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On Paraconsistent Extensions of C_1

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Abstract

We show that logic C_1 cannot be extended to a paraconsistent logic in which the substitution theorem is valid. We show that C_1 can be extended to larger paraconsistent logics by adding some desirable properties as axioms. We use three-valued logics to support our claims.

Keywords: multi-valued logics, substitution theorem, logic C_1, paraconsistent logic.

1 Introduction

Two main approaches are common to define a logic, the Hilbert axiomatic system and the use of multi-valued tables that define the connectives of the logic. In the first approach the validity of a formula is determined by a set of axioms and a family of inference rules, namely, if the formula can be derived from those axioms and the use of the inference rules, then the formula is valid, otherwise it is not valid. In general, there are many ways of choosing the family of axioms to define a logic and Modus Ponens is one of the most common inference rules appearing in the definition of logics. In the second approach, the tables used to define the logic are called truth

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tables, each connective is regarded as a function taking values in a set of numbers (usually integers) that are specified from the beginning and are called truth values. Some of the values are chosen as designated values. Any formula that evaluates to one of the designated values regardless of the truth values taken by the atoms that appear in the formula, is considered valid. In this paper, we combine both approaches.

In logic, as in any other area of mathematics, when choosing a family of axioms to define a logic, it is desirable to have independence of the axioms, that is, any formula chosen as an axiom should be independent from the other axioms. Multi-valued logics can be used for this purpose (see an example of this in [14]). This methodology sometimes can have limitations (see [10]), however it is useful to researchers interested in the study of logics, such as in our case.

One of the properties we are particularly interested in is paraconsistency. Following Béziau [2], a logic is paraconsistent if it has a negation $\neg$, which is paraconsistent in the sense that the formula $a, \neg a \vdash b$ is not valid, and at the same time has enough strong properties to be called a negation. Paraconsistent logics have important applications, specifically [7] mentions three applications in different fields: Mathematics, Artificial Intelligence and Philosophy. In relation to the second one, the authors mention that in certain domains, such as the construction of expert systems, the presence of inconsistencies is almost unavoidable (see for example [9]). An application that has not been fully recognized is the use of paraconsistent logics in non-monotonic reasoning. In this sense [21,20] illustrate such novel applications.

One example where intuition indicates that paraconsistent logics would be useful for describing abstract structures is provided by Birkhoff and Von Neumann's approach to quantum logic [5].

We emphasize the convenience of accepting local inconsistencies by mentioning Minsky's comment [4] [15]: "*But I do not believe that consistency is necessary or even desirable in a developing intelligent system. No one is ever completely consistent. What is important is how one handles paradox or conflict, how one learns from mistakes, how one turns aside from suspected inconsistencies*". We think that paraconsistent logics could help to give an answer to this important issue addressed by Minsky. In fact, in [16] an interesting approach for Knowledge Representation (KR) was proposed. This approach can be supported by any paraconsistent logic stronger than or equal to C_ω, the weakest paraconsistent logic introduced by Da Costa [8].

Therefore we must consider paraconsistent logics as a supplement to classical logic that deviates from it only in some of its principles (mainly the non-contradiction principle) but that might be applied to contradictory or inconsistent systems like those caused by vagueness or empirical theories whose postulates or basic assumptions are contradictory [5].

Thus, the research on paraconsistent logics is far from being over and, in this work we focus our attention on the paraconsistent logic C_1, which has been studied in [12].

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[4] "*Minsky's Frame paper*" (1975) in its original form had an appendix entitled "*Criticism of the Logistic approach*"

As mentioned before the applications of paraconsistency can be done by using any paraconsistent logic stronger than C_ω. There are many of such logics and their particular properties depend on which properties that are valid in Classical Logic they preserve. The reason to study C_1 is not just because the applications of its paraconsistency, but also because of the theoretical value in terms of studying possible ways of approaching Classical Logic while preserving paraconsistency.

In this paper we present two results, first we show that there is no paraconsistent logic that extends C_1 and for which the substitution theorem holds. Second we extend C_1 to larger paraconsistent logics by adding different axioms to the family of axioms that define it. Each of such axioms is a formula valid in Classical logic, so that by adding them, the logics we obtain are paraconsistent but at the same time closer to classical logic. For each of these extensions we present the tables of one or more three-valued logics that guarantee that each axiom added is independent of the axioms of C_1 and that the logic obtained is paraconsistent.

Our paper is structured as follows. In section 2, we summarize some basic concepts and definitions. In section 3, we show our results. Finally, in section 4, we present some conclusions.

1.1 Contribution

We can think of certain properties or theorems of classical logic that would enrich the scope of any paraconsistent logic. Some of this properties are: the De Morgan laws, the necessitation rule (if α is a theorem then $\neg\neg\alpha$ is a theorem), the week explosion principle: $\neg\alpha \to (\neg\neg\alpha \to \beta)$, the weak contrapositive: $(\neg\alpha \to \neg\beta) \leftrightarrow (\neg\neg\beta \to \neg\neg\alpha)$, and double negation equivalence: $\alpha \leftrightarrow \neg\neg\alpha$.

In this work we study the relation of logic C_1 with some of this properties after showing that C_1 can not be extended to a paraconsistent logic where the substitution property holds.

2 Background

There are two ways to define a logic: by giving a set of axioms and specifying a set of inference rules; and by the use of truth values and interpretations. In this section we summarize each of them and we present some basic concepts and definitions useful to understand this paper. From here on, when we refer to any logic, we understand that the only primitive connectives are $\wedge$, $\vee$, $\to$, $\neg$ and the biconditional $\leftrightarrow$ that is an abbreviation of $(A \to B) \wedge (B \to A)$.

2.1 Hilbert style

In Hilbert style proof systems, also known as axiomatic systems, a logic is specified by giving a set of axioms and a set of inference rules. In these systems, it is common to use the notation $\vdash_X F$ for provability of a logic formula F in the logic X. In that case we say that F is a theorem of X.

We say that a logic X is paraconsistent if the formula $(A \wedge \neg A) \to B$ is not a

Pos1: $A \to (B \to A)$	C_ω**1:** $A \vee \neg A$
Pos2: $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$	C_ω**2:** $\neg\neg A \to A$
Pos3: $A \wedge B \to A$	
Pos4: $A \wedge B \to B$	
Pos5: $A \to (B \to (A \wedge B))$	
Pos6: $A \to (A \vee B)$	
Pos7: $B \to (A \vee B)$	
Pos8: $(A \to C) \to ((B \to C) \to (A \vee B \to C))$	

Table 1
Axiomatization of C_ω.

theorem [5]. The relevance of logics for which the formula $(A \wedge \neg A) \to B$ is not a theorem, is that they are useful to define alternative semantics that can be applied in the study of non monotonic reasoning as we mentioned in the introduction.

A very important property satisfied by many logics is the substitution theorem which we present now.

Definition 2.1 A logic X satisfies the substitution property if: $\Gamma \vdash_X \alpha \leftrightarrow \beta$ then $\Gamma \vdash_X \Psi[\alpha/p] \leftrightarrow \Psi[\beta/p]$ for any formulas α, β, and Ψ and any atom p that appear in Ψ where $\Psi[\alpha/p]$ denotes the resulting formula that is left after every occurrence of p is substituted by the formula α.

As examples of axiomatic systems, we present two logics: the positive logic [17] and the C_ω logic which is a paraconsistent logic defined by daCosta [8]. In Table 1 we present a list of axioms, the first eight of them define positive logic. C_ω logic is defined by the axioms of positive logic plus axioms $\mathrm{C}_\omega 1$ and $\mathrm{C}_\omega 2$.

2.2 Multi-valued logics

An alternative way to define a logic is by the use of truth values and interpretations. Multi-valued logics generalize the idea of using truth tables to determine the validity of formulas in classical logic. The core of a multi-valued logic is its *domain* of values $\mathcal{D}$, where some of such values are special and identified as *designated* or `select` values. Logic connectives (e.g. $\wedge$, $\vee$, $\to$, $\neg$) are then introduced as operators over $\mathcal{D}$ according to the particular definition of the logic, see [14].

An *interpretation* is a function $I\colon \mathcal{L} \to \mathcal{D}$ that maps atoms to elements in the domain. The application of I is then extended to arbitrary formulas by mapping first the atoms to values in $\mathcal{D}$, and then evaluating the resulting expression in terms of the connectives of the logic (which are defined over $\mathcal{D}$). It is understood in general that, if I is an interpretation defined on the arbitrary formulas of a given program P, then $I(P)$ is defined as the function I applied to the conjunction of all the formulas in P. A formula F is said to be a *tautology*, denoted usually by $\models F$ if, for every possible interpretation, the formula F evaluates to a designated value. The simplest example of a multi-valued logic is classical logic where: $\mathcal{D} = \{0, 1\}$, 1

[5] For any logic X that contains Pos1 and Pos2 (axioms of positive logic defined in Table 1) among its axioms and Modus Ponens as its unique inference rule, the formula $(A \wedge \neg A) \to B$ is a theorem if and only if $A, \neg A \vdash_X B$.

$\wedge$	0	1	2
0	0	0	0
1	0	2	2
2	0	2	2

$\vee$	0	1	2
0	0	2	2
1	2	2	2
2	2	2	2

$\to$	0	1	2
0	2	2	2
1	0	2	2
2	0	2	2

x	$\neg x$
0	2
1	1
2	0

Table 2
Truth tables of connectives $\wedge$, $\vee$, $\to$, and $\neg$ in $\mathbf{P}^2$.

is the unique designated value, and the connectives are defined through the usual basic truth tables.

Note that in a multi-valued logic, so that it can truly be a *logic*, the implication connective has to satisfy the following property: for any given value $x \in \mathcal{D}$, and for every designated value $y \in \mathcal{D}$ such that $y \to x$ is designated, then x must also be a designated value. This restriction enforces the validity of Modus Ponens in the logic.

As an example of a multi-valued logic, we present the well known paraconsistent logic $\mathbf{P}^2$ [4] (also called **Cive**).

The truth values of logic $\mathbf{P}^2$ are in the domain $D = \{0, 1, 2\}$ where 1 and 2 are the designated values. The $\wedge$, $\vee$, $\to$, and $\neg$ connectives are defined according to the truth tables given in Table 2.

2.3 C_1 logic

C_1 is a paraconsistent logic defined by Béziau [12]. It is defined by the axioms of C_ω plus the following axioms:
$\neg_1 :\quad B^\circ \to ((A \to B) \to ((A \to \neg B) \to \neg A))$
$\neg_2 :\quad A^\circ \wedge B^\circ \to (A \wedge B)^\circ \wedge (A \vee B)^\circ \wedge (A \to B)^\circ$
where $B^\circ = \neg(B \wedge \neg B)$. We observe that formula $\neg_1$ makes valid reductio ad absurdum whenever B satisfies the principle of non contradiction.

- De Morgan law $\neg(A \wedge B) \to (\neg A \vee \neg B)$ is a theorem in C_1 [11].
- $\neg A \vee A^\circ$ is a theorem in C_1 [11].
- $A^{\circ\circ}$ is a theorem in C_1 [6].
- If we replace the axiom $\neg_2$ in C_1 by the following stronger axiom, called $\neg_3$, then we obtain logic C_1^+ [12], a paraconsistent logic that extends C_1:

$$\neg_3 : A^\circ \vee B^\circ \to (A \wedge B)^\circ \wedge (A \vee B)^\circ \wedge (A \to B)^\circ$$

- We define the *strong negation* as $\neg^* = \neg\alpha \wedge \alpha^\circ$. The connectives $\to$, $\wedge$, $\vee$, and $\neg^*$ satisfy all schemas and inference rules of classical propositional calculus [6]. Then, we can say the classical propositional calculus is contained in C_1, and at the same time C_1 is a subcalculus of it.

As we know, paraconsistent logics have been used as a tool in knowledge representation due to the fact that they allow local inconsistencies without being explosive, i.e., they accept formulas like α and $\neg\alpha$ as theorems without becoming explosive in the sense that every formula becomes a theorem, that is, the formula $\alpha \wedge \neg\alpha \rightarrow \beta$ is not valid in the paraconsistent logic.

According to some authors [1], a paraconsistent logic should retain as much of classical logic, but must allow non-trivial inconsistent theories. Also, it should not validate any inference which is not valid in classical logic. Then, it should be contained in classical logic.

According to some of the facts mentioned before, we obtain an increasing sequence of paraconsistent logics

$$C_\omega \subset C_1 \subset C_1^+ \subset Cive$$

where Cive is defined by the axioms of C_ω plus the next three formulas as axioms:

$\neg_1 :\quad B^\circ \rightarrow ((A \rightarrow B) \rightarrow ((A \rightarrow \neg B) \rightarrow \neg A))$

$\qquad A \rightarrow \neg\neg A$

$\qquad (A \wedge B)^\circ \wedge (A \vee B)^\circ \wedge (A \rightarrow B)^\circ$

Cive is an axiomatization of P_2, the tautologies of P_2 are the theorems of cive and vice-versa [13].

3 Main contribution

An interesting theoretical question that arises in the study of logics is whether a given logic satisfies the substitution theorem [22]. It is well known that there are several paraconsistent logics for which that theorem is not valid [3]. We show that logic C_1 can not be extended to a paraconsistent logic in which the substitution theorem is valid. Then we will explore the relation between C_1 and the De Morgan laws. Since one of our purposes is to extend C_1 to other paraconsistent logics, in what follows, whenever we provide a three-valued logic, it will be such that the formula $(A \wedge \neg A) \rightarrow B$ is not a tautology, and also will have the following property: if A and $A \rightarrow B$ evaluate both to designated values then B must also evaluate to a designated value. These two conditions guarantee that the extensions we are defining are paraconsistent and sound with respect to this three-valued logic. Besides we will need to define two three-valued logics for each axiom we add, in one of them the new axiom must be a tautology and in the other one must be not a tautology, this is to guarantee that the new axiom is independent of the family of axioms that define C_1.

The first result of our paper is that C_1 can not be extended to a paraconsistent logic in which the substitution theorem holds. In order to do this, we provide a definition and a theorem.

Definition 3.1 A logic X satisfies the weak substitution property if: $\Gamma \vdash_X A \leftrightarrow B$ then $\Gamma \vdash_X \neg A \leftrightarrow \neg B$.

Next theorem is presented in [18] in a slightly different form. Their proofs are similar.

Theorem 3.2 *Any logic stronger than* C_ω *satisfies the weak substitution property iff satisfies the substitution property.*

Theorem 3.3 *Any extension of logic* C_1 *to a logic where the substitution theorem holds, is not paraconsistent.*

Proof.

It is easy to see that $a, \neg a \vdash (a \wedge \neg a) \leftrightarrow a$.
By substitution theorem we have $a, \neg a \vdash \neg(a \wedge \neg a) \leftrightarrow \neg a$,
but $a, \neg a \vdash \neg a$,
therefore $a, \neg a \vdash \neg(a \wedge \neg a)$.
We also know by an instance of axiom $\neg_1$ that
$\vdash \neg(a \wedge \neg a) \to ((\neg b \to a) \to ((\neg b \to \neg a) \to \neg\neg b)))$,
then by using modus ponens in the previous line
$a, \neg a \vdash ((\neg b \to a) \to ((\neg b \to \neg a) \to \neg\neg b)))$,
now we use **Pos1** to obtain $a, \neg a \vdash (\neg b \to a)$,
and modus ponens to obtain $\vdash (\neg b \to \neg a) \to \neg\neg b$,
we also have $a, \neg a \vdash (\neg b \to a)$.
then $a, \neg a \vdash \neg\neg b$,
using $\mathrm{C}_\omega\mathbf{2}$, $\neg\neg b \vdash b$.
Finally, we have $a, \neg a \vdash b$.
This last line shows that paraconsistency does not hold. □ □

3.1 De Morgan laws and C_1

As we said before, the De Morgan laws are relevant properties which are desirable in any logic. C_ω, P_2 and Z fail to satisfy at least one of this laws. We explore these laws in the context of C_1. First, we prove that De Morgan law $\neg A \vee \neg B \to \neg(A \wedge B)$ is not provable in C_1.

Theorem 3.4 *The De Morgan law* $\neg A \vee \neg B \to \neg(A \wedge B)$ *does not hold in* C_1.

Proof. If we put B as $\neg A$ in the formula $\neg A \vee \neg B \to \neg(A \wedge B)$ we obtain $\neg A \vee \neg\neg A \to \neg(A \wedge \neg A)$ and then by using axiom $\mathrm{C}_\omega\mathbf{1}$ and Modus Ponens we obtain $\neg(A \wedge \neg A) = A^\circ$, and from this and theorem 2.1.5 in [6] we obtain classical logic. This contradicts the paraconsistency of C_1. □ □

As a consequence, we have three easy statements. The first one is a well known fact.

Corollary 3.5 C_1 *has the following properties:*

(i) C_1 *is not Classical logic.*

(ii) *If we add* $\neg A \vee \neg B \to \neg(A \wedge B)$ *as an axiom of* C_1, *the resulting logic is Classical logic.*

(iii) *If we add* $\neg A \vee \neg B \to \neg(A \wedge B)$ *as an axiom of* C_1, *paraconsistency does not hold any longer.*

$\wedge$	0	1	2
0	0	0	0
1	0	2	2
2	0	2	2

$\vee$	0	1	2
0	0	2	2
1	2	2	1
2	2	2	2

$\rightarrow$	0	1	2
0	2	2	2
1	0	2	2
2	0	2	2

x	$\neg x$
0	2
1	1
2	0

Table 3

$\wedge$	0	1	2
0	0	0	0
1	0	2	2
2	0	2	2

$\vee$	0	1	2
0	0	2	2
1	2	2	2
2	2	2	2

$\rightarrow$	0	1	2
0	2	2	2
1	0	2	2
2	0	2	2

x	$\neg x$
0	2
1	1
2	0

Table 4

$\wedge$	0	1	2
0	0	0	0
1	0	2	2
2	0	2	2

$\vee$	0	1	2
0	0	1	2
1	1	1	1
2	2	2	2

$\rightarrow$	0	1	2
0	2	2	2
1	0	2	2
2	0	2	2

x	$\neg x$
0	2
1	2
2	0

Table 5

3.2 Extending C_1 with De Morgan axioms.

We start by extending C_1 with some of the De Morgan laws that are consistent with paraconsistency in the context of the axioms that define C_1.

Theorem 3.6 *It is possible to add to C_1 any of the two following axioms or both of them at the same time to obtain three different paraconsistent extensions of C_1.*

$$\textbf{\textit{D1:}}\neg(A \vee B) \rightarrow (\neg A \wedge \neg B),$$
$$\textbf{\textit{D2:}}(\neg A \wedge \neg B) \rightarrow \neg(A \vee B).$$

Proof. We split the proof in four steps:

(i) Table 3 shows a three-valued logic with 1 and 2 as designated values, where the axioms of C_1 are tautologies and none of the axioms **D1** and **D2** is a tautology.

(ii) Table 4 shows a three-valued logic with 1 and 2 as designated values, where the axioms of C_1 are tautologies and the axiom **D1** is a tautology but axiom **D2** is not a tautology.

(iii) Table 5 shows a three-valued logic with 1 and 2 as designated values, where the axioms of C_1 are tautologies and the axiom **D2** is also a tautology but axiom **D1** is not a tautology.

(iv) Table 6 shows a three-valued logic with 1 and 2 as designated values, where

$\wedge$	0	1	2
0	0	0	0
1	0	2	2
2	0	2	2

$\vee$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

$\rightarrow$	0	1	2
0	2	2	2
1	0	2	2
2	0	2	2

x	$\neg x$
0	2
1	2
2	0

Table 6

$\wedge$	0	1	2
0	0	0	0
1	0	2	2
2	0	2	2

$\vee$	0	1	2
0	0	2	2
1	2	2	2
2	2	2	2

$\rightarrow$	0	1	2
0	0	2	2
1	2	2	2
2	0	2	2

x	$\neg x$
0	2
1	2
2	0

Table 7

the axioms of C_1 are tautologies and the axioms **D1** and **D2** are tautologies. □

3.3 Other ways of extending C_1

Now we search for more paraconsistent extensions of C_1 by adding as axioms to C_1 some formulas that express interesting properties and that are valid in a more general form in classical logic. These formulas are **A1:**$\neg\neg A \rightarrow (\neg A \rightarrow B)$, **A2:**$\neg\neg(A \rightarrow B) \rightarrow (\neg\neg A \rightarrow \neg\neg B)$ and **A3:**$\neg\neg(A \wedge B) \leftrightarrow (\neg\neg A \wedge \neg\neg B)$. In particular **A1** (week explosion principle) is equivalent to the formula $(\neg A \rightarrow \neg B) \leftrightarrow (\neg\neg B \rightarrow \neg\neg A)$ (weak contrapositive) in any paraconsistent logic that extends C_ω, as is the case of Z, G'_3 and $P - four$ [19,2].

Now we add axioms **A1** and **A2** to C_1.

Theorem 3.7 *C_1 can be extended by adding formula **A1** or formula **A2** or both as axioms to obtain three different paraconsistent extensions of C_1.*

Proof.

We show three truth tables for the connectives of three-valued logics with 1 and 2 as designated values.

(i) In the first one, table 7, all axioms of C_1 as well as **A1** are tautologies, but **A2** is not a tautology.

(ii) In the second one, table 8, all axioms of C_1 as well as **A2** are tautologies, but **A1** is not a tautology.

(iii) In the third one, table 9, all the axioms for C_1 as well as **A1** and **A2** are tautologies.

This shows that adding axiom **A1** or axiom **A2** or both to C_1, we obtain three different paraconsitent logics stronger than C_1. □ □

$\wedge$	0	1	2
0	0	0	0
1	0	1	1
2	0	1	1

$\vee$	0	1	2
0	0	1	2
1	2	1	1
2	2	2	2

$\to$	0	1	2
0	1	1	2
1	0	1	2
2	0	2	2

x	$\neg x$
0	1
1	0
2	2

Table 8

$\wedge$	0	1	2
0	0	0	0
1	0	2	2
2	0	2	2

$\vee$	0	1	2
0	0	2	2
1	2	2	2
2	2	2	2

$\to$	0	1	2
0	2	2	2
1	0	2	2
2	0	1	2

x	$\neg x$
0	2
1	2
2	0

Table 9

$\wedge$	0	1	2
0	0	0	0
1	0	2	2
2	0	2	2

$\vee$	0	1	2
0	0	2	2
1	2	2	2
2	2	2	2

$\to$	0	1	2
0	2	2	2
1	0	2	2
2	0	2	2

x	$\neg x$
0	2
1	1
2	0

Table 10

$\wedge$	0	1	2
0	0	0	0
1	0	2	2
2	0	2	2

$\vee$	0	1	2
0	0	2	2
1	2	2	2
2	2	2	2

$\to$	0	1	2
0	2	2	2
1	0	2	2
2	0	2	2

x	$\neg x$
0	2
1	2
2	0

Table 11

Corollary 3.8 *None of the three extensions presented in the previous theorem is* P_2.

Proof. In P_2 the formula $A \to \neg\neg A$ is a theorem which together with formula **A1** leads to $A \to (\neg A \to B)$ which contradicts the property of paraconsistency. □ □

Theorem 3.9 C_1 *can be extended to a paraconsistent logic by adding axiom* ***A3****.*

Proof. We show two truth tables for the connectives of three-valued logics with 1 and 2 as designated values.

(i) In the first one, table 10, all axioms of C_1 and also **A3** are tautologies.

(ii) In the second one, table 11, all axioms of C_1 are tautologies but **A3** is not.

This shows that by adding axiom **A3** to C_1, we obtain a paraconsistent extension of C_1. □ □

$\wedge$	0	1	2
0	0	0	0
1	0	2	2
2	0	2	2

$\vee$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

$\to$	0	1	2
0	2	2	2
1	0	2	2
2	0	2	2

x	$\neg x$
0	2
1	2
2	0

Table 12

$\wedge$	0	1	2
0	0	0	0
1	0	1	1
2	0	1	1

$\vee$	0	1	2
0	0	1	2
1	1	1	1
2	2	1	2

$\to$	0	1	2
0	1	1	1
1	0	1	1
2	0	2	2

x	$\neg x$
0	1
1	0
2	2

Table 13

$\wedge$	0	1	2
0	0	0	0
1	0	2	2
2	0	2	2

$\vee$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

$\to$	0	1	2
0	2	2	2
1	0	2	2
2	0	1	2

x	$\neg x$
0	2
1	2
2	0

Table 14

Theorem 3.10 *The three systems consisting of the axioms of*

- *C_1 plus formula **D1**, **D2** and **A1**,*
- *C_1 plus formula **D1**, **D2** and **A2**, and*
- *C_1 plus formula **D1**, **D2**, **A1** and **A2***

are different paraconsistent extensions of C_1.

Proof. We present three truth tables for the connectives of three-valued logics with 1 and 2 as designated values.

(i) In the first one, table 12, all axioms of C_1 and also **D1**, **D2** and **A1** are tautologies but **A2** is not.

(ii) In the second one, table 13, all axioms of C_1 and also **D1**, **D2** and **A2** are tautologies but **A1** is not.

(iii) In the third one, table 14, all axioms of C_1 and also **D1**, **D2**, **A1** and **A2** are tautologies.

This shows that we have three different paraconsistent extensions of C_1. □ □

Proposition 3.11 *There is no three-valued logic for which the axiomatic system consisting of the axioms of C_1 plus **D1**, **D2**, **A1**, **A2** and **A3** is sound.*

The next results are easy to prove.

Proposition 3.12 *Other comments we can make about the extensions of C_1 in relation to the formula $A \to \neg\neg A$ are:*

- *The logic that results from adding the formula $A \to \neg\neg A$ as an axiom to the system C_1, has formulas **A2** and **A3** as theorems.*
- *There is not three valued logic for which*

(i) *all axioms of C_1, formula **A2** and formula **A3** are tautologies and*
(ii) *the formula $A \to \neg\neg A$ is not a tautology.*

Figure 1 shows two ways of extending C_1, one leads to Cive and the other one contains logics that have **D2** as a theorem. We recall that **D2** is not a theorem in Cive [19].

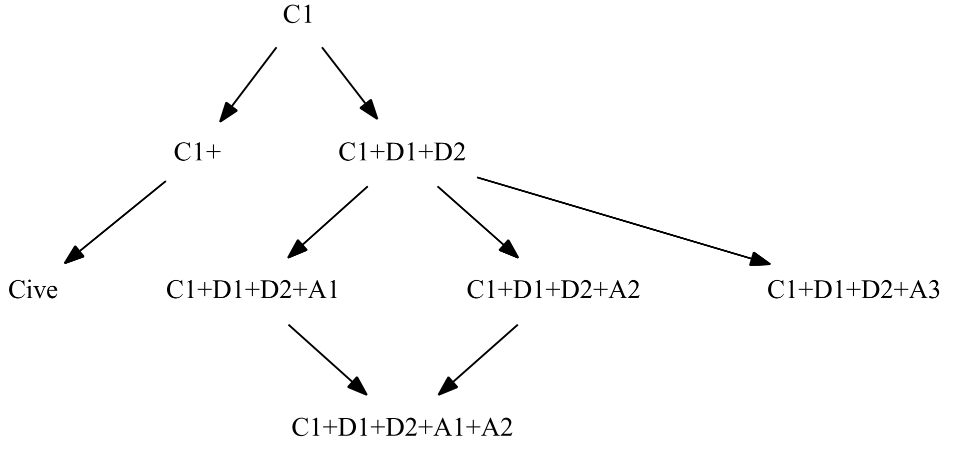

Fig. 1. Two ways of extending C_1

Proposition 3.13 ***D1**, **A2** and **A3** are theorems in Cive, **A1** is not.*

4 Conclusion

It is well known that Classical logic is very useful in applications related to Artificial Intelligence. In this paper we analyse different paraconsistent logics and their proximity with to Classical Logic. As we know, the substitution theorem is a property that meets Classical Logic and it is desirable that a logic that approximates Classical Logic satisfies it. In this paper we show that any paraconsistent logic that extends paraconsistent logic C_1 does not satisfy the substitution theorem. On the other hand, we also know that the De Morgan laws are relevant properties which are valid in Classical Logic, then we shown different paraconsistent extensions of C_1 where some of the De Morgan laws are valid.

At the end of this paper, we also present paraconsistent extensions of C_1 obtained by adding as axioms to C_1 some formulas that express interesting properties and that are valid in a more general form in classical logic, such as the week explosion principle, the weak contrapositive property, etc. We consider that having different paraconsistent extensions, each of them close to classical logic, would allow Artificial Intelligence developers to select the one that according to its properties is best suitable for the desired application.

We use three valued logics to support our claims. We have also presented a diagram to show different ways of extending C_1 to other paraconsistent logics. Besides the problem of discovering new paraconsistent logics, there is the problem of deciding which of them is a maximal paraconsistent logic. Exploring different paraconsistent logics and solving the problem of maximality for each of them is part of our future work.

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