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A Tool for Analysing Logics

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Abstract

We introduce and examine a tool for analysing logics. This algebraic tool, coming from some ideas introduced by J. Piaget, provides condensed information about a logic (with emphasis on the behavior of a unary symbol), as such, it can be employed for analysing and, to some extent, comparing logics.

Keywords: Logic, interpretation, transformations, groups, monoids, homomorphism.

1 Introduction

We introduce and examine an algebraic tool for analysing and comparing logics. This tool stems from some ideas introduced by Jean Piaget to analyse the behavior of classical propositional negation [10] [11]. We will extend them to a unary symbol (e. g. negation or a modality).

This algebraic tool provides condensed information about a logic, much as eigenvalues (or eigenvectors) give some information about matrices. As such, it can be

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employed for analysing and, to some extent, comparing logics. Comparing logics is not an easy task; our tool can be used to simplify this task, as we reduce it to comparing algebraic structures.

The structure of this paper is as follows. In Section 2, we revisit Piaget's analysis and indicate how to extend it to other similar cases. In Section 3, we extend Piaget's classical analysis to monoids of transformations: we give some examples of transformation monoids, introduce our method for constructing such monoids and give some bounds. In Section 4, we illustrate how our transformation monoids can be used for comparing logics. In Section 5, we extend these ideas to unary symbols other than negation (such as modalities), formulating them in the general context of universal logic and institutions. Finally, Section 6 presents some remarks about our approach and on-going work towards possible extensions.

2 Reverse Engineering

We will now revisit Piaget's classical analysis (in 2.1) and introduce some tools for extending it to other similar cases (in 2.2).

2.1 Piaget's analysis

We will now examine Piaget's analysis of the behavior of classical propositional negation [10] [11].

Piaget's analysis rests on a simple idea, namely regarding a proposition as a function of its propositional letters. There are three natural ways of applying negation: negate the result, negate the arguments or both. This gives rise to the three Piaget's transformations: inversion, reciprocal and correlative. For instance, the proposition $\mathrm{p} \wedge \mathrm{q}$ has inverse $\neg(\mathrm{p} \wedge \mathrm{q})$, reciprocal $(\neg\mathrm{p} \wedge \neg\mathrm{q})$ and correlative $\neg(\neg\mathrm{p} \wedge \neg\mathrm{q})$.

These three transformations are defined as follows.

($\mathcal{N}$) *Inversion* $\mathcal{N} : \varphi(\mathrm{p}_1, \ldots, \mathrm{p}_n) \mapsto \neg\,\varphi(\mathrm{p}_1, \ldots, \mathrm{p}_n)$

($\mathcal{R}$) *Reciprocal* $\mathcal{R} : \varphi(\mathrm{p}_1, \ldots, \mathrm{p}_n) \mapsto \varphi(\neg\,\mathrm{p}_1, \ldots, \neg\,\mathrm{p}_n)$

($\mathcal{C}$) *Correlative* $\mathcal{C} : \varphi(\mathrm{p}_1, \ldots, \mathrm{p}_n) \mapsto \neg\,\varphi(\neg\,\mathrm{p}_1, \ldots, \neg\,\mathrm{p}_n)$

Piaget worked in the context of logical equivalence $\equiv$. For instance, for the proposition $\mathrm{p} \to \mathrm{q}$, we have $\mathcal{N}(\mathrm{p} \to \mathrm{q}) := \neg(\mathrm{p} \to \mathrm{q}) \equiv (\mathrm{p} \wedge \neg\mathrm{q})$, $\mathcal{R}(\mathrm{p} \to \mathrm{q}) := (\neg\mathrm{p} \to \neg\mathrm{q}) \equiv (\mathrm{q} \to \mathrm{p})$, and $\mathcal{C}(\mathrm{p} \to \mathrm{q}) := \neg(\neg\mathrm{p} \to \neg\mathrm{q}) \equiv (\neg\mathrm{p} \wedge \mathrm{q})$ [10] [11]. By examining the effect of repeated applying the three transformations above, Piaget found that they form a 4-element group with the following table [4]:

$\cdot$	I	$\mathcal{N}$	$\mathcal{R}$	$\mathcal{C}$
I	I	$\mathcal{N}$	$\mathcal{R}$	$\mathcal{C}$
$\mathcal{N}$	$\mathcal{N}$	I	$\mathcal{C}$	$\mathcal{R}$
$\mathcal{R}$	$\mathcal{R}$	$\mathcal{C}$	I	$\mathcal{N}$
$\mathcal{C}$	$\mathcal{C}$	$\mathcal{R}$	$\mathcal{N}$	I

He also noticed that this group is isomorphic to a familiar and important group: the so called Klein group of symmetries of a plane rectangle [8] [9].

2.2 Piaget's study revisited

Piaget's analysis and the structure of his group, $\mathfrak{P}_{\mathsf{C}}$, can be explained by the underlying negation graph, as we will now indicate.

We will often use $\underline{\mathrm{p}}$ to abbreviate the n-tuple $\langle \mathrm{p}_1, \ldots, \mathrm{p}_n \rangle$, and, accordingly, $\neg\, \underline{\mathrm{p}}$ for the negated n-tuple $\langle \neg\mathrm{p}_1, \ldots, \neg\mathrm{p}_n \rangle$.

Remark 2.1 The following diagram commutes

$$\begin{array}{ccc} \varphi(\underline{\mathrm{p}}) & \overset{\mathcal{N}}{\to} & \neg\,\varphi(\underline{\mathrm{p}}) \\ & \mathcal{C} & \\ \mathcal{R}\downarrow & \searrow & \downarrow\mathcal{R} \\ \varphi(\neg\,\underline{\mathrm{p}}) & \overset{\mathcal{N}}{\to} & \neg\,\varphi(\neg\,\underline{\mathrm{p}}) \end{array}$$

Corollary 2.2 *The transformations have the following properties.*

(i) *Inversion and reciprocal commute:* $\mathcal{N}(\mathcal{R}(\varphi)) = \mathcal{R}(\mathcal{N}(\varphi))$.

(ii) *Correlative is derived:* $\mathcal{C}(\varphi) = \mathcal{N}(\mathcal{R}(\varphi))$.

The negation graph describes how negation acts on (the representatives of) the equivalence classes of propositions. In the classical case, we have two such representatives, φ and $\neg\varphi$, and negation flips them around. Thus, the underlying negation structure $\mathfrak{N}_{\mathsf{C}}$ has 2 elements, $\neg^0$ and $\neg^1$ behaving as $\leadsto \neg^0 \overset{\neg}{\leftrightarrow} \neg^1$. So, $\mathfrak{N}_{\mathsf{C}}$ is isomorphic to the group of integers modulo 2: $\mathbb{Z}_2$.

Thus, we have $4 = 2^2$ possible classical transformations: for $i, j \in \{0, 1\}$, $\mathcal{N}^i\mathcal{R}^j$: $\varphi(\underline{\mathrm{p}}) \mapsto \neg^i\, \varphi(\neg^j\, \underline{\mathrm{p}})$. These 4 possible classical transformations form a group $\mathfrak{T}_{\mathsf{C}}$ (isomorphic to the direct product $\mathbb{Z}_2 \times \mathbb{Z}_2$). Its structure is obtained by 2 actions of $\mathbb{Z}_2$: in the horizontal ($\mathcal{N}$) and vertical ($\mathcal{R}$) directions:

$$\begin{array}{ccccc} \leadsto \mathcal{N}^0\,\mathcal{R}^0 & \overset{\mathcal{N}}{\leftrightarrow} & \mathcal{N}^1\,\mathcal{R}^0 & & \text{vertical} \\ \mathcal{R}\updownarrow & & \updownarrow\mathcal{R} & & \updownarrow \\ \mathcal{N}^0\,\mathcal{R}^1 & \overset{\mathcal{N}}{\leftrightarrow} & \mathcal{N}^1\,\mathcal{R}^1 & & \mathbb{Z}_2 \\ & & & & \\ & \text{horizontal} \leftrightarrow \mathbb{Z}_2 & & & \end{array}$$

We have 4 possible classical transformations. Are they all really distinct? The answer depends on the available connectives.

If we have only negation, then $\mathcal{N}$ and $\mathcal{R}$ become the same transformation ($\mathcal{N}(\mathrm{p}) = \neg\,\mathrm{p} = \mathcal{R}(\mathrm{p})$). This leads to the collapse $\mathfrak{P}_{\mathsf{C}} \simeq \mathfrak{N}_{\mathsf{C}}$:

$$\begin{array}{ccc} \leadsto \mathcal{N}^0{\circ}\mathcal{R}^0 & \overset{\mathcal{N}}{\leftrightarrow} & \mathcal{N}^1{\circ}R^0 \\ || & & || \\ \mathcal{N}^1{\circ}\mathcal{R}^1 & \overset{\mathcal{N}}{\leftrightarrow} & \mathcal{N}^0{\circ}\mathcal{R}^1 \end{array}$$

We now introduce a tool for the analysis of the other cases. Considering the composite transformation, we note that $N^i \circ \mathcal{R}^j[\mathrm{p}] = \neg^i \neg^j \mathrm{p}$. The *classical weight* of a possible transformation is its total number of negations modulo 2, i. e. $\mathtt{w}_{\mathsf{C}}(\mathcal{N}^i \mathcal{R}^j) := i +_2 j$. The 4 possible classical transformations have weights as follows: $\mathtt{w}_{\mathsf{C}}(\mathcal{N}^0\mathcal{R}^0) = 0$, $\mathtt{w}_{\mathsf{C}}(\mathcal{N}^1\mathcal{R}^0) = 1$, $\mathtt{w}_{\mathsf{C}}(\mathcal{N}^0\mathcal{R}^1) = 1$ and $\mathtt{w}_{\mathsf{C}}(\mathcal{N}^1\mathcal{R}^1) = 0$.

We can now see some cases with other connectives besides negation.

(i) If we have the nullary constant $\perp$ ($\mathcal{R}(\perp) = \perp$), then we have no identifications in view of the following situation:

Weights	Composite	transformations
0	$\mathcal{N}^0 \circ \mathcal{R}^0(\perp) = \perp$	$\mathcal{N}^1 \circ \mathcal{R}^1(\perp) = \neg\perp$
1	$\mathcal{N}^1 \circ \mathcal{R}^0(\perp) = \neg\perp$	$\mathcal{N}^0 \circ \mathcal{R}^1(\perp) = \perp$

(ii) If we have binary connectives ($\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$), then we have no identifications as the transformations on $\varphi := \mathrm{p} \wedge \mathrm{q}$ behave as follows:

Weights	Composite	transformations
0	$\mathcal{N}^0 \circ \mathcal{R}^0(\varphi) = \mathrm{p} \wedge \mathrm{q}$	$\mathcal{N}^1 \circ \mathcal{R}^1(\varphi) = \dfrac{\neg(\neg\mathrm{p} \wedge \neg\mathrm{q})}{\mathrm{p} \vee \mathrm{q}}$
1	$\mathcal{N}^1 \circ \mathcal{R}^0(\varphi) = \dfrac{\neg(\mathrm{p} \wedge \mathrm{q})}{\neg\mathrm{p} \vee \neg\mathrm{q}}$	$\mathcal{N}^0 \circ \mathcal{R}^1(\varphi) = \neg\mathrm{p} \wedge \neg\mathrm{q}$

In both cases, we have no identifications, whence $\mathfrak{P}_{\mathsf{C}} \simeq \mathfrak{N}_{\mathsf{C}} \times \mathfrak{N}_{\mathsf{C}} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$.

We thus have an explanation for Piaget's analysis: the underlying classical negation structure $\mathfrak{N}_{\mathsf{C}}$ is isomorphic to $\mathbb{Z}_2$ and we have no identifications.

3 Direct Engineering

We will now extend Piaget's classical analysis to monoids of transformations. We will give some examples of logical monoids (in 3.1) and introduce our method for constructing such monoids giving some bounds (in 3.2).

Note that the definitions of the transformations (in 2.1) do not depend on the logic. So, we can examine them in other cases, such as intuitionistic logic.

3.1 Examples: intuitionistic negation

We will now consider the case of intuitionistic negation. We will examine the transformations much as in 2.2.

With intuitionistic negation, we no longer have the classical equivalence between φ and $\neg\neg\varphi$, but we do have the equivalence between $\neg\varphi$ and $\neg\neg\neg\varphi$ [12]. So, iterated applications of intuitionistic negation lead to an equivalence only after a delay. The underlying intuitionistic negation structure has 3 elements, $\neg^0$, $\neg^1$ and $\neg^2$ behaving as $\leadsto\neg^0 \overset{\neg}{\rightarrow} \neg^1 \overset{\neg}{\leftrightarrow} \neg^2$. So, $\mathfrak{N}_{\mathsf{I}}$ is isomorphic to the cyclic monoid with transient 1 and period 2: ${}_1\mathfrak{C}_2$ [5]. Its table is as follows:

$*$	0	1	2
0	0	1	2
1	1	2	1
2	2	1	2

Thus, we now have $9 = 3^2$ possible intuitionistic transformations, namely $\mathcal{N}^i\mathcal{R}^j : \varphi(\underline{\mathrm{p}}) \mapsto \neg^i \varphi(\neg^j \underline{\mathrm{p}})$, for $i, j \in \{0, 1, 2\}$. These 9 possible intuitionistic transformations form a monoid $\mathfrak{T}_\mathsf{I}$ (isomorphic to the direct product ${}_1\mathfrak{C}_2 \times {}_1\mathfrak{C}_2$). Its structure is obtained by 2 actions of ${}_1\mathfrak{C}_2$: in the $\mathcal{N}$ and $\mathcal{R}$ directions.

Now, we have 9 possible intuitionistic transformations. As before, whether they are all really distinct depends on the available connectives. If we have only negation, then $\mathcal{N} = \mathcal{R}$, leading to the collapse $\mathfrak{P}_\mathsf{I} \simeq \mathfrak{N}_\mathsf{I}$.

We now adapt our tool for the analysis of the other cases: the *intuitionistic weight* of a possible transformation is its total number of negations counted within ${}_1\mathfrak{C}_2$: $\mathtt{w}_\mathsf{I}(\mathcal{N}^i\mathcal{R}^j) := i * j$ (where $*$ is the operation of ${}_1\mathfrak{C}_2$). The partition of the 9 possible intuitionistic transformations by weights is as follows:

Weights	Possible transformations
0	$\mathcal{N}^0\,\mathcal{R}^0$
1	$\mathcal{N}^1\,\mathcal{R}^0$, $\mathcal{N}^0\,\mathcal{R}^1$, $\mathcal{N}^2\,\mathcal{R}^1$, $\mathcal{N}^1\,\mathcal{R}^2$
2	$\mathcal{N}^2\,\mathcal{R}^0$, $\mathcal{N}^1\,\mathcal{R}^1$, $\mathcal{N}^0\,\mathcal{R}^2$, $\mathcal{N}^2\,\mathcal{R}^2$

We can now see some cases with other connectives besides negation.

(i) If we only have the constant $\bot$, then we have the following situation:

Weights	Composite transformations $\mathcal{N}^i{\circ}\mathcal{R}^j(\bot)\,/\,\mathcal{N}^i{\circ}\mathcal{R}^j(\neg\bot)$
0	$\dfrac{\mathcal{N}^0{\circ}\mathcal{R}^0}{\bot/\neg\bot}$
1	$\boxed{\dfrac{\mathcal{N}^1{\circ}\mathcal{R}^0}{\neg\bot/\neg^2\bot}}$ $\dfrac{\mathcal{N}^0{\circ}\mathcal{R}^1}{\bot/\neg\bot} \equiv \dfrac{\mathcal{N}^2{\circ}R^1}{\neg^2\bot/\neg^3\bot}$ $\boxed{\dfrac{\mathcal{N}^1{\circ}\mathcal{R}^2}{\neg\bot/\neg^2\bot}}$
2	$\boxed{\dfrac{\mathcal{N}^2{\circ}\mathcal{R}^0}{\neg^2\bot/\neg^3\bot}}$ $\dfrac{\mathcal{N}^1{\circ}\mathcal{R}^1}{\neg\bot/\neg^2\bot}$ $\boxed{\dfrac{\mathcal{N}^0{\circ}\mathcal{R}^2}{\bot/\neg\bot} \equiv \dfrac{\mathcal{N}^2{\circ}\mathcal{R}^2}{\neg^2\bot/\neg^3\bot}}$

We then have 3 identifications: $\mathcal{N}^0{\circ}\mathcal{R}^1 = \mathcal{N}^2{\circ}\mathcal{R}^1$, $\mathcal{N}^1{\circ}\mathcal{R}^0 = \mathcal{N}^1{\circ}\mathcal{R}^2$ and $\mathcal{N}^2{\circ}\mathcal{R}^0 = \mathcal{N}^2{\circ}\mathcal{R}^2 = \mathcal{N}^0{\circ}\mathcal{R}^2$. So, the transformation monoid $\mathfrak{P}_\bot$ is a homomorphic image of ${}_1\mathfrak{C}_2 \times {}_1\mathfrak{C}_2$ with 5 elements.

(ii) If we have binary connectives ($\wedge$, $\vee$, $\to$, $\leftrightarrow$), then we can see that we have exactly 2 identifications: $\mathcal{N}^1{\circ}\mathcal{R}^0 = \mathcal{N}^1{\circ}\mathcal{R}^2$ and $\mathcal{N}^2{\circ}\mathcal{R}^0 = \mathcal{N}^2{\circ}\mathcal{R}^2$.[4] Thus,

[4] Double negation distributes over binary connectives other than $\vee$ and we have de Morgan's law $\neg(G{\vee}H) \equiv$

the transformation monoid $\mathfrak{P}_{\mathsf{I}}$ is a homomorphic image of ${}_1\mathfrak{C}_2 \times {}_1\mathfrak{C}_2$ with 7 elements, as follows:

$$
\begin{array}{ccccc}
\rightsquigarrow \mathcal{N}^0\circ\mathcal{R}^0 & \overset{\mathcal{N}}{\rightarrow} & \mathcal{N}^1\circ\mathcal{R}^0 & \overset{\mathcal{N}}{\leftrightarrow} & \mathcal{N}^2\circ\mathcal{R}^0 \; {\scriptstyle = \mathcal{N}^2\circ\mathcal{R}^2} \\
 & & \Vert & & \\
 & & \mathcal{N} \nearrow \; \mathcal{N}^1\circ\mathcal{R}^2 & & \\
\mathcal{R}\downarrow & \mathcal{N}^0\circ\mathcal{R}^2 & \updownarrow \mathcal{R} & & \updownarrow \mathcal{R} \\
 & \nearrow \mathcal{R} \swarrow & & & \\
\mathcal{N}^0\circ\mathcal{R}^1 & \overset{\mathcal{N}}{\rightarrow} & \mathcal{N}^1\circ\mathcal{R}^1 & \overset{\mathcal{N}}{\leftrightarrow} & \mathcal{N}^2\circ\mathcal{R}^1
\end{array}
$$

3.2 Method: monoid construction

We will now introduce our method for constructing Piaget monoids.

We will examine the general pattern in the negation and Piaget monoids of above logics, which involves equivalence classes of formulas.

($\mathfrak{N}$) Negation gives a transformation on formulas $\neg : \varphi \mapsto \neg\varphi$. Consider this transformation up to equivalence and its iterated compositions. This gives a monoid under composition $\circ$: the underlying *negation monoid* $\mathfrak{N}$. Its elements are of the form $\neg^n$, for $n \in \mathbb{N}$, with structure $\rightsquigarrow \neg^0 \overset{\neg}{\rightarrow} \neg^1 \overset{\neg}{\rightarrow} \cdots$. Since monoid $\mathfrak{N}$ is cyclic, it is either finite or isomorphic to $\mathbb{N}$ [5].

($\mathfrak{T}$) The *monoid of possible transformations* $\mathfrak{T}$ consists of the ordered pairs $\langle \mathcal{N}^i, \mathcal{R}^j \rangle$. So $\mathfrak{T}$ is isomorphic to $\mathfrak{N} \times \mathfrak{N}$, with the following structure:

$$
\begin{array}{ccccc}
\rightsquigarrow \mathcal{N}^0\,\mathcal{R}^0 & \overset{\mathcal{N}}{\rightarrow} & \mathcal{N}^1\,\mathcal{R}^0 & \overset{\mathcal{N}}{\rightarrow} & \cdots \\
\mathcal{R}\downarrow & & \downarrow\mathcal{R} & & \\
\mathcal{N}^0\,\mathcal{R}^1 & \overset{\mathcal{N}}{\rightarrow} & \mathcal{N}^1\,\mathcal{R}^1 & \overset{\mathcal{N}}{\rightarrow} & \cdots \\
\mathcal{R}\downarrow & & \downarrow\mathcal{R} & & \\
\vdots & & \vdots & &
\end{array}
\qquad
\begin{array}{c}
\text{vertical} \\ \downarrow \\ \mathfrak{N}
\end{array}
$$

$$\text{horizontal} \rightarrow \mathfrak{N}$$

($\mathfrak{P}$) The *Piaget monoid* $\mathfrak{P}$ consists of the compositions of inversion $\mathcal{N}$, reciprocal $\mathcal{R}$ and correlative $\mathcal{C}$. By Corollary 2.2, its elements are the composite transformations $\mathcal{N}^i\circ\mathcal{R}^j$. To determine its structure, we employ weights. The *weight* of a possible transformation is its total number of negations within underlying negation monoid $\mathfrak{N}$: $\mathtt{w}(\mathcal{N}^i\mathcal{R}^j) = k$ iff $\neg^i\circ\neg^j = \neg^k$ (in $\mathfrak{N}$).

($\neq$) We first partition by weights (which involves only $\neg$). If $\mathcal{N}^i\circ\mathcal{R}^j = \mathcal{N}^k\circ\mathcal{R}^l$, then $\neg^i\neg^j\mathrm{p} \equiv \neg^k\neg^l\mathrm{p}$, so $\neg^i\neg^j\varphi \equiv \neg^k\neg^l\varphi$, and $\neg^i\circ\neg^j = \neg^k\circ\neg^l$.

($=$) Next, we identify within weights (examining $\neg$ and other connectives). If, for every φ, $\neg^i\,\mathcal{R}^j(\varphi) \equiv \neg^k\,\mathcal{R}^l(\varphi)$, then $\mathcal{N}^i\circ\mathcal{R}^j = \mathcal{N}^k\circ\mathcal{R}^l$ (by definition).

This construction provides the Piaget monoid.

$(\neg G \wedge \neg H)$ [12]. So, an inductive proof shows that $\mathcal{R}^2[\varphi] \equiv \neg\varphi$.

In Section 2, we saw two classical cases (with weights $0, 1$) and no identifications: $\mathfrak{P} \simeq \mathfrak{N} \times \mathfrak{N}$ (cf. 2.2). We can use our method to construct the Piaget monoid of a logic whose underlying negation monoid is $\mathbb{Z}_n$. In Section 3, we saw two intuitionistic cases (with weights $0, 1, 2$) and some identifications: $\mathfrak{P}$ is a homomorphic image of $\mathfrak{N} \times \mathfrak{N}$ (cf. 3.1).

In Section 5, we will show that the Piaget monoid $\mathfrak{P}$ is a homomorphic image of $\mathfrak{N} \times \mathfrak{N}$ where the underlying negation monoid $\mathfrak{N}$ can be embedded (cf. Theorem 5.1 in 5.2). This will provide some bounds on the size of the Piaget monoid: $|\mathfrak{N}| \leq |\mathfrak{P}| \leq |\mathfrak{N}|^2$.

4 Comparing Logics

We will now illustrate how Piaget monoids can be used for comparing logics. We will examine some simple examples (in 4.1) and some examples involving interpretations (in 4.2).

4.1 Simple examples

We now see some simple examples of distinct logics with distinct monoids and with the same monoids. We will consider logics with only $\neg$ and $\bot$. Classical logic has underlying negation monoid $\mathfrak{N}_{\mathsf{C}} \simeq \mathbb{Z}_2$ and Piaget monoid $\mathfrak{P}_{\mathsf{C}} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ (cf. 2.2).

($\neq$) Distinct logics with distinct monoids. A trivalent logic with underlying negation monoid $\mathfrak{N}_3 \simeq \mathbb{Z}_3$ will have Piaget monoid $\mathfrak{P}_3 \simeq \mathbb{Z}_3 \times \mathbb{Z}_3$. So, the distinctions between classical logic and this trivalent logic are reflected in their negation and Piaget monoids.

Also, notice that the only possible homomorphisms between the underlying negation monoids $\mathfrak{P}_{\mathsf{C}}$ and $\mathfrak{P}_3$ are the trivial ones: erasing negations:

$$\rightsquigarrow \neg^0 \overset{\neg}{\leftrightarrows} \neg^1 \qquad\qquad \begin{array}{ccc} & & \neg^1 \\ & \overset{\neg}{\nearrow} & \\ \rightsquigarrow \neg^0 & & \downarrow \neg \\ & \overset{\neg}{\nwarrow} & \\ & & \neg^2 \end{array}$$

($=$) Distinct logics with the same monoids. Consider a Łukasiewicz logic with 3 values $0 < \frac{1}{2} < 1$ so that $\mathtt{v}(\bot) = 0$ and $\mathtt{v}(\neg\varphi) = 1 - \mathtt{v}(\varphi)$. The value table for this Łukasiewicz negation can be visualized as follows:

$$\begin{array}{cc} 0 & \\ \updownarrow & \tfrac{1}{2} \circlearrowleft \\ 1 & \end{array}$$

We then have $\neg\neg\varphi \equiv \varphi$. Thus, it has underlying negation monoid $\mathfrak{N}_{\text{Ł}} \simeq \mathbb{Z}_2$ and it will have Piaget monoid $\mathfrak{P}_{\text{Ł}} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$. Hence, $\mathfrak{N}_{\text{Ł}} \simeq \mathfrak{N}_{\mathsf{C}}$ and $\mathfrak{P}_{\text{Ł}} \simeq \mathfrak{P}_{\mathsf{C}}$. So, the distinctions between classical and Łukasiewicz logics are not reflected in their negation and Piaget monoids.

Negation and Piaget monoids (with sizes) for some logics are as follows:

Logics	connectives	Neg. $\mathfrak{N}$	Piag. $\mathfrak{P}$
Classical	$\neg$	$\mathbb{Z}_2 : 2$	$\mathbb{Z}_2 : 2$
Classical	$\neg, \bot$	$\mathbb{Z}_2 : 2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 : 4$
Classical	$\neg, \wedge, \vee, \rightarrow, \leftrightarrow$	$\mathbb{Z}_2 : 2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 : 4$
Łukasiewicz	$\neg, \bot$	$\mathbb{Z}_2 : 2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 : 4$
Modulo 3	$\neg, \bot$	$\mathbb{Z}_3 : 3$	$\mathbb{Z}_3 \times \mathbb{Z}_3 : 9$
Modulo 4	$\neg$	$\mathbb{Z}_4 : 4$	$\mathbb{Z}_4 : 4$
Intuitionistic	$\neg$	${}_1\mathfrak{C}_2 : 3$	${}_1\mathfrak{C}_2 : 3$
Intuitionistic	$\neg, \bot$	${}_1\mathfrak{C}_2 : 3$	$\mathsf{H}({}_1\mathfrak{C}_2 \times {}_1\mathfrak{C}_2) : 5$
Intuitionistic	$\neg, \wedge, \vee, \rightarrow, \leftrightarrow$	${}_1\mathfrak{C}_2 : 3$	$\mathsf{H}({}_1\mathfrak{C}_2 \times {}_1\mathfrak{C}_2) : 7$

4.2 Examples with interpretations

We will show some examples involving interpretations. We will consider logics with connectives $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$ and $\bot$, besides $\neg$ and examine their monoids.

Consider classical and intuitionistic logics. It is known that the former C is a non-conservative extension of the latter I and we have a faithful interpretration of the latter I into the former C [6] [12].

($\rightarrow$) First, consider the (non-conservative) extension $\mathsf{I} \subseteq \mathsf{C}$.

($\mathfrak{N}$) The assignment $\neg_{\mathsf{I}} \mapsto \neg_{\mathsf{C}}$ defines a homomorphism of underlying negation monoids: $\mathfrak{N}_{\mathsf{I}} \rightarrow \mathfrak{N}_{\mathsf{C}}$ (if $\neg^i\varphi \equiv_{\mathsf{I}} \neg^k\varphi$, then $\neg^i\varphi \equiv_{\mathsf{C}} \neg^k\varphi$).

$$\begin{array}{ccccc}
 & \rightsquigarrow & {\neg_{\mathsf{I}}}^0 & & \\
\mathfrak{N}_{\mathsf{I}} & & & \searrow & \\
 & & {\neg_{\mathsf{I}}}^2 & \leftrightarrow & {\neg_{\mathsf{I}}}^1 \\
\downarrow & & \downarrow & \downarrow & \downarrow \\
\mathfrak{N}_{\mathsf{C}} & \rightsquigarrow & {\neg_{\mathsf{C}}}^0 & \leftrightarrow & {\neg_{\mathsf{C}}}^1
\end{array}$$

($\mathfrak{P}$) The two assignments $\mathcal{N}_{\mathsf{I}} \mapsto \mathcal{N}_{\mathsf{C}}$ and $\mathcal{R}_{\mathsf{I}} \mapsto \mathcal{R}_{\mathsf{C}}$ define a homomorphism of Piaget monoids: $\mathfrak{P}_{\mathsf{I}} \rightarrow \mathfrak{P}_{\mathsf{C}}$. The argument is similar. [5]

($\leftarrow$) Next, consider Gödel's double-negation translation $\mathsf{d} : \mathsf{C} \rightarrow \mathsf{I}$ [6] [12].

($\mathfrak{N}$) The assignment $\neg_{\mathsf{I}} \mapsto \neg_{\mathsf{C}}$ defines a homomorphism $\mathfrak{N}_{\mathsf{d}} : \mathfrak{N}_{\mathsf{I}} \rightarrow \mathfrak{N}_{\mathsf{C}}$ of underlying

[5] If $\mathcal{N}^i \circ \mathcal{R}^j(\varphi) \equiv_{\mathsf{I}} \mathcal{N}^k \circ \mathcal{R}^l(\varphi)$, then $\mathcal{N}^i \circ \mathcal{R}^j(\varphi) \equiv_{\mathsf{C}} \mathcal{N}^k \circ \mathcal{R}^l(\varphi)$.

negation monoids (if $\mathsf{d}[\neg^i\varphi] \equiv_{\mathsf{I}} \mathsf{d}[\neg^k\varphi]$, then $\neg^i\varphi \equiv_{\mathsf{C}} \neg^k\varphi$).

$$
\begin{array}{rcccccc}
\mathfrak{N}_{\mathsf{C}} & & \rightsquigarrow & \neg_{\mathsf{C}}{}^0 & \leftrightarrow & \neg_{\mathsf{C}}{}^1 \\
\mathfrak{N}_{\mathsf{d}} \uparrow & & & \uparrow & \uparrow & \uparrow \\
 & & \rightsquigarrow & \neg_{\mathsf{I}}{}^0 & & \\
\mathfrak{N}_{\mathsf{I}} & & & & \searrow & \\
 & & & \neg_{\mathsf{I}}{}^2 & \leftrightarrow & \neg_{\mathsf{I}}{}^1
\end{array}
$$

($\mathfrak{P}$) The two assignments $\mathcal{N}_{\mathsf{I}} \mapsto \mathcal{N}_{\mathsf{C}}$ and $\mathcal{R}_{\mathsf{I}} \mapsto \mathcal{R}_{\mathsf{C}}$ define a homomorphism $\mathfrak{P}_{\mathsf{d}} : \mathfrak{P}_{\mathsf{I}} \to \mathfrak{P}_{\mathsf{C}}$ of Piaget monoid. The argument is similar.

5 General Formulation

We now extend our ideas to a unary symbol, formulating them in the general context of universal logic and institutions.

5.1 Context

In the context of universal logic, a logic consists of a set of formulas and a consequence relation [2]. We will impose some restrictions on both items.

A *general logic* G consists of a set F_{G} (of formulas) and a binary (consequence) relation $\vdash_{\mathsf{G}}$ on F_{G}. An *equivalence logic* is a general logic E whose binary (consequence) relation $\vdash_{\mathsf{E}}$ is reflexive and transitive. We then define the *equivalence relation* $\equiv_{\mathsf{E}}$ on F_{E} by $\psi \equiv_{\mathsf{E}} \theta$ iff $\psi \vdash_{\mathsf{E}} \theta$ and $\theta \vdash_{\mathsf{E}} \psi$.

Consider given sets P (of propositional letters) and K (of formula-building operations) [1]. We call a set *free* on P under K iff it is freely generated by K on P. We call an equivalence logic S *structural* (on P under K) iff

(F) its set F_{S} of formulas is free on P under K;

($\equiv$) its equivalence relation $\equiv_{\mathsf{S}}$ is a congruence for K that is closed under replacement [3] (so that if $\varphi \equiv_{\mathsf{S}} \varphi'$ and $\theta \equiv_{\mathsf{S}} \theta'$, then $\varphi[\mathrm{p}/\theta] \equiv_{\mathsf{S}} \varphi'[\mathrm{p}/\theta']$).

We wish to extend our ideas to unary symbols other than negation (such as modalities $\Box$ and $\Diamond$). Propositional and modal logics usually are structural. Thus, we will consider a fixed unary formula building operation ∂.

By a ∂ *logic* we mean a structural logic D with $\partial \in \mathrm{K}$. In such a ∂ logic we can reformulate the transformations introduced in 2.1, with ∂ in lieu of $\neg$, as follows (with $\underline{\mathrm{p}}$ as in 2.2 and $\partial\,\underline{\mathrm{p}} := \langle \partial\,\mathrm{p}_1, \ldots, \partial\,\mathrm{p}_n \rangle$): $\mathcal{N}(\varphi(\underline{\mathrm{p}})) := \partial\,\varphi(\underline{\mathrm{p}})$, $\mathcal{R}(\varphi(\underline{\mathrm{p}})) := \varphi(\partial\,\underline{\mathrm{p}})$ and $\mathcal{C}(\varphi(\underline{\mathrm{p}})) := \partial\,\varphi(\partial\,\underline{\mathrm{p}})$. Note that $\mathcal{R}$ is an endomorphism on F_{D} (e. g. $\mathcal{R}(\psi \bullet \theta) = \mathcal{R}(\psi) \bullet \mathcal{R}(\theta)$, for a binary $\bullet$).

5.2 Logics and monoids

We now extend our previous ideas to ∂ logics.

Consider a ∂ logic D. We can work with our reformulated transformations up to equivalence. So, using $[\varphi]_{\mathsf{D}}$ for equivalence class of formula $\varphi \in \mathrm{F}_{\mathsf{D}}$, we have well-defined transformations as follows: $\mathcal{N}_{\mathsf{D}}([\varphi]_{\mathsf{D}}) := [\mathcal{N}(\varphi)]_{\mathsf{D}}$, $\mathcal{R}_{\mathsf{D}}([\varphi]_{\mathsf{D}}) := [\mathcal{R}(\varphi)]_{\mathsf{D}}$ and $\mathcal{C}_{\mathsf{D}}([\varphi]_{\mathsf{D}}) := [\mathcal{C}(\varphi)]_{\mathsf{D}}$. We can also can transfer Remark 2.1 and Corollary 2.2 (in 2.2). We introduce monoids much as in 3.2.

($\mathfrak{N}$) Unary ∂ gives a transformation on equivalence classes $\partial_{\mathsf{D}} : [\varphi]_{\mathsf{D}} \mapsto [\partial\varphi]_{\mathsf{D}}$. Considering iterated compositions of this transformation, we have a monoid under composition $\circ$: the underlying ∂ *monoid* $\mathfrak{N}_{\mathsf{D}}$. Its elements are of the form ${\partial_{\mathsf{D}}}^n$, for $n \in \mathbb{N}$, with structure much as before.

($\mathfrak{T}$) The *monoid of possible transformations* $\mathfrak{T}_{\mathsf{D}}$ consists of the ordered pairs $\langle {\mathcal{N}_{\mathsf{D}}}^i, {\mathcal{R}_{\mathsf{D}}}^j \rangle$, being isomorphic to the direct product $\mathfrak{N}_{\mathsf{D}} \times \mathfrak{N}_{\mathsf{D}}$.

($\mathfrak{P}$) The *Piaget monoid* $\mathfrak{P}_{\mathsf{D}}$ consists of the compositions of $\mathcal{N}_{\mathsf{D}}$, $\mathcal{R}_{\mathsf{D}}$ and $\mathcal{C}_{\mathsf{D}}$. By Corollary 2.2, its elements are the composite transformations ${\mathcal{N}_{\mathsf{D}}}^i \circ {\mathcal{R}_{\mathsf{D}}}^j$.

We then have a method, much as in 3.2, for constructing these 3 monoids. Notice that it is not necessary to obtain the quotient logic: the Lindenbaum-Tarski algebra of formulas (which is often infinite).

We now connect our structures of underlying and Piaget monoids.

Theorem 5.1 *Given a ∂ logic D, consider its ∂ and Piaget monoids $\mathfrak{N}_{\mathsf{D}}$ and $\mathfrak{P}_{\mathsf{D}}$. Then, the Piaget monoid $\mathfrak{P}_{\mathsf{D}}$ is a homomorphic image of $\mathfrak{N}_{\mathsf{D}} \times \mathfrak{N}_{\mathsf{D}}$, where $\mathfrak{N}_{\mathsf{D}}$ can be embedded.*

$$\mathfrak{N}_{\mathsf{D}} \hookrightarrow \mathfrak{P}_{\mathsf{D}} \leftharpoondown \mathfrak{N}_{\mathsf{D}} \times \mathfrak{N}_{\mathsf{D}}$$

Proof. First, consider the assignment on the generator $\partial_{\mathsf{D}} \mapsto \langle {\mathcal{N}_{\mathsf{D}}}^1, {\mathcal{R}_{\mathsf{D}}}^0 \rangle$. It gives an embedding $\mathsf{i} : \mathfrak{N}_{\mathsf{D}} \hookrightarrow \mathfrak{P}_{\mathsf{D}}$.[6] Next, consider the assignment $\langle {\mathcal{N}_{\mathsf{D}}}^i, {\mathcal{R}_{\mathsf{D}}}^j \rangle \mapsto {\mathcal{N}_{\mathsf{D}}}^i \circ {\mathcal{R}_{\mathsf{D}}}^j$. It gives a surjective homomorphism $\mathsf{e} : \mathfrak{N}_{\mathsf{D}} \times \mathfrak{N}_{\mathsf{D}} \rightharpoonup \mathfrak{P}_{\mathsf{D}}$ (by Corollary 2.2 as $\mathsf{e}(\langle {\mathcal{N}_{\mathsf{D}}}^1, {\mathcal{R}_{\mathsf{D}}}^0 \rangle) = \mathcal{N}_{\mathsf{D}}$ and $\mathsf{e}(\langle {\mathcal{N}_{\mathsf{D}}}^0, {\mathcal{R}_{\mathsf{D}}}^1 \rangle) = \mathcal{R}_{\mathsf{D}}$). □

5.3 Comparison of logics

We now examine translation for comparing logics [7].

Consider logics: source $\mathsf{G_s} = \langle \mathrm{F_s}, \vdash_{\mathsf{s}} \rangle$ and target $\mathsf{G_t} = \langle \mathrm{F_t}, \vdash_{\mathsf{t}} \rangle$. A *translation* is a function $\mathsf{h} : \mathrm{F_s} \to \mathrm{F_t}$ translating formulas: $\varphi \mapsto \varphi^{\mathsf{h}}$. Now, a translation $\mathsf{h} : \mathrm{F_s} \to \mathrm{F_t}$ will be said to

($\vdash$) *interpret* $\mathsf{G_s}$ into $\mathsf{G_t}$ ($\mathsf{h} : \mathsf{G_s} \to \mathsf{G_t}$) iff $\psi^{\mathsf{h}} \vdash_{\mathsf{t}} \theta^{\mathsf{h}}$ whenever $\psi \vdash_{\mathsf{s}} \theta$;

($\rightharpoonup$) be *eq-surjective* ($\mathsf{h} : \mathsf{G_s} \rightharpoonup \mathsf{G_t}$) iff h is surjective up to equivalence: for each $\theta \in \mathrm{F_t}$, $\theta \equiv_{\mathsf{t}} \psi^{\mathsf{h}}$, for some $\psi \in \mathrm{F_s}$ (e. g. $\mathsf{i} : \mathsf{I} \rightharpoonup \mathsf{C}$ in 4.2);

($\hookrightarrow$) be *eq-injective* ($\mathsf{h} : \mathsf{G_s} \hookrightarrow \mathsf{G_t}$) iff is injective up to equivalences: $\psi \equiv_{\mathsf{s}} \theta$, whenever $\psi^{\mathsf{h}} \equiv_{\mathsf{t}} \theta^{\mathsf{h}}$ (e.g. $\mathsf{d} : \mathsf{C} \hookrightarrow \mathsf{I}$ in 4.2).

Consider ∂ logics on P under K: source $\mathsf{D_s}$ and target $\mathsf{D_t}$. Given naturals $\lambda, \delta \in \mathbb{N}$, we call a translation $\mathsf{h} : \mathrm{F_s} \to \mathrm{F_t}$ a ∂ *translation* of rank $\langle \lambda, \delta \rangle$ iff $\mathsf{h}(\mathrm{p}) =$

[6] Note that $\langle {\mathcal{N}_{\mathsf{D}}}^i, {\mathcal{R}_{\mathsf{D}}}^0 \rangle = \langle {\mathcal{N}_{\mathsf{D}}}^k, {\mathcal{R}_{\mathsf{D}}}^0 \rangle$ yields ${\partial_{\mathsf{D}}}^i = {\partial_{\mathsf{D}}}^k$.

∂^λp, $\mathsf{h}(\partial\varphi) = \partial^\delta \mathsf{h}(\varphi)$ and, for each n-ary formula-building operation k in K other than ∂, there exists a formula $\mathsf{H}_\mathrm{k}(\mathrm{p}_1, \ldots, \mathrm{p}_n) \in \mathrm{F}_\mathsf{t}$ such that $\mathsf{h}(\mathrm{k}(\varphi_1, \ldots, \varphi_n)) = \mathsf{H}_\mathrm{k}(\mathsf{h}(\varphi_1), \ldots, \mathsf{h}(\varphi_n))$ (e. g. $\mathsf{h}(\mathrm{p} \bullet \mathrm{q}) = \mathsf{H}_\bullet(\partial^\lambda \mathrm{p}, \partial^\lambda \mathrm{q})$). Gödel's double-negation translation is a $\neg$ translation of rank $\langle 2, 1\rangle$.

Now, call ∂ logics D' and D'' *isomorphic* ($\mathsf{D}' \cong \mathsf{D}''$) iff there exist ∂ interpretations $\mathsf{h}' : \mathsf{D}' \to \mathsf{D}''$ and $\mathsf{h}'' : \mathsf{D}'' \to \mathsf{D}'$ that are inverses up to equivalences: $\mathsf{h}''\circ\mathsf{h}'(\varphi') \equiv_{\mathsf{D}'} \varphi'$ and $\mathsf{h}'\circ\mathsf{h}''(\varphi'') \equiv_{\mathsf{D}''} \varphi''$.

Remark 5.2 For a ∂ translation of rank $\langle\lambda, \delta\rangle$, $\mathsf{h}\circ\mathcal{N} = \mathcal{N}^\delta\circ\mathsf{h}$.

Lemma 5.3 *For a ∂ translation of rank $\langle\lambda, \delta\rangle$, $\mathsf{h}\circ\mathcal{R} = \mathcal{R}^\delta\circ\mathsf{h}$.*

Proof. Induction on formulas, as $\mathcal{R}$ is an endomorphism. For $\mathrm{p} \in \mathrm{P}$: $\mathsf{h}(\mathcal{R}(\mathrm{p})) \overset{(\mathcal{R})}{=} \mathsf{h}(\partial\,\mathrm{p}) \overset{(\mathsf{h})}{=} \partial^\delta\,\mathsf{h}(\mathrm{p}) \overset{(\mathsf{h})}{=} \partial^\delta\,\partial^\lambda\,\mathrm{p} = \partial^\lambda\,\partial^\delta\,\mathrm{p} \overset{(\mathcal{R})}{=} \mathcal{R}^\delta(\partial^\lambda \mathrm{p}) \overset{(\mathsf{h})}{=} \mathcal{R}^\delta(\mathsf{h}(p))$ For ∂: $\mathsf{h}(\mathcal{R}(\partial\,\varphi)) \overset{(\mathcal{R})}{=} \mathsf{h}(\partial\,\mathcal{R}(\varphi)) \overset{(\mathsf{h})}{=} \partial^\delta\,\mathsf{h}(\mathcal{R}(\varphi)) \overset{(IH)}{=} \partial^\delta\,\mathcal{R}^\delta(\mathsf{h}(\varphi)) \overset{(\mathcal{R})}{=} \mathcal{R}^\delta(\partial^\delta\,\mathsf{h}(\varphi)) \overset{(\mathsf{h})}{=} \mathcal{R}^\delta(\mathsf{h}(\partial\,\varphi))$. For a binary $\bullet$: $\mathsf{h}(\mathcal{R}(\psi \bullet \theta)) \overset{(\mathcal{R})}{=} \mathsf{h}(\mathcal{R}(\psi) \bullet \mathcal{R}(\theta)) \overset{(\mathsf{h})}{=} \mathsf{H}_\bullet(\mathsf{h}(\mathcal{R}(\psi)), \mathsf{h}(\mathcal{R}(\theta)) \overset{(IH)}{=} \mathsf{H}_\bullet(\mathcal{R}^\delta(\mathsf{h}(\psi)), \mathcal{R}^\delta(\mathsf{h}(\theta))) \overset{(\mathcal{R})}{=} \mathcal{R}^\delta(\mathsf{H}_\bullet(\mathsf{h}(\psi), \mathsf{h}(\theta))) \overset{(\mathsf{h})}{=} \mathcal{R}^\delta(\mathsf{h}(\psi \bullet \theta))$. □

Theorem 5.4 *Consider ∂ logics on* P *under* K*: source D_s and target D_t. Each eq-surjective ∂ interpretation $\mathsf{h} : \mathsf{D}_\mathsf{s} \rightharpoonup \mathsf{D}_\mathsf{t}$ of rank $\langle\lambda, \delta\rangle$ induces monoid homomorphisms $\mathfrak{N}_\mathsf{h} : \mathfrak{N}_\mathsf{s} \to \mathfrak{N}_\mathsf{t}$ and $\mathfrak{P}_\mathsf{h} : \mathfrak{P}_\mathsf{s} \to \mathfrak{P}_\mathsf{t}$.*

$$\begin{array}{cccc} \mathsf{D}_\mathsf{s} & & \mathfrak{N}_\mathsf{s} & \mathfrak{P}_\mathsf{s} \\ \mathsf{h} \downharpoonleft & \mapsto & \downarrow \mathfrak{N}_\mathsf{h} & \downarrow \mathfrak{P}_\mathsf{h} \\ \mathsf{D}_\mathsf{t} & & \mathfrak{N}_\mathsf{t} & \mathfrak{P}_\mathsf{t} \end{array}$$

Proof. It suffices to define the monoid homomorphisms on the generators. Set $\mathfrak{N}_\mathsf{h}(\partial_\mathsf{s}) := {\partial_\mathsf{t}}^\delta$, $\mathfrak{P}_\mathsf{h}(\mathcal{N}_\mathsf{s}) := {\mathcal{N}_\mathsf{t}}^\delta$ and $\mathfrak{P}_\mathsf{h}(\mathcal{R}_\mathsf{s}) := {\mathcal{R}_\mathsf{t}}^\delta$. They are well defined by eq-sujectivity and Lemma 5.3. For each $\theta \in \mathrm{F}_\mathsf{t}$, $\theta \equiv_\mathsf{t} \mathsf{h}(\psi)$, for some $\psi \in \mathrm{F}_\mathsf{s}$. Now, if $\partial^i\,\psi \equiv_\mathsf{s} \partial^k\,\psi$, then $\mathsf{h}(\partial^i\,\psi) \equiv_\mathsf{t} \mathsf{h}(\partial^k\,\psi)$, i. e. $\partial^{(\delta\cdot i)}\,\theta \equiv_\mathsf{t} \partial^{(\delta\cdot k)}\,\theta$. Hence, ${\partial_\mathsf{s}}^i = {\partial_\mathsf{s}}^k$ yields ${\partial_\mathsf{t}}^{(\delta\cdot i)} = {\partial_\mathsf{t}}^{(\delta\cdot k)}$. Similarly, if $\mathcal{N}^i\circ\mathcal{R}^j(\psi) \equiv_\mathsf{s} \mathcal{N}^k\circ\mathcal{R}^l(\psi)$, then $\mathsf{h}(\mathcal{N}^i\circ\mathcal{R}^j(\psi)) \equiv_\mathsf{t} \mathsf{h}(\mathcal{N}^k\circ\mathcal{R}^l(\psi))$, i. e. $\mathcal{N}^{(\delta\cdot i)}\circ\mathcal{R}^{(\delta\cdot j)}(\theta) \equiv_\mathsf{t} \mathcal{N}^{(\delta\cdot k)}\circ\mathcal{R}^{(\delta\cdot l)}(\theta)$. Hence, ${\mathcal{N}_\mathsf{s}}^i\circ{\mathcal{R}_\mathsf{s}}^j = {\mathcal{N}_\mathsf{s}}^k\circ{\mathcal{R}_\mathsf{s}}^l$ yields ${\mathcal{N}_\mathsf{t}}^{(\delta\cdot i)}\circ{\mathcal{R}_\mathsf{t}}^{(\delta\cdot j)} = {\mathcal{N}_\mathsf{t}}^{(\delta\cdot k)}\circ{\mathcal{R}_\mathsf{t}}^{(\delta\cdot l)}$. □

Proposition 5.5 *If ∂ logics D' and D'' are isomorphic ($\mathsf{D}' \cong \mathsf{D}''$), then they have isomorphic monoids $\mathfrak{N}' \simeq \mathfrak{N}''$ and $\mathfrak{P}' \simeq \mathfrak{P}''$.*

Proof. By Theorem 5.4 as eq-surjectivity follows from having inverse up to equivalence. □

Proposition 5.6 *Consider ∂ logics on* P *under* K*: source D_s and target D_t. Each eq-injective ∂ translation $\mathsf{f} : \mathsf{D}_\mathsf{s} \hookrightarrow \mathsf{D}_\mathsf{t}$ of rank $\langle\lambda, 1\rangle$ induces monoid homomorphisms $\mathfrak{N}_\mathsf{f} : \mathfrak{N}_\mathsf{t} \to \mathfrak{N}_\mathsf{s}$ and $\mathfrak{P}_\mathsf{f} : \mathfrak{P}_\mathsf{t} \to \mathfrak{P}_\mathsf{s}$.*

$$\begin{array}{cccc} \mathsf{D_s} & & \mathfrak{N}_\mathsf{s} & \mathfrak{P}_\mathsf{s} \\ \mathsf{f}\downarrow & \mapsto & \uparrow \mathfrak{N}_\mathsf{f} & \uparrow \mathfrak{P}_\mathsf{f} \\ \mathsf{D_t} & & \mathfrak{N}_\mathsf{t} & \mathfrak{P}_\mathsf{t} \end{array}$$

Proof. It suffices to define the monoid homomorphisms on the generators. Set $\mathfrak{N}_\mathsf{f}(\partial_\mathsf{t}) := \partial_\mathsf{s}$, $\mathfrak{P}_\mathsf{f}(\mathcal{N}_\mathsf{t}) := \mathcal{N}_\mathsf{s}$ and $\mathfrak{P}_\mathsf{f}(\mathcal{R}_\mathsf{t}) := \mathcal{R}_\mathsf{s}$. They are well defined by eq-injectivity and Lemma 5.3. If ${\partial_\mathsf{t}}^i = {\partial_\mathsf{t}}^k$, then, for every formula $\varphi \in \mathrm{F}_\mathsf{s}$, $\mathsf{f}(\partial^i \varphi) = \partial^i \mathsf{f}(\varphi) \equiv_\mathsf{t} \partial^k \mathsf{f}(\varphi) = \mathsf{f}(\partial^k \varphi)$, so $\partial^i \varphi \equiv_\mathsf{s} \partial^k \varphi$, whence ${\partial_\mathsf{s}}^i = {\partial_\mathsf{s}}^k$. Similarly, if ${\mathcal{N}_\mathsf{t}}^i \circ {\mathcal{R}_\mathsf{t}}^j = {\mathcal{N}_\mathsf{t}}^k \circ {\mathcal{R}_\mathsf{t}}^l$, then, for every formula $\varphi \in \mathrm{F}_\mathsf{s}$, we have $\mathsf{f}(\mathcal{N}^i \circ \mathcal{R}^j(\varphi)) = \mathcal{N}^i \circ \mathcal{R}^j(\mathsf{f}(\varphi)) \equiv_\mathsf{t} \mathcal{N}^k \circ \mathcal{R}^l(\mathsf{f}(\varphi)) = \mathsf{f}(\mathcal{N}^k \circ \mathcal{R}^l(\varphi))$, thus $\mathcal{N}^i \circ \mathcal{R}^j(\varphi) \equiv_\mathsf{s} \mathcal{N}^k \circ \mathcal{R}^l(\varphi)$, whence ${\mathcal{N}_\mathsf{s}}^i \circ {\mathcal{R}_\mathsf{s}}^j = {\mathcal{N}_\mathsf{s}}^k \circ {\mathcal{R}_\mathsf{s}}^l$. □

6 Conclusion

We have introduced and examined an algebraic tool for analysing and comparing logics. This tool originates from some ideas introduced by Jean Piaget [10] [11], which we have extended to a unary symbol (e. g. negation or a modality) and formulated in a more general context.

We have provided a method for constructing such algebraic structures (monoids) in 3.2 and 5.2. Our method does not require obtaining the quotient logic: the Lindenbaum-Tarski algebra of formulas is often infinite, whereas our monoids are usually finite.

In this framework, comparing logics can be reduced to the existence of monoid homomorphisms. Such monoids provide condensed information about a logic and there is a wide range of algebraic machinery for checking the existence of monoid homomorphisms.

Piaget monoids are reminiscent of modality diagrams. We intend to extend this approach to such diagrams; this case seems somewhat different, but the ideas presented here provide a first step towards this goal.

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