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Electronic Notes in Theoretical Computer Science

Electronic Notes in Theoretical Computer Science 341 (2018) 277–295

www.elsevier.com/locate/entcs

Near Distributive Laws

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Abstract

Monads and their compositions can sometimes be generated from simpler data types and without necessarily requiring any monad axioms. Free monads and monad approximations provide two approaches to overcoming the constraints required by monad composition laws while generating near distributive laws.

Keywords: monad composition, free monad, monad approximation, near distributive law.

1 Introduction

This paper continues our study of monad composition in [10], [11]. We will use the same notations as in the second of these papers. We work in a category $\mathcal{V}$.

In working with monads in a programming language, there are two problems: It may be hard to define a monad using the data types available to the programmer, and it may be difficult to verify the monad axioms. The second of these is a very common situation in monad composition.

Given monads (H, μ, η) and $K, \nu, \rho)$ their composition should have form $(KH, \tau, \rho\eta)$. The problem is that there is no obvious τ. The solution is to provide a natural transformation $\lambda : HK \to KH$ which allows τ to be defined as

$$\tau = KHKH \xrightarrow{K\lambda H} KKHH \xrightarrow{\nu\mu} KH$$

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https://doi.org/10.1016/j.entcs.2018.11.014
1571-0661/

The axioms on λ equivalent to rendering $(KH, \tau, \rho\eta)$ a monad were discovered by [3] and are as follows.

$$\begin{tikzcd}
H \arrow[r, "H\rho"] \arrow[dr, "\rho H"'] & HK \arrow[d, "\lambda"] & HKK \arrow[l, "H\nu"'] \arrow[d, "\lambda K"] \\
 & KH & KHK \arrow[d, "K\lambda"] \\
 & & KKH \arrow[ul, "\nu H"]
\end{tikzcd}$$

$(DL\ A)$ $(DL\ B)$

$$\begin{tikzcd}
K \arrow[r, "\eta K"] \arrow[dr, "K\eta"'] & HK \arrow[d, "\lambda"] & HHK \arrow[l, "\mu K"'] \arrow[d, "H\lambda"] \\
 & KH & HKH \arrow[d, "\lambda H"] \\
 & & KHH \arrow[ul, "K\mu"]
\end{tikzcd}$$

$(DL\ C)$ $(DL\ D)$

Axioms (DL C, DL D) hold if and only if K lifts through the category $\mathcal{V}^{\mathbf{H}}$ of Eilenberg-Moore algebras of H and we say λ is a **near distributive law** in this case.

In practice, such axioms present an obstruction to the programmer. We recall [4, Page 34]:

> "all polymorphic functions in functional programming are natural transformations".

Thus the axioms are the obstruction and not the requirement of naturality. In this paper we present two approaches to getting around the obstruction and, in the process, discover various near distributive laws. In the first case, we consider endofunctors H, K and arbitrary natural transformations $\lambda : HK \to KH$ where there are no axioms on λ. If H generates a free monad $\mathbf{H}^@$ then λ induces a near distributive law $\lambda^@ : H^@K \to KH^@$ and say λ *generates* $\lambda^@$. We consider in detail near distributive laws for a common class of free monads namely those generated by algebraic signatures.

A different approach involves defining the notion of a **pre-monad** in $\mathcal{V}$ to be (H, η, μ) with $H : \mathcal{V} \to \mathcal{V}$ an endofunctor and $\mu : HH \to H$, $\eta : \mathrm{id} \to H$ natural transformations, with no further axioms. Pre-monads retain a surprising amount of structure. Analogous to the property of generating a free monad, we shall see that a pre-monad $\mathbf{H}$ usually has a monad approximation $\widehat{\mathbf{H}}$. We show that if (K, m, e) is a pre-monad, then a near distributive law $\lambda : HK \to KH$ induces a near distributive law $\widehat{\lambda} : \widehat{H}K \to K\widehat{H}$.

2 Free Monads

Our most relaxed model of a monad is a functor $H : \mathcal{V} \to \mathcal{V}$. In standard situations the **free monad generated by** H, $(H^@, \mu, \eta; \iota)$ exists where $\mathbf{H}^@ = (H^@, \mu, \eta)$ is a monad in $\mathcal{V}$ and $\iota : H \to H^@$ is a natural transformation, subject to the universal property [2]

$$\begin{tikzcd} H \arrow[r, "\iota"] \arrow[dr, "\alpha"'] & H^@ \arrow[d, "\psi"] \\ & K \end{tikzcd} \qquad\qquad \begin{tikzcd} (H^@, \mu, \eta) \arrow[d, "\psi"] \\ (K, \nu, \rho) \end{tikzcd}$$

that if (K, ν, ρ) is a monad in $\mathcal{V}$ and $\alpha : H \to K$ is a natural transformation then there exists a unique monad map ψ as shown with $\psi\iota = \alpha$.

Example 2.1 Assume that $\mathcal{V}$ has finite powers. For a finite ordinal $i \geq 1$, let $H_i : \mathcal{V} \to \mathcal{V}$ be the functor $H_i X = X^i$, the usual i-ary power functor. When $\mathcal{V} = \mathbf{Set}$, the data type $H_i^@ X$ is the set of all trees in which every node is either an element of X, denoted $L_i x$ (if it is a leaf) or has i subtrees beneath it, denoted $B_i t_1 \cdots t_i \in H_i^@ X$. The natural transformation $\eta_X : X \to H_i^@ X$ maps x to $L_i x$ while $\mu_X : H_i^@ H_i^@ X \to H_i^@$ maps $L_i t$ to t and $B_i\ t_1 \ldots\ t_i$ to $B_i\ (\mu_X t_1)\ \ldots (\mu_X t_i)$.

In what follows, we consider only H for which $\mathbf{H}^@$ exists.

Definition 2.2 An H**-algebra** is a pair (X, δ) where $\delta : HX \to X$ in $\mathcal{V}$. An H**-homomorphism** $f : (X, \delta) \to (Y, \epsilon)$ of H-algebras must satisfy

$$\begin{tikzcd} HX \arrow[r, "Hf"] \arrow[d, "\delta"'] & HY \arrow[d, "\epsilon"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

It is evident that $\mathrm{id}_X : (X, \delta) \to (X, \delta)$ is an H-homomorphism and that H-homomorphisms are closed under composition. This gives rise to a category $\mathcal{V}^H$ of H-algebras with underlying functor $\mathcal{V}^H \to \mathcal{V}$.

Theorem 2.3 *[2] If $\mathcal{V}^H \to \mathcal{V}$ is monadic, $\mathcal{V}^H$ is isomorphic over $\mathcal{V}$ to the category of Eilenberg-Moore algebras $\mathcal{V}^{\mathbf{H}^@}$, where $\mathbf{H}^@$ is the free monad generated by H. The isomorphism $\Phi : \mathcal{V}^{\mathbf{H}^@} \to \mathcal{V}^H$ is given by*

$$\Phi(X,\ H^@ X \xrightarrow{\xi} X) \ = \ (X,\ HX \xrightarrow{\iota_X} H^@ X \xrightarrow{\xi} X)$$

3 Functorial Lifts

Definition 3.1 Let $\mathbf{H} = (H, \mu, \eta)$ be a monad in $\mathcal{V}$, and let $K : \mathcal{V} \to \mathcal{V}$ be a functor. A functor $K^\star : \mathcal{V}^{\mathbf{H}} \to \mathcal{V}^{\mathbf{H}}$ is a **functorial lift** of K through the Eilenberg-Moore category $\mathcal{V}^{\mathbf{H}}$ if the following square commutes:

$$\begin{tikzcd} \mathcal{V}^{\mathbf{H}} \arrow[r, "K^\star"] \arrow[d] & \mathcal{V}^{\mathbf{H}} \arrow[d] \\ \mathcal{V} \arrow[r, "K"'] & \mathcal{V} \end{tikzcd}$$

The following result is due to [1]. Also, see [7].

Theorem 3.2 *For $\mathbf{H} = (H, \mu, \eta)$ a monad in $\mathcal{V}$ and $K : \mathcal{V} \to \mathcal{V}$ a functor, functorial lifts $K^\star : \mathcal{V}^{\mathbf{H}} \to \mathcal{V}^{\mathbf{H}}$ are in bijective correspondence with natural transformations $\lambda : HK \to KH$ which satisfy (DL C) and (DL D). The correspondences are*

(1) $K^\star(X, \xi) = (KX,\ HKX \xrightarrow{\lambda_X} KHX \xrightarrow{K\xi} KX)$

and

(2) $\lambda_X = HKX \xrightarrow{HK\eta_X} HKHX \xrightarrow{\omega_X} KHX$

where $K^\star(HX, \mu_X) = (KHX,\ HKHX \xrightarrow{\omega_X} KHX)$. *Further,*

(3) $\omega_X = HKHX \xrightarrow{\lambda_{HX}} KHHX \xrightarrow{K\mu_X} KHX$

It is immediate that $\lambda_X : (HKX, \mu_{KX}) \to K^\star(HX, \mu_X)$ is a $\mathbf{H}$-homomorphism; the homomorphism diagram is precisely (DL D). It follows (using (DL C)) that λ_X is the unique homomorphic extension of $K\eta$. There is more than one possible λ, however, because there is more than one possible $\mathbf{K}$-algebra structure for $K^\star(HX, \mu_X)$.

Definition 3.3 Let $H : \mathcal{V} \to \mathcal{V}$ generate a free monad $\mathbf{H}^@$ and let $K : \mathcal{V} \to \mathcal{V}$ be a functor. Let $K^\star : \mathcal{V}^{\mathbf{H}^@} \to \mathcal{V}^{\mathbf{H}^@}$ be a functorial lift of K with classifying natural transformation $\lambda^@ : H^@K \to KH^@$ as in Theorem 3.2. We say $K^\star$ is a **flat** functorial lift if there exists a natural transformation $\lambda : HK \to KH$ such that the following square commutes.

$$\begin{tikzcd} HK \arrow[r, "\iota K"] \arrow[d, "\lambda"'] & H^@K \arrow[d, "\lambda^@"] \\ KH \arrow[r, "K\iota"'] & KH^@ \end{tikzcd} \qquad (3.3)$$

We then say that λ **generates** $K^\star$, or λ **generates** $\lambda^@$, and when K is a monad that $\lambda^@$ is a **flat near distributive law**.

Theorem 3.4 *Given $H, K : \mathcal{V} \to \mathcal{V}$ such that $\mathbf{H}^@$ exists, every natural transformation $\lambda : HK \to KH$ generates a flat functorial lift of K through $\mathbf{H}^@$.*

Proof. Given λ, define $K^\dagger : \mathcal{V}^H \to \mathcal{V}^H$ over $\mathcal{V}$ by

$$K^\dagger(X, \delta) \ = \ (KX,\ HKX \xrightarrow{\lambda_X} KHX \xrightarrow{K\delta} KX)$$

If $f : (X, \delta) \to (Y, \epsilon)$ is an H-homomorphism, the diagram

$$\begin{tikzcd} HKX \arrow[r, "\lambda_X"] \arrow[d, "HKf"'] & KHX \arrow[r, "K\delta"] \arrow[d, "KHf"'] & KX \arrow[d, "Kf"] \\ HKY \arrow[r, "\lambda_Y"'] & KHY \arrow[r, "K\epsilon"'] & KY \end{tikzcd}$$

shows that $Kf : K^\dagger(X, \delta) \to K^\dagger(Y, \epsilon)$ is again an H-homomorphism. In the notations of Theorem 2.3 we then have the functorial lift

$$K^\star \;=\; \mathcal{V}^{\mathbf{H}^@} \xrightarrow{\Phi} \mathcal{V}^H \xrightarrow{K^\dagger} \mathcal{V}^H \xrightarrow{\Phi^{-1}} \mathcal{V}^{\mathbf{H}^@}$$

We leave the remaining details to the reader. □

Corollary 3.5 *Given $H, K : \mathcal{V} \to \mathcal{V}$ where $\mathbf{K}$ is a monad and $\mathbf{H}^@$ exists, then every natural transformation $\lambda : HK \to KH$ generates a flat near distributive law $\lambda^@ : H^@K \to KH^@$.*

4 Near Distributive Laws for Free Monads

4.1 Near Distributive Laws via Generic Prestrengths

The notion of prestrength on an endofunctor F of a category was defined and used in [11] and [12] as part of the process of working with Kleisli strength. For fixed $n \geq 1$, a prestrength of order n on the functor F is a natural transformation $\Gamma_n : FX_1 \times \cdots \times FX_n \to F(X_1 \times \cdots \times X_n)$. We note immediately that a special case of a prestrength of order n on the functor F is a natural transformation $\Gamma_n : H_nF \to FH_n$, where H_n is the n-ary power functor of Example 2.1. We exploit the existence of what could be suitably called a monad-induced *generic prestrength* to derive classes of near distributive laws on free monads $\mathbf{H}_n^@$. Later, in Section 4.3, we will identify alternative non-generic kinds of prestrengths which generate in turn different near distributive laws.

Lemma 4.1 *For any monad $\mathbf{K} = (K, \nu, \rho)$ in* **Set** *there exists a generic prestrength $\Gamma_n : KX_1 \times ...KX_n \to K(X_1 \times ...X_n)$ of order $n \geq 1$.*

Proof. Let $\mathbf{K} = (K, \nu, \rho)$ be a monad in **Set**. We show for any given $n \geq 1$ there exists a natural transformation $\Gamma_n : KX_1 \times ...KX_n \to K(X_1 \times ...X_n)$. Suppose there exists a natural transformation in two variables

$$KX \times KY \xrightarrow{\Gamma_{XY}} K(X \times Y)$$

Then for $n = 1$ define $\Gamma_1 = \mathrm{id}_{X_1}$, and for $n = 2$ let $\Gamma_2 = \Gamma_{X_1X_2}$. Proceeding inductively, if Γ_i is natural, we obtain a natural transformation Γ_{i+1} by

$$KX_1 \times (KX_2 ... \times KX_{i+1}) \xrightarrow{id_{X_1} \times \Gamma_i} KX_1 \times K(X_2 ... \times X_{i+1}) \xrightarrow{\Gamma_{X_1(X_2 ... \times X_{i+1})}} K(X_1 \times X_2 ... \times X_{i+1})$$

To construct Γ_2, for $x \in X$ let $\mathrm{in}_x : Y \to X \times Y$ be defined by $\mathrm{in}_x(y) = (x, y)$. It is obvious that the following square commutes for each $f : X \to X_1,\ g : Y \to Y_1$:

$$\begin{tikzcd} Y \arrow[r, "\mathrm{in}_x"] \arrow[d, "g"'] & X \times Y \arrow[d, "f \times g"] \\ Y_1 \arrow[r, "\mathrm{in}_{fx}"'] & X_1 \times Y_1 \end{tikzcd}$$

Define $\delta_{XY} : X \times KY \to K(X \times Y)$ by $\delta_{XY}(x, \tau) = (K\,\mathrm{in}_x)\,\tau$. From the preceding square and the functoriality of K we obtain

$$\begin{tikzcd} X \times KY \arrow[r, "\delta_{XY}"] \arrow[d, "f \times Kg"'] & K(X \times Y) \arrow[d, "K(f \times g)"] \\ X_1 \times KY_1 \arrow[r, "\delta_{X_1Y_1}"'] & K(X_1 \times Y_1) \end{tikzcd}$$

At this stage, we need that $\mathbf{K}$ is a monad. Any function from X to a $\mathbf{K}$-algebra admits a unique $\mathbf{K}$-homomorphic extension $f^{\#}$ from the free algebra (KX, μ_X). Define the desired $\Gamma_{XY} : KX \times KY \to K(X \times Y)$ by $\Gamma_{XY}(\sigma, \tau) = \delta_{XY}(\cdot, \tau)^{\#}\sigma$. The desired naturality square amounts to the commutativity of

$$\begin{tikzcd} KX \arrow[r, "(\delta_{XY}(\cdot,\tau))^{\#}"] \arrow[d, "Kf"'] & K(X \times Y) \arrow[d, "K(f \times g)"] \\ KX_1 \arrow[r, "(\delta_{X_1Y_1}(\cdot,(Kf)\tau))^{\#}"'] & K(X_1 \times Y_1) \end{tikzcd}$$

for each $\tau \in KY$. Since each of the four maps in this square is a $\mathbf{K}$-homomorphism, it suffices to check commutativity restricted to the generators ρ_X and this is clear from the square for δ_{XY} immediately above. $\square$

4.2 Amenable Monads

We know of no nontrivial monad which admits a distributive law with every monad. This places some constraint on the use of distributive laws in programming. We consider instead monads which admit near distributive laws with every monad, calling these *amenable* and provide examples.

Definition 4.2 A monad $\mathbf{H}$ in $\mathcal{V}$ is **amenable** if for every monad $\mathbf{K}$ in $\mathcal{V}$, K has a functorial lift through $\mathcal{V}^{\mathbf{H}}$.

Proposition 4.3 *The monads* $\mathbf{H}_\mathbf{i}^{@}$ *in* **Set** *of Example 2.1 are amenable.*

Proof. Let $\mathbf{K} = (K, \nu, \rho)$ be a monad in **Set**. By Lemma 4.1 there exists a generic natural transformation $\Gamma_i : H_iK \to KH_i$ for every $i \geq 1$ and so by Corollary 3.5 we are done. $\square$

Example 4.4 Let $(C, e, *)$ be a monoid and let $\mathbf{K}$ be the reader monad $KX = C \times X$. Then $\lambda = \Gamma_2 : H_2K \to KH_2$ in Proposition 4.3 becomes $\Gamma_2((c_1, x_1), (c_2, x_2)) = (c_1 * c_2, (x_1, x_2))$. Acting on a binary tree t of type $H_2^{@}KX$, $\lambda^{@}(t) = (p, t^*)$ where p is the product of the c_is found in the leaves and t^* is the corresponding tree in $H_2^{@}X$ consisting only of the elements of X.

Example 4.5 When K is itself a free monad of the form $H_j^{@}$, we can give a recursive construction of the functorial lift of $K = H_j^{@}$ through $\mathbf{Set}^{H_i^{@}}$ defining the near distributive law $\lambda : H_i^{@}H_j^{@} \to H_j^{@}H_i^{@}$ in cases. Details are straightforward and left to the reader (see Example 2.1 for notation). For $i, j \geq 1$:

$$\begin{aligned}
\lambda(L_iL_ja) &= L_jL_ia \\
\lambda L_i(B_jt_1 \cdots t_j) &= B_j(\lambda L_it_1) \cdots (\lambda L_it_j) \\
\lambda B_i(t_1 \cdots t_i) &= (H_j^{@}B_i)\Gamma_i((\lambda t_1), \cdots, (\lambda t_i))
\end{aligned}$$

Proposition 4.6 *Let $\mathcal{V}$ have small coproducts, let (H_α) be a small family of endofuctors and let $H = \coprod H_\alpha$ be the pointwise coproduct. Assume that the free monads $\mathbf{H}_\alpha^{@}$, $\mathbf{H}^{@}$ exist. Then if each $\mathbf{H}_\alpha^{@}$ is amenable, so is $\mathbf{H}^{@}$.*

Proof. An $\mathbf{H}$-algebra is determined by a family $(\delta_\alpha : H_\alpha X \to X)$. Let $\mathbf{K}$ be a monad in $\mathcal{V}$ and let $(X,\, H_\alpha X \xrightarrow{\delta_\alpha} X) \mapsto (KX, H_\alpha KX \xrightarrow{\epsilon_\alpha} X)$ under a functorial lift of K through $\mathcal{V}^{\mathbf{H}_\alpha}$. The remaining details are clear. □

Example 4.7 Let Σ be a finitary operator domain, that is, a disjoint sequence (Σ_n) of (possibly empty) sets. A Σ-algebra (as conventionally defined in universal algebra) is (X, δ) where X is a set and $\delta = (\delta_\sigma : \sigma \in \Sigma)$ with $\delta_\sigma : X^n \to X$ if $\sigma \in \Sigma_n$. Consider the coproduct functor

$$H_\Sigma X = \coprod_{\sigma \in \Sigma_n} X^n$$

so that an H_Σ-algebra is the same thing as a Σ-algebra.

A variety of universal algebras is obtained from $\mathbf{Set}^{H_\Sigma}$ by imposing equations. $H_\Sigma^{@}X$ is the usual free Σ-algebra generated by X. It is immediate from Propositions 4.3 and 4.6 that $\mathbf{H}_\Sigma^{@}$ is an amenable monad in $\mathbf{Set}$.

4.3 Prestrengths and Flat near distributive Laws

In this section we consider a different class of flat near distributive laws which generally differ from those of the previous section.

Lemma 4.8 *For any monad $\mathbf{K} = (K, \nu, \rho)$ in $\mathbf{Set}$, if there exists a natural transformation $\gamma : K \to id$ then there exists a prestrength $\Gamma_i : KA_1 \times ... \times KA_i \to K(A_1 \times ... \times A_i)$ for any $i \geq 1$.*

Proof. The construction is simple: for $i = 1$ define $\Gamma_1 = \rho \circ \gamma$. If $i \geq 2$ then $\Gamma_i = \rho \circ (\gamma \times \ldots \times \gamma)$. Since in each case Γ_i is a composition of natural transformations we are done. □

Proposition 4.9 *For any monad* $\mathbf{K} = (K, \nu, \rho)$ *with natural transformation* $\gamma : K \to id$*, there exists a flat near distributive law* $\lambda^{@} : \mathbf{H_i^{@}} K \to K\mathbf{H_i^{@}}$*.*

Proof. For any $i \geq 1$, the prestrength Γ_i of the previous lemma generates a natural transformation $H_i K \to K H_i$ and so the result follows immediately from Corollary 3.5. $\square$

Example 4.10 For $j \geq 1$ let π_j denote the j-th projection natural transformation $H_j \to \mathrm{id}$. By the previous proposition this generates a flat near distributive law $\lambda^{@} : \mathbf{H_i^{@}} H_j^{@} \to H_j^{@}\mathbf{H_i^{@}}$ which generally differs from that of Example 4.5.

Example 4.11 Let $\mathbf{K}$ be the reader monad of Example 4.4 with monoid $(C, e, *)$. Define $\gamma : K \to \mathrm{id}$ by $\gamma(c, a) = a$. Such γ is natural, generating $\Gamma_n : KA_1 \times ... KA_n \to K(A_1 \times ... A_n)$. For the monad of non-empty lists $\mathbf{L}$, the resulting flat distributive law $\lambda^{@} : \mathbf{L}(C \times A) \to C \times \mathbf{L}A$ takes $[(c_1, a_1), ...(c_n, a_n)]$ to $(e, [a_1, ...a_n])$.

4.4 Uniformly branching trees and non-flat near distributive laws.

For the free monad $\mathbf{H_i^{@}}$, $H_i^{@}X$ consists of trees in which every non-leaf has i branches. Due to their particular structure, these trees also generate a class of (*not necessarily flat*) near distributive laws of $\mathbf{H_i^{@}}$ over $\mathbf{H_j^{@}}$ for which very little underlying data is either created or destroyed in the process. Significantly these near distributive laws do not arise via flat liftings, but rather arise directly from the monad structure on $H_i^{@}$.

Recall that an algebra for $H_i^{@}$ is generated by $(A, [\ ]_i)$, where $[\ ]_i : A^i \to A$ is an i-ary operation on A. For $i, j \geq 1$, we build a recursive schema for canonical functorial liftings of $H_j^{@}$ over $\mathbf{Set}^{H_i^{@}}$. To do this, we define $(H_j^{@})^*$ in cases and expressly define $(H_j^{@})^*(A, [\ ]_i) = (H_j^{@}A, [\ ]_i)$. (Note that we use the same notation for the two i-ary operations). When $i = 1$

- $[(L_j\ a)]_1 = L_j([\ a]_1)$
- $[(B_j\ t_1 ... t_j)]_1 = B_j\ [t_1]_1 ... [t_j]_1$

Likewise when $j = 1$ we have

- $[L_1 a_1, ... L_1 a_i]_i = L_1[a_1, ... a_i]_i$
- $[L_1 a_1, ... L_1 a_{i-1}, (B_1\ t)]_i = B_1\ [L_1 a_1, ... L_1 a_{i-1}, t]_i$
- *etc*
- $[(B_1\ t_1)\ t_2 ... t_i]_i = B_1\ [t_1,\ t_2 ... t_i]_i$

Otherwise for $i, j \geq 2$

- $[L_j a_1, ... L_j a_i]_i = L_j[a_1, ... a_i]_i$
- $[L_j a_1, ... L_j a_{i-1}, (B_j\ t_{i,1} ... t_{i,j})]_i = B_j\ [L_j a_1, ... L_j a_{i-1}, t_{i,1}]_i\ t_{i,2} ... t_{i,j}$
- *etc*
- $[(B_j\ t_{1,1} ... t_{1,j})\ t_2 ... t_i]_i = B_j\ t_{1,1} ... t_{1,j-1}\ [t_{1,j},\ t_2 ... t_i]_i$

Theorem 4.12 *For $i, j \geq 1$, there exists a schema of recursively defined near distributive laws $\lambda : H_i^@ H_j^@ \to H_j^@ H_i^@$ between all free monads $H_i^@, H_j^@$ as defined above.*

Proof. A near distributive law λ is created via the lifting functor $(H_j^@)^*$ over $H_i^@$ algebras described above. Applying $(H_j^@)^*$ to $(H_i^@ A, B_i)$, the i-ary operation associated to the canonical algebra $(H_i^@ A, \mu)$ generates λ defined by the following set of equations:

- $\lambda(L_i L_j\ a) = L_j L_i\ a$
- $\lambda L_i(B_j t_1 ...\ t_j) = B_j(\lambda\ L_i t_1)...(\lambda\ L_i t_j)$
- $\lambda(B_i\ t_1 ...\ t_i) = [\lambda t_i]_i$ where $[\ ]_i$ is defined as in the previous result

Verifying that the two laws (DL C) and (DL D) hold follows from a straightforward argument via structural recursion and is left to the reader.

□

Example 4.13 For the special case of $i = 1$ of Theorem 4.12, $H_1^@$ is the writer monad $N \times \_$ where N is the commutative monoid of natural numbers $\{0, 1, 2, ...\}$ under addition. $\lambda : N \times H_j^@\ a \to H_j^@(N \times A)$ is actually a distributive law where $\lambda(n, t)$ distributes n to each leaf of tree t. Likewise for the special case of $j = 1$, $\lambda : H_i^@(N \times A) \to N \times H_i^@ A$ is again a full distributive law as follows. For an arbitrary tree t in $H_i^@(N \times A)$, $\lambda\ t = (k, t^*)$ where t^* is the tree in $H_i^@ A$, with the same shape as t, generated by replacing every leaf in t of the form $L_i(m, a)$ by $L_i a$ and where k equals the sum of all the m's found in the leaves.

Are the near distributive laws of Theorem 4.12 always distributive laws as in the two cases of the previous example? The answer is no in every other case.

Theorem 4.14 *For any $i, j \geq 2$ the near distributive law $\lambda : H_i^@ H_j^@ \to H_j^@ H_i^@$ of Theorem 4.12 fails to produce a distributive law as one can produce a tree $t \in H_i^@ H_j^@ H_j^@$ for which law (DL B) fails.*

Proof. For $\lambda : H_i^@ H_j^@ \to H_j^@ H_i^@$ we produce a tree $t \in H_i^@ H_j^@ H_j^@$ with $4(j-1)+i$ leaves for which (DL B) fails. Let

- $lt = B_j\ (L_j\ (L_j\ a_1))...(L_j\ (L_j\ a_{j-1}))\ (L_j(B_j(L_j a_j)...(L_j a_{2j-1})))$
- $rt = B_j\ (L_j(B_j(L_j a_{2j})...(L_j a_{3j-1})))\ (L_j\ (L_j\ a_{3j}))...(L_j\ (L_j\ a_{4j-2}))$
- $t = B_i(L_i(lt))\ (L_i(L_j(L_j b_1)))\ ...\ (L_i(L_j(L_j b_{i-2})))\ (L_i(rt))$

then (DL B) fails for this t. The details are left to the reader. □

5 Pre-Monads

Definition 5.1 A **pre-monad** in $\mathcal{V}$ is $\mathbf{H} = (H, \mu, \eta)$ with $H : \mathcal{V} \to \mathcal{V}$ a functor and with $\eta : \mathrm{id} \to H$, $\mu : HH \to H$ natural transformations.

Composition of pre-monads is easily obtained. If (H, μ, η) and (K, ν, ρ) are pre-monads and if $\lambda : HK \to KH$ is a natural transformation, we obtain the **composite** pre-monad

$$(KH,\ KHKH \xrightarrow{K\lambda H} KKHH \xrightarrow{\nu\mu} KH,\ \mathrm{id} \xrightarrow{\rho\eta} KH)$$

It develops that λ with additional axioms will classify a functorial lift of K through $\mathcal{V}^{\mathbf{H}}$. To make sense of this we will have to define the "Eilenberg-Moore" category $\mathcal{V}^{\mathbf{H}}$.

Definition 5.2 The axioms defining an **algebra** (X, ξ) for a pre-monad $\mathbf{H} = (H, \mu, \eta)$ and an **H**-homomorphism $f : (X, \xi) \to (Y, \theta)$ are exactly the same as for a monad, namely

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "\mathrm{id}_X"'] & HX \arrow[d, "\xi"'] & HHX \arrow[l, "\mu_X"'] \arrow[d, "H\xi"] \\ & X & HX \arrow[l, "\xi"] \end{tikzcd} \qquad \begin{tikzcd} HX \arrow[r, "Hf"] \arrow[d, "\xi"'] & HY \arrow[d, "\theta"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

As for monads, the resulting category of algebras is denoted $\mathcal{V}^{\mathbf{H}}$. It is well known that the theory of algebras for a monad provides an approach to developing universal algebra [9]. It is very frequently the case that for a pre-monad $\mathbf{K}$ there is an isomorphism $\Phi : \mathcal{V}^{\mathbf{K}} \to \mathcal{V}^{\mathbf{K}^\bullet}$ with $\mathbf{K}^\bullet$ a monad, i.e. that $\mathcal{V}^{\mathbf{K}} \to \mathcal{V}$ is monadic. By the well-known "Beck tripleableness theorem" it is enough that $\mathcal{V}$ is complete and that $\mathcal{V}^{\mathbf{K}} \to \mathcal{V}$ satisfies the solution set condition, since the coequalizer condition of the theorem always holds. This shows that pre-monads play a role in developing universal algebra. This idea will be developed elsewhere, but we present an example now, with emphasis on the idea that pre-monads may reduce complications for the programmer.

A **band** is a semigroup in which every element is idempotent. Bands arise as the variety of semigroups satisfying the additional equation $xx = x$, so the monad $\mathbf{B}$ for bands is a quotient monad $\theta : \mathbf{L} \to \mathbf{B}$ of the monad of non-empty lists. Note that throughout the paper the list monad will refer to non-empty lists. From the point of view of the monad programmer, the construction of the free band BX (despite the fact that BX is finite when X is, unlike the situation with lists) is not intuitive. An entire section of [6] is devoted to the word problem involved.

We next introduce an approach to bands which uses only the list data type. This illustrates our claim that, by relaxing axioms, we can sometimes describe what we need using readily available data types. We shall see in Theorem 6.2 below how this result leads to a simplification in specifying near distributive laws. In effect, we have gotten around the work to solve the word problem for free bands by simply not needing it!

Proposition 5.3 *Let (L, m, e) be the monad of non-empty lists in* **Set**. *Modify this to the pre-monad $(L, m, \hat{e})$ where $\hat{e}_X\, x = [x, x]$. Then $\mathbf{Set}^{(L,m,\hat{e})}$ is the category of* *bands.*

Proof. An algebra (X, ξ) satisfies

$$\begin{tikzcd}
X \arrow[r, "\hat{e}_X"] \arrow[dr, "\mathrm{id}_X"'] & LX \arrow[d, "\xi"'] & LLX \arrow[l, "m_X"'] \arrow[d, "L\xi"] \\
& X & LX \arrow[l, "\xi"]
\end{tikzcd}$$

We have

$$\begin{aligned} x = \xi[x,x] \;&=\; \xi(m_X[[x],[x]]) \;=\; \xi(L\xi)[[x],[x]]) \\ = \xi([\xi[x],\xi[x]] \;&=\; \xi\hat{e}_X(\xi[x]) \;=\; \xi[x] \end{aligned}$$

But then (X, ξ) is also an algebra of the list monad, that is a semigroup $(X, \cdot)$ with $\xi([x_1, \ldots, x_n]) = x_1 \cdots x_n$. This semigroup is a band because $x^2 = \xi[x,x] = x$. The remaining details are routine. □

Although every monad is a pre-monad, a pre-monad need not satisfy any of the three monad axioms. We do have two pre-monad laws or axioms (PME.1, PME.2) where “PM” stands for “pre-monad”.

Proposition 5.4 *Pre-monads may be equivalently described as $(H, (\cdot)^{\#}, \eta)$ where $H : \mathcal{V} \to \mathcal{V}$ is a functor, $\eta : id \to H$ is a natural transformation and $X \xrightarrow{f} HY \mapsto HX \xrightarrow{f^{\#}} HY$ is an operator subject to the axioms*

(PME.1) For $g : Y \to HZ$, $\quad g^{\#} \;=\; HY \xrightarrow{Hg} HHZ \xrightarrow{(\mathrm{id}_{HZ})^{\#}} HZ$

(PME.2) For $f : X \to HY$, $g : Y \to Z$, $\;(Hg)\, f^{\#} = ((Hg)f)^{\#}$

As for monads, the correspondences are

(4) $\quad f^{\#} = HX \xrightarrow{Hf} HHY \xrightarrow{\mu_Y} HY$

(5) $\quad \mu_X = (\mathrm{id}_{HX})^{\#}$

Proof. Let (H, μ, η) be a pre-monad. For $f : X \to HY$ define $f^{\#} : HX \to HY$ as in (4). Since $H(\mathrm{id}_{HX}) = \mathrm{id}_{HX}$, $(\mathrm{id}_{HX})^{\#} = \mu_X$, and this gives (PME.1). For (PME.2), let $f : X \to HY$, $g : Y \to Z$. Then

$$\begin{aligned} (Hg)f^{\#} &= (Hg)\,\mu_Y\,(Hf) = \mu_Z\,(HHg)\,(Hf) \quad (\mu \text{ natural}) \\ &= \mu_Z\, H((Hg)f) = ((Hg)f)^{\#} \end{aligned}$$

Conversely, let (PME.1, PME.2) hold and define μ by (5). (4) holds by PME.1. For $g : Y \to Z$,

$$\begin{aligned} (Hg)\,\mu_Y &= (Hg)\,(\mathrm{id}_{HY})^{\#} = (\,(Hg)\,\mathrm{id}_{HY})^{\#} \quad \text{(PME.2)} \\ &= (Hg)^{\#} = \mu_Z\,(HHg) \end{aligned}$$

which shows that μ is natural. To complete the proof, we show the two passages are inverse bijections. Start with $(\cdot)^\#$, define $\mu_Z = (\mathrm{id}_{HZ})^\#$ and then $(\cdot)^{\#\#}$ by (4). Then $(\cdot)^\# = (\cdot)^{\#\#}$ by (PME.1). Starting with μ, define $(\cdot)^\#$ as in (4) and then $\nu_Z = (\mathrm{id}_{HZ})^\#$. Then $\nu_Z = \mu_Z$ as is clear from setting $g = \mathrm{id}_{HZ}$ in (4). The proof is complete. □

Definition 5.5 Let $\mathbf{H} = (H, \mu, \eta)$, $\mathbf{K} = (K, \nu, \rho)$ be pre-monads. A **pre-monad map** $\sigma : \mathbf{H} \to \mathbf{K}$ is a natural transformation $\sigma : H \to K$ such that

$$\begin{tikzcd}
\mathrm{id} \arrow[r, "\eta"] \arrow[dr, "\rho"'] & H \arrow[d, "\sigma"] & HH \arrow[l, "\mu"'] \arrow[d, "\sigma\sigma"] \\
 & K & KK \arrow[l, "\nu"]
\end{tikzcd}$$

The definition is the same as the usual one for monad maps so that monads form a full subcategory of pre-monads.

Definition 5.6 Given a pre-monad $\mathbf{H}$ in $\mathcal{V}$, a **monad approximation** of $\mathbf{H}$ is a reflection $\sigma : \mathbf{H} \to \mathbf{K}$ of $\mathbf{H}$ in the full subcategory of monads.

Theorem 5.7 *Let $\mathbf{H} = (H, \mu, \eta)$, $\mathbf{K} = (K, \nu, \rho)$ be pre-monads in $\mathcal{V}$. Then a pre-monad map $\sigma : H \to K$ induces a functor $W : \mathcal{V}^{\mathbf{K}} \to \mathcal{V}^{\mathbf{H}}$ over $\mathcal{V}$ defined by*

(6) $W(X, \xi) = (X,\, HX \xrightarrow{\sigma_X} KX \xrightarrow{\xi} X)$

If, additionally, $\mathbf{K}$ is a monad, then $\sigma \mapsto W$ is bijective with inverse

(7) $\sigma_X = HX \xrightarrow{H\rho_X} HKX \xrightarrow{\gamma_X} KX$

where $(KX, \gamma_X) = W(KX, \nu_X)$.

Proof. Given σ, we first show $W(X, \xi)$ is an $\mathbf{H}$-algebra. This follows from $\xi\,\sigma_X\,\eta_X = \xi\,\rho_X = id_X$ and

$$\begin{aligned}
\xi\,\sigma_X\,\mu_X &= \xi\,\nu_X(\sigma\sigma)_X && (\sigma \text{ pre-monad map}) \\
&= \xi\,\nu_X\,\sigma_{KX}(H\sigma_X) \\
&= \xi(K\xi)\sigma_{KX}(H\sigma_X) && (\mathbf{K}\text{-algebra}) \\
&= \xi\,\sigma_X(H\xi)(H\sigma_X) && (\sigma \text{ natural}) \\
&= \xi\,\sigma_X H(\xi\sigma_X)
\end{aligned}$$

That W maps homomorphisms to homomorphisms is clear from the naturality of σ. Now assume that $\mathbf{K}$ is a monad, so that (KX, ν_X) is a $\mathbf{K}$-algebra. We next show that if $W \mapsto \sigma \mapsto \overline{W}$ then $\overline{W} = W$. (Of course we cannot assume here that σ is a pre-monad map since that has not yet been shown). Starting with a $\mathbf{K}$-algebra (X, ξ), $\overline{W}(X, \xi)$ is $(X,\, HX \xrightarrow{\sigma_X} KX \xrightarrow{\xi} X)$ where $\sigma_X = HX \xrightarrow{H\rho_X} HKX \xrightarrow{\gamma_X} KX$ and $(KX, \gamma_X) = W(KX, \nu_X)$. As $\xi : (KX, \nu_X) \to (X, \xi)$ is a $\mathbf{K}$-homomorphism, $\xi : (KX, \gamma_X) \to W(X, \xi)$ is an $\mathbf{H}$-homomorphism. Writing $W(X, \xi)$ as (X, δ), we have the commutative diagram

$$\begin{tikzcd} HX \arrow[r, "H\rho_X"] \arrow[dr, "\mathrm{id}_X"'] & HKX \arrow[r, "\gamma_X"] \arrow[d, "H\xi"] & KX \arrow[d, "\xi"] \\ & HX \arrow[r, "\delta"'] & X \end{tikzcd}$$

where the square is because ξ is a homomorphism and the triangle is a **K**-algebra axiom. But the top row is σ_X, so $\delta = \xi\,\sigma_X$ and $\overline{W}(X,\xi) = (X,\delta) = W(X,\xi)$. We may apply this, in particular, to the **K**-algebra (KX, ν_X) to establish that

(8) $\gamma_X = \nu_X\,\sigma_{KX}$

We turn to showing that $W \mapsto \sigma$ is well defined. For $f : X \to Y$ in $\mathcal{V}$, $Kf : (KX, \nu_X) \to (KY, \nu_Y)$ is a **K**-homomorphism. Applying W gives the square on the right in the diagram

$$\begin{tikzcd} HX \arrow[r, "H\rho_X"] \arrow[d, "Hf"'] & HKX \arrow[r, "\gamma_X"] \arrow[d, "HKf"'] & KX \arrow[d, "Kf"] \\ HY \arrow[r, "H\rho_Y"'] & HKY \arrow[r, "\gamma_Y"'] & KY \end{tikzcd}$$

But the square on the left commutes because ρ is natural. Since the rows are σ_X and σ_Y, the perimeter of the diagram then shows that σ is natural. The first pre-monad map law is shown by

$$\begin{aligned} \sigma_X\,\eta_X = \gamma_X(H\rho_X)\eta_X \ &= \ \gamma_X\,\eta_{KX}\,\rho_X \qquad (\eta \text{ natural}) \\ = \rho_X \qquad &((KX, \gamma_X) \text{ algebra}) \end{aligned}$$

For the second pre-monad map law,

$$\begin{aligned} \nu_X(\sigma\sigma)_X = \nu_X\,\sigma_{KX}(H\sigma_X) \ &= \ \gamma_X(H\sigma_X) \qquad (\text{by } (8)\,) \\ = \gamma_X(H\gamma_X)(HH\rho_X) \ &= \ \gamma_X\,\mu_{KX}(HH\rho_X) \qquad ((KX,\gamma) \text{ algebra}) \\ = \gamma_X(H\rho_X)\mu_X \qquad &(\mu \text{ natural}) \\ = \sigma_X\,\mu_X & \end{aligned}$$

Finally, we show that if $\sigma \mapsto W \mapsto \overline{\sigma}$ then $\overline{\sigma} = \sigma$.

$$\begin{aligned} \overline{\sigma}_X = \gamma_X(H\rho_X) \ &= \ \nu_X\,\sigma_{KX}(H\rho_X) \qquad (\text{ by } (8)) \\ = \nu_X(K\rho_X)\sigma_X \qquad &(\sigma \text{ natural}) \\ = \sigma_X \qquad &(\mathbf{K} \text{ monad}) \end{aligned}$$

□

Theorem 5.8 *Let* **H** *be a pre-monad such that* $U : \mathcal{V}^{\mathbf{H}} \to \mathcal{V}$ *is monadic so that there exists a monad* **K** *and an isomorphism of categories* $\Phi : \mathcal{V}^{\mathbf{K}} \to \mathcal{V}^{\mathbf{H}}$ *over* $\mathcal{V}$*. Then the corresponding pre-monad map* $\sigma : \mathbf{H} \to \mathbf{K}$ *of Theorem 5.7 is a monad approximation of* **H***.*

Proof. Let $\alpha : \mathbf{H} \to \mathbf{M}$ be a pre-monad map with $\mathbf{M}$ a monad and show that there exists a unique monad map β completing the triangle below.

$$\begin{tikzcd} H \arrow[r, "\sigma"] \arrow[dr, "\alpha"'] & K \arrow[d, "\beta"] \\ & M \end{tikzcd}$$

Let $W : \mathcal{V}^{\mathbf{M}} \to \mathcal{V}^{\mathbf{H}}$ correspond to α and let β be the unique monad map corresponding to $\mathcal{V}^{\mathbf{M}} \xrightarrow{W} \mathcal{V}^{\mathbf{H}} \xrightarrow{\Phi^{-1}} \mathcal{V}^{\mathbf{K}}$. We leave the remaining details to the reader. $\square$

Example 5.9 By the theorem, $\sigma : (L, m, \hat{e}) \to \mathbf{B}$ is a monad approximation of the band monad. This example can be easily generalized. Let $(L, m, \hat{e})$ be the list premonad where $\hat{e}(x) = [x, x, x, \cdots, x]$ n-times and m is the usual counit for lists. Then $\mathbf{Set}^{(L,m,\hat{e})}$ is the category of n-bands (semigroups in which $x^n = x$ for every element).

Example 5.10 Let $(L, \hat{m}, e)$ be the list premonad where $e(x) = [x]$ and $\hat{m}[l_1, ...l_k] = l{+}{+}\, l{+}{+}...\, l$ k-times where l has the same length as list l_1 and all of its elements are the first element of l_1. Then $\sigma : L \to N \times A$ where $\sigma\ [a_1...a_k] = (k, a_1)$ has as monad approximation the reader monad $N \times A$ where $N = (N, *, 1)$ is the monoid of natural numbers under multiplication.

Example 5.11 A rectangular band is a semigroup in which every element is idempotent and additionally the equation $xyz = xz$ holds. Let $(L, \hat{m}, \hat{e})$ be the list premonad with $\hat{e}(x) = [x, x]$ and $\hat{m}\ ll = [fst\ fst\ ll, lst\ lst\ ll]$ where *fst* and *lst* pick out the first and last elements of a non-empty list. As in the case of Proposition 5.3, an algebra (X, ξ) in $\mathbf{Set}^{(L,\hat{m},\hat{e})}$ satisfies $x = \xi[x]$. Additionally,

$$\begin{aligned} x(yz) &= \xi[\xi[x], \xi[y, z]] \\ &= \xi \circ \hat{m}[[x], [y, z]] = \xi[x, z] = xz \end{aligned}$$

Similarly, $(xy)z = xz$. The rectangular band monad $\mathbf{R} = (R, \mu, \eta)$ where $R\ A = A \times A$ is the monad approximation of $(L, \hat{m}, \hat{e})$ defined by the reflection $\sigma\ [x] = (x, x)$ and $\sigma\ [x_1, \ldots x_n] = (x_1, x_n)$. One can easily check that σ is a premonad map and that the monad properties of (R, μ, η) can be derived directly from $(L, \hat{m}, \hat{e})$. See Example 6.5.

Example 5.12 Let $(L, \hat{m}, e)$ be the list premonad where $e(x) = [x]$, $\hat{m}[[x]] = [x]$ and $\hat{m}[l_1, ...l_k] = [fst\ l_1, fst\ l_1]$. Then $\sigma : L \to B \times A$ defined by $\sigma\ [a] = (F, a)$, $\sigma\ [a_1...a_k] = (T, a_1)$ defines a monad approximation of the *Boolean*-set monad $B \times A$ where $\mathrm{Bool} = (T, F)$, $B = (\mathrm{Bool}, OR, F)$. The algebras (X, ξ) in $\mathbf{Set}^{(L,\hat{m},e)}$ are semigroups that satisfy $xy = x^2$ since

$$\begin{aligned} xy &= \xi[x, y] = \xi[\xi[x], \xi[y]] \\ &= \xi \circ \hat{m}[[x], [y]] = \xi[x, x] = x^2 \end{aligned}$$

6 Near Distributive Laws for Pre-Monads

We note that the laws DL A, DL B, DL C, DL D make sense whenever (H, μ, η), (K, ν, ρ) are pre-monads. In this section, we show how distributive laws and near distributive laws for pre-monads induce similar laws on their monad approximations. As shown in [10, Theorem 2.2.2] the following result is well known when $\mathbf{H}$ is a monad. The generalization to the case when $\mathbf{H}$ is a pre-monad must be proved with some care.

Theorem 6.1 *Let $K : \mathcal{V} \to \mathcal{V}$ be a functor, (M, m, e) a monad in $\mathcal{V}$ and let (H, μ, η) be a pre-monad in $\mathcal{V}$ such that $\mathcal{V}^{\mathbf{H}} \to \mathcal{V}$ is monadic. Functorial lifts $K^\star : \mathcal{V}^{\mathbf{M}} \to \mathcal{V}^{\mathbf{H}}$ correspond bijectively to natural transformations $\lambda : HK \to KM$ which satisfy $(K^\star\, A, K^\star\, B)$:*

$$\begin{tikzcd}
K \arrow[r, "\eta K"] \arrow[ddr, "Ke"'] \arrow[dr, phantom, "(K^\star\ A)"] & HK \arrow[dd, "\lambda"] \arrow[dr, phantom, "(K^\star\ B)"] & HHK \arrow[l, "\mu K"'] \arrow[d, "H\lambda"] \\
 & & HKM \arrow[d, "\lambda M"] \\
 & KM & KMM \arrow[l, "Km"]
\end{tikzcd}$$

The correspondences are

(9) $K^\star(X, MX \xrightarrow{\theta} X) = (KX, HKX \xrightarrow{\lambda_X} KMX \xrightarrow{K\theta} KX)$

and, if $K^\star(MX, m_X) = (KMX, \gamma_X)$,

(10) $\lambda_X = HKX \xrightarrow{HKe_X} HKMX \xrightarrow{\gamma_X} KMX$

Moreover, half of this result holds if $\mathbf{M}$ is only a pre-monad, namely if λ satisfies $(K^\star\, A)$ and $(K^\star\, B)$, then $K^\star$ as in (9) is a functorial lift $\mathcal{V}^{\mathbf{M}} \to \mathcal{V}^{\mathbf{H}}$ of K.

Proof. Given λ satisfying $(K^\star\, A)$ and $(K^\star\, B)$ and assuming that $\mathbf{M}$, $\mathbf{H}$ are arbitrary pre-monads, we first show that $(KMX, (K\theta)\,\lambda_X)$ is an $\mathbf{H}$-algebra by checking the properties in Definition 5.2. We have

$$\begin{aligned}
(K\theta)\lambda_X \eta_{KX} &= (K\theta)(Ke_X) \qquad ((K^*A) \text{ for } \lambda)\\
&= id_{KX} \qquad (\theta \text{ algebra})
\end{aligned}$$

Also,

$$\begin{aligned}
(K\theta)\lambda_X \mu_{KX} &= (K\theta)(Km_X)\lambda_{MX}(H\lambda_X) \qquad ((\mathrm{K}^* \text{ B}) \text{ for } \lambda)\\
&= (K\theta)(KM\theta)\lambda_{MX}(H\lambda_X) \qquad (\theta \text{ algebra})\\
&= (K\theta)\lambda_X(HK\theta)(H\lambda_X) \qquad (\lambda \text{ natural})
\end{aligned}$$

Further an $\mathbf{M}$-homomorphism $f : (X, \xi) \to (Y, \theta)$, Kf is an $\mathbf{H}$-homomorphism as follows:

$$\begin{aligned}
(Kf)(K\xi)\lambda_X &= (K\theta)(KMf)\lambda_X \qquad (\mathbf{M}\text{-homomorphism})\\
&= (K\theta)\lambda_Y(HKf) \qquad (\lambda \text{ natural})
\end{aligned}$$

Conversely, now assuming that $\mathbf{M}$ is a monad, let $K^\star : \mathcal{V}^{\mathbf{M}} \to \mathcal{V}^{\mathbf{H}}$ be a functorial lift of K. Let $\sigma : H \to \widehat{H}$ be such that $(\widehat{H}, \widehat{\mu}, \widehat{\eta})$ is the monad approximation corresponding to the isomorphism $\Phi : \mathcal{V}^{\widehat{\mathbf{H}}} \to \mathcal{V}^{\mathbf{H}}$, $(X, \xi) \mapsto (X, \xi\sigma_X)$ as in Theorem 5.8. This gives rise to a new functorial lift

$$\mathcal{V}^{\mathbf{M}} \xrightarrow{K^\star} \mathcal{V}^{\mathbf{H}} \xrightarrow{\Phi^{-1}} \mathcal{V}^{\widehat{\mathbf{H}}}$$

of K which, since the theorem holds for monads, corresponds to a natural transformation $\widehat{\lambda} : \widehat{H}K \to KM$ which satisfies $(K^\star\, A)$ and $(K^\star\, B)$. Define

$$\lambda \;=\; HK \xrightarrow{\sigma K} \widehat{H}K \xrightarrow{\widehat{\lambda}} KM$$

We first show that such λ satisfies (K* A) and (K* B).

$$\begin{aligned}
\widehat{\lambda}(\sigma K)(\eta K) &= \widehat{\lambda}(\widehat{\eta}K) \qquad (\sigma \text{ pre-monad map})\\
&= Ke \qquad ((K^* \text{ A}) \text{ for } \widehat{\lambda})
\end{aligned}$$

$$\begin{aligned}
\widehat{\lambda}(\sigma K)(\mu K) &= \widehat{\lambda}(\widehat{\mu}K)(\sigma\widehat{H}K)(H\sigma K) \qquad (\sigma \text{ pre-monad map})\\
&= (Km)(\widehat{\lambda}M)(\widehat{H}\widehat{\lambda})(\sigma\widehat{H}K)(H\sigma K) \qquad ((K^\star\, B) \text{ for } \widehat{\lambda})\\
&= (Km)(\widehat{\lambda}M)(\sigma KM)(H\widehat{\lambda})(H\sigma K) \qquad (\sigma \text{ natural})
\end{aligned}$$

as desired.

If $K^\star(MX, m_X) = (KMX, \gamma_X)$ there exists a unique $\widehat{\mathbf{H}}$-algebra $(KMX, \widehat{\gamma}_X)$ with $\gamma_X = HKMX \xrightarrow{\sigma_{KMX}} \widehat{H}KMX \xrightarrow{\widehat{\gamma}_X} KMX$. By (10), which holds since $\widehat{\mathbf{H}}$ is a monad, $\widehat{\lambda_X} = \widehat{H}KMX \xrightarrow{\widehat{H}Ke_X} \widehat{H}KMX \xrightarrow{\widehat{\gamma}_X} KMX$. We can then check that λ is also defined by (10) as follows.

$$\begin{aligned}
\widehat{\lambda}_X\, \sigma_{KX} &= \widehat{\gamma}_X\, (\widehat{H}Ke_X)\, \sigma_{KX}\\
&= \widehat{\gamma}_X\, \sigma_{KMX}\, (HKe_X) \qquad (\sigma \text{ natural})\\
&= \gamma_X\, (HKe_X)
\end{aligned}$$

So far, the passages of (9, 10) are well defined. If $\lambda \mapsto K^\star \mapsto \lambda_1$ then $\gamma_X = (Km_X)\, \lambda_{MX}$ so

$$\begin{aligned}
\lambda_{1,X} &= (Km_X)\, \lambda_{MX}\, (HKe_X) \;=\; (Km_X)\, (KMe_X)\, \lambda_X \qquad (\lambda \text{ natural})\\
&= id_{KMX}\, \lambda_X \qquad (m_X \text{ algebra})\\
&= \lambda_X
\end{aligned}$$

If $K^\star \mapsto \lambda \mapsto K^\bullet$, let $K^\star(X, \xi) = (KX, HKX \xrightarrow{\delta} KX)$. Taking $K^\star$ of the $\mathbf{M}$-homomorphism $\xi : (MX, m_X) \to (X, \xi)$ gives a commutative square

$$\begin{tikzcd}
HKMX \arrow[r, "HK\xi"] \arrow[d, "\gamma_X"'] & HKX \arrow[d, "\delta"] \\
KMX \arrow[r, "K\xi"'] & KX
\end{tikzcd}$$

Then $K^\bullet(X, \xi) = (KX, \epsilon)$ where

$$\begin{aligned}
\epsilon &= (K\xi)\, \lambda_X \;=\; (K\xi)\, \gamma_X\, (KHe_X)\\
&= \delta\, (HK(\xi e_X)) \;=\; \delta \qquad (\xi \text{ algebra})
\end{aligned}$$

□

We conclude the section with the promised theorems and point out the connection to Proposition 5.3 (which asserts that the band monad is the monad approximation of $(L, m, \widehat{e})$).

Theorem 6.2 *Let $\mathbf{K} = (K, \nu, \rho)$ be a pre-monad in $\mathcal{V}$ and let $\mathbf{H} = (H, \mu, \eta)$ be a pre-monad in $\mathcal{V}$ with monad approximation $\sigma : \mathbf{H} \to \widehat{\mathbf{H}}$, $\widehat{\mathbf{H}} = (\widehat{H}, \widehat{\mu}, \widehat{\eta})$. Let $\lambda : HK \to KH$ be a natural transformation satisfing (DL C, DL D). Then there exists a near distributive law $\widehat{\lambda} : \widehat{H}K \to K\widehat{H}$ of $\widehat{\mathbf{H}}$ over $\mathbf{K}$ such that the following square commutes. We say λ **generates** $\widehat{\lambda}$.*

$$\begin{tikzcd} HK \arrow[r, "\sigma K"] \arrow[d, "\lambda"'] & \widehat{H}K \arrow[d, "\widehat{\lambda}"] \\ KH \arrow[r, "K\sigma"'] & K\widehat{H} \end{tikzcd} \qquad (6)$$

Proof. For $\sigma : H \to \widehat{H}$ the monad approximation of H, the isomorphism $\Phi : \mathcal{V}^{\widehat{\mathbf{H}}} \to \mathcal{V}^{\mathbf{H}}$ maps the free algebra $(\widehat{H}X, \widehat{\mu}_X)$ to the $\mathbf{H}$-algebra $(\widehat{H}X, \widehat{\mu}_X \sigma_{\widehat{H}X})$, $K^\star$ of which is $(K\widehat{H}X, \omega_X)$ where

$$\omega_X \;=\; HK\widehat{H}X \xrightarrow{\lambda_{\widehat{H}X}} KH\widehat{H}X \xrightarrow{K\sigma_{\widehat{H}X}} K\widehat{H}\widehat{H}X \xrightarrow{K\widehat{\mu}_X} K\widehat{H}X$$

Applying Theorem 6.1 to the composite functorial lift through $\mathcal{V}^{\widehat{H}}$, $(\widehat{H}X, \widehat{\mu}_X)$ is mapped to $(K\widehat{H}X, K\widehat{\eta})$, the $\widehat{\mathbf{H}}$-algebra corresponding to the $\mathbf{H}$-algebra $(K\widehat{H}X, \omega_X)$. This lift is classified by

$$\widehat{\lambda}_X \;=\; \widehat{H}KX \xrightarrow{\widehat{H}K\widehat{\eta}_X} \widehat{H}K\widehat{H}X \xrightarrow{\omega_X^\bullet} K\widehat{H}X$$

Then $\widehat{\lambda}$ satisfies (DL C, DL D) because it arises from the formula corresponding to a functorial lift. To complete the proof, we must show that the square (6) commutes. We have

$$\begin{aligned} \widehat{\lambda}\,(\sigma K) &= \omega^\bullet\,(\widehat{H}K\widehat{\eta})\,(\sigma K) \;=\; \omega^\bullet(\sigma K\widehat{H})\,(HK\widehat{\eta}) \qquad (\sigma \text{ natural}) \\ &= \omega\,(HK\widehat{\eta}) \qquad (\text{by the definition of } \omega^\bullet) \\ &= (K\widehat{\mu})\,(K\sigma\widehat{H})\,(\lambda\widehat{H})(HK\widehat{\eta}) \;=\; (K\widehat{\mu})(K\sigma)\,(KH\widehat{\eta})\,\lambda \qquad (\lambda \text{ natural}) \\ &= (K\widehat{\mu})\,(K\widehat{H}\widehat{\eta})\,(K\sigma)\,\lambda \qquad (\sigma \text{ natural}) \\ &= (K\sigma)\,\lambda \qquad (\widehat{\mathbf{H}}\text{-algebra}) \end{aligned}$$

□

Lemma 6.3 *Let $\mathbf{H} = (H, \eta, (\cdot)^{\#})$ and $\mathbf{K} = (K, e, (\cdot)^{\#\#})$ be pre-monads in $\mathcal{V}$. Given a map $\sigma_X : HX \to KX$ (not necessarily assumed to be a natural transformation) for each X, σ is a pre-monad map if and only if $\sigma\eta = e$ and for each $f : X \to HY$ the following square commutes.*

$$\begin{tikzcd} HX \arrow[r, "\sigma_X"] \arrow[d, "f^{\#}"'] & KX \arrow[d, "(\sigma_Y f)^{\#\#}"] \\ HY \arrow[r, "\sigma_Y"'] & KY \end{tikzcd} \qquad (6.3)$$

Proof. We first show that if σ is a pre-monad map then (6.3) holds. $f^{\#} = HX \xrightarrow{Hf} HHY \xrightarrow{\mu_Y} HY$.

$$\begin{aligned}\sigma_Y f^{\#} &= \sigma_Y \mu_Y (Hf) \ = \ \mu_X (K\sigma_X)(\sigma_{HY})(Hf)\\ &= \mu_Y (K\sigma_Y)(Kf)\,\sigma_X \qquad (\sigma \text{ is natural})\\ &= (\sigma_Y f)^{\#\#}\,\sigma_X\end{aligned}$$

Now the other direction. Let $g : X \to Y$. Then

$$\begin{aligned}(Kg)\,\sigma_X = (e_Y g)^{\#\#}\,\sigma_X \ &= \ (\sigma_Y \eta_Y g)^{\#\#}\sigma_X\\ &= \sigma_Y (\eta_Y\, g)^{\#} \qquad (\text{by } (6.3))\\ &= \sigma_Y\,(Hg)\end{aligned}$$

shows that σ is natural. To see that σ is a pre-monad map,

$$\begin{aligned}m_X\,(\sigma\sigma)_X = m_X\,(K\sigma_X)\,\sigma_{HX} \ &= \ (\sigma_X\,\mathrm{id}_{HX})^{\#\#}\,\sigma_{HX}\\ &= \sigma_X\,(\mathrm{id}_{HX})^{\#} \qquad (\text{by } (6.3))\\ &= \sigma_X\,\mu_X\end{aligned}$$

□

Proposition 6.4 *Let $\sigma : (H, \eta, (\cdot)^{\#}) \to (K, e, (\cdot)^{\#\#})$ be a pre-monad map, let $f : X \to KY$ and let $s : KX \to HX$, $t : KY \to HY$ be morphisms such that $\sigma_X\, s = id_{KX}$, $\sigma_Y\, t = id_{KY}$. Then*

$$f^{\#\#} \ = \ \sigma_Y\,(t\,f)^{\#}\,s$$

Proof. As σ is a pre-monad map, we have by Lemma 6.3

$$f^{\#\#} = (\sigma_Y\, t\, f)^{\#\#}\,\sigma_X\, s \ = \ \sigma_Y\,(t\,f)^{\#}\,s$$

□

Example 6.5 The calculation of μ for the rectangular band monad (R, μ, η) is not immediately intuitive. We can derive it however applying Proposition 6.4 where R is the monad approximation of L of Example 5.11 so that $\sigma_X[x_1, \ldots, x_m] = (x_1, x_m)$. Define a splitting $s : R \to L$ by $s(x, y) = [x, y]$, so $\mu_X = (id_{RX})^{\#} = \sigma \circ (s_X \circ id)^{\#} \circ s_X = \sigma_X \circ \hat{m} \circ L(s_X) \circ s_X$ and thus $\mu(a, b, c, d) = \sigma_X \circ \hat{m} \circ L(s_X) \circ s_X(a, b, c, d) = \sigma_X \circ \hat{m} \circ L(s_X)[(a, b), (c, d)] = \sigma_X \circ \hat{m}[[a, b], [c, d]] = \sigma_X[a, d] = (a, d)$ as expected.

Example 6.6 The previous example can be generalized to other dimensions. For $n \geq 1$, let (L_n, μ, η) denote the monad of lists of length exactly n. For instance when $n = 3$, $\eta(a) = [a, a, a]$ while $\mu([[a, b, c], [d, e, f], [g, h, i]]) = [a, e, i]$ defines monad (L_3, μ, η). L_3 is the monad approximation associated with premonad $(L, \hat{m}, \hat{e})$ defined by $\hat{e}(x) = [x, x, x]$ and $\hat{m}(ll) = [p_1(p_1\ ll), p_2(p_2\ ll), p_3(p_3\ ll)]$ where p_i picks out the $i - th$ element in a list(or the last element if the list is too small). L_n, which is equivalent to the cartesian product of the identity monad (n-times), $(id)^n$, is the monad approximation of a premonad structure on lists L, similar to the previous example.

When $\mathcal{V} = \mathbf{Set}$, the image of $\sigma : H \to \widehat{H}$ is a submonad with the universal property. Thus all monad approximations are pointwise split epic in **Set**.

Theorem 6.7 *If $\sigma : \mathbf{H} \to \widehat{\mathbf{H}}$ is a pointwise split epic monad approximation then if $\lambda : HK \to KH$ is a distributive law then so too is $\widehat{\lambda} : \widehat{H}K \to K\widehat{H}$.*

Proof. For (DL A),

$$\begin{aligned}\widehat{\lambda}(\widehat{H}\rho)\sigma &= \widehat{\lambda}(\sigma_K H\rho) \quad (\sigma \text{ natural})\\ &= (K\sigma)\lambda(H\rho) \quad \text{by } (6)\\ &= (K\sigma)\rho_H \quad (\lambda \text{ a distributive law})\\ &= \rho_{\widehat{H}}\sigma \quad (\rho \text{ natural})\end{aligned}$$

so (DL A) holds as σ is pointwise epic. Similarly, for (DL B),

$$\begin{aligned}\widehat{\lambda}(\widehat{H}\nu)\sigma_{KK} &= \widehat{\lambda}(\sigma_K H\nu) \quad (\sigma \text{ natural})\\ &= (K\sigma)\lambda(H\nu) \quad \text{by } (6)\\ &= (K\sigma)(\nu_H)(K\lambda)(\lambda_K) \quad (\text{DL B for } \lambda)\\ &= (\nu_{\widehat{H}})(KK\sigma)(K\lambda)(\lambda_K) \quad (\nu \text{ natural})\\ &= (\nu_{\widehat{H}})(K\widehat{\lambda})(K\sigma_K)\lambda_K \quad \text{by } (6)\\ &= (\nu_{\widehat{H}})(K\widehat{\lambda})\widehat{\lambda_K}\sigma_{KK} \quad \text{by } (6)\end{aligned}$$

Since σ is a retraction, it is surjective, so (DLA) and (DL B) hold for $\widehat{\lambda}$. □

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