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Tuple Domination on Graphs with the Consecutive-zeros Property [1]

M.P. Dobson[2]

Universidad Nacional de Rosario - Argentina

V. Leoni[3]

Universidad Nacional de Rosario and CONICET - Argentina

M.I. Lopez Pujato[4]

Universidad Nacional de Rosario and CONICET - Argentina

Abstract

The k-*tuple domination problem*, for a fixed positive integer k, is to find in a given graph a minimum sized vertex subset such that every vertex in the graph is dominated by at least k vertices in this set. The k-tuple domination problem is NP-hard even for chordal graphs. For the class of circular-arc graphs, its complexity remains open for $k \geq 2$. A $0,1$-matrix has the *consecutive 0's property (C0P) for columns* if there is a permutation of its rows that places the 0's consecutively in every column. Due to A. Tucker, graphs whose augmented adjacency matrix has the C0P for columns are circular-arc. In this work we provide efficient algorithms to solve the k-tuple domination problem on graphs G whose augmented adjacency matrices have the C0P for columns, for each $2 \leq k \leq |U| + 3$, where U is the set of universal vertices of G.

Keywords: k-tuple dominating sets, stable sets, adjacency matrices, linear time

1 Preliminaries, definitions and notation

In this work we consider finite simple graphs G, where $V(G)$ and $E(G)$ denote its vertex and edge sets, respectively. G' is a (vertex) *induced subgraph* of G and write $G' \subseteq G$, if $E(G') = \{uv : uv \in E(G), \{u, v\} \subseteq V'\}$, for some $V' \subseteq V(G)$. When necessary, we use $G[V']$ to denote G'. Given $S \subseteq V(G)$, the induced subgraph

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[2] Email: pdobson@fceia.unr.edu.ar

[3] Email: valeoni@fceia.unr.edu.ar

[4] Email: lpujato@fceia.unr.edu.ar

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1571-0661/

$G[V(G) \setminus S]$ is denoted by $G - S$. For simplicity, we write $G - v$ instead of $G - \{v\}$, for $v \in V(G)$.

The *(closed) neighborhood* of $v \in V(G)$ is $N_G[v] = N_G(v) \cup \{v\}$, where $N_G(v) = \{u \in V(G) : uv \in E(G)\}$. The *minimum degree* of G is denoted by $\delta(G)$ and is the minimum between the cardinalities of $N_G(v)$ for all $v \in V(G)$.

A vertex $v \in V(G)$ is *universal* if $N_G[v] = V(G)$.

A *clique* in G is a subset of pairwise adjacent vertices in G.

A *stable set* in G is a subset of mutually non-adjacent vertices in G and the cardinality of a stable set of maximum cardinality in G is denoted by $\alpha(G)$ and called the *independence (or stability) number* of G.

A graph G is *circular-arc* if it has an intersection model consisting of arcs on a circle, that is, if there is a one-to-one correspondence between the vertices of G and a family of arcs on a circle such that two distinct vertices are adjacent in G when the corresponding arcs intersect. Figure 1 shows a circular-arc model for the drawn graph G. A graph G is an *interval graph* if it has an intersection model consisting of intervals on the real line, that is, if there exists a family $\mathcal{I}$ of intervals on the real line and a one-to-one correspondence between the set of vertices of G and the intervals of $\mathcal{I}$ such that two vertices are adjacent in G when the corresponding intervals intersect. A *proper interval graph* is an interval graph that has a *proper interval model*, that is, an intersection model in which no interval contains another one. Circular-arc graphs constitute a superclass of proper interval graphs and they are of interest to workers in coding theory because of their relation to "circular" codes [14] and in testing for circular arrangements of genetic molecules [10].

The square matrix whose entries are all 1's is denoted by J and the identity matrix by I, both of appropriate sizes.

The square matrix $M(G)$ associated with a graph G is defined with entry $m_{ij} = 1$ if vertices v_i and v_j are adjacent, and $m_{ij} = 0$ otherwise, it is called the *adjacency matrix* of G. Note that $M(G)$ is symmetric and has 0's on the main diagonal. The *augmented adjacency matrix* or *neighborhood matrix* $M^*(G)$ with entries m^*_{ij} is defined as $M^*(G) := M(G) + I$, i.e. $M(G)$ with 1's added on the main diagonal (see Fig. 2 which corresponds to the graph G of Fig. 1).

A $0,1$-matrix has the *consecutive 0's property* (C0P) for columns if there is a permutation of its rows that places the 0's consecutively in every column. This property was presented by Tucker in [14]. Figure 1 shows an example of a graph G whose augmented adjacency matrix —shown in Fig. 2— has this property. Tucker proved that graphs whose augmented adjacency matrix has the C0P for columns are circular-arc [14].

Fulkerson and Gross [5] have described an efficient algorithm to test whether a $0,1$-matrix has the C0P for columns and to obtain a desired row permutation when one exists.

For a non-negative integer k, $D \subseteq V(G)$ is a *k-tuple dominating set* of G if $|N_G[v] \cap D| \geq k$, for every $v \in V(G)$. Notice that G has k-tuple dominating sets if

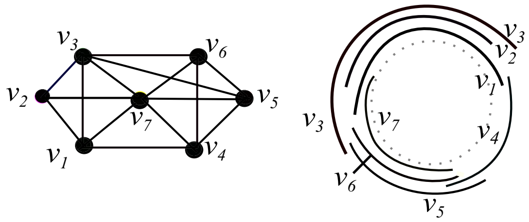

Fig. 1. A circular-arc graph G and a circular-arc model for it.

$$M^*(G) = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Fig. 2. The augmented adjacency matrix for graph G in Figure 1.

and only if $k \leq \delta(G) + 1$ and, if G has a k-tuple dominating set D, then $|D| \geq k$. When $k \leq \delta(G) + 1$, $\gamma_{\times k}(G)$ denotes the cardinality of a k-tuple dominating set of G of minimum size and $\gamma_{\times k}(G) = +\infty$, when $k > \delta(G) + 1$. $\gamma_{\times k}(G)$ is called the *k-tuple dominating number* of G [8]. Observe that $\gamma_{\times 1}(G) = \gamma(G)$, the usual domination number. Besides, note that $\gamma_{\times 0}(G) = 0$ for every graph G. When G is not connected, the k-tuple dominating number of G is defined as the sum of the k-tuple dominating numbers of its connected components. Thus in this work, G will always be connected and the number k will be less or equal $\delta(G) + 1$.

For a fixed positive integer k, the *k-tuple domination problem* is to find in a given graph G, a k-tuple dominating set of G of size $\gamma_{\times k}(G)$.

For a graph G, a positive integer t and $S \subseteq V(G)$ with $t \leq |S|$, we say that S *t-dominates* G if S is a t-tuple dominating set of G.

Concerning computational complexity results, the decision problem (fixed k) associated with this concept is NP-complete even for chordal graphs [11]. Efficient algorithms for every k are known only for strongly chordal graphs [11] and for P_4-tidy graphs [4]. It is natural and challenging then to try to find other graph classes where these problems are "tractable".

For circular-arc graphs, efficient algorithms are already presented in [1] and [9] only for the problem corresponding to $k = 1$. Besides, among the known polynomial time solvable instances of the problem for the case $k = 2$, proper interval graphs constitute the maximal subclass of chordal graphs already studied [13]. Proper interval graphs were characterized by Roberts [12] as those graphs whose augmented adjacency matrices have the consecutive 1's property for columns (defined also by Tucker [14] in a similar way as the C0P for columns).

With a different approach, polynomial algorithms were recently provided for some variations of domination, say k-domination and total k-domination (for fixed k) for proper interval graphs [2].

The slight difference involved in k-domination, k-tuple domination and total k-domination problems makes them useful in various applications, for example in forming sets of representatives or in resource allocation in distributed computing systems. However, the problems are all known to be NP-hard and also hard to approximate [3].

In this work we study 2- and 3-tuple domination on the subclass of circular-arc graphs whose augmented adjacency matrices have the C0P for columns. Our results then allow to solve the k-tuple domination problem in this graph class for $2 \leq k \leq |U| + 3$, where U is the set of universal vertices, if any, of the input graph. In Sections 2 and 3, we present some properties on k-tuple domination for graphs with universal vertices, for any positive integer k. The study of the problem for $k = 2$ and $k = 3$, the running time analysis of the algorithms and further study for the general case are developed in Section 4.

2 k-tuple dominating sets on graphs with universal vertices

From the definition, it is clear that $\gamma_{\times k}(G) \geq k$ for every graph G and positive integer k. Besides, for any $S \subseteq V(G)$ it is remarkable that it $|S|$-dominates G if and only if each vertex of S is a universal vertex. Thus, we can state the following.

Lemma 2.1 *Let G be a graph, U the set of its universal vertices and k a positive integer. Then $\gamma_{\times k}(G) = k$ if and only if $|U| \geq k$.*

Notice that, when u is a universal vertex of a graph G and $D \subset V(G)$ k-dominates G with $u \notin D$, then by interchanging u with any other vertex of D, we obtain another k-tuple dominating set containing u. Formally,

Remark 2.2 If G is a graph and u any universal vertex of G, there exists a k-tuple dominating set D of G such that $u \in D$.

From the above remark, it is easy to prove the following relationship:

Lemma 2.3 *Let G be a graph, u a universal vertex of G and k a positive integer. Then*

$$\gamma_{\times k}(G) = \gamma_{\times (k-1)}(G - u) + 1.$$

Proof. Let D be a k-tuple dominating set of G with $|D| = \gamma_{\times k}(G)$.

If $u \in D$, then $D - u$ is a $(k-1)$-tuple dominating set of $G - u$, thus $\gamma_{\times (k-1)}(G - u) + 1 \leq |D| = \gamma_{\times k}(G)$. If $u \notin D$, from Remark 2.2 we can build a k-tuple dominating set D' of G with $|D'| = \gamma_{\times k}(G)$ and $u \in D'$ and proceed as above with D' instead of D.

On the other side, let D be a minimum $(k-1)$-tuple dominating set of $G-u$. It is clear that $D \cup \{u\}$ is a k-tuple dominating set of G since u is a universal vertex. Then $\gamma_{\times k}(G) \leq |D \cup \{u\}| = |D| + 1 = \gamma_{\times (k-1)}(G-u) + 1$, and the proof is complete. □

The above lemma can be generalized as follows:

Proposition 2.4 *Let G be a graph, U the set of its universal vertices and k a positive integer with $|U| \leq k-1$. Then*

$$\gamma_{\times k}(G) = \gamma_{\times (k-|U|)}(G-U) + |U|.$$

The following corollary is clear from Lemma 2.1 and Proposition 2.4.

Corollary 2.5 *Let G be a graph and U the set of its universal vertices with $U \neq \emptyset$. If $\gamma_{\times i}(G-U)$ can be found in polynomial time for $i = 1, 2, 3$, then $\gamma_{\times k}(G)$ can be found in polynomial time for every k with $1 \leq k \leq |U| + 3$.*

3 C0P-graphs. General properties

Recall that a 0, 1-matrix has the C0P for columns if there is a permutation of its rows that places the 0's consecutively in every column. We introduce the following definition:

Definition 3.1 A graph G whose augmented adjancency matrix, $M^*(G)$, has the C0P for columns is called a *C0P*-graph.

Remark 3.2 If G is a C0P-graph then $G-U$ is a C0P-graph, where U is the set of universal vertices of G.

Let G be a C0P-graph with its vertices indexed so that the 0's occur consecutively in each column of $M^*(G)$. Let C_1 be the set of columns whose 0's are below the main diagonal, C_2 the set of columns whose 0's are above the main diagonal, and C_3 the set of columns without 0's (see Fig. 3 for an example). Sets C_1, C_2 and C_3 partition $V(G)$, C_3 corresponds to the set U of universal vertices of G and $G[C_1]$ and $G[C_2]$ are cliques in G (if a vertex $v_i \in C_1$ is not adjacent to a vertex v_j, the corresponding 0 in column j and row i has to be above the diagonal since the 0 in column i, row j is below). We denote this partition by (C_1, C_2, U), or simply (C_1, C_2) when $U = \emptyset$ and then $|C_1| \geq 2$ and $|C_2| \geq 2$. Also for simplicity, we denote $G_1 := G[C_1]$ and $G_2 := G[C_2]$.

From now on, G is a C0P-graph and (C_1, C_2, U) (or (C_1, C_2) when $U = \emptyset$) is the above mentioned partition of $V(G)$.

It is easy to prove the following upper bound on the size of a minimum k-tuple dominating set of a C0P-graph:

Lemma 3.3 *Let G be a C0P-graph and k a positive integer. If $|C_i| \geq k$ for $i = 1, 2$, then*

$$\gamma_{\times k}(G) \leq 2k.$$

Proof. Let $D_i \subseteq C_i$ with $|D_i| = k$, for $i = 1, 2$ and consider the set $D_1 \cup D_2$. Take $v \in V(G)$. If $v \in C_i$, then $D_i \subseteq N_G[v]$, thus $|N_G[v] \cap (D_1 \cup D_2)| \geq |D_i| = k$, for $i = 1, 2$. If $v \in U$, clearly $D_1 \cup D_2 \subseteq N_G[v] = V(G)$ and thus $|N_G[v] \cap (D_1 \cup D_2)| = |D_1 \cup D_2| = 2k \geq k$. Then $D_1 \cup D_2$ is a k-tuple dominating set of G and the upper bound follows. $\square$

Proposition 2.4 allows us to restrict our study of C0P-graphs to those with partition (C_1, C_2) and C_1 and C_2 are non-empty sets. Under these assumptions and Lemmas 2.1 and 3.3, we have $k + 1 \leq \gamma_{\times k}(G) \leq 2k$ for any C0P-graph G.

Following the notation in [14], let us denote $V(G) = \{v_1, v_2, \cdots, v_n\}$, $C_1 = \{v_1, v_2, \cdots, v_r\}$ and $C_2 = \{v_{r+1}, v_{r+2}, \cdots, v_n\}$ for a given C0P-graph G with partition (C_1, C_2). Also let us denote by $M^*_{C_iC_j}$, the submatrix of $M^*(G)$ with rows indexed by C_i and columns by C_j. Notice that $M^*_{C_1C_1}$ and $M^*_{C_2C_2}$ are both equal to a matrix J of appropriate size.

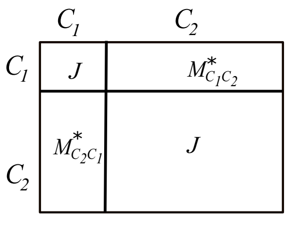

Fig. 3. Scheme of $M^*(G)$ for a C0P-graph G with $U = \emptyset$.

3.1 Construction of auxiliary interval graphs H_i

Let G be a C0P-graph and (C_1, C_2) the above mentioned partition of $V(G)$.

We construct two interval graphs H_1 and H_2 as follows:

- for each vertex $v_i \in C_1$, define an interval I_i from $[r + 1, n]_{\mathbb{N}}$ such that, if the consecutive 0's of column v_i correspond to the vertices $v_p, ..., v_{p+s}$ where $p \geq r + 1$ and $p + s \leq n$, then $I_i = [p, p + s]_{\mathbb{N}}$;
- for each vertex $v_i \in C_2$, define an interval I_i from $[1, r]_{\mathbb{N}}$ such that, if the consecutive 0's of column v_i correspond to the vertices $v_p, ..., v_{p+s}$ with $p \geq 1$ and $p + s \leq r$, then $I_i = [p, p + s]_{\mathbb{N}}$.

We will consider that v_i represents the interval I_i, for each $i = 1, ..., n$.

The two interval graphs H_1 and H_2 constructed as above have interval models $\mathcal{I}_1 = \{I_1, I_2, ..., I_r\}$ and $\mathcal{I}_2 = \{I_{r+1}, I_{r+2}, ..., I_n\}$, respectively.

An example of the above construction is shown in Fig. 4:

Fig. 4. Graphs H_1 and H_2 related to graph G of Figure 1.

Remark 3.4 For a C0P-graph G with partition (C_1, C_2, U) and $U \neq \emptyset$, graphs H_1 and H_2 are defined as above from the subgraph $G - U$ of G.

It is clear that given two intersecting intervals I_i and I_j of H_1 for $1 \leq i \neq j \leq r$, there exists q with $r+1 \leq q \leq n$ such that $m^*_{qi} = m^*_{qj} = 0$. This means that $v_q v_i \notin E(G)$ and $v_q v_j \notin E(G)$. In other words, given two non-intersecting intervals I_i and I_j of H_1 for $1 \leq i \neq j \leq r$, we have $m^*_{qi} = 1$ or $m^*_{qj} = 1$ for all q with $r+1 \leq q \leq n$. Therefore in each row of $M^*_{C_2 C_1}$ there exists at least one 1 in the columns corresponding to vertex v_i or v_j, and then $v_q v_i \in E(G)$ or $v_q v_j \in E(G)$ for all q with $r+1 \leq q \leq n$.

In a similar way, the above argument clearly holds for the interval graph H_2.

3.2 Stable sets of H_i and tuple-dominating sets of G

We will denote by α_i, the independence number of the interval graphs H_i constructed as in the previous subsection, for $i \in \{1, 2\}$. Let us remark that the independence number of an interval graph can be found in linear time [6]. We first have:

Lemma 3.5 *Let G be a C0P-graph with partition (C_1, C_2), $S \subseteq C_j$, and t be a positive integer such that S t-dominates G_i, for $i \neq j$. Then $|S| \geq t + 1$.*

Proof. Since $U = \emptyset$, it is clear that for each vertex $v \in C_i$, there is a non-adjacent vertex $w \in C_j$, for $i \neq j$ and $i \in \{1, 2\}$. The inequality easily follows. □

Thus, it is straightforward that any subset $S \subseteq C_i$ $|S|$-dominates G_i, for each $i \in \{1, 2\}$ and at most $(|S| - 1)$-dominates the whole graph G. When considering stable sets of H_i, the following interesting fact will be the key of the results in the next section:

Proposition 3.6 *Let G be a C0P-graph with partition (C_1, C_2) and $S \subseteq C_i$, for some $i \in \{1, 2\}$. Then S is a stable set of H_i if and only if S $(|S| - 1)$-dominates G_j, for $i \neq j$.*

Proof. S is an $(|S| - 1)$-tuple dominating set of G_j if and only if for every vertex $v \in C_j$, $|N_G[v] \cap S| \geq |S| - 1$. In other words, S is an $(|S| - 1)$-tuple dominating set of G_j if and only if for each row of $M^*_{C_j C_i}$ there exists at most one zero in the columns corresponding to vertices in S. This is equivalent to say that the set of intervals $\{I_t\}_{t:v_t \in S}$ are pairwise non-adjacent, i.e S is a stable set of H_i. □

4 k-tuple domination on C0P-graphs

The relationship exhibited in the previous section between tuple dominating sets of a given C0P-graph and stable sets of the auxiliary interval graphs H_1 and H_2 defined from it allows us to state the following general result for k-tuple domination on C0P-graphs.

Theorem 4.1 *Let G be a $C0P$-graph with partition (C_1, C_2), interval graphs H_i defined as in the previous section, and α_i be the independence number of H_i, for each $i \in \{1, 2\}$. Then*

(i) *if $\alpha_i = 1$ and D is a k-tuple dominating set of G, then $|D \cap C_j| \geq k$ with $1 \leq i \neq j \leq 2$;*

(ii) *if $\alpha_1 + \alpha_2 = 2$ then $\gamma_{\times k}(G) = 2k$;*

(iii) *if $\alpha_1 + \alpha_2 > k$ then $\gamma_{\times k}(G) = k + 1$;*

(iv) *if $\alpha_1 + \alpha_2 = k$ and $|C_i| \geq \alpha_i + 1$ for $i \in \{1, 2\}$ then $\gamma_{\times k}(G) = k + 2$.*

Proof.

(i) W.l.o.g., assume $i = 1$. Then $\alpha_1 = 1$ implies that the vertices in H_1 are pairwise adjacent. Hence, the corresponding intervals are pairwise overlapping. It is known that the interval model of an interval graph fulfills the Helly property (for the definition of this property, see for example [7]). It follows that there is a point that is part of every interval. Hence, there is a row j in $M^*_{C_2C_1}$ that contains only 0's and thus vertex $v_j \in C_2$ is non-adjacent to every vertex in C_1. This implies that $|D \cap C_2| \geq k$ for each k-tuple dominating set D of G.

(ii) If $\alpha_1 = \alpha_2 = 1$, then the previous item implies that any k-tuple dominating set of G has at leat $2k$ vertices. Thus $\gamma_{\times k}(G) \geq 2k$. The equality follows from Lemma 3.3.

(iii) Let S_1 and S_2 be stable sets of H_1 and H_2 respectively, with $|S_1 \cup S_2| = k+1$. From Proposition 3.6 S_i $|S_i|$-dominates G_i and also $(|S_i|-1)$-dominates G_j for each $i, j \in \{1, 2\}$ and $i \neq j$. Thus $S_1 \cup S_2$ is a k-tuple dominating set of G and then $\gamma_{\times k}(G) \leq k + 1$. Since $\gamma_{\times k}(G) > k$ from Lemma 2.1, we conclude that $\gamma_{\times k}(G) = k + 1$.

(iv) Let S_1 and S_2 be maximum stable sets of H_1 and H_2 respectively. It is clear that $S_1 \cup S_2$ is a $(\alpha_1+\alpha_2-1)$-dominating set of G, i.e a $(k-1)$-dominating set of G. Take two vertices $w_1 \in C_1 - S_1$ and $w_2 \in C_2 - S_2$. The set $S_1 \cup S_2 \cup \{w_1, w_2\}$ is a k-tuple dominating set of G with cardinality $k+2$, implying $\gamma_{\times k}(G) \leq k+2$.

Now, since $\gamma_{\times k}(G) \geq k+1$ $(U = \emptyset)$, it is enough to show that $\gamma_{\times k}(G) \neq k+1$. Suppose D is a minimum k-tuple dominating set of G with $|D| = k + 1$ and denote $D_1 = D \cap C_1$, $D_2 = D \cap C_2$, $d_1 = |D_1|$ and $d_2 = |D_2|$. W.l.o.g. we assume $\alpha_1 < d_1$ and $\alpha_2 \geq d_2$. It follows that D_1 is not a stable set of H_1. Thus, by Proposition 3.6, D_1 dominates at best $d_1 - 2$ vertices in C_2. Therefore, for each vertex $v \in C_2$, it holds $|N[v] \cap D| \leq d_1 - 2 + d_2 = k - 1$ contradicting the fact that D is a k-tuple dominating set of G.

□

The results up to now allow us to completely solve the problems for the cases $k = 2$ and $k = 3$, as shown in the following two subsections.

4.1 2-tuple domination

Theorem 4.2 *Let G be a $C0P$-graph with partition (C_1, C_2, U), interval graphs H_i defined as in the previous section, and α_i the independence number of H_i for each $i \in \{1, 2\}$.*

(i) *If $|U| = 1$ then $\gamma_{\times 2}(G) = 3$.*

(ii) *If $|U| \geq 2$ then $\gamma_{\times 2}(G) = 2$.*

(iii) *If $|U| = 0$ and $\alpha_1 + \alpha_2 \geq 3$ then $\gamma_{\times 2}(G) = 3$.*

(iv) *If $|U| = 0$ and $\alpha_1 = \alpha_2 = 1$ then $\gamma_{\times 2}(G) = 4$.*

Proof.

(i) From Lemma 2.3 we have $\gamma_{\times k}(G) = \gamma_{\times k}(G - U) + 1$. Since $G - U$ is not a complete graph, Lemma 2.1 implies $\gamma_{\times 1}(G - U) \geq 2$. The set $\{w_1, w_2\}$ is a 2-tuple dominating set of $G - U$, where $w_1 \in C_1$ and $w_2 \in C_2$ are any two vertices. Thus $\gamma_{\times 1}(G - U) = 2$.

(ii) Follows from Lemma 2.1.

(iii) Follows from Proposition 4.1 item iii.

(iv) Follows from Proposition 4.1 item iv., taking into account that $|U| = 0$ implies $|C_i| \geq 2$ for each $i \in \{1, 2\}$. □

4.2 3-tuple domination

Theorem 4.3 *Let G be a $C0P$-graph with partition (C_1, C_2, U), interval graphs H_i defined as in the previous section, and α_i the independence number of H_i for each $i \in \{1, 2\}$.*

(i) *If $|U| = 1$, then $\gamma_{\times 3}(G) = 4$ if $\alpha_1 + \alpha_2 \geq 3$, and $\gamma_{\times 3}(G) = 5$ if $\alpha_1 + \alpha_2 = 2$.*

(ii) *If $|U| = 2$ then $\gamma_{\times 3}(G) = 4$.*

(iii) *If $|U| \geq 3$ then $\gamma_{\times 3}(G) = 3$.*

(iv) *If $|U| = 0$ and $\alpha_1 + \alpha_2 \geq 4$ then $\gamma_{\times 3}(G) = 4$.*

(v) *If $|U| = 0$ and $\alpha_1 = \alpha_2 = 1$ then $\gamma_{\times 3}(G) = 6$.*

(vi) *If $|U| = 0$ and $\alpha_1 + \alpha_2 = 3$ then $\gamma_{\times 3}(G) = 5$.*

Proof. Similarly as in the previous theorem, the proof follows by applying accordingly Lemmas 2.1 and 2.3, Proposition 2.4, Theorem 4.1 and Theorem 4.2. □

Corollary 4.4 *The k-tuple domination problem can be solved efficiently on a C0P-graph G for each $2 \leq k \leq |U| + 3$, where U is the set of universal vertices of G.*

Proof. Given a C0P-graph G, follow the next scheme:

1. Construct the augmented adjacency matrix $M^*(G)$.
2. Apply the $O(n^2)$-time algorithm in [5] to permute accordingly rows and columns of $M^*(G)$ to ensure the structure shown in Fig. 3, where $n = |V(G)|$.
3. Build the interval graphs H_1 and H_2 as explained in Section 3.1 and find the independence number of H_i for $i = 1, 2$. As already pointed out, this can be done in lineal time [6].
4. Apply Theorem 4.2 and Theorem 4.3.

Following Proposition 2.4 the proof is completed. □

We conclude by applying the previous findings to the graph G of Figure 1.

Example 4.5 Recall graph G from Figure 1 and the auxiliary graphs H_1 and H_2 of Figure 4. The results exposed in this section can be applied appropriately in order to calculate the values of $\gamma_{\times i}(G)$ for each $i \in \{1, 2, 3, 4\}$. Actually, since $\alpha_1 = 2$ and $\alpha_2 = 1$, we have:

$$\gamma_{\times 4}(G) = \gamma_{\times 3}(G - v_7) + 1 = 5 + 1 = 6,$$

$$\gamma_{\times 3}(G) = \gamma_{\times 2}(G - v_7) + 1 = 3 + 1 = 4,$$

$$\gamma_{\times 2}(G) = \gamma_{\times 1}(G - v_7) + 1 = 2 + 1 = 3$$

and

$$\gamma_{\times 1}(G) = 1.$$

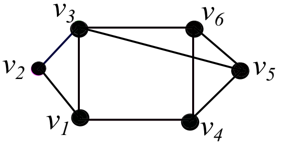

Fig. 5. Graph $G - U$, where G is the graph of Figure 1 and $U = \{v_7\}$.

5 Conclusions

In this work we solved efficiently the k-tuple domination problem on the subclass of circular-arc graphs given by C0P-graphs, for each $2 \leq k \leq |U| + 3$, where U is the set of universal vertices of the input graph. Notice that, when the augmented adjacency matrix of the input graph is given in the form of Fig. 3, our algorithm runs in linear time. We think that —under a suitable implementation— the techniques used in this paper together with the more general result in Theorem 4.1 can be further developed to solve the problem for the remaining values of k, even for other subclasses or moreover, the whole class of circular-arc graphs where the problems remain unsolved.

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