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Probabilistic Logic over Paths

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Abstract

We introduce a probabilistic modal logic **PPL** extending the work of [13,12] by allowing arbitrary nesting of a path probabilistic operator and we prove its completeness. We prove that our logic is strictly more expressive than other logics such as the logics cited above. By considering a probabilistic extension of CTL we show that this additional expressive power is really needed in some applications.

Keywords: Probabilistic Logics, Modal Logic, Completeness

There are many papers combining logic and probability, the ones mentioned here are far from exhaustive [4,18,19,10,11,1,14,15], [16,3,8,22,7,5,20]. Some of these papers deal with probabilistic model checking and some deal with theorem proving for probabilistic logics. There are very few probabilistic logics with complete proof systems, the only ones we are aware of are based on [12,13,21] and we will focus on this type of logic in this paper. One serious restriction to these languages is their inability to express non-linear probabilistic terms. The lack of multiplication in the signature means that they cannot express such basic notions as independence of events and conditional probability. The other limitation of these languages is that the probability operator applies to single formulas, rather than sequences of formulas. But there are probabilistic versions of CTL which really require us to express probabilities of sequences of formulas (or branches in a probabilistic model), rather than probabilities of single individual formulas. Unfortunately no deduction system is known to be complete for these probabilistic versions of CTL. The focus of the current paper is to extend known probabilistic logics so as to allow us to express probabilities of sequences of probabilistic formulas. We provide axioms and rules for our logic and prove their soundness and completeness. We also show that

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our logic is strictly more expressive than the previously cited probabilistic logics, which lack path expressiveness.

The previously mentioned logic of [12] is actually a probabilistic epistemic logic (we call this logic **PEL** following [17]). Since our interest here has more to do with probabilistic reasoning than epistemic reasoning, we consider the non-epistemic part of **PEL** and name this $\mathbf{PEL}_\times$. $\mathbf{PEL}_\times$ is a probabilistic propositional logic in which we can refer to and compare probabilities of formulas (see definition 2.3 below), for example we can write $\mathbf{P}(p) > \mathbf{P}(\neg q)$. The probability operators can be nested, so we can write $\mathbf{P}(\mathbf{P}(p) > \mathbf{P}(\neg q)) < 2 \times \mathbf{P}(\neg p)$. The language also allows us to form arbitrary linear combinations of terms involving probabilities.

The outline of the paper is as follows: Over the next section we define a probabilistic modal logic called **PPL** (**P**ath **P**robabilistic **L**ogic). **PPL** is a probabilistic branching logic with some similarities to the logic in [16,3]. In section 3 we present a number of worked examples illustrating the use of this logic and we relate our logic with other known logics of the field. We show that **PPL** is strictly more expressive than $\mathbf{PEL}_\times$ and we illustrate also the different expressive power between **PPL** and the probabilistic version of CTL [16]. In section 4 we prove that **PPL** has the finite model property, we provide axioms and inference rules and prove their completeness, thereby solving an important open problem. To quote [21, page 2], "To the best of our knowledge, so far no sound and complete proof system for the branching-time probabilistic logics has been proposed".

1 Probabilistic Path Logic PPL

We start by setting up the syntax of **PPL**, we define its semantics and we present examples for illustrating its use. Let $\mathbf{PROP} = \{p, q, r, \ldots\}$ be a non empty set of propositional variables.

Definition 1.1

- Formulas of the language **PPL** are built out of numerical terms τ as follows. In the recursive definitions below, $p \in \mathbf{PROP}$, $\phi, \phi_1, \ldots$ represent **PPL** formulas and τ, τ_1, τ_2 represent terms.

  $$\tau := r \mid \mathbf{P}^n(\phi_1, \ldots, \phi_n) \mid (\tau_1 + \tau_2) \mid (\tau_1 \times r) \quad (r \in \mathbb{Q},\ n \in \mathbb{N})$$
  $$\phi := p \mid \tau > 0 \mid \neg\phi \mid \phi \wedge \psi$$

- $deg(\phi)$ denotes the degree of a formula ϕ and $deg(\tau)$ is the degree of a term τ, defined as follows: $deg(p) = deg(r) = 0 \quad (p \in Prop,\ r \in \mathbb{Q})$, $deg(\neg\phi) = deg(\phi)$, $deg(\phi \wedge \psi) = max\{deg(\phi), deg(\psi)\}$, $deg(\tau_1 + \tau_2) = max\{deg(\tau_1), deg(\tau_2)\}$, $deg(\tau \times r) = deg(\tau)$, $deg(\tau > 0) = deg(\tau)$, $deg(\mathbf{P}^k(\phi_1, \ldots, \phi_k)) = max\{0, 1 + deg(\phi_1), 2 + deg(\phi_2), \ldots, k + deg(\phi_k)\}$

Notation

Throughout the paper a sequence of formulas $(\phi_1, \ldots, \phi_k)$ will be denoted $\overline{\phi}$ (some $k \in \mathbb{N}$, some formulas $\phi_1, \ldots, \phi_k$), in other words it will be implicit that the

m'th formula in the sequence $\overline{\phi}$ is ϕ_m (for $1 \leq m \leq k$). Therefore we write $\mathbf{P}^n\overline{\phi}$ instead of $\mathbf{P}^n(\phi_1, \ldots, \phi_n)$. When we wish to refer to a number of different sequences of formulas, we will use superscripts (e.g. $\overline{\phi^1}, \overline{\phi^2}, \ldots$) to avoid confusion with this convention. The length of the sequence $\overline{\phi}$ is $|\overline{\phi}|$. For $0 \leq i \leq |\overline{\phi}|$, we write $\overline{\phi \restriction_i}$ for the restriction $(\phi_1, \ldots, \phi_i)$ of the sequence, noting that $\overline{\phi \restriction_0}$ is the empty sequence. $\overline{\phi}[m/\psi]$ is obtained from $\overline{\phi}$ by substituting the formula ψ for the m'th element ϕ_m. We refer to the last element of a non-empty sequence $\overline{\phi}$ as $last(\overline{\phi}) = \phi_{|\overline{\phi}|}$. If f is a function with finite domain X we may write $(f(x) : x \in X)$ for the sequence $(f(x_0), \ldots, f(x_{k-1}))$, where $x_0, \ldots, x_{k-1}$ is some arbitrary but fixed enumeration of X.

A term of the form r or $\mathbf{P}^n\overline{\phi}$ is called a *primitive term*, and $\mathbf{P}^n\overline{\phi}$ should be thought of as 'the probability of the sequence $\overline{\phi}$'. Formulas of the form $\tau > 0$, p are called *primitive formulas*. Primitive formulas and negated primitive formulas are called *literals*. We write $(\phi \vee \psi)$ as an abbreviation of $\neg(\neg\phi \wedge \neg\psi)$ and $(\phi \to \psi)$ for $\neg(\phi \wedge \neg\psi)$. A *disjunctive normal form* (DNF) is a disjunction of conjunctions of literals. We write $\tau_1 > \tau_2$ as an abbreviation for the primitive formula $(\tau_1 + (-1) \times \tau_2) > 0$, and $\tau_1 = \tau_2$ abbreviates the non-primitive formula $\neg(\tau_1 > \tau_2) \wedge \neg(\tau_2 > \tau_1)$. We may write $\tau_2 < \tau_1$ instead of $\tau_1 > \tau_2$, when convenient. We will usually write $\mathbf{P}\overline{\phi}$ instead of $\mathbf{P}^k\overline{\phi}$, where k is implicitly defined to be the length of the sequence $\overline{\phi}$. The empty sequence is written (). For a sequence (ϕ) of length one, we may write $\mathbf{P}\phi$ instead of $\mathbf{P}^1(\phi)$. Given two sequences of formulas $\overline{\phi}$ and $\overline{\psi}$, we denote the concatenation of the two sequences by $\overline{\phi\psi}$. So, for $1 \leq l \leq |\overline{\psi}| + |\overline{\psi}|$, we have:

$$(\overline{\phi\psi})_l = \begin{cases} \phi_l, & \text{if } 1 \leq l \leq |\overline{\phi}|; \\ \psi_{l-|\overline{\phi}|} & \text{if } |\overline{\phi}| < l \leq |\overline{\phi}| + |\overline{\psi}| \end{cases}$$

We may write $\phi\overline{\phi}$ or $(\phi : \overline{\phi})$ for the concatenation of the one element sequence (ϕ) with the sequence $\overline{\phi}$, and similarly $\overline{\phi}\phi$ is the concatenation of $\overline{\phi}$ and the one element sequence (ϕ). Now we define the semantics.

Definition 1.2 [Models] A *structure* $\mathcal{M}$ for **PPL** is $\mathcal{M} = (W, f, V)$ such that, $W \neq \emptyset$, is a countable [4] set of possible worlds, $V : Props \to \wp(W)$ assigns a set of worlds to each proposition and $f : W \times W \to [0, 1]$ satisfies [5] : $\sum_{v \in W} f(w, v) = 1$ for all $w \in W$.

PPL-terms can be evaluated in a structure $\mathcal{M} = (W, f, V)$ by:

(i) $[r]^{\mathcal{M},w} = r$ where r $\in \mathbb{Q}$

(ii) $[\mathbf{P}()]^{\mathcal{M},w} = 1$

(iii) $[\mathbf{P}(\phi : \overline{\phi})]^{\mathcal{M},w} = \sum_{v:\mathcal{M},v \models \phi} f(w, v) \cdot [\mathbf{P}\overline{\phi}]^{\mathcal{M},v}$

[4] There are problems in defining sums of probabilities if there are uncountably many worlds. A generalisation to uncountable but measurable semantics can be carried forward as in [12] where f is required to be integrable with respect to each of its arguments.

[5] Another generalisation of these structures can be obtained by letting the range of f be the non-negative real numbers and replacing the requirement $\sum_{f \in W} f(w, v) = 1$ by the bounded sum condition: for any $w \in W$ the supremum of $\{\sum_{x \in X} f(w, x) : X \subseteq W \text{ is finite}\}$ exists. Such a logic could be useful for counting or measuring various quantities. In the set of axioms given in figure 2 it would be necessary to delete **W0** to axiomatise these more general structures.

(iv) $[\tau_1 + \tau_2]^{\mathcal{M},w} = [\tau_1]^{\mathcal{M},w} + [\tau_2]^{\mathcal{M},w}$.

(v) $[\tau \times r]^{\mathcal{M},w} = [\tau]^{\mathcal{M},w} \cdot r$.

Formulas of this logic can be evaluated by

$$\begin{aligned}
\mathcal{M}, w \models p &\text{ iff } w \in V(p) \\
\mathcal{M}, w \models \tau > 0 &\text{ iff } [\tau]^{\mathcal{M},w} > 0 \\
\mathcal{M}, w \models \neg\varphi &\text{ iff } \mathcal{M}, w \not\models \varphi \\
\mathcal{M}, w \models \varphi \wedge \psi &\text{ iff } \mathcal{M}, w \models \varphi \text{ and } \mathcal{M}, w \models \psi
\end{aligned}$$

Note: syntactic terms of **PPL** are built up from rational numbers but $[\mathbf{P}(\phi : \overline{\phi})]^{\mathcal{M},w}$, above, could potentially result in terms evaluating to irrational, real numbers, even if the range of the function f were contained in the set of rational numbers. If $\mathcal{M}$ is a structure, ϕ is a **PPL**-formula and $\mathcal{M}, w \models \phi$ then $(\mathcal{M}, w)$ is a *model* of ϕ.

Lemma 1.3 *Let $\overline{\phi}$ be a non-empty sequence and let $\overline{\psi}$ be an arbitrary sequence of* **PPL** *formulas, and let $\mathcal{M} = (W, f, V)$ be any* **PPL***-structure.*

$$[\mathbf{P}(\overline{\phi\psi})]^{\mathcal{M},v_0} = \sum_{\textit{for } 1\leq i\leq|\overline{\phi}| \quad \overline{v}\in W^{|\overline{\phi\psi}|}:\mathcal{M},v_i\models\phi_i} \quad \prod_{0\leq i<|\overline{\phi}|} f(v_i, v_{i+1}) \cdot [\mathbf{P}\overline{\psi}]^{\mathcal{M},v_{|\overline{\phi}|}}$$

Proof. The proof is by induction over $|\overline{\phi}|$. By definition 1.2, the case $|\overline{\phi}| = 1$ holds. Now let $\overline{\phi} = \phi_1 : \overline{\phi_{/1}}$ where $\overline{\phi_{/1}} = (\phi_2, \phi_3, \ldots, \phi_{|\overline{\phi}|})$ is non-empty. By definition 1.2 again, and by our inductive hypothesis,

$$\begin{aligned}
[\mathbf{P}(\overline{\phi\psi})]^{\mathcal{M},v_0} &= [\mathbf{P}(\phi_1 : \overline{\phi_{/1}\psi})]^{\mathcal{M},v_0} \\
&= \sum_{v_1:\mathcal{M},v_1\models\phi_1} f(v_0, v_1) \cdot [\mathbf{P}(\overline{\phi_{/1}\psi})]^{\mathcal{M},v_1} \\
&= \sum_{v_1:\mathcal{M},v_1\models\phi_1} f(v_0, v_1) \cdot \sum_{\mathcal{M},v_i\models\phi_i,\ 2\leq i\leq|\overline{\phi}|} \prod_{1\leq i<|\overline{\phi}|} f(v_i, v_{i+1}) \cdot [\mathbf{P}\overline{\psi}]^{\mathcal{M},v_{|\overline{\phi}|}} \\
&= \sum_{\overline{v}:\mathcal{M},v_1\models\phi_1,\ \mathcal{M},v_i\models\phi_i,\ (2\leq i\leq|\overline{\phi}|)} f(v_0, v_1) \cdot \prod_{1\leq i<|\overline{\phi}|} f(v_i, v_{i+1}) \cdot [\mathbf{P}\overline{\psi}]^{\mathcal{M},v_{|\overline{\phi}|}} \\
&= \sum_{\overline{v}:\mathcal{M},v_i\models\phi,\ (1\leq i\leq|\overline{\phi}|)} \prod_{0\leq i<|\overline{\phi}|} f(v_i, v_{i+1}) \cdot [\mathbf{P}]^{\mathcal{M},v_{|\overline{\phi}|}}
\end{aligned}$$

as required. □

2 Examples and related logics

We start this section with some examples of **PPL** formulas for illustrating the semantics of our logic.

Example 2.1 Some formulas of **PPL** are:

- $\mathbf{P}(p, p) > 0.19$. We will show later that formulas like this have no equivalent in $\mathbf{PEL}_\times$.

- $\mathbf{P}(p \wedge \mathbf{P}p > 0.49) > 0.39$ As we note below, this formula is (apart from minor notational changes) a formula of $\mathbf{PEL}_\times$ [12].
- We could say that **PPL** is a logic for stating properties such as, "the probability that q will hold continuously over 2 seconds is greater than the probability that p will hold over 2 seconds". Such properties, as we will discuss later, are not expressible in the probabilistic version of **CTL** [16]. In **PPL** we could write something like that: $\mathbf{P}(q, q) > \mathbf{P}(p, p)$

2.1 PEL

We continue by surveying a number of well known logics of the field of probabilistic logics. In [12] a probabilistic epistemic logic was given and proved to be complete and decidable, based on a number of results of [13]. This logic allows nesting of the probabilistic operator, but the operator is applied to single formulas, not sequences.

Definition 2.2 [**PEL** in [17]] Let $\mathcal{P}$ be a countable set of propositional letters and let $\mathcal{A}$ be a finite set of agents. **PEL** formulas are defined by the following rule.

$$\phi := p \mid \neg\phi \mid \phi \wedge \psi \mid \Box_a\phi \mid q_1 \times \mathbf{P}_a(\phi_1) + \ldots + q_n \times \mathbf{P}_a(\phi_n) \geq q$$

where $p \in \mathcal{P}$, $a \in \mathcal{A}$ and $q_1, \ldots, q_n, q$ are rationals.

A sentence of the form $\mathbf{P}_a(\phi) \geq q$ should be read as *the probability that agent a assigns to ϕ is greater than or equal to q*. Also **PEL** expresses higher order probabilities such as $\mathbf{P}_a(\mathbf{P}_b(\phi) \geq q_1) \geq q_2$. We could read this sentence as the probability a assigns to the sentence that the probability b assigns to ϕ is greater or equal to q_1, is greater or equal q_2.

The semantics of this logic as formulated in [12], is probabilistic epistemic models. These models $\mathcal{M} = (W, f, V)$ have a function f that assigns a probability function f_a to each agent a at each world. In more detail we have: $f : (\mathcal{A} \times W) \to (W \to [0, 1])$ where $\mathcal{A}$ is the set of agents. To interpret $\mathbf{P}_a(\phi)$ we have: $[\mathbf{P}_a(\phi)]^{\mathcal{M},w} = \sum_{w:\mathcal{M},v\models\phi} f(a, w, v)$

Our logic has no epistemic operators, but we can eliminate the epistemic aspects of **PEL** if we assume that the probability function $f_a : (w, v) \mapsto f(a, w, v)$ is the same for all agents. For this non-epistemic case we rewrite **PEL**'s syntax as follows:

Definition 2.3 $\mathbf{PEL}_\times$ is the non-epistemic sublanguage of **PEL**, defined by,

$$\phi := p \mid \neg\phi \mid (\phi \wedge \psi) \mid q_1 \times \mathbf{P}(\phi_1) + \ldots + q_n \times \mathbf{P}(\phi_n) \geq q$$

where p is a propositional letter, $q, q_1, \ldots, q_n$ are rational numbers and $\phi, \psi, \phi_1, \ldots, \phi_n$ are formulas. A $\mathbf{PEL}_\times$-structure $\mathcal{M} = (W, f, V)$ is defined just as in definition 1.2, but where a term $\mathbf{P}(\phi)$ is evaluated by: $[\mathbf{P}(\phi)]^{\mathcal{M},w} = \sum_{v:\mathcal{M},v\models\phi} f(w, v)$

Thus, $\mathbf{PEL}_\times$ can be obtained from our language **PPL** be restricting terms to

$$\tau := \quad r|\mathbf{P}^1\phi|(\tau_1 + \tau_2)|\tau \times r \quad (r \in \mathbb{Q})$$

Thus, the **P** operator can now only be applied to single formulas, rather than sequences of formulas. Below we show that this restriction really diminishes the expressive power of the language.

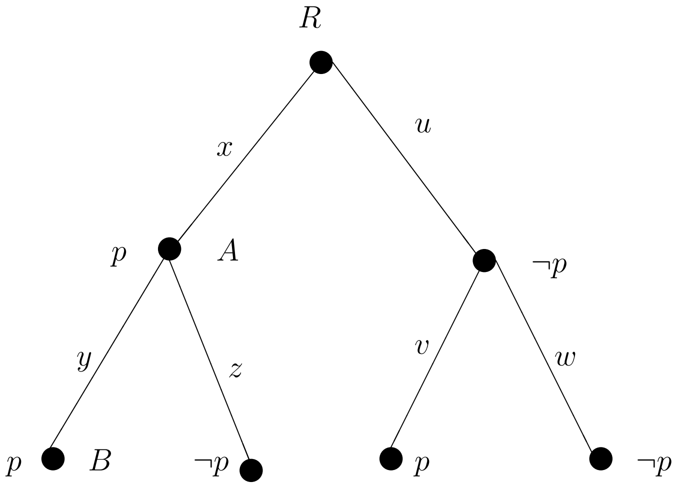

Fig. 1. The binary tree $\mathcal{T}(x, y, z, u, v, w)$.

We have just seen that the set of $\mathbf{PEL}_\times$ formulas forms a fragment of **PPL**. Here we will show that this is a proper fragment, by showing that there are **PPL** formulas that have no equivalent in $\mathbf{PEL}_\times$. We will use a result due to [12, Theorem 4.1] (based on a theorem of [13, Theorem 2.2]) that the satisfiability of a $\mathbf{PEL}_\times$ formula is reduced to satisfiability of a set of linear inequalities over the variables representing the weight of the edge from a node to one of its successors.

Definition 2.4 Let ψ be a non-epistemic $\mathbf{PEL}_\times$ formula and let ϕ be a **PPL** formula. We say that ψ is equivalent to ϕ if for all **PPL** structures $\mathcal{M}$ and all worlds w we have $\mathcal{M}, w \models \psi \iff \mathcal{M}, w \models \phi$. Since $\mathbf{PEL}_\times$-structures have the same form as **PPL**-structures, this makes sense.

Theorem 2.5 *There is a* **PPL** *formula for which there is no equivalent* $\mathbf{PEL}_\times$ *formula.*

Proof. Consider the formula $\phi = (\mathbf{P}(p, p) = \frac{1}{2})$. This is about the simplest **PPL** formula you could write, that does not directly translate into $\mathbf{PEL}_\times$, because the **P** applies to a sequence of formulas of length greater than one. Consider the class of **PPL**-structures of the form of figure 1, i.e. binary trees of depth 2. Denote the tree in the diagram as $\mathcal{T}(x, y, z, u, v, w)$, where x is the weight of the edge from the root to the successor (A) where p holds and y, z, u, v, w are the weights of the other edges, as shown in the diagram. [Note that the variables we are using here, representing the weight of the edge from a node to one of its successors, are the same sort of variables used in [12] for **PEL**. In our completeness proof, below, we use different variables representing the weight of a branch from the root to a given node of the tree.] It is clear that $\mathcal{T}(x, y, z, u, v, w), R \models \phi$ iff $xy = \frac{1}{2}$. Now for any $\mathbf{PEL}_\times$ formula ψ the set $\{(x, y, z, u, v, w) : \mathcal{T}(x, y, z, u, v, w), R \models \psi\}$ can easily be shown along the lines of [13, Theorem 2.2] to be a subset of $\mathbb{R}^6$ defined by linear inequalities only. But $\{(x, y, z, u, v, w) : xy = \frac{1}{2}\}$ cannot be defined by linear inequalities. Therefore there is no $\mathbf{PEL}_\times$ formula equivalent to ϕ. $\square$

2.2 Related work

For probabilistic modal logics closely related with the investigations of Halpern and Fagin we refer to [9,23,2]. The first two papers study a logic with an operator $\mathbf{P}_r^{>}(\phi)$ meaning that 'the probability of ϕ is greater than r'. Clearly, such formulas can be expressed in the sublanguage of **PEL** given in definition 2.3, but unlike **PEL**, these languages cannot express non-trivial linear combinations of probabilities. In [2] a complete axiomatization was proved by using an infinitary rule for languages containing an operator $\mathbf{P}_r(\phi)$ standing for 'the probability of ϕ is equal to r'. A complete axiomatization was proved regarding the logics in [9,23] with respect to the class of models where probabilities are taken from a finite set subset of $[0,1]$.

2.3 PCTL

During the 1990s various probabilistic temporal logics were studied, combining temporal logics with probabilities. A probabilistic extension of CTL was given in [16,3] where a number of results concerning the decidability of model checking were also proved. The definition given here is a minor variation on the one given in [6].

Definition 2.6 The syntax of **PCTL** is defined by the following grammar:

(1) $\phi := p \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \mathbf{P}_{\rhd d}\ \psi$

(2) $\psi := \phi_1 \mathcal{U} \phi_2 \mid \phi_1 \mathcal{U}^{\leq t} \phi_2$

where p is a propositional letter, d is a rational from $[0,1]$, $\rhd \in \{>, \geq\}$ [6] and $t \in \mathbb{N}$. Formulas defined in (1) are called *state formulas* and formulas defined in (2) are called *path formulas*. In the right hand side of the definitions, ϕ, ϕ_1, ϕ_2 stand for arbitrary state formulas and ψ is an arbitrary path formula.

The syntax of **WPCTL** is the same as that of **PCTL**, except that in the definition of path formulas, only the 'restricted until' is allowed.

$$\psi := \phi_1 \mathcal{U}^{\leq t} \phi_2$$

Here $\mathbf{P}_{\rhd d}$ is a probabilistic operator, while the temporal operator $\mathcal{U}^{\leq t}$ is a restricted version of the 'until' operator. $\mathcal{U}^{\leq t}$ is like the standard until operator except that, in $\psi_1 \mathcal{U}^{\leq t} \psi_2$, ψ_2 should be true within t time units. A time unit is one transition in the model of the formulas, and as models they consider discrete time Markov chains.

The semantics of **PCTL** is a structure (S, s_0, f, V) where, S is a finite set of states, s_0 is the initial state, f is the probability function $f : S \times S \to [0,1]$ such that for any state s we have: $\sum_{s'} f(s, s') = 1$, and V is a propositional valuation. For $s \in S$, an *s-path* $\sigma : \omega \to S$ is a countable sequence of states $\sigma[0], \sigma[1], \ldots$, such that $\sigma[0] = s$. We write $\sigma\restriction_n = (\sigma[0], \sigma[1], \ldots, \sigma[n])$ for the initial segment of σ with $n+1$ terms. We can define a σ-algebra generated by certain sets of s-paths and a measure on such sets, as follows. For any finite sequence of states $(s_0, s_1, \ldots, s_n)$,

[6] In [6], $\rhd$ can also be $<$ or $\leq$, but these relations are not needed since $\tau < d \equiv \neg(\tau \geq d)$ and $\tau \leq d \equiv \neg(\tau > d)$.

let $(s_0, \ldots, s_n) \uparrow$ be the set of all s_0-paths σ such that $\sigma[i] = s_i$, for each $i \leq n$.

$$\{(s_0, \ldots, s_n) \uparrow : s_0 = s,\ n \in \mathbb{N},\ s_i \in S\ (i \leq n)\}$$

generates a σ algebra, by taking countable unions and complements. If X is a set of s-paths, let $-X$ be the complement of X in the set of all s-paths. We define a measure μ_s by letting

$$\begin{aligned} \mu_s(s_0, s_1, \ldots, s_n) \uparrow &= \prod_{i<n} f(s_i, s_{i+1}),\ \text{where } s_0 = s \\ \mu_s(-X) &= 1 - \mu_s(X) \\ \mu_s(\bigcup_{i<\omega} X_i) &= \sum_{i<\omega} \mu_S(X_i),\ \text{if the } X_i \text{ are disjoint} \end{aligned}$$

It can be shown, for any path formula ψ, that $\{s\text{-paths } \sigma \text{ such that } \sigma \models\equiv \psi\}$ is measurable.

Formulas of this logic can be evaluated over states (relation $\models$)and paths (relation $\models\equiv$):

$$\begin{aligned} S, s \models p &\ \text{ iff }\ s \in V(p) \\ S, s \models \neg\varphi &\ \text{ iff }\ S, s \not\models \varphi \\ S, s \models \varphi \wedge \varphi' &\ \text{ iff }\ S, s \models \varphi \text{ and } S, s \models \varphi' \\ S, s \models \mathbf{P}_{\triangleright d}\ \psi &\ \text{ iff }\ \mu_s\{\sigma : \sigma[0] = s \wedge \sigma \models\equiv \psi\} \triangleright d \\ \sigma \models\equiv \varphi\ \mathcal{U}\ \varphi' &\ \text{ iff }\ \exists i \in \mathbb{N},\ (S, \sigma[i] \models \varphi' \text{ and } \forall j < i,\ S, \sigma[j] \models \varphi) \\ \sigma \models\equiv \varphi\ \mathcal{U}^{\leq t}\ \varphi' &\ \text{ iff }\ \exists i \leq t,\ (S, \sigma[i] \models \varphi' \text{ and } \forall j < i,\ S, \sigma[j] \models \varphi) \end{aligned}$$

PCTL is able to express soft deadline properties, such as 'after a request for service, there is at least a 98 percent probability that the service will be carried out within 2 seconds'. For example, the property: 'with at least 0.8 probability q will hold continuously for the next 7 time units' is expressed by the formula

$$\neg\mathbf{P}_{\geq 0.2}(\top\mathcal{U}^{\leq 7}\neg q). \tag{3}$$

The unrestricted until is beyond the scope of our language **PPL**, but the restricted language **WPCTL** can be expressed in **PPL**. According to [21] a complete set of axioms for **WPCTL** is not known and its complexity is also unknown. Based on our completeness proof in Section 4, we could show that there is a **2 − EXPTIME** upper bound for the validity problem of **WPCTL**.

Definition 2.7 The translation **Tr** takes state formulas of **WPCTL** to **PPL** is defined as follows: $\mathbf{Tr}(p) = p$, $\mathbf{Tr}(\neg\phi) = \neg\mathbf{Tr}(\phi)$, $\mathbf{Tr}(\phi \wedge \psi) = \mathbf{Tr}(\phi) \wedge \mathbf{Tr}(\psi)$ and

$$\mathbf{Tr}(\mathbf{P}_{>d}(\phi\mathcal{U}^{\leq t}\psi)) = \begin{cases} \psi \vee (\phi \wedge [\mathbf{P}(\psi) + \mathbf{P}((\phi \wedge \neg\psi), \psi) + \ldots & \\ \ldots + \mathbf{P}^t((\phi \wedge \neg\psi), \ldots, (\phi \wedge \neg\psi), \psi)] > d) & d < 1 \\ \perp & d \geq 1 \end{cases}$$

and the translation of $\mathbf{P}_{\geq d}(\phi\mathcal{U}^{\leq t}\psi)$ is similar. Note that (3) can be expressed in **PPL** as

$$q \wedge (\mathbf{P}(q, q, q, q, q, q, q) \geq 0.8)$$

We have seen that such expressions have no equivalents in logics such as $\mathbf{PEL}_\times$.

Lemma 2.8 *Suppose* $(S, s_0, f, V) \models \phi$*, for some state formula* ϕ *of* **WPCTL**. *Then* $(S, f, V), s_0 \models \mathbf{Tr}(\phi)$.

Conversely, if $(W, f, V), w \models \mathbf{Tr}(\phi)$ *for some state formula* ϕ *of* **WPCTL** *then* $(W, w, f, V) \models \phi$.

The proof is entirely routine and omitted. So **WPCTL** is expressively equivalent to a fragment of **PPL**. In fact this fragment is a proper fragment since **PPL**-formulas such as

$$3 \times \mathbf{P}(p, p) + 2 \times \mathbf{P}(p, q) > 0.5$$

have no equivalent in **WPCTL**.

3 Completeness

Before we really start, we should mention that **PPL** is not compact; this could be proved along the lines of [17]. This compactness failure indicates that a strong completeness result is not provable for this logic; we prove weak completeness below.

In figure 2 we give a set of inference rules and axioms for **PPL**. We write $\vdash \psi$ if ψ can be proved by the rules and axioms included in the figure. If ψ can be proved using only modus ponens and **Prop** we may write $\vdash_{\mathbf{Prop}} \psi$. We will show that the rules and axioms of figure 2 are sound and complete. As is often the case, the proof of soundness is relatively straightforward. To prove completeness we will show that an arbitrary consistent formula has a model, in fact the model we construct will have only finitely many worlds, and the number of worlds will be bound by an exponential function, in terms of the size of the formula. Some notation first. From definition 1.1 we see that every term is a linear sum of primitive terms. A term $\tau = \sum_{i=1}^{k} c_i \times \mathbf{P}\overline{\psi^i} + d$ ($c_i, d \in \mathbb{Q}$) can be written as

$$\mathbf{L}(\mathbf{P}\overline{\psi^1}, \mathbf{P}\overline{\psi^2}, \ldots, \mathbf{P}\overline{\psi^k}) + d \tag{4}$$

where $\mathbf{L}$ is the linear operator $\mathbf{L} : (x_1, \ldots, x_k) \mapsto \sum_{i=1}^{k} c_i \times x_i$. Thus, every term has the form (4), for some linear operator $\mathbf{L}$, some primitive terms $\mathbf{P}\overline{\psi^1}, \ldots, \mathbf{P}\overline{\psi^k}$ and some $d \in \mathbb{Q}$. Figure 2 is divided in to two parts: inference rules and axioms. All but one of these are rephrased versions of the axioms and rules given by [13] for reasoning about probabilities, but the inference rule **Extension** is a brand new rule for reasoning about probabilistic paths.

Lemma 3.1 *The rules and axioms of figure 2 are sound.*

Proof. We prove that **Extension** is sound, the other cases are routine (and covered by [13]). Suppose that

$$last(\overline{\phi}) \rightarrow \mathbf{L}(\mathbf{P}\overline{\psi^1}, \ldots, \mathbf{P}\overline{\psi^k}) \ \triangle \ d \tag{5}$$

is valid, for some sequences $\overline{\phi}$, $\overline{\psi^1}$, ..., $\overline{\psi^k}$, some $\triangle \in \{<, =, >\}$, some $d \in \mathbb{Q}$ and some $\mathbf{L}$. Let $\mathcal{M}$ be any **PPL**-structure with a node v_0 such that

$$\mathcal{M}, v_0 \models \ \mathbf{P}\overline{\phi} > 0 \tag{6}$$

Inference Rules	
Modus Ponens	$\dfrac{\phi,\ (\phi{\rightarrow}\psi)}{\psi}$
Generalisation	$\dfrac{\phi{\rightarrow}\psi}{\mathbf{P}\overline{\phi}[m/\phi] \le \mathbf{P}\overline{\phi}[m/\psi]}$
Extension	$\dfrac{last(\overline{\phi}){\rightarrow}(\mathbf{L}(\mathbf{P}\overline{\psi^1},...,\mathbf{P}\overline{\psi^m}) \bigtriangleup d)}{\mathbf{P}\overline{\phi}{>}0 \rightarrow (\mathbf{L}(\mathbf{P}\overline{\phi\psi^1},...,\mathbf{P}\overline{\phi\psi^m}) \bigtriangleup (\mathbf{P}\overline{\phi}{\times}d))}$
Axioms	
Prop	An axiomatisation of propositional logic
Reals	All instances of valid formulas about reals (see [13, section 4])
W0	$\mathbf{P}() = 1$ $\qquad \mathbf{P}(\overline{\phi}\top) = \mathbf{P}\overline{\phi}$
W1	$\mathbf{P}\overline{\phi} \geq 0$
W2	$\mathbf{P}(\overline{\phi}[m/(\phi_m \wedge \psi)]) + \mathbf{P}(\overline{\phi}[m/(\phi_m \wedge \neg\psi)]) = \mathbf{P}\overline{\phi}$

Fig. 2. Quantitative Rules and Axioms. In **‘Extension’**, $d \in \mathbb{Q}$, $\bigtriangleup$ stands for $<$, $=$ or $>$. $\mathbf{L}$ is an arbitrary linear operator.

We must show that

$$\mathcal{M}, v_0 \models \mathbf{L}(\mathbf{P}\overline{\phi\psi^1}, \ldots, \mathbf{P}\overline{\phi\psi^k}) \ \bigtriangleup\ (\mathbf{P}\overline{\phi} \times d) \tag{7}$$

Let $n = |\overline{\phi}|$. Let $\overline{v} = (v_0, v_1, \ldots, v_n)$ be a sequence of nodes in $\mathcal{M}$ such that for $1 \leq i \leq n$ we have $\mathcal{M}, v_i \models \phi_i$ (such a sequence must exist, by (6)).So $\mathcal{M}, v_n \models last(\overline{\phi})$ and by (5),

$$[\mathbf{L}(\mathbf{P}\overline{\psi^1}, \ldots, \mathbf{P}\overline{\psi^k})]^{\mathcal{M},v_n} \ \bigtriangleup\ d \tag{8}$$

Write $f(\overline{v})$ for $\prod_{i<n} f(v_i, v_{i+1})$. By lemma 1.3, for $1 \leq m \leq k$,

$$[\mathbf{P}\overline{\phi\psi^m}]^{\mathcal{M},v_0} = \sum_{\overline{v}:\mathcal{M},v_i\models\phi_i,\ (1\leq i\leq n)} f(\overline{v}) \cdot [\mathbf{P}\overline{\psi^m}]^{\mathcal{M},v_n} \tag{9}$$

and by (6), we can choose the sequence $\overline{v}$ so that $f(\overline{v}) > 0$. By linearity of $\mathbf{L}$ and by (9),

$$\begin{aligned}
&[\mathbf{L}(\mathbf{P}\overline{\phi\psi^1}, \ldots, \mathbf{P}\overline{\phi\psi^k})]^{\mathcal{M},v_0} \\
&= \mathbf{L}([\mathbf{P}\overline{\phi\psi^1}]^{\mathcal{M},v_0}, \ldots, [\mathbf{P}\overline{\phi\psi^k}]^{\mathcal{M},v_0}) \\
&= \mathbf{L}(\sum_{\overline{v}:\mathcal{M},v_i\models\phi_i\ (1\leq i\leq n)} f(\overline{v}) \cdot [\mathbf{P}\overline{\psi^1}]^{\mathcal{M},v_n}, \ldots, \sum_{\overline{v}:\mathcal{M},v_i\models\phi_i\ (1\leq i\leq n)} f(\overline{v}) \cdot [\mathbf{P}\overline{\psi^k}]^{\mathcal{M},v_n}) \\
&= \sum_{\overline{v}:\mathcal{M},v_i\models\phi_i\ (1\leq i\leq n)} f(\overline{v}) \cdot [\mathbf{L}(\mathbf{P}\overline{\psi^1}, \ldots, \mathbf{P}\overline{\psi^k})]^{\mathcal{M},v_n}
\end{aligned}$$

$$\triangle \sum_{\overline{v}:\mathcal{M},v_i\models\phi_i\ (1\le i\le n)} f(\overline{v})\cdot d \quad \text{(by (8) since } f(\overline{v})>0, \text{ some } \overline{v} \text{ included in sum)}$$
$$= [\mathbf{P}\overline{\phi}]^{\mathcal{M},v_0}\cdot d$$

which proves (7), as required. □

Lemma 3.2 *For each formula ϕ there is a DNF formula ϕ' with $\vdash_{\mathbf{Prop}} \phi \leftrightarrow \phi'$, where ϕ' is a disjunction of conjunctive clauses and each clause is a conjunction of propositions, negated propositions and inequalities*[7] $\mathbf{L}(\mathbf{P}\overline{\psi^1},\ldots,\mathbf{P}\overline{\psi^k}) \triangle d$, *for* $\triangle \in \{<,=,>\}$.

Proof. By propositional reasoning, ϕ is equivalent to DNF. A literal of **PPL** $\neg(\mathbf{L}(\mathbf{P}\overline{\psi^1},\ldots,\mathbf{P}\overline{\psi^k})>d)$ is equivalent (using **Reals**) to $(\mathbf{L}(\mathbf{P}\overline{\psi^1},\ldots,\mathbf{P}\overline{\psi^k})=d) \vee (\mathbf{L}(\mathbf{P}\overline{\psi^1},\ldots,\mathbf{P}\overline{\psi^k})<d)$.

Similarly, literals $\neg(\mathbf{L}(\mathbf{P}\overline{\psi^1},\ldots,\mathbf{P}\overline{\psi^k})=d)$ and $\neg(\mathbf{L}(\mathbf{P}\overline{\psi^1},\ldots,\mathbf{P}\overline{\psi^k})<d)$ can each be replaced by the disjunction of two primitive formulas. Let ϕ^* be obtained from ϕ by first finding an equivalent DNF and then replacing each such negated primitive formula by an equivalent disjunction of two primitive formulas. Now repeatedly apply the distribution law

$$\vdash A\wedge(B\vee C)\leftrightarrow(A\wedge B)\vee(A\wedge C)$$

to transform ϕ^* to an equivalent DNF formula ϕ'. Note that literals involving $\mathbf{L}(\mathbf{P}\overline{\psi^1},\ldots,\mathbf{P}\overline{\psi^k}) \triangle d$ (for $\triangle\in\{<,=,>\}$) only occur positively in ϕ'. □

Before we proceed with completeness, we need to prove some lemmas. The ones that follow, Lemma 3.3 and Lemma 3.7 are similar to [13, Lemma 2.3]. That is, we define certain 'atoms' in an algebra generated by subformulas of a given formula ϕ. The proof of completeness of our logic follows the outline of the corresponding proof in [13], but with the added complication of handling paths of formulas rather than single formulas. Here will use sequences of atoms as worlds in the model we construct.

Lemma 3.3 *Let $1\le m\le|\overline{\phi}|$. If $\vdash \phi_m \leftrightarrow \psi_1\vee\ldots\vee\psi_n$ and $\vdash\neg(\psi_l\wedge\psi_{l'})$ for all distinct $1\le l,l'\le n$ $(l\ne l')$ (in other words, it is provable that $\psi_1,\ldots,\psi_n$ are mutually exclusive and their disjunction is equivalent to ϕ_m) then $\vdash \mathbf{P}\overline{\phi} = \mathbf{P}(\overline{\phi}[m/\psi_1])+\ldots+\mathbf{P}(\overline{\phi}[m/\psi_n])$.*

Proof. For $n=1$ we have $\vdash\phi_m\leftrightarrow\psi_1$. By two applications of rule **Generalisation** and an instance of **Reals**, we deduce that
$\mathbf{P}(\phi_1,\ldots,\phi_{m-1},\phi_m,\phi_{m+1},\ldots,\phi_k)=\mathbf{P}(\phi_1,\ldots,\phi_{m-1},\psi_1,\phi_{m+1},\ldots,\phi_k)$, as required. Assume the lemma holds for some $n\ge1$. For the inductive step, we must prove the lemma when $\vdash\phi_m\leftrightarrow(\psi_1\vee\ldots\vee\psi_{n+1})$ and $\vdash\neg(\psi_l\wedge\psi_{l'})$ (any distinct $1\le l,l'\le n+1$). By the axiom **W2** of figure 2, $\vdash\mathbf{P}\overline{\phi}=\mathbf{P}\overline{\phi}[m/\phi_m\wedge\psi_{n+1}]+\mathbf{P}\overline{\phi}[m/\phi_m\wedge\neg\psi_{n+1}]$. By our assumptions, $\vdash(\phi_m\wedge\psi_{n+1})\leftrightarrow\psi_{n+1}$ and $\vdash(\phi_m\wedge\neg\psi_{n+1})\leftrightarrow(\psi_1\vee\ldots\vee\psi_n)$. Hence, using **Generalisation**: $\vdash\mathbf{P}\overline{\phi}=\mathbf{P}\overline{\phi}[m/\psi_{n+1}]+\mathbf{P}\overline{\phi}[m/(\psi_1\vee\ldots\vee\psi_n)]$.

[7] In fact we could even make the restriction $\triangle\in\{<,=\}$, since the primitive formula $\mathbf{L}(\mathbf{P}\overline{\psi^1},\ldots,\mathbf{P}\overline{\psi^k})<d$ is equivalent to $(-\mathbf{L})(\mathbf{P}\overline{\psi^1},\ldots,\mathbf{P}\overline{\psi^k})>(-d)$, where $(-\mathbf{L})$ is the linear operator whose coefficients are the negations of those of $\mathbf{L}$. But we do not need this restriction in the proofs that follow.

By our induction hypothesis $\vdash \mathbf{P}\overline{\phi}[m/(\psi_1 \vee \ldots \vee \psi_n)] = \sum_{i=1}^{n} \mathbf{P}\overline{\phi}[m/\psi_i]$. Hence $\vdash \mathbf{P}\overline{\phi} = \sum_{i=1}^{n+1} \mathbf{P}\overline{\phi}[m/\psi_i]$ □

For the remainder of this section we fix a formula ϕ and let $n = deg(\phi)$.

Definition 3.4 Let $m \in \mathbb{N}$.

(i) $S_m(\phi)$ denotes the set of primitive subformulas of ϕ of degree at most m (see definition 1.1). Clearly $|S_m(\phi)| \leq |\phi|$.

(ii) $S^+_m(\phi) \supseteq S_m(\phi)$ is obtained from $S_m(\phi)$ by replacing each proposition $p \in S_m(\phi)$ by both p and $\neg p$, and each primitive formula $(\tau \bigtriangleup d) \in S_m(\phi)$ by all three formulas $(\tau < d)$, $(\tau = d)$, $(\tau > d)$. We have $|S^+_m(\phi)| \leq 3 \cdot |\phi|$.

(iii) A *ϕ-formula* is any formula whose primitive subformulas belong to $S^+_n(\phi)$. A *ϕ-term* is a linear combination of terms $\mathbf{P}\overline{\psi}$, where each ψ_i is a ϕ-formula.

(iv) A subset σ of $S^+_m(\phi)$ is said to be *complete* if

$$|\{p, \neg p\} \cap \sigma| = 1$$
$$|\{(\tau < d), (\tau = d), (\tau > d)\} \cap \sigma| = 1$$

whenever the proposition p occurs in ϕ and $(\tau \bigtriangleup d)$ occurs in ϕ.

(v) If $\sigma \subseteq S^+_m(\phi)$ is complete then $\bigwedge \sigma \wedge \bigwedge\{\neg x : x \in S^+_m(\phi) \setminus \sigma\}$ is called an *atom of degree m of ϕ*. The set of atoms of degree m of ϕ is defined to be

$$A_m(\phi) = \{\bigwedge \sigma \wedge \bigwedge\{\neg x : x \in S^+_m(\phi) \setminus \sigma\} : \sigma \subseteq S^+_m(\phi) \text{ is complete}\}$$

$|A_m(\phi)| \leq 2^{3\cdot|\phi|}$.

(vi) For $0 \leq k \leq m \leq n$, let $X_{(m,k)} = A_{m-1}(\phi) \times \ldots \times A_k(\phi)$. Write X_m for $X_{(m,0)}$. We have $|X_{(m,k)}| \leq 2^{3\cdot(m-k)\cdot|\phi|}$.

(vii) Let $\overline{\theta}, \overline{\psi}$ be sequences of formulas with $|\overline{\theta}| \geq |\overline{\psi}|$. We write $\vdash_{\mathbf{Prop}} \overline{\theta} \to \overline{\psi}$ if for each i with $1 \leq i \leq |\overline{\psi}|$ we have $\vdash_{\mathbf{Prop}} \theta_i \to \psi_i$.

(viii) We define the degree $deg(\overline{\psi})$ of a sequence $\overline{\psi}$ of formulas to be $max\{0, i + deg(\psi_i) : 1 \leq i \leq \overline{\psi}\}$.

(ix) We define a map α_m from terms of degree m to terms of the form $\mathbf{L}(\mathbf{P}\overline{a} : \overline{a} \in X_m)$. For any sequence of formulas $\overline{\psi}$ with $deg(\overline{\psi}) \leq m$, let $\alpha_m(\mathbf{P}\overline{\psi}) = \sum_{\overline{a}\in X_m : \vdash_{\mathbf{Prop}} \overline{a}\to\overline{\psi}} \mathbf{P}\overline{a}$ and for an arbitrary linear combination $\sum_{i=1}^{k} c_i \cdot \mathbf{P}\overline{\psi^i}$ of such terms let $\alpha_m(\sum_{i=1}^{k} c_i \cdot \mathbf{P}\overline{\psi^i}) = \sum_{i=1}^{k} c_i \cdot \alpha_m(\mathbf{P}\overline{\psi^i}) = \sum_{\overline{a}\in X} (\sum_{i : \vdash_{\mathbf{Prop}} \overline{a}\to\overline{\psi^i}} c_i) \cdot \mathbf{P}\overline{a}$

Propositional axioms suffice to prove that distinct atoms of the same degree are disjoint. Also, since any subformula ψ of ϕ is a boolean combination of primitive subformulas of degree at most $deg(\psi)$, propositional reasoning suffices to prove that ψ is equivalent to a disjunction of atoms of degree $deg(\psi)$.

Our completeness theorem will build a tree-like model for ϕ out of sets $X_{(m,k)}$. We first define this tree structure:

Definition 3.5 We define a model $\mathcal{X}$ for ϕ as the triple $\mathcal{X} = (X, V, f)$ where:

- the root node of $\mathcal{X}$ is the empty string ().

- X is the union of all $X_{(n,m)}$: $\bigcup_{0\le m\le n} X_{(n,m)}$.
- In our tree, we draw an edge from a sequence s to a sequence t if and only if $|s|+1=|t|\le n$ and $t\restriction_{|s|}=s$.
- V is the valuation defined by: Let $()\in V(p) \iff \vdash_{\mathbf{Prop}} \phi\to p$ and for non-empty $\overline{a}\in X$ let $\overline{a}\in V(p) \iff \vdash_{\mathbf{Prop}} last(\overline{a})\to p$.

We comment on all four clauses. First, the root of the model will be the empty sequence, its 'children' will be sequences of length one and their elements atoms of degree $n-1$, its grandchildren will be sequences of length two where their first elements are atoms of degree $n-1$ and their second elements are atoms of degree $n-2$, etc. Regarding the second clause, note that $X_{(n,0)}=\{()\}$ and $|X_{(n-i)}|\le 2^{3\cdot(n-i)\cdot|\phi|}$. Thus $|X|\le\sum_{i=0}^{n} 2^{3\cdot(n-i)\cdot|\phi|}\le(n+1)\cdot 2^{3\cdot n\cdot|\phi|}$. The third clause ensures that we draw an edge from s to t, only when t extends s with a single atom. Regarding the function f we need also to deduce various constraints on the weights of the branches of this tree, evaluated at the root. Valuation V equates the truth of a propositional symbol at $\overline{a}$ with its membership at the last element of $\overline{a}$. Also, for this construction to yield a genuine model of ϕ it will also be necessary (in order to prove a 'truth lemma') that certain constraints hold concerning the weights of branches evaluated away from the root. The following two lemmas will allow us to translate such constraints to constraints on branch weights evaluated back at the root.

Lemma 3.6 *Let $\overline{a}\in X_{(n,n-|\overline{a}|)}$. Let $\sum_{i=1}^{k} c_i\cdot\mathbf{P}\overline{\psi^i}$ be a term of degree at most $n-|\overline{a}|$. If $\alpha_{(n-|\overline{a}|)}(\sum_{i=1}^{l} c_i\cdot\mathbf{P}\overline{\psi^i})=\mathbf{L}(\mathbf{P}\overline{b}:\overline{b}\in X_{n-|\overline{a}|})$ then:*

$$\alpha_n(\sum_{i=1}^{k} c_i\cdot\mathbf{P}\overline{a\psi^i})=\mathbf{L}(\mathbf{P}\overline{ab}:\overline{b}\in X_{n-|\overline{a}|})$$

Proof. For any atoms $a,b\in A_m(\phi)$ (any $m\in\mathbb{N}$) we have $\vdash_{\mathbf{Prop}} a\to b$ if and only if $a=b$. Hence, if $\overline{d}\in X_n$ and $\overline{a\psi}$ is a sequence of degree at most n with $a_i\in A_{n-i}(\phi)$ for $1\le i\le|\overline{a}|$, then $\vdash_{\mathbf{Prop}} \overline{d}\to\overline{a\psi} \iff \exists\overline{b}:\overline{d}=\overline{ab}$ and $\vdash_{\mathbf{Prop}} \overline{b}\to\overline{\psi}$. $\alpha_{n-|\overline{a}|}(\sum_{i=1}^{k} c_i\cdot\mathbf{P}\overline{\psi^i})$ is, by definition, a linear combination of terms $(\mathbf{P}\overline{b}:\overline{b}\in X_{n-|\overline{a}|})$. For $\overline{b}\in X_{n-|\overline{a}|}$, the coefficient of $\mathbf{P}\overline{b}$ in this linear combination is $\sum_{i\,:\,\vdash_{\mathbf{Prop}}\overline{b}\to\overline{\psi^i}} c_i$ (definition 3.4(ix)). The coefficient of $\overline{ab}$ in $\alpha_n(\sum_{i=1}^{k}\mathbf{P}\overline{a\psi^i})$ is $\sum_{i\,:\,\vdash_{\mathbf{Prop}}\overline{d}\to\overline{a\psi^i}} c_i=\sum_{i\,:\,\vdash_{\mathbf{Prop}}\overline{b}\to\overline{\psi^i}} c_i$, by the previous paragraph. Thus, the coefficient of $\mathbf{P}\overline{b}$ in $\alpha_{n-|\overline{a}|}(\sum_{i=1}^{k} c_i\cdot\mathbf{P}\overline{\psi^i})$ is the same as the coefficient of $\mathbf{P}\overline{ab}$ in $\alpha_n(\sum_{i=1}^{k}\mathbf{P}\overline{a\psi^i})$. This proves the lemma. $\square$

Lemma 3.7 *Let $m\in\mathbb{N}$ and let τ be a ϕ-term of degree at most m. Then: $\vdash\tau=\alpha_m(\tau)$*

Proof. First consider a primitive term $\tau=\mathbf{P}\overline{\psi}$, of degree not more than m. Let ψ' be obtained by concatenating $\overline{\psi}$ with a sequence of $(m-|\overline{\psi}|)$ $\top$s, so that $|\overline{\psi'}|=m$. By axiom **W0**, $\vdash\mathbf{P}\overline{\psi}=\mathbf{P}\overline{\psi'}$. For $1\le i\le m$, let $A_i=\{a\in A_{m-i}(\phi):\ \vdash_{\mathbf{Prop}} a\to\psi'_i\}$. Then $\overline{a}\in A_1\times\ldots\times A_m$ iff $\vdash_{\mathbf{Prop}} \overline{a}\to\overline{\psi'}$. By propositional reasoning, since ψ'_i is a ϕ-formula, $\vdash_{\mathbf{Prop}} \psi_i' \leftrightarrow \bigvee_{a \in A_i} a$, for $1 \leq i \leq m$. By lemma 3.3, for $1 \leq i \leq m$, we have $\vdash \mathbf{P}\overline{\psi'} = \sum_{a_i \in A_i} \mathbf{P}(\overline{\psi'}[i/a_i])$. Repeating this, for each dimension i,

$$\begin{aligned}\vdash \mathbf{P}\overline{\psi'} &= \sum_{a_1 \in A_1, \ldots, a_m \in A_m} \mathbf{P}\overline{\psi'}[1/a_1]\ldots[m/a_m] \\ &= \sum_{\overline{a} \in A_1 \times \ldots \times A_m} \mathbf{P}\overline{a} \\ &= \sum_{\vdash_{\mathbf{Prop}} \overline{a} \to \overline{\psi'}} \mathbf{P}\overline{a} = \alpha_m(\mathbf{P}\overline{\psi})\end{aligned}$$

which yields the result, for primitive terms. The result follows also for linear combinations of primitive terms, by linearity of α_m. □

Theorem 3.8 *The formula ϕ ($deg(\phi) = n$) is either inconsistent or it is satisfied at the root w of a tree like model $\mathcal{M}$ (defined below) of depth n and size not more than: $(n+1) \cdot 2^{3 \cdot n \cdot |\phi|}$.*

Proof. Suppose ϕ is consistent. Since ϕ is consistent: $\phi \wedge \bigwedge_{\overline{a} \in X}(\mathbf{P}\overline{a} = 0 \vee \mathbf{P}\overline{a} > 0)$ is also consistent. By propositional reasoning, there is a set $Z \subseteq X$ such that $\phi \wedge \bigwedge_{\overline{a} \in Z} \mathbf{P}\overline{a} = 0 \wedge \bigwedge_{\overline{a} \in N} \mathbf{P}\overline{a} > 0$ is consistent too, where $N = X \setminus Z$. By axioms **W0** and **W2**, $\vdash \mathbf{P}\overline{a} = 0 \to \mathbf{P}\overline{ab} = 0$, so $\overline{ab} \in N \Rightarrow \overline{a} \in N$. By lemma 3.2, there is a DNF formula $\bigvee_i \gamma_i \equiv \phi$, where each clause γ_i is a conjunction of propositions, negated propositions and primitive formulas $\tau \bigtriangleup d$. Furthermore, there is a clause $\gamma = \gamma_i$ (some i) such that

$$\phi_0 = \gamma \wedge \bigwedge_{\overline{a} \in Z} \mathbf{P}\overline{a} = 0 \wedge \bigwedge_{\overline{a} \in N} \mathbf{P}\overline{a} > 0$$

is consistent.

We will define a model $\mathcal{X} = (X, V, f)$ for ϕ_0 (hence a model for $\bigvee_i \gamma_i \equiv \phi$) according to the lines of definition 3.5. We aim to build this model in such a way that for any $\overline{a} \in N$ and any $\psi \in S^+_{n-|\overline{a}|}(\phi)$,

$$(10) \qquad \vdash_{\mathbf{Prop}} last(\overline{a}) \to \psi \iff \mathcal{X}, \overline{a} \models \psi$$

and for an arbitrary $\psi \in S^+_n(\phi)$,

$$(11) \qquad \text{If } \vdash_{\mathbf{Prop}} \gamma \to \psi \text{ then } \mathcal{X}, () \models \psi$$

(11) will show that $\mathcal{X}$ is a model for γ, hence a model for ϕ. The definition of the valuation V of $\mathcal{X}$ (3.5, fourth clause) ensures that (10) and (11) hold for propositions and negated propositions occurring in ϕ_0.

To complete the definition of $\mathcal{X}$ it remains to define the weight function f. Each term $\mathbf{P}\overline{a}$ for $\overline{a} \in X$ will be considered as a real-valued variable and we will define a consistent set of linear constraints in these variables. A solution to these constraints will then be used to define f. The set of constraints is

$$\begin{aligned}(12)\qquad C = &\{\mathbf{P}\overline{a} = 0 : \overline{a} \in Z\} \cup \{0 < \mathbf{P}\overline{a} : \overline{a} \in N\} \\ &\cup \{\alpha_n(\tau) \triangle d : \vdash_{\mathbf{Prop}} \gamma \to (\tau \triangle d)\} \\ &\cup \{\alpha_n(\mathbf{L}(\mathbf{P}\overline{a\psi^1}, \ldots, \mathbf{P}\overline{a\psi^k})) \triangle (\mathbf{P}\overline{a} \times d) : \\ &\qquad \overline{a} \in N \setminus \{()\},\ \vdash_{\mathbf{Prop}} last(\overline{a}) \to (\mathbf{L}(\mathbf{P}\overline{\psi^1}, \ldots, \mathbf{P}\overline{\psi^k}) \triangle d)\}\end{aligned}$$

and we let $\bar{C}$ be the set of all linear constraints $\mathbf{L}(\mathbf{P}\overline{a^1}, \ldots, \mathbf{P}\overline{a^k}) \triangle d$ ($\mathbf{L}$ a linear operator, $\overline{a^i} \in X$, $\triangle \in \{<, =, >\}$, $d \in \mathbb{Q}$) such that $\vdash C \to \mathbf{L}(\mathbf{P}\overline{a^1}, \ldots, \mathbf{P}\overline{a^k}) \triangle d$.

We claim that $\vdash \phi_0 \to \bigwedge C$. For this claim, each element of the first two sets is a conjunct of ϕ_0. For the third set, let $\vdash_{\mathbf{Prop}} \gamma \to (\tau \triangle d)$. By lemma 3.7, $\vdash (\tau = \alpha_n(\tau))$ and by **Reals** $\vdash \gamma \to (\alpha_n(\tau) \triangle d)$, so $\vdash \phi_0 \to (\alpha_n(\tau) \triangle d)$. For the final set, let $\overline{a} \in N$ be non-empty and let $\vdash_{\mathbf{Prop}} last(\overline{a}) \to \mathbf{L}(\mathbf{P}\overline{\psi^1}, \ldots, \mathbf{P}\overline{\psi^k}) \triangle d$. Note that $deg(\mathbf{L}(\mathbf{P}\overline{\psi^1}, \ldots, \mathbf{P}\overline{\psi^k})) \leq n - |\overline{a}|$, since $last(\overline{a}) \in A_{n-|\overline{a}|}(\phi)$. By inference rule Extension we have $\vdash \mathbf{P}\overline{a} > 0 \to \mathbf{L}(\mathbf{P}\overline{a\psi^1}, \ldots, \mathbf{P}\overline{a\psi^k}) \triangle \mathbf{P}\overline{a} \times d$. Since $\overline{a} \in N$ we know that $\vdash \phi_0 \to \mathbf{P}\overline{a} > 0$, hence $\vdash \phi_0 \to \mathbf{L}(\mathbf{P}\overline{a\psi^1}, \ldots, \mathbf{P}\overline{a\psi^k}) \triangle \mathbf{P}\overline{a} \times d$. By lemma 3.7 and **Reals** we get $\vdash \phi_0 \to \alpha_n(\mathbf{L}(\mathbf{P}\overline{a\psi^1}, \ldots, \mathbf{P}\overline{a\psi^k})) \triangle \mathbf{P}\overline{a} \times d$. This proves the claim.

Hence the set C is a consistent set of constraints, and consequently the 'provable closure' $\bar{C}$ of C is also consistent. By completeness of **Reals** there is a map $g : X \to \mathbb{R}$ such that

$$(13)\qquad \mathbf{L}(\mathbf{P}\overline{a^1}, \ldots, \mathbf{P}\overline{a^m}) \triangle d \in \bar{C} \Rightarrow \mathbf{L}(g(\overline{a^1}), \ldots, g(\overline{a^m})) \triangle d \quad \text{holds}$$

Observe that $g(\overline{a}) > 0$ for all $\overline{a} \in N$, since $\mathbf{P}\overline{a} > 0$ is a constraint in C.

Now we can complete the definition of $\mathcal{X}$ by defining the weight function f. We let $f(\overline{a}, \overline{b}) = 0$, unless $\overline{b} = \overline{a}a$ for some $a \in A_{n-(1+|\overline{a}|)}(\phi)$ such that $\overline{b} \in N$, i.e. the edges with non-zero weight are contained in a tree with root () with edges defined by extending a sequence by a single atom producing a sequence in N. To define the weights of such edges, let

$$f(\overline{a}, \overline{a}b) = \frac{g(\overline{a}b)}{g(\overline{a})}$$

(since $\overline{a}b \in N$ we have $\overline{a} \in N$ so $g(\overline{a}) > 0$). This completes the definition of $\mathcal{X}$.

It remains to verify (10) and (11). For the left to right implication in (10), suppose $\overline{a} \in N$ is non-empty, $(\mathbf{L}(\mathbf{P}\overline{\psi^1}, \ldots, \mathbf{P}\overline{\psi^k}) \triangle d) \in S^+_{n-|\overline{a}|}(\phi)$ and

$$(14)\qquad \vdash_{\mathbf{Prop}} last(\overline{a}) \to \mathbf{L}(\mathbf{P}\overline{\psi^1}, \ldots, \mathbf{P}\overline{\psi^k}) \triangle d$$

We must show that $\mathcal{X}, \overline{a} \models \mathbf{L}(\mathbf{P}\overline{\psi^1}, \ldots, \mathbf{P}\overline{\psi^k}) \triangle d$. Assume, as an induction hypothesis over $n - |\overline{a}|$, that (10) holds at $\overline{w}$, whenever $|\overline{w}| > |\overline{a}|$ and $\overline{w} \in N$. (This holds when $|\overline{w}| = n$ since in this case we are only required to verify (10) for subformulas of ϕ of degree 0, i.e. propositional formulas, and we have already seen that (10) holds in this case, by definition of the valuation V.) It follows, for any $\overline{w} \in N$ with $|\overline{w}| > |\overline{a}|$ and any atom $a \in A_{n-|\overline{w}|}(\phi), \mathcal{X}, \overline{w} \models a \iff a = last(\overline{w})$

Hence by lemma 1.3,

$$\begin{aligned}[\mathbf{P}\bar{b}]^{\mathcal{X},\bar{a}} &= f(\bar{a},\overline{a\bar{b}\restriction_1})\cdot f(\overline{a\bar{b}\restriction_1},\overline{a\bar{b}\restriction_2})\cdots f(\overline{a\bar{b}\restriction_{|\bar{b}|-1}},\overline{ab})\cdot 1\\ &= \frac{g(\overline{ab})}{g(\bar{a})}\end{aligned}$$

when $\overline{ab}\in N$. Therefore

$$(15)\qquad g(\bar{a})\times[\mathbf{L}^*(\mathbf{P}\overline{b^1},\ldots,\mathbf{P}\overline{b^k})]^{\mathcal{X},\bar{a}} = \mathbf{L}^*(g(\overline{ab^1}),\ldots,g(\overline{ab^k}))$$

for any linear operator $\mathbf{L}^*$, provided $\overline{ab^1},\ldots,\overline{ab^k}\in N$.

By (14) and (12),

$$(16)\qquad \alpha_n(\mathbf{L}(\mathbf{P}\overline{a\psi^1},\ldots,\mathbf{P}\overline{a\psi^k}))\bigtriangleup\mathbf{P}\bar{a}\times d \quad\in C$$

Let

$$(17)\qquad \alpha_{n-|\bar{a}|}(\mathbf{L}(\mathbf{P}\overline{\psi^1},\ldots,\mathbf{P}\overline{\psi^k})) = \mathbf{L}'(\mathbf{P}\bar{b}:\bar{b}\in X_{n-|\bar{a}|})$$

(some linear operator $\mathbf{L}'$). By lemma 3.6 $\alpha_n(\mathbf{L}(\mathbf{P}\overline{a\psi^1},\ldots,\mathbf{P}\overline{a\psi^k})) = \mathbf{L}'(\mathbf{P}\overline{ab}:\bar{b}\in X_{n-|\bar{a}|})$. Hence

$$\begin{array}{rll}\mathbf{L}'(g(\overline{ab}):\bar{b}\in X_{n-|\bar{a}|}) & \bigtriangleup\ g(\bar{a})\cdot d & \text{by (13) and (16)}\\ {[\mathbf{L}'(\mathbf{P}\bar{b}:\bar{b}\in X_{n-|\bar{a}|})]}^{\mathcal{X},\bar{a}} & \bigtriangleup\ d & \text{by (15), } g(\bar{a})>0\\ {[\alpha_{n-|\bar{a}|}(\mathbf{L}(\mathbf{P}\overline{\psi^1},\ldots,\mathbf{P}\overline{\psi^k}))]}^{\mathcal{X},\bar{a}} & \bigtriangleup\ d & \text{by (17)}\\ {[\mathbf{L}(\mathbf{P}\overline{\psi^1},\ldots,\mathbf{P}\overline{\psi^k})]}^{\mathcal{X},\bar{a}} & \bigtriangleup\ d & \text{by lemma 3.7}\end{array}$$

and so $\mathcal{X},\bar{a}\models\mathbf{L}(\mathbf{P}\overline{\psi^1},\ldots,\mathbf{P}\overline{\psi^k})\bigtriangleup d$. We have shown that $\vdash_{\mathbf{Prop}} last(\bar{a})\to\psi\Rightarrow\mathcal{X},\bar{a}\models\psi$, for $\psi=\mathbf{L}(\mathbf{P}\overline{\psi^1},\ldots,\mathbf{P}\overline{\psi^k})\bigtriangleup d\ \in S^+_{n-|\bar{a}|}(\phi)$. The right to left implication in (10) is now easy.

If $\not\vdash_{\mathbf{Prop}}\ last(\bar{a})\to\mathbf{L}(\mathbf{P}\overline{\psi^1},\ldots,\mathbf{P}\overline{\psi^k})\bigtriangleup d$ (where $(\mathbf{L}(\mathbf{P}\overline{\psi^1},\ldots,\mathbf{P}\overline{\psi^k})\bigtriangleup d)\in S^+_{n-|\bar{a}|}(\phi)$) then by definition 3.4(iv) and3.4(v), $\vdash_{\mathbf{Prop}} last(\bar{a})\to\mathbf{L}(\mathbf{P}\overline{\psi^1},\ldots,\mathbf{P}\overline{\psi^k})\bigtriangleup'\ d$ for some $\bigtriangleup'\in\{<,=,>\}\setminus\{\bigtriangleup\}$. By the already proved left to right implication, we deduce $\mathcal{X},\bar{a}\models\mathbf{L}(\mathbf{P}\overline{\psi^1},\ldots,\mathbf{P}\overline{\psi^k})\bigtriangleup'\ d$ and therefore $\mathcal{X},\bar{a}\not\models\mathbf{L}(\mathbf{P}\overline{\psi^1},\ldots,\mathbf{P}\overline{\psi^k})\bigtriangleup d$, as required. This proves (10) at $\bar{a}$; by induction it holds throughout $\mathcal{X}$.

It follows that

$$(18)\qquad [\mathbf{P}\bar{a}]^{\mathcal{X},()} = g(\bar{a})$$

for any $\bar{a}\in N$. The equation also holds for $\bar{a}\in Z$, more trivially, since in this case $[\mathbf{P}\bar{a}]^{\mathcal{X},()}=g(\bar{a})=0$. For (11) recall that γ is a conjunction of propositions, negated propositions and primitive formulas $\mathbf{L}(\mathbf{P}\overline{\psi^1},\ldots,\mathbf{P}\overline{\psi^k})\bigtriangleup d$. We have already dealt with propositional formulas. Let $\vdash_{\mathbf{Prop}}\gamma\to(\mathbf{L}(\mathbf{P}\overline{\psi^1},\ldots,\mathbf{P}\overline{\psi^k})\bigtriangleup d)$. Let $\alpha_n(\mathbf{L}(\mathbf{P}\overline{\psi^1},\ldots,\mathbf{P}\overline{\psi^k}))=\mathbf{L}'(\mathbf{P}\bar{a}:\bar{a}\in X_n)$ for some linear operator $\mathbf{L}'$.

$$\begin{array}{rll}\mathbf{L}'(\mathbf{P}\bar{a}:\bar{a}\in X_n)\bigtriangleup d\ \in\ C & & \text{by (12)}\\ \mathbf{L}'(g(\bar{a}):\bar{a}\in X_n)\ \bigtriangleup\ d & & \text{by (13)}\\ {[\mathbf{L}'(\mathbf{P}\bar{a}:\bar{a}\in X_n)]}^{\mathcal{X},()} & = \mathbf{L}'([\mathbf{P}\bar{a}]^{\mathcal{X},()}:\bar{a}\in X_n) & \text{by linearity of } \mathbf{L}\\ & = \mathbf{L}'(g(\bar{a}):\bar{a}\in X_n) & \text{by (18)}\end{array}$$

so $[\mathbf{L}'(\mathbf{P}\overline{a} : \overline{a} \in X_n)]^{\mathcal{X},()} \triangle d$. By lemma 3.7, $\vdash \mathbf{L}(\mathbf{P}\overline{\psi^1}, \ldots, \mathbf{P}\overline{\psi^k}) = \mathbf{L}'(\mathbf{P}\overline{a} : \overline{a} \in X_n)$ and by soundness (lemma 3.1) it follows that $[\mathbf{L}(\mathbf{P}\overline{\psi^1}, \ldots, \mathbf{P}\overline{\psi^k})]^{\mathcal{X},()} \triangle d$. Hence $\mathcal{X}, () \models \mathbf{L}(\mathbf{P}\overline{\psi^1}, \ldots, \mathbf{P}\overline{\psi^k}) \triangle d$. This proves (11). Therefore $\mathcal{X}, () \models \gamma$, so $\mathcal{X}, () \models \phi$.□

4 Conclusion

We have provided a sound and complete set of axioms and rules for a probabilistic path logic, **PPL**, and we have shown that this logic has the finite model property, thereby solving an open problem in the context of branching probabilistic logics [21]. Based on that completeness proof we could show that the validity problem for **PPL** has **2-EXPTIME** complexity, at worst. We have shown that the fragment of **PCTL** where only the restricted until is allowed for path formulas, is a proper fragment of **PPL**. We named this fragment as **WPCTL**. Thus, the **2-EXPTIME** upper bound of **PPL** is an upper bound of **WPCTL**, therefore we found a fragment of **PCTL** which is decidable. The ability of the logic **PPL** to represent properties of paths rather than mere points makes it more expressive than other probabilistic logics, such as $\mathbf{PEL}_\times$. We introduced also a formula of **PPL** which is not expressible in a probabilistic version of **CTL**.

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