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A Computable Version of the Daniell-Stone Theorem on Integration and Linear Functionals

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Abstract

For every measure μ, the integral $I : f \mapsto \int f \, d\mu$ is a linear functional on the set of real measurable functions. By the *Daniell-Stone theorem*, for every abstract integral $\Lambda : F \to \mathbb{R}$ on a stone vector lattice F of real functions $f : \Omega \to \mathbb{R}$ there is a measure μ such that $\int f \, d\mu = \Lambda(f)$ for all $f \in F$. In this paper we prove a computable version of this theorem.

Keywords: computable analysis, measure theory, Daniell-Stone theorem

1 Introduction and Mathematical Preliminaries

In this section we summarize some notations, definitions and facts from measure theory and computable analysis.

As a reference to measure theory we use the book [1]. A *ring* in a set Ω is a set $\mathcal{R}$ of subsets of Ω such that $\emptyset \in \mathcal{R}$ and $A \cup B \in \mathcal{R}$ and $A \setminus B \in \mathcal{R}$

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if $A, B \in \mathcal{R}$. A *σ-algebra* in Ω is a set $\mathcal{A}$ of subsets of Ω such that $\Omega \in \mathcal{A}$, $\Omega \setminus A \in \mathcal{A}$ if $A \in \mathcal{A}$ and $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$, if $A_1, A_2, \ldots \in \mathcal{A}$. For any system $\mathcal{E}$ of subsets of Ω let $\mathcal{A}(\mathcal{E})$ be the smallest σ-algebra in Ω containing $\mathcal{E}$.

A *premeasure* on a ring $\mathcal{R}$ is a function $\mu : \mathcal{R} \to \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ such that $\mu(\emptyset) = 0$, $\mu(A) \geq 0$ for $A \in \mathcal{R}$ and

$$\mu(\bigcup_{i=1}^{\infty} A_n) = \sum_{i=1}^{\infty} \mu(A_n)$$

if $A_1, A_2, \ldots \in \mathcal{A}$ are pairwise disjoint and $\bigcup_{i=1}^{\infty} A_n \in \mathcal{A}$. A premeasure on an algebra is called a *measure*. A premeasure μ on a ring $\mathcal{R}$ is called *σ-finite*, if there is a sequence $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$ in $\mathcal{R}$ such that $A_1 \cup A_2 \cup \ldots = \Omega$ and $\mu(A_i) < \infty$ for all i.

Theorem 1.1 ([1]) *Every σ-finite premeasure μ on a ring $\mathcal{R}$ in Ω has a unique extension to a measure on $\mathcal{A}(\mathcal{R})$ which (for convenience) we also denote by μ.*

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. A function $f : \Omega \to \mathbb{R}$ is called *measurable*, if $\{x \mid f(x) > a\} \in \mathcal{A}$ for all $a \in \mathbb{R}$. The following condition is equivalent:

$$(\forall a \in D)\ \{x \mid f(x) > a\} \in \mathcal{A} \quad \text{for some set } D \text{ dense in } \mathbb{R}\,. \tag{1}$$

As usual we will abbreviate $\{f > a\} := \{x \in \Omega \mid f(x) > a\}$. In (1) the relation "$>$" can be replaced by "$\leq$", "$\geq$" or "$<$". A function $f : \Omega \to \mathbb{R}$ is *simple*, if there are *non-negative* real numbers $a_1, \ldots, a_n$ and pairwise disjoint sets $A_1, \ldots, A_n \in \mathcal{A}$ of finite measure such that $f(x) = \sum_{i=1}^{n} a_i \chi_{A_i}$, where χ_A is the characteristic function of A. For a simple function the integral is defined by

$$\int \sum_{i=1}^{n} a_i \chi_{A_i} := \sum_{i=1}^{n} a_i \mu(A_i)\,. \tag{2}$$

For functions $u, u_0, u_1, \ldots \Omega \to \mathbb{R}$, $u_i \nearrow u$ means: For all $x \in \omega$, $u_0(x) \leq u_1(x) \leq \ldots$ and $\sup_i u_i(x) = u(x)$. For a non-negative measurable real function $f : \Omega \to \mathbb{R}$ and $b \in \mathbb{R}$, $\int f\, d\mu = b$, iff there is some increasing sequence $(u_i)_{i \in \mathbb{N}}$ of simple functions such that

$$u_i \nearrow f \quad \text{and} \quad \sup_i \int u_i\, d\mu = b \tag{3}$$

[1]. In particular, $\int f\, d\mu$ does not exist (in $\mathbb{R}$), if the sequence $(\int u_i\, d\mu)_i$ is unbounded. For an arbitrary real function $f : \Omega \to \mathbb{R}$ let $f_+ := \sup(0, f)$ (the positive part of f) and $f_- := \sup(0, -f)$ (the negative part of f). By

definition, a measurable function f is integrable, if $\int f_+ \, d\mu$ and $\int f_- \, d\mu$ exist and its integral is defined by

$$\int f \, d\mu := \int f_+ \, d\mu - \int f_- \, d\mu \,. \tag{4}$$

For the following concepts from computable analysis see [4]. Let $\mathbb{N} := \{0, 1, 2, \ldots\}$ be the set of natural numbers. A partial function from X to Y is denoted by $f :\subseteq X \to Y$, a multifunction by $f :\subseteq X \rightrightarrows Y$. Let Σ be a sufficiently large finite alphabet such that $\{0, 1\} \subseteq \Sigma$. The set of finite words over Σ is denoted by Σ^*, the set of infinite sequences by Σ^ω. Computability of functions on Σ^* and Σ^ω is defined by Turing machines which can read and write finite and infinite sequences, respectively. Standard pairing functions on Σ^* are denoted by $\langle\, ; \rangle$. For $w \in \Sigma^*$ let $\xi_w :\subseteq \Sigma^* \to \Sigma^*$ be the word function computed by the Turing machine with canonical code $w \in \Sigma^*$. Like the "effective Gödel numbering" $\phi : \mathbb{N} \to P^{(1)}$ of the partial recursive functions the notation ξ satisfies the utm-theorem and the smn-theorem.

Computability on other sets is introduced by using finite or infinite sequences of symbols as "names". For the natural numbers let $\nu_{\mathbb{N}} :\subseteq \Sigma^* \to \mathbb{N}$ be the notation by binary numbers and let bn_i be the binary name of $i \in \mathbb{N}$. Let $\nu_{\mathbb{Q}} :\subseteq \Sigma^* \to \mathbb{Q}$ be some standard notation of the rational numbers. For the real numbers we use the standard Cauchy representation $\rho :\subseteq \Sigma^\omega \to \mathbb{R}$, where $\rho(p) = x$, iff p encodes a sequence $(a_i)_i$ of rational numbers such that $|a_i - x| \le 2^{-i}$. For naming systems $\delta_i :\subseteq Y_i \to M_i$, $Y_i \subseteq \{\Sigma^*, \Sigma^\omega\}$ for $i = 1, 2$, a multifunction $f :\subseteq M_1 \rightrightarrows M_2$ is (δ_1, δ_2)-computable, iff there is a computable function $h :\subseteq Y_1 \to Y_2$ such that $\delta_2 \circ h(p) \in f(\delta_1(p))$ for all $p \in \mathrm{dom}(\delta_1)$ such that $f(\delta_1(p)) \neq \emptyset$.

In this article we will consider computability on factorizations of several pseudometric spaces [2]. We generalize the definition of a computable metric space with Cauchy representation from [4] straightforwardly as follows: A computable pseudometric space is a quadruple $\mathcal{M} = (M, d, A, \alpha)$ such that (M, d) is a pseudometric space, $A \subseteq M$ is dense and $\alpha :\subseteq \Sigma^* \to A$ is a notation of A such that $\mathrm{dom}(\alpha)$ is recursive and the restriction of the pseudometric d to A is (α, α, ρ)-computable. (In [4], $\mathrm{dom}(\alpha)$ is assumed to be r.e. Notice that for every notation with r.e. domain there is an equivalent one with recursive domain.) In our applications, $\mathcal{M}$ is a linear space and the pseudometric is derived from a seminorm $||.||$, $d(x, y) = ||x - y||$.

The factorization $(\overline{M}, \overline{d})$ of the pseudometric space (M, d) is a metric space defined canonically as follows: $\overline{x} := \{y \in M \mid d(x, y) = 0\}$, $\overline{M} := \{\overline{x} \mid x \in M\}$, $\overline{d}(\overline{x}, \overline{y}) := d(x, y)$. We define the Cauchy representation $\delta_{\mathcal{M}}$ of the factorization of a computable pseudometric space as follows: $\delta_{\mathcal{M}}(p) = \overline{x}$, if $p \in \Sigma^\omega$ encodes a sequence $(a_i)_i$ (of α-names) of elements of A such that $d(a_i, x) \leq 2^{-i}$ for all i. If $\mathcal{M}$ is a linear space with seminorm $||.||$, by $a\overline{x} := \overline{ax}$ and $\overline{x} + \overline{y} := \overline{x + y}$ the factor space becomes a linear space with norm $||\overline{x}|| := ||x||$. In this case, $\overline{d}(\overline{x}, \overline{y}) = ||x - y||$.

2 Computable Measure Spaces

In this section let $(\Omega, \mathcal{A}, \mu)$ be a measure space. For any $\mathcal{D} \subseteq \mathcal{A}$ let $\mathcal{D}^f := \{A \in \mathcal{D} \mid \mu(A) < \infty\}$. In computable measure theory we want to identify two sets $A, B \in \mathcal{A}$, if their symmetric difference $A\Delta B := (A \setminus B) \cup (B \setminus A)$ has measure 0 and distinguish them otherwise. Since $A\Delta B \subseteq A\Delta C \cup C\Delta B$, on the set $\mathcal{A}^f$ the mapping $d : (A, B) \mapsto \mu(A\Delta B)$ is a pseudometric.

Lemma 2.1 *Let $\mathcal{R}$ be a ring such that $\mathcal{A}(\mathcal{R}) = \mathcal{A}$ and μ is a σ-finite premeasure on $\mathcal{R}$. Then $(\mathcal{A}^f, d)$, $d : (A, B) \mapsto \mu(A\Delta B)$, is a complete pseudometric space with $\mathcal{R}^f$ as a dense subset.*

Proof: Straightforward. □

For including sets with infinite measure consider the mapping $d_\infty : (A, B) \mapsto \mu(A\Delta B)/(1 + \mu(A\Delta B))$ which is a pseudometric on $\mathcal{A}$ (notice: $\infty/(1 + \infty) = 1$). Its restriction to $\mathcal{A}^f$ is equivalent to d. For introducing computability on a pseudometric space we need a countable dense subset [4,3]. Unfortunately, there are important measure spaces such that the pseudometric space $(\mathcal{A}, d_\infty)$ is not separable.

Example: Consider the measure space $(\mathbb{R}, \mathcal{B}, \lambda)$ where $\mathcal{B}$ is the set of Borel subsets of the real numbers and λ is the Lebesgue-Borel measure. Let $(E_i)_{i\in\mathbb{N}}$ be any countable sequence in $\mathcal{B}$. Define $B := \bigcup_i (i; i + 1) \setminus E_i$. Then for all i, $\lambda(B\Delta E_i) \geq 1$ and hence $d_\infty(B, E_i) \geq 1/2$. Therefore, the set of all E_i cannot be dense. Since this is true for every sequence $(E_i)_{i\in\mathbb{N}}$, the pseudometric space $(\mathcal{B}, d_\infty)$ is not separable.

We will consider measures which are completions of σ-finite premeasures on *countable* rings consisting of sets with finite measure. We assume that the operations on the ring and the premeasure are computable.

Definition 2.2 A *computable measure space* is a quintuple $\mathcal{M} = (\Omega, \mathcal{A}, \mu, \mathcal{R}, \alpha)$ such that

(i) $\mathcal{A}$ is a σ-algebra in Ω and μ is a measure on it,

(ii) $\mathcal{R}$ is a countable ring such that $\mathcal{A} = \mathcal{A}(\mathcal{R})$,

(iii) $\mu(A) < \infty$ for all $A \in \mathcal{R}$,

(iv) the restriction of μ to $\mathcal{R}$ is σ-finite,

(v) $\alpha :\subseteq \Sigma^* \to \mathcal{R}$ is a notation of $\mathcal{R}$ with recursive domain,

(vi) $(A, B) \mapsto A \cup B$ and $(A, B) \mapsto A \setminus B$ are (α, α, α)-computable,

(vii) μ is (α, ρ)-computable on $\mathcal{R}$.

By (iv), $\Omega = \bigcup \mathcal{R}$. If $\bigcup \mathcal{R}$ is a proper subset of Ω, then for obtaining a σ-finite measure, either restrict Ω to $\bigcup \mathcal{R}$ or add the set $\Omega \setminus \bigcup \mathcal{R}$ to $\mathcal{R}$ and define $\mu(\Omega \setminus \bigcup \mathcal{R}) = 0$.

Theorem 2.3 *Let $(\Omega, \mathcal{A}, \mu, \mathcal{R}, \alpha)$ be a computable measure space. Then the quadruple $(\mathcal{A}^f, d, \mathcal{R}, \alpha)$ is a computable complete pseudometric space, where $\mathcal{A}^f = \{A \in \mathcal{A} \mid \mu(A) < \infty\}$ and $d(A, B) = \mu(A \Delta B)$.*

Proof: By Lemma 2.1, $(\mathcal{A}^f, d)$ is a complete pseudometric space with $\mathcal{R}$ as a dense subset. By Def. 2.2(v)-(vii) the notation α has recursive domain and the distance d is (α, α, ρ)-computable. □

Computability on the computable measure space can be defined via the Cauchy representation of the joined pseudometric space.

Example 2.4 [Lebesgue-Borel measure on $\mathbb{R}$] Let $\Omega = \mathbb{R}$, let $D \subseteq \mathbb{R}$ be dense in $\mathbb{R}$ and let $\nu_D :\subseteq \Sigma^* \to D$ be a notation such that $\mathrm{dom}(\nu_D)$ is recursive and $\nu \leq \rho$. Let $\tilde{I}_D$ be the set of all intervals $[a; b) \subseteq \mathbb{R}$ such that $a, b \in D$ and $a < b$. Let $\mathcal{R}_\mathcal{D}$ be the set of all finite unions of intervals from $\tilde{I}_D$ and let α_D be some notation of $\mathcal{R}_\mathcal{D}$ canonically derived from ν_D. Then $\mathcal{B} := \mathcal{A}(\mathcal{R}_D)$ is the set of Borel-subsets of $\mathbb{R}$. The Lebesgue-Borel measure λ on $\mathcal{B}$ is defined uniquely by setting $\lambda([a; b)) := b - a$ for all $a, b \in D$, $a < b$ [1]. $\mathcal{M}_D := (\mathbb{R}, \mathcal{B}, \lambda, \mathcal{R}_D, \alpha_D)$ is a computable measure space.

3 Computability on the Integrable Functions

In this section we assume that $\mathcal{M} = (\Omega, \mathcal{A}, \mu, \mathcal{R}, \alpha)$ is a computable measure space. We introduce a computable pseudometric space for the integrable functions. On the set $\mathcal{I}(\mathcal{M})$ of μ-integrable functions $f : \Omega \to \mathbb{R}$ a seminorm and a pseudometric are defined by

$$||f||_\mathcal{M} := \int |f| \, d\mu, \quad d_\mathcal{M}(f, g) := ||f - g||_\mathcal{M}. \tag{5}$$

(see [1]). For introducing computability on $\mathcal{I}(\mathcal{M})$ we consider a countable dense set.

Definition 3.1 (i) A function $u : \Omega \to \mathbb{R}$ is a *rational step function*, iff there are rational numbers $a_1, \ldots, a_n$ and pairwise disjoint sets

$A_1, \ldots, A_n \in \mathcal{R}$ such that $u = \sum_{i=1}^n a_i \cdot \chi_{A_i}$.

(ii) Let $\beta : \subseteq \Sigma^* \to \mathrm{RSF}$ be a canonical notation of the set RSF of rational step functions derived from the notation α such that $\mathrm{dom}(\beta)$ is recursive.

In contrast to a simple function (see Sec. 1), for a rational step function $f = \sum_{i=1}^n a_i \cdot \chi_{A_i}$ the sets A_i must be in $\mathcal{R}$ and the coefficients must be rational, but may be negative. For a rational step function $u = \sum_{i=1}^n a_i \cdot \chi_{A_i}$, $\int u \, d\mu = \sum_{i=1}^n a_i \cdot \mu(A_i)$ and $||u||_{\mathcal{M}} = \sum_{i=1}^n |a_i| \cdot \mu(A_i)$.

Lemma 3.2 *For rational step functions u, v and $a \in \mathbb{Q}$ the functions*

(i) $(a, u) \mapsto a \cdot u$, $(u, v) \mapsto u + v$, $u \mapsto |u|$, $u \mapsto \inf(u, 1)$, $u \mapsto \int u \, d\mu$,

(ii) $(u, v) \mapsto \sup(u, v)$, $(u, v) \mapsto \inf(u, v)$, $u \mapsto u_+$, $u \mapsto u_-$, $(u, a) \mapsto \inf(u, a)$, $u \mapsto ||u||_{\mathcal{M}}$

are computable w.r.t. the notations β, $\nu_{\mathbb{Q}}$ and ρ.

Proof: Straightforward. □

In Def. 3.1(i) the condition "$A_1, \ldots, A_n$ are pairwise disjoint" is not restrictive.

Lemma 3.3 *Let β' be a canonical notation of all $u = \sum_{i=1}^n a_i \cdot \chi_{A_i}$ such that $a_i \in \mathbb{Q}$ and $A_i \in \mathcal{R}$ (but the A_i are not necessarily disjoint). Then $\beta' \equiv \beta$.*

Proof: "$\leq$": From the sets A_i by determining intersections and differences a finite set $B_1, \ldots, B_m$ of pairwise disjoint sets can be computed such that each A_i is a finite union of B_js. Then coefficinets $b_j \in \mathbb{Q}$ can be computed such that $\sum_{i=1}^n a_i \cdot \chi_{A_i} = \sum_{j=1}^m b_j \cdot \chi_{B_j}$. This procedure is computable w.r.t the representations $\beta, \beta', \alpha, \nu_{\mathbb{Q}}$ and $\nu_{\mathbb{N}}$.

"$\geq$": Obvious. □

Theorem 3.4 $(\mathcal{I}(\mathcal{M}), d_{\mathcal{M}}, \mathrm{RSF}, \beta)$ *is a computable complete peudometric space.*

Proof: By Th. 15.5 in [1], $(\mathcal{I}(\mathcal{M}), d_{\mathcal{M}})$ is complete.

Consider $f \in \mathcal{I}(\mathcal{M})$ and $\varepsilon > 0$. Then $\int f \, d\mu = \int f_+ \, d\mu - \int f_- \, d\mu$. By (3) there is a simple function $u \leq f_+$ such that $0 \leq \int f_+ \, d\mu - \int u \, d\mu < \varepsilon/4$, hence $d_{\mathcal{M}}(f_+, u) = \int |f_+ - u| \, d\mu = \int f_+ \, d\mu - \int u \, d\mu < \varepsilon/4$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$ and $\mathcal{R}$ is dense in $\mathcal{A}^f$ by Thm. 2.3, there is a rational step function v such that $d_{\mathcal{M}}(u, v) < \varepsilon/4$. We obtain $d_{\mathcal{M}}(f_+, v) \leq d_{\mathcal{M}}(f_+, u) + d_{\mathcal{M}}(u, v) \leq \varepsilon/2$. Correspondingly, there is a rational step function w such that $d_{\mathcal{M}}(f_-, w) \leq \varepsilon/2$. We obtain $d_{\mathcal{M}}(f, v - w) = ||f_+ - f_- - (v - w)|| \leq ||f_+ - v|| + ||f_- - w|| < \varepsilon$. Therefore, $v - w$ is a rational step function which is ε-close to f.

On RSF the distance $d_{\mathcal{M}}$ is (β, β, ρ)-computable. This follows from Lemma 3.2.
□

Let $\delta_{\mathcal{M}} :\subseteq \Sigma^\omega \to \mathcal{I}(\mathcal{M})/_\equiv$ be the Cauchy representation of the set of equivalence classes of integrable functions (see Sec. 1).

4 The Computable Daniell-Stone Theorem

For two real-valued functions let $(f \wedge g)(x) := \inf(f(x), g(x))$. A *Stone vector lattice* of real functions is a vector space $\mathcal{F}$ of functions $f : \Omega \to \mathbb{R}$ such that the functions $x \mapsto |f(x)|$ and $x \mapsto \inf(f(x), 1)$ (denoted by $|f|$ and $f \wedge 1$, resp.) are in $\mathcal{F}$ if $f \in \mathcal{F}$.

Let $\mathcal{F}_+$ be the set of non-negative functions in $\mathcal{F}$. Let us call $\mathcal{F}$ *complete*, if $f \in \mathcal{F}$ whenever $u_i \nearrow f$ for $u_i \in \mathcal{F}_+$ and $f : \Omega \to \mathbb{R}$.

An *abstract integral* on a Stone vector lattice $\mathcal{F}$ of real functions is a linear functional $I : \mathcal{F} \to \mathbb{R}$ such that for all $f, f_0, f_1, \ldots \in \mathcal{F}_+$,

$$I(f) \geq 0 \quad \text{and} \quad I(f) = I(\sup_n f_n) = \sup_n I(f_n) \quad \text{if} \quad f_i \nearrow f\,. \tag{6}$$

Let $\mathcal{A}(\mathcal{F})$ be the smallest σ-algebra in Ω such that every function $f \in \mathcal{F}$ is measurable.

Theorem 4.1 (Daniell-Stone [1]) *Let $\mathcal{F}$ be a Stone vector lattice with abstract integral I. Then there is a measure μ on $\mathcal{A}(\mathcal{F})$ such that f is μ-integrable and $I(f) = \int f \, d\mu$ for all $f \in \mathcal{F}$. Furthermore, if there is a sequence $(f_i)_i$ in $\mathcal{F}$ such that $(\forall x \in \Omega)(\exists i) f_i(x) > 0$, then the measure μ is uniquely defined.*

For a proof see Thms. 39.4 and Cor. 39.6 in [1]. On a Stone vector lattice with abstract integral a seminorm $||.||_{\mathcal{S}}$ and a pseudometric $d_{\mathcal{S}}$ can be defined by

$$||f||_{\mathcal{S}} := I(|f|) \quad \text{and} \quad d_{\mathcal{S}}(f, g) := ||f - g||_{\mathcal{S}} = I(|f - g|)\,. \tag{7}$$

For an effective version of Thm. 4.1 we consider a notation γ of a dense subset $\mathcal{D}$ such that $(\mathcal{F}, d_{\mathcal{S}}, \mathcal{D}, \gamma)$ is a computable pseudometric space. Furthermore, we assume that $|f|,\ f \wedge 1 \in \mathcal{D}$ if $f \in \mathcal{D}$ and that $\mathcal{D}$ is closed under rational linear combination.

Definition 4.2 A *computable Stone vector lattice with abstract integral* is a tuple $\mathcal{S} = (\Omega, \mathcal{F}, I, \mathcal{D}, \gamma)$ such that

(i) $\mathcal{F}$ is a Stone vector lattice with abstract integral I,

(ii) $\mathcal{D} \subseteq \mathcal{F}$ is dense w.r.t the pseudometric $d_{\mathcal{S}} : (f, g) \mapsto I(|f - g|)$,

(iii) γ is a notation of $\mathcal{D}$ with recursive domain,

(iv) if $a \in \mathbb{Q}$ and $f, g \in \mathcal{D}$, then $\{af,\ f+g,\ |f|,\ f \wedge 1\} \subseteq \mathcal{D}$,

(v) for $a \in \mathbb{Q}$ and $f, g \in \mathcal{D}$, the functions $(a, f) \mapsto af$, $(f, g) \mapsto f+g$, $f \mapsto |f|$ and $f \mapsto f \wedge 1$ are computable w.r.t. $\nu_{\mathbb{Q}}$, γ and ρ.

(vi) the restriction of I to $\mathcal{D}$ is (γ, ρ)-computable.

Let $\delta_{\mathcal{S}} :\subseteq \Sigma^\omega \to \mathcal{F}/_\equiv$ be the canonical Cauchy representation of the factorization of the computable pseudometric space $(\mathcal{F}, d_{\mathcal{S}}, \mathcal{D}, \gamma)$.

It can be shown easily that $(\mathcal{F}, d_{\mathcal{S}}, \mathcal{D}, \gamma)$ is a computable pseudometric space. For a computable measure space, the integrable functions with the integral as linear operator form a computable Stone vector lattice with abstract integral.

Proposition 4.3 *Let $\mathcal{M} = (\Omega, \mathcal{A}, \mu, \mathcal{R}, \alpha)$ be a computable measure space. Then $(\Omega, \mathcal{I}(\mathcal{M}), (f \mapsto \int f\, d\mu), \mathrm{RSF}, \beta)$ (see Def. 3.1(ii)) is a computable complete Stone vector lattice with abstract integral.*

Proof: Straightforward. □

For two metric spaces (M_i, d_i) $(i = 0, 1)$ call $\psi : M_0 \to M_1$ a *metric embedding*, iff $d_1(\psi(x), \psi(y)) = d_0(x, y)$ for all $x, y \in M_0$. Obviously, a metric embedding ψ is injective, i.e., (M_0, d_0) is, up to renaming, a subspace of (M_1, d_1). For computable metric spaces $(M_i, d_i, A_i, \alpha_i)$ $(i = 0, 1)$ with Cauchy representaions δ_i $(i = 0, 1)$, if $\psi : M_0 \to M_1$ is a (δ_0, δ_1)-computable embedding, then its inverse $\psi^{-1} :\subseteq M_1 \to M_0$ is (δ_1, δ_0)-computable. In this case, the first space is, up to renaming, a very well behaved subspace of the second one.

We can now formulate and prove our computational version of the Daniell-Stone theorem. (We use the Cauchy representation $\delta_{\mathcal{M}}$ of a factorized pseudometric space of the integrable functions, see Thm. 3.4 and the end of Sec. 3.)

Theorem 4.4 (computable Daniell-Stone) *Let $\mathcal{S} = (\Omega, \mathcal{F}, I, \mathcal{D}, \gamma)$ be a computable Stone vector lattice with abstract integral such that $(\forall x \in \Omega)(\exists f \in \mathcal{D}) f(x) > 0$. Then there exist a computable measure space $\mathcal{M} = (\Omega, \mathcal{A}, \mu, \mathcal{R}, \alpha)$ and a funtion ψ such that*

(i) *ψ is a $(\delta_{\mathcal{S}}, \delta_{\mathcal{M}})$ computable metric embedding $\psi : \mathcal{F}/_\equiv \to \mathcal{I}(\mathcal{M})/_\equiv$;*

(ii) *$I(f) = \int g\, d\mu$ for all $f \in \mathcal{F}$ and $g \in \psi(f/_\equiv)$;*

where $\delta_{\mathcal{S}}$ is the Cauchy representation of the factorized pseudometric space derived from $\mathcal{S}$ (Def. 4.2) and $\delta_{\mathcal{M}}$ is the Cauchy representation of the factorized pseudometric space derived from $\mathcal{M}$ (Thm. 3.4).

For the main proof we need a number of auxiliary propositions. Because of the space limit their proofs are omitted. First, a ring $\mathcal{R}$ on Ω must be defined. Consider $f \in \mathcal{D}$. Since f must be μ-integrable by (i) and hence $\mathcal{A}$-measurable, we must have $\{f > a\} \in \mathcal{A} = \mathcal{A}(\mathcal{R})$ for all $a \in \mathbb{R}$. Since $\{f > a\} = \bigcup_{a<b\in\mathbb{Q}}\{f > b\}$, it would suffice to require $\{f > b\} \in \mathcal{R}$ for all $f \in \mathcal{D}$ and $b \in \mathbb{Q}$. Unfortunately, some of the values $\mu(\{f > b\})$, $b \in \mathbb{Q}$, (which will be defined canonically) might become non-computable. In order to avoid this problem, for every function $f \in \mathcal{D}_+$ (the non-negative functions from $\mathcal{D}$) we construct a new countable dense set C_f of computable real numbers (see (1)) such that $\mu(\{f > c\})$ becomes computable for each $c \in C_f$. $\mathcal{R}$ will be the smallest ring containing all the sets $\{f > c\}$ ($f \in \mathcal{D}_+$, $c \in C_f$) for which we define $\mu\{f > c\} := \sup\{I(h) \mid h \in \mathcal{D}_+,\ h \le \chi_{\{f>c\}}\}$. Moreover, we define a notation $\alpha :\subseteq \Sigma^* \to \mathcal{R}$ such that (v) - (vii) from Def. 2.2 are satisfied. A further crucial step is to show that for every function $f \in \mathcal{D}_+$ and every $n \in \mathbb{N}$ a rational step function t in $\mathcal{M} = (\Omega, \mathcal{A}, \mu, \mathcal{R}, \alpha)$ with non-negative coefficients can be computed w.r.t. the notations γ, $\nu_\mathbb{N}$ and β (from Def. 3.1) such that $t \le f$ and $0 \le I(f) - \int t\,d\mu \le 2^{-n}$.

Define a notation γ_+ of $\mathcal{D}_+ := \mathcal{D} \cap \mathcal{F}_+$ by $\gamma_+(v) := |\gamma(v)|$. From Def. 4.2 we can conclude that γ_+ is reducible to γ ($\gamma_+ \le \gamma$). Define a notation $\nu_\to$ of the computable sequences in $\mathcal{D}_+$ by

$$\nu_\to(s) = (f_0, f_1, \ldots) \iff (\forall w \in \mathrm{dom}(\nu_\mathbb{N}))\ \ f_{\nu_\mathbb{N}(w)} = \gamma_+ \circ \xi_s(w)\,, \tag{8}$$

that is, iff ξ_s is a $(\nu_\mathbb{N}, \gamma_+)$-realization of $i \mapsto f_i$ (see Section 1).

As a first step, for each $f = \gamma_+(v) \in \mathcal{D}_+$ we compute some dense set $D_v \subseteq \mathbb{R}_+$ such that $\mu(\{f > a\})$ is a computable real number for all $a \in D_v$ (and show how to compute these values).

Proposition 4.5 *For every $f \in \mathcal{D}_+$ and every $a_0, b_0 \in \mathbb{Q}$, $0 < a_0 < b_0$, a real number c and two sequences $(g_n)_n$ and $(h_n)_n$ in $\mathcal{D}_+$ can be computed w.r.t. the notations γ, $\nu_\mathbb{Q}$, $\nu_\to$ and ρ such that*

$$a_0 < c < b_0 \tag{9}$$

$$0 \le h_0 \le h_1 \le \ldots \le \chi_{\{f>c\}} \le \chi_{\{f\ge c\}} \le \ldots \le g_1 \le g_0 \tag{10}$$

$$\sup I(h_n) = \inf I(g_n). \tag{11}$$

Notice that for every fixed $v \in \mathrm{dom}(\gamma) = \mathrm{dom}(\gamma_+)$, the set of constants c,

$$D_v := \{\rho \circ H_0(v, u_l, u_r) \mid 0 < \nu_\mathbb{Q}(u_l) < \nu_\mathbb{Q}(u_r)\} \ \text{ is dense in } \ \mathbb{R}_+\,. \tag{12}$$

We define the ring and the σ-algebra for the measure space $\mathcal{M}$.

Definition 4.6

$$\mathcal{R}_0 := \{\{\gamma_+(v) > \rho \circ H_0(v, u_l, u_r)\} \mid v \in \mathrm{dom}(\gamma_+), 0 < \nu_\mathbb{Q}(u_l) < \nu_\mathbb{Q}(u_r)\}$$

$$\mathcal{R} := \text{the smallest ring containing } \mathcal{R}_0$$

$$\mathcal{A} := \mathcal{A}(\mathcal{R}) = \mathcal{A}(\mathcal{R}_0)$$

Notice that $\mathcal{R}_0$ is not a ring in general. By Prop. 4.8 for every set $A \in \mathcal{R}_0$ there are sequences (h_i) and (g_i) in $\mathcal{D}_+$ such that

$$0 \le h_0 \le h_1 \le \ldots \le \chi_A \le \ldots \le g_1 \le g_0 \quad \text{and} \quad \sup I(h_n) = \inf I(g_n)\,. \tag{13}$$

In the following we prove that this is true also for all $A \in \mathcal{R}$. Additionally we introduce a notation α of $\mathcal{R}$ such that the sequences (h_i) and (g_i) can be computed from $A \in \mathcal{R}$.

Proposition 4.7 *For functions $h_n, g_n, h'_n, g'_n \in \mathcal{D}_+$ and $A, A' \subseteq \Omega$ let*

$$\begin{aligned} 0 \le h_0 \le h_1 \le \ldots \le \chi_A \le \ldots \le g_1 \le g_0,\\ \sup I(h_n) = \inf I(g_n),\\ 0 \le h'_0 \le h'_1 \le \ldots \le \chi_{A'} \le \ldots \le g'_1 \le g'_0,\\ \sup I(h'_n) = \inf I(g'_n). \end{aligned}$$

Then for $h_n^+ := \sup(h_n, h'_n)$, $g_n^+ := \sup(g_n, g'_n)$, $h_n^- := (h_n - g'_n)_+$ *and* $g_n^- := (g_n - h'_n)_+$,

$$\begin{aligned} 0 \le h_0^+ \le h_1^+ \le \ldots \le \chi_{A\cup A'} \le \ldots \le g_1^+ \le g_0^+,\\ \sup I(h_n^+) = \inf I(g_n^+),\\ 0 \le h_0^- \le h_1^- \le \ldots \le \chi_{A\setminus A'} \le \ldots \le g_1^- \le g_0^-,\\ \sup I(h_n^-) = \inf I(g_n^-). \end{aligned}$$

By the next proposition the constructions in Prop. 4.7 are computable. Let us say that $t = \langle s_-, s_+\rangle$ encloses a set $A \subseteq \Omega$, if (13) for the sequences $(h_0, h_1, \ldots) := \nu_\to(s_-)$ and $(g_0, g_1, \ldots) := \nu_\to(s_+)$.

Proposition 4.8 *There are computable functions G_1 and G_2 such that $G_1(t, t')$ encloses $A \cup A'$ and $G_2(t, t')$ encloses $A \setminus A'$, if t encloses A and t' encloses A'.*

Proposition 4.9 *There is a computable function L such that $\rho \circ L(\langle s_-, s_+\rangle) = \sup I(h_n)$, if $\nu_\to(s_-) = (h_i)_i$ and $\nu_\to(s_+) = (g_i)_i$ such that (13).*

Proof: This follows by standard arguments from Def. 4.2(vi). $\square$(Prop. 4.9)

We define a notation α of $\mathcal{R}$ inductively as follows. By Prop. 4.5 there is a computable function H_0 such that

$$c; \rho \circ -09v, u_l, u_r)$$

if $f = \gamma_+(v)$, $a_0 = \nu_\mathbb{Q}(u_l)$ and $b_0 = \nu_\mathbb{Q}(u_r)$. (For convenience we assume $\mathrm{dom}(\gamma), \mathrm{dom}(\nu_\mathbb{Q}) \subseteq (\Sigma \setminus \Sigma')^*$ and $\Sigma' \subseteq \Sigma \setminus \{0, 1\}$ for $\Sigma' := \{(,), \cup, \setminus\}$.)

$$\alpha(\langle v, u_l, u_r\rangle) := \{\gamma_+(v) \; > \; \rho \circ H_0(v, u_l, u_r)\} \in \mathcal{R}_0\,, \tag{14}$$
$$\alpha(\,(w \cup w')\,) := \alpha(w) \cup \alpha(w')\,, \tag{15}$$
$$\alpha(\,(w \setminus w')\,) := \alpha(w) \setminus \alpha(w') \tag{16}$$

for $v \in \mathrm{dom}(\gamma) = \mathrm{dom}(\gamma_+)$, $u_l, u_r \in \mathrm{dom}(\nu_\mathbb{Q})$ such that $0 < \nu_\mathbb{Q}(u_l) < \nu_\mathbb{Q}(u_r)$ and $w, w' \in \mathrm{dom}(\alpha)$. Let $\alpha(x)$ be undefined for all other $x \in \Sigma^*$. Then α is a notation of $\mathcal{R}$ such that $\mathrm{dom}(\alpha)$ is recursive. Obviously, union and difference on $\mathcal{R}$ are (α, α, α)-computable.

Thus we have proved (v) and (vi) in Def. 2.2:

Proposition 4.10 *$\alpha :\subseteq \Sigma^* \to \mathcal{R}$ is a notation of $\mathcal{R}$ with recursive domain and $(A, B) \mapsto A \cup B$ and $(A, B) \mapsto A \setminus B$ are (α, α, α)-computable,*

Next, we define the function μ on $\mathcal{A} = \mathcal{A}(\mathcal{R})$. For finding a σ-additive measure we apply the non-effective theorem 4.1 since $\mathcal{R}_0 \subseteq \mathcal{F}$, $\mathcal{A}(\mathcal{R}) \subseteq \mathcal{A}(\mathcal{F})$.

Definition 4.11 Let μ' be the unique measure on $\mathcal{A}(\mathcal{F})$ such that f is μ'-integrable and $I(f) = \int f \, d\mu'$ for all $f \in \mathcal{F}$ (Thm. 4.1). Let μ be the restriction of μ' to $\mathcal{A}(\mathcal{R})$.

Since $\mathcal{A}(\mathcal{R})$ is a σ-algebra, μ is a measure. Therefore, (i), (ii), (v) and (vi) from Def. 2.2 are true. It remains to prove (iii) and (vii). From Prop. 4.7 we obtain:

Proposition 4.12 *For every $A \in \mathcal{R}$ and sequences (h_i) and (g_i) in $\mathcal{D}_+$ such that (13), $\int \chi_A \, d\mu = \mu(A) = \sup_i I(h_i) = \inf_i I(g_i)$. Furthermore, appropriate sequences (h_i) and (g_i) in $\mathcal{D}_+$ can be computed from A w.r.t. the notations α and $\nu_\to$.*

Proof: For all i we obtain: $I(h_i) = \int h_i \, d\mu' \le \int \chi_A \, d\mu' \le \int g_i \, d\mu' = I(g_i)$. Therefore, $\sup_i I(h_i) = \int \chi_A \, d\mu' = \mu'(A) = \mu(A)$. □(Prop. 4.12)

Using the functions G_1 and G_2 from Prop. 4.8 and the function L from Prop. 4.9 we prove that the measure μ is (α, ρ)-computable on $\mathcal{R}$.

Proposition 4.13 *The measure μ is (α, ρ)-computable on $\mathcal{R}$, in particular, $\mu(A) < \infty$ for all $A \in \mathcal{R}$.*

Thus we have proved Def. 2.2(iii) and (vii). Finally we prove Def. 2.2(iv).

Proposition 4.14 *The restriction of μ to $\mathcal{R}$ is σ-finite.*

Proof: Since $\mathcal{R}$ is countable and $\mu(A) < \infty$ for all $A \in \mathcal{R}$, it suffices to show $(\forall x \in \Omega)(\exists A \in \mathcal{R})\, x \in A$. Consider $x \in \Omega$. By assumption $f(x) \neq 0$ for some $f \in \mathcal{D}$. Then $|f| = \gamma_+(v) \in \mathcal{D}_+$ for some v and $|f|(x) > 0$. Therefore, there is

some $c \in D_v$ (see (12)) such that $|f|(x) > c$. Therefore, $x \in \{|f| > c\} \in \mathcal{R}$.
□(Prop. 4.14)

Altogether, we have a defined a computable measure space $\mathcal{M} = (\Omega, \mathcal{A}, \mu, \mathcal{R}, \alpha)$.

Finally, we consider integration. First, we generalize Prop. 4.12 from characterictic functions χ_A, $A \subseteq \mathcal{R}$ to rational linear combinations of such functions, i.e., rational step functions. A notation β for the rational step functions is defined in Def. 3.1.

Proposition 4.15 *For every rational step function t with non-negative coefficients and every $m \in \mathbb{N}$, functions $H, G \in \mathcal{D}_+$ can be computed (w.r.t. β and γ) such that $H \leq t \leq G$ and*

$$\int t\,d\mu - 2^{-m} \leq I(H) \leq \int t\,d\mu \leq I(G) \leq \int t\,d\mu + 2^{-m}\,.$$

Proof: Straightforward from Prop. 4.12 . □

Notice that a μ'-integrable function $f \in \mathcal{F}$ (see Def. 4.11) which is μ-measurable may be not μ-integrable. We prove the converse of Prop. 4.15.

Proposition 4.16 *For every function $f \in \mathcal{D}_+$ and every $n \in \mathbb{N}$ a rational step function t in $\mathcal{M} = (\Omega, \mathcal{A}, \mu, \mathcal{R}, \alpha)$ with non-negative coefficients can be computed w.r.t. the notations γ, $\nu_{\mathbb{N}}$ and β (from Def. 3.1) such that*

$$t \leq f \quad \textit{and} \quad 0 \leq I(f) - \int t\,d\mu \leq 2^{-n}\,.$$

Let $\mathcal{F}_+^*$ be the set of all $f : \Omega \to \overline{\mathbb{R}}$ such that $f_i \nearrow f$ for some sequence of functions in $\mathcal{F}_+$.

Define $I^* : \mathcal{F}_+^* \to \overline{\mathbb{R}}$ by

$$I^*(f) := \sup_i I(u_i) \;\text{ if }\; u_i \nearrow f\,.$$

In [1] p. 189 it is proved that I^* is well-defined (i.e., $\sup_i I(u_i) = \sup_i I(v_i)$ if $u_i \nearrow f$ and $v_i \nearrow f$) and that I^* extends I on $\mathcal{F}_+$ such that $I^*(af) = aI^*(f)$ $(a \geq 0)$, $I^*(f+g) = I^*(f) + I^*(g)$ $(f, g, \in \mathcal{F}_+^*)$ and $I^*(\sup_i f_i) = \sup_i I^*(f_i)$ if $f_i \nearrow f$ in $\mathcal{F}_+^*$.

For every $A \in \mathcal{R}$, there is a sequence $(h_i)_i$ in $\mathcal{D}_+$ such that $h_i \nearrow \chi_A$, hence by Prop. 4.12, $\int \chi_A\,d\mu = \mu(A) = I^*(\chi_A)$, therefore

$$\int t \, d\mu = I^*(t) \quad \text{for every non-negative rational step function} \quad t. \tag{17}$$

Now define the embedding $\psi : \mathcal{F}/_{\equiv} \to \mathcal{I}(\mathcal{M})/_{\equiv}$. First, we define $\psi(\overline{f})$ for $f \in \mathcal{F}_+$ by a $(\delta_{\mathcal{S}}, \delta_{\mathcal{M}})$-realization on names as follows.

Suppose $\delta_{\mathcal{S}}(p) = \overline{f}$. Then p encodes (γ-names of) elements $f_i \in \mathcal{D}_+$ such $I(|f - f_i|) \leq 2^{-i}$. By Prop 4.16, for each i a rational step function s_i can be computed such that $0 \leq s_i \leq f_{i+2}$ and $0 \leq I(f_{i+2}) - \int s_i \, d\mu \leq 2^{-i-2}$, and hence

$$\begin{aligned} 0 &\leq I^*(|f_{i+2} - s_i|) \\ &= I^*(f_{i+2}) - I^*(s_i) \\ &\leq 2^{-i-2}. \end{aligned}$$

Then for any $k > i$,

$$\begin{aligned} \int |s_i - s_k| d\mu &= I^*(|s_i - s_k|) \quad \text{ by (17)} \\ &\leq I^*(|s_i - f_{i+2}|) + I^*(|f_{i+2} - f|) \\ &\qquad + I^*(|f - f_{k+2}|) + I^*(|f_{k+2} - s_k|) \\ &\leq 2^{-i-2} + 2^{-i-2} + 2^{-k-2} + 2^{-k-2} \\ &\leq 2^{-i}. \end{aligned}$$

By Thm 15.5 in [1], the sequence (s_i) of rational step functions converges to some $h \in \mathcal{I}(\mathcal{M})$ such that $d_{\mathcal{S}}(s_i, h) \leq 2^{-i}$.

Define $\psi(\overline{f}) := \overline{h}$.

We show that ψ is well-defined on $\mathcal{F}_+$. Suppose $\overline{f} = \overline{g}$ and $\delta_{\mathcal{S}}(q) = \overline{g}$. The computation specified above gives a sequence $(g_i)_i$ of functions in $\mathcal{D}_+$ and a sequence $(t_i)_i$ of rational step functions such that

$$I(|g - g_i|) \leq 2^{-i}, \quad 0 \leq t_i \leq g_{i+2} \quad \text{and} \quad 0 \leq I(g_{i+2}) - \int t_i \, d\mu \leq 2^{-i-2}$$

and $d_{\mathcal{S}}(t_i, h') \leq 2^{-i}$ for some $h' \in \mathcal{I}(\mathcal{M})$. Therefore for all i,

$$\begin{aligned} d_{\mathcal{S}}(h, h') &\leq d_{\mathcal{S}}(h, s_i) + d_{\mathcal{S}}(s_i, t_i) + d_{\mathcal{S}}(t_i, h') \\ &\leq 2^{-i} + \int |s_i - t_i| \, d\mu + 2^{-i} \\ &= 2^{-i+1} + I^*(|s_i - t_i|) \\ &\leq 2^{-i+1} + I^*(|s_i - f_{i+2}| + |f_{i+2} - f| \\ &\qquad + |f - g| + |g - g_{i+2}| + |g_{i+2} - t_i|) \\ &\leq 2^{-i+1} + 2^{-i-2} + 2^{-i-2} + 0 + 2^{-i-2} + 2^{-i-2} \\ &\leq 2^{-i+2}, \end{aligned}$$

and hence, $\overline{h} = \overline{h'}$.

We extend ψ from $\mathcal{F}_+/_\equiv$ to $\mathcal{F}/_\equiv$. For $f = f_+ - f_-$, $(f_+, f_- \in \mathcal{F}_+)$, define

$$\psi(\overline{f}) := \psi(\overline{f}_+) - \psi(\overline{f}_-)\,.$$

The definition is sound since f_+ and f_- are uniquely defined.

We show that ψ is norm-preserving. Let $f = f_+ - f_- \in \mathcal{F}$. Let f_i^+, s_i^+, h^+ and f_i^-, s_i^-, h^- be the functions used in the computation of $\psi(\overline{f}_+)$ and $\psi(\overline{f}_-)$, respectively. Then

$$||\psi(\overline{f})|| = ||\psi(\overline{f}_+) - \psi(\overline{f}_-)|| = ||h^+ - h^-|| = I^*(|h^+ - h^-|)$$

and for all i,

$$\begin{aligned} h^+ - h^- &= (h^+ - s_i^+) + (s_i^+ - f_{i+2}^+) + (f_{i+2}^+ - f_+) \\ &\quad +(f_+ - f_-) \\ &\quad +(f_- - f_{i+2}^-) + (f_{i+2}^- - s_i^-) + (s_i^- - h^-) \\ &=: (f_+ - f_-) + v_i\,. \end{aligned}$$

Then $I^*(v_i) \leq 2^{-i+2}$. Since in general $|I^*(|g|) - I^*(|g+u|)| \leq I^*(|u|)$ we can conclude

$$I^*(|h^+ - h^-|) - I^*(|f_+ - f_-|)| \leq 2^{-i+2}$$

and therefore,

$$||\psi(\overline{f})|| = I^*(|h^+ - h^-|) = I^*(|f_+ - f_-|) = I^*(|f|) = ||f|| = ||\overline{f}||\,.$$

Similar considerations show that ψ is a linear mapping and that $I(f) = \int g d\mu$ for all $f \in \mathcal{F}$ and $g \in \psi(\overline{f})$.

This ends the proof of the computable Daniell-Stone Theorem.

The complete 6 pages longer version of this article is available from the authors. The authors want to thank the unknown referee for careful proofreading and valuable comments.

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