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Electronic Notes in Theoretical Computer Science
ELSEVIER Electronic Notes in Theoretical Computer Science 73 (2004) 141–147
www.elsevier.com/locate/entcs

A Nonstandard Characterisation of the Type-structure of Continuous Functionals Over the Reals

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Abstract

We extend a hyperfinite discretisation of the real line to a typed structure of hyperfinite functionals, and we show that the hereditarily near-standard functionals correspond to the continuous functionals over the reals obtained from domain theory.

Keywords: continuous functional, nonstandard, hyperfinite, total

1 Introduction

If we view domain theory as the mathematical analysis of ordered sets of finitary information via completions and topology, an alternative tool suggests itself, the use of nonstandard analysis. One way of viewing nonstandard analysis is that we go beyond completion, each element of the completion is represented by several *hyperfinitary* objects sharing the algebraic properties of the finitary objects.

In nonstandard analysis one works with the interplay between two typed structures, the full type-structure $\{Tp(n, \mathbb{R})\}_{n\in\mathbb{N}}$ and the extension $\{Tp(n, \mathbb{R})^*\}_{n\in\mathbb{N}}$. The non-standard extension is *elementary* in the sense of

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doi:10.1016/j.entcs.2004.08.007

first order logic. This fact is called *the transfer principle*. The objects in the extended type-structure are called *internal*.

One way of constructing the non-standard extension goes via *ultraproducts*: Let S_n be the set of infinite sequences $\bar{\alpha} = \{\alpha_n\}_{n\in\mathbb{N}}$ of elements in $Tp(n,\mathbb{R})$.
Let F be a non-principal ultrafilter on $\mathbb{N}$.
If $\bar{\alpha}$ and $\bar{\beta}$ are elements of S_n, we let $\bar{\alpha} \approx \bar{\beta}$ if $\{n \mid \alpha_n = \beta_n\} \in F$.
This is an equivalence relation, and $Tp(n,\mathbb{R})^*$ is (up to isomorphism) the set of equivalence classes in S_n.
If $\bar{\alpha} \in S_{n+1}$ and $\bar{\beta} \in S_n$, we let $\bar{\alpha}(\bar{\beta})$ denote $\lambda n.\alpha_n(\beta_n)$. This induces an application operator from $Tp(n+1,\mathbb{R})^* \times Tp(n,\mathbb{R})^*$ to $\mathbb{R}^*$.
The *-map is an embedding of $Tp(n,\mathbb{R})$ into $Tp(n,\mathbb{R})^*$ where x^* is the equivalence class of $\lambda n.x$, i.e. the constant sequence. This is an elementary embedding with respect to the many-sorted language for $\{Tp(n,\mathbb{R})\}_{n\in\mathbb{N}}$ with application.
We will identify a set with its characteristic function. Thus we have nonstandard versions $\mathbb{N}^*$ and $\mathbb{Q}^*$ of e.g. $\mathbb{N}$ and $\mathbb{Q}$ resp. If x consists of finite objects, the elements of x^* will be called *hyperfinite*. In domain theory, we often call an element *finitary* if the information about it can be given in a finite way. Nonstandard versions will be called *hyperfinitary*.

Let c_0 be the equivalence class of $\lambda n.n$. Then $c_0 \in \mathbb{N}^*$
Let c be the equivalence class of $\lambda n.n!$ Then we also have that $c \in \mathbb{N}^*$. This is the number we will use in the construction.
Let

$$X_n = \{\frac{k}{n!} \mid -(n!)^2 < k < (n!)^2\}.$$

Then the equivalence class of $\{X_n\}_{n\in\mathbb{N}}$ is an internal object that can be described as

$$\{\frac{k}{c} \mid -c^2 < k < c^2\}.$$

This set will be used in the construction.

Normally definitions and proofs are not based on a particular construction of the non-standard extension, but only from the assumption that $\{Tp(n,\mathbb{R})^*\}_{n\in\mathbb{N}}$ is a proper elementary extension of $\{Tp(n,\mathbb{R})\}_{n\in\mathbb{N}}$.
One fact that we will use is that $\mathbb{N}$ is not internal, and thus not definable in the non-standard extension from internal objects. We see this by observing that $\mathbb{N}$ is a bounded subset of $\mathbb{N}^*$, and by the transfer principle, each internal bounded subset of $\mathbb{N}^*$ will have a maximal element. This is used in the proofs of Claims 2 and 3.

Example 1.1 Let $(D,\sqsubseteq)$ be an algebraic, separable domain where $\{d_n\}_{n\in\mathbb{N}}$ is the enumerated set of finitary (compact) elements.

Let $(D^*, \sqsubseteq^*)$ be the non-standard version (we only need the hyperfinitary objects here).
Then each $x \in D$ can be obtained as

$$\sqcup\{d_n \mid n \in \mathbb{N} \wedge d_n \sqsubseteq^* d\}$$

for some hyperfinitary d.

Proof. If d is hyperfinitary, we see that $\{d_n \mid n \in \mathbb{N} \wedge d_n \sqsubseteq^* d\}$ is an ideal of finitary objects, so $x \in D$ is well defined as its least upper bound.
If $x \in D$, and x is not itself finitary , let $\{m_n\}_{n\in\mathbb{N}}$ be such that $\{d_{m_n}\}_{n\in\mathbb{N}}$ is a strictly increasing sequence of finitary objects with x as its least upper bound.
Then $(\{d_{m_n}\}_{n\in\mathbb{N}})^*$ is of the form $\{d_{m_n}\}_{n\in\mathbb{N}^*}$.
If we let $c \in \mathbb{N}^* \setminus \mathbb{N}$, and let $d = d_{m_c}$, x will satisfy the equation above. □

2 The Continuous Functionals Over the Reals

The main result of this note is a characterisation of one of the typestructures of total, continuous functionals over the reals. In [2], Bauer, Escardó and Simpson discuss alternative constructions of such type-structures. One of them is based on the continuous domain of closed real intervals and the Cartesian closed category of continuous domains. It will be this type-structure that we will give an alternative characterisation of. The language *RealPCF* use this typestructure for its semantics, see Escardó [4].
An alternative approach to this hierarchy is via algebraic domains with totality. The base domain will then be generated from the set of closed rational intervals. Normann [5,6] proved a density theorem for this hierarchy, and showed that it may be characterised in the Cartesian closed category of Kuratowski limit spaces.

Our starting point will be one of the standard hyperfinite representations of the real line, used e.g. in nonstandard analysis to replace differential equations by difference equations. Thus there is nothing ad hoc in the construction itself. From now on c will be a fixed element of $\mathbb{N}^* \setminus \mathbb{N}$, e.g. the one constructed in Section 1. We will let the *types* be formal objects generated from 0 by $\to$.

Definition 2.1 Let $H_0 = \{\frac{k}{c} \mid -c^2 < k < c^2\}$. H_0 is an internal set.
For a type $\sigma \to \tau$ we let $H_{\sigma\to\tau}$ consist of all internal maps from H_σ to H_τ.

$H_{\sigma\to\tau}$ will be a hyperfinite, internal set.
We now define *the hereditarily near-standard objects*.

Definition 2.2 For each type σ we isolate a set H^{ns}_σ of near-standard objects together with an equivalence relation $\approx_\sigma$ on H^{ns}_σ as follows:

(i) • $H_0^{ns} = \{\frac{k}{c} \mid \exists n \in \mathbb{N}(-nc < k < nc)\}$.
- $\frac{k_1}{c} \approx_0 \frac{k_2}{c} \Leftrightarrow |\frac{k_1}{c} - \frac{k_2}{c}|$ is infinitesimal.

(ii) If $f \in H_{\sigma\to\tau}$, we let $f \in H^{ns}_{\sigma\to\tau}$ if
- $f(x) \in H_\tau^{ns}$ whenever $x \in H_\sigma^{ns}$
- $x \approx_\sigma y \Rightarrow f(x) \approx_\tau f(y)$.

We let $f \approx_{\sigma\to\tau} g$ if $x \approx_\sigma y \Rightarrow f(x) \approx_\tau g(y)$ for all $x, y \in H_\sigma^{ns}$.

We observe that $\approx_\sigma$ will be an equivalence relation on H_σ^{ns} for all types σ.

Definition 2.3 For each $x \in H_\sigma^{ns}$ we define *the standard part* $st(x)$, and the set T_σ of standard parts, by induction:
For $x \in H_0^{ns}$ we use the usual standard part, $st(\frac{k}{c})$ is the unique real infinitesimally close to $\frac{k}{c}$. (The existence of $st(\frac{k}{c})$ is a consequence of the completenes of the real line.)
For $f \in H^{ns}_{\sigma\to\tau}$ we define $st(f) : T_\sigma \to T_\tau$ by $st(f)(st(x)) = st(f(x))$.

We observe that $T_0 = \mathbb{R}$. Thus this defines a type structure over the reals.

Definition 2.4 Let D_σ be the interpretation of the type σ as an algebraic domain, where D_0 is the algebraic domain based on closed rational intervals.

Definition 2.5 For each type σ, let C_σ be the quotient space of the hereditarily total objects of type σ organised as a function-space in the natural way (with $C_0 = \mathbb{R}$).

The main result of this note is

Theorem 2.6 *For each type σ, $C_\sigma = T_\sigma$.*

Proof. Let F_σ be the set of finitary elements of D_σ, F_σ^* the non-standard extension and F_σ^c the elements in F_σ^* using only intervals $[\frac{k_1}{c}, \frac{k_2}{c}]$ as base elements.
If $p \in F_\sigma^c$ and $x \in H_\sigma$ we define the relation $p \prec_\sigma x$ by recursion on σ as follows:

(i) $[\frac{k_1}{c}, \frac{k_2}{c}] \prec_0 \frac{k}{c}$ if $k_1 \leq k \leq k_2$.

(ii) Here we use the standard notation for describing finitary (and hyperfinitary) elements in $D_{\sigma\to\tau}$
$\{(p_1, q_1), \ldots, (p_k, q_k)\} \prec_{\sigma\to\tau} f$ if $q_i \prec_\tau f(x)$ whenever $p_i \prec_\sigma x$.

Note that in 2., k may be hyperfinite, but $\{(p_1, q_1), \ldots, (p_k, q_k)\}$ will be an internal object.

Claim 1

If $p \in F^c_\sigma$ there is an $f \in H_\sigma$ such that $p \prec_\sigma f$.

Proof
We will use induction on the type.
For type 0, choose e.g. the left hand endpoint of the interval.
Now consider

$$p = \{(q_1, r_1), \ldots, (q_k, r_k)\} \in F^c_{\sigma \to \tau}$$

and $x \in H_\sigma$.
Then $\{r_i \mid q_i \prec_\sigma x\}$ is a consistent set, and $\bigsqcup\{r_i \mid q_i \prec_\sigma x\} \in F^c_\tau$. By the induction hypothesis there is a $y_x \in H_\tau$ such that $\bigsqcup\{r_i \mid q_i \prec_\sigma x\} \prec y_x$. Let $f(x) = y_x$.
It is easy to verify that this f does the trick.

In the proof of this claim, we are relying on the transfer principle. The argument would have been unproblematic if c had been a genuine natural number. The transfer principle tells us that the argument then is unproblematic for hyperfinite c.

Each element of H_σ will determine an object in D_σ, the *weak standard part*, by

$$wst(f) = \bigsqcup\{p \in F_\sigma \mid p \prec_\sigma f\}.$$

We will show that $f \in H^{ns}_\sigma$ if and only if $wst(f)$ is total, and that two objects are equivalent if and only if their weak standard parts are consistent. Moreover we will show that each total object is consistent with the weak standard part of some $f \in H^{ns}_\sigma$.

From now on we will assume that the nonstandard number c has all standard numbers n as factors. This is not essential for the result, but simplifies the argument. We use this assumption to show:

Claim 2
If $x \in D_\sigma$, there is a $y \in F^c_\sigma$ such that $x_0 \sqsubseteq^*_\sigma y$ for all finitary $x_0 \sqsubseteq x$.
We write $x \sqsubseteq_\sigma y$ when this is the case.

Proof
Let $\{x_n\}_{n \in \mathbb{N}}$ be an increasing sequence of finitary objects with x as the limit. Since all end-points in all rational intervals used to describe any element of x_n have denominators that are factors in c, the same must be the case for a hyperfinitary $x_{c'}$, and then $x_{c'} \in F^c_\sigma$, with the required property.

Claim 3 Let σ be any type.

a) Let $f \in H_\sigma$. Then $f \in H^{ns}_\sigma$ if and only if $wst(f)$ is hereditarily total. Moreover, $f \approx_\sigma g$ if and only if $wst(f)$ and $wst(g)$ are consistent.

b) If $x \in D_\sigma$ is total, then there is an $f \in H^{ns}_\sigma$ such that $wst(f)$ is consistent with x.

Proof

We will only prove this claim for pure types. Using the Curry-isomorphism the proof will work for mixed types as well, but will require more unpleasant notation. We will use induction on the type.

For $\sigma = 0$, a) and b) are trivial. So assume that the claim holds for τ and consider the case $\sigma = \tau \to 0$.

a): Let $f \in H^{ns}_\sigma$. Let $x \in D_\tau$ be total, $\{x_n\}_{n\in\mathbb{N}}$ an increasing sequence of finitary objects with x as the limit.

If $d \in \mathbb{N}^* \setminus \mathbb{N}$ and $x_d \prec^*_\tau a \in H_\tau$, then, by the induction hypothesis, $a \in H^{ns}_\tau$. Moreover, if $x_d \prec^*_\tau a_1$ and $x_d \prec^*_\tau a_2$ then $a_1 \approx_\tau a_2$. It follows that $f(a_1)$ and $f(a_2)$ are infinitesimally close to each other.

Now, let $\epsilon > 0$ be standard. Then

$$\{d \in \mathbb{N}^* \mid x_d \in F^c_\tau \wedge \forall a, b \in H_\tau(x_d \prec d \wedge x_d \prec b \Rightarrow |f(a) - f(b)| < \frac{\epsilon}{2})\}$$

is internal, since it is definable from internal objects only. Since this set contains an initial segment of the nonstandard natural numbers, it must contain a standard number n.

But this means that $\{(x_n, [p, q])\} \prec_\sigma f$ for some rational numbers p and q with $p - q < \epsilon$.

The consequence is that $wst(f)$ is total.

Using Claim 1 and the induction hypothesis, it is easy to prove the other part of a).

b): Let $x \in D_\sigma$ be total. By Claim 2 there is a $y \in F^c_\sigma$ such that $x \sqsubseteq_\sigma y$. By Claim 1 there is an $f \in H_\sigma$ such that $y \prec_\sigma f$. Let $a \approx_\tau b$. Then $wst(a)$ and $wst(b)$ are consistent, total objects, so $x(wst(a)) = x(wst(b))$. It follows that $f(a)$ and $f(b)$ are equivalent, so $f \in H^{ns}_\sigma$.

Claim 3 establishes the 1-1-correspondance between equivalence classes in H^{ns}_σ and equivalence classes of hereditarily total objects. Clearly this correspondance commutes with application, and the theorem is proved. $\square$

3 Discussion

The motivation for considering this example was to look at a natural construction of a type structure over the reals, based on methods from outside domain theory, and to see if the result would correspond to one of the standard constuctions. The question now is if this use of nonstandard analysis gives insight to effectivity in a way expressible in domain theory. The strength of nonstandard analysis is mainly that it opens for the use of constructions that

are alternatives to constructions in analysis, not representatives of them. Examples are nonstandard approaches to distributions and stochastic processes, see Albeverio, Fenstad, Høegh-Krohn and Lindstrøm [1]. The analysis of these constructions often leads to internal algorithms of a simple kind, but effectivity is lost when we go to the standard parts.
One example is the solving of differential equations.

Let $F : \mathbb{R} \times \mathbb{R} \to [-1, 1]$ be continuous. Then we may solve the differential equation with initial value

$$y' = F(x, y) \ \ , \ \ y(0) = 0$$

by replacing F by a representative G and solving the difference equations

$$Y_G(0) = 0 \ \ , \ \ Y_G(\frac{k+1}{c}) = Y_G(\frac{k}{c}) + \frac{1}{c} G(\frac{k}{c}, Y_G(\frac{k}{c})).$$

Then

$$y(st(\frac{k}{c})) = st(Y_G(\frac{k}{c}))$$

is well defined and solves the equation. However, the map $\Phi(G) = Y_G$ is not near standard, and there is no way to select a solution to the equation in a continuous way. In fact, Birkeland and Normann [3] essentially showed that by choosing an appropriate representative G for F, all solutions of the equation can be obtained as the standard part of some Y_G. It is not clear how this relates to domain theory, but one may for instance now ask the question: Will the set of solutions be represented as an effective element of some power domain over $\mathbb{R} \to \mathbb{R}$, and can this be extracted from the nonstandard approach?

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