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A Note On the Turing Degrees of Divergence Bounded Computable Reals

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Abstract

The Turing degree of a real number is defined as the Turing degree of its binary expansion. In this note we apply the double witnesses technique recently developed by Downey, Wu and Zheng [2] and show that there exists a Δ^0_2-Turing degree which contains no divergence bounded computable real numbers. This extends the result of [2] that not every Δ^0_2-Turing degree contains a d-c.e. real.

Keywords: Turing degree of reals, divergence bounded computable reals.

1 Introduction

The computability of real numbers was introduced by Turing [11] by means of decimal expansions. Namely, a real number x is *computable* if it has a computable decimal expansion, i.e., $x = \sum_{n\in\mathbb{N}} f(n) \cdot 10^{-n}$ for a computable function $f : \mathbb{N} \to \{0, 1, \cdots, 9\}$. This notion is independent of the representations of the reals as shown by Robinson [9] and others. Concretely, a real $x \in [0; 1]$ is computable iff the Dedekind cut $L_x := \{r \in \mathbb{Q} : r < x\}$ is a computable set; iff x has a computable binary expansion, i.e., there is a computable set A such that $x = x_A := \sum_{n\in A} 2^{-(n+1)}$; and iff there is a computable

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sequence (x_s) of rational numbers which converges to x effectively in the sense that $|x_s - x_{s+1}| \leq 2^{-s}$ for all s, and so on. Similarly, the Turing degree of a real number and the Turing reducibility between real numbers can be naturally defined based on binary expansion, Dedekind cut or Cauchy sequence representations of the reals, respectively and it is not difficult to see that they are equivalent (see e.g., [5]). For example, x_A is Turing reducible to x_B (denoted by $x_A \leq_T x_B$) if $A \leq_T B$ and $x_A \equiv_T x_B$ if $x_A \leq_T x_B$ & $x_B \leq_T x_A$. In addition, the Turing degree of a real x is defined as the class of all reals which are Turing equivalent to x, namely, $\deg_T(x) := \{y \in \mathbb{R} : y \equiv_T x\}$. Especially, the degree of a computable real number consists of all computable reals. On the other hand, Ho [6] shows that a real number is Turing reducible to $\mathbf{0}'$, the Turing degree of the halting problem, if and only if it is computably approximable, where the computably approximable reals are simply the limits of computable sequences of rational numbers (see [1]). The Turing degree of a computably approximable real is also called Δ^0_2-Turing degree or simply Δ^0_2-degree.

Between the classes of computable and computably approximable real numbers there are a lot of interesting classes of real numbers introduced in literatures ([1,7,12]). For example, x is left (right) computable if it is the limit of an increasing (decreasing) computable sequence of rational numbers. The left computable reals are also called computably enumerable (c.e., for short) (see [3,4]) because their left Dedekind cuts are c.e. sets of rational numbers. x is d-c.e. if it is the difference of two c.e. reals. The class of d-c.e. real numbers (denoted by **WC**) is a very interesting class. Ambos-Spies, Weihrauch and the first author showed in [1] that the class **WC** is the arithmetical closure of the c.e. reals and hence is a field and that x is d-c.e. if and only if there is a computable sequence (x_s) of rational numbers which converges to x *weakly effectively* in the sense that the sum $\sum_{s\in\mathbb{N}} |x_s - x_{s+1}|$ is bounded.

Since any c.e. real has a c.e. left Dedekind cut, the Turing degree of any c.e. real is c.e., although not every c.e. real has a c.e. binary expansion as observed by Jockusch (see [10]). On the other hand, every non-computable c.e. degree contains a non-c.e. real. For the d-c.e. reals, the situation is a little bit complicated. The case for the d-c.e. reals is different. The first author shows in [13] that there is a d-c.e. real whose Turing degree is not even ω-c.e. Here a set $A \subseteq \mathbb{N}$ is ω-c.e. if there are a computable function h and a computable sequence (A_s) of finite sets which converges to A such that $|\{s \in \mathbb{N} : n \in A_s \Delta A_{s+1}\}| \leq h(n)$ for all n. A Turing degree is called ω-c.e. if it contains an ω-c.e. set. Recently, Downey, Wu and the first author showed in [2] that any ω-c.e. Turing degree contains at least a d-c.e. real, but there exists a Δ^0_2-Turing degree which does not contain any d-c.e. real.

In this paper, we explore the Turing degrees of another class of reals, namely, the divergence bounded computable reals which were introduced by the authors together with Gengler and von Braunmühl in [7]. A real number x is called *divergence bounded computable* (dbc, for short) if there are a total computable function h and a computable sequence (x_s) of rational numbers which converges to x *h-bounded effectively* in the sense that there are at most $h(n)$ non-overlapping 2^{-n}-jumps (i, j) for all n. Here an index pair (i, j) is a 2^{-n}-jump of the sequence (x_s) if $|x_i - x_j| \geq 2^{-n}$. It is shown that, every d-c.e. real is dbc but there is computably approximable real which is not dbc, i.e., the class of all dbc reals (denoted by $\mathbf{DBC}$) is strictly between the classes of d-c.e. and computably approximable reals. Like $\mathbf{WC}$, the class $\mathbf{DBC}$ is also a field. Furthermore, the class $\mathbf{DBC}$ is also closed under total computable real functions while $\mathbf{WC}$ is not so. Actually, $\mathbf{DBC}$ is the closure of $\mathbf{WC}$ under computable total real functions (see [8]). We will show in this paper that, even the Turing degrees of dbc reals does not exhaust all the Δ_2-Turing degrees.

2 Main Result

This section gives a construction which shows that not every Δ^0_2-degree contains a divergence bounded computable real. The construction is another interesting example of the "double witnesses" technique introduced in [2].

In the following, the use function of computable functionals Φ and Ψ are denoted by the lower case φ and ψ, respectively. W.l.o.g. we assume that $n \leq \varphi_{e,s}(n) \leq s$ for all n and s, where $\varphi_{e,s}(n)$ denotes the use function of Φ_e for the input n up to stage s. We identify a set A with its characteristic function. That is, $A(n) = 1 \iff n \in A$ and $A(n) = 0 \iff n \notin A$ for all n. For convenience, we write $A(n)(m) := A(n)A(m)$.

Theorem 2.1 *There exists a Δ^0_2-degree which contains no divergence bounded computable reals.*

Proof. We will construct a computable sequence (A_s) of finite subsets of natural numbers which converges to A such that A is not Turing equivalent to any divergence bounded computable real. To this end, let $(b_e, h_e, \Phi_e, \Psi_e)$ be an effective enumeration of all tuples of computable functions $b_e :\subseteq \mathbb{N} \to \mathbb{D}$, $h_e :\subseteq \mathbb{N} \to \mathbb{N}$, and computable functionals Φ_e, Ψ_e. For any $e, s \in \mathbb{N}$, if $b_e(s)$ is defined, then let $B_{e,s}$ be a finite set of natural numbers such that $b_e(s) = x_{B_{e,s}}$. Thus, the set A has to satisfy all the following requirements.

$$R_e: \quad \left.\begin{array}{l} b_e \text{ and } h_e \text{ are total and } (b_e(s)) \text{ con-} \\ \text{verges } h_e\text{-bounded effectively to } x_{B_e} \end{array}\right\} \Longrightarrow A \neq \Phi_e^{B_e} \vee B_e \neq \Psi_e^A.$$

The sequence (A_s) is constructed in stages such that A_s is the approximation of A at the end of stage s and $A = \lim_{s\to\infty} A_s$. We define a length function l as follows:

$$l(e,s) := \max\{x : A_s \upharpoonright x = \Phi_{e,s}^{B_{e,s}} \upharpoonright x \ \& \ B_{e,s} \upharpoonright \varphi_{e,s}(x) = \Psi_{e,s}^{A_s} \upharpoonright \varphi_{e,s}(x)\},$$

where φ_e is the use function of the functional Φ_e. Thus, to satisfy a requirement R_e, it suffices to guarantee that $l(e,s)$ is bounded from above, if the premisses of R_e hold.

Now let's describe the strategy to satisfy a single requirement R_e. We first choose a witness n_e large enough. At the beginning, let $A(n_e-1)(n_e) = 00$, i.e., both $n_e - 1$ and n_e are not in A. Then we wait for a stage s such that $l(e,s) > n_e$. If there does not exist such s at all, then $l(e,s) \leq n_e$ for all s and we are done. Otherwise, suppose that s_1 is the first stage such that $l(e,s_1) > n_e$. In this case, both computations $\Phi_{e,s_1}^{B_{e,s_1}}(n_e)$ and $\Phi_{e,s_1}^{B_{e,s_1}}(n_e-1)$ halt and hence $\varphi_{e,s_1}(n_e-1)$ and $\varphi_{e,s_1}(n_e)$ are defined. Furthermore, the initial segment $\Psi_{e,s_1}^{A_{s_1}} \upharpoonright \varphi_{e,s_1}(n_e)$ is defined too. Let $m_e := \psi_{e,s_1}(\varphi_{e,s_1}(n_e))$. Assume w.l.o.g. that $n_e < m_e$. If $h_{e,s_1}(m_e)$ is also defined, then we put $n_e - 1$ into A (the number n_e remains out of A) to destroy the agreement. That is, we define $A_{s_1+1}(n_e-1)(n_e) = 10$. Then we wait for a new stage $s_2 > s_1$ such that $l(e,s_2) > n_e$ holds again. If no such a stage exists, then we are done again. Otherwise, we put n_e into A, i.e., let $A_{s_2+1}(n_e-1)(n_e) := 11$. If there exists another stage $s_3 > s_2$ such that $l(e,s_3) > n_e$, then we take both $n_e - 1$ and n_e out of A, i.e., let $A_{s_3+1}(n_e-1)(n_e) := 00$. In this case, the set A_{s_3+1} is recovered to that of stage s_1, i.e., $A_{s_3+1} = A_{s_1}$. This closes a cycle in which the values $A(n_e-1)(n_2)$ change in the order of $00 \to 10 \to 11 \to 00$. This process will continue as long as the number of 2^{-m_e}-jumps of the sequence $(x_{B_{e,s}})$ (i.e., the sequence $(b_e(s))$) does not exceed $h_e(m_e)$ yet.

Thus, we achieve a temporary disagreement between A and $\Phi_e^{B_e}$ by changing the values $A(n_e-1)(n_e)$ whenever the length of agreement goes beyond the witness n_e. After that, if the agreement becomes bigger than n_e again, then the corresponding value $\Phi_e^{B_e}(n_e-1)(n_e)$ has to be changed too and this forces the initial segment $B_e \upharpoonright \varphi_e(n_e)$ to be changed, say, $B_{e,s} \upharpoonright \varphi_{e,s}(n_e) \neq B_{e,t} \upharpoonright \varphi_{e,t}(n_e)$. There are two possibilities now.

Case 1. This corresponds to a 2^{-m_e}-jump, i.e., $|x_{B_{e,s}} - x_{B_{e,t}}| \geq 2^{-m_e}$. If the sequence $(b_e(s))$ converges h_e-bounded effectively, then $(b_e(s))$ has at most $h_e(m_e)$ non-overlapping 2^{-m_e}-jumps. Thus, this can happen at most $h_e(m_e)$ times

Case 2. The change of the initial segment $B_e \upharpoonright \varphi_e(n_e)$ does not lead to a 2^{-m_e}-jump. That is, $|x_{B_{e,s}} - x_{B_{e,t}}| = 2^{-m} < 2^{-m_e}$ for a natural number $m > m_e$. In this case, there exists a (least) natural number $n < m_e$ such that $B_{e,s}(n) \neq B_{e,t}(n)$ because $B_{e,s} \upharpoonright m_e \neq B_{e,t} \upharpoonright m_e$ (remember that $m_e \geq \varphi_e(n_e)$).

In this case, as binary word, $B_{e,s}$ has one of the following forms

$$\begin{array}{rllllllll} \text{form } 1 := & 0\ .\ w & 1 & 0 \cdots & 0 & \cdots & 0 & v \\ \text{form } 2 := & 0\ .\ w & 0 & 1 \cdots & 1 & \cdots & 1 & v \\ & & \uparrow & & \uparrow & & \uparrow & \\ (\text{positions}\ : & & n & & m_e & & m &) \end{array}$$

for some $w, v \in \{0,1\}^*$ and $B_{e,t}$ takes another one. Here n, m_e and m indicate the corresponding positions. This implies that, if the sequence $(x_{B_e(s)})$ does not have 2^{-m_e}-jumps after some stage s any more, then the initial segment $B_{e,s} \upharpoonright m_e$ can have only two possible forms: $0.w10\cdots0$ or $0.w01\cdots1$. Correspondingly, the combination $\Phi_{e,s}^{B_{e,s}}(n_e-1)(n_e)$ can have at most two possibilities too. However, in every circle described above, $A(n_e-1)(n_e)$ takes three different forms, i.e., $00, 10$ and 11. In other words, we can always achieve a disagreement $A \neq \Phi_e^B$ at some stage and hence the requirement R_e is satisfied eventually.

To satisfy all requirements simultaneously, we apply a finite injury priority construction. In this case, R_e has higher priority than R_i if $e < i$. To preserve the requirement R_e from the disturbance by lower priority R_i, the initial segment $A \upharpoonright \psi_e\varphi_e(n_e)$ should be preserved. To this end, only the elements which are larger than $\psi_e\varphi_e(n_e)$ are allowed to be appointed as witnesses of R_i for $i > e$ afterward. $m_e := \psi_e(\varphi_e(n_e))$ is called a restriction of R_e. A requirement R_e being *initialized* means that n_e and m_e (if any) are set to be undefined.

The following is a formal construction of the sequence (A_s).

Stage 0: Let $A_0 := \emptyset$ and all requirements R_e are initialized.

Stage $s+1$: A requirement R_e *requires attention* if the following conditions hold.

(i) the witness $n_{e,s}$ is defined such that $n_{e,s} < l(e,s)$ holds (in this case the restriction $m_{e,s} := \psi_{e,s}(\varphi_{e,s}(n_{e,s}))$ is defined too);

(ii) $h_{e,s}(m_{e,s})$ is defined and the sequence $(b_e(t))_{t\leq s}$ does not make more than $h_{e,s}(m_{e,s})$ non-overlapping $2^{-m_{e,s}}$-jumps so far.

If no requirement requires attention at this stage, then choose a least e such that n_e is currently not defined and let $n_{e,s+1} := s+2$ (remember the convention that $\varphi_{e,s}(n) \leq s$ for all n, e). Otherwise, suppose that R_e is the requirement of highest priority (i.e., of the least index) which requires attention. Then we define

$$A_{s+1}(n_e-1)(n_e) := \begin{cases} 01, & \text{if } A_s(n_e-1)(n_e) = 00; \\ 11, & \text{if } A_s(n_e-1)(n_e) = 01; \\ 00, & \text{if } A_s(n_e-1)(n_e) = 11. \end{cases}$$

In addition, all requirements R_i of lower priority (i.e., $i > e$) are *initialized*. In this case, we say that R_e *receives attention*.

This completes the construction.

To show that the constructed sequence (A_s) converges to a set A which satisfies all requirements R_e, it suffices to prove the following claim.

Claim 2.2 *For any $e \in \mathbb{N}$, the requirement R_e receives attention finitely many times and is eventually satisfied.*

Proof of Claim: The claim can be proved by an induction on e. Assume that, for any $i < e$, the requirements R_i receive attention only finitely many times and are satisfied eventually. Let s_0 be the first stage such that no requirement R_i for $i < e$ requires and receives attention after stage s_0 any more. By the minimality of s_0, R_e is initialized at stage s_0. By construction, there is a stage $s_1 > s_0$ at which the witness n_{e,s_1} is defined. This witness will not be changed any more, i.e., $n_{e,s_1} = n_{e,s}$ for all $s \geq s_1$, because R_e will never be initialized after stage s_1 again. For convenience, let $n_e := n_{e,s_1}$.

If R_e does not require attention after stage s_1, then either the length $l(e, s)$ does not go beyond n_e, or $l(e, s) > n_e$ for some s but either $h_e(m_e)$ (for $m_e := \psi_{e,s}(\varphi_{e,s}(n_e))$) is not defined or the sequence $(b_e(s))_s$ has already more 2^{-m_e}-jumps than $h_e(m_e)$. In any of these cases, R_e is satisfied.

Otherwise, suppose that R_e requires and receives attention after stage s_1 at stages $t_0 + 1, t_1 + 1, t_2 + 1, \cdots$. Thus, both $m_e := \psi_{e,t_0}(\varphi_{e,t_0}(n_e))$ and $h_e(m_e) := h_{e,t_0}(m_e)$ are defined. At stage $t_0 + 1$, all requirements R_i for $i > e$ are initialized. If a witness n_i for R_i is appointed later, then we have $n_i > t_0 + 1 > m_e$. This implies that the initial segment $A_s \upharpoonright m_e$ will not be changed after stage $t_0 + 1$ except the elements $n_e - 1$ and n_e. Furthermore, since $A(n_e - 1)(n_e)$ changes always in the order $00 \to 10 \to 11 \to 00$, we can prove by a simple induction on n that $A_{t_i} \upharpoonright m_e = A_{t_{3n+i}} \upharpoonright m_e$ for all $n \in \mathbb{N}$ and $i = 0, 1, 2$. This implies obviously that

$$\Psi_{e,t_i}^{A_{t_i}} \upharpoonright \varphi_{e,t_i}(n_e) = \Psi_{e,t_{3n+i}}^{A_{t_{3n+i}}} \upharpoonright \varphi_{e,t_{3n+i}}(n_e)$$

and, because of $n_e < l(e, t_{3n+i})$, hence

$$B_{e,t_i} \upharpoonright \varphi_{e,t_i}(n_e) = B_{e,t_{3n+i}} \upharpoonright \varphi_{e,t_{3n+i}}(n_e).$$

Thus, we have $\varphi_{e,t_0}(n_e) = \varphi_{e,t_{3n}}(n_e)$ and $\psi_{e,t_0}(\varphi_{e,t_0}(n_e)) = \psi_{e,t_{3n}}(\varphi_{e,t_{3n}}(n_e))$ for all n. Since $A(n_e-1)(n_e)$ changes whenever R_e receives attention, $\Phi_e^{B_e}(n_e-1)(n_e)$ has to be changed before R_e receives a new attention. That is, $B_{e,t_n} \upharpoonright m_e \neq B_{e,t_{n+1}} \upharpoonright m_e$ for all n. By construction, the case $|x_{B_{e,t_n}} - x_{B_{e,t_{n+1}}}| \geq 2^{-m_e}$ can happen at most $h_e(m_e)$ times. If $|x_{B_{e,t_n}} - x_{B_{e,t_{n+1}}}| < 2^{-m_e}$, then we have $B_{e,t_n} \upharpoonright m_e = 0.w10\cdots0$ or $B_{e,t_n} \upharpoonright m_e = 0.w01\cdots1$ for some binary word w. That is, $B_{e,t_n} \upharpoonright m_e$ switches between two different forms. On the other hand, $A(n_e-1)(n_e)$ takes three different values consecutively. This means that after some stage, an agreement between A and $\Phi_e^{B_e}$ of a length bigger than n_e is impossible and R_e stops requiring further attention. Therefore, R_e receives attention only finitely often and it is satisfied eventually. $\square$

References

[1] K. Ambos-Spies, K. Weihrauch, and X. Zheng. Weakly computable real numbers. *Journal of Complexity*, 16(4):676–690, 2000.

[2] R. Downey, G. Wu, and X. Zheng. Degrees of d.c.e. reals. *Mathematical Logic Quartely*, 2004. (to appear).

[3] R. G. Downey. Some computability-theoretical aspects of real and randomness. Preprint, September 2001.

[4] R. G. Downey and D. R. Hirschfeldt. *Algorithmic Randomness and Complexity.* Springer-Verlag, 200? monograph to be published.

[5] A. J. Dunlop and M. B. Pour-El. The degree of unsolvability of a real number. In J. Blanck, V. Brattka, and P. Hertling, editors, *Computability and Complexity in Analysis*, volume 2064 of *LNCS*, pages 16–29, Berlin, 2001. Springer. CCA 2000, Swansea, UK, September 2000.

[6] C.-K. Ho. Relatively recursive reals and real functions. *Theoretical Computer Science*, 210:99–120, 1999.

[7] R. Rettinger, X. Zheng, R. Gengler, and B. von Braunmühl. Weakly computable real numbers and total computable real functions. In *Proceedings of COCOON 2001, Guilin, China, August 20-23, 2001*, volume 2108 of *LNCS*, pages 586–595. Springer, 2001.

[8] R. Rettinger, X. Zheng, and B. von Braunmühl. Computable real functions of bounded variation and semi-computable real numbers. In *Proceedings of COCOON 2002, Singapore, Aug. 15-17, 2002*, volume 2387 of *LNCS*, pages 47–56. Springer, 2002.

[9] R. M. Robinson. Review of "Peter, R., Rekursive Funktionen". *The Journal of Symbolic Logic*, 16:280–282, 1951.

[10] R. I. Soare. Cohesive sets and recursively enumerable Dedekind cuts. *Pacific J. Math.*, 31:215–231, 1969.

[11] A. M. Turing. On computable numbers, with an application to the "Entscheidungsproblem". *Proceedings of the London Mathematical Society*, 42(2):230–265, 1936.

[12] X. Zheng. Recursive approximability of real numbers. *Mathematical Logic Quarterly*, 48(Suppl. 1):131–156, 2002.

[13] X. Zheng. On the Turing degrees of weakly computable real numbers. *Journal of Logic and Computation*, 13(2):159–172, 2003.