Available online at www.sciencedirect.com

ScienceDirect

Electronic Notes in Theoretical Computer Science 345 (2019) 261–269

Electronic Notes in Theoretical Computer Science

www.elsevier.com/locate/entcs

Induced Topologies on the Poset of Finitely Generated Saturated Sets

Xiaoquan Xu[1,2]

School of Mathematics and Statistics
Minnan Normal University
Zhangzhou, China

and

Department of Mathematics
Nanchang Normal University
Nanchang, China

Wenfeng Zhang[3]

School of Mathematics and Computer Science
Jiangxi Science and Technology Normal University
Nanchang, China

Abstract

In [7], Heckmann and Keimel proved that a dcpo P is quasicontinuous iff the poset $\mathbf{Fin}\ P$ of nonempty finitely generated upper sets ordered by reverse inclusion is continuous. We generalize this result to general topological spaces in this paper. More precisely, for any T_0 space (X,τ) and $U \in \tau$, we construct a topology $\tau_{\mathcal{F}}$ generated by the basic open subsets $U_{\mathcal{F}} = \{\uparrow F \in \mathbf{Fin}\ X\colon F \subseteq U\}$. It is shown that a T_0 space (X,τ) is a hypercontinuous lattice iff $\tau_{\mathcal{F}}$ is a completely distributive lattice. In particular, we prove that if a poset P satisfies property DINT^{op}, then P is quasi-hypercontinuous iff $\mathbf{Fin}\ P$ is hypercontinuous.

Keywords: Hypercontinuous poset, quasicontinuous domain, Scott topology, upper topology

1 Introduction and Preliminaries

Quasicontinuous domains were introduced by Gierz, Lawson and Stralka (see [4]) as a common generalization of both generalized continuous lattices (see [5]) and continuous domains (see [6]). It was proved that quasicontinuous domains equipped with the Scott topologies are precisely the spectra of distributive hypercontinuous

[1] Supported by the National Natural Science Foundation of China (Nos. 11661057, 11701238, 11626121) and the Natural Science Foundation of Jiangxi Province (Nos. 20161BAB2061004, 20161BAB211017)

[2] Corresponding author. Email: xiqxu2002@163.com

[3] Email: zhangwenfeng2100@163.com

https://doi.org/10.1016/j.entcs.2019.07.028

lattices. In [7], Heckmann and Keimel proved that a dcpo P is quasicontinuous iff the poset **Fin** P of nonempty finitely generated upper sets ordered by reverse inclusion is continuous. In this paper, we generalize this result to general topological spaces. Firstly, for any T_0 space (X, τ) and $U \in \tau$, we construct a topology $\tau_{\mathcal{F}}$ generated by the basic open subsets $U_{\mathcal{F}} = \{\uparrow F \in \mathbf{Fin}\ X \colon F \subseteq U\}$. Then we show that a T_0 space (X, τ) is a hypercontinuous lattice iff $\tau_{\mathcal{F}}$ is a completely distributive lattice. In particular, we prove that for a dcpo P, if the Scott topology $\sigma(P)$ is hypercontinuous or $\sigma(\mathbf{Fin}\ P)$ is completely distributive, then $\sigma(P)_{\mathcal{F}}$=$\sigma(\mathbf{Fin}\ P)$. Furthermore, it is proved that if a poset P satisfies property DINT^{op}, then P is quasi-hypercontinuous iff **Fin** P is hypercontinuous.

For a poset P, let $P^{(<\omega)} = \{F \subseteq P : F \text{ is finite}\}$ and **Fin** P=$\{\uparrow F : F \in P^{(<\omega)}\}$. For all $x \in P$, $A \subseteq P$, let $\uparrow x = \{y \in P : x \leq y\}$ and $\uparrow A = \bigcup_{a \in A} \uparrow a$; $\downarrow x$ and $\downarrow A$ are defined dually. For a poset P, the topology generated by the collection of sets $P \setminus \downarrow x$ (as a subbase) is called the *upper topology* and denoted by $\upsilon(P)$; the *lower topology* on P is dually defined and denoted by $\omega(P)$. A subset U of P is called *Scott open* provided that $U = \uparrow U$ and $D \cap U \neq \emptyset$ for all directed sets $D \subseteq P$ with $\bigvee D \in U$ whenever $\bigvee D$ exists. The topology formed by all the Scott open sets of P is called the *Scott topology* on P, written as $\sigma(P)$.

If P is a poset, more generally a preordered set, we introduce a preorder $\leq$ on the powerset of P, sometimes called the *Smyth preorder*, by $A \leq B$ iff $\uparrow B \subseteq \uparrow A$. Throughout the paper, **Fin** P is always endowed with the Smyth preorder.

Definition 1.1 ([6,11]) Let P be a poset.

(1) For any two elements x and y in P, we write $x \ll y$, if for each directed subset $D \subseteq P$ with $\bigvee D$ existing, $y \leq \bigvee D$ implies $x \leq d$ for some $d \in D$. The set $\{y \in P : y \ll x\}$ will be denoted $\Downarrow x$ and $\{y \in P : x \ll y\}$ denoted $\Uparrow x$.

(2) P is called a *continuous poset* if $x = \bigvee \Downarrow x$ and $\Downarrow x$ is directed for all $x \in P$.

(3) P is called an *algebraic poset* if $x = \bigvee \{y \in P : y \ll y \leq x\}$ for all $x \in P$ and the set $\{y \in P : y \ll y \leq x\}$ is directed.

Definition 1.2 ([5,6]) Let P be a poset.

(1) We define a relation $\prec$ on P by $x \prec y \Leftrightarrow y \in int_{\upsilon(P)} \uparrow x$.

(2) P is called a *hypercontinuous poset* if $\{u \in P : u \prec x\}$ is directed and $x = \bigvee \{u \in P : u \prec x\}$ for each $x \in P$. A complete lattice which is hypercontinuous as a poset is called a *hypercontinuous lattice*.

(3) P is called a *hyperalgebraic poset* if $\{u \in P : u \prec u \leq x\}$ is directed and $x = \bigvee \{u \in P : u \prec u \leq x\}$ for each $x \in P$. A complete lattice which is hyperalgebraic as a poset is called a *hyperalgebraic lattice*.

Theorem 1.3 ([1,11]) *Let P be a poset. Then the following conditions are equivalent:*

(1) *P is a continuous poset;*

(2) *For all $x \in U \in \sigma(P)$, there exists $y \in P$ such that $x \in int_{\sigma(P)} \uparrow y \subseteq \uparrow y \subseteq U$;*

(2) *$\sigma(P)$ is a completely distributive lattice.*

Theorem 1.4 ([5,6]) *Let P be a poset. Then the following conditions are equivalent:*

(1) *P is a hypercontinuous poset;*

(2) *For all $x \in U \in \upsilon(P)$, there exists $y \in P$ such that $x \in int_{\upsilon(P)}\uparrow y \subseteq \uparrow y \subseteq U$;*

(2) *$\upsilon(P)$ is a completely distributive lattice.*

Definition 1.5 ([2,3]) A T_0 space (X, τ) is called a *web space* if for each $x \in X$ and $Y \subseteq X$ with $x \in cl_\tau Y$, one has $x \in cl_\tau(\downarrow x \cap \downarrow Y)$.

Definition 1.6 ([10]) A poset P is called *meet continuous* if for any $x \in P$ and any directed set D, if $\bigvee D$ exists and $x \leq \bigvee D$, then $x \in cl_{\sigma(P)}(\downarrow x \cap \downarrow D)$.

Theorem 1.7 ([2,3]) *Let P be a poset. Then the following conditions are equivalent:*

(1) *P is meet continuous;*

(2) *P is a web space endowed with the Scott topology;*

(3) *For any Scott open set U and any $x \in P$, $\uparrow(U \cap \downarrow x)$ is Scott open.*

The proof of the following lemma is similar to that of the analogous results for dcpos in [6].

Lemma 1.8 *If F is a finite set in a meet continuous poset P, then we have*

$$int_{\sigma(P)}\uparrow F \subseteq \bigcup\{\Uparrow x : x \in F\}.$$

2 Quasicontinuous domains and quasihypercontinuous posets

Definition 2.1 ([4,6]) Let P be a dcpo.

(1) For all $F, G \subseteq P$, we say that G is *way below* F and write $G \ll F$ if for every directed set $D \subseteq P$, $\bigvee D \in \uparrow F$ implies $d \in \uparrow G$ for some $d \in D$.

(2) P is called a *quasicontinuous domain* if $\{\uparrow F \in \mathbf{Fin}\ P : F \ll x\}$ is directed and $\uparrow x = \bigcap\{\uparrow F \in \mathbf{Fin}\ P : F \ll x\}$ for each $x \in P$.

(3) P is called a *quasialgebraic domain* if $\{\uparrow F \in \mathbf{Fin}\ P : F \ll F \ll x\}$ is directed and $\uparrow x = \bigcap\{\uparrow F \in \mathbf{Fin}\ P : F \ll F \ll x\}$ for each $x \in P$.

Definition 2.2 ([12]) Let P be a poset.

(1) We define a relation $\prec$ on 2^P by $F \prec G \Leftrightarrow G \subseteq int_{\upsilon(P)}\uparrow F$.

(2) P is called a *quasi-hypercontinuous poset* if $\{\uparrow F \in \mathbf{Fin}\ P : F \prec x\}$ is directed and $\uparrow x = \bigcap\{\uparrow F \in \mathbf{Fin}\ P : F \prec x\}$ for each $x \in P$.

(3) P is called a *quasi-hyperalgebraic poset* if $\{\uparrow F \in \mathbf{Fin}\ P : F \prec F \prec x\}$ is directed and $\uparrow x = \bigcap\{\uparrow F \in \mathbf{Fin}\ P : F \prec F \prec x\}$ for each $x \in P$.

Theorem 2.3 ([4,6]) *Let P be a dcpo. Then the following conditions are equivalent:*

(1) *P is a quasicontinuous domain;*

(2) *For all $x \in U \in \sigma(P)$, there exists $F \in P^{(<\omega)}$ such that $x \in int_{\sigma(P)}\uparrow F \subseteq \uparrow F \subseteq U$;*

(3) *$\sigma(P)$ is a hypercontinuous lattice.*

Theorem 2.4 ([6]) *Let P be a dcpo. Then the following conditions are equivalent:*

(1) *P is a quasialgebraic domain;*

(2) *For all $x \in U \in \sigma(P)$, there exists $F \in P^{(<\omega)}$ such that $x \in int_{\sigma(P)}\uparrow F = \uparrow F \subseteq U$;*

(3) *$\sigma(P)$ is a hyperalgebraic lattice.*

Theorem 2.5 ([12]) *Let P be a poset. Then the following conditions are equivalent:*

(1) *P is a quasi-hypercontinuous poset;*

(2) *For all $x \in U \in \upsilon(P)$, there exists $F \in P^{(<\omega)}$ such that $x \in int_{\upsilon(P)}\uparrow F \subseteq \uparrow F \subseteq U$;*

(3) *$\upsilon(P)$ is a hypercontinuous lattice.*

According to [10], a poset P is called a *quasicontinuous poset* (resp., *quasialgebraic poset)* if for all $x \in U \in \sigma(P)$, there exists $F \in P^{(<\omega)}$ such that $x \in int_{\sigma(P)}\uparrow F \subseteq \uparrow F \subseteq U$ (resp., $x \in int_{\sigma(P)}\uparrow F = \uparrow F \subseteq U$).

Theorem 2.6 *Let P be a poset. Then the following two conditions are equivalent:*

(1) *P is an algebraic poset;*

(2) *P is a meet continuous and quasialgebraic poset.*

Proof. (1) $\Rightarrow$ (2): Obviously.

(2) $\Rightarrow$ (1): CLAIM: Let $x \in P$ and $F \subseteq P$ be finite. If $x \in int_{\sigma(P)}\uparrow F = \uparrow F$, then there exists $t \in F$ with $t \in \downarrow x \cap K(P)$.

Proof of Claim. Since F is finite, $\uparrow F = \uparrow \mathrm{Min(F)}$ where Min(F) is the set of all minimal elements in F. By Lemma 1.8, $x \in \uparrow\mathrm{Min(F)} = \uparrow F = int_{\sigma(P)}\uparrow F = int_{\sigma(P)}\uparrow\mathrm{Min(F)} \subseteq \bigcup\{\Uparrow \mathrm{t} : \mathrm{t} \in \mathrm{Min(F)}\}$. So there exists $t \in \mathrm{Min(F)}$ with $t \ll x$. Since $\uparrow\mathrm{Min(F)} \subseteq \bigcup\{\Uparrow \mathrm{t} : \mathrm{t} \in \mathrm{Min(F)}\}$, there exists $s \in \mathrm{Min(F)}$ with $s \ll t$, hence $s \leq t$. So $s = t$ since s, $t \in \mathrm{Min(F)}$. Thus $t \in \downarrow x \cap K(P)$.

Firstly, we show that $x = \bigvee(\downarrow x \cap K(P))$ for all $x \in P$. Clearly, x is an upper bound of $\downarrow x \cap K(P)$. Let y be any upper bound of $\downarrow x \cap K(P)$ and assume $x \not\leq y$. Then $x \in P\backslash\downarrow y \in \sigma(P)$. By (2), there exists $F \in P^{(<\omega)}$ such that $x \in int_{\sigma(P)}\uparrow F = \uparrow F \subseteq P\backslash\downarrow y$. By Claim, there exists $t \in F$ with $t \in \downarrow x \cap K(P)$, a contradiction to $\downarrow x \cap K(P) \subseteq \downarrow y$.

Then we show that $\downarrow x \cap K(P)$ is directed for all $x \in P$. On the one hand, since P is quasialgebraic, there exists $G \in P^{(<\omega)}$ such that $x \in int_{\sigma(P)}\uparrow G = \uparrow G \subseteq P$. By Claim, there is a $y \in G$ with $y \in \downarrow x \cap K(P)$. Thus $\downarrow x \cap K(P) \neq \emptyset$. On the other hand, let u, $v \in \downarrow x \cap K(P)$. Then $x \in \uparrow u \cap \uparrow v \in \sigma(P)$. By (2), there exists $H \in P^{(<\omega)}$ such that $x \in int_{\sigma(P)}\uparrow H = \uparrow H \subseteq \uparrow u \cap \uparrow v$. By Claim, there exists $m \in H$ with $m \in \downarrow x \cap K(P)$. Whence $m \in \uparrow u \cap \uparrow v$. Hence $\downarrow x \cap K(P)$ is directed.□

Proposition 2.7 *Let P be a poset. Then the following two conditions are equivalent:*

(1) *P is a quasi-hyperalgebraic poset;*

(2) *For all $x \in U \in \upsilon(P)$, there exists $F \in P^{(<\omega)}$ such that $x \in int_{\upsilon(P)}\uparrow F = \uparrow F \subseteq U$;*

(3) *$\upsilon(P)$ is a hyperalgebraic lattice.*

Proof. (1) $\Rightarrow$ (2): For all $U \in \upsilon(P)$ with $x \in U$, there exists $H \in P^{(<\omega)}$ such that $x \in P\backslash\downarrow H \subseteq U$. For all $h \in H$, by (1), there exists $F_h \in P^{(<\omega)}$ such that $x \in int_{\upsilon(P)}\uparrow F_h = \uparrow F_h \subseteq P\backslash\downarrow h$. Since $H \in P^{(<\omega)}$ and $\{\uparrow F \in \mathbf{Fin}\ P : x \in int_{\upsilon(P)}\uparrow F = \uparrow F\}$ is directed, there exists $G \in P^{(<\omega)}$ such that $x \in int_{\upsilon(P)}\uparrow G = \uparrow G \subseteq \bigcap_{h\in H} \uparrow F_h \subseteq \bigcap_{h\in H} P\backslash\downarrow h = P\backslash\downarrow H \subseteq U$.

(2) $\Rightarrow$ (1): Suppose $\uparrow F_1, \uparrow F_2 \in \{\uparrow F \in \mathbf{Fin}\ P : x \in int_{\upsilon(P)}\uparrow F = \uparrow F\}$. Then $x \in int_{\upsilon(P)}\uparrow F_1 \cap int_{\upsilon(P)}\uparrow F_2 \in \upsilon(P)$. By (2), there is $F_3 \in P^{(<\omega)}$ such that $x \in int_{\upsilon(P)}\uparrow F_3 = \uparrow F_3 \subseteq int_{\upsilon(P)}\uparrow F_1 \cap int_{\upsilon(P)}\uparrow F_2 \subseteq \uparrow F_1 \cap \uparrow F_2$. Therefore, $\{\uparrow F \in \mathbf{Fin}\ P : x \in int_{\upsilon(P)}\uparrow F = \uparrow F\}$ is directed. Clearly, $\uparrow x \subseteq \bigcap\{\uparrow F \in \mathbf{Fin}\ P : x \in int_{\upsilon(P)}\uparrow F = \uparrow F\}$. If $z \notin \uparrow x$, then $x \in P\backslash\downarrow z \in \upsilon(P)$. By (2), there is $G \in P^{(<\omega)}$ with $x \in int_{\upsilon(P)}\uparrow G = \uparrow G \subseteq P\backslash\downarrow z$. It follows that $z \notin \bigcap\{\uparrow F \in \mathbf{Fin}\ P : x \in int_{\upsilon(P)}\uparrow F = \uparrow F\}$. Therefore, $\uparrow x = \bigcap\{\uparrow F \in \mathbf{Fin}\ P : x \in int_{\upsilon(P)}\uparrow F = \uparrow F\}$.

(2) $\Leftrightarrow$ (3): This follows from Lemma 3.3 of [13]. $\square$

It is similar to the proof of Theorem 2.6, we have the following

Theorem 2.8 *Let P be a poset. Then the following two conditions are equivalent:*

(1) *P is a hyperalgebraic poset;*

(2) *P is a meet continuous and quasi-hyperalgebraic poset.*

3 Induced topologies on the poset of finitely generated saturated sets

Definition 3.1 ([2]) Let (X, τ) be a T_0 space.

(1) (X, τ) is called a *c-space* if for all $x \in U \in \tau$, there exist $y \in X$ and $V \in \tau$ such that $x \in V \subseteq \uparrow y \subseteq U$.

(2) (X, τ) is called a *locally hypercompact space* if for all $x \in U \in \tau$, there exists $F \in X^{(<\omega)}$ and $V \in \tau$ such that $x \in V \subseteq \uparrow F \subseteq U$.

Theorem 3.2 ([1]) *Let (X, τ) be a T_0 space. The following conditions are equivalent:*

(1) *X is a c-space;*

(2) *τ is a completely distributive lattice.*

Theorem 3.3 ([2,8]) *Let (X, τ) be a T_0 space. The following conditions are equivalent:*

(1) *X is locally hypercompact;*

(2) *τ is a hypercontinuous lattice.*

Let (X, τ) be a T_0 space. For all $U \in \tau$, let $U_{\mathcal{F}} = \{\uparrow F \in \mathbf{Fin}\ X\colon\ F \subseteq U\}$. The topology generated by the basic open subsets $U_{\mathcal{F}}$ is denoted by $\tau_{\mathcal{F}}$.

It is easy to get the following

Proposition 3.4 *Let (X, τ) be a T_0 space.*

(1) *$\emptyset_{\mathcal{F}} = \emptyset$, $X_{\mathcal{F}} = \mathbf{Fin}\ X$.*

(2) *For all U, $V \in \tau$, $(U \cap V)_{\mathcal{F}} = U_{\mathcal{F}} \cap V_{\mathcal{F}}$.*

Theorem 3.5 *Let (X, τ) be a T_0 space. Then the following two conditions are equivalent:*

(1) *τ is a hypercontinuous lattice;*

(2) *$\tau_{\mathcal{F}}$ is a completely distributive lattice.*

Proof. (1) $\Rightarrow$ (2): For any $\uparrow G \in \mathcal{U} = \bigcup\limits_{i \in I} (U_i)_{\mathcal{F}}$, there exists $i \in I$ such that $\uparrow G \subseteq U_i$. By Theorem 3.3, for each $g \in G$, there exists $F_g \in X^{(<\omega)}$ such that $g \in int_{\tau} \uparrow F_g \subseteq \uparrow F_g \subseteq U_i$. Let $F = \bigcup\limits_{g \in G} F_g$ and $V = int_{\tau} \uparrow F$. Obviously, F is finite. Thus $\uparrow G \in V_{\mathcal{F}} \subseteq \uparrow_{\mathbf{Fin} X}(\uparrow F) \subseteq (U_i)_{\mathcal{F}} \subseteq \mathcal{U}$. Thus $\tau_{\mathcal{F}}$ is completely distributive by Theorem 3.2.

(2) $\Rightarrow$ (1): Let $U \in \tau$ with $x \in U$. Then $\uparrow x \in U_{\mathcal{F}}$. By (2), there exists $\uparrow F \in \mathbf{Fin}$ P such that $\uparrow x \in int_{\tau_{\mathcal{F}}} \uparrow_{\mathbf{Fin} X}(\uparrow F) \subseteq \uparrow_{\mathbf{Fin} X}(\uparrow F) \subseteq U_{\mathcal{F}}$. Thus there exists $V \in \tau$ such that $\uparrow x \in V_{\mathcal{F}} \subseteq \uparrow_{\mathbf{Fin} X}(\uparrow F) \subseteq U_{\mathcal{F}}$. Hence $x \in V \subseteq \uparrow F \subseteq U$. Therefore, τ is hypercontinuous by Theorem 3.3. $\square$

Similarly, we have the following

Theorem 3.6 *Let (X, τ) be a T_0 space. Then the following two conditions are equivalent:*

(1) *τ is a hyperalgebraic lattice;*

(2) *$\tau_{\mathcal{F}}$ is a completely distributive and algebraic lattice.*

Lemma 3.7 *Let P be a poset. Then $\bigvee\limits_{d \in D} \uparrow F_d$ exists in $\mathbf{Fin}\ P$ iff $\bigcap\limits_{d \in D} \uparrow F_d \in \mathbf{Fin}\ P$ for all $\{\uparrow F_d : d \in D\} \subseteq \mathbf{Fin}\ P$. In that case $\bigcap\limits_{d \in D} \uparrow F_d = \bigvee\limits_{d \in D} \uparrow F_d$.*

Proof. Obviously, $\bigcap\limits_{d \in D} \uparrow F_d \in \mathbf{Fin}\ P$ implies that $\bigvee\limits_{d \in D} \uparrow F_d$ exists in $\mathbf{Fin}\ P$.

Conversely, let $\uparrow F \in \mathbf{Fin}\ P$ with $\bigvee\limits_{d \in D} \uparrow F_d = \uparrow F$. Then for all $d \in D$, $\uparrow F_d \leq \uparrow F$, i.e., $\uparrow F \subseteq \uparrow F_d$. Thus $\uparrow F \subseteq \bigcap\limits_{d \in D} \uparrow F_d$. On the other hand, if $\bigcap\limits_{d \in D} \uparrow F_d \nsubseteq \uparrow F$, then there exists $x \in \bigcap\limits_{d \in D} \uparrow F_d$ but $x \notin \uparrow F$. Thus $\uparrow x$ is an upper bound of $\{\uparrow F_d : d \in D\}$ in $\mathbf{Fin}\ P$ and $\uparrow F \nleq \uparrow x$, a contradiction. $\square$

Proposition 3.8 *For any poset P, $\mathbf{Fin}\ P$ is a meet continuous poset.*

Proof. For all $\uparrow F \in \mathbf{Fin}\ P$ and $\mathcal{U} \in \sigma(\mathbf{Fin}\ P)$, we show that $\uparrow_{\mathbf{Fin}P}(\downarrow_{\mathbf{Fin}P}(\uparrow F) \cap \mathcal{U}) \in \sigma(\mathbf{Fin}\ P)$. For all directed sets $\{\uparrow F_d : d \in D\} \subseteq \mathbf{Fin}\ P$ with $\bigvee_{d\in D} \uparrow F_d \in \uparrow_{\mathbf{Fin}P}(\downarrow_{\mathbf{Fin}P}(\uparrow F) \cap \mathcal{U})$, there exists $\uparrow G \in \mathcal{U}$ with $\uparrow G \leq \uparrow F$ such that $\uparrow G \leq \bigvee_{d\in D} \uparrow F_d$. By Lemma 3.7, we have $\bigcap_{d\in D} \uparrow F_d \subseteq \uparrow G$. Thus $\uparrow G = \uparrow G \cup \bigcap_{d\in D} \uparrow F_d = \bigcap_{d\in D} (\uparrow G \cup \uparrow F_d) \in \mathcal{U}$. Hence there exists $d \in D$ such that $\uparrow G \cup \uparrow F_d \in \mathcal{U}$. Thus $\uparrow G \cup \uparrow F_d \leq \uparrow F_d, \uparrow G$, which implies $\uparrow G \cup \uparrow F_d \in \mathcal{U} \cap \downarrow_{\mathbf{Fin}P}(\uparrow F)$. Whence $\uparrow F_d \in \uparrow_{\mathbf{Fin}P}(\downarrow_{\mathbf{Fin}P}(\uparrow F) \cap \mathcal{U})$. Hence $\mathbf{Fin}\ P$ is meet continuous by Theorem 1.7. $\square$

Lemma 3.9 *Let P be a dcpo. Then $\sigma(P)_{\mathcal{F}} \subseteq \sigma(\mathbf{Fin}\ P)$.*

Proof. For all $U \in \sigma(P)$, we show $U_{\mathcal{F}} = \{\uparrow F \in \mathbf{Fin}\ P : \uparrow F \subseteq U\} \in \sigma(\mathbf{Fin}\ P)$. Obviously, $U_{\mathcal{F}} = \uparrow_{\mathbf{Fin}P} U_{\mathcal{F}}$. For all directed sets $\{\uparrow F_d : d \in D\} \subseteq \mathbf{Fin}\ P$ with $\bigvee_{d\in D} \uparrow F_d \in U_{\mathcal{F}}$, by Lemma 3.7, we have $\bigcap_{d\in D} \uparrow F_d = \uparrow H \in \mathbf{Fin}\ P$ and $\uparrow H \subseteq U$. By Rudin's Lemma [6, III-3.3], there exists $d \in D$ such that $\uparrow F_d \subseteq U$. Whence $\uparrow F_d \in U_{\mathcal{F}}$. Hence $U_{\mathcal{F}} \in \sigma(\mathbf{Fin}\ P)$. $\square$

Lemma 3.10 *Let P be a poset and $\mathcal{U} \in \sigma(\mathbf{Fin}\ P)$. Then $U = \bigcup \mathcal{U} = \bigcup\{\uparrow F \in \mathbf{Fin}\ P : \uparrow F \in \mathcal{U}\} \in \sigma(P)$.*

Proof. Let $y \in \uparrow U$. Let $x \in U$ such that $x \leq y$. Then there exists $\uparrow F \in \mathcal{U}$ such that $x \in \uparrow F$. Thus $\uparrow y \subseteq \uparrow x \subseteq \uparrow F$, i.e., $\uparrow F \leq \uparrow x \leq \uparrow y$. Since $\mathcal{U} \in \sigma(\mathbf{Fin}\ P)$, $\uparrow y \in \mathcal{U}$. Whence $y \in U$. Hence $\uparrow U = U$.

For all directed sets $D \subseteq P$ with $\bigvee D \in U$, we have $\bigcap_{d\in D} \uparrow d = \uparrow \bigvee D \in \mathcal{U}$. Thus there exists $d \in D$ such that $\uparrow d \in \mathcal{U}$. So $d \in U$. $\square$

Theorem 3.11 *Let P be a dcpo. If $\sigma(P)$ is hypercontinuous or $\sigma(\mathbf{Fin}\ P)$ is completely distributive, then $\sigma(P)_{\mathcal{F}}$=$\sigma(\mathbf{Fin}\ P)$.*

Proof. Let $\uparrow F \in \mathcal{U} \in \sigma(\mathbf{Fin}\ P)$. If $\sigma(P)$ is hypercontinuous, then $\uparrow F = \bigcap\{\uparrow G \in \mathbf{Fin}\ P : \uparrow F \subseteq int_{\sigma(P)} \uparrow G\}$ and $\{\uparrow G \in \mathbf{Fin}\ P : \uparrow F \subseteq int_{\sigma(P)} \uparrow G\}$ is directed. Thus there exists $\uparrow G \in \mathbf{Fin}\ P$ such that $\uparrow F \subseteq int_{\sigma(P)} \uparrow G \subseteq \uparrow G \in \mathcal{U}$. Let $V = int_{\sigma(P)} \uparrow G$. Then $\uparrow F \in V_{\mathcal{F}} \subseteq \uparrow_{\mathbf{Fin}P}(\uparrow G) \subseteq \mathcal{U}$. Hence $\mathcal{U} \in \sigma(P)_{\mathcal{F}}$.

If $\sigma(\mathbf{Fin}\ P)$ is completely distributive, then there is $\uparrow H \in \mathbf{Fin}\ P$ with $\uparrow F \in int_{\sigma(\mathbf{Fin}P)} \uparrow_{\mathbf{FinP}}(\uparrow H) \subseteq \uparrow_{\mathbf{FinP}}(\uparrow H) \subseteq \mathcal{U}$. Let $W = \bigcup int_{\sigma(\mathbf{Fin}P)} \uparrow_{\mathbf{FinP}}(\uparrow H)$. Then by Lemma 3.10, $W \in \sigma(P)$ and $\uparrow F \in W_{\mathcal{F}} \subseteq \uparrow_{\mathbf{Fin}P}(\uparrow H) \subseteq \mathcal{U}$. Therefore, $\mathcal{U} \in \sigma(P)_{\mathcal{F}}$. By Lemma 3.9, we have $\sigma(P)_{\mathcal{F}}$=$\sigma(\mathbf{Fin}\ P)$. $\square$

By Theorem 3.5 and Theorem 3.11, we get the following

Corollary 3.12 *Let P be a dcpo. Then the following two conditions are equivalent:*

(1) *$\sigma(P)$ is a hypercontinuous lattice;*

(2) *$\sigma(\mathbf{Fin}\ P)$ is a completely distributive lattice.*

By Theorem 1.3, Theorem 2.3 and Corollary 3.12, we have the following

Corollary 3.13 ([7]) *Let P be a dcpo. Then the following two conditions are equivalent:*

(1) *P is a quasicontinuous domain;*

(2) **Fin** *P is a continuous poset.*

By Theorem 5.6 of [10] and Proposition 3.8, we have the following

Corollary 3.14 *Let P be a poset. Then the following two conditions are equivalent:*

(1) **Fin** *P is a continuous poset;*

(2) **Fin** *P is a quasicontinuous poset.*

By Theorem 2.6 and Proposition 3.8, we have the following

Corollary 3.15 *Let P be a poset. Then the following two conditions are equivalent:*

(1) **Fin** *P is an algebraic poset;*

(2) **Fin** *P is a quasialgebraic poset.*

A poset P is said to have property $DINT$ (see [9]) if every set closed in the lower topology is a directed intersection of finitely generated upper sets.

Theorem 3.16 *Let P be a poset satisfying property DINT^{op}. Then $\upsilon(\mathbf{Fin}\ P)=\upsilon(P)_{\mathcal{F}}$.*

Proof. For all $\uparrow F \in \mathbf{Fin}\ P$, we have $\mathbf{Fin}\ P \setminus{\downarrow_{\mathbf{Fin}P}}(\uparrow F) = \bigcup_{u\in F}(P\setminus{\downarrow u})_{\mathcal{F}}$. Thus $\upsilon(\mathbf{Fin}\ P) \subseteq \upsilon(P)_{\mathcal{F}}$.

Conversely, it is clear that $P_{\mathcal{F}} = \mathbf{Fin}\ P \in \upsilon(\mathbf{Fin}\ P)$. For any nonempty set $U \in \upsilon(P)$ with $U \neq P$, we show $U_{\mathcal{F}} \in \upsilon(\mathbf{Fin}\ P)$. Since P satisfies property DINT^{op}, there exists a directed family $\{{\downarrow F_d} : F_d \in P^{(<\omega)} \text{ and } d \in D\}$ such that $U = P\setminus \bigcap_{d\in D}{\downarrow F_d} = \bigcup_{d\in D}(P\setminus{\downarrow F_d})$. Thus $U_{\mathcal{F}} = (\bigcup_{d\in D}(P\setminus{\downarrow F_d}))_{\mathcal{F}} = \bigcup_{d\in D}(P\setminus{\downarrow F_d})_{\mathcal{F}}$. We claim that $(P\setminus{\downarrow F_d})_{\mathcal{F}} \in \upsilon(\mathbf{Fin}\ P)$ for all $d \in D$. For all $\uparrow G \in (P\setminus{\downarrow F_d})_{\mathcal{F}}$, we have $\uparrow G \subseteq P\setminus{\downarrow F_d}$, i.e., $\uparrow h \nsubseteq \uparrow G$ for all $h \in F_d$. Thus $\uparrow G \in \bigcap_{h\in F_d}(\mathbf{Fin}\ P\setminus{\downarrow_{\mathbf{Fin}P}}(\uparrow h)) = \mathbf{Fin}\ P\setminus \bigcup_{h\in F_d}{\downarrow_{\mathbf{Fin}P}}(\uparrow h) = \mathbf{Fin}\ P\setminus{\downarrow_{\mathbf{Fin}P}}\{\uparrow h : h \in F_d\} \in \upsilon(\mathbf{Fin}\ P)$. Therefore, $U_{\mathcal{F}} \in \upsilon(\mathbf{Fin}\ P)$. □

Problem 3.17 *Is property DINT^{op} necessary to derive Theorem 3.16?*

By Theorem 1.4, Theorem 2.5, Theorem 3.5 and Theorem 3.16, we have the following

Corollary 3.18 *Let P be a poset satisfying property DINT^{op}. Then the following two conditions are equivalent:*

(1) *P is a quasi-hypercontinuous poset;*

(2) **Fin** *P is a hypercontinuous poset.*

Corollary 3.19 *Let P be a semilattice. Then the following two conditions are equivalent:*

(1) *P is a quasi-hypercontinuous poset;*

(2) **Fin** *P is a hypercontinuous poset.*

Similarly, we have the following two corollaries.

Corollary 3.20 *Let P be a poset. Then the following two conditions are equivalent:*

(1) **Fin** *P is a hypercontinuous poset;*

(2) **Fin** *P is a quasi-hypercontinuous poset.*

Corollary 3.21 *Let P be a poset. Then the following two conditions are equivalent:*

(1) **Fin** *P is a hyperalgebraic poset;*

(2) **Fin** *P is a quasi-hyperalgebraic poset.*

References

[1] Erné, M., *Scott convergence and Scott topology on partially ordered sets II*, in: B. Banaschewski, R.-E. Hoffman (Eds.), Continuous Lattices, Bremen 1979, in: Lecture Notes in Math., **871**, Springer-Verlag, Berlin, Heidelberg, New York, 1981, pp.61-96.

[2] Erné, M., *Infinite distributive laws versus local connectedness and compactness properties*, Topology and its Applications **156**, (2009), 2054-2069.

[3] Erné, M., A. Pultr and J. Sichler, *Closure Frames and Web Spaces*, KAM-DIMATIA Series 2000-501, DAM and ITI, Charles University, Prague, 2000.

[4] Gierz, G., J.D. Lawson and A.R. Stralka, *Quasicontinuous posets*, Houston J. Math. **9** (1983), 191-208.

[5] Gierz, G., and J.D. Lawson, *Generalized continuous and hypercontinuous lattices*, Rocky Mt. J. Math. **11** (1981), 271-296.

[6] Gierz, G. et al., *Continuous Lattices and Domains*, Cambridge University Press, Cambridge, 2003.

[7] Heckmann, R., and K. Keimel, *Quasicontinuous Domains and the Smyth Powerdomain*, Electronic Notes in Theoretical Computer Science **298**)2013), 215-232.

[8] Lawson, J.D., *T_0-spaces and pointwise convergence*, Topology and its Applications **21** (1985), 73-76.

[9] Lawson, J.D., *The upper interval topology, Property $\mathcal{M}$, and compactness*, Electronic Notes in Theoretical Computer Science, 1998.

[10] Mao, X. X., and L. S. Xu, *Quasicontinuity of Posets via Scott Topology and Sobrification*, Order **23** (2006), 359-369.

[11] Xu, L. S., *Continuity of posets via Scott topology and sobrification*, Topology and its Applications **153** (2006), 1886-1894.

[12] Xu, X. Q., and Y. M. Liu, *The Scott topology and Lawson topology on a $\mathcal{Z}$-quasicontinuous domain* (in Chinese), Chin. Ann. of Math. **3** (2003), 365-376.

[13] Xu, X. Q., and J. B. Yang, *Topological representations of distributive hypercontinuous lattices*, Chin. Ann. Math. **2** (2009), 199-206.