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An Algebra of Automata That Includes Both Classical and Quantum Entities

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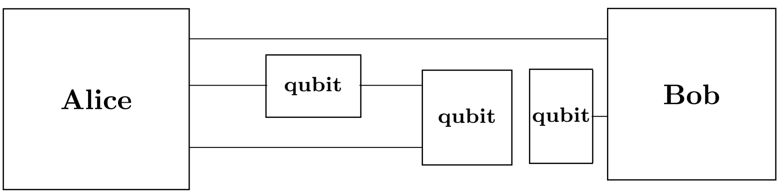
Abstract

We describe an algebra for composing automata which includes both classical and quantum entities. We illustrate by describing in detail a quantum protocol.

Keywords: Monoidal category, Span, Quantum automaton, Classical automaton, Compact closed, Frobenius equations, Teleportation

1 Introduction

We propose in this paper a categorical algebra which gives meaning to diagrams like the following one, which represents a teleportation protocol [3] in terms of communicating automata (and is explained in more detail in section 4.1):



The idea is to extend the algebra of automata, **Span**(**Graph**) [12], an algebra introduced to describe concurrent systems in a compositional way, in order to per-

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mit the inclusion of quantum components. This extension permits a compositional description of quantum protocols, in which quantum components interact with classical finite state components. In our view, the inclusion of explicit components of finite state classical control adds conceptual clarity and precision to quantum protocols.

The idea for this algebra came from an earlier extension [10] of **Span**(**Graph**) to probabilistic automata. An important thing to note is that in order to decompose the above system into parts, the algebra contains much more than just classical and quantum components - it is only in combination that the qubits form quantum components: perhaps the main question about our proposal is whether this wider context is an advantage or not.

We have mentioned that the algebra of [12] is the monoidal category of spans of graphs. Spans were introduced in [2] and are a generalization of relations. For example, in the category of sets a span R from A to B may be thought of as a $A \times B$ indexed family of sets $R_{a,b}$ $(a \in A,\ b \in B)$, or alternatively as a matrix of sets, whereas relations are matrices of boolean values. Span composition of R from A to B and S from B to C is given by pullback, or by the formula $\Sigma_{b \in B} R_{a,b} \times S_{b,c}$, $(a \in A,\ c \in C)$ clearly reminiscent of the formula for composition of relations. The monoidal categories of spans, and of relations, have been intensively studied beginning in [5] where the Frobenius equation was discovered. Recent papers are [4], and [24]. The paper [12] in considering spans of graphs, in particular between one vertex graphs, introduced a new aspect. Such spans may be considered to be doubly indexed families of graphs; but a graph may be thought of as a *square matrix* of sets.

This paper introduces a natural extension of the algebra of spans of graphs, namely doubly indexed families $\varphi_{a,b}$ $(a \in A, b \in B)$ of *operators* on finite dimensional vector spaces, the main operation being $\Sigma_{b \in B} \varphi_{a,b} \otimes \psi_{b,c}$, $(a \in A,\ c \in C)$.

A mixed algebra of quantum and classical phenomena has already been introduced by Coecke and Pavlovic in [6], with further work in [7], following the categorical twist on quantum logic introduced in [1]. The idea of those works is to describe data flow in quantum protocols, involving also classical measurements, as expressions in a symmetric monoidal category with extra structure. Such a formulation yields geometric pictures (following [19],[15]) of the flow in protocols, as well as pictorial equations which may be used to prove correctness. Another mixed algebra of quantum and classical phenomena was introduced by Selinger in [21], in the context of a functional programming language, which incorporates flow of data and flow of classical control. Neither of these approaches is an algebra of entities with states and actions, and their communication, and neither can explain the diagram we introduced in the first paragraph.

At the level of entities the importance of the *distributive law* of tensor product over direct sum in making classical choices becomes evident, a fact first observed for data by Selinger [21]. The situation is entirely analogous to classical Turing machines where an infinite state tape interacts with, and is controlled by, a finite state automaton (see the (non-compositional) description of Turing machines in

[23]; and also [20],[22] for relations with the Blum-Shub-Smale theory of computable functions).

Another point of interest is that **Span**(**Graph**) has been used for compositional model checking (in which non-determinism and the state explosion are considered the main problem) whereas here our extension is used for quantum computing (in which linearity and the expanded state space are the cited advantage).

Our automata are not to be confused with the quantum automata of [17] or [18], and hence we use the name $\mathbb{C}$-*automaton* for the general notion and *quantum or classical* C*-automaton* for those which represent respectively quantum or classical components.

We define a $\mathbb{C}$-automaton $\mathbf{Q}$ with a given set A of "signals on the left interface", and set B of "signals on the right interface" to consist of a finite dimensional complex vector space V and a family of linear transformations $\varphi_{a,b} : V \to V$ $(a \in A, b \in B)$. A quantum $\mathbb{C}$-automaton is one in which the space V has the extra structure of an hermitian inner product, and in which the linear transformations are unitary transformations or orthogonal projections. A classical $\mathbb{C}$-automaton is one with the extra structure that the space V is of the form $\mathbb{C}^X$ for a given finite set X and for which the matrices of the linear transformations are zero-one matrices induced by binary relations on X.

The idea of [12] was to introduce two-sided automata, in order to permit operations analogous to the parallel, series and feedback of classical circuits, in particular in concurrency theory. We have more recently described a similar algebra for automata with *probability* in [9],[10].

As an illustration of the algebra we will give details of the teleportation protocol of [3], proving its correctness with these tools.

2 $\mathbb{C}$-automata

Definition 2.1 Consider two finite alphabets A and B. A $\mathbb{C}$*-automaton* $\mathbf{Q}$ *with left interface* A *and right interface* B consists of a finite dimensional complex vector space V of states, and an $A \times B$ indexed family $\varphi = \varphi_{a,b(a \in A, b \in B)}$ of linear transformations from V to V.

Definition 2.2 A $\mathbb{C}$-automaton $\mathbf{Q}$ with the extra structure that the space V is endowed with an hermitian inner product $< | >$ and for which the linear transformations are either unitary or orthogonal projections is called a *quantum* $\mathbb{C}$*-automaton*.

Definition 2.3 A $\mathbb{C}$-automaton $\mathbf{Q}$ with the extra structure that the space V is $\mathbb{C}^X$ for a given finite set X, and for which the linear transformations $\varphi_{a,b}$ are of the form $\varphi_{a,b}(e_x, e_y) = 0$ or 1 $(x \in X)$ (where e_x $(x \in X)$ is the standard basis of $\mathbb{C}^X$ defined by $e_x(y) = 1$ if $y = x$, and 0 otherwise) is called a *classical (finite state)* $\mathbb{C}$ *-automaton*. Note: we will often write just x instead of e_x for a basis element.

The idea is that in a given state various transitions to other states are possible; the transitions that occur have effects, which we may think of a *signals*, on the two interfaces of the automaton, which signals are represented by letters in the alphabets. It is fundamental *not* to think of the letters in A and B in general as inputs or outputs, but rather signals induced by transitions of the automaton on the interfaces. For examples see a later section.

Definition 2.4 Consider a $\mathbb{C}$-automaton $\mathbf{Q}$ with interfaces A and B. A *behaviour* of length k of $\mathbf{Q}$ consists of a two words of length k, one $w_1 = a_1 a_2 \cdots a_k$ in A^* and the other $w_2 = b_1 b_2 \cdots b_k$ in B^* and a sequence of vectors

$$\mathbf{x}_0, \mathbf{x}_1 = \varphi_{a_1,b_1}(\mathbf{x}_0), \mathbf{x}_2 = \varphi_{a_2,b_2}(\mathbf{x}_1), \cdots, \mathbf{x}_k = \varphi_{a_k,b_k}(\mathbf{x}_{k-1}).$$

3 Graphical representation

Although the definitions above are mathematically straightforward, in practice a graphical notation is more intuitive. Given a chosen basis for the state space of an automaton (or, more generally, a decomposition of the state space as a direct sum) we may compress the description of an automaton with interfaces A and B, which requires $A \times B$ matrices of scalars (or, more generally, matrices of linear transformations), into a single labelled graph, like the ones introduced in [12]. Further, expressions of automata in this algebra may be drawn as "tensor diagrams" also as in [12]. We indicate both of these matters by describing some examples.

3.1 Qubits

Qubit automata are a $\mathbb{C}$-automata with state space $\mathbb{C}^2$ which singly, or combined, form quantum automata. We will describe three particular qubit automata which will need for our discussion of teleportation. One of the qubit automata is a quantum automaton; the others will be combined to form a 2 qubit quantum automaton.

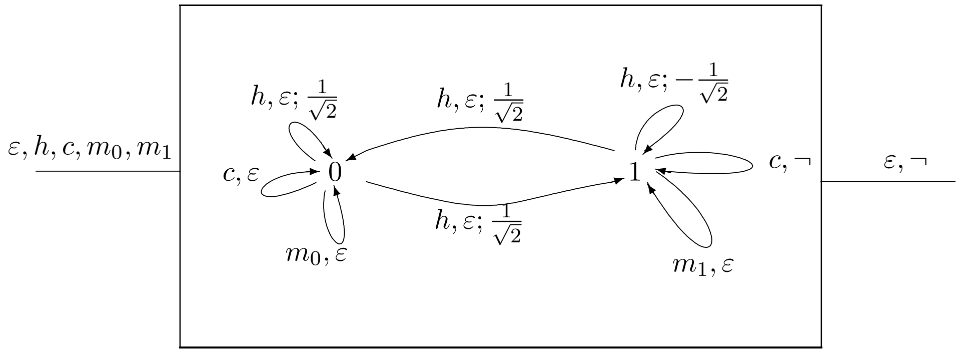
3.1.1 Qubit Q_1

Consider the alphabets $A_1 = \{\varepsilon, c, h, m_0, m_1\}$ and $B_1 = \{\varepsilon, \neg\}$. Then $\mathbf{Q}_1$ is the automaton with left interface A_1 and right interface B_1, state space $\mathbb{C}^2$ and transition matrices

$$\varphi_{\varepsilon,\varepsilon} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \varphi_{c,\neg} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$
$$\varphi_{c,\varepsilon} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ \varphi_{h,\varepsilon} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$
$$\varphi_{m_0,\varepsilon} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ \varphi_{m_1,\varepsilon} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The other four transition matrices are zero matrices.

The intention behind these matrices is as follows: $\mathbf{Q}_1$ may do a transition labelled ε, ε (*idle transition*); $\mathbf{Q}_1$ may receive a signal h and perform a transition determined by the unitary Hadamard matrix; $\mathbf{Q}_1$ may receive a signal c (*do* $\mathrm{C}_{\mathrm{not}}$) and if it is in state 1 pass on a signal $\neg$ with the intention to perform a *not* on another qubit; the signal m_0 means that a *measurement* with result 0 has occurred on $\mathbf{Q}_1$; the signal m_1 means that a *measurement* with result 1 has occurred on $\mathbf{Q}_1$. All this information may be put in the following diagram, noting that (i) the basis elements of $\mathbb{C}^2$ are called 0 and 1, and occur in the diagram as vertices, (ii) labels of transitions indicate which matrix is involved, (iii) the absence of an edge from i to j means that the i,jth element of the matrix is 0, (iv) we have in any case omitted loops labelled ε, ε, (v) we have included the value of the matrix element only when it is not 1.



3.1.2 *Qubit Q_2*

Consider the alphabets $A_2 = \{\varepsilon, \neg\} \times \{\varepsilon, m_0, m_1\} = A_{21} \times A_{22} = B_1 \times A_{22}$ and $B_2 = \{\varepsilon\}$. Then $\mathbf{Q}_2$ is the automaton with left interface A_2 and right interface B_2, state space $\mathbb{C}^2$ and transition matrices

$$\varphi_{(\varepsilon,\varepsilon),,\varepsilon} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \varphi_{(\neg,\varepsilon),\varepsilon} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
$$\varphi_{(\varepsilon,m_0),\varepsilon} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ \varphi_{(\varepsilon,m_1),\varepsilon} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The remaining matrices are zero.

The intention behind these matrices is as follows: $\mathbf{Q}_2$ may do a transition labelled ε, ε (*idle transition*); $\mathbf{Q}_2$ may receive a signal $\neg$ and perform a *not* transition; the signal m_0 means that a *measurement* with result 0 has occurred on $\mathbf{Q}_2$; the signal m_1 means that a *measurement* with result 1 has occurred on $\mathbf{Q}_2$.

3.1.3 *Qubit Q_3*

Consider the alphabets $A_3 = \{\varepsilon\}$, and $B_3 = \{\varepsilon, 00, 01, 10, 11\}$. Then $\mathbf{Q}_3$ is the automaton with left interface A_3 and right interface B_3, state space $\mathbb{C}^2$ and transition matrices

$$\varphi_{\varepsilon,\varepsilon} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\varphi_{\varepsilon,00} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \varphi_{\varepsilon,10} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$\varphi_{\varepsilon,01} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ \varphi_{\varepsilon,11} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

The intention behind these matrices is as follows: $\mathbf{Q}_3$ may do a transition labelled ε, ε (*idle transition*); $\mathbf{Q}_3$ may receive one of four signal $00, 01, 10, 11$ and perform the given unitary transformations.

3.2 Alice and Bob

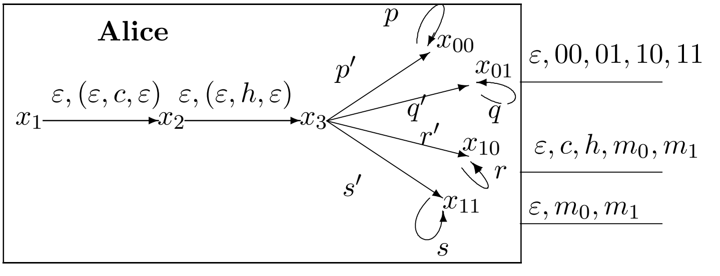
We now describe two classical $\mathbb{C}$-automata **Alice** and **Bob** which represent, respectively, the sender and the receiver of teleportation.

3.2.1 Alice

Let $X = \{x_1, x_2, x_3, x_{00}, x_{01}, x_{10}, x_{11}\}$. Then **Alice** is the classical $\mathbb{C}$-automaton with state space $\mathbb{C}^X$ with left interface $A_{Alice} = \{\varepsilon\}$ and right interface

$$\begin{aligned} B_{Alice} &= \{\varepsilon, 00, 01, 10, 11\} \times \{\varepsilon, c, h, m_0, m_1\} \times \{\varepsilon, m_0, m_1\} \\ &= B_{Alice,1} \times A_1 \times A_{22}. \end{aligned}$$

and transformations as indicated in the diagram



where $p' = \varepsilon, (\varepsilon, m_0, m_0)$, $q' = \varepsilon, (\varepsilon, m_0, m_1)$, $r' = \varepsilon, (\varepsilon, m_1, m_0)$, $s' = \varepsilon, (\varepsilon, m_1, m_1)$, $p = \varepsilon, (00, \varepsilon, \varepsilon)$, $q = \varepsilon, (01, \varepsilon, \varepsilon)$, $r = \varepsilon, (10, \varepsilon, \varepsilon)$, $s = \varepsilon, (11, \varepsilon, \varepsilon)$.

3.2.2 Bob

Let $Y = \{y_1, y_2\}$. Then **Bob** is the classical $\mathbb{C}$-automaton with state space $\mathbb{C}^Y$ with left interface $A_{Bob} = \{\varepsilon, 00, 01, 10, 11\} \times \{\varepsilon, 00, 01, 10, 11\}$ and right interface

$B_{Bob} = \{\varepsilon\}$ and transformations relative to the standard basis e_{y_1}, e_{y_2} having the following non-zero elements:

$$\varphi_{(\varepsilon,\varepsilon),\varepsilon}(e_{y_1}) = e_{y_1},$$
$$\varphi_{(00,00),\varepsilon}(e_{y_1}) = e_{y_2}, \varphi_{(01,01),\varepsilon}(e_{y_1}) = e_{y_2},$$
$$\varphi_{(10,10),\varepsilon}(e_{y_1}) = e_{y_2}, \varphi_{(11,11),\varepsilon}(e_{y_1}) = e_{y_2}.$$

4 The algebra of $\mathbb{C}$-automata

Now we define operations on $\mathbb{C}$-automata analogous (in a precise sense) to those defined in [12].

Definition 4.1 Given a $\mathbb{C}$-automata $\mathbf{Q}$ with left and right interfaces A and B, state space V, and family of transformations φ, and $\mathbf{S}$ with interfaces C and D, state space W, transformations ψ, the *parallel composite* $\mathbf{Q} \otimes \mathbf{R}$ is the $\mathbb{C}$-automaton which has state space $V \otimes W$, left interfaces $A \times C$, right interface $B \times D$, and transformations

$$(\varphi \otimes \psi)_{(a,c),(b,d)} = \varphi_{a,b} \otimes \psi_{c,d}.$$

Definition 4.2 Given $\mathbb{C}$-automata $\mathbf{Q}$ with left and right interfaces A and B, state space V, and family of transformations φ, and $\mathbf{R}$ with interfaces B and C, state space W, and family of transformations ψ the *series (communicating parallel) composite of* $\mathbb{C}$*-automata* $\mathbf{Q} \circ \mathbf{R}$ has state space $V \otimes W$, left interfaces A, right interface C, and transition maps

$$(\varphi \circ \psi)_{a,c} = \sum_{b \in B} \phi_{a,b} \otimes \psi_{b,c}.$$

Notice that when the state spaces of the $\mathbf{C}$-automata have direct sum decompositions, and hence the operators have matrix representations, the tensor products in the above definitions may be calculated (via distributivity isomorphisms) using tensor products of matrices. This gives a way of calculating the operations analogous to the operations on automata in [12].

Definition 4.3 Given a relation $\rho \subset A \times B$ we define a $\mathbb{C}$-automaton ρ as follows: it has state space $\mathbb{C}$. The transition matrices $\rho_{a,b}$ are 1×1 matrices, that is, complex numbers. Then $\rho_{a,b} = 1$ if ρ relates a and b, and $\rho_{a,b} = 0$ otherwise.

Some special cases, all described in [12], have particular importance:

(i) the automaton corresponding to the identity function 1_A, considered as a relation on $A \times A$ is called 1_A;

(ii) the automaton corresponding to the diagonal function $\Delta : A \to A \times A$ (considered as a relation) is called Δ_A; the automaton corresponding to the opposite relation of Δ is called ∇_A.

(iii) the automaton corresponding to the function $twist : A \times B \to B \times A$ is called $twist_{A,B}$.

(iv) the automaton corresponding to the relation $\eta = \{(*,(a,a)); a \in A\} \subset \{*\} \times (A \times A)$ is called η_A; the automaton corresponding to the opposite of η is called ϵ_A.

4.1 The teleportation protocol

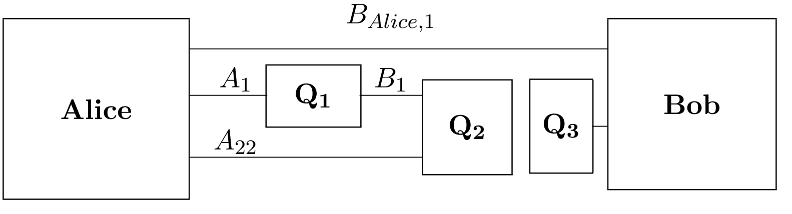
4.1.1 The protocol TP

Now the model of the teleportation protocol we consider is an expression in the algebra, involving also the automata $\mathbf{Q}_1, \mathbf{Q}_2, \mathbf{Q}_3$, **Alice**, and **Bob**. The protocol is

$$\mathbf{TP} = \mathbf{Alice} \circ (1_{A_3} \otimes ((\mathbf{Q}_1 \otimes 1_{A_{22}}) \circ \mathbf{Q}_2)) \circ (1_{A_3} \otimes \mathbf{Q}_3) \circ \mathbf{Bob}.$$

Notice that $(\mathbf{Q}_1 \otimes 1_{A_{22}}) \circ \mathbf{Q}_2$ and $\mathbf{Q}_3$ are quantum $\mathbb{C}$-automata.

As explained in [12], we may represent this system by the following diagram:



4.1.2 The behaviour of **TP**

Consider the following initial state of **TP**

$$x_1 \otimes (\alpha 0 + \beta 1) \otimes \frac{1}{\sqrt{2}}(0 \otimes 0 + 1 \otimes 1) \otimes y_1;$$

that is that state of $\mathbf{Q}_1$ is arbitrary and $\mathbf{Q}_2$ and $\mathbf{Q}_3$ are in Bell state. Since the combined system **TP** is closed it consists of a single linear transformation θ acting on the state space $\mathbb{C}^X \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^Y$. A behaviour consists of a sequence of applications of θ to the initial state. However, in view of the construction of θ from parts, we may give a more explicit description of behaviours beginning in this initial state. In the following calculation it is critical that $\mathbb{C}^X$ and C^Y break up into a direct sums $\mathbb{C} \oplus \mathbb{C} \oplus \cdots \oplus \mathbb{C}$ so that, using the distributive law of tensor over direct sum, **Alice** and **Bob** can do different actions on the qubits in different summands. This is entirely analogous to the use of sums and the distributive law in sequential programming, in particular in defining "if then else" [8],[23].

Simplifying the notation, writing for example 00 instead of $0 \otimes 0$, a four step behaviour is:

$$
\begin{aligned}
&x_1 \otimes (\alpha 0 + \beta 1) \otimes \frac{1}{\sqrt{2}}(00 + 11) \otimes y_1 \\
&\mapsto x_2 \otimes \frac{1}{\sqrt{2}}(\alpha 000 + \alpha 011 + \beta 110 + \beta 101) \otimes y_1 \\
&\mapsto \frac{1}{2} x_3 \otimes (\alpha(0+1)00 + \alpha(0+1)11 + \beta(0-1)10 + \beta(0-1)01) \otimes y_1 \\
&= \frac{1}{2} x_3 \otimes (\alpha(000 + 100 + 011 + 111) + \beta(010 - 110 + 001 - 101)) \otimes y_1 \\
&\mapsto \frac{1}{2}(x_{00} \otimes (\alpha 000 + \beta 001) \otimes y_1 + x_{01} \otimes (\alpha 011 + \beta 010) \otimes y_1 + \\
&x_{10} \otimes (\alpha 100 - \beta 101) \otimes y_1 + x_{11} \otimes (\alpha 111 - \beta 110) \otimes y_1) \\
&\mapsto \frac{1}{2}(x_{00} \otimes (\alpha 000 + \beta 001) \otimes y_2 + x_{01} \otimes (\alpha 010 + \beta 011) \otimes y_2 + \\
&x_{10} \otimes (\alpha 100 + \beta 101) \otimes y_2 + x_{11} \otimes (\alpha 110 + \beta 111) \otimes y_2) \\
&= \frac{1}{2}(x_{00} \otimes 00 + x_{01} \otimes 01 + x_{10} \otimes 10 + x_{11} \otimes 11) \otimes (\alpha 0 + \beta 1) \otimes y_2.
\end{aligned}
$$

4.2 The algebra of automata: further work

There is clearly much more to develop about the algebraic structure. We mention only that the constants Δ_A, ∇_A satisfy the Frobenius equations [5], namely that

$$(\Delta_A \otimes 1_A) \circ (1_A \otimes \nabla_A) = \nabla_A \circ \Delta_A.$$

Notice that relations on X also exist as closed classical automata with state space $\mathbb{C}^X$ and there the Frobenius equations are also satisfied, which fact has been used in axiomatizing classical data in [7].

There is another sense in which the algebra is incomplete. We have not described the relation between our diagrammatic representation, which concern parallel operations, and those of Coecke, Selinger and others in which the diagrams represent flow of data, that is, involve sequential operations. We hope to apply the ideas of [13], [11] to study this relation.

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