If, not when

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Abstract

We present a logic of verified and unverified assertions and prove it sound and complete with respect to its possible-worlds semantics. The logic, a constructive modal logic, is motivated by considerations of the interpretation of conditionals in natural language semantics, but, we claim, is of independent interest.

1 Motivation

One of us wrote a dissertation on the patterns of temporal reference exhibited by conditional and modal sentences in English a long time ago [1]. The other one was always fascinated by the verificationist logic that emerged, whether you were concerned with the modelling of conditionals and modals, or not. This one decided that the logic that emerged, a 'temporal' logic of verified and unverified assertions with curious properties, deserved a writing up of its own. Hence this note.

Whether one is concerned with the original application of this logic or not, it makes sense to explain some of the issues that motivated it. Operators that refer to tenses like **past** and **pres**ent are self-explanatory, but logicians by and large are blissfully unaware of the subtle issues of interpretation surrounding English sentences. Consider the sentence:

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(1)   If I smile when I get out, the interview went well.

This is an example of *deictic shift*: deictic shift occurs when a tense locates an event as being past or present with respect to some time other than the speech time. At speech time both the presumed interview and smiling (or not) are in the future, but the sentence orders then as first the interview event and then the smiling (or not) event.

Assume we want to define meaning as the potential to *change states of information*, instead of the conditions under which something is true. If meaning is to be explicated in terms of information change, there are two levels under which temporal reference operates. First temporal references must describe how the world changes over time. But secondly they have also to explain the way the information about the world changes over time.

Typically the tenses are taken to state a relation between the time at which some assertion is made (the speech time) and the time at which the event being described occurs (the event time). The speech time then serves as the (single) deictic centre for the tenses.

An alternative is to centre tenses on the time at which an update is made to one's stock of information, where this update is the result of the utterance of the sentence. Some kind of *update time* would provide the tenses deictic centre rather than the speech time. In most cases the move from speech time to update time will make no difference: normally the update occurs as soon as the utterance is made. However modal and conditional sentences do not behave in this simple fashion. They place constraints on the way that updates may be made in the future. Tenses within the scope of the modal or conditional are centred relative to these future update times leading to deictic shift. Or so goes the story in the thesis.

Moreover the update should not be considered a single operation, occurring at a single time. Instead it is necessary to decompose the update into two operations: *assertion* and *verification*. Typically, assertion and verification occur simultaneously, and most sentences convey verified information. Modals and conditionals by contrast allow delayed assertion and verification. *If A, then B* means roughly: suppose you were now to assert A; if and when A is verified, you will be in a position to assert B, and in due course this assertion will also be verified. Since A and B will both be tensed clauses, allowing for both assertion and verification times will lead to several possible shifted interpretations of the clauses.

Establishing the usefulness of this setting for the analysis of modals and conditionals was the main goal of the thesis, but our objective here is much more restricted. We simply describe the logic that emerged from the analysis, prove some properties and speculate about more general uses for it.

We shall motivate the logic in terms of information states and its application to dealing with conditionals in natural language and then define a (somewhat complicated) semantics for it. Then we provide a proof theory for this logic and show the required sourness and completeness. We conclude with some speculation on further uses.

2   Information States, Tenses and Conditionals

We will first take information states as primitive. They will not be construed as sets of propositions or formulas. It is customary in tense logic to view propositions as functions from times (and perhaps possible worlds) to truth values. But this is not how propositions in our information states behave. A proposition in an information state p is really a short-hand for saying that "p is true at t".

It makes sense to talk about being in (possession of) a certain information state at a certain time. All being well, as time goes by this state will grow into other states that are informational extensions of the original. In reality some of the initial information may prove to be mistaken, so sometimes information may decrease over time. Just because information grows over time one should not conclude that the ordering of information extension (which we write as $\sqsubseteq$) is a temporal order. To do so would make time forward and backwards branching, but we will assume instead that time is linear, so there is only one past and one future. [4]

Information states and sets of them are essentially non-temporal structures. Information states and their ordering into sets of possible extensions give a gods-eye, atemporal view of the world. But if information states are atemporal the same cannot be said for English sentences. The role of the past and present tenses can be seen as one of tying the time varying nature of sentences down to the static assertions found in the information states. We need to show how, via the tenses, the temporarily varying update potential of sentences maps onto the temporally fixed relations in information states.

We will assume a logical language L consisting of atomic sentence letters (p, q, etc . . . plus the absurd sentence $\bot$), four connectives ($\wedge, \vee, \rightarrow$ and $\Rightarrow$) and three one place sentence operators (**past**, **pres** and $\neg$) with parentheses and usual formation rules.

We will write

$$s \models \phi$$

to say that the state s can be the result of an update by a sentence ϕ. This means that if $s \models \phi$ then updating s with ϕ will have no effect, it will add no further information, hence leave one still in s. If $s \models \phi$ holds we say that ϕ is *supported by* s. The notation $s, t \models \phi$ will be used to say that an assertion corresponding to an utterance ϕ made at time t is supported by a state s. Using **past**(p) and **pres**(p) to represent simple past and present tense sentences, we can describe the update effects of utterances of simple tensed sentences as follows:

  (i) $s, t \models$ **past**(p) iff there is some time $t_p \leq t$ such that the assertion $p(t_p)$ is contained in s.

 (ii) $s, t \models$ **pres**(p) iff $p(t)$ is contained in s.

Now an English conditional will consist of two tensed clauses, e.g. (1) above is of

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[4] You don't have to agree with this, you can just take it as a simplifying assumption.

the form

$$\mathbf{pres}(p) \rightarrow \mathbf{past}(q)$$

the tenses of these clauses will be interpreted relative to the future states and times s' and t'.

Back to our semantic definition, for conditional sentences we will say, to begin with, that a conditional $\phi \rightarrow \psi$ uttered at time t is supported in s just in case for any state s' extending s that supports an assertion of the antecedent ϕ made at time t' after t, the state s' also supports an assertion of the consequent ψ made at t'.

3. $s, t \models \phi \rightarrow \psi$ iff for all $s' \sqsupseteq s$ and all $t' \geq t$, if $s', t' \models \phi$ then $s', t' \models \psi$

Our information states consist of sets of assertions. We now divide the assertions contained in an information state into verified and unverified assertions at time t. As time goes by more unverified assertions become verified, but no verified assertion can become unverified.

What is to verify an assertion and what is the difference between verified and unverified assertion? To answer this it helps to note an obvious asymetry between the future and the past. While we can have knowledge about both past and future, the origin of our information about the past is quite different from the origin of our information about the future. With information about the past there is usually some causal chain linking a past event to present information. As for information about the future, at best, one can extrapolate on the basis of what one knows about the past and the present.

With verified information, it is normally assumed that there is some causal chain linking the information to what the information is about. When a causal chain is involved, verification of information about an event cannot precede the occurrence of the event. However, there are some cases where verification is not causal in origin. In particular, a case where a degree of prescience seems to apply is in talking about plans or pre-determined events. When talking about plans, it is not really the world that we are discussing but what is laid out in the plan. Of the plan, we have a gods-eye view, so that we can survey in a glance which events are supposed to happen when. If the plan provides our source of information about these events, then we can verify that certain events will occur, according to the plan, well in advance of the time at which those events are supposed to occur. When one says

(3)   The sun rises at 05:47 tomorrow.

one is in fact reporting on what is stated in a set of sunrise and sunset tables.

Simple, tensed sentences are normally used to convey verified information. Updates consist of first making an assertion and then marking it as verified. For simple sentences, these two operations take place at the same time. This means that the update effect of a sentence ϕ at a time t needs to be expanded out as follows:

$s, t \models \phi$ iff $s, t, t \models \phi$

The two occurrences of t on the right hand side correspond to the *assertion* and

verification times respectively. As we can see, to begin with the two times are assumed to be identical, and correspond to the time at which the sentence is uttered.

The effects of the tenses on atomic sentences may be described as follows:

- $s, a, v \models \mathbf{past}(p)$ iff there is a time $t_p < a$ such that the assertion $p(t_p)$ is verified in s at time v.

- $s, a, v \models \mathbf{pres}(p)$ iff there is a time $t_p \geq a$ such that the assertion $p(t_p)$ is verified in s at time v.

Here, a stands for the assertion time and v for the verification time. An assertion $p(t_p)$ is verified in a state s at a time v if $p(t_p)$ is contained in the set of assertions that count as verified in s at v. (If we represent states as sets of assertions, at any one time this set will be split into two subsets of verified and unverified assertions).

3 Verified and Unverified Assertions

We now forget our motivation in terms of tenses and conditionals while still taking account of the times at which formulas are verified. In these information models, each information state can be seen as a linearly ordered sequence of temporal 'snapshots' of the state, where different formulas are forced at different time points.

An information model M is now a quintuple

$$M = \langle S, \subseteq_t, T, \leq, V \rangle$$

where S is a set of information states, s
$\qquad \subseteq_t$ is a relation in $S \times S \times T$
$\qquad\qquad\qquad$ and is transitive and reflexive over S for any t
$\qquad T$ is a set of time instants, t
$\qquad \leq$ is a (linear) temporal order over T, and
$\qquad V$ is a valuation function

The valuation function V is a function from states, times and atomic sentences in some language L onto the (verification) values 1 or 0. This specifies for a state, time and sentence whether the sentence is verified at that time in that state.

We impose two monotonicity conditions on the valuation function V:

Monotonicity of direct verification ('in-state' monotonicity):
For every state s and atomic sentence p
$t_1 \leq t_2$ implies if $V(s, t_1, p) = 1$ then $V(s, t_2, p) = 1$

Monotonicity of information growth ('out-of-state' monotonicity):
If $s_1 \subseteq_t s_2$ then for atomic sentences p
(a) $\{p \mid V(s_1, t, p) = 1\} \subseteq \{p \mid V(s_2, t, p) = 1\}$
(b) $\{p \mid \exists t : V(s_1, t, p) = 1\} \subseteq \{p \mid \exists t : V(s_2, t, p) = 1\}$

Monotonicity of direct verification says that once an atomic sentence is verified within a state, if remains verified in that state. Monotonicity of direct verification will turn out to apply to all sentences, and not just to atomic ones.

Monotonicity of information growth says that moving from s_1 to a more fully informed state s_2 at some time t verifies the same atomic sentences in s_2 at t as were verified in s_1 at t (condition (a)). It also says (condition (b)) that the sentences asserted in s_1 but not yet verified in at t continue to be asserted in s_2, though the times at which they becomes verified in s_2 may differ form those in s_1. Indeed, some unverified assertions in s_1 may count as already verified in s_2. In addition, s_2 may include verified and unverified assertions that were not present at all in s_1. If s_2 does not contain any extra assertions (i.e. the $\subseteq_t$ becomes $\approx_t$ in (a) and (b)), then the two states are *informationally equivalent* at t. Monotonicity of information growth does not hold for all non-atomic sentences.

In addition, a convergence of verification property is imposed:

Convergence of Verification:
If $s_1 \subseteq_{t_1} s_2 \subseteq_{t_2} s_3$,
then there is a time t_3 such that $t_3 \geq t_1$, $t_3 \geq t_2$ and $\forall t_4 \geq t_3$ $s_1 \subseteq_{t_4} s_3$

as well as the constraint on absurdity:

No Absurdity:
For no s or t is it the case that $V(s, t, \bot) = 1$

3.0.1   Semantic Definitions of Connectives

We now give semantic definitions for a propositional language involving the connectives $\wedge, \vee, \rightarrow, \neg$ and $\sim$. These specify what is required for a sentence to be verified as true at a time t in a state s.

(i) $s, t \Vdash p$ iff $V(s, t, p) = 1$ if p is atomic

(ii) $s, t \Vdash \phi \wedge \psi$ iff $s, t \Vdash \phi$ and $s, t \Vdash \psi$

(iii) $s, t \Vdash \phi \vee \psi$ iff $s, t \Vdash \phi$ or $s, t \Vdash \psi$

(iv) $s, t \Vdash \phi \rightarrow \psi$ iff $\forall t_1 \geq t, s_1 \supseteq_{t_1}^{\phi,t} s : \exists t_2 \geq t_1$ such that $s_1, t_2 \Vdash \psi$

(v) $s, t \Vdash \neg\phi$ iff $\forall t_1 \geq t, s_1 \supseteq_{t_1}^{\phi,t} s : \exists t_2 \geq t_1$ such that $s_1, t_2 \Vdash \bot$

(vi) $s, t \Vdash\sim \phi$ iff $\forall t_1 \geq t : s, t_1 \not\Vdash \phi$

Minimal Information Extensions

The semantic definitions for $\rightarrow$ and $\neg$ rely on minimal information extensions. A minimal information extension can be defined as follows:

- $s_1 \supseteq_{t_1}^{\phi,t} s$ iff
  a) $s_1 \supseteq_{t_1} s$
  b) $s_1, t_1 \Vdash \phi$, and
  c) $\nexists t_2, s_2$ such that $t \leq t_2 < t_1$, $s \supseteq_{t_2} s_2 \supseteq_{t_2} s_1$, and $s_2, t_2 \Vdash \phi$

That is, if s_1 is a minimal extension of s with respect to ϕ at time t, then s_2 is the first state extending s that verifies ϕ at the earliest time t_1.

Negation

Two types of negation are defined: 'out-of-state' negation, $\neg$, and 'in-state' negation $\sim$. Out-of-state negation says that a sentence will never be verified in any future state at any future time. In-state negation says that a sentence will never be verified in the current state at any future time.

If a sentence $\sim \phi$ holds in a state s at a time t, the sentence ϕ is *not* asserted by state s. (Note that the monotonicity of direct verification will ensure that if $\sim \phi$ holds in s at any time t, it holds at all times). If $\sim\sim \phi$ holds in s at t, then it follows that at some later time t_1, ϕ will hold in s. Thus $\sim\sim \phi$ gives a way of saying that ϕ is asserted by a state, though not necessarily as yet verified in that state.

As well as referring to the two negations as in-state and out-of-state negation, we can also say that $\sim$ amounts to a denial of assertion, while $\neg$ amounts to an assertion of denial.

3.0.2 *Monotonicity*

A significant feature of the forcing relation in intuitionistic logic is its monotonicity: once a sentence is forced in one state, it remains forced in all subsequent states. This holds for all sentences.

Within the information models currently being dealt with, we need to consider two distinct kinds of monotonicity: in-state monotonicity, and out-of-state monotonicity. In-state monotonicity holds for all sentences, but out-of-state monotonicity holds only for a restricted set of *stable* sentences.

Stable sentences are defined below, and are approximately those whose sub-formulas contain no odd-numbered sequences of in-state negations — thus $\sim\sim \phi$ is stable, but $\psi \wedge \sim \phi$ is not. Unstable sentences are ones that state that a particular sentence is not asserted in a given information state. That a sentence is not asserted in one state does not preclude it from coming to be asserted in a later state.

In-State Monotonicity

First, we show that all sentences satisfy in-state monotonicity:

In-State Monotonicity:
If $s, t \Vdash \phi$ then $\forall t_1 \geq t : s, t_1 \Vdash \phi$

This is a direct consequence of the condition of monotonicity of direct verification (in-state monotonicity for atomic sentences) that was imposed on the valuation function in information models. For conjunction and disjunction, it is easy to see that monotonicity carries across from monotonic sentences.

For in-state negation, note that $\sim \phi$ is only forced in state s at t if ϕ fails to be forced in s at any time t_1 at or after t. Assuming that ϕ itself behaves monotonically within the state, this means that ϕ is forced at no time within the state, and so $\sim \phi$ is forced at all times within the state. Exactly similar arguments apply to $\neg\phi$ — just ignore the reference to future information states in the semantic definition, and concentrate on reference to future times within the current state.

For implication, $\phi \to \psi$, assume both ϕ and ψ behave monotonically. Restricting attention to what must go on within the state if the semantic conditions for the implication hold at time t, we find that if ϕ holds at any time t_1 after t, then ψ must hold at some time t_2 after t_1. Since ϕ and ψ are monotonic, if this holds at t, it will also hold at all times t' after t.

Stable Sentences

Next we define what stable sentences are.

- If p is atomic, then p is stable.

- If ϕ and ψ are stable, then $\phi \wedge \phi$ and $\phi \vee \psi$ are stable.

- $\phi \to \psi$ is stable if ψ is stable. (Otherwise, it is semi-stable.)

- $\neg \phi$ is stable.

- If ϕ is stable, then $\sim\sim \phi$ is stable.

- Anything not classified as stable by the above is unstable.

A piece of notation that will be used below is that if Γ is a set of formulas or sentences, then $Stable(\Gamma)$ denotes the stable subset of Γ.

Out-of-State Monotonicity

It can be shown that all stable sentences satisfy out-of-state monotonicity:

Out-of-State Monotonicity:
If $s, t \Vdash \phi$ then $\forall s_1 \sqsupseteq_t s, t_1 \geq t : s_1, t_1 \Vdash \phi$

(Proof: by induction on the complexity of ϕ — given in Appendix.

The converse does not quite hold. There are some unstable formulas that satisfy out-of-state monotonicity: $\sim \phi \vee \sim\sim \phi$ is an example.

4   Proof theory

The semantic definitions presented above are sound and complete with respect to the following sequent calculus system (Proof: Appendix).

$$\wedge I \; \frac{\Gamma \vdash \phi; \;\; \Gamma \vdash \psi}{\Gamma \vdash \phi \wedge \psi} \qquad\qquad \wedge E \; \frac{\Gamma \vdash \phi \wedge \psi}{\Gamma \vdash \phi}$$

$$\vee I \; \frac{\Gamma \vdash \phi}{\Gamma \vdash \phi \vee \psi} \qquad\qquad \vee E \; \frac{\Gamma \vdash \phi \vee \psi; \;\; \Gamma, \phi \vdash \chi; \;\; \Gamma, \psi \vdash \chi}{\Gamma \vdash \chi}$$

$$\rightarrow I \; \frac{Stable(\Gamma), \phi \vdash \sim\sim \psi}{\Gamma \vdash \phi \rightarrow \psi} \qquad\qquad \rightarrow E \; \frac{\Gamma \vdash \phi; \;\; \Gamma \vdash \phi \rightarrow \psi}{\Gamma \vdash \sim\sim \psi}$$

$$\neg I \; \frac{Stable(\Gamma), \phi \vdash \sim\sim \bot}{\Gamma \vdash \neg\phi} \qquad\qquad \bot \; \frac{\Gamma \vdash \bot}{\Gamma \vdash \phi}$$

$$\sim I \; \frac{\Gamma, \phi \vdash \bot}{\Gamma \vdash \sim \phi} \qquad\qquad \sim Ax \; \frac{}{\Gamma \vdash \sim \phi \vee \sim\sim \phi}$$

$$\sim\rightarrow \; \frac{\Gamma \vdash \sim\sim \phi; \;\; \Gamma \vdash \phi \rightarrow \psi}{\Gamma \vdash \sim\sim \psi} \qquad\qquad \sim\sim \bot \; \frac{\Gamma \vdash \sim\sim \bot}{\Gamma \vdash \bot}$$

The axiom stating that $\vdash \sim \phi \vee \sim\sim \phi$ is a direct result of the fact that the ordering over temporal snapshots is linear, giving rise to the characteristic axiom of Gabbay's logic **KC**.

5 Conclusions

We have shown that our notion of information models with verified and unverified assertions, originally conceived to explain conditionals in natural language semantics does not lead to something that is logically absurd. By the contrary the logic of verified and unverified assertions can be shown to have a respectable proof system, satisfying soundness and completeness. It remains to be seen how useful the logic of verified and unverified assertion is as a logic of (non-temporal) conditionals and modals.

References

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A   Proof of Soundness and Completeness

This appendix shows that the proof theory for a logic of verified and unverified assertions presented is sound and complete with respect to the intended semantics.

Soundness

To prove soundness we must show that

$\Gamma \vdash \phi \Rightarrow \Gamma \models \phi$

where $\Gamma \models \phi$ is a short hand for saying that

for any model M, state s and time t such that $M, s, t \Vdash \sigma$ for every $\sigma \in \Gamma$, $M, s, t \Vdash \phi$

and $\Gamma \vdash \phi$ means that ϕ is derivable from premises Γ in the inference system

The proof proceeds by induction on the complexity of the derivation $\Gamma \vdash \phi$. We will assume a fixed model M.

Atomic Derivations

Soundness of atomic derivations, $\phi \vdash \phi$, is trivial. Any model of the premise is a model of the conclusion. This provides the base case for the induction hypothesis used below.

Conjunction Introduction

Suppose we have $\Gamma \vdash \phi$ and $\Gamma \vdash \psi$. By the induction hypothesis, $\Gamma \models \phi$ and $\Gamma \models \psi$. Choose any state s and time t such that $s, t \Vdash \Gamma$, from which it follows that $s, t \Vdash \phi$ and $s, t \Vdash \psi$. By the semantic definition of conjunction, $s, t \Vdash \phi \wedge \psi$. Hence it is safe to assume that $\Gamma \vdash \phi \wedge \psi$.

Conjunction Elimination

Trivial.

Disjunction Introduction

Trivial.

Disjunction Elimination

We have three induction hypotheses: $\Gamma \models \phi \vee \psi$, $\Gamma, \phi \models \chi$, and $\Gamma, \psi \models \chi$.

Let s and t be such that $s, t \Vdash \Gamma$. Hence $s, t \Vdash \phi \vee \psi$. Hence either $s, t \Vdash \phi$ or $s, t \Vdash \psi$. In the first case, $s, t \Vdash \Gamma, \phi$, and so by induction hypothesis, $s, t \Vdash \chi$. Similarly in the second case. Either way, $s, t \Vdash \chi$, and so $\Gamma \models \chi$. Hence it is safe to assume that $\Gamma \vdash \chi$.

In-State Negation Introduction

Suppose $\Gamma, \phi \vdash \perp$. By the induction hypothesis, $\Gamma, \phi \models \perp$. Since $\perp$ is never verified in any state, we may therefore assume that for no state and time does $s, t \Vdash \Gamma, \phi$.

Suppose that $s, t \Vdash \Gamma$. By monotonicity of direct verification, for all $t_1 \geq t$, $s, t_1 \Vdash \Gamma$. Since no state and time force Γ and ϕ, it follows that $\forall t_1 \geq t$, $s, t_1 \nVdash \phi$. By the semantics of in-state negation, this means that $s, t \Vdash\sim \phi$. Hence $\Gamma \models\sim \phi$ and so it is safe to assume that $\Gamma \vdash\sim \phi$

In-State Negation Axiom

This is not a rule of inference but an axiom. We must show that $s, t \Vdash\sim \phi\vee \sim\sim \phi$ for all s, t.

According to semantic definitions,
(1) $s, t \Vdash\sim \phi$ iff $\forall t_1 \geq t : s, t_1 \nVdash \phi$, and
(2) $s, t \Vdash\sim\sim \phi$ iff $\forall t_1 \geq t, \exists t_2 \geq t_1 : s, t_2 \Vdash \phi$
Since the temporal order is linear, (2) is equivalent to
(2a) $s, t \Vdash\sim\sim \phi$ iff $\exists t_2 \geq t : s, t_2 \Vdash \phi$
It can be seen that (1) and (2a) are complements of one another, and so it follows that either $s, t \Vdash\sim \phi$ or $s, t \Vdash\sim\sim \phi$, for any s and t. Hence by the semantic definition of disjunction, $s, t \Vdash\sim \phi\vee \sim\sim \phi$ for all s, t.

Note

The in-state negation axiom is in fact characteristic of ordinary intuitionistic logic with linear frames. The logic **KC** (Gabbay 1981:66) adds $\vdash \neg\phi \vee \neg\neg\phi$ as a single extra axiom governing intuitionistic negation. **KC** is sound and complete with respect to intuitionistic frames where $\subseteq$ is reflexive, transitive and linear. Since the temporal order $\leq$ is reflexive, transitive and linear, the in-state negation axiom is inevitable.

Implication Introduction

Suppose that $Stable(\Gamma), \phi \vdash\sim\sim \psi$. By the induction hypothesis, $Stable(\Gamma), \phi \models\sim\sim \psi$. That is, if $s, t \Vdash Stable(\Gamma), \phi$, then $\exists t_1 \geq t : s, t_1 \Vdash \psi$.

Let $s_0, t_0 \Vdash \Gamma$. Let s and t be a state and time forming a minimal extension of s_0 and t_0 with respect to ϕ. By out-of-state monotonicity, $\forall t' \geq t_0, s' \supseteq_{t'} s_0 : s', t' \Vdash Stable(\Gamma)$, and so $s, t \Vdash Stable(\Gamma)$. By the induction hypothesis $s, t \Vdash \phi$ and so $\exists t_1 \geq t : s, t_1 \Vdash \psi$.

These are the semantic conditions required for $\phi \to \psi$ to hold in s_0, t_0. So it is safe to assume that $\Gamma \vdash \phi \to \psi$.

Implication Elimination

Suppose $\Gamma \vdash \phi$ and $\Gamma \vdash \phi \to \psi$. By the induction hypothesis, $\Gamma \models \phi$ and $\Gamma \models \phi \to \psi$. According to the semantic definition of implication, if $s, t \Vdash \phi$ and

$s, t \Vdash \phi \to \psi$, then there is a $t_1 \geq t$ such that $s, t_1 \Vdash \psi$. From which we conclude that $s, t \Vdash \sim\sim \psi$. Hence, $\Gamma \models \sim\sim \psi$, and so it is safe to assume that $\Gamma \vdash \sim\sim \psi$.

Out-of-State Negation Introduction

It is possible to define out-of-state negation as $\neg\phi =_{df} \phi \to \bot$, eliminating the need for a separate introduction rule. Its introduction rule can be shown to be sound analogously to implication introduction, noting that no state or time is ever such that $s, t \Vdash \sim\sim \bot$.

Absurdity

Suppose $\Gamma \vdash \bot$, and so by induction hypothesis $\Gamma \models \bot$. Since there are no states s or times t such that $s, t \Vdash \bot$, one can safely assume that $\Gamma \models \phi$ for any ϕ.

Future Absurdity

Suppose $\Gamma \vdash \sim\sim \bot$, and so by induction hypothesis $\Gamma \models \sim\sim \bot$. By the semantic of $\sim$ if $s, t \Vdash \sim\sim \bot$, then $\exists t_1 \geq t$ such that $s, t_1 \bot$. Since there are no states or time that force $\bot$, there can be no states of times that force $\sim\sim \bot$. So it is safe to conclude that $\Gamma \models \bot$ and so $\Gamma \vdash \bot$.

Future Implication Elimination

Suppose $\Gamma \vdash \sim\sim \phi$ and $\Gamma \vdash \phi \to \psi$. By the induction hypothesis, $\Gamma \models \sim\sim \phi$ and $\Gamma \models \phi \to \psi$. According to the semantics of $\sim$, if $s, t \Vdash \sim\sim \phi$, then $\exists t_1 \geq t$ such that $s, t_1 \Vdash \phi$. By in-state monotonicity, if $s, t \Vdash \phi \to \psi$ then $s, t_1 \Vdash \phi \to \psi$ as well. By semantics of implication, $\exists t_2 \geq t_1$ such that $s, t_2 \Vdash \psi$. And so $\exists t_2 \geq t$ such that $s, t_2 \Vdash \psi$. Thus $s, t \Vdash \sim\sim \psi$. Hence $\Gamma \models \sim\sim \psi$, and so $\Gamma \vdash \sim\sim \psi$.

Completeness

The proof of completeness is a variation on the Henkin-style completeness proofs for intuitionistic used by Aczel, Fitting and Thomason (van Dalen 1984), and also by Veltman (1985) for Data Semantics. The method involves constructing information states that are represented as saturated sets of formulas. A number of definitions and proofs concerning saturated sets are therefore required before proving completeness.

Saturated Theories

Suppose that we have a language L containing a set of sentences Γ, such that

$$\Gamma \nvdash \phi$$

We can construct a *saturated extension* of Γ relative to (an enumeration of) the sentences in L and ϕ, $Sat(\Gamma, \phi, L)$, such that

(a) $Sat(\Gamma, \phi, L)$ is closed under $\vdash$

(b) $\Gamma \subseteq Sat(\Gamma, \phi, L)$

(c) If $\psi \vee \chi \in Sat(\Gamma, \phi, L)$, then either $\psi \in Sat(\Gamma, \phi, L)$ or $\chi \in Sat(\Gamma, \phi, L)$

(d) $\phi \notin Sat(\Gamma, \phi, L)$

The construction proceeds inductively as follows.

(i) Stage 0:
Let $\Delta_0 = \Gamma$

(ii) Stage $k + 1$:
Let $\psi_1 \vee \psi_2$ be the first disjunction in the enumeration of L that has not so far been dealt with, where $\Delta_k \vdash \psi_1 \vee \psi_2$.

Then $\Delta_{k+1} = \Delta_k \cup \{\psi_1\}$ if $\Delta_k \cup \{\psi_1\} \nvdash \phi$
$\Delta_k \cup \{\psi_2\}$ otherwise.

(iii) $Sat(\Gamma, \phi, L) = \bigcup_{k \geq 0} \Delta_k$

The definition and construction, as well as the proofs that (a)–(d) hold, are taken from van Dalen 1984. We start by establishing the disjunction property, (c).

(c)

Let $\psi \vee \chi \in Sat(\Gamma, \phi, L)$, and let k be the least number such that $\Delta_k \vdash \psi \vee \chi$. Then $\Delta_i \vdash \psi \vee \chi$ for all $i \geq k$. $\psi \vee \chi$ cannot be treated before stage k, so suppose that it is treated at state $j \geq k$. Then either $\psi \in \Delta_{j+1}$ or $\chi \in \Delta_{j+1}$. And $\Delta_{j+1} \subseteq Sat(\Gamma, \phi, L)$

(a)

$Sat(\Gamma, \phi, L)$ is closed under $\vdash$. For is $Sat(\Gamma, \phi, L) \vdash \psi$, then $Sat(\Gamma, \phi, L) \vdash \psi \vee \psi$. So by property (c), $\psi \in Sat(\Gamma, \phi, L)$.

(b)

That $\Gamma \subseteq Sat(\Gamma, \phi, L)$ is evident from the construction.

(d)

Clearly, $\Delta_0 \nvdash \phi$. If $\Delta_k \nvdash \phi$, then $\Delta_{k+1} \nvdash \phi$. For let $\phi \vee \psi$ be the $k + 1$th disjunction treated, i.e. $\Delta_k \vdash \phi \vee \psi$. Clearly, $\Delta_k \cup \{\phi\} \vdash \phi$, so ψ must be added. Moreover, $\Delta_k, \psi \nvdash \phi$, for if it did and $\Delta_k \vdash \phi \vee \psi$, then it could not be that $\Delta_k \nvdash \phi$.

Which saturated extension of Γ one gets is dependent on the order in which the sentences in L are enumerated. When constructing sets of saturated extensions for Γ, as we will do below, it is assumed that all possible enumerations are employed.

Saturated theories will form the state snapshots used to build canonical models used to establish completeness.

Statement of Completeness Problem

To prove completeness, we must show that

$$\Gamma \models \phi \Rightarrow \Gamma \vdash \phi$$

Put in contrapositive form, and expanding out the meaning of $\models$, this amounts to

$\Gamma \nvdash \phi \Rightarrow \Gamma \nvDash \phi$, i.e.
$\qquad \Gamma \nvdash \phi \Rightarrow$ there is a model, state and time such that $M, s, t \Vdash \Gamma$ but $M, s, t \nVdash \phi$.

    The general strategy for proving completeness is to construct a saturated extension relative to Γ, ϕ and some enumeration of the language. This forms the state and time that forces Γ. We then build up temporal extensions of the state, as well as new states plus their temporal extensions, all of which are also saturated theories. We show that for any such theory, Σ_s^t, (i.e. state s taken at time t), $s, t \Vdash \psi$ iff $\psi \in \Sigma_s^t$. Since Σ_0^0 is just a saturated extension of Γ relative to ϕ, we have $\Gamma \subseteq \Sigma_0^0$, but $\phi \notin \Sigma_0^0$. This establishes the contrapositive of completeness.

Constructing State Snapshots

Suppose that we have some saturated extension of a set of sentences Γ relative to ϕ and a given enumeration of L. Call it Σ^0, and take it as the initial snapshot of the state. Subsequent snapshots, Σ^i are constructed relative to the same enumeration of the language as follows:

- Σ^{i+1}: Let $\sim\sim \psi$ be the first double negation not so far dealt with, such that $\Sigma^i \vdash \sim\sim \psi$.

- Construct a saturated theory from $\Sigma^i \cup \{\psi\}$ as follows
  - Step 0:
    Let $\Delta_0 = \Sigma^i \cup \{\psi\}$
  - Step $k+1$:
    Let $\psi_1 \vee \psi_2$ be the first disjunction not so far dealt with such that $\Delta_k \vdash \psi_1 \vee \psi_2$.
    $\Delta_{k+1} = \Delta_k \cup \{\psi_1\}$ iff $\Sigma^i \vdash \sim\sim \psi_1$ or $\Sigma^i \vdash \psi_1$
    $\qquad\qquad \Delta_k \cup \{\psi_2\}$ otherwise.
  - $\Sigma^{i+1} = \bigcup_{k \geq 0} \Delta_k$

It is straightforward to show that the conditions (a)–(c) of saturated sets apply to these temporal snapshots. The final property (d), that ϕ is not contained in any Σ^i, where ϕ is the sentence used in the construction of the saturated extension to Γ, does not always hold; if ϕ is a stable sentence, it may well get added to some Σ^i. But in exchange for this condition, it can be shown that the following holds:

    For all Σ^i, if $\psi \in \Sigma^i$, then $\sim\sim \psi \in \Sigma^0$

Constructing an Information Model

To construct an information model where $\Gamma \nvDash \phi$ given that $\Gamma \nvdash \phi$

(i) Construct a saturated extension, Σ_0^0, of Γ relative to ϕ and some enumeration of L, and construct a sequence of temporal snapshots, Σ_0^i relative to the same

language enumeration.

(ii) Take $Stable(\Sigma_0^0)$ and construct saturated extensions from it relative to the absurd proposition $\bot$, and temporal extensions of these saturated extensions. These form the information states of the canonical information model.

(iii) For any Σ_s^t and $\Sigma_{s'}^t$ formed in this, set

$s \subseteq_t s'$ iff
  (a) $Stable(\Sigma_s^t) \subseteq Stable(\Sigma_{s'}^t)$, and
  (b) for any semi stable implication $\phi \rightarrow \psi \in \Sigma_s^t$, if $\neg\psi \in \Sigma_{s'}^t$, then $\phi \rightarrow \psi \notin \Sigma_{s'}^t$

(iv) The valuation function for atomic sentence letters coincides with the atomic sentences contained in a given state s at time t, i.e. Σ_s^t.

We need to establish the models constructed in this way are legitimate information models. It is easy to show that the relation $\subseteq_t$ is transitive and reflexive from the transitivity and reflexivity of the $\subseteq$ relation. That the relation converges (i.e if $s_0 \subseteq_{t_0} s_1 \subseteq t_1 s_2$, then $s_0 \subseteq_{t_2} s_2$ for some t_2 such that $t_2 \geq t_0$ and $t_2 \geq t_1$) can be seen (for a denumerable language L) by picking a time large enough that all the temporal extensions of s_0, s_1 and s_2 become fixed and unchanging. The monotonicity of the valuation function with respect to subsequent verification times follows from the fact that all the sentences contained in Σ_s^t are also included in Σ_s^{t+1}. Monotonicity of information growth for atomic sentences arises from the fact that all atomic sentences are classed as stable, and so will be included in all states extending a state that contains them.

Before showing that complex formulas also satisfy monotonicity of information growth where appropriate, it is necessary to establish one other thing about these canonical information models.

Partial monotonicity of semi-stable implications:
If $\phi \rightarrow \psi \in \Sigma_s^t$, then $\forall s', t'$ if $s' \supseteq_{t'} s$, $\phi \in \Sigma_{s'}^{t'}$ and $\neg(\exists s'', t''$ such that $t \leq t'' < t'$, $s \subseteq_{t''} s'' \subseteq_{t''} s'$ and $\phi \in \Sigma_{s''}^{t''}$,
then $\phi \rightarrow \psi \in \Sigma_{s'}^{t'}$

Proof: (a) Suppose that $\neg\psi \in \Sigma_s^t$. It can be proved that $\{\phi \rightarrow \psi, \neg\psi\} \vdash \neg\phi$ and so Σ_s^t has no ϕ-supporting extensions, minimal or otherwise. And so the condition is vacuously true. (b) Suppose $s' \supseteq_{t'} s$, $\phi \in \Sigma_{s'}^{t'}$ and $\phi \rightarrow \psi \notin \Sigma_{s'}^{t'}$. By the construction of the model it follows that $\neg\psi \in \Sigma_{s'}^{t'}$. By (a) it can be seen that $\neg\psi \notin \Sigma_s^t$, which means that $\Sigma_{s'}^{t'}$ cannot be a minimal ϕ-extension of Σ_s^t. That is, by the construction of the model, there will be some $\Sigma_{s''}^{t''}$ preceding $\Sigma_{s'}^{t'}$ and extending Σ_s^t in which ϕ holds, and where $\neg\psi$ does not.

In other words, semi-stable implications persist until their consequents become false. Any minimal extension of a state in which a semi-stable implication holds to support the implication's antecedent will lead to a state in which the implication still holds.

This result helps to nullify problems caused by the monotonicity of some unstable formulas. Recall that unstable formulas can sometimes be valid or invalid (and

hence monotonic), conjunctions or disjunctions of stable formulas with valid or invalid unstable formulas, in-state negations of these formulas, or implications into them. Taking $Stable(\Sigma)$ to form the basis of extending information states might seem to miss out on some of the monotonic formulas holding in Σ.

Only implication is of any concern. Since Σ is saturated, all conjunctions and disjunctions and Σ are broken down into their component parts, and will be included in $Stable(\Sigma)$, and so will be included in states extending Σ. Valid (or invalid) unstable formulas will be included (or excluded) in extensions of Σ (by the logical closure of saturated sets), allowing monotonic conjunctions and disjunctions in Σ to be reformed by logical closure. In-state negation can be pushed in so that it has scope over atomic formulas, or out-of-state negations ($\sim (\phi \to \psi) \equiv \sim \neg(\phi \wedge \sim \psi)$). Out of state negations, $\neg\phi$ are monotonic, and may either be valid or invalid, in which case $\sim \neg\phi$ is monotonic, or contingent, in which case $\sim \neg\phi$ is non-monotonic. In the former case, validities will be included and invalidities included in states built on $Stable(\Sigma)$ by logical closure, causing $\sim \neg\phi$ to be excluded or included in saturations of $Stable(\Sigma)$ as required. When $\neg\phi$ is contingent, it will not and should not be included in $Stable(\Sigma)$.

With implication, note that $\phi \to (\psi \wedge \chi)$ is equivalent to $(\phi \to \psi) \wedge (\phi \to \chi)$, so problematic unstable monotonic conjunctions in the consequent get broken down into their component parts. This leaves $\phi \to (\psi \vee \chi)$, where χ is a valid or invalid unstable formula. This is where the partial monotonicity of semi-stable implications comes to the rescue. First, let χ be valid. Then $\psi \vee \chi$ will never become false, and so any extension of Σ will include $\phi \to (\psi \vee \chi)$ by the partial monotonicity of semi-stable implications. Now let χ be invalid, and note that $\phi \to (\psi \vee \bot)$ is equivalent to $\phi \to \psi$, which is stable if ψ is stable, and so will be included in $Stable(\Sigma)$.

Forcing and Containment

To finish the completeness proof, we need to establish that

$s, t \Vdash \psi$ iff $\psi \in \Sigma_s^t$

To set the ball rolling, for atomic sentences we define the forcing relation as

$s, t \Vdash p$ iff $p \in \Sigma_s^t$, atomic p

Then by an inductive argument we show that the same holds for complex formulas. Note that Σ_s^t is deductively closed, and so $\psi \in \Sigma_s^t$ iff $\Sigma_s^t \vdash \phi$.

Conjunction

$s, t \Vdash \phi \wedge \psi$ iff $\phi \wedge \psi \in \Sigma_s^t$

Only if: Assume $\phi \wedge \psi \in \Sigma_s^t$.
By $\wedge$-E, $\phi \in \Sigma_s^t$, $\psi \in \Sigma_s^t$.
By inductive hyp., $s, t \Vdash \phi$, $s, t \Vdash \psi$.
By semantics of $\wedge$, $s, t \Vdash \phi \wedge \psi$.

If: Assume $s, t \Vdash \phi \wedge \psi$.

By semantics of $\wedge$, $s, t \Vdash \phi$, $s, t \Vdash \psi$.
By inductive hyp., $\phi \in \Sigma_s^t$, $\psi \in \Sigma_s^t$.
By $\wedge$-I, $\phi \wedge \psi \in \Sigma_s^t$

Disjunction
$\quad s, t \Vdash \phi \vee \psi$ iff $\phi \vee \psi \in \Sigma_s^t$

Only if: Assume $\phi \vee \psi \in \Sigma_s^t$.
By saturation of Σ_s^t, $\phi \in \Sigma_s^t$, or $\psi \in \Sigma_s^t$.
By inductive hyp., $s, t \Vdash \phi$ or $s, t \Vdash \psi$.
By semantics of $\vee$, $s, t \Vdash \phi \vee \psi$.

If: Assume $s, t \Vdash \phi \vee \psi$.
By semantics of $\vee$, $s, t \Vdash \phi$, or $s, t \Vdash \psi$.
By inductive hyp., $\phi \in \Sigma_s^t$, or $\psi \in \Sigma_s^t$.
By $\vee$-I, $\phi \vee \psi \in \Sigma_s^t$

In-State Negation
$\quad s, t \Vdash \sim \phi$ iff $\sim \phi \in \Sigma_s^t$

Only if: Assume $\sim \phi \in \Sigma_s^t$
By construction of model, for all $t' \geq t$, $\sim \phi \in \Sigma_s^{t'}$
By $\sim$-I, $\phi \notin \Sigma_s^{t'}$
By inductive hyp., $s, t' \not\Vdash \phi$
By semantics of $\sim$, $s, t \Vdash \sim \phi$

If: Assume $s, t \Vdash \sim \phi$
By semantics of $\sim$, for all $t' \geq t$, $s, t' \not\Vdash \phi$
Suppose $\sim \phi \notin \Sigma_s^t$ and derive a contradiction:
By $\sim$-Ax, if $\sim \phi \notin \Sigma_s^t$, then $\sim\sim \phi \in \Sigma_s^t$
By semantics of $\sim\sim$, $\exists t' \geq t.\, s, t' \Vdash \phi$
Contradiction.

Implication
$\quad s, t \Vdash \phi \to \psi$ iff $\phi \to \psi \in \Sigma_s^t$

Only if: Assume $\phi \to \psi \in \Sigma_s^t$
Consider a minimal extension of Σ_s^t, $\Sigma_{s'}^{t'}$, with respect to ϕ.
By the partial monotonicity of semi-stable implications, $\phi \to \psi \in \Sigma_{s'}^{t'}$ (If $\phi \to \psi$ is stable it will be in $\Sigma_{s'}^{t'}$ as a matter of course).
By $\to$-E, $\sim\sim \psi \in \Sigma_{s'}^{t'}$
By ind. hyp., $s', t' \Vdash \phi$, $s', t' \Vdash \sim\sim \psi$
By semantics of $\sim\sim$, $\exists t'' \geq t'.\, s', t'' \Vdash \psi$
Since $s' \approx_{t'} s'$, $\exists s'' \supseteq_{t'} s', \exists t'' \geq t'.\, s'', t'' \Vdash \psi$
Since s' and t' were arbitrary, by semantics of $\to$, $s, t \Vdash \phi \to \psi$

If: Assume $s, t \Vdash \phi \to \psi$

Suppose $\phi \to \psi \notin \Sigma_s^t$ and derive a contradiction:

By $\to$-I, $Stable(\Sigma_s^t), \phi \nvdash \sim\sim \psi$

Let $\Sigma_{s'}^{t'}$ be a saturated theory constructed from $Stable(\Gamma) \cup \{\phi\}$ relative to $\sim\sim \psi$ and some enumeration of L

By construction of model, there will be such a $\Sigma_{s'}^{t'}$, where $\Sigma_s^t \subseteq_t \Sigma_{s'}^{t'}$

$\sim\sim \psi \notin \Sigma_{s'}^{t'}$, and likewise for any $\Sigma_{s''}^{t'} \approx_{t'} \Sigma_{s'}^{t'}$

By ind. hyp., $\exists s' \supseteq_t s, t' \geq t. \, s', t' \Vdash \phi$ but not $\exists t'' \geq t'. \, s', t'' \Vdash \psi$

Contradiction.

Out-of-State Negation

By analogy with implication: $\neg\phi =_{df} \phi \to \bot$.

Completeness

To sum up: Constructing a model from $\Gamma \nvdash \phi$, the state snapshot Σ_0^0 contains Γ but not ϕ. Consequently $0, 0 \Vdash \Gamma$ but $0, 0 \nVdash \phi$. So there is a model falsifying $\Gamma \models \phi$. Hence $\Gamma \nvDash \phi$.