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Computing the Solution of the m-Korteweg-de Vries Equation on Turing Machines

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Abstract

In this paper, we study the initial value problem of the mKdV equation, and convert the constant coefficients mKdV equation into its standard norm by linear transformation. Then, we define a nonlinear map $K_R : H^s(\mathbb{R}) \to C(\mathbb{R}, H^s(\mathbb{R}))$ $(s \geqslant \frac{1}{4})$, from the initial data to the solution of the equation, and prove K_R is Turing computable for any integer $s \geqslant 3$. Therefore, the solution of the mKdV equation with arbitrary precision on Turing machines can be satisfied.

Keywords: mKdV equation, conservation equation, computable, Turing machines.

1 Introduction

In engineering and other scientific computation there are a lot of practical applications which are related to finding solutions of some kind of differential equations. It is always a great challenge for mathematicians to determine whether the equations of certain type have solution and, if it is the case, how these solutions can be computed. Unfortunately, this task cannot always be satisfactorily fulfilled for all equations. However, there are a lot of special equations whose solutions do exist and can be exactly and precisely specified. Those equations are usually called exactly solvable equations. The Korteweg-de Vries equation (KdV equation for short) $u_t + uu_x + u_{xxx} = 0$ is a good example of such kind of equations. Actually, earlier, Gay, Zhang and Zhong[4] studied the Cauchy of the Korteweg-de Vries equation posed on a periodic domain, defined the nonlinear operator and proved that the operator is computable. Later, Klaus Weihrauch and Ning Zhong [12]have shown that

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the solution operator of the KdV equation is computable on real line in the framework of Type-2 computability theory. In this paper, we extend the investigations of [12] to the modified Korteweg-de Vries (mKdV) equations $u_t + \alpha u^2 u_x + \beta u_{xxx} = 0$ for $\alpha,\ \beta \in \mathbb{R}$.

The mKdV equations are used to describe the acoustic spread of non-harmonic Lattice and the Alfen wave sport of non-collision plasma in plasma physics, solid physics, atomic physics, hydrodynamics and the theory of quantum, etc. The mKdV equations have been intensively studied from various aspects of both physics and mathematics. Anne Boutet de Monvel and Dmitry Shepelsky [9] have analyzed an initial-boundary value problem for the mKdV equation in a finite interval. F. Gesztesy and B. Simon [5]have constructed solutions of the mKdV equation. Jing Yu and Ruguang Zhou[15] have presented two kinds of integrable decompositions of the mKdV equation. Turabi Geyikliand and Dogan Kaya [6]have obtained a numerical solution to mKdV equation. In[1,2,3,14], more new kinds of solutions such as periodic wave solutions and solitary wave solutions have been obtained. The mKdV equation has widespread application. Therefore, it is very significant to study the solution operator of the equation.

Our goal is to show that the solution operator of mKdV equations is also computable in the same framework as in [12]. Although a similar approach to that of [12] is used in this paper, the construction is more complicated because the nonlinearity of the mKdV equation is stronger than the KdV equation. Especially, we need conservation equations of the mKdV equation and use properties of Banach algebra to prove our main result. It seems that this method can be further applied to study the operator of the generalized KdV equations.

The paper is organized as follows. In Section 2, we recall the Turing machine, Type 2 theory of effectivity, basic spaces, and representations. In Section 3, we prove the main result and put the tedious inferential process of the proof to Section 4 which contains also three estimates and their proofs for the purpose of effectively determining a computable subsequence.

2 Preliminaries

The computability of subsets and functions on the discrete (countable) sets is usually defined by means of Turing machines. Both inputs and outputs of a Turing machine are finite words. In order to investigate the computability on uncountable sets, the Turing machines have been extended by Klaus Weihrauch [11] so that their inputs and outputs can be infinite sequences as well. These machines are usually called *Type 2 Turing machines* and they can be used to define the computability on the set Σ^ω of infinite sequences in an analogous way while the (classic) Turing machines introduce the computability to the set Σ^* of finite sequences on the alphabet Σ. If we want to introduce the computability to other set D of a cardinality up to continuum, we can choose a representation $\delta : \Sigma^\omega \to D$ of D which is simply a surjective function. That is, the representation δ assigns (possibly infinite) names (δ-names) to each element $x \in D$ and transfers the computability on Σ^ω straightforwardly to the set D. For example, an element $x \in D$ is called δ-computable if it has a computable δ-name.

In order to investigate the computability of the solution operation of various differential equations, we have to introduce the corresponding computability notion to the function spaces at first. In this section, we recall the definitions of computability on several function spaces which are necessary for our discussions. They essentially belong to Klaus Weihrauch and Ning Zhong [12].

Usually, we are interested in the computability on some metric spaces. If a metric space (M, d) has a countable dense subset, then we can define its effectivization as a *computable quadruple metric space* $M = (M, d, A, \nu)$ in which (1) A is a dense subset of M, (2) $\nu :\subseteq \Sigma^* \to A$ is a surjective function (so-called *notation* of A); and (3) the set $\{u, v, w, x \in \Sigma^* : \nu_Q(w) < d(\nu(u), \nu(v)) < \nu_Q(x)\}$ is a recursively enumerable set, where $\nu_Q : \Sigma^* \to \mathbb{Q}$ is the standard notation of the rational numbers. In a computable metric space (M, d, A, ν) we can introduce the computability to the following Cauchy representation $\delta_C :\subseteq \Sigma^\omega \to M$ which is a surjective function such that $\delta_C(p) = x$ if and only if $p = w_0\#w_1\#w_2\#\cdots$ for $w_i \in dom(\nu)$ and the sequence $\{\nu(w_i)\}$ converges effectively to x in the sense that $d(x, \nu(w_i)) \leq 2^{-i}$ for all $i \in \mathbb{N}$.

For example, let $L^2(\mathbb{R})$ be the set of all L^2-functions, i.e., the functions f meet the condition that $(\int_{\mathbb{R}} |f(x)|^2 dx)^{1/2} < \infty$ and let d_{L^2} be the standard L^2-norm defined by $d_{L^2}(f, g) = ||f - g|| = (\int_{\mathbb{R}} \mid f(x) - g(x) \mid^2 dx)^{1/2}$. Then the L^2-function space $(L^2(\mathbb{R}), d_{L^2})$ has a countable dense subset σ consisting of all rational finite step functions. Let v_{L^2} be a canonical notation of σ. Thus we achieve the computable metric space $(L^2(\mathbb{R}), d_{L^2}, \sigma, v_{L^2})$. On this computable metric space we can define the Cauchy representation δ_{L^2} as follows: a sequence $p \in \Sigma^\omega$ is a δ_{L^2}-name of $g \in L^2(\mathbb{R})$ (i.e., $\delta_{L^2}(p) = g$) if and only if $p = w_0\#w_1\#w_2\#\cdots$ with $w_i \in dom(v_{L^2})$ and $||v_{L^2}(w_i) - g|| \leq 2^{-i}$ for all $i \in \mathbb{N}$.

In order to prove our main theorem, we need to introduce some representations of the Schwardtz space. Let the Schwardtz space $S(\mathbb{R})$ be the set of all functions $\phi \in C^\infty(\mathbb{R})$, such that $\sup_{x\in\mathbb{R}} |x^\alpha \phi^{(\beta)}(x)| < \infty$ for all $\alpha, \beta \in \mathbb{N}$,where $C^\infty(\mathbb{R})$ is the space of complex-valued functions of class C^∞ equipped with the compact open topology. Let d_s be the metric defined by

$$d_s(\phi, \varphi) = \sum_{\alpha,\beta=0}^{\infty} 2^{-<\alpha,\beta>} \frac{\| \phi - \varphi \|_{\alpha,\beta}}{1 + \| \phi - \varphi \|_{\alpha,\beta}} \qquad \forall \alpha, \beta \in \mathbb{N}$$

where $\langle \alpha, \beta \rangle$ is the bijective Cantor pairing function, defined by $< \alpha, \beta >:= \beta + (\alpha + \beta)(\alpha + \beta + 1)/2$, and $\| \phi \|_{\alpha,\beta} := \sup_{x\in\mathbb{R}} \mid x^\alpha \phi^{(\beta)}(x) \mid$. Then the space $(S(\mathbb{R}), d_s)$ has a dense subset $\mathcal{P}^*$ consisting of the set of truncated polynomials with rational coefficients. Let ν_∞^p be the canonical notation of $\mathcal{P}^*$. Thus we obtain the computable metric space $(S(\mathbb{R}), d_s, \mathcal{P}^*, \nu_\infty^p)$. Let $\delta_{sc} :\subseteq \Sigma^\omega \to S(\mathbb{R})$ be the Cauchy representation. However, we have to introduce another representation of $S(\mathbb{R})$ for the proof, denoted as δ_s, defined by

$$\delta_s(\langle q,p\rangle)=\phi \Leftrightarrow \begin{cases} \delta_\infty^P(p)=\phi \quad and \\ q=u_0\#u_1\#u_2\cdots, where\ u_k\in dom(\upsilon_{\mathbb{N}}) \\ and \quad \sup\limits_{|x|\geqslant \upsilon_{\mathbb{N}}(u_{<i,j,n>})} |x^i\phi^{(j)}(x)|\leqslant 2^{-n} \end{cases}$$

Remark 2.1 δ_∞^P is the Cauchy representation of the computable metric space $(C^\infty(\mathbb{R}),d_c,P,\nu^p)$, where d_c is the metric defined as follows:

$$d_c(\phi,\varphi)=\sum_{\alpha,\beta=0}^{\infty}2^{-<\alpha,\beta>}\frac{\parallel \phi-\varphi\parallel_{\alpha,\beta}}{1+\parallel \phi-\varphi\parallel_{\alpha,\beta}},$$

P is the set of polynomials with rational coefficients, and ν^p is a canonical notation of P. $\nu_{\mathbb{N}}$ is a canonical notation of the set $\mathbb{N}$.

The representations on $L^2(\mathbb{R})$ and $S(\mathbb{R})$ lead straightforwardly to a representation of the Sobolev space $H^s(\mathbb{R})$. By definition, the Sobolev space consisits of all functions $f\in L^2(\mathbb{R})$ such that $T_s(f)\in L^2(\mathbb{R})$, where

$$T_s(f)(\xi):=(1+|\xi|^2)^{s/2}\mathcal{F}(f)(\xi)$$

is a weighted Fourier transform of f with weight $(1+|\xi|^2)^{s/2}$ and $\mathcal{F}(f)$ denotes the Fourier transform of f. An infinite word $p\in\Sigma^\omega$ is a δ_{H^s}-name of $f\in H^s(\mathbb{R})$, iff p is a δ_{L^2}-name of the weighted transform $T_s(f)\in L^2(\mathbb{R})$(i.e $\delta_{H^s}(p)=T_s^{-1}\circ\delta_{L^2}(p)$). During the proof, we also need another representation of $H^s(\mathbb{R})$. When $s\geqslant 0$ is an integer, $H^s(\mathbb{R})$ is the same as the set: $\{f\in L^2(\mathbb{R}):$ the k^{th}order derivative f^k of f is in $L^2(\mathbb{R})$ for all $0\le k\le s\}$, define the norm as following:

$$\parallel f(x,t)\parallel_s=(\parallel f(x,t)\parallel^2+\parallel f'(x,t)\parallel^2+\cdots+\parallel f^{(s)}(x,t)\parallel)^{1/2}$$

An infinite word $p\in\Sigma^\omega$ is $\tilde{\delta}_{H^s}$-name of f, iff $p=\langle p_0,p_1,p_2,\cdots\rangle$ with $p_i\in dom(\delta_{sc})$ and $\parallel \delta_{sc}(p_i)-f\parallel_s\leqslant 2^{-i}$.

Remark 2.2 $S(\mathbb{R})$ is dense in $H^j(\mathbb{R})$ for all $j\in\mathbb{N}$. If $f(t)\in S(\mathbb{R})$,we can see that $f(t)\in H^j(\mathbb{R})$, in particular , $D_x^j f(t)\in L^2(\mathbb{R})$ for all $j\in\mathbb{N}$ and $t\in\mathbb{R}$

Finally, we consider the representation of functions. Suppose that we have two sets M and M' with the representations $\delta :\subseteq \Sigma^\omega\to M$ and $\delta' :\subseteq\Sigma^\omega\to M'$, respectively. We say that a function $f:M\to M'$ is (δ,δ')-computable if there is a type-2 Turing machine which transfers any δ-name of an $x\in dom(f)$ to a δ'-name of $f(x)$ for any $x\in dom(x)$. It is well known that, any (δ,δ')-computable function is (δ,δ')-continuous. Here a function $f:M\to M'$ is (δ,δ')-continuous if there is a continuous function $g:\Sigma^\omega\to\Sigma^\omega$ which transfers any δ-name of $x\in dom(f)$ to a δ'-name of $f(x)$. Denote by $C(M,M')$ the set of all (δ,δ')-continuous functions from $M\to M'$. For the set $C(M,M')$, there is a canonical representation (see [11])

$[\delta \to \delta'] : \Sigma^\omega \to C(M, M')$ such that a function f is (δ, δ')-computable iff it has a computable $[\delta \to \delta']$-name.

3 Main result

In this section, we will prove our main result, that is, the solution operator of the mKdV equations is computable. We explain in this section the essential idea of how this result can be proved. Some technical details will be given in Section 4.

Precisely, we are interested in the following initial value problem (IVP, for short) of mKdV equation on the real line $\mathbb{R}$,

$$(1) \quad \begin{cases} u_t + \alpha u^2 u_x + \beta u_{xxx} = 0, & (t, x \in \mathbb{R}) \\ u(x, 0) = \varphi(x) \in H^s(\mathbb{R}) \end{cases}$$

By a computable linear transformation $u = \frac{1}{\sqrt{\alpha\beta^{-1/3}}} u'$, $x = \beta^{1/3} x'$ and $t = t'$, the IVP (1) can be transformed computably to the following

$$(2) \quad \begin{cases} u_t + u^2 u_x + u_{xxx} = 0, & (t, x \in \mathbb{R}) \\ u(x, 0) = \varphi(x) \in H^s(\mathbb{R}) \end{cases}$$

Thus, it suffices to consider only the IVP (2) because any computable solution of (2) leads straightforwardly by a computable reverse transformation to a computable solution of (1). Originally, the IVP (2) looks for a function u of two arguments which satisfies both equations from any given function φ. However, if we write the function u as a functional $u(x,t) := u(t)(x)$, then the IVP (2) can be regarded equivalently as a solution operator mapping a function φ to the functional $u : t \mapsto u(t)$ where $u(t) : \mathbb{R} \to \mathbb{R}$ is a real function of one argument. Furthermore, the initial function φ is in the Sobolev space $H^s(\mathbb{R})$, then (2) does have a solution u which, as a functional, is a continuous function from $\mathbb{R}$ to $H^s(\mathbb{R})$ for any $s \geq \frac{1}{4}$ (see[7]). In other words, the IVP (2) has a solution operator $K_{\mathbb{R}} : H^s(\mathbb{R}) \to C(\mathbb{R}; H^s(\mathbb{R}))$.The special solidary wave solution possessed by the mKdV equation in [2] is

$$u(x,t) = \sqrt{\frac{6c}{\alpha}} sech(\sqrt{\frac{c}{\beta}}(x - ct))$$

where c is wave speed. The solution is obviously computable when c is computable real number. Our main result asserts that this solution operator is actually computable in the framework of type-2 computability theory.

Theorem 3.1 *The solution operator $K_{\mathbb{R}} : H^s(\mathbb{R}) \to C(\mathbb{R}; H^s(\mathbb{R}))$ of the initial value problem (2) is $(\delta_{H^s}, [\rho \to \delta_{H^s}])$-computable for any integer $s \geqslant 3$.*

During the proof, firstly we will give the equivalent integral equation of the IVP(2), and use the iterative approach to solute this integral equation. Secondly, we prove that this iterative sequence is computable, and converges uniformly in $t \in [0, T]$. Thus, the limit of the converging sequence will also be computable, and satisfy the equivalent integral equation of the IVP(2). In other words, the limit will be the solution of the IVP(2), and then the solution will be computable.

The following is the equivalent integral equation of the initial value problem (2)

$$(3)\quad u(t) = \mathcal{F}^{-1}\left(E(t)\cdot\mathcal{F}(\varphi)\right) - \frac{1}{3}\int_0^t \mathcal{F}^{-1}\left(E(t-\tau)\cdot\mathcal{F}\left(\tfrac{d}{dx}\left(u(\tau)\right)^3\right)\right)\mathrm{d}\tau$$

Where $u(t)(x) := u(x,t)\,, E(t)(x) := e^{ix^3 t}$, and $\mathcal{F}(\varphi)(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathrm{R}} e^{-ix\xi}\varphi(\xi)\,\mathrm{d}\xi$.

We show that the following iterative sequence with the initial data φ as the seed:

$$(4)\quad \begin{cases} v_0(t) = \mathcal{F}^{-1}\left(E(t)\cdot\mathcal{F}(\varphi)\right) \\ v_{j+1}(t) = v_0(t) - \frac{1}{3}\int_0^t \mathcal{F}^{-1}\left(E(t-\tau)\cdot\mathcal{F}\left(\frac{\mathrm{d}}{\mathrm{dx}}\left(v_j(\tau)\right)^3\right)\right)\mathrm{d}\tau. \end{cases}$$

The iterative sequence (4) is contracting near $t = 0$, thus the sequence converges to a unique limit. Since the limit satisfies the integral equation (3), it is the solution of the initial value problem (2) near $t = 0$. To prove that the solution operator is computable, we need to construct a type-2 Turing machine which computes fast approximations to the solution $u(t)$ when given enough information on the initial data φ.The machine will be designed in such a way that is capable of computing the iterative sequence in (4) when inputting Schwartz function. Thus for any given initial data $\varphi \in H^s(\mathbb{R})$ and any $\tilde{\delta}_{H^s}$-name $\langle p_0, p_1 \ldots\rangle$ of φ, the machine is able to compute the iterative sequence for each seed $\delta_{sc}(p_i)$.We recall that $\delta_{sc}(p_i)$ is an approximation to φ in $H^s(\mathbb{R})$ with accuracy 2^{-i}.

Firstly, we define the operator

$$S(u,\varphi,t) = \mathcal{F}^{-1}(E(t)\cdot\mathcal{F}(\varphi)) - \frac{1}{3}\int_0^t \mathcal{F}^{-1}\left(E(t-\tau)\cdot\mathcal{F}\left(\tfrac{d}{dx}(u(\tau))^3\right)\right)\mathrm{d}\tau$$

which is $([\rho \to \delta_s], \delta_s, \rho, \delta_s)$-computable. This follows from Lemma 3.2 in [12] straightforwardly. Therefore, the function $\bar{S}(u,\varphi)(t) := S(u,\varphi,t)$ is $([\rho \to \delta_s], \delta_s, [\rho \to \delta_s])$-computable. Then we define the function $v : S(\mathbb{R}) \times \mathbb{N} \to C(\mathbb{R} : S(\mathbb{R}))$ by

$$v(\psi,0) = \bar{S}(0,\psi)$$
$$v(\psi,j+1) = \bar{S}(v(\psi,j),\psi).$$

It is easy to verify that v is $(\delta_S, \gamma_{\mathbb{N}}, [\rho \to \delta_S])$-computable.

Proof. (of Theorem 3.1) For a given initial value $\varphi \in H^s(\mathbb{R})$ and a rational number $\bar{T} > 0$ we will show how to compute the solution $u(t)$ of the initial value problem (2) at the time interval $0 \leqslant t \leqslant \bar{T}$. For this purpose, we first find some appropriate rational number T such that $0 < T < \bar{T}$, and show how to compute $u(t)$ from t' and $\psi := u(t')$ at the time interval $[t', t'+T]$, $0 \leqslant t' \leqslant \bar{T}$, by a fixed point iteration.Using this method, we can compute the values $u(T/2m)$ successively for $m = 1, 2, \cdots$ and finally $u(t)$ for any $0 \leqslant t \leqslant \bar{T}$.

If $u_t+u^2u_x+u_{xxx}=0, u(x,t')=\psi(x)$, and v is defined by $v(x,t):=u(x,t+t')$, then

$$(5)\quad \begin{cases} v_t+v^2v_x+v_{xxx}=0 & x\in\mathbb{R}, t\geqslant 0,\\ v(x,0)=\psi(x). \end{cases}$$

We assume that the initial value $\psi\in H^s(\mathbb{R})$ is given by a $\tilde{\delta}_{H^s}$-name, i.e., by a sequence $\psi_0,\psi_1,\cdots$ of Schwartz functions such that $\parallel \psi-\psi_n \parallel_s\leqslant 2^{-n}$. For any $n\in\mathbb{N}$, we define function $v_n^0, v_n^1,\cdots\in C(\mathbb{R}:S(\mathbb{R}))$ by

$$(6)\quad v_n^0:=\bar{S}(0,\psi_n),\quad v_n^{j+1}:=\bar{S}(v_n^j,\psi_n).$$

We note that the sequence $\{v_n^j\}$ can be computed from ψ_n. If the iterative sequence $v_n^0, v_n^1,\cdots$ converges to some v_n, then v_n is the fixed point of the iteration $\bar{S}$ and satisfies the following internal equation:

$$(7)\quad v_n(t)=\bar{S}(v_n,\psi_n)$$

$$(8)\quad =\mathcal{F}^{-1}(E(t)\cdot\mathcal{F}(\psi_n))-\frac{1}{3}\int_0^t \mathcal{F}^{-1}\left(E(t-\tau)\cdot\mathcal{F}\left(\frac{\mathrm{d}}{\mathrm{dx}}(v_n(\tau))^3\right)\right)\mathrm{d}\tau$$

hence v_n solves the initial value problem:

$$(9)\quad \frac{\partial v_n}{\partial t}+v_n^2\frac{\partial v_n}{\partial x}+\frac{\partial v_n}{\partial x^3}=0,\qquad\text{and}\qquad v_n(x,0)=\psi_n(x).$$

We will show that, by a contraction argument, for some sufficiently small computable real number $T>0$ (depending only on φ and $\bar{T}$), $v_n^j(t)\to v_n(t)$ as $j\to\infty$ for all n, and $v_n(t)\to v(t)$ as $n\to\infty$, sufficiently fast and uniformly in $t\in[0,T]$. We recall that v is the solution of the initial value problem (5). Then we can effectively determine a computable subsequence of the double sequence$\{v_n^j\}$ which will converge fast to v uniformly in $t\in[0,T]$.The inferential process is tedious,we will finish it in next section.

Since v is the limit of a fast convergent computable sequence, v itself is computable. So $K_{\mathbb{R}}:(\varphi,t)\mapsto u(t)$ is $(\delta_{H^s},\rho,\delta_{H^s})$-computable for $t\geqslant 0$, the reflection $R:S(\mathbb{R})\to S(\mathbb{R}), R(\psi)(x):=\psi(-x)$, is $(\delta_{H^s},\delta_{H^s})$-computable. Define $u'(t)(x):=u(-t)(-x)$. Then $u'_t+u'^2u'_x+u'_{xxx}=0$ and for $t\geqslant 0, u(-t)=R\circ u'(t)=R\circ K_{\mathbb{R}}(u'(0),t)=R\circ K_{\mathbb{R}}(R(\varphi),t)$, i.e., $u(t)=R\circ K_{\mathbb{R}}(R(\varphi),-t)$ for $t\leqslant 0$. Therefore, as the two computable functions join at 0, $K_{\mathbb{R}}$ is computable for $t\in\mathbb{R}$. (see e.g. Lemma 4.35 in [12]). □

4 Three estimates

From the above section, we obtain the result that v_n is the solution of the IVP(9), and $\|\psi-\psi_n\|_s\leq 2^{-n}$. If the sequence $\|v-v_n\|_s$ is controlled by $\|\psi-\psi_n\|_s$, we can obtain the result that the sequence $\{v_n\}$ converges uniformly. For the purpose of effectively determining a computable subsequence $\{v_n^j\}$ that is convergent fast and uniformly, we need three estimates listed below as Propositions 4.1,4.6,4.9. In the first proposition, we prove the solution of IVP(2) can be controlled by the initial function $\varphi(x)$.

Proposition 4.1 *If $u(x,t)$ is the solution of the IVP (2), Then there is a computable function $e:\mathbb{N}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ which is non-decreasing in the second and third argument such that*

$$\sup_{0\leqslant t\leqslant T}\parallel u(x,t)\parallel_s\leqslant e_T^s(\parallel\varphi\parallel_s),$$

where $e_T^s(r):=e(s,T,r)$, s is an integer and $s\geqslant 3$.

In order to prove the proposition, we need to introduce the conservation equation. As we know, there are three significant conservation laws in the area of physics, which are conservation of mass, of energy and of momentum. In the area of mathematics, when a physical problem can be described by a differential equation, like $u_t=k(u)$, the conservations law of this equation can be described as the following form:

$$\frac{\partial T}{\partial t}+\frac{\partial X}{\partial x}=0. \tag{10}$$

Where T and X are relative to $u(x,t)$, $X=0$ on the boundary of the field of definition. From (10) we know that $I=\int T\mathrm{dx}$ is irrelative to t. From the mKdV equation, we have the following:

$$[u^2]_t+[\frac{1}{2}u^4+2uu_{xx}-u_x^2]_x=0$$

$$[-\frac{3}{2}u_x^2+\frac{1}{4}u^4]_t+[-3u^2u_x^2-3u_xu_{xxx}+\frac{3}{2}u_{xx}^2+\frac{1}{6}u^6+u^3u_{xx}]_x=0$$

$$[\frac{5}{3}u^2u_x^2-u_{xx}^2-\frac{1}{18}u^6]_t+[\frac{5}{2}u^4u_x^2+\frac{10}{3}u^2u_xu_xxx-\frac{8}{3}u^2u_xx-\frac{10}{3}uu_x^2u_{xx}$$
$$-\frac{1}{6}u_x^4-2u_{xx}u_{xxxx}+u_{xxx}^2-\frac{1}{24}u^8-\frac{1}{3}u^5u_{xx}]_x=0$$

Define

$$\Phi_0(u)=\int\limits_{\mathbb{R}}u^2(x,t)\mathrm{d}x$$

$$\Phi_1(u)=\int\limits_{\mathbb{R}}\left(-\frac{3}{2}u_x^2+\frac{1}{4}u^4\right)\mathrm{d}x$$

$$\Phi_2(u)=\int\limits_{\mathbb{R}}[\frac{5}{3}u^2(x,t)u_x^2(x,t)-u_{xx}^2(x,t)-\frac{1}{18}u^6(x,t)]\mathrm{d}x.$$

It is easy to verify that

$$\frac{\mathrm{d}}{\mathrm{dt}}\Phi_j(u)=0\quad\text{for}\quad j=0,1,2\quad\text{and}\quad t\in\mathbb{R}.$$

Consequently, for any $t\in\mathbb{R}$,

$$\int\limits_{\mathbb{R}}u^2(x,t)dx=\int\limits_{\mathbb{R}}\varphi^2(x)\mathrm{d}x\quad\text{ thus }\|u(\cdot,t)\|=\|\varphi\|$$

$$\int_{\mathbb{R}} \left(-\frac{3}{2}u_x^2(x,t)+\frac{1}{4}u^4(x,t)\right)\mathrm{d}x = -\frac{3}{2}\int_{\mathbb{R}} \varphi_x^2(x)\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}} \varphi^4(x)\mathrm{d}x$$

$$\int_{\mathbb{R}} [\frac{5}{3}u^2(x,t)u_x^2(x,t) - u_{xx}^2(x,t) - \frac{1}{18}u^6(x,t)]\mathrm{d}x$$

$$= \int_{\mathbb{R}} [\frac{5}{3}\varphi^2(x)\varphi_x^2(x) - \varphi_{xx}^2(x) - \frac{1}{18}\varphi^6(x)]\mathrm{d}x$$

During the proof, we need the following inequalities, which are in common use and whose proofs are seen in [8].

Lemma 4.2 (Young's inequality) *For $p, q > 1, p^{-1}+q^{-1} = 1$ and $a, b \geqslant 0$,*

$$ab \leqslant \varepsilon a^p + c(\varepsilon)b^q,$$

where $c(\varepsilon) = (p-1)\,/\left(p^q \varepsilon^{\frac{1}{p-1}}\right)$.

Lemma 4.3 *For any $f \in H^1(\mathbb{R})$,*

$$\| f \|_{L^\infty} \leqslant \| f \|_1 \quad \text{and} \quad \| f \|_{L^\infty} \leqslant \sqrt{2} \, \| f \|^{1/2} \| f' \|^{1/2}$$

Lemma 4.4 (Gronwall's inequality) *If, for any $t \geqslant 0$ and $a, b > 0$, $D_t x(t) \leqslant ax(t) + b$, then*

$$x(t) \leqslant x(0)\, e^{at} + \frac{b}{a}\left(e^{at} - 1\right),$$

for any $t \geqslant 0$, in particular,

$$x(t) \leqslant x(0)\, e^{aT} + \frac{b}{a}\left(e^{aT} - 1\right)$$

for any $0 \leqslant t \leqslant T$.

Since $\| u(x,t) \|_s = (\| u(x,t) \|^2 + \| u'(x,t) \|^2 + \cdots + \| u^{(s)}(x,t) \|)^{1/2}$, we give the following lemma 4.5 firstly.

Lemma 4.5 *There is a computable function $C : \mathbb{N}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$, such that for any $\varphi \in H^s(\mathbb{R})$, any integer $s \geqslant 1$ and any $T > 0$,*

$$\left\| u^{(s)}(\cdot, t) \right\| \leqslant C_T^s\left(\|\varphi\|_{s-1}\right)\|\varphi\|_s, \quad (0 \leqslant t \leqslant T),$$

where u is the solution of the IVP (2), $u^{(s)} := \partial_x^s u$ and $C_T^s(r) := C(s, T, r)$.

Proof. In the following let $t \in \mathbb{R}$ and abbreviate $\|u\|_\infty := \|u(t)\|_{L^\infty}$. From the second conservation, we obtain

$$\|u_x\|^2 = \int_{\mathbb{R}} u_x^2(x,t)dx = \int_{\mathbb{R}} \varphi_x^2 dx - \frac{1}{6}\int_{\mathbb{R}} \varphi^4(x)dx + \frac{1}{6}\int_{\mathbb{R}} u^4 dx$$

$$\leqslant \|\varphi_x\|^2 + \frac{1}{6}\|u\|_\infty^2 \|u\|^2$$

$$\leqslant \|\varphi_x\|^2 + \frac{1}{6}\cdot 2 \cdot \|u\| \|u_x\| \|u\|^2$$

$$\leqslant \frac{1}{2}\|u_x\|^2 + \frac{1}{18}\|u\|^6 + \|\varphi_x\|^2 \qquad \text{(Young's inequality)}$$

Subtracting both sides by $\frac{1}{2}\|u_x\|^2$ we have

$$\|u_x\|^2 \leqslant \frac{1}{9}\|\varphi\|^6 + 2\|\varphi_x\|^2 \leqslant 2\left(\|\varphi\|^4 + 1\right) \cdot \|\varphi\|_1^2 .$$

Thus $\|u_x\| \leqslant \sqrt{2}\left(\|\varphi\|^4+1\right)^{\frac{1}{2}} \cdot \|\varphi\|_1 \leqslant C_1(\varphi). \|\varphi\|_1$, where $C_1(r) = \sqrt{2}\left(r^4+1\right)^{\frac{1}{2}}$. When $s = 1$, we prove the result.

we observe that when $s > \frac{1}{2}$, $H^s(\mathbb{R})$ is a Banach algebra. Therefore, $\| u^m \|_s \leqslant \| u \|_s^m$, from the third conservation, we have

$$\|u_{xx}\|^2 = \int\limits_{\mathbb{R}} u_{xx}^2(x,t)dx$$

$$= \int\limits_{\mathbb{R}} \varphi_{xx}^2(x,t)dx - \frac{5}{3}\int\limits_{\mathbb{R}} \varphi^2(x)\varphi_x^2(x)\mathrm{d}x + \frac{1}{18}\int\limits_{\mathbb{R}} \varphi^6(x)\mathrm{d}x$$

$$+\frac{5}{3}\int\limits_{\mathbb{R}} u^2(x,t)u_x^2(x,t)\mathrm{d}x - \frac{1}{18}\int\limits_{\mathbb{R}} u^6(x,t)\mathrm{d}x$$

$$\leqslant \|\varphi_{xx}\|^2 + \frac{1}{18}\|\varphi\|_1^6 + \frac{5}{3}\|u\|_\infty^2\|u_x\|^2$$

$$\leqslant \|\varphi_{xx}\|^2 + \frac{1}{18}\|\varphi\|_1^6 + \frac{10}{3}\|u\|\|u_x\|^3$$

$$\leqslant \|\varphi_{xx}\|^2 + \frac{1}{18}\|\varphi\|_1^6 + \frac{5}{3}\|u\|^2 + \frac{5}{3}\|u_x\|^6$$

$$\leqslant [2 + 2(1 + c_1^6(\| \varphi \|))\cdot \| \varphi \|_1^4] \cdot (\|\varphi\|^2 + \|\varphi_x\|^2 + \|\varphi_{xx}\|^2)$$

$$\leqslant C_2(\|\varphi\|_1)\|\varphi\|_2$$

where $C_2(r) = 2+2[1+c_1^6(r)r^4]$, $c_1^6(r) = [c_1(r)]^6$. When $s = 2$, we obtain the result.

We prove the lemma by induction on s . The inequality holds for $s = 1, 2$ $\left(C_T^1 := C_1, C_T^2 := C_2\right)$. Assume $s \geqslant 3$ and that the inequality holds for all $s\prime < s$. Differentiating (2) s times with respect to x , we obtain

$$\begin{cases} u_t^{(s)} + u_{xxx}^{(s)} = -D_x^s\left(u^2 u_x\right), \\ u^{(s)}(x,0) = \varphi^{(s)}(x). \end{cases}$$

Since

$$D_x^s\left(u^2u_x\right) = \sum_{j=0}^{s} \tbinom{s}{j} \left(u^2\right)^{(j)} u^{(s-j+1)}$$

$$= \sum_{j=0}^{s} \tbinom{s}{j} \sum_{i=0}^{j} \binom{j}{i} u^{(i)} u^{(j-i)} u^{(s-j+1)}$$

$$= \sum_{j=2}^{s-2} \tbinom{s}{j} \sum_{i=0}^{j} \binom{j}{i} u^{(i)} u^{(j-i)} u^{(s-j+1)} + \sum_{i=1}^{s-2} \tbinom{s}{i} u^{(i)} u^{(s-i)} u_x + u^2 u^{(s+1)}$$

$$+2\left(s+1\right) u u_x u^{(s)} + s u^{(s-1)} u_x u_x + 2 s u u^{(s-1)} u_{xx} + s \sum_{i=1}^{s-2} \tbinom{s-1}{i} u^{(i)} u^{(s-i-1)} u_{xx}$$

we have

$$\frac{d}{dt} \int\limits_{\mathbb{R}} \left(u^{(s)}\right)^2 dx = -2 \int\limits_{\mathbb{R}} u^{(s)}_{xxx} \cdot u^{(s)} dx - 2 \int\limits_{\mathbb{R}} D_x^s \left(u^2 u_x\right) u^{(s)} dx$$

$$= -2 \int\limits_{\mathbb{R}} u^2 u^{(s+1)} u^{(s)} dx - 4\left(s+1\right) \int\limits_{\mathbb{R}} u u_x \left(u^{(s)}\right)^2 dx - 2s \int\limits_{\mathbb{R}} u^{(s-1)} u_x u_x u^{(s)} dx$$

$$-4s \int\limits_{\mathbb{R}} u u^{(s-1)} u_{xx} u^{(s)} dx - 2 \sum_{i=1}^{s-2} \tbinom{s}{i} \int\limits_{\mathbb{R}} u^{(i)} u^{(s-i)} u_x u^{(s)} dx$$

$$-2 \sum_{j=2}^{s-2} \tbinom{s}{j} \sum_{i=0}^{j} \binom{j}{i} \cdot \int\limits_{\mathbb{R}} u^{(i)} u^{(j-i)} u^{(s-j+1)} u^{(s)} dx$$

$$-2s \sum_{i=1}^{s-2} \tbinom{s-1}{i} \int\limits_{\mathbb{R}} u^{(i)} u^{(s-i-1)} u_{xx} u^{(s)} dx = A + B$$

where

$$A = -2 \int\limits_{\mathbb{R}} u^2 u^{(s+1)} u^{(s)} dx - 4\left(s+1\right) \int\limits_{\mathbb{R}} u u_x \left(u^{(s)}\right)^2 dx$$

$$-2s \int\limits_{\mathbb{R}} u^{(s-1)} u_x u_x u^{(s)} dx - 4s \int\limits_{\mathbb{R}} u u^{(s-1)} u_{xx} u^{(s)} dx$$

$$B = -2 \sum_{i=1}^{s-2} \tbinom{s}{i} \int\limits_{\mathbb{R}} u^{(i)} u^{(s-i)} u_x u^{(s)} dx$$

$$-2 \sum_{j=2}^{s-2} \tbinom{s}{j} \sum_{i=0}^{j} \binom{j}{i} \cdot \int\limits_{\mathbb{R}} u^{(i)} u^{(j-i)} u^{(s-j+1)} u^{(s)} dx$$

$$-2s \sum_{i=1}^{s-2} \tbinom{s-1}{i} \int\limits_{\mathbb{R}} u^{(i)} u^{(s-i-1)} u_{xx} u^{(s)} dx$$

First , consider

$$A = -\int\limits_{\mathbb{R}} u^2 d\left(u^{(s)}\right)^2 - 4\left(s+1\right) \cdot \int\limits_{\mathbb{R}} u u_x \left(u^{(s)}\right)^2 dx$$

$$
\begin{aligned}
&-2s\int\limits_{\mathbb{R}} u^{(s-1)}u_xu_xu^{(s)}dx-4s\int\limits_{\mathbb{R}} uu^{(s-1)}u_{xx}u^{(s)}dx\\
=&-2\left(2s+1\right)\cdot\int\limits_{\mathbb{R}} uu_x\left(u^{(s)}\right)^2dx-2s\int\limits_{\mathbb{R}} u^{(s-1)}u_xu_xu^{(s)}dx\\
&\quad-4s\int\limits_{\mathbb{R}} uu^{(s-1)}u_{xx}u^{(s)}dx\\
\leqslant&\, 2\left(2s+1\right)\|u_x\|\cdot\|u\|_\infty\cdot\left\|u^{(s)}\right\|^2+2s\left\|u^{(s-1)}\right\|\cdot\|u_x\|_\infty^2\cdot\left\|u^{(s)}\right\|\\
&+4s\left\|u^{(s-1)}\right\|_\infty\cdot\left\|u^{(s)}\right\|\cdot\|u\|_\infty\cdot\|u_{xx}\|\\
\leqslant&\, 2\left(2s+1\right)\|u_x\|\cdot\left(\|u\|+\|u_x\|\right)\cdot\left\|u^{(s)}\right\|^2+2s\left\|u^{(s-1)}\right\|\cdot\left(\|u_x\|+\|u_{xx}\|\right)^2\\
&\quad\cdot\left\|u^{(s)}\right\|+4s\left(\left\|u^{(s-1)}\right\|+\left\|u^{(s)}\right\|\right)\cdot\left(\|u\|+\|u_x\|\right)\cdot\|u_{xx}\|\cdot\left\|u^{(s)}\right\|\\
\leqslant&\, 2\left(2s+1\right)\|u_x\|\cdot\left(\|u_x\|+\|u\|\right)\cdot\left\|u^{(s)}\right\|^2+4s\left(\|u\|+\|u_x\|\right)\cdot\|u_{xx}\|\cdot\left\|u^{(s)}\right\|^2\\
&\quad+2s\left(\|u_x\|+\|u_{xx}\|\right)^2\cdot\left\|u^{(s-1)}\right\|\cdot\left\|u^{(s)}\right\|\\
&\quad+4s\left\|u^{(s-1)}\right\|\left(\|u\|+\|u_x\|\right)\cdot\|u_{xx}\|\cdot\left\|u^{(s)}\right\|\\
\leqslant&\, 2(4s+1)(\|u\|+\|u_x\|+\|u_x\|)^2\cdot\left\|u^{(s)}\right\|^2+6s(\|u\|+\|u_x\|+\|u_x\|)^2\\
&\cdot\left\|u^{(s-1)}\right\|\cdot\left\|u^{(s)}\right\|
\end{aligned}
$$

Then

$$
\begin{aligned}
\frac{A}{2\left\|u^{(s)}\right\|}\leqslant&\, 2\left(4s+1\right)\left(\|\varphi\|+C_1\left(\|\varphi\|\right)\cdot\|\varphi\|_1+C_2\left(\|\varphi\|_1\right)\cdot\|\varphi\|_2\right)^2\left\|u^{(s)}\right\|\\
&+3s\left(\|\varphi\|+C_1\left(\|\varphi\|\right)\cdot\|\varphi\|_1+C_2\left(\|\varphi\|_1\right)\cdot\|\varphi\|_2\right)^2\cdot C_T^{s-1}\left(\|\varphi\|_{s-2}\right)\\
&\quad\cdot\|\varphi\|_{s-1}
\end{aligned}
$$

Secondly, consider

$$
\begin{aligned}
B\leqslant&\, 2\sum_{i=1}^{s-2}\tbinom{s}{i}\left\|u^{(s-i)}\right\|\cdot\left\|u^{(s)}\right\|\cdot\left\|u^{(i)}\right\|_\infty\cdot\|u_x\|_\infty\\
&+2\sum_{j=2}^{s-2}\tbinom{s}{j}\sum_{i=0}^{j}\binom{j}{i}\cdot\left\|u^{(i)}\right\|_\infty\cdot\left\|u^{(j-i)}\right\|_\infty\cdot\left\|u^{(s-j+1)}\right\|\cdot\left\|u^{(s)}\right\|\\
&+2s\sum_{i=1}^{s-2}\tbinom{s-1}{i}\left\|u^{(i)}\right\|\cdot\left\|u^{(s)}\right\|\cdot\|u_{xx}\|_\infty\cdot\left\|u^{(s-i-1)}\right\|_\infty
\end{aligned}
$$

Thus

$$\frac{B}{2\left\|u^{(s)}\right\|} \leqslant \sum_{i=1}^{s-2} \tbinom{s}{i} \left\|u^{(s-i)}\right\| \cdot (\|u_x\| + \|u_{xx}\|) \left(\left\|u^{(i+1)}\right\| + \left\|u^{(i)}\right\|\right)$$
$$+ \sum_{j=2}^{s-2} \binom{s}{j} \sum_{i=0}^{j} \binom{j}{i} \left(\left\|u^{(j-i+1)}\right\| + \left\|u^{(j-i)}\right\|\right) \cdot \left\|u^{(s-j+1)}\right\| \cdot \left(\left\|u^{(i+1)}\right\| + \left\|u^{(i)}\right\|\right)$$
$$+ s \sum_{i=1}^{s-2} \tbinom{s-1}{i} \left(\left\|u^{(i+1)}\right\| + \left\|u^{(i)}\right\|\right) \cdot \|u_{xx}\| \cdot \left(\left\|u^{(s-i-1)}\right\| + \left\|u^{(s-i)}\right\|\right)$$

$$\leqslant \sum_{i=1}^{s-2} \tbinom{s}{i} \left(C_T^{i+1} \left(\|\varphi\|_i\right) \|\varphi\|_{i+1} + C_T^{i} \left(\|\varphi\|_{i-1}\right) \|\varphi\|_i\right)$$
$$\cdot \left(C_1 \left(\|\varphi\|\right) \|\varphi\|_1 + C_2 \left(\|\varphi\|_1\right) \|\varphi\|_2\right) \cdot C_T^{s-i} \left(\|\varphi\|_{s-i-1}\right) \|\varphi\|_{s-i}$$
$$+ \sum_{j=2}^{s-1} \tbinom{s}{j} \sum_{i=0}^{j} \binom{j}{i} \left(C_T^{j-i+1} \left(\|\varphi\|_{j-i}\right) \|\varphi\|_{j-i+1} + C_T^{j-i} \left(\|\varphi\|_{j-i-1}\right) \|\varphi\|_{j-i}\right)$$
$$\cdot \left(C_T^{i+1} \left(\|\varphi\|_i\right) \|\varphi\|_{i+1} + C_T^{i} \left(\|\varphi\|_{i-1}\right) \|\varphi\|_i\right) \cdot C_T^{s-j+1} \left(\|\varphi\|_{s-j}\right) \|\varphi\|_{s-j+1}$$
$$+ 2s \sum_{i=1}^{s-2} \tbinom{s-1}{i} \left(C_T^{i+1} \left(\|\varphi\|_i\right) \|\varphi\|_{i+1} + C_T^{i} \left(\|\varphi\|_{i-1}\right) \|\varphi\|_i\right)$$
$$\cdot C_2 \left(\|\varphi\|_1\right) \|\varphi\|_2 \cdot [C_T^{s-i-1} \left(\|\varphi\|_{s-i-2}\right) \|\varphi\|_{s-i-1} + C_T^{s-i} \left(\|\varphi\|_{s-i-1}\right) \|\varphi\|_{s-i}]$$

Therefore

$$\frac{d}{dt} \left\|u^{(s)}\right\| = \frac{1}{2\left\|u^{(s)}\right\|} \cdot \frac{d}{dt} \left(\left\|u^{(s)}\right\|\right)^2 = \frac{1}{2\left\|u^{(s)}\right\|} \cdot \frac{d}{dt} \int_{\mathrm{R}} \left(u^{(s)}\right)^2 dx = \frac{A+B}{2\left\|u^{(s)}\right\|}$$
$$\leqslant a(\|\varphi\|_{s-1})\|u^{(s)}\| + b(\|\varphi\|_{s-1})\|\varphi\|_s$$

where

$$a\left(r\right) := (4s+1)[r + C_1(r) \cdot r + C_2(r) \cdot r]^2 + 1$$
$$b\left(r\right) := \sum_{i=1}^{s-2} \left(C_T^{i+1}\left(r\right) + C_T^{i}\left(r\right)\right) \left(C_1(r) + C_2(r)\right) \cdot C_T^{s-i}\left(r\right) \cdot r^3$$
$$+ \sum_{i=2}^{s-2} \tbinom{s}{j} \sum_{i=0}^{j} \binom{j}{i} \left(C_T^{j-i+1}\left(r\right) + C_T^{j-i}\left(r\right)\right) \cdot \left(C_T^{i+1}\left(r\right) + C_T^{i}\left(r\right)\right) \cdot C_T^{s-j+1}\left(r\right) \cdot r^3$$
$$+ s \sum_{i=1}^{s-2} \tbinom{s-1}{i} \left(C_T^{i+1}\left(r\right) + C_T^{i}\left(r\right)\right) \cdot C_2\left(r\right) \cdot \left(C_T^{s-i-1}\left(r\right) + C_T^{s-i}\left(r\right)\right) \cdot r^3$$
$$+ 3s[r + C_1(r) \cdot r + C_2(r) \cdot r]^2 \cdot C_T^{(s-1)}(r) \cdot r$$

Applying the Gronwall inequality, we have for any $0 \leqslant t \leqslant T$

$$\left\|u^{(s)}(\cdot,\ t)\right\| \leqslant e^{a\left(\|\varphi\|_{s-1}\right)T}\left\|u^{(s)}(\cdot,\ 0)\right\| + \frac{b\left(\|\varphi\|_{s-1}\right)\|\varphi\|_s}{a\left(\|\varphi\|_{s-1}\right)} e^{a\left(\|\varphi\|_{s-1}\right)T}$$
$$= e^{a\left(\|\varphi\|_{s-1}\right)T}\left\|\varphi^{(s)}\right\| + b\left(\|\varphi\|_{s-1}\right)\|\varphi\|_s e^{a\left(\|\varphi\|_{s-1}\right)T}$$
$$\leqslant e^{a\left(\|\varphi\|_{s-1}\right)T}\left(1 + b\left(\|\varphi\|_{s-1}\right)\right)\|\varphi\|_s \leqslant C_T^s(\|\varphi\|_{s-1})\,\|\varphi\|_s$$

Where $C_T^s(r) := e^{a(r)T}(1 + b(r))$. □

Now we can prove the Proposition 4.1

Proof. (of Proposition 4.1) Let $e_T^s(y) = (1 + C_T^1 + C_T^2 + \cdots + C_T^s)\cdot y$. By Lemma 4.5, the function $d(s,T,r) := d_T^s(r)$, is computable and for any $0 \leqslant t \leqslant T$,

$$\| u(\cdot,t) \|_s = \{\| u \|^2 + \| u^{(1)} \|^2 + \cdots + \| u^{(s)} \|^2\}^{1/2}$$
$$\leq \{\| \varphi \|^2 + \left(C_T^1(\| \varphi \|)\| \varphi \|_1\right)^2 + \cdots + (C_T^s(\| \varphi \|_{s-1})\| \varphi \|_s)^2\}^{1/2}$$
$$\leq [\left(1 + C_T^1(\| \varphi \|_{s-1}) + \cdots + C_T^s(\| \varphi \|_{s-1})\right)^2]^{1/2}\| \varphi \|_s$$
$$\leq d_T^s(\| \varphi \|_s)\| \varphi \|_s$$

This proves Proposition 4.1 □

We will prove the convergence of the iterative sequence $\{v_n^j\}$ about j.

Proposition 4.6 *Let* $v^0 := \bar{S}(0,\varphi)$, *and* $v^{j+1} := \bar{S}(v^j,\varphi)$. *If* $\alpha_T^s T^{1/2}(3+T)^{3/2}\| \varphi \|_s^2 + 8(3+T)\alpha_T^s T^{1/2}\| \varphi \|_s \leqslant 1$, *then we have*

$$\| v^{j+1}(t) - v^j(t) \|_s \leqslant 2^{-j}(3+T)^{1/2}\| \varphi \|_s,$$

where $\alpha_T^s = (e_T^s(\|\varphi\|_s) + 1)\cdot\sqrt{s}\cdot 2^s\cdot(2^s+1)\cdot T^{1/2} + 1$, *for all* $0 \leqslant t \leqslant T$, $\varphi \in H^s(\mathbb{R})$.

Firstly, we will construct the space to obtain the estimate of the nonlinearity of the mKdV equation, and then prove the Proposition 4.6.

Definition 4.7 Let $T > 0$, and continuous functions $u : Y \to H^s(\mathbb{R})$ with $[0\ ,T] \subseteq Y$ define

$$\Lambda_{1,T}^s(u) = \sup_{0\leqslant t\leqslant T}\|u(\cdot\ ,t)\|_s$$
$$\Lambda_{2,T}^s(u) = \left(\sup_{x\in\mathbb{R}}\int_0^T\left|D_x^{s+1}u(x,t)\right|^2dt\right)^{1/2}$$
$$\Lambda_{3,T}^s(u) = \left(\int_{\mathbb{R}}\sup_{0\leqslant t\leqslant T}|u(x,t)|^2dx\right)^{1/2}$$

and

$$\|u\|_{X_T^s} := \Lambda_T^s(u) := \left(\left(\Lambda_{1,T}^s(u)\right)^2 + \left(\Lambda_{2,T}^s(u)\right)^2 + \left(\Lambda_{3,T}^s(u)\right)^2\right)^{1/2}$$

Then $X_T^s = \{u \in C([0,T]; H^s(\mathbb{R}); \Lambda_T^s(u) < \infty)\}$ is a Banach space with the norm $\|u\|_{X_T^s}$.

Lemma 4.8 *If* $T > 0$, $u \in X_T^s(\mathbb{R})$, $\sup\limits_{0 \leqslant t \leqslant T} \| u(x,t) \|_s \leqslant e_T^s(\| \varphi \|_s)$, *then we have*

$$\int_0^T \left\|u^2 u_x\right\|_s dt \leqslant \alpha_T^s T^{1/2} \|u\|_{X_T^s} \|u\|_{X_T^s}$$

where $\alpha_T^s = (e_T^s(\|\varphi\|_s) + 1) \cdot \sqrt{s} \cdot 2^s \cdot (2^s + 1) \cdot T^{1/2} + 1$ *,and* $e_T^s(\| \varphi \|_s)$ *is the same form as Proposition* 4.1*.*

Proof. For $s \geqslant 3$,

$$\begin{aligned} D_x^s\left(u^2 u_x\right) \leqslant & \sum_{i=1}^{s-1} \tbinom{s}{i} \left|u^{(i+1)} u^{(s-i-1)} u_x\right| + \sum_{j=2}^{s-1} \tbinom{s}{j} \sum_{i=0}^{j} \binom{j}{i} \left|u^{(i+1)} u^{(j-i)} u^{(s-j)}\right| \\ & + |u^2 u^{s+1}| + 2(s+1)|u u_x u^{(s)}| \end{aligned}$$

$$\begin{aligned} \left\|D_x^s\left(u^2 u_x\right)\right\| \leqslant & \sum_{i=1}^{s-1} \tbinom{s}{i} \left\|u^{(s-i-1)}\right\|_\infty \|u_x\|_\infty \left\|u^{(i+1)}\right\| \\ & + \sum_{j=2}^{s-1} \tbinom{s}{j} \sum_{i=0}^{j} \binom{j}{i} \left\|u^{(i+1)}\right\| \cdot \left\|u^{(j-i)}\right\|_\infty \left\|u^{(s-j)}\right\|_\infty \\ & + \left\|u^2 u^{s+1}\right\| + 2(s+1)\|u\|_\infty \|u_x\|_\infty \left\|u^{(s)}\right\| \end{aligned}$$

Since $\left\|f^{(k)}\right\| = \left\|F f^{(k)}\right\| = |\xi|^k \|F f\|$, $\quad 1 + |\xi|^2 + \ldots + |\xi|^{2s} \leq s\left(1 + |\xi|^s\right)^2$

$$\begin{aligned} \left\|u^2 u_x\right\|_s & \leqslant \sqrt{s}\left(\left\|u^2 u_x\right\| + \left\|D_x^s\left(u^2 u_x\right)\right\|\right) \\ & \leqslant \sqrt{s}\left(\sum_{i=1}^{s-1} \tbinom{s}{i} + \sum_{j=2}^{s-1} \binom{s}{j} \sum_{i=0}^{j} \binom{j}{i} + 2(s+1) + 1\right) \|u\|_s \cdot \|u\|_s \cdot \|u\|_s \\ & \quad + \sqrt{s}\left\|u^2 u^{s+1}\right\| \end{aligned}$$

Since $\left(\int_0^T f(t)dt\right)^2 \leqslant T \int_0^T \left(f(t)^2\right)dt$, we have

$$\int_0^T \left\|u^2u_x\right\|_s dt$$

$$\leqslant \sqrt{s}\cdot T\cdot\left(\sum_{i=1}^{s-1}\binom{s}{i}+\sum_{j=2}^{s-1}\binom{s}{j}\sum_{i=0}^{j}\binom{j}{i}+2(s+1)+1\right)\cdot\sup_{0\le t\le T}\|u\|_s\cdot\sup_{0\le t\le T}\|u\|_s$$

$$\cdot\sup_{0\le t\le T}\|u\|_s\quad+\sqrt{s}T^{1/2}\left(\sup_{x\in\mathbb{R}}\int_0^T\left|u^{(s+1)}(x,t)\right|^2\right)^{1/2}\left(\sup_{0\leqslant t\leqslant T}\int_{\|\mathbb{R}}\left|u^2(x,t)\right|^2dt\right)^{1/2}$$

$$\leqslant\sqrt{s}\sup_{0\leqslant t\leqslant T}\|u\|_s\,[T\left(\sum_{i=1}^{s-1}\binom{s}{i}+\sum_{j=2}^{s-1}\binom{s}{j}\sum_{i=0}^{j}\binom{j}{i}+2(s+1)+1\right)\cdot\sup_{0\leqslant t\leqslant T}\|u\|_s$$

$$\cdot\sup_{0\leqslant t\leqslant T}\|u\|_s+T^{1/2}\left(\sup_{x\in\mathbb{R}}\int_0^T\left|u^{(s+1)}(x,t)\right|^2\right)^{1/2}\left(\sup_{0\leqslant t\leqslant T}\int_{\mathbb{R}}|u(x,t)|^2dt\right)^{1/2}]$$

The Lemma follows straightforwardly. □

Proof. (of Proposition 4.6) Let $W(t)\varphi=\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{\mathbb{R}}e^{ix\xi}e^{i\xi^3t}\hat{\varphi}(\xi)\mathrm{d}\xi.$

Since $v^0:=\bar{S}(0,\varphi),v^{j+1}:=\bar{S}\left(v^j,\varphi\right)$, by Lemma 4.8 and Lemma 4.8,4.9,4.10 in [8], for $j\geqslant1$,

$$\|\,v^j\,\|_{X_T^s}=\|\,W(t)\,\varphi-\frac{1}{3}\int_0^t W(t-\tau)\left[(v^{j-1})^3\right]_x d\tau\,\|_{X_T^s}$$

$$\leqslant(3+T)^{1/2}\,\|\,\varphi\,\|_s+(3+T)^{1/2}\int_0^T\|\,\left(v^{j-1}\right)^2v_x^{j-1}\,\|_s\,d\tau$$

$$\leqslant(3+T)^{1/2}\,\|\,\varphi\,\|_s+(3+T)^{1/2}\,\alpha_T^sT^{1/2}\,\|\,v^{j-1}\,\|_{X_T^s}^2$$

Let $T>0$, such that

$$24\alpha_T^sT^{1/2}(3+T)^{3/2}\,\|\,\varphi\,\|_s^2+8(3+T)\,\alpha_T^sT^{1/2}\,\|\,\varphi\,\|_s\leqslant1$$

From $\|\,v^0\,\|_{X_T^s}=\|\,W(t)\,\varphi\,\|_{X_T^s}\leq(3+T)^{1/2}\,\|\,\varphi\,\|_s$ we obtain by induction

$$\|\,v^j\,\|_{X_T^s}\leqslant2(3+T)^{1/2}\,\|\,\varphi\,\|_s\;(for\ all\ j\in\mathbb{N})$$

For $j\geqslant2$

$$\|\,v^j-v^{j-1}\,\|_{X_T^s}=\|\,\frac{1}{3}\int_0^t W(t-\tau)\left[\left(v^{j-1}\right)^3\right]_x-\left[\left(v^{j-2}\right)^3\right]_x d\tau\,\|_{X_T^s}$$

$$\leqslant(3+T)^{1/2}\,\alpha_T^sT^{1/2}\,\|\,\left[\left(v^{j-1}\right)^2+v^{j-1}v^{j-2}+\left(v^{j-2}\right)^2\right]\cdot\left[v^{j-1}-v^{j-2}\right]\,\|_{X_T^s}$$

$$\leqslant(3+T)^{1/2}\,12\alpha_T^sT^{1/2}\,\|\varphi\|_s^2\,\|\,v^{j-1}-v^{j-2}\,\|_{X_T^s}$$

$$\leqslant\frac{1}{2}\,\|\,\left[v^{j-1}-v^{j-2}\right]\,\|_{X_T^s}$$

Therefore, for $0\leqslant t\leqslant T$,

$$\| v^{j+1}(t) - v^j(t) \|_s \leqslant \| v^{j+1}(t) - v^j(t) \|_{X_T^s} \leq 2^{-j}(3+T)^{1/2} \| \varphi \|_s$$

if $24\alpha_T^s T^{1/2}(3+T)^{3/2} \| \varphi \|_s^2 + 8(3+T)\alpha_T^s T^{1/2} \| \varphi \|_s \leqslant 1$. This proves Proposition 4.6. □

From the above proof, if we use v_n^j instead of v_n, when $j \to \infty$, we can obtain the result $v_n^j \to v_n$. Then we will prove the uniform convergence of the sequence $\{v_n\}$ or $\{v_n^j\}$.

Proposition 4.9 $v(t) = \bar{S}(v, \psi)(t)$, $v_n(t) = \bar{S}(v_n, \psi_n)(t)$,

$$\|v(t) - v_n(t)\|_s \leqslant 2(3+T)^{1/2} \|\psi - \psi_n\|_s$$

for all $0 \leqslant t \leqslant T$, *if* $24\alpha_T^s T^{1/2}(3+T)^{3/2}(\| \psi \|_s + 1)^2 + 8\alpha_T^s T^{1/2}(3+T)(\| \psi \|_s + 1) \leqslant 1$, *where* $\alpha_T^s = (e_T^s \|\varphi\|_s + 1) \cdot \sqrt{s} \cdot 2^s \cdot (2^s + 1) \cdot T^{1/2} + 1$.

Proof. Since $v(t) = \bar{S}(v, \psi)(t)$, $v_n(t) = \bar{S}(v_n, \psi_n)(t)$, by Lemma 4.8 in [8],we obtain the result as following:

$$\| v - v_n \|_{X_T^s} = \| W(t)(\psi - \psi_n) - \frac{1}{3}\int_0^t W(t-\tau)\left[v^3 - v_n^3\right]_x d\tau \|_{X_T^s}$$

$$\leqslant (3+T)^{1/2} \| \psi - \psi_n \|_s + (3+T)^{1/2} a_T^s T^{1/2} \| v^2 + vv_n + v_n^2 \|_{X_T^s} \| v - v_n \|_{X_T^s}$$

where $a_T^s = \sqrt{s} \cdot 2^s \cdot T^{1/2} + 1(see[8])$. By Proposition 4.6 (notice that $\| \psi_n \|_s \leqslant \| \psi \|_s + 1,$) if $24\alpha_T^s T^{1/2}(3+T)^{3/2}(\| \psi \|_s + 1)^2 + 8\alpha_T^s T^{1/2}(3+T)(\| \psi \|_s + 1) \leqslant 1$, then

$$\begin{aligned}
&\| v - v_n \|_{X_T^s} \\
&\leqslant (3+T)^{1/2} \| \psi - \psi_n \|_s + (3+T)^{1/2} \alpha_T^s T^{1/2} \| v^2 + vv_n + v_n^2 \|_{X_T^s} \| v - v_n \|_{X_T^s} \\
&\leqslant (3+T)^{1/2} \| \psi - \psi_n \|_s + 12(3+T)^{3/2} \alpha_T^s T^{1/2} (\| \psi \|_s + 1)^2 \| v - v_n \|_{X_T^s} \\
&\leqslant (3+T)^{1/2} \| \psi - \psi_n \|_s + \frac{1}{2} \| v - v_n \|_{X_T^s}
\end{aligned}$$

Therefore $\| v - v_n \|_{X_T^s} \leqslant 2(3+T)^{1/2} \| \psi - \psi_n \|_s$, the sequence $\{v_n\}$ is uniform convergence. □

Thus, by Proposition 4.6 and 4.9, the sequence $\{v_n^j\}$ converges fast to v uniformly in $t \in [0 \ , \ T]$. We can see that the machine searches fast approximations to $u(x,t)$, and computes the solutions of mKdV equation with arbitrary precision. This approach can be extended to other nonlinear equations.

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