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Nonlinear Gaussian Mixture Approximation Smoother

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Abstract

This paper presents a novel Gaussian mixture approximation smoother (GMA) for estimation the nonlinear system state. The smoothing implementation is transformed into computing some Gaussian weighted integrals, which triggers the development of the new GMA-URTSS algorithm by applying unscented transformation (UT). Simulation application demonstrates the superior performance of the proposed method.

Keywords: Nonlinear state, smoother, Gaussian mixture approximation, unscented transformation

1. Introduction

Nonlinear state smoothing estimation has been attracting considerable research interests (see, e,g., [1]-[3] and the references therein) because of its widespread applications in signal processing, navigation and communication. The smoothing problem involves computing the probability density function (PDF) of a stochastic dynamic state $\boldsymbol{x}_k$ conditioned on a sequence of noisy measurements $\boldsymbol{z}_N \in \{z_1, z_2, \cdots, z_k, \cdots z_N\}$. The general optimal fixed-interval smoothing scheme in conjunction with the Bayesian rule [4], is given by the functional recursive relations for computation of the PDFs $p(\boldsymbol{x}_k|\boldsymbol{z}_N)$ from $p(\boldsymbol{x}_{k+1}|\boldsymbol{z}_N)$ for all $k \in \{0, 1, \cdots, N\}$. For the nonlinear system, the exact description of such posterior smoothing PDFs by way of the Bayesian recursive equations is unavailable and intractable since the Bayesian solution requires the propagation of full probability density which results in the computation intensive in numerical implementation [5]. Approximations

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are thus necessary, and the Gaussian approximation (GA) to such PDFs is widely accepted by the fact that the correspondingly derived GA smoother is always cost-effective with acceptable accuracy desirable in practical applications. So far, a large number of GA smoother have been proposed for smoothing the nonlinear state from a reentry tracking problem [1]-[2], including the extended Kalman smoother (EKS) based on the first-order linearization, the unscented Rauch–Tung–Striebel smoother (URTSS) based on the unscented transformation (UT) , the quadrature Kalman smoother (QKS) based on the quadrature rule, the central difference Kalman smoother (CDKS) based on the polynomial interpolation and the cubature Klaman smoother (CKS) based on the cubature rule.

In general, the above-mentioned GA smoothers approximate the state PDF using a single Gaussian, such that they are only suitable for the case that the state PDF is the gaussianity or unimodality, and that the system model is at most weakly nonlinear. However, a state which is initially Gaussian will inevitably become significantly non-Gaussian if propagated over a sufficiently long time span. In this situation, although the single GA smoother is better than the corresponding GA filter, it still inevitably results in inaccurate state estimate. Consequently, a Gaussian mixture approximation (GMA) to the smoothing PDF is more appropriate than the GA as a result that: 1) it can effectively approximate any PDF as closely as desired (in the L^1 norm) [6] by using a weighted sum of Gaussian densities, and 2) it enables a more accurate representation of the nonlinearities in the dynamics and measurement models. Therefore, there is a great demand to develop the corresponding GMA smoother for nonlinear system in a reentry tracking problem. Up to the present, an explicit and systematic solution to the derivation of the GMA smoother has been seldom reported.

Here, motivated by the fact that the GMA is more accurate than the GA, we present the GMA smoother for the ballistic reentry tracking problem. The rest of the paper is organized as follows. Section II formulates the investigated problem and recalls the standard GA filter and smoother. Section III derives the GMA smoother on the basis of the standard GA estimators. Section IV gives the simulation application for the reentry tracking problem. Some conclusions are drawn in section V.

2. Problem Formulation and Standard GA Estimator

We consider the following discrete-time nonlinear stochastic system for generally describing the problem of tracking a ballistic target on reentry

$$\boldsymbol{x}_{k+1} = \boldsymbol{f}_k(\boldsymbol{x}_k) + \boldsymbol{w}_k , \tag{1}$$

$$\boldsymbol{z}_k = \boldsymbol{h}_k(\boldsymbol{x}_k) + \boldsymbol{v}_k , \tag{2}$$

where $\boldsymbol{x}_k \in \mathbb{R}^n$ is the state vector, $\boldsymbol{z}_k \in \mathbb{R}^m$ is the measurement vector, $\boldsymbol{f}_k(\bullet)$ and $\boldsymbol{h}_k(\bullet)$ are known nonlinear functions, $\boldsymbol{w}_k \in \mathbb{R}^n$ and $\boldsymbol{v}_k \in \mathbb{R}^m$ are uncorrelated zero-mean Gaussian white noises satisfying $\mathrm{E}[\boldsymbol{w}_k \boldsymbol{w}_l^{\mathrm{T}}] = \boldsymbol{Q}_k \delta_{kl}$ and $\mathrm{E}[\boldsymbol{v}_k \boldsymbol{v}_l^{\mathrm{T}}] = \boldsymbol{R}_k \delta_{kl}$ where δ_{kl} is the Kronecker delta function, and the initial state is assumed to be Gaussian distribution with mean $\hat{\boldsymbol{x}}_{0|0}$ and covariance $\boldsymbol{P}_{0|0}$, which is independent of $\boldsymbol{w}_k$ and $\boldsymbol{v}_k$.

Our aim is to design the GMA smoother for the system in (1)-(2). In other words, we need to find the Gaussian mixture approximation to the smoothing PDFs $p_{\mathrm{GMA}}(\boldsymbol{x}_k \mid \boldsymbol{Z}_N)$ for all $k \in \{0, \ 1, 2, \cdots, N\}$, where $\boldsymbol{Z}_N = \{\boldsymbol{z}_i\}_{i=1}^N$. Then, the smoothing estimation and the associated covariance in the GMA form can be obtained by computing the first two moments of $p_{\mathrm{GMA}}(\boldsymbol{x}_k \mid \boldsymbol{Y}_N)$.

In [6], Ito and Xiong presented a general GA filter, whose implementation can be separated into two steps including the state prediction and correction. Afterwards, [2] and [3] provided the GA smoother, which is implemented as follows.

$$\begin{cases} \hat{\boldsymbol{x}}_{k|N} = \hat{\boldsymbol{x}}_{k|k} + \boldsymbol{A}_k [\hat{\boldsymbol{x}}_{k+1|N} - \hat{\boldsymbol{x}}_{k+1|k}], \ \boldsymbol{A}_k = \boldsymbol{P}_{k,k+1|k} \boldsymbol{P}_{k+1|k}^{-1}, \\ \boldsymbol{P}_{k|N} = \boldsymbol{P}_{k|k} + \boldsymbol{A}_k [\boldsymbol{P}_{k+1|N} - \boldsymbol{P}_{k+1|k}] \boldsymbol{A}_k^{\mathrm{T}}, \end{cases} \tag{3}$$

$$\boldsymbol{P}_{k,k+1|k} = \int \boldsymbol{x}_k \boldsymbol{f}_k^{\mathrm{T}}(\boldsymbol{x}_k) \mathcal{N}(\boldsymbol{x}_k; \hat{\boldsymbol{x}}_{k|k}, \boldsymbol{P}_{k|k}) d\boldsymbol{x}_k - \hat{\boldsymbol{x}}_{k|k} \hat{\boldsymbol{x}}_{k+1|k}^{\mathrm{T}}. \tag{4}$$

In general, the GA smoother in (3) is a forward-backward smoother [3]. As seen from Fig. 1, in the forward pass, the GA smoother uses the GA filter to compute the filtering density up to $p(\boldsymbol{x}_N \mid \boldsymbol{Z}_N)$. In the backward pass, it implements (3) for rolling backwards from $p_{\mathrm{GA}}(\boldsymbol{x}_{N-1} \mid \boldsymbol{Z}_N)$ to $p_{\mathrm{GA}}(\boldsymbol{x}_0 \mid \boldsymbol{Z}_N)$. Through initialization of $p_{\mathrm{GA}}(\boldsymbol{x}_N \mid \boldsymbol{Y}_N) = \mathcal{N}(\boldsymbol{x}_N; \hat{\boldsymbol{x}}_{N|N}, \boldsymbol{P}_{N|N})$, the GA smoother recursively operates by joint the analytical computation in (3) and the Gaussian weighted integrals in (4).

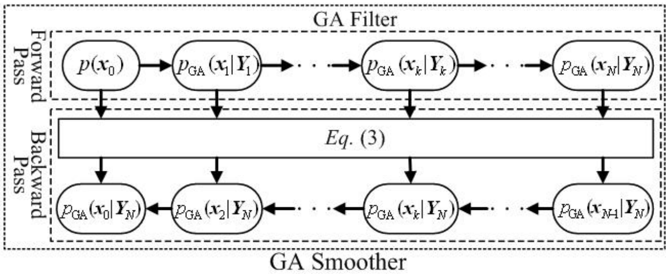

Fig. 1 The forward and backward passes in the GA smoother

The heart of implementing the GA smoother is how to compute the nonlinear Gaussian weighted integrals in (4). However, for the nonlinearity of $\boldsymbol{f}_k(\bullet)$ and $\boldsymbol{h}_k(\bullet)$, the GA smoother is only formal in the sense that it rarely can be directly used in practice since the model nonlinearity causes the intractability and infeasibility of analytically computing such integrals. For this reason, some numerical methods are required for approximating such integrals, e.g., the linearization and the corresponding GA smoother is the existing GA-EKS in [2]. At the same time, as a result of the better precision of unscented transformation (UT) than the linearization, we can also develop the existing GA-URTSS in [2] by using UT for computing such integrals.

3. Problem Formulation and Standard GA Estimator

For the derivation of GMA smoother, a standard result about Gaussian functions in [7] is recalled.

***Lemma* 1**. Given $\boldsymbol{F}$, $\boldsymbol{d}$, $\boldsymbol{Q}$, $\boldsymbol{m}$ and $\boldsymbol{P}$ of appropriate dimensions, and that $\boldsymbol{Q}$ and $\boldsymbol{P}$ are positive definition, it can be obtained that

$$\int \mathcal{N}(\boldsymbol{x}, \boldsymbol{F}\boldsymbol{\zeta} + \boldsymbol{d}, \boldsymbol{Q}) \mathcal{N}(\boldsymbol{\zeta}, \boldsymbol{m}, \boldsymbol{P}) \mathrm{d}\boldsymbol{\zeta} = \mathcal{N}(\boldsymbol{x}, \boldsymbol{F}\boldsymbol{m} + \boldsymbol{d}, \boldsymbol{Q} + \boldsymbol{F}\boldsymbol{P}\boldsymbol{F}^{\mathrm{T}}). \tag{5}$$

Firstly, we have to find the GMA filter.

***Step* 1**: The GMA filter. According to [6], given the Gaussian sum approximation

$$p_{\mathrm{GMA}}(\boldsymbol{x}_k \mid \boldsymbol{Z}_k) = \sum_{i=1}^{M} \alpha_k^i p_{\mathrm{GA}}^i(\boldsymbol{x}_k \mid \boldsymbol{Z}_k) = \sum_{i=1}^{M} \alpha_k^i \mathcal{N}(\boldsymbol{x}_k; \hat{\boldsymbol{x}}_{k|k}^i, \boldsymbol{P}_{k|k}^i), \sum_{i=1}^{M} \alpha_k^i = 1, \tag{6}$$

the GMA filtering PDF can be obtained as follows

$$p_{\mathrm{GMA}}(\boldsymbol{x}_{k+1} \mid \boldsymbol{Z}_{k+1}) = \sum_{i=1}^{M} \alpha_{k+1}^i p_{\mathrm{GA}}^i(\boldsymbol{x}_{k+1} \mid \boldsymbol{Z}_{k+1}) = \sum_{i=1}^{M} \alpha_{k+1}^i \mathcal{N}(\boldsymbol{x}_{k+1}; \hat{\boldsymbol{x}}_{k+1|k+1}^i, \boldsymbol{P}_{k+1|k+1}^i), \tag{7}$$

where each filter is independent of the others and can be performed in a parallel manner, and the weights α_{k+1}^i

are updated and normalized by learning from Eqs. (5.1)-(5.2) in [6]

$$\alpha_{k+1}^{i}=\frac{\alpha_{k}^{i}\mathcal{N}(\boldsymbol{z}_{k+1};\hat{\boldsymbol{z}}_{k+1|k}^{i},\boldsymbol{P}_{k+1|k}^{zz,i})}{\sum_{i=1}^{M}\alpha_{k}^{i}\mathcal{N}(\boldsymbol{z}_{k+1};\hat{\boldsymbol{z}}_{k+1|k}^{i},\boldsymbol{P}_{k+1|k}^{zz,i})}, \tag{8}$$

Hence, the GMA filtering estimate and the corresponding covariance are obtained by computing the first two moments of the PDF in (7)

Step 2: The GMA smoother. The PDF $p_{\mathrm{GMA}}(\boldsymbol{x}_k \mid \boldsymbol{x}_{k+1},\boldsymbol{z}_k)$ can be expressed by the GMA method, i.e.

$$p_{\mathrm{GMA}}(\boldsymbol{x}_k \mid \boldsymbol{x}_{k+1},\boldsymbol{Z}_k)=\sum_{i=1}^{M}\beta_k^i p_{\mathrm{GA}}^i(\boldsymbol{x}_k \mid \boldsymbol{x}_{k+1},\boldsymbol{Z}_k)=\sum_{i=1}^{M}\beta_k^i\mathcal{N}(\boldsymbol{x}_k;\hat{\boldsymbol{x}}_{k|k+1,k}^i,\boldsymbol{P}_{k|k+1,k}^i), \tag{9}$$

where $\hat{\boldsymbol{x}}_{k|k+1,k}^i$ and $\boldsymbol{P}_{k|k+1,k}^i$ are given in (10), and the weights β_k^i are updated and normalized by learning from Eqs. (5.1)-(5.2) in [6]

$$\begin{cases}\hat{\boldsymbol{x}}_{k|k+1,k}^i=\hat{\boldsymbol{x}}_{k|k}+\boldsymbol{A}_k(\boldsymbol{x}_{k+1}-\hat{\boldsymbol{x}}_{k+1|k}),\\ \boldsymbol{P}_{k|k+1,k}^i=\boldsymbol{P}_{k|k}-\boldsymbol{A}_k\boldsymbol{P}_{k+1|k}\boldsymbol{A}_k^{\mathrm{T}},\end{cases} \tag{10}$$

$$\beta_k^i=\frac{\alpha_k^i\mathcal{N}(\boldsymbol{x}_{k+1};\hat{\boldsymbol{x}}_{k+1|k}^i,\boldsymbol{P}_{k+1|k}^i)}{\sum_{i=1}^{M}\alpha_k^i\mathcal{N}(\boldsymbol{x}_{k+1};\hat{\boldsymbol{x}}_{k+1|k}^i,\boldsymbol{P}_{k+1|k}^i)}. \tag{11}$$

Assume that the GMA smoothing PDF at $k+1$ has been given

$$p_{\mathrm{GMA}}(\boldsymbol{x}_{k+1} \mid \boldsymbol{Z}_N)=\sum_{j=1}^{L}\varphi_{k+1}^j p_{\mathrm{GA}}^j(\boldsymbol{x}_{k+1} \mid \boldsymbol{Z}_N)=\sum_{j=1}^{L}\varphi_{k+1}^j\mathcal{N}(\boldsymbol{x}_{k+1};\hat{\boldsymbol{x}}_{k+1|N}^j,\boldsymbol{P}_{k+1|N}^j),\ \sum_{j=1}^{L}\varphi_{k+1}^j=1. \tag{12}$$

Then through using *Lemma* 1, the smoothed PDF $p_{\mathrm{GMA}}(\boldsymbol{x}_k \mid \boldsymbol{Y}_N)$ at k can be updated from (9) and (12)

$$\begin{aligned}p_{\mathrm{GMA}}(\boldsymbol{x}_k \mid \boldsymbol{Z}_N)&=\int p_{\mathrm{GMA}}(\boldsymbol{x}_k,\boldsymbol{x}_{k+1} \mid \boldsymbol{Z}_N)\mathrm{d}\boldsymbol{x}_{k+1}=\int p_{\mathrm{GMA}}(\boldsymbol{x}_{k+1} \mid \boldsymbol{Z}_N)p_{\mathrm{GMA}}(\boldsymbol{x}_k \mid \boldsymbol{x}_{k+1},\boldsymbol{Z}_k)\mathrm{d}\boldsymbol{x}_{k+1}\\ &=\sum_{i=1}^{M}\sum_{j=1}^{L}\beta_k^i\varphi_{k+1}^j\mathcal{N}\{\boldsymbol{x}_k;\hat{\boldsymbol{x}}_{k|k}^i+\boldsymbol{A}_k^i[\hat{\boldsymbol{x}}_{k+1|N}^j-\hat{\boldsymbol{x}}_{k+1|k}^i],\ \ \boldsymbol{P}_{k|k}^i+\boldsymbol{A}_k^i[\boldsymbol{P}_{k+1|N}^j-\boldsymbol{P}_{k+1|k}^i](\boldsymbol{A}_k^i)^{\mathrm{T}}\}\\ &=\sum_{s=1}^{ML}\varphi_k^s p_{\mathrm{GA}}^s(\boldsymbol{x}_k \mid \boldsymbol{Z}_N)=\sum_{s=1}^{ML}\varphi_k^s\mathcal{N}(\boldsymbol{x}_k;\hat{\boldsymbol{x}}_{k|N}^s,\boldsymbol{P}_{k|N}^s).\end{aligned} \tag{13}$$

Hence, the GMA smoothing estimate and the corresponding covariance are obtained by computing the first two moments of the PDF in (13)

Note that as shown in (3), the GA smoother starts from the initial density $p_{\mathrm{GA}}(\boldsymbol{x}_N \mid \boldsymbol{Y}_N)$. Similarly, the GA smoother is the foundation of developing the GMA smoother, hence, according to (7), the GMA smoother is initialized as follows, i.e.

$$p_{\mathrm{GMA}}(\boldsymbol{x}_N \mid \boldsymbol{Y}_N)=\sum_{j=1}^{L}\varphi_N^j p_{\mathrm{GA}}^j(\boldsymbol{x}_N \mid \boldsymbol{Y}_N)=\sum_{j=1}^{L}\varphi_N^j\mathcal{N}(\boldsymbol{x}_N;\hat{\boldsymbol{x}}_{N|N}^j,\boldsymbol{P}_{N|N}^j), \tag{14}$$

where $1 \le i = j \le L = M$ and $\varphi_N^j = \alpha_N^i$.

In addition, the number of the GA terms in (13) grows exponentially, due to the fact that the posterior PDFs in (9) and (12) have a Gaussian mixture form. Therefore, in order to decrease the computational complexity, it is necessary to reduce the number of the GA terms. One simplest way of mixture reduction is to retain only a few Gaussians with highest weights, where, if $\{\varphi_k^j,\ j=1,\cdots,L\}$ are the L highest weights all of $\{\varphi_k^s,\ s=1,\cdots,ML\}$, the GMA smoother in (13) becomes

$$p_{\text{GMA}}(\boldsymbol{x}_k \mid \boldsymbol{Z}_N) = \sum_{j=1}^{L} \varphi_k^j \mathcal{N}(\boldsymbol{x}_k ; \hat{\boldsymbol{x}}_{k|N}^j, \boldsymbol{P}_{k|N}^j),\ \varphi_k^j = \frac{\varphi_k^j}{\sum_{j=1}^{L} \varphi_k^j}. \tag{15}$$

About the other reduction ways, please see [8].

4. Simulation

We describe the problem of tracking a ballistic target on reentry as the following nonlinear dynamic model [9]

$$\begin{cases} x_{1,k+1} = x_{1,k} - \delta x_{2,k}, \\ x_{2,k+1} = x_{2,k} - \delta[e^{-\gamma x_{1,k}} (x_{2,k})^2 x_{3,k} + \mathrm{g}], \\ x_{3,k+1} = x_{3,k}. \end{cases} \tag{16}$$

where x_1 and x_2 are altitude and velocity, respectively; x_3 is a constant ballistic coefficient that depends on the target's mass, shape, cross-sectional area and air density; g is gravity ($\mathrm{g}=9.81\mathrm{m/s}^2$) and the constant $\gamma=5\times10^{-4}$.

For the model problem at hand, a radar is located at (0, H) and equipped to measure the range z at a sampling time interval of T. Hence, the measurement equation is given by

$$z_k = \sqrt{M^2 + (x_{1,k} - H)^2} + v_k, \tag{17}$$

where the measurement noise $v_k \sim \mathcal{N}(0, R_k)$, and M is the horizontal distance.

Here, we select the accumulative root mean square error (ARMSE) for comparing the GA and GMA smoothing performance. For the GMA in (7) and (15), the number of the GA terms is $M = L = 50$. κ in UT is chosen as 0. Each trajectory was simulated for 100s and a total of $N=1000$.The experimental data is that $H=1.2\text{km}$, $M=6\text{km}$, $R=(20\text{m})^2$, $\delta=0.5\text{s}$, the initial true state $x_0=[50\text{km}, 2010\text{m/s}, 7\times10^{-4}]$ and in simulation, the initial state estimation $\hat{\boldsymbol{x}}_{0|0}=[50\text{km}, 2010\text{m/s}, 10^{-4}]$ and covariance $\boldsymbol{P}_{0|0}=\text{diag}[10^6, 10^5, 10^{-4}]$.

The ARMSE results of altitude and velocity are shown in Fig. 3 and Fig. 4, respectively.

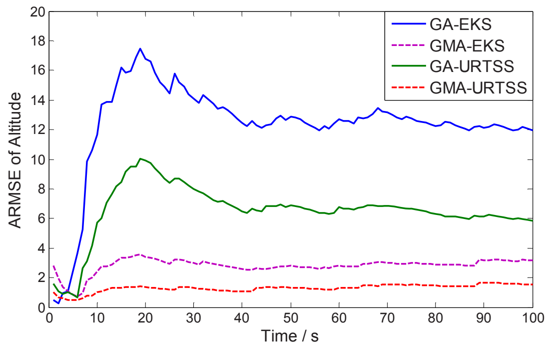

Fig. 2 ARMSE of altitude

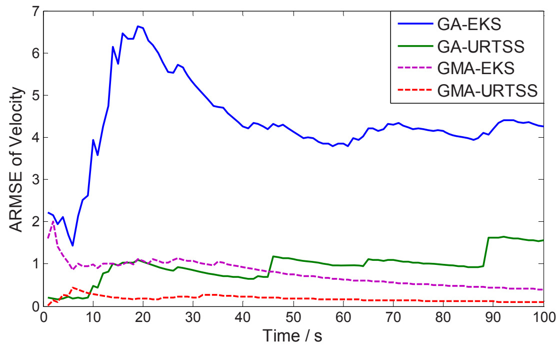

Fig. 3 ARMSE of velocity

In estimation accuracy, the proposed GMA-URTSS and GMA-EKS algorithms outperform the existing GA-URTSS and GA-EKS in [2], respectively. Moreover, expect for some time steps, the ARMSE of the GMA-URTSS is less than that of the GMA-EKS since the UT technique is superior to the first-order linearization in approximating the posterior mean and covariance of the nonlinear state. Fig. 2 and Fig. 3 make us believe that the GMA smoother performs significantly better than the corresponding GA smoother in estimation precision.

5. Conclusion

Motivated by the fact that the GMA is more accurate than the GA, based on the existing standard GA filter and smoother, we propose a general GMA smoother for estimating the nonlinear state from racking a ballistic target on reentry. Furthermore, the new GMA-URTSS algorithm is developed by using UT for numerically computing the Gaussian weighted integrals in GMA smoother. The simulations show that the proposed GMA-URTSS outperforms the corresponding GA-URSS by its superior estimation accuracy.

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