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A Single Proof of Classical Behaviour in da Costa's C_n Systems

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Abstract

A strong negation in da Costa's C_n systems can be naturally extended from the strong negation $(\neg^*)$ of C_1. In [6] Newton da Costa proved the connectives $\{\to, \wedge, \vee, \neg^*\}$ in C_1 satisfy all schemas and inference rules of classical logic. In the following paper we present a proof that all logics in the C_n herarchy also behave classically as C_1. This result tell us the existance of a common property among the paraconsistent family of logics created by da Costa.

Keywords: Paraconsistent logic, C_n systems, Strong negation

1 Introduction

According to the authors in [6] a paraconsistent logic is the underlying logic for inconsistent but non-trivial theories. In fact, many authors [2,1] have pointed out *paraconsistency* is mainly due to the construction of a negation operator which satisfies some properties about classical logic, but at the same time do not hold the so called law of explosion $\alpha, \neg\alpha \vdash \beta$ for arbitrary formulas α, β, as well as others [6].

A common misconception related to paraconsistent logics is the confusion between triviality and contradiction. A theory T is *trivial* when any of the sentences in the language of T can be proven. We say that a theory T is *contradictory* if exists a sentence α in the language of T such that T proves α and $\neg\alpha$. Finally, a theory T is *explosive* if and only if T is *trivial* in the presence of a *contradiction*. We can see that *contradictoriness* and *triviality* are equivalent if and only if for the underlying logic the law of explosion is valid[4]. One of the greatest achievements of

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paraconsistent logic is to provide a general framework to the study of inconsistent theories based on the distinction of contradiction and triviality.

Paraconsistent logics were born in two different ways. In 1948, Jaskowski gave the following conditions that any paraconsistent logic should satisfy [8]:

J1. When applied to inconsistent systems it should not always entail their trivialization;

J2. It should be rich enough to enable practical inferences;

J3. It should have an intuitive justification.

Also, in 1963, we can find a new approach given by da Costa, who independently defined a set of conditions that a paraconsistent logic should satisfy. These conditions are the following:

dC1. In these calculi the principle of non-contradiction, in the form $\neg(\alpha \wedge \neg\alpha)$, should not be a valid schema;

dC2. From two contradictory formulae, α and $\neg\alpha$, it would not in general be possible to deduce any arbitrary formula β;

dC3. It should be simple to extend these calculi to corresponding predicate calculi;

dC4. They should contain the most part of the schemata and rules of the classical propositional calculus which do not interfere with the first conditions.

Nowadays we can find paraconsistent logics applications in many fields such as informatics, physics, medicine, etc. From Minsky's comment we can see that paraconsistent ideas are an approach in Artificial Intelligence [10]: *"But I do not believe that consistency is necessary or even desirable in a developing intelligent system. No one is ever completely consistent. What is important is how one handles paradox or conflict, how one learns from mistakes, how one turns aside from suspected inconsistencies"*.

In physics the authors in [11] have established an approach to formalize concepts in quantum mechanics, the so called principle of superposition, via paraconsistent methods. In general most of scientific knowledge as theories can have inconsistencies. Most of the time scientist do not throw away these theories if they are successful in predicting results and describing phenomena [4].

In the literature we can find many proper paraconsistent logics [3] in the sense of da Costa. The most known paraconsistent logic is C_1 which in [6] the author also introduces an increasingly weaker family/hierarchy of logics called C_n, for $1 \leq n \leq \omega$. Also the authors mention that the strong negation defined in the da Costa's C_n systems has all properties of the propositional classical negation.

Finding a strong negation in the C_n hierarchy is interesting because we can collapse a fragment of these logics into classical logic, that is, we can have a translation which provides an embedding of classical logic into any logic of this C_n system. This fact is mentioned in many papers [6,5], on the other hand the proof does not explicitly appears. In this paper we present an inductive proof about the relation between strong negation and classical behaviour in the C_n systems. The proof follows from three lemmas and two theorems. From this proof we can see that many properties in C_1 can also hold in C_n, excluding the obvious ones.

The organization of this document is as follows: In Section 2 we present basic background in logic, including definitions of some basic properties (monotonicity, cut-elimination, deduction theorem) of the paraconsistent logic C_ω that we are going to work with; In Section 3 we present a inductive proof about the classical behavior of the strong negation defined in the C_n systems; In Section 4 we study an extension of C_ω called $C_\omega^{'}$ where we show that we can find a same increasingly chain of weaker logics as in C_n systems from the new extension; Finally, in Section 5, we present some conclusions about the proof presented.

2 Background

We first introduce the syntax of logical formulas considered in this paper. Then we present a few basic definitions of how logics can be built to interpret the meaning of such formulas.

2.1 Logic Systems

We consider a formal (propositional) language built from: an enumerable set $\mathcal{L}$ of elements called *atoms* (denoted *a, b, c, ...*); the binary connectives $\wedge$ (*conjuntion*), $\vee$ (*disjunction*) and $\rightarrow$ (*implication*); and the unary connective $\neg$ (*negation*). Formulas (denoted $\alpha, \beta, \gamma, ...$) are constructed as usual by combining these basic connectives together with the help of parentheses. We also use $\alpha \leftrightarrow \beta$ to abbreviate $(\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)$. Finally, it is useful to agree on some conventions to avoid the use of many parenthesis when writing formulas in order to make easier the reading of complicated expressions. First, we may omit the outer pair of parenthesis of a formula. Second, the connectives are ordered as follows: $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$, and parentheses are eliminated according to the rule that, first, $\neg$ applies to the smallest formula following it, then $\wedge$ is to connect the smallest formulas surrounding it, and so on.

We consider a *logic* simply as a set of formulas that (i) is closed under Modus Ponens (i.e. if α and $\alpha \rightarrow \beta$ are in the logic, then so is β) and (ii) is closed under substitution (i.e. if a formula α is in the logic, then any other formula obtained by replacing all occurrences of an atom b in α with another formula β is also in the logic). The elements of a logic are called *theorems* and the notation $\vdash_X \alpha$ is used to state that the formula α is a theorem of X (i.e. $\alpha \in X$). We say that a logic X is *weaker than or equal to a logic* Y if $X \subseteq Y$, similarly we say that X is *stronger than or equal to* Y if $Y \subseteq X$.

2.1.1 Hilbert proof systems

There are many different approaches that have been used to specify the meaning of logic formulas or, in other words, to define logics. In Hilbert style proof systems, also known as axiomatic systems, a logic is specified by giving a set of axioms (which is usually assumed to be closed under substitution). This set of axioms specifies, so to speak, the "kernel" of the logic. The actual logic is obtained when this "kernel" is closed with respect to some given inference rules which include Modus Ponens.

The notation $\vdash_X \alpha$ for provability of a logic formula α in the logic X is usually extended within Hilbert style systems; given a theory Γ, we use $\Gamma \vdash_X \alpha$ to denote the fact that the formula α can be derived from the axioms of the logic and the formulas contained in Γ by a sequence of applications of the inference rules.

As a example of a Hilbert style system we present next a logic that is relevant for our work.

C_ω [6] is defined by the following set of axiom schemata:

Pos1: $\alpha \to (\beta \to \alpha)$
Pos2: $(\alpha \to \beta) \to ((\alpha \to (\beta \to \gamma)) \to (\alpha \to \gamma))$
Pos3: $\alpha \wedge \beta \to \alpha$
Pos4: $\alpha \wedge \beta \to \beta$
Pos5: $\alpha \to (\beta \to \alpha \wedge \beta)$
Pos6: $\alpha \to (\alpha \vee \beta)$
Pos7: $\beta \to (\alpha \vee \beta)$
Pos8: $(\alpha \to \gamma) \to ((\beta \to \gamma) \to (\alpha \vee \beta \to \gamma))$
$C_\omega 1$: $\alpha \vee \neg\alpha$
$C_\omega 2$: $\neg\neg\alpha \to \alpha$

Note that the first 8 axiom schemata somewhat constrain the meaning of the $\to, \wedge$ and $\vee$ connectives to match our usual intuitions. It is a well known result that in any logic satisfying Pos1 and Pos2, and with Modus Ponens as its unique inference rule, the *deduction theorem* holds [9].

Theorem 2.1 *[12] Let Γ and Δ be two sets of formulas. Let θ, θ_1, θ_2, α and ψ be arbitrary formulas. Let $\vdash$ be the deductive inference operator of C_ω. Then the following basic properties hold.*

1. $\Gamma \vdash \alpha \to \alpha$ *(identity theorem)*
2. $\Gamma \vdash \alpha$ *implies* $\Gamma \cup \Delta \vdash \alpha$ *(monotonicity)*
3. $\Gamma \vdash \alpha$ *and* $\Delta, \alpha \vdash \psi$ *then* $\Gamma \cup \Delta \vdash \psi$ *(cut)*
4. $\Gamma, \theta \vdash \alpha$ *if and only if* $\Gamma \vdash \theta \to \alpha$ *(deduction theorem)*
5. $\Gamma \vdash \theta_1 \wedge \theta_2$ *if and only if* $\Gamma \vdash \theta_1$ *and* $\Gamma \vdash \theta_2$ *($\wedge$ - rules)*
6. $\Gamma, \theta \vdash \alpha$ *and* $\Gamma, \neg\theta \vdash \alpha$ *if and only if* $\Gamma \vdash \alpha$ *(strong proof by cases)*

3 Strong negation in C_n systems

We will start giving some basic definitions in order to understand concepts needed in the C_n hierarchy.

Definition 3.1 ([6]) $\alpha^o =_{def} \neg(\alpha \wedge \neg\alpha)$. We will refer to $(^o)$ as the consistency operator.

In fact α^o can be seen as a modal operator to the formula α that captures the idea of consistency/well - behavior in C_1.

Definition 3.2 ([5]) We recursively define α^n, $0 \leq n < \omega$ as follows:

(i) $\alpha^0 =_{def} \alpha$

(ii) $\alpha^{n+1} =_{def} (\alpha^n)^o$

Definition 3.3 ([5]) We recursively define $\alpha^{(n)}$, $1 \leq n < \omega$ as follows:

(i) $\alpha^{(1)} =_{def} \alpha^1$

(ii) $\alpha^{(n+1)} =_{def} \alpha^{(n)} \wedge \alpha^{n+1}$

For the careful reader should not confuse α^0 with α^o. Basically α^n represents n applications of the consistency operator (^{o}) to the formula α, and $\alpha^{(n)}$ represents a conjunction of $\alpha^1, \ldots, \alpha^n$.

Definition 3.4 ([6]) We define C_n as an extension of C_ω, which includes the following axiom schemas:

$C_n1:\ \beta^{(n)} \rightarrow ((\alpha \rightarrow \beta) \rightarrow ((\alpha \rightarrow \neg\beta) \rightarrow \neg\alpha)$
$C_n2:\ (\alpha^{(n)} \wedge \beta^{(n)}) \rightarrow ((\alpha \rightarrow \beta)^{(n)} \wedge (\alpha \vee \beta)^{(n)} \wedge (\alpha \wedge \beta)^{(n)})$

Also, we can see that in C_n, the axiom C_n1 can be replaced by the axiom schema $(\beta \wedge \neg\beta \wedge \beta^{(n)}) \rightarrow \alpha$. Intuitively from C_n2 we see that $\alpha^{(n)}$ propagates what we call *n-consisteny* in C_n. Finally we define a strong negation in both C_1 and C_n.

Definition 3.5 ([6]) The strong negations for C_1 and C_n are defined as:

(i) For C_1: $\neg^*\alpha =_{def} \neg\alpha \wedge \alpha^o$

(ii) For C_n: $\neg^{(n)}\alpha =_{def} \neg\alpha \wedge \alpha^{(n)}$

Lemma 3.6 *For all $n \in \mathbb{N}$ we have that $\neg(\alpha^n) \vdash_{C_\omega} \alpha$*

Proof. By induction on n.

Base case (n = 1). By Definition 3.2 we have that $\vdash_{C_\omega} \neg(\alpha^1) \leftrightarrow \neg(\alpha^o)$. Also by Definition 3.1, $\vdash_{C_\omega} \neg(\alpha^o) \leftrightarrow \neg(\neg(\alpha \wedge \neg\alpha))$, we can expand the last formula to $\vdash_{C_\omega} \neg(\alpha^1) \leftrightarrow \neg(\neg(\alpha \wedge \neg\alpha))$. We can use axiom schema $\neg\neg\alpha \rightarrow \alpha$ to prove $\vdash_{C_\omega} \neg(\alpha^1) \rightarrow \alpha \wedge \neg\alpha$, which is by axiom schema Pos3 we have $\vdash_{C_\omega} \neg(\alpha^1) \rightarrow \alpha$. From this we apply *deduction theorem* to obtain $\neg(\alpha^1) \vdash_{C_\omega} \alpha$ as desired.

Inductive step. We assume by induction hypothesis that $\neg(\alpha^n) \vdash_{C_\omega} \alpha$ holds. Accordingly to Definition 3.2 we have that $\vdash_{C_\omega} \neg(\alpha^{n+1}) \leftrightarrow \neg(\alpha^n)^o$, which in fact is $\vdash_{C_\omega} \neg\neg(\alpha^n \wedge \neg(\alpha^n)) \leftrightarrow \neg(\alpha^n)^o$. From the latter and using $C_\omega 2$ axiom and transitivity property we can prove that $\neg(\alpha^{n+1}) \vdash_{C_\omega} \neg(\alpha^n)$, and with the inductive hypothesis we have that $\neg(\alpha^{n+1}) \vdash_{C_\omega} \alpha$. □ □

Lemma 3.7 *For all $n \in \mathbb{N}$ we have that $\vdash_{C_\omega} \alpha \vee \alpha^n$*

Proof. We can see that $\alpha^n \vdash_{C_\omega} \alpha \vee \alpha^n$. On the other hand, due to Lemma 3.6 we have that $\neg(\alpha^n) \vdash_{C_\omega} \alpha$, therefore $\neg(\alpha^n) \vdash_{C_\omega} \alpha \vee \alpha^n$. Applying strong proof by cases (Theorem 2.1) we have that $\vdash_{C_\omega} \alpha \vee \alpha^n$. □ □

Lemma 3.8 *For all $n \in \mathbb{N}$ we have that $\vdash_{C_\omega} \alpha \vee \alpha^{(n)}$*

Proof. By induction on n.

Base case (n = 1). From Lemma 3.6 we have that $\vdash_{C_\omega} \alpha \vee \alpha^o$ holds when n = 1.

Inductive step. We assume by induction hypothesis that $\vdash_{C_\omega} \alpha \vee \alpha^{(n)}$ holds. We know from Lemma 3.7 that $\vdash_{C_\omega} \alpha \vee \alpha^{n+1}$. Thus $\vdash_{C_\omega} (\alpha \vee \alpha^{(n)}) \wedge (\alpha \vee \alpha^{n+1})$. Applying the *distributive law* to the last formula we have that $\vdash_{C_\omega} \alpha \vee (\alpha^{n+1} \wedge \alpha^{(n)})$, which in fact it is by definition $\vdash_{C_\omega} \alpha \vee \alpha^{(n+1)}$. □ □

Theorem 3.9 (Excluded Middle) *In C_ω, we have that $\vdash_{C_\omega} \alpha \vee \neg^{(n)}\alpha$*

Proof. In C_ω we have the following:

$\vdash_{C_\omega} (\alpha \vee \neg^{(n)}\alpha) \leftrightarrow (\alpha \vee (\alpha \wedge \alpha^{(n)}))$

$\vdash_{C_\omega} (\alpha \vee \neg^{(n)}\alpha) \leftrightarrow (\alpha \vee \neg\alpha) \wedge (\alpha \vee \alpha^{(n)})$

$\vdash_{C_\omega} (\alpha \vee \neg^{(n)}\alpha) \leftrightarrow \alpha \vee \alpha^{(n)}$

Therefore it is only necessary to check that $\alpha \vee \alpha^{(n)}$ holds, but accordingly to the *Lemma 3.8* this is true. □ □

The next two theorems follows from a similar proof in [6] where the author proved the same theorems in C_1.

Theorem 3.10 (Reductio Ad Absurdum) *In C_n we have that:*

$$(\Gamma \cup \{\alpha\} \vdash_{C_n} \beta), (\Gamma \cup \{\alpha\} \vdash_{C_n} \neg\beta), (\Gamma \cup \{\alpha\} \vdash_{C_n} \beta^{(n)}) \Rightarrow \Gamma \vdash_{C_n} \neg\alpha$$

Proof. Using *Deduction Theorem* we can prove the following from the hypothesis given: $\Gamma \vdash_{C_n} \alpha \to \beta^{(n)}$, $\Gamma \vdash_{C_n} \alpha \to \beta$ and $\Gamma \vdash_{C_n} \alpha \to \neg\beta$. By the transitive rule and the axiom schema $\vdash_{C_n} \beta^{(n)} \to ((\alpha \to \beta) \to ((\alpha \to \neg\beta) \to \neg\alpha))$ we have that $\Gamma \vdash_{C_n} \alpha \to ((\alpha \to \beta) \to ((\alpha \to \neg\beta) \to \neg\alpha))$. By the application of *Modus Ponens* twice we have that $\Gamma \vdash_{C_n} \alpha \to \neg\alpha$. From this, using theorem $\vdash_{C_n} \neg\alpha \to \neg\alpha$ (as an instance of Identity theorem), and axiom schemas $\vdash_{C_n} \alpha \vee \neg\alpha$ and $\vdash_{C_n} (\alpha \to \neg\alpha) \to ((\neg\alpha \to \neg\alpha) \to ((\alpha \vee \neg\alpha) \to \neg\alpha))$ we can conclude that $\Gamma \vdash_{C_n} \neg\alpha$. □ □

Theorem 3.11 (Explosive Principle) *In C_n we have that:*

$$\vdash_{C_n} \alpha \to (\neg^{(n)}\alpha \to \beta)$$

Proof. According to the strong negation definition 3.5, we have that: $\alpha, \neg^{(n)}\alpha, \neg\beta \vdash_{C_n} \neg\alpha \wedge \alpha^{(n)}$, therefore $\alpha, \neg^{(n)}\alpha, \neg\beta \vdash_{C_n} \neg\alpha$ and $\alpha, \neg^{(n)}\alpha, \neg\beta \vdash_{C_n} \alpha^{(n)}$. Also we have that $\alpha, \neg^{(n)}\alpha, \neg\beta \vdash_{C_n} \alpha$. By the theorem 3.10 is easy to prove that $\alpha, \neg^{(n)}\alpha \vdash_{C_n} \neg\neg\beta$. C_n contains the axiom schemata $\neg\neg\alpha \to \alpha$, which it let us prove that $\alpha, \neg^{(n)}\alpha \vdash_{C_n} \beta$. Finally, applying two times *deduction theorem* to the last formula we have that $\vdash_{C_n} \alpha \to (\neg^{(n)}\alpha \to \beta)$. □ □

Theorem 3.12 *The connectives $\{\to, \wedge, \vee, \neg^{(n)}\}$ in C_n satisfy all the axiom schemata and inference rules in classical propositional calculus.*

Proof. Any logic in C_n extends the positive logic axioms from C_ω. Then, it is only necessary observe that the following axiom $(\neg^{(n)}\alpha \to \neg^{(n)}\beta) \to (\beta \to \alpha)$ holds in C_n

1.	$\neg^{(n)}\alpha \rightarrow \neg^{(n)}\beta$	Hypothesis
2.	β	Hypothesis
3.	$\beta \rightarrow (\neg^{(n)}\beta \rightarrow \alpha)$	From *Theorem 3.11*
4.	$\neg^{(n)}\beta \rightarrow \alpha$	Modus Ponens (2, 3)
5.	$\neg^{(n)}\alpha \rightarrow \alpha$	Transitivity (1, 4)
6.	$\alpha \rightarrow \alpha$	Identity theorem
7.	$(\alpha \rightarrow \alpha) \rightarrow ((\neg^{(n)}\alpha \rightarrow \alpha) \rightarrow ((\alpha \vee \neg^{(n)}\alpha) \rightarrow \alpha))$	Axiom Pos8
8.	$(\neg^{(n)}\alpha \rightarrow \alpha) \rightarrow ((\alpha \vee \neg^{(n)}\alpha) \rightarrow \alpha)$	Modus Ponens (6, 7)
9.	$(\alpha \vee \neg^{(n)}\alpha) \rightarrow \alpha$	Modus Ponens (5, 8)
10.	$\alpha \vee \neg^{(n)}\alpha$	From Theorem 3.9
11.	α	Modus Ponens (10, 9)
12.	$(\neg^{(n)}\alpha \rightarrow \neg^{(n)}\beta), \beta \vdash_{C_n} \alpha$	1-11
13.	$(\neg^{(n)}\alpha \rightarrow \neg^{(n)}\beta) \vdash_{C_n} \beta \rightarrow \alpha$	Deduction Theorem(12)
14.	$\vdash_{C_n} (\neg^{(n)}\alpha \rightarrow \neg^{(n)}\beta) \rightarrow (\beta \rightarrow \alpha)$	Deduction Theorem(13)

□ □

The author in [6] shows the strong negation in C_1 has all the properties of the classical negation. With the above proof we extend this result to the hierarchy of logics in C_n in the sense that also the strong negation defined for each logic in the hierarchy behaves like the classical negation. This result is interesting because strong negations exhibit the possibility to develop theories in these paraconsistent logics to be equivalent to the classical counterpart. People with the desire to study paraconsistent theories that distinguishes between triviallity and explosiveness should avoid strong negations in their theories.

4 Some Additional Results

In this section we studied a new hierarchy that we called $C_n^{'}$. This hierarchy includes $\alpha \rightarrow \neg\neg\alpha$ to each calculi of the hierarchy C_n. Since $\neg\neg\alpha \rightarrow \alpha$ is valid in each C_n due to $C_\omega 1$, then the so called Double Negation Elimination [12] is valid in this new hierarchy. This property allows us to introduce or eliminate a negation from a proof. It is interesting to notice that the Double Negation Elimination is not valid in Intuitionistic Logic due the lack of constructivism of the proof; on the other hand only one side of the property $(\alpha \rightarrow \neg\neg\alpha)$ is valid in Intuitionism.

We consider really important to study extensions of paraconsistent logics in order to develop richer and stronger paraconsistent systems that could be used for any purpose, from application in artificial intelligence to quantum physics; opening more possibilities to engineers and scientist respectively. In this section we proved in a similar way of [7] that $C_n^{'}$ is indeed a hierarchy. In [7] the authors recursively defined valuations T_n starting with the logic $\mathbf{P}^1$. For the purpose of our proof we used $\mathbf{P}^2$ [8], in which $\alpha \rightarrow \neg\neg\alpha$ is a valid formula.

Definition 4.1 ([8]) Let $\mathbf{P}^2$ be the logic defined by the following truth tables, where 1 and 2 are the designated values:

$\wedge$	1	2	3
1	1	1	3
2	1	1	3
3	3	3	3

$\vee$	1	2	3
1	1	1	1
2	1	1	1
3	3	1	3

$\rightarrow$	1	2	3
1	1	1	3
2	1	1	3
3	1	1	1

	$\neg$
1	3
2	2
3	1

Definition 4.2 Let $C_\omega^{'}$ be an extension of C_ω where the following axiom is included:
$C_\omega 2^{'} : \alpha \rightarrow \neg\neg\alpha$

Also, we define the hierarchy $C_n^{'}$ just adding $\alpha \rightarrow \neg\neg\alpha$ to each calculi of C_n.

In the following definition we introduce a valuation of the form of truth tables called $T_n^{'}$. These tables are valuations of $n+2$ values (in $\mathbb{N}$) being the only non designated the greatest (n + 2). We recursively define these tables beginning with the logic $\mathbf{P}^2$, so $T_1^{'}$ is $\mathbf{P}^2$. For $T_2^{'}$ we keep the values from the previous table $T_1^{'}$ and we add one more value, in this case 4, which will be the only non - designated value, notice that the value 3 in $T_2^{'}$ is no longer a non - designated value. The valuation with formulas involving this new value are stated in the following definition. We repeated this process to generate the table $T_n^{'}$ from the table $T_{n-1}^{'}$.

Definition 4.3 ([7]) We define $T_n^{'}$ as truth tables in the following way:
(Base Case) $T_1^{'} = \mathbf{P}^2$
(Inductive Step) $T_n^{'}$ is obtained from $T_{n-1}^{'}$ adding a new value $n+2$ in the current table as the only non - designated value. The mapping of formulas involving this new value is defined as follows:

(1) The element in row α and column β of the conjunction table of $T_n^{'}$ is $max(\alpha, \beta)$.
(2) The element in row α and column β of the disjunction table of $T_n^{'}$ is $min(\alpha, \beta)$.
(3) The element in row α and column β of the implication table of $T_n^{'}$ is 1 if $\alpha = \beta$, and β otherwise.
(4) Negation: $T_n^{'}$ gives us the following table:

α	1	2	3	...	n	n + 1	n + 2
$\neg\alpha$	n + 2	2	2	...	n - 1	n	1

In the table above, $n = 1, 2, \ldots, n, n+1$ are the designated values and $n+2$ is the only non - designated one. We will use the notation $v_n(\alpha)$ to indicate the valuation of the formula α in the $T_n{'}$ valuation.

We remind the reader to notice that the definition of conjunction and disjunction for the new value is the maximum and minimum element respectively because the greatest value in the valuation is the non - designated value.

The intuitive idea behind the above negation table is basically permute the values 1 and $n+2$ in the table and "shift" one value for the middle values, except from 2 which remains the same.

Lemma 4.4 *In T_n' we have that $v_n(\alpha) = v_n(\alpha \wedge \neg\alpha)$ if $v_n(\alpha) \in \{3, \dots, n+1\}$ and $n \geq 2$.*

Proof. By Induction on n:
(Base Case $n = 2$) It is just enough to ckeck $T_2{}'$ valuation.
(Inductive Step) Assume $v_n(\alpha) = v_n(\alpha \wedge \neg\alpha)$. We see two possible cases.

(1) Case $v_{n+1}(\alpha) \in \{3, \dots, n+2\}$. In this case, since T_{n+1}' is obtained from T_n', we have that $v_{n+1}(\alpha) = v_{n+1}(\alpha \wedge \neg\alpha)$ from the inductive hypothesis.

(2) Case $v_{n+1}(\alpha) = n+3$. If $v_{n+1}(\alpha) = n+3$ then $v_{n+1}(\neg\alpha) = 1$. From this applying the new rule for conjunction $v_{n+1}(\alpha \wedge \neg\alpha) = n+3$, which is $v_{n+1}(\alpha)$, therefore $v_{n+1}(\alpha) = v_{n+1}(\alpha \wedge \neg\alpha)$.

□

Lemma 4.5 *In T_n' we have that:*

$$v_n(\alpha^o) = \begin{cases} 1 & \text{if } v_n(\alpha) = 1 \\ n+2 & \text{if } v_n(\alpha) = 2 \\ v_n(\neg\alpha) & \text{otherwise} \end{cases}$$

Proof. By cases:
Case $v_n(\alpha) = 1$. If $v_n(\alpha) = 1$ then $v_n(\neg\alpha) = n+2$. The new value for conjunction is applied, so we get $v_n(\alpha \wedge \neg\alpha) = n+2$. From this $v_n(\neg(\alpha \wedge \neg\alpha)) = 1$, which is $v_n(\alpha^o) = 1$.
Case $v_n(\alpha) = 2$. If $v_n(\alpha) = 2$ then $v_n(\neg\alpha) = 2$. Looking up $\mathbf{P}^2$ - valuation we can see that $v_n(\alpha \wedge \neg\alpha) = 1$, from this we have that $v_n(\neg(\alpha \wedge \neg\alpha)) = n+2$, which is $v_n(\alpha^o) = 1$.
Case $v_n(\alpha) = n+2$. If $v_n(\alpha) = n+2$ then $v_n(\neg\alpha) = 1$. The new value for conjunction is applied, hence $v_n(\alpha \wedge \neg\alpha) = n+2$. From this $v_n(\alpha^o) = 1$ as in the case $v_n(\alpha) = 1$. But also $v_n(\neg\alpha) = 1$, so $v_n(\alpha^o) = v_n(\neg\alpha)$ as desired.
Case $v_n(\alpha) \in \{3, \dots, n+1\}$. If $n = 1$ it is just enough to check T_1'. For $n \geq 2$ we can see from lemma 4.4 that $v_n(\alpha) = v_n(\alpha \wedge \neg\alpha)$, hence $v_n(\neg(\alpha \wedge \neg\alpha)) = v_n(\neg\alpha)$, which is $v_n(\alpha^o) = v_n(\neg\alpha)$ as desired. □

Lemma 4.6 *In T_n' we have that:*

$$v_n(\alpha^{(n)}) = \begin{cases} 1 & \text{if } v_n(\alpha) = 1 \text{ or } v_n(\alpha) = n+2 \\ n+2 & \text{otherwise} \end{cases}$$

Proof. By cases:
Case $v_n(\alpha) = 1$. Due to lemma 4.5 $v_n(\alpha^o) = 1$ we observe that the consistency operator maps 1 to 1. Furthermore, the conjunction $v_n(\alpha^o \wedge \cdots \wedge \alpha^n)$ only involves the value 1. Using T_1' we can see that the latter evaluates to 1.
Case $v_n(\alpha) = n+2$. From lemma 4.5 $v_n(\alpha^o) = v_n(\neg\alpha) = 1$. Therefore $v_n(\alpha^{(n)}) = 1$ by the same reason of last case.

Case $v_n(\alpha) = 2$. We can see that $v_n(\alpha^{(n)}) = n+2$, because $v_n(\alpha^o)$ involves the new value $n+2$.
Case $v_n(\alpha) \in \{3, \dots, n+1\}$. From lemma 4.5 $v_n(\alpha^o) = v_n(\alpha) - 1$. Let $v_n(\alpha) = \lambda$; from this applying $(\lambda - 2) - \mathit{times}$ the consistency operator to α we obtain $v_n(\alpha^{\lambda-2}) = 2$. In the next application of the consistency operator we will obtain $v_n(\alpha^{\lambda-1}) = n+2$ due to lemma 4.5. Hence $n+2$ is involved in one of the conjuncts of $\alpha^o \wedge \cdots \wedge \alpha^n$. Therefore $v_n(\alpha^{(n)}) = n+2$. □

Lemma 4.7 *$(\forall n \in \mathbb{N})$ $(\nvdash_{C'_n} \alpha^n)$*

Proof. By cases:
Case $n = 1$. It is only necessary to check T'_1.
Case $n \in \mathbb{N} \setminus \{1\}$. We claim that if $v_n(\alpha) = n+1$ then $v_n(\alpha^n) = n+2$. If $v_n(\alpha) = n+1$ then $v_n(\alpha^o) = v_n(\neg\alpha) = v_n(\alpha) - 1 = n$, due to lemma 4.5. To evaluate to 2 we will need to apply $(n-1) - \mathit{times}$ more the consistency operator. In the next application of the consistency operator we evaluate $v_n(\alpha^n) = n+2$ due to lemma 4.5. Therefore $v_n(\alpha^n) = n+2$. □

Theorem 4.8 $\vdash_{C'_{n+1}} x$ *entails* $\vdash_{C'_n} x$

Proof. It is only necessary to verify that $(A9)^{n+1}$ and $(A10)^{n+1}$ are provable in C'_n

We wil show that $\vdash_{C'_n} \beta^{(n+1)} \to ((\alpha \to \beta) \to ((\neg\alpha \to \beta) \to \beta))$

1. $\beta^{(n)} \to ((\alpha \to \beta) \to ((\neg\alpha \to \beta) \to \beta))$ $(C'_n 1)$
2. $\beta^{(n+1)}$ Hypothesis
3. $\beta^{(n+1)} \leftrightarrow (\beta^{n+1} \wedge \beta^{(n)})$
4. $(\beta^{n+1} \wedge \beta^{(n)}) \to \beta^{(n)}$ (Pos4)
5. $\beta^{(n)}$ Transitivity (3, 4) and Modus Ponens with 2
6. $((\alpha \to \beta) \to ((\neg\alpha \to \beta) \to \beta))$ Modus Ponens(5, 1)
7. $\vdash_{C'_n} \beta^{(n+1)} \to ((\alpha \to \beta) \to ((\neg\alpha \to \beta) \to \beta))$ 1 - 6

We wil show that $\vdash_{C'_n} (\alpha^{(n+1)} \wedge \beta^{(n+1)}) \to (\alpha \circledast \beta)^{(n+1)}$, where $\circledast \in \{\wedge, \vee, \to\}$

1. $(\alpha^{(n+1)} \wedge \beta^{(n+1)})$ Hypothesis
2. $(\alpha^{(n+1)} \wedge \beta^{(n+1)}) \to \alpha^{(n+1)}$ (Pos3)
3. $(\alpha^{(n+1)} \wedge \beta^{(n+1)}) \to \beta^{(n+1)}$ (Pos4)
4. $\alpha^{(n+1)}$ Modus Ponens(1, 2)
5. $\beta^{(n+1)}$ Modus Ponens(1, 3)
6. $\alpha^{(n+1)} \leftrightarrow (\alpha^{n+1} \wedge \alpha^{(n)})$
7. $\beta^{(n+1)} \leftrightarrow (\beta^{n+1} \wedge \beta^{(n)})$
8. $\alpha^{n+1} \wedge \alpha^{(n)}$ Modus Ponens(4, 6)
9. $\beta^{n+1} \wedge \beta^{(n)}$ Modus Ponens(5, 7)
10. $(\alpha^{n+1} \wedge \alpha^{(n)}) \to \alpha^{(n)}$ (Pos4)

11. $(\beta^{n+1} \wedge \beta^{(n)}) \to \beta^{(n)}$ (Pos4)
12. $\alpha^{(n)}$ Modus Ponens(8, 10)
13. $\beta^{(n)}$ Modus Ponens(9, 11)
14. $\alpha^{(n)} \to (\beta^{(n)} \to (\alpha^{(n)} \wedge \beta^{(n)}))$ (Pos5)
15. $\beta^{(n)} \to (\alpha^{(n)} \wedge \beta^{(n)})$ Modus Ponens(12, 14)
16. $\alpha^{(n)} \wedge \beta^{(n)}$ Modus Ponens(13, 15)
17. $(\alpha^{(n)} \wedge \beta^{(n)}) \to (\alpha \circledast \beta)^{(n)}$ $(C_n'2)$
18. $(\alpha \circledast \beta)^{(n)}$ Modus Ponens(16, 17)
19. $(\alpha \circledast \beta)^{n+1}$ Lemma 4.4
20. $(\alpha \circledast \beta)^{(n)} \to ((\alpha \circledast \beta)^{n+1} \to ((\alpha \circledast \beta)^{(n)} \wedge (\alpha \circledast \beta)^{n+1}))$ (Pos5)
21. $(\alpha \circledast \beta)^{n+1} \to ((\alpha \circledast \beta)^{(n)} \wedge (\alpha \circledast \beta)^{n+1})$ Modus Ponens(18, 20)
22. $(\alpha \circledast \beta)^{(n)} \wedge (\alpha \circledast \beta)^{n+1}$ Modus Ponens(19, 21)
23. $(\alpha \circledast \beta)^{(n+1)}$
24. $\vdash_{C_n'} (\alpha^{(n+1)} \wedge \beta^{(n+1)}) \to (\alpha \circledast \beta)^{(n+1)}$ 1 - 23

□

Theorem 4.9 *T_n' is sound w.r.t. C_n'.*

Proof. We will prove that inference rules and all axiom schema in C_n' are tautologies in the T_n' valuation. All proofs in this sections are done by contradiction.

Modus Ponens: We will show that T_n' preserves Modus Ponens. We assume $v_n(\alpha)$ and $v_n(\alpha \to \beta)$ to be designated values. Suppose $v(\beta)$ evaluates to $n+2$ (the only non-designated value in T_n'). Since $v_n(\alpha) \neq v_n(\beta)$ then $v_n(\alpha \to \beta) = n+2$. But $v_n(\alpha \to \beta)$ evaluates a designated value. Contradiction, therefore T_n' preserves Modus Ponens.

Pos1: We claim that: $(\forall \alpha, \beta)(v_n(\alpha \to (\beta \to \alpha)) \neq n+2)$.

We assume $(\exists \alpha, \beta)(v_n(\alpha \to (\beta \to \alpha)) = n+2)$. Hence $v_n(\alpha) \neq v_n(\beta \to \alpha) = n+2$. From the latter $v_n(\beta) \neq v_n(\alpha) = n+2$. But $v_n(\alpha) \neq n+2$. Contradiction. Therefore $(\forall \alpha, \beta)(v_n(\alpha \to (\beta \to \alpha)) \neq n+2)$.

Pos2: We claim that: $(\forall \alpha, \beta, \gamma)(v_n((\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))) \neq n+2)$.

We assume $(\exists \alpha, \beta, \gamma)(v_n((\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))) = n+2)$. Hence $v_n(\alpha \to (\beta \to \gamma)) \neq v_n((\alpha \to \beta) \to (\alpha \to \gamma)) = n+2$. Also, from the last formula $v_n(\alpha \to \beta) \neq v_n(\alpha \to \gamma) = n+2$. From the latter $v_n(\alpha) \neq v_n(\gamma) = n+2$. Since $v_n(\alpha) \neq n+2$ and $v_n(\alpha \to (\beta \to \gamma)) \neq n+2$ then $v_n(\beta \to \gamma) \neq n+2$ due to Modus Ponens. Also, since $v_n(\alpha) \neq n+2$ and $v_n(\alpha \to \beta) \neq n+2$ then $v_n(\beta) \neq n+2$. Finally, because $v_n(\beta) \neq n+2$ and $v_n(\beta \to \gamma) \neq n+2$ then $v_n(\gamma) \neq n+2$. But $v_n(\gamma) = n+2$. Contradiction, therefore $(\forall \alpha, \beta, \gamma)(v_n((\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))) \neq n+2)$.

Pos3 and Pos4: We claim that: $(\forall \alpha, \beta)(v_n((\alpha \wedge \beta) \to \alpha) \neq n+2$ and $v_n((\alpha \wedge \beta) \to \beta) \neq n+2)$

Suppose $(\exists\alpha,\beta)(v_n((\alpha\wedge\beta)\to\alpha)=n+2)$. Hence $v_n(\alpha\wedge\beta)\neq v_n(\alpha)=n+2$. Since the conjunction of α and β involves the non-designated value then $v_n(\alpha\wedge\beta)=n+2$, but $v_n((\alpha\wedge\beta)\to\alpha)\neq n+2$. Contradiction, therefore $(\forall\alpha,\beta)(v_n((\alpha\wedge\beta)\to\alpha)=n+2)$. The proof for $(\forall\alpha,\beta)(v_n((\alpha\wedge\beta)\to\beta)=n+2)$ is similar to the above proof.

Pos5: We claim that: $(\forall\alpha,\beta)(v_n(\alpha\to(\beta\to(\alpha\wedge\beta)))\neq n+2)$.

Suppse $(\exists\alpha,\beta)(v_n(\alpha\to(\beta\to(\alpha\wedge\beta)))=n+2)$. Then $v_n(\alpha)\neq v_n(\beta\to(\alpha\wedge\beta))=n+2$. From the latter $v_n(\beta)\neq v_n(\alpha\wedge\beta)=n+2$. Since $v_n(\alpha)\neq n+2$ and $v_n(\beta)\neq n+2$ then $v_n(\alpha\wedge\beta)\neq n+2$. Contradiction, therefore $(\forall\alpha,\beta)(v_n(\alpha\to(\beta\to(\alpha\wedge\beta)))\neq n+2)$.

Pos6 and Pos7: We claim that: $(\forall\alpha,\beta)(v_n(\alpha\to(\alpha\vee\beta))\neq n+2$ and $v_n(\beta\to(\alpha\vee\beta))\neq n+2)$.

Suppose $(\exists\alpha,\beta)(v_n(\alpha\to(\alpha\vee\beta))=n+2)$. That is $v_n(\alpha)\neq v_n(\alpha\vee\beta)=n+2$. From the latter we can see that both $v_n(\alpha)$ and $v_n(\beta)$ evaluate to $n+2$. But $v_n(\alpha)\neq n+2$. Contradiction. Therefore $(\forall\alpha,\beta)(v_n(\alpha\to(\alpha\vee\beta))\neq n+2$. Furthemore a similar reasoning we can prove that $(\forall\alpha,\beta)((v_n(\beta\to(\alpha\vee\beta))\neq n+2))$.

Pos8: We claim that: $(\forall\alpha,\beta,\gamma)(v_n((\alpha\to\gamma)\to((\beta\to\gamma)\to((\alpha\vee\beta)\to\gamma)))\neq n+2)$.

We assume that $(\exists\alpha,\beta,\gamma)(v_n((\alpha\to\gamma)\to((\beta\to\gamma)\to((\alpha\vee\beta)\to\gamma)))=n+2)$. Hence $v_n(\alpha\to\gamma)\neq v_n((\beta\to\gamma)\to((\alpha\vee\beta)\to\gamma))=n+2$. From the latter $v_n(\beta\to\gamma)\neq v_n((\alpha\vee\beta)\to\gamma)=n+2$. Also, from this last formula we have that $v_n(\alpha\vee\beta)\neq v_n(\gamma)=n+2$. From $v_n(\alpha\vee\beta)\neq n+2$ at least one disjunct evaluates to different to $n+2$. Suppose $v_n(\alpha)\neq n+2$, then because of $v_n(\alpha\to\gamma)\neq n+2$ we have that $v_n(\gamma)\neq n+2$. Also if $v_n(\beta)\neq n+2$ then $v_n(\gamma)\neq n+2$ due to $v_n(\beta\to\gamma)\neq n+2$. Contradiction, so $(\forall\alpha,\beta,\gamma)(v_n((\alpha\to\gamma)\to((\beta\to\gamma)\to((\alpha\vee\beta)\to\gamma)))\neq n+2)$.

$(C_\omega 1)$: We claim that: $(\forall\alpha)(v_n(\alpha\vee\neg\alpha)\neq n+2)$.

We assume that $(\exists\alpha)(v_n(\alpha\vee\neg\alpha)=n+2)$. We see two possible cases:

Case 1: We assume that $v_n(\alpha)=v_n(\neg\alpha)=n+2$. But this is impossible from T_n' definition.

Case 2: We assume that $v_n(\alpha)\neq v_n(\neg\alpha)$. We can see two possible sub cases:

Sub Case 1: We assume that $v_n(\alpha)=n+2$. From T_n' we have that $v_n(\neg\alpha)=1$. Thefore the disjunction evaluates to $v_n(\alpha\vee\neg\alpha)=1$. Contradiction.

Sub Case 2:]We assume that $v_n(\neg\alpha)=n+2$. From T_n' we have that $v_n(\alpha)=1$. Thefore the disjunction evaluates to $v_n(\alpha\vee\neg\alpha)=1$. Contradiction.

In all possible cases we reached a contradiction. Therefore $(\forall\alpha)(v_n(\alpha\vee\neg\alpha)\neq n+2)$.

$(C_\omega 2)$: We claim that: $(\forall\alpha)(v_n(\neg\neg\alpha\to\alpha)\neq n+2)$.

We assume that $(\exists\alpha)(v_n(\neg\neg\alpha\to\alpha)=n+2)$. From T_n' we have that $v_n(\neg\neg\alpha)\neq v_n(\alpha)$ and $v(\alpha)=n+2$. From the latter we have that $v_n(\neg\alpha)=1$. From the latter and T_n' we have that $v_n(\neg\neg\alpha)=n+2$, but $v_n(\neg\neg\alpha)\neq v_n(\alpha)$. Contradiction. Therefore $(\forall\alpha)(v_n(\neg\neg\alpha\to\alpha)\neq n+2)$.

$(C'_\omega 2)$: We claim that: $(\forall\alpha)(v_n(\alpha \to \neg\neg\alpha) \neq n+2)$.

We assume that $(\exists\alpha)(v_n(\alpha \to \neg\neg\alpha) = n+2)$. From T'_n we have that $v_n(\alpha) \neq v_n(\neg\neg\alpha)$ and $v(\neg\neg\alpha) = n+2$. From the latter we have that $v_n(\neg\alpha) = 1$. From the latter and T'_n we have that $v_n(\alpha) = n+2$, but $v_n(\neg\neg\alpha) \neq v_n(\alpha)$. Contradiction. Therefore $(\forall\alpha)(v_n(\alpha \to \neg\neg\alpha) \neq n+2)$.

$(C'_n 1)$: We claim that $(\forall\alpha,\beta)(v_n(\beta^{(n)} \to ((\alpha \to \beta) \to ((\alpha \to \neg\beta) \to \neg\alpha))) \neq n+2)$.

We assume $(\exists\alpha,\beta)(v_n(\beta^{(n)} \to ((\alpha \to \beta) \to ((\alpha \to \neg\beta) \to \neg\alpha))) = n+2)$. From T'_n we have that $v_n(\beta^{(n)}) \neq v_n((\alpha \to \beta) \to ((\alpha \to \neg\alpha) \to \neg\alpha))$ and $v_n((\alpha \to \beta) \to ((\alpha \to \neg\alpha) \to \neg\alpha)) = n+2$. From the latter and T'_n definition we have that $v_n(\alpha \to \beta) \neq v_n((\alpha \to \neg\beta) \to \neg\alpha)$ and $v_n((\alpha \to \neg\beta) \to \neg\alpha) = n+2$. Also by T'_n definition and the latter we have that $v_n(\alpha \to \neg\beta) \neq v_n(\neg\alpha)$ and that $v_n(\neg\alpha) = n+2$. From the latter $v_n(\alpha) = 1$. Due to $v_n(\beta^{(n)}) \neq n+2$, then due to lemma 4.6 we have that $v_n(\beta^{(n)}) = 1$. The latter implies that $v_n(\beta) = 1$ or $v_n(\beta) = n+2$. From this we distinguish two possible cases:

Case 1: $v_n(\beta) = 1$. If $v_n(\beta) = 1$ then $v_n(\neg\beta) = n+2$. we can easily see that $v_n(\alpha \to \neg\beta) = n+2$. But $v_n(\alpha \to \neg\beta) \neq v_n(\neg\alpha)$. Contradiction.

Case 2: $v_n(\beta) = n+2$. If $v_n(\beta) = n+2$ then $v_n(\alpha \to \beta) = n+2$. But $v_n(\alpha \to \beta) \neq v_n((\alpha \to \neg\beta) \to \neg\alpha)$. Contradiction.

In all possible cases we reached a contradiction, therefore $(\forall\alpha,\beta)(v_n(\beta^{(n)} \to ((\alpha \to \beta) \to ((\alpha \to \neg\beta) \to \neg\alpha))) \neq n+2)$.

$(C'_n 2)$ We claim that $(\forall\alpha,\beta)(v_n((\alpha^{(n)} \wedge \beta^{(n)}) \to (\alpha \circledast \beta)^{(n)}) \neq n+2)$, where $\circledast \in \{\wedge, \vee, \to\}$.

We assume that $(\exists\alpha,\beta)(v_n((\alpha^{(n)} \wedge \beta^{(n)}) \to (\alpha \circledast \beta)^{(n)}) = n+2)$. From T'_n definition we have that $v_n(\alpha^{(n)} \wedge \beta^{(n)}) \neq v_n((\alpha \circledast \beta)^{(n)})$ and that $v_n((\alpha \circledast \beta)^{(n)}) = n+2$. From the latter and lemma 4.6 we have that $v_n(\alpha \circledast \beta) \in \{2, \ldots, n+1\}$. Since $\circledast \in \{\wedge, \vee, \to\}$ we distinguish three possible cases:

Case 1: $\circledast = \to$. In this case we see that $v_n(\alpha) \neq v_n(\beta)$, otherwise the implication would evaluate to 1. Furthermore we see that $v_n(\alpha) \in \{1, \ldots, n+2\}$ and $v_n(\beta) \in \{2, \ldots, n+1\}$. From the latter and lemma 4.6 we have that $v_n(\beta^{(n)}) = n+2$. Hence the conjunction evaluates to $v_n(\alpha^{(n)} \wedge \beta^{(n)}) = n+2$. But $v_n(\alpha^{(n)} \wedge \beta^{(n)}) \neq v_n((\alpha \to \beta)^{(n)})$. Contradiction.

Case 2: $\circledast = \wedge$. We can see that: $v_n(\alpha) \in \{1, \ldots, n+1\}$ y $v_n(\beta) \in \{1, \ldots, n+1\}$ and that is not the case $v_n(\alpha) = v_n(\beta) = 1$. From the latter at least one of the conjuncts of $v_n(\alpha^{(n)} \wedge \beta^{(n)})$ evaluates something different of 1. Due to lemma 4.6 at least one of the conjuncts of $v_n(\alpha^{(n)} \wedge \beta^{(n)})$ evaluates to $n+2$, hence $v_n(\alpha^{(n)} \wedge \beta^{(n)}) = n+2$. But $v_n(\alpha^{(n)} \wedge \beta^{(n)}) \neq v_n((\alpha \to \beta)^{(n)})$. Contradiction.

Case 3: $\circledast = \vee$. We can see that: $v_n(\alpha) \in \{2, \ldots, n+2\}$ y $v_n(\beta) \in \{2, \ldots, n+2\}$ and that is not the case $v_n(\alpha) = v_n(\beta) = n+2$. From the latter we see that at least one of the conjuncts of $v_n(\alpha^{(n)} \wedge \beta^{(n)})$ evaluates something differents of $n+2$. Due to lemma 4.6 at least one of the conjuncts $v_n(\alpha^{(n)} \wedge \beta^{(n)})$ evaluates to $n+2$, hence $v_n(\alpha^{(n)} \wedge \beta^{(n)}) = n+2$. But $v_n(\alpha^{(n)} \wedge \beta^{(n)}) \neq v_n((\alpha \to \beta)^{(n)})$. Contradiction.

□

Theorem 4.10 *C_n' is a hierarchy of paraconsistent logics.*

Proof. The result holds from the following reasons:

(1) For each formula x such that $\vdash_{C_{n+1}'} x$ then $\vdash_{C_n'} x$. (Theorem 4.8).

(2) Exists a sound valuation for C_n', in this case T_n' (Theorem 4.9), where $(\exists x)(\vdash_{C_n'} x)$ and $\nvdash_{C_{n+1}'} x$. That formula is for instance α^{n+1} (Lemma 4.7).

□

5 Conclusions

The presented work gives a general idea how to extend a property in C_1 to C_n, mainly using inductive proofs. We know that all logics in the C_n system are strictly weaker than C_1 [6], perhaps many of them share many things in common as a strong negation. The section 4 introduce a new hierarchy of paraconsistent logics called C_n' which is a stronger chaing than C_n. This new hierarchy could be useful for theories and applications where the axiom $\alpha \to \neg\neg\alpha$ is crucial. In the future should be interesting to investigate how much these logics are related each other among relevant properties.

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