Completely Precontinuous Posets

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Abstract

In this paper, concepts of strongly way below relations, completely precontinuous posets, coprimes and Heyting posets are introduced. The main results are: (1) The strongly way below relations of completely precontinuous posets have the interpolation property; (2) A poset P is a completely precontinuous poset iff its normal completion is a completely distributive lattice; (3) An ω-chain complete P is completely precontinuous iff P and P^{op} are precontinuous and its normal completion is distributive iff P is precontinuous and has enough coprimes; (4) A poset P is completely precontinuous iff the strongly way below relation is the smallest approximating auxiliary relation on P iff P is a Heyting poset and there is a smallest approximating auxiliary relation on P. Finally, given a poset P and an auxiliary relation on P, we characterize those join-dense subsets of P whose strongly way-below relation agrees with the given auxiliary relation.

Keywords: Completely precontinuous poset, completely distributive lattice, normal completion, coprime, auxiliary relation.

1 Introduction

The theory of continuous domains arose independently in a variety of mathematical contexts. Due to their strong connections to computer science, general topology and topological algebra, continuous domains have been extensively studied by people coming from various areas [1,9]. There are several different equivalent ways to define continuous domains, the most straightforward one is formulated by using the way below relation. We say *x way below y*, denoted by $x \ll y$, iff $x \in \Downarrow y := \bigcap\{\downarrow D : D$ is directed and $y \leq \vee D\}$. A dcpo P is called a *continuous domain* if each of

the sets $\Downarrow y$ is directed and has join y. This definition has turned out to be very fruitful for many categorical and topological developments generalizing the theory of continuous lattices, but it is still rather restrictive, taking into consideration only dcpos. However, some naturally arising posets are important but fail to be directed complete. So, there are more and more occasions to study posets which lack suprema of directed sets (see [7,10,15,17]). Though continuous posets inherit some good properties of continuous domains, but they also lose some hopeful properties, for example, they are not *completion-invariant* [5], i.e. the normal completion of a continuous poset is not always a continuous lattice (see [3]). In [3], Erné introduced a new way below relation and the concept of precontinuous posets by taking Frink ideals [8] instead of directed lower sets, which are not restricted to dcpos and have the desired completion-invariant property.

A complete lattice is completely distributive if it satisfies the most general distributivity [9]. Taking arbitrary subsets instead of directed ones leads to the strong way below relation $\lhd$, where $x \lhd y$ iff $x \in \Downarrow y := \bigcap\{\downarrow Y : y \le \vee Y\}$. Since Raney [13] proved that a complete lattice L is completely distributive iff $x = \vee\{y \in L : y \lhd x\}$, many people have studied this structure (see [4,6,16]). In [18], Zhao generalized the concept of completely distributive lattices to general posets and introduced the concept of supercontinuous posets, which inherit some good properties of completely distributive lattices, but they fail to be completion-invariant. A simple counterexample will be given in Section 2.

Now questions naturally arise: What is the counterpart of a precontinuous poset in the realm of completely distributive lattices and to what extent does the counterpart possess the completion-invariant property? In this paper, we answer these questions. First, in the manner of Erné, we introduce a strongly way below relation for arbitrary posets. The fundamental idea is to replace each inequality of the form $x \le \vee Y$ by the condition that x be less or equal to every upper bound of Y. Thereby, we avoid the use of any join. Then we introduce the concept of completely precontinuous posets, which are precisely those in which every element is the join of its strong way below elements. It is proved that the strongly way below relations of completely precontinuous posets have the interpolation property and that a poset is completely precontinuous iff its normal completion is a completely distributive lattice. Considering relationships of precontinuous posets and completely precontinuous posets, we show that an ω-chain complete is completely precontinuous iff it and its dual are precontinuous and its normal completion is distributive iff it is precontinuous and has enough coprimes. We also locate the strong way below relation within auxiliary relations, it is proved that a poset is completely precontinuous iff the strong way below relation is the smallest approximating auxiliary relation iff it is a Heyting poset and there is a smallest approximating auxiliary relation. At the end of this paper, given a poset P with an auxiliary relation, we characterize those join-dense subsets of P whose strongly way-below relation agrees with the given auxiliary relation.

2   Preliminaries

Let P be a poset. For all $x \in P$, $A \subseteq P$, let $\uparrow x = \{y \in P : x \leq y\}$ and $\uparrow A = \bigcup_{a \in A} \uparrow a$; $\downarrow x$ and $\downarrow A$ are defined dually. Let $D(P) = \{A \subseteq P : A = \downarrow A\}$. $A^{\uparrow}$ and $A^{\downarrow}$ denote the sets of all upper and lower bounds of A, respectively. Let $A^{\delta} = (A^{\uparrow})^{\downarrow}$ and $\delta(P) = \{A^{\delta} : A \subseteq P\}$. $(\delta(P), \subseteq)$ is called the normal completion, or the Dedekind-MacNeille completion of P. By a *completion-invariant property* we mean a property that holds for P iff it holds for any normal completion of P. Let $\delta(P)^c = \{B : P \backslash B \in \delta(P)\}$. A subset $I \subseteq P$ is called a Frink ideal if for all finite subsets $Z \subseteq I$, we have $Z^{\delta} \subseteq I$. Let $\mathrm{Fid}(P)$ denote the set of all Frink ideals of P.

The following lemma is well-known (see [5]).

Lemma 2.1 *Let P be a poset.*

(1) *The maps $(-)^{\uparrow} : (2^P)^{op} \to 2^P$, $A \mapsto A^{\uparrow}$ and $(-)^{\downarrow} : 2^P \to (2^P)^{op}$, $A \mapsto A^{\downarrow}$ are order preserving.*

(2) *$((-)^{\uparrow}, (-)^{\downarrow})$ is a Galois connection between $(2^P)^{op}$ and 2^P, that is, for all A, $B \subseteq P$, $B^{\uparrow} \supseteq A \Leftrightarrow B \subseteq A^{\downarrow}$. Thus both $\delta : 2^P \to 2^P$, $A \mapsto A^{\delta} = (A^{\uparrow})^{\downarrow}$ and $\delta^* : 2^P \to 2^P$, $A \mapsto (A^{\downarrow})^{\uparrow}$ are closure operators.*

(3) *For all $\{C_j : j \in J\} \subseteq 2^P$, $(\bigcup_{j \in J} C_j)^{\uparrow} = \bigcap_{j \in J} C_j^{\uparrow}$, $(\bigcup_{j \in J} C_j)^{\downarrow} = \bigcap_{j \in J} C_j^{\downarrow}$.*

(4) *Let $L = \delta(P)$. For all $\{A_i^{\delta} : i \in I\} \subseteq L$, $\bigwedge_L \{A_i^{\delta} : i \in I\} = \bigcap \{A_i^{\delta} : i \in I\}$, $\bigvee_L \{A_i^{\delta} : i \in I\} = (\bigcup \{A_i^{\delta} : i \in I\})^{\delta} = (\bigcup_{i \in I} A_i)^{\delta}$.*

Corollary 2.2 *Let P be a poset. Then the map $j : P \to \delta(P)$, $x \mapsto \downarrow x$ is an order embedding of P in normal completion $\delta(P)$ and*

(1) *j preserves all existing joins and meets;*

(2) *For all $A^{\delta} \in \delta(P)$, $A^{\delta} = \bigvee_{a \in A} j(a) = \bigvee_{a \in A^{\delta}} j(a)$.*

For two sets X and Y, we call $\rho : X \rightharpoonup Y$ a binary relation if $\rho \subseteq X \times Y$. When $X = Y$, ρ is usually called a binary relation on X. For $\rho : X \rightharpoonup Y$ and $\tau : Y \rightharpoonup Z$, define $\tau \circ \rho = \{(x, z) \in X \times Z : \text{there exists } y \in Y \text{ with } (x, y) \in \rho \text{ and } (y, z) \in \tau\}$. The relation $\tau \circ \rho : X \rightharpoonup Z$ is called the composition of ρ and τ.

Definition 2.3 ([14]) A binary relation $\rho : X \rightharpoonup Y$ is called *regular* if there exists $\tau : Y \rightharpoonup X$ such that $\rho \circ \tau \circ \rho = \rho$.

Definition 2.4 ([9]) A binary relation $\sqsubset$ on a nonempty set X is said to have the *interpolation property*, if for all $x, y \in X$ with $x \sqsubset y$, there exists $z \in X$ such that $x \sqsubset z \sqsubset y$.

3   Completely Precontinuous Posets

In [18], Zhao generalized the concept of completely distributive lattices to general posets and introduced the following

Definition 3.1 Let P be a poset.

(1) Given any two elements x and y in P , we write $x \blacktriangleleft y$ if for any subset $A \subseteq P$ with $\vee A$ existing and $y \leq \vee A$, there exists $z \in A$ with $x \leq z$.

(2) A poset P is called *supercontinuous* if for all $x \in P$, $x = \vee \{y \in P : y \blacktriangleleft x\}$.

The supercontinuity fails to be completion-invariant, as following simple example shows.

Example 3.2 Let $P = \{a, b, c\}$ with discrete order. Obviously, P is supercontinuous. But $\delta(P) = \{\emptyset, X, \{a\}, \{b\}, \{c\}\}$ is not distributive. Thus $\delta(P)$ is not a completely distributive lattice.

To obtain the desired completion-invariance property, we will introduce a new continuity concept.

Definition 3.3 Let P be a poset. Given any two elements x and y in P , we say x is *strongly way below* y (in symbols: $x \lhd y$) if $x \in \Downarrow y := \bigcap \{I \in D(P) : y \in I^\delta\}$.

It is easy to get the following two propositions

Proposition 3.4 *In a poset P, the following statements hold for all u, x, y, $z \in P$.*

(1) $x \lhd y$ *implies* $x \leq y$;

(2) $u \leq x \lhd y \leq z$ *implies* $u \lhd z$.

(3) *if a smallest element 0 exists, then* $0 \lhd x$.

Proposition 3.5 *For a poset P, x, $y \in P$, the following conditions are equivalent.*

(1) $x \lhd y$;

(2) $y \notin (P \backslash \uparrow x)^\delta$;

(3) $y \in P \backslash (P \backslash \uparrow x)^\delta \subseteq \uparrow x$.

Remark 3.6 If $X \subseteq \downarrow x$, then $x = \sup X$ iff $x \in X^\delta$.

Definition 3.7 A poset P is called *completely precontinuous* if for all $x \in P$, $x \in (\Downarrow x)^\delta$.

By Remark 3.6, we have that a poset P is completely precontinuous iff $x = \sup \Downarrow x$ for all $x \in P$.

Obviously, complete precontinuity implies supercontinuity. However, the converse is not true by Example 3.2 and Theorem 3.10.

Proposition 3.8 *P is a chain iff $\delta(P)$ is a chain.*

Proof. If P is a chain, then its set of lower sets ordered by inclusion is a chain, and thus $\delta(P)$ is a chain, since it consists of lower sets. Conversely P order embeds in $\delta(P)$ by Corollary 2.2, so is a chain if $\delta(P)$ is. $\qquad \square$

Proposition 3.9 *Let P be a poset. Then for all x, $y \in P$, $x \lhd y$ in P iff $\downarrow x \blacktriangleleft \downarrow y$ in $\delta(P)$.*

Proof. For all $\{B_i^\delta : i \in I\} \subseteq \delta(P)$ with $\downarrow y \subseteq \bigvee_{\delta(P)} B_i^\delta$, we have $\downarrow y \subseteq (\bigcup_{i \in I} B_i)^\delta$ by Lemma 2.1. Since $x \lhd y$, $x \in \bigcup_{i \in I} \downarrow B_i$. Thus there exists $i \in I$ such that $x \in \downarrow B_i$. Therefore, $\downarrow x \subseteq B_i^\delta$.

Conversely, for all $I \in D(P)$ with $y \in I^\delta = \bigvee_{\delta(P)} \{\downarrow z : z \in I\}$, since $\downarrow x \blacktriangleleft \downarrow y$, there exists $z \in I$ such that $\downarrow x \subseteq \downarrow z$. Thus $x \in I$. $\qquad\square$

It will be convenient to introduce the following three operators:

$$\delta : 2^P \to 2^P, \ \delta(Y) = Y^\delta;$$

$$\Downarrow : 2^P \to 2^P, \ \Downarrow Y = \{x : x \lhd y \text{ for some } y \in Y\}.$$

$$\Uparrow : 2^P \to 2^P, \ \Uparrow Y = \{x : y \lhd x \text{ for some } y \in Y\}.$$

Notice that $\delta(Y)$ and $\Downarrow Y$ are always lower sets so that the restrictions

$$\triangle = \delta|_{D(P)} \text{ and } \triangledown = \Downarrow|_{D(P)}$$

are self-maps of $D(P)$. The following inclusions hold throughout:

$$\Downarrow Y = \Downarrow\Downarrow Y \subseteq \Downarrow Y^\delta \subseteq \downarrow Y \subseteq \downarrow Y^\delta = Y^\delta.$$

By Lemma 2.1, Proposition 3.5 and Theorem 2.2 in [14], we can get the following

Theorem 3.10 *For a poset P, the following conditions are equivalent:*

(1) *P is completely precontinuous;*

(2) *The relation $\not\leq$ on P is regular;*

(3) *For all $x, y \in P$ with $x \not\leq y$, there are $u, v \in P$ such that*
    (i) *$x \not\leq v$, $u \not\leq y$, and*
    (ii) *for all $z \in P$, $u \leq z$ or $z \leq v$.*

(4) *$Y \subseteq (\Downarrow Y)^\delta$ for all $Y \subseteq P$;*

(5) *$Y^\delta = (\Downarrow Y)^\delta$ for all $Y \subseteq P$;*

(6) *The pair $(\triangle, \triangledown)$ is a Galois connection of the complete lattice $D(P)$, i.e. $\triangledown \triangle Y \subseteq Y \subseteq \triangle \triangledown Y$ for all $Y \in D(P)$;*

(7) *$\triangle : D(P) \to D(P)$ preserves arbitrary intersections;*

(8) *$\triangle$ induces a complete homomorphism from $D(P)$ onto the normal completion $\delta(P)$;*

(9) *The normal completion $\delta(P)$ is a completely distributive lattice.*

Remark 3.11 The equivalence of (3) and (9) was firstly obtained by H.J.Bandelt in [2].

From Proposition 3.8 and Theorem 3.10, we can immediately get the following two corollaries.

Corollary 3.12 *Completely precontinuous is self-dual.*

Corollary 3.13 *Every chain is completely precontinuous.*

Proposition 3.14 *For a poset P, the following two conditions are equivalent:*

(1) *P is completely precontinuous;*

(2) *For all $a \in P$, $P\backslash \downarrow a = \bigcup\{\Uparrow x : x \in P\backslash \downarrow a\}$.*

Proof. $(1) \Rightarrow (2)$: Obviously.

$(2) \Rightarrow (1)$: If there is $x \in P$ with $x \notin (\Downarrow x)^{\delta}$, then there exists $y \in P$ with $\Downarrow x \subseteq \downarrow y$ such that $x \not\leq y$. Thus $x \in P\backslash \downarrow y$. So, there exists $a \in P\backslash \downarrow y$ such that $x \in \Uparrow a$, that is, $a \not\leq y$ and $a \triangleleft x$, a contradiction to $\Downarrow x \subseteq \downarrow y$. $\qquad\square$

Theorem 3.10 also yields several criteria of complete distributivity for complete lattice: Observing that a complete lattice is isomorphic to its completion, we get (cf.2,12,13,14)

Corollary 3.15 *For a complete lattice L, the following conditions are equivalent:*

(1) *L is a completely distributive lattice;*

(2) *L^{op} is a completely distributive lattice;*

(3) *$y = \vee \Downarrow y$ for all $y \in L$;*

(4) *$\vee Y = \vee \Downarrow Y$ for all $Y \subseteq L$;*

(5) *$Y \mapsto \downarrow \vee Y : D(L) \to D(L)$ preserves arbitrary intersections;*

(6) *$Y \mapsto \vee Y$ induces a complete homomorphism from $D(L)$ onto L;*

(7) *L is a complete homomorphic image of a complete ring of sets;*

(8) *The relation $\not\leq$ on L is regular;*

(9) *For all $x, y \in L$ with $x \not\leq y$, there are $u, v \in L$ such that*
    (i) *$x \not\leq v$, $u \not\leq y$, and*
    (ii) *for all $z \in P$, $u \leq z$ or $z \leq v$;*

(10) *For all $a \in L$, $L\backslash \downarrow a = \bigcup\{\Uparrow x : x \in L\backslash \downarrow a\}$.*

By Theorem 3.10, we can get the following

Theorem 3.16 *If P is a completely precontinuous poset, then $\triangleleft \circ \triangleleft = \triangleleft$, i.e. $\triangleleft$ has the interpolation property.*

The next theorem clarifies the relationships between complete precontinuity of a poset P and the system $\delta(P)^c$.

Theorem 3.17 *For a poset P, the following conditions are equivalent:*

(1) *P is completely precontinuous;*

(2) *$(P\backslash U)^{\delta} = P\backslash \Uparrow U$ for all upper sets U;*

(3) *$P\backslash A = \Uparrow (P\backslash A)$ for all $A \in \delta(P)$;*

(4) *For all $x \in P$, $P\backslash \downarrow x = \Uparrow (P\backslash \downarrow x)$;*

(5) *$\delta(P)^c = \{\Uparrow Y : Y \subseteq P\}$ and $\triangleleft$ has the interpolation property.*

4 Coprimes

Definition 4.1 An elements p in a poset P is called *coprime* if for all finite $F \subseteq P$, $p \in F^\delta$ implies $p \in\downarrow F$. The set of coprimes is denoted by Coprime(P).

Remark 4.2 For sup semilattices the preceding definition of coprime is equivalent to the standard one ([9, Definition I-3.11]).

Proposition 4.3 *For a poset P and $p \in P$, the following conditions are equivalent:*

(1) $p \in$ *Coprime(P);*

(2) *For all x_1, $x_2 \in P$, $p \in (\{x_1, x_2\})^\delta$ implies $p \in\downarrow x_1\cup\downarrow x_2$.*

Proof. (1) $\Rightarrow$ (2): Trivial.

(2) $\Rightarrow$ (1): For all finite $F = \{x_1, \cdots, x_n\} \subseteq P$ with $p \in F^\delta = (\bigcup_{i=1}^n \downarrow x_i)^\delta$. For all $x \in (\bigcup_{i=2}^n \downarrow x_i)^\uparrow$, we have $\bigcup_{i=1}^n \downarrow x_i \subseteq\downarrow x_1\cup\downarrow x$. Thus $p \in (\downarrow x_1\cup\downarrow x)^\delta$. By (2), $p \leq x_1$ or $p \leq x$. If $p \not\leq x_1$, then $p \leq x$. Thus $p \in (\bigcup_{i=2}^n \downarrow x_i)^\delta$. Repeating the above steps, we can obtain $p \in\downarrow F$. $\square$

Definition 4.4 ([9]) A poset P is called *ω-chain complete* if every countable chain has a sup.

Lemma 4.5 *Let P be ω-chain complete and $\lhd$ have the interpolation property, x, $y \in P$, $x \not\leq y$. If there exists $u \in P$ such that $u \lhd x$ and $u \not\leq y$, then there exists $p \in$ Coprime(P) such that $p \lhd x$ and $p \not\leq y$.*

Definition 4.6 ([3]) Let P be a poset.

(1) Given two elements x, $y \in P$, we say x is *way-below* y, in symbols $x \ll y$, if $y \in I^\delta$ implies $x \in I$ for all $I \in \text{Fid}(P)$.

(2) P is called precontinuous if for each $x \in P$, $x \in (\{y \in P : y \ll x\})^\delta$.

Theorem 4.7 ([3]) *For a poset P, the following two conditions are equivalent:*

(1) *P is precontinuous;*

(2) *$\delta(P)$ is a continuous lattice.*

Theorem 4.8 ([9,11]) *Let L be a complete lattice. Then the following two conditions are equivalent:*

(1) *L is completely distributive;*

(2) *L and L^{op} are continuous lattices, and L is distributive.*

By Lemma 4.5, Theorem 4.7 and Theorem 4.8, we can get the following

Theorem 4.9 *For a poset P, consider the following conditions:*

(1) *P is completely precontinuous;*

(2) *P and P^{op} are precontinuous, and $\delta(P)$ is distributive;*

(3) *P is precontinuous and has enough coprimes.*

Then (3) $\Rightarrow$ (1) $\Leftrightarrow$ (2); If P is an ω-chain complete, then (1) $\Rightarrow$ (3).

5   Locating the strongly way below relation ⊲ within Aux(P)

In this section, we firstly characterize completely precontinuous posets by auxiliary relations. Then we construct those posets which are completely precontinuous and whose strongly way below relation ⊲ agrees with ≺.

Definition 5.1 ([9]) We say that a binary relation ≺ on a poset P is an *auxiliary relation*, if it satisfies the following conditions for all u, x, y, $z \in P$,

(i) $x \prec y$ implies $x \leq y$;
(ii) $u \leq x \prec y \leq z$ implies $u \prec z$;
(iii) if a smallest element 0 exists, then $0 \prec x$.

The set of all auxiliary relations on a poset P will be denoted by $Aux(P)$. As $Aux(P)$ is closed under arbitrary intersections in $2^{P \times P}$, it is therefore a complete lattice. Let $\upharpoonright x = \{y \in P : x \prec y\}$ and $\downharpoonright x = \{y \in P : y \prec x\}$.

Definition 5.2 ([9]) An auxiliary relation ≺ on a poset P is *approximating* if for all $x \in P$, $x \in (\downharpoonright x)^{\delta}$. The set of all approximating auxiliary relations is denoted by $App(P)$.

Proposition 5.3 ([9]) *Let P be a poset and let M be the set of all monotone functions $s : P \to D(P))$ satisfying $s(x) \subseteq \downharpoonright x$ for all $x \in P$ - considered as a poset relative to the ordering $s \leq t$ iff $s(x) \subseteq t(x)$ for all $x \in P$. Then the mapping*

$$\phi : Aux(L) \to M, \ \phi(\prec) = s_{\prec} = (x \mapsto \{y : y \prec x\})$$

is a well-defined isomorphism from $Aux(P)$ onto M, whose inverse associates to each function $s \in M$ the relation $\prec_s$ given by

$$x \prec_s y \Leftrightarrow x \in s(y).$$

From the definition of strongly way below relation ⊲, we have $s_{\triangleleft}(x) = \bigcap\{T \in D(P) : x \in T^{\delta}\}$ for all $x \in P$.

By Proposition 5.3, we can get the following

Lemma 5.4 *Let P be a poset. For each $T \in D(P)$, we define the function $m_T : P \to D(P)$ by*

$$m_T(x) = \begin{cases} \downarrow x \cap T, & \text{if } x \in T^{\delta}, \\ \downarrow x, & \text{otherwise.} \end{cases}$$

Then $m_T \in M$ for all $T \in D(P)$ and $\triangleleft = \bigcap\{\prec_{m_T} : T \in D(P)\}$.

Definition 5.5 A poset P is called *Heyting* if for each $x \in P$ and $A \subseteq P$, $x \in A^{\delta}$ implies $x \in (\downarrow x \cap \downarrow A)^{\delta}$.

Obviously, in a complete lattice, the definition of a Heyting poset agrees with that of a frame [8].

Lemma 5.6 *In a Heyting poset P, all relations $s_{\prec_{m_T}}$ belonging to the functions m_T for $T \in D(P)$ are approximating. This holds, in particular, for completely precontinuous posets, as these are Heyting posets.*

By Lemma 5.4 and Lemma 5.6, we can get the following

Lemma 5.7 *In a poset P, the strong way below relation $\lhd$ is contained in all approximating auxiliary relations, and is equal to their intersection, if P is a Heyting poset.*

By Lemma 5.7, we can get the following

Theorem 5.8 *For a poset P, the following conditions are equivalent:*

(1) *P is completely precontinuous;*

(2) *$\lhd$ is the smallest approximating auxiliary relation on P;*

(3) *P is a Heyting poset and there is a smallest approximating auxiliary relation on P.*

To conclude the paper, we construct posets which are completely precontinuous and whose strongly way below relation $\lhd$ agrees with $\prec$. In cases of possible ambiguity, we shall use superscripts to distinguish between concepts related to two posets P and Q , here $\lhd^P$ from $\lhd^Q$.

Definition 5.9 ([5]) Let P be a poset. We say that $S \subseteq P$ is *join-dense* in P if $x \in (\downarrow x \cap S)^{\delta}$ for all $x \in P$.

Theorem 5.10 *Let S be join-dense in a poset P and $\prec$ an auxiliary relation on S. Then the following two conditions are equivalent:*

(1) *P is completely precontinuous and $\prec$ agrees with $\lhd^P$ on S.*

(2) *The conjunction of*
    (a) *For all x, $y \in S$, $x \prec y$ implies $x \lhd^P y$.*
    (b) *For all $x \in S$, $x \in (\downarrow x \cap S)^{\delta}$.*

6   Summary

In this paper, based on cut operator, the concept of completely precontinuous posets is introduced as a generalization of completely distributive lattices. Some properties and characterizations of completely precontinuous posets are investigated. In particular, we discuss the relationships between completely precontinuous posets and precontinuous posets. The important contribution of this paper is that we show that a poset is completely precontinuous iff its normal completion is a completely distributive lattice. Therefore, from the sense of Erné in [3], our generalization is a good generalization.

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