delta \equiv \delta'$, where δ and δ' are the standard representations of (X, β, ϑ) and (X, β', ϑ'), respectively.

(iii) (X, β', ϑ') is computably regular.

(iv) (X, β', ϑ') is computably compact if (X, β, ϑ) is computably compact.

---

[6] This is because Schröder's representation only contains information on the values of the measure on open sets. Unbounded measures, however, are not necessarily defined uniquely by these values.

[7] For example: Let $(x_n)_n$ be a computable sequence of non-negative rationals such that $c := \sum_n x_n < 1$ is not computable from the right. Now consider the measure ν defined by $\nu(A) := (1-c)\chi_A(0) + \sum_n x_n \chi_A((n+1)^{-1})$.

Proof. Put

$$Q := \{(a, r, s) \in \mathrm{range}(\alpha) \times \mathbb{Q} \times \mathbb{Q} \ : \ 0 < r < s\}.$$

For any $(a, r, s) \in Q$ put $R(a, r, s) := \overline{B}(a, s) \setminus B(a, r)$. It is easy to verify that $(a, r, s) \mapsto X \setminus R(a, r, s)$ is $([\alpha, \nu_{\mathbb{Q}}, \nu_{\mathbb{Q}}]|^{Q}, \vartheta_{<})$-computable. From this fact, in combination with the $\rho_{>}$-computability of $\nu(X)$, it follows that

$$Q \to \mathbb{R}, \qquad (a, r, s) \mapsto \nu(R(a, r, s)) \ (\ = \nu(X) - \nu(X \setminus R(a, r, s)) \ ),$$

is $([\alpha, \nu_{\mathbb{Q}}, \nu_{\mathbb{Q}}]|^{Q}, \rho_{>})$-computable. It is clear that for all $(a, r, s) \in Q$, we have

$$\inf_{\substack{r', s' \in \mathbb{Q}, \\ r \leq r' < s' \leq s}} \nu(R(a, r', s')) = 0,$$

and hence there is an $([[\alpha, \nu_{\mathbb{Q}}, \nu_{\mathbb{Q}}]|^{Q}, \nu_{\mathbb{N}}], [\nu_{\mathbb{Q}}, \nu_{\mathbb{Q}}])$-computable mapping $h : Q \times \mathbb{N} \to \mathbb{Q} \times \mathbb{Q}$ such that

$$h((a, r, s), k) = (r', s') \Longrightarrow [r < r' < s' \leq s \text{ and } \nu(R(a, r', s')) \leq 2^{-k}].$$

Now let an input $(a, r, s) \in Q$ be given. By repeated use of h, we can compute a sequence $(r'_k, s'_k)_k$ in $\mathbb{Q} \times \mathbb{Q}$ such that $r \leq r'_1 < s'_1 \leq s$ and for all $k \in \mathbb{N}$

$$r_k \leq r'_{k+1} < s'_{k+1} \leq s_k,$$
$$s_k - r_k \leq 2^{-k},$$
$$\nu(R(a, r_k, s_k)) \leq 2^{-k}.$$

We can hence ρ-compute $g(a, r, s) := \lim_{k \to \infty} r_k = \lim_{k \to \infty} s_k$; one has $r \leq g(a, r, s) \leq s$ and

$$\nu(\overline{B}(a, g(a, r, s)) \setminus B(a, g(a, r, s))) = \nu\left(\bigcap_k R(a, r_k, s_k)\right) = 0.$$

Now choose

$$\beta' := \{B(a, g(a, r, s)) \ : \ (a, r, s) \in Q\},$$
$$\vartheta'(w) = B(a, g(a, r, s)) \ :\Longleftrightarrow \ [\alpha, \nu_{\mathbb{Q}}, \nu_{\mathbb{Q}}]|^{Q}(w) = (a, r, s).$$

It is clear that (X, β', ϑ') is a computable topological space. (ii) can be derived easily from the fact that the metric d is both (δ, δ, ρ)- and (δ', δ', ρ)-computable. The simple proof for the computable regularity of (X, β, ϑ) (as omitted from Lemma 2.7) goes through almost identically for (X, β', ϑ'), so we have (iii). (iv) follows easily from the fact that from every $[(\vartheta')^{\cap}]^{\omega}$-input $(U_n)_n$, one can $[[\vartheta^{\cap}]^{\omega}]^{\omega}$-compute a double sequence $(U_{n,m})_{n,m}$ such that $U_n = \bigcup_m U_{n,m}$.

It remains to show that $\nu|_{\mathcal{A}(\beta)}$ is $(\vartheta'_{\mathrm{alg}}, \rho)$-computable.

We consider the condition

$$U_1 \subseteq V, \ U_2 \subseteq X \setminus V, \ \nu(U_1) + \nu(U_2) = \nu(X) \tag{1}$$

for Borel sets $V, U_1, U_2 \subseteq X$. The multi-valued mapping $h' : \beta' \rightrightarrows (\mathcal{O}(X) \times \mathcal{O}(X))$ defined by the graph

$$G = \{(V, (U_1, U_2)) \ : \ V, U_1, U_2 \text{ fulfill } (1)\}$$

is $(\vartheta', [\vartheta_<, \vartheta_<])$-computable, because by construction one has

$$(\forall\, (a, r, s) \in Q)\ (B(a, g(a, r, s)), (B(a, g(a, r, s)), X \setminus B(a, g(a, r, s))) \in G.$$

(Note that $\vartheta_<$ is the representation of $\mathcal{O}(X)$ derived from the original computable topology). We inductively extend G to a graph $G' \subseteq \mathcal{A}(\beta') \times (\mathcal{O}(X) \times \mathcal{O}(X))$:

$$\begin{aligned}
(V, (U_1, U_2)) \in G &\Rightarrow (V, (U_1, U_2)) \in G', \\
(V, (U_1, U_2)) \in G' &\Rightarrow (X \setminus V, (U_2, U_1)) \in G', \\
(V, (U_1, U_2)),\ (V', (U_1', U_2')) \in G' &\Rightarrow (V \cup V', (U_1 \cup U_1', U_2 \cap U_2') \in G', \\
(V, (U_1, U_2)),\ (V', (U_1', U_2')) \in G' &\Rightarrow (V \cap V', (U_1 \cap U_1', U_2 \cup U_2') \in G'.
\end{aligned}$$

It is elementary to verify by induction that

$$(V, (U_1, U_2)) \in G' \implies V, U_1, U_2 \text{ fulfill } (1).$$

Lemma 2.3 yields that the mapping $h'' : \mathcal{A}(\beta') \rightrightarrows (\mathcal{O}(X) \times \mathcal{O}(X))$ defined by G' is $(\vartheta'_{\mathrm{alg}}, [\vartheta_<, \vartheta_<])$-computable.

Let an $\vartheta'_{\mathrm{alg}}$-input $V \in \mathcal{A}$ be given. Using h'', we can $\vartheta_<$-compute sets $U_1, U_2 \in \mathcal{O}(X)$ such that (1) holds. We can $\rho_<$-compute $\nu(U_1)$ and $\nu(U_2)$ by assumption. Because $\nu(U_1) = \nu(X) - \nu(U_2)$, we can also $\rho_>$-compute $\nu(U_1)$. It is clear that $\nu(V) = \nu(U_1)$. □

2.5 From measure theory

2.5.1 Completion of a measure space

Let $(X, \mathcal{S}, \nu)$ be a measure space. A set $N \subseteq X$ is called *ν-null* if there is a set $B \in \mathcal{S}$ with $\nu(B) = 0$ and $N \subseteq B$. A property $P \subseteq X$ is said to *hold ν-almost everywhere (ν-a.e.)* if $X \setminus P$ is ν-null. The σ-algebra $\mathcal{S}_\nu$ generated by $\mathcal{S}$ and all ν-null sets is called the *completion of $\mathcal{S}$ w.r.t. ν*. $\mathcal{S}_\nu$ contains exactly the sets of the form $A \cup N$ with $A \in \mathcal{S}$ and N ν-null. We call the elements of $\mathcal{S}_\nu$ the *ν-measurable* sets. The measure ν extends to a measure $\overline{\nu}$ on $\mathcal{S}_\nu$ by putting $\overline{\nu}(A \cup N) = \nu(A)$. A measure space that is identical to its completion is called *complete*.

Lemma 2.12 *Let $(X, \mathcal{S}, \nu)$ be a complete measure space and $(Y, \mathcal{S}')$ a measurable space. Let $f : X \to Y$ be a mapping such that $f|_{X \setminus N}$ is $(\mathcal{S} \cap (X \setminus N), \mathcal{S}')$-measurable for some ν-null set N. Then f is $(\mathcal{S}, \mathcal{S}')$-measurable.* □

2.5.2 Outer measures

An *outer measure* on a set X is a set function $\mu^* : 2^X \to [0,\infty]$ such that

$$\mu^*(\emptyset) = 0, \qquad A \subseteq B \Rightarrow \mu^*(A) \leq \mu^*(B), \qquad \mu^*(\bigcup_n A_n) \leq \sum_n \mu^*(A_n).$$

A set $A \subseteq X$ is called μ^*-*measurable* if

$$(\forall\, E \subseteq X)\ \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A).$$

The μ^*-measurable sets form a σ-algebra MEAS_{μ^*}. Restricting μ^* to MEAS_{μ^*} yields a complete measure space.

Let $(X, \mathcal{S}, \nu)$ be a measure space. The measure ν induces an outer measure ν^* via

$$\nu^*(A) := \inf\{\nu(B) \ :\ B \in \mathcal{S},\ A \subseteq B\}.$$

If ν is σ-finite, it turns out that $\mathrm{MEAS}_{\nu^*} = \mathcal{S}_\nu$, and that $\overline{\nu}$ and ν^* coincide on this σ-algebra. It is known that not every outer measure is induced by a measure.

The following two results are actually well-known but usually not stated for outer measures. We will use the second one in the proof of Proposition 3.8.

Lemma 2.13 (Cantelli-Theorem) *Let X be a set with an outer measure μ^*. Then for every sequence $(A_n)_{n\in\mathbb{N}}$ of subsets of X with $\sum_n \mu^*(A_n) < \infty$ we have*

$$\mu^*(\limsup_n A_n) = 0,$$

where $\limsup_n A_n := \bigcap_n \bigcup_{k\geq n} A_k$.

Proof. Follows from the fact that for every $m \in \mathbb{N}$ one has

$$\mu^*\left(\bigcap_n \bigcup_{k\geq n} A_k\right) \leq \mu^*\left(\bigcup_{k\geq m} A_k\right) \leq \sum_{k\geq m} \mu^*(A_k).$$

□

For a topological space Y, let $\mathcal{B}(Y)$ denote the *Borel σ-algebra* on Y, i.e. the σ-algebra generated by the topology.

Lemma 2.14 *Let X be a set with an outer measure μ^*. If $(f_n)_{n\in\mathbb{N}}$ is a sequence of $(\mathrm{MEAS}_{\mu^*}, \mathcal{B}(Y))$-measurable mappings from X into a metric space (Y,d), and $f : X \to Y$ is an arbitrary mapping with*

$$(\forall\, n \in \mathbb{N})\ \mu^*([d(f_n, f) > 2^{-n}]) \leq 2^{-n}, \tag{2}$$

then f is $(\mathrm{MEAS}_{\mu^*}, \mathcal{B}(Y))$-measurable.

Proof. Define $G := \{x \in X \ :\ f_n(x) \to f(x)\}$. By Cantelli's Theorem and (2), we have

$$\mu^*(X \setminus G) \leq \mu^*([(\forall\, n)(\exists k \geq n)\ d(f_k, f) > 2^{-k}]) = 0.$$

From the completeness of $(X, \mathrm{MEAS}_{\mu^*}, \mu^*)$, we especially have that G is μ^*-measurable. It follows from [7, Lemma 1.10(ii)] that $f|_G = \lim f_n|_G$ is $(\mathrm{MEAS}_{\mu^*} \cap G, \mathcal{B}(Y))$-measurable. The claim now follows from Lemma 2.12. □

2.5.3 Outer regularity

Let X be a topological space and let $\mathcal{S}$ be a σ-algebra on X that includes $\mathcal{B}(X)$. A measure μ on $\mathcal{S}$ is called *outer-regular* if[8]

$$(\forall\, A \in \mathcal{S}) \;\; \inf\{\mu(G \setminus A) \;:\; G \supseteq A,\, G \text{ open}\} = 0.$$

An outer measure μ^* on 2^X is called *outer-regular* if $\mathcal{B}(X) \subseteq \mathrm{MEAS}_{\mu^*}$ and the restriction of μ^* to MEAS_{μ^*} is an outer-regular measure.

On metric spaces, all finite Borel measures are outer-regular (see [7, Lemma 1.34]).

The following lemma will be needed in the proof of Theorem 3.19 below:

Lemma 2.15 *Let X be a topological space, and let $\mathcal{S}$ be a σ-algebra on X that includes $\mathcal{B}(X)$. Let μ be an outer-regular measure on $\mathcal{S}$ and let $f : X \to [0, \infty]$ be a μ-integrable function. Then the measure ν on $\mathcal{S}$ defined by*

$$\nu(A) := \int_A f \, d\mu.$$

is outer-regular.

Proof. Let $A \in \mathcal{S}$ be arbitrary and consider a descending sequence $(G_n)_{n \in \mathbb{N}}$ of open sets such that $G_n \supseteq A$ and $\mu(G_n \setminus A) \to 0$. The set $C := \bigcap_n G_n \setminus A$ has measure 0 and so $\int_C f \, d\mu = 0$. Dominated Convergence now yields $\int_{G_n \setminus A} f \, d\mu \to 0$. □

3 Three probabilistically weakened concepts of computability

Assumption 3.1 *Throughout the remaining of this article, we denote by*

- *X a nonempty set,*
- *δ a naming system of X,*
- *(Y, d, α) an effective metric space with Cauchy representation δ_Y,*
- *μ^* an outer measure on 2^X,*
- *$\mathcal{S}$ a σ-algebra on X,*
- *ν a measure on $(X, \mathcal{S})$,*
- *ν^* the outer measure induced by ν.*

[8] In many textbooks, a measure μ is called outer-regular if it fulfills the weaker condition that $\mu(A) = \inf\{\mu(G) \;:\; G \supseteq A,\, G \text{ open}\}$ for all $A \in \mathcal{S}$. It will be crucial for some of the results below that regularity is understood in the strong sense!

3.1 The local error

Definition 3.2 For any total operation $f : X \rightrightarrows Y$ and any $\phi : \mathrm{dom}(\delta) \to \mathrm{dom}(\alpha)$ define

$$e(f, \delta, \phi, \cdot) : X \to [0, \infty],$$
$$e(f, \delta, \phi, x) := \sup_{p \in \delta^{-1}\{x\}} d((\alpha \circ \phi)(p), f(x)).$$

Note that here the second argument of d is a set; the usual convention

$$(\forall\, y \in Y)(\forall\, A \in 2^Y \setminus \{\emptyset\})\ d(y, A) := \inf\{d(y, a)\ :\ a \in A\}$$

applies.

For any mapping $\phi :\subseteq \mathbb{N} \times A \to B$ (for sets A, B) and any $n \in \mathbb{N}$, we shall denote by $\phi_n \subseteq A \to B$ the mapping defined by $\mathrm{dom}(\phi_n) = \{a \in A\ :\ (n, a) \in \mathrm{dom}(\phi)\}$ and $\phi_n(a) = \phi(n, a)$.

The following observation will be useful below:

Lemma 3.3 *Consider the assumptions of Definition 3.2, and additionally, let $g :\subseteq W \to \mathrm{dom}(\delta)$ ($W \in \{\Sigma^*, \Sigma^\omega\}$) be a mapping such that $\delta \circ g$ is a naming system of X. Then*

$$(\forall\, x \in X)\ e(f, \delta \circ g, \phi \circ g, x) \le e(f, \delta, \phi, x)$$

□

3.2 Concept (I): Computability almost everywhere

The concept to be defined now is a rather straight-forward generalization of "decidability up to measure zero" as defined by Parker [11,12].

Definition 3.4 A total operation $f : X \rightrightarrows Y$ is $(\delta, \delta_Y)^\nu_{\mathrm{AE}}$*-continuous (-computable)* if there is ν-null set $N \subseteq X$ such that $f|_{X\setminus N}$ is $(\delta|^{X\setminus N}, \delta_Y)$-continuous (-computable).

Proposition 3.5 *Assume that X is endowed with a topology w.r.t. which δ is admissible. Then a mapping $f : X \to Y$ is $(\delta, \delta_Y)^\nu_{\mathrm{AE}}$-continuous iff there is a ν-null set N such that $f|_{X\setminus N}$ is sequentially continuous.*

Proof. Let N be an arbitrary subset of X. By [13, Subsection 4.1], $\delta|^{X\setminus N}$ is an admissible representation of $X\setminus N$. By [20, Theorem 8.1.4], δ_Y admissibly represents Y. The claim hence follows from the Representation Theorem. □

3.3 Concept (II): Computable approximation

The definitions in this subsection generalize a definition of Ko (cf. [9, Definition 5.10]).

Definition 3.6 Let $f : X \rightrightarrows Y$ be a total operation.

(i) f is $(\delta,\alpha)^{\mu^*}_{\mathrm{APP}}$-*continuous (-computable)* if there is a continuous (computable) mapping

$$\phi : \mathbb{N} \times \mathrm{dom}(\delta) \to \mathrm{dom}(\alpha)$$

such [9] that

$$(\forall\, n \in \mathbb{N})\ \mu^*([e(f,\delta,\phi_n,\cdot) > 2^{-n}]) \le 2^{-n}.$$

(ii) f is $(\delta,\alpha)^{\nu}_{\mathrm{APP}}$-*continuous (-computable)* if f is $(\delta,\alpha)^{\nu^*}_{\mathrm{APP}}$-continuous (-computable).

Concerning this definition, we assent to the following statement of Parker [11, p. 8]:

> Why require a machine that always halts? Assuming we have a machine that sometimes gives incorrect output, the epistemological situation would seem no worse if in principle that machine could also fail to halt, but with probability zero.

This leads to a combination of concepts (I) and (II):

Definition 3.7 Let $f : X \rightrightarrows Y$ be a total operation.

(i) f is $(\delta,\alpha)^{\mu^*}_{\mathrm{APP/AE}}$-*continuous (-computable)* if there is a μ^*-null set $N \subseteq X$ such that $f|_{X\setminus N}$ is $(\delta|^{X\setminus N},\alpha)^{\mu^*}_{\mathrm{APP}}$-continuous (-computable).

(ii) f is $(\delta,\alpha)^{\nu}_{\mathrm{APP/AE}}$-*continuous (-computable)* if f is $(\delta,\alpha)^{\nu^*}_{\mathrm{APP/AE}}$-continuous (-computable).

In the spirit of the Representation Theorem, we now seek for connections between classical properties of mappings and their APP-continuity.

Proposition 3.8 *Assume that $\sigma(\delta^{-1}) \subseteq \mathrm{MEAS}_{\mu^*}$. If a single-valued $f : X \to Y$ is $(\delta,\alpha)^{\mu^*}_{\mathrm{APP/AE}}$-continuous, then f is $(\mathrm{MEAS}_{\mu^*},\mathcal{B}(Y))$-measurable.*

Proof. It follows from Lemma 2.12 that it is sufficient to prove the claim for $(\delta,\alpha)^{\mu^*}_{\mathrm{APP}}$-continuous f. Let ϕ be a corresponding realizer as in Definition 3.6(i). Let $(a_m)_{m\in\mathbb{N}}$ be an enumeration of $\mathrm{dom}(\alpha)$. For every $n,m \in \mathbb{N}$ put $A_{n,m} := \phi_n^{-1}\{a_m\}$. Clearly, every $A_{n,m}$ is open in $\mathrm{dom}(\delta)$, and $\mathrm{dom}(\delta) \subseteq \bigcup_m A_{n,m}$. The assumption $\sigma(\delta^{-1}) \subseteq \mathrm{MEAS}_{\mu^*}$ implies that all sets $D_{n,m} := \delta(A_{n,m})$ are μ^*-measurable. Define

$$c(n,x) := \min\{m \in \mathbb{N}\ :\ x \in D_{n,m}\},$$
$$f_n(x) := \alpha(a_{c(n,x)}).$$

The f_n are clearly $(\mathrm{MEAS}_{\mu^*},\mathcal{B}(Y))$-measurable. It follows from the definition of the local error that $d(f_n(x),f(x)) \le e(f,\delta,\phi_n,x)$ for all $x \in X$, so $\mu^*([d(f,f_n) > 2^{-n}]) \le 2^{-n}$ for every $n \in \mathbb{N}$. The claim now follows with Lemma 2.14. □

[9] Of course, input from $\mathbb{N}$ also has to be encoded and tupled with the other input, so the domain of ϕ is actually $\mathrm{dom}([\nu_{\mathbb{N}},\delta])$. For convenience, we will ignore this formal difference here and below.

Proposition 3.9 *Suppose X is endowed with a topology w.r.t. which δ is continuous and μ^* is outer-regular. Then every $(\mathrm{MEAS}_{\mu^*}, \mathcal{B}(Y))$-measurable operation $f : X \rightrightarrows Y$ is $(\delta, \alpha)^{\mu^*}_{\mathrm{APP}}$-continuous.*

Proof. Let $(a_m)_{m\in\mathbb{N}}$ be an enumeration of $\mathrm{dom}(\alpha)$. For all $m, n \in \mathbb{N}$, put

$$A_{m,n} := f^{-1}(B(\alpha(a_m), 2^{-n})).$$

Note that $X = \bigcup_m A_{m,n}$. By the outer regularity of μ^*, there are open sets $G_{m,n}$, $m, n \in \mathbb{N}$, with $A_{m,n} \subseteq G_{m,n}$ and $\mu^*(G_{m,n} \setminus A_{m,n}) \leq 2^{-(n+m+1)}$. Now for every $n \in \mathbb{N}$, there is a continuous "selector" $c_n : \mathrm{dom}(\delta) \to \mathbb{N}$ such that $\delta(p) \in G_{c_n(p),n}$ for every $p \in \mathrm{dom}(\delta)$. Put $\phi(n,p) := a_{c_n(p)}$. It is easy to see that

$$[e(f, \delta, \phi_n, \cdot) > 2^{-n}] \subseteq \bigcup_{m\in\mathbb{N}} (G_{m,n} \setminus A_{m,n})$$

and that the set on the right hand side has μ^*-content at most 2^{-n}. □

Combining the last two propositions yields:

Corollary 3.10 *Suppose that X is topological, δ is continuous, μ^* is outer-regular, and $\sigma(\delta^{-1}) \subseteq \mathrm{MEAS}_{\mu^*}$. Then for every mapping $f : X \to Y$, the following statements are equivalent:*

(i) f is $(\delta, \alpha)^{\mu^*}_{\mathrm{APP}}$-continuous.

(ii) f is $(\delta, \alpha)^{\mu^*}_{\mathrm{APP/AE}}$-continuous.

(ii) f is $(\mathrm{MEAS}_{\mu^*}, \mathcal{B}(Y))$-measurable.

□

It is a natural question, under what conditions APP-computability is preserved by composition. We are able to give at least a partial answer to this question (for further remarks see Subsection 3.5):

Theorem 3.11 *Suppose that (Z, d', α') is a computable metric space, $f : X \to Y$ is a mapping, and $g : Y \rightrightarrows Z$ is an operation. If f is $(\delta, \alpha)^{\mu^*}_{\mathrm{APP}}$-computable and g is $(\delta_Y, \alpha')^{\mu^*\circ f^{-1}}_{\mathrm{APP}}$-computable, then $g \circ f$ is $(\delta, \alpha')^{\mu^*}_{\mathrm{APP/AE}}$-computable.*

Proof. Let ϕ be a $(\delta, \alpha)^{\mu^*}_{\mathrm{APP}}$-realizer for f. First, we demonstrate how to compute a mapping $a : \mathbb{N} \times \mathrm{dom}(\delta) \to \Sigma^\omega$ such that $\mu^*(R_n) \leq 2^{-n}$ for all $n \in \mathbb{N}$, where

$$R_n := \{x \in X \ : \ (\exists\, p \in f^{-1}\{x\})\ a(n,p) \notin \delta_Y^{-1}\{f(x)\}\}).$$

Put $a(n,p) := \iota(\phi(n+1,p))\iota(\phi(n+2,p))\iota(\phi(n+3,p))\cdots$. By the definition of the Cauchy representation δ_Y:

$$R_n \subseteq \bigcup_k [e(f, \delta, \phi_{n+k+1}, \cdot) > 2^{-(k+1)}] \subseteq \bigcup_k [e(f, \delta, \phi_{n+k+1}, \cdot) > 2^{-(n+k+1)}].$$

(Here the single-valuedness of f goes in.) By assumption, the set on the right hand side has μ^*-content at most $\sum_k 2^{-(n+k+1)} = 2^{-n}$.

Now let ϕ' be a $(\delta_Y, \alpha')^{\mu^* \circ f^{-1}}_{\text{APP}}$-realizer of g. Consider the following procedure: "On input $(n,p) \in \mathbb{N} \times \text{dom}(\delta)$, run a dovetailed process that simulates the computation of a machine for ϕ' on all inputs $(n+1, a(n+m+2,p))$, $m \geq 0$. Each time one of these threads of simulation halts, try to verify that its output is in the domain of α', and once this succeeds, halt and put it out." Put $N := \bigcup_n \bigcap_m R_{n+m+2}$ and note that $\mu^*(N) = 0$. For given (n,p), the procedure just described will surely halt, if $a(n+m+2,p) \in \text{dom}(\delta_Y)$ for at least one m. Hence, if the procedure does not halt, then $\delta(p) \in \bigcap_m R_{n+m+2}$. So the procedure defines a computable mapping $\widetilde{\phi} : \mathbb{N} \times \text{dom}(\delta|^{X\setminus N}) \to \text{dom}(\alpha')$. It is sufficient to show that $\widetilde{\phi}$ is a $(\delta|^{X\setminus N}, \alpha')^{\mu^* \circ f^{-1}}_{\text{APP}}$-realizer of $g \circ f|_{X\setminus N}$.

If for some $n \in \mathbb{N}$, $x \in X$, we have that both the conditions

$$(\forall\, p \in \delta^{-1}\{x\})(\forall\, m \in \mathbb{N})\ a(n+m+2,p) \in \delta_Y^{-1}(f(x))$$

and

$$(\forall\, q \in \delta_Y^{-1}\{f(x)\})\ d((\alpha' \circ \phi')(n+1,q), (g \circ f)(x)) \leq 2^{-(n+1)}$$

are fufilled, then it follows from the construction of our procedure for $\widetilde{\phi}$ that

$$(\forall\, p \in \delta^{-1}\{x\})\ d((\alpha' \circ \widetilde{\phi})(n,p), (g \circ f)(x)) \leq 2^{-(n+1)} \leq 2^{-n}.$$

This implies

$$[e(g \circ f|_{X\setminus N}, \delta|^{X\setminus N}, \widetilde{\phi}_n, \cdot) > 2^{-n}] \subseteq \bigcup_m R_{n+m+2} \cup f^{-1}[e(g, \delta_Y, \phi'_{n+1}, \cdot) > 2^{-(n+1)}].$$

Finally note that $\mu^*\left(\bigcup_m R_{n+m+2}\right) \leq 2^{-(n+1)}$ by construction, and $(\mu^* \circ f^{-1})[e(g, \delta_Y, \phi'_{n+1}, \cdot) > 2^{-(n+1)}] \leq 2^{-(n+1)}$ by assumption. □

3.4 Concept (III): Computability in the mean

We now come to a notion that has been proposed in a talk by Hertling [6], but has apparently not been treated in the literature so far.

Definition 3.12 Let $h :\subseteq X \to [0,\infty]$ be an arbitrary function that is defined ν-almost everywhere. We define the *outer integral* of h as

$$\int^* h\, d\nu := \inf\left\{ \int g\, d\nu \ :\ g \text{ is } (\mathcal{S}, \mathcal{B}(\mathbb{R}))\text{-measurable, } h \leq g|_{\text{dom}(h)} \right\}.$$

One easily verifies:

Lemma 3.13 *(i) The outer integral is monotone, i.e.* $h_1 \leq h_2 \Rightarrow \int^* h_1\, d\nu \leq \int^* h_2\, d\nu$.

(ii) The outer integral is sublinear, i.e. $\int^* (h_1 + h_2)\, d\nu \leq \int^* h_1\, d\nu + \int^* h_2\, d\nu$ *and* $\int^* th\, d\nu = t \int^* h\, d\nu$ *for all* $t \in [0,\infty)$.

(iii) For every $A \subseteq X$, one has $\nu^*(A) = \int^* \chi_A \, d\nu$.

□

Definition 3.14 A total operation $f : X \rightrightarrows Y$ is $(\delta, \alpha)^\nu_{\text{MEAN}}$-continuous (-computable) if there is a continuous (computable) mapping

$$\Phi : \mathbb{N} \times \operatorname{dom}(\delta) \to \operatorname{dom}(\alpha)$$

such that

$$(\forall\, n \in \mathbb{N}) \ \int^* e(f, \delta, \Phi_n, x)\, \nu(dx) \leq 2^{-n}.$$

We will also consider the mixed setting here:

Definition 3.15 A total operation $f : X \rightrightarrows Y$ is $(\delta, \alpha)^\nu_{\text{MEAN/AE}}$-continuous (-computable) if there is a ν-null set N such that $f|_{X \setminus N}$ is $(\delta|^{X \setminus N}, \alpha)^\nu_{\text{MEAN}}$-continuous (-computable).

Recall the setting described in the introduction and suppose now that our agent is supplied with a sequence of independent identically distributed measurements of the physical magnitude and has the task to compute an approximation to f on each of them. Using a MEAN-algorithm ensures a good average error on the long run: [10]

Proposition 3.16 *Suppose that ν is a probability measure. Let $f : X \to Y$ be a $(\delta, \alpha)^\nu_{\text{MEAN}}$-continuous mapping, and let Φ be a corresponding realizer as in Definition 3.14. Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $(w_i)_i$ be a sequence of mappings $w_i : \Omega \to \operatorname{dom}(\delta)$ such that the mappings $\delta \circ w_i$ are independent ν-distributed random variables. Then for every $n \in \mathbb{N}$ one has*

$$\limsup_{m \to \infty} \frac{1}{m} \sum_{i \leq m} e_i \leq 2^{-n} \qquad P\text{-almost surely}$$

where $e_i := d((\alpha \circ \Phi_n)(w_i), (f \circ \delta)(w_i))$.

Proof. For all i, we have $e_i \leq e(f, \delta, \Phi_n, \delta(w_i))$. It follows from Definitions 3.12 and 3.14 that there is a sequence $(g_k)_k$ of measurable functions $g_k : X \to [0, \infty]$ such that $e(f, \delta, \Phi_n, \cdot) \leq g_k$ and $\int g_k \, d\nu \leq 2^{-n} + 2^{-k}$ for every k. The Strong Law of Large Numbers (see [7, Theorem 4.23]) now yields that for every k

$$\limsup_{m \to \infty} \frac{1}{m} \sum_{i \leq m} e_i \leq 2^{-n} + 2^{-k} \qquad P\text{-almost surely.}$$

Intersecting over k yields the claim. □

See Corollary 3.29 below for a result on the measurability of MEAN/AE-continuous mappings.

[10] If only an APP-algorithm is used, the average error may tend to infinity.

We now look for an analogon of Proposition 3.9, i.e. for a natural condition on f to assure its MEAN(/AE)-continuity. We first ask, whether conditions such as those of Proposition 3.9 (δ continuous, ν outer-regular, f measurable) are actually already sufficient. We will see below (Proposition 3.32(ii)) that this is not the case. The next natural step is to consider integrability. This makes sense only if Y is a normed space.

Assumption 3.17 *Throughout the remaining of this subsection, we additionally assume that*

- *Y is a normed space, and d is the metric induced by the norm.*
- *$0 \in \mathrm{range}(\alpha)$.*
- *X is endowed with a topology.*

Proposition 3.18 *Suppose that δ is open and ν^* is locally finite. If a mapping $f : X \to Y$ is $(\delta, \alpha)^\nu_{MEAN}$-continuous, then $\|f\|$ is locally outer-ν-integrable, i.e. for every $x \in X$ there is an open neighbourhood $G \subseteq X$ of x such that $\int^*_G \|f\|\, d\nu < \infty$.*

Proof. Let Φ be a continuous realiser as in Definition 3.14. Let $x_0 \in X$ be arbitrary, and let p be an arbitrary δ-name of x_0. Φ_n is constantly equal to $\Phi_n(p)$ on an open (in $\mathrm{dom}(\delta)$) neighbourhood $U \subseteq \mathrm{dom}(\delta)$ of p. Put $a := (\alpha \circ \Phi_0)(p)$. By the definition of the local error, we have

$$(\forall\, x \in \delta(U))\;\; e(f, \delta, \Phi_0, x) \geq \|a - f(x)\|.$$

$\delta(U)$ is open, and by the local finiteness of ν^*, we can find an open neighbourhood $G \subseteq \delta(U)$ of x_0 such that $\nu^*(G) < \infty$. We finally have

$$\begin{aligned} 1 &\geq \int^* e(f, \delta, \Phi_0, x)\, \nu(dx) \geq \int^*_G e(f, \delta, \Phi_0, x)\, \nu(dx) \geq \int^*_G \|a - f(x)\|\, \nu(dx) \\ &\geq \int^*_G \|f\|\, d\nu - \int^*_G \|a\|\, d\nu \geq \int^*_G \|f\|\, d\nu - \nu^*(G)\|a\|. \end{aligned}$$

□

Theorem 3.19 *Suppose that δ is continuous, $\mathcal{B}(X) \subseteq \mathcal{S}$, ν is outer-regular, f is $(\mathcal{S}, \mathcal{B}(Y))$-measurable, and $\|f\|$ is locally ν-integrable. Then f is $(\delta, \alpha)^\nu_{\mathrm{MEAN}}$-continuous.*

Proof. We first assume that $\|f\|$ is integrable over the whole space. Let $(a_m)_{m \in \mathbb{N}}$ be an enumeration of $\mathrm{dom}(\alpha)$. For all $m, n \in \mathbb{N}$ put

$$A_{m,n} := \begin{cases} f^{-1}(\; B(\alpha(a_m), \min\{2^{-n}, \|\alpha(a_m)\|/2\})\;) & \text{if } \alpha(a_m) \neq 0 \\ f^{-1}\{0\} & \text{else.} \end{cases}$$

Note that $X = \bigcup_m A_{m,n}$. Put $C_{m,n} := A_{m,n} \setminus \bigcup_{k<m} A_{k,n}$ and

$$g_n := \sum_m \alpha(a_m) \chi_{C_{m,n}},$$

and note that (g_n) converges to f pointwise and that $\|g_n\| \leq 2\|f\|$; so (g_n) converges to f in $L^1(\nu)$ by Dominated Convergence. By transition to a subsequence, we can assume that $\int \|f - g_n\| \, d\nu < 2^{-(n+1)}$ for all $n \in \mathbb{N}$. The measures ν_n on $\mathcal{S}$ defined by

$$\nu_n(A) := \int_A \|g_n\| \, d\nu$$

are outer-regular by Lemma 2.15. So there are open sets $G_{m,n}$ with $G_{m,n} \supseteq C_{m,n}$ and $\nu(G_{m,n} \setminus C_{m,n}) \leq (2^{n+m+3} \cdot \max\{1, \|\alpha(a_m)\|\})^{-1}$ and $\nu_n(G_{m,n} \setminus C_{m,n}) \leq 2^{-(n+m+3)}$. Now for every $n \in \mathbb{N}$ there is a continuous $m_n : \mathrm{dom}(\delta) \to \mathbb{N}$ such that $\delta(p) \in C_{m_n(p),n}$ for every $p \in \mathrm{dom}(\delta)$. Put $\Phi_n(p) = a_{m_n(p)}$. We have

$$\begin{aligned}
&\int^* e(f, \delta, \Phi_n, x) \, \nu(dx) \\
&\leq \int \|f - g_n\| \, d\nu + \int^* e(g_n, \delta, \Phi_n, x) \, \nu(dx) \\
&\leq 2^{-(n+1)} + \sum_m \int_{G_{m,n} \setminus C_{m,n}} \|\alpha(a_m) - g_n(x)\| \nu(dx) \\
&\leq 2^{-(n+1)} + \sum_m \nu(G_{m,n} \setminus C_{m,n}) \|\alpha(a_m)\| + \sum_m \nu_n(G_{m,n} \setminus C_{m,n}) \\
&\leq 2^{-n}.
\end{aligned}$$

We have hence shown that f is $(\delta, \alpha)^{\nu}_{\mathrm{MEAN}}$-continuous.

Now assume that $\|f\|$ is only locally integrable. Remember that X is Lindelöf (because it allows a continuous representation). There hence is a countable open cover $(G_\ell)_\ell$ of X, such that $\|f\|$ is integrable on each G_ℓ. By the first part of the proof, each mapping $f|_{G_\ell}$ is $(\delta|^{G_\ell}, \alpha)^{\nu}_{\mathrm{MEAN}}$-continuous; let $\Phi^{(\ell)}$ be the corresponding realizer. On input (n, p), continuously choose an ℓ such that $\delta(p) \in G_\ell$. Then put out $\Phi^{(\ell)}(n + \ell + 1, p)$. One immediately estimates that the average error of this algorithm is bounded by 2^{-n}. $\square$

It would be desirable to investigate the question under what conditions MEAN-computability is preserved by composition (cp. Theorem 3.11). We postpone this question to the future.

3.5 Alternative definitions

We are going to give a brief (and unfortunately incomplete) discussion of a possible modification to the definitions from Subsection 3.3.[11] We start by asking the reader to recall what it means for a multi-valued mapping $f : X \rightrightarrows Y$ to be (δ, δ_Y)-computable: there is a Turing machine that runs infinitely and transduces every δ-name p into a sequence of rapidly converging range(α)-approximations to some $y \in f(\delta(p))$. This is clearly equivalent to having a machine that takes as input a δ-name p and a precision parameter $k \in \mathbb{N}$, halts in finite time, and puts out

[11] Parts of this subsection have been added in answer to a remark from an anonymous referee.

a range(α)-approximation of quality 2^{-k} to some $y \in f(\delta(p))$, where y *must be independ of* k. Relaxing this last requirement leads to:

Definition 3.20 A total operation $f : X \rightrightarrows Y$ is $(\delta, \alpha)_\sim$*-continuous (-computable)* if there is a continuous (computable) mapping

$$\phi : \mathbb{N} \times \mathrm{dom}(\delta) \to \mathrm{dom}(\alpha)$$

such that

$$(\forall\, n \in \mathbb{N})(\forall p \in \mathrm{dom}(\delta))\ d((\alpha \circ \phi)(n, p), (f \circ \delta)(p)) \leq 2^{-n}.$$

We observe the following:

Lemma 3.21 *Suppose X is a set with naming system δ, and (Y, d, α) is an effective metric space with Cauchy representation δ_Y.*

(i) If $f : X \rightrightarrows Y$ is (δ, δ_Y)-continuous (-computable) then f is $(\delta, \alpha)_\sim$-continuous (-computable).

(ii) For single-valued f, the converse of (i) also holds true.

□

We shall demonstrate that the converse of Lemma 3.21(i) is not true in general:

Proposition 3.22 *There is an operation $f : [0, 1] \rightrightarrows \mathbb{R}$ which is $(\rho|^{[0,1]}, \nu_\mathbb{Q})_\sim$-computable but not $(\rho|^{[0,1]}, \rho)$-continuous. f can be chosen such that* range(f) *is compact and the cardinality of $f(x)$ is at most two for each $x \in [0, 1]$.*

Proof. For $k = 0, 1, 2, \ldots$ put $A_k := (2^{-(k+1)}, 2^{-k}]$. Consider the functions f_1, f_{-1} : $[0, 1] \to \mathbb{R}$ given by

$$f_1(x) = \begin{cases} 1 + 2^{-k} & \text{for } x \in A_k, \\ 1 & \text{for } x = 0, \end{cases} \qquad f_{-1}(x) = \begin{cases} -1 & \text{for } x > 0, \\ 1 & \text{for } x = 0. \end{cases}$$

Now define f by putting $f(x) := \{f_1(x), f_{-1}(x)\}$ for all $x \in [0, 1]$.

We first show that f is $(\rho|^{[0,1]}, \nu_\mathbb{Q})_\sim$-computable, i.e. we give an algorithm for a mapping ϕ as in Definition 3.20. Let the input $(n, p) \in \mathbb{N} \times \mathrm{dom}(\rho|^{[0,1]})$ be given. At least one of the sets $[0, 2^{-(n+2)}), (2^{-(n+3)}, 1]$ contains $\rho(p)$, and one can effectively pick one such set. If the chosen set is $[0, 2^{-(n+2)})$, put out 1; this choice is correct, because $\rho(p)$ is either 0 (and $f(0) = \{1\}$) or in A_k for some $k \geq n$ (and then $1 + 2^k \in f(\rho(p))$). If the chosen set is $(2^{-(n+3)}, 1]$, put out -1 (which is exactly $f_{-1}(\rho(p))$).

Now we show that f is not $(\rho|^{[0,1]}, \rho)$-continuous. Assume the contrary and let ϕ be a corresponding continuous realizer. Let p be a ρ-name of 0. It follows from $f(0) = \{1\}$ that there is a prefix w of p such that $\rho(\phi(wp')) > 0$ for all $wp' \in \mathrm{dom}(\rho|^{[0,1]})$. Then necessarily $\rho(\phi(wp')) = f_1(\rho(wp'))$ for all $wp' \in \mathrm{dom}(\rho|^{[0,1]})$. From this it follows that f_1 must be continuous on $W(\rho|^{[0,1]}, w)$. But $W(\rho|^{[0,1]}, w)$

contains an open neighbourhood of 0, and f_1 is not continuous on any open neighbourhood of 0. □

In a certain sense, our definitions of APP- and MEAN-continuity (-computability) can be derived in a natural way as weakenings of $(\delta, \alpha)_\sim$-continuity (-computability):

Lemma 3.23 *A total operation $f : X \rightrightarrows Y$ is $(\delta, \alpha)_\sim$-continuous (-computable) iff there is a continuous (computable) mapping $\phi : \mathbb{N} \times \mathrm{dom}(\delta) \to \mathrm{dom}(\alpha)$ such that*

$$(\forall\, n \in \mathbb{N})(\forall\, x \in X)\ e(f, \delta, \phi_n, x) \leq 2^{-n}. \tag{3}$$

□

Now the definitions of APP-continuity (-computability) and MEAN-continuity (-computability), respectively, are obtained by replacing the ∀-quantification over x by a probabilistic criterion.

In general, $(\delta, \alpha)_\sim$-computability is not an adequate alternative to (δ, δ_Y)-computability, because (in contrast to the latter) $(\delta, \alpha)_\sim$-computability is not preserved under composition of multi-valued mappings. It seems like APP-computability inherits this flaw, as we were only able to prove Theorem 3.11 for the inner mapping being single-valued. Can we "repair" the definition of APP-computability in this respect? A definition that achieves this would have to judge the quality of an approximation $\phi(n, p)$ not by its distance to the set $f(\delta(p))$, but by its distance to a fixed element $\zeta(p) \in f(\delta(p))$ that does not depend on the precision level n:

Definition 3.24 Let $f : X \rightrightarrows Y$ be a total operation. f is $(\delta, \alpha)^{\mu^*}_{\mathrm{APP}+}$*-continuous (-computable)* if there is a mapping $\zeta : \mathrm{dom}(\delta) \to Y$ with

$$(\forall\, p \in \mathrm{dom}(\delta))\ \zeta(p) \in (f \circ \delta)(p)$$

and a continuous (computable) mapping $\phi : \mathbb{N} \times \mathrm{dom}(\delta) \to \mathrm{dom}(\alpha)$ such that

$$(\forall\, n \in \mathbb{N})\ \mu^*\big(\big[\sup_{p \in \delta^{-1}\{x\}} d((\alpha \circ \phi)(p), \zeta(p)) > 2^{-n}\big]\big) \leq 2^{-n}.$$

If one requires the inner mapping f in Theorem 3.11 to be $(\delta, \alpha)^{\mu^*}_{\mathrm{APP}+}$-computable, then one can in return allow it to be multivalued (the proof goes through nearly unchanged). But does Proposition 3.9 hold analogously for $(\delta, \alpha)^{\mu^*}_{\mathrm{APP}+}$-continuity? Unfortunately, we do not yet have an answer to this question.

It remains open at this time, whether Definition 3.24 should be favoured over Definition 3.6. In the present article, we will restrict ourselves to working with the latter.

In the future, an analogue of Definition 3.24 for MEAN-continuity (-computability) will probably be useful when the composition of MEAN-computable mappings is investigated.

3.6 (Multi-)Representations corresponding to the probabilistic continuity notions

The definitions from the preceding subsections suggest definitions of multi-representations. We will not work out all of them here, but only give one example.

Definition 3.25 Suppose that δ is a representation of X.

(i) The multi-representation $(\delta, \alpha)^{\nu}_{\mathrm{MEAN}}$ of the set of all $(\delta, \alpha)^{\nu}_{\mathrm{MEAN}}$-continuous total operations from X into Y is defined by

$$f \in (\delta, \alpha)^{\nu}_{\mathrm{MEAN}}(p) \;:\Longleftrightarrow\; \eta(p)|_{\mathbb{N}\times\mathrm{dom}(\delta)} = \Phi \text{ for some } \Phi \text{ as in Definition 3.14},$$

where

$$\eta := \begin{cases} \eta^{**} & \text{if } \delta \text{ is a notation,} \\ \eta^{\omega *} & \text{if } \delta \text{ is a representation,} \end{cases}$$

and $\eta^{**}, \eta^{\omega *}$ are defined in [20, Definition 2.3.10].

(ii) The multi-representation $(\delta, \alpha)^{\nu}_{\mathrm{MEAN/AE}}$ of the set of all $(\delta, \alpha)^{\nu}_{\mathrm{MEAN/AE}}$-continuous total operations from X into Y is defined by

$$\begin{aligned} & f \in (\delta, \alpha)^{\nu}_{\mathrm{MEAN/AE}}(p) \\ :\Longleftrightarrow\; & \text{there is a } \nu\text{-null set } N \text{ and a } h \in (\delta|^{X\setminus N}, \alpha)^{\nu}_{\mathrm{MEAN}}(p) \text{ such that } f|_{X\setminus N} = h. \end{aligned}$$

Following the pattern of Definition 3.25, one can also turn Definitions 3.20, 3.4, 3.6, 3.7 into definitions of multi-representations.

Remark 3.26 Under the assumptions of Definition 3.25, an operation is $(\delta, \alpha)^{\nu}_{\mathrm{MEAN}}$-computable iff it has a computable $(\delta, \alpha)^{\nu}_{\mathrm{MEAN}}$-name. The same holds analogously for the other multi-representations formed by the pattern of Definition 3.25. So our terminology is consistent with [20, Definition 3.1.3.1]. □

We have the following immediate consequence of Lemma 3.3:

Lemma 3.27 *Suppose that δ' is another naming system of X.*

(i) If $\delta' \leq_t \delta$, then every $(\delta, \alpha)^{\nu}_{\mathrm{MEAN/AE}}$-continuous operation is $(\delta', \alpha)^{\nu}_{\mathrm{MEAN/AE}}$-continuous and $(\delta, \alpha)^{\nu}_{\mathrm{MEAN/AE}} \leq_t (\delta', \alpha)^{\nu}_{\mathrm{MEAN/AE}}$.

(ii) If $\delta' \leq \delta$, then $(\delta, \alpha)^{\nu}_{\mathrm{MEAN/AE}} \leq (\delta', \alpha)^{\nu}_{\mathrm{MEAN/AE}}$.

Items (i) and (ii) hold analogously for other representations formed by the pattern of Definition 3.25. □

3.7 General relations between (I), (II) and (III)

We will now clarify the mutual relations between the concepts defined above. The first proposition sums up the cases where there is a computable reduction of one multi-representation to the other. Then we give some strong counter-examples – i.e. examples envolving functions from $[0, 1]$ to $\mathbb{R}$ and the Lebesgue measure – for other cases. The remaining cases are treated in the next subsection.

Proposition 3.28 *(i)* $(\delta, \delta_Y) \leq (\delta, \delta_Y)^\nu_{\mathrm{AE}}$.

(ii) $(\delta, \delta_Y)^\nu_{\mathrm{AE}} \leq (\delta, \alpha)^\nu_{\mathrm{APP/AE}}$.

(iii) $(\delta, \alpha)^\nu_{\mathrm{MEAN}} \leq (\delta, \alpha)^\nu_{\mathrm{APP}}$.

(iv) $(\delta, \alpha)^\nu_{\mathrm{MEAN/AE}} \leq (\delta, \alpha)^\nu_{\mathrm{APP/AE}}$.

Proof. (i) and (ii) are obvious. (iv) is a corollary of (iii). We prove (iii): Lemma 3.13 yields that for every $h : X \to [0, \infty]$ and every $\epsilon > 0$, one has

$$\int^* h\, d\nu \geq \int^* \epsilon \cdot \chi_{[h>\epsilon]} d\nu \geq \epsilon \cdot \nu^*([h > \epsilon]).$$

From this it easily follows: If Φ is a mapping as in Definition 3.14, then a suitable mapping ϕ as in Definition 3.6 is given by $\phi(n, p) := \Phi(2n, p)$; this yields a computable reduction. □

Combining Propositions 3.28 and 3.8 yields:

Corollary 3.29 *Assume that $\sigma(\delta^{-1}) \subseteq \mathrm{MEAS}_{\nu^*}$. If a single-valued $f : X \to Y$ is $(\delta, \alpha)^\nu_{\mathrm{MEAN/AE}}$-continuous, then f is $(\mathrm{MEAS}_{\nu^*}, \mathcal{B}(Y))$-measurable.* □

Lemma 3.30 *Suppose that Y is a normed space, the mapping $a \mapsto \|a\|$ is (α, ρ)-computable, and ν is finite. For an arbitrary constant $N \in \mathbb{N}$, let B be the set of all operations $f : X \rightrightarrows Y$ with*

$$\sup\{\|y\| \ : \ y \in \mathrm{range}(f)\} \leq N.$$

Then $(\delta, \alpha)^\nu_{\mathrm{APP}}|^B \leq (\delta, \alpha)^\nu_{\mathrm{MEAN}}$.

Proof. We can assume $N > 0$. There is an $a_0 \in \mathrm{dom}(\alpha)$ such that $\|\alpha(a_0)\| < N$. Given a mapping ϕ as in Definition 3.6, we can compute a $\phi' : \mathbb{N} \times \mathrm{dom}(\delta) \to \mathrm{dom}(\alpha)$ such that for all $(n, p) \in \mathrm{dom}(\phi)$, one has

$$\begin{gathered}\phi'(n, p) \in \{\phi(n, p), a_0\},\\ \|(\alpha \circ \phi)(n, p)\| \geq 4N \implies \phi'(n, p) = a_0,\\ \|(\alpha \circ \phi)(n, p)\| \leq 3N \implies \phi'(n, p) = \phi(n, p).\end{gathered}$$

By distinguishing the cases

(i) $\|(\alpha \circ \phi)(n, p)\| \leq 3N$, (ii) $3N < \|(\alpha \circ \phi)(n, p)\| < 4N$, (iii) $4N \leq \|(\alpha \circ \phi)(n, p)\|$,

one finds that

$$(\forall\, a \in B(0, N))\ \|(\alpha \circ \phi')(n, p) - a\| \leq \min\{\|(\alpha \circ \phi)(n, p) - a\|, 5N\};$$

hence one has $e(f, \delta, \phi'_n, \cdot) \leq \min\{e(f, \delta, \phi_n, \cdot), 5N\}$ for every $f \in B$. A standard estimate yields that a suitable mapping Φ as in Definition 3.14 is given by $\Phi(n, p) := \phi'(n + \lceil \log(5N + \nu(X)) \rceil, p)$. This yields a computable reduction. □

Proposition 3.31 *There is a set $S \subseteq [0,1]$ such that χ_S is $(\rho, \nu_{\mathbb{Q}})^{\lambda}_{\mathrm{MEAN}}$-computable but not $(\rho, \rho_C)^{\lambda}_{\mathrm{AE}}$-continuous.*

Proof. Parker [11, Theorem IV] defines a set $S \subseteq [0,1]$ and proves that χ_S is $(\rho, \nu_{\mathbb{Q}})^{\lambda}_{\mathrm{APP}}$-computable but not $(\rho, \rho_C)^{\lambda}_{\mathrm{AE}}$-continuous (although he does not use these terms). By the previous lemma, χ_S is even $(\rho, \nu_{\mathbb{Q}})^{\lambda}_{\mathrm{MEAN}}$-computable. □

Proposition 3.32*(i) There is a function $f : [0,1] \to \mathbb{R}$ which is $(\rho|^{[0,1]}, \rho_C)^{\lambda}_{\mathrm{AE}}$- and $(\rho|^{[0,1]}, \nu_{\mathbb{Q}})^{\lambda}_{\mathrm{MEAN/AE}}$-computable but not $(\rho|^{[0,1]}, \nu_{\mathbb{Q}})^{\lambda}_{\mathrm{MEAN}}$-continuous.*

(ii) There is a function $f : [0,1] \to \mathbb{R}$ which is $(\rho|^{[0,1]}, \nu_{\mathbb{Q}})^{\lambda}_{\mathrm{APP}}$-computable but not $(\rho|^{[0,1]}, \nu_{\mathbb{Q}})^{\lambda}_{\mathrm{MEAN/AE}}$-continuous.

Proof. Recall that $\rho|^{[0,1]}$ is an open representation of $[0,1]$. We can hence apply Proposition 3.18.

For (i), simply consider $f(x) := x^{-1} \cdot \chi_{(0,1]}$, which clearly is computable and MEAN-computable on $(0,1]$, but not locally integrable in 0.

For (ii), we need a more elaborate example: For every $a \in [0,1], n \in \mathbb{N}$, define

$$f_{a,n}(x) := (x-a)^{-1}\chi_{(a,a+2^{-n}]\cap[0,1]}(x).$$

Let $(a_n)_{n\in\mathbb{N}}$ be a computable dense sequence of rationals in $[0,1]$. Choose $\widetilde{f} := \sup_{n\in\mathbb{N}} f_{a_n,n}$. Clearly, $\widetilde{f}$ is a measurable function into $\overline{\mathbb{R}}$, that is not integrable on any open subset of $[0,1]$. Obviously, $\widetilde{f}(x) = \infty$ implies that x is contained in infinitely many of the $(a, a+2^{-n}]$, and hence Cantelli's Theorem yields $\lambda([\widetilde{f} = \infty]) = 0$. So, the function $f := \widetilde{f} \cdot \chi_{[\widetilde{f}\neq\infty]}$ is into $\mathbb{R}$ and is still measurable and nowhere integrable. Clearly, $f|_{X\setminus N}$ is still nowhere integrable for any ν-null set N. So f is not $(\rho|^{[0,1]}, \nu_{\mathbb{Q}})^{\lambda}_{\mathrm{MEAN/AE}}$-continuous. On the other hand, it is not hard to see that f is $(\rho|^{[0,1]}, \nu_{\mathbb{Q}})^{\lambda}_{\mathrm{APP}}$-computable. □

3.8 Reductions that require certain effectivity assumptions

Only the following relations have not been covered yet: AE $\rightsquigarrow$ MEAN/AE, AE $\rightsquigarrow$ APP, APP/AE $\rightsquigarrow$ APP, MEAN/AE $\rightsquigarrow$ APP. For them, computable reductions do not exist in general, but under some additional assumptions, which are in particular fulfilled for mappings defined on $\mathbb{R}^n$ with its standard representation and Lebesgue measure.

We first turn to AE $\rightsquigarrow$ MEAN/AE:

Definition 3.33 Suppose that X is topological, δ is continuous, and θ is a representation of the hyperspace $\mathcal{O}(X)$ of open subsets of X. δ and θ are said to be *compatible*, if the relation $\{(x,U) \in X \times \mathcal{O}(X) \;:\; x \in U\}$ is (δ,θ)-r.e.

Proposition 3.34 *Suppose that X is topological, δ is continuous, and θ is a compatible representation of $\mathcal{O}(X)$. Further suppose that there is a $[\theta, \nu_{\mathbb{N}}]^{\omega}$-computable sequence $(U_r, M_r)_r$ in $\mathcal{O}(X) \times \mathbb{N}$ such that $X = \bigcup_r U_r$ and $\nu^*(U_r) \leq M_r$ for all $r \in \mathbb{N}$. Then*

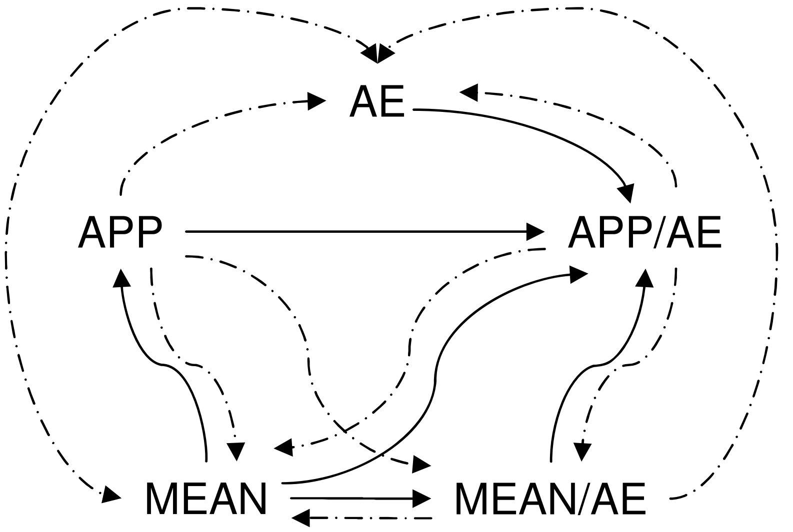

Fig. 1. The graphic subsumes the results of this subsection. A solid arrow indicates computable reduction, a dashed arrow indicates a strong counter-example.

(i) $(\delta, \alpha)_\sim \leq (\delta, \alpha)^\nu_{\mathrm{MEAN}}$.

(ii) $(\delta, \delta_Y)^\nu_{\mathrm{AE}} \leq (\delta, \alpha)^\nu_{\mathrm{MEAN/AE}}$.

Proof. It is sufficient to show (i): From the input $(\delta, \alpha)_\sim$-name we can compute a ϕ as in Lemma 3.23. By type conversion, it is sufficient to demonstrate how to compute a mapping Φ as in Definition 3.14, so suppose we are additionally given as input an $n \in \mathbb{N}$ and a $p \in \mathrm{dom}(\delta)$. Effectively determine an r such that $\delta(p) \in U_r$. Then put out $\phi(n + \lceil \log M_r \rceil + 1, p)$.

We have to verify that the Φ computed by this algorithm is correct. Let $f : X \rightrightarrows Y$ be an operation corresponding to the input $(\delta, \alpha)_\sim$-name. We then have

$$\begin{aligned}\int^* e(f, \delta, \Phi_n, x)\,\nu(dx) &\leq \int^* \sup_r \chi_{U_r}(x) e(f, \delta, \phi_{n+\lceil \log M_r \rceil + 1}, x)\,\nu(dx)\\ &\leq \sum_r M_r 2^{-(n+\lceil \log M_r \rceil + 1)}\\ &\leq 2^{-n}.\end{aligned}$$

□

We now look for assumptions that imply computable reducibility from APP/AE to APP (and hence from AE to APP and from MEAN/AE to APP). The next lemma is intended as preparation for the proof of Theorem 3.36. [12]

Lemma 3.35 *Suppose the following:*

(i) X is topological, δ is continuous, and θ is a compatible representation of $\mathcal{O}(X)$.

[12] But Lemma 3.35 might also be interesting in its own right, because the assumptions it makes are somewhat weaker than needed for the proof of Theorem 3.36.

(ii) There is a $[\theta]^\omega$-computable sequence (U_r) such that $X = \bigcup_r U_r$, and μ^* is finite on the U_r.

(iii) From any prefix-free sequence $(w_\ell)_\ell$ in Σ^* with

$$\mu^*\left(X \setminus \bigcup_\ell W(\delta, w_\ell)\right) = 0$$

and any $r, k \in \mathbb{N}$, one can $[\theta]^\omega$-compute a sequence $(V_\ell)_\ell$ and θ-compute a set $\widetilde{V}$, such that

$$U_r \subseteq \bigcup_\ell V_\ell \cup \widetilde{V}$$

and $\mu^*(L) \leq 2^{-k}$ where

$$L := U_r \cap \left(\widetilde{V} \cup \bigcup_\ell (V_\ell \setminus W(\delta, w_\ell))\right).$$

Then $(\delta, \alpha)^{\mu^*}_{\mathrm{APP/AE}} \leq (\delta, \alpha)^{\mu^*}_{\mathrm{APP}}$.

Proof. Let ϕ' be the mapping encoded in the input $(\delta, \alpha)^{\mu^*}_{\mathrm{APP/AE}}$-name (see Definitions 3.6, 3.7 and 3.25), and let $f : X \rightrightarrows Y$ be an arbitrary operation described by that name. By type conversion, it is sufficient to demonstrate how to compute a mapping ϕ as in Definition 3.6, so suppose we are additionally given as input an $n \in \mathbb{N}$ and a $p \in \mathrm{dom}(\delta)$. The $(\delta, \alpha)^{\mu^*}_{\mathrm{APP/AE}}$-name of ϕ' encodes a double-sequence $(w_{m,\ell}, a_{m,\ell})_{m,\ell}$ in $\Sigma^* \times \mathrm{dom}(\alpha)$ such that $\bigcup_\ell w_{m,\ell}\Sigma^\omega \supseteq \delta^{-1}(X \setminus N)$ (where N is as in Definition 3.7) and $\phi'(m, p) = a_{m,\ell}$ whenever $w_{m,\ell} \sqsubseteq p$. Each sequence $(w_{m,\ell})_\ell$ can assumed to be prefix-free. By definition, $\mu^*(H_m) \leq 2^{-m}$ for every m where

$$H_m := \bigcup_\ell ([d(f, \alpha(a_{m,\ell})) > 2^{-m}] \cap W(\delta, w_{m,\ell})).$$

We can now apply assumption (iii) to each sequence $(w_{m,\ell})_\ell$, and compute sequences $(V_{m,\ell,r,k})_{m,\ell,r,k}$ und $(\widetilde{V}_{m,r,k})_{m,r,k}$ such that

$$U_r \subseteq \bigcup_\ell V_{m,\ell,r,k} \cup \widetilde{V}_{m,r,k}$$

and $\mu^*(L_{m,r,k}) \leq 2^{-k}$, where

$$L_{m,r,k} := U_r \cap \left(\widetilde{V}_{m,r,k} \cup \bigcup_\ell (V_{m,\ell,r,k} \setminus W(\delta, w_{m,\ell}))\right).$$

Now first find an r_0 such that $\delta(p) \in U_{r_0}$, then put $m_0 := n + 1$, $k_0 := n + r_0 + 2$ and effectively determine a set

$$A \in \{V_{m_0,\ell,r_0,k_0}\}_\ell \cup \{\widetilde{V}_{m_0,r_0,k_0}\}$$

with $\delta(p) \in A$. In case that A is $\widetilde{V}_{m_0,r_0,k_0}$, put out an arbitrary $a \in \mathrm{dom}(\alpha)$; in case that A is V_{m_0,ℓ,r_0,k_0} for some ℓ, put out $a_{m_0,\ell}$.

We have to verify that the ϕ computed by this algorithm is correct. From the construction it follows that if $d((\alpha \circ \phi)(p,n),(f \circ \delta)(p)) > 2^{-n}$ for some $p \in \mathrm{dom}(\delta), n \in \mathbb{N}$, then this must be because of one of the following:

- $\delta(p) \in N$,
- $\delta(p) \in H_{n+1} \setminus N$ (i.e. $\delta(p)$ is in the set where ϕ'_{n+1} does not work well),
- there is an $r \in \mathbb{N}$ such that $\delta(p) \in L_{n+1,r,n+r+2} \setminus (N \cup H_{n+1})$ (i.e. ϕ'_{n+1} would work well on p, but ϕ_{n+1} possibly differs from it here).

We can hence estimate:

$$\begin{aligned}\mu^*([e(f,\delta,\phi_n,\cdot)] > 2^{-n}) &\leq \mu^*(H_{n+1}) + \sum_r \mu^*(L_{n+1,r,n+r+2}) \\ &\leq 2^{-(n+1)} + \sum_r 2^{-(n+r+2)} = 2^{-n}.\end{aligned}$$

□

Theorem 3.36 *Let (X,β,ϑ) be a computably regular computable topological space, and let δ be its standard representation. Let ν be a $\vartheta_{\mathcal{M}<}$-computable Borel measure on X with the additional property:*

$$\nu|_\beta \textit{ takes only finite values and is } (\vartheta,\rho_>)\textit{-computable.} \tag{4}$$

Then $(\delta,\alpha)^\nu_{\mathrm{APP/AE}} \leq (\delta,\alpha)^\nu_{\mathrm{APP}}$.

We start with an auxiliary lemma:

Lemma 3.37 *Under the assumptions of Theorem 3.36, the multi-valued mapping defined by the graph*

$$\{((V,k),U) \in (\beta \times \mathbb{N}) \times \mathcal{O}(X) \ : \ X \setminus V \subseteq U,\, \nu(V \cap U) \leq 2^{-k}\}$$

is $(\vartheta,\nu_\mathbb{N},\vartheta_<)$-computable.

Proof. From the ϑ-input V one can $[\vartheta^\cap,\vartheta_<]^\omega$-compute a sequence $(V_n,U_n)_n$ as in Definition 2.5. As the number $\nu(V)$ can be $\rho_>$-computed and the numbers $\nu(V_0 \cup \ldots \cup V_m)$ can be $\rho_<$-computed for every m, we can effectively find some m such that

$$\nu(V \cap U_0 \cap \cdots \cap U_m) \leq \nu(V \setminus (V_0 \cup \ldots \cup V_m)) = \nu(V) - \nu(V_0 \cup \ldots \cup V_m) \leq 2^{-k}.$$

So put out $U_0 \cap \cdots \cap U_m$. □

Proof. [Proof of Theorem 3.36] It is sufficient to check that assumptions (i) to (iii) from Lemma 3.35 are fulfilled for $\mu^* = \nu^*$ and $\theta = \vartheta_<$. It is easy to check that (i) is fulfilled. For assumption (ii), let $(u_r)_r$ be a computable enumeration of $\mathrm{dom}(\vartheta)$ and choose $U_r := \vartheta(u_r)$; then the assumption is fulfilled by (4). Let us turn to (iii): Suppose we are given $(w_\ell)_\ell$ and r, k. If D is defined as in Lemma 2.4, we can compute a sequence $(w'_\ell)_\ell$ such that $\{w'_\ell\}_\ell = \{w_\ell\}_\ell \cap D$. For $w \notin D$, one has $W(\delta, w) = \emptyset$, so $(w'_\ell)_\ell$ still has the property $\nu\left(X \setminus \bigcup_\ell W(\delta, w'_\ell)\right) = 0$. Let us w.l.o.g. assume that $\{w_\ell\}_\ell \subseteq D$. By the second assertion of Lemma 2.4, we can $[\vartheta^\cap]^\omega$-compute the sequence $(W(\delta, w_\ell))_\ell$. We have $\nu\left(U_r \setminus \bigcup_\ell W(\delta, w'_\ell)\right) = 0$, and hence, in view of the computability of ν and (4), we can effectively find a number $s \in \mathbb{N}$ such that

$$\nu\left(U_r \setminus \bigcup_{\ell \leq s} W(\delta, w_\ell)\right) \leq 2^{-(k+1)}.$$

Choose

$$V_\ell := \begin{cases} W(\delta, w_\ell) & \text{for } \ell \leq s \\ \emptyset & \text{for } \ell > s. \end{cases}$$

Resolving the definition of $\vartheta^\cap$, we have a $[[\vartheta]^{<\omega}]^s$-computable tuple

$$((V_{1,1}, \ldots, V_{1,t(1)}), \ldots, (V_{s,1}, \ldots, V_{s,t(s)}))$$

such that $W(\delta, w_\ell) = V_{\ell,1} \cap \cdots \cap V_{\ell,t(\ell)}$ for all $\ell \leq s$. For all $\ell \leq s$ and $i \leq t(\ell)$, apply the auxiliary lemma to the pair $(V_{\ell,i}, \lceil \log s + \log t(\ell) \rceil + k + 1)$ and let

$$((\widetilde{V}_{1,1}, \ldots, \widetilde{V}_{1,t(1)}), \ldots, (\widetilde{V}_{s,1}, \ldots, \widetilde{V}_{s,t(s)}))$$

be the tuple $[[\vartheta_<]^{<\omega}]^s$-computed that way. Choose

$$\widetilde{V} := \bigcap_{\ell \leq s} \bigcup_{i \leq t(\ell)} \widetilde{V}_{\ell,i}$$

and note that we can $\vartheta_<$-compute $\widetilde{V}$. One easily verifies that $X \subseteq \bigcup_\ell V_\ell \cup \widetilde{V}$, and hence the first part of assumption (iii) is fulfilled. The second part is fulfilled because

$$U_r \cap \big(\widetilde{V} \cup \bigcup_\ell (V_\ell \setminus W(\delta, w_\ell))\big) = U_r \cap \widetilde{V}$$

and

$$\begin{aligned}
\nu(U_r \cap \widetilde{V}) &\leq \nu\Big(\big(\bigcup_{\ell \leq s} W(\delta, w_\ell)\big) \cap \widetilde{V} \Big) + 2^{-k+1} \\
&= \nu\Big(\big(\bigcup_{\ell \leq s} \bigcap_{i \leq t(\ell)} V_{\ell,i}\big) \cap \big(\bigcap_{\ell \leq s} \bigcup_{i \leq t(\ell)} \widetilde{V}_{\ell,i}\big) \Big) + 2^{-k+1} \\
&\leq \sum_{\ell \leq s} \nu\Big(\big(\bigcap_{i \leq t(\ell)} V_{\ell,i}\big) \cap \big(\bigcup_{i \leq t(\ell)} \widetilde{V}_{\ell,i}\big) \Big) + 2^{-k+1} \\
&\leq \sum_{\ell \leq s} \sum_{i \leq t(\ell)} \nu(V_{\ell,i} \cap \widetilde{V}_{\ell,i}) + 2^{-k+1} \\
&\leq 2^{-k}.
\end{aligned}$$

□

Theorem 3.36 is in fact a generalization of [11, Theorem II], where it was proved that the characteristic function of a subset of Euclidean space is APP-computable if it is AE-computable with respect to Lebesgue measure.

We have the following corollary for finite measures on metric spaces:

Theorem 3.38 *Let (X, d', α') be a computable metric space. Let (X, β, ϑ) be the computable topological space derived from it as in Lemma 2.7, and let δ be its standard representation. Suppose that ν is a $\vartheta^0_{\mathcal{M}<}$-computable measure. Then $(\delta, \alpha)^\nu_{\mathrm{APP/AE}} \leq (\delta, \alpha)^\nu_{\mathrm{APP}}$.*

Proof. Apply Theorem 2.11 to (X, β, ϑ), and note that the resulting computable topological space (X, β', ϑ') (with standard representation δ') fulfills the assumptions of Theorem 3.36, hence $(\delta', \alpha)^\nu_{\mathrm{APP/AE}} \leq (\delta', \alpha)^\nu_{\mathrm{APP}}$. δ and δ' are equivalent, hence Lemma 3.27 yields the claim. □

4 Computability of integration

Assumption 4.1 *Throughout this section we assume that*

- *ν is finite,*
- *Y is a normed space over $\mathbb{R}$, δ_Y is its Cauchy representation,*
- *the norm on Y is (δ_Y, ρ)-computable,*
- *vector addition is $(\delta_Y, \delta_Y, \delta_Y)$-computable,*
- *scalar multiplication is $(\rho, \delta_Y, \delta_Y)$-computable.*

The following definitions and basic results are taken from [17, Section II.3.1]: Let Y^* denote the topological dual of Y, and let $\mathcal{C}(Y)$ be the *cylindrical σ-algebra* on Y, i.e. the coarsest σ-algebra w.r.t. which all elements of Y^* are measurable. Suppose that $f : X \to Y$ is an $(\mathcal{S}, \mathcal{C}(Y))$-measurable mapping. We say that f is of *weak order p* (for $0 < p < \infty$) if $\int |g \circ f|^p \, d\mu < \infty$ for every $g \in Y^*$. If f is of weak order one, then we call an element y_f of Y *(Pettis) integral of f w.r.t. ν* if

$$(\forall\, g \in Y^*) \int g \circ f \, d\nu = g(y_f).$$

If there is an integral of f, then it is unique and we denote it by $\mathbb{E}(f; \nu)$. The mappings for which the integral exists form a vector space on which $\mathbb{E}(\cdot; \nu)$ is linear. For real-valued mappings, the Pettis integral is equal to the usual integral. Now suppose that $f : X \to Y$ is $(\mathcal{S}, \mathcal{B}(Y))$-measurable. We say that f is of *strong order p* (for $0 < p < \infty$) if $\int \|f\|^p \, d\nu < \infty$. Every mapping of strong order p is of weak order p. If f is of strong order one and $\mathbb{E}(f; \nu)$ exists, then $\|\mathbb{E}(f; \nu)\| \leq \mathbb{E}(\|f\|; \nu)$. For the existence of $\mathbb{E}(f; \nu)$, it is sufficient that f is of strong order one and Y is complete.

Under what circumstances and from what input is $\mathbb{E}(f; \nu)$ δ_Y-computable? Consider for example $\nu = \gamma$ with γ being the standard Gaussian distribution on $\mathbb{R}$. It is an easy exercise to make up a γ-integrable $(\rho, \nu_{\mathbb{Q}})^\gamma_{\mathrm{MEAN}}$-computable function

$f : \mathbb{R} \to \mathbb{R}$ such that $\mathbb{E}(f;\gamma)$ is not $\rho_>$-computable and hence no computable real.

This example makes clear that integrals cannot be computed from MEAN-names in general, not even for computable probability measures on the real line. The next theorem, however, shows that integration becomes computable under the additional assumption of the computable compactness of X, or if certain stronger information on the input is provided.

Theorem 4.2 *Let (X, β, ϑ) be a computable topological space , and let δ be its standard representation. Suppose that ν is a $\vartheta_{\mathcal{M}=}$-computable finite Borel measure on X. Put*

$$L := \{f : X \to Y \ : \ f \text{ is } (\mathcal{S}, \mathcal{B}(Y))\text{-measurable}, \quad (\delta,\alpha)^{\nu}_{\mathrm{MEAN}}\text{-continuous, and } \mathbb{E}(f;\nu) \text{ exists}\}.$$

(i) If X is computably compact, then $f \mapsto \mathbb{E}(f;\nu)$ is $((\delta,\alpha)^{\nu}_{\mathrm{MEAN}}|^{L}, \delta_Y)$-computable.

(ii) Define the set

$$B := \{(f, b) \in L \times \mathbb{N} \ : \ \|f\| \leq b\}.$$

Then $(f, b) \mapsto \mathbb{E}(f;\nu)$ is $([(\delta,\alpha)^{\nu}_{\mathrm{MEAN}}, \nu_{\mathbb{N}}]|^{B}, \delta_Y)$-computable.

(iii) Define the set

$$I := \{(f, c) \in L \times \mathbb{R} \ : \ \mathbb{E}(\|f\|;\nu) = c\}.$$

Then $(f, c) \mapsto \mathbb{E}(f;\nu)$ is $([(\delta,\alpha)^{\nu}_{\mathrm{MEAN}}, \rho_>]|^{I}, \delta_Y)$-computable.

Proof. The proofs for (i), (ii) and (iii) start the same: Let p be the $(\delta,\alpha)^{\nu}_{\mathrm{MEAN}}$-name given as input and let $f : X \to Y$ be an arbitrary mapping described by it. It is sufficient to demonstrate how to δ_Y-compute a 2^{-k}-approximation to $\mathbb{E}(f;\nu)$ for $k = 0, 1, 2, \ldots$. So fix an arbitrary k. From p, we can draw a sequence $(w_\ell, a_\ell)_\ell$ in $\Sigma^* \times \mathrm{dom}(\alpha)$ such that $\bigcup_\ell w_\ell \Sigma^\omega \supseteq \mathrm{dom}(\delta)$ and

$$\int \sup_\ell \chi_{W(\delta,w_\ell)}(x) \cdot \|f(x) - \alpha(a_\ell)\| \, \nu(dx) \leq 2^{-(k+2)}.$$

By Lemma 2.4, we can assume[13] that we can $[\vartheta^{\cap}]^{\omega}$-compute the sequence $(W(\delta, w_\ell))_\ell$. Put $A_0 = W(\delta, w_0)$ and $A_\ell = W(\delta, w_\ell) \setminus \bigcup_{j<\ell} A_j$ for $\ell \geq 1$. Note that $(A_\ell)_\ell$ can be $[\vartheta_{\mathrm{alg}}]^{\omega}$-computed. Put

$$s(x) := \sum_\ell \chi_{A_\ell}(x) \cdot \alpha(a_\ell).$$

[13] This argument is carried out in more detail in the proof of Theorem 3.36.

One has

$$\begin{aligned}\mathbb{E}(\|f-s\|;\nu) &= \int \left\| f(x) - \sum_{\ell} \chi_{A_\ell}(x)\cdot\alpha(a_\ell)\right\| \nu(dx) \\ &= \int \sup_{\ell} \chi_{A_\ell}(x)\|f(x)-\alpha(a_\ell)\|\,\nu(dx) \\ &\le \int \sup_{\ell} \chi_{W(\delta,w_\ell)}(x)\cdot\|f(x)-\alpha(a_\ell)\|\,\nu(dx) \\ &\le 2^{-(k+2)}.\end{aligned} \tag{5}$$

For every $m \in \mathbb{N}$ put $B_m := \bigcup_{\ell\le m} A_\ell = \bigcup_{\ell\le m} W(\delta, w_\ell)$ and

$$y_m := \sum_{\ell\le m} \nu(A_\ell)\alpha(a_\ell), \qquad s_m(x) := \chi_{B_m}(x)\cdot s(x) = \sum_{\ell\le m}\chi_{A_\ell}(x)\cdot\alpha(a_\ell).$$

One immediately verifys that $\mathbb{E}(s_m;\nu) = y_m$, and that the sequence $(y_m)_m$ is $[\delta_Y]^\omega$-computable. Combining this with (5) yields:

$$\begin{aligned}\|\mathbb{E}(f;\nu)-y_m\| &\le \mathbb{E}(\|f-s_m\|;\nu) = \mathbb{E}(\chi_{X\setminus B_m}\cdot\|f\|;\nu) + \mathbb{E}(\chi_{B_m}\cdot\|f-s\|;\nu) \\ &\le \mathbb{E}(\chi_{X\setminus B_m}\cdot\|f\|;\nu) + 2^{-(k+2)}.\end{aligned}$$

So it is sufficient to compute an m such that $\mathbb{E}(\chi_{X\setminus B_m}\cdot\|f\|;\nu) \le 2^{-(k+1)} + 2^{-(k+2)}$.

For (i): By the computable compactness of X, we can compute an m such that $B_m = X$.

For (ii): We can effectively find an m such that $\nu(X\setminus B_m) \le b^{-1}(2^{-(k+1)} + 2^{-(k+2)})$.

For (iii): From (5), it follows that

$$2^{-(k+2)} \ge \mathbb{E}(\|f\|;\nu) - \mathbb{E}(\|s\|;\nu) = \mathbb{E}(\|f\|;\nu) - \lim_{m\to\infty}\mathbb{E}(\chi_{B_m}\cdot\|s\|;\nu).$$

The sequence $(\mathbb{E}(\chi_{B_m}\cdot\|s\|;\nu))_m$ can be $[\rho]^\omega$-computed, because

$$\mathbb{E}(\chi_{B_m}\cdot\|s\|;\nu) = \mathbb{E}(\|s_m\|;\nu) = \sum_{\ell\le m}\nu(A_\ell)\|\alpha(a_\ell)\|.$$

As we are given a $\rho_>$-name of $\mathbb{E}(\|f\|;\nu)$, we can effectively find an m such that $2^{-(k+1)} \ge \mathbb{E}(\|f\|;\nu) - \mathbb{E}(\|s_m\|;\nu)$. We then have

$$\begin{aligned}&\mathbb{E}(\chi_{X\setminus B_m}\cdot\|f\|;\nu) \\ &\le (\mathbb{E}(\|f\|;\nu) - \mathbb{E}(\chi_{B_m}\cdot\|s\|;\nu)) + (\mathbb{E}(\chi_{B_m}\cdot\|s\|;\nu) - \mathbb{E}(\chi_{B_m}\cdot\|f\|;\nu) \\ &\le 2^{-(k+1)} + 2^{-(k+2)}.\end{aligned}$$

□

Corollary 4.3 *If the space (X,β,ϑ) in the previous theorem is derived from a computable metric space as in Lemma 3.27, then the theorem still holds true if ν is merely $\vartheta^0_{\mathcal{M}<}$-computable.*

Proof. Same idea as in the proof of Theorem 3.38. □

5 Measurability of the local error

In Definitions 3.6(ii) and 3.14, we used the outer measure ν^* induced by a measure ν and the outer integral $\int^* d\nu$, respectively. The reason for using those instead of ν itself and the proper integral $\int d\nu$ is, that one cannot be sure that the local error is always $(\mathcal{S}, \mathcal{B}(\mathbb{R}))$-measurable. So far, this has not turned out to have any negative consequences for our theory; we especially have that the outer integral grants us sufficient access to the Strong Law of Large Numbers to prove Theorem 3.16, whose statement is essentially what we expect from a notion of "computability in the mean". Anyway, we consider it an interesting question, under what conditions the measurability of f implies the measurability of the local error $e(f, \delta, \Phi, \cdot)$.

Proposition 5.1 *Suppose that $\sigma(\delta^{-1}) \subseteq \mathcal{S}$. Let $\Phi : \mathrm{dom}(\delta) \to \mathrm{dom}(\rho)$ be continuous, and let $f : X \rightrightarrows Y$ be $(\mathcal{S}, \mathcal{B}(Y))$-measurable. Then $e(f, \delta, \Phi, \cdot)$ is $(\mathcal{S}, \mathcal{B}(\mathbb{R}))$-measurable.*

Proof. There is a prefix-free set $\{w_\ell\}_\ell \subseteq \Sigma^*$ such that $\mathrm{dom}(\delta) \subseteq \bigcup_\ell w_\ell \Sigma^\omega$ and Φ is constantly equal to some $a_\ell \in \mathrm{dom}(\alpha)$ on each set $w_\ell \Sigma^\omega \cap \mathrm{dom}(\delta)$. One then has

$$e(f, \delta, \Phi, x) = \sup_\ell \chi_{W(\delta, w_\ell)}(x) d((\alpha \circ \Phi)(p), f(x)).$$

$e(f, \delta, \Phi, \cdot)$ is a countable supremum of measurable functions and hence itself measurable. □

Lemma 5.2 *Suppose that Y contains at least two distinct points and that $\sigma(\delta^{-1}) \not\subseteq \mathcal{S}$. Then there is a constant mapping $f : X \to Y$ and a computable $\Phi : \mathrm{dom}(\delta) \to \mathrm{dom}(\alpha)$ such that $e(f, \delta, \Phi, .)$ is not $(\mathcal{S}, \mathcal{B}(\mathbb{R}))$-measurable.*

Proof. There must be at least two distinct points $\alpha(a_0), \alpha(a_1)$ in range(α). Choose $f \equiv \alpha(a_0)$. There must be a $w \in \Sigma^*$ such that $W(\delta, w) \notin \mathcal{S}$. Define Φ by

$$\Phi(p) := \begin{cases} a_1 & \text{if } w \sqsubseteq p \\ a_0 & \text{else.} \end{cases}$$

We then have $e(f, \delta, \Phi, \cdot)^{-1}((0, \infty)) = W(\delta, w)$. □

We combine the last two results:

Corollary 5.3 *Suppose that Y contains at least two distinct points. Then the following two statements are equivalent:*

(i) *For every $(\mathcal{S}, \mathcal{B}(\mathbb{R}))$-measurable $f : X \rightrightarrows Y$ and every continuous $\Phi : \mathrm{dom}(\delta) \to \mathrm{dom}(\alpha)$, we have that $e(f, \delta, \Phi, \cdot)$ is $(\mathcal{S}, \mathcal{B}(\mathbb{R}))$-measurable.*

(ii) $\sigma(\delta^{-1}) \subseteq \mathcal{S}$.

□

We have found that $\sigma(\delta^{-1}) \subseteq \mathcal{S}$ is the crucial condition. This condition has already appeared before in a different context (see Proposition 3.8 and Corollary 3.29). It is time to ask for conditions under which it is fulfilled. In fact, we believe that $\sigma(\delta^{-1}) \subseteq \mathcal{S}$ will be fulfilled in all common applications. For example, we have already seen (Lemma 2.4) that we have $\sigma(\delta^{-1}) \subseteq \mathcal{B}(X)$ if X is an effective topological space and δ is its standard representation. For the Cauchy representation of an effective metric space, there are no complications, either:

Proposition 5.4 *Let (X, d', α') be an effective metric space with Cauchy representation δ_X. Then $\sigma(\delta_X^{-1}) \subseteq \mathcal{B}(X)$.*

Proof. Let $w \in \Sigma^*$ be arbitrary. Let us first suppose that w has the form

$$\iota(w_0)\iota(w_1)\ldots\iota(w_k). \tag{6}$$

Either $W(\delta_X, w) = \emptyset$ (and is hence measurable), or w can be extended to an element of $\mathrm{dom}(\delta_X)$. In the latter case, we know that $w_0, \ldots, w_k \in \mathrm{dom}(\alpha')$ and $d'(\alpha'(w_i), \alpha(w_j)) \leq 2^{-i}$ for $0 \leq i < j \leq k$. It is easy to see that

$$W(\delta, w) = \overline{\bigcap_{0 \leq i \leq k} \{a \in \mathrm{range}(\alpha') : d'(\alpha'(w_i), a) \leq 2^{-i}\}}$$

and hence is a closed set. If w is not necessarily of the form (6), we still have

$$W(\delta, w) = \bigcup \{W(\delta_X, wv) \ : \ wv \text{ is of the form } (6)\}$$

So $W(\delta, w)$ is an at most countable union of closed sets and hence Borel. $\square$

A different different type of sufficient condition for $\sigma(\delta^{-1}) \subseteq \mathcal{S}$ is presented in following:

Proposition 5.5 *Suppose $(X, \mathcal{A})$ is a standard Borel space*[14]*, μ is a σ-finite measure on $(X, \mathcal{A})$, D is a Borel subset of Σ^ω ,and $\delta : D \to X$ is a Borel measurable representation of X. Put $\mathcal{S} = \mathcal{A}_\mu$. Then $\sigma(\delta^{-1}) \subseteq \mathcal{S}$.*

Proof. By [8, Corollary 13.4], all Borel subsets of a standard Borel space are again standard Borel, hence all $w\Sigma^\omega \cap D$ are. From [8, Exercise 14.6] we have that Borel images of standard Borel spaces are analytic (see [8, Definition 14.1]); so all sets of the form $\delta(w\Sigma^\omega \cap D)$ are analytic. Finally, [8, Theorem 21.10] asserts that every analytic subset of a standard Borel space is universally measurable, which means μ-measurable with respect to any σ-finite Borel measure μ. $\square$

We finally ask whether $\delta_1 \equiv \delta_2 \ \wedge \ \sigma(\delta_1^{-1}) \subseteq \mathcal{S}$ implies $\sigma(\delta_2^{-1}) \subseteq \mathcal{S}$. Surprisingly, the answer is "No", as can be seen by combining Proposition 5.4 with the next proposition:

Proposition 5.6 *Suppose that (X, d', α') is a perfect*[15] *and Polish effective metric*

[14] A measurable space is called a (standard) *Borel space* if it is isomorphic to $(Y, \mathcal{B}(Y))$ for some Polish space Y. A *Polish space* is a separable completely metrizable topological space.

[15] A topological space is called *perfect* if it has no isolated points.

space. Let δ_X denote its Cauchy representation. There is a representation δ of X such that $\mathrm{dom}(\delta) \in \mathcal{B}(\Sigma^\omega)$*,* $\delta \equiv \delta_X$*, and* $\sigma(\delta^{-1}) \not\subseteq \mathcal{B}(X)$*.*

Proof. Let $\mathcal{N}$ be the Baire space and let $\delta_{\mathcal{N}}$ be its representation as defined in [20, Definition 3.1.2.8]); one easily verifies that

- $\delta_{\mathcal{N}}$ is a homeomorphism between $\mathrm{dom}(\delta_{\mathcal{N}})$ and $\mathcal{N}$,
- $\delta_{\mathcal{N}}^{-1} : \mathcal{N} \to \Sigma^\omega$ is $(\delta_{\mathcal{N}}, \mathrm{id}_{\Sigma^\omega})$-computable,
- $\mathrm{dom}(\delta_{\mathcal{N}}) \in \mathcal{B}(\Sigma^\omega)$.

It is also clear that the projection $\pi_{1,2} : \mathcal{N}^3 \to \mathcal{N}^2$ onto the first two coordinates as well as the standard homeomorphic tuplings

$$\underbrace{\langle \cdot, \cdot, \ldots, \cdot, \cdot \rangle}_{n} : \mathcal{N}^n \to \mathcal{N},\ n \geq 1,$$

are $([\delta_{\mathcal{N}}]^3, [\delta_{\mathcal{N}}]^2)$- and $([\delta_{\mathcal{N}}]^n, \delta_{\mathcal{N}})$-computable, respectively.

From [8, Proof of Theorem 14.2], we have that there is a closed set $\mathcal{F} \subseteq \mathcal{N}^3$ such that $\pi_{1,2}(\mathcal{F})$ is not Borel. Clearly,

$$F := (\delta_{\mathcal{N}}^{-1} \circ \langle \cdot, \cdot, \cdot \rangle)(\mathcal{F})$$

is closed in $\mathrm{dom}(\delta_{\mathcal{N}})$ and is hence Borel.

By a straightforward effectivization of [8, Theorem 6.2], there is an $(\mathrm{id}_{\Sigma^\omega}, \delta_X)$-computable injective mapping $\iota : \Sigma^\omega \to X$. We have that the composition

$$\mathcal{N}^3 \overset{\pi_{1,2}}{\to} \mathcal{N}^2 \overset{\langle \cdot, \cdot \rangle}{\to} \mathcal{N} \overset{(\delta_{\mathcal{N}})^{-1}}{\to} \Sigma^\omega \overset{\iota}{\to} X,$$

which we shall call H, is injective and $([\delta_{\mathcal{N}}]^3, \delta_X)$-computable. We especially have that $A := H(\mathcal{F})$ is non-Borel in X (because A is the continuous injective image of a non-Borel set). We also have that

$$\widetilde{\delta} := H \circ \langle \cdot, \cdot, \cdot \rangle^{-1} \circ \delta_{\mathcal{N}} : F \to A$$

is a representation of A with Borel-domain and $\widetilde{\delta} \leq \delta_X$.

It is easy to verify that $\mathrm{dom}(\delta_X) =: D$ itself is Borel. So we can define δ by $\mathrm{dom}(\delta) = \mathtt{0}D \cup \mathtt{1}F$, $\delta(\mathtt{0}p) = \delta_X(p)$ and $\delta(\mathtt{1}p) = \widetilde{\delta}(p)$. So of course δ has Borel domain, $\delta_X \equiv \delta$, and $W(\delta, \mathtt{1}) = A$ is not Borel. □

6 Conclusion

We considered several notions of probabilistic computability via representations and were able to answer a number of natural questions about them. Some open questions (e.g. concerning the composition of MEAN-computable mappings and the "right" definition of multi-valued APP- and MEAN-computability) remain and can hopefully be treated in the near future.

Our next aim is to apply the concepts developed in this paper to study the probabilistic computability and complexity of concrete operators from numerical analysis.

References

[1] Bishop, E. and D. Bridges, “Constructive Analysis,” Springer, Berlin, 1985.

[2] Edalat, A., *Dynamical systems, measures and fractals via domain theory*, in: G. L. Burn, S. J. Gay and M. Ryan, editors, *Theory and Formal Methods*, Workshops in Computing (1993), pp. 82 – 99.

[3] Edalat, A., *Domain theory and integration*, Theoretical Computer Science **151** (1995), pp. 163 – 193.

[4] Gács, P., *Uniform test of algorithmic randomness over a general space*, Theoretical Computer Science **341** (2005), pp. 91 – 137.

[5] Grubba, T. and K. Weihrauch, *On computable metrization*, Electronic Notes in Theoretical Computer Science **167** (2007), pp. 345 – 364.

[6] Hertling, P., *On the solvability of ill-posed problems in two models of computation* (2005), talk presented at the Foundations of Computational Mathematics conference, Santander, Spain.

[7] Kallenberg, O., “Foundations of Modern Probability,” Springer, New York, 2002, second edition.

[8] Kechris, A. S., “Classical Descriptive Set Theory,” Springer, New York, 1995.

[9] Ko, K.-I., “Complexity Theory of Real Functions,” Birkhäuser, Boston, 1991.

[10] Müller, N. T., *Computability on random variables*, Theoretical Computer Science **219** (1999), pp. 287 – 299.

[11] Parker, M. W., *Undecidability in $\mathbb{R}^n$: Riddled basins, the KAM tori, and the stability of the solar system*, Philosophy of Science **70** (2003), pp. 359–382.

[12] Parker, M. W., *Three concepts of decidability for general subsets of uncountable spaces*, Theoretical Computer Science **351** (2006), pp. 2 – 13.

[13] Schröder, M., *Extended admissibility*, Theoretical Computer Science **284** (2002), pp. 519 – 538.

[14] Schröder, M., *Admissible representations of probability measures*, Electronic Notes in Theoretical Computer Science **167** (2007), pp. 61 – 78.

[15] Schröder, M. and A. Simpson, *Representing probability measures using probabilistic processes*, Journal of Complexity **22** (2006), pp. 768 – 782.

[16] Traub, J. F., G. W. Wasilkowski and H. Woźniakowski, “Information-based complexity,” Academic Press, New York, 1988.

[17] Vakhania, N. N., V. I. Tarieladze and S. A. Chobanian, “Probability Distributions on Banach Spaces,” Kluwer Academic, Dordrecht/Norwell, MA, 1987.

[18] Webster, J., *Finite approximation of measure and integration*, Annals of Pure and Applied Logic **137** (2006), pp. 439 – 449.

[19] Weihrauch, K., *Computability on the probability measures on the Borel sets of the unit interval.*, Theoretical Computer Science **219** (1999), pp. 421–437.

[20] Weihrauch, K., “Computable Analysis: An Introduction,” Springer, Berlin, 2000.

[21] Wu, Y. and D. Ding, *Computability of measurable sets via effective metrics*, Mathematical Logic Quarterly **51** (2005), pp. 543 – 559.

[22] Wu, Y. and D. Ding, *Computability of measurable sets via effective topologies*, Archive for Mathematical Logic **45** (2006), pp. 365 – 379.

[23] Wu, Y. and K. Weihrauch, *A computable version of the Daniell-Stone theorem on integration and linear functionals*, Theoretical Computer Science **359** (2006), pp. 28 – 42.