Available online at www.sciencedirect.com

ScienceDirect

Electronic Notes in Theoretical Computer Science

ELSEVIER Electronic Notes in Theoretical Computer Science 230 (2009) 129–140

www.elsevier.com/locate/entcs

A Fibration Category of Local Pospaces

Thomas Kahl [1]

Centro de Matemática
Universidade do Minho
Campus de Gualtar
4710-057 Braga, Portugal

Abstract

L. Fajstrup, E. Goubault, and M. Raussen have introduced local pospaces as a model for concurrent systems. In this paper it is shown that the category of local pospaces under a fixed local pospace is a fibration category in the sense of H. Baues. The homotopy notion in this fibration category is relative directed homotopy.

Keywords: Local pospaces, dihomotopy, concurrency, fibration category, model category

1 Introduction

Homotopy theoretical methods have been used successfully in the recent past to study problems in concurrency theory, the domain of theoretical computer science that deals with parallel computing. Various topological models have been introduced to describe concurrent systems. Examples are partially ordered spaces (or pospaces) and local pospaces [4], flows [5], globular CW-complexes [6], and d-spaces [8]. The reader is referred to E. Goubault [7] for a recent introduction to different topological models for concurrency. The purpose of this paper is to study the homotopy theory or, more precisely, the relative directed homotopy theory of local pospaces.

Many concurrent systems can be modeled as pospaces. A pospace is a space X with a partial order $\leq$ on it which is closed as a subspace of $X \times X$. The space X is interpreted as the state space of the system and the partial order represents the time flow. The idea here is that the execution of a system is a process in time so that a system in each state x can only proceed to subsequent states $y \geq x$ and not go back to preceding states $y < x$.

[1] Email: kahl@math.uminho.pt

1571-0661/
doi:10.1016/j.entcs.2009.02.021

The pospace conception of concurrent systems is too restrictive if one wishes to consider systems which contain loops in the sense that they might return various times to the same state during the execution. Such systems with loops can be modeled as local pospaces. Local pospaces have been introduced in the late 1990's by L. Fajstrup, E. Goubault, and M. Raussen in a preprint version of [4] (available at `http://www.di.ens.fr/~goubault`). In the meantime some alternative definitions of local pospaces have been proposed (cf. [6], [3]). Note also that the definition given in [4] is not the original one. In this paper we shall work with still another definition of local pospaces. We define a local pospace to be a space X together with a relation $\leq$ which locally is a partial order. It can be shown that this definition is equivalent to the original one in the sense that the two concepts give rise to equivalent categories.

A natural question is whether a system in a given state x can reach another state y or, in other words, whether there is an "execution path" from x to y. Such problems can be formalized appropriately using the following notion of maps between local pospaces. A dimap (short for directed map) from a local pospace $(X, \leq)$ to a local pospace $(Y, \leq)$ is a continuous map $f : X \to Y$ such that each point of X has a neighborhood on which f is compatible with the relations. An execution path from a state x of a local pospace $(X, \leq)$ to a state y can now formally be defined to be a dimap ω from the unit interval $I = [0, 1]$ with the natural order to $(X, \leq)$ such that $\omega(0) = x$ and $\omega(1) = y$.

If there exists an execution path from one state of a system to another, there will, in general, exist a lot. Many of them will actually not be qualitatively different and correspond to computer scientifically equivalent executions. From a computer scientific point of view it makes sense to consider two execution paths $\omega, \nu\colon (I, \leq) \to (X, \leq)$ from a state x to a state y as equivalent if there exists a homotopy $H : I \times I \to X$ from ω to ν which is a dimap with respect to the partial order on $I \times I$ given by $(t, s) \leq (t', s') \Leftrightarrow t \leq t',\ s = s'$ and which satisfies $H(0, t) = x$ and $H(1, t) = y$ for each $t \in I$. Such a homotopy is called a directed homotopy (dihomotopy) from ω to ν relative to the sub pospace $(\{0, 1\}, \leq)$ of $(I, \leq)$. Relative directed homotopy theory plays thus a fundamental role in the study of execution paths. As P. Bubenik [2] has pointed out, relative directed homotopy theory is also useful for the task to decide to what extent two local pospaces can be considered as models of the same concurrent system. Note that some authors work with a stronger notion of dihomotopy where the time parameter interval is equipped with the natural order (cf. [8], [3]).

The best known general framework for homotopy theory is certainly the one of closed model categories in the sense of D. Quillen [10]. A closed model category is a category with three classes of morphisms, called weak equivalences, fibrations, and cofibrations, which are subject to certain axioms. The structure of a closed model category splits up into two dual structures which are essentially the structure of a cofibration category and the structure of a fibration category. Cofibration and fibration categories have been introduced by H. Baues [1] who has developed an extensive homotopy theory for these categories. The main ingredient of this homotopy theory is of course a notion of homotopy. In this paper we show that the category of local pospaces under a fixed local pospace is a fibration category such that the homotopy notion is relative directed homotopy.

In [9] it has been shown that the category of pospaces (with a not necessarily closed partial order) under a fixed pospace is both a fibration and a cofibration category. Unfortunately, the author does not know whether the category of local pospaces (under a fixed local pospace) is a cofibration category. The main problem is that it is not known whether the category of local pospaces has enough colimits. Note in this context that P. Bubenik and K. Worytkiewicz [3] have constructed a closed model category containing the category of local pospaces (essentially in the original sense) under a fixed local pospace as a subcategory.

2 Local pospaces

Definition 2.1 A *pospace* (*po* is short for *partially ordered*) is a pair $(X, \leq)$ consisting of a space X and a partial order $\leq$ on X which is closed as a subset of $X \times X$. A pair $(X, \leq)$ consisting of a space X and a relation $\leq$ on X is called a *local pospace* if each point $x \in X$ has a neighborhood U such that $(U, \leq)$ is a pospace. A *dimap* (short for *directed map*) $f : (X, \leq) \to (Y, \leq)$ is a continuous map $f : X \to Y$ such that each point $x \in X$ has a neighborhood restricted to which f is compatible with the relation $\leq$.

Remark 2.2 (i) Note that a pospace is a local pospace. Note also that the relation $\leq$ of a local pospace $(X, \leq)$ is necessarily reflexive. When this is helpful we shall denote this relation by $\leq_X$ instead of $\leq$.

(ii) Recall that a space X is a Hausdorff space if and only if the diagonal $\Delta = \{(x, y) \in X \times X \,|\, x = y\}$ is closed in $X \times X$. Therefore the pair (X, Δ) is a pospace if and only if X is a Hausdorff space.

(iii) For a Hausdorff space X and a local pospace $(Y, \leq)$ the set of dimaps from (X, Δ) to $(Y, \leq)$ coincides with the set of continuous maps from X to Y.

Example 2.3 (i) The circle S^1 is a local pospace with respect to the relation $\leq$ defined by

$$x \leq y \Leftrightarrow \exists\, \theta \in [0, \pi[: y = xe^{i\theta}.$$

For $x \in S^1$,

$$U = \{xe^{i\theta} \,|\, \theta \in] - \pi/2, \pi/2[\}$$

is an open neighborhood such that $(U, \leq)$ is a global pospace.

(ii) The unit interval $I = [0, 1]$ together with the natural order $\leq$ is a pospace and hence a local pospace. Consider $x, y \in S^1$ and $\theta \in [0, +\infty[$ such that $y = xe^{i\theta}$. An execution path from x to y, i.e., a dimap $\omega : (I, \leq) \to (S^1, \leq)$ with $\omega(0) = x$ and $\omega(1) = y$, is given by $\omega(t) = xe^{it\theta}$. For $t \in I$, $U = \{s \in I \,|\, |s - t| < \frac{\pi}{2\theta}\}$ is a neighborhood of t restricted to which ω is compatible with $\leq$.

Proposition 2.4 *The composite of two dimaps* $f : (X, \leq) \to (Y, \leq)$ *and* $g : (Y, \leq) \to (Z, \leq)$ *is a dimap.*

Proof Let $x \in X$. Since f and g are dimaps, there exist neighborhoods $U \subset X$ of x and $V \subset Y$ of $f(x)$ such that f is compatible with $\leq$ on U and g is compatible with $\leq$ on V. Since f is continuous, there exists a neighborhood $W \subset X$ of x such that $f(W) \subset V$. The intersection $U \cap W$ is a neighborhood of x on which the composite $g \circ f$ is compatible with $\leq$. $\square$

It follows from the preceding proposition that local pospaces and dimaps form a category. This category will be denoted by **LPS**.

Proposition 2.5 *The category* **LPS** *is finitely complete.*

Proof We show that **LPS** has a final object and is closed under pullbacks. The final object is $(*, \Delta)$. Let $f : (X, \leq_X) \to (B, \leq_B)$ and $p : (E, \leq_E) \to (B, \leq_B)$ be two dimaps. Define a relation $\leq$ on the product $X \times E$ by

$$(x, e) \leq (x', e') \Leftrightarrow x \leq_X x' \text{ and } e \leq_E e'.$$

Then the fiber product $X \times_B E$ is a local pospace with respect to the restriction of $\leq$ to $X \times_B E$. Indeed, let $(x, e) \in X \times_B E$. Since X and E are local pospaces, there exist open neighborhoods $U \subset X$ of x and $V \subset E$ of e such that $(U, \leq_X)$ and $(V, \leq_E)$ are pospaces. The subspace $U \times_B V = (U \times V) \cap (X \times_B E)$ of $X \times_B E$ is an open neighborhood of (x, e) and $\leq$ is a partial order on $U \times_B V$. Since $\leq_X \cap (U \times U)$ is closed in $U \times U$ and $\leq_E \cap (V \times V)$ is closed in $V \times V$, $\leq_X \times \leq_E \cap (U \times U \times V \times V)$ is closed in $U \times U \times V \times V$. It follows that $\leq \cap (U \times V \times U \times V)$ is closed in $U \times V \times U \times V$ and hence that $\leq \cap (U \times_B V \times U \times_B V)$ is closed in $U \times_B V \times U \times_B V$. Therefore $(X \times_B E, \leq)$ is a local pospace. It is clear that the projections $pr_X : X \times_B E \to X$ and $pr_E : X \times_B E \to E$ are dimaps. We check that $(X \times_B E, \leq)$ has the universal property of the pullback. Consider dimaps $\phi : (Z \leq_Z) \to (X, \leq_X)$ and $\psi : (Z, \leq_Z) \to (E, \leq_E)$ such that $f \circ \phi = p \circ \psi$. Let $h : Z \to X \times_B E$ be the unique continuous map such that $pr_X \circ h = \phi$ and $pr_E \circ h = \psi$. We check that h is a dimap. Let $z \in Z$. Since both ϕ and ψ are dimaps, there exist neighborhoods U and V of z such that ϕ is compatible with the relations on U and ψ is compatible with the relations on V. The intersection $U \cap V$ is a neighborhood of z. Since ϕ and ψ are compatible with the relations on $U \cap V$, h is compatible with the relations on $U \cap V$. This shows that h is a dimap. It follows that $(X \times_B E, \leq)$ is the pullback of f and p in **LPS**. $\square$

Let $(X, \leq_X)$ be a local pospace. We define a relation on the path space X^I (i.e., the set of all continuous maps $\omega : I \to X$ with the compact-open topology) by

$$\omega \leq_{X^I} \nu \Leftrightarrow \forall\, t \in I\ \omega(t) \leq_X \nu(t).$$

Proposition 2.6 $(X^I, \leq_{X^I})$ *is a local pospace.*

Proof Let $\omega \in X^I$. For each $t \in I$ choose an open neighborhood U_t of $\omega(t)$ such that $(U_t, \leq_X)$ is a pospace. Since ω is continuous, for all $t \in I$ there exists $\varepsilon_t > 0$ such that $\omega(I \cap]t - 2\varepsilon_t, t + 2\varepsilon_t[) \subset U_t$. Since I is compact, there exist $t_1, \ldots, t_m \in I$ such that $I = \cup_{j=1}^m I \cap [t_j - \varepsilon_{t_j}, t_j + \varepsilon_{t_j}]$. Set

$$W_j = \{\nu \in X^I \,|\, \nu(I \cap [t_j - \varepsilon_{t_j}, t_j + \varepsilon_{t_j}]) \subset U_{t_j}\} \quad (j = 1, \ldots, m)$$

and $W = \cap_{j=1}^{m} W_j$. Then W is an open neighborhood of ω in X^I. One checks that $\leq_{X^I}$ is a partial order on W. We show that $\leq_{X^I} \cap (W \times W)$ is a closed subset of $W \times W$. Let $\alpha, \beta \in W$ such that $\alpha \not\leq_{X^I} \beta$. Then there exists $t \in I$ such that $\alpha(t) \not\leq_X \beta(t)$. Let $j \in \{1, \ldots, m\}$ such that $t \in [t_j - \varepsilon_{t_j}, t_j + \varepsilon_{t_j}]$. Then $\alpha(t), \beta(t) \in U_{t_j}$. Since $\leq_X \cap (U_{t_j} \times U_{t_j})$ is a closed subset of $U_{t_j} \times U_{t_j}$, there exists an open neighborhood N of $(\alpha(t), \beta(t))$ in $U_{t_j} \times U_{t_j}$ such that $a \not\leq_X b$ for all $(a, b) \in N$. Consider the continuos map $f : W \to U_{t_j}$ given by $f(\gamma) = \gamma(t)$. The set $(f \times f)^{-1}(N)$ is an open neighborhood of (α, β) in $W \times W$. For all $(\gamma, \delta) \in (f \times f)^{-1}(N)$, $\gamma(t) \not\leq_X \delta(t)$ and hence $\gamma \not\leq_{X^I} \delta$. This shows that the complement of $\leq_{X^I} \cap (W \times W)$ is open in $W \times W$ and hence that $\leq_{X^I} \cap (W \times W)$ is closed in $W \times W$. □

Definition 2.7 Let $(C, \leq)$ be a local pospace. A *local pospace under* $(C, \leq)$ is a triple $(X, \leq, \xi)$ consisting of a local pospace $(X, \leq)$ and a dimap $\xi \colon (C, \leq) \to (X, \leq)$. A *dimap under* $(C, \leq)$ from $(X, \leq, \xi)$ to $(Y, \leq, \theta)$ is a dimap $f \colon (X, \leq) \to (Y, \leq)$ such that $f \circ \xi = \theta$. The category of local pospaces under $(C, \leq)$ is denoted by $\mathbf{LPS}^{(C,\leq)}$.

Note that a local pospace is the same as a local pospace under $(\emptyset, \Delta)$.

Proposition 2.8 *For any pospace $(C, \leq)$ the category $\mathbf{LPS}^{(C,\leq)}$ is finitely complete.*

Proof This follows from 2.5. □

Let $(X, \leq_X, \xi)$ be a local pospace under $(C, \leq)$. Consider the dimap $\mathsf{c}_\xi : (C, \leq) \to (X^I, \leq_{X^I})$, $z \mapsto \mathsf{c}_{\xi(z)}$ where c_y is the constant path $t \mapsto y$. Then $(X^I, \leq_{X^I}, \mathsf{c}_\xi)$ is a local pospace under $(C, \leq)$. Moreover, for each $t \in I$, the evaluation map $ev_t : X^I \to X$, $\omega \mapsto \omega(t)$ is a dimap under $(C, \leq)$.

3 Dihomotopy

Throughout this section we work under a fixed local pospace $(C, \leq)$.

Definition 3.1 Two dimaps $f, g : (X, \leq, \xi) \to (Y, \leq, \theta)$ under $(C, \leq)$ are said to be *dihomotopic relative to* $(C, \leq)$, $f \simeq g$ *rel.* $(C, \leq)$, if there exists a *dihomotopy relative to* $(C, \leq)$ from f to g, i.e., a dimap $H : (X, \leq) \times (I, \Delta) \to (Y, \leq)$ such that $H(x, 0) = f(x)$, $H(x, 1) = g(x)$ $(x \in X)$, and $H(\xi(c), t) = \theta(c)$ $(c \in C,\ t \in I)$. If $C = \emptyset$ we simply talk of *dihomotopies* and *dihomotopic maps*.

We shall need the following lemma concerning the compatibility of dihomotopies with the relations.

Lemma 3.2 *Let $H : (X, \leq) \times (I, \Delta) \to (Y, \leq)$ be a dimap. Then each point $x_0 \in X$ admits an open neighborhood U such that H is compatible with the relations on $U \times I$.*

Proof Let $x_0 \in X$. For each $t \in I$ there exist an open neighborhood $V_t \subset X$ of x_0 and an open neighborhood $W_t \subset I$ of t such that H is compatible with the relations on $V_t \times W_t$. Since I is compact, there exist $t_1, \ldots, t_n \in I$ such that $I = \cup_{i=1}^{n} W_{t_i}$. Set $U = \cap_{i=1}^{n} V_{t_i}$. Then U is an open neighborhood of x_0. Let $(x, t), (x', t') \in U \times I$ such that $(x,t) \leq (x',t')$. Then $t = t'$. There exists $i \in \{1,\dots,n\}$ such that $t = t' \in W_{t_i}$. It follows that $(x,t),(x',t') \in V_{t_i} \times W_{t_i}$ and hence that $H(x,t) \leq H(x',t')$. □

Proposition 3.3 *Dihomotopy relative to $(C,\leq)$ is a natural equivalence relation.*

Proof We only show transitivity. Let $f,g,h : (X,\leq,\xi) \to (Y,\leq,\theta)$ be three dimaps under $(C,\leq)$, $F : (X,\leq) \times (I,\Delta) \to (Y,\leq)$ be a dihomotopy relative to $(C,\leq)$ from f to g, and $G : (X,\leq) \times (I,\Delta) \to (Y,\leq)$ be a dihomotopy relative to $(C,\leq)$ from g to h. Consider the continuous map $H : X \times I \to Y$ defined by

$$H(x,t) = \begin{cases} F(x,2t), & 0 \leq t \leq \frac{1}{2}, \\ G(x,2t-1), & \frac{1}{2} \leq t \leq 1 \end{cases}$$

We have $H(x,0) = f(x)$, $H(x,1) = h(x)$, and $H(\xi(c),t) = \theta(c)$ for all $c \in C$ and $t \in I$. We check that H is a dimap $(X,\leq) \times (I,\Delta) \to (Y,\leq)$. Let $(x_0,t_0) \in X \times I$. Since F and G are dimaps there exist, by 3.2, open neighborhoods U and V of x such that F is compatible with $\leq$ on $U \times I$ and G is compatible with $\leq$ on $V \times I$. The set $(U \cap V) \times I$ is an open neighborhood of (x_0,t_0). Let $(x,t),(x',t') \in (U \cap V) \times I$ such that $(x,t) \leq (x',t')$. Then $t = t'$ and $H(x,t) \leq H(x',t')$. Thus H is a dihomotopy relative to $(C,\leq)$ from f to h. □

Definition 3.4 The equivalence class of a dimap under $(C,\leq)$ with respect to dihomotopy relative to $(C,\leq)$ is called its *dihomotopy class relative to* $(C,\leq)$. The quotient category $\mathbf{LPS}^{(C,\leq)}/\simeq$ *rel.* $(C,\leq)$ is the *dihomotopy category relative to* $(C,\leq)$. A *dihomotopy equivalence relative to* $(C,\leq)$ is a dimap $f : (X,\leq,\xi) \to (Y,\leq,\theta)$ under $(C,\leq)$ such that there exists a *dihomotopy inverse relative to* $(C,\leq)$, i.e., a dimap $g\colon (Y,\leq,\theta) \to (X,\leq,\xi)$ under $(C,\leq)$ satisfying $f \circ g \simeq id_{(Y,\leq,\theta)}$ *rel.*$(C,\leq)$ and $g \circ f \simeq id_{(X,\leq,\xi)}$ *rel.*$(C,\leq)$. Two local pospaces under $(C,\leq)$, $(X,\leq,\xi)$ and $(Y,\leq,\theta)$, are said to be *dihomotopy equivalent relative to* $(C,\leq)$ or of the *same dihomotopy type relative to* $(C,\leq)$ if there exists a dihomotopy equivalence relative to $(C,\leq)$ from $(X,\leq,\xi)$ to $(Y,\leq,\theta)$.

Note that a dimap under $(C,\leq)$ is a dihomotopy equivalence relative to $(C,\leq)$ if and only if its dihomotopy class relative to $(C,\leq)$ is an isomorphism in the dihomotopy category relative to $(C,\leq)$. Similarly, two local pospaces under $(C,\leq)$ are dihomotopy equivalent relative to $(C,\leq)$ if and only if they are isomorphic in the dihomotopy category relative to $(C,\leq)$.

Proposition 3.5 *Any isomorphism of local pospaces is a dihomotopy equivalence relative to $(C,\leq)$. Let $f : (X,\leq,\xi) \to (Y,\leq,\theta)$ and $g : (Y,\leq,\theta) \to (Z,\leq,\zeta)$ be two dimaps under $(C,\leq)$. If two of f, g, and $g \circ f$ are dihomotopy equivalences relative to $(C,\leq)$, so is the third.*

Proof The first assertion is obvious and the other follows from the corresponding fact for isomorphisms. □

As one would expect, relative dihomotopy can be characterized using path spaces:

Proposition 3.6 *Two dimaps $f, g : (X, \leq, \xi) \to (Y, \leq, \theta)$ under $(C, \leq)$ are dihomotopic relative to $(C, \leq)$ if and only if there exists a dimap $h\colon (X, \leq, \xi) \to (Y^I, \leq, \mathsf{c}_\theta)$ under $(C, \leq)$ such that $f = ev_0 \circ h$ and $g = ev_1 \circ h$.*

Proof Suppose first that $f \simeq g$ *rel.* $(C, \leq)$. Let $H : (X, \leq) \times (I, \Delta) \to (Y, \leq)$ be a dihomotopy relative to $(C, \leq)$ from f to g. Consider the continuous map $h : X \to Y^I$ defined by $h(x)(t) = H(x, t)$. This is a dimap. Indeed, let $x_0 \in X$. By 3.2, there exists an open neighborhood U of x_0 such that H is compatible with $\leq$ on $U \times I$. Let $x \leq x'$ be two elements of U. Then for each $t \in I$, $(x, t) \leq (x', t)$ and hence $h(x)(t) = H(x, t) \leq H(x', t) = h(x')(t)$. It follows that $h(x) \leq h(x')$. Since $h(\xi(c))(t) = H(\xi(c), t) = \theta(c)$, we have $h(\xi(c)) = \mathsf{c}_{\theta(c)}$ so that h is a dimap under $(C, \leq)$. We have $ev_0(h(x)) = H(x, 0) = f(x)$ and $ev_1(h(x)) = H(x, 1) = g(x)$.

Suppose now that we are given a dimap $h : (X, \leq, \xi) \to (Y^I, \leq, \mathsf{c}_\theta)$ under $(C, \leq)$ such that $f = ev_0 \circ h$ and $g = ev_1 \circ h$. Define a continuous map $H : X \times I \to Y$ by $H(x, t) = h(x)(t)$. Let $(x_0, t_0) \in X \times I$. Since h is a dimap, there exists an open neighborhood U of x_0 such that h is compatible with $\leq$ on U. Let $(x, t), (x', t') \in U \times I$ such that $(x, t) \leq (x', t')$. Then $t = t'$ and therefore $H(x, t) = h(x)(t) \leq h(x')(t) = H(x', t')$. This shows that H is a dimap. We have $H(x, 0) = h(x)(0) = f(x)$, $H(x, 1) = h(x)(1) = g(x)$, and $H(\xi(c), t) = h(\xi(c))(t) = \mathsf{c}_{\theta(c)}(t) = \theta(c)$. It follows that $f \simeq g$ *rel.* $(C, \leq)$. □

4 Difibrations

As in the preceding section we work under a fixed local pospace $(C, \leq)$. We define difibrations relative to $(C, \leq)$ and establish their fundamental properties.

Definition 4.1 A *difibration relative to* $(C, \leq)$ is a dimap $p\colon (E, \leq, \varepsilon) \to (B, \leq, \beta)$ under $(C, \leq)$ such that for every local pospace $(X, \leq, \xi)$ under $(C, \leq)$, every Hausdorff space Y, every dimap $f : (X, \leq) \times (Y, \Delta) \to (E, \leq)$ satisfying $f(\xi(c), y) = \varepsilon(c)$ and every dimap $H : (X, \leq) \times (Y, \Delta) \times (I, \Delta) \to (B, \leq)$ satisfying $H(x, y, 0) = (p \circ f)(x, y)$ and $H(\xi(c), y, t) = \beta(c)$ there exists a dimap $G : (X, \leq) \times (Y, \Delta) \times (I, \Delta) \to (E, \leq)$ such that $G(x, y, 0) = f(x, y)$, $p \circ G = H$, and $G(\xi(c), y, t) = \varepsilon(c)$.

Proposition 4.2 *The class of difibrations relative to $(C, \leq)$ is closed under composition and base change. Every isomorphism of local pospaces under $(C, \leq)$ is a difibration relative to $(C, \leq)$.*

Proof The proof is by standard left lifting property arguments. □

Proposition 4.3 *Every dimap f under $(C, \leq)$ admits a factorization $f = p \circ i$ where p is a difibration relative to $(C, \leq)$ and i is a dihomotopy equivalence relative to $(C, \leq)$.*

Proof Let $f : (X, \leq, \xi) \to (Y, \leq, \theta)$ be a dimap under $(C, \leq)$. Form the pullback of local pospaces under $(C, \leq)$

$$\begin{tikzcd}
(X \times_Y Y^I, \leq, (\xi, \mathsf{c}_\theta)) \arrow[r, "pr_{Y^I}"] \arrow[d, "pr_X"'] & (Y^I, \leq, \mathsf{c}_\theta) \arrow[d, "ev_0"] \\
(X, \leq, \xi) \arrow[r, "f"'] & (Y, \leq, \theta).
\end{tikzcd}$$

Let $p\colon (X \times_Y Y^I, \leq, (\xi, \mathsf{c}_\theta)) \to (Y, \leq, \theta)$ and $i\colon (X, \leq, \xi) \to (X \times_Y Y^I, \leq, (\xi, \mathsf{c}_\theta))$ be the dimaps under $(C, \leq)$ defined by $p(x, \omega) = \omega(1)$ and $i(x) = (x, \mathsf{c}_{f(x)})$. We have $p \circ i = f$. We show that i is a dihomotopy equivalence relative to $(C, \leq)$ and that p is a difibration relative to $(C, \leq)$.

The projection $pr_X : (X \times_Y Y^I, \leq, (\xi, \mathsf{c}_\theta)) \to (X, \leq, \xi)$ is a dihomotopy inverse relative to $(C, \leq)$ of i. Indeed, $pr_X \circ i = id_X$ and a dihomotopy relative to $(C, \leq)$ from $id_{X \times_Y Y^I}$ to $i \circ pr_X$ is given by $F(x, \omega, t) = (x, \omega_{1-t})$. Here, ω_s is the path given by $t \to \omega(st)$.

We check that p is a difibration relative to $(C, \leq)$. Let $(W, \leq, \psi)$ be a local pospace under $(C, \leq)$, Z be a Hausdorff space, $g\colon (W, \leq) \times (Z, \Delta) \to (X \times_Y Y^I, \leq)$ be a dimap satisfying $g(\psi(c), z) = (\xi(c), \mathsf{c}_{\theta(c)})$, and $G\colon (W, \leq) \times (Z, \Delta) \times (I, \Delta) \to (Y, \leq)$ be a dimap such that $G(w, z, 0) = (p \circ g)(w, z)$ and $G(\psi(c), z, t) = \theta(c)$. Define a continuous map $H : W \times Z \times I \to X \times_Y Y^I$ by

$$H(w, z, t) = ((pr_X \circ g)(w, z), h(w, z, t))$$

where

$$h(w, z, t)(s) = \begin{cases} (pr_{Y^I} \circ g)(w, z) \left(\frac{2s}{2-t}\right), & 2s \leq 2 - t, \\ G(w, z, 2s + t - 2), & 2 - t \leq 2s. \end{cases}$$

We have $H(w, z, 0) = g(w, z)$ and $(p \circ H)(w, z, t) = h(w, z, t)(1) = G(w, z, t)$. Since

$$h(\psi(c), z, t)(s) = \begin{cases} (pr_{Y^I} \circ g)(\psi(c), z) \left(\frac{2s}{2-t}\right) = \theta(c), & 2s \leq 2 - t, \\ G(\psi(c), z, 2s + t - 2) = \theta(c), & 2 - t \leq 2s, \end{cases}$$

we have $H(\psi(c), z, t) = ((pr_X \circ g)(\psi(c), z), h(\psi(c), z, t)) = (\xi(c), \mathsf{c}_{\theta(c)})$. We verify that H is a dimap. It is clear that the first component of H is a dimap. So we only have to check that h is a dimap. Let $(w_0, z_0, t_0) \in W \times Z \times I$. Since $pr_{Y^I} \circ g$ and G are dimaps, there exists an open neighborhood $U \subset W \times Z$ of (w_0, z_0) such that $pr_{Y^I} \circ g$ is compatible with $\leq$ on U and G is compatible with $\leq$ on $U \times I$. Let $(w, z, t), (w', z', t') \in U \times I$ such that $(w, z, t) \leq (w', z', t')$. Then $(w, z) \leq (w', z')$ and $t = t'$. Since $pr_{Y^I} \circ g$ and G are dimaps, we have $h(w, z, t)(s) \leq h(w', z', t')(s)$ for all $s \in I$. This shows that $h(w, z, t) \leq h(w', z', t')$. It follows that p is a difibration relative to $(C, \leq)$. □

The proof of the following proposition is an easy exercise and is left to the reader:

Proposition 4.4 *For every local pospace $(X, \leq, \xi)$ under $(C, \leq)$ the final dimap under $(C, \leq)$, $* : (X, \leq, \xi) \to (*, \Delta, *)$, is a difibration relative to $(C, \leq)$.*

Definition 4.5 A *trivial difibration relative to* $(C,\leq)$ is a dimap under $(C,\leq)$ which is both a difibration relative to $(C,\leq)$ and a dihomotopy equivalence relative to $(C,\leq)$.

Proposition 4.6 *Let* $p:(E,\leq,\varepsilon)\to(B,\leq,\beta)$ *be a trivial difibration relative to* $(C,\leq)$. *Then* p *admits a section* s *such that* $s\circ p\simeq id_{(E,\leq,\varepsilon)}$ *rel.* $(C,\leq)$.

Proof Let $f:(B,\leq,\beta)\to(E,\leq,\varepsilon)$ be a dihomotopy inverse relative to $(C,\leq)$ of p. Let $F:(B,\leq)\times(I,\Delta)\to(B,\leq)$ be a dihomotopy relative to $(C,\leq)$ from $p\circ f$ to $id_{(B,\leq,\beta)}$. Since p is a difibration relative to $(C,\leq)$, there exists a dimap $H:(B,\leq)\times(I,\Delta)\to(E,\leq)$ such that $H(b,0)=f(b)$, $p\circ H=F$, and $H(\beta(c),t)=\varepsilon(c)$. Let $s:(B,\leq,\beta)\to(E,\leq,\varepsilon)$ be the dimap under $(C,\leq)$ defined by $s(b)=H(b,1)$. Then $s\simeq f$ *rel.* $(C,\leq)$ and hence $s\circ p\simeq f\circ p\simeq id_{(E,\leq,\varepsilon)}$ *rel.* $(C,\leq)$. We have $(p\circ s)(b)=p(H(b,1))=F(b,1)=b$. □

By a *trivial cofibration* of spaces we mean a closed cofibration which is also a homotopy equivalence. The proof of the following important characterization of difibrations relative to $(C,\leq)$ is a straightforward adaptation of the proof of [9, 4.7].

Proposition 4.7 *A dimap* $p:(E,\leq,\varepsilon)\to(B,\leq,\beta)$ *under* $(C,\leq)$ *is a difibration relative to* $(C,\leq)$ *if and only if for every local pospace* $(Z,\leq,\zeta)$ *under* $(C,\leq)$, *every trivial cofibration of Hausdorff spaces* $i\colon A\to X$, *every dimap* $f:(Z,\leq)\times(A,\Delta)\to(E,\leq)$ *satisfying* $f(\zeta(c),a)=\varepsilon(c)$, *and every dimap* $g:(Z,\leq)\times(X,\Delta)\to(B,\leq)$ *satisfying* $g(z,i(a))=p(f(z,a))$ *and* $g(\zeta(c),x)=\beta(c)$, *there exists a dimap* $\lambda:(Z,\leq)\times(X,\Delta)\to(E,\leq)$ *such that* $\lambda(z,i(a))=f(z,a)$, $p\circ\lambda=g$, *and* $\lambda(\zeta(c),x)=\varepsilon(c)$.

Proposition 4.8 *The class of trivial difibrations relative to* $(C,\leq)$ *is closed under base change.*

Proof Let $p:(E,\leq,\varepsilon))\to(B,\leq,\beta)$ be a trivial difibration and consider a pullback diagram of local pospaces under $(C,\leq)$

$$\begin{tikzcd}
(X\times_B E,\leq,(\xi,\varepsilon)) \arrow[r, "\bar f"] \arrow[d, "\bar p"'] & (E,\leq,\varepsilon) \arrow[d, "p"] \\
(X,\leq,\xi) \arrow[r, "f"'] & (B,\leq,\beta).
\end{tikzcd}$$

By 4.2, $\bar p$ is a difibration relative to $(C,\leq)$. It remains to show that $\bar p$ is a dihomotopy equivalence relative to $(C,\leq)$. By 4.6, there exists a section s of p such that $s\circ p\simeq id_{(E,\leq,\varepsilon)}$ *rel.* $(C,\leq)$. Let $F:(E,\leq)\times(I,\Delta)\to(E,\leq)$ be a dihomotopy relative to $(C,\leq)$ from $s\circ p$ to $id_{(E,\leq,\varepsilon)}$. Consider the following commutative diagram of spaces where i is the obvious inclusion and h and H are given by $h(e,t,0)=F(e,t)$,

$h(e,t,1) = (s \circ p \circ F)(e,t)$, $h(e,0,\tau) = (s \circ p)(e)$, and $H(e,t,\tau) = (p \circ F)(e,t)$:

$$\begin{tikzcd} E \times (I \times \{0,1\} \cup \{0\} \times I) \arrow[r, "h"] \arrow[d, "id_E \times i"'] & E \arrow[d, "p"] \\ E \times I \times I \arrow[r, "H"'] & B \end{tikzcd}$$

Since $p \circ F$ is a dimap, H is a dimap $(E, \leq) \times (I \times I, \Delta) \to (B, \leq)$. Consider an element $(e_0, t_0, \tau_0) \in E \times (I \times \{0,1\} \cup \{0\} \times I)$. Since F and $s \circ p$ are dimaps, there exists an open neighborhood $U \subset E$ of e_0 such that $s \circ p$ is compatible with $\leq$ on U and F and $s \circ p \circ F$ are compatible with $\leq$ on $U \times I$. The set $U \times (I \times \{0,1\} \cup \{0\} \times I)$ is an open neighborhood of (e_0, t_0, τ_0). Let $(e,t,\tau), (e',t',\tau') \in U \times (I \times \{0,1\} \cup \{0\} \times I)$ such that $(e,t,\tau) \leq (e',t',\tau')$ in $(E, \leq) \times (I \times \{0,1\} \cup \{0\} \times I, \Delta)$. Then $e \leq e'$, $t = t'$, and $\tau = \tau'$ and hence $h(e,t,\tau) \leq h(e',t',\tau')$ in $(E, \leq)$. Thus h is a dimap $(E, \leq) \times (I \times \{0,1\} \cup \{0\} \times I, \Delta) \to (E, \leq)$. We have $h(\varepsilon(c), t, \tau) = \varepsilon(c)$ and $H(\varepsilon(c), t, \tau) = \beta(c)$. Since i is a trivial cofibration of Hausdorff spaces there exists, by 4.7, a dimap $G : (E, \leq) \times (I \times I, \Delta) \to (E, \leq)$ such that $G \circ (id_E \times i) = h$, $p \circ G = H$, and $G(\varepsilon(c), t, \tau) = \varepsilon(c)$. Let $\Phi : (E, \leq) \times (I, \Delta) \to (E, \leq)$ be the dimap given by $\Phi(e, \tau) = G(e, 1, \tau)$. We have $\Phi(\varepsilon(c), \tau) = \varepsilon(c)$,

$$\Phi(e,0) = G(e,1,0) = h(e,1,0) = F(e,1) = e,$$

$$\Phi(e,1) = G(e,1,1) = h(e,1,1) = (s \circ p \circ F)(e,1) = (s \circ p)(e),$$

and

$$(p \circ \Phi)(e,\tau) = (p \circ G)(e,1,\tau) = H(e,1,\tau) = (p \circ F)(e,1) = p(e).$$

Let $\sigma : (X, \leq, \xi) \to (X \times_B E, \leq, (\xi, \varepsilon))$ be the dimap under $(C, \leq)$ defined by $\sigma(x) = (x, (s \circ f)(x))$. Then $\bar{p} \circ \sigma = id_{(X, \leq, \xi)}$. Consider the dimap

$$\Psi : (X \times_B E, \leq) \times (I, \Delta) \to (X \times_B E, \leq)$$

given by $\Psi((x,e),\tau) = (x, \Phi(e,\tau))$. Since $f(x) = p(e) = p\Phi(e,\tau)$, Ψ is well-defined. We have $\Psi((x,e),0) = (x, \Phi(e,0)) = (x,e)$,

$$\Psi((x,e),1) = (x, \Phi(e,1)) = (x, (s \circ p)(e)) = (x, (s \circ f)(x)) = \sigma(x) = (\sigma \circ \bar{p})(x,e),$$

and $\Psi((\xi(c), \varepsilon(c)), \tau) = (\xi(c), \Phi(\varepsilon(c), \tau)) = (\xi(c), \varepsilon(c))$. This shows that

$$id_{(X \times_B E, \leq, (\xi, \varepsilon))} \simeq \sigma \circ \bar{p} \quad rel.\ (C, \leq)$$

and hence that $\bar{p}$ is a dihomotopy equivalence relative to $(C, \leq)$. □

5 The fibration category structure

In this section we put the results of the preceding sections together and show that the category of local pospaces under a fixed local pospace is a fibration category in the sense of H. Baues [1]. The homotopy theory of this fibration category is relative directed homotopy theory.

Definition 5.1 [1, I.1a] A category $\mathbf{F}$ equipped with two classes of morphisms, *weak equivalences* and *fibrations*, is a *fibration category* if it has a final object $*$ and if the following axioms are satisfied:

(F1) An isomorphism is a *trivial fibration*, i.e., a morphism which is both a fibration and a weak equivalence. The composite of two fibrations is a fibration. If two of the morphisms $f : X \to Y$, $g : Y \to Z$, and $g \circ f : X \to Z$ are weak equivalences, so is the third.

(F2) The pullback of two morphisms one of which is a fibration exists. The fibrations are stable under base change. The base extension of a weak equivalence along a fibration is a weak equivalence.

(F3) Every morphism f admits a factorization $f = p \circ j$ where p is a fibration and j is weak equivalence.

(F4) For each object X there exists a trivial fibration $Y \to X$ such that Y is *cofibrant*, i.e., every trivial fibration $E \to Y$ admits a section.

An object X is said to be $*$-*fibrant* if the final morphism $X \to *$ is a fibration.

Theorem 5.2 *Let $(C, \leq)$ be a local pospace. The category $\mathbf{LPS}^{(C,\leq)}$ of local pospaces under $(C, \leq)$ is a fibration category. The weak equivalences are the dihomotopy equivalences relative to $(C, \leq)$ and the fibrations are the difibrations relative to $(C, \leq)$. All objects are $(*, \Delta, *)$-fibrant and cofibrant.*

Proof By 2.8, $\mathbf{LPS}^{(C,\leq)}$ is finitely complete. By 4.4, all objects are $(*, \Delta, *)$-fibrant. The fact that all objects are cofibrant (and hence F4) is proved in 4.6. F1 follows from 3.5 and 4.2. F3 is 4.3. By 4.2, fibrations are stable under base change. By 4.8, the trivial fibrations are stable under base change. Since all objects are fibrant, this implies that weak equivalences are stable under base change along fibrations (cf. [1, I.1.4]). □

There is an extensive homotopy theory available for fibration categories (cf. [1]). The homotopy relation is defined as follows:

Definition 5.3 Let $\mathbf{F}$ be a fibration category, Y be a $*$-fibrant object, and X be a cofibrant object. Two morphisms $f, g : X \to Y$ are *homotopic* if for some factorization of the diagonal $Y \to Y \times Y$ into a weak equivalence $Y \to P$ and a fibration $e : P \to Y \times Y$ there exists a morphism $h : X \to P$ such that $e \circ h = (f, g)$.

Proposition 5.4 *Let $(C, \leq)$ be a local pospace and $(E, \leq, \varepsilon)$ be a local pospace under $(C, \leq)$. Then the dimap under $(C, \leq)$ $i : (E, \leq, \varepsilon) \to (E^I, \leq, \mathsf{c}_\varepsilon)$ given by $i(e) = \mathsf{c}_e$ is a dihomotopy equivalence relative to $(C, \leq)$ and the dimap under $(C, \leq)$ $ev : (E^I, \leq, \varepsilon) \to (E, \leq, \varepsilon) \times (E, \leq, \varepsilon)$ given by $ev(\omega) = (\omega(0), \omega(1))$ is a difibration relative to $(C, \leq)$.*

Proof Consider the dimap under $(C, \leq)$

$$f : (E \times_{E\times E} (E \times E)^I, \leq, (\varepsilon, \mathsf{c}_{(\varepsilon,\varepsilon)})) \to (E^I, \leq, \mathsf{c}_\varepsilon)$$

given by

$$f(e, (\alpha, \beta))(t) = \begin{cases} \alpha(1-2t),\ 0 \leq t \leq \frac{1}{2}, \\ \beta(2t-1),\ \frac{1}{2} \leq t \leq 1. \end{cases}$$

This is an isomorphism of local pospaces under $(C, \leq)$. The inverse is given by $f^{-1}(\omega) = (\omega(\frac{1}{2}), (\omega^-, \omega^+))$ where $\omega^-(t) = \omega(\frac{1}{2} - \frac{1}{2}t)$ and $\omega^+(t) = \omega(\frac{1}{2} + \frac{1}{2}t)$. We have seen in 4.3 that $f^{-1} \circ i$ is a dihomotopy equivalence relative to $(C, \leq)$ and that $ev \circ f$ is a difibration relative to $(C, \leq)$. The result follows. □

Proposition 5.5 *Let $(C, \leq)$ be a local pospace. Two dimaps under $(C, \leq)$ are homotopic in the fibration category* $\mathbf{LPS}^{(C,\leq)}$ *if and only if they are dihomotopic relative to $(C, \leq)$.*

Proof By [1, II.2.2], we may replace the word "some" in Definition 5.3 by "any". The result follows from 5.4 and 3.6. □

References

[1] Baues, H.J., "Algebraic Homotopy", Cambridge University Press (1989).

[2] Bubenik, P., *Context for models of concurrency*, in Preliminary Proceedings of the Workshop on Geometry and Topology in Concurrency and Distributed Computing GETCO 2004, BRICS Notes **NS-04-2** (2004), 33-49.

[3] Bubenik, P. and K. Worytkiewicz, *A model category for local po-spaces*, Homology, Homotopy and Applications **8** (1) (2006), 263-292.

[4] Fajstrup, L., E. Goubault, and M. Raussen, *Algebraic Topology and Concurrency*, Theoretical Computer Science **357** (2006), 241-278.

[5] Gaucher, P. *A model category for the homotopy theory of concurrency*, Homology, Homotopy and Applications **5** (2) (2003), 549-599.

[6] Gaucher, P. and E. Goubault, *Topological deformation of higher dimensional automata*, Homology, Homotopy and Applications **5** (2) (2003), 39-82.

[7] Goubault, E., *Some geometric perspectives in concurrency theory*, Homology, Homotopy and Applications **5** (2) (2003), 95-136.

[8] Grandis, M., *Directed homotopy theory I*, Cah. Topol. Géom. Différ. Catég. **44** (4) (2003), 281-316.

[9] Kahl, T., *Relative directed homotopy theory of partially ordered spaces*, Journal of Homotopy and Related Structures **1** (1) (2006), 79-100.

[10] Quillen, D., "Homotopical Algebra", LNM 43, Springer-Verlag (1967).