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On Graphs for Intuitionistic Modal Logics [1]

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Abstract

We present a graph approach to intuitionistic modal logics, which provides uniform formalisms for expressing, analysing and comparing Kripke-like semantics. This approach uses the flexibility of graph calculi to express directly and intuitively possible-world semantics for intuitionistic modal logics. We illustrate the benefits of these ideas by applying them to some familiar cases of intuitionistic multi-modal semantics.

Keywords: Intuitionistic modal logics, semantics, graph formulations, calculi, refutation, special relations.

1 Introduction

We present a graph approach to intuitionistic modal logics, which provides a flexible and uniform tool for expressing, analysing and comparing possible-world semantics.

This graph approach can be regarded as a version of diagrammatic reasoning, where we can express formulas by diagrams, which can be manipulated to unveil properties (like consequence and satisfiability). Graph representations and transformations, having precise syntax and semantics, give proof methods. An interesting feature of this graph approach is its 2-dimensional notation providing pictorial representations that support visual manipulations [4]. These ideas have been adapted to refutational reasoning [14] and applied to multi-modal classical logics [15].

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Modal logics and graphs are closely connected. Kripke semantics can be presented via directed labelled graphs for the accessibility relation of each modality [2]. It is natural to represent that a is related to b via relation r by an arrow $a \xrightarrow{r} b$.

Intuitionistic modal logic is an interesting subject [6,11]. There seems to be little consensus on the appropriate approach to its semantics, as indicated by the diversity of Kripke-like semantics proposed (see [5,13] and references therein).

We provide graph calculi, having diagrams as terms and whose rules transform diagrams, capturing graphically the semantics of the modal operators and accessibility relations. These calculi provide uniform and flexible formalisms where one can explore Kripe-like semantics for intuitionistic modal logics: satisfaction conditions, valid formulas, etc. We illustrate these ideas by 2 case studies: logics as in [13,5].

We will consider a modal language ML, with set Φ of formulas, given by sets $\mathbb{PL}$, of propositional letters, and $\mathbb{RS}$, of 2-ary relation symbols. The formulas of ML are generated by the grammar $\varphi :\!= \bot \mid \mathrm{p} \mid \varphi' \wedge \varphi'' \mid \varphi' \vee \varphi'' \mid \varphi' \rightarrow \varphi'' \mid \langle \mathrm{r} \rangle \varphi \mid [\mathrm{r}]\varphi$. [4]

2 Graphs and Modalities: Basic Ideas

We now introduce informally some basic ideas about graphs and modalities. [5]

A graph amounts to a finite set of (alternative) slices. A slice S consists of an underlying draft $\underline{\mathrm{S}}$ together with a distinguished node (marked, e. g. $\widehat{\mathrm{w}}$). A draft amounts to finite sets of nodes and arcs. Slices and graphs represent sets of states, whereas drafts (and sketches, see Section 3) will describe restrictions on states.

Arcs may be binary or unary. A binary arc stands for accessibility between states; we represent that node v is accessible from node u by the relation of r by a solid arrow labelled r from u to v: $\mathrm{u} \xrightarrow{\mathrm{r}} \mathrm{v}$ (abbreviated u r v). A unary arc is meant to capture the fact that a formula holds at a state; we represent that formula φ holds at node w by a dashed line from w to φ: $\mathrm{w} - - \prec \varphi$ (abbreviated $\mathrm{w}|\varphi$).

Expressions will encompass slices, graphs and their complements (noted by an overbar). As such, an expression represents a set of states; so we can also use unary arcs of the form $\mathrm{w} - - \prec \mathrm{E}$, where E is an expression.

We now introduce some concepts to be used and illustrated in Example 2.1.

A (draft) morphism is a node mapping that preserves arcs. A (slice) homomorphism is a morphism of their underlying drafts that preserves distinguished nodes. A draft may have conflicts that prevent its satisfaction. We consider two kinds of conflicts. One concerns contradictory 1-ary arcs: if draft D has the pattern $\mathrm{E} \succ - - \mathrm{w} - - \prec \overline{\mathrm{E}}$, then expression E is a witness of a conflict at node w. If D has 1-ary arc $\mathrm{w} - - \prec \overline{\mathrm{Q}}$, slice Q will be a witness of a conflict at node w if there is a morphism from $\underline{\mathrm{Q}}$ to D mapping the distinguished node of Q to w.

To reason about modal formulas, we convert them to expressions (with the same meaning) and reason graphically about these. We reduce consequence to unsatisfiablility: "every state satisfying $\psi_1, \ldots, \psi_n$ also satisfies θ" (noted $\{ \psi_1, \ldots, \psi_n \} \models$

[4] As usual, $\neg\varphi$ abbreviates $\varphi \rightarrow \bot$.

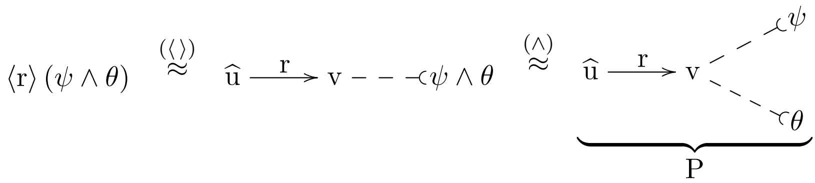
[5] These and other ideas will be formulated more precisely later on: in Section 3.

θ) is equivalent to "there is no state satisfying $\psi_1, \ldots, \psi_n$ and failing to satisfy θ" (noted $\{\psi_1, \ldots, \psi_n, \overline{\theta}\} \models \bot$). Notice that $\overline{\theta}$ is not a formula: $^-$ is complementation rather than intuitionistic negation. The next example illustrates this approach.

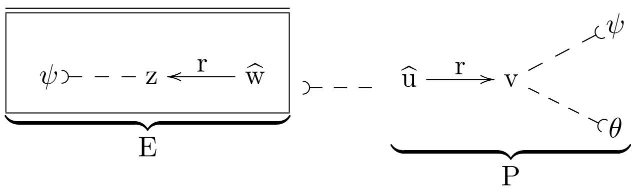
Example 2.1 (Consequence via slice conversions) We reduce $\langle r\rangle(\psi \wedge \theta) \models \langle r\rangle \psi$ to $\{\langle r\rangle(\psi \wedge \theta), \overline{\langle r\rangle \psi}\} \models \bot$. We first indicate the graph-calculus steps.

(i) We convert $\overline{\langle r\rangle \psi}$ to expression E: $\overline{\langle r\rangle \psi} \overset{(\langle\rangle)}{\approx} \underbrace{\overline{\boxed{\hat{w} \xrightarrow{r} z \dashrightarrow \psi}}}_{E}$.

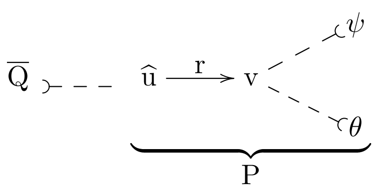
(ii) We also convert $\langle r\rangle(\psi \wedge \theta)$ to slice P as follows:



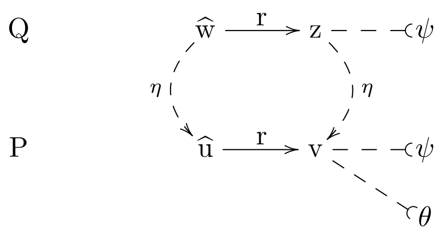
(iii) We now obtain, from P and E, the following slice P′ (for $\{\langle r\rangle(\psi \wedge \theta), \overline{\langle r\rangle\psi}\}$):



(iv) Call $Q \coloneqq \hat{w} \xrightarrow{r} z \dashrightarrow \psi$ the slice under complement in E. So, slice P′ is:



We have a homomorphism η from Q to P (cf. p. 2), given by $w \mapsto u$, $z \mapsto v$:



Slice P′ has conflict at node u, with slice witness Q (cf. p. 2).

We now provide intuitive explanations for these steps, using R for the relation of r.

(i) The states pertaining to E are those not pertaining to slice $\hat{w} \xrightarrow{r} z \dashrightarrow \psi$. The states s pertaining to this slice Q are those for which there is a state t such that $\mathsf{s}\,\mathsf{R}\,\mathsf{t}$ and t satisfies the unary arc $z \dashrightarrow \psi$ (i. e., t satisfies formula ψ).

(ii) The states s pertaining to P are those for which there is a state t such that $\mathsf{s}\,\mathsf{R}\,\mathsf{t}$ and t satisfies both arcs $\mathrm{v} \dashrightarrow \psi$ and $\mathrm{v} \dashrightarrow \theta$ (i. e., t satisfies $\psi \wedge \theta$).

(iii) The states pertaining to slice P′ are those pertaining to slice P that satisfy the unary arc $\mathrm{u} \dashrightarrow \mathrm{E} \;=\; \mathrm{u} \dashrightarrow \overline{\mathrm{Q}}$.

(iv) Any state pertaining to P must (as $\eta : \mathrm{Q} \to \mathrm{P}$) pertain to Q, whence it does not pertain to $\overline{\mathrm{Q}}$. Thus, there is no state pertaining to P′, so it is not satisfiable.

Hence, set $\{\langle \mathrm{r}\rangle\,(\psi \wedge \theta), \overline{\langle \mathrm{r}\rangle\,\psi}\}$ is unsatisfiable and $\langle \mathrm{r}\rangle\,(\psi \wedge \theta) \;\models\; \langle \mathrm{r}\rangle\,\psi$. ♮

3 Graph Concepts and Results

We now introduce some basic concepts and results about graphs.[6] We will use an infinite set $\mathbb{N}\mathrm{d}$ of nodes; the *first 3 nodes* being x, y and z.

A *graph language* $\mathbb{GL}$ is characterized by two sets of symbols: $\mathbb{S}\mathrm{b}_1$, of unary ones, and $\mathbb{S}\mathrm{b}_2$, of binary ones. Its syntax is defined by mutual recursion as follows.

(E) The *expressions* are the 1-ary symbols $\mathrm{s} \in \mathbb{S}\mathrm{b}_1$, the slices and the graphs (see below), as well as $\overline{\mathrm{E}}$, for an expression E.

(a) The *arcs over* a set $\mathrm{N} \subseteq \mathbb{N}\mathrm{d}$ of nodes are as follows.
 (1) A unary arc $\mathrm{w}|\mathrm{E}$ over N consists of a node $\mathrm{w} \in \mathrm{N}$ and an expression E.
 (2) A binary arc $\mathrm{u}\,\mathrm{L}\,\mathrm{v}$ over N consists of nodes $\mathrm{u}, \mathrm{v} \in \mathrm{N}$ and 2-ary symbol $\mathrm{L} \in \mathbb{S}\mathrm{b}_2$.

(Σ) A *sketch* $\Sigma = \langle \mathrm{N}; \mathtt{A}\rangle$ consists of 2 sets: $\mathrm{N} \subseteq \mathbb{N}\mathrm{d}$ (of nodes) and $\mathtt{A}$ of arcs over N.

(D) A *draft* is a sketch with finite sets of nodes and arcs.

(S) A *slice* $\mathrm{S} = \langle \mathrm{N}; \mathtt{A} : \mathrm{w}\rangle$ consists of an *underlying draft* $\underline{\mathrm{S}} \coloneqq \langle \mathrm{N}; \mathtt{A}\rangle$ and a *distinguised node* $\mathrm{w} \in \mathrm{N}$. We often use the notation $\mathrm{S} = (\underline{\mathrm{S}} : \mathrm{w})$.

(G) A *graph* is a finite set of slices.

A *proper* sketch has non-empty node set. The *positive part* of a sketch consists of its nodes and its complement-free arcs. The *empty graph* $\{\,\}$ has no slices.

A *structure* $\mathfrak{M}$ for graph language $\mathbb{GL}$ consists of a universe $\mathsf{M} \neq \emptyset$, as well as a subset $\mathrm{s}^{\mathfrak{M}} \subseteq \mathsf{M}$, for each $\mathrm{s} \in \mathbb{S}\mathrm{b}_1$, and a binary relation $\mathrm{L}^{\mathfrak{M}}$ on M, for each $\mathrm{L} \in \mathbb{S}\mathrm{b}_2$.

We now define semantics also by mutual recursion.

(E) The *extension* $[\mathrm{E}]_{\mathfrak{M}}$ of expression E is defined as follows. For a 1-ary symbol $\mathrm{s} \in \mathbb{S}\mathrm{b}_1$: $[\mathrm{s}]_{\mathfrak{M}} \coloneqq \mathrm{s}^{\mathfrak{M}}$; if E is a slice or a graph, then we use its behaviour: $[\mathrm{E}]_{\mathfrak{M}} \coloneqq [\![\mathrm{E}]\!]_{\mathfrak{M}}$ (see below); and, for $\overline{\mathrm{E}}$, we use complement: $[\overline{\mathrm{E}}]_{\mathfrak{M}} \coloneqq \mathsf{M} \setminus [\mathrm{E}]_{\mathfrak{M}}$.

(g) An *assignment* for $\mathrm{N} \subseteq \mathbb{N}\mathrm{d}$ is a function $\mathsf{g} : \mathrm{N} \to \mathsf{M}$ (so $\mathrm{w} \in \mathrm{N} \mapsto \mathrm{w}^{\mathsf{g}} \in \mathsf{M}$).

(a) We define *satisfaction* (in $\mathfrak{M}$) for an arc over set N as follows.
 (1) Assignment g satisfies unary arc $\mathrm{w}|\mathrm{E}$ (in $\mathfrak{M}$) iff $\mathrm{w}^{\mathsf{g}} \in [\mathrm{E}]_{\mathfrak{M}}$.
 (2) Assignment g satisfies binary arc $\mathrm{u}\,\mathrm{L}\,\mathrm{v}$ (in $\mathfrak{M}$) iff $(\mathrm{u}^{\mathsf{g}}, \mathrm{v}^{\mathsf{g}}) \in \mathrm{L}^{\mathfrak{M}}$.

(Σ) Assignment g satisfies a sketch (in $\mathfrak{M}$) iff it satisfies all its arcs.

[6] For more details about graphs see, e. g. [14,15] and references therein.

(S) For a slice $\mathrm{S} = (\underline{\mathrm{S}} : \mathrm{w})$, its *behaviour* (in $\mathfrak{M}$) is the set $[\![\mathrm{S}]\!]_{\mathfrak{M}}$ consisting of the values $\mathrm{w}^{\mathsf{g}} \in \mathsf{M}$ for the assignments g satisfying its underlying draft $\underline{\mathrm{S}}$.

(G) For a graph G, its *behaviour* (in $\mathfrak{M}$) is $[\![\mathrm{G}]\!]_{\mathfrak{M}} \coloneqq \bigcup_{\mathrm{S}\in\mathrm{G}} [\![\mathrm{S}]\!]_{\mathfrak{M}}$.

We define *satisfiability*, *equivalence* and *nullity* as follows. Consider a class of models $\mathfrak{K}$. A sketch Σ is satisfiable in $\mathfrak{K}$ iff there exist a model $\mathfrak{M} \in \mathfrak{K}$ and an assignment satisfying Σ in $\mathfrak{M}$. A slice S is satisfiable in $\mathfrak{K}$ iff its underlying draft $\underline{\mathrm{S}}$ is so; and a graph G is satisfiable in $\mathfrak{K}$ iff some slice $\mathrm{S} \in \mathrm{G}$ is so. Expressions E and F are equivalent in $\mathfrak{K}$ iff, for every model $\mathfrak{M} \in \mathfrak{K}$: $[\mathrm{E}]_{\mathfrak{M}} = [\mathrm{F}]_{\mathfrak{M}}$. An expression set $\mathbb{E}$ is null in $\mathfrak{K}$ iff, for every model $\mathfrak{M} \in \mathfrak{K}$: $\bigcap_{\mathrm{E}\in\mathbb{E}} [\mathrm{E}]_{\mathfrak{M}} = \emptyset$. We use simply *satisfiable*, *equivalent* (noted $\equiv$) and *null* when referring to the class of all models. For instance, a singleton graph $\{\mathrm{S}\}$ and its slice S are equivalent (so we can identify them); the empty graph $\{\,\}$ is null, as is the formula $\bot$.

We now define structural comparison and conflicts, introduced in Section 2.

For sketches Δ and Σ, a *morphism* from Δ to Σ is a function $\mu : \mathrm{N}_\Delta \to \mathrm{N}_\Sigma$ (noted $\mu : \Delta \dashrightarrow \Sigma$), for which we have $\mu(\mathtt{a}) \in \mathtt{A}_\Sigma$, for every arc $\mathtt{a} \in \mathtt{A}_\Delta$ (with $\mu(\mathrm{w}|\mathrm{E}) \coloneqq \mathrm{w}^\mu|\mathrm{E}$ and $\mu(\mathrm{u}\,\mathrm{L}\,\mathrm{v}) \coloneqq \mathrm{u}^\mu\,\mathrm{L}\,\mathrm{v}^\mu$). Now, given slices $\mathrm{Q} = (\underline{\mathrm{Q}} : \mathrm{v})$ and $\mathrm{P} = (\underline{\mathrm{P}} : \mathrm{u})$, a *homomorphism* from Q to P is a function $\eta : \mathrm{N}_\mathrm{Q} \to \mathrm{N}_\mathrm{P}$ (noted $\eta : \mathrm{Q} \to \mathrm{P}$) that is a morphism $\eta : \underline{\mathrm{Q}} \dashrightarrow \underline{\mathrm{P}}$ and $\mathrm{v}^\eta = \mathrm{u}$.

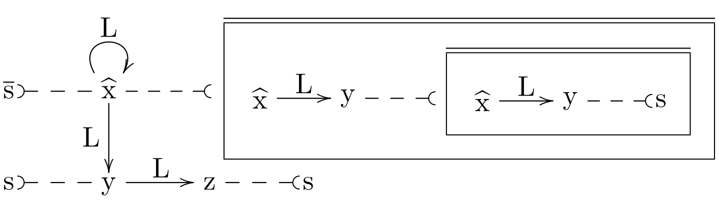
Morphisms transfer satisfying assignments by composition. Given a morphism $\mu : \Delta \dashrightarrow \Sigma$, if g satisfies Σ in $\mathfrak{M}$, then the composite $\mathsf{g} \cdot \mu$ satisfies Δ in $\mathfrak{M}$. Thus, if there exists a homomorphism $\eta : \mathrm{Q} \to \mathrm{P}$, then $[\![\mathrm{Q}]\!]_{\mathfrak{M}} \supseteq [\![\mathrm{P}]\!]_{\mathfrak{M}}$.

Consider a sketch $\Sigma = \langle \mathrm{N}; \mathtt{A}\rangle$. An expression E with $\mathrm{u}|\mathrm{E}, \mathrm{u}|\overline{\mathrm{E}} \in \mathtt{A}$ is an *expression witness* of Σ at node $\mathrm{u} \in \mathrm{N}$. A slice $\mathrm{Q} = (\underline{\mathrm{Q}} : \mathrm{v})$ for which there is a morphism $\mu : \underline{\mathrm{Q}} \dashrightarrow \Sigma$ such that $\mathrm{v}^\mu|\overline{\mathrm{Q}} \in \mathtt{A}$ is a *slice witness* of Σ at node $\mathrm{v}^\mu \in \mathrm{N}$. A sketch is *zero* iff it has some witness. A slice S is *zero* iff $\underline{\mathrm{S}}$ is zero. A graph is *zero* iff all its slices are zero. Clearly, a zero sketch is not satisfiable; so zero slices and graphs are null. One can effectively decide whether a draft, a slice or a graph is zero.

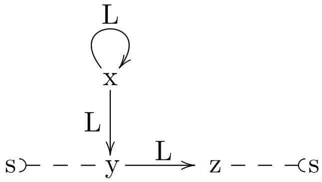
We use '+' for adding arcs (and their nodes). To glue a slice S on node w of slice P, we take a copy S' of S having only its distinguished node in common with P and add $\underline{\mathrm{S}'}$ to P, thereby obtaining a *glued slice* $\mathrm{P}_\mathrm{w}\mathrm{S}$. We glue a graph by gluing its slices: $\mathrm{P}^\mathrm{w}\mathrm{G} \coloneqq \{\mathrm{P}_\mathrm{w}\mathrm{S} \,/\, \mathrm{S} \in \mathrm{G}\}$. For instance, $\mathrm{P} \coloneqq \widehat{\mathrm{v}} \xrightarrow{\mathrm{L}} \mathrm{w} \dashrightarrow\!\!\prec \mathrm{E}$ and $\mathrm{S} \coloneqq \widehat{\mathrm{u}} \xrightarrow{\mathrm{K}} \mathrm{v} \dashrightarrow\!\!\prec \mathrm{F}$ have $\mathrm{P}_\mathrm{w}\mathrm{S} = \widehat{\mathrm{v}} \xrightarrow{\mathrm{L}} \mathrm{w} \xrightarrow{\mathrm{K}} \mathrm{v}^* \dashrightarrow\!\!\prec \mathrm{F}$, $\mathrm{w} \dashrightarrow\!\!\prec \mathrm{E}$ as glued slice.

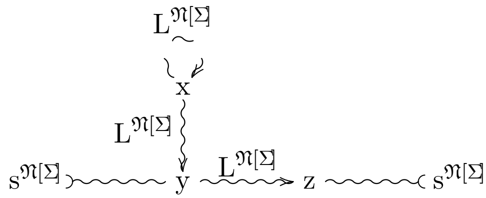
A proper sketch $\Sigma = \langle \mathrm{N}; \mathtt{A}\rangle$ gives a *natural structure* $\mathfrak{N}[\Sigma]$: with universe N, $\mathrm{s}^{\mathfrak{N}[\Sigma]} \coloneqq \{\mathrm{w} \in \mathrm{N} \,/\, \mathrm{w}|\mathrm{s} \in \mathtt{A}\}$ ($\mathrm{s} \in \mathbb{S}\mathrm{b}_1$), $\mathrm{L}^{\mathfrak{N}[\Sigma]} \coloneqq \{(\mathrm{u},\mathrm{v}) \in \mathrm{N}^2 \,/\, \mathrm{u}\,\mathrm{L}\,\mathrm{v} \in \mathtt{A}\}$ ($\mathrm{L} \in \mathbb{S}\mathrm{b}_2$).

Example 3.1 (Natural construction) Consider the following draft D:



The positive part D_+ of D and the natural structure $\mathfrak{N}[\mathsf{D}]$ are as follows:





Consider the identity assignment $\mathtt{1}$ on set $\{\mathrm{x}, \mathrm{y}\}$.

(i) Assignment $\mathtt{1}$ satisfies the arcs of D_+ as well as the 1-ary arc $\mathrm{x}|\overline{\mathrm{s}}$.

(ii) We can see that assignment $\mathtt{1}$ also satisfies the unary arc

$$\mathrm{x} \dashrightarrow \overline{\boxed{\hat{\mathrm{x}} \xrightarrow{\mathrm{L}} \mathrm{y} \dashrightarrow \overline{\boxed{\hat{\mathrm{x}} \xrightarrow{\mathrm{L}} \mathrm{y} \dashrightarrow \mathrm{s}}}}}\,.$$

Thus, assignment $\mathtt{1}$ satisfies draft D in natural structure $\mathfrak{N}[\mathsf{D}]$. $\natural$

In the sequel, we will show how one can represent a modal formula by an expression (of a graph language) with the same meaning, thus reducing questions about formulas to questions on expressions. We will be able to eliminate logical symbols from a modal formula, converting it to an equivalent expression. The elimination rules mimic the semantics of the modal language. The following ones are general.

We can eliminate double complement. We can also move complement inside by rules like De Morgan laws: *rule* $(\overline{\cup})$ converts a complemented graph $\overline{\mathrm{G}}$ to the slice with arcs $\hat{\mathrm{x}}|\overline{\mathrm{S}}$, for $\mathrm{S} \in \mathrm{G}$; for a slice S having only arcs of the form $\hat{\mathrm{w}}|\mathrm{E}_i$, *rule* $(\overline{\cap})$ converts $\overline{\mathrm{S}}$ to a graph with slices $\hat{\mathrm{w}}|\overline{\mathrm{E}_i}$. So, $\overline{\{\,\}} \overset{(\overline{\cup})}{\approx} \hat{\mathrm{x}}$, $\overline{\hat{\mathrm{w}}} \overset{(\overline{\cap})}{\approx} \{\,\}$ and $\overline{\hat{\mathrm{w}}|\mathrm{E}} \overset{(\overline{\cap})}{\approx} \{\hat{\mathrm{w}}|\overline{\mathrm{E}}\}$.

Structural rules transform slices and graphs. The *singleton rules* convert a slice to its singleton graph and vice-versa. The *promotion rule* converts an expression E to the slice $\hat{\mathrm{x}}|\mathrm{E}$. *Rule* $(\leftarrow_\vee)$ converts slice $\mathrm{P}+\mathrm{w}|\mathrm{G}$ to the graph $\{\mathrm{P}+\mathrm{w}|\mathrm{S} \,/\, \mathrm{S} \in \mathrm{G}\}$, and *rule* $(\leftarrow_\wedge)$ converts slice $\mathrm{P}+\mathrm{w}|\mathrm{S}$ to the glued slice $\mathrm{P}_\mathrm{w}\mathrm{S}$. We thus have a derived rule $(\leftarrow)$ converting slice $\mathrm{P}+\mathrm{w}|\mathrm{G}$ to the glued graph $\mathrm{P}^\mathrm{w}\mathrm{G}$. So, $\mathrm{P}+\mathrm{w}|\{\,\} \overset{(\leftarrow)}{\approx} \{\,\}$.

The *zero rule* $(\mathcal{Z})$ erases a zero slice. The *alternative expansion rule* $(\mathrm{w}\,|\,\mathrm{E})$ expands a slice S to graph $\{\mathrm{S}+\mathrm{w}|\mathrm{E}, \mathrm{S}+\mathrm{w}|\overline{\mathrm{E}}\}$. From $(\mathrm{w}\,|\,\mathrm{E})$ and $(\mathcal{Z})$, we can derive the *shift* expansion rule: a slice P with 1-ary arc $\mathrm{u} \dashrightarrow \overline{\boxed{\hat{\mathrm{w}} \xrightarrow{\mathrm{L}} \mathrm{w}' \dashrightarrow \mathrm{E}}}$ and 2-ary arc $\mathrm{u} \xrightarrow{\mathrm{L}} \mathrm{u}'$, expands to $\mathrm{P}+\mathrm{u}'|\overline{\mathrm{E}}$. So, a slice with slice witness $\hat{\mathrm{w}} \xrightarrow{\mathrm{L}} \mathrm{v} \dashrightarrow \mathrm{E}$ shifts to a slice with expression witness E.[7]

The *expression-set* rule converts set $\{\mathrm{E}_1, \ldots, \mathrm{E}_n\}$ to the slice $\mathrm{E}_1 \dashleftarrow \hat{\mathrm{x}} \dashrightarrow \ldots \dashrightarrow \mathrm{E}_n$.

Tables 1 and 2 summarize these conversion rules.

[7] This shift expansion rule can be used to simulate the modal [ ] transfer [7].

Table 1
General graph calculus conversion rules (cf. p. 5)

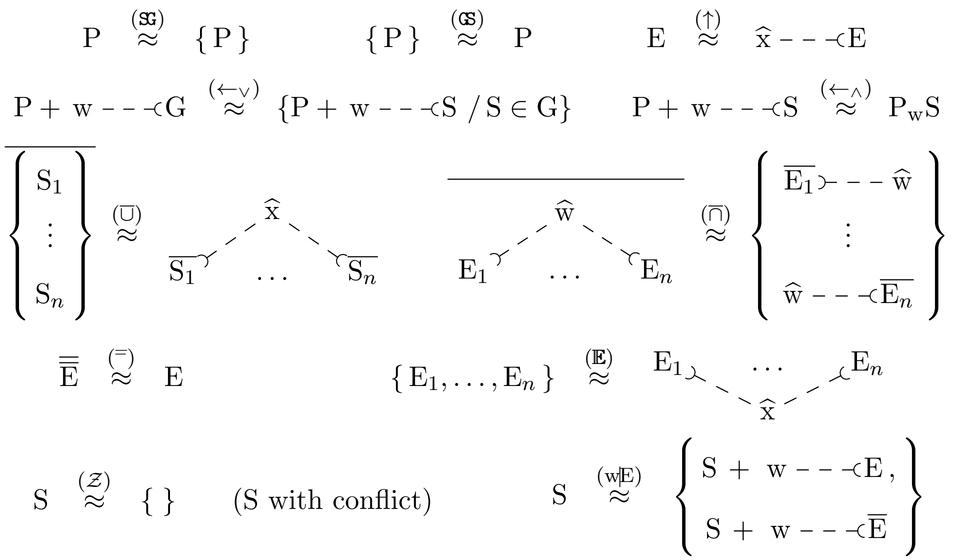
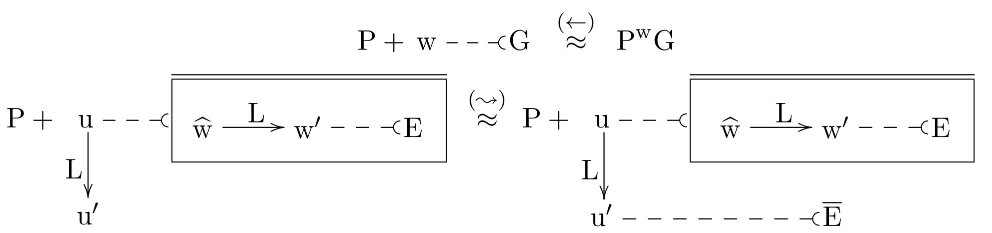


Table 2
Derived graph calculus rules



We also have rules for capturing special properties of a relation.

(Rf[L]) For a reflexive relation L: expand node w to $\begin{tikzcd} w \arrow[loop right, "L"] \end{tikzcd}$.

(Tr[L]) For a transitive relation L: expand $u \xrightarrow{L} v \xrightarrow{L} w$ to $\begin{tikzcd} u \arrow[r, "L"] \arrow[rr, bend right, "L"'] & v \arrow[r, "L"] & w \end{tikzcd}$.

(Sm[L]) For a symmmetric relation L: expand $u \xrightarrow{L} v$ to $\begin{tikzcd} u \arrow[r, bend left, "L"] & v \arrow[l, bend left, "L"] \end{tikzcd}$.

(As[L]) For an anti-symmmetric relation L: identify nodes u and v with $u \xleftrightarrow{L} v$.

We often use $u \xleftrightarrow{L} v$ as short for $\begin{tikzcd} u \arrow[r, bend left, "L"] & v \arrow[l, bend left, "L"] \end{tikzcd}$.

The *general graph calculus* consists of the conversion rules in Table 1 (p. 7). A *graph calculus* extends the general one by some rules for properties of relations (as above). A *derivation* is a finite sequence of rule applications. The general calculus is sound for equivalence: if E derives F, then $E \equiv F$.

Proposition 3.2 (Graph calculi) *Each graph calculus is refutationally sound and complete: a finite expression set $\mathbb{E}$ is null iff $\mathbb{E}$ derives the empty graph $\{\ \}$.* $\flat$

Proof. Soundness is clear: the rules involve equivalent expressions. For completeness: if $\mathbb{E}$ does not derives $\{\ \}$, then we can obtain a chain of non-zero slices, whose underlying drafts have as co-limit a non-zero sketch Σ; $\mathbb{E}$ is non-null in the natural structure $\mathfrak{N}[\Sigma]$. If a rule like ($\mathbb{Tr}$[L]) is present, then $\mathfrak{N}[\Sigma]$ will be L-transitive, as Σ is "saturated". For more details, see [14,15]. $\square$

4 Intuitionistic Modal Logic: Flat Semantics

We now examine flat semantics for intuitionistic modal logic, akin to that in [13]. It is convenient to consider a hierarchy of structures.

A *pre-relational structure* $\mathfrak{B}$ consists of a set $\mathsf{W} \neq \emptyset$ (of worlds) with a special binary relation $\leq$ on W, together with a binary relation $\mathrm{r}^{\mathfrak{B}}$ on W (for $\mathrm{r} \in \mathbb{RS}$) and a valuation $\mathsf{M} : \mathbb{PL} \to \wp(\mathsf{W})$. We use the abbreviations P for $\mathrm{p}^{\mathfrak{B}} \coloneqq \mathsf{M}(\mathrm{p})$ and R for $\mathrm{r}^{\mathfrak{B}}$. We also introduce function $\mathsf{Lv} : \mathsf{W} \to \wp(\mathbb{PL})$ by $\mathsf{u} \in \mathsf{W} \mapsto \{\mathrm{p} \in \mathbb{PL} \,/\, \mathsf{u} \in \mathsf{M}(\mathrm{p})\}$.

Formula *satisfaction* (with $\Vdash$ as short for $\Vdash_{\mathfrak{B}}$) is as follows. The local cases are: $\mathsf{u} \nVdash \bot$; $\mathsf{u} \Vdash \mathrm{p}$ iff $\mathsf{u} \in \mathsf{M}(\mathrm{p})$ (i. e., $\mathrm{p} \in \mathsf{Lv}(\mathsf{u})$); $\mathsf{u} \Vdash \psi \wedge \theta$ iff $\mathsf{u} \Vdash \psi$ and $\mathsf{u} \Vdash \theta$; $\mathsf{u} \Vdash \psi \vee \theta$ iff $\mathsf{u} \Vdash \psi$ or $\mathsf{u} \Vdash \theta$. For $\langle\,\rangle$: $\mathsf{u} \Vdash \langle \mathrm{r}\rangle\varphi$ iff, for some $\mathsf{v} \in \mathsf{W}$, $\mathsf{u}\,\mathsf{R}\,\mathsf{v}$ and $\mathsf{v} \Vdash \varphi$. For $\to$: $\mathsf{u} \Vdash \psi \to \theta$ iff, for every $\mathsf{v} \geq \mathsf{u}$, if $\mathsf{v} \Vdash \psi$, then $\mathsf{v} \Vdash \theta$ (i. e., there exists no $\mathsf{v} \geq \mathsf{u}$ such that $\mathsf{v} \Vdash \psi$ and $\mathsf{v} \nVdash \theta$). For $[\,]$: $\mathsf{u} \Vdash [\mathrm{r}]\varphi$ iff, for all $\mathsf{v}, \mathsf{w} \in \mathsf{W}$, if $\mathsf{v} \geq \mathsf{u}$ and $\mathsf{v}\,\mathsf{R}\,\mathsf{w}$, then $\mathsf{w} \Vdash \varphi$ (i. e., there exist no $\mathsf{v}, \mathsf{w} \in \mathsf{W}$ such that $\mathsf{v} \geq \mathsf{u}$, $\mathsf{v}\,\mathsf{R}\,\mathsf{w}$ and $\mathsf{w} \nVdash \varphi$). Thus, for $\neg$: $\mathsf{u} \Vdash \neg\varphi$ iff, for every $\mathsf{v} \geq \mathsf{u}$, $\mathsf{v} \nVdash \varphi$ (i. e., there exists no $\mathsf{v} \geq \mathsf{u}$ such that $\mathsf{v} \Vdash \varphi$).

A *relational structure* is a pre-relational structure $\mathfrak{B}$ where relation $\leq$ is a partial order on W. To have monotonicity of satisfaction, one restricts relational structures to *birelational structures* by imposing 3 extra requirements. Monotone valuation: if $\mathsf{u} \leq \mathsf{u}'$, then $\mathsf{Lv}(\mathsf{u}) \subseteq \mathsf{Lv}(\mathsf{u}')$. (F1): given $\mathsf{u}', \mathsf{u}, \mathsf{v} \in \mathsf{W}$, such that $\mathsf{u}' \geq \mathsf{u}$ and $\mathsf{u}\,\mathsf{R}\,\mathsf{v}$, there exists $\mathsf{v}' \in \mathsf{W}$, such that $\mathsf{u}'\,\mathsf{R}\,\mathsf{v}'$ and $\mathsf{v} \leq \mathsf{v}'$. (F2): given $\mathsf{u}, \mathsf{v}, \mathsf{v}' \in \mathsf{W}$, such that $\mathsf{u}\,\mathsf{R}\,\mathsf{v}$ and $\mathsf{v} \leq \mathsf{v}'$, there exists $\mathsf{u}' \in \mathsf{W}$, such that $\mathsf{u} \leq \mathsf{u}'$ and $\mathsf{u}'\,\mathsf{R}\,\mathsf{v}'$. [13, p. 50]

To reason graphically about flat semantics with a symbol wc for $\leq$, we consider a graph language $\mathbb{GL}_{\mathfrak{f}}$ with $\mathbb{Sb}_1 \coloneqq \Phi$ and $\mathbb{Sb}_2 \coloneqq \mathbb{RS} \cup \{\mathrm{wc}\}$. We draw wc-arrows as $\twoheadrightarrow$. A pre-relational structure $\mathfrak{B}$ gives a structure for $\mathbb{GL}_{\mathfrak{f}}$ with $\varphi^{\mathfrak{B}} \coloneqq \{\mathsf{u} \in \mathsf{W} \,/\, \mathsf{u} \Vdash_{\mathfrak{B}} \varphi\}$.

Then, we can handle logical symbols by the 6 *pre-relational elimination rules* converting formulas to equivalent expressions given in Table 3.

The pre-relational elimination rules in Table 3 transcribe formula satisfaction in graph terms, which guarantees their soundness. For instance, for $(\langle\,\rangle)$, we have: $\mathsf{s} \in (\langle \mathrm{r}\rangle\varphi)^{\mathfrak{B}}$ iff, for some $\mathsf{v} \in \mathsf{W}$, $\mathsf{u}\,\mathsf{R}\,\mathsf{v}$ and $\mathsf{v} \Vdash_{\mathfrak{B}} \varphi$, i. e., assignment g with $\mathrm{x}^{\mathsf{g}} = \mathsf{u}$ and $\mathrm{y}^{\mathsf{g}} = \mathsf{v}$ satisfies draft $\mathrm{x} \xrightarrow{\mathrm{r}} \mathrm{y} - - \!\!\prec \varphi$ iff $\mathsf{u} \in [\,\widehat{\mathrm{x}} \xrightarrow{\mathrm{r}} \mathrm{y} - - \!\!\prec \varphi\,]_{\mathfrak{B}}$. Also, for $(\to)$, we have: $\mathsf{u} \notin (\psi \to \theta)^{\mathfrak{B}}$ iff $\mathsf{s} \nVdash_{\mathfrak{B}} \psi \to \theta$ iff there is some $\mathsf{v} \geq \mathsf{u}$ such that $\mathsf{v} \Vdash_{\mathfrak{B}} \psi$ and $\mathsf{v} \nVdash_{\mathfrak{B}} \theta$, i. e., assignment g with $\mathrm{x}^{\mathsf{g}} = \mathsf{u}$ and $\mathrm{y}^{\mathsf{g}} = \mathsf{v}$ satisfies draft

Table 3
Pre-relational elimination rules

Formula	$\approx$	Expression	Comment
$\bot$	$\overset{(\bot)}{\approx}$	$\{\,\}$	empty graph
$\psi \wedge \theta$	$\overset{(\wedge)}{\approx}$	$\psi \succ\!- - - \widehat{\mathrm{x}} - - -\!\prec \theta$	single-node slice
$\psi \vee \theta$	$\overset{(\vee)}{\approx}$	$\left\{ \begin{array}{l} \psi \succ\!- - - \widehat{\mathrm{x}}\ , \\ \widehat{\mathrm{x}} - - -\!\prec \theta \end{array} \right\}$	graph with single-node slices
$\langle \mathrm{r} \rangle\, \varphi$	$\overset{(\langle\rangle)}{\approx}$	$\widehat{\mathrm{x}} \xrightarrow{\mathrm{r}} \mathrm{y} - - -\!\prec \varphi$	2-node slice
$\psi \to \theta$	$\overset{(\to)}{\approx}$	$\overline{\widehat{\mathrm{x}} \twoheadrightarrow \mathrm{y} \begin{array}{l} - - -\!\prec \psi \\ - - -\!\prec \overline{\theta} \end{array}}$	complemented 2-node slice
$[\mathrm{r}]\, \varphi$	$\overset{([\,])}{\approx}$	$\overline{\widehat{\mathrm{x}} \twoheadrightarrow \mathrm{y} \xrightarrow{\mathrm{r}} \mathrm{z} - - -\!\prec \overline{\varphi}}$	complemented 3-node slice

$\mathrm{x} \twoheadrightarrow \mathrm{y} \begin{array}{l} - - -\!\prec \psi \\ - - -\!\prec \overline{\theta} \end{array}$ iff u is in the extension of slice $\widehat{\mathrm{x}} \twoheadrightarrow \mathrm{y} \begin{array}{l} - - -\!\prec \psi \\ - - -\!\prec \overline{\theta} \end{array}$ in $\mathfrak{B}$.

One can also consider some variations. For the condition "$\mathsf{w} \Vdash \langle \mathrm{r} \rangle \varphi$ iff, there are $\mathsf{w}_0, \mathsf{v}_0 \in \mathsf{W}$ such that $\mathsf{w} \geq \mathsf{w}_0$, $\mathsf{w}_0\, \mathsf{R}\, \mathsf{v}_0$ and $\mathsf{v}_0 \Vdash \varphi$" (attributed to Plotkin and Stirling [13, p. 49]), we obtain the slice $\widehat{\mathrm{x}} \twoheadleftarrow \mathrm{y} \xrightarrow{\mathrm{r}} \mathrm{z} - - -\!\prec \varphi$. We could similarly handle a condition like "$\mathsf{w} \Vdash \langle \mathrm{r} \rangle \varphi$ iff, for all $\mathsf{w}' \in \mathsf{W}$, if $\mathsf{w}' \geq \mathsf{w}$ there exists $\mathsf{v}' \in \mathsf{W}$, such that $\mathsf{w}'\, \mathsf{R}\, \mathsf{v}'$ and $\mathsf{v}' \Vdash \varphi$".

The next result illustrates how one can obtain expressions for complex formulas from those of its immediate sub-formulas.

Proposition 4.1 (Derived pre-relational conversions) *The pre-relational conversions in Tables 4 and 5 are derived.* ♭

Proof. By graph rules and pre-relational elimination rules. For instance, $(\neg)$ is clear; for $(\overline{[\,]})$: $\overline{[\mathrm{r}]\, \varphi} \overset{([\,])}{\approx} \overline{\overline{\widehat{\mathrm{x}} \twoheadrightarrow \mathrm{y} \xrightarrow{\mathrm{r}} \mathrm{z} - - -\!\prec \overline{\varphi}}} \overset{(=)}{\approx} \widehat{\mathrm{x}} \twoheadrightarrow \mathrm{y} \xrightarrow{\mathrm{r}} \mathrm{z} - - -\!\prec \overline{\varphi}$; for $(\overline{\neg\neg})$: $\overline{\neg\neg\varphi} \overset{(\neg)}{\approx} \overline{\overline{\boxed{\widehat{\mathrm{x}} \twoheadrightarrow \mathrm{y} - - -\!\prec \neg\varphi}}} \overset{(=;\neg)}{\approx} \widehat{\mathrm{x}} \twoheadrightarrow \mathrm{y} - - -\!\prec \overline{\overline{\boxed{\widehat{\mathrm{x}} \twoheadrightarrow \mathrm{y} - - -\!\prec \varphi}}}$.

For $([\,]\wedge)$: $[\mathrm{r}]\,(\psi \wedge \theta) \overset{([\,])}{\approx} \overline{\widehat{\mathrm{x}} \twoheadrightarrow \mathrm{y} \xrightarrow{\mathrm{r}} \mathrm{z} - - -\!\prec \overline{\psi \wedge \theta}} \overset{(\wedge)}{\approx}$

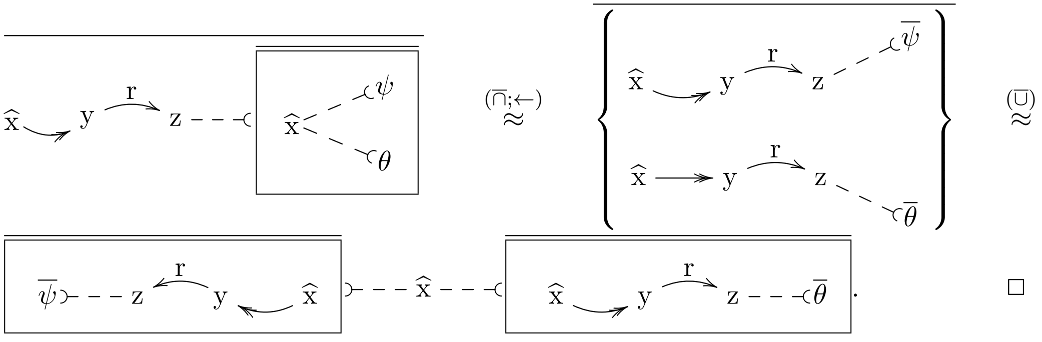
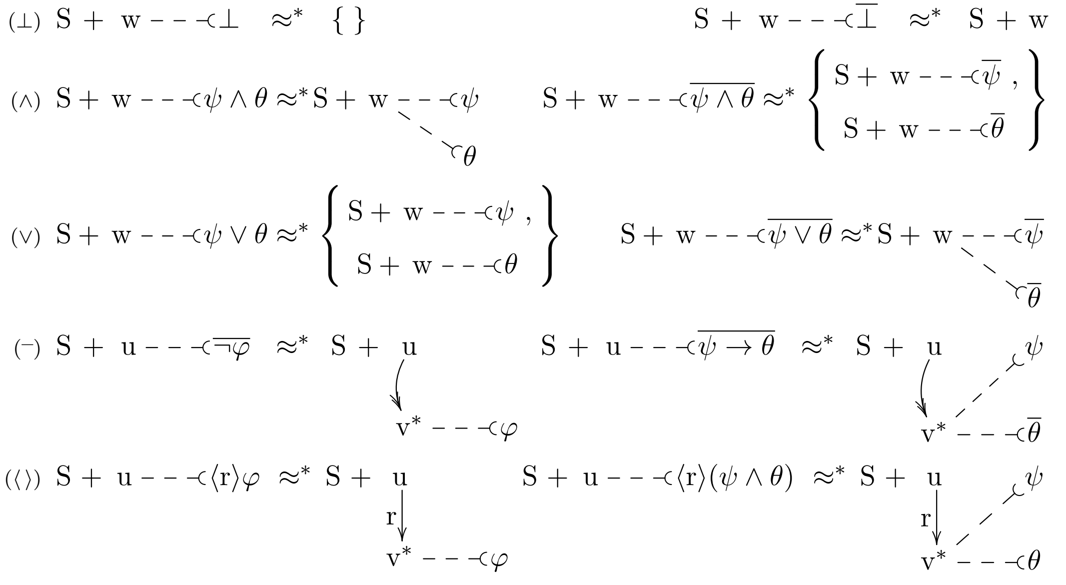
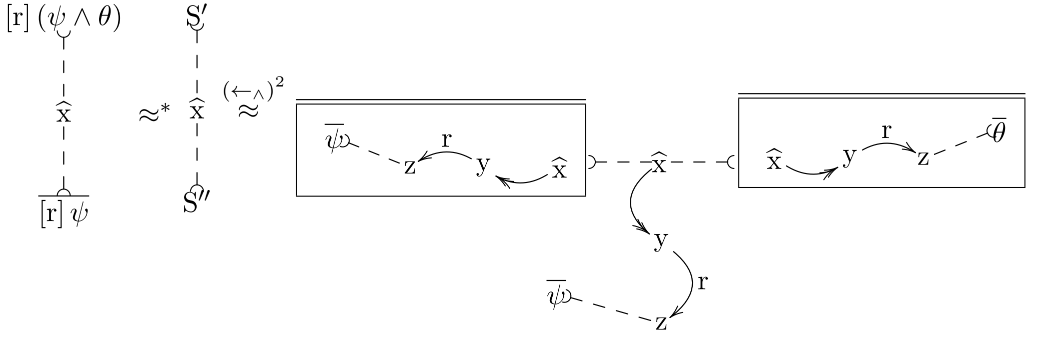


Table 4
Derived pre-relational slice conversions (the nodes with '*' are new)



Example 4.2 (Pre-relational consequence) To show that $[\mathrm{r}]\,\psi$ is a consequence of $[\mathrm{r}]\,(\psi \wedge \theta)$, we first convert $[\mathrm{r}]\,(\psi \wedge \theta) \approx^* \mathrm{S}'$ and $\overline{[\mathrm{r}]\,\psi} \approx^* \mathrm{S}''$ (cf. Table 5). We have:



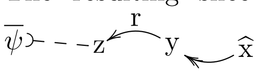
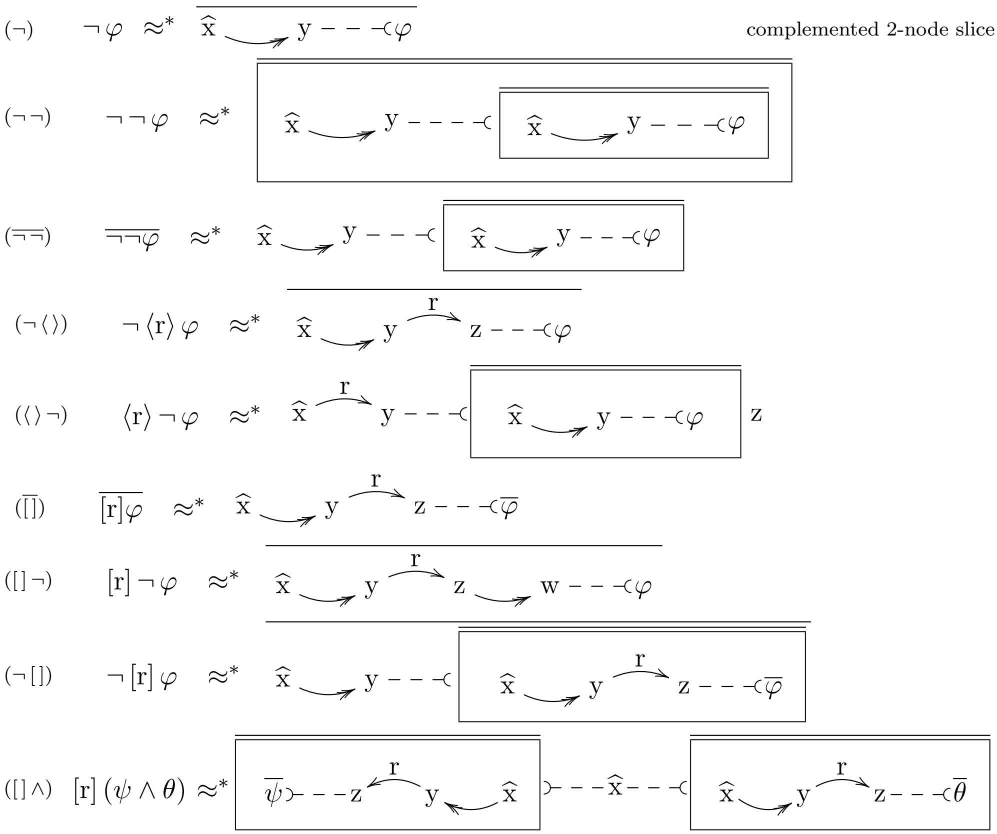
The resulting slice S is zero: slice S has as witness at node x the slice  under morphism $\mathrm{x} \mapsto \mathrm{x}$, $\mathrm{y} \mapsto \mathrm{y}$, $\mathrm{z} \mapsto \mathrm{z}$. (Notice that S can be

Table 5
Derived pre-relational expression conversions



shifted to a slice with expression witness ψ at z.) Thus S is unsatisfiable. Hence, set $\{[\mathrm{r}](\psi \wedge \theta), \overline{[\mathrm{r}]\,\psi}\}$ cannot be satisfied in a pre-relational structure. ♮

We now indicate how our calculus handles the necessitation rule, namely: "from theorem ψ, infer theorem $[\mathrm{r}]\,\psi$". Its refutational analogue can be formulated as: "if $\overline{\varphi} \approx^* \{\ \}$, then $\overline{[\mathrm{r}]\,\varphi} \approx^* \{\ \}$". Now, if $\overline{\varphi} \approx^* \{\ \}$, then we will have the conversions: $\overline{[\mathrm{r}]\,\varphi} \overset{(\overline{[]})}{\approx} \hat{\mathrm{x}} \rightarrow \mathrm{y} \xrightarrow{\mathrm{r}} \mathrm{z} - - \prec \overline{\varphi} \approx^* \hat{\mathrm{x}} \rightarrow \mathrm{y} \xrightarrow{\mathrm{r}} \mathrm{z} - - \prec \{\ \} \overset{(\leftarrow)}{\approx} \{\ \}$.

We also have rules coming from the intended meaning of wc as $\leq$. The *relational operational rules* are (Rf[wc]), (As[wc]) and (Tr[wc]) (cf. Section 3, p. 7).

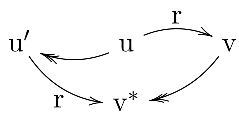
A sketch, or slice, of $\mathbb{GL}_{\mathrm{f}}$ is wc-*reduced* iff u = v, whenever it has arcs $\mathrm{u} \longleftrightarrow \mathrm{v}$.
Every $\mathbb{GL}_{\mathrm{f}}$-slice can be contracted to a wc-reduced slice.[8]

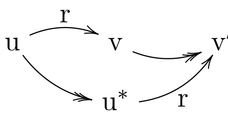
For birelational structures, we also have the 3 *birelational transformation rules*:

(p) Contract slice S + $\mathrm{p} \succ - - \mathrm{u} \rightarrow \mathrm{u}' - - \prec \overline{\mathrm{p}}$ to the empty graph $\{\ \}$.

---

[8] For instance, $\hat{\mathrm{x}} \underset{}{\overset{\mathrm{r}}{\rightleftarrows}} \mathrm{y} - - \prec \mathrm{p}$ is not wc-reduced, but it contracts to the wc-reduced $\hat{\mathrm{z}} - - \prec \mathrm{p}$.

(F1) Expand slice $S + u' \twoheadleftarrow u \xrightarrow{r} v$ to S +  (with new v^*).

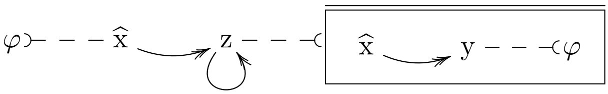
(F2) Expand slice $S + u \xrightarrow{r} v \twoheadrightarrow v'$ to S +  (with new u^*).

Then, we can derive the following *birelational formula transfer* conversion: $S + \varphi \succ - - u \twoheadrightarrow u' \approx^* S + \varphi \succ - - u \twoheadrightarrow u' - - \prec \varphi$. (By the alternative expansion rule (w|E): case $\langle r \rangle \varphi$ follows from (F1) and case $[r]\varphi$ follows from ($\mathfrak{Tr}$[wc])).

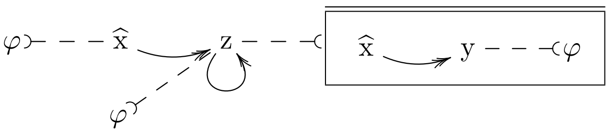
Example 4.3 (Birelational consequence) To show that $\neg\neg\varphi$ is a birelational consequence of φ, we consider the set $\{\varphi, \overline{\neg\neg\varphi}\}$ and use ($\overline{\neg\neg}$) (cf. Table 5, p. 11).

(i) $\{\varphi, \overline{\neg\neg\varphi}\} \approx^* S_1$, with $S_1 = \varphi \succ - - \hat{x} \twoheadrightarrow z - - \prec \boxed{\hat{x} \twoheadrightarrow y - - \prec \varphi}$.

(ii) By reflexivity ($\mathfrak{Rf}$[wc]), slice S_1 expands to the following slice S_2:



(iii) Transfer formula φ from x to z to obtain the following slice S_3:



Slice S_3 is zero: slice S_3 has as witness at node z the slice $\hat{x} \twoheadrightarrow y - - \prec \varphi$ under morphism $x, y \mapsto z$. (Notice that S_3 can be shifted (cf. p. 6) to a slice with expression witness φ at z.) So, S_3 is unsatisfiable.

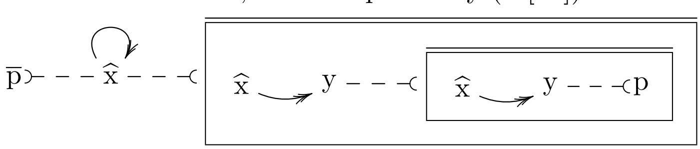
Hence, set $\{\varphi, \overline{\neg\neg\varphi}\}$ cannot be satisfied in a birelational structure. ♮

We can show graphically that the following formulas (cf. [13, p. 51, 52]) are birelationally valid: $[r](\psi \to \theta) \to ([r]\psi \to [r]\theta)$, $[r](\psi \to \theta) \to (\langle r \rangle\psi \to \langle r \rangle\theta)$, $\neg\langle r \rangle\bot$, $\langle r \rangle(\psi \vee \theta) \to (\langle r \rangle\psi \vee \langle r \rangle\theta)$, $(\langle r \rangle\psi \to [r]\theta) \to [r](\psi \to \theta)$ and $\neg\langle r \rangle\varphi \to [r]\neg\varphi$. (In fact, $\neg\langle r \rangle\bot$ can be seen to be pre-relationally valid.)

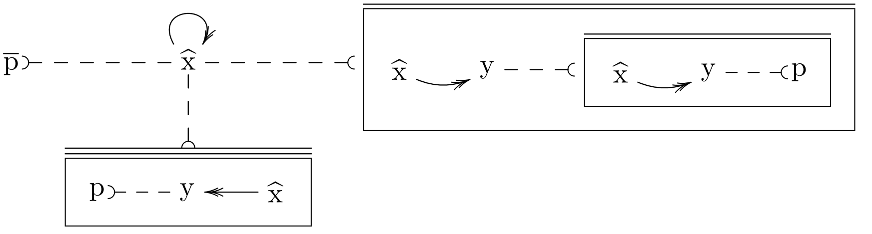
The natural construction (cf. Section 3, p. 5) applied to a proper wc-reduced $\mathfrak{GL}_{\mathfrak{f}}$-sketch Σ gives a pre-relational structure $\mathfrak{B}[\Sigma]$.

Example 4.4 (Birelational non-consequence) To show that p is not a birelational consequence of $\neg\neg p$, we consider the set $\{\neg\neg p, \overline{p}\}$ and use ($\neg\neg$) (cf. Table 5, p. 11).

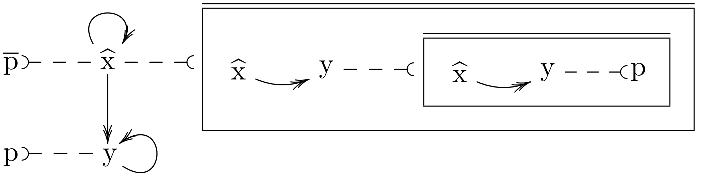
(i) Set $\{\neg\neg p, \overline{p}\}$ converts to a slice, which expands by ($\mathfrak{Rf}$[wc]) to slice S_1:



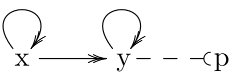
(ii) Now, slice S_1 shifts (cf. Section 3, p. 6) to the following slice S_2:

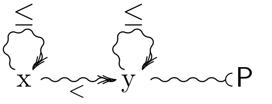


(iii) By ($^{=}$), lowering ($\leftarrow_{\wedge}$) (cf. Table 1) and reflexivity ($\mathtt{Rf}[\mathrm{wc}]$) , we have slice S_3:



(iv) The positive part of $\underline{S_3}$ and corresponding natural structure $\mathfrak{B}$ are:





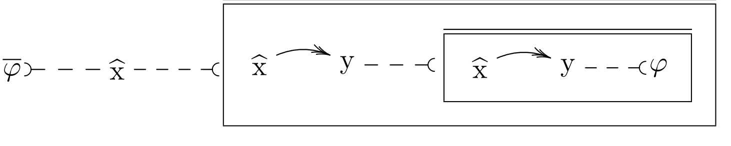
Note that structure $\mathfrak{B}$ is birelational. Much as in Example 3.1 (Natural construction), we see that the identity assignment $\mathtt{1}$ satisfies draft $\underline{S_3}$ in $\mathfrak{B}$.

Hence, these slices and set $\{\neg\neg p, \overline{p}\}$ are satisfiable in a birelational structure. Thus, p is not a birelational consequence of $\neg\neg p$. ♮

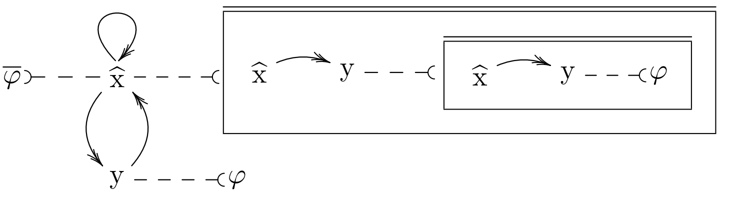
The special binary relation $\leq$ of a pre-relational structure may be symmetric. For such cases, we use the rule ($\mathtt{Sm}[\mathrm{wc}]$) (cf. Section 3, p, 7).

Example 4.5 (Symmetric birelational consequence) To show that φ is a symmetric birelational consequence of $\neg\neg\varphi$, we consider the set $\{\neg\neg\varphi, \overline{\varphi}\}$ and use ($\neg\,\neg$).

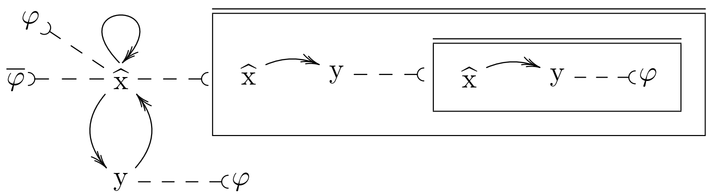
(i) Set $\{\neg\neg\varphi, \overline{\varphi}\}$ converts to the following slice S_1:



(ii) By ($\mathtt{Rf}[\mathrm{wc}]$), graph rules and symmetry ($\mathtt{Sm}[\mathrm{wc}]$), we transform S_1 to slice S_2:



(iii) Now, transfer formula φ from y to x (cf. p. 12), expanding S_2 to the slice S_3:



This slice S_3 has a conflict at node x (with formula φ as expression witness). So, in a birelational structure with symmetric $\leq$, one cannot satisfy $\{\neg\neg\varphi, \overline{\varphi}\}$. $\natural$

We can similarly show that $\varphi \vee \neg\varphi$ is valid in symmetric birelational structures.

The flat graph calculi are as follows. The *pre-relational calculus* is the graph calculus for graph language $\mathbb{GL}_{\mathsf{f}}$ with the elimination rules in Table 3. The *relational calculus* is the extension of the pre-relational calculus by the rules ($\mathbb{Rf}$[wc]), ($\mathbb{As}$[wc]) and ($\mathbb{Tr}$[wc]) (cf. p. 11). The *birelational calculus* is the extension of the relational calculus by the rules (p), ($F1$) and ($F2$) (cf. p. 11). The *symmetric flat calculus* is the extension of the birelational calculus by the symmetric rule ($\mathbb{Sm}$[wc]) (cf. p. 13).

Theorem 4.6 (Flat calculi) *The flat graph calculi are sound and complete.* $\flat$

Proof. By Proposition 3.2: Graph calculi (p. 8). □

5 Intuitionistic Modal Logic: Decoupled Semantics

We now examine another semantics for intuitionistic modal logic.

The motivation comes from decoupling objects and stages, as in [5]. A *stratified structure* consists of a set I (of stages) partially ordered by $\preceq$ with, for each $i \in I$: a universe $C_i \neq \emptyset$ (of objects), a subset $\mathrm{p}_i^{\mathfrak{C}} \subseteq C_i$ (for $\mathrm{p} \in \mathbb{PL}$) and binary relation $\mathrm{r}_i^{\mathfrak{C}}$ on C_i (for $\mathrm{r} \in \mathbb{RS}$). We use the abbreviations: P_i for $\mathrm{p}_i^{\mathfrak{C}}$ and R_i for $\mathrm{r}_i^{\mathfrak{C}}$.

We prefer another formulation as follows. A *pre-graded structure* $\mathfrak{C}$ consists of 2 sets I (of stages), with a special binary relation $\preceq$ on it, and $C \neq \emptyset$; it has as domain a non-empty subset $C_\times$ of $C \times I$ and (with abbreviations P for $\mathrm{p}^{\mathfrak{C}}$ and R for $\mathrm{r}^{\mathfrak{C}}$) a subset $\mathsf{P} \subseteq C_\times$ (for $\mathrm{p} \in \mathbb{PL}$) and 2-ary relation R on $C_\times$ (for $\mathrm{r} \in \mathbb{RS}$) such that $i = j$ whenever $\langle a, i\rangle \,\mathsf{R}\, \langle b, j\rangle$. We can introduce a special relation on ordered pairs by $\langle a, i\rangle \leq \langle b, j\rangle$ iff $i \preceq j$. We then obtain a pre-relational structure.

Satisfaction (with $\Vdash$ as short for $\Vdash_{\mathfrak{C}}$) is as follows. For $\bot$, p, $\wedge$, $\vee$ and $\langle\,\rangle$, it is as in Section 4, with $\mathsf{u} = \langle a, i\rangle$. For $\to$: $\langle a, i\rangle \Vdash \psi \to \theta$ iff , for every $j \succeq i$, if $\langle a, j\rangle \Vdash \psi$ then $\langle a, j\rangle \Vdash \theta$ (i. e., there exists no $j \succeq i$ such that $\langle a, j\rangle \Vdash \psi$ and $\langle a, j\rangle \nVdash \theta$). For $[\,]$: $\langle a, i\rangle \Vdash [\mathrm{r}]\varphi$ iff, for all $j \succeq i$ and $b \in C$, if $\langle a, j\rangle \,\mathsf{R}\, \langle b, j\rangle$ then $\langle b, j\rangle \Vdash \varphi$ (i. e., there exist no $j \succeq i$ and b such that $\langle a, j\rangle \,\mathsf{R}\, \langle b, j\rangle$ and $\langle b, j\rangle \nVdash \varphi$). Thus, for $\neg$: $\langle a, i\rangle \Vdash \neg\varphi$ iff, for every $j \succeq i$, $\langle a, j\rangle \nVdash \varphi$ (i. e., there exists no $j \succeq i$ such that $\langle a, j\rangle \Vdash \varphi$).

As in Section 4 (p. 8), we consider some restrictions. A *graded structure* is a pre-graded structure $\mathfrak{C}$ where special relation $\preceq$ is a partial order on I. A *growing-graded* structure is a graded one with growing universes, predicates and relations. For $i \preceq j \in I$: if $\langle a, i\rangle \in C_\times$ then $\langle a, j\rangle \in C_\times$ (i. e., $C_i \subseteq C_j$); if $\langle a, i\rangle \in \mathsf{P}$ then

$\langle a, j\rangle \in \mathsf{P}$ (i. e., $\mathsf{P}_i \subseteq \mathsf{P}_j$); if $\langle a, i\rangle\, \mathsf{R}\, \langle b, i\rangle$ then $\langle a, j\rangle\, \mathsf{R}\, \langle b, j\rangle$ (i. e., $\mathsf{R}_i \subseteq \mathsf{R}_j$).[9] On a growing-graded structure, satisfaction can be seen to be monotonic.

We wish to reason graphically about decoupled semantics with symbols ic (for $\preceq$) and eo (with intended meaning $\langle a, i\rangle \mathrm{eo} \langle b, j\rangle$ iff $a = b$). For this purpose, we consider a graph language $\mathbb{GL}_\mathrm{d}$ with $\mathbb{Sb}_1 \coloneqq \Phi$ and $\mathbb{Sb}_2 \coloneqq \mathbb{RS} \cup \{\mathrm{ic}, \mathrm{eo}\}$. We draw ic-arrows as $\dashrightarrow$ and eo-arrows as $\cdots\!\!>$. A pre-graded structure $\mathfrak{C}$ gives a structure for $\mathbb{GL}_\mathrm{d}$ with $\varphi^{\mathfrak{C}} \coloneqq \{\langle a, i\rangle \in C_\times \,/\, \langle a, i\rangle \Vdash_{\mathfrak{C}} \varphi\}$.

Then, we can handle logical symbols by 6 *pre-graded elimination rules* converting formulas to equivalent expressions, much as before. The rules for $\perp$, p, $\wedge$, $\vee$ and $\langle\,\rangle$ are as in Table 3 (p. 9). The other 2 rules convert formulas $\psi \rightarrow \theta$ and $[\mathrm{r}]\,\varphi$, respectively, to the expressions
$$\overline{\begin{tikzcd}[row sep=small] & & \prec\psi \\ \widehat{\mathrm{x}} \arrow[r, dotted, bend left] \arrow[r, dashed, bend right] & \mathrm{y} \arrow[ur, dashed, -] \arrow[dr, dashed, -] & \\ & & \prec\overline{\theta} \end{tikzcd}} \quad \text{and} \quad \overline{\widehat{\mathrm{x}} \;\substack{\cdots\rightarrow \\ \dashrightarrow}\; \mathrm{y} \xrightarrow{\mathrm{r}} \mathrm{z} - - \prec\overline{\varphi}}\ .$$
Also, formula $\neg\,\varphi$ converts to the expression $\overline{\widehat{\mathrm{x}} \;\substack{\cdots\rightarrow \\ \dashrightarrow}\; \mathrm{y} - - \prec\varphi}$.

Thus, we have derived pre-graded conversions much as those in Proposition 4.1.

Also, the intended meanings of eo and ic lead to some operational rules as follows (cf. Section 3, p. 7). For eo, we have ($\mathtt{Rf}$[eo]), ($\mathtt{Sm}$[eo]) and ($\mathtt{Tr}$[eo]). For graded ic, we have ($\mathtt{Rf}$[ic]), ($\mathtt{As}$[ic]) and ($\mathtt{Tr}$[ic]), as well as the rule identifying nodes u and v such that $\mathrm{u} \;\substack{\leftarrow\cdots\rightarrow \\ \dashleftarrow\dashrightarrow}\; \mathrm{v}$. For a symmetric $\preceq$, we use ($\mathtt{Sm}$[ic]).

For growing-graded structures, we also have the following 3 *growing transformation rules*. For domain: given $\mathrm{u} \dashrightarrow \mathrm{v}$, add $\mathrm{u} \overset{\cdots}{\rightarrow} \mathrm{u}^* \dashleftarrow\!\!\dashrightarrow \mathrm{v}$ (with new node u*). For $\mathrm{p} \in \mathbb{PL}$: erase slice with $\mathrm{p}\succ - - \mathrm{u} \;\substack{\cdots\rightarrow \\ \dashrightarrow}\; \mathrm{v} - - \prec\overline{\mathrm{p}}$. For $\mathrm{r} \in \mathbb{RS}$: given $\mathrm{u}' \;\substack{\leftarrow\cdots \\ \dashleftarrow}\; \mathrm{u} \xrightarrow{\mathrm{r}} \mathrm{v} \;\substack{\cdots\rightarrow \\ \dashrightarrow}\; \mathrm{v}'$, add $\mathrm{u}' \xrightarrow[\mathrm{r}]{} \mathrm{v}'$. Then, we can derive the growing formula transfer: $\mathrm{S} + \varphi\succ - - \mathrm{u} \;\substack{\cdots\rightarrow \\ \dashrightarrow}\; \mathrm{v} \;\approx^*\; \mathrm{S} + \varphi\succ - - \mathrm{u} \;\substack{\cdots\rightarrow \\ \dashrightarrow}\; \mathrm{v} - - \prec\varphi$.

We can establish consequence as in Section 4, with ic and eo in lieu of wc. We can show that $[\mathrm{r}]\,\psi$ is a pre-graded consequence of $[\mathrm{r}]\,(\psi\wedge\theta)$ as in Example 4.2, that $\neg\neg\varphi$ is a growing-graded consequence of φ as in Example 4.3, and that φ is a symmetric growing-graded consequence of $\neg\neg\varphi$ as in Example 4.5 (notice that symmetric growing-graded structures have constant universes, predicates and relations).

We can also establish non-consequence much as in Section 4, even though the natural construction is now more involved. A sketch is $\cap$-*reduced* iff $\mathrm{u} = \mathrm{v}$, whenever it has arcs $\mathrm{u} \;\substack{\leftarrow\cdots\rightarrow \\ \dashleftarrow\dashrightarrow}\; \mathrm{v}$. The natural construction applied to a proper $\cap$-reduced $\mathbb{GL}_\mathrm{d}$-sketch $\Sigma = \langle \mathrm{N}; \mathtt{A}\rangle$ gives a pre-graded structure $\mathfrak{C}[\Sigma]$ and assignment $\mathtt{h}_\Sigma$ as follows.

(i) Define 2-ary relations on N: $\mathrm{u} \overset{\mathrm{eo}}{\sim} \mathrm{v}$ iff $\mathrm{u} \overset{\cdots}{\leftarrow\!\rightarrow} \mathrm{v} \in \mathtt{A}$ and $\mathrm{u} \overset{\mathrm{ic}}{\sim} \mathrm{v}$ iff $\mathrm{u} \dashleftarrow\!\!\dashrightarrow \mathrm{v} \in \mathtt{A}$. We have equivalences, with quotient sets $I \coloneqq \mathrm{N}/_{\overset{\mathrm{ic}}{\sim}}$ and $C \coloneqq \mathrm{N}/_{\overset{\mathrm{eo}}{\sim}}$.

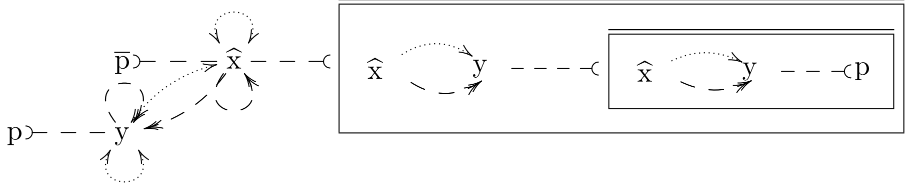
(ii) Define domain $C_\times \coloneqq \{\langle [\mathrm{w}]_{\mathrm{eo}}, [\mathrm{w}]_{\mathrm{ic}}\rangle \in C \times I \,/\, \mathrm{w} \in \mathrm{N}\}$ and special relation $\preceq$ on I by $[\mathrm{u}]_{\mathrm{ic}} \preceq [\mathrm{v}]_{\mathrm{ic}}$ iff $\mathrm{u} \dashrightarrow \mathrm{v} \in \mathtt{A}$.

[9] Notice that these restrictions are simpler and more intuitive than requirements (F1) and (F2) in Section 4.

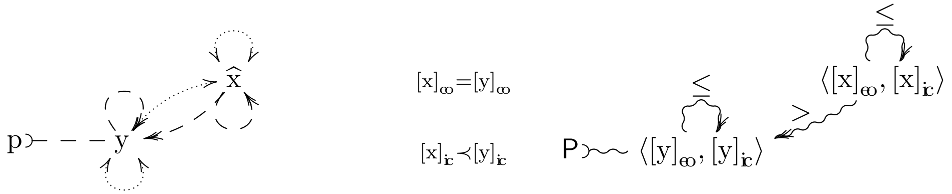
(iii) Define subsets by $\langle [\mathrm{w}]_{\text{ꝏ}}, [\mathrm{w}]_{\text{ɨc}} \rangle \in \mathsf{P}$ iff $\mathrm{w} - - \prec \mathrm{p} \in \mathtt{A}$ and relations by $\langle [\mathrm{u}]_{\text{ꝏ}}, [\mathrm{v}]_{\text{ɨc}} \rangle \, \mathsf{R} \, \langle [\mathrm{v}]_{\text{ꝏ}}, [\mathrm{v}]_{\text{ɨc}} \rangle$ iff $\mathrm{u} \xrightarrow{\mathrm{r}} \mathrm{v} \in \mathtt{A}$.

(iv) Define natural assignment $\mathtt{h}_\Sigma : \mathrm{N} \to C_\times$ by $\mathrm{w} \mapsto \langle [\mathrm{w}]_{\text{ꝏ}}, [\mathrm{w}]_{\text{ɨc}} \rangle$.

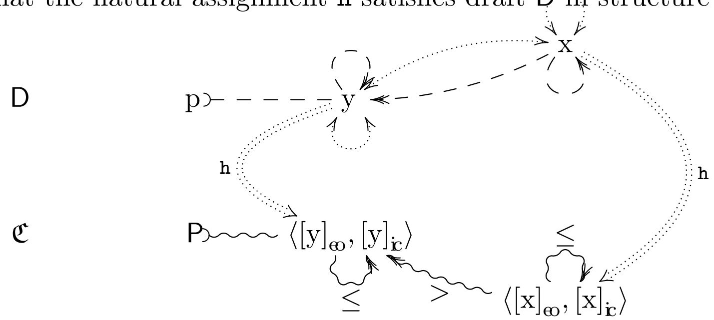
To see that p is not a growing-graded consequence of ¬¬p, we proceed as in Example 4.4 (Birelational non-consequence, p. 12), with ɨc and ꝏ in lieu of wc, as well as ($\mathtt{Rf}$[ɨc]), ($\mathtt{Rf}$[ꝏ]) and ($\mathtt{Sm}$[ꝏ]). We obtain the following final slice S'_3:



The positive part D of $\underline{\mathrm{S}'_3}$ and corresponding natural structure $\mathfrak{C}$ are:



Notice that structure $\mathfrak{C}$ is growing-graded.
We can see that the natural assignment $\mathtt{h}$ satisfies draft D in structure $\mathfrak{C}$:



Much as before, we can see that the natural assignment $\mathtt{h}$ satisfies draft $\underline{\mathrm{S}'_3}$ in structure $\mathfrak{C}$. Thus, these slices and set $\{\neg\neg\mathrm{p}, \overline{\mathrm{p}}\}$ are satisfiable in a growing-graded structure. Hence, p is not a growing-graded consequence of ¬¬p.

The *decoupled graph calculi* are as follows (cf. p. 15). The *pre-graded calculus* is the graph calculus for graph language $\mathbb{G}_{\mathrm{d}}$ with the 6 pre-graded elimination rules and the 3 equivalence rules for ꝏ. The *graded calculus* is the extension of the pre-graded calculus by the rules ($\mathtt{Rf}$[ɨc]), ($\mathtt{As}$[ɨc]), ($\mathtt{Tr}$[ɨc]) and the rule identifying nodes u and v with u ⇄ v. The *growing calculus* is the extension of the graded calculus by the growing rules. The *symmetric decoupled calculus* is the extension of the growing calculus by the symmetric rule ($\mathtt{Sm}$[ɨc]).

Theorem 5.1 ($\mathbb{G}_{\mathrm{d}}$ calculi) *The decoupled calculi are sound and complete.* ♭

Proof. By Proposition 3.2: Graph calculi (p. 8). □

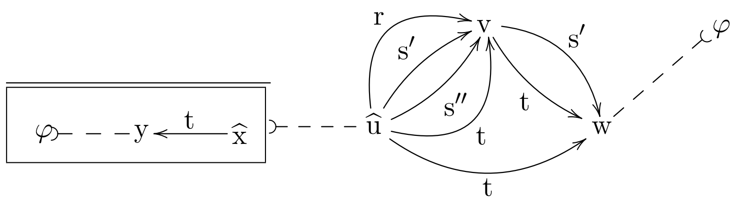
6 Extension to Multi-modal Logics

We have sound and complete graph calculi for flat and decoupled intuitionistic modal logics. We now indicate how to extend these calculi to multi-modal logics.

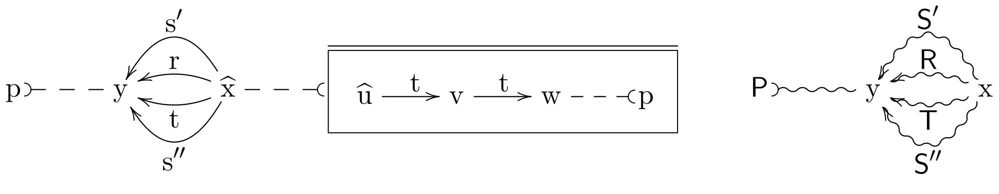
We can also allow some connections as well as some operations on relations (much as in [3]). For instance, we can express inclusion of relations by a rule $(\mathrm{L} \sqsubseteq \mathrm{K})$ adding $\mathrm{u} \xrightarrow{\mathrm{K}} \mathrm{v}$ whenever we have $\mathrm{u} \xrightarrow{\mathrm{L}} \mathrm{v}$ and intersection of relations by a rule $(\mathrm{L} \sqcap \mathrm{K})$ adding $\mathrm{u} \overset{\mathrm{L}}{\underset{\mathrm{K}}{\rightrightarrows}} \mathrm{v}$ whenever we have $\mathrm{u} \xrightarrow{\mathrm{L} \sqcap \mathrm{K}} \mathrm{v}$. We can similarly express composition (by consecutive arrows), transposal (by arrow reversal) and identity (by node identification) [14,15]. For a set Δ of constraints, a Δ-*derivation* is a finite sequence of rule applications and constraints in set Δ.

Consider relation symbols r, s′, s″ and t, subject to the restrictions: "$\mathrm{r} \subseteq \mathrm{s}' \cap \mathrm{s}''$, $\mathrm{s}' \subseteq \mathrm{t}$ and t is transitive". We construct a graph calculus by adding to our basic rules the set Δ consisting of the rules $(\mathrm{r} \sqsubseteq \mathrm{s}' \sqcap \mathrm{s}'')$, $(\mathrm{s}' \sqsubseteq \mathrm{t})$, $(\mathrm{s}' \sqcap \mathrm{s}'')$ and $(\mathbb{Tr}[\mathrm{t}])$.

(+) We can show that $\langle \mathrm{t} \rangle\, \varphi$ is a Δ-consequence of $\langle \mathrm{r} \rangle \langle \mathrm{s}' \rangle\, \varphi$, much as before: we transform the set $\{ \langle \mathrm{r} \rangle \langle \mathrm{s}' \rangle\, \varphi, \overline{\langle \mathrm{t} \rangle\, \varphi} \}$ to the following slice:



(−) We can also obtain a Δ-model for $\{ \langle \mathrm{r} \rangle\, \mathrm{p}, \overline{\langle \mathrm{t} \rangle \langle \mathrm{t} \rangle\, \mathrm{p}} \}$, much as before. We transform this set to a slice S, which gives a model $\mathfrak{N}$, as follows:



Now, consider graph languages $\mathbb{GL}_{\mathfrak{f}}$ (cf. Section 4) and $\mathbb{GL}_{\mathfrak{d}}$ (cf. Section 5).

Lemma 6.1 (Equivalent calculi) *A modal formula is flat (relational or birelational) derivable iff it is decoupled (graded or growing) derivable.*

Proof. We can transform derivations back and forth. [10] □

Theorem 6.2 (Equivalent semantics) *The same modal formulas hold in flat (relational or birelational) and decoupled (graded or growing) structures.*

[10] Call expressions E of $\mathbb{GL}_{\mathfrak{f}}$ and F of $\mathbb{GL}_{\mathfrak{d}}$ are associated ($\mathrm{E} \simeq \mathrm{F}$) iff F is the result of replacing everywhere in E $\mathbb{wc}$ by $\mathbb{ic}$ and $\mathbb{ec}$, and similarly for constraints. For sets of constraints $\Delta_{\mathfrak{f}}$ in $\mathbb{GL}_{\mathfrak{f}}$ and $\Delta_{\mathfrak{d}}$ in $\mathbb{GL}_{\mathfrak{d}}$, $\Delta_{\mathfrak{f}} \simeq \Delta_{\mathfrak{d}}$ iff each $\delta \in \Delta_{\mathfrak{f}}$ has some associated $\delta' \in \Delta_{\mathfrak{d}}$ and vice-versa. Call derivations $\mathrm{E}_1, \dots, \mathrm{E}_n$ and $\mathrm{F}_1, \dots, \mathrm{F}_n$ associated iff $\mathrm{E}_i \simeq \mathrm{F}_i$, for $i = 1, \dots, n$. Call an expression of $\mathbb{GL}_{\mathfrak{d}}$ neat iff $\mathbb{ic}$ and $\mathbb{ec}$ occur only in parallel arcs, and similarly for (sets of) constraints and derivations. By $\leq \mapsto \mathbb{ic}, \mathbb{ec}$ we transform flat rules to decoupled ones and vice-versa. So, given associated constraint sets $\Delta_{\mathfrak{f}} \simeq \Delta_{\mathfrak{d}}$, every flat $\Delta_{\mathfrak{f}}$-derivation $\Pi_{\mathfrak{f}}$ has an associated neat decoupled $\Delta_{\mathfrak{d}}$-derivation $\Pi_{\mathfrak{d}}$ (which will be graded or growing whenever $\Pi_{\mathfrak{f}}$ is relational or birelational) and, similarly, every neat decoupled $\Delta_{\mathfrak{d}}$-derivation $\Pi_{\mathfrak{d}}$ has an associated flat $\Delta_{\mathfrak{f}}$-derivation $\Pi_{\mathfrak{f}}$.

Proof. By Lemma 6.1 and completeness: Theorems 4.6, p. 14, and 5.1, p. 16. □

7 Concluding Remarks

We have presented a flexible and uniform formalism for intuitionistic modal logics where one can express, analyse and compare possible-world semantics. Our approach explores the flexibility of graph caluli [14,15] to express directly and graphically Kripke-based semantics of intuitionistic modal logics.

We have illustrated these ideas by applying them to two semantics (in Sections 4 and 5) and indicated their extension to multi-modal logics in Section 6. Our approach is uniform: once we have expressed the semantics (including connections among relations), we apply the corresponding (sound and complete) graph-calculus. For flat and decoupled semantics, we have transcribed their satisfaction conditions graphically to expressions and used this to show that they give equivalent semantics (in Section 6). We have also illustrated (in Section 4) how one can express simple variations of the satisfaction conditions, which give different semantics on relational structures, though some of them may coincide on birelational structures.

We would like to stress some distinctions between graph calculi and other methods for handling logics. Natural deduction relies on rules for introducing and eliminating logical operators (connectives, etc.) and its aim is building derivation trees [13]. Sequent calculi uses rules for left and right introduction of logical operators and its aim is building sequent trees [8,9,10,12]. In tableaux, the emphasis is on rules that describe truth/falsity conditions for logical operators and the aim is constructing refutation trees [1,7]. Graph calculi employ graphical interpretations of logical operators and the aim is building graphical objects that represent conditions on models; their visual features render them attractive to human users.

We thus have a flexible, uniform, rigourous and intuitive formalism for visual exploration of intuitionistic multi-modal logics.

References

[1] Amati, G., and F. Pirri, *A uniform tableau method for intuitionistic modal logics I*, Studia Logica **53**: 1 (1994), 29–60.

[2] Blackburn, P., M. de Rijke and Y. Venema, "Modal Logic", Cambridge U. Press, Cambridge, 2001.

[3] Catach, R., Normal modal logics, In: Proc. AAAI-88 (1988), 491–495.

[4] Curtis, S., and G. Lowe, *Proofs with graphs*, Science of Computer Programming **26** (1996), 197–216.

[5] Ewald, W. B., *Intuitionistic tense and modal logic*, J. of Symbolic Logic **51** (1986), 166–179.

[6] Fischer Servi, G., Semantics for a class of intuitionistic modal calculi, In: dalla Chiara, M. L. (ed.) Italian Studies in the Philosophy of Science, 59–72, D. Reidel, Dordrecht, 1981.

[7] Fitting, M., "Proof Methods for Modal and Intuitionistic Logics", D. Reidel, Dordrecht, 1983.

[8] Galmiche, D., and Y. Salhi, *Sequent calculi and decidability for intuitionistic hybrid logic*, Information and Computation **209** (2011), 1447–1463.

[9] Garg, D., V. Genovese and S. Negri, Countermodels from sequent calculi in multi-modal logics, In: Proc. LICS, 2012.

[10] Marin, S., and L. Straßburger, Label-free modular systems for classical and intuitionistic logics, In: Proc. AiML, 2013.

[11] Plotkin, G. D., and C. P. Stirling, A framework for intuitionistic modal logic, In: Halpern, J. Y. (ed.), Theoretical Aspects of Reasoning about Knowledge (1986).

[12] Poggiolesi, F., The method of tree-hypersequents for modal propositional logic, In: Makinson, D., J. Malinowski and H. Wansing (eds.), Towards Mathematical Philosophy, Trends in Logic **28** (2009), 31–51.

[13] Simpson, A., "The Proof Theory and Semantics of Intuitionistic Modal Logic", Ph.D. thesis, Edinburgh University, Edinburgh, 1994.

[14] Veloso, P. A. S., and S. R. M. Veloso, On graph refutation for relational inclusions, In: Rocca, S. R. D. and E. Pimentel (eds.), EPTCS **81** (2011), 47–62.

[15] Veloso, P. A. S., S. R. M. Veloso and M. R. F Benevides, On graph calculi for multi-modal logics, In: Ayala-Rincón, M. and I. Mackie (eds.) LSFA 2014, 223–241, Brasília, D. F., 2014.