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Towards Computability over Effectively Enumerable Topological Spaces ⋆

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Abstract

In this paper we study different approaches to computability over *effectively enumerable topological spaces*. We introduce and investigate the notions of computable function, strongly-computable function and weakly-computable function. Under natural assumptions on effectively enumerable topological spaces the notions of computability and weakly-computability coincide.

Keywords: Computably enumerable topological space, computability, effective continuity.

1 Introduction

In this paper we approach the problem of computability over effectively enumerable spaces. Since the class of effectively enumerable topological spaces contains effective ω-continuous domains, computable metric spaces, and abstract structures with computably enumerable $\exists$-theory as proper subclasses, computability over effectively enumerable spaces is crucial problem to investigate. We introduce and study different natural approaches to computability based on well-known enumeration operators [16]. These approaches lead to nonequivalent classes of computable functions over effectively enumerable spaces. The paper is structured as follows. In Section 2 we recall notion and properties of effectively enumerable spaces [11].

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In Section 3 we propose and study approaches to computability over effectively enumerable spaces.

2 Basic notions and Definitions

Let (X, τ, ν) be a topological space, where X is a non-empty set, $\tau^* \subseteq 2^X$ is a base of the topology τ and $\nu : \omega \to \tau^*$ is a numbering. Let D_k denote the k-th finite set with respect to the standard numbering of the finite sets.

Definition 2.1 A topological space (X, τ, ν) is **effectively enumerable** if the following conditions hold.

(i) There exists a computable function $g : \omega \times \omega \times \omega \to \omega$ such that

$$\nu i \cap \nu j = \bigcup_{n \in \omega} \nu g(i, j, n).$$

(ii) The set $\{i | \nu i \neq \emptyset\}$ is computably enumerable.

Definition 2.2 An effectively enumerable topological space (X, τ, ν) is **strongly effectively enumerable** if there exists a computable function $h : \omega \times \omega \to \omega$ such that

$$X \setminus cl(\nu i) = \bigcup_{j \in \omega} \nu h(i, j).$$

Now we show that the topological spaces corresponding to computable metric spaces likewise corresponding to effective ω-continuous domains are proper natural subclasses of effectively enumerable topological spaces.

For the definition of computable metric space we refer to [14,23,2].

Theorem 2.3 *If $\mathcal{M} = (M, \nu, \mathbf{B}, d)$ is a computable metric space then (M, τ_d, ν^*) is a strongly effectively enumerable topological space.*

Proof. Let $\mathcal{M} = (M, \nu, \mathbf{B}, d)$ be a computable metric space, where $\mathbf{B} \subseteq M$ is countable and dense in M, $\nu : \omega \to \mathbf{B}$ is a numbering, and $d : M \times M \to \mathbb{R}$ is a distance function computable on $(\mathbf{B}, \nu)$. We use a computable representation of the rational numbers $(\mathbb{Q}^+, \mu)$, the standard pairing function $c : \omega \times \omega \to \omega$, and the inverse function $(l, r) : \omega \to \omega \times \omega$. Let τ_d be topology induced by d, ν^* be a numbering of the base of τ_d such that $\nu^*(n) = B(\nu l(n), \mu r(n))$, where $B(x, y)$ is an open ball with the center x and the radius y.
It is easy to see that

$$\nu^* n \cap \nu^* m = \cup\{B(x, q) | x \in \mathbf{B}, q \in \mathbb{Q}^+, d(\nu l(n), x) + q < d(\nu l(n), \mu r(n)) \text{ and } d(\nu l(m), x) + q < d(\nu l(m), \mu r(m))\}$$

is an effectively open set. So,

$$\nu^* n \cap \nu^* m = \bigcup_{k \in \omega} \nu^* \chi(n, m, k) \text{ for a computable function } \chi.$$

Since $\nu^* n \neq \emptyset \leftrightarrow \mu r(n) > 0$, the set $\{n | \nu^* n \neq \emptyset\}$ is effectively open. Finally, since

$$M \setminus cl(\nu^* n) M \setminus \bar{B}(\nu l(n), \mu r(n)) =$$
$$\cup\{B(x,q) | x \in \mathbf{B}, q \in \mathbb{Q}^+, d(\nu l(n), x) > q + \mu r(n)\}$$

is an effectively open set, we have

$$M \setminus cl(\nu^* i) = \bigcup_{j\in\omega} \nu^* h(i,j) \text{ for a computable function } h.$$

So, (M, τ_d, ν^*) is a strongly effectively enumerable topological space. □

The following proposition shows that the condition of computably enumerability for the set $\{(i,j) \,|\, \alpha(i)\alpha(j)\}$ considered in [23] is too restrictive in the case of metric spaces.

Proposition 2.4 *There exists a computable metric space (M, B, d) such that the set $\{(i,j) \,|\, \nu^*(i) = \nu^*(j)\}$ is not c.e.*

Proof. In [9,10] it was constructed some computable closed set $A \subset \mathbb{R}$ that its interior is not effectively open. We put $X = \mathbb{R} \setminus A$ and consider it as a computable metric space since X is effectively open, $B = X \cap \mathbb{Q}$. It is easy to see that

$$x \in int(A) \leftrightarrow \exists a, b \in B \exists r_1, r_2 \in \mathbb{Q}(B_X(a, r_1) = B_X(b, r_2) \wedge |x - a| < r_1 \wedge$$
$$|x - b| > r_2).$$

Hence, if the set $\{(i,j) \,|\, \nu^*(i) = \nu^*(j)\}$ is c.e. for this space X, $int(A)$ is effectively open, a contradiction completes the proof. □

Now we compare effectively enumerable topological spaces with ω-continuous domains (c.f. [18,1,4]). First we recall well-known properties of ω-continuous domains.

Lemma 2.5 *For an ω-continuous domain $\mathcal{D} = (D, \{b_i\}_{i\in\omega}, \sqsubseteq)$ the following properties hold.*

(i) *If $a \ll x$ then there exists $n \in \omega$ such that $a \ll b_n \ll x$.*

(ii) *(D, τ, ν) is a T_0-space, where τ is generated by the base $\tau^* = \{U_{b_n}\} \cup \{\emptyset\}$ and the numbering $\nu : \omega \to \tau^*$ is defined as follows: $\nu 0 = \emptyset$, $\nu k = U_{b_{k-1}} = \{x | b_{k-1} \ll x\}$, $k > 0$.*

Definition 2.6 An ω-continuous domain $\mathcal{D} = (D, \{b_i\}_{i\in\omega}, \sqsubseteq)$ is called **weakly effective** if $\{< n, m > | b_n \ll b_m\}$ is computably enumerable.

Theorem 2.7 *Every weakly effective ω-continuous domain is an effectively enumerable topological space.*

Proof. Let $\mathcal{D} = (D, \{b_i\}_{i\in\omega}, \sqsubseteq)$ be a weakly effective ω-continuous domain. The topology τ is generated by the base $\tau^* = \{U_{b_n} | n \in \omega\} \cup \{\emptyset\}$, where $U_a = \{x | a \ll x\}$, and $\nu : \omega \to \tau^*$ is the standard numbering. We show now that

$$U_{b_n} \cap U_{b_m} = \bigcup_{b_s \gg b_n, b_m} U_{b_s}.$$

If $x \in U_{b_s}$ for $b_s \gg b_n, b_m$ then, by definition, $x \gg b_s$. So, $x \in U_{b_n} \cap U_{b_m}$. Suppose $x \in U_{b_n} \cap U_{b_m}$. By definition, $x \gg b_n$ and $x \gg b_m$. So, there exist s_1 and s_2 such that $x \gg b_{s_1} \gg b_n$ and $x \gg b_{s_2} \gg b_m$.

Since $\{b_i | b_i \ll x\}$ is directed, there exists $b_s \gg b_n, b_m$ such that $x \in U_{b_s}$. By weak effectiveness, the set $\{n | U_{b_n} \neq \emptyset\}$ is computably enumerable. □

The following results show that the effectively enumerable spaces enlarge the effective ω-continuous domains and the computable metric spaces. We consider structures with topologies induced by $\exists$-formulas. Suppose $\mathcal{A} = \langle A, \sigma_0 \rangle = \langle A, \sigma_P, \neq \rangle$ is an abstract structure, where A contains more than one element, σ_P is a countable set of basic predicates.
The topology $\tau_\Sigma^\mathcal{A}$ is formed by the base which is the set of subsets definable by existential formulas with positive occurrences of predicates from σ_0. The following proposition is straightforward from the definition of effectively enumerable topological space.

Theorem 2.8 *[11] The topological space $\left(X, \tau_\Sigma^\mathcal{A}\right)$ is effectively enumerable if and only if $Th_\exists(X)$ is computable enumerable.*

As the example of a structure which is an effectively enumerable space we consider the set of continuous functions $C(\mathbb{R})$. Let us note that $C(\mathbb{R})$ does not belong to the metric spaces and to the ω-continuous domains as well.

We consider the structure $\mathcal{C}(\mathbb{R}) = (C(\mathbb{R}), P_1, \ldots, P_{12}, \neq)$, where the predicates $P_1, \ldots, P_{12}$ are interpreted for every $f,\ g \in C(\mathbb{R})$ as follows.
The first group formalises relations between infimum and sumpemum of two functions on $[0, 1]$:

$$\mathcal{C}(\mathbb{R}) \models P_1(f, g) \leftrightarrow \sup f|_{[0,1]} < \sup g|_{[0,1]};$$

$$\mathcal{C}(\mathbb{R}) \models P_2(f, g) \leftrightarrow \sup f|_{[0,1]} < \inf g|_{[0,1]};$$

$$\mathcal{C}(\mathbb{R}) \models P_3(f, g) \leftrightarrow \sup f|_{[0,1]} > \inf g|_{[0,1]};$$

$$\mathcal{C}(\mathbb{R}) \models P_4(f, g) \leftrightarrow \inf f|_{[0,1]} > \inf g|_{[0,1]}.$$

The second group formalises properties of operations on $C(\mathbb{R})$.

$$\mathcal{C}(\mathbb{R}) \models P_5(f, g, h) \leftrightarrow f(x) + g(x) < h(x);\ \text{for every } x \in [0, 1];$$

$$\mathcal{C}(\mathbb{R}) \models P_6(f, g, h) \leftrightarrow f(x) \cdot g(x) < h(x) \text{ for every } x \in [0, 1];$$

$$\mathcal{C}(\mathbb{R}) \models P_7(f, g, h) \leftrightarrow f(x) + g(x) > h(x) \text{ for every } x \in [0, 1];$$

$$\mathcal{C}(\mathbb{R}) \models P_8(f, g, h) \leftrightarrow f(x) \cdot g(x) > h(x) \text{ for every } x \in [0, 1].$$

The third group formalises relations between functions f and $\lambda x.x$.

$$\mathcal{C}(\mathbb{R}) \models P_9(f) \ \leftrightarrow f(x) > x;\ \text{for every } x \in [0, 1];$$

$$\mathcal{C}(\mathbb{R}) \models P_{10}(f) \leftrightarrow f(x) < x \text{ for every } x \in [0, 1].$$

The fourth group formalises relations between a function h and the composition of functions f and g.

$$\mathcal{C}(\mathbb{R}) \models P_{11}(f,g,h) \leftrightarrow f(g(x)) < h(x) \text{ for every } x \in [0,1];$$
$$\mathcal{C}(\mathbb{R}) \models P_{12}(f,g,h) \leftrightarrow f(g(x)) > h(x) \text{ for every } x \in [0,1].$$

We recall the notion of compact open topology τ_{c-o} on $C(X,Y)$. Let (X,α) and (Y,β) be topological spaces, $\mathcal{K} \subseteq \mathcal{X}$ be a compact set, and $\mathcal{O} \subseteq \mathcal{Y}$ be an open set. Then subbase of the compact open topology is defined by sets of the type

$$U^{\mathcal{K}}_{\mathcal{O}} = \{f \in C(X,Y) | f(K) \subset O\}.$$

Since, by Weierstrass Theorem [21], $\mathbb{Q}[x]$ is dense in $C(\mathbb{R})$, the base τ^*_{c-o} of the topology τ_{c-o} and its numbering are defined as follows:

(i) The base τ^*_{c-o} is the finite intersections of the following sets

$$U^{a,b}_{p,n} = \{f | p - \frac{1}{n} < f|_{[a,b]} < p + \frac{1}{n}\}, \text{ where } b \in \mathbb{Q}, p \in \mathbb{Q}[x] \text{ and } \deg(p) = n.$$

(ii) The numbering $\nu : \omega \to \tau^*$ is standard.

Proposition 2.9 *On the structure $\mathcal{C} = (C(\mathbb{R}), P_1, \ldots, P_{12}, \neq)$ the compact open topology τ_{c-o} coincides with $\tau^{\mathcal{C}}_{\Sigma}$.*

Proof. $\subseteq$). It is easy to see that, for $1 \leq i \leq 12$ the sets $\{\bar{f} | C(\mathbb{R}) \models P_i(\bar{f})\}$ and projections of them belong to τ_{c-o}. By induction, $\tau^{\mathcal{C}(\mathbb{R})}_{\Sigma} \subseteq \tau_{c_o}$.
$\supseteq$). By definition, it is sufficient to show that the relations $f|_{[a,b]} > g|_{[a,b]}$ and $f|_{[a,b]} < g|_{[a,b]}$ are $\exists$–definable. Note that $W_{a,b} = \{\chi | \chi(0) < a \text{ and } \chi(1) > b\} \subseteq C[0,1]$ is $\exists$–definable set in the language $\{P_i, \neq\}_{i \leq 12}$. Since,

$$f|_{[a,b]} < g|_{[a,b]} \leftrightarrow \exists \chi \in W_{a,b} \exists h \left(f \circ \chi < h < g \circ \chi\right),$$

the relations $f|_{[a,b]} > g|_{[a,b]}$ and $f|_{[a,b]} < g|_{[a,b]}$ are $\exists$–definable. □

Theorem 2.10 *The topological space $(C(\mathbb{R}), \tau_{c-o}, \nu)$ is effectively enumerable.*

Proof. Existence of a computable function $g : \omega \times \omega \times \omega \to \omega$, such that

$$\nu i \cap \nu j = \bigcup_{n \in \omega} \nu g(i,j,n),$$

follows from the definition of ν. By quantifier elimination on $\mathbb{R}$, the set $\{i | \nu i \neq \emptyset\}$ ic computably enumerable. Indeed, by Weierstrass Theorem [21], existence of $g \in C(\mathbb{R})$ such that $g \in \bigcup_{i \in I} U^{a_i,b_i}_{p_i,n_i}$ is equivalent to existence of $m \in \omega$ and polynomial $p \in \mathbb{Q}[x]$ of degree m such that $p \in \bigcup_{i \in I} U^{a_i,b_i}_{p_i,n_i}$. By quantifier elimination on $\mathbb{R}$, we can effectively check this property.d □

We recall the notion of specialisation order on T_0-spaces.

Definition 2.11 Let (X,τ) be a T_0-space. A binary relation $\leq$ on X is called *specialisation order* if $y \leq x \leftrightarrow y \in cl(\{x\})$.

Remark 2.12 Let us note that every partial continuous function f on a T_0-space is monotone on dom$f$ with respect to the specialisation order.

We recall the notion of core-compact topological space.

Definition 2.13 A topological space (X,τ) is said to be *core-compact* iff the lattice $\mathbb{O}(X)$ of the open subsets is continuous.

It is well-known that locally compact spaces and continuous domains are core-compact [8]. Below we slightly modify the definition of strong inclusion. Let $\leq$ be the specialisation order. Denote $\check{y} = \{z \in X | y \leq z\} = \bigcap_{k: y\in\beta k} \beta k$.

Definition 2.14 Let (X,τ,ν) be an effectively enumerable core-compact T_0-space, where X is a non-empty set, $\tau^* \subseteq 2^X$ is a base of the topology τ and $\alpha: \omega \to \tau^*$ is a numbering. Let $E \subseteq \omega^2$ be a computably enumerable relation. We say that E is compact-like strong inclusion (abbreviated as *clsi*) if the following conditions hold.

(E 1). If kEm, then $\bigcap_{s\in D_k} \alpha s \ll \alpha m$.

(E 2). $\alpha n = \bigcup_{mE'n} \alpha m$ for every $n, m \in \omega$ where $E' = \{< n, m >\in \omega^2 | \exists k (D_k = \{n\} \wedge kEm)\}$.

(E 3). If $\bigcap_{j\in J} \alpha j = \check{x} \rightleftharpoons \{y \in X | x \leq y\}$ for $x \in \alpha m$ and $J \subseteq \omega$, then kEm for a finite $D_k \subseteq J$.

(E 4). If kEn and for all $j \in D_k$ $l_j E j$ and $D_s = \bigcup_{j\in D_k} D_{l_j}$, then sEn.

(E 5). If sEn and sEm, then $\exists k\,(kE'n \wedge kE'm \wedge sEk)$.

The basic examples are Euclidian spaces $(\mathbb{R}^n,\tau)$, where the topology τ is formed by the base which is the set of balls $B(p,r)$ with $p \in \mathbb{Q}^n$ and $r \in \mathbb{Q}^+$. It is easy to see that $\bigcap_{s\in D_k} \alpha s \ll \alpha m$ if and only if $cl(\bigcap \alpha_{s\in D_k} s) \subseteq \alpha m$. Put $kEm \rightleftharpoons cl(\bigcap_{s\in D_k} \alpha s) \subseteq \alpha m$. By decidability of $Th(\mathbb{R})$, the properties $(E1)-(E5)$ hold.

3 Computability on Effectively Enumerable Topological Spaces

Now we introduce notions of computable function over effectively enumerable topological spaces based on the well-known definition of enumeration operator.

Definition 3.1 [16] A function $\Gamma_e : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ is called **enumeration operator** if

$$\Gamma_e(A) = B \leftrightarrow B = \{j | \exists i\, c(i,j) \in W_e,\ D_i \subseteq A\},$$

where W_e is the e-th computably enumerable set, and D_i is the i-th finite set.

Definition 3.2 Let $\mathcal{X} = (X,\tau,\alpha)$ be an effectively enumerable topological space and $\mathcal{Y} = (Y,\lambda,\beta)$ be an effectively enumerable T_0-space.
A partial function $F : X \to Y$ is called **computable** if there exists an enumeration operator $\Gamma_e : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ such that, for every $x \in X$,

(i) If $x \in dom(F)$ then

$$\Gamma_e(\{i \in \omega | x \in \alpha i\}) = \{j \in \omega | F(x) \in \beta j\}.$$

(ii) If $x \notin dom(F)$ then, for all $y \in Y$

$$\bigcap_{j\in\omega} \{\beta j | j \in \Gamma_e(A_x)\} \neq \bigcap_{j\in\omega} \{\beta j | j \in B_y\},$$

where $A_x = \{i \in \omega | x \in \alpha i\}$ and $B_y = \{j \in \omega | y \in \beta j\}$.

Theorem 3.3 *Let $\mathcal{X} = (X, \tau, \alpha)$ be an effectively enumerable topological space and $\mathcal{Y} = (Y, \lambda, \beta)$ be an effectively enumerable T_0-space. For a total function $F : X \to Y$ the following are equivalent.*

(i) *F is computable;*

(ii) *There exists a computable function $h : \omega \times \omega \to \omega$ such that $F^{-1}(\beta j) = \bigcup_{i\in\omega} \alpha h(i, j)$.*

Proof. Let $F : X \to Y$ be computable. By definition, we have $\Gamma_e (\{i | x \in \alpha i\}) = \{j | F(x) \in \beta j\}$. Since $\mathcal{X}$ is effectively enumerable, there exists a computable function $H : \omega \times \omega \to \omega$ such that

$$\bigcap_{i\in D_k} \alpha i = \bigcup_{s\in\omega} \alpha H(k, s).$$

So,

$$x \in F^{-1}(\beta j) \leftrightarrow F(x) \in \beta j \leftrightarrow \exists k\, (D_k \subseteq \{i | x \in \alpha i\} \wedge c(k, j) \in W_e) \leftrightarrow$$
$$\bigvee_{c(k,j)\in W_e} x \in \alpha i \leftrightarrow \bigvee_{c(k,j)\in W_e} \exists s x \in \alpha H(k, s) \leftrightarrow$$
$$x \in \bigcup_{c(k,j)\in W_e, s\in\omega} \alpha H(k, s) \leftrightarrow x \in \bigcup_{m\in\omega} \alpha h(j, m)$$

for a computable function $h : \omega \times \omega \to \omega$.
Now suppose $F^{-1}(\beta j) = \bigcup_{i\in\omega} \alpha h(i, j)$. Then, there exists e such that, for $A_x = \{x | x \in \alpha i\}$,

$$\Gamma_e(A_x) = \{j | \exists s\, h(j, s) \in A_x\} = \{j | x \in F^{-1}(\beta j)\} = \{j | F(x) \in \beta j\}.$$

□

Proposition 3.4 *Let $\mathcal{X} = (X, \tau, \alpha)$ be an effectively enumerable topological space and $\mathcal{Y} = (Y, \lambda, \beta)$ be an effectively enumerable T_0-space.*

(i) *If $F : X \to Y$ is a computable function, then F is continuous at every points of* dom F.

(ii) *A total function $F : X \to Y$ is computable if and only if F is effectively continuous.*

Proof. The first claim is straightforward form Definition 3.2. The second claim is based on Theorem 3.3. □

Definition 3.5 Let $\mathcal{X} = (X, \tau, \alpha)$ be an effectively enumerable topological space and $\mathcal{Y} = (Y, \lambda, \beta)$ be an effectively enumerable T_0-space.
A partial function $F : X \to Y$ is called **strongly computable** if there exists an enumeration operator $\Gamma_e : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ such that

(i) If $x \in \mathrm{dom}F$, then $\Gamma_e(A_x) = B_{F(x)}$, where $A_x = \{i \in \omega | x \in \alpha i\}$, $B_y = \{j \in \omega | y \in \beta j\}$.

(ii) If $x \notin \mathrm{dom}F$ and $\Gamma_e(A_x) = J$, then $\bigcap\{\beta_i | j \in J\} \not\subseteq \check{y}$ for every $y \in Y$.

Remark 3.6 Let us note that the notion of strongly computability is invariant under computably equivalent numberings of topologies bases.

Now we compare our notion of strongly computability with strongly (ρ^c_X, ρ^c_Y)-computability for $F : X \to Y$, where X and Y are computable metric spaces, and ρ^c_X, ρ^c_Y are Cauchy-representations of them. For the definitions of Cauchy-representation and strongly (ρ^c_X, ρ^c_Y)-computability we refer to [23].

Theorem 3.7 *Let $\mathcal{X} = (X, \lambda, B_X, d_X)$ and $\mathcal{Y} = (Y, \beta, B_Y, d_Y)$ be computable metric spaces and (X, τ_X, α^*), (Y, τ_Y, β^*) be corresponding them effectively enumerable topological spaces. For every total function $F : X \to Y$, the following are equivalent.*

(i) *F is strongly (ρ^c_X, ρ^c_Y)-computable;*

(ii) *F is strongly computable as a function from one effectively enumerable topological space to another (c.f. Definition 3.5).*

Proof. It is easy to see that there exists an effective procedure which given a Cauchy-representation $\rho^c_X(z)$ produces $A_z = \{i | z \in \alpha^* i\}$ as well as there exists an effective procedure which given A_z produces a Cauchy-representation $\rho^c_X(z)$ for every $z \in X$. By Definition 3.5 and the definition of (ρ^c_X, ρ^c_Y)-computability, both computabilities coincide, details are routine. □

Theorem 3.8 *For total functions the notions of computability and strongly computability coincide.*

Remark 3.9 Below in the case of total functions we use notation "computable" for both computable and strongly computable functions.

Let $(\mathbb{N}, \tau, \nu)$, be a T_0–space, where $\mathbb{N}$ is the natural numbers, τ is the discrete topology and ν is its numbering defined as follows:

$$\nu\, 0 = \emptyset; \nu\, n + 1 = \{n\}.$$

Proposition 3.10 *For $(\mathbb{N}, \tau, \nu)$, the class of partial strongly computable functions coincides with the partial recursive functions.*

Proof. Suppose $f : \mathbb{N} \to \mathbb{N}$ is strongly computable. Since the specialisation order on $\mathbb{N}$ coincides with the equality on $\mathbb{N}$, there exists an enumeration operator $\Gamma_e : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ such that

$$n+1 \in \Gamma_e(D) \leftrightarrow \exists x\, (x+1 \in D \wedge f(x) = n)\,.$$

Suppose D is finite. Note that if $x \notin \mathrm{dom} f$, then for all $y \in Y$,

$$\bigcap_{j\in\omega} \{\beta j | j \in \Gamma_e(A_x)\} \not\subseteq \{y\}.$$

Hence, $f(x) = n \leftrightarrow \exists D\, (D \text{ is finite } \wedge x = 1 \in D \wedge n+1 \in \Gamma_e(D))$, i.e., f is a partial recursive function.

Suppose f is a partial recursive function. Put $\Gamma_e(A) = \{f(x)+1 | x+1 \in A\}$. It is easy to see that $\Gamma_e(A)$ is a required enumeration operator. □

Theorem 3.11 *For partial functions, the strongly computable functions is a proper subclass of the computable functions.*

Proof. Let us consider T_0–space $(\mathbb{N}, \tau, \nu)$. It is easy to see that a computable function is representable as $h_1 \backslash h_2$ for some partial recursive functions h_1, h_2 whereas the strongly computable functions coincide with the partial recursive functions. □

Definition 3.12 Let $\mathcal{X} = (X, \tau, \alpha)$ be an effectively enumerable topological space and $\mathcal{Y} = (Y, \lambda, \beta)$ be an effectively enumerable T_0-space.

A partial function $F : X \to Y$ is called **weakly computable** if there exists an enumeration operator $\Gamma_e : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ such that, for every $x \in X$,

(i) If $x \in dom(F)$, then

$$\Gamma_e(A_x) = J \text{ and } \bigcap_{j\in J} \beta_j = \check{F}(x)$$

(ii) If $x \notin dom(F)$, then

$$\Gamma_e(A_x) = J \text{ and } \bigcap_{j\in J} \beta_j \neq \check{y} \text{ for any } y \in Y.$$

Proposition 3.13 *The computable functions is a proper subclass of weakly computable functions.*

Let us consider the real numbers with two topologies $\tau_{\mathbb{R}}$ and τ_A, where $\tau_{\mathbb{R}}$ is the standard topology and τ_A is defined as follows. We fix a se t A which is open but not effectively open. The topology τ_A is induced by the base

$$\tau_A^* = \{(a,b) | a,\, b \in \mathbb{Q}\} \cup \{(a,b) \cap A | a,\, b \in \mathbb{Q}\} \cup \{(-\infty, +\infty)\}.$$

We take $f = id : (\mathbb{R}, \tau_{\mathbb{R}}, \alpha) \to (\mathbb{R}, \tau_A, \beta)$, where β is defined as follows.

$$\beta(2n) = \alpha n; \beta(2n+1) = A \cap \alpha n.$$

Since preimage of A is not effectively open, f is not computable whereas f is weakly computable. Indeed, it is easy to see that $\Gamma_e(Y) = 2Y = \{2m | m \in Y\}$ is a corresponding enumeration operator.

Theorem 3.14 *Let $\mathcal{X} = (X, \tau, \alpha)$ be an effectively enumerable topological space and $\mathcal{Y} = (Y, \lambda, \beta)$ be an effectively enumerable core-compact T_0-space endowed by some clsi-relation $E \subseteq \omega^2$. A partial function $F : X \to Y$ is computable if and only if F is weakly computable.*

Proof. If F is computable it is easy to see that the corresponding operator Γ_e satisfy the conditions of Definition 3.12.
Let F be a weakly computable function and Γ_e be a corresponding enumeration operator. We construct a new enumeration operator $\Gamma_{e'}$ as follows.

$$m \in \Gamma_{e'}(A) \leftrightarrow m \in \Gamma_e(A) \vee \exists k \exists s [D_s \subseteq A \wedge D_k \subseteq \Gamma_e(D_s) \wedge kEm].$$

By the properties $(E1)$ and $(E3)$ of the clsi-relation E it follows that

$$\bigcap \{\alpha_j | j \in \Gamma_{e'}(A)\} = \bigcap \{\alpha_j | j \in \Gamma_e(A)\}.$$

Hence,

$$\text{if } x \in \mathrm{dom} F, \text{ then } \alpha m \in F(x) \leftrightarrow m \in \Gamma_{e'}(A_x),$$

whereas,

$$\text{if } x \notin \mathrm{dom} F, then \bigcap \{\alpha_j | j \in \Gamma_{e'}(A_x)\} \neq \check{z} \text{ for any } z \in Y.$$

So, F is computable. □

4 Conclusion and Related work

We investigated computability over effectively enumerable topological spaces which contain computable metric spaces and effective ω-continuous domains as proper subclasses. It has been shown that computability over effectively enumerable topological spaces corresponds to effective continuity. There has been a considerable interest in computability theory in the question of whether computable maps are continuous with respect to natural topologies. Myhill and Shepherdson [15] have shown that every computable operator on the set of partial recursive functions is effectively continuous and vice versa. Kreisel, Lacombe and Shoenfield [13] have proven analogous results for the total recursive functions. These results have been generalised to effectively given Scott domains [6,17,22], recursive metric spaces [14], separable countable T_0-spaces with a witness for noninclusion [20]. It was shown that in general the correspondence between computability and effective continuity does not hold [7,13,24]. For historical remarks we refer to [19].

The main advantages of the class of effectively enumerable topological spaces are the following:

- The class of effectively enumerable topological spaces is not restricted to countable spaces.
- The class of effectively enumerable topological spaces contains computable metric spaces, ω-continuous domains.
- Different notions of computability of partial functions is formalised and investigated.

- For total functions, computability is equivalent to effective continuity.

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