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# Parametrized Fixed Points and Their Applications to Session Types

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**Abstract**

Parametrized fixed points are of particular interest to denotational semantics and are often given by "dagger operations" [10–12]. Dagger operations that satisfy the Conway identities [10] are particularly useful, because these identities imply a large class of identities used in semantic reasoning. We generalize existing techniques to define dagger operations on $\omega$-categories and on $\mathbf{O}$-categories. These operations enjoy a 2-categorical structure that implies the Conway identities. We illustrate these operators by considering applications to the semantics of session-typed languages.

*Keywords:* $\mathbf{O}$-categories, $\omega$-categories, fixed points, dagger operations, Conway identities.

## 1 Introduction

Recursive types are ubiquitous in functional languages. For example, in Standard ML we can define the type of (unary) natural numbers as:

```
datatype nat = Zero | Succ of nat
```

This declaration specifies that a `nat` is either zero or the successor of some natural number. Semantically, we can think of `nat` as a domain $D$ satisfying the domain equation $D \cong (\texttt{Zero} : \{\bot\}) \uplus (\texttt{Succ} : D)$, where $\uplus$ forms the labelled disjoint union of domains. Equivalently, we can think of $D$ as a fixed point of the functor $F_{\texttt{nat}}(X) = (\texttt{Zero} : \{\bot\}) \uplus (\texttt{Succ} : X)$ on a category of domains.

Mutually-recursive data types give rise to a similar interpretation. Consider, for example, the types of even and odd natural numbers:

```
datatype even = Zero | E of odd
     and odd  = O of even
```

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This declaration specifies that an even number is either zero or the successor of an odd number, and that an odd number is the successor of an even one. The types `even` and `odd` respectively denote solutions $D_{\mathsf{e}}$ and $D_{\mathsf{o}}$ to the system of domain equations:

$$D_{\mathsf{e}} \cong (\mathtt{Zero} : \{\bot\}) \uplus (\mathtt{E} : D_{\mathsf{o}}) \quad \text{and} \quad D_{\mathsf{o}} \cong (\mathtt{O} : D_{\mathsf{e}}).$$

These are solutions to the system of equations:

$$X_{\mathsf{e}} \cong F_{\mathtt{even}}(X_{\mathsf{e}}, X_{\mathsf{o}}) \tag{1}$$

$$X_{\mathsf{o}} \cong F_{\mathtt{odd}}(X_{\mathsf{e}}, X_{\mathsf{o}}) \tag{2}$$

where $F_{\mathtt{even}}$ and $F_{\mathtt{odd}}$ are the functors $F_{\mathtt{even}}(X_{\mathsf{e}}, X_{\mathsf{o}}) = (\mathtt{Zero} : \{\bot\}) \uplus (\mathtt{E} : X_{\mathsf{o}})$ and $F_{\mathtt{odd}}(X_{\mathsf{e}}, X_{\mathsf{o}}) = (\mathtt{O} : X_{\mathsf{e}})$. We can use Bekič's rule [4, § 2] to solve this system of equations. To do so, we think of eq. (1) as a family of equations parametrized by $X_{\mathsf{o}}$. If we could solve for $X_{\mathsf{e}}$, then we would get a parametrized family of solutions $F^{\dagger}_{\mathtt{even}}(X_{\mathsf{o}})$ such that:

$$F^{\dagger}_{\mathtt{even}}(X_{\mathsf{o}}) \cong F_{\mathtt{even}}(F^{\dagger}_{\mathtt{even}}(X_{\mathsf{o}}), X_{\mathsf{o}}) \tag{3}$$

for all domains $X_{\mathsf{o}}$. Substituting this for $X_{\mathsf{e}}$ in eq. (2) gives the domain equation

$$X_{\mathsf{o}} \cong F_{\mathtt{odd}}(F^{\dagger}_{\mathtt{even}}(X_{\mathsf{o}}), X_{\mathsf{o}}).$$

Solving for $X_{\mathsf{o}}$ gives the solution $D_{\mathsf{o}}$. Substituting $D_{\mathsf{o}}$ for $X_{\mathsf{o}}$ in eq. (3), we see that $D_{\mathsf{e}} = F^{\dagger}_{\mathtt{even}}(D_{\mathsf{o}})$ is the other part of the solution.

The above example motivates techniques for solving parametrized domain equations. These are well understood. For example, given a suitable functor $F : \mathbf{D} \times \mathbf{E} \to \mathbf{E}$ on suitable categories of domains, [1, Proposition 5.2.7] gives a recipe for constructing a functor $F^{\dagger} : \mathbf{D} \to \mathbf{E}$ such that for all objects $D$ of $\mathbf{D}$, $F^{\dagger}D \cong F(D, F^{\dagger}D)$. Is the mapping $F \mapsto F^{\dagger}$ functorial? Semantically, substitution is typically interpreted as composition [15, § 3.4]. If the interpretations of recursive types are to respect substitution, then the mapping $F \mapsto F^{\dagger}$ must be natural in $\mathbf{D}$. Is it? What other properties does it satisfy?

Families of parametric fixed points arise elsewhere in mathematics. An external **dagger operation** [12, Definition 2.6, 10, p. 7] on a cartesian closed category $\mathbf{C}$ is a family $\dagger_{A,B} : \mathbf{C}(A \times B, B) \to \mathbf{C}(A, B)$ of set-theoretic functions for each pair of objects $A$ and $B$ in $\mathbf{C}$. Of particular interest are dagger operations that satisfy the (cartesian) Conway identities. These identities imply many other identities [10, § 3.3] useful for semantic reasoning, such as Bekič's rule. They are also of independent interest. They axiomatize a decidable theory [9], and dagger operations that satisfy them are closely related to the trace operator [22, 20, Theorem 7.1, 5, p. 281]. Does the above dagger operation satisfy the Conway identities?

In this paper, we present dagger operators in two different categorical settings. In sections 3 to 5, we work with $\omega$-cocontinuous functors between categories with sufficiently many $\omega$-colimits. In section 6, we work with locally continuous functors between $\mathbf{O}$-categories. $\mathbf{O}$-categories [37] generalize categories of domains to provide just the structure required to compute fixed points of functors. In both cases, these dagger operators will enjoy a 2-categorical structure that will imply the Conway identities. As an application, in section 7 we see that properties of our dagger operation are essential for defining and reasoning about semantics of session-typed languages with recursion. An extended version of this paper is available at https://arxiv.org/abs/2006.08479.

## 2 Background and Notation

We review some standard definitions and fix our notation. Our results use concepts from 2-category theory. Readers unfamiliar with 2-category theory may replace the words "2-category", "2-functor", and "2-natural transformation" by "category", "functor", and "natural transformation" throughout to obtain weaker forms of our results. Fiore [16, Chapter 2] and Kelly and Street [25] give surveys of 2-category theory.

A **2-category** $\mathbf{C}$ has **objects** $A, B, \dots$, **arrows** (horizontal morphisms) $f : A \to B$, and **2-cells** (vertical morphisms) $\alpha : f \Rightarrow g : A \to B$ drawn as:

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd}$$

Objects and arrows form a category $\mathbf{C}_0$ called the **underlying category** of $\mathbf{C}$; we write $\circ$ for its composition. Each pair of objects $A$ and $B$ gives rise to a category $\mathbf{C}(A, B)$ whose objects are arrows $A \to B$ and whose morphisms are 2-cells between them; we call its composition operator "$\cdot$" **vertical composition**. Objects and 2-cells form a category $\mathbf{Cell}_{\mathbf{C}}$; we call its composition operator "$*$" **horizontal composition**. Vertical and horizontal composition satisfy the **middle four interchange** and **identity** laws: whenever

$$\begin{tikzcd} A \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=2.5ex, "\Downarrow\alpha"] \arrow[r, phantom, shift right=2.5ex, "\Downarrow\beta"] & B \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=2.5ex, "\Downarrow\gamma"] \arrow[r, phantom, shift right=2.5ex, "\Downarrow\delta"] & C \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Downarrow\mathsf{id}_f"] & B \arrow[r, bend left, "g"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\mathsf{id}_g"] & C \end{tikzcd}$$

we have $(\delta \cdot \gamma) * (\beta \cdot \alpha) = (\delta * \beta) \cdot (\gamma * \alpha)$ and $\mathsf{id}_g * \mathsf{id}_f = \mathsf{id}_{g \circ f}$, respectively. Thanks to the identity law, we can adopt the convention of writing $f$ for the identity 2-cell $\mathsf{id}_f : f \Rightarrow f : A \to B$.

The prototypical 2-category is **Cat**, the category of small categories, where objects are small categories, horizontal morphisms are functors, and vertical morphisms are natural transformations. Given 2-cells $\epsilon : F \Rightarrow G : \mathbf{C} \to \mathbf{D}$ and $\eta : H \Rightarrow I : \mathbf{D} \to \mathbf{E}$ in **Cat**, their horizontal composition $\eta * \epsilon : HF \Rightarrow IG$ is given by the equal natural transformations $I\epsilon \circ \eta F = \eta G \circ H\epsilon$. Given a morphism $f : K \to L$ in $\mathbf{C}$, we abuse notation and write $\eta * f : FK \to GL$ for the naturality square $FK \xrightarrow{Ff} FL \xrightarrow{\eta_L} GL = FK \xrightarrow{\eta_K} GK \xrightarrow{Gf} GK$.

Let $\mathbf{C}$ and $\mathbf{D}$ be 2-categories. A **2-functor** $F : \mathbf{C} \to \mathbf{D}$ sends objects of $\mathbf{C}$ to objects of $\mathbf{D}$, arrows of $\mathbf{C}$ to arrows of $\mathbf{D}$, and 2-cells of $\mathbf{C}$ to 2-cells of $\mathbf{D}$ while preserving all identities, compositions, domains, and codomains. A **2-natural transformation** $\eta : F \Rightarrow G : \mathbf{C} \to \mathbf{D}$ is a natural transformation $\eta : F \Rightarrow G$ that is 2-natural, i.e., such that for each 2-cell $\alpha : f \Rightarrow g : A \to B$ in $\mathbf{C}$, we have the following equality in $\mathbf{D}$:

$$\begin{tikzcd} FA \arrow[r, bend left, "Ff"] \arrow[r, bend right, "Fg"'] \arrow[r, phantom, "\Downarrow F\alpha"] & FB \arrow[r, "\eta_B"] & GB \end{tikzcd} = \begin{tikzcd} FA \arrow[r, "\eta_A"] & GA \arrow[r, bend left, "Gf"] \arrow[r, bend right, "Gg"'] \arrow[r, phantom, "\Downarrow G\alpha"] & GB. \end{tikzcd}$$

A **modification** $\rho : \alpha \to \beta : F \Rightarrow G : \mathbf{C} \to \mathbf{D}$ is a morphism of 2-natural transformations. It assigns to each object $A$ of $\mathbf{C}$ a 2-cell $\rho_A : \alpha_A \Rightarrow \beta_A$ such that for all $f : A \to B$ we have the following equality in $\mathbf{D}$:

$$\begin{tikzcd} FA \arrow[r, bend left, "\alpha_A"] \arrow[r, bend right, "\beta_A"'] \arrow[r, phantom, "\Downarrow \rho_A"] & GA \arrow[r, "Gf"] & GB \end{tikzcd} = \begin{tikzcd} FA \arrow[r, "Ff"] & FB \arrow[r, bend left, "\alpha_B"] \arrow[r, bend right, "\beta_B"'] \arrow[r, phantom, "\Downarrow \rho_B"] & GB. \end{tikzcd}$$

Various constructions give new 2-categories from old. The **dual** 2-category $\mathbf{C}^{\mathrm{op}}$ of a 2-category $\mathbf{C}$ is determined by $\mathbf{C}^{\mathrm{op}}(A, B) = \mathbf{C}(B, A)$, where arrows are reversed but not 2-cells between them. The **product 2-category** $\mathbf{C} \times \mathbf{D}$ is given by the usual product-category construction, where objects are pairs $(C, D)$ of objects $C$ of $\mathbf{C}$ and $D$ of $\mathbf{D}$, and all morphisms, compositions, and identities are given component-wise.

Every 2-category $\mathbf{C}$ is equipped with a **hom 2-functor** $\mathbf{C}(-,-) : \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{CAT}$, where $\mathbf{CAT}$ is the 2-category of locally small categories. It takes objects $(A, B)$ to categories $\mathbf{C}(A, B)$, arrows $(f, g) : (A, B) \to (A', B')$ to functors $g \circ - \circ f : \mathbf{C}(A, B) \to \mathbf{C}(A', B')$, and 2-cells $(\alpha, \beta) : (f, g) \Rightarrow (f', g') : (A, B) \to (A', B')$ to natural transformations $\alpha * \mathsf{id}_{-} * \beta : g \circ - \circ f \Rightarrow g' \circ - \circ f' : \mathbf{C}(A, B) \to \mathbf{C}(A', B')$. If $\mathbf{D}$ is a locally small category, then we overload notation and write $\mathbf{D}(-,-) : \mathbf{D}^{\mathrm{op}} \times \mathbf{D} \to \mathbf{Set}$ for the **hom functor**. If a category $\mathbf{E}$ has an **internal hom**, then we write $\mathbf{E}[- \to -]$ or just $[- \to -]$ for it. Given category $\mathbf{C}$ and $\mathbf{D}$ and an object $D$ of $\mathbf{D}$, let $\Delta D : \mathbf{C} \to \mathbf{D}$ be the constant functor onto the object $D$. When $\mathbf{C}$ and $\mathbf{D}$ are the same small category, this action on objects induces the **diagonal functor** $\Delta : \mathbf{C} \to \mathbf{Cat}[\mathbf{C} \to \mathbf{C}]$.

A 2-category $\mathbf{C}$ is **2-cartesian closed** if $\mathbf{Cell}_{\mathbf{C}}$ is cartesian closed [12, p. 97]. In elementary terms [16, p. 24], this means that $\mathbf{C}$ has a terminal object, binary 2-products, and 2-exponentials, where

- the **terminal object** of $\mathbf{C}$ is an object $1$ of $\mathbf{C}$ with a 2-natural isomorphism $\mathbf{C}(-, 1) \cong \Delta\mathbf{1}$, where $\mathbf{1}$ is the terminal category;
- the **2-product** of objects $A$ and $B$ of $\mathbf{C}$ is an object $A \times B$ of $\mathbf{C}$ with a 2-natural transformation $\mathbf{C}(-, A) \times \mathbf{C}(-, B) \cong \mathbf{C}(-, A \times B)$;
- the **2-exponential** of objects $A$ and $B$ of $\mathbf{C}$ is an object $[A \to B]$ of $\mathbf{C}$ with a 2-natural transformation $\mathbf{C}(- \times A, B) \cong \mathbf{C}(-, [A \to B])$.

We write $\perp_{\mathbf{C}}$ or just $\perp$ for the **initial object** of a category $\mathbf{C}$. We also write $\perp$ for the unique cone $\perp_{\mathbf{C}} \Rightarrow \mathsf{id}_{\mathbf{C}}$ witnessing the initiality of $\perp_{\mathbf{C}}$.

Let $\omega$ be the category with natural numbers as objects and with at most one arrow between each pair of objects, where $n \to m$ if and only if $n \leq m$. An $\omega$-**chain** in $\mathbf{K}$ is a diagram $J : \omega \to \mathbf{K}$. An $\omega$-**category** [26, Definition 5] is a category with all colimits of $\omega$-diagrams. We warn the reader that the definition of $\omega$-category varies in the literature. Some [28, Definition 2.4] additionally require the existence of an initial object; we call such categories **IFP**-categories. This name stems from the fact that **IFP**-categories have the structure required for $\omega$-functors to have initial fixed points (see corollary 3.9 below). An $\omega$-**functor** [28, Definition 2.5] is a functor that preserves all existing colimits of $\omega$-diagrams. Small $\omega$-categories and $\omega$-functors between them form a 2-cartesian-closed subcategory $\omega$-**Cat** of **Cat**. Small **IFP**-categories and $\omega$-functors between them form a 2-cartesian-closed subcategory **IFP** of $\omega$-**Cat**. A **parameterized** $\omega$-**functor** is a functor $F : \mathbf{C} \times \mathbf{D} \to \mathbf{E}$ such that $F(C, -) : \mathbf{D} \to \mathbf{E}$ is an $\omega$-functor for all objects $C$ of $\mathbf{C}$.

Many categories of interest are not small, so they are not objects in **Cat**. We can work around this by using a hierarchy of universes [33, § 3] to treat them as small categories.

We frequently work with cocones, morphisms of cocones, and colimits. Given a small category $\mathbf{J}$, a locally small category $\mathbf{C}$, and a functor $F : \mathbf{J} \to \mathbf{C}$, there exists [32, Definition 3.1.5] the **cocone functor** $\mathrm{Cone}(F, -) : \mathbf{C} \to \mathbf{Set}$ take an objects $C$ of $\mathbf{C}$ to the set of cocones on $F$ with summit $C$. Given a morphism $f : C \to C'$ and a cocone $(\lambda : F \Rightarrow C) \in \mathrm{Cone}(F, C)$, $\mathrm{Cone}(F, f)(\lambda) = f \circ \lambda$. Given a diagram $F : \mathbf{J} \to \mathbf{C}$, the **category of cocones on** $F$ is the category of elements $\int \mathrm{Cone}(F, -)$. Its objects are pairs $(\alpha, A)$ where $\alpha \in \mathrm{Cone}(F, A)$. Morphisms $f : (\alpha, A) \to (\beta, B)$ are morphisms $f : C \to D$ in $\mathbf{C}$ such that $\mathrm{Cone}(F, f)(\alpha) = \beta$, i.e., such that $f \circ \alpha = \beta$. The **colimit** [32, Definition 3.1.6] of $F$ is the initial object of $\int \mathrm{Cone}(F, -)$.

## 3 Functoriality of Fixed Points

We show that constructing fixed points of $\omega$-functors is itself a functorial operation. The initial fixed point of an $\omega$-functor $F$ on an $\omega$-category is given by the colimit of the $\omega$-chain $\perp \to F\perp \to F^2\perp \to \cdots$. Other fixed points can be constructed using a different "first link", i.e., by taking the colimit of a chain $K \to FK \to F^2K \to \cdots$ generated by a link $k : K \to FK$.

**Definition 3.1** Fix an $\omega$-category $\mathbf{K}$. Links form a category $\mathbf{Links}_{\mathbf{K}}$ where

- objects are triples $(K, k, F)$ called "links", where $K$ is an object of $\mathbf{K}$, $F : \mathbf{K} \to \mathbf{K}$ is an $\omega$-functor, and $k : K \to FK$ is a morphism in $\mathbf{K}$;
- morphisms $(K, k, F) \to (L, l, G)$ are pairs $(f, \eta)$ where $f : K \to L$ is a morphism of $\mathbf{K}$, $\eta : F \Rightarrow G$ is a natural transformation, and $f$ and $\eta$ satisfy $l \circ f = (\eta * f) \circ k : K \to GL$;
- composition is given component-wise: $(g, \rho) \circ (f, \eta) = (g \circ f, \rho \circ \eta)$.

The condition on morphisms between links provide the structure required to define morphisms between the $\omega$-chains they generate. If $\mathbf{K}$ has an initial object $\perp$, then the initial object in $\mathbf{Links_K}$ is the link $(\perp, \perp, \Delta\perp)$. The category $\mathbf{Links_K}$ is $\omega$-cocomplete, and the following proposition characterizes its $\omega$-colimits:

**Proposition 3.2** *Let $J = (K_0, k_0, F_0) \xrightarrow{(f_0,\eta_0)} (K_1, k_1, F_1) \xrightarrow{(f_1,\eta_1)} \cdots$ be an $\omega$-chain in $\mathbf{Links_K}$. Let $(\kappa, K_\infty)$ be the colimit of the $\omega$-chain $K = K_0 \xrightarrow{f_0} K_1 \xrightarrow{F_1} \cdots$ in $\mathbf{K}$. Let $(\phi, F_\infty)$ be the colimit of the $\omega$-chain $\Phi = F_0 \xrightarrow{\eta_0} F_1 \xrightarrow{\eta_1} \cdots$ in $\omega$-$\mathbf{Cat}[\mathbf{K} \to \mathbf{K}]$. Then there exists a $k_\infty : K_\infty \to F_\infty K_\infty$ such that $(\kappa, \phi) : J \Rightarrow (K_\infty, k_\infty, F_\infty)$ is colimiting in $\mathbf{Links_K}$.*

**Proof (sketch).** Let $\mathbf{2}$ be the category $(\bullet \to \bullet)$. We can recognize each link $(K_n, k_n, F_n)$ as a $\mathbf{2}$-diagram $\Delta K_n \overset{k_n}{\Longrightarrow} F_n \circ (\Delta K_n)$ in $\omega$-$\mathbf{Cat}[\mathbf{K} \to \mathbf{K}]$. Then $J$ induces a diagram $\hat{J} : \omega \to \mathbf{Cat}[\mathbf{2} \to \omega\text{-}\mathbf{Cat}[\mathbf{K} \to \mathbf{K}]]$:

$$\begin{tikzcd}
\Delta K_0 \arrow[r, Rightarrow, "\Delta f_0"] \arrow[d, Rightarrow, "k_0"] & \Delta K_1 \arrow[r, Rightarrow, "\Delta f_1"] \arrow[d, Rightarrow, "k_1"] & \Delta K_2 \arrow[r, Rightarrow, "\Delta f_2"] \arrow[d, Rightarrow, "k_2"] & \cdots \\
F_0 \circ (\Delta K_0) \arrow[r, Rightarrow, "\eta_0 * (\Delta f_0)"] & F_1 \circ (\Delta K_1) \arrow[r, Rightarrow, "\eta_1 * (\Delta f_1)"] & F_2 \circ (\Delta K_2) \arrow[r, Rightarrow, "\eta_2 * (\Delta f_2)"] & \cdots
\end{tikzcd}$$

Colimits in functor categories are determined point-wise, so the top row has colimit $(\Delta\kappa, \Delta K_\infty)$. The bottom row has colimit $(\phi * \kappa, F_\infty \circ (\Delta K_\infty))$ by [28, Lemma 4.3]. Let $k = (k_n : \Delta K_n \to F_n \circ (\Delta K_n))$ be the natural transformation from the top row to the bottom row. Then $((\phi * \kappa) \circ k, F_\infty \circ (\Delta K_\infty))$ is a cocone on the top row, so there exists a unique cocone morphism $k_\infty : (\Delta\kappa, \Delta K_\infty) \to ((\phi * \kappa) \circ k, F_\infty \circ (\Delta K_\infty))$. In particular, $k_\infty : \Delta K_\infty \Rightarrow F_\infty \circ (\Delta K_\infty)$ is a natural transformation between two constant functors, so it is given by a single morphism $K_\infty \to F_\infty K_\infty$ in $\mathbf{K}$. We conclude that $(K_\infty, k_\infty, F_\infty)$ is an object of $\mathbf{Links_K}$. A straightforward check shows that $(\kappa, \phi) : J \Rightarrow (K_\infty, k_\infty, F_\infty)$ is colimiting in $\mathbf{Links_K}$. □

There exists a functor $\Omega : \mathbf{Links_K} \to [\omega \to \mathbf{K}]$ that produces the $\omega$-chain $K \xrightarrow{k} FK \xrightarrow{Fk} F^2K \xrightarrow{F^2k} \cdots$ from a link $(K, k, F)$. Its action on morphisms uses the horizontal iteration of natural transformations. Consider functors $H, G : \mathbf{C} \to \mathbf{C}$ and a natural transformation $\eta : H \Rightarrow G$. We define the family of **horizontal iterates** $\eta^{(i)} : H^i \Rightarrow G^i$, $i \in \mathbb{N}$, by recursion on $i$. When $i = 0$, $H^0 = G^0 = \mathsf{id}_\mathbf{C}$ and we define $\eta^{(0)}$ to be the identity natural transformation on $\mathsf{id}_\mathbf{C}$. Given $\eta^{(i)}$, we set $\eta^{(i+1)} = \eta * \eta^{(i)}$.

We define the functor $\Omega : \mathbf{Links_K} \to [\omega \to \mathbf{K}]$. The action of $\Omega(K, k, F)$ on morphisms $n \to n + k$ is defined by induction on $k$.

$$\Omega(K, k, F)(n) = F^n K \tag{4}$$
$$\Omega(K, k, F)(n \to n) = \mathsf{id}_{F^n K} \tag{5}$$
$$\Omega(K, k, F)(n \to n + k + 1) = F^{n+k} k \circ \Omega(K, k, F)(n \to n + k) \tag{6}$$
$$\Omega(f : K \to L, \eta : F \Rightarrow G)_n = \eta^{(n)} * f : F^n K \to G^n L \tag{7}$$

Proposition 3.3 generalizes the functor $S : [\mathbf{C} \to \mathbf{C}] \to [\omega \to \mathbf{C}]$ of [29, § 3, 28, Lemma 4.2] to form chains with an arbitrary initial link in an $\omega$-cocontinuous manner.

**Proposition 3.3** *Equations (4) to (7) define an $\omega$-functor $\Omega : \mathbf{Links_K} \to \omega\text{-}\mathbf{Cat}[\omega \to \mathbf{K}]$.*

### *3.1 General Fixed Points*

We define a generalized-fixed-point $\omega$-functor, i.e., an $\omega$-functor $\mathsf{GFIX} : \mathbf{Links_K} \to \mathbf{K}$ such that for each link $(K, k, F)$, there is an isomorphism $\mathsf{GFIX}(K, k, F) \cong F(\mathsf{GFIX}(K, k, F))$. We assume that whenever $\mathbf{K}$ is an $\omega$-category, an $\omega$-colimit has been chosen for each $\omega$-chain. This choice determines an $\omega$-colimit functor $\mathrm{colim}_\omega : \omega\text{-}\mathbf{Cat}[\omega \to \mathbf{K}] \to \mathbf{K}$, itself an $\omega$-functor. The following result generalizes [28, Theorem 4.1]:

**Proposition 3.4** *Let $\mathbf{K}$ be an $\omega$-category. The following composition defines an $\omega$-functor:*

$$\mathsf{GFIX} = \mathrm{colim}_\omega \circ \Omega : \mathbf{Links_K} \to \mathbf{K}.$$

We claim that the isomorphism $\mathsf{GFIX}(K, k, F) \cong F(\mathsf{GFIX}(K, k, F))$ is natural in the link $(K, k, F)$. To show this, we begin by defining an "unfolding" functor:

**Proposition 3.5** *Let $\mathbf{K}$ be an $\omega$-category. The following defines an $\omega$-functor $\mathsf{UNF} : \mathbf{Links_K} \to \mathbf{K}$:*

- *On objects:* $\mathsf{UNF}(K, k, F) = F(\mathsf{GFIX}(K, k, F))$,
- *on morphisms:* $\mathsf{UNF}(f, \eta) = \eta * \mathsf{GFIX}(f, \eta)$.

We now construct a natural isomorphism $\mathsf{GFIX} \cong \mathsf{UNF}$. Each of its components is a mediating morphism of cocones induced by "shifting" the $\omega$-chain the link generates. When $J : \omega \to \mathbf{K}$, write $\blacktriangleright J$ for the $\omega$-chain induced by shifting $J$ by one, i.e., by taking $\blacktriangleright J(n) = J(n + 1)$. Observe that inclusion determines a natural transformation $\triangleright_J : J \Rightarrow \blacktriangleright J$, and every cocone $(\gamma, G)$ on $\blacktriangleright J$ induces a cocone $(\gamma \circ \triangleright_J, G)$ on $J$. Also observe that $\blacktriangleright\Omega(K, k, F) = F\Omega(K, k, F)$. Building on these observations, we get the desired natural isomorphism:

**Proposition 3.6** *Let $\mathbf{K}$ be an $\omega$-category. There exists a natural isomorphism $\mathsf{unfold} : \mathsf{GFIX} \Rightarrow \mathsf{UNF}$ with inverse $\mathsf{fold} : \mathsf{UNF} \Rightarrow \mathsf{GFIX}$. Where $\kappa : \Omega(K, k, F) \Rightarrow \mathsf{GFIX}(K, k, F))$ is colimiting, the $(K, k, F)$-component of $\mathsf{unfold}$ is the unique morphism $(\kappa, \mathsf{GFIX}(K, k, F)) \to (F\kappa \circ \triangleright_{\Omega(K,k,F)}, F(\mathsf{GFIX}(K, k, F)))$ in $\int \mathrm{Cone}(\Omega(K, k, F), -)$.*

**Proof (sketch).** Fix some arbitrary link $(K, k, F)$ and let $\mathsf{unfold}_{(K,k,F)}$ be as in the statement. Let $\kappa^+$ be the restriction of $\kappa$ to $F\Omega(K, k, F)$, i.e., $\kappa^+_n = \kappa_{n+1}$. Because $F$ is an $\omega$-functor, $F\kappa : F\Omega(K, k, F) \Rightarrow F(\mathsf{GFIX}(K, k, F))$ is also colimiting. Let $\mathsf{fold}_{(K,k,F)} : (F\kappa, F(\mathsf{GFIX}(K, k, F))) \to (\kappa^+, \mathsf{GFIX}(K, k, F))$ be given by initiality in $\int \mathrm{Cone}(F\Omega(K, k, F), -)$. To show that that $\mathsf{fold}_{(K,k,F)} \circ \mathsf{unfold}_{(K,k,F)} = \mathsf{id}_{\mathsf{GFIX}(K,k,F)}$, we observe that $\mathsf{fold}_{(K,k,F)} \circ \mathsf{unfold}_{(K,k,F)}$ is a cocone morphism $(\kappa, \mathsf{GFIX}(K, k, F)) \to$

$(\kappa, \mathsf{GFIX}(K,k,F))$ in $\int \mathrm{Cone}(\Omega(K,k,F),-)$. The desired equality follows from the fact that $(\kappa, \mathsf{GFIX}(K,k,F))$ is initial and that $\mathsf{id}_{\mathsf{GFIX}(K,k,F)}$ is also such a cocone morphism. An analogous argument gives that $\mathsf{unfold}_{(K,k,F)} \circ \mathsf{fold}_{(K,k,F)} = \mathsf{id}_{F(\mathsf{GFIX}(K,k,F))}$. So $\mathsf{unfold}_{(K,k,F)}$ is an isomorphism for all $(K,k,F)$.

To see that $\mathsf{unfold}$ is natural, let $(f,\eta) : (K,k,F) \to (L,l,G)$ be an arbitrary morphism of $\mathbf{Links_k}$, and let $(\kappa, \mathsf{GFIX}(K,k,F))$ and $(\lambda, \mathsf{GFIX}(L,l,G))$ respectively be the colimiting cocones of $\Omega(K,k,F)$ and $\Omega(L,l,G)$. Consider the following diagram in $\mathbf{K}$:

$$\begin{tikzcd}
\Omega(K,k,F) \arrow[rrr, Rightarrow, "\triangleright_{\Omega(K,k,F)}"] \arrow[dr, Rightarrow, "\kappa"] \arrow[ddd, Rightarrow, "\Omega(f,\eta)"'] & & & F\Omega(K,k,F) \arrow[dl, Rightarrow, "F\kappa"'] \arrow[ddd, Rightarrow, "\eta*\Omega(f,\eta)"] \\
& \mathsf{GFIX}(K,k,F) \arrow[r, "\mathsf{unfold}_{(K,k,F)}"] \arrow[d, "\mathsf{GFIX}(f,\eta)"'] & F(\mathsf{GFIX}(K,k,F)) \arrow[d, "\mathsf{UNF}(f,\eta)=\eta*\mathsf{GFIX}(f,\eta)"] & \\
& \mathsf{GFIX}(L,l,G) \arrow[r, "\mathsf{unfold}_{(L,l,G)}"] & G(\mathsf{GFIX}(L,l,G)) & \\
\Omega(L,l,G) \arrow[rrr, Rightarrow, "\triangleright_{\Omega(L,l,G)}"] \arrow[ur, Rightarrow, "\lambda"] & & & G\Omega(L,l,G) \arrow[ul, Rightarrow, "G\lambda"']
\end{tikzcd}$$

Each of the trapezoids commutes, as does the outer perimeter. The inner rectangle then describes two cocone morphisms $(\kappa, \mathsf{GFIX}(K,k,F)) \to (G\lambda \circ \triangleright_{\Omega(L,l,G)} \circ \Omega(f,n), G(\mathsf{GFIX}(L,l,G)))$ in $\int \mathrm{Cone}(\Omega(K,k,F),-)$. Because $(\kappa, \mathsf{GFIX}(K,k,F))$ is initial, these two cocone morphisms must be equal, i.e., the inner rectangle commutes. We conclude that $\mathsf{unfold}$ is natural. □

Applications to semantics motivate functor algebras. Given a functor $F : \mathbf{C} \to \mathbf{C}$, an $F$-**algebra** is a pair $(A,a)$ where $A$ and $a$ are respectively an object and a morphism $FA \to A$ in $\mathbf{C}$. A morphism $f : (A,a) \to (B,b)$ of $F$-algebras is a morphism $f : A \to B$ in $\mathbf{C}$ such that $f \circ a = b \circ Ff$. Such a morphism is called an $F$-**algebra homomorphism**. These objects and morphisms form a category $\mathbf{C}^F$ of $F$-algebras.

We can specialize the above constructions to produce initial algebras. Indeed, given an **IFP**-category $\mathbf{K}$, the category $\mathbf{IFP}[\mathbf{K} \to \mathbf{K}]$ embeds fully and faithfully into $\mathbf{Links_K}$ via the functor that maps objects $F : \mathbf{K} \to \mathbf{K}$ to the link $(\perp, \perp, F)$ and natural transformations $\eta : F \Rightarrow G$ to the morphism $(\mathsf{id}_\perp, \eta)$. We define the **initial-fixed-point functor** $\mathsf{FIX} : \mathbf{IFP}[\mathbf{K} \to \mathbf{K}] \to \mathbf{K}$ as the composition $[\mathbf{K} \to \mathbf{K}] \hookrightarrow \mathbf{Links_K} \xrightarrow{\mathsf{GFIX}} \mathbf{K}$. The following proposition is standard:

**Proposition 3.7 ([28, Theorem 4.1])** *The initial-fixed-point functor* $\mathsf{FIX} : \mathbf{IFP}[\mathbf{K} \to \mathbf{K}] \to \mathbf{K}$ *is an* $\omega$*-functor.*

We use the close relationship between cocones and functor algebras to show that $\mathsf{FIX}$ does indeed produce initial fixed points. Write $F^\omega : \omega \to \mathbf{K}$ for $\Omega(\perp, \perp, F)$. The following proposition tells us that every $F$-algebra induces a cocone on $F^\omega$. This construction of cocones from algebras is not new, and it appears in the proofs of [37, Lemma 2, 2, Theorem 3.5]. However, to the best of our knowledge, the fact that this action on objects extends to a full and faithful functor, and the initiality result are new. These facts will be used repeatedly in proofs below.

**Proposition 3.8** *Let* $\mathbf{K}$ *a category with an initial object, and let* $F : \mathbf{K} \to \mathbf{K}$ *be a functor. The following defines a full and faithful functor* $\mathrm{Cone}^F : \mathbf{K}^F \to \int \mathrm{Cone}\,(F^\omega, -)$*:*

- *on objects:* $\mathrm{Cone}^F(A, a) = (\alpha, A)$ *where* $\alpha : F^\omega \Rightarrow A$ *is inductively defined by* $\alpha_0 = \perp_A$ *and* $\alpha_{n+1} = a \circ F\alpha_n$
- *on morphisms:* $\mathrm{Cone}^F f = f$.

*If* $F$ *is an* $\omega$*-functor and* $\mathbf{K}$ *is an* **IFP***-category, then* $\mathrm{Cone}^F\left(\mathsf{FIX}(F), \mathsf{fold}_{(\perp,\perp,F)}\right)$ *is initial.*

**Proof (sketch).** An inductive argument shows $\mathrm{Cone}^F(A, a)$ is a cocone, and it is easy to check that the action on morphisms defines a faithful functor. To see that it is full, let $(A, a)$ and $(B, b)$ be arbitrary $F$-algebras, and let $f : \mathrm{Cone}^F(A, a) \to \mathrm{Cone}^F(B, b)$ be arbitrary. We show that $f : (A, a) \to (B, b)$ is an $F$-algebra homomorphism, i.e., that $f \circ a = b \circ Ff$. Consider the following diagram:

$$\begin{tikzcd}
FA \arrow[d, "a"] \arrow[rr, bend left, "Ff"] & F\perp \arrow[l, "F\perp_A"'] \arrow[r, "F\perp_B"'] & FB \arrow[d, "b"'] \\
A \arrow[rr, bend right, "f"'] & \perp \arrow[l, "\perp_A"'] \arrow[r, "\perp_B"] \arrow[u, "\perp_{F\perp}"'] & B
\end{tikzcd}$$

The left and right squares commute by definition of $\mathrm{Cone}^F(A, a)$ and $\mathrm{Cone}^F(B, b)$. The bottom circular segment commutes by initiality, while the top circular segment commutes because functors preserve commuting diagrams. So the whole diagram commutes. We conclude that $f : (A, a) \to (B, b)$ is an $F$-algebra homomorphism and that $\mathrm{Cone}^F$ is full. To show that $\mathrm{Cone}^F\left(\mathsf{FIX}(F), \mathsf{fold}_{(\perp,\perp,F)}\right)$ is initial, we show that it is a colimiting cocone. That it is colimiting follows by induction and the definition of $\mathsf{fold}_{(\perp,\perp,F)}$. □

The following corollary is again standard, but its proof is new and its statement clarifies the nature of the mediating F-algebra homomorphism:

**Corollary 3.9 ([37, Lemma 2, 2, Theorem 3.5])** *Let* $\mathbf{K}$ *be an* **IFP***-category. The initial algebra of an* $\omega$*-functor* $F : \mathbf{K} \to \mathbf{K}$ *is* $(\mathsf{FIX}(F), \mathsf{fold}_{(\perp,\perp,F)})$*. Given any other* $F$*-algebra* $(A, a)$*, the unique* $F$*-algebra homomorphism* $(\mathsf{FIX}(F), \mathsf{fold}_{(\perp,\perp,F)}) \to (A, a)$ *is the unique cocone morphism* $\mathrm{Cone}^F(\mathsf{FIX}(F), \mathsf{fold}_{(\perp,\perp,F)}) \to \mathrm{Cone}^F(A, a)$.

**Proof.** The cocone $\mathrm{Cone}^F(\mathsf{FIX}(F), \mathsf{fold}_{(\perp,\perp,F)})$ is initial in $\int \mathrm{Cone}\,(F^\omega, -)$ by proposition 3.8. Recall that initial objects are given by the limit of the identity functor [32, Lemma 3.7.1], and that full and faithful functors reflect any limits that are present in its codomain [32, Lemma 3.3.5]. Because $\mathrm{Cone}^F$ is full and faithful, it follows that $\left(\mathsf{FIX}(F), \mathsf{fold}_{(\perp,\perp,F)}\right)$ is initial and that the unique morphism is as described.□

# 4 2-Categorical Structure of Parametrized Fixed Points

In this section, we explore the 2-categorical properties of the parametrized-fixed-point functor given by Lehmann and Smyth [29, § 3]. This 2-categorical structure is, to the best of our knowledge, new. From these properties, we deduce that the parametrized-fixed-point functor defines a dagger operation that satisfies the Conway identities.

We begin by observing that their parametrized-fixed-point functor is a 2-natural transformation. This answers the first question of the introduction: the definition of $(\cdot)^\dagger : \mathbf{IFP}[\mathbf{D} \times \mathbf{E} \to \mathbf{E}] \to \mathbf{IFP}[\mathbf{D} \to \mathbf{E}]$ is natural in $\mathbf{D}$. In fact, naturality does not require $\mathbf{D}$ to be an $\omega$-category. Given a category $\mathbf{D}$ and an $\mathbf{IFP}$-category $\mathbf{E}$, let $\mathbf{Cat}[\mathbf{D} \times \mathbf{E} \to_\omega \mathbf{E}]$ be the category of parametrized $\omega$-functors $F : \mathbf{D} \times \mathbf{E} \to \mathbf{E}$, i.e., functors such that $F(D, -) : \mathbf{E} \to \mathbf{E}$ is an $\omega$-functor for all objects $D$ of $\mathbf{D}$.

**Proposition 4.1** *Let* $\mathbf{E}$ *an* $\omega$*-category. The following family of functors forms a 2-natural transformation* $(\cdot)^\dagger : \mathbf{Cat}[- \times \mathbf{E} \to_\omega \mathbf{E}] \Rightarrow \mathbf{Cat}[- \to \mathbf{E}] : \mathbf{Cat}^{\mathrm{op}} \to \mathbf{Cat}$*:*

$$(\cdot)^\dagger_{\mathbf{D}} = \mathbf{Cat}[\mathsf{id}_{\mathbf{D}} \to \mathsf{FIX}] \circ \Lambda : \mathbf{Cat}[\mathbf{D} \times \mathbf{E} \to_\omega \mathbf{E}] \to \mathbf{Cat}[\mathbf{D} \to \mathbf{E}].$$

*It restricts to a 2-natural transformation* $(\cdot)^\dagger : \mathbf{IFP}[- \times \mathbf{E} \to \mathbf{E}] \Rightarrow \mathbf{IFP}[- \to \mathbf{E}] : \mathbf{IFP}^{\mathrm{op}} \to \mathbf{IFP}$*.*

Explicitly, given an $F : \mathbf{D} \times \mathbf{E} \to_\omega \mathbf{E}$ and an object $D$ of $\mathbf{D}$, $F^\dagger_{\mathbf{D}} D = \mathsf{FIX}(F(D, -))$. Proposition 4.1 implies that $(\cdot)^\dagger$ defines an external dagger operation on horizontal morphisms in $\mathbf{IFP}$:

**Definition 4.2** Let $\mathbf{C}$ be a cartesian category. An **external dagger operation** in product form [10, § 3.1, 12, Definition 2.6] is a family of set-theoretic functions $\dagger = (\dagger_{A,B})$ indexed by pairs of objects $A, B$ in $\mathbf{C}$, where $\dagger_{A,B} : \mathbf{C}(A \times B, A) \to \mathbf{C}(B, A)$ is a function of hom-sets. Given an external dagger operation $\dagger_{A,B} : \mathbf{C}(A \times B, B) \to \mathbf{C}(A, B)$ and a morphism $f : A \times B \to B$, we write $f^\dagger$ for $\dagger_{A,B}(f)$.

We will show that $(\cdot)^\dagger$ produces parametrized fixed points. To do so, we begin by defining a family of functors that gives their unrollings:

**Proposition 4.3** *Let* $\mathbf{E}$ *an* $\omega$*-category. The following family of functors forms a 2-natural transformation* $\mathsf{UNR}(\cdot) : \mathbf{Cat}[- \times \mathbf{E} \to_\omega \mathbf{E}] \Rightarrow \mathbf{Cat}[- \to \mathbf{E}] : \mathbf{Cat}^{\mathrm{op}} \to \mathbf{Cat}$*, where* $\mathbf{D}$ *ranges over small categories:*

$$\mathsf{UNR}_{\mathbf{D}}(F) = F \circ \langle \mathsf{id}_{\mathbf{D}}, F^\dagger \rangle$$
$$\mathsf{UNR}_{\mathbf{D}}(\eta) = \eta * \langle \mathsf{id}_{\mathbf{D}}, \eta^\dagger \rangle$$

*It restricts to a 2-natural transformation* $\mathsf{UNR}(\cdot) : \mathbf{IFP}[- \times \mathbf{E} \to \mathbf{E}] \Rightarrow \mathbf{IFP}[- \to \mathbf{E}] : \mathbf{IFP}^{\mathrm{op}} \to \mathbf{IFP}$*.*

We usually expect a dagger operations to satisfy the fixed-point identity [10, p. 7]. It states that $f^\dagger = f \circ \langle \mathsf{id}_A, f^\dagger \rangle$ for all $f : A \times B \to B$, i.e., that a dagger operation gives parametrized fixed points. The fixed-point identity does not hold in general for dagger operations on functors: $F^\dagger$ and $F \circ \langle \mathsf{id}, F^\dagger \rangle$ need not be equal on the nose. However, it holds up to natural isomorphism, giving an analog of proposition 3.6 for solutions to parametrized equations. Proposition 4.4 gives a new 2-categorical formulation of the fixed-point identity. Not only do we have a natural isomorphism $F^\dagger \cong F \circ \langle \mathsf{id}, F^\dagger \rangle$ for each $F$, but these natural isomorphisms assemble to form a modification, i.e., a morphism between the 2-natural transformations $(\cdot)^\dagger$ and $\mathsf{UNR}$.

**Proposition 4.4 (Fixed-Point Identity)** *Let* $\mathbf{E}$ *be an* $\mathbf{IFP}$*-category. There is a modification*

$$\mathsf{Unfold} : (\cdot)^\dagger \to \mathsf{UNR} : \mathbf{Cat}[- \times \mathbf{E} \to_\omega \mathbf{E}] \Rightarrow \mathbf{Cat}[- \to \mathbf{E}] : \mathbf{Cat}^{\mathrm{op}} \to \mathbf{Cat}$$

*that is an isomorphism; we call its inverse* $\mathsf{Fold}$*. For each category* $\mathbf{D}$*, parametrized* $\omega$*-functor* $F : \mathbf{D} \times \mathbf{E} \to_\omega \mathbf{E}$*, and object* $D$ *of* $\mathbf{D}$*, the corresponding component is the isomorphism*

$$(\mathsf{Unfold}^F_{\mathbf{D}})_D = \mathsf{unfold}_{(\bot,\bot,F(D,-))} : F^\dagger D \to F(D, F^\dagger D)$$

*given by proposition 3.6.* $\mathsf{Unfold}$ *restricts to a modificative isomorphism* $(\cdot)^\dagger \to \mathsf{UNR} : \mathbf{IFP}[- \times \mathbf{E} \to \mathbf{E}] \Rightarrow \mathbf{IFP}[- \to \mathbf{E}] : \mathbf{IFP}^{\mathrm{op}} \to \mathbf{IFP}$*.*

**Proof (sketch).** We must show that for each small category $\mathbf{D}$, we have a 2-cell

$$\mathsf{Unfold}_{\mathbf{D}} : (\cdot)^\dagger_{\mathbf{D}} \Rightarrow \mathsf{UNR}_{\mathbf{D}} : \mathbf{Cat}[\mathbf{D} \times \mathbf{E} \to_\omega \mathbf{E}] \to \mathbf{Cat}[\mathbf{D} \to \mathbf{E}]$$

such that for all $G : \mathbf{C} \to \mathbf{D}$, the two following 2-cells are equal:

$$\begin{tikzcd} \mathbf{Cat}[\mathbf{D} \times \mathbf{E} \to_\omega \mathbf{E}] \arrow[r, bend left, "(\cdot)^\dagger_{\mathbf{D}}"] \arrow[r, bend right, "\mathsf{UNR}_{\mathbf{D}}"'] \arrow[r, phantom, "\Downarrow\mathsf{Unfold}_{\mathbf{D}}"] & \mathbf{Cat}[\mathbf{D} \to \mathbf{E}] \arrow[r, "{\mathbf{Cat}[G \to \mathbf{E}]}"] & \mathbf{Cat}[\mathbf{C} \to \mathbf{E}], \end{tikzcd}$$

$$\begin{tikzcd} \mathbf{Cat}[\mathbf{D} \times \mathbf{E} \to_\omega \mathbf{E}] \arrow[r, "{\mathbf{Cat}[G \times \mathbf{E} \to_\omega \mathbf{E}]}"] & \mathbf{Cat}[\mathbf{C} \times \mathbf{E} \to_\omega \mathbf{E}] \arrow[r, bend left, "(\cdot)^\dagger_{\mathbf{C}}"] \arrow[r, bend right, "\mathsf{UNR}_{\mathbf{C}}"'] \arrow[r, phantom, "\Downarrow\mathsf{Unfold}_{\mathbf{C}}"] & \mathbf{Cat}[\mathbf{C} \to \mathbf{E}]. \end{tikzcd}$$

It follows easily from proposition 3.6 that $\mathsf{Unfold}_{\mathbf{D}}$ is a 2-cell. To see that it satisfies the desired equality, consider some arbitrary $F : \mathbf{D} \times \mathbf{E} \to_\omega \mathbf{E}$ and object $C$ of $\mathbf{C}$. Then the $F, C$-component of the top 2-cell is

$$\begin{aligned} \left(\mathsf{Unfold}^F_{\mathbf{D}} G\right)_C = \left(\mathsf{Unfold}^F_{\mathbf{D}}\right)_{GC} &= \mathsf{unfold}_{(\bot,\bot,F(GC,-))} \\ &= \mathsf{unfold}_{(\bot,\bot,(F\circ(G\times\mathsf{id}_{\mathbf{E}}))(C,-))} = \left(\mathsf{Unfold}^{F\circ(G\times\mathsf{id}_{\mathbf{E}})}_{\mathbf{C}}\right)_C, \end{aligned}$$

which we recognize as the $F, C$-component of the bottom 2-cell. Because $F$, $C$, and $G$ were chosen arbitrarily, we conclude the desired equality and that $\mathsf{Unfold}$ is a modification. It is clearly an isomorphism, and the restriction clearly has the desired properties. □

Proposition 4.4 abstracts considerable information. We unpack its definitions to get several corollaries. The first corollary is a special case of [30, Theorem 4.4.8] when $N$ and $M$ are identity functors. It will be key to defining the interpretations of recursive session types in section 7.

**Corollary 4.5** *Let $\mathbf{D}$ be a small category. Then $\mathsf{Unfold}^F_{\mathbf{D}}$ and $\mathsf{Fold}^F_{\mathbf{D}}$ are natural in $F$, i.e., given any natural transformation $\eta : F \Rightarrow G : \mathbf{D} \times \mathbf{E} \to \mathbf{E}$, the two following squares commute:*

$$\begin{tikzcd}
F^\dagger \arrow[r, Rightarrow, "\mathsf{Unfold}^F_{\mathbf{D}}"] \arrow[d, Rightarrow, "\eta^\dagger"'] & F \circ \langle \mathsf{id}, F^\dagger \rangle \arrow[d, Rightarrow, "\eta * \langle \mathsf{id}, \eta^\dagger \rangle"] \\
G^\dagger \arrow[r, Rightarrow, "\mathsf{Unfold}^G_{\mathbf{D}}"] & G \circ \langle \mathsf{id}, G^\dagger \rangle
\end{tikzcd}
\qquad
\begin{tikzcd}
F \circ \langle \mathsf{id}, F^\dagger \rangle \arrow[r, Rightarrow, "\mathsf{Fold}^F_{\mathbf{D}}"] \arrow[d, Rightarrow, "\eta * \langle \mathsf{id}, \eta^\dagger \rangle"'] & F^\dagger \arrow[d, Rightarrow, "\eta^\dagger"] \\
G \circ \langle \mathsf{id}, G^\dagger \rangle \arrow[r, Rightarrow, "\mathsf{Fold}^G_{\mathbf{D}}"] & G^\dagger
\end{tikzcd}$$

Corollary 4.6 gives identities that will be useful in section 7. Equations (8) to (10) are immediate from the definitions of 2-natural transformation and proposition 4.1. Equations (11) and (12) are immediate from the definition of modification and proposition 4.4.

**Corollary 4.6 (Parameter Identity)** *Let $\mathbf{C}$ and $\mathbf{D}$ be small categories and let $\mathbf{E}$ be an $\mathbf{IFP}$-category. Let $F, H : \mathbf{D} \times \mathbf{E} \to \mathbf{E}$ be parametrized $\omega$-functors and let $G, I : \mathbf{C} \to \mathbf{D}$ be functors. Set $F_G = F \circ (G \times \mathsf{id}_E) : \mathbf{C} \times \mathbf{E} \to \mathbf{E}$, and analogously for $H_I$. Let $\phi : F \Rightarrow H$ and $\gamma : G \Rightarrow I$ be natural transformations. Then*

$$F_G^\dagger = F^\dagger \circ G : \mathbf{C} \to \mathbf{E}, \tag{8}$$

$$F_G \circ \langle \mathsf{id}_{\mathbf{C}}, F_G^\dagger \rangle = F \circ \langle \mathsf{id}_{\mathbf{D}}, F^\dagger \rangle \circ G : \mathbf{C} \to \mathbf{E}, \tag{9}$$

$$(\phi * (\gamma \times \mathsf{id}_{\mathbf{E}}))^\dagger = \phi^\dagger * \gamma : F_G^\dagger \Rightarrow H_I^\dagger, \tag{10}$$

$$\mathsf{Fold}^{F_G}_{\mathbf{C}} = \mathsf{Fold}^F_{\mathbf{D}} G : F_G \circ \langle \mathsf{id}_{\mathbf{C}}, F_G^\dagger \rangle \Rightarrow F_G^\dagger, \tag{11}$$

$$\mathsf{Unfold}^{F_G}_{\mathbf{C}} = \mathsf{Unfold}^F_{\mathbf{D}} G : F_G^\dagger \Rightarrow F_G \circ \langle \mathsf{id}_{\mathbf{C}}, F_G^\dagger \rangle. \tag{12}$$

Proposition 4.7 generalizes corollary 3.9 to parametrized fixed points. Given a horizontal morphism $f : A \times B \to B$ in a 2-cartesian category, an $f$-**algebra** [12, Definition 2.3] is a pair $(g, u)$ where $g : A \to B$ is a horizontal morphism and $u : f * \langle \mathsf{id}_A, g \rangle \Rightarrow g$ is vertical. An $f$-**algebra homomorphism** $(g, u) \to (h, v)$ is a vertical morphism $w : g \Rightarrow h$ such that $w \cdot u = v \cdot (f * \langle \mathsf{id}_A, w \rangle)$. These $f$-algebras and $f$-algebra homomorphisms form a category. If we restrict our attention to the 2-cartesian category **Cat**, we get the parametrized $F$-algebras of [16, Definition 6.1.8]. By additionally requiring $A = \mathbf{1}$, we recover the usual notion of $F$-algebras.

**Proposition 4.7** *Let $\mathbf{D}$ be a category and $\mathbf{E}$ be an $\mathbf{IFP}$-category. Let $F : \mathbf{D} \times \mathbf{E} \to_\omega \mathbf{E}$ be a parametrized $\omega$-functor. The initial $F$-algebra is $\left(F^\dagger, \mathsf{Fold}^F_{\mathbf{D}}\right)$. Given any other $F$-algebra $(G, \gamma)$, the mediating morphism $\phi : F^\dagger \to G$ is a natural transformation. The component $\phi_D$ is the unique $F(D, -)$-algebra homomorphism $\left(F^\dagger D, \left(\mathsf{Fold}^F_{\mathbf{D}}\right)_D\right) \to (GD, \gamma_D)$ given by corollary 3.9.*

**Proof (sketch).** Let $(G, \gamma)$ be an arbitrary $F$-algebra. We must show that $\phi : F^\dagger \Rightarrow G$ is natural and unique. We begin by naturality. Let $f : A \to B$ be arbitrary in $\mathbf{D}$; we must show that $Gf \circ \phi_A = \phi_B \circ F^\dagger f$. Write $F_D$ for the partial application $F(D, -)$. Let $(\nu^A, GA) = \mathrm{Cone}^{F_A}(GA, \gamma_A)$ and $(\nu^B, GB) = \mathrm{Cone}^{F_B}(GB, \gamma_B)$ be cocones on $F_A^\omega$ and $F_B^\omega$, respectively, induced by proposition 3.8, and let $\alpha : F_A^\omega \Rightarrow F^\dagger A$ and $\beta : F_B^\omega \to F^\dagger B$ be colimiting. By proposition 3.8, $\phi_A$ and $\phi_B$ are cocone morphisms $\phi_A : (\alpha, F^\dagger A) \to (\nu^A, GA)$ and $\phi_B : (\beta, F^\dagger B) \to (\nu^B, GB)$. Write $F_f$ for the natural transformation $\Lambda F f : F_A \Rightarrow F_B$. We then have the following diagram in $\mathbf{E}$:

$$\begin{tikzcd}
F^\dagger A \arrow[rrr, "F^\dagger f"] \arrow[dd, "\phi_A"'] & & & F^\dagger B \arrow[dd, "\phi_B"] \\
& F_A^\omega \arrow[ul, Rightarrow, "\alpha"] \arrow[dl, Rightarrow, "\nu^A"] \arrow[r, Rightarrow, "F_f^\omega"] & F_B^\omega \arrow[ur, Rightarrow, "\beta"'] \arrow[dr, Rightarrow, "\nu^B"'] & \\
GA \arrow[rrr, "Gf"] & & & GB.
\end{tikzcd}$$

The triangles and trapezoids all commute. It follows that $\phi_B \circ F^\dagger f$ and $Gf \circ \phi_A$ are both mediating morphisms from the cocone $(\alpha, F^\dagger A)$ to the cocone $(\nu^B \circ F_f^\omega, GB)$. Because $(\alpha, F^\dagger A)$ is initial, it follows that they are equal, i.e., that $\phi$ is natural. To see that $\phi$ is unique, observe that every $F$-algebra homomorphism $\rho : \left(F^\dagger, \mathsf{Fold}_{\mathbf{D}}^F\right) \to (G, \gamma)$ determines an $F_D$-algebra homomorphism $\rho_D : \left(F^\dagger D, \left(\mathsf{Fold}_{\mathbf{D}}^F\right)_D\right) \to (GD, \gamma_D)$ for each object $D$ of $\mathbf{D}$. Because $\left(F^\dagger D, \left(\mathsf{Fold}_{\mathbf{D}}^F\right)_D\right)$ is the initial $F_D$-algebra, it follows that $\rho_D = \phi_D$ and $\rho = \phi$. $\square$

Proposition 4.7 presents the converse of a class of external daggers on horizontal morphisms considered in [12, § 2.2]. Given a horizontal morphism $f : A \times B \to B$ in a 2-cartesian category, they define $f^\dagger = g$ where $(g, v)$ is the initial $f$-algebra. They do not consider the action of this dagger on vertical morphisms. In contrast, we gave a dagger operation that determines initial $f$-algebras. It induces an action on both horizontal and vertical morphisms. By proposition 4.4 and corollary 4.6, its action on vertical morphisms coheres with its action on horizontal morphisms.

# 5 Conway Identities

Semantics of programming languages should, ultimately, help users reason about programs. To this end, it is useful to have an arsenal of identities for the mathematical objects used to define the semantics. In our case, the semantics of recursive types motivated the definition of a dagger operation in section 4. In that section, we studied its 2-categorical properties. We now show how these 2-categorical properties imply a large class of identities useful for reasoning about recursive types. In particular, we show that they imply the *Conway identities* [10, 12] up to isomorphism. The Conway identities in turn imply a class of identities useful in the semantics of programming languages.

The Conway identities are also of independent interest. For example, the (cartesian) Conway identities together with an additional identity axiomatize the class of iteration theories [10, Remark 3.4]. Moreover, Hasegawa [20, Theorem 7.1] and Hyland independently discovered [5, p. 281] that a cartesian category has a trace operator [22] if and only if it has an external dagger operator satisfying the (cartesian) Conway identities.

We begin by presenting the Conway identities. The identities' names vary in the literature. We give those of Bloom and Ésik [10, 12] and of Simpson and Plotkin [35, Definitions 2.2 and 2.4]. An external dagger † satisfies:

(i) the **parameter identity** or **naturality** if for all $f : B \times C \to C$ and $g : A \to B$, $(f \circ (g \times \mathsf{id}_C))^\dagger = f^\dagger \circ g$.

(ii) the **composition identity** or **parametrized dinaturality** if for all $f : P \times A \to B$ and $g : P \times B \to A$, $(g \circ \langle \pi_P^{P\times A}, f\rangle)^\dagger = g \circ \langle \mathsf{id}_P, (f \circ \langle \pi_P^{P\times B}, g\rangle)^\dagger\rangle$.

(iii) the **double dagger identity** or **diagonal property** if for all $f : A \times B \times B \to B$, $(f^\dagger)^\dagger = (f \circ (\mathsf{id}_A \times \langle \mathsf{id}_B, \mathsf{id}_B\rangle))^\dagger$.

(iv) the **abstraction identity** if the following diagram commutes:

$$\begin{tikzcd}
{[A \times B \times C \to C]} \arrow[rr, "{[\mathsf{id}_A \times \langle \pi_B, \mathsf{ev}_{B,C}\rangle \to \mathsf{id}_C]}"] \arrow[d, "\dagger_{A\times B,C}"'] & & {[A \times [B \to C] \times B \to C]} \arrow[d, "\Lambda"] \\
{[A \times B \to C]} \arrow[r, "\Lambda"] & {[A \to [B \to C]]} & {[A \times [B \to C] \to [B \to C]]} \arrow[l, "\dagger_{A,[B\to C]}"']
\end{tikzcd}$$

(v) the **power identities** if for all $f : A \times B \to B$ and $n > 1$, $(f^n)^\dagger = f^\dagger$, where $f^n : A \times B \to B$ is inductively defined by $f^0 = \pi_B^{A\times B}$ and $f^{n+1} = f \circ \langle \pi_A^{A\times B}, f^n\rangle$.

An external dagger satisfies the cartesian Conway identities if it satisfies properties i to iii. It satisfies **Conway identities** if it additionally satisfies property iv. Theorem 5.1 answers the last question of section 1:

**Theorem 5.1** *The external dagger operation of proposition 4.1 satisfies the Conway identities and the power identities up to isomorphism.*

**Proof.** The category **IFP** is 2-cartesian closed. By proposition 4.7, each $\omega$-functor $F : \mathbf{D} \times \mathbf{E} \to \mathbf{E}$ in **IFP** has an initial $F$-algebra $\left(F^\dagger, \mathsf{Fold}^F_{\mathbf{D}}\right)$. By corollary 4.6, these initial algebras are related such that $\left(F^\dagger \circ G, \mathsf{Fold}^F_{\mathbf{D}} G\right)$ is the initial $(F \circ (G \times \mathsf{id}_{\mathbf{E}}))$-algebra for each $G : \mathbf{C} \to \mathbf{D}$. It then follows by [12, Theorem 7.1] that the external dagger operator induced by proposition 4.1 satisfies the Conway identities and the power identities up to isomorphism. □

The Conway identities imply the **pairing identity**, sometimes called Bekič's identity [10, p. 10], which relates the two main approaches for solving systems of simultaneous equations. Consider such a system

$$B \cong F(A, B, C)$$
$$C \cong G(A, B, C).$$

We can solve it by pairing $F$ and $G$, and solving the single equation $(B, C) \cong \langle F, G\rangle(A, B, C)$. Alternatively, we can use a Gaussian-elimination-style approach, e.g., as we did in section 1 for the functors defining data types `even` and `odd`. The pairing identity tells us that these two approaches yield isomorphic solutions:

**Corollary 5.2 (Pairing Identity)** *Let* $\mathbf{A}$, $\mathbf{B}$, *and* $\mathbf{C}$ *be small* $\mathbf{IFP}$*-categories, and let* $F : \mathbf{A} \times \mathbf{B} \times \mathbf{C} \to \mathbf{B}$ *and* $G : \mathbf{A} \times \mathbf{B} \times \mathbf{C} \to \mathbf{C}$ *be* $\omega$*-functors. Set* $H = \mathbf{A} \times \mathbf{B} \xrightarrow{\langle \mathsf{id}_{\mathbf{A}}, G^\dagger \rangle} \mathbf{A} \times \mathbf{B} \times \mathbf{C} \xrightarrow{F} \mathbf{B}$. *Then*

$$\langle F, G \rangle^\dagger \cong \langle G^\dagger \circ \langle \mathsf{id}_{\mathbf{A}}, H^\dagger \rangle, H^\dagger \rangle : \mathbf{A} \to \mathbf{B} \times \mathbf{C}.$$

The Conway identities also imply the **left zero identity** [10, p. 10]. Semantically, it describes the interplay between weakening and the formation of recursive types.

**Corollary 5.3 (Left Zero Identity)** *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be small* $\mathbf{IFP}$*-categories, and let* $F : \mathbf{B} \to \mathbf{A}$ *be an* $\omega$*-functor. Then*

$$\left( \mathbf{A} \times \mathbf{B} \xrightarrow{\pi_{\mathbf{B}}} \mathbf{B} \xrightarrow{F} \mathbf{A} \right)^\dagger \cong \mathbf{B} \xrightarrow{F} \mathbf{A}.$$

# 6 Canonical and Parametrized Fixed Points for O-Categories

In this section, we consider an order-theoretic of variation sections 3 and 4. We do so in the setting of $\mathbf{O}$-categories and locally continuous functors, which generalize categories of domains to provide just the amount of order-theoretic structure required for taking fixed points of functors. $\mathbf{O}$-categories are more concrete than $\omega$-categories, and their order-theoretic characterization of $\omega$-colimits and of $\omega$-functors is useful in applications. They also have enough structure to have *canonical* fixed points.

## *6.1 Background on* $\mathbf{O}$*-categories*

We refer the reader to [1, 19, Chapter 10, 37] for additional background on $\mathbf{O}$-categories and domain theory.

We write $\bigsqcup^{\uparrow} A$ for the **directed supremum** of a directed set $A$. If $\mathbf{C}$ has a terminal object isomorphic to its initial object, we call the initial object the **zero object** $0_{\mathbf{C}}$. $\mathbf{C}$ has **zero morphisms** if for all objects $A$ and $D$ there exists a fixed morphism $0_{AD} : A \to D$, and if this family of morphisms satisfies $0_{BD} \circ f = 0_{AD} = g \circ 0_{AC}$ for all morphisms $f : A \to B$ and $g : C \to D$. $\mathbf{C}$ has zero morphisms whenever it has a zero object: $0_{AB} = A \to 0 \to B$.

An **O-category** [37, Definition 5] is a category $\mathbf{K}$ where every hom-set $\mathbf{K}(C, D)$ is a dcpo, and where composition of morphisms is continuous with respect to the partial ordering on morphisms. A functor $F : \mathbf{D} \to \mathbf{E}$ between $\mathbf{O}$-categories is **locally continuous** if the maps $f \mapsto F(f) : \mathbf{D}(D_1, D_2) \to \mathbf{E}(F(D_1), F(D_2))$ are continuous for all objects $D_1, D_2$ of $\mathbf{D}$. Small $\mathbf{O}$-categories form a 2-cartesian closed category $\mathbf{O}$, where horizontal morphisms are locally continuous functors and vertical morphisms are natural transformations. Examples of $\mathbf{O}$-categories include $\mathbf{DCPO}$ and functor categories $\mathbf{Cat}[\mathbf{C} \to \mathbf{D}]$ whenever $\mathbf{D}$ is an $\mathbf{O}$-category.

An **embedding-projection pair** (e-p-pair) [37, Definition 6] $(e, p)$ is a pair of morphisms $e : D \to E$ and $p : E \to D$ such that $p \circ e = \mathsf{id}_D$ and $e \circ p \sqsubseteq \mathsf{id}_E$. We call $e$ an **embedding** and $p$ a **projection**. Given an embedding $e : D \to E$, we write $e^p : E \to D$ for its associated projection. Given a projection $p : E \to D$, we write $p^e : D \to E$ for its associated embedding. When $\mathbf{K}$ is an $\mathbf{O}$-category, we write $\mathbf{K}^e$ for the subcategory of $\mathbf{K}$ whose morphisms are embeddings. The category $\mathbf{K}^e$ is not in general an $\mathbf{O}$-category under the induced ordering [37, p. 768].

A cocone $\kappa : J \Rightarrow A$ in $\mathbf{K}^e$ is an **O-colimit** [37, Definition 7] if $(\kappa_n \circ \kappa_n^p)_n$ is an ascending chain in $\mathbf{K}(A, A)$ and $\bigsqcup^{\uparrow}_{n\in\mathbb{N}} \kappa_n \circ \kappa_n^p = \mathsf{id}_A$. $\mathbf{K}$ is **O-cocomplete** if every $\omega$-chain in $\mathbf{K}^e$ has an $\mathbf{O}$-colimit in $\mathbf{K}$. Our interest in $\mathbf{O}$-colimits is due to proposition 6.1, which appears as [37, Propositions A and D] and as part of the proof of [37, Proposition A]. Parts can also be found in the proof of [19, Theorem 10.4] or specialized to $\mathbf{DCPO}$ as [1, Theorem 3.3.7]. This result gives us an explicit characterization of colimits in $\mathbf{O}$-categories.

**Proposition 6.1 ([37, Propositions A and D])** *Let $\mathbf{K}$ be an $\mathbf{O}$-category, $\Phi$ an $\omega$-chain in $\mathbf{K}^e$, and $\alpha : \Phi \Rightarrow A$ a cocone in $\mathbf{K}$.*

(i) *If $\beta : \Phi \Rightarrow B$ is a cocone in $\mathbf{K}^e$, then $(\alpha_n \circ \beta_n^p)_{n\in\mathbb{N}}$ is an ascending chain in $\mathbf{K}(B, A)$ and the morphism $\theta = \bigsqcup^{\uparrow}_{n\in\mathbb{N}} \alpha_n \circ \beta_n^p$ is mediating from $\beta$ to $\alpha$.*

(ii) *If $\beta : \Phi \Rightarrow B$ is an $\mathbf{O}$-colimit, then $\theta$ is an embedding.*

(iii) *If $\alpha$ is an $\mathbf{O}$-colimit, then $\alpha$ is colimiting in both $\mathbf{K}$ and $\mathbf{K}^e$.*

(iv) *If $\alpha$ is colimiting in $\mathbf{K}$, then $\alpha$ lies in $\mathbf{K}^e$ and is an $\mathbf{O}$-colimit.*

The concept of $\mathbf{O}$-colimit dualizes to give $\mathbf{O}$-limits, and proposition 6.1 dualizes to give the corresponding result for projections, cones, limits, and $\mathbf{O}$-limits.

### *6.2 Local Continuity and $\omega$-Continuity*

Locally continuous functors preserve $\mathbf{O}$-colimits. Every locally continuous functor $F : \mathbf{D} \to \mathbf{E}$ restricts to a functor $F^e : \mathbf{D}^e \to \mathbf{E}^e$ [37, Lemma 4]. When $\mathbf{D}$ is $\mathbf{O}$-cocomplete, $F^e$ is an $\omega$-functor [37, Theorem 3] and $\mathbf{D}^e$ is an $\omega$-category by proposition 6.1. These observations raise the question: why not use $\mathbf{Links}_{\mathbf{K}^e}$ and the results of sections 3 and 4 to study fixed points of locally continuous functors?

The reason is that such an approach does not handle all natural transformations between locally continuous functors, but only for those between functors on $\mathbf{K}^e$. This is because natural transformations $\eta : F \Rightarrow G$ do not in general restrict to natural transformations $F^e \Rightarrow G^e$. By adapting the techniques of the previous sections to $\mathbf{O}$-categories and locally continuous functors, we get fixed-point operators defined on all natural transformations between locally continuous functors. The fixed-point operators are also themselves locally continuous.

### *6.3 Canonical Fixed Points*

By slightly modifying our category of links, we can construct canonical fixed points. Given a functor $F : \mathbf{K} \to \mathbf{K}$, we say that a fixed point $f : FX \cong X$ is **canonical** if $(X, f)$ is an initial $F$-algebra and $(X, f^{-1})$ is a terminal $F$-coalgebra. Given an $\mathbf{O}$-category $\mathbf{K}$, let $\mathbf{OLinks_K}$ be the category where

- objects are triples $(K, k, F)$ called "links", where $K$ is an object of $\mathbf{K}$, $F : \mathbf{K} \to \mathbf{K}$ is locally continuous, and $k : K \to FK$ is an embedding;
- morphisms and composition are defined as before.

**Proposition 6.2** *Equations (4) to (7) define a locally continuous functor* $\Omega : \mathbf{OLinks_K} \to \mathbf{O}[\omega \to \mathbf{K}]$. *For all links* $(K, k, F)$, $\Omega(K, k, F) : \omega \to \mathbf{K}^e$. *The natural transformation* $\Omega(f, \eta)$ *lies in* $\mathbf{K}^e$ *whenever* $f$ *and* $\eta$ *do.*

Let $\mathbf{O}[\omega \to \mathbf{K}^e \hookrightarrow \mathbf{K}]$ be the subcategory of $\mathbf{O}[\omega \to \mathbf{K}]$ whose objects are functors $\omega \to \mathbf{K}^e$ and whose morphisms are natural transformations in $\mathbf{K}$. It is an $\mathbf{O}$-category.

**Proposition 6.3** *Let* $\mathbf{K}$ *be an* $\mathbf{O}$*-cocomplete* $\mathbf{O}$*-category. A choice of* $\mathbf{O}$*-colimit in* $\mathbf{K}$ *for each diagram* $\omega \to \mathbf{K}^e \hookrightarrow \mathbf{K}$ *defines the action on objects of a locally continuous functor* $\mathrm{colim}_\omega : [\omega \to \mathbf{K}^e \hookrightarrow \mathbf{K}] \to \mathbf{K}$.

**Proof (sketch).** The action of $\mathrm{colim}_\omega$ on morphisms follows immediately from proposition 6.1. Indeed, where $\phi : \Phi \Rightarrow \mathrm{colim}_\omega \Phi$ and $\gamma : \Gamma \Rightarrow \mathrm{colim}_\omega \Gamma$ are the chosen $\mathbf{O}$-colimits in $\mathbf{K}$, a natural transformation $\eta : \Phi \Rightarrow \Gamma$ induces a cocone $\gamma \circ \eta : \Phi \Rightarrow \mathrm{colim}_\omega \Gamma$. By proposition 6.1, the unique mediating morphism of cocones is then:

$$\mathrm{colim}_\omega(\eta : \Phi \Rightarrow \Gamma) = \bigsqcup_{n\in\mathbb{N}}^{\uparrow} \gamma_n \circ \eta_n \circ \phi_n^p.$$

This action on morphisms is easily seen to be locally continuous. □

**Proposition 6.4** *Let* $\mathbf{K}$ *be an* $\mathbf{O}$*-cocomplete* $\mathbf{O}$*-category. The functor* $\mathsf{GFIX} = \mathrm{colim}_\omega \circ \Omega : \mathbf{OLinks_K} \to \mathbf{K}$ *is locally continuous and its image lies in* $\mathbf{K}^e$.

The recipe given by proposition 3.5 gives a locally continuous functor $\mathsf{UNF} : \mathbf{OLinks_K} \to \mathbf{K}$ whose image lies in $\mathbf{K}^e$. These functors $\mathsf{GFIX}$ and $\mathsf{UNF}$ are related by the same natural isomorphism as proposition 3.6.

We say that an $\mathbf{O}$-category $\mathbf{K}$ has **strict morphisms** if it has zero morphisms and $0_{AB}$ is the least element of $\mathbf{K}(A, B)$ for all objects $A$ and $B$. We say that $\mathbf{K}$ **supports canonical fixed points** if it has an initial object, strict morphisms, and is $\mathbf{O}$-cocomplete. Let $\mathbf{CFP}$ be the full subcategory of $\mathbf{O}$ whose objects are $\mathbf{O}$-categories that support canonical fixed points. It is 2-cartesian closed [16, Theorem 7.3.11].

Assume $\mathbf{K}$ supports canonical fixed points. Then $\perp$ is also the initial object of $\mathbf{K}^e$, and we can fully and faithfully embed $\mathbf{O}[\mathbf{K} \to \mathbf{K}]$ into $\mathbf{Links_K}$ using the same approach as before. This embedding is locally continuous. We define the locally continuous **canonical-fixed-point functor** $\mathsf{CFIX} : \mathbf{CFP}[\mathbf{K} \to \mathbf{K}] \to \mathbf{K}$ as the composition $\mathbf{CFP}[\mathbf{K} \to \mathbf{K}] \hookrightarrow \mathbf{Links_K} \xrightarrow{\mathsf{GFIX}} \mathbf{K}$. The following result is standard:

**Proposition 6.5** *If* $\mathbf{K}$ *supports canonical fixed points and* $F$ *is a locally continuous functor on* $\mathbf{K}$*, then* $\mathsf{fold} : F(\mathsf{CFIX}(F)) \to \mathsf{FIX}(F)$ *is a canonical fixed point.*

We can mimic the results of sections 4 and 5, generally replacing $\mathsf{FIX}$ by $\mathsf{CFIX}$, $\omega$-**Cat** by **O**, **IFP** by **CFP**, and $\omega$-functor by locally continuous functor. In particular, the parametrized fixed point functor $(\cdot)^\dagger$ is locally continuous and again satisfies the Conway identities up to isomorphism. It also produces canonical parametrized families of fixed points:

**Proposition 6.6** *Let* $\mathbf{D}$ *and* $\mathbf{E}$ *be* $\mathbf{O}$*-categories, and assume* $\mathbf{E}$ *supports canonical fixed points. Let* $F : \mathbf{D} \times \mathbf{E} \to \mathbf{E}$ *be a locally continuous functor. Then* $(F^\dagger, \mathsf{Fold})$ *and* $(F^\dagger, \mathsf{Unfold})$ *are respectively the initial* $F$*-algebra and terminal* $F$*-coalgebra.*

(i) *Given any other* $F$*-algebra* $(G, \gamma)$*, the mediating morphism* $\phi : F^\dagger \to G$ *is a natural family of embeddings. The component* $\phi_D$ *is the unique* $F_D$*-algebra homomorphism* $(F^\dagger D, \mathsf{Fold}_D) \to (GD, \gamma_D)$*.*

(ii) *Given any other* $F$*-coalgebra* $(\Gamma, \gamma)$*, the mediating morphism* $\rho : \Gamma \to F^\dagger$ *is a natural family of projections. The component* $\rho_D$ *is the unique* $F_D$*-coalgebra homomorphism* $(GD, \gamma_D) \to (F^\dagger D, \mathsf{Unfold}_D)$*.*

## 7 Applications to Semantics of Session Types

We illustrate our results by applying them to denotational semantics for session-typed languages. In particular, we show that they are essential both for defining and for reasoning about the denotations of recursive session types. Session types specify communication protocols between processes. We consider the restricted setting of two processes $S$ (the server) and $C$ (the client). They independently perform computation and communicate with each other over a wire $c$ (a channel). This bidirectional communication on $c$ satisfies a protocol specified by a session type $A$ that evolves over the course of execution. We can think of $S$ as a function from its input on $c$ to its output on $c$, and of $C$ as a function from its input on $c$ to its output on $c$. To do so, we imagine $c$ as a pair $(c^-, c^+)$ of wires carrying unidirectional communications: a wire $c^-$ that carries communications from $C$ to $S$, and a wire $c^+$ that carries communication from $S$ to $C$. This gives rise to the picture $S \underset{c^-}{\overset{c^+}{\rightleftarrows}} C$.

We interpret a session type $A$ as a Scott domain $[\![A]\!]$ whose elements are the bidirectional communications permitted by the protocol $A$. To interpret the picture, we decompose $[\![A]\!]$ into a pair of Scott domains $[\![A]\!]^-$ and $[\![A]\!]^+$. The domain $[\![A]\!]^+$ contains the left-to-right unidirectional communications on $c^+$ that $A$ permits, and symmetrically for $[\![A]\!]^-$. We then interpret $S$ and $C$ as continuous functions $[\![S]\!] : [\![A]\!]^- \to [\![A]\!]^+$ and $[\![C]\!] : [\![A]\!]^+ \to [\![A]\!]^-$. We can think of this interpretation as a generalization of Kahn's stream-based semantics for deterministic networks [23] to allow for bidirectional, session-typed communication channels. The details of process interpretations are beyond the scope of this paper and can be found in [24]. However, these process interpretations ensure that channels are used linearly: channels cannot be duplicated or discarded. In the more general setting of [24], processes can also communicate over multiple channels.

The decomposition of $[\![A]\!]$ into $[\![A]\!]^- \times [\![A]\!]^+$ introduces "semantic junk". Indeed,

$[\![A]\!]^- \times [\![A]\!]^+$ contains pairs $(a^-, a^+)$ of unidirectional communications that do not correspond to bidirectional communications $a \in [\![A]\!]$. We use an embedding $(\!| A |\!) : [\![A]\!] \to [\![A]\!]^- \times [\![A]\!]^+$ to pick out the pairs $(a^-, a^+)$ that correspond to genuine bidirectional communications.

We illustrate this semantic approach by giving interpretations to recursive session types. Though we only consider recursive session types here, we note that this approach works for a rich class of session types, including internal choice $(A \oplus B)$, external choice $(A \,\&\, B)$, channel transmission $(A \otimes B$ and $A \multimap B)$, synchronization $(\uparrow A$ and $\downarrow B)$, etc.

Assume that recursive session types $\rho\alpha.A$ are formed using the following rules. The judgment $\Xi \vdash A$ means that $A$ is a session type in the presence type variables $\Xi = \alpha_1, \ldots, \alpha_n$ $(n \geq 0)$.

$$\frac{}{\Xi, \alpha \vdash \alpha}\ (\textsc{CVar}) \qquad \frac{\Xi, \alpha \vdash A}{\Xi \vdash \rho\alpha.A}\ (\mathrm{C}\rho)$$

To handle open types, we generalize from triples of domains and a single embedding to a 2-cell whose components are embeddings. Let $\mathbf{M}$ be the $\mathbf{O}$-category of Scott domains where morphisms are strict continuous meet-preserving functions. It supports canonical fixed points. Let $[\![\alpha_1, \ldots, \alpha_n]\!]$ be the category $\prod_{i=1}^{n} \mathbf{M}$. Then $\Xi \vdash A$ denotes a 2-cell $\langle (\!| \Xi \vdash A |\!)^-, (\!| \Xi \vdash A |\!)^+ \rangle : [\![\Xi \vdash A]\!] \Rightarrow [\![\Xi \vdash A]\!]^- \times [\![\Xi \vdash A]\!]^+ : [\![\Xi]\!] \to \mathbf{M}$ in $\mathbf{CFP}$ where each component of the natural transformation $\langle (\!| \Xi \vdash A |\!)^-, (\!| \Xi \vdash A |\!)^+ \rangle$ is an embedding in $\mathbf{M}$. The interpretation of $\Xi \vdash A$ is defined by induction on its derivation. We note that all session types, including $A \multimap B$, are interpreted using *covariant* functors. This is in contrast to what happens with computational interpretations of linear logic, where $A \multimap B$ denotes a bifunctor that is contravariant in one component. The difference in variance stems from the fact that closed session types denote domains of communications instead of domains of values.

The functors interpreting $(\textsc{CVar})$ are projection of the $\alpha$ component:

$$[\![\Xi, \alpha \vdash \alpha]\!] = [\![\Xi, \alpha \vdash \alpha]\!]^- = [\![\Xi, \alpha \vdash \alpha]\!]^+ = \pi_{n+1}, \quad (\!| \Xi, \alpha \vdash \alpha |\!)^- = (\!| \Xi, \alpha \vdash \alpha |\!)^+ = \mathsf{id}.$$

The functors interpreting $\Xi \vdash \rho\alpha.A$ are defined using the dagger operation of section 6. We cannot simply apply the dagger operation to the interpretation of the premise $\Xi, \alpha \vdash A$. Doing so would give a 2-cell with the wrong codomain: its codomain would be $\left([\![\Xi, \alpha \vdash A]\!]^- \times [\![\Xi, \alpha \vdash A]\!]^+\right)^\dagger$, and it is not true general that $(F \times G)^\dagger \cong F^\dagger \times G^\dagger$. Instead, set

$$[\![\Xi \vdash \rho\alpha.A]\!] = [\![\Xi, \alpha \vdash A]\!]^\dagger, \quad [\![\Xi \vdash \rho\alpha.A]\!]^p = ([\![\Xi, \alpha \vdash A]\!]^p)^\dagger \ (p \in \{-, +\}).$$

Abbreviate $\Xi \vdash \rho\alpha.A$ by $\vdash \rho\alpha.A$ and $\Xi, \alpha \vdash A$ by $\alpha \vdash A$. Instantiating $\eta$ in the right diagram of corollary 4.5 by $(\!| \alpha \vdash A |\!)^p$ for $p \in \{-, +\}$ and expanding the definition of the horizontal composition $\eta * \langle \mathsf{id}, \eta^\dagger \rangle$ gives:

$$\begin{tikzcd}
\llbracket \alpha \vdash A \rrbracket \circ \langle \mathsf{id}_{\llbracket \Xi \rrbracket}, \llbracket \vdash \rho\alpha.A \rrbracket \rangle \arrow[rr, Rightarrow, "\mathsf{Fold}^{\llbracket \alpha \vdash A \rrbracket}"] \arrow[d, Rightarrow, "\llbracket \alpha \vdash A \rrbracket \langle \mathsf{id}, (\llparenthesis \alpha \vdash A \rrparenthesis^p)^\dagger \rangle"'] & & \llbracket \vdash \rho\alpha.A \rrbracket \arrow[d, Rightarrow, "(\llparenthesis \alpha \vdash A \rrparenthesis^p)^\dagger"] \\
\llbracket \alpha \vdash A \rrbracket \circ \langle \mathsf{id}_{\llbracket \Xi \rrbracket}, \llbracket \vdash \rho\alpha.A \rrbracket^p \rangle \arrow[rr, Rightarrow, "\mathsf{Fold}^{\llbracket \vdash \rho\alpha.A \rrbracket^p} \circ \llparenthesis \alpha \vdash A \rrparenthesis^p \langle \mathsf{id}, \llbracket \vdash \rho\alpha.A \rrbracket^p \rangle"] & & \llbracket \vdash \rho\alpha.A \rrbracket^p
\end{tikzcd}$$

The category of $\llbracket \alpha \vdash A \rrbracket$-algebras has products, so there exists a mediating morphism

$$\langle (\llparenthesis \alpha \vdash A \rrparenthesis^-)^\dagger, (\llparenthesis \alpha \vdash A \rrparenthesis^+)^\dagger \rangle : \llbracket \vdash \rho\alpha.A \rrbracket \Rightarrow \llbracket \vdash \rho\alpha.A \rrbracket^- \times \llbracket \vdash \rho\alpha.A \rrbracket^+ .$$

It is a natural family of embeddings by proposition 6.6, so we define:

$$\llparenthesis \vdash \rho\alpha.A \rrparenthesis = \langle (\llparenthesis \alpha \vdash A \rrparenthesis^-)^\dagger, (\llparenthesis \alpha \vdash A \rrparenthesis^+)^\dagger \rangle.$$

The parameter identity implies that these denotations respect substitution (cf. [24, Proposition 41]):

**Proposition 7.1** *Let $\Xi = \alpha_1, \ldots, \alpha_n$. If $\Xi \vdash A$ and $\Theta \vdash B_i$ for $1 \leq i \leq n$, then for $p \in \{-, +\}$,*

$$\llparenthesis \Theta \vdash [\boldsymbol{B}/\boldsymbol{\alpha}]A \rrparenthesis^p = \llparenthesis \Xi \vdash A \rrparenthesis^p * \langle \llparenthesis \Theta \vdash A_i \rrparenthesis^p \rangle_{1 \leq i \leq n}.$$

**Proof (sketch).** By induction on the derivation $\Xi \vdash A$. The variable case is clear. Consider the case where $\Xi \vdash \rho\alpha_{n+1}.A$ is formed by (C$\rho$). Let $B_{n+1} = \alpha_{n+1}$, so $\Theta, \alpha_{n+1} \vdash \alpha_{n+1}$ by (CVar). By the induction hypothesis, $\llparenthesis \Theta, \alpha_{n+1} \vdash [\boldsymbol{B}, \alpha_{n+1}/\boldsymbol{\alpha}, \alpha_{n+1}]A \rrparenthesis^p = \llparenthesis \Xi, \alpha_{n+1} \vdash A \rrparenthesis^p * \langle \llparenthesis \Theta, \alpha_{n+1} \vdash B_i \rrparenthesis^p \rangle_{1 \leq i \leq n+1}$. By a weakening lemma, $\llparenthesis \Theta, \alpha_{n+1} \vdash B_i \rrparenthesis^p = \llparenthesis \Theta \vdash B_i \rrparenthesis^p \pi_\Theta$ for $1 \leq i \leq n$. By a property of products, it follows that $\llparenthesis \Theta, \alpha_{n+1} \vdash [\boldsymbol{B}, \alpha_{n+1}/\boldsymbol{\alpha}, \alpha_{n+1}]A \rrparenthesis^p = \llparenthesis \Xi, \alpha_{n+1} \vdash A \rrparenthesis^p * (\langle \llparenthesis \Theta \vdash B_i \rrparenthesis^p \rangle_{1 \leq i \leq n} \times \mathsf{id})$. By the parameter identity and the interpretation of (C$\rho$), we conclude that $\llparenthesis \Theta \vdash [\boldsymbol{B}/\boldsymbol{\alpha}]\rho\alpha_{n+1}.A \rrparenthesis^p = \llparenthesis \Xi \vdash \rho\alpha_{n+1}.A \rrparenthesis^p * \langle \llparenthesis \Theta \vdash B_i \rrparenthesis^p \rangle_{1 \leq i \leq n}$. ☐

Propositions 4.4 and 7.1 and corollary 4.6 imply that the above denotations respect syntactic folding and unfolding of recursive up-to-isomorphism, e.g.,

$$\llbracket \Xi \vdash \rho\alpha.A \rrbracket \cong \llbracket \Xi, \alpha \vdash A \rrbracket \circ \langle \mathsf{id}_{\llbracket \Xi \rrbracket}, \llbracket \Xi \vdash \rho\alpha.A \rrbracket \rangle = \llbracket \Xi \vdash [\rho\alpha.A/\alpha]A \rrbracket .$$

# 8 Related Work

Scott [34] introduced inverse limit constructions to construct fixed points of functors. In particular, Scott used an inverse limit of a chain of projections to construct a continuous lattice $D \cong [D \to D]$. Until this point, the only tools for constructing fixed points were variations on Tarski's least fixed-point theorem [27, p. 9]. Lehmann [27] generalized these ideas to find fixed points of $\omega$-cocontinuous functors on $\omega$-cocomplete categories. These ideas were further explored by Lehmann and Smyth [28, 29] to give semantics to data types. We built on these ideas to define a general fixed-point functor $\mathsf{GFIX}$. Using $\mathsf{GFIX}$, we were able to show that a functor's fixed points assemble into a natural isomorphism. Their fixed-point functor is exactly $\mathsf{FIX}$, while their parametrized fixed-point functor is our $(\cdot)^\dagger$.

Wand [40] introduced the definitions of **O**-categories and locally continuous functors. Smyth and Plotkin [36, 37] introduced **O**-(co)limits and generalized Scott's limit-colimit coincidence theorem to **O**-categories. **O**-categories generalize categories of domains to provide just the structure required to solve recursive domain equations in a categorical setting. Smyth and Plotkin's "basic lemma" [37, Lemma 2] gives a recipe for constructing fixed points of covariant locally continuous functors on **O**-categories.

Some took the existence of fixed points of functors as their starting point. Freyd [17] studied algebraically complete categories, that is, categories $\mathbf{C}$ where every covariant functor $T : \mathbf{C} \to \mathbf{C}$ has a an initial $T$-algebras. Freyd also studied properties of functors on algebraically complete categories. Freyd [18] extended this analysis to algebraically compact categories, i.e., algebraically complete categories where initial algebra and terminal co-algebras are canonically isomorphic.

Fiore [16] investigated axiomatic categorical domain theory for application to the denotational semantics of deterministic programming languages. In chapter 6, Fiore used initiality to define a dagger operation on functors between certain algebraically complete **O**-categories. Under certain conditions, this dagger operation is functorial. It satisfies the parameter identity on functors, i.e., it satisfies eq. (8) above. Our category **CFP** appears as the category **Kind** [16, Definition 7.3.11].

Dagger operation and the Conway identities arose in a separate line of research. Iteration theories [11] were introduced to study the syntax and semantics of flowchart algorithms, and they are defined in terms of a dagger operation. Bloom and Ésik [10] studied external dagger operations on cartesian closed categories and showed that for many of the categories used in semantics, the least fixed point operator induces a dagger operation satisfying the Conway identities. They generalized this work to 2-cartesian closed categories in [12] and gave sufficient conditions for a dagger on horizontal morphisms to satisfy the Conway identities. They did not explore the 2-cartesian structure of daggers or the action of daggers on vertical morphisms.

Simpson and Plotkin [35] gave an axiomatic treatment of dagger operations satisfying Conway identities. They gave a purely syntactic account of free iteration theories. They give a precise characterization of the circumstances in which the iteration theory axioms are complete for categories with an iteration operator.

Honda [21] and Takeuchi, Honda, and Kubo [38] introduced session types to describe sessions of interaction. Caires and Pfenning [13] observed a proofs-as-programs correspondence between the session-typed $\pi$-calculus and intuitionistic linear logic. Wadler [39] built on this correspondence to give "Classical Processes" (CP), a proofs-as-programs interpretation of classical linear logic. These proofs-as-programs correspondences have several denotational semantics. For example, Atkey [3] gave a denotational semantics for CP, where types are interpreted as sets and processes are interpreted as relations over these. Castellan and Yoshida [14] gave a game semantics interpretation of the session $\pi$-calculus with recursion. Kavanagh [24] gave the first denotational semantics for a full-featured, functional language with session-typed concurrency and general recursion.

Linear logic enjoys other proofs-as-programs interpretations. Benton [7, 8] introduced the LNL calculus, a mixed linear and non-linear calculus. It is interpreted by an "LNL" or "adjoint" model: a symmetric monoidal closed category and a cartesian closed category related by a pair of adjoint functors. Benton and Wadler [6] used this model to relate translation of the $\lambda$-calculus in Moggi's computational metalanguage [31] and translations of intuitionistic logic into intuitionistic linear logic. Lindenhovius, Mislove, and Zamdzhiev [30] introduced the "linear/non-linear fixpoint calculus" (LNL-FPC), a type system with mixed linear and non-linear recursive types. They use the dagger operator of [28] to model arbitrary recursive types in a linear category and non-linear recursive types in a cartesian category. These two interpretations are strongly related by suitable mediating functors and natural isomorphisms, which allow them to define substructural operations on non-linear types. To give fixed points to contravariant functors, they used standard order-theoretic techniques [28, Theorem 3] to reduce contravariant functors to covariant functors.

## 9 Conclusion and Acknowledgments

We gave a generalized-fixed-point functor and a functorial dagger operation on $\omega$-categories. We explored the 2-categorical structure of the dagger operation, and showed how the Conway identities follow from this 2-categorical structure. We showed that these constructions also hold for **O**-categories, and we explored their order-theoretic properties. In section 7, we saw that the Conway identities and the dagger operation's order-theoretic properties were essential for defining the semantics of recursive session types.

The presentation of these results has changed considerably since the conference. The original version frequently relied on order-theoretic reasoning in **O**-categories to establish results. MFPS attendees, Vladimir Zamdzhiev in particular, suggested that it might be possible to generalize from locally continuous functors on **O**-categories to $\omega$-functors on $\omega$-categories. Doing so greatly elucidated our results, and we are grateful to the attendees for their helpful comments and suggestions.

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