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A Universal Cellular Automaton on the Ternary Heptagrid

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Abstract

In this paper, we construct the first weakly universal cellular automaton on the ternary heptagrid. It requires six states only. It provides a universal automaton with less states than in the case of the pentagrid where the best result is nine states, a result also recently established by the authors.

Keywords: Cellular automata, hyperbolic plane, tessellations, universality

1 Introduction

As indicated in the abstract, this paper is the first result about a universal cellular automaton on the ternary heptagrid. Although the somehow bigger number of neighbours for one cell is an argument in favor for a smaller number of states than in the case of the pentagrid, this was never proved before. Also note that a direct translation of the result obtained in the pentagrid, using the continuous transformation constructed in [6,7] would certainly produce an automaton with at least the same number of states as in the pentagrid. This is why proving a better result needs a definite amount of work. As the reader will see, it is not at all trivial. Moreover, as this is performed in another grid which has its own properties, this result cannot be simply considered as an improvement of [10].

Our simulation uses the same model of a railway circuit as performed in [1] and in [10]. We refer the reader to [3,1] for the simulation of a register machine by a railway circuit. In order to help the reader to better understand the paper, we

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sketch out the main lines of this simulation in Section 2. In Section 3, we remind the reader about hyperbolic geometry and cellular automata on the ternary heptagrid. Still in Section 3, we give the general features of the implementation of a railway circuit in the ternary heptagrid and in Section 4, we study it very precisely. In Section 5, we give the format of the rules and the transition table of the automaton whose action is described in Section 4. We also indicate how a computer program contributed to the construction of the table.

2 The railway circuit

As initially devised in [11] and then mentioned in [3,1,10], the circuit uses tracks represented by lines and quarters of circles and switches. There are three kinds of switches: the **fixed**, the **memory** and the **flip-flop** switches. They are represented by the schemes given in Fig. 1.

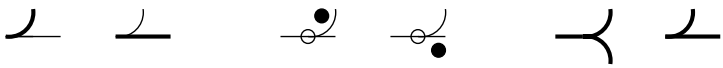


Figure 1 *The three kinds of switches. From left to right: fixed, flip-flop and memory switches.*

Note that a switch is an oriented structure: on one side, it has a single track u and, on the the other side, it has two tracks a and b. This defines two ways of crossing a switch. Call the way from u to a or b **active**. Call the other way, from a or b to u **passive**. The names comes from the fact that in a passive way, the switch plays no role on the trajectory of the locomotive. On the contrary, in an active crossing, the switch indicates which track between a and b will be followed by the locomotive after running on u: the new track is called the **selected** track.

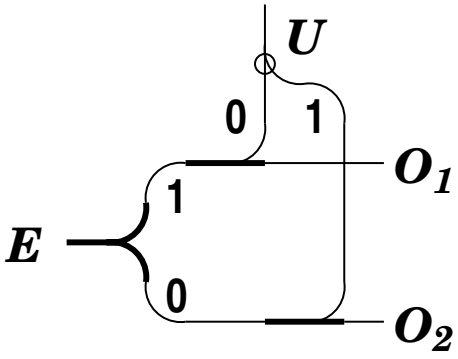


Figure 2 *The elementary circuit.*

With the help of these three kind of switches, we define an **elementary circuit** as in [11], which exactly contains one bit of information. The circuit is illustrated by Fig. 2, above. It can be remarked that the working of the circuit strongly depends on

how the locomotive enters it. If the locomotive enters the circuit through E, it leaves the circuit through O_1 or O_2, depending on the selected track of the memory switch which stands near E. If the locomotive enters through U, the application of the given definitions shows that the selected track at the switches near E and U are both changed: the switch at U is a flip-flop which is changed by the very active passage of the locomotive and the switch at E is a memory one which is changed because it is passively crossed by the locomotive and through the non-selected track. The just described actions of the locomotive correspond to a **read** and a **write** operation on the bit contained by the circuit which consists of the configurations of the switches at E and at U. It is assumed that the write operation is triggered when we know that we have to change the bit which we wish to rewrite.

From this element, it is easy to devise circuits which represent different parts of a register machine. As an example, Fig. 3 illustrates an implementation of a unit of a register.

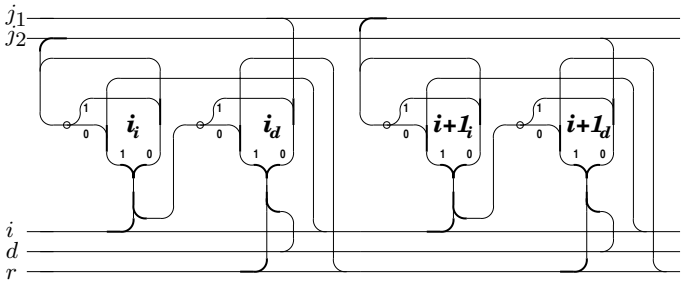


Figure 3 *Here, we have two consecutive units of a register. A register contains infinitely many copies of units. Note the tracks i, d, r, j_1 and j_2. For incrementing, the locomotive arrives at a unit through i and it leaves the unit through r. For decrementing, it arrives though d and it leaves also through r if decrementing the register was possible, otherwise, it leaves through j_1 or j_2.*

As indicated by its name, the **fixed switch** is left unchanged by the passage of the locomotive. It always remains in the same position: when actively crossed by the locomotive, the switch always sends it onto the same track. The flip-flop switch is assumed to be crossed actively only. Now, after each crossing by the locomotive, it changes the selected track. The memory switch can be crossed by the locomotive actively and passively. In an active passage, the locomotive is sent onto the selected track. Now, the selected track is defined by the track of the last passive crossing by the locomotive. Of course, at initial time, the selected track is fixed.

Other parts of the needed circuitry are described in [3,1]. The main idea in these different parts is to organize the circuit in possibly visiting several elementary circuits which represent the bits of a configuration which allow the whole system to remember the last visit of the locomotive. The use of this technique is needed for the following two operations.

When the locomotive arrives to a register R, it arrives either to increment R or to decrement it. As can be seen on Fig. 3, when the instruction is performed, the locomotive goes back from the register by the same track. Accordingly, we need

somewhere to keep track of the fact whether the locomotive incremented R or it decremented R. This is one type of control. The other control comes from the fact that several instructions usually apply to the same register. Again, when the locomotive goes back from R, in general it goes back to perform a new instruction which depends on the one it has just performed on R. Again this can be controlled by what we called the **selector** in [3,1].

At last, the dispatching of the locomotive on the right track for the next instruction is performed by the **sequencer**, a circuit whose main structure looks like its implementation in the classical models of cellular automata such as the game of life or the billiard ball model. The reader is referred to the already quoted papers for full details on the circuit. Remember that this implementation is performed in the Euclidean plane, as clear from Fig. 4 which illustrates the case of a few lines of a program of a register machine.

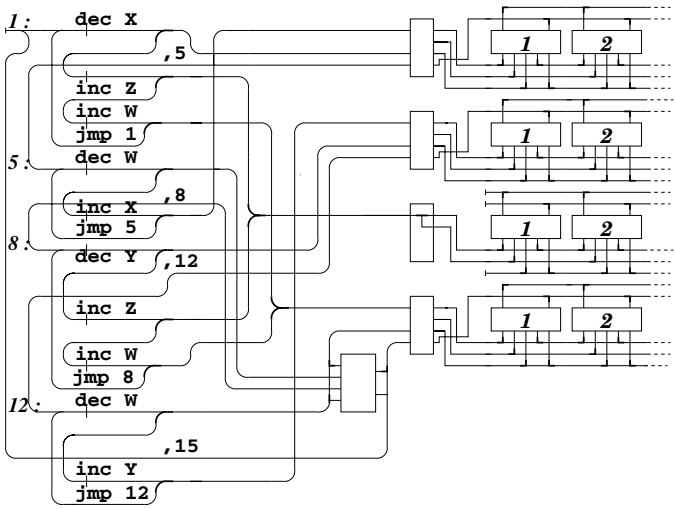


Figure 4 *An example of the implementation of a small program of a register machine. On the left-hand side of the figure, the part of the sequencer. It can be noticed how the tracks are attached to each instruction of the program. Note that there are four decrementing instructions for W: this is why a selector gathers the arriving tracks before sending the locomotive to the control of the register. On the way back, the locomotive is sent on the right track.*

Now, we turn to the implementation in the hyperbolic plane, which first requires some features of hyperbolic geometry.

3 Implementation in the hyperbolic plane

Hyperbolic geometry appeared in the first half of the 19$^{\text{th}}$ century, in the last attempts to prove the famous parallel axiom of Euclid's *Elements* from the remaining ones. Independently, Lobachevsky and Bolyai discovered a new geometry by assuming that in the plane, from a point out of a given line, there are at least two lines which are parallel to the given line. Later, models of the new geometry were

found, in particular Poincaré's model, which is the frame of all this study.

In this model, the hyperbolic plane is the set of points which lie in the open unit disc of the Euclidean plane whose border is the unit circle. The lines of the hyperbolic plane in Poincaré's disc model are either the trace of diametral lines or the trace of circles which are orthogonal to the unit circle, see Fig. 5. We say that the considered lines or circles **support** the hyperbolic line, simply **line** for short, when there is no ambiguity. Fig. 5 illustrates the notion of **parallel** and **non-secant** lines in this setting.

The angle between two h-lines are defined as the Euclidean angle between the tangents to their support. The reason for choosing the Poincaré's model is that hyperbolic angles between h-lines are, in a natural way, the Euclidean angle between the corresponding supports. In particular, orthogonal circles support perpendicular h-lines.

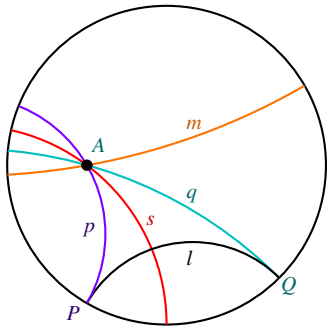


Figure 5 *The lines p and q are* **parallel** *to the line ℓ, with points at infinity P and Q, on the border of the unit disc. The h-line m is* **non-secant** *with ℓ: it can be seen that there are infinitely many such lines.*

3.1 The ternary heptagrid

Remember that in the Euclidean plane and up to similarities, there are only three kinds of tilings based on the recursive replication of a regular polygon by reflection in its sides and of the images in their sides. In the hyperbolic plane, where the notion of similarity is meaningless, there are infinitely many such tilings. In this paper, we consider the smallest regular polygon defined by the property that three copies of it can be put around a vertex in order to cover a neighbourhood of the vertex with no overlapping. This tiling is called the **ternary heptagrid**, see Fig. 6 and 7 for an illustrative representation. Here, we give a rough explanation of these objects, referring to [7] and to [5] for more details and references.

The left-hand side of Fig. 6 illustrates the ternary heptagrid. But, besides the occurrence of a lot of symmetries, nothing can be grasped on the structure of the tiling from this mere picture. The right-hand side picture of Fig. 6 illustrates the main tool to make the structure visible. There, we can see two lines which we call **mid-point lines** as they join mid-points of edges of heptagons belonging to the

tiling. On the figure, a half of each line is drawn with a thicker stroke. It is a **ray** issued from the common point of these lines: here, a mid-point of an edge of the central heptagon of the figure. We shall say a **ray of mid-points**. These two rays define an angle, and the set of tiles whose all mid-points of the edges fall inside the angle is called a **sector**.

Fig. 6 and 7 sketchily remember that the tiling is spanned by a generating tree. In fact, as can be noticed on both the right-hand side of Fig. 6 and the left-hand side of Fig. 7, the set of tiles constituting a sector is spanned by a Fibonacci tree.

Now, as indicated in Fig. 7, seven sectors around a central tile allow us to exactly cover the hyperbolic plane with the ternary heptagrid which is the tessellation obtained from the regular heptagon described above and easily seen on the figures.

In the left-hand side picture of Fig. 7, we represent the sectors in terms of tiles. The tiles are in bijection with the tree which is represented on the right-hand side part of the figure. This allows to define the coordinates in a sector of the ternary heptagrid, see [7]. We number the nodes of the tree, starting from the root and going on, level by level and, on each level, from the left to the right. Then, we represent each number in the basis defined by the Fibonacci sequence with $f_1 = 1$, $f_2 = 2$, taking the maximal representation, see[2,7].

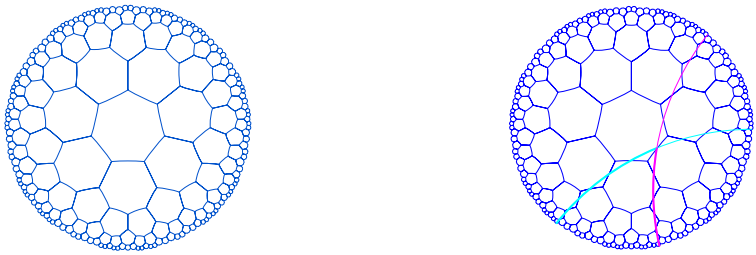


Figure 6 *On the left: the tiling; on the right: the delimitation of the sectors which are spanned by a tree. Note the rays of mid-points. They are issued from the same point: a mid-point of an edge of the central cell of the figure.*

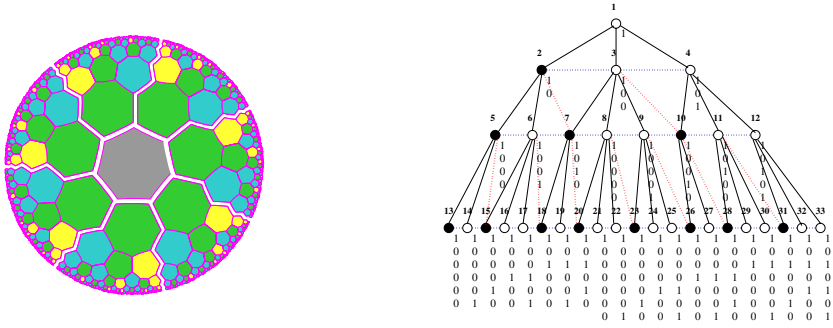


Figure 7 *On the left: seven sector around a central tile; on the right: the representations of the numbers attached to the nodes of the Fibonacci tree.*

One of the reasons to use this system of coordinates is that from any cell, we

can find out the coordinates of its neighbours in linear time with respect to the coordinate of the cell. Also in linear time from the coordinate of the cell, we can compute the path which goes from the central cell to the cell. These properties are established in [4,7] and they rely on a particular property of the coordinates in the tree which allow to compute the coordinate of the father of a node in constant time from the coordinate of the node. In the paper, the coordinate of a cell is of the form $\nu(\sigma)$ where σ is the number of the sector where the cell is and ν is its number in the Fibonacci tree which spans the sector. Now, as the system of coordinates is fixed, we can turn to the application to the implementation of cellular automata on the ternary heptagrid, we shall say **heptagrid** for short.

3.2 Cellular automata on the ternary heptagrid

A cellular automaton on the heptagrid is defined by a **local transition function** which can be put in form of a table. Each row of the table defines a **rule** and the table has nine columns numbered from 0 to 8, each entry of the table containing a state of the automaton. On each row, column 0 contains the state of the cell to which the rule applies. The rule applies because columns 1 to 7 contain the states of the neighbours of the cell defined in the following way. For the central cell, its neighbour 1 is fixed once and for all. For another cell, its neighbour 1 is its father. In all cases, the other neighbours are increasingly numbered from 2 to 7 while counter-clockwise turning around the cell starting from side 1. The representation mentioned in Subsection 3.1 allows to find the coordinates of the neighbours from that of the coordinate of the cell in linear time. The list of states on a row, from column 0 to 7 is called the **context** of a rule. It is required that two different rules have different contexts. We say that the cellular automaton is **deterministic**. As there is a single row to which a rule can be applied to a given cell, the state of column 8 defines the **new state** of the cell. The local transition function is the function which transforms the state of a cell into its new one, also depending on the states of the neighbours as just mentioned.

An important case in the study of cellular automata is what are called **rotation invariant** cellular automata. To define this notion, we consider the following transformation on the rules. Say that the context of a rule is the **rotated image** of another one if and only if both contexts have the same state in column 0 and if one context is obtained from the other by a **circular** permutation on the contents of columns 1 to 7. Now, a cellular automaton is **rotation invariant** if and only if its table of transition T possesses the following properties:

- for each row ρ of T, T also contains six rules exactly whose contexts are the rotated image of that of ρ and whose new state is that of ρ;
- if ρ_1 and ρ_2 are two rules of T whose contexts are the rotated image of each other, then their column 8 contains the same state.

The name of rotation invariance comes from the fact that a rotation around a tile T leaving the heptagrid globally invariant is characterized by a circular permutation on the neighbours of T defined as above.

Note that the universal cellular automata devised in [1,10] are rotation invariant

while the one of [9] is not. For the question of rotation invariance for cellular automata on the heptagrid, we refer the reader to [8].

Now, we can turn to the simulation of the railway circuit by a cellular automaton.

3.3 The implementation of the railway circuit

In [1], the various elements of the circuit mentioned in [3] are implemented. In fact, the paper does not give an exact description of the implementation: it only gives the guidelines, but with enough details, so that an exact implementation is useless. In this paper, we take the same implementation exactly. This is why we do not repeat it in this paper and we refer the reader to [1] for further details. Just to help the reader to better see things, Fig. 8 provides a simplified illustration of the implementation of the example given by Fig. 4.

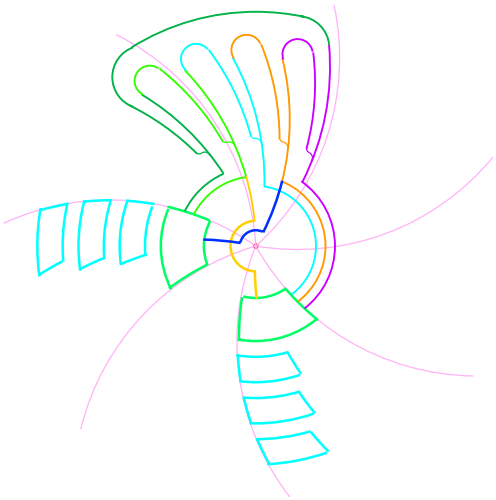


Figure 8 *The implementation, on the heptagrid, of the example of Fig. 4. In sector* 1*, also overlapping onto sector* 2*, the sequencing of the instructions of the program of the register machine. In sector* 3*, we can see the first register and, in sector* 5*, the second one. For simplicity, the figure represents two registers only.*

Note the instructions which arrive to the control of the register through tracks in the shape of an arc of circle. Also note the return from the controller of the register when decrementing a register fails, because its content was zero.

Here, we just mention the general principle of motion of the locomotive which is again taken in the new setting which we define in the next sub-section.

The tracks are implemented by portions of paths between two switches. Such a portion consists in a finite sequence of tiles which are called **blue** ones. We require that each blue tile which is in contact neither with a switch nor a crossing has two blue neighbours exactly, where a neighbour of a tile T is another tile sharing an edge with T. Its other neighbours are in the quiescent state.

From now on, we call tiles **cells**. Most cells are in a quiescent state which we also call the **blank**. In the following figures of the paper, it is represented by a light blue colour and it is not marked by a letter, contrary to the cells which are under

another state. The state of the cells which are on a blue tile is also **blue**, unless the locomotive is present on the cell. The locomotive is represented by two contiguous cells on a path. One cell is **green**, also represented by the letter G in the figures, the other is **red**, represented by the letter R. The green cell represents the front of the locomotive and the red cell represents its rear.

Now, the motion of the locomotive on a track is represented by Fig. 9. The easy rules can be found in [1].

Fig. 9 is a space-time diagram of the evolution of the states of the cells which are along a track.

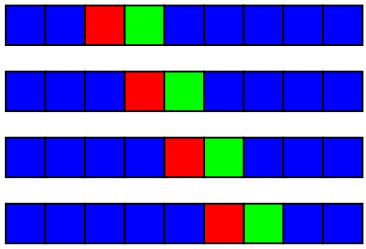


Figure 9 *The motion of the locomotive on its tracks: first approximation.*

In the heptagrid, the rules become a bit more complex. However, we can formulate them in a concise way with the help of Fig. 9. Table 1 gives a synthetic representation of the concerned rules. It is assumed that there are exactly five blank neighbours of the cell c. Then, the table gives the new state of c depending on the state of c and the states of the other non-blank neighbours.

Table 1 *Table for the definition of the rules of the basic motion.*

state	number of neighbours in			new
	B	G	R	
B	2	_	_	B
B	1	1	_	G
G	1	_	1	R
R	1	1	_	B
B	1	_	1	B

Any configuration which respects these constraints defines a rule of the cellular automaton accordingly. We call these rules the **rules of the basic motion**.

The switches organize the meeting of tracks according to the principles defined in the previous section. By definition, four paths meet at a crossing and three ones meet at a switch. As in [1,10], such a meeting is materialized by a tile which plays a special role and which we call the **centre** of the meeting. The other parts of the tracks abut to a neighbour of the centre. This neighbour is called the **first cell** of the considered track.

In the case of a crossing, we have four tracks attached to two different paths. As

in [10], one path is marked by another blue colour for its first cell, see the picture (a) of Fig 10. We take advantage of the ternary heptagrid to put in contact two tracks which arrive at a neighbour of the centre when they do not belong to the same path. This allows each track to know what it has to do when the locomotive is passing through the crossing.

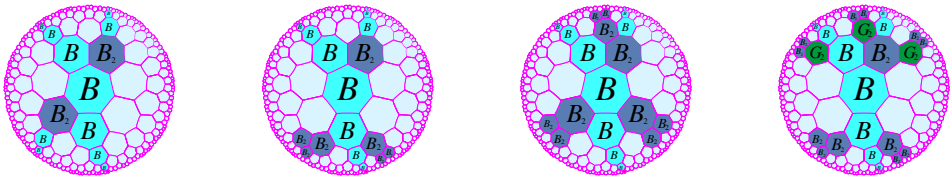
In [10], we have more systematically followed the pattern of the basic motion than in [1], even in a context where the cell has less than five blank neighbours. This is the reason why we could reduce the number of states of a weakly universal cellular automaton on the pentagrid. Now, in the heptagrid, the advantage we have mentioned for the crossings can be maintained for the switches. For them, the simulation of [10] makes use of all the nine states. As here we have less states, we have to simulate the differences by different configurations of cells. This is what is performed for the three types of switches where these configurations allow to differentiate the first cell. For the centre, it is an additional neighbour with the same colour. For the cell itself, the additional neighbours allow to differentiate its behaviour depending on its place with respect to the tracks.

4 The simulation in the heptagrid

In this section and in the next one, we prove the following:

Theorem 4.1 (Margenstern-Song) − *There is a cellular automaton on the ternary heptagrid which is weakly universal and which has six states. Moreover, the rules of the cellular automaton are rotation invariant. The cellular automaton has an infinite initial configuration which is ultimately periodic along two different rays of mid-points r_1 and r_2 of the ternary heptagrid and finite in the complement of the parts attached to r_1 and r_2.*

The configurations at a crossing and at the different kinds of switches are given in Fig. 10. Say that such a configuration is **idle** if the locomotive is not near its centre. As in [10], the centre of an idle configuration is blue. The difference between a crossing and the different switches are given by the configuration itself.



(a) (b) (c) (d)

Figure 10 *The idle configurations. From left to right: crossing and then fixed, memory and flip-flop switches.*

Now, we shall consider that the locomotive arrives at such a meeting. In the next sub-sections, we look at what happens in each case.

4.1 The crossings

There are four cases of crossing from the different sectors. We shall consider two cases only, from sectors 1 and 7 illustrated by Fig. 11 and 12 respectively. These pictures show that we need the six states for the crossing: `W`, `B`, `G`, `R`, `B2` and `G2`. Five of them are enough for one path. The additional state, `G2`, is required to distinguish the arrival of the front of the locomotive from the other path.

Fig. 12 illustrates what happens when the locomotive comes from the other path. Note that the situation from the other tracks is the same as from the ones we have studied: the other tracks are obtained by a rotation around the centre.

The second path, from sectors 7 to 3, is signalized by the first cells of its tracks: there are in the state `B2` while the similar cells for the other tracks are in the state `B`.

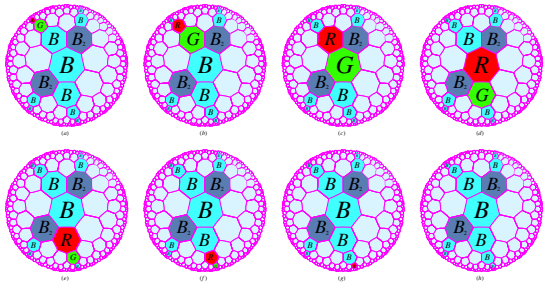


Figure 11 *The locomotive goes through a crossing: here from sector* 1 *to sector* 4.

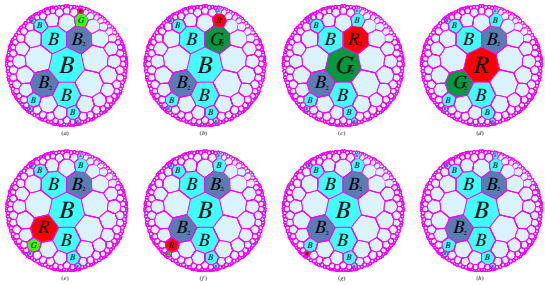


Figure 12 *The locomotive goes through a crossing: here from sector* 7 *to sector* 3.

Without loss of generality, we may assume that all crossings are rotated images of the configuration described by picture (a) of Fig. 10. Note that we require the track to go along the rightmost branch of the Fibonacci tree spanning the corresponding sectors only up to the third level. On the complement of a ball of radius 4 around the centre of the crossing, the four paths may be very different. In particular, from the assumption of rotation invariance of the configurations, we may always number the sectors by counter-clockwise turning around the centre in such a way that the cells marked by `B2` are in sectors 7 and 3.

4.2 The switches

All switches are built according a similar pattern. They are already marked by the fact that the cell 2(1) is in a different state which is specific to each switch. This state is blank, `B2` and `G2` for the fixed, the memory and the flip-flop switch

respectively. We can see that the different local configurations around the centre allow to have the cells (0), 1(1) and 1(7) play different roles, see Fig. 10, pictures (*b*), (*c*) and (*d*). Three tracks meet at a centre, the central cell of the pictures of Fig. 10, 13, 14 and 15. Here also, the track arriving to the switch goes along the rightmost branch of the Fibonacci tree spanning the corresponding sector, passing through cells 33, 13, 4 and 1. Moreover, the unique arriving track is in sector 4, and the two tracks emanating from the switch are in sectors 1 and 7. On the figures, the selected path is in sector 1. Of course, symmetric figures where the selected paths is in sector 7 are also possible for the memory and flip-flop switches. For the fixed one, this is not mandatory: indeed, a fixed switch to the other direction can be simulated by the fixed switch of Fig. 13 followed by a crossing which allows to exchange the directions between the tracks which go out from the switch. Now, any switch can be considered as a rotated image of the switches represented in Fig. 10, 13, 14 and 15 and their symmetrical counterparts. Now, for all switches, cell 1(4) has two neighbours permanently in the state `B2`. This permanence is guaranteed by the two additional cells in the same state. These additional cells have a unique non-blank neighbour, so that this condition is easy to satisfy.

Fixed switch

Its working is illustrated by Fig. 13.

The fixed switch uses the rules of the crossings too. Contrarily to what we have in the pentagrid, here the fact that the first cells of the selected and non-selected tracks can see each other simplifies the situation a lot. Also note that the configuration of the central cell is clearly different from the one of a crossing.

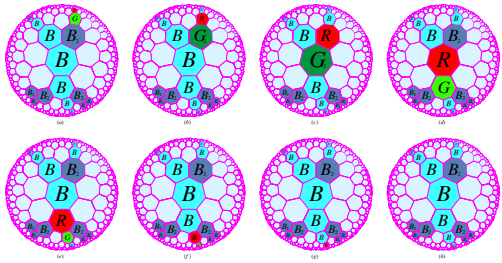


Figure 13 *The locomotive crosses a fixed switch from the non-selected track, here in sector* 7.

Memory switch

The memory switch is the most complicate item among the three kinds of switches.

The crossing from track 1 or 4 is not very difficult and it mainly uses the rules of a fixed switch, taking into account the difference of configuration of the central cell. However, specific rules are also in use due to the state `B2` which is permanently present in the cell 2(1).

The memory switch is characterized by the configuration defined by the cells 1(1), 2(1) and 1(7). One of the cells 1(1) and 1(7) has the same state as

the cell 2(1): it is the first cell of the track which is not selected by the switch. It constitutes a **mark** which changes after the visit of the locomotive from this track.

Fig. 14 illustrates the working of the switch when the locomotive comes from the non-selected track, here track 7. As the colours of the illustrations have been placed on the heptagrid from the execution of the cellular automaton by a computer program, the representation is exact. We can see that the mark is changed. The change of selected track occurs from picture c to picture d of the figure. It is triggered when the front of the locomotive is at the centre of the switch. At this moment, both cells 1(1) and 1(7) can see this state in cell 0 at the same time and the cell 1(1) can see the red state of the cell 1(7). This allows the mark to switch from one track to the other.

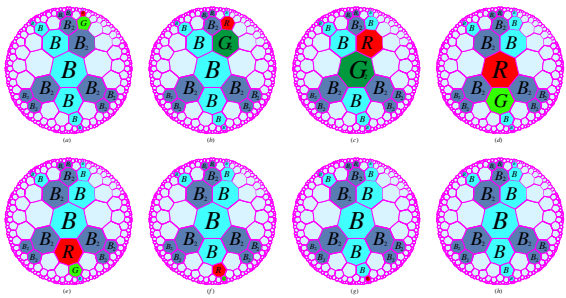


Figure 14 *The locomotive crosses a memory switch from the non-selected track, here in sector* 7*. Note the change of the selected track when the rear of the locomotive leaves the first cell of the non-selected track.*

Flip-flop switch

Here, the switch is again recognized by the state at the cell 2(1): this time, `G2`. Note that here, the transformation is easier than in the case of the pentagrid, see [10].

It can be remarked that the configuration around the central cell and around the cell 1(4) is exactly the same as the configurations of the same cells in a fixed switch. This means that the rules used for the motion of the locomotive actively arriving to a fixed switch are also in use here when the locomotive arrives to this switch. The situation changes when the locomotive arrives to the cells 1(1) or 1(7) only. Now there, the neighbouring of the cells 1(1) and 1(7) of a flip-flop switch are clearly different from the neighbouring cells in both the fixed and the memory switches.

On Fig. 15, we can see how the change of marks occurs to indicate the new selected track. Again, it makes use that the cells 1(1) and 1(7) can see each other until the very last moment: when the rear of the locomotive is already about to leave the cell 1(1).

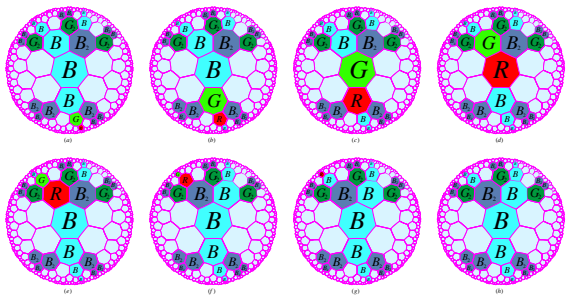


Figure 15 *The locomotive crosses a flip-flop switch, necessarily actively, here from sector* 4. *Note the change of the selected track when the rear of the locomotive leaves the switch.*

Note that we could control the behaviour of the switches without any additional state, mainly because we succeeded to make the neighbouring around the key cells of each case different from their neighbouring in the other cases.

The tracks

In the case of the pentagrid, we had a problem of **sharp turns** on a track. To fix things, a sharp turn at the cell c means that the track arrives at a neighbour n and leaves at the next neighbour $n+1$ while turning around the cell. In the pentagrid, such turns are required and they raised a problem which was solved, see [10]. Here we have no more this problem: first, even if sharp turns occurred, this would raise no problem. But in fact, sharp turns can be avoided: in the previous setting, as neighbours n and $n+1$ are contiguous, the track goes from n to $n+1$ without passing through c. Also here, we do not need any additional state. Consequently, the simulation requires six states only.

5 The rules

The set of rules of the automaton is displayed in table 2. Here, we indicate how the rules where computed by a computer program and how the rules are represented and dispatched in the table.

5.1 The format of the rules and rotation invariance

Now, we complete the information given in Subsection 3.2 about the format of the rules.

Each cell has a numbering of its sides from 1 to 7. The central cell excepted,for the other cells, side 1 is the side shared with the father. In the central cell, side 1 is a side fixed once and for all. Once this side is fixed, all the others are also fixed: the numbers of the side increase when counter-clockwise turning around the tile which supports the cell. Note that the sons of a black node in the Fibonacci tree are given by sides 4 and 5 and that those of a white node are given by sides 3, 4 and 5.

A rule can be denoted as follows:

$$\eta_0,\ \eta_1,\ \eta_2,\ \eta_3,\ \eta_4,\ \eta_5,\ \eta_6,\ \eta_7 \rightarrow \eta_0^1,$$

where η_0 is the state of the cell, η_i the state of its neighbour i and η_0^1 is its new state.

In Subsection 3.2, we defined the context of a rule which is the word $\eta_0\eta_1\eta_2\eta_3\eta_4\eta_5\eta_6\eta_7$. In table 2, we represent the rules by a word too. The just above form of the rule is represented by the following word: $\underline{\eta_0}\eta_1\eta_2\eta_3\eta_4\eta_5\eta_6\eta_7\underline{\eta_0^1}$, in the same notations as the rule. We can see that the context is the biggest proper prefix of this word.

To define the features of the computer program, it can be useful to devise a test in order to check the rotation invariance of the automaton. The notion of rotated form can be used for that purpose in the following way.

Consider a rule (η) of the cellular automaton which we represent by the word $\underline{\eta_0}\eta_1\eta_2\eta_3\eta_4\eta_5\eta_6\eta_7\underline{\eta_0^1}$, as above indicated. Its rotated forms are defined by $\underline{\eta_0}\eta_{\pi(1)}\eta_{\pi(2)}\eta_{\pi(3)}\eta_{\pi(4)}\eta_{\pi(5)}\eta_{\pi(6)}\eta_{\pi(7)}\underline{\eta_0^1}$, where π is a circular permutation on 1..7. We can see that the difference between the rotated forms of a rule and the rule itself lies in their contexts: they can be deduced from each other by a suitable circular permutation on $\{1..7\}$. There are 7 rotated forms for each rule. Now, as we consider the contexts as words, we can order them lexicographically. And so, there is a **minimal** rotated form for each rule (η): that whose context is the minimum of the contexts of the rotated forms of (η) with respect to the lexicographical order which is a linear order. Denote $\min(\eta)$ the minimal rotated form of the rule (η). Remember that the current state of the cell is the first letter of the context.

Now, it is plain that we have the following property:

Lemma 5.1 *The set of rules of a cellular automaton on the heptagrid is invariant by rotation if and only if for any pair of rules* (η) *and* (ϵ) *of the automaton, if* $\min(\eta)$ *and* $\min(\epsilon)$ *have equal contexts then* $\eta_0^1 = \epsilon_0^1$.

5.2 The program and the table of the rules

The computer program was written in *ADA*.

The program uploads the initial configuration of the crossings and of the switches from a file and puts the corresponding information into a table 0. In this table, each row represents a cell. The entries of the row indicate the coordinates of the neighbours of the cell as well as the states of the cell and of its neighbours. The program also contains a copy of table 0 with no state in the cells which we call table 1. The set of rules is in a file under an appropriate format, close to the one which was depicted in Subsection 5.1.

During the construction of the set of rules, the program works as follows. When we run the program, it reads the file of the rules which, initially contains the rule WWWWWWWWW which says that a cell in the quiescent state whose neighbours are all

Table 2 *The rules of the cellular automaton of Theorem 4.1:*

crossing:

track 1:

1: WWWWWWWWW
2: BBWB2BWWB2B
3: BBB2WWBWWB
4: WBB2BWWWWW
5: WBWWWWWBW
6: BBWWWGWWG
7: WWBBWWWWW
8: WBWWWWWWW
9: WBWWWWGW
10: GBWWWRWWR
11: WGWWWWWWW
12: WGWWWWWRW
13: RGWWBWWB
14: WBBWWWWB2W
15: WWBGWWWWW
16: WWGRWWWWW
17: WWWWBWWW
18: B2BWWWBWBB2
19: WB2WWWWWWW
20: WB2WWWWBW
21: BB2WWBWWB
22: BBWWBWWB
23: WBBWWWWWW
24: WBWWWWB2W
25: BBB2WWGWWG
26: WRWWWWWW
27: WRWWWWWBW
28: BRWWBWWB
29: WWRBWWWWW
30: BGWB2BWWB2G
31: GBB2WWRWWR
32: WGB2BWWWWW
33: WBGWWWWB2W
34: B2BWWWBWGB2
35: GRWB2BWWB2R
36: RGB2WWBWWB
37: WRB2BWWWWW
38: WGRWWWWB2W
39: B2GWWWBWBB2
40: BGB2WWBWWG
41: WGBWWWWW
42: WGWWWWWB2W
43: B2GWWWBWRB2
44: RBWB2GWWB2B
45: BRB2WWBWWB
46: WRBWWWWB2W
47: B2RWWWBWGB2
48: GRB2WWBWWR
49: WGWWWWWBW
50: BGWWWBWWG
51: WRGWWWWWW
52: WWGBWWWWW
53: WRWWWWWB2W
54: B2RWWWBWBB2
55: BBWB2RWWB2B
56: B2BWWWBWRB2
57: RBB2WWGWWB
58: WRWWWWWGW
59: GRWWBWWR
60: WBRWWWWWW
61: WWRGWWWWW
62: BBB2WWRWWB
63: WBWWWWWRW
64: RBWWWGWWB
65: WWBRWWWWW
66: BBWWWRWWB

track 4:

67: BBWB2GWWB2G
68: WBGWWWWWW
69: GBWB2RWWB2R

crossing:

track 4:

70: WGBWWWWB2W
71: WGRWWWWWW
72: RGWB2BWWB2B
73: WRGWWWWB2W
74: WRBWWWWWW
75: BRWB2BWWB2B
76: WBRWWWWB2W
77: BB2WWWGWWG
78: WBB2GWWWWW

track 7:

79: B2BWWWGWBG2
80: WB2WWWWWGW
81: GB2WWWRWWR
82: BBWB2BWWG2G2
83: BBG2WWBWWB
84: WBG2RWWWWW
85: WBWWWWG2W
86: G2BWWWRWBR
87: WG2WWWWWWW
88: WG2WWWWWRW
89: RG2WWWBWWB
90: G2BWB2BWWRR
91: BG2RWWBWWB
92: WBRBWWWWW
93: WG2BWWWWB2W
94: B2G2WWWBWBG2
95: BG2B2WWBWWB
96: WG2BWWWWWW
97: RG2WWWBWBB2
98: RBWG2BWWB2B
99: WRBWWWWG2W
100: G2RWWWBWBR
101: WG2WWWWWBW
102: BG2WWWBWWG
103: BRG2WWBWWB
104: WBG2BWWWWW
105: BBWRBWWB2B
106: WBBWWWWRW
107: RBWWWGWBB2
108: BBRWWBWWB
109: WBRGWWWWW
110: B2BWWWRWBB2
111: WB2WWWWRW
112: RB2WWWGWWB
113: WBB2RWWWWW
114: BB2WWWRWWB
115: BBWG2BWWB2G2
116: WBBWWWWG2W

track 3:

117: G2BWRBWWB2R
118: WG2BWWWWRW
119: WG2WWWWB2W
120: RBWB2BWWG2B
121: WRWWWWWG2W
122: BBWB2BWWRB
123: WBWWWB2B2BW
124: BBWWBWWB2B

fixed switch:

track 1:

125: WWWWWWWB2W
126: B2WWWWWB2B2B2
127: WB2WWWWWB2W
128: BBWB2WBB2WB
129: B2BWB2B2WWWB2
130: WBB2WWWWBW
131: BBWWWBWB2B
132: B2B2B2WWWWWB2
133: WB2B2WWWWWW

fixed switch:

track 1:

134: WWB2WWWWWW
135: WBBB2WWWWW
136: B2WBBWB2B2WB2
137: WWB2B2WWWWW
138: WB2BBWWWB2W
139: B2B2WWWWWB2B2
140: B2WB2B2WWWWB2
141: BGWWBWWB2G
142: GRWWBWWB2R
143: WGWWWB2B2BW
144: BGWB2WBB2WG
145: WGBB2WWWWW
146: RBWWGWWB2B
147: WRWWWB2B2GW
148: GRWB2WBB2WR
149: B2GWB2B2WWWB2
150: WGB2WWWWBW
151: BGWWWBWB2G
152: WRGB2WWWWW
153: B2WGBWB2B2WB2
154: BBWWRWWB2B
155: WBWWWB2B2RW
156: RBWB2WGB2WB
157: B2RWB2B2WWWB2
158: WRB2WWWWGW
159: GRWWWBWB2R
160: WBRB2WWWWW
161: B2WRGWB2B2WB2
162: WB2GBWWWB2W
163: BBWB2WRB2WB
164: WBB2WWWWRW
165: RBWWWGWB2B
166: B2WBRWB2B2WB2
167: WB2RGWWWB2W
168: BBWWWRWB2B
169: WB2BRWWWB2W
170: BBWWWGWB2G
171: WB2BGWWWB2W

track 4:

172: BBWB2WGB2WG
173: WBB2WWWWGW
174: GBWWWRWB2R
175: B2WBGWB2B2WB2
176: WB2GRWWWB2W
177: BBWWGWWB2G
178: WBWWWB2B2GW
179: GBWB2WRB2WR
180: WGB2WWWWRW
181: RGWWWBWB2B
182: WBGB2WWWWW
183: B2WGRWB2B2WB2
184: WB2RBWWWB2W
185: GBWWRWWB2R
186: WGWWWB2B2RW
187: RGWB2WBB2WB
188: WRB2WWWWBW
189: BRWWWBWB2B
190: WGRB2WWWWW
191: B2WRBWB2B2WB2
192: RGWWBWWB2B
193: WRWWWB2B2BW
194: BRWB2WBB2WB
195: WRBB2WWWWW
196: BRWWBWWB2B

track 7:

197: BBWWBWWG2G2
198: G2BWWBWWRR
199: WG2WWWB2B2BW
200: BG2WB2WBB2WG
201: WG2BB2WWWWW

memory switch:

track 1:

202: BBWB2BB2WB2B
203: BBB2B2WBWWB
204: B2BB2BWB2B2WB2
205: WBB2B2WWBW
206: B2BWWB2B2WBB2
207: BBB2WWBWB2B
208: WBB2B2WWWWW
209: B2BBWB2B2WWB2
210: WBB2WWWWB2W
211: B2BWWWBB2BB2
212: BB2WWWBWB2B
213: BBB2B2WGWWG
214: WBB2B2WWWGW
215: BGWB2BB2WB2G
216: GBB2B2WRWWR
217: B2GB2BWB2B2WB2
218: WGB2B2WWWRW
219: B2BWWWBB2GB2
220: GRWB2BB2WB2R
221: RGB2B2WBWWB
222: B2RB2BWB2B2WB2
223: WRB2B2WWWBW
224: B2GWWB2B2WBB2
225: BGB2WWBWB2G
226: B2GBWB2B2WWB2
227: WGB2WWWWB2W
228: B2GWWWBB2RB2
229: RBWB2GB2WB2B
230: BRB2B2WBWWB
231: B2RWWB2B2WGB2
232: GRB2WWBWB2R
233: WGB2B2WWWWW
234: B2RGWB2B2WWB2
235: WRB2WWWWB2W
236: B2RWWWBB2BB2
237: BBWB2RB2WB2B
238: B2BWWB2B2WRB2
239: RBB2WWGWB2B
240: WRB2B2WWWWW
241: B2BRWB2B2WWB2
242: BBB2WWRWB2B

track 4:

243: BBB2WWGWB2G
244: BBWB2GB2WB2G
245: B2BWWB2B2WGB2
246: GBB2WWRWB2R
247: B2BGWB2B2WWB2
248: GBWB2RB2WB2R
249: BGB2B2WBWWG
250: B2GWWB2B2WRB2
251: RGB2WWBWB2B
252: B2GRWB2B2WWB2
253: B2GWWWBB2BB2
254: RGWB2BB2WB2B
255: GRB2B2WBWWR
256: WGB2B2WWWBW
257: B2RWB2B2WBB2
258: BRB2WWBWB2B
259: B2RBWB2B2WWB2
260: B2RWWWBB2GB2
261: BRWB2BB2WB2B
262: RBB2B2WGWWB
263: WRB2B2WWWGW
264: B2BWWWBB2RB2
265: BBB2B2WRWWB
266: WBB2B2WWWRW

track 7:

267: BB2WWWGWB2G
268: B2BB2GWB2B2WB2

memory switch:

track 7:

269: B2BWWGB2BG2
270: GB2WWWRWB2R
271: BBWB2BB2WG2G2
272: BBG2B2WBWWB
273: B2BG2RWB2B2WB2
274: WBB2WWWWG2W
275: G2BWWRB2BR
276: RG2WWWBWB2B
277: G2BWB2BB2WRR
278: BG2RB2WBWWB2
279: B2BRBWB2B2WB2
280: B2G2WWB2B2WBB2
281: BG2B2WWBWB2G
282: B2G2BWB2B2WWB2
283: WG2B2WWWWRW
284: RG2WWWBB2BB
285: RB2WB2GB2WBB
286: B2RBB2WBWWB2
287: B2B2BBWB2B2WB2
288: WB2B2B2WWWBW
289: WWB2BWWWWW
290: BRWWWBB2B2B
291: BB2WB2RB2WBB
292: B2BBB2WBWWB2
293: BBWWWBB2B2B
294: BB2WB2BB2WBB

memory switch L

track 7:

295: B2B2BGWB2B2WB2
296: BBWWWGB2B2G
297: BB2WB2BB2WGG
298: B2BGB2WBWWB2
299: B2B2GRWB2B2WB2
300: GBWWWRB2B2R
301: GB2WB2BB2WRR
302: B2GRB2WBWWB2
303: B2B2RBWB2B2WB2
304: RGWWWBB2B2B

track 4:

305: BB2WB2GB2WBG
306: GB2WB2RB2WBR
307: B2GBB2WBWWB2
308: BGWWWBB2B2G
309: RB2WB2BB2WGB
310: B2RGB2WBWWB2
311: B2B2GBWB2B2WB2
312: GRWWWBB2B2R
313: BB2WB2BB2WRB
314: B2BRB2WBWWB2
315: B2B2RGWB2B2WB2
316: RBWWWGB2B2B
317: B2B2BRWB2B2WB2
318: BBWWWRB2B2B

track 1:

319: B2BBB2WGWWG2
320: WB2B2B2WWWGW
321: WWB2GWWWWW
322: BG2WB2BB2WBG2
323: G2BBB2WRWWR
324: B2G2BBWB2B2WB2
325: WG2B2B2WWWRW
326: WBG2WWWWB2W
327: WWG2RWWWWW
328: BBWWWBB2G2B
329: G2RWB2BB2WBR
330: RG2BB2WBWWB
331: B2RBBWB2B2WB2
332: WG2RWWWWB2W
333: WG2B2WWWWBW
334: BG2WWWBB2RB2

flip-flop switch:

track 4:

335: BBB2G2WBG2WB
336: G2BB2BWB2B2WG2
337: WBG2B2WWWBW
338: BBWWWBWG2B
339: WG2BBWWWB2W
340: B2G2WWWWWB2B2
341: B2WG2B2WWWWB2
342: WBBG2WWWWW
343: G2WBBWB2B2WG2
344: WWG2B2WWWWW
345: WBWWWWG2B2W
346: WWWWWWG2W
347: B2BWG2WBG2BB2
348: G2B2WWWB2B2WG2
349: WB2G2B2WWWBW
350: BB2WWBWG2B
351: BGB2G2WBG2WG
352: WGBG2WWWWW
353: WGWWWWG2B2W
354: B2GWG2WBG2BB2
355: GRB2G2WBG2WR
356: G2GB2BWB2B2WG2
357: WGG2B2WWWBW
358: BGWWWBWG2G
359: WRGG2WWWWW
360: G2WGBWB2B2WG2
361: WRWWWWG2B2W
362: B2RWG2WBG2GB2
363: RBB2G2WGG2WB2
364: G2RB2BWB2B2WG2
365: WRG2B2WWWGW
366: GRWWWBWG2R
367: WBRG2WWWWW
368: G2WRGWB2B2WG2
369: WG2GBWWWB2W
370: B2BWG2WBG2RB
371: BB2WWBWWBB
372: B2BBG2WRG2WB2
373: G2B2BBWB2B2WG2
374: WB2G2B2WWWRW
375: RB2WWWGWG2B
376: WBB2G2WWWWW
377: G2WB2RWB2B2WG2
378: WG2RGWWWB2W
379: WBWWWWG2BW
380: BBWG2WBG2B2B
381: G2BWWWB2B2WG2
382: B2BBG2WBG2WB2
383: BB2WWWRWG2B
384: G2WB2BWB2B2WG2
385: WG2BRWWWB2W

flip-flop switch L:

track 4:

386: BB2WWGWWBG
387: GB2WWRWWBR
388: B2GBG2WBG2WB2
389: WGB2G2WWWWW
390: WGWWWWG2BW
391: BGWG2WBG2B2G
392: RB2WWBWWGB
393: B2RGG2WBG2WB2
394: G2B2GBWB2B2WG2
395: WRB2G2WWWWW
396: WRWWWWG2GW
397: GRWG2WBG2B2R
398: G2GWWWB2B2WG2
399: BB2WWBWWRB
400: B2BRG2WBG2WB
401: G2B2RGWB2B2WG2
402: WBWWWWG2RW
403: RBWG2WGG2B2B2
404: G2RWWWB2B2WG2
405: G2BB2RWB2B2WG2
406: B2BWG2WRG2BB2

tracks alone:

407: WWBBBWWBW
408: WWBWWWWBW
409: BWWWWBWBB
410: BWBWWWWBB
411: WBBBWWWWW
412: BWBWWWBWB
413: WWBBBBBWW
414: WWWBBWBBW
415: BWWBWWWBB
416: WWBBBBBBW
417: BWWBWBWWB
418: BBWWBWWWB
419: WBBWWWWBW
420: BBWBWWWWB
421: WWBWWWWWW
422: BWWBWGWWG
423: WBBWWWWGW
424: WWBBBWWGW
425: WWGWWWWBW
426: WWWBBWBGW
427: GWWBWRWWR
428: BGWWBWWWG
429: WGBWWWWRW
430: WWBBBWWRW
431: WWRWWWWBW
432: WWWBBWGRW
433: WWBBBBBGW
434: RWWGWBWWB
435: GRWWBWWWR
436: WRGWWWWBW
437: BGWBWWWWG
438: WWGWWWWWW
439: WWWBBWRBW
440: WWBBBBGRW
441: BWBWWWWGG
442: BWWRWBWWB
443: RBWWGWWWB
444: WBRWWWWBW
445: GRWBWWWWR
446: WWRWWWWWW
447: WWBBBGRBW
448: GWBWWWWRR
449: BBWWRWWWB
450: RBWGWWWWB
451: WWBBGRBBW
452: RWGWWWWBB
453: BBWRWWWWB
454: BWBWWWGWG
455: WWBGRBBBW
456: BWRWWWWBB
457: WWWBGWBBW
458: BWWBWWWGG
459: GWBWWWRWR
460: WWGRBBBBW
461: WWBBBBGWW
462: WWWGRWBBW
463: GWWBWWWRR
464: RWGWWWBWB
465: WWRBBBBBW
466: WWBBBGRWW
467: WWWRBWBBW
468: RWWGWWWBB
469: BWRWWWBWB
470: WWBBGRBWW
471: BWWRWWWBB
472: WWBGRBBWW
473: WWBBGWWBW
474: WWGRBBBWW
475: WWBGRWWBW
476: BWWWWBWGG
477: WGBBWWWWW

tracks alone:

478: WWRBBBBWW
479: WWGRBWWBW
480: WWBWWWWGW
481: GWWWWBWRR
482: WRGBWWWWW
483: WWRBBWWBW
484: WWBWWWWRW
485: RWWWWGWBB
486: BWWWWRWBB
487: WBBRWWWWW
488: BWWWWGWBG
489: WBBGWWWWW
490: WWGBBWWBW
491: GWWWWRWBR
492: BWGWWWWBG
493: WBGRWWWWW
494: WWRGBWWBW
495: RWWWWBWGB
496: GWRWWWWBR
497: WGRBWWWWW
498: BWGWWWBWG
499: WWBRGWWBW
500: BWWWWBWRB
501: RWBWWWWGB
502: WRBBWWWWW
503: GWRWWWBWR
504: WWGBBBBWW
505: WWBBRWWBW
506: BWBWWWWRB
507: RWBWWWGWB
508: WWRGBBBWW
509: BWBWWWRWB
510: WWBRGBBWW
511: WWBBRGBWW
512: BWWGWWWBG
513: WWBBBRGWW
514: WWWGBWBBW
515: GWWRWWWBR
516: WWBBBBRWW
517: WWWRGWBBW
518: RWWBWWWGB
519: WWGBBBBBW
520: WWWBRWBBW
521: BWWBWWWRB
522: WWRGBBBBW
523: WWBRGBBBW
524: WWBBRGBBW
525: BBWGWWWWG
526: WWBBBRGBW
527: BBWWGWWWG
528: GBWRWWWWR
529: WWWBBWGBW
530: WWBBBBRGW
531: BWWGWBWWG
532: GBWWRWWWR
533: WBGWWWWBW
534: RGWBWWWWB
535: WWWBBWRGW
536: WWBBBBBRW
537: GWWRWBWWR
538: RGWWBWWWB
539: WGRWWWWBW
540: BRWBWWWWB
541: WWWBBWBRW
542: RWWBWGWWB
543: BRWWBWWWB
544: WRBWWWWGW
545: BWWBWRWWB

quiescent remains quiescent. Starting from the central cell the program scans the sectors one after the other and in each one, from the root to the last level, level by level. For each cell c, call **context** of the cell the sequence of states constituted of the state of c and those of the neighbours of c, starting from its neighbour 1 and going on by counter-clockwise turning around c. The program takes the context κ of c in table 0. Then, it compares κ with the contexts of the rules of the file. If it finds a match, it copies the new state of the rule at the address of c in table 1, under column 0. If it does not find a match, it asks for a new rule which the user writes into the file. To help the user, the program indicates the context of the cell. The user enters the new state only. Then the program resumes its computation: it reads again table 0 from the initial configuration and performs the computation as far as possible. If it can compute the new state of all cells of table 0, it completes table 1 by computing the new states of the neighbours of each cell. When this task is over, the program copies table 1 onto table 0: a new step of the computation of the cellular automaton can be processed. This cycle is repeated until no new rule is required and until the fixed in advance number of steps is reached.

Now, when a new rule is entered by the user on a cell c, it may happen that the new rule is in conflict with the previously entered rules. This happens when there is a rule η whose context is a rotated form of the context of c, but the state suggested by the user is not the new state of the rule. In this case, the program stops with an error message which also displays the rule with which the program have found a mismatch. If the rule constructed on the context of the cell and the state indicated by the user is a rotated form of an already existing rule, it is appended to the set of rules.

When the program can be run without asking a new rule nor indicating any error, we know that the set of rules is computed.

The program also contributes to check that the rules are rotation invariant, using lemma 5.1 which is very easy to implement: this is left to the reader. The same property was used in [10]. In the display of the rules in Table 2, we assume that the minimal rotated forms of the rules are pairwise distinct. This means that on the seven rotated forms which should be present for each rule, we keep one form only for the table. As we have 545 rules in the table, this is an important simplification.

Note that we have obtained 545 rules instead of 299 in the case of the pentagrid, see [10]. In Table 2, the rules are displayed in the format indicated in Subsection 5.1. Also, the rules are gathered according to the order in which they were entered in the set of rules during the execution of the program. This order also corresponds to the different configurations which we described in Section 4 and to the figures which are displayed in this section as well as figures which are not displayed but which correspond to the motions described in the section. The reader may check in the table that for all crossings and switches, the motion of the locomotive was checked for all possible arrivals. Also, the program have checked configurations which are not represented in the figures of the paper: as an example, the picture (h) of Fig. 14 represents the idle configuration of the memory switch in which the selected track is 7. But the figure represents a motion in which, initially, the selected track is 1.

No figure of the paper represents the motion of the locomotive when the selected track is 7. However, in order to get a correct set of rules, the program had to test this situation too. As the reader can check it on the table, all the needed tests were performed. Indeed, each test on a type of motion induced new rules where at least one of them was not a rotated form of a previously obtained rule.

This completes the proof of Theorem 4.1. □

6 Conclusion

It is probably possible to do better, but it will require some effort and changes in the configurations themselves.

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