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s_2-C-continuous Poset

Chongxia Lu[1] Qingguo Li[2]

College of Mathematics and Econometrics
Hunan University
Changsha, Hunan, 410082, P.R. China

Abstract

In this paper, we introduce the concept of s_2-C-continuous poset by cut operator. The main results are: (1) A sup-semilattice is both s_2-C-continuous and s_2-continuous if and only if it is s_2-CD; (2) A sup-semilattice is both s_2-C-continuous and hypercontinuous if and only if it is s_2-CD; (3) A sup-semilattice is both s_2-QC-continuous and s_2-quasicontinuous if and only if it is s_2-GCD; (4) A sup-semilattice is both s_2-QC-continuous and quasi-hypercontinuous if and only if it is s_2-GCD; (5) A poset is s_2-C-continuous if and only if it is both s_2-MC-continuous and s_2-QC-continuous; (6) A poset is s_2-CD if and only if its order dual is s_2-CD; (7) A semi-lattice is s_2-GCD if and only if its order dual is hypercontinuous; (8) The lattice of all σ_2-closed subsets of a poset is C-continuous; (9) A poset P is s_2-continuous if and only if the lattice $C_2(P)$ of all σ_2-closed subsets of P is a continuous lattice if and only if $C_2(P)$ is a CD lattice; (10) A poset P is s_2-quasicontinuous if and only if the lattice $\sigma_2(P)$ of all σ_2-open subsets of P is a hypercontinuous lattice if and only if $C_2(P)$ is a GCD lattice if and only if $C_2(P)$ is a quasicontinuous lattice.

Keywords: s_2-C-continuous poset, s_2-QC-continuous poset, s_2-MC-continuous poset, s_2-CD poset, s_2-GCD poset.

1 Introduction

Domain theory was introduced by Dana Scott in the late sixties as models for the denotational semantics of programming languages, due to its strong background in computer science, general topology and topological algebra has been extensively studied in various areas. An important approach in the study of domains is to extend the theory of domains to that of posets as much as possible.

In domain theory, there are some categories which gained particularly wide attention, that is continuous lattice category, completely distributive (for short, CD) lattice category ([10]), Domain category, as well as hypercontinuous lattice category ([6]) etc. Recently, Ho and Zhao defined the binary relation $\prec$ by all Scott closed subsets instead of the directed subsets in the definition of the way below

[1] Email: lcx19882012@163.com (C. Lu)

[2] Email: liqingguoli@aliyun.com (Q. Li)

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relation, and proposed the concept of a C-continuous poset ([8]). The special case of the C-continuous poset is the C-continuous lattice. To know more about these research objects, the most important studying method is to generalize them. One of the generalizations of the continuous lattices is called quasicontinuous lattice ([5]), they were introduced by Gierz, Lawson and Stralka in 1983. The basic idea is to generalize the way below relation to that on the collection of all subsets of a complete lattice. Based on this idea: Yang and Xu introduced the concept of a generalized completely distributive (for short, GCD) lattice in 2010 ([11]); Xu gave the concepts of the quasi-hypercontinuous lattice in 2016; He, Xu and Yang provided the concept of a quasi-C-continuous (for short, QC-continuous) poset in 2016 ([7]). Another generalizations of the continuous lattices is called s_2-continuous poset (s_1-continuous poset) ([3]), they were proposed by Erné in 1981. The basic idea is by making use of the cut operator instead of joins in the definition of the way below relation. The notion of s_2-continuty admits to generalize most important characterizations of continuity from dcpos to arbitrary posets and has the advantage that the existence of directed joins is not even required. Afterwards Yao gave the concept of the $\mathcal{M}$-continuous in 2011 ([12]) and Zhang and Xu gave the concepts of the s_2-quasicontinuous in 2015 ([13]) and s_1-quasicontinuous poset in 2016 ([14]) in the manner of Erné. In addition, Mao and Xu introduced the concept of meet-C-continuous (for short, MC-continuous) poset by the semi-topological structure ([1]) and discussed the relation among C-continuous, QC-continuous and MC-continuous in 2016([9]).

In this paper, we will also generalize the binary relations $\prec$ and $\lhd$ by making use of the cut operator instead of joins, and define s_2-C-continuous (s_2-QC-continuous) and s_2-CD (s_2-GCD) posets, respectively. We obtain some characterization for the continuity of the poset: a poset is s_2-CD if and only if its order dual is s_2-CD; a semilattice is s_2-GCD if and only if its order dual is hypercontinuous, and a sup-semilattice is both s_2-C-continuous and s_2-continuous if and only if it is s_2-CD. We also prove that the lattice of all σ_2-closed subsets of a poset is C-continuous. In last section, we will propose the concept of s_2-meet-C-continuous (for short, s_2-MC-continuous) poset by semi-topological structure and give the characterization theorem that a poset is s_2-C-continuous if and only if it is both s_2-MC-continuous and s_2-QC-continuous.

2 Preliminaries

In this paper, the order dual of the poset P is written as P^{op}. For a poset P and for all $x \in P$, $A \subseteq P$, let $\uparrow x = \{y \in P \mid x \leq y\}$ and $\uparrow A = \bigcup_{a \in A} \uparrow a$; $\downarrow x$ and $\downarrow A$ are defined dually. A^l and A^u denote the sets of all upper and lower bounds of A are defined by

$$A^u = \{x \in P \mid a \leq x \textit{ for all } a \in A\} \quad \textit{and} \quad A^l = \{x \in P \mid x \leq a \textit{ for all } a \in A\},$$

respectively. Let $A^\delta = (A^u)^l$ be the cut of A. Further, A^u is an up-set and A^l is a down-set.

Lemma 2.1 [2] *Let P be a poset. For subsets A and B of P, we have*
(1) $A \subseteq A^\delta$;
(2) if $A \subseteq B$, then $A^u \supseteq B^u$ and $A^l \supseteq B^l$, which implies that $A^\delta \subseteq B^\delta$;
(3) $A^u = A^{ulu}$, i.e., $A^\delta = (A^\delta)^\delta$;
(4) $(\downarrow x)^\delta = \downarrow x$ for all $x \in P$;
(5) If $\sup A$ exists in P, then $A^\delta = \downarrow \sup A$.

Definition 2.2 [13] Let P be a poset. A subset $U \subseteq P$ is called σ_2-open if it satisfies:
(i) $U = \uparrow U$.
(ii) $D^\delta \cap U \neq \emptyset$ implies $D \cap U \neq \emptyset$ for all directed sets $D \subseteq P$.

The collection of all σ_2-open subsets of P forms a topology. It will be called σ_2-topology of P and will be denoted by $\sigma_2(P)$. The topology $\lambda_2(P) = \sigma_2(P) \vee \omega(P)$ is called the λ_2-topology on P. Obviously, $\upsilon(P) \subseteq \sigma_2(P) \subseteq \sigma(P)$.

The complements of σ_2-open sets of the poset P are the σ_2-*closed* sets. We use $C_2(P)$ and $C_2^*(P)$ to denote the set of all σ_2-closed sets and the set of all nonempty σ_2-closed sets of P, respectively. Thus a subset $S \subseteq P$ is σ_2-closed if and only if $S = \downarrow S$ and for all directed subsets $D \subseteq S$ implies $D^\delta \subseteq S$. Both $\sigma_2(P)$ and $C_2(P)$ are distributive complete lattices with respect to the inclusion order. For a poset P and $x \in P$, let $\mathcal{S}(x) = \{S \in C_2^*(P) \mid x \in S^\delta\}$.

Remark 2.3 (i) Let P be a poset. Then $\downarrow x, A^\delta \in C_2(P)$ for all $x \in P, A \subseteq P$.
(ii) If P is a dcpo, then σ_2-closed subsets are precisely Scott-closed. We use $C(P)$ to denote the set of all Scott-closed sets of P.

3 s_2-C-continuous posets

Definition 3.1 [13] Let P be a poset.
(i) The way-below relation $\ll_2$ on P is defined by $x \ll_2 y$ for all $x, y \in P$, if for all directed subsets $D \subseteq P$ with $y \in D^\delta$, which implies that $x \in \downarrow D$. The set $\{y \in P \mid y \ll_2 x\}$ will be denoted $\Downarrow x$ and $\{y \in P \mid x \ll_2 y\}$ denoted $\Uparrow x$.
(ii) P is called s_2-continuous if for all $x \in P$, the set $\Downarrow x$ is directed and $x = \sup \Downarrow x$.

Remark 3.2 Let P be a poset. Then $x \ll_2 z$ and $y \ll_2 z$ imply $x \vee y \ll_2 z$ whenever the least upper bound $x \vee y$ exists in P.

Proposition 3.3 *Let P be a poset. If $x \ll_2 z$ and if $z \in D^\delta$ for a directed subset $D \subseteq P$ which $\Downarrow d$ is directed and $d = \sup \Downarrow d$ for all $d \in D$, then $x \ll_2 d$ for some element $d \in D$, that is, $\bigcup \{\Downarrow z \mid z \in D^\delta\} = \bigcup \{\Downarrow d \mid d \in D\}$. Further, If P is s_2-continuous, then $\Uparrow x$ is σ_2-open for all $x \in P$.*

Proof. Assume that $x \notin \bigcup \{\Downarrow d \mid d \in D\}$. Let $I = \bigcup_{d \in D} \Downarrow d$. Then I is a directed subset of P and $I^\delta = (\bigcup_{d \in D} \Downarrow d)^\delta \supseteq \bigcup_{d \in D} (\Downarrow d)^\delta \supseteq D$, which implies that $D^\delta \subseteq (I^\delta)^\delta = I^\delta$. Since $x \ll_2 z$ and $z \in D^\delta$, so $x \in \downarrow I = I = \bigcup_{d \in D} \Downarrow d$, this is a contradiction. Hence $x \ll_2 d$ for some element $d \in D$. Further,

$\bigcup\{\Downarrow z \mid z \in D^\delta\} \subseteq \bigcup\{\Downarrow d \mid d \in D\}$. Since $D \subseteq D^\delta$ for a directed subset $D \subseteq P$, we have $\bigcup\{\Downarrow z \mid z \in D^\delta\} \supseteq \bigcup\{\Downarrow d \mid d \in D\}$. □

Definition 3.4 [8] Let P be a poset. For any two elements x and y in P, we write $x \prec y$, if for each nonempty Scott-closed subset $S \subseteq P$ for which $\sup S$ exists, $y \leq \sup S$ implies $x \in S$.

Given a poset P, we now define a new binary relation on P which is crucial for us to formulate the properties of lattices of σ_2-closed sets.

Definition 3.5 Let P be a poset. For any two elements x and y in P, we write $x \prec_2 y$, if for each nonempty σ_2-closed subset $S \subseteq P$, $y \in S^\delta$ implies $x \in S$. The set $\{y \in P \mid y \prec_2 x\}$ will be denoted $\downarrow^{\prec} x$ and $\{y \in P \mid x \prec_2 y\}$ denoted $\uparrow^{\prec} x$.

Remark 3.6 (i) If P is a poset, then $x \prec_2 y$ implies $x \prec y$ for all $x, y \in P$.
(ii) If P is a complete lattice, then $x \prec_2 y$ is equal to $x \prec y$ for all $x, y \in P$.
(iii) $x \prec_2 y$ if and only if $x \in \bigcap \mathcal{S}(y)$ for all $x, y \in P$.

Now it is routine to verify the following properties of the relation $\prec_2$.

Proposition 3.7 *Let P be a poset and $u, v, x, y \in P$. Then the following statements hold:*
(i) $x \prec_2 y$ implies $x \leq y$;
(ii) $u \leq x \prec_2 y \leq v$ implies $u \prec_2 v$;
(iii) if P has a smallest element 0, then $0 \prec_2 x$ always holds.

Proof. This follows immediately from Definition 3.5. □

Proposition 3.8 *Let P be a poset and D a directed subset of P. If $D \subseteq \downarrow^{\prec} x$, then $D^\delta \subseteq \downarrow^{\prec} x$.*

Proof. Suppose $S \in C_2^*(P)$ with $x \in S^\delta$. Since $d \prec_2 x$ for all $d \in D$, it follows that $D \subseteq S$. Because S is σ_2-closed and D is directed, we have $D^\delta \subseteq S$. Thus $D^\delta \subseteq \downarrow^{\prec} x$. □

Propositions 3.7 and 3.8 together imply the following corollary.

Corollary 3.9 *Let P be a poset. Then the set $\downarrow^{\prec} x$ is a σ_2-closed subset of P for all $x \in P$.*

Definition 3.10 Let P be a poset. P is called s_2-C-continuous if for all $x \in P$, $x = \sup \downarrow^{\prec} x$.

Remark 3.11 Notice that one of the requirements of a s_2-continuous poset P is that for every $x \in P$ the set $\Downarrow x$ is directed. In contrast, for any poset P, the set $\downarrow^{\prec} x$ is automatically σ_2-closed for all $x \in P$ by virtue of Corollary 3.9.

Lemma 3.12 *Let P be a poset. Then for any collection $\{S_i \mid i \in I\}$ of the nonempty σ_2-closed subsets of P, $\bigcap_{i \in I} S_i \neq \emptyset$ if and only if $(\bigcap_{i \in I} S_i)^\delta \neq \emptyset$.*

Proof. Since $\bigcap_{i\in I} S_i \subseteq (\bigcap_{i\in I} S_i)^\delta$, so $\bigcap_{i\in I} S_i \neq \emptyset$ implies $(\bigcap_{i\in I} S_i)^\delta \neq \emptyset$. Conversely, assume that $\bigcap_{i\in I} S_i = \emptyset$. Then P has no a smallest element, which implies that $(\bigcap_{i\in I} S_i)^\delta = P^l = \emptyset$, this is a a contradiction. □

The following proposition is similar to Theorem I-1.10 in [4].

Proposition 3.13 *For a poset P, the following conditions are equivalent:*
(1) P is s_2-C-continuous;
(2) for each $x \in P$, the set $\downarrow^{\prec} x$ is the smallest nonempty σ_2-closed set $S \subseteq P$ with $x \in S^\delta$;
(3) for each $x \in P$, there is a smallest nonempty σ_2-closed set $S \subseteq P$ with $x \in S^\delta$;
(4) for any collection $\{S_i \mid i \in I\}$ of σ_2-closed subsets of P, the following equation hold:

$$\bigcap_{i\in I} S_i^\delta = (\bigcap_{i\in I} S_i)^\delta.$$

Proof. (1) ⇒ (2): Condition (1) holds if and only if for each $x \in P$, $\downarrow^{\prec} x \in C_2^*(P)$ and $x \in (\downarrow^{\prec} x)^\delta$ by Definition 3.10. Since for any nonempty σ_2-closed set $S \subseteq P$ with $x \in S^\delta$, $\downarrow^{\prec} x \subseteq S$ by Definition 3.5, so (2) hold.

Condition (2) trivially implies (3).

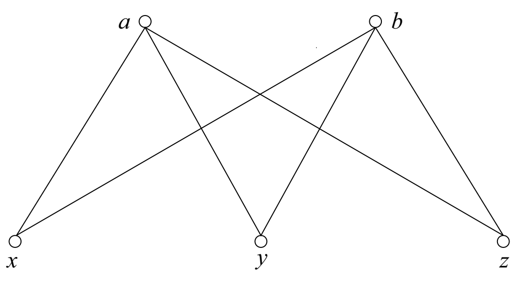
(3) ⇒ (1): For each $x \in P$, if $\mathcal{S}(x)$ has a smallest element S, then $S \subseteq \bigcap \mathcal{S}(x) \subseteq S$, and thus $S = \bigcap \mathcal{S}(x) = \downarrow^{\prec} x$ by Remark 3.6(iii). Since $x \in S^\delta$, we have $x = \sup \downarrow^{\prec} x$. Hence P is s_2-C-continuous.

(1) ⇒ (4): Since $\bigcap_{i\in I} S_i \subseteq S_i$ for all $i \in I$, we have $(\bigcap_{i\in I} S_i)^\delta \subseteq \bigcap_{i\in I} S_i^\delta$. For the reverse, suppose $x \in \bigcap_{i\in I} S_i^\delta$. Then $x \in S_i^\delta$, which implies that $\downarrow^{\prec} x \subseteq S_i$ for all $i \in I$, since S_i is σ_2-closed for all $i \in I$. Thus $\downarrow^{\prec} x \subseteq \bigcap_{i\in I} S_i$. Therefore, $x \in (\downarrow^{\prec} x)^\delta \subseteq (\bigcap_{i\in I} S_i)^\delta$, because P is s_2-C-continuous. Hence $(\bigcap_{i\in I} S_i)^\delta \supseteq \bigcap_{i\in I} S_i^\delta$.

(4) ⇒ (3): Suppose (4) hold. Then for each $x \in P$, $(\bigcap_{S\in\mathcal{S}(x)} S)^\delta = \bigcap_{S\in\mathcal{S}(x)} S^\delta$. Therefore $x \in (\bigcap_{S\in\mathcal{S}(x)} S)^\delta$, i.e., $(\bigcap_{S\in\mathcal{S}(x)} S)^\delta \neq \emptyset$ and $\bigcap_{S\in\mathcal{S}(x)} S \in \mathcal{S}(x)$ by Lemma 3.12. Hence $\bigcap_{S\in\mathcal{S}(x)} S$ a smallest element $\mathcal{S}(x)$. □

The following example illustrates that s_2-continuous and s_2-C-continuous can be quite different.

Example 3.14 (1) Consider the poset $P = \{a, b, x, y, z\}$, where the order as defined by $x, y, z < a, b$. Since P is a finite poset, so P is a s_2-continuous. Let $S_1 = \{x\}$ and $S_2 = \{y, z\}$. Then S_1 and S_2 are σ_2-closed by Remark 2.3(i), and $(S_1 \cap S_2)^\delta = (\{x\} \cap \{y, z\})^\delta = (\emptyset)^\delta = \emptyset$, $S_1^\delta \cap S_2^\delta = \{x\}^\delta \cap \{y, z\}^\delta = \{x\} \cap \{x, y, z\} = \{x\} \neq \emptyset$. Thus $(S_1 \cap S_2)^\delta \neq S_1^\delta \cap S_2^\delta$, hence P is not s_2-C-continuous by Proposition 3.13.



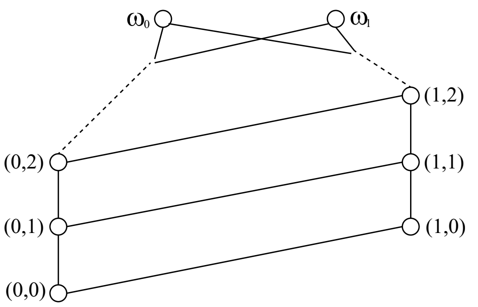
(2) Consider the poset $P = \{(m,n) \mid m \in \{0,1\} \textit{ and } n \in \mathbb{N}\} \cup \{\omega_0, \omega_1\}$ with ordering defined by $(m_1,n_1) \leq (m_2,n_2)$ if and only if $m_1 \leq m_2$ and $n_1 \leq n_2$ for all $m_1, m_2 \in \{0,1\}$, $n_1, n_2 \in \mathbb{N}$, and $(m,n) \leq \omega_0, \omega_1$ for all $m \in \{0,1\}$, $n \in \mathbb{N}$. It is easy to prove that

(i) $\downarrow^{\prec}(0,n) = \{(0,n') \mid n' \leq n\} = \Downarrow(0,n)$ for all $n \in \mathbb{N}$;

(ii) $\downarrow^{\prec}(1,n) = \{(0,n') \mid n' \leq n\} \cup \{(1,0)\}$, $\Downarrow(1,n) = \{(0,n') \mid n' \leq n\}$ for all $n \in \mathbb{N}$;

(iii) $\downarrow^{\prec}\omega_i = \downarrow\omega_i = \Downarrow\omega_i$, $i = 0, 1$.

Hence P is s_2-C-continuous, but it is not s_2-continuous.



Proposition 3.15 *Let P be a s_2-C-continuous poset. Then for any collection $\{F_i \mid i \in I\}$ of finite subsets of P, the following equation holds:*

$$\bigcap_{i\in I} F_i^{\delta} = \left(\bigcap_{f\in\prod_{i\in I} F_i} \{f(i) \mid i \in I\}^{lu}\right)^{l}.$$

Proof. For each $i \in I$ and any $f \in \prod_{i\in I} F_i$, $f(i) \in F_i$ implies $\downarrow f(i) \subseteq \downarrow F_i$. Thus $\{f(i) \mid i \in I\}^{l} = \bigcap_{i\in I} \downarrow f(i) \subseteq \bigcap_{i\in I} \downarrow F_i$, which implies that $\{f(i)|i \in I\}^{lu} \supseteq \left(\bigcap_{i\in I} \downarrow F_i\right)^{u}$. Therefore,

$$\bigcap_{f\in\prod_{i\in I} F_i} \{f(i) \mid i \in I\}^{lu} \supseteq \left(\bigcap_{i\in I} \downarrow F_i\right)^{u},$$

that is,

$$\left(\bigcap_{f\in\prod_{i\in I} F_i} \{f(i) \mid i \in I\}^{lu}\right)^{l} \subseteq \left(\bigcap_{i\in I} \downarrow F_i\right)^{\delta}.$$

Since F_i is finite for all $i \in I$, we have $\downarrow F_i$ is σ_2-closed for all $i \in I$ by Remark 3.6. Hence by Proposition 3.13, we know that

$$\left(\bigcap_{f\in\prod_{i\in I} F_i} \{f(i) \mid i \in I\}^{lu}\right)^{l} \subseteq \bigcap_{i\in I} (\downarrow F_i)^{\delta} = \bigcap_{i\in I} F_i^{\delta}.$$

Conversely, suppose $x \in \bigcap_{i\in I} F_i^\delta = \bigcap_{i\in I}(\downarrow F_i)^\delta$ and $u \prec_2 x$. For each $i \in I$, the set $\downarrow F_i$ is σ_2-closed for all $i \in I$ by Remark 3.6, since F_i is finite for all $i \in I$, so there exists $y_i \in F_i$ such that $u \le y_i$. Let $f \in \prod_{i\in I} F_i$ be defined by $f(i) = y_i$ for all $i \in I$. Then $u \in \{f(i) \mid i \in I\}^l$, which implies that $\uparrow u \supseteq \{f(i) \mid i \in I\}^{lu}$, that is, $(\downarrow^{\prec} x)^u \supseteq \bigcap_{f\in\prod_{i\in I} F_i} \{f(i) \mid i \in I\}^{lu}$. Therefore,

$$(\downarrow^{\prec} x)^\delta \subseteq \left(\bigcap_{f\in\prod_{i\in I} F_i} \{f(i) \mid i \in I\}^{lu}\right)^l.$$

Since P is s_2-C-continuous, so $x \in (\downarrow^{\prec} x)^\delta$ for all $x \in P$. Hence

$$\left(\bigcap_{f\in\prod_{i\in I} F_i} \{f(i) \mid i \in I\}^{lu}\right)^l \supseteq \bigcap_{i\in I} F_i^\delta.$$

□

Remark 3.16 Let P be a s_2-C-continuous poset which is also a complete lattice. Then for any collection $\{F_i \mid i \in I\}$ of finite subsets of P, the following equation holds:

$$\bigwedge_{i\in I} \bigvee F_i = \bigvee_{f\in\prod_{i\in I} F_i} \bigwedge_{i\in I} f(i).$$

Definition 3.17 Let P be a poset.
(i) The binary relation $\lhd_2$ on P is defined by $x \lhd_2 y$ for all $x, y \in P$, if for all subsets $A \subseteq P$ with $y \in A^\delta$, which implies that $x \in \downarrow A$. The set $\{y \in P \mid y \lhd_2 x\}$ will be denoted $\downarrow^{\lhd} x$ and $\{y \in P \mid x \lhd_2 y\}$ denoted $\uparrow^{\lhd} x$.
(ii) P is called s_2-completely distributive (for short, s_2-CD) poset, if for all $x \in P$, $x = \sup \downarrow^{\lhd} x$.

Remark 3.18 (i) $y \lhd_2 x$ if and only if $x \notin (P \backslash \uparrow y)^\delta$, for all $x, y \in P$;
(ii) If P is a s_2-CD poset which is also a complete lattice, then P is a CD lattice.

Theorem 3.19 *Let P be a sup-semilattice. Then the following are equivalent:*
(1) P is s_2-C-continuous and s_2-continuous;
(2) P is s_2-CD.

Proof. (2) ⇒ (1) Follows immediately from Definitions 3.1, 3.10, 3.17 and Remark 3.2. For (1) ⇒ (2): Suppose that P is both s_2-C-continuous and s_2-continuous. Since P is s_2-continuous, for each $x \in P$, $x = \sup\{y \in P \mid y \ll_2 x\}$. Now for each $y \ll_2 x$, $y = \sup\{z \in P \mid z \prec_2 y\}$ because P is s_2-C-continuous. It follows that

$$x = \sup\{z \in P \mid \textit{ there exists } y \in P \textit{ such that } z \prec_2 y \ll_2 x\}.$$

Next, suppose $z \prec_2 y \ll_2 x$, we shall show that $z \lhd_2 x$. Let $A \subseteq P$ with $x \in A^\delta$. Construct the set $D = \{\sup F \mid F\ is\ a\ finite\ subset\ of\ A\}$. Then D is a directed set and $x \in A^\delta = D^\delta$. Since $y \ll_2 x$, so there is a finite subset $F \subseteq A$ such that $y \in \downarrow \sup F = (\downarrow F)^\delta$. Note that the last set $\downarrow F$ is σ_2-closed. So it follows from $z \prec_2 y$ that $z \in \downarrow F \subseteq \downarrow A$, this implies that $z \lhd_2 x$. Hence P is s_2-CD. □

Proposition 3.20 *Let P be a poset and $\mathcal{S} \in C(C_2(P))$. Then $\bigvee_{C_2(P)} \mathcal{S} = \bigcup \mathcal{S}$.*

Proof. Note that each member of $\mathcal{S}$ is a σ_2-closed subset of P. So to prove the equation, it suffices to show that

$$\bigcup \mathcal{S} \in C_2(P).$$

Obviously $\bigcup \mathcal{S}$ is a lower subset of P. Now let D be any directed subset of P with $D \subseteq \bigcup \mathcal{S}$. We want to prove that $D^\delta \in \mathcal{S}$. Since $\mathcal{D} = \{\downarrow d \mid d \in D\}$ is a directed subset of $C_2(P)$. Moreover, $\mathcal{D} \subseteq \mathcal{S}$ because $\mathcal{S}$ is a lower set in $C_2(P)$. Since $\mathcal{S}$ is a Scott-closed set of $C_2(P)$, we have $\bigvee_{C_2(P)} \mathcal{D} \in \mathcal{S}$. But $\bigvee_{C_2(P)} \mathcal{D}$ is precisely D^δ by Remark 2.3(i). Hence $D^\delta \in \mathcal{S}$. $\square$

Definition 3.21 (i) An element x of a poset P is called C-compact if $x \prec x$. We use $KC(P)$ to denote the set of all the C-compact elements of P.
(ii) An element x of a poset P is called s_2-C-compact if $x \prec_2 x$. We use $KC_2(P)$ to denote the set of all the s_2-C-compact elements of P.

Remark 3.22 If P is a complete lattice, then $KC(P) = KC_2(P)$.

Recall that an element $q \neq 0$ of a poset P is called co-prime if $P \backslash \uparrow q$ is directed in P.

Proposition 3.23 *Let P be a poset. If $k \in KC_2(P)$, then k is co-prime.*

Proof. Assume that k is not co-prime. Then there exist $u, v \in P \backslash \uparrow k$ such that $\{u, v\}^u \subseteq \uparrow k$, this implies that $k \in \{u, v\}^\delta$. Let $S = \downarrow u \cup \downarrow v$. Then $S \in C_2(P)$ by Remark 2.3(i) and $S \subseteq P \backslash \uparrow k$ with $S^\delta = \{u, v\}^\delta$. Since $k \in KC_2(P)$, we have $k \in S$ implies $k \in P \backslash \uparrow k$, this is a contradiction. Hence k is co-prime. $\square$

Proposition 3.24 *Let P be a poset and $S_0 \in C_2^*(P)$. Then for each $x \in S_0$, $\downarrow x \prec S_0$ holds in $C_2(P)$.*

Proof. Let $x \in S_0$. Suppose $\mathcal{S} \in C(C_2(P))$ with $S_0 \subseteq \bigvee_{C_2(P)} \mathcal{S}$. Then by Proposition 3.20, $S_0 \subseteq \bigcup \mathcal{S}$. Hence there exists $S \in \mathcal{S}$ such that $x \in S$. Therefore, $\downarrow x \subseteq S$, and thus $\downarrow x \in \mathcal{S}$. $\square$

Corollary 3.25 *Let P be a poset. Then for each $x \in P$, it holds that $\downarrow x \in KC(C_2(P))$*

Proof. Since $x \in \downarrow x$ and $\downarrow x \in C_2^*(P)$, so $\downarrow x \prec \downarrow x$, i.e., $\downarrow x \in KC(C_2(P))$ for all $x \in P$ by Proposition 3.24. $\square$

Definition 3.26 A poset P is said to be s_2-C-prealgebraic if for each $x \in P$,

$$x = \sup\{k \in KC_2(P) \mid k \leq x\}.$$

A s_2-C-prealgebraic poset P is s_2-C-algebraic if for any $x \in P$,

$$\downarrow \{k \in KC_2(P) \mid k \leq x\} \in C_2(P).$$

Obviously every s_2-C-prealgebraic poset is s_2-C-continuous.

Proposition 3.27 *For any poset P, the lattice $C_2(P)$ is s_2-C-prealgebraic. Hence the lattice $C_2(P)$ is s_2-C-continuous, i.e., $C_2(P)$ is C-continuous.*

Proof. This follows from Corollary 3.25 and the fact $S = \bigvee_{C_2(P)}\{\downarrow x \mid x \in S\}$ holds for every $S \in C_2(P)$. □

It is well-known that a poset P is continuous if and only if $C(P)$ is completely distributive. From Theorem 3.19, we obtain the following theorem.

Theorem 3.28 *For any poset P, the following statements are equivalent:*
(1) P is a s_2-continuous;
(2) $C_2(P)$ is a continuous lattice;
(3) $C_2(P)$ is a completely distributive lattice.

Lemma 3.29 *Let P be a poset. Then for any $x \in P$ and $H \subseteq P$, $x \notin (P\backslash\uparrow H)^\delta$ if and only if there exists $u \in P$ such that $x \notin \downarrow u$ and $\downarrow u \cup \uparrow H = P$.*

Proof. Suppose $x \notin (P\backslash\uparrow H)^\delta$. Then $(P\backslash\uparrow H)^u \neq \emptyset$ and there is $u \in (P\backslash\uparrow H)^u$ such that $x \notin \downarrow u$, which implies that $\downarrow u \cup \uparrow H = P$. Conversely, if $\downarrow u \cup \uparrow H = P$, then $(P\backslash\uparrow H)^\delta \subseteq (\downarrow u)^\delta = \downarrow u$. □

Proposition 3.30 *A poset P is s_2-CD if and only if for any $x, y \in P$ with $x \not\leq y$, there exists $v \in P$ such that $y \notin \uparrow v$ and $x \notin (P\backslash\uparrow v)^\delta$.*

Proof. Suppose P is s_2-CD and $x, y \in P$ with $x \not\leq y$. Then there exists $v \in P$ such that $y \notin \uparrow v$ and $v \lhd_2 x$. Thus $x \notin (P\backslash\uparrow v)^\delta$ by Remark 3.18(i).

Conversely, we now need to prove that $x = \sup\downarrow^{\lhd} x$ for all $x \in P$. Suppose there exists $x \in P$ such that $x \neq \sup\downarrow^{\lhd} x$, i.e., there is $y \in (\downarrow^{\lhd} x)^u$ such that $x \not\leq y$. Then there exists $v \in P$ such that $y \notin \uparrow v$ and $x \notin (P\backslash\uparrow v)^\delta$, i.e., $v \lhd_2 x$. Thus $y \in \uparrow v$, this is a contradiction. □

Proposition 3.30 and Lemma 3.29 together imply the following corollary.

Corollary 3.31 *A poset P is s_2-CD if and only if for any $x, y \in P$ with $x \not\leq y$, there exist $u, v \in P$ such that $x \notin \downarrow u, y \notin \uparrow v$ and $\downarrow u \cup \uparrow v = P$. Further, a poset P is s_2-CD if and only if so is P^{op}.*

For a poset P, we define $x \preccurlyeq y$ if and only if $y \in \mathrm{int}_\upsilon \uparrow x$. A poset P is called *hypercontinuous* if and only if for all $x \in P$, the set $\downarrow^{\preccurlyeq} x = \{y \in P \mid y \preccurlyeq x\}$ is directed and $x = \sup\downarrow^{\preccurlyeq} x$.

Proposition 3.32 *Let P be a poset. The following statements are equivalent:*
(1) P is hypercontinuous;
(2) P is s_2-continuous in which $y \ll_2 x$ if and only if $y \preccurlyeq x$.

Proof. Suppose P is a hypercontinuous poset and $y \ll_2 x$. Then there exists $z \in P$ such that $y \leq z \preccurlyeq x$. If $y \preccurlyeq x$, then $x \in \mathrm{int}_\upsilon \uparrow y \subseteq \mathrm{int}_{\sigma_2} \uparrow y$, i.e., $y \ll_2 x$. Hence P is s_2-continuous.

Conversely if (2) holds, then $\preccurlyeq$ is approximating since $\ll_2$ is. Hence P is hypercontinuous. □

Proposition 3.33 *Let P is a hypercontinuous poset. Then for any $x, y \in P$ with $x \not\leq y$, there exist $u \in P$ and a finite subset F of P such that $y \notin \uparrow u, x \notin \downarrow F$ and $\uparrow u \cup \downarrow F = P$.*

Proof. Suppose P is hypercontinuous and $x, y \in P$ with $x \not\leq y$. Then there exists $u \in P$ such that $y \notin \uparrow u$ and $u \preccurlyeq x$. Since $u \preccurlyeq x$ if and only if $x \in \mathrm{int}_\upsilon \uparrow u$, so there is a finite subset $F \subseteq P$ such that $x \in P \backslash \downarrow F \subseteq \uparrow u$, i.e., $x \notin \downarrow F$ and $\uparrow u \cup \downarrow F = P$.□

Proposition 3.34 *A sup-semilattice P is hypercontinuous if and only if for any $x, y \in P$ with $x \not\leq y$, there exist $u \in P$ and a finite subset F of P such that $y \notin \uparrow u, x \notin \downarrow F$ and $\uparrow u \cup \downarrow F = P$.*

Proof. If P is hypercontinuous, it is obvious that for any $x, y \in L$ with $x \not\leq y$, there exist $u \in P$ and a finite subset F of P such that $y \notin \uparrow u, x \notin \downarrow F$ and $\uparrow u \cup \downarrow F = P$ by Proposition 3.33.

Conversely, for any fixed $x \in P$, if $x \leq y$ for all $y \in P$, then x is the least element of P, and thus $x \preccurlyeq x$; If x is not the least element of P, then there exists $y \in P$ such that $x \not\leq y$, thus there exist $u \in P$ and a finite subset F of P such that $y \notin \uparrow u, x \notin \downarrow F$ and $\uparrow u \cup \downarrow F = P$, implies $x \in P \backslash \downarrow F \subseteq \uparrow u$, i.e., $u \preccurlyeq x$. Hence $\downarrow^{\preccurlyeq} x \neq \emptyset$ for all $x \in P$. Since P is a sup-semilattice, we have $\downarrow^{\preccurlyeq} x$ is directed for all $x \in P$.

We now need to prove that $x = \sup \downarrow^{\preccurlyeq} x$ for all $x \in P$. Suppose there exists $x \in P$ such that $x \neq \sup \downarrow^{\preccurlyeq} x$, i.e., there is $y \in (\downarrow^{\preccurlyeq} x)^u$, such that $x \not\leq y$. Then there exist $u \in P$ and a finite subset F of P such that $y \notin \uparrow u, x \notin \downarrow F$ and $\uparrow u \cup \downarrow F = P$, i.e., $x \in P \backslash \downarrow F \subseteq \uparrow u$ implies $u \preccurlyeq x$. Thus $y \in \uparrow u$, this is a contradiction. □

From Theorem 3.19, Propositions 3.32, 3.34 and Corollary 3.31 we have the following.

Theorem 3.35 *Let P be a sup-semilattice. Then*
(1) P is s_2-CD implies P is hypercontinuous.
(2) P is s_2-C-continuous and hypercontinuous if and only if P is s_2-CD.

4 s_2-QC-continuous posets

For a set X, we use $\mathcal{P}(X)$ to denote the power set of X. We consider the order between subsets G, H of a poset P by $G \leq H$ if $H \subseteq \uparrow G$. This implies that a family $\mathcal{F}$ of subsets is *directed* if the corresponding family $\{\uparrow F \mid F \in \mathcal{F}\}$ is a filter base. Generalizing the way below relation $\ll_2$ on points of a poset P to the nonempty subsets of P, one obtains the following concept of s_2-quasicontinuous poset.

Definition 4.1 [13] Let P be a poset.
(i) The way-below relation $\ll_2$ on $\mathcal{P}(P) \backslash \{\emptyset\}$ is defined by $G \ll_2 H$ for all $G, H \subseteq P$, if for all directed subsets $D \subseteq P$ with $\uparrow H \cap D^\delta \neq \emptyset$, which implies that $\uparrow G \cap D \neq \emptyset$. We write $G \ll_2 x$ for $G \ll_2 \{x\}$. The set $\{x \in P \mid H \ll_2 x\}$ will be denoted $\Uparrow H$.
(ii) P is called s_2-quasicontinuous if for all $x \in P$, the family

$$Q_{fin}(x) = \{F \subseteq P \mid F \text{ } is \text{ } finite \text{ } and \text{ } F \ll_2 x\}$$

is a directed family and whenever $x \not\leq y$, there exists a finite subset $F \in Q_{fin}(x)$ with $y \notin \uparrow F$, i.e., $\uparrow x = \bigcap\{\uparrow F \mid F \in Q_{fin}(x)\}$.
(iii) P is called s_2-quasialgebraic if for all $x \in P$, the family

$$KQ_{fin}(x) = \{F \subseteq P \mid F \; is \; finite, \; F \ll_2 F \; and \; x \in \uparrow F\}$$

is a directed family and whenever $x \not\leq y$, there exists $F \in KQ_{fin}(x)$ with $y \notin \uparrow F$, i.e., $\uparrow x = \bigcap\{\uparrow F \mid F \in KQ_{fin}(x)\}$.

Remark 4.2 Let P be a sup-semilattice. Then $F_1 \ll_2 x$ and $F_2 \ll_2 x$ imply $F_1 \vee F_2 \ll_2 x$, where F_1, F_2 are the nonempty finite subsets of P and $F_1 \vee F_2 = \{y_1 \vee y_2 \mid y_i \in F_i, \; i = 1, 2\}$.

Lemma 4.3 *[13] Let $\mathcal{F}$ be a directed family of nonempty finite sets in a poset. If $H \ll_2 x$ and $\bigcap_{F \in \mathcal{F}} \uparrow F \subseteq \uparrow x$, then $F \subseteq \uparrow H$ for some $F \in \mathcal{F}$.*

Proposition 4.4 *Let P be a poset. If $H \ll_2 x$ and if $x \in D^\delta$ for a directed subset $D \in P$ which $Q_{fin}(d)$ is directed and $\uparrow d = \bigcap\{\uparrow F \mid F \in Q_{fin}(d)\}$ for all $d \in D$, then $H \ll_2 d$ for some element $d \in D$. Further, If P is s_2-quasicontinuous, then $\Uparrow H$ is σ_2-open for all $H \subseteq P$.*

Proof. Let $\mathcal{F} = \bigcup\{Q_{fin}(d) \mid d \in D\}$. Then for any $F_1, F_2 \in \mathcal{F}$, there exist $d_1, d_2 \in D$ such that $F_i \ll_2 d_i, i = 1, 2$. Since D is directed, so there is $d \in D$ such that $F_1, F_2 \ll_2 d$, i.e., $F_1, F_2 \in Q_{fin}(d)$, which implies that $F \subseteq \uparrow F_1 \cap \uparrow F_2$ for some $F \in Q_{fin}(d)$. Thus $\mathcal{F}$ is a directed family. Since $\bigcap_{F \in \mathcal{F}} \uparrow F = \bigcap_{d \in D} \uparrow d = D^u \subseteq \uparrow x$ and $H \ll_2 x$, so there exists $F \in \mathcal{F}$ such that $F \subseteq \uparrow H$ by Lemma 4.3. For F, there is $d \in D$ such that $F \ll_2 d$, and thus $H \ll_2 d$. □

Definition 4.5 [7] Let P be a poset. For any two subsets G and H of P, we write $G \prec H$, if for each nonempty Scott-closed subset $S \subseteq P$ for which $\sup S$ exists, $\sup S \in \uparrow H$ implies $S \cap \uparrow G \neq \emptyset$. We write $G \prec x$ for $G \prec \{x\}$.

Given a poset P, we now define a new binary relation on $\mathcal{P}(P)$ which is crucial for us to formulate the properties of lattices of σ_2-closed sets.

Definition 4.6 Let P be a poset. For any two subsets G and H of P, we write $G \prec_2 H$, if for each nonempty σ_2-closed subset $S \subseteq P$, $S^\delta \cap \uparrow H \neq \emptyset$ implies $S \cap \uparrow G \neq \emptyset$. We write $G \prec_2 x$ for $G \prec_2 \{x\}$ and $y \prec_2 H$ for $\{y\} \prec_2 H$. The set $\{x \in P \mid H \prec_2 x\}$ will be denoted $\uparrow^{\prec} H$.

Remark 4.7 (i) If P is a poset, then $G \prec_2 H$ implies $G \prec H$ for all $G, H \subseteq P$.
(ii) If P is a complete lattice, then $G \prec_2 H$ is equal to $G \prec H$ for all $G, H \subseteq P$.

The next proposition is basic and the proof is omitted.

Proposition 4.8 *Let P be a poset and $E, F, G, H \subseteq P$. Then the following statements hold:*
(1) $G \prec_2 H$ implies $G \leq H$;
(2) $G \prec_2 H$ if and only if $G \prec_2 h$ for all $h \in H$;
(3) $E \leq G \prec_2 H \leq F$ implies $E \prec_2 F$;

(4) $\{x\}\prec_2\{y\}$ *if and only if* $x\prec_2 y$*;*
(5) if P *has a smallest element* 0*, then* $0\prec_2 H$ *always holds.*

Definition 4.9 Let P be a poset.
(i) P is called s_2-quasi-C-continuous (for short, s_2-QC-continuous), if for all $x\in P$,

$$\uparrow x=\bigcap\{\uparrow F\mid F\in C_{fin}(x)\},$$

where $C_{fin}(x)=\{F\subseteq P\mid F\ is\ finite\ and\ F\prec_2 x\}$;
(ii) P is called s_2-quasi-C-prealgebraic (for short, s_2-QC-prealgebraic), if for all $x\in P$,

$$\uparrow x=\bigcap\{\uparrow F\mid F\in KC_{fin}(x)\},$$

where $KC_{fin}(x)=\{F\subseteq P\mid F\ is\ finite,\ F\prec_2 F\ and\ x\in\uparrow F\}$.

Proposition 4.10 *Every* s_2*-C-continuous (resp.,* s_2*-C-prealgebraic) poset is a* s_2*-QC-continuous (resp.,* s_2*-QC-prealgebraic) poset.*

Proof. Suppose P is a s_2-C-continuous poset and $x\in P$. Then $\{\{y\}\subseteq P\mid y\prec_2 x\}\subseteq C_{fin}(x)$ and $\bigcap\{\uparrow y\mid y\prec_2 x\}=\uparrow x$. Therefore, $\uparrow x\subseteq\bigcap\{\uparrow F\mid F\in C_{fin}(x)\}\subseteq\bigcap\{\uparrow y\mid y\prec_2 x\}=\uparrow x$. Hence P is s_2-QC-continuous. The s_2-C-prealgebraic case can be similarly proved. □

Definition 4.11 Let P be a poset.
(i) The binary relation $\lhd_2$ on $\mathcal{P}(P)$ is defined by $G\lhd_2 H$ for all $G,H\subseteq P$, if for all subsets $A\subseteq P$ with $A^\delta\cap\uparrow H\neq\emptyset$, which implies that $\uparrow G\cap A\neq\emptyset$. We write $G\lhd_2 x$ for $G\lhd_2\{x\}$ and $y\lhd_2 H$ for $\{y\}\lhd_2 H$. The set $\{x\in P\mid H\lhd_2 x\}$ will be denoted $\uparrow^{\lhd}H$.
(ii) P is called s_2-generalized completely distributive (for short, s_2-GCD), if for all $x\in P$,

$$\uparrow x=\bigcap\{\uparrow F\mid F\in G_{fin}(x)\},$$

where $G_{fin}(x)=\{F\subseteq P\mid F\ is\ finite\ and\ F\lhd_2 x\}$;
(iii) P is called s_2-strongly pseudoalgebraic (for short, s_2-SPA), if for all $x\in P$,

$$\uparrow x=\bigcap\{\uparrow F\mid F\in KG_{fin}(x)\},$$

where $KG_{fin}(x)=\{F\subseteq P\mid F\ is\ finite,\ F\lhd_2 F\ and\ x\in\uparrow F\}$.

Remark 4.12 (i) $G\lhd_2 H$ if and only if $\uparrow H\cap(P\backslash\uparrow G)^\delta=\emptyset$, for all $G,H\subseteq P$;
(ii) If P is a s_2-GCD poset which is also a complete lattice, then P is a GCD lattice ([11]).

Theorem 4.13 *Let* P *be a sup-semilattice. Then the following are equivalent:*
(1) P *is* s_2*-QC-continuous and* s_2*-quasicontinuous;*
(2) P *is* s_2*-GCD.*

Proof. (2) $\Rightarrow$ (1) follows immediately from Definitions 4.1, 4.9, 4.11 and Remark 4.2. For (1) $\Rightarrow$ (2): Suppose that P is both s_2-QC-continuous and s_2-quasicontinuous. Since P is s_2-quasicontinuous, for each $x\in P$, $\uparrow x=\bigcap\{\uparrow F\mid F\in$

$Q_{fin}(x)\}$. Now for each $F \in Q_{fin}(x)$,

$$\uparrow F = \bigcup_{x\in F} \bigcap\{\uparrow E_x \mid E_x \in C_{fin}(x)\} = \bigcap\{\uparrow(\bigcup_{x\in F} E_x) \mid E_x \in C_{fin}(x) \ and \ x \in F\}$$

because P is s_2-QC-continuous. By Proposition 4.8, we have $\{\bigcup_{x\in F} E_x \mid E_x \in C_{fin}(x) \ and \ x \in F\} \subseteq \{E \mid E \ is \ finit \ and \ E \prec_2 F\}$. Therefore, $\uparrow F = \bigcap\{\uparrow E \mid E \ is \ finit \ and \ E \prec_2 F\}$. It follows that

$$\uparrow x = \bigcap\{\uparrow E \mid \ there \ exists \ F \in Q_{fin}(x) \ such \ that \ E \prec_2 F \ll_2 x\}.$$

Next, suppose $E \prec_2 F \ll_2 x$, we shall show that $E \lhd_2 x$. Let $A \subseteq P$ with $x \in A^\delta$. Construct the set $D = \{\sup G \mid G \ is \ a \ finite \ subset \ of \ A\}$. Then D is a directed set and $x \in A^\delta = D^\delta$. Since $F \ll_2 x$, so there is a finite subset $G \subseteq A$ such that $\sup G \subseteq \uparrow F$, i.e., $(\downarrow G)^\delta \cap \uparrow F \neq \emptyset$. Note that the last set $\downarrow G$ is σ_2-closed. So it follows from $E \prec_2 F$ that $\uparrow E \cap \downarrow G \neq \emptyset$, i.e., $\uparrow E \cap A \supseteq \uparrow E \cap G \neq \emptyset$, this implies that $E \lhd_2 x$. Hence P is s_2-GCD. □

For the algebraic case, similarly, we have the following statements:

Theorem 4.14 *Let P be a sup-semilattice. Then the following are equivalent:*
(1) P is s_2-QC-prealgebraic and s_2-quasialgebraic;
(2) P is s_2-SPA.

Proposition 4.15 *A poset P is s_2-GCD if and only if for any $x, y \in P$ with $x \nleq y$, there exists a finite subset F of P such that $y \notin \uparrow F$ and $x \notin (P \backslash \uparrow F)^\delta$.*

Proof. Suppose P is s_2-GCD and $x, y \in P$ with $x \nleq y$. Then there exists a finite subset $F \subseteq P$ such that $y \notin \uparrow F$ and $F \lhd_2 x$. Thus $x \notin (P \backslash \uparrow F)^\delta$ by Remark 4.12(i).

Conversely, for any fixed $x \in P$, if $x \leq y$ for all $y \in P$, then x is the least element of P, and thus $\{x\} \in G_{fin}(x)$; If x is not the least element of P, then there exists $y \in P$ such that $x \nleq y$, thus there exists a finite subset F of P such that $y \notin \uparrow F$ and $x \notin (P \backslash \uparrow F)^\delta$, implies $F \in G_{fin}(x)$ by Remark 4.12(i). Hence $G_{fin}(x) \neq \emptyset$ for all $x \in P$.

We now need to prove that $\uparrow x = \bigcap \{\uparrow F \mid F \in G_{fin}(x)\}$ for all $x \in P$. Suppose there exists $x \in P$ such that $\uparrow x \neq \bigcap \{\uparrow F \mid F \in G_{fin}(x)\}$, i.e., there exists $y \in \bigcap \{\uparrow F \mid F \in G_{fin}(x)\}$, such that $x \nleq y$. Then there exists a finite subset F of P such that $y \notin \uparrow F$ and $x \notin (P \backslash \uparrow F)^\delta$, i.e., $F \in G_{fin}(x)$ by Remark 4.12(i). Thus $y \in \uparrow F$, this is a contradiction. □

Proposition 4.15 and Lemma 3.29 together imply the following corollary.

Corollary 4.16 *A poset P is s_2-GCD if and only if for any $x, y \in P$ with $x \nleq y$, there exist $u \in P$ and a finite subset F of P such that $x \notin \downarrow u, y \notin \uparrow F$ and $\downarrow u \cup \uparrow F = P$.*

From Proposition 3.34 and Corollary 4.16 we have the following.

Corollary 4.17 *A semi-lattice P is s_2-GCD if and only if P^{op} is hypercontinuous.*

It is easy to see that for a finite poset P, P^{op} is hypercontnious, hence by Corollary 4.17, P is s_2-GCD, especially P is a s_2-QC-continuous poset by Definitions 4.9 and 4.11. By this observation, we have

Corollary 4.18 *Every finite poset is a s_2-QC-continuous poset.*

Observe that a s_2-QC-continuous poset is generally not actually a s_2-C-continuous poset.

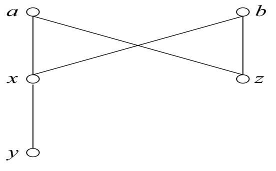
Example 4.19 Consider the poset $P = \{a, b, x, y, z\}$, where the order is defined by $x, z \leq a, b$ and $y \leq x$. Since P is a finite poset, so P is s_2-QC-continuous by Corollary 4.18. Let $S_1 = \downarrow x$ and $S_2 = \downarrow y \cup \downarrow z$. Then S_1 and S_2 are σ_2-closed by Remark 2.3(i), and

$$(S_1 \cap S_2)^\delta = (\downarrow x \cap (\downarrow y \cup \downarrow z))^\delta = (\downarrow y)^\delta = \downarrow y,$$

as well as

$$S_1^\delta \cap S_2^\delta = (\downarrow x)^\delta \cap (\downarrow y \cup \downarrow z)^\delta = \downarrow x \cap (\downarrow x \cup \downarrow z) = \downarrow x.$$

Thus $(S_1 \cap S_2)^\delta \neq S_1^\delta \cap S_2^\delta$, hence P is not s_2-C-continuous by Proposition 3.13.



For a poset P, a binary relation $\preccurlyeq$ on the set of all subsets of P is defined as follows: $G \preccurlyeq H$ if and only if $H \subseteq \mathrm{int}_\upsilon \uparrow G$. We write $G \preccurlyeq x$ for $G \preccurlyeq \{x\}$ and $y \preccurlyeq H$ for $\{y\} \preccurlyeq H$. Note that $x \preccurlyeq y$ is unambiguously defined. A poset P is called *quasi-hypercontinuous* if for each $x \in P$ the family

$$H_{fin}(x) = \{F \subseteq P \mid F \textit{ is finite and } F \preccurlyeq x\}$$

is a directed family and whenever $x \not\leq y$, then there exists a finite subset $F \in H_{fin}(x)$ with $y \notin \uparrow F$, i.e., $\uparrow x = \cap\{\uparrow F \mid F \in H_{fin}(x)\}$.

Proposition 4.20 *Let P be a poset. The following statements are equivalent:*
(1) P is quasi-hypercontinuous;
(2) P is s_2-quasicontinuous in which $H \ll_2 x$ implies $H \preccurlyeq x$.

Proof. Suppose P is a quasi-hypercontinuous poset and $H \ll_2 x$. Then there exists a finite subset $F \in P$ such that $H \leq F \preccurlyeq x$. If $H \preccurlyeq x$, then $x \in \mathrm{int}_\upsilon \uparrow H \subseteq \mathrm{int}_{\sigma_2} \uparrow H$, i.e., $H \ll_2 x$. Hence P is s_2-quasicontinuous.

Conversely if (2) holds, then $\preccurlyeq$ is approximating since $\ll_2$ is. Hence P is hypercontinuous. $\square$

Proposition 4.21 *Let P is a quasi-hypercontinuous poset. Then for any $x, y \in P$ with $x \not\leq y$, there exist finite subsets F_1, F_2 of P such that $y \notin \uparrow F_1, x \notin \downarrow F_2$ and $\uparrow F_1 \cup \downarrow F_2 = P$.*

Proof. Suppose P is quasi-hypercontinuous and $x, y \in P$ with $x \not\leq y$. Then there exists a finite subset $F_1 \subseteq P$ such that $y \notin \uparrow F_1$ and $F_1 \preccurlyeq x$. Since $F_1 \preccurlyeq x$ if and only if $x \in \mathrm{int}_\upsilon \uparrow F_1$, so there is a finite subset $F_2 \subseteq P$ such that $x \in P \backslash \downarrow F_2 \subseteq \uparrow F_1$, i.e., $x \notin \downarrow F_2$ and $\uparrow F_1 \cup \downarrow F_2 = P$. □

Proposition 4.22 *A sup-semilattice P is quasi-hypercontinuous if and only if for any $x, y \in P$ with $x \not\leq y$, there exist finite subsets F_1, F_2 of P such that $y \notin \uparrow F_1$, $x \notin \downarrow F_2$ and $\uparrow F_1 \cup \downarrow F_2 = P$.*

Proof. Suppose that P is quasi-hypercontinuous. It is obvious that for any $x, y \in P$ with $x \not\leq y$, there exist finite subsets F_1, F_2 of P such that $y \notin \uparrow F_1$, $x \notin \downarrow F_2$ and $\uparrow F_1 \cup \downarrow F_2 = P$ by Proposition 4.21.

Conversely, for any fixed $x \in P$, if $x \leq y$ for all $y \in P$, then x is the least element of P, and thus $\{x\} \in H_{fin}(x)$; If x is not the least element of P, then there exists $y \in P$ such that $x \not\leq y$, thus there exist finite subsets F_1, F_2 of P such that $y \notin \uparrow F_1, x \notin \downarrow F_2$ and $\uparrow F_1 \cup \downarrow F_2 = P$, implies $x \in P \backslash \downarrow F_2 \subseteq \uparrow F_1$, i.e., $F_1 \in H_{fin}(x)$. Hence $H_{fin}(x) \neq \emptyset$ for all $x \in P$. Since P is a sup-semilattice, we have $\{\uparrow F \mid F \in H_{fin}(x)\}$ is filtered for all $x \in P$.

We now need to prove that $\uparrow x = \bigcap \{\uparrow F \mid F \in H_{fin}(x)\}$ for all $x \in P$. Suppose there exists $x \in P$ such that $\uparrow x \neq \bigcap \{\uparrow F \mid F \in H_{fin}(x)\}$, i.e., there is $y \in \bigcap \{\uparrow F \mid F \in H_{fin}(x)\}$, such that $x \not\leq y$. Then there exist finite subsets F_1, F_2 of L such that $y \notin \uparrow F_1, x \notin \downarrow F_2$ and $\uparrow F_1 \cup \downarrow F_2 = P$, i.e., $x \in P \backslash \downarrow F_2 \subseteq \uparrow F_1$ implies $F_1 \in H_{fin}(x)$. Thus $y \in \uparrow F_1$, this is a contradiction. □

From Theorem 4.13, Propositions 4.20, 4.22 and Corollary 4.16 we have the following.

Theorem 4.23 *Let P be a sup-semilattice. Then*
(1) P is s_2-GCD implies P is quasi-hypercontinuous.
(2) P is s_2-QC-continuous and quasi-hypercontinuous if and only if P is s_2-GCD.

Lemma 4.24 *[13] A poset P is s_2-quasicontinuous if and only if the lattice $\sigma_2(P)$ of all σ_2-open sets is hypercontinuous.*

So, in view of Corollary 4.17 and Remark 4.12(ii), a poset P is s_2-quasicontinuous if and only if $C_2(P)$ is a GCD lattice.

Theorem 4.25 *For any poset P, the following statements are equivalent:*
(1) P is a s_2-quasicontinuous poset;
(2) $\sigma_2(P)$ is a hypercontinuous lattice;
(3) $C_2(P)$ is a GCD lattice;
(4) $C_2(P)$ is a quasicontinuous lattice.

Proof: (1) ⇔ (2) By Lemma 4.24. (2) ⇔ (3) By Remark 4.12(ii) and Corollary 4.17. (3) ⇔ (4) By Propositions 3.27, 4.10, 4.15 and Remark 4.12(ii). □

5 s_2-MC-continuous posets

We first give a definition of the σ_2-C-set of a poset P, and then introduce the notion of s_2-meet-C-continuous (for short, s_2-MC-continuous) poset by the σ_2-C-set.

Definition 5.1 Let P be a poset. A subset $A \subseteq P$ is called σ_2-C-set if it satisfies:
(i) $A = \downarrow A$;
(ii) $S \subseteq A$ implies $S^\delta \subseteq A$ for all $S \in C_2^*(P)$.

The collection of all σ_2-C-sets of P will be denoted by $SC_2(P)$ and let $SO_2(P) = \{U \subseteq P \mid P \backslash U \in SC_2(P)\}$.

Proposition 5.2 *Let P be a poset. Then the following statements hold:*
(i) $L, \emptyset \in SC_2(P)$;
(ii) $\downarrow x \in SC_2(P)$ for all $x \in P$;
(iii) For any the family $\{A_i \mid i \in I\} \subseteq SC_2(P)$, $\bigcap_{i\in I} A_i \in SC_2(P)$;
(iv) $A^\delta \in SC_2(P)$ for all $A \subseteq P$;
(v) $U \in SO_2(P)$ if and only if $U = \uparrow U$ and $S^\delta \cap U \neq \emptyset$ implies $S \cap U \neq \emptyset$ for all $S \in C_2^*(P)$.

Proof. This follows immediately from Definition 5.1 and Lemma 2.1. □

Remark 5.3 For a poset P and $A_1, A_2 \in SC_2(P)$, in general $A_1 \cup A_2 \in SC_2(P)$ does not hold. Thus the dual $SO_2(P)$ of $SC_2(P)$ cannot compose a topology on P. But $SO_2(P)$ can compose a semi-topology of P by Proposition 5.2.

Definition 5.4 A poset P is s_2-meet-C-continuous (for short, s_2-MC-continuous), if for any $x \in P$ and any $S \in C_2^*(P)$ with $x \in S^\delta$, then $x \in \bigcap\{A \in SC_2(P) \mid \downarrow x \cap S \subseteq A\}$.

Proposition 5.5 *If a poset P is s_2-C-continuous, then it is also s_2-MC-continuous.*

Proof. Suppose that P is a s_2-C-continuous poset. Then for any $x \in P$ and $S \in C_2^*(P)$ with $x \in S^\delta$, we know that $\downarrow^{\prec} x \subseteq S \cap \downarrow x$. Therefore, for any $A \in SC_2(P)$ with $\downarrow x \cap S \subseteq A$, we have $\downarrow^{\prec} x \subseteq A$. Hence by Corollary 3.9 and Definition 5.1, we have $x \in (\downarrow^{\prec} x)^\delta \subseteq A$. Thus $x \in \bigcap\{A \in SC_2(P) \mid \downarrow x \cap S \subseteq A\}$, that is, P is s_2-MC-continuous. □

Proposition 5.6 *Let P be a s_2-MC-continuous poset. Then for any $x, y, z \in P$, the following equation hold:*

$$(\downarrow x \cap (\downarrow y \cup \downarrow z))^\delta = \downarrow x \cap (\downarrow y \cup \downarrow z)^\delta.$$

Proof. Since $\downarrow x \cap (\downarrow y \cup \downarrow z) \subseteq \downarrow y \cup \downarrow z$ and $\downarrow x \cap (\downarrow y \cup \downarrow z) \subseteq \downarrow x$, we know that

$$(\downarrow x \cap (\downarrow y \cup \downarrow z))^\delta \subseteq \downarrow x \cap (\downarrow y \cup \downarrow z)^\delta,$$

by Lemma 2.1. Conversely, for any $u \in \downarrow x \cap (\downarrow y \cup \downarrow z)^\delta$, we know that $\downarrow u \subseteq \downarrow x$ and $u \in (\downarrow y \cup \downarrow z)^\delta$. Thus $\downarrow u \cap (\downarrow y \cup \downarrow z) \subseteq \downarrow x \cap (\downarrow y \cup \downarrow z) \subseteq (\downarrow x \cap (\downarrow y \cup \downarrow z))^\delta$. Since $\downarrow y \cup \downarrow z \in C_2^*(P)$ and $(\downarrow x \cap (\downarrow y \cup \downarrow z))^\delta \in SC_2(P)$, so $u \in (\downarrow x \cap (\downarrow y \cup \downarrow z))^\delta$. □

Corollary 5.7 *Let P be a s_2-MC-continuous poset. If P is also a lattice, then P is a distributive lattice.*

Proof. For any $x, y, z \in P$,

$$\downarrow x \cap (\downarrow y \cup \downarrow z) = (\downarrow x \cap \downarrow y) \cup (\downarrow x \cap \downarrow z) = \downarrow(x \wedge y) \cup \downarrow(x \wedge z).$$

Thus $(\downarrow x \cap (\downarrow y \cup \downarrow z))^\delta = \downarrow((x \wedge y) \vee (x \wedge z))$ by Lemma 2.1(5). Since $\downarrow x \cap (\downarrow y \cup \downarrow z)^\delta = \downarrow(x \wedge (y \vee z))$ by Lemma 2.1(5), and $(\downarrow x \cap (\downarrow y \cup \downarrow z))^\delta = \downarrow x \cap (\downarrow y \cup \downarrow z)^\delta$ by Proposition 5.6, we know that $\downarrow((x \wedge y) \vee (x \wedge z)) = \downarrow(x \wedge (y \vee z))$, that is, $((x \wedge y) \vee (x \wedge z)) = (x \wedge (y \vee z))$. □

Theorem 5.8 *A poset P is s_2-MC-continuous if and only if for any $U \in SO_2(P)$ and any $x \in P$, $\uparrow(U \cap \downarrow x) \in SO_2(P)$.*

Proof. Suppose that P is a s_2-MC-continuous poset, $x \in P$ and $U \in SO_2(P)$ with $S^\delta \cap \uparrow(U \cap \downarrow x) \neq \emptyset$ for some $S \in C_2^*(P)$. Then there exists $z \in U \cap \downarrow x$ such that $z \in S^\delta$. Assume that $S \cap \downarrow z \cap U = \emptyset$. Then $S \cap \downarrow z \subseteq P \backslash U \in SC_2(P)$. Since P is s_2-MC-continuous, so $z \in P \backslash U$, this is a contradiction. Thus $S \cap \downarrow z \cap U \neq \emptyset$, which implies that $S \cap \uparrow(U \cap \downarrow x) \supseteq S \cap \uparrow(U \cap \downarrow z) \neq \emptyset$. Therefore, $\uparrow(U \cap \downarrow x) \in SO_2(P)$.

Conversely, assume that there exist $x \in P, S \in C_2^*(P)$ with $x \in S^\delta$ and $A \in SC_2(P)$ with $\downarrow x \cap S \subseteq A$ such that $x \notin A$. Then $x \in P \backslash A = U \in SO_2(P)$, and thus $x \in S^\delta \cap U \cap \downarrow x$ implies $S^\delta \cap \uparrow(U \cap \downarrow x) \neq \emptyset$. Since $\uparrow(U \cap \downarrow x) \in SO_2(P)$, so $S \cap \uparrow(U \cap \downarrow x) \neq \emptyset$ implies $S \cap U \cap \downarrow x \neq \emptyset$, this is a contradiction. Therefore, for any $x \in P$ and any $S \in C_2^*(P)$ with $x \in S^\delta$, $x \in \bigcap\{A \in SC_2(P) \mid \downarrow x \cap S \subseteq A\}$, that is, P is s_2-MC-continuous. □

Corollary 5.9 *A poset P is s_2-MC-continuous if and only if for any $U \in SO_2(P)$ and any nonempty subset $A \subseteq P$, $\uparrow(U \cap \downarrow A) \in SO_2(P)$.*

Proof. This follows immediately from Theorem 5.8. □

Theorem 5.10 *For any poset P, the following statements are equivalent:*
(1) P is a s_2-MC-continuous poset;
(2) $\downarrow x \cap S^\delta = (\downarrow x \cap S)^\delta$ for all $x \in P$ and $S \in C_2^*(P)$;
(3) $S_1^\delta \cap S_2^\delta = (S_1 \cap S_2)^\delta$ for all $S_1, S_2 \in C_2^*(P)$;
(4) $x \in S^\delta$ implies $x = \sup(\downarrow x \cap S)$ for all $x \in P$ and $S \in C_2^*(P)$.

Proof. (1) $\Rightarrow$ (2) Since $\downarrow x \cap S \subseteq S$ and $\downarrow x \cap S \subseteq \downarrow x$, we know that $(\downarrow x \cap S)^\delta \subseteq \downarrow x \cap S^\delta$, by Lemma 2.1. Conversely, for any $u \in \downarrow x \cap S^\delta$, we have $\downarrow u \subseteq \downarrow x$ and $u \in S^\delta$. Thus $\downarrow u \cap S \subseteq \downarrow x \cap S \subseteq (\downarrow x \cap S)^\delta$. Since $S \in C_2^*(P)$ and $(\downarrow x \cap S)^\delta \in SC_2(P)$, so $u \in (\downarrow x \cap S)^\delta$. Hence $\downarrow x \cap S^\delta = (\downarrow x \cap S)^\delta$.

(2) $\Leftrightarrow$ (3) Suppose (3) holds. Then (2) is also holds, because $\downarrow x \in C_2^*(P)$ and $(\downarrow x)^\delta = \downarrow x$ for all $x \in P$. Suppose (2) holds. Then

$$\begin{aligned}
S_1^\delta \cap S_2^\delta &= \left(\bigcup\nolimits_{x\in S_1^\delta} \downarrow x\right) \cap S_2^\delta = \bigcup\nolimits_{x\in S_1^\delta} (\downarrow x \cap S_2^\delta)\\
&= \bigcup\nolimits_{x\in S_1^\delta} (\downarrow x \cap S_2)^\delta \subseteq \left(\bigcup\nolimits_{x\in S_1^\delta} (\downarrow x \cap S_2)\right)^\delta\\
&= \left(\left(\bigcup\nolimits_{x\in S_1^\delta} \downarrow x\right) \cap S_2\right)^\delta = \left(S_1^\delta \cap \left(\bigcup\nolimits_{x\in S_2} \downarrow x\right)\right)^\delta\\
&= \left(\bigcup\nolimits_{x\in S_2} (S_1^\delta \cap \downarrow x)\right)^\delta = \left(\bigcup\nolimits_{x\in S_2} (S_1 \cap \downarrow x)^\delta\right)^\delta\\
&\subseteq \left(\left(S_1 \cap \left(\bigcup\nolimits_{x\in S_2} \downarrow x\right)\right)^\delta\right)^\delta = ((S_1\cap S_2)^\delta)^\delta = (S_1 \cap S_2)^\delta.
\end{aligned}$$

Since $S_1 \cap S_2 \subseteq S_i,\ i = 1, 2$, so $(S_1 \cap S_2)^\delta \subseteq S_i^\delta,\ i = 1, 2$. Thus $(S_1 \cap S_2)^\delta \subseteq S_1^\delta \cap S_2^\delta$.

(2) $\Rightarrow$ (4) For any $x \in P$ and $S \in C_2^*(P)$ with $x \in S^\delta$, by (2), we have $\downarrow x = \downarrow x \cap S^\delta = (\downarrow x \cap S)^\delta$. Thus $x = \sup(\downarrow x \cap S)$.

(4) $\Rightarrow$ (1) For any $x \in P$, $S \in C_2^*(P)$ with $x \in S^\delta$, by Lemma 3.12 we have $\downarrow x \cap S \in C_2^*(P)$ because $\downarrow x = (\downarrow x \cap S)^\delta \neq \emptyset$. Thus for every $A \in SC_2(P)$ with $\downarrow x \cap S \subseteq A$, we know that $x \in (\downarrow x \cap S)^\delta \subseteq A$. Hence P is s_2-MC-continuous. □

Lemma 5.11 *Let P be a s_2-MC-continuous poset, $x \in P$ and $F \subseteq P$ finite. Then $F \prec_2 x$ if and only if there exists $y \in F$ such that $y \prec_2 x$.*

Proof: Suppose $F \prec_2 x$. Assume that $x \notin \bigcup_{y\in F} \uparrow^\prec y$. Then by Definition 3.5, we know that for each $y \in F$, there exists $S_y \in C_2^*(P)$ such that $x \in S_y^\delta$ and $y \notin S_y$. Let $S = \bigcap_{y\in F} S_y$. Then $x \in \bigcap_{y\in F} S_y^\delta = S^\delta$ by Theorem 5.10, and $S \in C_2^*(P)$ by Lemma 3.12. Since $F \prec_2 x$, so $\uparrow F \cap S \neq \emptyset$. Thus there exists $y_0 \in F$ such that $y_0 \in S \subseteq S_{y_0}$, this is a contradiction. Therefore, there exists $y \in F$ such that $y \prec_2 x$.

Conversely, if there exists $y \in F$ such that $y \prec_2 x$, then for any $S \in C_2^*(P)$ with $x \in S^\delta$, $y \in S$, which implies that $\uparrow F \cap S \neq \emptyset$. Thus $F \prec_2 x$. □

Theorem 5.12 *Let P be a poset. Then P is s_2-C-continuous if and only if P is both s_2-MC-continuous and s_2-QC-continuous.*

Proof. Suppose that P is s_2-C-continuous. Then P is both s_2-MC-continuous and s_2-QC-continuous by Propositions 4.10 and 5.5.

Suppose that P is both s_2-MC-continuous and s_2-QC-continuous. Assume that there exist $x \in P$ and $z \in (\downarrow^\prec x)^u$ such that $x \nleq z$. Since P is s_2-QC-continuous, so there exists a finite subset F of P such that $F \prec_2 x$ and $z \notin \uparrow F$. By Lemma 5.1, there exists $y \in F$ such that $y \prec_2 x$, because P is s_2-MC-continuous. Therefore $y \leq z$, this is a contradiction. Hence $x = \sup \downarrow^\prec x$ since $x \in (\downarrow^\prec x)^u$ for all $x \in P$, i.e., P is s_2-C-continuous. □

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