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A Hofmann-Mislove theorem for Bitopological Spaces

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Abstract

We present a Stone duality for bitopological spaces in analogy to the duality between topological spaces and frames, and discuss the resulting notions of sobriety and spatiality. Under the additional assumption of regularity, we prove a characterisation theorem for subsets of a bisober space that are compact in one and closed in the other topology. This is in analogy to the celebrated Hofmann-Mislove theorem for sober spaces. We link the characterisation to Taylor's and Escardó's reading of the Hofmann-Mislove theorem as continuous quantification over a subspace. As an application, we define locally compact d-frames and show that these are always spatial.

Keywords: Bitopological spaces, d-frames, Stone duality, sober spaces, Hofmann-Mislove theorem

1 Introduction

The Hofmann-Mislove theorem, first published as [10], states that in a sober space the open neighbourhood filters of compact saturated sets are precisely the Scott-open filters in the corresponding frame of opens. Mathematically, it has some remarkable consequences, such as the fact that the set of compact saturated subsets of a sober space form a dcpo when ordered by reverse inclusion, and it links Lawson duality (applied to the frame of opens) to the idea of the co-compact topology on the space, [16]. Its significance in Computer Science took some time to emerge, and credit in this respect is due to Plotkin, [17,18], Smyth, [19], and Vickers, [21],

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who pointed out that it is at the core of the proof that the upper powerdomain (defined as a free algebraic theory) has a concrete representation as a set of subsets of the given domain. Quite unexpectedly, it was also required in the classification of cartesian closed categories of domains, [12].

More recently, Taylor, [20], and Escardó, [6], have interpreted the theorem as expressing the idea that the compact saturated sets are precisely those for which there is a continuous universal quantifier. To this end, they read "open set" as "predicate" and "Scott-open filter of opens" as a map from predicates to Sierpiński space that is Scott-continuous and finite meet preserving, that is, as a "quantifier" which tells us whether a predicate is true for all elements of the corresponding compact set.

Below we present a Stone duality for bitopological spaces motivated by the idea that a predicate may not only be true for some states, but in general will be false for others, and that the mechanisms for establishing falsehood will in general be different from those that establish truth. As Smyth has stressed, the positive extents of *observable* predicates form a topology, and so all we do is to add a second topology for the negative extents. However, in semantics we are already quite familiar with dealing with two topologies: Early on in the study of continuous lattices it was discovered by Lawson that the "weak lower topology" is a natural partner for the Scott-topology, their join being the (compact Hausdorff) Lawson topology. On hyperspaces $Y \subseteq \mathcal{P}X$ one naturally has the upper topology generated by sets of the form $\Box O := \{A \in Y \mid A \subseteq O\}$ (O an open in the original space), and the lower topology generated by sets of the form $\Diamond O := \{A \in Y \mid A \cap O \neq \emptyset\}$. Abramsky, [1], showed that the three powerdomains can be obtained systematically from this (bi-)topological point of view.

Our interest in bitopological spaces was driven by these examples and also by a desire to analyse various Stone dualities, but there is no room here to expand on this latter aspect; instead we refer the reader to the report [13].

2 Stone duality and the Hofmann-Mislove theorem

We briefly review the duality between topological spaces and frames. For more details see [2, Chapter 7], and [11,8].

Definition 2.1 A *frame* is a complete lattice in which finite meets distribute over arbitrary joins. We denote with $\sqsubseteq$, $\sqcap$, $\bigsqcup$, 0, and 1 the order, finite meets, arbitrary joins, least and largest element, respectively.

A *frame homomorphism* preserves finite meets and arbitrary joins; thus we have the category **Frm**.

For $(X;\tau)$ a topological space, $(\tau;\subseteq)$ is a frame; for $f\colon (X;\tau) \to (X';\tau')$ a continuous function, $f^{-1}\colon \tau' \to \tau$ is a frame homomorphism. These are the constituents of the contravariant functor $\Omega\colon \mathbf{Top} \to \mathbf{Frm}$. It is represented by $\mathbf{Top}(-,\mathbb{S})$ where $\mathbb{S}$ is *Sierpiński space*.

The collection $\mathcal{N}(a)$ of open neighbourhoods of a point a in a topological space $(X;\tau)$ forms a *completely prime filter* in the frame $\Omega\, X$, that is, it is an upper set, closed under finite intersections, and whenever $\bigcup \mathcal{O} \in \mathcal{N}(a)$ then $\mathcal{O} \cap \mathcal{N}(a) \neq \emptyset$. This leads one to consider the set of *points* (sometimes called "abstract points" for emphasis) of a frame L to be the collection $\mathsf{spec}\, L$ of completely prime filters. Abstract points are exactly the pre-images of $\{1\}$ under homomorphisms from L to $2 = \{0 < 1\}$.

A frame L induces a topology on $\mathsf{spec}\, L$ whose opens are of the form $\Phi(x) = \{F \in \mathsf{spec}\, L \mid x \in F\}$ with $x \in L$. A frame homomorphism $h\colon L \to L'$ induces a continuous function $\mathsf{spec}\, h\colon \mathsf{spec}\, L' \to \mathsf{spec}\, L$ by letting $\mathsf{spec}\, h(F) := h^{-1}(F)$ for $F \in \mathsf{spec}\, L'$. These are the components of the contravariant functor spec from **Frm** to **Top**, represented by $\mathbf{Frm}(-, 2)$.

Theorem 2.2 *The functors Ω and* spec *constitute a dual adjunction between* **Top** *and* **Frm**.

The unit and co-unit of this adjunction are simply $\mathcal{N}$ and Φ. That is, for any space $(X;\tau)$ the map $\eta_X\colon X \to \mathsf{spec}\,\Omega\, X$, given by $a \mapsto \mathcal{N}(a)$, is continuous; it is also open onto its image. Likewise, for any frame L the map $\epsilon_L\colon L \to \Omega\,\mathsf{spec}\, L$, given by $x \mapsto \Phi(x)$ is a frame homomorphism; it is also surjective.

We can ask when a frame L is *spatial* in the sense that it is isomorphic to $\Omega\, X$ for some space X. The adjunction transfers isomorphisms: $L \cong \Omega\, X$ if and only if $X \cong \mathsf{spec}\, L$. So L is spatial if and only if $L \cong \Omega\,\mathsf{spec}\, L$, that is, ϵ_L is a frame isomorphism. Because ϵ_L is already a surjective frame homomorphism, this holds if and only if ϵ_L is injective.

Similarly, we can ask when a space X is *sober* in the sense that it is homeomorphic to $\mathsf{spec}\, L$ for some frame L. By the same reasoning as in frames, this holds if and only if η_X is a homeomorphism. Because η_X is already continuous and open onto its image, it suffices for η_X to be a bijection. Injectivity is precisely the T_0 axiom and surjectivity says that every completely prime filter of opens is the neighbourhood filter of a point.

Theorem 2.3 *The functors Ω and* spec *restrict to a dual equivalence between sober spaces and spatial frames.*

This is the setting for the *Hofmann-Mislove theorem*, [10], which we are now ready to state.

Theorem 2.4 *In a sober space (X,τ), there is a bijection between the set of compact saturated subsets of X and the set of Scott-open filters in τ.*

Although a direct proof is possible, [15], it more useful for us to refer to Stone duality, as in the original paper [10]:

Lemma 2.5 *A Scott-open filter in a frame L is equal to the intersection of the collection of completely prime filters containing it.*

Proof. (Sketch) Let $\mathcal{S}$ be the Scott-open filter and a an element not in $\mathcal{S}$. Extend a to a maximal chain outside $\mathcal{S}$ and take its supremum v, which by Scott openness is a maximal element of $L \setminus \mathcal{S}$. Because $\mathcal{S}$ is a filter, v is irreducible, and because L is distributive, it is furthermore prime. The set $L \setminus {\downarrow}v$ is completely prime and separates a from $\mathcal{S}$. $\Box$

Proof. (of 2.4) Clearly, the neighbourhoods of a compact subset form a Scott-open filter. For the converse, let A be the intersection of a Scott-open filter $\mathcal{S}$ of opens. By the lemma, every open neighbourhood of A belongs to $\mathcal{S}$. Because $\mathcal{S}$ is assumed to be Scott-open, A is compact (and obviously saturated).

A saturated set is the intersection of its open neighbourhoods by definition, and a Scott-open filter is the intersection of the completely prime filters containing it by the lemma, so the two translations are inverses of each other. $\Box$

3 Stone duality for bitopological spaces

Without spending too much time on motivation, we now sketch a Stone duality for bitopological spaces; for the full picture we refer to [13].

A *bitopological space* is a set X together with two topologies τ_+ and τ_-. No connection between the two topologies is assumed. Morphisms between bitopological spaces are required to be continuous with respect to each of the two topologies; this gives rise to the category **biTop**.

For a Stone dual it is natural to consider pairs (L_+, L_-) of frames (and pairs of frame homomorphisms) but for some purposes it is more convenient to axiomatise the product $\tau_+ \times \tau_-$, that is, to have a single-sorted algebraic structure. In fact, the two views are completely equivalent:

Proposition 3.1 *The category* **Frm**$\times$**Frm** *is equivalent to the category whose objects are frames which contain a pair of complemented elements* tt *and* ff*, and whose morphisms are frame homomorphisms that preserve* tt *and* ff*.*

Proof. In one direction, one assigns to a pair (L_+, L_-) the product $L_+ \times L_-$ and the constants $tt := (1, 0)$ and $ff := (0, 1)$. In the other direction, one assigns to $(L; tt, ff)$ the two frames $L_+ := [0, tt]$ and $L_- := [0, ff]$. The isomorphism from L to $[0, tt] \times [0, ff]$ is given by $\alpha \mapsto \langle \alpha_+, \alpha_- \rangle := \langle \alpha \sqcap tt, \alpha \sqcap ff \rangle$. The isomorphism from $L_+ \times L_-$ to L is given by $\langle x, y \rangle \mapsto x \sqcup y$. $\Box$

In addition to the notation $\langle \alpha_+, \alpha_- \rangle$ introduced in the proof above we will also use $\alpha \sqsubseteq_+ \beta$ in case $\alpha_+ \sqsubseteq \beta_+$, and similarly $\sqsubseteq_-$. One has $\alpha \sqsubseteq \beta$ if and only if $\alpha \sqsubseteq_+ \beta$ and $\alpha \sqsubseteq_- \beta$.

Having two frames is not enough, however, as we also need to express the fact that they represent topologies *on the same set*. One approach for achieving this was introduced by Banaschewski, Brümmer, and Hardie in [3]; their *biframes* axiomatise the two topologies and the joint refinement $\tau_+ \vee \tau_-$. Our proposal is different; we only record when two open sets $O_+ \in \tau_+$ and $O_- \in \tau_-$ are disjoint from each other, and when they cover the whole space X. In the first case we say that they are *consistent*, in the second that they are *total*.

Definition 3.2 A *d-frame* consists of a frame L, a pair of complemented elements tt and ff, and two unary predicates con and tot. Morphisms between d-frames are required to preserve all of this structure. The resulting category is denoted by **dFrm**.

As we have already explained informally, the contravariant functor Ω from bitopological spaces to d-frames assigns to a space $(X;\tau_+,\tau_-)$ the d-frame $(\tau_+\times\tau_-;(X,\emptyset),(\emptyset,X),\mathsf{con},\mathsf{tot})$ where $(U,V)\in\mathsf{con}$ if and only if $U\cap V=\emptyset$ and $(U,V)\in\mathsf{tot}$ if and only if $U\cup V=X$. The functor associates with a bicontinuous function f the map $(U,V)\mapsto(f^{-1}(U),f^{-1}(V))$. A trivial bit of set theory will convince the reader that the consistency and totality predicates are preserved. Figure 1 shows some small examples. The bitopological space $\mathbb{S}.\mathbb{S}$, which looks like a product of two copies of Sierpinski space, allows us to represent the functor Ω as $\mathbf{biTop}(-,\mathbb{S}.\mathbb{S})$. Note how the four elements of $\mathbb{S}.\mathbb{S}$ correspond to the four ways in which an element of the space can be related to an open from τ_+ and an open from τ_-: it can be in one of the two but not the other, it can be in both, or it can be in neither.

For a functor in the reverse direction, we continue to follow the theory of frames by considering d-frame morphisms from $\mathcal{L}=(L;\mathit{tt},\mathit{ff};\mathsf{con},\mathsf{tot})$ to 2.2, depicted in the upper right corner of Figure 1. Such morphisms are determined by pairs of frame homomorphisms $p_+\colon L_+\to 2$ and $p_-\colon L_-\to 2$ that together preserve con and tot. So they correspond to pairs of completely prime filters $F_+\subset L_+$, $F_-\subset L_-$ such that

$$(\mathrm{dp}_{\mathsf{con}})\qquad \alpha\in\mathsf{con}\implies\alpha_+\notin F_+\text{ or }\alpha_-\notin F_-;$$

$$(\mathrm{dp}_{\mathsf{tot}})\qquad \alpha\in\mathsf{tot}\implies\alpha_+\in F_+\text{ or }\alpha_-\in F_-.$$

The reader should pause at this point to assure himself that the pair of neighbourhood filters $(\mathcal{N}_+(x),\mathcal{N}_-(x))$ of a point x in a bitopological space satisfies these two axioms.

On $\mathcal{L}$ itself, a point manifests itself as a pair (F_+^*,F_-^*) of completely prime filters that satisfy the analogue of $(\mathrm{dp}_{\mathsf{con}})$ and $(\mathrm{dp}_{\mathsf{tot}})$, plus

$$(\mathrm{dp}_+)\qquad \mathit{tt}\in F_+^*;$$

$$(\mathrm{dp}_-)\qquad \mathit{ff}\in F_-^*;$$

Figure 2 illustrates the idea that (F_+^*,F_-^*) determines four "quadrants" so that con does not intersect with the "upper quadrant" and tot does not intersect with the 'lower."

The set of d-points becomes a bitopological space by considering the collection of $\Phi_+(x):=\{(F_+,F_-)\mid x\in F_+\}$, $x\in L$, as the first topology $\mathcal{T}_+$, and the collection of $\Phi_-(x):=\{(F_+,F_-)\mid x\in F_-\}$, $x\in L$, as the second topology $\mathcal{T}_-$. Together, this is the *spectrum* of the d-frame $\mathcal{L}$, which we denote as $\mathsf{spec}\,\mathcal{L}$, following the notation for frames. The construction for objects is extended to a (contravariant) functor $\mathsf{spec}\colon\mathbf{dFrm}\to\mathbf{biTop}$ in the usual way, that is, by noting that the inverse image of a point under a d-frame morphism is again a point.

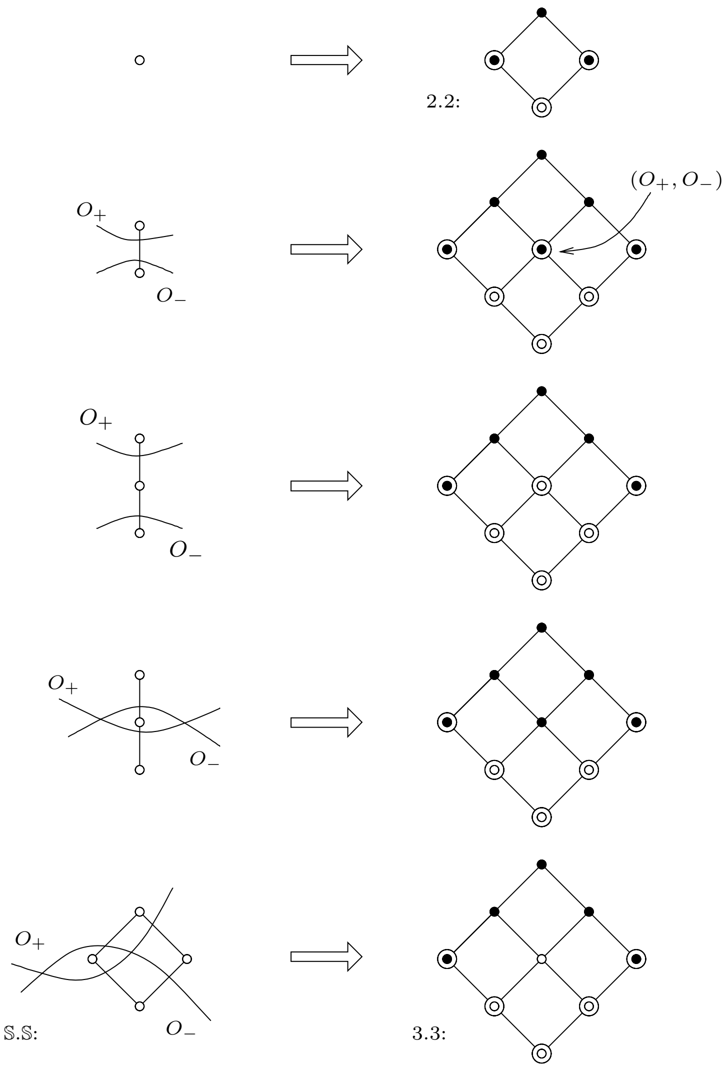

Fig. 1. Some bitopological spaces and their concrete d-frames. (D-frame elements in the con-predicate are indicated by an additional circle, those in the tot-predicate are filled in.)

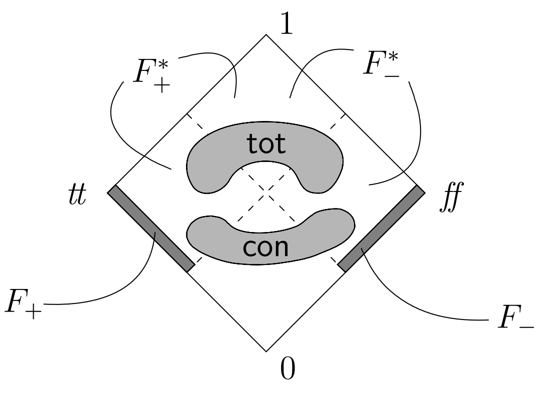

Fig. 2. An abstract point in a d-frame.

Theorem 3.3 *The functors Ω and spec establish a dual adjunction between* **biTop** *and* **dFrm**.

We say that a bitopological space X is *(d-) sober* if it is bihomeomorphic to $\mathsf{spec}\,\mathcal{L}$ for some d-frame $\mathcal{L}$. As with frames and topological spaces, d-sobriety is equivalent to the unit $x \mapsto (\mathcal{N}_+(x), \mathcal{N}_-(x))$ being a bijection.

Example 3.4 All the bitopological spaces in Figure 1 are d-sober. For the one-point space this is clear, as the associated d-frame admits only one point. For the other four spaces one argues as follows: The underlying frame is the same in each case and it admits four completely prime filters:

$$\begin{array}{ll} F_+^1 := \uparrow tt & F_-^1 := \uparrow ff \\ F_+^2 := \uparrow(O_+, \emptyset) & F_-^2 := \uparrow(\emptyset, O_-) \end{array}$$

The notation already indicates which of these can be used as the first, respectively second, component of a point. From this we get four possible combinations, and these are indeed all available in the last example. In the other three examples, the $\mathsf{con}/\mathsf{tot}$ labelling of the element (O_+, O_-) in the centre of the d-frame excludes certain combinations: if it belongs to con, then F_+^2 cannot be paired with F_-^2, and if it belongs to tot then F_+^1 cannot be paired with F_-^1.

For an exploration into the concept of d-sobriety we refer to [13]; here we confine ourselves to one particular class of examples.

Definition 3.5 A bitopological space $(X; \tau_+, \tau_-)$ is called *order-separated* if $\leq\, = \,\leq_+ \cap \geq_-$ is a partial order and $x \not\leq y$ implies that there are disjoint open sets $O_+ \in \tau_+$ and $O_- \in \tau_-$ such that $x \in O_+$ and $y \in O_-$. (The relations $\leq_+$ and $\leq_-$ refer to the specialisation orders on X with respect to τ_+ and τ_-, respectively.)

Lemma 3.6 *In an order-separated bitopological space the following are true:*

(1) $\leq_+ \,=\, \geq_-$;

(2) $\leq_+ \cap \leq_- \,=\,$ '$=$'.

Proof. For the first claim assume $x \not\leq_+ y$. This implies $x \not\leq y$ and we get a separating consistent pair (O_+, O_-). Since $y \in O_-$ but $x \notin O_-$ we conclude $x \not\geq_- y$. So $\not\leq_+ \,=\, \not\geq_-$ and this is equivalent to the first claim.

The second claim follows immediately from (1) and anti-symmetry of $\leq$. $\square$

Theorem 3.7 *Order-separated bitopological spaces are sober.*

Proof. Order separation clearly implies that the canonical map $\eta\colon X \to \mathsf{spec}\,\Omega\, X$ is injective; the real issue is surjectivity. So assume that (F_+, F_-) is a point of $\Omega\, X$. Consider the two sets

$$V_+ := \bigcup \{O_+ \in \tau_+ \mid O_+ \notin F_+\} \qquad V_- := \bigcup \{O_- \in \tau_- \mid O_- \notin F_-\}$$

and their complements V_+^c, V_-^c. Because of condition ($\mathrm{dp}_{\mathsf{tot}}$), $V_+ \cup V_-$ cannot be the whole space, in other words, the intersection $V_+^c \cap V_-^c$ is non-empty.

Next we show that every element of V_+^c is below every element of V_-^c in the specialisation order $\leq = \leq_+ \cap \geq_-$. Indeed, if $x \in V_+^c$, $y \in V_-^c$, and $x \not\leq y$, then by order separation there is a partial predicate (O_+, O_-) with $x \in O_+$ and $y \in O_-$. By definition of V_+, V_- we have $O_+ \in F_+$ and $O_- \in F_-$, contradicting condition ($\mathrm{dp}_{\mathsf{con}}$) of d-points.

Finally, let a be an element in the intersection $V_+^c \cap V_-^c$. We show that F_+ is the neighbourhood filter of a in τ_+. Assume $a \in O_+$; this implies $O_+ \not\subseteq V_+$ and the latter is equivalent to $O_+ \in F_+$. For the converse we start at $O_+ \not\subseteq V_+$, which gives us an element $b \in V_+^c \cap O_+$ about which we already know that $b \leq a$. It follows that $b \leq_+ a$ and hence $a \in O_+$. $\square$

From this result it follows immediately that the real line together with the usual upper and lower topology is d-sober. Likewise, one sees that the *punctured unit interval* $[0,1] \setminus \{\frac{1}{2}\}$ is d-sober with respect to the same two topologies. Note that neither is sober in the traditional sense when equipped with only one of the topologies.

4 The logical structure of d-frames

Before we consider spatiality for d-frames let us have a look at the duality from the point of view of logic. For this we interpret the elements of a d-frame $\mathcal{L}$ as *logical propositions*. An abstract point (F_+, F_-) is then a *model*, and F_+ consists of those propositions which are true in the model, F_- of those that are false. If a proposition belongs to con then for no model is it both true and false (and may be neither); if it belongs to tot then in every model it is either true or false (or indeed both). The set of all models (i.e., $\mathsf{spec}\,\mathcal{L}$) becomes a bitopological space by collecting into one topology all sets of models in which some proposition is true (the "positive extents") and in the other the sets of models where some proposition is false (the "negative extents").

From this perspective it is natural to consider an order between propositions which increases the positive extent and shrinks the negative one. As it turns out, this additional relation is always present in a d-frame, and in fact it follows from the distributive lattice structure and the two complemented elements alone. The earliest reference to this appears to be [5], but the proof is entirely straightforward and can be left as an exercise.

Proposition 4.1 *Let $(L; \sqcap, \sqcup, 1, 0)$ be a bounded distributive lattice, and (t, f) a complemented pair in L, that is, $t \sqcap f = 0$ and $t \sqcup f = 1$. Then by defining*

$$x \wedge y := (x \sqcap f) \sqcup (y \sqcap f) \sqcup (x \sqcap y) = (x \sqcup f) \sqcap (y \sqcup f) \sqcap (x \sqcup y)$$

$$x \vee y := (x \sqcup t) \sqcap (y \sqcup t) \sqcap (x \sqcup y) = (x \sqcap t) \sqcup (y \sqcap t) \sqcup (x \sqcap y)$$

one obtains another bounded distributive lattice $(L; \wedge, \vee, t, f)$, in which $(1, 0)$ is a *complemented pair. The original operations are recovered from it as*

$$x \sqcap y = (x \wedge 0) \vee (y \wedge 0) \vee (x \wedge y) = (x \vee 0) \wedge (y \vee 0) \wedge (x \vee y)$$
$$x \sqcup y = (x \vee 1) \wedge (y \vee 1) \wedge (x \vee y) = (x \wedge 1) \vee (y \wedge 1) \vee (x \wedge y)$$

Furthermore, any two of the operations $\sqcap$, $\sqcup$, $\wedge$, and $\vee$ distribute over each other. If L is a frame, then $\wedge$ and $\vee$ are also Scott continuous.

This justifies our choice of symbols tt and ff in a d-frame, and suggests that we regard $(L; \wedge, \vee, tt, ff)$ as the *logical structure* of a d-frame. Altogether, then, we see that d-frames are special "bilattices," which were introduced by Ginsberg, [9], as a generalisation of Belnap's four-valued logic [4].

Exploiting Proposition 3.1 we can easily compute conjunction and disjunction in terms of the representation of a d-frame as $L_+ \times L_-$:

$$\langle x, y\rangle \wedge \langle x', y'\rangle := \langle x \sqcap x', y \sqcup y'\rangle$$
$$\langle x, y\rangle \vee \langle x', y'\rangle := \langle x \sqcup x', y \sqcap y'\rangle$$

Note the reversal of order in the second component. This makes sense, as we think of the second frame as providing negative answers.

5 Reasonable d-frames and spatiality

We say that a d-frame $\mathcal{L}$ is *spatial* if it is isomorphic to $\Omega\, X$ for some bitopological space X. As with d-sobriety, this is equivalent to the co-unit $\epsilon\colon \alpha \mapsto (\Phi_+(\alpha), \Phi_-(\alpha))$ being an isomorphism of d-frames. As it is always surjective by definition, the condition boils down to ϵ being injective and reflecting con and tot. If this is spelt out concretely, one arrives at the following:

Proposition 5.1 *A d-frame $\mathcal{L}$ is spatial if and only if the following four conditions are satisfied:*

$$\begin{array}{rl}
(\mathrm{s}_+) & \forall \alpha \not\sqsubseteq_+ \beta\ \ \exists (F_+, F_-) \in \mathsf{spec}\,\mathcal{L}.\ \ \alpha \in F_+, \beta \notin F_+; \\
(\mathrm{s}_-) & \forall \alpha \not\sqsubseteq_- \beta\ \ \exists (F_+, F_-) \in \mathsf{spec}\,\mathcal{L}.\ \ \alpha \in F_-, \beta \notin F_-; \\
(\mathrm{s}_{\mathsf{con}}) & \forall \alpha \notin \mathsf{con}\ \ \exists (F_+, F_-) \in \mathsf{spec}\,\mathcal{L}.\ \ \alpha_+ \in F_+, \alpha_- \in F_-; \\
(\mathrm{s}_{\mathsf{tot}}) & \forall \alpha \notin \mathsf{tot}\ \ \exists (F_+, F_-) \in \mathsf{spec}\,\mathcal{L}.\ \ \alpha_+ \notin F_+, \alpha_- \notin F_-;
\end{array}$$

The following lemma is very easy to prove for concrete d-frames that arise from a bitopological space, and it confirms the intuition of con as the set of pairs of open sets that do not intersect, and tot as those pairs that cover the whole space.

Lemma 5.2 *Let $(L; tt, ff; \mathsf{con}, \mathsf{tot})$ be a spatial d-frame. The following properties hold:*

$$
\begin{array}{rl}
(\mathsf{con}-\!\downarrow) & \alpha \sqsubseteq \beta \;\&\; \beta \in \mathsf{con} \quad\Longrightarrow\quad \alpha \in \mathsf{con} \\
(\mathsf{tot}-\!\uparrow) & \alpha \sqsubseteq \beta \;\&\; \alpha \in \mathsf{tot} \quad\Longrightarrow\quad \beta \in \mathsf{tot} \\
(\mathsf{con}-\mathit{tt}) & \mathit{tt} \in \mathsf{con} \\
(\mathsf{con}-\mathit{ff}) & \mathit{ff} \in \mathsf{con} \\
(\mathsf{con}-\wedge) & \alpha \in \mathsf{con} \;\&\; \beta \in \mathsf{con} \quad\Longrightarrow\quad (\alpha \wedge \beta) \in \mathsf{con} \\
(\mathsf{con}-\vee) & \alpha \in \mathsf{con} \;\&\; \beta \in \mathsf{con} \quad\Longrightarrow\quad (\alpha \vee \beta) \in \mathsf{con} \\
(\mathsf{tot}-\mathit{tt}) & \mathit{tt} \in \mathsf{tot} \\
(\mathsf{tot}-\mathit{ff}) & \mathit{ff} \in \mathsf{tot} \\
(\mathsf{tot}-\wedge) & \alpha \in \mathsf{tot} \;\&\; \beta \in \mathsf{tot} \quad\Longrightarrow\quad (\alpha \wedge \beta) \in \mathsf{tot} \\
(\mathsf{tot}-\vee) & \alpha \in \mathsf{tot} \;\&\; \beta \in \mathsf{tot} \quad\Longrightarrow\quad (\alpha \vee \beta) \in \mathsf{tot} \\
(\mathsf{con}-\bigsqcup^{\uparrow}) & A \subseteq \mathsf{con} \textit{ directed w.r.t. } \sqsubseteq \quad\Longrightarrow\quad \bigsqcup^{\uparrow} A \in \mathsf{con} \\
(\mathsf{con}-\mathsf{tot}) & \alpha \in \mathsf{con}, \beta \in \mathsf{tot}, \quad (\alpha =_{+} \beta \quad \textit{or} \quad \alpha =_{-} \beta) \quad\Longrightarrow\quad \alpha \sqsubseteq \beta
\end{array}
$$

Definition 5.3 A d-frame which satisfies the properties stated in Lemma 5.2 is called *reasonable*. The category of reasonable d-frames is denoted by **rdFrm**.

Note that the converse of Lemma 5.2 does not hold, i.e., a reasonable d-frame need not be spatial: take a frame L without any points and consider $(L\times L; (1,0), (0,1), \mathsf{con}, \mathsf{tot})$ where $\langle x, y\rangle \in \mathsf{con}$ if $x \sqcap y = 0$, and $\langle x, y\rangle \in \mathsf{tot}$ if $x \sqcup y = 1$. It is a trivial exercise to prove that the resulting d-frame is reasonable, but it obviously can't have any points.

Proposition 5.4 *The forgetful functor from* **rdFrm** *to* **Set** *has a left adjoint.*

Proof. The free reasonable d-frame over a set A is $(FA\times FA; (1,0), (0,1), \mathsf{con}, \mathsf{tot})$ where FA is the free frame over A. Generators are the pairs (a, a), $a \in A$. The two relations are chosen minimally: $\langle x, y\rangle \in \mathsf{con}$ if and only if $x = 0$ or $y = 0$; $\langle x, y\rangle \in \mathsf{tot}$ if and only if $x = 1$ or $y = 1$. The conditions for a reasonable d-frame are proved by case analysis. □

As an example, the structure labelled 3.3 in Figure 1 is the free reasonable d-frame generated by a one-element set.

The following additional property of spatial d-frames will also play a part in our presentation of a Hofmann-Mislove theorem for sober bitopological spaces, but we do not consider it elementary enough to be included in the definition of "reasonable." The proof-theoretic terminology used in its label refers to a presentation of d-frames that places more emphasis on the logical structure, see [13, Section 7].

Proposition 5.5 *Every spatial d-frame satisfies the following property:*

$$(\mathrm{CUT}_r) \quad \langle x, y \sqcup \bigsqcup_{i\in I} b_i\rangle \in \mathsf{tot} \;\&\; \forall i \in I.\ \langle x \sqcup a_i, y\rangle \in \mathsf{tot} \;\&\; \langle a_i, b_i\rangle \in \mathsf{con} \;\Rightarrow\; \langle x, y\rangle \in \mathsf{tot}$$

6 Regularity and the Hofmann-Mislove theorem

A major practical problem with d-frames is that it is very difficult to construct abstract points for them. For example, consider the proof of the Hofmann-Mislove lemma 2.5, where we exploited the fact that in a frame there is a one-to-one correspondence between completely prime filters F and $\sqcap$-prime elements v (given by the translations $v \mapsto L \setminus \downarrow v$ and $F \mapsto \bigsqcup L \setminus F$). The analogue for d-frames is not very helpful. The situation improves if we also require regularity.

Definition 6.1 Let $(L; tt, ff; \mathsf{con}, \mathsf{tot})$ be a reasonable d-frame. For two elements $x, x' \in L_+$ we say that x' is *well-inside* x (and write $x' \lhd x$) if there is $y \in L_-$ such that $\langle x', y\rangle \in \mathsf{con}$ and $\langle x, y\rangle \in \mathsf{tot}$. To avoid lengthy verbiage, we will usually write $r_{x' \lhd x}$ for the "witnessing" element y (although it is not uniquely determined). On L_- the well-inside relation is defined analogously.

A d-frame is called *regular* if every $x \in L_+$ is the supremum of elements well-inside it, and similarly for elements of L_-.

For a bitopological space to be regular we require that at least one of the two topologies is T_0 and that the corresponding d-frame is regular.

We note that the elements well-inside a fixed element x of a reasonable d-frame form a directed set; this follows from $(\mathsf{con}{-}\vee)$ and $(\mathsf{tot}{-}\vee)$. That they are all below x is a consequence of $(\mathsf{con}{-}\mathsf{tot})$. $1 \lhd 1$ is always true as 0 can be chosen as the witness in the other frame. It is an easy exercise to show that a regular bitopological space is order-separated (and hence d-sober), but a regular d-frame need not be spatial.

Lemma 6.2 *Let $\mathcal{L}$ be a reasonable d-frame and $x \in L_+$. Define*

$$\mathsf{P}(x) := \{b \in L_- \mid \exists a \not\sqsubseteq x.\ \langle a, b\rangle \in \mathsf{con}\} \text{ and } \mathsf{C}(x) := \{b \in L_- \mid \langle x, b\rangle \notin \mathsf{tot}\}$$

(1) $\mathsf{P}(x) \subseteq \mathsf{C}(x)$;

(2) If $\mathcal{L}$ is regular then $\bigsqcup \mathsf{P}(x) = \bigsqcup \mathsf{C}(x)$.

Proof. (1) is a direct consequence of $(\mathsf{con}{-}\mathsf{tot})$: if we have $\langle a, b\rangle \in \mathsf{con}$ and $\langle x, b\rangle \in \mathsf{tot}$ then $a \sqsubseteq x$ follows.

For (2) let $b' \lhd b \in \mathsf{C}(x)$. The witness $r_{b' \lhd b}$ cannot be below x as otherwise we could conclude $\langle x, b\rangle \in \mathsf{tot}$ from $\langle r_{b' \lhd b}, b\rangle \in \mathsf{tot}$. We also have $\langle r_{b' \lhd b}, b'\rangle \in \mathsf{con}$ and so find that $b' \in \mathsf{P}(x)$. By regularity, $\bigsqcup \mathsf{P}(x)$ is above b itself. It follows that $\bigsqcup \mathsf{P}(x) \sqsupseteq \bigsqcup \mathsf{C}(x)$, and by (1) the two suprema are in fact the same. □

Lemma 6.3 *Let $\mathcal{L}$ be a reasonable d-frame and $v_+ \in L_+$, $v_- \in L_-$. Consider the following statements:*

(i) *$v_- = \max \mathsf{C}(v_+)$ and $v_+ = \max \mathsf{C}(v_-)$;*

(ii) *$(L_+ \setminus \downarrow v_+, L_- \setminus \downarrow v_-)$ is a d-point;*

(iii) *$\langle v_+, v_-\rangle \notin \mathsf{tot}$ and $v_- \sqsupseteq \bigsqcup^{\uparrow} \mathsf{P}(v_+)$;*

(iv) *$\langle v_+, v_-\rangle$ is a maximal element of $(L_+ \times L_-) \setminus \mathsf{tot}$.*

The following are true:

(1) *(i) $\Rightarrow$ (ii) $\Rightarrow$ (iii), and (i) $\Rightarrow$ (iv).*

(2) *If $\mathcal{L}$ is regular then (iii) $\Rightarrow$ (i).*

(3) *If $\mathcal{L}$ satisfies the* (CUT_r) *rule then (iv) $\Rightarrow$ (ii).*

Proof. Part (1), (i) $\Rightarrow$ (ii): If $\langle x, y\rangle \in \mathsf{tot}$ then either $x \not\sqsubseteq v_+$ or $y \not\sqsubseteq v_-$ as otherwise we would have $\langle v_+, v_-\rangle \in \mathsf{tot}$ by ($\mathsf{tot}{-}\uparrow$). If $\langle x, y\rangle \in \mathsf{con}$ and $x \not\sqsubseteq v_+$ then $y \in \mathsf{P}(v_+) \subseteq \mathsf{C}(v_+)$ by the previous lemma; hence $y \sqsubseteq v_-$. Thus we have shown that the pair $(L_+ \setminus \downarrow v_+, L_- \setminus \downarrow v_-)$ satisfies conditions ($\mathrm{dp}_{\mathsf{tot}}$) and ($\mathrm{dp}_{\mathsf{con}}$) for d-points and it remains to show that we have two completely prime filters. This will hold if v_+ and v_- are $\sqcap$-irreducible. So let $v_- = y \sqcap y'$; by ($\mathsf{tot}{-}\vee$) either $\langle v_+, y\rangle \notin \mathsf{tot}$ or $\langle v_+, y'\rangle \notin \mathsf{tot}$, which means that either $y = v_-$ or $y' = v_-$.

(ii) $\Rightarrow$ (iii): If $x \not\sqsubseteq v_+$ and $\langle x, y\rangle \in \mathsf{con}$ then $y \sqsubseteq v_-$ by ($\mathrm{dp}_{\mathsf{con}}$). So we have $v_- \sqsupseteq \bigsqcup \mathsf{P}(v_+)$. $\langle v_+, v_-\rangle \notin \mathsf{tot}$ follows from ($\mathrm{dp}_{\mathsf{tot}}$). The set $\mathsf{P}(v_+)$ is directed because $L_+ \setminus \downarrow v_+$ is a filter and ($\mathsf{con}{-}\wedge$) is assumed for reasonable d-frames.

(i) $\Rightarrow$ (iv) is trivial.

Part (2), (iii) $\Rightarrow$ (i): On the side of L_- we already have $v_- \sqsupseteq \bigsqcup \mathsf{C}(v_+)$ by the previous lemma. For the other direction, assume $x \not\sqsubseteq v_+$. By regularity there is $x' \lhd x$ with $x' \not\sqsubseteq v_+$. Because of $\langle x', r_{x'\lhd x}\rangle \in \mathsf{con}$ we have $r_{x'\lhd x} \sqsubseteq v_-$ by assumption, and then from $\langle x, r_{x'\lhd x}\rangle \in \mathsf{tot}$ we infer $\langle x, v_-\rangle \in \mathsf{tot}$ by ($\mathsf{tot}{-}\uparrow$). It follows that $\mathsf{C}(v_-) \subseteq \downarrow v_+$. Together with $\langle v_+, v_-\rangle \notin \mathsf{tot}$ this is exactly (i).

Part (3), (iv) $\Rightarrow$ (ii): As in (i) $\Rightarrow$ (ii) we get that v_+ and v_- are $\sqcap$-prime, and that condition ($\mathrm{dp}_{\mathsf{tot}}$) is satisfied for $(L_+ \setminus \downarrow v_+, L_- \setminus \downarrow v_-)$. In order to show ($\mathrm{dp}_{\mathsf{con}}$) assume $\langle x, y\rangle \in \mathsf{con}$. If we had $x \not\sqsubseteq v_+$ and $y \not\sqsubseteq v_-$ then by (the contrapositive of) the ($\mathrm{cut}_{\mathsf{tot}}$) rule we would have either $\langle v_+, v_- \sqcup y\rangle \notin \mathsf{tot}$ or $\langle v_+ \sqcup x, v_-\rangle \notin \mathsf{tot}$, contradicting the maximality of $\langle v_+, v_-\rangle$. $\square$

We are ready to formulate and prove the d-frame analogue to the Hofmann-Mislove lemma 2.5:

Lemma 6.4 *Let $\mathcal{L}$ be a regular d-frame that satisfies* (CUT_r)*. Assume that $\mathcal{S}_+$ is a Scott-open filter in L_+ and $\mathcal{U}_- = L_- \setminus \downarrow u_-$ is a completely prime upper set in L_- such that:*

$$(\mathrm{hm}_{\mathsf{con}}) \quad \alpha \in \mathsf{con} \quad\Longrightarrow\quad \alpha_+ \notin \mathcal{S}_+ \text{ or } \alpha_- \notin \mathcal{U}_-$$

$$(\mathrm{hm}_{\mathsf{tot}}) \quad \alpha \in \mathsf{tot} \quad\Longrightarrow\quad \alpha_+ \in \mathcal{S}_+ \text{ or } \alpha_- \in \mathcal{U}_-$$

Then the following are true:

(1) $u_- = \bigsqcup^{\uparrow}\{b \mid \exists a \in \mathcal{S}_+ .\, \langle a, b\rangle \in \mathsf{con}\}$*, that is, $\mathcal{U}_-$ is uniquely determined by $\mathcal{S}_+$.*

(2) $\mathcal{S}_+ = \{a \mid \langle a, u_-\rangle \in \mathsf{tot}\}$*, that is, $\mathcal{S}_+$ is uniquely determined by $\mathcal{U}_-$.*

(3) $x \sqsubseteq \mathcal{S}_+ \quad\Leftrightarrow\quad (x, u_-) \in \mathsf{con}$.

(4) *For any point $(F_+, F_-) \in \mathsf{spec}\,\mathcal{L}$, $\quad \mathcal{S}_+ \subseteq F_+ \quad\Leftrightarrow\quad F_- \subseteq \mathcal{U}_-$.*

(5) *$\mathcal{S}_+$ is the intersection of all F_+ where (F_+, F_-) is a point and $\mathcal{S}_+ \subseteq F_+$.*

(6) *$\mathcal{U}_-$ is the union of all F_- where (F_+, F_-) is a point and $F_- \subseteq \mathcal{U}_-$.*

(7) *The set* $A := \{(F_+, F_-) \mid \mathcal{S}_+ \subseteq F_+\} = \{(F_+, F_-) \mid F_- \subseteq \mathcal{U}_-\}$ *is* $\mathcal{T}_+$*-compact saturated and* $\mathcal{T}_-$*-closed in the bitopological space* $(\mathsf{spec}\,\mathcal{L}; \mathcal{T}_+, \mathcal{T}_-)$.

Proof. (1) The element u_- can not be any smaller because of $(\mathrm{hm}_{\mathsf{con}})$. For the converse assume $y \lhd u_-$. The corresponding witness $r_{y \lhd u_-}$ belongs to $\mathcal{S}_+$ by $(\mathrm{hm}_{\mathsf{tot}})$ and so $y \in \{b \mid \exists a \in \mathcal{S}_+. \langle a, b\rangle \in \mathsf{con}\}$. By regularity, then, $u_- \sqsubseteq \bigsqcup^{\uparrow}\{b \mid \exists a \in \mathcal{S}_+. \langle a, b\rangle \in \mathsf{con}\}$.

(2) Because of $(\mathrm{hm}_{\mathsf{tot}})$ it is clear that $\mathcal{S}_+$ must contain all $a \in L_+$ with $\langle a, u_-\rangle \in \mathsf{tot}$. For the converse let $x \in \mathcal{S}_+$. By regularity and Scott-openness of $\mathcal{S}_+$ there is $x' \lhd x$ still in $\mathcal{S}_+$. The corresponding witness $r_{x' \lhd x}$ must be below u_- because of $(\mathrm{hm}_{\mathsf{con}})$, but then $\langle x, u_-\rangle \in \mathsf{tot}$ by $(\mathsf{tot}{-}\uparrow)$.

(3) Assume $x \sqsubseteq a$ for all $a \in \mathcal{S}_+$. By $(\mathsf{con}{-}\downarrow)$ we have $(x, b) \in \mathsf{con}$ for all $b \in \{b \mid \exists a \in \mathcal{S}_+. (a, b) \in \mathsf{con}\}$, so $(x, u_-) \in \mathsf{con}$ by $(\mathsf{con}{-}\bigsqcup^{\uparrow})$ and part (1). For the converse, remember that $(a, u_-) \in \mathsf{tot}$ for all $a \in \mathcal{S}_+$ by (2), so $(x, u_-) \in \mathsf{con}$ implies $x \sqsubseteq a$ by $(\mathsf{con}{-}\mathsf{tot})$.

(4) Let $v_+ = \bigsqcup(L_+ \setminus F_+)$. From $\mathcal{S}_+ \subseteq F_+$ and $(\mathrm{hm}_{\mathsf{con}})$ we get $\mathsf{P}(v_+) \supseteq (L_- \setminus \mathcal{U}_-)$, so $v_- = \bigsqcup \mathsf{P}(v_+) \sqsupseteq u_-$ and hence $F_- \subseteq \mathcal{U}_-$.

Starting with the right hand side, $F_- \subseteq \mathcal{U}_-$, we let $v_- = \bigsqcup(L_- \setminus F_-)$. From $(\mathrm{hm}_{\mathsf{con}})$ we get $\mathsf{P}(v_-) \cap \mathcal{S}_+ = \emptyset$. So $v_+ = \bigsqcup^{\uparrow}\mathsf{P}(v_-) \notin \mathcal{S}_+$ and hence $\mathcal{S}_+ \subseteq F_+$.

(5) Assume that $x \notin \mathcal{S}_+$. Because $\mathcal{S}_+$ is assumed to be Scott-open, we can apply Zorn's Lemma to obtain a maximal element v_+ above x that does not belong to $\mathcal{S}_+$. The set $F_+ := L_+ \setminus {\downarrow} v_+$ is a completely prime filter that separates x from $\mathcal{S}_+$, and it remains to show that it is the first component of a d-point. According to Lemma 6.3 the right candidate is $F_- = L_- \setminus {\downarrow} v_-$ where $v_- = \bigsqcup^{\uparrow}\mathsf{P}(v_+) = \bigsqcup \mathsf{C}(v_+)$. Note that $u_- \sqsubseteq v_-$ as $u_- \in \mathsf{C}(v_+)$ by $(\mathrm{hm}_{\mathsf{tot}})$. Using Lemma 6.3(iii) we only need to show that $\langle v_+, v_-\rangle \notin \mathsf{tot}$. For this we employ (CUT_r): for all $\langle a, b\rangle \in \mathsf{con}$ with $a \in F_+$ we have $\langle v_+ \sqcup a, v_-\rangle \in \mathsf{tot}$ by (2); if it was the case that $\langle v_+, v_-\rangle = \langle v_+, u_- \sqcup \bigsqcup^{\uparrow}\mathsf{P}(v_+)\rangle \in \mathsf{tot}$, then $\langle v_+, u_-\rangle \in \mathsf{tot}$ would follow, contradicting $(\mathrm{hm}_{\mathsf{tot}})$.

For part (6) let $y \in \mathcal{U}_-$. By regularity and the assumption that $\mathcal{U}_-$ is completely prime, some $y' \lhd y$ also belongs to $\mathcal{U}_-$. The witness $r_{y' \lhd y}$ is not in $\mathcal{S}_+$ because of $\langle r_{y' \lhd y}, y'\rangle \in \mathsf{con}$ and assumption $(\mathrm{hm}_{\mathsf{con}})$. By part (5) there is a point (F_+, F_-) that separates $r_{y' \lhd y}$ from $\mathcal{S}_+$. By (4) we have that $F_- \subseteq \mathcal{U}_-$ and because of $\langle r_{y' \lhd y}, y\rangle \in \mathsf{tot}$ it must also be the case that $y \in F_-$.

Finally, consider the last claim; the two descriptions of A are equal because of (4). Any $\mathcal{T}_+$-open neighbourhood of A has the form $\Phi_+(x)$ with $x \in \mathcal{S}_+$ by (5). It follows that A is $\mathcal{T}_+$-compact. Only the maximality of u_- in $L_- \setminus \mathcal{U}_-$ is required to see that A is the complement of $\Phi_-(u_-)$. □

Theorem 6.5 *For a regular d-frame* $\mathcal{L}$ *that satisfies* (CUT_r) *there is a one-to-one correspondence between*

(i) *maps* q *from* L *to the four-element d-frame* 2.2 *which preserve* tt, $\bigsqcup^{\uparrow}$, con, tot, *and the logical operation* $\wedge$, *and*

(ii) *subsets* A *of* $\mathsf{spec}\,\mathcal{L}$ *which are compact saturated in the positive and closed in the negative topology.*

Proof. Given a map q as described in part (i), consider $\mathcal{S}_+ = q^{-1}(tt) \cap L_+$ and $\mathcal{U}_- = q^{-1}(ff) \cap L_-$. It is straightforward to show that the pair $(\mathcal{S}_+, \mathcal{U}_-)$ satisfies the assumptions of Lemma 6.4. The translation in the opposite direction is equally easy. $\square$

A few comments on this result are in order: Given a *consistent predicate* φ, that is, $\varphi \in \mathsf{con}$, the value of q at φ can only be tt, ff, or 0. The first outcome indicates that *all* elements of A satisfy φ, the second that *some* element of A fails φ, and the last that neither holds (which is a possibility because a consistent predicate does not need to be Boolean). This means that q acts like a *universal quantifier* for partial predicates.

Generally, one would expect a universal quantifier to preserve tt but not necessarily ff, because A could be the empty set. Also, one would expect it to preserve conjunction ($\wedge$) but not disjunction ($\vee$), and certainly one would not want it to be inconsistent (returning 1) for a consistent predicate, or to be undecided (returning 0) for a total predicate, that is, one expects it to preserve con and tot.

The preservation of $\bigsqcup^{\uparrow}$ can be seen as a *computability* condition on the universal quantifier: If a (partial) predicate φ is the directed supremum of (partial) predicates φ_i, and if the universal quantifier applied to φ returns a definite answer, that is, either tt or ff, then computability requires the same answer be obtained from an approximant φ_i already.

All in all, then, Theorem 6.5 is a generalisation of the theory of continuous quantification on topological spaces, discovered by Taylor [20] and Escardó [6], to a logic in which predicates are allowed to have value ff as well as tt.

For a version of Theorem 6.5 on the side of bitopological spaces we first observe that regularity implies that the space is order-separated, so by Theorem 3.7 it is automatically d-sober. In an order-separated space a τ_+-compact saturated set is also τ_--closed. Furthermore, the corresponding d-frame $\Omega\, X$ satisfies (CUT_r) by Proposition 5.5, and so 6.5 applies:

Theorem 6.6 *If $(X; \tau_+, \tau_-)$ is a regular bitopological space then there is a one-to-one correspondence between*

(i) *maps from $\Omega\, X$ to 2.2 which preserve tt, $\bigsqcup^{\uparrow}$, con, tot and $\wedge$, and*

(ii) *subsets A of X which are compact saturated with respect to τ_+.*

7 An application: local compactness

We use the machinery of the previous section to define a notion of local compactness for regular bitopological spaces.

Definition 7.1 Let $\mathcal{S}$ be a Scott-open filter of L_+ and $\mathcal{U}_-$ a completely prime upper set of L_-. We say that $(\mathcal{S}_+, \mathcal{U}_-)$ is an *HM-pair* if it satisfies the conditions $(\mathrm{hm}_{\mathsf{con}})$ and $(\mathrm{hm}_{\mathsf{tot}})$ of Lemma 6.4.

For $x', x \in L_+$ we set $x' \blacktriangleleft x$ if there is an HM-pair $(\mathcal{S}_+, \mathcal{U}_-)$ such that $x' \sqsubseteq \mathcal{S}_+ \ni x$.

A d–frame is called *locally compact* if it is regular, satisfies (CUT_r), and the following two conditions hold:

$$(\mathrm{lc}_+) \quad \forall x \in L_+.\ x = \bigsqcup \{x' \mid x' \blacktriangleleft x\}$$

$$(\mathrm{lc}_{\mathsf{tot}}) \quad \forall \alpha.\ (\forall (\mathcal{S}_+, \mathcal{U}_-).\ \alpha_+ \in \mathcal{S}_+ \text{ or } \alpha_- \in \mathcal{U}_-) \ \Rightarrow \ \alpha \in \mathsf{tot}$$

We note that ($\mathrm{lc}_{\mathsf{tot}}$) is just the converse of ($\mathrm{hm}_{\mathsf{tot}}$).

Proposition 7.2 *Locally compact d-frames are spatial.*

Proof. We check the conditions of Proposition 5.1. For (s_+) assume $x \not\sqsubseteq a \in L_+$; by local compactness there is $x' \blacktriangleleft x$ with $x' \not\sqsubseteq a$. Let $(\mathcal{S}_+, \mathcal{U}_-)$ be the corresponding HM-pair with $x' \sqsubseteq \mathcal{S}_+ \ni x$. The element a can not be contained in $\mathcal{S}_+$, so by Lemma 6.4(5) there exists a point (F_+, F_-) such that $\mathcal{S}_+ \subseteq F_+$ and $a \notin F_+$.

Next we tackle ($\mathrm{s}_{\mathsf{tot}}$), so assume $\alpha \notin \mathsf{tot}$. By the contrapositive of (lc_-) there exists an HM-pair $(\mathcal{S}_+, \mathcal{U}_-)$ such that $\alpha_- \notin \mathcal{U}_-$ and $\alpha_+ \notin \mathcal{S}_+$. By 6.4(5) we obtain a point (F_+, F_-) with $\mathcal{S}_+ \subseteq F_+ \not\ni \alpha_+$ and from 6.4(4) we get that $\alpha_- \notin F_- \subseteq \mathcal{U}_-$.

For (s_-) assume $y \not\sqsubseteq b \in L_-$. By regularity, there exists $y' \in L_-$ with $y' \lhd y$ and $y' \not\sqsubseteq b$. The witness $r_{y' \lhd y}$ satisfies $\langle r_{y' \lhd y}, b\rangle \notin \mathsf{tot}$ by ($\mathsf{con}{-}\mathsf{tot}$). From ($\mathrm{s}_{\mathsf{tot}}$) we obtain a point (F_+, F_-) such that $r_{y' \lhd y} \notin F_+$, $b \notin F_-$. Because $\langle r_{y' \lhd y}, y\rangle \in \mathsf{tot}$, we must have $y \in F_-$.

For ($\mathrm{s}_{\mathsf{con}}$) assume $\langle x, y\rangle \notin \mathsf{con}$. Because of local compactness and Lemma 5.4 (together with ($\mathsf{con}{-}\vee$)) there exists $x' \blacktriangleleft x$ such that $\langle x', y\rangle \notin \mathsf{con}$. Let $(\mathcal{S}_+, \mathcal{U}_-)$ be the corresponding HM-pair. By Lemma 6.4(3), $x' \sqsubseteq \mathcal{S}_+$ forces $\langle x', u_-\rangle \in \mathsf{con}$, hence y must belong to $\mathcal{U}_-$. Using 6.4(6) we obtain a point (F_+, F_-) such that $y \in F_- \subseteq \mathcal{U}_-$ and by 6.4(4) we also have $x \in \mathcal{S}_+ \subseteq F_+$. $\square$

Note that we did not need that the sets $\{x' \mid x' \blacktriangleleft x\}$ are directed, but this is in fact the case: If $x'_1, x'_2 \blacktriangleleft x$ with witnessing HM-pairs $(\mathcal{S}^1_+, \mathcal{U}^1_-)$, $(\mathcal{S}^2_+, \mathcal{U}^2_-)$, then $(\mathcal{S}^1_+ \cap \mathcal{S}^2_+, \mathcal{U}^1_- \cup \mathcal{U}^2_-)$ witnesses $x'_1 \sqcup x'_2 \blacktriangleleft x$.

Definition 7.3 A bitopological space $(X; \tau_+, \tau_-)$ is called *locally compact* if it is regular and τ_+ is locally compact in the usual T_0 sense.

Proposition 7.4 *For $(X; \tau_+, \tau_-)$ a locally compact bitopological space, the d–frame ΩX is locally compact.*

Proof. Obviously, an HM-set on X gives rise to an HM-pair in ΩX, and only ($\mathrm{lc}_{\mathsf{tot}}$) needs checking. For this assume that the union of $O_+ \in \tau_+$ and $O_- \in \tau_-$ does not cover X, that is, there is $p \in X \setminus O_+ \cup O_-$. Then by order-separation $\uparrow p$ is τ_+-compact and τ_--closed, that is, an HM-set. Neither is O_+ a neighbourhood of it, nor does O_- intersect with it, so we conclude the contrapositive of ($\mathrm{lc}_{\mathsf{tot}}$). $\square$

Theorem 7.5 *The functors Ω and* spec *restrict to a dual equivalence between locally compact bitopological spaces and locally compact d-frames.*

8 Discussion

As we pointed out in the introduction, a corollary of the classical Hofmann-Mislove theorem is that the collection of compact saturated sets forms a dcpo under reverse inclusion. The analogue for bitopological spaces need not be true:

Example 8.1 The punctured unit interval $[0,1] \setminus \{\frac{1}{2}\}$ is locally compact when equipped with the usual τ_+ and τ_-. Each set of the form $[r,1] \setminus \{\frac{1}{2}\}$, $0 \leq r < \frac{1}{2}$ is HM but their intersection is $(\frac{1}{2}, 1]$ which is not.

However, our motivation for studying this problem was based on the view of HM-sets as the continuously "quantifiable" ones, as explained in the text after Theorem 6.5 above, and this part of the story works out in a most satisfying way.

Another motivation was the desire to extend the duality between stably compact spaces and strong proximity lattices, [14]. There, it is the case that the two topologies determine each other (each being the co-compact topology with respect to the other), but this is no longer true in the locally compact case:

Example 8.2 Let $(X;\tau)$ be a locally compact Hausdorff space. Then $(X;\tau,\tau)$ is a locally compact bispace in the sense of Definition 7.3. However, this is also true of $(X;\tau,\tau_{cc})$ where τ_{cc} is the co-compact topology with respect to τ. In general, τ and τ_{cc} are different; for a concrete example consider $\mathbb{R}$ with its usual metric topology.

Still, we believe that our definition of "locally compact bispace" is very promising as a generalisation of "stably compact" and that it warrants further investigation.

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