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Bi-parametric distance and similarity measures of picture fuzzy sets and their applications in medical diagnosis

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ABSTRACT

The concept of picture fuzzy sets (PFS) is a generalization of ordinary fuzzy sets and intuitionistic fuzzy sets, which is characterized by positive membership, neutral membership, and negative membership functions. Keeping in mind the importance of similarity measures and applications in data mining, medical diagnosis, decision making, and pattern recognition, several studies have been proposed in the literature. Some of those, however, cannot satisfy the axioms of similarity and provide counter-intuitive cases. In this paper, we propose new similarity measures for *PFSs* based on two parameters t and p, where t identifies the level of uncertainty and p is the L_p norm. The properties of the bi-parametric similarity and distance measures are discussed. We provide some counterexamples for existing similarity measures in the literature and show how our proposed similarity measure is important and applicable to the pattern recognition problems. In the end, we provide an application of a proposed similarity measure for medical diagnosis.

1. Introduction

Many classical theories such as fuzzy set theory [1], probability theory, vague set theory [2], rough set theory [3], intuitionistic fuzzy set (IFS) [4], and the interval mathematics [5] are well known and effectively model uncertainties. These approaches show their inherent difficulties because of intensive quantity and type of uncertainties as pointed out by Molodtsov [6]. In [6], Molodtsov defined the soft set which is an absolutely new logical instrument for dealing uncertainties. Nowadays, many authors work to hybridize the different models with the soft set and achieved results in many applicable theories. The fuzzy soft set and intuitionistic fuzzy soft set (IFSS) were defined by Maji et al. [7,8]. Then the further extensions of soft sets like the generalized fuzzy soft set [9], the interval-valued fuzzy soft set [10], the soft rough set [11], the vague soft set [12], the trapezoidal fuzzy soft set [13], the neutrosophic soft set [14], the intuitionistic neutrosophic soft set [15], the multi-fuzzy soft set [16], and the hesitant fuzzy soft set [17] were introduced. Agarwal defined the generalized intuitionistic fuzzy soft set (GIFSS) as the cross product of IFSS and IFS [18]. Later, Feng el al. [19] pointed out that this cross product was not well defined. Feng el al. clarified and reformulated the GIFSS as the combination of an IFSS over the universe of discourse and the IFS in the parameter set. The soft decernibility matrix approach for GIFSS was used to solve decision making problems in reference [20].

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In [21], Coung introduced the notion of PFS by including an extra membership; the "neutral membership degree". The PFS is the generalization of fuzzy sets and IFSs [22]. Coung and Kreinovich discussed PFS for computational intelligence problems [22]. The PFS model uncertainties more effectively and directly applied to solve the daily life problems. Coung and Kreinovich explained t-norm operators for PFSs [23]. Fuzzy logic operators for PFSs were determined by Cuong and Hai [24]. Correlation coefficients of PFS and their applications in clustering analysis were introduced by Sing [25]. With the help of novel fuzzy calculations based on the PFS domain time arrangement gauging and climate estimating was given by Son and Thong [26]. Son [27], defined picture fuzzy separation, generalized distance and association measures, and connected them to tackle group investigation under the PFS condition. Wei [28], exhibited picture fuzzy aggregation operators method and applied it to multi attribute decision making (MADM) for ranking enterprise resource planning (ERP) structures. Wei [29], researched a basic leadership technique in light of the picture fuzzy weighted cross-entropy and used this to rank the choices. Garg [30], contemplated aggregation operations on PFSs and applied it to multi criteria decision-making (MCDM) problems. Picture fuzzy hamarcher, geometric and Dombi aggregation operators and their applications to the MADM were demonstrated by Wei, Wang and Jana, respectively [31–33]. For more about aggregation operators of PFSs, we refer to [34–37].

Yang et al. [38] defined the hybrid model of soft set and PFS and called it picture fuzzy soft set (PFSS). The generalized picture fuzzy soft set (GPFSS) and their applications in decision making were discussed by Khan et al. [39]. The soft decernibility matrix approach for PFSSs and GPFSSs approach was used to solve decision making problems [38,40]. The decernibility matrix approach is important when you have to start the process with predefined conditions on membership functions, i.e., threshold values. The applications of GPFSS in concept selection were discussed in [41].

In this paragraph, we discuss the distance and similarity measures of PFSs and the methods derived based on them. The distance and similarity measures have application in data mining, medical diagnosis, decision making and pattern recognition. In [42], Wei defined the picture fuzzy cross entropy and use it for MADM problems. Similarity measures for PFSs based on cosine and cotangent functions and applied them for strategic decision making [43]. Peng [44], determined an algorithm for PFS that was implemented to decision making problems. In [45], Wei interpreted cosine, weighted cosine, set theoretic, weighted set theoretic, grey and weighted grey similarity measure for picture fuzzy sets and applied them to the pattern recognition problems. Wang et al. [46] used projection based VIKOR method for risk evaluation of construction project by using PFSs. Meksavang et al. [47] used extended VIKOR method for sustainable supplier selection using PFSs and discussed its application in beef industry. An applications of PFSs in cleaner production of gold mines by using integrated EDAS-ELECTRE was discussed by Liang [48]. Picture fuzzy TOPSIS method based on the linear programming and distance measure was discussed by Sindhu et al. [49]. The picture fuzzy entropy based similarity measures and its applications in MADM were practiced by Thao [50]. Nhung et al. [51] adopted the dissimilarity measures for PFS and discussed its applications in MADM. A projection model was exercised by Wei et al. [52], in order to measure the similarity. The picture fuzzy generalized dice similarity measures were presented by Wei and Gao [53]. The generalized picture distance measures and picture association measures, and their application in clustering analysis were discussed by Son [54]. Ganei et al. [55] defined some new correlation coefficients of PFSs and discussed their applications in pattern recognition, medical diagnosis and clustering analysis.

In this paragraph, we discuss the applications of IFSs and type-2 fuzzy sets in medical diagnosis. Szmidt and Kacprzyk discuss the medical diagnosis problems using IFSs based on the similarity and distance measures [56–58]. De et al. [59] extended the Sanchez's approach for medical diagnosis using max–min-max composition based on IFSs. The pattern recognition and medical diagnosis problems were discussed by Vlachos et al. [60] using cross-entropy in IF settings. Wei et al. [61] introduced the entropy measure for interval valued IFSs and construct similarity measures based on the proposed entropy. Then similarity measures were used to solve pattern recognition and medical diagnosis problems. The distance measures for interval values IFSs were used for medical diagnosis of headache in [63]. Boran and Akay introduced the bi-parametric distance and similarity measures for IFSs and discussed their applications in pattern recognition and medical diagnosis problems in [62]. In reference [64] Own studied advantages of type-2 fuzzy and switching relation between type-2 fuzzy sets and IFSs. Then the switching results are applied in medical diagnosis. For advance study about medical diagnosis by using type-2 fuzzy sets, we refer you to [65–70].

We adopt the generalize model of fuzzy set (PFS) for medical diagnosis. The PFSs are comparatively a new extension of fuzzy sets which describe the human opinions that has more answers like acceptance, rejection, neutral and desist, which cannot be correctly presented in fuzzy sets and IFSs. The PFSs are categorized by three objects, the degree of belonging (membership), the degree of neutral belonging and the degree of non-belonging (non-membership) such that the total of these three degrees must not be more than one. Basically, PFSs play an important role in situations involving more types of answers like no, yes, refusal and abstain. To model such situations, a PFS is an appropriate choice.

The main contributions of our work are:

- We extend the idea of bi-parametric distance and similarity measures for PFSs. The properties of the proposed similarity measures are discussed.
- The interpretation of the novel distance measure and the functionality of level of uncertainty t are discussed.
- Applications in pattern recognition for the proposed bi-parametric distance and similarity measures are discussed, as well as, the counter intuitive cases for existing distance and similarity measures are discussed.
- The medical diagnosis problem is discussed based on the bi-parametric distance and similarity measures.

Some of the proposed similarity measure for *PFSs* has some problems which are pointed out in Section 5. To improve the idea of the similarity measure, we proposed the new similarity measure based on two parameters p and t, where p is the L_p norm and t identifies the level of uncertainty. The remaining paper is organized as follows: The preliminaries are presented in Sections 2. In Section 3, we proposed the new distance and similarity measures for *PFSs*. In Section 4, we give the explanation of both parameters p and t, and interpretation of novel distance measure. In Section 5, we provide some counterexamples for already proposed similarity measures. To support the proposed similarity measure a numerical example of medical diagnosis is presented in Section 6. Comparison analysis and conclusion are presented in Sections 7 and 8.

2. Preliminaries

In this section, we provide some basic definitions of fuzzy set, intuitionistic fuzzy set, picture fuzzy set and some similarity measures.

A fuzzy set is defined by Zadeh [1], which handles uncertainty based on the view of gradualness effectively.

Definition 2.1. [1] A membership function $\xi_{\hat{\mathcal{A}}} : \hat{\mathcal{Y}} \to [0,1]$ defines the fuzzy set $\hat{\mathcal{A}}$ over the $\hat{\mathcal{Y}}$, where $\xi_{\hat{\mathcal{A}}}(y)$ particularized the membership of an element $y \in \hat{\mathcal{Y}}$ in fuzzy set $\hat{\mathcal{A}}$.

In [21], Cuong defines the PFS, which is an extension of fuzzy set and applicable in many real life problems. By adding an extra membership function, namely, the degree of the neutral membership function, the picture fuzzy set is obtained. Basically, the model of the picture fuzzy set may be adequate in situations when we face human opinions involving more answers of the type: yes, abstain, no, refusal. Voting can be a good example of picture fuzzy set because it involves the situation of more answers of the type: yes, abstain, no, refusal.

Definition 2.2. [21] A PFS $\hat{\mathcal{A}}$ over the universe $\hat{\mathcal{Y}}$ is defined as

$$\hat{\mathcal{A}} = \{(y, \xi_{\hat{\mathcal{A}}}, \eta_{\hat{\mathcal{A}}}, \upsilon_{\hat{\mathcal{A}}}) | y \in \hat{\mathcal{Y}}\},$$

where $\xi_{\hat{\mathcal{A}}} : \hat{\mathcal{Y}} \to [0,1], \eta_{\hat{\mathcal{A}}} : \hat{\mathcal{Y}} \to [0,1]$ and $\vartheta_{\hat{\mathcal{A}}} : \hat{\mathcal{Y}} \to [0,1]$ are the degree of positive membership, neutral membership and degree of negative membership, respectively. Furthermore, it is required that $0 \leqslant \xi_{\hat{\mathcal{A}}} + \eta_{\hat{\mathcal{A}}} + \upsilon_{\hat{\mathcal{A}}} \leqslant 1$. Then for $y \in \hat{\mathcal{Y}}, \pi_{\hat{\mathcal{A}}}(y) = 1 - (\xi_{\hat{\mathcal{A}}}(y) + \eta_{\hat{\mathcal{A}}}(y) + \upsilon_{\hat{\mathcal{A}}}(y))$ is called the degree of refusal membership of y in $\hat{\mathcal{A}}$. For PFS $(\xi_{\hat{\mathcal{A}}}(y), \eta_{\hat{\mathcal{A}}}(y), \upsilon_{\hat{\mathcal{A}}}(y))$ are said to picture fuzzy value (*PFV*) or picture fuzzy number (*PFN*) and each *PFV* can be denoted by $q = (\xi_q, \eta_q, \upsilon_q)$, where ξ_q, η_q and $\upsilon_q \in [0,1]$, with condition that $0 \leqslant \xi_q + \eta_q + \upsilon_q \leqslant 1$.

In [43], Wei defines some similarity measures for *PFSs* based on cosine and cotangent functions.

Definition 2.3. [43] For two *PFSs* $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$ in $\hat{\mathcal{Y}}$, a cosine similarity measure between $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$ is defined as follows:

$$CS^1(\hat{\mathcal{A}}, \hat{\mathcal{B}}) = \frac{1}{m}\sum_{j=1}^{m} \frac{\xi_{\hat{\mathcal{A}}}(y_j)\xi_{\hat{\mathcal{B}}}(y_j) + \eta_{\hat{\mathcal{A}}}(y_j)\eta_{\hat{\mathcal{B}}}(y_j) + \nu_{\hat{\mathcal{A}}}(y_j)\nu_{\hat{\mathcal{B}}}(y_j)}{\sqrt{\xi^2_{\hat{\mathcal{A}}}(y_j) + \eta^2_{\hat{\mathcal{A}}}(y_j) + \nu^2_{\hat{\mathcal{A}}}(y_j)}\sqrt{\xi^2_{\hat{\mathcal{B}}}(y_j) + \eta^2_{\hat{\mathcal{B}}}(y_j) + \nu^2_{\hat{\mathcal{B}}}(y_j)}}.$$

Definition 2.4. [43] For two *PFSs* $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$ in $\hat{\mathcal{Y}}$, a cosine similarity measure between $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$ is defined as follows:

$$CS^2(\hat{\mathcal{A}}, \hat{\mathcal{B}}) = \frac{1}{m}\sum_{j=1}^{m} cos\left\{\frac{\pi}{2}[|\xi_{\hat{\mathcal{A}}}(y_j) - \xi_{\hat{\mathcal{B}}}(y_j)| \vee |\eta_{\hat{\mathcal{A}}}(y_j) - \eta_{\hat{\mathcal{B}}}(y_j)| \vee |\nu_{\hat{\mathcal{A}}}(y_j) - \nu_{\hat{\mathcal{B}}}(y_j)|]\right\},$$

where $\vee$ is the maximum operation.

Definition 2.5. [43] For two *PFSs* $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$ in $\hat{\mathcal{Y}}$, a cosine similarity measure between $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$ is defined as follows:

$$CS^3(\hat{\mathcal{A}}, \hat{\mathcal{B}}) = \frac{1}{m}\sum_{j=1}^{m} cos\left\{\frac{\pi}{4}[|\xi_{\hat{\mathcal{A}}}(y_j) - \xi_{\hat{\mathcal{B}}}(y_j)| + |\eta_{\hat{\mathcal{A}}}(y_j) - \eta_{\hat{\mathcal{B}}}(y_j)| + |\nu_{\hat{\mathcal{A}}}(y_j) - \nu_{\hat{\mathcal{B}}}(y_j)|]\right\}.$$

Definition 2.6. [43] For two *PFSs* $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$ in $\hat{\mathcal{Y}}$, a cotangent similarity measure between $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$ is defined as follows:

$$CS^4(\hat{\mathcal{A}}, \hat{\mathcal{B}}) = \frac{1}{m}\sum_{j=1}^{m} cot\left\{\frac{\pi}{4} + \frac{\pi}{4}[|\xi_{\hat{\mathcal{A}}}(y_j) - \xi_{\hat{\mathcal{B}}}(y_j)| \vee |\eta_{\hat{\mathcal{A}}}(y_j) - \eta_{\hat{\mathcal{B}}}(y_j)| \vee |\nu_{\hat{\mathcal{A}}}(y_j) - \nu_{\hat{\mathcal{B}}}(y_j)|]\right\},$$

where $\vee$ is the maximum operation.

Definition 2.7. [43] For two *PFSs* $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$ in $\hat{\mathcal{Y}}$, a cosine similarity measure by using degree of refusal membership between $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$ is defined as follows:

$$CS^5(\hat{\mathcal{A}}, \hat{\mathcal{B}}) = \frac{1}{m}\sum_{j=1}^{m} cos\left\{\frac{\pi}{2}[|\xi_{\hat{\mathcal{A}}}(y_j) - \xi_{\hat{\mathcal{B}}}(y_j)| \vee |\eta_{\hat{\mathcal{A}}}(y_j) - \eta_{\hat{\mathcal{B}}}(y_j)| \vee |\nu_{\hat{\mathcal{A}}}(y_j) - \nu_{\hat{\mathcal{B}}}(y_j)| \vee |\pi_{\hat{\mathcal{A}}}(y_j) - \pi_{\hat{\mathcal{B}}}(y_j)|]\right\},$$

where $\vee$ is the maximum operation.

Definition 2.8. [43] For two *PFSs* $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$ in $\hat{\mathcal{Y}}$, a cotangent similarity measure by using degree of refusal membership between $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$ is defined as follows:

$$CS^6(\hat{\mathcal{A}}, \hat{\mathcal{B}}) = \frac{1}{m}\sum_{j=1}^{m} cot\left\{\frac{\pi}{4} + \frac{\pi}{4}[|\xi_{\hat{\mathcal{A}}}(y_j) - \xi_{\hat{\mathcal{B}}}(y_j)| \vee |\eta_{\hat{\mathcal{A}}}(y_j) - \eta_{\hat{\mathcal{B}}}(y_j)| \vee |\nu_{\hat{\mathcal{A}}}(y_j) - \nu_{\hat{\mathcal{B}}}(y_j)| \vee |\pi_{\hat{\mathcal{A}}}(y_j) - \pi_{\hat{\mathcal{B}}}(y_j)|]\right\},$$

where $\vee$ is the maximum operation.

Definition 2.9. [43] For two $\mathcal{PFS}s$ $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$ in $\hat{\mathcal{Y}}$, the similarity measure based on cosine function by using degree of refusal membership are defined as follows:

$$C^7(\hat{\mathcal{A}}, \hat{\mathcal{B}}) = \frac{1}{m}\sum_{j=1}^{m} cos\left\{\frac{\pi}{2}[|\xi_{\hat{\mathcal{A}}}(y_j) - \xi_{\hat{\mathcal{B}}}(y_j)| + |\eta_{\hat{\mathcal{A}}}(y_j) - \eta_{\hat{\mathcal{B}}}(y_j)| + |\nu_{\hat{\mathcal{A}}}(y_j) - \nu_{\hat{\mathcal{B}}}(y_j)| + |\pi_{\hat{\mathcal{A}}}(y_j) - \pi_{\hat{\mathcal{B}}}(y_j)|]\right\}$$

where $\vee$ and π are the maximum operation and refusal membership degree, respectively.

Definition 2.10. [45] For two *PFSs* $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$ in $\hat{\mathcal{Y}}$, a set theoretic similarity measure between $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$ is defined as follows:

$$STS(\hat{\mathcal{A}}, \hat{\mathcal{B}}) = \frac{1}{m}\sum_{j=1}^{m} \frac{\xi_{\hat{\mathcal{A}}}(y_j)\xi_{\hat{\mathcal{B}}}(y_j) + \eta_{\hat{\mathcal{A}}}(y_j)\eta_{\hat{\mathcal{B}}}(y_j) + \nu_{\hat{\mathcal{A}}}(y_j)\nu_{\hat{\mathcal{B}}}(y_j)}{max\{\xi^2_{\hat{\mathcal{A}}}(y_j) + \eta^2_{\hat{\mathcal{A}}}(y_j) + \nu^2_{\hat{\mathcal{A}}}(y_j), \xi^2_{\hat{\mathcal{B}}}(y_j) + \eta^2_{\hat{\mathcal{B}}}(y_j) + \nu^2_{\hat{\mathcal{B}}}(y_j)\}}.$$

Definition 2.11. [53] For two $\mathcal{PFS}s$ $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$ in $\hat{\mathcal{Y}}$, Dice similarity measures between $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$ is defined as follows:

$$\mathcal{S}^1_d(\hat{\mathcal{A}}, \hat{\mathcal{B}}) = \frac{1}{m}\sum_{j=1}^{m} \frac{2(\xi_{\hat{\mathcal{A}}}(y_j)\xi_{\hat{\mathcal{B}}}(y_j) + \eta_{\hat{\mathcal{A}}}(y_j)\eta_{\hat{\mathcal{B}}}(y_j) + \nu_{\hat{\mathcal{A}}}(y_j)\nu_{\hat{\mathcal{B}}}(y_j))}{\xi^2_{\hat{\mathcal{A}}}(y_j) + \eta^2_{\hat{\mathcal{A}}}(y_j) + \nu^2_{\hat{\mathcal{A}}}(y_j) + \xi^2_{\hat{\mathcal{B}}}(y_j) + \eta^2_{\hat{\mathcal{B}}}(y_j) + \nu^2_{\hat{\mathcal{B}}}(y_j)}$$

$$\mathcal{S}^2_d(\hat{\mathcal{A}}, \hat{\mathcal{B}}) = \frac{1}{m}\sum_{j=1}^{m} \frac{2(\xi_{\hat{\mathcal{A}}}(y_j)\xi_{\hat{\mathcal{B}}}(y_j) + \eta_{\hat{\mathcal{A}}}(y_j)\eta_{\hat{\mathcal{B}}}(y_j) + \nu_{\hat{\mathcal{A}}}(y_j)\nu_{\hat{\mathcal{B}}}(y_j)) + \pi_{\hat{\mathcal{A}}}(y_j)\pi_{\hat{\mathcal{B}}}(y_j)}{[\xi^2_{\hat{\mathcal{A}}}(y_j) + \eta^2_{\hat{\mathcal{A}}}(y_j) + \nu^2_{\hat{\mathcal{A}}}(y_j) + \pi^2_{\hat{\mathcal{A}}}(y_j)] + [\xi^2_{\hat{\mathcal{B}}}(y_j) + \eta^2_{\hat{\mathcal{B}}}(y_j) + \nu^2_{\hat{\mathcal{B}}}(y_j) + \pi^2_{\hat{\mathcal{B}}}(y_j)]}$$

$$\mathcal{S}^3_d(\hat{\mathcal{A}}, \hat{\mathcal{B}}) = \frac{\sum_{j=1}^{m} 2(\xi_{\hat{\mathcal{A}}}(y_j)\xi_{\hat{\mathcal{B}}}(y_j) + \eta_{\hat{\mathcal{A}}}(y_j)\eta_{\hat{\mathcal{B}}}(y_j) + \nu_{\hat{\mathcal{A}}}(y_j)\nu_{\hat{\mathcal{B}}}(y_j))}{\sum_{j=1}^{m}(\xi^2_{\hat{\mathcal{A}}}(y_j) + \eta^2_{\hat{\mathcal{A}}}(y_j) + \nu^2_{\hat{\mathcal{A}}}(y_j)) + \sum_{j=1}^{m}(\xi^2_{\hat{\mathcal{B}}}(y_j) + \eta^2_{\hat{\mathcal{B}}}(y_j) + \nu^2_{\hat{\mathcal{B}}}(y_j))}$$

3. Biparametric distance and similarity measures for *PFSs*

In this section, we define new distance and similarity measure for *PFSs* based on two parameters and give their proof.

Definition 3.1. A distance measure between *PFSs* $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$ is a mapping $\hat{\mathcal{D}} : PFS \times PFS \to [0,1]$, which satisfies the following properties:

(D1) $0 \leqslant \hat{\mathcal{D}}(\hat{\mathcal{A}}, \hat{\mathcal{B}}) \leqslant 1$
(D2) $\hat{\mathcal{D}}(\hat{\mathcal{A}}, \hat{\mathcal{B}}) = 0 \Longleftrightarrow \hat{\mathcal{A}} = \hat{\mathcal{B}}$
(D3) $\hat{\mathcal{D}}(\hat{\mathcal{A}}, \hat{\mathcal{B}}) = \hat{\mathcal{D}}(\hat{\mathcal{B}}, \hat{\mathcal{A}})$

(D4) If $\hat{\mathcal{A}} \subseteq \hat{\mathcal{B}} \subseteq \hat{\mathcal{C}}$ then $\hat{\mathcal{D}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) \geqslant \hat{\mathcal{D}}(\hat{\mathcal{A}},\hat{\mathcal{B}})$ and $\hat{\mathcal{D}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) \geqslant \hat{\mathcal{D}}(\hat{\mathcal{B}},\hat{\mathcal{C}})$.

Definition 3.2. A similarity measure between *PFSs* $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$ is a mapping $\hat{\mathcal{S}} : PFS \times PFS \to [0,1]$, which satisfies the following properties:

(S1) $0 \leqslant \hat{\mathcal{S}}(\hat{\mathcal{A}},\hat{\mathcal{B}}) \leqslant 1$
(S2) $\hat{\mathcal{S}}(\hat{\mathcal{A}},\hat{\mathcal{B}}) = 1 \Longleftrightarrow \hat{\mathcal{A}} = \hat{\mathcal{B}}$
(S3) $\hat{\mathcal{S}}(\hat{\mathcal{A}},\hat{\mathcal{B}}) = \hat{\mathcal{S}}(\hat{\mathcal{B}},\hat{\mathcal{A}})$
(S4) If $\hat{\mathcal{A}} \subseteq \hat{\mathcal{B}} \subseteq \hat{\mathcal{C}}$ then $\hat{\mathcal{S}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) \leqslant \hat{\mathcal{S}}(\hat{\mathcal{A}},\hat{\mathcal{B}})$ and $\hat{\mathcal{S}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) \leqslant \hat{\mathcal{S}}(\hat{\mathcal{B}},\hat{\mathcal{C}})$.

Definition 3.3. A bi-parametric distance measure between two *PFSs* $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$ is a mapping $\hat{\mathcal{D}} : \hat{\mathcal{A}} \times \hat{\mathcal{B}} \to [0,1]$ which is defined as

$$\hat{\mathcal{D}}(\hat{\mathcal{A}},\hat{\mathcal{B}}) = \Bigg[\frac{1}{3m(t+1)^p}\sum_{j=1}^{m}\big(|t(\xi_{\hat{\mathcal{A}}}(y_j) - \xi_{\hat{\mathcal{B}}}(y_j)) - (\eta_{\hat{\mathcal{A}}}(y_j) - \eta_{\hat{\mathcal{B}}}(y_j)) - (\nu_{\hat{\mathcal{A}}}(y_j) - \nu_{\hat{\mathcal{B}}}(y_j))|^p + |t(\eta_{\hat{\mathcal{A}}}(y_j) - \eta_{\hat{\mathcal{B}}}(y_j)) - (\xi_{\hat{\mathcal{A}}}(y_j) - \xi_{\hat{\mathcal{B}}}(y_j)) + (\nu_{\hat{\mathcal{A}}}(y_j) - \nu_{\hat{\mathcal{B}}}(y_j))|^p + |t(\nu_{\hat{\mathcal{A}}}(y_j) - \nu_{\hat{\mathcal{B}}}(y_j)) - (\xi_{\hat{\mathcal{A}}}(y_j) - \xi_{\hat{\mathcal{B}}}(y_j)) + (\eta_{\hat{\mathcal{A}}}(y_j) - \eta_{\hat{\mathcal{B}}}(y_j))|^p\big)\Bigg]^{\frac{1}{p}}, \quad (1)$$

where $t = 3,4,\ldots$ and $p = 1,2,3,\ldots$ are identifies the level of uncertainty and L_p norm, respectively.

Theorem 3.4. $\hat{\mathcal{D}}(\hat{\mathcal{A}},\hat{\mathcal{B}})$ is the distance measure between two *PFSs* $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$ in $\hat{\mathcal{Y}}$.

Proof. (**D1**) Let $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$ be two *PFSs*.

We can write the following equations:

$$|t(\xi_{\hat{\mathcal{A}}}(y_j) - \xi_{\hat{\mathcal{B}}}(y_j)) - (\eta_{\hat{\mathcal{A}}}(y_j) - \eta_{\hat{\mathcal{B}}}(y_j)) - (\nu_{\hat{\mathcal{A}}}(y_j) - \nu_{\hat{\mathcal{B}}}(y_j))| = |(t\xi_{\hat{\mathcal{A}}}(y_j) - \eta_{\hat{\mathcal{A}}}(y_j) - \nu_{\hat{\mathcal{A}}}(y_j)) - (t\xi_{\hat{\mathcal{B}}}(y_j) - \eta_{\hat{\mathcal{B}}}(y_j) - \nu_{\hat{\mathcal{B}}}(y_j))|,$$

$$|t(\eta_{\hat{\mathcal{A}}}(y_j) - \eta_{\hat{\mathcal{B}}}(y_j)) - (\xi_{\hat{\mathcal{A}}}(y_j) - \xi_{\hat{\mathcal{B}}}(y_j)) + (\nu_{\hat{\mathcal{A}}}(y_j) - \nu_{\hat{\mathcal{B}}}(y_j))| = |(t\eta_{\hat{\mathcal{A}}}(y_j) + \nu_{\hat{\mathcal{A}}}(y_j) - \xi_{\hat{\mathcal{A}}}(y_j)) - (t\eta_{\hat{\mathcal{B}}}(y_j)) + \nu_{\hat{\mathcal{B}}}(y_j) - \xi_{\hat{\mathcal{B}}}(y_j))|,$$

$$|t(\nu_{\hat{\mathcal{A}}}(y_j) - \nu_{\hat{\mathcal{B}}}(y_j)) - (\xi_{\hat{\mathcal{A}}}(y_j) - \xi_{\hat{\mathcal{B}}}(y_j)) + (\eta_{\hat{\mathcal{A}}}(y_j) - \eta_{\hat{\mathcal{B}}}(y_j))| = |(t\nu_{\hat{\mathcal{A}}}(y_j) + \eta_{\hat{\mathcal{A}}}(y_j) - \xi_{\hat{\mathcal{A}}}(y_j)) - (t\nu_{\hat{\mathcal{B}}}(y_j)) + \eta_{\hat{\mathcal{B}}}(y_j) - \xi_{\hat{\mathcal{B}}}(y_j))|.$$

Since from the definition of PFS, we have $0 \leqslant \xi_{\hat{\mathcal{A}}}(y_j) \leqslant 1, 0 \leqslant \eta_{\hat{\mathcal{A}}}(y_j) \leqslant 1, 0 \leqslant \nu_{\hat{\mathcal{A}}}(y_j) \leqslant 1, 0 \leqslant \xi_{\hat{\mathcal{B}}}(y_j) \leqslant 1, 0 \leqslant \eta_{\hat{\mathcal{B}}}(y_j) \leqslant 1$ and $0 \leqslant \nu_{\hat{\mathcal{B}}}(y_j) \leqslant 1$, and therefore we have the following inequalities:

$$-1 \leqslant (t\xi_{\hat{\mathcal{A}}}(y_j) - \eta_{\hat{\mathcal{A}}}(y_j) - \nu_{\hat{\mathcal{A}}}(y_j)) \leqslant t$$
$$-t \leqslant -(t\xi_{\hat{\mathcal{B}}}(y_j) - \eta_{\hat{\mathcal{B}}}(y_j) - \nu_{\hat{\mathcal{B}}}(y_j)) \leqslant 1$$

then we have

$$-(1+t) \leqslant (t\xi_{\hat{\mathcal{A}}}(y_j) - \eta_{\hat{\mathcal{A}}}(y_j) - \nu_{\hat{\mathcal{A}}}(y_j)) - (t\xi_{\hat{\mathcal{B}}}(y_j) - \eta_{\hat{\mathcal{B}}}(y_j) - \nu_{\hat{\mathcal{B}}}(y_j)) \leqslant (1+t).$$

It means that

$$0 \leqslant |(t\xi_{\hat{\mathcal{A}}}(y_j) - \eta_{\hat{\mathcal{A}}}(y_j) - \nu_{\hat{\mathcal{A}}}(y_j)) - (t\xi_{\hat{\mathcal{B}}}(y_j) - \eta_{\hat{\mathcal{B}}}(y_j) - \nu_{\hat{\mathcal{B}}}(y_j))|^p \leqslant (1+t)^p. \quad (2)$$

Similarly, we have the following inequalities:

$$-1 \leqslant (t\eta_{\hat{\mathcal{A}}}(y_j) + \nu_{\hat{\mathcal{A}}}(y_j) - \xi_{\hat{\mathcal{A}}}(y_j)) \leqslant t$$
$$-t \leqslant -(t\eta_{\hat{\mathcal{B}}}(y_j) + \nu_{\hat{\mathcal{B}}}(y_j) - \xi_{\hat{\mathcal{B}}}(y_j)) \leqslant 1$$

then we have

$$-(1+t) \leqslant (t\eta_{\hat{\mathcal{A}}}(y_j) + \nu_{\hat{\mathcal{A}}}(y_j) - \xi_{\hat{\mathcal{A}}}(y_j)) - (t\eta_{\hat{\mathcal{B}}}(y_j) + \nu_{\hat{\mathcal{B}}}(y_j) - \xi_{\hat{\mathcal{B}}}(y_j)) \leqslant (1+t).$$

It means that

$$0 \leqslant |(t\eta_{\hat{\mathcal{A}}}(y_j) + \nu_{\hat{\mathcal{A}}}(y_j) - \xi_{\hat{\mathcal{A}}}(y_j)) - (t\eta_{\hat{\mathcal{B}}}(y_j) + \nu_{\hat{\mathcal{B}}}(y_j) - \xi_{\hat{\mathcal{B}}}(y_j))|^p \leqslant (1+t)^p. \quad (3)$$

Also for remaining equation we have

$$-1 \leqslant (t\nu_{\hat{\mathcal{A}}}(y_j) + \eta_{\hat{\mathcal{A}}}(y_j) - \xi_{\hat{\mathcal{A}}}(y_j)) \leqslant t$$
$$-t \leqslant -(t\nu_{\hat{\mathcal{B}}}(y_j) + \eta_{\hat{\mathcal{B}}}(y_j) - \xi_{\hat{\mathcal{B}}}(y_j)) \leqslant 1$$

then we have

$$; -(1+t) \leqslant (t\nu_{\hat{\mathcal{A}}}(y_j) + \eta_{\hat{\mathcal{A}}}(y_j) - \xi_{\hat{\mathcal{A}}}(y_j)) - (t\nu_{\hat{\mathcal{B}}}(y_j) + \eta_{\hat{\mathcal{B}}}(y_j) - \xi_{\hat{\mathcal{B}}}(y_j)) \leqslant (1+t).$$

It means that

$$0 \leqslant |(t\nu_{\hat{\mathcal{A}}}(y_j) + \eta_{\hat{\mathcal{A}}}(y_j) - \xi_{\hat{\mathcal{A}}}(y_j)) - (t\nu_{\hat{\mathcal{B}}}(y_j) + \eta_{\hat{\mathcal{B}}}(y_j) - \xi_{\hat{\mathcal{B}}}(y_j))|^p \leqslant (1+t)^p. \quad (4)$$

Finally we have the following inequality:

$$0 \leqslant \Bigg[\frac{1}{3m(t+1)^p}\sum_{j=1}^{m}\big(|t(\xi_{\hat{\mathcal{A}}}(y_j) - \xi_{\hat{\mathcal{B}}}(y_j)) - (\eta_{\hat{\mathcal{A}}}(y_j) - \eta_{\hat{\mathcal{B}}}(y_j)) - (\nu_{\hat{\mathcal{A}}}(y_j) - \nu_{\hat{\mathcal{B}}}(y_j))|^p + |t(\eta_{\hat{\mathcal{A}}}(y_j) - \eta_{\hat{\mathcal{B}}}(y_j)) - (\xi_{\hat{\mathcal{A}}}(y_j) - \xi_{\hat{\mathcal{B}}}(y_j)) + (\nu_{\hat{\mathcal{A}}}(y_j) - \nu_{\hat{\mathcal{B}}}(y_j))|^p + |t(\nu_{\hat{\mathcal{A}}}(y_j) - \nu_{\hat{\mathcal{B}}}(y_j)) - (\xi_{\hat{\mathcal{A}}}(y_j) - \xi_{\hat{\mathcal{B}}}(y_j)) + (\eta_{\hat{\mathcal{A}}}(y_j) - \eta_{\hat{\mathcal{B}}}(y_j))|^p$$

$$0 \leqslant \hat{\mathcal{D}}(\hat{\mathcal{A}},\hat{\mathcal{B}}) \leqslant 1.$$

(**D2**) Since for two *PFSs* $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$, if $\hat{\mathcal{A}} = \hat{\mathcal{B}}$, then $\xi_{\hat{\mathcal{A}}}(y_j) = \xi_{\hat{\mathcal{B}}}(y_j), \eta_{\hat{\mathcal{A}}}(y_j) = \eta_{\hat{\mathcal{B}}}(y_j)$ and $\nu_{\hat{\mathcal{A}}}(y_j) = \nu_{\hat{\mathcal{B}}}(y_j)$, therefore, $\xi_{\hat{\mathcal{A}}}(y_j) - \xi_{\hat{\mathcal{B}}}(y_j) = 0, \eta_{\hat{\mathcal{A}}}(y_j) - \eta_{\hat{\mathcal{B}}}(y_j) = 0$ and $\nu_{\hat{\mathcal{A}}}(y_j) - \nu_{\hat{\mathcal{B}}}(y_j) = 0$. Therefore, the distance measure is equal to zero.

(**D3**) For two *PFSs* $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$, we can write the following equations:

$$|t(\xi_{\hat{\mathcal{A}}}(y_j) - \xi_{\hat{\mathcal{B}}}(y_j)) - (\eta_{\hat{\mathcal{A}}}(y_j) - \eta_{\hat{\mathcal{B}}}(y_j)) - (\nu_{\hat{\mathcal{A}}}(y_j) - \nu_{\hat{\mathcal{B}}}(y_j))|^p = |(-1)\{t(\xi_{\hat{\mathcal{B}}}(y_j) - \xi_{\hat{\mathcal{A}}}(y_j)) - (\eta_{\hat{\mathcal{B}}}(y_j) - \eta_{\hat{\mathcal{A}}}(y_j)) - (\nu_{\hat{\mathcal{B}}}(y_j) - \nu_{\hat{\mathcal{A}}}(y_j))\}|^p,$$

$$|t(\eta_{\hat{\mathcal{A}}}(y_j) - \eta_{\hat{\mathcal{B}}}(y_j)) - (\xi_{\hat{\mathcal{A}}}(y_j) - \xi_{\hat{\mathcal{B}}}(y_j)) + (\nu_{\hat{\mathcal{A}}}(y_j) - \nu_{\hat{\mathcal{B}}}(y_j))|^p = |(-1)\{t(\eta_{\hat{\mathcal{B}}}(y_j) - \eta_{\hat{\mathcal{A}}}(y_j)) - (\xi_{\hat{\mathcal{B}}}(y_j) - \xi_{\hat{\mathcal{A}}}(y_j)) + (\nu_{\hat{\mathcal{B}}}(y_j) - \nu_{\hat{\mathcal{A}}}(y_j))\}|^p,$$

$$|t(\nu_{\hat{\mathcal{A}}}(y_j) - \nu_{\hat{\mathcal{B}}}(y_j)) - (\xi_{\hat{\mathcal{A}}}(y_j) - \xi_{\hat{\mathcal{B}}}(y_j)) + (\eta_{\hat{\mathcal{A}}}(y_j) - \eta_{\hat{\mathcal{B}}}(y_j))|^p = |(-1)\{t(\nu_{\hat{\mathcal{B}}}(y_j) - \nu_{\hat{\mathcal{A}}}(y_j)) - (\xi_{\hat{\mathcal{B}}}(y_j) - \xi_{\hat{\mathcal{A}}}(y_j)) + (\eta_{\hat{\mathcal{B}}}(y_j) - \eta_{\hat{\mathcal{A}}}(y_j))\}|^p.$$

Based on the definition of absolute value, we have

$$|t(\xi_{\hat{\mathcal{A}}}(y_j) - \xi_{\hat{\mathcal{B}}}(y_j)) - (\eta_{\hat{\mathcal{A}}}(y_j) - \eta_{\hat{\mathcal{B}}}(y_j)) - (\nu_{\hat{\mathcal{A}}}(y_j) - \nu_{\hat{\mathcal{B}}}(y_j))|^p = |\{t(\xi_{\hat{\mathcal{B}}}(y_j) - \xi_{\hat{\mathcal{A}}}(y_j)) - (\eta_{\hat{\mathcal{B}}}(y_j) - \eta_{\hat{\mathcal{A}}}(y_j)) - (\nu_{\hat{\mathcal{B}}}(y_j) - \nu_{\hat{\mathcal{A}}}(y_j))\}|^p, \quad (5)$$

$$|t(\eta_{\hat{\mathcal{A}}}(y_j) - \eta_{\hat{\mathcal{B}}}(y_j)) - (\xi_{\hat{\mathcal{A}}}(y_j) - \xi_{\hat{\mathcal{B}}}(y_j)) + (\nu_{\hat{\mathcal{A}}}(y_j) - \nu_{\hat{\mathcal{B}}}(y_j))|^p = |\{t(\eta_{\hat{\mathcal{B}}}(y_j) - \eta_{\hat{\mathcal{A}}}(y_j)) - (\xi_{\hat{\mathcal{B}}}(y_j) - \xi_{\hat{\mathcal{A}}}(y_j)) + (\nu_{\hat{\mathcal{B}}}(y_j) - \nu_{\hat{\mathcal{A}}}(y_j))\}|^p, \quad (6)$$

$$|t(\nu_{\hat{\mathcal{A}}}(y_j) - \nu_{\hat{\mathcal{B}}}(y_j)) - (\xi_{\hat{\mathcal{A}}}(y_j) - \xi_{\hat{\mathcal{B}}}(y_j)) + (\eta_{\hat{\mathcal{A}}}(y_j) - \eta_{\hat{\mathcal{B}}}(y_j))|^p = |\{t(\nu_{\hat{\mathcal{B}}}(y_j) - \nu_{\hat{\mathcal{A}}}(y_j)) - (\xi_{\hat{\mathcal{B}}}(y_j) - \xi_{\hat{\mathcal{A}}}(y_j)) + (\eta_{\hat{\mathcal{B}}}(y_j) - \eta_{\hat{\mathcal{A}}}(y_j))\}|^p. \quad (7)$$

Thus, $\hat{\mathcal{D}}(\hat{\mathcal{A}},\hat{\mathcal{B}}) = \hat{\mathcal{D}}(\hat{\mathcal{B}},\hat{\mathcal{A}})$.

(**D4**) Let $\hat{\mathcal{A}},\hat{\mathcal{B}}$ and $\hat{\mathcal{C}}$ are three *PFSs*. The distance between $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$, and $\hat{\mathcal{A}}$ and $\hat{\mathcal{C}}$ are the following:

$$\hat{\mathcal{D}}(\hat{\mathcal{A}},\hat{\mathcal{B}}) = \left[\frac{1}{3m(t+1)^p}\sum_{j=1}^{m}\left(|t(\xi_{\hat{\mathcal{A}}}(y_j)-\xi_{\hat{\mathcal{B}}}(y_j)) - (\eta_{\hat{\mathcal{A}}}(y_j)-\eta_{\hat{\mathcal{B}}}(y_j)) -(\nu_{\hat{\mathcal{A}}}(y_j)-\nu_{\hat{\mathcal{B}}}(y_j))|^p + |t(\eta_{\hat{\mathcal{A}}}(y_j)-\eta_{\hat{\mathcal{B}}}(y_j)) - (\xi_{\hat{\mathcal{A}}}(y_j)-\xi_{\hat{\mathcal{B}}}(y_j)) +(\nu_{\hat{\mathcal{A}}}(y_j)-\nu_{\hat{\mathcal{B}}}(y_j))|^p + |t(\nu_{\hat{\mathcal{A}}}(y_j)-\nu_{\hat{\mathcal{B}}}(y_j)) - (\xi_{\hat{\mathcal{A}}}(y_j) -\xi_{\hat{\mathcal{B}}}(y_j)) + (\eta_{\hat{\mathcal{A}}}(y_j)-\eta_{\hat{\mathcal{B}}}(y_j))|^p\right)\right.$$

$$\hat{\mathcal{D}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) = \left[\frac{1}{3m(t+1)^p}\sum_{j=1}^{m}\left(|t(\xi_{\hat{\mathcal{A}}}(y_j)-\xi_{\hat{\mathcal{C}}}(y_j)) - (\eta_{\hat{\mathcal{A}}}(y_j)-\eta_{\hat{\mathcal{C}}}(y_j)) -(\nu_{\hat{\mathcal{A}}}(y_j)-\nu_{\hat{\mathcal{C}}}(y_j))|^p + |t(\eta_{\hat{\mathcal{A}}}(y_j)-\eta_{\hat{\mathcal{C}}}(y_j)) - (\xi_{\hat{\mathcal{A}}}(y_j) -\xi_{\hat{\mathcal{C}}}(y_j)) + (\nu_{\hat{\mathcal{A}}}(y_j)-\nu_{\hat{\mathcal{C}}}(y_j))|^p + |t(\nu_{\hat{\mathcal{A}}}(y_j)-\nu_{\hat{\mathcal{C}}}(y_j)) - (\xi_{\hat{\mathcal{A}}}(y_j) -\xi_{\hat{\mathcal{C}}}(y_j)) + (\eta_{\hat{\mathcal{A}}}(y_j)-\eta_{\hat{\mathcal{C}}}(y_j))|^p\right)\right.$$

We can write the following equations:

$$|t(\xi_{\hat{\mathcal{A}}}(y_j)-\xi_{\hat{\mathcal{B}}}(y_j)) - (\eta_{\hat{\mathcal{A}}}(y_j)-\eta_{\hat{\mathcal{B}}}(y_j)) - (\nu_{\hat{\mathcal{A}}}(y_j)-\nu_{\hat{\mathcal{B}}}(y_j))| = |(t\xi_{\hat{\mathcal{A}}}(y_j)-\eta_{\hat{\mathcal{A}}}(y_j)-\nu_{\hat{\mathcal{A}}}(y_j)) - (t\xi_{\hat{\mathcal{B}}}(y_j)-\eta_{\hat{\mathcal{B}}}(y_j)-\nu_{\hat{\mathcal{B}}}(y_j))|,$$

$$|t(\eta_{\hat{\mathcal{A}}}(y_j)-\eta_{\hat{\mathcal{B}}}(y_j)) - (\xi_{\hat{\mathcal{A}}}(y_j)-\xi_{\hat{\mathcal{B}}}(y_j)) + (\nu_{\hat{\mathcal{A}}}(y_j)-\nu_{\hat{\mathcal{B}}}(y_j))| = |(t\eta_{\hat{\mathcal{A}}}(y_j)+\nu_{\hat{\mathcal{A}}}(y_j)-\xi_{\hat{\mathcal{A}}}(y_j)) - (t\eta_{\hat{\mathcal{B}}}(y_j))+\nu_{\hat{\mathcal{B}}}(y_j)-\xi_{\hat{\mathcal{B}}}(y_j))|,$$

$$|t(\nu_{\hat{\mathcal{A}}}(y_j)-\nu_{\hat{\mathcal{B}}}(y_j)) - (\xi_{\hat{\mathcal{A}}}(y_j)-\xi_{\hat{\mathcal{B}}}(y_j)) + (\eta_{\hat{\mathcal{A}}}(y_j)-\eta_{\hat{\mathcal{B}}}(y_j))| = |(t\nu_{\hat{\mathcal{A}}}(y_j)+\eta_{\hat{\mathcal{A}}}(y_j)-\xi_{\hat{\mathcal{A}}}(y_j)) - (t\nu_{\hat{\mathcal{B}}}(y_j))+\eta_{\hat{\mathcal{B}}}(y_j)-\xi_{\hat{\mathcal{B}}}(y_j))|,$$

$$|t(\xi_{\hat{\mathcal{A}}}(y_j)-\xi_{\hat{\mathcal{C}}}(y_j)) - (\eta_{\hat{\mathcal{A}}}(y_j)-\eta_{\hat{\mathcal{C}}}(y_j)) - (\nu_{\hat{\mathcal{A}}}(y_j)-\nu_{\hat{\mathcal{C}}}(y_j))| = |(t\xi_{\hat{\mathcal{A}}}(y_j)-\eta_{\hat{\mathcal{A}}}(y_j)-\nu_{\hat{\mathcal{A}}}(y_j)) - (t\xi_{\hat{\mathcal{C}}}(y_j)-\eta_{\hat{\mathcal{C}}}(y_j)-\nu_{\hat{\mathcal{C}}}(y_j))|,$$

$$|t(\eta_{\hat{\mathcal{A}}}(y_j)-\eta_{\hat{\mathcal{C}}}(y_j)) - (\xi_{\hat{\mathcal{A}}}(y_j)-\xi_{\hat{\mathcal{C}}}(y_j)) + (\nu_{\hat{\mathcal{A}}}(y_j)-\nu_{\hat{\mathcal{C}}}(y_j))| = |(t\eta_{\hat{\mathcal{A}}}(y_j)+\nu_{\hat{\mathcal{A}}}(y_j)-\xi_{\hat{\mathcal{A}}}(y_j)) - (t\eta_{\hat{\mathcal{C}}}(y_j))+\nu_{\hat{\mathcal{C}}}(y_j)-\xi_{\hat{\mathcal{C}}}(y_j))|,$$

$$|t(\nu_{\hat{\mathcal{A}}}(y_j)-\nu_{\hat{\mathcal{C}}}(y_j)) - (\xi_{\hat{\mathcal{A}}}(y_j)-\xi_{\hat{\mathcal{C}}}(y_j)) + (\eta_{\hat{\mathcal{A}}}(y_j)-\eta_{\hat{\mathcal{C}}}(y_j))| = |(t\nu_{\hat{\mathcal{A}}}(y_j)+\eta_{\hat{\mathcal{A}}}(y_j)-\xi_{\hat{\mathcal{A}}}(y_j)) - (t\nu_{\hat{\mathcal{C}}}(y_j))+\eta_{\hat{\mathcal{C}}}(y_j)-\xi_{\hat{\mathcal{C}}}(y_j))|.$$

If $\hat{\mathcal{A}}\subseteq\hat{\mathcal{B}}\subseteq\hat{\mathcal{C}}$, then $0 \leqslant \xi_{\hat{\mathcal{A}}}(y_j) \leqslant \xi_{\hat{\mathcal{B}}}(y_j) \leqslant \xi_{\hat{\mathcal{C}}}(y_j) \leqslant 1, 1 \geqslant \eta_{\hat{\mathcal{A}}}(y_j) \geqslant \eta_{\hat{\mathcal{B}}}(y_j) \geqslant \eta_{\hat{\mathcal{C}}}(y_j) \geqslant 0$ and $1 \geqslant \nu_{\hat{\mathcal{A}}}(y_j) \geqslant \nu_{\hat{\mathcal{B}}}(y_j) \geqslant \nu_{\hat{\mathcal{C}}}(y_j) \geqslant 0$. Therefore we have the following inequalities:

$$(t\xi_{\hat{\mathcal{C}}}(y_j)-\eta_{\hat{\mathcal{C}}}(y_j)-\nu_{\hat{\mathcal{C}}}(y_j)) \geqslant (t\xi_{\hat{\mathcal{B}}}(y_j)-\eta_{\hat{\mathcal{B}}}(y_j)-\nu_{\hat{\mathcal{B}}}(y_j)) \geqslant (t\xi_{\hat{\mathcal{A}}}(y_j)-\eta_{\hat{\mathcal{A}}}(y_j)-\nu_{\hat{\mathcal{A}}}(y_j))$$

$$(t\eta_{\hat{\mathcal{A}}}(y_j)+\nu_{\hat{\mathcal{A}}}(y_j)-\xi_{\hat{\mathcal{A}}}(y_j)) \geqslant (t\eta_{\hat{\mathcal{B}}}(y_j))+\nu_{\hat{\mathcal{B}}}(y_j)-\xi_{\hat{\mathcal{B}}}(y_j)) \geqslant (t\eta_{\hat{\mathcal{C}}}(y_j))+\nu_{\hat{\mathcal{C}}}(y_j)-\xi_{\hat{\mathcal{C}}}(y_j))$$

$$(t\nu_{\hat{\mathcal{A}}}(y_j)+\eta_{\hat{\mathcal{A}}}(y_j)-\xi_{\hat{\mathcal{A}}}(y_j)) \geqslant (t\nu_{\hat{\mathcal{B}}}(y_j))+\eta_{\hat{\mathcal{B}}}(y_j)-\xi_{\hat{\mathcal{B}}}(y_j)) \geqslant (t\nu_{\hat{\mathcal{C}}}(y_j))+\eta_{\hat{\mathcal{C}}}(y_j)-\xi_{\hat{\mathcal{C}}}(y_j)).$$

So it is easy to see that:

$$|(t\xi_{\hat{\mathcal{A}}}(y_j)-\eta_{\hat{\mathcal{A}}}(y_j)-\nu_{\hat{\mathcal{A}}}(y_j)) - (t\xi_{\hat{\mathcal{C}}}(y_j)-\eta_{\hat{\mathcal{C}}}(y_j)-\nu_{\hat{\mathcal{C}}}(y_j))|^p \geqslant |(t\xi_{\hat{\mathcal{A}}}(y_j)-\eta_{\hat{\mathcal{A}}}(y_j)-\nu_{\hat{\mathcal{A}}}(y_j)) - (t\xi_{\hat{\mathcal{B}}}(y_j)-\eta_{\hat{\mathcal{B}}}(y_j)-\nu_{\hat{\mathcal{B}}}(y_j))|^p, \quad (8)$$

$$|(t\eta_{\hat{\mathcal{A}}}(y_j)+\nu_{\hat{\mathcal{A}}}(y_j)-\xi_{\hat{\mathcal{A}}}(y_j)) - (t\eta_{\hat{\mathcal{C}}}(y_j))+\nu_{\hat{\mathcal{C}}}(y_j)-\xi_{\hat{\mathcal{C}}}(y_j))|^p \geqslant |(t\eta_{\hat{\mathcal{A}}}(y_j)+\nu_{\hat{\mathcal{A}}}(y_j)-\xi_{\hat{\mathcal{A}}}(y_j)) - (t\eta_{\hat{\mathcal{B}}}(y_j))+\nu_{\hat{\mathcal{B}}}(y_j) - \xi_{\hat{\mathcal{B}}}(y_j))|^p, \quad (9)$$

$$|(t\nu_{\hat{\mathcal{A}}}(y_j)+\eta_{\hat{\mathcal{A}}}(y_j)-\xi_{\hat{\mathcal{A}}}(y_j)) - (t\nu_{\hat{\mathcal{C}}}(y_j))+\eta_{\hat{\mathcal{C}}}(y_j)-\xi_{\hat{\mathcal{C}}}(y_j))|^p \geqslant |(t\nu_{\hat{\mathcal{A}}}(y_j)+\eta_{\hat{\mathcal{A}}}(y_j)-\xi_{\hat{\mathcal{A}}}(y_j)) - (t\nu_{\hat{\mathcal{B}}}(y_j))+\eta_{\hat{\mathcal{B}}}(y_j) - \xi_{\hat{\mathcal{B}}}(y_j))|^p, \quad (10)$$

and this completes the fourth property for distance measure, i.e., $\hat{\mathcal{D}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) \geqslant \hat{\mathcal{D}}(\hat{\mathcal{A}},\hat{\mathcal{B}})$ and $\hat{\mathcal{D}}(\hat{\mathcal{A}},\hat{\mathcal{C}}) \geqslant \hat{\mathcal{D}}(\hat{\mathcal{B}},\hat{\mathcal{C}})$. □

Definition 3.5. The weighted bi-parametric distance measure between two *PFSs* $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$ is a mapping $\hat{\mathcal{D}} : \hat{\mathcal{A}}\times\hat{\mathcal{B}} \to [0,1]$ which is defined as

$$\hat{\mathcal{D}}_\omega(\hat{\mathcal{A}},\hat{\mathcal{B}}) = \left[\frac{1}{3m(t+1)^p}\sum_{j=1}^{m}\hat{\omega}_j\left(|t(\xi_{\hat{\mathcal{A}}}(y_j)-\xi_{\hat{\mathcal{B}}}(y_j)) - (\eta_{\hat{\mathcal{A}}}(y_j) -\eta_{\hat{\mathcal{B}}}(y_j)) - (\nu_{\hat{\mathcal{A}}}(y_j)-\nu_{\hat{\mathcal{B}}}(y_j))|^p + |t(\eta_{\hat{\mathcal{A}}}(y_j)-\eta_{\hat{\mathcal{B}}}(y_j)) -(\xi_{\hat{\mathcal{A}}}(y_j)-\xi_{\hat{\mathcal{B}}}(y_j)) + (\nu_{\hat{\mathcal{A}}}(y_j)-\nu_{\hat{\mathcal{B}}}(y_j))|^p + |t(\nu_{\hat{\mathcal{A}}}(y_j) -\nu_{\hat{\mathcal{B}}}(y_j)) - (\xi_{\hat{\mathcal{A}}}(y_j)-\xi_{\hat{\mathcal{B}}}(y_j)) + (\eta_{\hat{\mathcal{A}}}(y_j)-\eta_{\hat{\mathcal{B}}}(y_j))|^p\right)\right.$$

where $t = 3,4,\ldots, p = 1,2,3,\ldots$ are the level of uncertainty and L_p norm and $\hat{\omega} = \{\hat{\omega}_1,\hat{\omega}_2,\ldots,\hat{\omega}_m\}$ is the weight vector with each $\hat{\omega}_j \in [0,1]$ and $\sum_{j=1}^{m}\hat{\omega}_j = 1$.

Theorem 3.6. $\hat{\mathcal{D}}_\omega(\hat{\mathcal{A}},\hat{\mathcal{B}})$ is the distance measure between two *PFSs* $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$ in $\hat{\mathcal{Y}}$.

Proof. (**D1**) If we multiply $\hat{\omega}_j$ with inequalities (2)–(4), then we have

$$0 \leqslant \hat{\omega}_j|(t\xi_{\hat{\mathcal{A}}}(y_j)-\eta_{\hat{\mathcal{A}}}(y_j)-\nu_{\hat{\mathcal{A}}}(y_j)) - (t\xi_{\hat{\mathcal{B}}}(y_j)-\eta_{\hat{\mathcal{B}}}(y_j)-\nu_{\hat{\mathcal{B}}}(y_j))|^p \leqslant \hat{\omega}_j(1+t)^p$$

$$0 \leqslant \hat{\omega}_j|(t\eta_{\hat{\mathcal{A}}}(y_j)+\nu_{\hat{\mathcal{A}}}(y_j)-\xi_{\hat{\mathcal{A}}}(y_j)) - (t\eta_{\hat{\mathcal{B}}}(y_j)+\nu_{\hat{\mathcal{B}}}(y_j)-\xi_{\hat{\mathcal{B}}}(y_j))|^p \leqslant \hat{\omega}_j(1+t)^p$$

$$0 \leqslant \hat{\omega}_j|(t\eta_{\hat{\mathcal{A}}}(y_j)+\nu_{\hat{\mathcal{A}}}(y_j)-\xi_{\hat{\mathcal{A}}}(y_j)) - (t\eta_{\hat{\mathcal{B}}}(y_j)+\nu_{\hat{\mathcal{B}}}(y_j)-\xi_{\hat{\mathcal{B}}}(y_j))|^p \leqslant \hat{\omega}_j(1+t)^p.$$

For all $y_j \in \hat{\mathcal{Y}}$ $j = 1,\ldots,m$, we have

$$0 \leqslant \sum_{j=1}^{m}\hat{\omega}_j|(t\xi_{\hat{\mathcal{A}}}(y_j)-\eta_{\hat{\mathcal{A}}}(y_j)-\nu_{\hat{\mathcal{A}}}(y_j)) - (t\xi_{\hat{\mathcal{B}}}(y_j)-\eta_{\hat{\mathcal{B}}}(y_j) - \nu_{\hat{\mathcal{B}}}(y_j))|^p \leqslant \sum_{j=1}^{m}\hat{\omega}_j(1+t)^p 0 \leqslant \sum_{j=1}^{m}\hat{\omega}_j|(t\eta_{\hat{\mathcal{A}}}(y_j)+\nu_{\hat{\mathcal{A}}}(y_j)-\xi_{\hat{\mathcal{A}}}(y_j)) - (t\eta_{\hat{\mathcal{B}}}(y_j)+\nu_{\hat{\mathcal{B}}}(y_j)-\xi_{\hat{\mathcal{B}}}(y_j))|^p \leqslant \sum_{j=1}^{m}\hat{\omega}_j(1+t)^p 0$$

$$\leqslant \sum_{j=1}^{m}\hat{\omega}_j|(t\eta_{\hat{\mathcal{A}}}(y_j)+\nu_{\hat{\mathcal{A}}}(y_j)-\xi_{\hat{\mathcal{A}}}(y_j)) - (t\eta_{\hat{\mathcal{B}}}(y_j)+\nu_{\hat{\mathcal{B}}}(y_j) - \xi_{\hat{\mathcal{B}}}(y_j))|^p \leqslant \sum_{j=1}^{m}\hat{\omega}_j(1+t)^p.$$

Since $\sum_{j=1}^{m}\hat{\omega}_j = 1$, therefore $\sum_{j=1}^{m}\hat{\omega}_j(1+t)^p$ is equal to the $(1+t)^p$ and after addition of the above equations, we get

$$0 \leqslant \left[\frac{1}{3m(t+1)^p}\sum_{j=1}^{m}\hat{\omega}_j\left(|t(\xi_{\hat{\mathcal{A}}}(y_j)-\xi_{\hat{\mathcal{B}}}(y_j)) - (\eta_{\hat{\mathcal{A}}}(y_j)-\eta_{\hat{\mathcal{B}}}(y_j)) -(\nu_{\hat{\mathcal{A}}}(y_j)-\nu_{\hat{\mathcal{B}}}(y_j))|^p + |t(\eta_{\hat{\mathcal{A}}}(y_j)-\eta_{\hat{\mathcal{B}}}(y_j)) - (\xi_{\hat{\mathcal{A}}}(y_j)-\xi_{\hat{\mathcal{B}}}(y_j)) +(\nu_{\hat{\mathcal{A}}}(y_j)-\nu_{\hat{\mathcal{B}}}(y_j))|^p + |t(\nu_{\hat{\mathcal{A}}}(y_j)-\nu_{\hat{\mathcal{B}}}(y_j)) - (\xi_{\hat{\mathcal{A}}}(y_j)-\xi_{\hat{\mathcal{B}}}(y_j)) +(\eta_{\hat{\mathcal{A}}}(y_j)-\eta_{\hat{\mathcal{B}}}(y_j))|^p\right)\right]^{\frac{1}{p}} \leqslant 1$$

$0 \leqslant \hat{\mathcal{D}}_{\omega}(\hat{\mathcal{A}}, \hat{\mathcal{B}}) \leqslant 1$.

(**D2**) Since for two *PFSs* $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$, if $\hat{\mathcal{A}} = \hat{\mathcal{B}}$, then $\xi_{\hat{\mathcal{A}}}(y_j) = \xi_{\hat{\mathcal{B}}}(y_j), \eta_{\hat{\mathcal{A}}}(y_j) = \eta_{\hat{\mathcal{B}}}(y_j)$ and $\nu_{\hat{\mathcal{A}}}(y_j) = \nu_{\hat{\mathcal{B}}}(y_j)$, therefore, $\xi_{\hat{\mathcal{A}}}(y_j) - \xi_{\hat{\mathcal{B}}}(y_j) = 0, \eta_{\hat{\mathcal{A}}}(y_j) - \eta_{\hat{\mathcal{B}}}(y_j) = 0$ and $\nu_{\hat{\mathcal{A}}}(y_j) - \nu_{\hat{\mathcal{B}}}(y_j) = 0$. Therefore, the weighted distance measure, $\hat{\mathcal{D}}_{\omega}(\hat{\mathcal{A}}, \hat{\mathcal{B}})$, is equal to zero.

(**D3**) If we multiply $\hat{\omega}_j$ with Eqs. (5)–(7), then we have

$$\hat{\omega}_j|t(\xi_{\hat{\mathcal{A}}}(y_j) - \xi_{\hat{\mathcal{B}}}(y_j)) - (\eta_{\hat{\mathcal{A}}}(y_j) - \eta_{\hat{\mathcal{B}}}(y_j)) - (\nu_{\hat{\mathcal{A}}}(y_j) - \nu_{\hat{\mathcal{B}}}(y_j))|^p$$
$$= \hat{\omega}_j|\{t(\xi_{\hat{\mathcal{B}}}(y_j) - \xi_{\hat{\mathcal{A}}}(y_j)) - (\eta_{\hat{\mathcal{B}}}(y_j) - \eta_{\hat{\mathcal{A}}}(y_j)) - (\nu_{\hat{\mathcal{B}}}(y_j) - \nu_{\hat{\mathcal{A}}}(y_j))\}|^p,$$

$$\hat{\omega}_j|t(\eta_{\hat{\mathcal{A}}}(y_j) - \eta_{\hat{\mathcal{B}}}(y_j)) - (\xi_{\hat{\mathcal{A}}}(y_j) - \xi_{\hat{\mathcal{B}}}(y_j)) + (\nu_{\hat{\mathcal{A}}}(y_j) - \nu_{\hat{\mathcal{B}}}(y_j))|^p$$
$$= \hat{\omega}_j|\{t(\eta_{\hat{\mathcal{B}}}(y_j) - \eta_{\hat{\mathcal{A}}}(y_j)) - (\xi_{\hat{\mathcal{B}}}(y_j) - \xi_{\hat{\mathcal{A}}}(y_j)) + (\nu_{\hat{\mathcal{B}}}(y_j) - \nu_{\hat{\mathcal{A}}}(y_j))\}|^p,$$

$$\hat{\omega}_j|t(\nu_{\hat{\mathcal{A}}}(y_j) - \nu_{\hat{\mathcal{B}}}(y_j)) - (\xi_{\hat{\mathcal{A}}}(y_j) - \xi_{\hat{\mathcal{B}}}(y_j)) + (\eta_{\hat{\mathcal{A}}}(y_j) - \eta_{\hat{\mathcal{B}}}(y_j))|^p$$
$$= \hat{\omega}_j|\{t(\nu_{\hat{\mathcal{B}}}(y_j) - \nu_{\hat{\mathcal{A}}}(y_j)) - (\xi_{\hat{\mathcal{B}}}(y_j) - \xi_{\hat{\mathcal{A}}}(y_j)) + (\eta_{\hat{\mathcal{B}}}(y_j) - \eta_{\hat{\mathcal{A}}}(y_j))\}|^p.$$

Thus, $\hat{\mathcal{D}}_{\omega}(\hat{\mathcal{A}}, \hat{\mathcal{B}}) = \hat{\mathcal{D}}_{\omega}(\hat{\mathcal{B}}, \hat{\mathcal{A}})$.

(**D4**) If we multiply $\hat{\omega}_j$ with inequalities (8)–(10), then we have

$$\hat{\omega}_j|(t\xi_{\hat{\mathcal{A}}}(y_j) - \eta_{\hat{\mathcal{A}}}(y_j) - \nu_{\hat{\mathcal{A}}}(y_j)) - (t\xi_{\hat{\mathcal{C}}}(y_j) - \eta_{\hat{\mathcal{C}}}(y_j) - \nu_{\hat{\mathcal{C}}}(y_j))|^p$$
$$\geqslant \hat{\omega}_j|(t\xi_{\hat{\mathcal{A}}}(y_j) - \eta_{\hat{\mathcal{A}}}(y_j) - \nu_{\hat{\mathcal{A}}}(y_j)) - (t\xi_{\hat{\mathcal{B}}}(y_j) - \eta_{\hat{\mathcal{B}}}(y_j) - \nu_{\hat{\mathcal{B}}}(y_j))|^p,$$

$$\hat{\omega}_j|(t\eta_{\hat{\mathcal{A}}}(y_j) + \nu_{\hat{\mathcal{A}}}(y_j) - \xi_{\hat{\mathcal{A}}}(y_j)) - (t\eta_{\hat{\mathcal{C}}}(y_j)) + \nu_{\hat{\mathcal{C}}}(y_j) - \xi_{\hat{\mathcal{C}}}(y_j))|^p$$
$$\geqslant \hat{\omega}_j|(t\eta_{\hat{\mathcal{A}}}(y_j) + \nu_{\hat{\mathcal{A}}}(y_j) - \xi_{\hat{\mathcal{A}}}(y_j)) - (t\eta_{\hat{\mathcal{B}}}(y_j)) + \nu_{\hat{\mathcal{B}}}(y_j) - \xi_{\hat{\mathcal{B}}}(y_j))|^p,$$

$$\hat{\omega}_j|(t\nu_{\hat{\mathcal{A}}}(y_j) + \eta_{\hat{\mathcal{A}}}(y_j) - \xi_{\hat{\mathcal{A}}}(y_j)) - (t\nu_{\hat{\mathcal{C}}}(y_j)) + \eta_{\hat{\mathcal{C}}}(y_j) - \xi_{\hat{\mathcal{C}}}(y_j))|^p$$
$$\geqslant \hat{\omega}_j|(t\nu_{\hat{\mathcal{A}}}(y_j) + \eta_{\hat{\mathcal{A}}}(y_j) - \xi_{\hat{\mathcal{A}}}(y_j)) - (t\nu_{\hat{\mathcal{B}}}(y_j)) + \eta_{\hat{\mathcal{B}}}(y_j) - \xi_{\hat{\mathcal{B}}}(y_j))|^p,$$

since $\hat{\omega}_j \geqslant 0$, therefore, we have the fourth property for the weighted distance measure i.e., $\hat{\mathcal{D}}_{\omega}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) \geqslant \hat{\mathcal{D}}_{\omega}(\hat{\mathcal{A}}, \hat{\mathcal{B}})$ and $\hat{\mathcal{D}}_{\omega}(\hat{\mathcal{A}}, \hat{\mathcal{C}}) \geqslant \hat{\mathcal{D}}_{\omega}(\hat{\mathcal{B}}, \hat{\mathcal{C}})$. Hence from (**D1**) – (**D4**), $\hat{\mathcal{D}}_{\omega}(\hat{\mathcal{A}}, \hat{\mathcal{B}})$ is a distance measure. □

Definition 3.7. A bi-parametric similarity measure between two *PFSs* $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$ is a mapping $\hat{\mathcal{S}} : \hat{\mathcal{A}} \times \hat{\mathcal{B}} \to [0, 1]$ which is defined as

$$\hat{\mathcal{S}}(\hat{\mathcal{A}}, \hat{\mathcal{B}}) = 1 - \left[\frac{1}{3m(t+1)^p}\sum_{j=1}^{m}\left(|t(\xi_{\hat{\mathcal{A}}}(y_j) - \xi_{\hat{\mathcal{B}}}(y_j)) - (\eta_{\hat{\mathcal{A}}}(y_j) - \eta_{\hat{\mathcal{B}}}(y_j)) - (\nu_{\hat{\mathcal{A}}}(y_j) - \nu_{\hat{\mathcal{B}}}(y_j))|^p + |t(\eta_{\hat{\mathcal{A}}}(y_j) - \eta_{\hat{\mathcal{B}}}(y_j)) - (\xi_{\hat{\mathcal{A}}}(y_j) - \xi_{\hat{\mathcal{B}}}(y_j)) + (\nu_{\hat{\mathcal{A}}}(y_j) - \nu_{\hat{\mathcal{B}}}(y_j))|^p + |t(\nu_{\hat{\mathcal{A}}}(y_j) - \nu_{\hat{\mathcal{B}}}(y_j)) - (\xi_{\hat{\mathcal{A}}}(y_j) - \xi_{\hat{\mathcal{B}}}(y_j)) + (\eta_{\hat{\mathcal{A}}}(y_j) - \eta_{\hat{\mathcal{B}}}(y_j))|^p\right)\right]^{\frac{1}{p}}, \quad (11)$$

where $t = 3, 4, \ldots$ and $p = 1, 2, 3, \ldots$ are called the level of uncertainty and L_p norm, respectively.

Definition 3.8. The weighted bi-parametric similarity measure between two *PFSs* $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$ is a mapping $\hat{\mathcal{S}} : \hat{\mathcal{A}} \times \hat{\mathcal{B}} \to [0, 1]$ which is defined as

$$\hat{\mathcal{S}}_{\omega}(\hat{\mathcal{A}}, \hat{\mathcal{B}}) = 1 - \left[\frac{1}{3m(t+1)^p}\sum_{j=1}^{m}\hat{\omega}_j\left(|t(\xi_{\hat{\mathcal{A}}}(y_j) - \xi_{\hat{\mathcal{B}}}(y_j)) - (\eta_{\hat{\mathcal{A}}}(y_j) - \eta_{\hat{\mathcal{B}}}(y_j)) - (\nu_{\hat{\mathcal{A}}}(y_j) - \nu_{\hat{\mathcal{B}}}(y_j))|^p + |t(\eta_{\hat{\mathcal{A}}}(y_j) - \eta_{\hat{\mathcal{B}}}(y_j)) - (\xi_{\hat{\mathcal{A}}}(y_j)) - \xi_{\hat{\mathcal{B}}}(y_j)) + (\nu_{\hat{\mathcal{A}}}(y_j) - \nu_{\hat{\mathcal{B}}}(y_j))|^p + |t(\nu_{\hat{\mathcal{A}}}(y_j) - \nu_{\hat{\mathcal{B}}}(y_j)) - (\xi_{\hat{\mathcal{A}}}(y_j) - \xi_{\hat{\mathcal{B}}}(y_j)) + (\eta_{\hat{\mathcal{A}}}(y_j) - \eta_{\hat{\mathcal{B}}}(y_j))|^p\right)\right]^{\frac{1}{p}},$$

where $t = 3, 4, \ldots, p = 1, 2, 3, \ldots$ are called the level of uncertainty and L_p norm, respectively and $\hat{\omega} = \{\hat{\omega}_1, \hat{\omega}_2, \ldots, \hat{\omega}_m\}$ is the weight vector with each $\hat{\omega}_j \in [0, 1]$ and $\sum_{j=1}^{m}\hat{\omega}_j = 1$.

Theorem 3.9. $\hat{\mathcal{S}}(\hat{\mathcal{A}}, \hat{\mathcal{B}})$ and $\hat{\mathcal{S}}_{\omega}(\hat{\mathcal{A}}, \hat{\mathcal{B}})$ are similarity measure between two *PFSs* $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$ in $\hat{\mathcal{Y}}$.

4. Interpretation of novel distance measure for *PFSs*

The interpretation of novel distance measure and functionality of level of uncertainty t are explained in this section.

Let $(\xi_{\hat{\mathcal{A}}}, \eta_{\hat{\mathcal{A}}}, \nu_{\hat{\mathcal{A}}})$ and $(\xi_{\hat{\mathcal{B}}}, \eta_{\hat{\mathcal{B}}}, \nu_{\hat{\mathcal{B}}})$ be two *PFNs*. $\xi_{\hat{\mathcal{A}}}$ takes the value from $[\xi_{\hat{\mathcal{A}}}, \pi_{\hat{\mathcal{A}}}], \eta_{\hat{\mathcal{A}}}$ takes the value from $[\eta_{\hat{\mathcal{A}}}, \pi_{\hat{\mathcal{A}}}]$ and $\nu_{\hat{\mathcal{A}}}$ takes the value from $[\nu_{\hat{\mathcal{A}}}, \pi_{\hat{\mathcal{A}}}]$. Shaded area of Fig. 1 represents the all possible values for $(\xi_{\hat{\mathcal{A}}}, \eta_{\hat{\mathcal{A}}}, \nu_{\hat{\mathcal{A}}})$.

Undoubtedly, the center of gravity (centroid) G of tetrahedron *ABCD* in Fig. 1 is the most informative point among all the possible points. The center of gravity $(\xi'_{\hat{\mathcal{A}}}, \eta'_{\hat{\mathcal{A}}}, \nu'_{\hat{\mathcal{A}}})$ can be easily found as $(\xi_{\hat{\mathcal{A}}} + \frac{\pi_{\hat{\mathcal{A}}}}{4}, \eta_{\hat{\mathcal{A}}} + \frac{\pi_{\hat{\mathcal{A}}}}{4}, \nu_{\hat{\mathcal{A}}} + \frac{\pi_{\hat{\mathcal{A}}}}{4})$. Since $\pi_{\hat{\mathcal{A}}} = 1 - (\xi_{\hat{\mathcal{A}}} + \eta_{\hat{\mathcal{A}}} + \nu_{\hat{\mathcal{A}}})$, therefore, $(\xi'_{\hat{\mathcal{A}}}, \eta'_{\hat{\mathcal{A}}}, \nu'_{\hat{\mathcal{A}}})$ takes the form

$$\left(\xi_{\hat{\mathcal{A}}} + \frac{\pi_{\hat{\mathcal{A}}}}{4}, \eta_{\hat{\mathcal{A}}} + \frac{\pi_{\hat{\mathcal{A}}}}{4}, \nu_{\hat{\mathcal{A}}} + \frac{\pi_{\hat{\mathcal{A}}}}{4}\right) = \left(\frac{1 + 3\xi_{\hat{\mathcal{A}}} - \eta_{\hat{\mathcal{A}}} - \nu_{\hat{\mathcal{A}}}}{4}, \frac{1 + 3\eta_{\hat{\mathcal{A}}} - \xi_{\hat{\mathcal{A}}} - \nu_{\hat{\mathcal{A}}}}{4}, \frac{1 + 3\nu_{\hat{\mathcal{A}}} - \eta_{\hat{\mathcal{A}}} - \xi_{\hat{\mathcal{A}}}}{4}\right).$$

Similarly, we also have the pair $(\xi'_{\hat{\mathcal{B}}}, \eta'_{\hat{\mathcal{B}}}, \nu'_{\hat{\mathcal{B}}})$ for *PFN* $(\xi_{\hat{\mathcal{B}}}, \eta_{\hat{\mathcal{B}}}, \nu_{\hat{\mathcal{B}}})$ as follows:

$$\left(\xi_{\hat{\mathcal{B}}} + \frac{\pi_{\hat{\mathcal{B}}}}{4}, \eta_{\hat{\mathcal{B}}} + \frac{\pi_{\hat{\mathcal{B}}}}{4}, \nu_{\hat{\mathcal{B}}} + \frac{\pi_{\hat{\mathcal{B}}}}{4}\right) = \left(\frac{1 + 3\xi_{\hat{\mathcal{B}}} - \eta_{\hat{\mathcal{B}}} - \nu_{\hat{\mathcal{B}}}}{4}, \frac{1 + 3\eta_{\hat{\mathcal{B}}} - \xi_{\hat{\mathcal{B}}} - \nu_{\hat{\mathcal{B}}}}{4}, \frac{1 + 3\nu_{\hat{\mathcal{B}}} - \eta_{\hat{\mathcal{B}}} - \xi_{\hat{\mathcal{B}}}}{4}\right).$$

The absolute difference between $(\xi'_{\hat{\mathcal{A}}}, \eta'_{\hat{\mathcal{A}}}, \nu'_{\hat{\mathcal{A}}})$ and $(\xi'_{\hat{\mathcal{B}}}, \eta'_{\hat{\mathcal{B}}}, \nu'_{\hat{\mathcal{B}}})$ is defined in Eqs. (12)–(14):

$$|\xi'_{\hat{\mathcal{A}}} - \xi'_{\hat{\mathcal{B}}}| = \left|\frac{1 + 3\xi_{\hat{\mathcal{A}}} - \eta_{\hat{\mathcal{A}}} - \nu_{\hat{\mathcal{A}}}}{4} - \frac{1 + 3\xi_{\hat{\mathcal{B}}} - \eta_{\hat{\mathcal{B}}} - \nu_{\hat{\mathcal{B}}}}{4}\right| = \left|\frac{3(\xi_{\hat{\mathcal{A}}} - \xi_{\hat{\mathcal{B}}}) - (\eta_{\hat{\mathcal{A}}} - \eta_{\hat{\mathcal{B}}}) - (\nu_{\hat{\mathcal{A}}} - \nu_{\hat{\mathcal{B}}})}{4}\right| \quad (12)$$

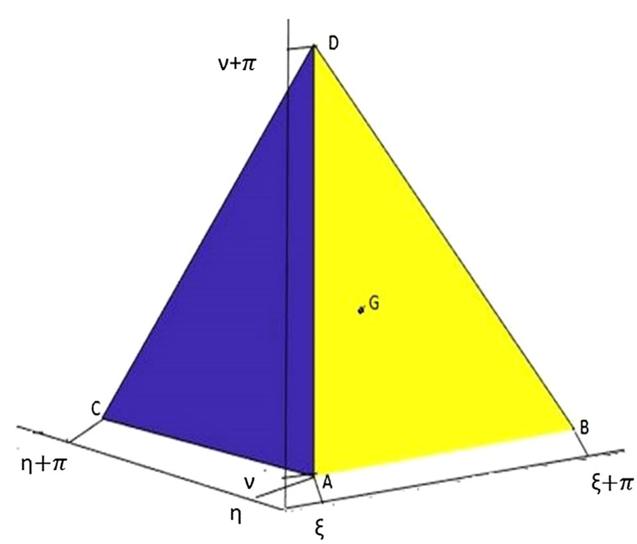

Fig. 1. The Possible Values for (ξ, η, ν).

$$\left|\eta'_{\hat{\mathcal{A}}}-\eta'_{\hat{\mathcal{B}}}\right| = \left|\frac{1+3\eta_{\hat{\mathcal{A}}}-\xi_{\hat{\mathcal{A}}}-\nu_{\hat{\mathcal{A}}}}{4}-\frac{1+3\eta_{\hat{\mathcal{B}}}-\xi_{\hat{\mathcal{B}}}-\nu_{\hat{\mathcal{B}}}}{4}\right| = \left|\frac{3(\eta_{\hat{\mathcal{A}}}-\eta_{\hat{\mathcal{B}}})-(\xi_{\hat{\mathcal{A}}}-\xi_{\hat{\mathcal{B}}})-(\nu_{\hat{\mathcal{A}}}-\nu_{\hat{\mathcal{B}}})}{4}\right| \quad (13)$$

$$\left|\nu'_{\hat{\mathcal{A}}}-\nu'_{\hat{\mathcal{B}}}\right| = \left|\frac{1+3\nu_{\hat{\mathcal{A}}}-\eta_{\hat{\mathcal{A}}}-\xi_{\hat{\mathcal{A}}}}{4}-\frac{1+3\nu_{\hat{\mathcal{B}}}-\eta_{\hat{\mathcal{B}}}-\xi_{\hat{\mathcal{B}}}}{4}\right| = \left|\frac{3(\nu_{\hat{\mathcal{A}}}-\nu_{\hat{\mathcal{B}}})-(\xi_{\hat{\mathcal{A}}}-\xi_{\hat{\mathcal{B}}})-(\eta_{\hat{\mathcal{A}}}-\eta_{\hat{\mathcal{B}}})}{4}\right| \quad (14)$$

After tanking power p to the $|\xi'_{\hat{\mathcal{A}}}-\xi'_{\hat{\mathcal{B}}}|, |\eta'_{\hat{\mathcal{A}}}-\eta'_{\hat{\mathcal{B}}}|$ and $|\nu'_{\hat{\mathcal{A}}}-\nu'_{\hat{\mathcal{B}}}|$, we get

$$|\xi'_{\hat{\mathcal{A}}}-\xi'_{\hat{\mathcal{B}}}|^p = \frac{1}{4^p}\left|3(\xi_{\hat{\mathcal{A}}}-\xi_{\hat{\mathcal{B}}})-(\eta_{\hat{\mathcal{A}}}-\eta_{\hat{\mathcal{B}}})-(\nu_{\hat{\mathcal{A}}}-\nu_{\hat{\mathcal{B}}})\right|^p$$

$$\left|\eta'_{\hat{\mathcal{A}}}-\eta'_{\hat{\mathcal{B}}}\right|^p = \frac{1}{4^p}\left|3(\eta_{\hat{\mathcal{A}}}-\eta_{\hat{\mathcal{B}}})-(\xi_{\hat{\mathcal{A}}}-\xi_{\hat{\mathcal{B}}})-(\nu_{\hat{\mathcal{A}}}-\nu_{\hat{\mathcal{B}}})\right|^p$$

$$\left|\nu'_{\hat{\mathcal{A}}}-\nu'_{\hat{\mathcal{B}}}\right|^p = \frac{1}{4^p}\left|3(\nu_{\hat{\mathcal{A}}}-\nu_{\hat{\mathcal{B}}})-(\xi_{\hat{\mathcal{A}}}-\xi_{\hat{\mathcal{B}}})-(\eta_{\hat{\mathcal{A}}}-\eta_{\hat{\mathcal{B}}})\right|^p$$

The mean of $|\xi'_{\hat{\mathcal{A}}}-\xi'_{\hat{\mathcal{B}}}|^p, |\eta'_{\hat{\mathcal{A}}}-\eta'_{\hat{\mathcal{B}}}|^p$ and $|\nu'_{\hat{\mathcal{A}}}-\nu'_{\hat{\mathcal{B}}}|^p$ is obtained in Eq. (15) as follows:

$$\begin{aligned}&\frac{1}{3}\left\{|\xi'_{\hat{\mathcal{A}}}-\xi'_{\hat{\mathcal{B}}}|^p+|\eta'_{\hat{\mathcal{A}}}-\eta'_{\hat{\mathcal{B}}}|^p+|\nu'_{\hat{\mathcal{A}}}-\nu'_{\hat{\mathcal{B}}}|^p\right\}\\ &=\frac{1}{3\times 4^p}\Big[\left|3(\xi_{\hat{\mathcal{A}}}-\xi_{\hat{\mathcal{B}}})-(\eta_{\hat{\mathcal{A}}}-\eta_{\hat{\mathcal{B}}})-(\nu_{\hat{\mathcal{A}}}-\nu_{\hat{\mathcal{B}}})\right|^p\\ &+\left|3(\eta_{\hat{\mathcal{A}}}-\eta_{\hat{\mathcal{B}}})-(\xi_{\hat{\mathcal{A}}}-\xi_{\hat{\mathcal{B}}})-(\nu_{\hat{\mathcal{A}}}-\nu_{\hat{\mathcal{B}}})\right|^p\\ &+\left|3(\nu_{\hat{\mathcal{A}}}-\nu_{\hat{\mathcal{B}}})-(\xi_{\hat{\mathcal{A}}}-\xi_{\hat{\mathcal{B}}})-(\eta_{\hat{\mathcal{A}}}-\eta_{\hat{\mathcal{B}}})\right|^p\Big]\end{aligned} \quad (15)$$

The p root of Eq. (15) is found as:

$$\begin{aligned}&\left[\frac{1}{3}\left\{|\xi'_{\hat{\mathcal{A}}}-\xi'_{\hat{\mathcal{B}}}|^p+|\eta'_{\hat{\mathcal{A}}}-\eta'_{\hat{\mathcal{B}}}|^p+|\nu'_{\hat{\mathcal{A}}}-\nu'_{\hat{\mathcal{B}}}|^p\right\}\right]^{\frac{1}{p}}\\ &=\Big[\frac{1}{3\times 4^p}\Big(\left|3(\xi_{\hat{\mathcal{A}}}-\xi_{\hat{\mathcal{B}}})-(\eta_{\hat{\mathcal{A}}}-\eta_{\hat{\mathcal{B}}})-(\nu_{\hat{\mathcal{A}}}-\nu_{\hat{\mathcal{B}}})\right|^p\\ &\quad+\left|3(\eta_{\hat{\mathcal{A}}}-\eta_{\hat{\mathcal{B}}})-(\xi_{\hat{\mathcal{A}}}-\xi_{\hat{\mathcal{B}}})-(\nu_{\hat{\mathcal{A}}}-\nu_{\hat{\mathcal{B}}})\right|^p\\ &\quad+\left|3(\nu_{\hat{\mathcal{A}}}-\nu_{\hat{\mathcal{B}}})-(\xi_{\hat{\mathcal{A}}}-\xi_{\hat{\mathcal{B}}})-(\eta_{\hat{\mathcal{A}}}-\eta_{\hat{\mathcal{B}}})\right|^p\Big)\Big]^{\frac{1}{p}}\end{aligned} \quad (16)$$

For more than one feature such as $(y_j), 1 \leqslant j \leqslant m$, Eq. (16) can be defined in Eq. (17) as follows:

$$\begin{aligned}&\Big[\frac{1}{3m}\sum_{j=1}^{m}\{|\xi_{\hat{\mathcal{A}}}(y_j)'-\xi_{\hat{\mathcal{B}}}(y_j)'|^p+|\eta_{\hat{\mathcal{A}}}(y_j)'-\eta_{\hat{\mathcal{B}}}(y_j)'|^p\\ &+|\nu_{\hat{\mathcal{A}}}(y_j)'-\nu_{\hat{\mathcal{B}}}(y_j)'|^p\}\Big]^{\frac{1}{p}}\\ &=\Big[\frac{1}{3m\times 4^p}\sum_{j=1}^{m}\Big(|3(\xi_{\hat{\mathcal{A}}}(y_j)-\xi_{\hat{\mathcal{B}}}(y_j))-(\eta_{\hat{\mathcal{A}}}(y_j)-\eta_{\hat{\mathcal{B}}}(y_j))\\ &-(\nu_{\hat{\mathcal{A}}}(y_j)-\nu_{\hat{\mathcal{B}}}(y_j))|^p+|3(\eta_{\hat{\mathcal{A}}}(y_j)-\eta_{\hat{\mathcal{B}}}(y_j))-(\xi_{\hat{\mathcal{A}}}(y_j)-\xi_{\hat{\mathcal{B}}}(y_j))\\ &-(\nu_{\hat{\mathcal{A}}}(y_j)-\nu_{\hat{\mathcal{B}}}(y_j))|^p+|3(\nu_{\hat{\mathcal{A}}}(y_j)-\nu_{\hat{\mathcal{B}}}(y_j))-(\xi_{\hat{\mathcal{A}}}(y_j)-\xi_{\hat{\mathcal{B}}}(y_j))\\ &-(\eta_{\hat{\mathcal{A}}}(y_j)-\eta_{\hat{\mathcal{B}}}(y_j))|^p\Big)\Big]^{\frac{1}{p}}\end{aligned} \quad (17)$$

To obtain distance measure and the reliability of the proposed method, we replace the negative sign from positive sign in second and third term in Eq. (17) for neutral membership function and negative membership function. Hence the Eq. (17) takes the form:

$$\begin{aligned}&\left[\frac{1}{3m}\sum_{j=1}^{m}\{|\xi_{\hat{\mathcal{A}}}(y_j)'-\xi_{\hat{\mathcal{B}}}(y_j)'|^p+|\eta_{\hat{\mathcal{A}}}(y_j)'-\eta_{\hat{\mathcal{B}}}(y_j)'|^p+|\nu_{\hat{\mathcal{A}}}(y_j)'-\nu_{\hat{\mathcal{B}}}(y_j)'|^p\}\right]^{\frac{1}{p}}\\ &=\Big[\frac{1}{3m\times 4^p}\sum_{j=1}^{m}\Big(|3(\xi_{\hat{\mathcal{A}}}(y_j)-\xi_{\hat{\mathcal{B}}}(y_j))-(\eta_{\hat{\mathcal{A}}}(y_j)-\eta_{\hat{\mathcal{B}}}(y_j))-(\nu_{\hat{\mathcal{A}}}(y_j)-\nu_{\hat{\mathcal{B}}}(y_j))|^p\\ &+|3(\eta_{\hat{\mathcal{A}}}(y_j)-\eta_{\hat{\mathcal{B}}}(y_j))-(\xi_{\hat{\mathcal{A}}}(y_j)-\xi_{\hat{\mathcal{B}}}(y_j))+(\nu_{\hat{\mathcal{A}}}(y_j)-\nu_{\hat{\mathcal{B}}}(y_j))|^p\\ &+|3(\nu_{\hat{\mathcal{A}}}(y_j)-\nu_{\hat{\mathcal{B}}}(y_j))-(\xi_{\hat{\mathcal{A}}}(y_j)-\xi_{\hat{\mathcal{B}}}(y_j))+(\eta_{\hat{\mathcal{A}}}(y_j)-\eta_{\hat{\mathcal{B}}}(y_j))|^p\Big)\Big]^{\frac{1}{p}}\end{aligned} \quad (18)$$

It is easy to see that Eq. (18) is the special case of Eq. (1) when $t=3$. We shall have Eq. (18) as the distance measure between $(\xi_{\hat{\mathcal{A}}},\eta_{\hat{\mathcal{A}}},\nu_{\hat{\mathcal{A}}})$ and $(\xi_{\hat{\mathcal{B}}},\eta_{\hat{\mathcal{B}}},\nu_{\hat{\mathcal{B}}})$ when we parametrize $(\xi'_{\hat{\mathcal{A}}},\eta'_{\hat{\mathcal{A}}},\nu'_{\hat{\mathcal{A}}}) = (\xi_{\hat{\mathcal{A}}}+\frac{\pi_{\hat{\mathcal{A}}}}{4},\eta_{\hat{\mathcal{A}}}+\frac{\pi_{\hat{\mathcal{A}}}}{4},\nu_{\hat{\mathcal{A}}}+\frac{\pi_{\hat{\mathcal{A}}}}{4})$ and $(\xi'_{\hat{\mathcal{B}}},\eta'_{\hat{\mathcal{B}}},\nu'_{\hat{\mathcal{B}}}) = (\xi_{\hat{\mathcal{B}}}+\frac{\pi_{\hat{\mathcal{B}}}}{4},\eta_{\hat{\mathcal{B}}}+\frac{\pi_{\hat{\mathcal{B}}}}{4},\nu_{\hat{\mathcal{B}}}+\frac{\pi_{\hat{\mathcal{B}}}}{4})$. In Eq. (1), the effect of hesitation margin in the computation is adjusted by the parameter t. When we need to neglect the effect of hesitation margin, the value of t should be high and the effect of hesitation margin is taken into account when the value of t is very low in the computation.

5. Applications in pattern recognition and counter examples

In this section, we provide several counterexamples for existing similarity measures. We have seen that the already proposed measures cannot classify the unknown pattern while the bi-parametric similarity measure classifies the unknown pattern. This guarantee that our proposed similarity measure is applicable to pattern recognition problems.

Example 5.1. In this example, we have seen that the second condition of Definition 3.2 (S_2) is not satisfied for cosine similarity measure CS^1 (Definition 2.3), i.e., if $\hat{\mathcal{A}} = \{(a,a,a)/y_j | y_j \in \hat{\mathcal{Y}}, 1 \leqslant j \leqslant m\}$ and $\hat{\mathcal{B}} = \{(b,b,b)/y_j | y_j \in \hat{\mathcal{Y}}, 1 \leqslant j \leqslant m\}$ are two *PFSs* in $\hat{\mathcal{Y}}$ with $0 \leqslant a,b \leq 1, 0 \leqslant a+a+a \leq 1, 0 \leqslant b+b+b \leq 1$ and $a \neq b$, then $\hat{\mathcal{A}} \neq \hat{\mathcal{B}}$. But, we have seen that $CS^1(\hat{\mathcal{A}},\hat{\mathcal{B}}) = 1$.

For example, let $\hat{\mathcal{Y}} = \{y_1,y_2,y_3\}$ and *PFSs* in $\hat{\mathcal{Y}}$ are

$$\begin{aligned}\hat{\mathcal{A}} &= \{(0.20,0.20,0.20)/y_1,(0.30,0.30,0.30)/y_2,(0.25,0.25,0.25)/y_3\}\\ \hat{\mathcal{B}} &= \{(0.31,0.31,0.31)/y_1,(0.27,0.27,0.27)/y_2,(0.33,0.33,0.33)/y_3\}.\end{aligned}$$

Clearly, $\hat{\mathcal{A}} \neq \hat{\mathcal{B}}$ but $CS^1(\hat{\mathcal{A}},\hat{\mathcal{B}}) = 1$. Hence CS^1 is not effective for these cases and not reliable to find the similarity measure between *PFSs*. But when we find the similarity measure from proposed bi-parametric similarity measure with $t=3$ and $p=2$, we get $S^p_t(\hat{\mathcal{A}},\hat{\mathcal{B}}) = 0.949406$.

Example 5.2. Let Q_1 and Q_2 be two known patterns with class labels Z_1 and Z_2, respectively, are given. The *PFSs* are used to represents the patterns in $\hat{\mathcal{Y}} = \{y_1,y_2,y_3\}$ as follows:

$$\begin{aligned}Q_1 &= \{(0.20,0.20,0.10)/y_1,(0.15,0.15,0.15)/y_2,(0.25,0.15,0.10)/y_3\}\\ Q_2 &= \{(0.30,0.10,0.10)/y_1,(0.20,0.20,0.05)/y_2,(0.20,0.30,0.00)/y_3\}.\end{aligned}$$

P is the unknown pattern which is given as follows:

$$P = \{(0.40,0.40,0.20)/y_1,(0.30,0.30,0.30)/y_2,(0.50,0.30,0.20)/y_3\}.$$

Our aimed to find out the class of unknown pattern P belongs to. When we use set theoretic similarity measure STM (Definition 2.10), we get the same similarity measure i.e., $STM(P,Q_1) = STM(P,Q_2) = 0.5$. Hence in this case we can't decide the class of unknown pattern P. But when we find the similarity measure from proposed bi-parametric similarity measure with $t = 3$ and $p = 2$, we get $S_t^p(P, Q_1) = 0.907298$ and $S_t^p(P, Q_2) = 0.870163$. Since $S_t^p(P, Q_1) > S_t^p(P, Q_2)$, therefore, the unknown pattern P belongs to class Z_1.

Example 5.3. Let Q_1 and Q_2 be two known patterns with class labels Z_1 and Z_2, respectively, are given. The *PFSs* are used to represents the patterns in $\hat{\mathcal{Y}} = \{y_1, y_2, y_3\}$ as follows:

$$Q_1 = \{(0.3, 0.2, 0.1)/y_1, (0.5, 0.1, 0.2)/y_2, (0.6, 0.1, 0.3)/y_3\}$$
$$Q_2 = \{(0.6, 0.1, 0.3)/y_1, (0.1, 0.2, 0.5)/y_2, (0.6, 0.3, 0.1)/y_3\}.$$

P is the unknown pattern which is given as follows:

$$P = \{(0.5, 0.3, 0.2)/y_1, (0.3, 0.4, 0.2)/y_2, (0.4, 0.3, 0.2)/y_3\}.$$

Our aimed to find out the class of unknown pattern P belongs to. When we use cosine similarity measure CS^2 (Definition 2.4), we get the same similarity measure i.e., $CS^2(P, Q_1) = CS^2(P, Q_2) = 0.93104$. Hence in this case we can't decide the class of unknown pattern P. Also when we use the cotangent similarity measure CS^4, we get the same similarity measures i.e., $CS^4(P, Q_1) = CS^4(P, Q_2) = 0.688629$. Hence in this case we can't decide the class of unknown pattern P by using cotangent similarity measure CS^4. The bi-parametric similarity measure successfully applies for $t = 3$ and $p = 2$. We get $S_t^p(P, Q_1) = 0.844098$ and $S_t^p(P, Q_2) = 0.864087$. Since $S_t^p(P, Q_2) > S_t^p(P, Q_1)$, therefore, the unknown pattern P belongs to class Z_2.

Example 5.4. Let Q_1 and Q_2 be two known patterns with class labels Z_1 and Z_2, respectively, are given. The *PFSs* are used to represents the patterns in $\hat{\mathcal{Y}} = \{y_1, y_2, y_3\}$ as follows:

$$Q_1 = \{(0.3, 0.2, 0.1)/y_1, (0.5, 0.1, 0.2)/y_2, (0.4, 0.3, 0.3)/y_3\}$$
$$Q_2 = \{(0.6, 0.1, 0.3)/y_1, (0.1, 0.3, 0.4)/y_2, (0.3, 0.1, 0.2)/y_3\}.$$

P is the unknown pattern which is given as follows:

$$P = \{(0.5, 0.3, 0.2)/y_1, (0.3, 0.4, 0.2)/y_2, (0.6, 0.2, 0.2)/y_3\}.$$

Our aimed to find out the class of unknown pattern P belongs to. But when we use cosine similarity measure CS^3 (Definition 2.5), we get the same similarity measure i.e., $CS^3(P, Q_1) = CS^3(P, Q_2) =0.941998$. Hence in this case we can't decide the class of unknown pattern P. But when we find the similarity measure from proposed bi-parametric similarity measure with $t = 3$ and $p = 2$, we get $S_t^p(P, Q_1) = 0.836064$ and $S_t^p(P, Q_2) = 0.876678$. Since $S_t^p(P, Q_2) > S_t^p(P, Q_1)$, therefore, the unknown pattern P belongs to class Z_2.

Example 5.5. Let Q_1 and Q_2 be two known patterns with class labels Z_1 and Z_2, respectively, are given. The *PFSs* are used to represents the patterns in $\hat{\mathcal{Y}} = \{y_1, y_2, y_3\}$ as follows:

$$Q_1 = \{(0.4, 0.1, 0.3)/y_1, (0.5, 0.1, 0.2)/y_2, (0.6, 0.1, 0.3)/y_3\}$$
$$Q_2 = \{(0.6, 0.1, 0.3)/y_1, (0.1, 0.2, 0.5)/y_2, (0.5, 0.3, 0.0)/y_3\}.$$

P is the unknown pattern which is given as follows:

$$P = \{(0.5, 0.3, 0.2)/y_1, (0.3, 0.4, 0.2)/y_2, (0.4, 0.3, 0.2)/y_3\}.$$

Our aimed to find out the class of unknown pattern P belongs to. But when we use cosine similarity measure CS^5 with degree of refusal membership (Definition 2.7), we get the same similarity measure i.e., $CS^5(P, Q_1) = CS^5(P, Q_2) = 0.93104$. Also when we use the cotangent similarity measure CS^6 with degree of refusal membership (Definition 2.8), we get the same similarity measures i.e., $CS^6(P, Q_1) = CS^6(P, Q_2) = 0.688629$. Hence in this case we can't decide the class of unknown pattern P by using cotangent similarity measure CS^6 with degree of refusal membership function. Hence in this case we can't decide the class of unknown pattern P. But when we find the similarity measure from proposed bi-parametric similarity measure with $t = 3$ and $p = 2$, we get $S_t^p(P, Q_1) = 0.844098$ and $S_t^p(P, Q_2) = 0.864087$. Since $S_t^p(P, Q_2) > S_t^p(P, Q_1)$, therefore, the unknown pattern P belongs to class Z_2.

6. Application in medical diagnoses

Let $\hat{U} = \{Joy, Pob, Aya, Ted\}$ represents the set of four patients, $\hat{V}$ represent their symptoms, where $\hat{V} = \{$*Chest Pain*, *Cough*, *Stomach Pain*, *Headche*, *Temprature*$\}$ and the set $\hat{W}$ represents the diagnosis, where $\hat{W}$ =$\{$*Stomach Problem*, *Malaria*, *Viral Fever*, *Heart Problem*, *Typhoid*$\}$. *PFNs* are given as relation $\hat{U} \to \hat{V} \to \hat{W}$ in Tables 1 and 2.

For a proper diagnosis, we calculated similarity measure between patients and diagnosis in the context of symptoms. For all diagnosis this process is done. If similarity measure is high then the patient suffer from that particular disease. The similarity measure between patients and diagnosis is presented in Table 3. According to the similarity measure in Table 3, Joy suffers from Stomach problem, Pob suffers from Heart problem, Aya suffers from Viral fever and Ted suffers from Stomach problem.

When we compare our result with already proposed methods in Table 4, we have seen that the results are same but only using CS^2 and CS^4, we can't discriminate that Aya suffers from stomach problem or headache.

Table 1
Symptoms Characteristics for the Patients.

	Chest pain	Cough	Stomach pain	Headache	Temperature
Joy	(0.8,0.1,0.1)	(0.6,0.2,0.1)	(0.1,0.1,0.8)	(0.6,0.1,0.2)	(0.2,0.1,0.6)
Pob	(0.0,0.2,0.8)	(0.4,0.2,0.4)	(0.6,0.1,0.2)	(0.1,0.1,0.7)	(0.1,0.1,0.8)
Aya	(0.8,0.1,0.1)	(0.8,0.1,0.1)	(0.0,0.2,0.6)	(0.2,0.1,0.7)	(0.0,0.2,0.5)
Ted	(0.6,0.1,0.2)	(0.5,0.1,0.4)	(0.3,0.1,0.4)	(0.7,0.1,0.2)	(0.3,0.2,0.4)

Table 2
Symptoms Characteristics for the Diagnosis.

	Chest pain	Cough	Stomach pain	Headache	Temperature
Stomach problem	(0.6,0.0,0.0)	(0.3,0.2,0.5)	(0.1,0.1,0.7)	(0.4,0.1,0.3)	(0.1,0.1,0.7)
Malaria	(0.7,0.0,0.1)	(0.2,0.2,0.6)	(0.0,0.1,0.9)	(0.7,0.0,0.0)	(0.1,0.1,0.8)
Viral fever	(0.3,0.3,0.3)	(0.6,0.1,0.1)	(0.2,0.1,0.7)	(0.2,0.1,0.6)	(0.1,0.0,0.9)
Heart problem	(0.1,0.1,0.7)	(0.2,0.1,0.4)	(0.8,0.0,0.1)	(0.2,0.1,0.7)	(0.2,0.1,0.7)
Typhoid	(0.1,0.1,0.8)	(0.0,0.1,0.8)	(0.2,0.0,0.8)	(0.2,0.0,0.8)	(0.8,0.1,0.1)

Table 3
Symptoms Characteristics for the Diagnosis.

	Stomach problem	Malaria	Viral fever	Heart problem	Typhoid
Joy	**0.842282**	0.796489	0.755988	0.562583	0.542561
Pob	0.612809	0.517688	0.744559	**0.891028**	0.642171
Aya	0.770236	0.654433	**0.806674**	0.578693	0.52645
Ted	**0.821348**	0.766458	0.737798	0.663783	0.609434

Table 4
All the Considered Results.

		Stomach problem	Malaria	Viral fever	Heart problem	Typhoid
	Joy	**0.920408**	0.883993	0.852514	0.541238	0.481780
	Pob	0.629256	0.501889	0.812292	**0.975644**	0.692161
CS^1	Aya	0.868131	0.729510	**0.915494**	0.578503	0.493098
	Ted	**0.906828**	0.881773	0.799051	0.699248	0.631653
	Joy	**0.937301**	0.916919	0.876501	0.682359	0.614248
	Pob	0.760103	0.662606	0.856119	**0.973036**	0.765234
CS^2	Aya	**0.881185**	0.762295	**0.881185**	0.660370	0.602154
	Ted	**0.915027**	0.869175	0.817447	0.780667	0.67976
	Joy	**0.952134**	0.931842	0.895582	0.694304	0.638913
	Pob	0.770674	0.647313	0.877439	**0.976080**	0.764781
CS^3	Aya	0.894519	0.794582	**0.909742**	0.717893	0.647573
	Ted	**0.937874**	0.899481	0.862786	0.814174	0.72981
	Joy	**0.734154**	0.715092	0.64894	0.451595	0.396816
	Pob	0.558057	0.489356	0.629878	**0.803065**	0.556537
CS^4	Aya	**0.646181**	0.528936	**0.646181**	0.489985	0.427494
	Ted	**0.658297**	0.623433	0.533366	0.493648	0.397558
	Joy	**0.908893**	0.909593	0.876501	0.682359	0.614248
	Pob	0.760103	0.662606	0.856119	**0.961026**	0.765234
CS^5	Aya	0.852777	0.754969	**0.881185**	0.64836	0.602154
	Ted	**0.903017**	0.857165	0.817447	0.780667	0.67976
	Joy	**0.690751**	0.689584	0.64894	0.451595	0.396816
	Pob	0.558057	0.489356	0.629878	**0.780317**	0.556537
CS^6	Aya	0.602778	0.503429	**0.646181**	0.467237	0.427494
	Ted	**0.635549**	0.600684	0.533366	0.493648	0.397558
	Joy	**0.780610**	0.776750	0.718477	0.458849	0.428452
	Pob	0.549545	0.460958	0.700649	**0.840259**	0.603189
STS	Aya	0.668152	0.561164	**0.689217**	0.467620	0.417647
	Ted	**0.736332**	0.723483	0.623126	0.537819	0.462145

To observe the consistency and effect of the parameter p, we have prepared Table 5 and Fig. 2. For different values of p, we have seen that the values of similarity measures decreases when we increase the value of p but the order of the diagnosis remains same. From Fig. 2, we have seen that the graph for $p = 2$ remains above and for $p = 8$ remains lower. No graph intersect or cut each other and hence consistent and reliable for all value of p. This shows that if we increase or decrease the value for p, we get the same diagnosis (order). For different values of t, we get the same diagnosis but their value depends on the data because t identifies the level of uncertainty and purely depends on the data set that we have provided.

Table 5
Bi-parametric Similarity Measure for $p = 2$ to $p = 8$.

	Joy	Pob	Aya	Ted
$p = 2$	$\begin{Bmatrix} \textit{Stomach problem} \\ 0.842282 \end{Bmatrix}$	$\begin{Bmatrix} \textit{Heart problem} \\ 0.891028 \end{Bmatrix}$	$\begin{Bmatrix} \textit{Viral fever} \\ 0.806674 \end{Bmatrix}$	$\begin{Bmatrix} \textit{Stomach problem} \\ 0.821348 \end{Bmatrix}$
$p = 3$	$\begin{Bmatrix} \textit{Stomach problem} \\ 0.809326 \end{Bmatrix}$	$\begin{Bmatrix} \textit{Heart problem} \\ 0.881166 \end{Bmatrix}$	$\begin{Bmatrix} \textit{Viral fever} \\ 0.764421 \end{Bmatrix}$	$\begin{Bmatrix} \textit{Stomach problem} \\ 0.811558 \end{Bmatrix}$
$p = 4$	$\begin{Bmatrix} \textit{Stomach problem} \\ 0.782716 \end{Bmatrix}$	$\begin{Bmatrix} \textit{Heart problem} \\ 0.873604 \end{Bmatrix}$	$\begin{Bmatrix} \textit{Viral fever} \\ 0.732354 \end{Bmatrix}$	$\begin{Bmatrix} \textit{Stomach problem} \\ 0.80347 \end{Bmatrix}$
$p = 5$	$\begin{Bmatrix} \textit{Stomach problem} \\ 0.762076 \end{Bmatrix}$	$\begin{Bmatrix} \textit{Heart problem} \\ 0.867324 \end{Bmatrix}$	$\begin{Bmatrix} \textit{Viral fever} \\ 0.70739 \end{Bmatrix}$	$\begin{Bmatrix} \textit{Stomach problem} \\ 0.796708 \end{Bmatrix}$
$p = 6$	$\begin{Bmatrix} \textit{Stomach problem} \\ 0.746042 \end{Bmatrix}$	$\begin{Bmatrix} \textit{Heart problem} \\ 0.86186 \end{Bmatrix}$	$\begin{Bmatrix} \textit{Viral fever} \\ 0.687400 \end{Bmatrix}$	$\begin{Bmatrix} \textit{Stomach problem} \\ 0.790998 \end{Bmatrix}$
$p = 7$	$\begin{Bmatrix} \textit{Stomach problem} \\ 0.733378 \end{Bmatrix}$	$\begin{Bmatrix} \textit{Heart problem} \\ 0.857004 \end{Bmatrix}$	$\begin{Bmatrix} \textit{Viral fever} \\ 0.671029 \end{Bmatrix}$	$\begin{Bmatrix} \textit{Stomach problem} \\ 0.786128 \end{Bmatrix}$
$p = 8$	$\begin{Bmatrix} \textit{Stomach problem} \\ 0.733378 \end{Bmatrix}$	$\begin{Bmatrix} \textit{Heart problem} \\ 0.857004 \end{Bmatrix}$	$\begin{Bmatrix} \textit{Viral fever} \\ 0.671029 \end{Bmatrix}$	$\begin{Bmatrix} \textit{Stomach problem} \\ 0.786128 \end{Bmatrix}$

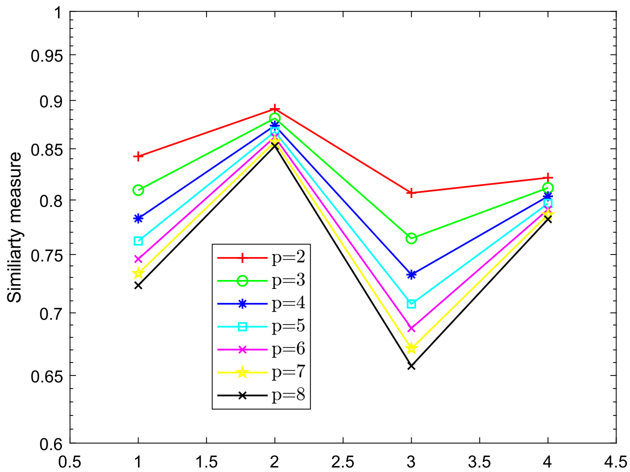

Fig. 2. Bi-parametric Similarity Measure for $p = 2$ to $p = 8$.

7. Comparison analysis

A comparison between bi-parametric similarity measure and already proposed similarity measure is conducted to illustrate the superiority of the bi-parametric similarity measure.

We have seen from Example 5.1 that the second condition of Definition 3.2 (S_2) is not satisfied for cosine similarity measure CS^1, i.e., $\hat{\mathcal{S}}(\hat{\mathcal{A}}, \hat{\mathcal{B}}) = 1$ even $\hat{\mathcal{A}} \neq \hat{\mathcal{B}}$. Also we provide a general criteria when second condition of Definition 3.2 (S_2) is not satisfied for cosine similarity measure CS^1. In Example 5.2, we have seen that the STM can not classify the unknown pattern and i.e., $STM(P, Q_1) = STM(P, Q_2)$ even $Q_1 \neq Q_2$. In Example 5.3, we have seen that the CS^2 can not classify the unknown pattern i.e., $CS^2(P, Q_1) = CS^2(P, Q_2)$ even $Q_1 \neq Q_2$. Also in Example 5.3, we have seen that the similarity measure CS^4 fails to classify unknown pattern i.e., $CS^4(P, Q_1) = CS^4(P, Q_2)$ even $Q_1 \neq Q_2$. In Example 5.4, we have seen that the CS^3 can not classify the unknown pattern and i.e., $CS^3(P, Q_1) = CS^3(P, Q_2)$ even $Q_1 \neq Q_2$. In Example 5.5, we have seen that the CS^5 can not classify the unknown pattern i.e., $CS^5(P, Q_1) = CS^5(P, Q_2)$ even $Q_1 \neq Q_2$. Also in Example 5.5, we have seen that the similarity measure CS^6 fails to classify unknown pattern i.e., $CS^6(P, Q_1) = CS^6(P, Q_2)$ even $Q_1 \neq Q_2$. But in all Examples 5.1–5.5, the bi-parametric similarity measure classify the unknown pattern and hence successfully applicable to the pattern recognition problems. Also in medical diagnosis problems CS^2 and CS^4 can not diagnosis Aya (see Table 4).

A comprehensive comparison with the existing similarity measures is discussed by considering four cases. In first case, suppose $A = (0.4, 0.4, 0.2), B = (0.2, 0.2, 0.1)$ and $C = (0.3, 0.1, 0.1)$ be three distinct PFVs. We have calculated the similarity measure between A and B, as well as, A and C. The second condition for similarity measure is not satisfied for CS^1, i.e., $A \neq B$ but $CS^1(A, B) = 1$. The similarity measures CS^3, CS^5, CS^6, CS^7 and STS have no capabilities to distinguish between the PFVs B and C from the PFV A perspective. In second case, we suppose $D = (0.5, 0.3, 0.2), E = (0.3, 0.2, 0.1))$ and $F = (0.6, 0.1, 0.3)$ be three distinct PFVs. The similarity measures CS^2, CS^3 and CS^4 have no capabilities to distinguish between the PFVs E and F from the PFV D perspective. Similarly, in third case, the PFVs $G = (0.5, 0.3, 0.2), H = (0.3, 0.2, 0.1))$ and $I = (0.6, 0.1, 0.3)$ are considered. All the previously proposed similarity measures for PFSs have no capabilities to distinguish between the PFVs H and I from the PFV G perspective. The fourth case consist of two distinct PFVs $K = (0.2, 0.2, 0.2), L = (0.3, 0.3, 0.3)$ but $CS^1(K, L) = 1$. The bi-parametric similarity measure remains consistent for all cases and gave us a satisfactory results. The summary of the four cases presented in Table 6.

8. Conclusion

In this paper, we have proposed new similarity measures for *PFSs* called bi-parametric similarity measures based on two parameters t and p, where t is the level of uncertainty and p is the L_p norm. We have defined bi-parametric distance measures, weighted bi-parametric distance measures, bi-parametric similarity measures and weighted bi-parametric similarity measures and discussed their properties. Also, we discussed the comprehensive interpretation of the proposed similarity measure. We discuss some special cases ((5.1)–(5.5)), where already proposed similarity measure fails to classify the unknown pattern while the proposed similarity measure successfully applied to the pattern recognition problems. A numerical example is proposed to solve medical diagnosis problems and we have seen that the results are consistent and does not change with the values of the parameters (See Table 5 and Fig. 2).

In the future direction, we will apply the bi-parametric similarity measure to data mining, medical diagnosis, decision making, complex group decision making, linguistic summarization risk analysis, pattern recognition, clustering analysis, image processing and image restoration problems.

Table 6
Comparison Table.

	S(A, B)	S(A, C)	S(D, E)	S(D, F)	S(G, H)	S(G, I)	S(K, L)
	(0.4,0.4,0.2) (0.2,0.2,0.1)	(0.4,0.4,0.2) (0.3,0.1,0.1)	(0.5,0.3,0.2) (0.3,0.2,0.1)	(0.5,0.3,0.2) (0.6,0.1,0.3)	(0.3,0.3,0.3) (0.2,0.2,0.1)	(0.3,0.3,0.3) (0.1,0.2,0.2)	(0.2,0.2,0.2) (0.3,0.3,0.3)
CS^1	**1.**	0.9045	0.9972	0.9328	**0.9623**	**0.9623**	**1.**
CS^2	0.9511	0.891	**0.9511**	**0.9511**	**0.9511**	**0.9511**	0.9877
CS^3	**0.9239**	**0.9239**	**0.9511**	**0.9511**	**0.9511**	**0.9511**	0.9724
CS^4	0.7265	0.6128	**0.7265**	**0.7265**	**0.7265**	**0.7265**	0.8541
CS^5	**0.7071**	**0.7071**	0.809	0.9511	**0.809**	**0.809**	0.891
CS^6	**0.4142**	**0.4142**	0.5095	0.7265	**0.5095**	**0.5095**	0.6128
CS^7	**0.7071**	**0.7071**	0.809	0.9511	**0.809**	**0.809**	0.891
STS	**0.5**	**0.5**	0.6053	0.7222	**0.5556**	**0.5556**	0.6667
$\mathcal{S}_d^1$	0.8	0.766	0.8846	0.9286	**0.8333**	**0.8333**	0.9231
$\mathcal{S}_d^2$	0.5143	0.5	0.6765	0.9286	**0.6452**	**0.6452**	0.7857
$\mathcal{S}_d^3$	0.8	0.766	0.8846	0.9286	**0.8333**	**0.8333**	0.9231
S_t^p	0.0947	0.1493	0.0707	0.1041	0.1041	0.0707	0.0629

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