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# From Varieties of Algebras to Covarieties of Coalgebras

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**Abstract**

Varieties of $F$-algebras with respect to an endofunctor $F$ on an arbitrary cocomplete category $\mathsf{C}$ are defined as equational classes admitting free algebras. They are shown to correspond precisely to the monadic categories over $\mathsf{C}$. Under suitable assumptions satisfied in particular by any endofunctor on $\mathsf{Set}$ and $\mathsf{Set}^{op}$ the Birkhoff Variety Theorem holds. By dualization, covarieties over complete categories $\mathsf{C}$ are introduced, which then correspond to the comonadic categories over $\mathsf{C}$, and allow for a characterization in dual terms of the Birkhoff Variety Theorem. Moreover, the well known conditions of accessibilitly and boundedness for $\mathsf{Set}$-functors $F$, sufficient for the existence of cofree $F$-coalgebras, are shown to be equivalent.

## Introduction

What is a variety? A classical answer is: an equationally presented class of finitary algebras (such as groups, lattices, etc). Less classical answer: an equationally presented class of algebras with infinitary operations of possibly unbounded arities (such as complete semilattices or compact Hausdorff spaces). The first, classical, case corresponds precisely to algebras of a finitary monad

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over $\mathsf{Set}$. The latter one, then, to algebras of an arbitrary monad over $\mathsf{Set}$. This, however, has nothing to do with $\mathsf{Set}$ as a base category: we are going to introduce equations for —and equational classes of—$F$-algebras, where $F$ is an endofunctor on any cocomplete category $\mathsf{C}$. Those equational classes that have free algebras are called *varieties*. They are proved to precisely correspond to monadic categories over $\mathsf{C}$. Although this is a folklore fact, it seems that it has never been really formulated. Our formulation is based on the construction of free $F$-algebras as a colimit of "term-objects" (which, in general, diverges: we do *not* assume that free $F$-algebras exist) presented by the first author in [2]. Functors $F$ for which free $F$-algebras exist are called *varietors* in [6]. For all varietors on $\mathsf{Set}$ (and all varietors on "reasonable" categories preserving regular epimorphisms) the Birkhoff Variety Theorem generalizes to the present context: varieties are precisely the full subcategories of $\mathsf{Alg}(F)$ which are closed under products, subalgebras and quotients.

What is a covariety? It is a simple but important observation that for every endofunctor on a category $\mathsf{C}$ the category $\mathsf{Coalg}(F)$ of all $F$-coalgebras is the dual of the category of $F^{op}$-algebras, where $F^{op}$ is the endofunctor on $\mathsf{C}^{op}$ acting as $F$. Consequently, by simple dualization we obtain the concept of coequation and coequational class of coalgebras. Those coequational classes that have cofree coalgebras (i.e., which are varieties of $F^{op}$-algebras) are called *covarieties*. And, for complete base categories, those are precisely the comonadic categories. If $F$ is a *covarietor*, i.e., if all cofree coalgebras exist, and $\mathsf{C} = \mathsf{Set}$ or $\mathsf{C}$ is "reasonable" and $F$ preserves regular monomorphisms, then covarieties are precisely the full subcategories of $\mathsf{Coalg}(F)$ which are closed under coproducts, quotients and subcoalgebras.

Covarieties—for bounded endofunctors on $\mathsf{Set}$ only—have been considered by various authors. The equivalence of our approach, when specialized to this particular case, to most of these concepts will be shown below.

Which Functors are (Co-)Varietors? Varietors on $\mathsf{Set}$ have been completely characterized in [6] by the existence of arbitrarily large fixed points. A full characterization of covarietors on $\mathsf{Set}$ is not known, but several sufficient conditions are known: M. Barr shows in [9] that every accessible functor is a covarietor, and Y. Kawahara and M. Mori [17] prove that every bounded functor has a final coalgebra (from which it follows that every bounded functor is a covarietor). In the present note we prove that accessibility is, in fact, equivalent to boundedness, and both are equivalent to $F$ being *small*, i.e., a small colimit of hom-functors. This seems to be the first time that the implication *accessible* $\Longrightarrow$ *small* has been properly proved (but see e.g. P. Freyd [11], which contains this result implicitly).

## 1 Algebras and coalgebras with respect to a functor

Let $F\colon \mathsf{C} \to \mathsf{C}$ be an endofunctor of some category $\mathsf{C}$. Categories $\mathsf{Alg}(F)$ and $\mathsf{Coalg}(F)$ are defined as follows.

Objects of $\mathsf{Alg}(F)$, called *$F$-algebras (over $\mathsf{C}$)*, are pairs $(C, \alpha_C)$ where $C$ is a $\mathsf{C}$-object and $\alpha_C \colon FC \to C$ is a $\mathsf{C}$-morphism. Morphisms $f \colon (C, \alpha_C) \to (D, \alpha_D)$ of $\mathsf{Alg}(F)$, called *$F$-algebra homomorphisms*, are $\mathsf{C}$-morphisms $f \colon C \to D$ making the diagram

$$\begin{tikzcd} FC \arrow[r, "Ff"] \arrow[d, "\alpha_C"'] & FD \arrow[d, "\alpha_D"] \\ C \arrow[r, "f"'] & D \end{tikzcd}$$

commute.

Objects of $\mathsf{Coalg}(F)$, called *$F$-coalgebras (over $\mathsf{C}$)*, are pairs $(C, \alpha_C)$ where $C$ is a $\mathsf{C}$-object and $\alpha_C \colon C \to FC$ is a $\mathsf{C}$-morphism. Morphisms $f \colon (C, \alpha_C) \to (D, \alpha_D)$ of $\mathsf{Coalg}(F)$, called *$F$-coalgebra homomorphisms*, are $\mathsf{C}$-morphisms $f \colon C \to D$ making the diagram

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\alpha_C"'] & D \arrow[d, "\alpha_D"] \\ FC \arrow[r, "Ff"'] & FD \end{tikzcd}$$

commute.

Composition and identities in $\mathsf{Alg}(F)$ and $\mathsf{Coalg}(F)$ respectively are those of $\mathsf{C}$.

$\mathsf{Alg}(F)$ and $\mathsf{Coalg}(F)$ are concrete categories over $\mathsf{C}$ in that they are equipped with canonical underlying functors

$${}_FU \colon \mathsf{Alg}(F) \to \mathsf{C} \quad \text{and} \quad U_F \colon \mathsf{Coalg}(F) \to \mathsf{C}$$

respectively[5].

The dual of a functor $F \colon \mathsf{C} \to \mathsf{D}$ is the functor $F^{op} \colon \mathsf{C}^{op} \to \mathsf{D}^{op}$ acting on objects and morphisms as $F$. With these notations one has

**1.1 Lemma** *For any functor $F \colon \mathsf{C} \to \mathsf{C}$ the following hold:*

1. $\mathsf{Coalg}(F) = (\mathsf{Alg}(F^{op}))^{op}$
2. $U_F = ({}_{F^{op}}U)^{op}$

**1.2 Example** Let $\Omega$ be a signature in Birkhoff's sense, i.e., $\Omega = (\Omega_n)_{n \in \mathbb{N}}$ is a countable family of sets $\Omega_n$. We shall describe the category $\mathsf{Alg}\Omega$ of $\Omega$-algebras—up to a concrete isomorphism— as $(\mathsf{Alg}(F),\ U)$ for a functor $F = F_\Omega \colon \mathsf{Set} \to \mathsf{Set}$ as follows:

$F_\Omega$ assigns to a set $X$ the set $\sum_{n \in \mathbb{N}} \Omega_n \times X^n$. Correspondingly $F_\Omega$ assigns to a map $f \colon X \to Y$ the map $\sum_{n \in \mathbb{N}} \Omega_n \times f^n$, i.e., the map $\sum_{n \in \mathbb{N}} \Omega_n \times X^n \to \sum_{n \in \mathbb{N}} \Omega_n \times Y^n$ mapping a pair $(\omega, (x_1, \ldots, x_n))$ to the pair $(\omega, (fx_1, \ldots, fx_n))$.

[5] Whenever confusion is unlikely to arise we will omit the subscript $F$.

Functors of the form $F_\Omega$ are called *polynomial functors.*

We collect a number of well known properties of categories of $F$-algebras in the case $\mathsf{C} = \mathsf{Set}$ as follows:

**1.3 Theorem** *For every endofunctor $F$ of* $\mathsf{Set}$ *the following hold:*

1. $\mathsf{Alg}(F)$ *has all limits and these are created by $U$.*
2. $\mathsf{Alg}(F)$ *has all colimits. Those which are preserved by $F$ are created by $U$.*
3. $\mathsf{Alg}(F)$ *has regular factorizations of homomorphisms; these are created by $U$.*

**1.4 Remark** The above properties hold in more general situations than just over $\mathsf{Set}$: as an inspection of the essentially standard proofs (see e.g. [3] or [6]) shows, the following properties of the category $\mathsf{C} = \mathsf{Set}$ and the functor $F$ respectively are needed only:

- $\mathsf{C}$ is complete, cocomplete, (regularly) co-wellpowered and has (regular epi, mono)-factorizations of homomorphisms.
- $F$ preserves (regular) epimorphisms.

These conditions can be assumed to be satisfied in particular for every endofunctor on the category $\mathsf{Set}^{op}$, too. One only has to recall the following result of V.Trnková (see [6, III.4.5-6]):

*Every endofunctor $F$ on* $\mathsf{Set}$ *either preserves monomorphisms, or there is a monomorphism-preserving functor $F'$ which coincides with $F$ on all non-empty sets and functions, and $F'\emptyset \neq \emptyset \neq F\emptyset$.*

It follows that, concerning categories $\mathsf{Alg}(F)$ and $\mathsf{Coalg}(F)$ over $\mathsf{Set}$, one always may assume that $F$ preserves monomorphisms: $\mathsf{Coalg}(F)$ and $\mathsf{Coalg}(F')$ are (concretely) isomorphic, while $\mathsf{Alg}(F) = \mathsf{Alg}(F')$ whenever $F$ fails to preserve monomorphisms.

By means of Lemma 1.1 one thus gets by dualization the following properties of categories of $F$-coalgebras:

**1.5 Theorem** *Let $F$ be an endofunctor of* $\mathsf{Set}$. *Then the following hold:*

1. $\mathsf{Coalg}(F)$ *has all colimits and these are created by $U$.*
2. $\mathsf{Coalg}(F)$ *has all limits. Those which are preserved by $F$ are created by $U$.*
3. $\mathsf{Coalg}(F)$ *has regular factorizations for homorphisms; these are created by $U$.*

## 2 Free algebras and cofree coalgebras

**2.1 Example** For polynomial endofunctors $F_\Omega$ on $\mathsf{Set}$, the concept of free algebra $X^\sharp$ on a set $X$ of generators is well known. We can describe it either recursively as $X^\sharp = \cup_{i<\omega} X_i^\sharp$ where

$$\begin{aligned} X_0^\sharp &= X + \Omega_0 \\ &= X + F_\Omega\emptyset \quad \text{terms of depths 0 are variables and nullary operations} \\ X_{i+1}^\sharp &= X + \{(\omega, t_0, \dots, t_{n-1}) \mid \omega \in \Omega_n, t_0, \dots, t_{n-1} \in X_i^\sharp\} \\ &= X + F_\Omega X_i^\sharp \quad \text{terms of depths i+1} \end{aligned}$$

Or directly: $X^\sharp$ is the algebra of all finite "properly" labeled trees. "Properly" means that a node with $n > 0$ children is labeled by an $n$-ary operation, and a leaf is labeled by a variable or a nullary operation. We have the universal arrow $\eta_X \colon X \to X^\sharp$, embedding $X$ into $X^\sharp$.

**2.2 Remark** Free $F$-algebras on $X$ for an object $X$ (of "variables") of $\mathsf{C}$ can be defined for all functors $F \colon \mathsf{C} \to \mathsf{C}$ as pairs consisting of an $F$-algebra

$$FX^\sharp \xrightarrow{\varphi_X} X^\sharp \quad \text{and a morphism} \quad \eta_X \colon X \to X^\sharp$$

with the universal property that given an $F$-algebra $(C, \alpha_C)$ and a morphism $f \colon X \to C$ of $\mathsf{C}$ there exists a unique $F$-homomorphism $f^\sharp$ extending $f$, i.e., such that the following diagram commutes.

$$\begin{tikzcd} FX^\sharp \arrow[r, "\varphi_X"] \arrow[d, "Ff^\sharp"'] & X^\sharp \arrow[d, "f^\sharp"] & X \arrow[l, "\eta_X"'] \arrow[dl, "f"] \\ FC \arrow[r, "\alpha_C"] & C & \end{tikzcd}$$

In other words, $\eta_X$ is a universal arrow of the forgetful functor $U \colon \mathsf{Alg}(F) \to \mathsf{C}$.

**2.3 Lemma** *Let $F \colon \mathsf{C} \to \mathsf{C}$ be a functor where $\mathsf{C}$ has finite coproducts and $X$ a $\mathsf{C}$-object. The following are equivalent for a morphism $\iota_X \colon X + FI_X \to I_X$ with components $\eta_X \colon X \to I_X$ and $\alpha_X \colon FI_X \to I_X$.*

*(i) $(I_X, \iota_X)$ is the initial algebra of type $X + F(-)$.*

*(ii) $(I_X, \alpha_X)$ is the free $F$-algebra on $X$ with universal morphism $\eta_X$.*

**2.4 Corollary** *For a free $F$-algebra $X^\sharp$ one has $X^\sharp \simeq X + FX^\sharp$.*

This is Lambek's Lemma (saying that initial $F$-algebras are fixed points of $F$) applied to $F_X = X + F(-)$.

It is good to have a name for endofunctors $F$ such that every object of $\mathsf{C}$ admits a free $F$-algebra, that is, such that $U$ has a left adjoint.

**2.5 Definition ([6])** An endofunctor $F \colon \mathsf{C} \to \mathsf{C}$ is called a *varietor* provided that a free $F$-algebra exists on every $\mathsf{C}$-object.

**2.6 Examples** 1. Polynomial endofunctors on $\mathsf{Set}$ are varietors.

2. If, more generally, $\mathsf{C}$ has colimits and finite products such that colimits of $\omega$-chains commute with finite products, then every polynomial endofunctor $F_\Omega$ on $\mathsf{C}$ is a varietor. In fact, it is easy to see that, for all $\Omega$, $F_\Omega$ preserves colimits of $\omega$-chains. And then free algebras can be obtained by the following

**2.7 Finitary Free-Algebra Construction** (see [2]): This is an application of the famous construction of an *initial $F$-algebra* (the free $F$-algebra on 0, an initial object of $\mathsf{C}$) as a colimit of the chain

$$0 \xrightarrow{!} F0 \xrightarrow{F!} F^2 0 \xrightarrow{F^2!} F^3 0 \cdots$$

to the functor $F_X = X + F(-)$ (see Lemma 2.3 above). Let $\mathsf{C}$ have countable colimits. Given an object $X$ in $\mathsf{C}$ we define an $\omega$-chain $X_i^\sharp$ $(i < \omega)$ as follows:

$$0 \xrightarrow{!} X + F0 \xrightarrow{X+F!} X + F(X + F0) \xrightarrow{X+F(X+F!)} (X + F(X + F(X + F0)) \cdots$$

That is:

- $X_0^\sharp = 0$, $X_1^\sharp = X + F0 = X + FX_0^\sharp$ and $x_{0,1}^\sharp = 0 \xrightarrow{!} X + F0$ is the unique morphism
- $X_{i+1}^\sharp = X + FX_i^\sharp$ and $x_{i+1,j+1}^\sharp = X + Fx_{i,j}^\sharp$ for all $i \leq j$

**Claim:** If $F$—and thus $X + F$—preserve a colimit $X^\sharp = \mathsf{colim}_{i<\omega} X_i^\sharp$ of the above chain, then $X^\sharp$ is a free $F$-algebra on $X$. More detailed: suppose $(X_i^\sharp \xrightarrow{x_i} X^\sharp)$ is a colimit cocone. If $X + F$ preserves that colimit we have a unique morphism

$$\varphi_X \colon X + FX^\sharp \to X^\sharp \quad \text{with} \quad \varphi_X \circ (X + Fx_i) = x_{i+1}$$

The two components $\eta_X \colon X \to X^\sharp$ and $\alpha_X \colon FX^\sharp \to X^\sharp$ of $\varphi_X$ form a free $F$-algebra on $X$.

**Proof** For every $F$-algebra $(C, \alpha_C)$ and any morphism ("assignment to variables") $f \colon X \to C$ define a cocone of the above chain ( *computation of terms*) recursively as follows:

$$f_0^\sharp = ! \quad \text{and} \quad f_{i+1}^\sharp = [f, \alpha_C \circ Ff_i^\sharp]$$

Then the (unique) factorization $X_i^\sharp \xrightarrow{x_i} X^\sharp \xrightarrow{f^\sharp} C = f_i^\sharp$ gives the (unique) homomorphism $f^\sharp \colon (X^\sharp, \alpha_X) \to (C, \alpha_C)$ with $f = f^\sharp \circ \eta_X$. ◇

**2.8 Examples** 1. For the endofunctor $FY = 1 + Y \times Y$ (i.e., one constant and one binary operation) on $\mathsf{Set}$, we know that the terms in $X_i^\sharp$ are just the binary trees of depths $\leq i$ labelled in $X + 1$. This corresponds precisely to the construction above.

2. For the endofunctor $FY = Y^{\mathbb{N}}$ (i.e., one $\omega$-ary operation) on $\mathsf{Set}$ we again might form the sets $X_i^\sharp$ of terms, but here the colimit after $\omega$ steps does not give a free $F$-algebra, of course: we need $\omega_1$ steps of the following

**2.9 Free-Algebra Construction** (see [2] or [6, IV.3.2]): Let $\mathsf{C}$ be a cocomplete category. For every endofunctor $F$ on $\mathsf{C}$ and every object $X$ ("of variables") in $\mathsf{C}$ define a transfinite chain of objects $X_i^\sharp$ ($i$ any ordinal) and connecting morphisms

$$x_{i,j}^\sharp \colon X_i^\sharp \to X_j^\sharp \ (i \leq j)$$

by the following transfinite induction:

- $X_0^\sharp = 0$, $X_1^\sharp = X + F0$ with $x_{0,1}^\sharp$ being the unique morphism $0 \xrightarrow{!} X + F0$
- $X_{i+1}^\sharp = X + FX_i^\sharp$ for all ordinals $i$, $x_{i+1,j+1}^\sharp = X + Fx_{i,j}^\sharp$ for all $i \leq j$
- $X_j^\sharp = \mathsf{colim}_{i<j} X_i^\sharp$ for all limit ordinals $j$ with colimit cocone $x_{i,j}^\sharp$, $i < j$.

**Claim:** If the above chain construction *stops after $k$ steps*, i.e, if $k$ is an ordinal such that $x_{k,k+1}^\sharp \colon X_k^\sharp \to X + FX_k^\sharp$ is an isomorphism, then $X_k^\sharp$ is a free $F$-algebra on $X$. More detailed: Denoting the inverse of $x_{k,k+1}$ by $\varphi_X$ with components

$$\alpha_X \colon FX_k^\sharp \to X_k^\sharp \quad \text{and} \quad \eta_X \colon X \to X_k^\sharp$$

these form a free $F$-algebra on $X$.

**Proof** Given an $F$-algebra $(C, \alpha_C)$ and a morphism $f \colon X \to C$ we define a cocone $f_i^\sharp \colon X_i^\sharp \to C$ ($i$ an ordinal) by transfinite induction as above (leaving out the limit steps; compatibility $f_j \circ x_{i,j}^\sharp = f_i$ $(i < j)$ implies that the $f_i$ $(i < j)$ determine $f_j$ for limit ordinals $j$):

$$f_0^\sharp = ! \quad \text{and} \quad f_{i+1}^\sharp = [f, \alpha_C \circ Ff_i^\sharp]$$

Now $f_k^\sharp \colon X_k^\sharp \to C$ is the unique homomorphism with $f = f_k^\sharp \circ \eta_X$. ◇

**2.10 Definition ([6])** A functor $F \colon \mathsf{C} \to \mathsf{C}$ is called *constructive varietor* provided that its Free-Algebra Construction 2.9 stops for each $\mathsf{C}$-object $X$.

A functor can be a varietor, though the above chain-construction fails to stop for every $X$ (see e.g. [6, IV.3.A]). However we have the following results:

**2.11** *An endofunctor $F$ on a cocomplete category which preserves colimits of $\lambda$-chains for some infinite cardinal $\lambda$ is a constructive varietor.*

In fact, if $F$ preserves $X_\lambda^\sharp = \mathsf{colim}_{j<\lambda} X_j^\sharp$, then the free algebra construction stops after $\lambda$ steps.

**2.12 Theorem ([6, 4.3], [4])** *Every varietor on each of the categories* $\mathsf{Set}$, $\mathsf{Set}^{op}$, $\mathsf{Vec}_k$ [6] *and* $\mathsf{Vec}_k^{op}$ *is a constructive varietor.*

[6] This is the category of vector spaces over some field $k$.

Calling an endofunctor $F$ on $\mathsf{Set}$ *trivial* iff $F$ is constant on nonempty sets one can prove (note that trivial endofunctors clearly are varietors):

**2.13 Theorem ([6])** *A non-trivial endofunctor $F$ on $\mathsf{Set}$ is a (constructive) varietor iff $F$ has arbitrarily large fixed points.*

**2.14 Cofree coalgebras** are the corresponding dualization of free algebras. A cofree $F$-coalgebra (with respect to a functor $F\colon \mathsf{C} \to \mathsf{C}$) on a $\mathsf{C}$- object $X$ ("of colours") is a coalgebra $\psi_X\colon X_\sharp \to FX_\sharp$ together with a ("colouring") morphism $\rho_X\colon X_\sharp \to X$ having the universal property that given an $F$-coalgebra $(C, \alpha_C)$ and a morphism $f\colon C \to X$ of $\mathsf{C}$ there exists a unique $F$-coalgebra homomorphism $f_\sharp$ extending $f$, i.e., such that the diagram

$$\begin{tikzcd} & C \arrow[r, "\alpha_C"] \arrow[dl, "f"'] \arrow[d, "f_\sharp"] & FC \arrow[d, "Ff_\sharp"] \\ X & X_\sharp \arrow[l, "\rho_X"] \arrow[r, "\psi_X"] & FX_\sharp \end{tikzcd}$$

commutes. In other words, $\rho_X$ is a couniversal arrow of the forgetful functor $U\colon \mathsf{Coalg}(F) \to \mathsf{C}$.

**2.15 Definition** An endofunctor $F\colon \mathsf{C} \to \mathsf{C}$ is called a *covarietor* provided that a cofree $F$-coalgebra exists on every $\mathsf{C}$-object.

This terminology is justified by the following remark based on 1.1 and 2.3.

**2.16 Remark** The following are equivalent for any $F\colon \mathsf{C} \to \mathsf{C}$:

- $F$ is a covarietor.
- $F^{op}$ is a varietor.

In case $\mathsf{C}$ has finite products, another equivalent condition is:

- For every object $X$ in $\mathsf{C}$ the functor $F^X = X \times F$ has a terminal (= final) coalgebra.

Dualization of the free-algebra construction above gives the following

**2.17 Cofree Coalgebra Construction:** Let $\mathsf{C}$ be a complete category. For every endofunctor $F$ on $\mathsf{C}$ and every object $X$ ("of colours") in $\mathsf{C}$ define a transfinite cochain of objects $X_\sharp^i$ ($i$ any ordinal) and connecting morphisms $x_\sharp^{i,j}\colon X_\sharp^i \to X_\sharp^j$ ($i \geq j$) as follows (where 1 denotes a terminal object of $\mathsf{C}$):

- $X_\sharp^0 = 1,\ X_\sharp^i = X \times F1$ with $x_\sharp^{1,0}\colon X \times F1 \xrightarrow{!} 1$ the unique morphism
- $X_\sharp^{i+1} = X \times FX_\sharp^i$ for all ordinals $i$, $x_\sharp^{i+1,j+1} = X \times Fx_\sharp^{i,j}$ for all $i \geq j$
- $X_\sharp^j = \mathsf{lim}_{i<j} X_\sharp^i$ for all limit ordinals $j$ with limit cone $x_\sharp^{j,i}$, $i < j$.

If this cochain construction *stops after $k$ steps*, i.e, if $k$ is an ordinal such that $x_\sharp^{k,k+1} \colon X \times FX_\sharp^k \to X_\sharp^k$ is an isomorphism, then $X_\sharp^k$ is a cofree $F$-coalgebra on $X$. More detailed: Denoting the inverse of $x^{k,k+1}$ by $\varphi_X \colon X_\sharp^k \to X \times FX_\sharp^k$ with components

$$\alpha_X \colon X_\sharp^k \to FX_\sharp^k \quad \text{and} \quad \rho_X \colon X_\sharp^k \to X$$

these form a cofree $F$-coalgebra on $X$. For an $F$-coalgebra $(C, \alpha_C)$ and a morphism $f \colon C \to X$ the extension $f_\sharp$ of $f$ is the $k$-th member of the cocone $f_\sharp^i \colon C \to X_\sharp^i$ which is defined by transfinite induction (leaving out the limit steps) as follows:

$$f_\sharp^0 = ! \quad \text{and} \quad f_\sharp^{i+1} = \langle f, Ff_\sharp^i \circ \alpha_C \rangle.$$

**2.18 Definition** A functor $F \colon \mathsf{C} \to \mathsf{C}$ is called *constructive covarietor* provided that its Cofree-Coalgebra Construction 2.17 stops for each $\mathsf{C}$-object $X$.

As the dual of Corollary 2.11 the following holds:

**2.19 Corollary** *An endofunctor $F$ on a complete category which preserves limits of $\lambda$-cochains for some infinite cardinal $\lambda$ is a constructive covarietor.*

**2.20 Examples** 1. Polynomial functors on $\mathsf{Set}$ are covarietors (here $\lambda = \omega$).

2. *Generalized polynomial functors on* $\mathsf{Set}$, i.e., functors $FY = \sum_{i \in I} Y^{C_i}$ for a given family $(C_i)_{i \in I}$ of (not necessarily finite) sets are covarietors (again, $\lambda = \omega$).
3. Every endofunctor on $\mathsf{Set}$ which has arbitrarily large *exponential fixed points* (i.e., there are arbitrarily large sets $X$ such that each set $Y$ with $\mathsf{card} X \leq \mathsf{card} Y \leq \mathsf{card}\ \mathsf{exp} X$ is a fixed point of $F$) is a covarietor (see [4]). Compare with Theorem 2.13.

## 3 Varieties and Covarieties

The following definition generalizes concepts from [1]:

**3.1 Definitions** Let $F$ be an endofunctor of a cocomplete category $\mathsf{C}$. Using the notation $X_i^\sharp$ and $f_i^\sharp$ as in 2.9 we define:

1. An *equation arrow over* $X$ is a regular epimorphism $e \colon X_i^\sharp \to E$ for some ordinal $i$. An $F$-algebra $(C, \alpha_C)$ is said to *satisfy* $e$ provided that for every morphism $f \colon X \to C$ the morphism $f_i^\sharp$ factors through $e$:

$$\begin{tikzcd} X_i^\sharp \arrow[r, "e"] \arrow[dr, "f_i^\sharp"'] & E \arrow[d, dashed] \\ & C \end{tikzcd}$$

2. For any class $\mathcal{E}$ of equation arrows, $\mathsf{Alg}(F, \mathcal{E})$ denotes the full subcategory of $\mathsf{Alg}(F)$ spanned by all $F$-algebras satisfying every $e \in \mathcal{E}$. Such categories are called *equational categories (of $F$-algebras)* over $\mathsf{C}$.
3. An equational category $\mathsf{Alg}(F, \mathcal{E})$ over $\mathsf{C}$ will be called a *variety (of $F$-algebras)* over $\mathsf{C}$ provided that the underlying functor

$$U_{\mathcal{E}} = U|_{\mathsf{Alg}(F,\mathcal{E})} \colon \mathsf{Alg}(F, \mathcal{E}) \to \mathsf{C}$$

has a left adjoint.

**3.2 Remarks** 1. Equations in classical (finitary) universal algebra are pairs of terms, i.e, parallel pairs of morphisms

$$u, v \colon 1 \to X^\sharp = X^\sharp_\omega$$

An algebra $(C, \alpha_C)$ satisfies this equation iff for every morphism $f \colon X \to C$ the unique homomorphism $f^\sharp = f^\sharp_\omega$ extending $f$ merges $u$ and $v$, i.e,

$$f^\sharp \circ u = f^\sharp \circ v.$$

This is equivalent to the satisfaction, in the above sense, of the equation arrow $e \colon X^\sharp \to E$ which is a coequalizer of $u$ and $v$.

In general, every pair of parallel morphisms (with $\mathsf{C}$-objects $A, X$ and an ordinal $i$)

$$u, v \colon A \to X^\sharp_i$$

in $\mathsf{C}$ defines an equation arrow $e \colon X^\sharp_i \to E$, a coequalizer of $u$ and $v$. An algebra $(C, \alpha_C)$ "satisfies $u = v$" (in the expected sense: for every morphism $f \colon X \to C$ we have $f^\sharp_i \circ u = f^\sharp_i \circ v$) iff $(C, \alpha_C)$ satisfies $e$ in the above sense.

2. If the base category $\mathsf{C}$ has kernel pairs, then, conversely, equation arrows can be substituted by parallel pairs: given a regular epimorphism $e \colon X^\sharp_i \to E$, denote by $u, v \colon A \to X^\sharp_i$ a kernel pair of $e$. Then an algebra satisfies $u = v$ iff it satisfies $e$.

   Observe here that the index $i$ can be upgraded arbitrarily: given a parallel pair $u, v \colon A \to X^\sharp_i$ and an ordinal $j > i$, put $u' = x^\sharp_{i,j} u$ and $v' = x^\sharp_{i,j} v$. Then the equations $u = v$ and $u' = v'$ are satisfied by the same algebras.
3. Let $\mathsf{C}$ have kernel pairs and let $F$ be a *constructive* varietor. It follows from 2. that all equations we have to consider are of the form

$$u, v \colon A \to X^\sharp$$

for objects $A, X$ in $\mathsf{C}$: in fact, upgrade any given parallel pair to an ordinal $j$ such that $X^\sharp = X^\sharp_j$. Here, the satisfaction of $u = v$ by $(C, \alpha_C)$ means that for every $f \colon X \to C$ the unique homomorphism $f^\sharp \colon (X^\sharp, \varphi_X) \to$

$(C, \alpha_C)$ merges $u$ and $v$. Since all homomorphisms $h$ on $(X^\sharp, \varphi_X)$ have the form $h = f^\sharp$ (for $f = h \circ \eta_X$), we see that $(C, \alpha_C)$ satisfies $u = v$ iff $hu = hv$ for all homorphisms $h\colon (X^\sharp, \varphi_X) \to (C, \alpha_C)$. Thus we can, equivalently, work with equation arrows

$$e\colon X^\sharp \to E$$

which are regular quotients, in $\mathsf{C}$, of free $F$-algebras.

4. Suppose that $F$ is a constructive varietor such that $\mathsf{Alg}(F)$ has coequalizers (e.g., whenever $\mathsf{C}$ has kernel pairs and regular factorizations of morphisms, is regularly cowellpowered and $F$ preserves regular epimorphisms, see Remark 1.4). Then instead of regular epimorphisms $e\colon X^\sharp \to E$ in $\mathsf{C}$ we can work with regular epimorphisms in $\mathsf{Alg}(F)$. For that purpose recall the following notions:
   (a) An object $I$ in a category $\mathsf{C}$ is called *injective* w.r.t. a given morphism $e\colon C \to D$ provided that each morphism $h\colon C \to I$ factorizes over $e$:

$$\begin{tikzcd} h\colon C \arrow[r, "e"] \arrow[dr, "h"'] & D \arrow[d, dashed] \\ & I \end{tikzcd}$$

   (b) Given a class $\mathcal{E}$ of homomorphisms, we denote by $\mathsf{Inj}\mathcal{E}$ the *injectivity class of* $\mathcal{E}$, i.e., the full subcategory of $\mathsf{Alg}(F)$ spanned by all algebras injective w.r.t. each $e \in \mathcal{E}$.
   (c) The injectives w.r.t. all regular monomorphisms are called the *regular injectives.*
   (d) The dual notions are *(regular) projective* and *projectivity class* $\mathsf{Proj}\mathcal{E}$.

   Now observe that for every equation $u, v\colon A \to X^\sharp$ we have a new equation $u^\sharp, v^\sharp\colon A^\sharp \to X^\sharp$ which is satisfied by precisely the same algebras $(C, \alpha_C)$ (because, given a homomorphism $h\colon (X^\sharp, \varphi_X) \to (C, \alpha_C)$, then $(hu)^\sharp = hu^\sharp$ and $(hv)^\sharp = hv^\sharp$). Thus, if $\bar{e}\colon (X^\sharp, \varphi_X) \to (\bar{E}, \alpha_{\bar{E}})$ denotes a coequalizer of $u^\sharp$ and $v^\sharp$ in $\mathsf{Alg}(F)$, then for every algebra $(C, \alpha_C)$ we have

$$(C, \alpha_C) \text{ satisfies } e \iff (C, \alpha_C) \text{ is injective w.r.t. } \bar{e}.$$

5. Consider the base category $\mathsf{C} = \mathsf{Set}$. It then follows from 4. that, for any varietor $F\colon \mathsf{Set} \to \mathsf{Set}$,

   every variety of $F$-algebras is specified by injectivity to regular epimorphisms of $\mathsf{Alg}(F)$ with regularly projective domains[7].

   The converse is also true:

   every class of $F$-algebras specified by injectivity to regular epimorphisms of $\mathsf{Alg}(F)$ with regularly projective domains is a variety.

[7] Every free $F$-algebra over $\mathsf{Set}$ is regularly projective by the axiom of choice.

In fact, consider such an epimorphism, $e\colon (D,\alpha_D)\to(E,\alpha_E)$, in $\mathsf{Alg}(F)$. Since Since the homomorphism $id^\sharp\colon (D^\sharp,\varphi_D)\to(D,\alpha_D)$ is a regular epimorphism in $\mathsf{Alg}(F)$ and $(D,\alpha_D)$ is regularly projective we have a homomorphism

$$m\colon (D,\alpha_D)\to(D^\sharp,\varphi_D)\ \text{ with }\ id^\sharp\circ m=id.$$

Choose a pair of homomorphisms $u,v$ with coequalizer $e$ in $\mathsf{Alg}(F)$. Then an algebra $(C,\alpha_C)$ is orthogonal to $e$ iff for every homomorphism $h\colon (D,\alpha_D)\to(C,\alpha_C)$ we have $h\circ u=h\circ v$. This is equivalent to stating that for every homomorphism $k\colon (D^\sharp,\varphi_D)\to(C,\alpha_C)$ we have $k\circ(m\circ u)=k\circ(m\circ v)$: given $k$, put $h=k\circ m$, and given $h$, put $k=h\circ id^\sharp$. Thus, if $\bar e\colon (D^\sharp,\varphi_D)\to(\bar E,\alpha_{\bar E})$ denotes a coequalizer of $m\circ u$ and $m\circ v$ in $\mathsf{Alg}(F)$, then injectivity to $\bar e$ and $e$, respectively, is equivalent. (And the former can be substituted by the equations $u_0=v_0$ obtained by the kernel pair of $\bar e$.)

This concept of equation and its satisfaction has already been considered by H. Herrlich and his co-authors in [16] and [8].

**3.3 Example** The power-set functor $\mathcal{P}$ on $\mathsf{Set}$ is not a varietor. However, we can consider equational categories of $\mathcal{P}$-algebras. Complete semilattices are an example. In fact, the join-operation of a complete (upper) semilattice C is an arrow $\alpha_C:\mathcal{P}C\to C$ satisfying (i) $\alpha_C\{x\}=x$, and (ii) $\alpha_C\bigcup M_i=\alpha_C\{\alpha_C M_i\mid\ i\in I\}$ for any collection $M_i$ in $\mathcal{P}C$. Conversely, every $\mathcal{P}$-algebra satisfying (i) and (ii) is a (join operation of a unique) complete semilattice. Now (i) can be expressed by the equation arrow $e\colon X_2^\sharp\to E$ where $X=\{x\}$ and $e$ just merges $x$ and $\{x\}$, whereas (ii) corresponds to the equation arrows $f\colon X_3^\sharp\to F$ where $X$ is an arbitrary set and, for a given collection $M_i$ in $\mathcal{P}X$, $f$ merges $\bigcup M_i$ with $\{M_i\mid i\in I\}$. The homomorphisms are precisely the functions preserving all joins.

The following lemma—to be proven by an easy transfinite induction—will be used frequently:

**3.4 Lemma** *Homomorphisms of $F$-algebras preserve computation of terms, i.e., given a homomorphism $h\colon (C,\alpha_C)\to(D,\alpha_D)$ and an assignment of variables $f\colon X\to C$ then, for all ordinals $i$, $(h\circ f)_i^\sharp=h\circ f_i^\sharp$.*

**3.5 Proposition** $\mathsf{Alg}(F,\mathcal{E})$ *is always closed in* $\mathsf{Alg}(F)$ *under*

1. *subalgebras and all limits which exist;*
2. *homomorphic images carried by split epimorphisms in* $\mathsf{C}$.

**Proof** 1. is trivial by an obvious diagonal fill-in argument.
2. Let $r\colon (C,\alpha_C)\to(D,\alpha_D)$ be a homomorphism with coretraction $s$ in $\mathsf{C}$, where $(C,\alpha_C)$ satisfies the equation arrow $e\colon X_i^\sharp\to E$. Given $f\colon X\to D$, one

has $r \circ (s \circ f)_i^\sharp = (r \circ s \circ f)_i^\sharp = f_i^\sharp$. Thus, since $(s \circ f)_i^\sharp$ factorizes through $e$ so does $f_i^\sharp$. $\diamond$

**3.6 Theorem** *Monadic categories over a cocomplete category* $\mathsf{C}$ *are precisely the categories concretely equivalent to varieties over* $\mathsf{C}$.

**Proof** I. Sufficiency: By Beck's Theorem we have to verify that the underlying functor of a variety $\mathsf{Alg}(F, \mathcal{E})$ creates split coequalizers. Since $\mathsf{Alg}(F, \mathcal{E})$ is closed under quotients splitting in $\mathsf{C}$, it suffices to prove that $U \colon \mathsf{Alg}(F) \to \mathsf{C}$ creates absolute coequalizers. This is proved exactly as in the proof of Becks's Theorem (see [18]).

II. For the converse it suffices to show that, for any monad $\mathbb{T} = (T, \eta, \mu)$ on a cocomplete category $\mathsf{C}$, the Eilenberg-Moore category $\mathsf{C}^{\mathbb{T}}$ of $\mathbb{T}$-algebras coincides with the subcategory $\mathsf{Alg}(T, \mathcal{E})$ of $\mathsf{Alg}(T)$ for a suitable class $\mathcal{E}$ of equation arrows. For doing so consider, for every $\mathsf{C}$-object $X$, the coproduct

$$X \xrightarrow{m_{i+1}} X + TX_i^\sharp = X_{i+1}^\sharp \xleftarrow{n_{i+1}} TX_i^\sharp.$$

A class $\mathcal{E}_1$ of equation arrows now is defined as follows: for every $\mathsf{C}$-object $X$ let $e_X \colon X_2^\sharp \to E_X$ be a coequalizer of the pair $m_2, n_2 \circ \eta_{X_1^\sharp} \circ m_1$. A $T$-algebra $(C, \alpha_C)$ satisfies $e_X$ iff, for every morphism $f \colon X \to C$, the morphism

$$f_2^\sharp = [f, \alpha_C \circ Tf_1^\sharp] \colon X + TX_1^\sharp \to C$$

satisfies $f_2^\sharp \circ m_2 = f_2^\sharp \circ n_2 \circ \eta_{X_1^\sharp} \circ m_1$ or, equivalently, $f = \alpha_C \circ Tf_1^\sharp \circ \eta_{X_1^\sharp} \circ m_1$. Since $\eta$ is natural, this is equivalent to $f = \alpha_C \circ \eta_C \circ f$ which, for $X = C$ and $f = 1_C$ yields satisfaction of the $\mathbb{T}$-algebra axiom $\alpha_C \circ \eta_C = 1_C$. Conversely, $\alpha_C \circ \eta_C = 1_C$ yields $f = \alpha_C \circ \eta_C \circ f$ by composition with $f$. Thus, the satisfaction of $\mathcal{E}_1$ is equivalent $\alpha_C \circ \eta_C = 1_C$.

Next we define a class $\mathcal{E}_2$ of equation arrows as follows: for every $\mathsf{C}$-object $X$ let $d_X \colon X + TX_2^\sharp = X_3^\sharp \to D_X$ be a coequalizer of the pair $n_3 \circ Tm_2 \circ \mu_X, n_3 \circ Tn_2 \circ T^2 m_1$.

$$\begin{tikzcd}
& TX \arrow[dr, "Tm_2"] & & & \\
T^2X \arrow[ur, "\mu_X"] \arrow[dr, "T^2 m_1"'] & & TX_2^\sharp \arrow[r, "n_3"] & X + TX_2^\sharp \arrow[r, "d_X"] & D_X \\
& T^2 X_1^\sharp \arrow[ur, "Tn_2"'] & & &
\end{tikzcd}$$

A $T$-algebra $(C, \alpha_C)$ satisfies $d_X$ iff, for every morphism $f \colon X \to C$, the morphism

$$f_3^\sharp = [f, \alpha_C \circ T[f, \alpha_C \circ Tf_1^\sharp]] \colon X + TX_2^\sharp \to C$$

satisfies $f_3^\sharp \circ n_3 \circ Tn_2 \circ T^2 m_1 = f_3^\sharp \circ n_3 \circ Tm_2 \circ \mu_X$. This is equivalent to $\alpha_C \circ T\alpha_C \circ T^2 f = \alpha_C \circ Tf \circ \mu_X$ or, since $\mu$ is natural, to $\alpha_C \circ T\alpha_C \circ T^2 f =$

$\alpha_C \circ \mu_C \circ T^2 f$. Choosing $f = 1_C$ this yields satisfaction of the $\mathbb{T}$-algebra axiom $\alpha_C \circ T\alpha_C = \alpha_C \circ \mu_C$. Conversely, $\alpha_C \circ T\alpha_C = \alpha_C \circ \mu_C$ yields $\alpha_C \circ T\alpha_C \circ T^2 f = \alpha_C \circ \mu_C \circ T^2 f$ by composition with $T^2 f$. Thus, the satisfaction of $\mathcal{E}_2$ is equivalent $\alpha_C \circ T\alpha_C = \alpha_C \circ \mu_C$.

Chosing $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$ one thus gets $\mathsf{C}^{\mathbb{T}} = \mathsf{Alg}(T, \mathcal{E})$. $\diamond$

**3.7 Theorem** *Let $\mathsf{C}$ be a cocomplete, regularly co-wellpowered category with regular factorizations and kernel pairs. If $F\colon \mathsf{C} \to \mathsf{C}$ is a constructive varietor preserving regular epimorphisms, the following are equivalent for any full subcategory $\mathsf{K}$ of $\mathsf{Alg}(F)$:*

*(i)* *$\mathsf{K}$ is a variety.*

*(ii)* *$\mathsf{K}$ is closed under subalgebras, products and homomorphic images carried by split epimorphisms.*

**Proof** In view of Proposition 3.5 we only have to show that (ii) implies (i). By Remark 1.4, $\mathsf{Alg}(F)$ has regular factorizations. Thus (ii) implies that $\mathsf{K}$ is a reflective subcategory whose reflection-arrows are regular epimorphisms in $\mathsf{Alg}(F)$, see [3, 16.8]. Let now $\mathcal{E}$ be the class of all reflection-arrows of free algebras. We claim $\mathsf{K} = \mathsf{Inj}\mathcal{E}$. Trivially each algebra in $\mathsf{K}$ is injective w.r.t. all reflection arrows. Conversely, if $(C, \alpha_C)$ is injective w.r.t. the reflection $r$ of the free algebra $(F, \alpha_F)$ over $C$, then the homomorphic extension $id_C^\sharp$ of the identity of $C$ factors as $id_C^\sharp = g \circ r$. This shows that $g$ is—as a $\mathsf{C}$-morphism—a retraction. Thus, $(C, \alpha_C)$ is a split-epi carried quotient of the $\mathsf{K}$-reflection of $(F, \alpha_F)$, hence belonging to $\mathsf{K}$ by hypothesis. Thus, $\mathsf{K} = \mathsf{Inj}\mathcal{E}$. From Remark 3.2.4 above we conclude that $\mathsf{K}$ is a variety. $\diamond$

**3.8 Corollary (Birkhoff Variety Theorem)** *For every varietor $F$ on the category $\mathsf{Set}$, varieties of $F$-algebras are precisely the full subcategories closed under products, subalgebras and homomorphic images.*

**3.9 Remark** In the special case $\mathsf{C} = \mathsf{Set}$ it suffices, in the proof of Theorem 3.7 above, to take as $\mathcal{E}$ the *set* of reflection arrows of free algebras on sets of cardinality less than $\lambda$ ($\lambda$ a regular cardinal), provided that $F$ preserves $\lambda$-directed colimits (see e.g. proof of [5, 3.9]).

By formally dualizing Definition 3.1, see Lemma 1.1, we obtain the following

**3.10 Definitions** Let $F$ be an endofunctor of a complete category $\mathsf{C}$.

1. An *coequation arrow over* $X$ is a regular monomorphism $m\colon M \to X^i_\sharp$ for some ordinal $i$. An $F$-coalgebra $(C, \alpha_C)$ is said to *satisfy* $m$ provided that for every morphism $f\colon C \to X$ the morphism $f^i_\sharp$ factors through $m$.

2. For any class $\mathcal{M}$ of coequation arrows $\mathsf{Coalg}(F,\mathcal{M})$ is the full subcategory of $\mathsf{Coalg}(F)$ spanned by all $F$-coalgebras satisfying every $m \in \mathcal{M}$. Such categories are called *coequational categories (of $F$-coalgebras)* over $\mathsf{C}$.
3. A coequational category $\mathsf{Coalg}(F,\mathcal{M})$ will be called a *covariety (of $F$-coalgebras)* over $\mathsf{C}$ provided that the underlying functor

$$U^{\mathcal{M}} = U|_{\mathsf{Coalg}(F,\mathcal{M})}\colon \mathsf{Coalg}(F,\mathcal{M}) \to \mathsf{C}$$

has a right adjoint (that is, if $\mathsf{Alg}(F^{op},\mathcal{M})$ is a variety over $\mathsf{C}^{op}$).

By dualizing the respective results on equational categories and varieties we immediately obtain the following results.

**3.11 Corollary** *Comonadic categories over a complete category* $\mathsf{C}$ *are precisely the categories concretely equivalent to covarieties over* $\mathsf{C}$.

**3.12 Corollary (Birkhoff Covariety Theorem)** *For every covarietor $F$ on the category* $\mathsf{Set}$*, covarieties of $F$-algebras are precisely the full subcategories closed under coproducts, subalgebras and homomorphic images*[8] *.*

**3.13 Remarks** 1. Clearly, by duality, an equivalent condition for a full subcategory of $\mathsf{Coalg}(F)$ (over $\mathsf{Set}$) to be a covariety is to be a projectivity class w.r.t. of some class of regular monomorphisms $\mathsf{M}$ with cofree codomains.

2. Moreover, as in the case of varieties over $\mathsf{Set}$ (see Remark 3.9), also covarieties of $F$-coalgebras over $\mathsf{Set}$ can be specified by projectivity w.r.t. a *set* of regular monomorphisms—in fact a single one—with cofree codomains, provided that the functor $F$ preserves $\lambda$-directed colimits for some regular cardinal $\lambda$. This follows easily from the boundedness property (see Theorem 4.1 below) of these functors (see [20,12]). Note, however, that this observation *cannot* be obtained by dualization of Remark 3.9.

While Theorem 3.11 shows that the dual of a covariety over $\mathsf{Set}$ is a variety over $\mathsf{Set}^{op}$ it moreover implies the following additional dualization principle:

**3.14 Proposition** *The dual of a covariety over* $\mathsf{Set}$ *is equivalent to a variety over* $\mathsf{Set}$.

**Proof** By means of the contravariant power-set functor $\mathcal{P}'$ the category $\mathsf{Set}^{op}$ is monadic over $\mathsf{Set}$. Let $V\colon \mathsf{Coalg}(F,\mathcal{M}) \to \mathsf{Set}$ be the composite of $(U^{\mathcal{M}})^{op}$ and $\mathcal{P}'$. We need to show that $V$ is monadic. Since $V$ has a left adjoint and creates limits it suffices to prove that $V$ creates coequalizers of congruence relations (= kernel pairs). Hence let $r, s\colon (C,\alpha_C) \to (D,\alpha_D)$ be a pair of $\mathsf{Coalg}(F,\mathcal{M})$-morphisms sich that $Vr, Vs$ is a congruence relation and

[8] Observe that, in $\mathsf{Coalg}(F)$, the homorphic images are given by (plain) epimorphims while the embeddings of subalgebras are the regular monomorphisms.

let $q\colon \mathcal{P}'(D) \to X$ be its coequalizer. Since $\mathcal{P}'$ reflects congruence relations and creates their coequalizers there is a unique $\mathsf{Set}^{op}$-morphism $q'\colon D \to X'$ with $\mathcal{P}'(q') = q$ and this is a coequalizer of the congruence relation $U^{\mathcal{M}}r, U^{\mathcal{M}}s$. If $X' \neq \emptyset$ this will even be a split coequalizer such that $U^{\mathcal{M}}$ creates from it a coequalizer of $r, s$. The remaining case $X' = \emptyset$ is trivial: the unique $F$-coalgebra structure on $\emptyset$ obviously does the job. $\diamond$

**3.15 Remarks** 1. Coequations and their satisfaction have the following simple interpretation in the case of coalgebras over $\mathsf{Set}$: define, for every "coterm" $x \in X^i_\sharp$, the coequation $[x]$ as the following embedding

$$X^i_\sharp \setminus \{x\} \hookrightarrow X^i_\sharp.$$

A coalgebra $(C, \alpha_C)$ satisfies $[x]$ iff $x$ does not lie in the image of $f^i_\sharp\colon C \to X^i_\sharp$ for any colouring $f\colon C \to X$. These are all the coequations needed: we can substitute an arbitrary coequation

$$m\colon M \to X^i_\sharp$$

by the set of coequations $\{[x] \mid x \in X^i_\sharp \setminus m[M]\}$.

2. Various concepts of covariety of $F$-coalgebras—all restricted to the case of a bounded endofunctor on $\mathsf{Set}$ (thus, a varietor—see Section 4)—have already been discussed in the literature:
   - subcategories of $\mathsf{Alg}(F)$ closed w.r.t. coproducts, subcoalgebras and homomorphic images ([20]).
   - projectivity classes in $\mathsf{Alg}(F)$ w.r.t. collections of embeddings of subcoalgebras of cofree coalgebras ([13]).
   - projectivity classes in $\mathsf{Alg}(F)$ w.r.t. collections of embeddings of subcoalgebras of regularly injective coalgebras ([14] [9] , [7]).

   Theorem 3.12 (in connection with Remark 3.2) shows in particular that all of them are equivalent to the concept introduced here if specialized to bounded $\mathsf{Set}$-functors.

3. A more complicated concept of coequation appears in [10]; this probably is not equivalent to the one above.

# 4 Properties of Set-functors

Every endofunctor $F$ of Set preserving colimits of $\lambda$-chains (or, equivalently, $\lambda$-filtered colimits) for some regular cardinal $\lambda$ is a varietor by 2.11. Such functors are called *accessible*, see [19]. An accessible functor is also a covarietor, as observed by M. Barr [9]. A different criterion is due to Y. Kawahara and M. Mori (see [17] or also [20] and [15]): recall that $F$ is called *bounded* if there

[9] Here, regular injectivity is called *extension property*

exists an infinite cardinal $\lambda$ such that for every $F$-coalgebra $(C, \alpha_C)$ and every element $x$ of $C$ there is a coalgebra homomorphism $h\colon (D, \alpha_D) \to (C, \alpha_C)$ with $x \in h[D]$ and $\mathsf{card} D \leq \lambda$. We are going to prove that this, however, is equivalent to accessibility and both are equivalent to $F$ being *small*, i.e., a small colimit of hom-functors:

**4.1 Theorem** *For an endofunctor $F$ of* $\mathsf{Set}$ *the following conditions are equivalent:*

*(i)* *$F$ is small;*

*(ii)* *$F$ is accessible;*

*(iii)* *$F$ is bounded.*

**Proof** I. Suppose first that the given endofunctor $F$ preserves finite intersections (i.e., pullbacks of monomorphisms).

(iii) $\Longrightarrow$ (i): For the above cardinal $\lambda$ let $\mathsf{D}$ be the (essentially small) category of all pairs $(X, x)$ where $X$ is a set of cardinality $\leq \lambda$ and $x \in FX$, with morphisms $f : (X, x) \to (X^{'}, x^{'})$ all functions $f : X^{'} \to X$ with $Ff(x^{'}) = x$. We prove that $F$ is a colimit of the diagram $V\colon \mathsf{D} \to [\mathsf{Set}, \mathsf{Set}]$ where $V(X, x) = \mathsf{hom}(X, -)$ with the colimit cocone $f_{(X,x)}$ having components

$$f^{Y}_{(X,x)}\colon \mathsf{hom}(X, Y) \to FY, \quad q \longmapsto Fq(x) \ \text{ for all } \ q\colon X \to Y.$$

That is, we prove that for every set $Y$

(a) the maps $f^{Y}_{(X,x)}$ are collectively epimorphic, and

(b) whenever $f^{Y}_{(X,x)}(q) = f^{Y}_{(X^{'},x^{'})}(q^{'})$ then $q$ is connected with $q^{'}$ by a zig–zag in the diagram of elements of $V$ composed with the evaluation–at–$Y$, $eval_Y\colon [\mathsf{Set}, \mathsf{Set}] \to \mathsf{Set}$.

Proof of (a): Given $y \in FY$, for the coalgebra $(Y, const(y))$ there exists a homomorphism $h : (D, \alpha_D) \to (Y, const(y))$ with card $D \leq \lambda$ which fulfills $D \neq \emptyset$ if $Y \neq \emptyset$. For $Y \neq \emptyset$ choose $d_0 \in D$, then $(D, d) \in \mathsf{D}$ with $d = \alpha_D(d_0)$
$f^{Y}_{(D,d)}(h) = Fh(\alpha_D(d)) = const(y) \cdot h(d) = y$.
The case $Y = \emptyset$ is trivial since $(Y, y) \in \mathsf{D}$.

Proof of (b): We have $Fq(x) = Fq^{'}(x^{'})$ for some $q\colon X \to Y$ and $q'\colon X' \to Y$. Factor $q$ as an epimorphism $e\colon X \to Z$ followed by a monomorphism $m\colon Z \to Y$ and put $z = Fq(x)$; analogously $e'$, $m'$, and $z'$. By assumtion, $F$ preserves the pullback

$$\begin{tikzcd} & P \arrow[dl, "u"'] \arrow[dr, "u'"] & \\ Z \arrow[dr, "m"'] & & Z' \arrow[dl, "m'"] \\ & Y & \end{tikzcd}$$

The equality $Fm(z) = Fq(x) = Fq'(x') = Fm'(z')$ thus guarantees that there exists $p \in FP$ with $z = Fu(p)$ and $z' = Fu'(p')$. And since $\mathsf{card}P \leq \mathsf{card}(Z \times Z') \leq \mathsf{card}(X \times X') \leq \lambda^2 = \lambda$, we obtain an object $(P, p)$ of $\mathsf{D}$ with morphisms

$$(X, x) \xleftarrow{q} (Z, z) \xrightarrow{u} (P, p) \xleftarrow{u'} (Z', z') \xrightarrow{q} (X', x')$$

forming the desired zig-zag.

(i) $\implies$ (ii): Every hom–functor is accessible, and a small colimit of accessible functors is accessible, see [11].

(ii) $\implies$ (iii): Let F preserve $\lambda$–filtered colimits for some regular cardinal $\lambda$. Since every set is a $\lambda$–filtered colimit of all subsets of cardinality less than $\lambda$, we see that

$(*)$ given sets $C$ and $T \subseteq FC$ with card $T < \lambda$ there exists a subset $m\colon B \hookrightarrow C$ with $\mathsf{card}B < \lambda$ and $T \subseteq Fm[FB]$.

We prove that $F$ is bounded: given $(C, \alpha_C)$ and $x \in C$ define a $\lambda$–chain $m_i\colon B_i \hookrightarrow C \quad (i < \lambda)$ of subsets of cardinality less than $\lambda$ by transfinite induction as follows:

- $B_0 = \{x\}$;
- given $B_i$, apply $(*)$ to $T = \alpha_C[B_i]$ to get $m_{i+1}\colon B_{i+1} \to C$ with $m_i \subseteq m_{i+1}, \alpha_C[B_i] \subseteq Fm_{i+1}[FB_{i+1}]$, and $\mathsf{card}B_{i+1} < \lambda$;
- given a limit ordinal $i$ define $B_i = \bigcup\limits_{j<i} B_j$ – due to the regularity of $\lambda$, if $\mathsf{card}B_j < \lambda$ for all $j < i$, then $\mathsf{card}B_i < \lambda$.

Define $D = \bigcup\limits_{i<\lambda} B_i$ and $h = \mathsf{colim}\ m_i\colon D \to A$, then since $F$ preserves the colimit $D = \mathsf{colim}B_i$, and since $\alpha_C[B_i]$ is contained in the image of $Fm_{i+1}$ for each $i < \lambda_i$ we see that $\alpha_C[D]$ is contained in the image of $Fh$. Thus, we have $\alpha_D\colon D \to FD$ for which $h\colon (D, \alpha_D) \to (C, \alpha_C)$ is a coalgebra homomorphism. And $\mathsf{card}D \leq \sum_{i<\lambda} \mathsf{card}B_i = \lambda$. Since $x \in B_0 \subseteq D$, this proves that $F$ is bounded.

II. For $F\colon \mathsf{Set} \to \mathsf{Set}$ arbitrary we use the result of V.Trnková (see [6, III.4.5-6]) that there exists a functor $F'$ preserving finite intersections and such that $FX = F'X$ for all nonempty sets $X$ (and $Fh = F'h$ for all nonempty functions h). It is easy to verify that $F$ satisfies one of the properties (i)–(iii) iff so does $F'$. $\diamond$

**4.2 Example** of a covarietor which is not small. Given a class $M$ of cardinal numbers, define $\mathcal{P}_M\colon \mathsf{Set} \to \mathsf{Set}$ on objects $X$ by $\mathcal{P}_M X = \{A \subseteq X; A = \emptyset$ or $\mathsf{card}A \in M\}$ and an morphismus $f\colon X \to Y$ by $\mathcal{P}_M f(A) = f[A]$ if $f/A$ is injective, else $= \emptyset$. Then $\mathcal{P}_M$ is small iff $M$ is small ( = bounded). But every infinite set $X$ with $\mathsf{card}X \notin M$ is, obviously, a fixed point of $\mathcal{P}_M$. It is easy to find an unbounded class $M$ for which $\mathcal{P}_M$ has arbitrarily large exponential fixed points (i.e., $M$ is unbounded but has arbitrarily large "exponential holes"). Then $\mathcal{P}_M$ is a varietor and covarietor but is not small.

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