Product of Graphs and Hybrid Logic

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Abstract

Left and right commutativity and the Church-Rosser and reverse Church-Rosser properties are necessary conditions for a graph (frame) to be a (non-trivial) product of two other graphs, but their conjunction is not a sufficient condition. This work presents a fifth property, called H-V intransitivity, that, when added to the four previous properties, results in a necessary and sufficient condition for a finite and connected graph to be a product. Then, we show that although the first four properties can be defined in a modal logic (the reverse Church-Rosser property requires a converse modality), H-V intransitivity is not modally definable. We also show that no necessary and sufficient condition for a graph to be a product can be modally definable. Finally, we present a formula in a hybrid language that defines H-V intransitivity.

Keywords: Product of Graphs, H-V Intransitivity, Modal Languages, Modal Definability, Hybrid Languages

1 Introduction

The goal of this work is to find a necessary and sufficient condition for a graph (frame) to be isomorphic to a (cartesian) product [3] of two non-trivial graphs and to verify whether this condition can be expressed in a modal language or in some hybrid language.

In [7], three properties that are satisfied in graphs that are products are presented: *left commutativity, right commutativity and the Church-Rosser property.*

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[3] In graph theoretical terminology, a product of two graphs would be called a *bigraph*, since it has two sets of edges. More generally, in graph theory, graph-like structures with multiple sets of edges are called *multi-graphs*. In the context of modal logics and Kripke semantics, this notational difference is often lost and all these structures are called simply *labeled graphs*.

However, although these properties, together with the *reverse Church-Rosser property*, are necessary for a graph to be a product, they are not sufficient (as illustrated by an example in [7]). There are graphs that satisfy left and right commutativity and the Church-Rosser and reverse Church-Rosser properties, but cannot be decomposed as a product of two other graphs.

In this work, we introduce a new property called H-V intransitivity that, together with the previous ones, form a necessary and sufficient condition for a finite and connected graph to be a product. The proof of the necessity of these properties is fairly simple and is done directly, without the need to assume that the graph is finite or connected. On the other hand, the proof of the sufficiency is done in two steps. First, we prove that if a finite and connected graph satisfies the five properties stated above, then its horizontal and vertical components must satisfy a particular isomorphism. Then, we show that if a finite and connected graph satisfies H-V intransitivity and its components satisfy this particular isomorphism, then the graph is a product.

The limits to the expressive power of basic modal languages are fairly well known. There are a series of standard results that state that frames that are "similar" in a number of ways must agree on the validity of formulas [3]. Using these techniques, we show that the property of H-V intransitivity is not modally definable. In fact, we also show that no condition that is necessary and sufficient for a graph to be a product can be modally definable.

Hybrid logics are extensions of modal logics that allow explicit references to individual states of a model. Their goal is to extend the expressive power of ordinary modal logics. Besides proposition symbols, they have a second set of atomic formulas, called *nominals*, which have the property of being satisfied at exactly one state [1,2]. Using a hybrid language, we are able to express H-V intransitivity.

Products of graphs and multi-graphs in general come up naturally as an extension of ordinary Kripke semantics to multi-dimensional modal logics. [7] presents a good textbook discussion of multi-dimensional modal logics and provides many examples of products of modal logics, where the semantics is built using products of graphs. As an example of a concrete application of multi-dimensional modal logics with products of graphs, in [6] and [5], a multi-dimensional modal logic is used to express properties of distributed systems. This logic uses the transitive closure of a product of modal logics.

The paper is organized as follows. In section 2, we introduce the definition of a product of graphs and present four properties related to this definition: left and right commutativity and the Church-Rosser and reverse Church-Rosser properties. We also introduce a new property called H-V intransitivity. In section 3, we present the concept of graph decompositions and use it to prove that the five properties presented in the previous section form a necessary and sufficient condition for a finite and connected graph to be a product. Section 4 shows that the property of H-V intransitivity is not modally definable and that no necessary and sufficient condition for a graph to be a product can be modally definable. In section 5, we extend the modal language of the previous section to a hybrid language and show

that H-V intransitivity can be expressed by a hybrid formula. We summarize our results and present potential future works in section 6.

2 Product of Graphs

In this section, we define the product of two graphs, following [7]. The subscripts h and v refer to the geometrical intuition of the horizontal and vertical accessibility relations.

Definition 2.1 Given two directed graphs $G_1 = \langle V_1, E_1 \rangle$ and $G_2 = \langle V_2, E_2 \rangle$, we define their product G, notation $G = G_1 \times G_2$, as the graph $G = \langle V_1 \times V_2, E_h, E_v \rangle$, where for all $x, u \in V_1$ and $y, v \in V_2$

(i) $\langle x, y \rangle E_h \langle u, v \rangle$ iff xE_1u and $y = v$ and

(ii) $\langle x, y \rangle E_v \langle u, v \rangle$ iff yE_2v and $x = u$.

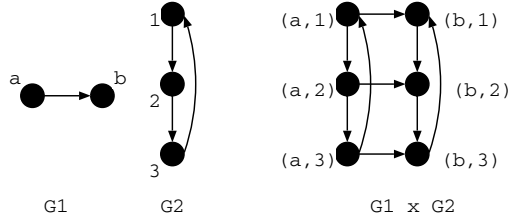An example of a product graph is shown in figure 1.

Fig. 1. Product of Graphs

In this work, we would like to identify a necessary and sufficient condition for a graph to be a product of two other graphs. In [7], three properties that are satisfied in graphs that are products are presented. These properties, together with the reverse Church-Rosser property, are necessary for a graph to be a product (figure 2):

(i)   Left Commutativity: $\forall x \forall y \forall z (xE_vy \land yE_hz \rightarrow \exists u(xE_hu \land uE_vz))$

(ii) Right Commutativity: $\forall x \forall y \forall z (xE_hy \land yE_vz \rightarrow \exists u(xE_vu \land uE_hz))$

(iii) Church-Rosser Property: $\forall x \forall y \forall z (xE_vy \land xE_hz \rightarrow \exists u(yE_hu \land zE_vu))$

(iv) Reverse Church-Rosser Property: $\forall x \forall y \forall z (yE_vx \land zE_hx \rightarrow \exists u(uE_hy \land uE_vz))$
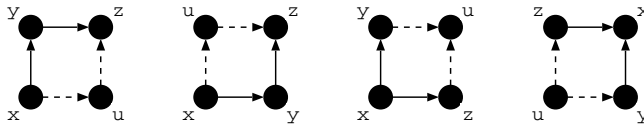

Fig. 2. Left and Right Commutativity and Church-Rosser and Reverse Church-Rosser Properties

However, although these properties are necessary for a graph to be a product, they are not sufficient: there are graphs that satisfy left and right commutativity and the Church-Rosser and reverse Church-Rosser properties, but cannot be decomposed as a product of two other graphs, as shows an example from [7] in figure 3.
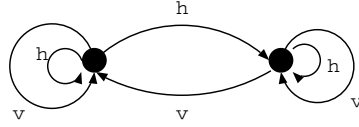
Fig. 3. Counterexample to the sufficiency of the basic properties

In order to obtain a necessary and sufficient condition we need to add a fifth property to the four stated before. We call it H-V intransitivity.

Definition 2.2 We say that a graph G satisfies H-V intransitivity if and only if every triple $\langle u, v, w \rangle$ of vertices of G such that $u \neq v$, $v \neq w$, there is an undirected path through edges of E_v from u to v and there is an undirected path through edges of E_h from v to w also satisfies the following three conditions:

(i) $u \neq w$;

(ii) $\langle u, w \rangle \notin E_h$;

(iii) $\langle u, w \rangle \notin E_v$.

Let $xU_h y$ and $xU_v y$ denote that there is an undirected path through edges of E_h (E_v, respectively) from x to y. H-V intransitivity is illustrated in figure 4.
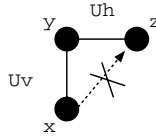

Fig. 4. H-V Intransitivity

Definition 2.2 lists the three conditions that we need for H-V intransitivity. However, it turns out that they can be simplified, as, under the hypotheses in definition 2.2, the first condition implies the other two. Suppose that all triples $\langle u, v, w \rangle$ that satisfy the hypotheses in definition 2.2 also satisfy the first condition ($u \neq w$). Now, suppose that there is one such triple $\langle a, b, c \rangle$ such that $\langle a, c \rangle \in E_h$ (does not satisfy the second condition). Then, $aU_v b$, $bU_h c$ and $aE_h c$. This implies that $bU_h a$. But then, $\langle a, b, a \rangle$ is a triple that satisfies the hypotheses in definition 2.2 but does not satisfy the first condition ($u \neq w$), which is a contradiction to our initial assumption. An analogous argument can be made for the third condition as well.

Thus, when we need to test whether a graph satisfies H-V intransitivity, we just need to verify the first condition in definition 2.2. On the other hand, when we know that a graph satisfies H-V intransitivity, we may use any one of the three conditions according to our needs.

Following the above simplification, H-V intransitivity can be described in the following way:

$$\forall x \forall y \forall z ((xU_v y \wedge yU_h z \wedge x \neq y \wedge y \neq z) \rightarrow (x \neq z))\,[4].$$

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[4] It is important to notice that H-V intransitivity cannot be expressed by a first order formula, since the

3 Graph Decomposition

The problem of graph decomposition consists of, given a graph, to determine whether this graph can be decomposed in a product of two non-trivial graphs. A graph is said to be *trivial* if it has only one vertex and no edges. Every graph can be described as a product of itself with a trivial graph. In this work, we consider a restricted version of this problem.

Problem 3.1 *Given a finite, directed and weakly connected[5] (called just* connected *from now on) graph $G = \langle V, E_h, E_v \rangle$, determine whether G is isomorphic to a product $G' = G_1 \times G_2$, where G_1 and G_2 are both non-trivial.*

In the general problem, the graph would not have to be necessarily finite or connected and the labelling of the edges into E_h and E_v would not necessarily be given *a priori*. In the case where the graph $G = \langle V, E \rangle$ is directed, finite and connected, but the labelling of the edges is not given, G is a product if and only if *there is* a suitable labelling of the edges into E_h and E_v such that $G' = \langle V, E_h, E_v \rangle$ is a product in the restricted case above.

Hypothesis 3.2 *From now on, all the graphs G are considered to be directed, finite and connected and to be given in the form $G = \langle V, E_h, E_v \rangle$.*

Remark 3.3 We denote by $\mathcal{V}(G)$ the set of vertices of a graph G.

In this section, we want to prove that a finite and connected graph G is a product if and only if it satisfies left and right commutativity, the Church-Rosser and reverse Church-Rosser properties and H-V intransitivity. We start with the simpler direction.

Theorem 3.4 *If G is a product, then G satisfies left and right commutativity, the Church-Rosser and reverse Church-Rosser properties and H-V intransitivity.*

Proof. We start with left commutativity. Let us take three vertices u, v and w of G such that uE_vv and vE_hw. As G is a product $G_1 \times G_2$, $u = (u_1, u_2)$, $v = (v_1, v_2)$ and $w = (w_1, w_2)$. Then, as uE_vv, $u_1 = v_1$ and $u_2E_2v_2$ and, as vE_hw, $v_2 = w_2$ and $v_1E_1w_1$. Now, take the vertex $x = (x_1, x_2) = (w_1, u_2)$ (this vertex exists, since $\mathcal{V}(G) = \mathcal{V}(G_1) \times \mathcal{V}(G_2)$). Then, as $u_1 = v_1$, $x_1 = w_1$ and $v_1E_1w_1$, then $u_1E_1x_1$. This, together with $u_2 = x_2$, implies that uE_hx. Now, as $u_2 = x_2$, $v_2 = w_2$ and $u_2E_2v_2$, then $x_2E_2w_2$. This, together with $x_1 = w_1$, implies that xE_vw. Right commutativity and the Church-Rosser and reverse Church-Rosser properties follow by analogous arguments.

Now, suppose that G does not satisfy H-V intransitivity. Then, we have vertices x, y and z, such that $x \neq y$, $y \neq z$, there is an undirected E_v-path from x to y and an undirected E_h-path from y to z and $x = z$. As G is a product, $x = (x_1, x_2)$, $y = (y_1, y_2)$ and $z = (z_1, z_2)$. Then, as there is an undirected E_v-path from x to

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definitions of U_h and U_v depend on transitive closures. Nevertheless, this property is still elementary, as it can be defined by a set of first order formulas.

[5] A graph G is weakly connected if, for any pair of vertices u and v of G, there is an undirected path from u to v in G.

y, then $x_1 = y_1$. Also, as there is an undirected E_h-path from y to z, $y_2 = z_2$. As $x = z$, then $z_1 = x_1 = y_1$ and $x_2 = z_2 = y_2$, which implies that the three vertices are the same, contradicting the fact that $x \neq y$ and $y \neq z$.   □

Notice that, in this direction of the proof, we make no use of hypothesis 3.2. This means that theorem 3.4 holds for any graph G. Now, we proceed to prove the other direction.

Definition 3.5 Let G be a graph $G = \langle V, E_h, E_v \rangle$ and let $G_h = \langle V, E_h \rangle$ and $G_v = \langle V, E_v \rangle$ be subgraphs of G. The *horizontal components* are the maximal connected subgraphs of G_h, $\{G_h^1, ..., G_h^m\}$. Analogously, the *vertical components* are the maximal connected subgraphs of G_v, $\{G_v^1, ..., G_v^n\}$. When it is not relevant whether the component or edge is horizontal or vertical, we use x to stand for either h or v and $\overline{x}$ to stand for the other direction.

Remark 3.6 From now on, every time that we need to consider a pair of x-components G_x^i and G_x^j, they *do not* need to be distinct, unless explicitly mentioned.

Definition 3.7 We say that a x-component G_x^i is a *neighbor to* the x-component G_x^j if there is an edge $\langle u, w \rangle \in E_{\overline{x}}$ such that $u \in G_x^i$ and $w \in G_x^j$. Notice that it is possible for a component to be a neighbor to itself.

Definition 3.8 Let f_x^{ij} be the (possibly partial and multi-valued) map that associates to each vertex $u \in G_x^i$ the set of vertices w such that $\langle u, w \rangle \in E_{\overline{x}}$ and $w \in G_x^j$. We say that f_x^{ij} is the *induced map* from G_x^i to G_x^j.

Proposition 3.9 *If a graph satisfies left and right commutativity and the Church-Rosser and reverse Church-Rosser properties, it also satisfies the following properties:*

(i) *Extended Left Commutativity:* $\forall x \forall y \forall z (xU_v^l y \wedge yU_h^k z \to \exists u(xU_h^k u \wedge uU_v^l z))$;

(ii) *Extended Right Commutativity:* $\forall x \forall y \forall z (xU_h^k y \wedge yU_v^l z \to \exists u(xU_v^l u \wedge uU_h^k z))$.

where $uU_h^k v$ and $uU_v^l v$ denote that there is an undirected path through edges of E_h (E_v, respectively) of length k (l) from u to v.

Proof. These properties follow by an straightforward induction on the length of the paths, using one of the four hypotheses for each of the four possible cases of edge incidences in the "corner" vertices: horizontal and vertical inward (reverse Church-Rosser), horizontal inward and vertical outward (right commutativity), horizontal outward and vertical inward (left commutativity) and horizontal and vertical outward (Church-Rosser).   □

Proposition 3.10 *Let G be a graph that satisfies left and right commutativity, the Church-Rosser and reverse Church-Rosser properties and H-V intransitivity and G_x^i be a neighbor to G_x^j. Then, the induced map f_x^{ij} from G_x^i to G_x^j is an isomorphism.*

Proof. We show the proof for the case of f_v^{ij}.

(i) f_v^{ij} *is a function*: Suppose that there are vertices u, v and w, such that $v \neq w$, $u \in G_v^i$, $v, w \in G_v^j$ and $f_v^{ij}(u) = \{v, w\}$ ($\langle u, v \rangle, \langle u, w \rangle \in E_h$). If $u \neq v$, then

we have an undirected E_h-path from u to v, an undirected E_v-path from v to w (since they are in the same vertical component) and an edge from u to w, contradicting H-V intransitivity. If $u = v$, then we have an undirected E_h path from v to w and an undirected E_v-path from w to v, also contradicting H-V intransitivity.

(ii) f_v^{ij} *is injective*: Analogous to the previous item.

(iii) $\text{Im}(f_v^{ij}) = \mathcal{V}(G_v^j)$ (f_v^{ij} is surjective): Let v in G_v^j. We need to find a vertex u in G_v^i such that $\langle u, v \rangle \in E_h$. As G_v^i is a neighbor to G_v^j, there are vertices x in G_v^i and y in G_v^j such that $\langle x, y \rangle \in E_h$. We may assume that $y \neq v$, otherwise the proof is over. Now, v and y are in G_v^j, so yU_vv. Then, by extended right commutativity, there is u such that xU_vu, which means that $u \in G_v^i$, and uE_hv.

(iv) $\text{Dom}(f_v^{ij}) = \mathcal{V}(G_v^i)$ (f_v^{ij} is total): Analogous to the previous item, using extended left commutativity instead.

(v) uE_vw if and only if $f_v^{ij}(u)E_vf_v^{ij}(w)$: First of all, $uE_hf_v^{ij}(u)$ and $wE_hf_v^{ij}(w)$. If uE_vw, we can use left commutativity to conclude that $f_v^{ij}(u)E_vf_v^{ij}(w)$. On the other hand, if $f_v^{ij}(u)E_vf_v^{ij}(w)$, we can use right commutativity to conclude that uE_vw.

The case of f_h^{ij} is entirely analogous.   □

Definition 3.11 If G_x^i is a neighbor to G_x^j and the induced map f_x^{ij} is an isomorphism between G_x^i and G_x^j, we call f_x^{ij} a *primitive isomorphism*.

Now, in a case such as the one in the above proposition, where the induced maps are isomorphisms, we can easily extend them beyond neighbor components.

Remark 3.12 If f_x^{ij} is an isomorphism, then its inverse is also an isomorphism and is denoted by f_x^{ji}.

Remark 3.13 If all the elements in the set $\{f_x^{i,i+1}, f_x^{i+1,i+2}, \ldots, f_x^{j-1,j}\}$ are isomorphisms, then

$$f_x^{ij} = f_x^{j-1,j} \circ \cdots f_x^{i+1,i+2} \circ f_x^{i,i+1}$$

is also an isomorphism.

Definition 3.14 If f_x^{ij} is a primitive isomorphism or is obtained from primitive isomorphisms using composition and inverse, we call f_x^{ij} an *orthogonal isomorphism* or *O-isomorphism*. By this definition, if f_x^{ij} is an O-isomorphism and $w = f_x^{ij}(u)$, then there is an undirected $E_{\overline{x}}$-path from u to w.

Lemma 3.15 *Let G be a graph that satisfies left and right Commutativity, the Church-Rosser and reverse Church-Rosser properties and H-V intransitivity. Then, for all pairs G_x^i and G_x^j of x-components, there is an O-isomorphism f_x^{ij} between them.*

Proof. The proof follows from proposition 3.10, remarks 3.12 and 3.13 and the fact that G is finite and connected.   □

Proposition 3.16 *Let G be a graph with k horizontal components and l vertical components. If G satisfies H-V intransitivity and, for every pair G_x^i and G_x^j of x-components such that G_x^i is a neighbor to G_x^j, the induced map f_x^{ij} is an isomorphism between them, then*

 (i) *All horizontal components are isomorphic between themselves and contain l vertices each and*

 (ii) *All vertical components are isomorphic between themselves and contain k vertices each.*

Proof. First, as G is connected, for every x-component G_x^i there must be a component G_x^j such that either G_x^i is a neighbor to G_x^j or G_x^j is a neighbor to G_x^i. Then, since the induced maps are isomorphisms, it is straightforward to conclude that all x-components are isomorphic between themselves.

Now, to show that each horizontal component contains l vertices, where l is the number of vertical components, and each vertical component contains k vertices, where k is the number of horizontal components, we proceed by a double induction on k and l. Start with $k = l = 1$. Every graph in this case must have only one vertex. Since we have only one component of each kind, if we had two distinct vertices u and w in the graph, we would have to have both an undirected E_h-path and an undirected E_v-path between u and w, contradicting H-V intransitivity. The only edges that this graph may have are horizontal and vertical loops. It is important to notice that the addition or deletion of loops does not change the number of components. In this basic case, the proposition is true.

Now, suppose that the proposition is true for $k = 1$ and $l < n$. Consider a graph with 1 horizontal component and n vertical components that satisfies the hypotheses. If we take the subgraph generated by the vertices in the $n - 1$ first vertical components, the proposition holds in it. Then, the horizontal component contains $n - 1$ vertices and the vertical components contain 1 vertex each. Now, consider the n-th vertical component G_v^n. As G is connected, there must be a vertical component G_v^i such that either G_v^i is a neighbor to G_v^n or G_v^n is a neighbor to G_v^i. Besides that, we have an isomorphism between these components. This means that this vertical component also contains 1 vertex and this vertex is contained in the horizontal component. Thus, the proposition is true.

Finally, suppose that the proposition is true for $k < m$ and $l = n$. Consider a graph with m horizontal components and n vertical components that satisfies the hypotheses. If we take the subgraph generated by the vertices in the $m - 1$ first horizontal components, the proposition holds in it. Then, the horizontal components contain n vertices each and the vertical components contain $m - 1$ vertices each. Now, consider the m-th horizontal component G_h^m. As G is connected, there must be an horizontal component G_h^i such that either G_h^i is a neighbor to G_h^m or G_h^m is a neighbor to G_h^i. Besides that, we have an isomorphism between these components. This means that this horizontal component also contains n vertices and each one of these vertices is contained in a distinct vertical component. Thus, the proposition is true. □

Lemma 3.17 *Let G be a graph. If G satisfies H-V intransitivity and, for all pairs G_x^i and G_x^j of x-components, there is an O-isomorphism f_x^{ij} between them, then G is (isomorphic to) a product.*

Proof. Using proposition 3.16, let $G_h^* = \langle V_h, E_h \rangle$ be an arbitrary horizontal component (among a total of k) and $G_v^* = \langle V_v, E_v \rangle$ be an arbitrary vertical component (among a total of l), with $V_h = \{h_1, \ldots, h_l\}$ and $V_v = \{v_1, \ldots, v_k\}$. Let $P = G_h^* \times G_v^* = \langle V_h \times V_v, E_h^P, E_v^P \rangle$. We want to prove that there is an isomorphism between G and P.

Let us consider the function $L(u) = (h_j, v_i)$, if $u \in G_h^i$ and $u \in G_v^j$.

(i) L is injective: Suppose that there are two distinct vertices u and w in G such that $L(u) = L(w) = (h_j, v_i)$. Then, they are in the same horizontal component G_h^i and in the same vertical component G_v^j. This means that we have both an undirected E_h-path and an undirected E_v-path between u and w, contradicting H-V intransitivity.

(ii) L is surjective: Suppose that there is no u in G such that $L(u) = (h_j, v_i)$. Let $x \in G_h^i$ and $y \in G_v^j$. Now, y must belong to a horizontal component, let us say G_h^k. Because of the existence of an O-isomorphism between G_h^k and G_h^i, there is $w \in G_h^i$ such that there is an undirected E_v-path from y to w. This implies that w is in the same vertical component of y. Then, $w \in G_h^i$ and $w \in G_v^j$, which is a contradiction.

(iii) $uE_h w$ if and only if $L(u)E_h^P L(w)$: Suppose that $u \in G_h^i$ and $u \in G_v^j$. Then, $L(u) = (u_1, u_2) = (h_j, v_i)$. Now, w must also be in G_h^i. Then, $L(w) = (w_1, w_2) = (h_l, v_i)$ for some l. Notice that the vertices u and w of G_h^i are, respectively, on the same vertical components in G as the vertices h_j and h_l of G_h^*. Using the O-isomorphism between G_h^i and G_h^*, as $uE_h w$ in G_h^i, then $u_1 E_h w_1$ in G_h^*. This, together with $u_2 = w_2$ implies that $L(u)E_h^P L(w)$. Now, suppose that $L(u) = (u_1, u_2)$, $L(w) = (w_1, w_2)$ and $(u_1, u_2)E_h^P(w_1, w_2)$. Then, $u_2 = w_2 = v_i$, which means that $u, w \in G_h^i$. Besides that, $u_1 E_h w_1$ in G_h^*. Thus, using the O-isomorphism between G_h^i and G_h^*, we get that $uE_h w$.

(iv) $uE_v w$ if and only if $L(u)E_v^P L(w)$: Analogous to the previous item. $\square$

Theorem 3.18 *If G satisfies left and right commutativity, the Church-Rosser and reverse Church-Rosser properties and H-V intransitivity, then G is a product.*

Proof. Straightforward from lemmas 3.15 and 3.17. $\square$

Theorem 3.19 *Let G be a graph. G is a product if and only if G satisfies left and right commutativity, the Church-Rosser and reverse Church-Rosser properties and H-V intransitivity.*

Proof. Straightforward from theorems 3.4 and 3.18. $\square$

4   Modal Definability

In this section, we show that the property of H-V intransitivity is not modally definable. In fact, we also show that no condition that is necessary and sufficient for a graph to be a product can be modally definable. Even tough we restricted ourselves to finite and connected graphs in the previous section, this restriction is not necessary for the undefinability results presented in this section.

4.1   A Basic Modal Language

In this section, we define a modal language with a family of modal operators: $\Diamond_h$, $\Diamond_v$, $\Diamond_h^{-1}$, $\Diamond_v^{-1}$, $\blacklozenge_h$ and $\blacklozenge_v$.

Definition 4.1 Let us consider a modal language consisting of a set Φ of countably many proposition symbols (the elements of Φ are denoted by $p, q, \ldots$), the boolean connectives $\neg$ and $\wedge$ and the modal operators $\Diamond_h$, $\Diamond_v$, $\Diamond_h^{-1}$, $\Diamond_v^{-1}$, $\blacklozenge_h$ and $\blacklozenge_v$. The formulas are defined as follows:

$$\varphi ::= p \mid \top \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \Diamond_h\varphi \mid \Diamond_v\varphi \mid \Diamond_h^{-1}\varphi \mid \Diamond_v^{-1}\varphi \mid \blacklozenge_h\varphi \mid \blacklozenge_v\varphi.$$

We freely use the standard boolean abbreviations $\vee$, $\rightarrow$, $\leftrightarrow$ and $\perp$ and also the abbreviation $\widetilde{\Box}_x\varphi = \neg\widetilde{\Diamond}_x\neg\varphi$, where $\widetilde{\Diamond} \in \{\Diamond, \Diamond^{-1}, \blacklozenge\}$, $\widetilde{\Box}$ is the correspondent $\Box$ with appropriate labels and color and $x \in \{h, v\}$.

We now define the structures in which we evaluate our formulas: *frames* and *models*.

Definition 4.2 A *frame* is a tuple $\mathcal{F} = (V, R_h, R_v, U_h, U_v)$, where V is a set (finite or not) of vertices, R_h and R_v are binary relations over V, i.e., $R_x \subseteq V \times V$, where $x \in \{h, v\}$, and U_h and U_v are the transitive closures of the relations $R_h \cup R_h^{-1}$ and $R_v \cup R_v^{-1}$, respectively.

As we can see, a frame is essentially a graph with the two distinct sets of edges E_h and E_v.

Definition 4.3 A *model* is a pair $\mathcal{M} = (\mathcal{F}, \mathbf{V})$, where $\mathcal{F}$ is a frame and $\mathbf{V}$ is a valuation function mapping proposition symbols into subsets of V, i.e., $\mathbf{V} : \Phi \mapsto \mathcal{P}(V)$.

The notion of satisfaction is defined as follows:

Definition 4.4 Let $\mathcal{M} = (\mathcal{F}, \mathbf{V})$ be a model. The notion of *satisfaction* of a formula φ in a model $\mathcal{M}$ at a vertex v, notation $\mathcal{M}, v \Vdash \varphi$, can be inductively defined as follows:

(i) $\mathcal{M}, v \Vdash p$ iff $v \in \mathbf{V}(p)$;

(ii) $\mathcal{M}, v \Vdash \top$ always;

(iii) $\mathcal{M}, v \Vdash \neg\varphi$ iff $\mathcal{M}, v \nVdash \varphi$;

(iv) $\mathcal{M}, v \Vdash \varphi_1 \wedge \varphi_2$ iff $\mathcal{M}, v \Vdash \varphi_1$ and $\mathcal{M}, v \Vdash \varphi_2$;

(v) $\mathcal{M}, v \Vdash \Diamond_x \varphi$ iff there is a $w \in V$ such that vR_xw and $\mathcal{M}, w \Vdash \varphi$;

(vi) $\mathcal{M}, v \Vdash \Diamond_x^{-1} \varphi$ iff there is a $w \in V$ such that wR_xv and $\mathcal{M}, w \Vdash \varphi$;

(vii) $\mathcal{M}, v \Vdash \blacklozenge_x \varphi$ iff there is a $w \in V$ such that vU_xw and $\mathcal{M}, w \Vdash \varphi$.

Here, $x \in \{h, v\}$.

If $\mathcal{M}, v \Vdash \varphi$ for every vertex v in a model $\mathcal{M}$, we say that φ is *globally satisfied* in $\mathcal{M}$, notation $\mathcal{M} \Vdash \varphi$. And if φ is globally satisfied in all models $\mathcal{M}$ of a frame $\mathcal{F}$, we say that φ is *valid* in $\mathcal{F}$, notation $\mathcal{F} \Vdash \varphi$.

When we say that a formula ϕ defines or describes some graph property, this means that a graph G has the desired property if and only if $\mathcal{F} \Vdash \phi$, where $\mathcal{F}$ is the frame that represents G.

As shown in [7], left commutativity, right commutativity and the Church-Rosser property can be defined in the present logic by the following formulas:

(i) $\phi_1 = \Diamond_v \Diamond_h \varphi \leftrightarrow \Diamond_h \Diamond_v \varphi$ (left and right commutativity);

(ii) $\phi_2 = \Diamond_h \Box_v \varphi \rightarrow \Box_v \Diamond_h \varphi$ (Church-Rosser property).

The reverse Church-Rosser property can then be defined by the following formula, analogous to ϕ_2:

(iii) $\phi_3 = \Diamond_h^{-1} \Box_v^{-1} \varphi \rightarrow \Box_v^{-1} \Diamond_h^{-1} \varphi$.

4.2 A Limitative Result

The limits to the expressive power of basic modal languages are fairly well known. There are a series of standard results that state that frames that are "similar" in a number of ways must agree on the validity of formulas. We can then use these results to prove that a certain property *cannot* be expressed by any modal formula. To do this, we take two frames that are "similar" and show that in one the desired property holds, while in the other it does not. We present one of these "similarity" results (more details about it and other related results may be found in [3]), and then we prove two results for graph products using it.

Definition 4.5 Let $\mathcal{M} = (W, R_h, R_v, U_h, U_v, \mathbf{V})$ and $\mathcal{M}' = (W', R_h', R_v', U_h', U_v', \mathbf{V}')$ be two models. A function $f : W \rightarrow W'$ is a *bounded morphism* from $\mathcal{M}$ to $\mathcal{M}'$ if it satisfies the following conditions:

(i) w and $f(w)$ satisfy the same proposition symbols;

(ii) f is a homomorphism with respect to R_x, $x \in \{h, v\}$ (if wR_xv, then $f(w)R_x' f(v)$);

(iii) if $f(w)R_x'v'$, $x \in \{h, v\}$, then there is a v such that wR_xv and $f(v) = v'$;

(iv) if $w'R_x'f(v)$, $x \in \{h, v\}$, then there is a w such that wR_xv and $f(w) = w'$.

A similar definition can be given for a bounded morphism of frames, just removing the part of the above definition that deals with valuations (item (i)). If there is a bounded morphism from a model (frame) $\mathcal{M}$ ($\mathcal{F}$) to a model (frame) $\mathcal{M}'$ ($\mathcal{F}'$), we use the notation $\mathcal{M} \rightarrow \mathcal{M}'$ ($\mathcal{F} \rightarrow \mathcal{F}'$). If there is a surjective bounded morphism,

then we say that $\mathcal{M}'$ ($\mathcal{F}'$) is a *bounded morphic image* of $\mathcal{M}$ ($\mathcal{F}$) and use the notation $\mathcal{M} \Rightarrow \mathcal{M}'$ ($\mathcal{F} \Rightarrow \mathcal{F}'$).

The last item of the previous definition is usually not necessary. However, as the modalities $\Diamond_x^{-1}$ and $\blacklozenge_x$ deal with the inverses of the relations R_x, we have to enforce it to get the preservation result that we want. It may seem like conditions such as "if wU_xv, then $f(w)U_x' f(v)$", which is analogous to condition (ii), and others analogous to conditions (iii) and (iv) should also be added. However, this is not necessary, as the definition of U_x, with its use of transitive closure, and conditions (ii), (iii) and (iv) already imply such conditions.

Below is a basic theorem about modal definability that is going to be used to prove our results. Its proof for a language that contains only one modality can be found at [3]. It is not difficult to extend that proof to a language that contains a family of modalities, each with its accessibility relation.

Theorem 4.6 *Let $\mathcal{M}$ and $\mathcal{M}'$ be two models such that $\mathcal{M} \to \mathcal{M}'$. Then, $\mathcal{M}, w \Vdash \phi$ if and only if $\mathcal{M}', f(w) \Vdash \phi$.*

Corollary 4.7 *Let $\mathcal{F}$ and $\mathcal{F}'$ be two frames such that $\mathcal{F} \Rightarrow \mathcal{F}'$. If $\mathcal{F} \Vdash \phi$, then $\mathcal{F}' \Vdash \phi$.*

Theorem 4.8 *Neither H-V intransitivity nor its negation are modally definable.*

Proof. In figure 5, let $f = \{(1,a),(2,b),(3,a),(4,b)\}$ and $g = \{(a,A),(b,A)\}$. It is straightforward to prove that f and g are surjective bounded morphisms. It is also not difficult to see that the first and third graphs respect H-V intransitivity, while the second does not. By corollary 4.7, since neither H-V intransitivity nor its negation are preserved under bounded morphic images, they are not modally definable. □
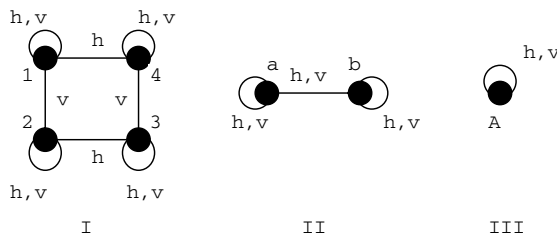

Fig. 5. Graph III is a bounded morphic image of graph II, which is a bounded morphic image of graph I (each undirected edge represents a pair of symmetric edges)

Theorem 4.9 *No necessary and sufficient condition for a graph to be a product or for a graph not to be a product can be modally definable.*

Proof. We take again the same bounded morphisms between the graphs in figure 5. It is not difficult to see that the first and third graphs are products while the second is not. By corollary 4.7, since neither the property of being a product nor the property of not being a product are preserved under bounded morphic images, they are not modally definable. □

This is not the only possible proof of theorem 4.9. Readers familiar with products of logics may view the above theorem as a natural consequence of the completeness proof of the axiomatization for the logic $K \times K$. However, as products of logics in general and their axiomatizations in particular are not thoroughly presented in this work, as the proof presented above for theorem 4.8 could not be substituted in a similar way and as the counter-example used in theorem 4.8 could also be used in theorem 4.9 without any change, it was our choice to prove both theorems through the use of bounded morphic images.

5 A Hybrid Extension

As was shown in the previous section, a basic modal language does not have enough expressive power to describe the properties that we want. In order to achieve our goal, we need a language that is more expressive. In this section we describe a simple hybrid language and then use it to define H-V intransitivity.

5.1 Language

A good way to improve the expressive power of a modal logic is to consider hybrid extensions of it. The fundamental resource that allows a logic to be called "hybrid" is a set of *nominals*. Nominals are a new kind of atomic symbol and they behave similarly to proposition symbols. The key difference between a nominal and a proposition symbol is related to their valuation in a model. While the set $\mathbf{V}(p)$ for a proposition symbol p can be any element of $\mathcal{P}(V)$, the set $\mathbf{V}(i)$ for a nominal i has to be a singleton set. This way, each nominal is true at exactly one state of the model, and thus, can be used to refer to this unique state. This is why these logics are called "hybrid": they are still modal logics, but they have the capacity to refer to specific states of the model, like in first-order logic.

The expressive power and computational complexity of a hybrid extension of a given modal logic usually lie between the ones of the original modal logic and the ones of first-order logic. This, however, depends on which operators, besides the nominals, are added to build the hybrid logic. With the addition of state-variables and quantifiers, it is possible to achieve full first-order expressivity and complexity (undecidability). For a general introduction to hybrid logics, [1] and [2] can be consulted.

Here, we consider the smallest hybrid extension of the modal logic presented in the previous section. We add only nominals and nothing else to the language.

Definition 5.1 Let us consider a hybrid language consisting of a set Φ of countably many proposition symbols (the elements of Φ are denoted by $p, q, \ldots$), a set Ω of countably many nominals (the elements of Ω are denoted by $i, j, \ldots$), such that $\Phi \cap \Omega = \emptyset$, the boolean connectives $\neg$ and $\wedge$ and the modal operators $\Diamond_h$, $\Diamond_v$, $\Diamond_h^{-1}$, $\Diamond_v^{-1}$, $\blacklozenge_h$ and $\blacklozenge_v$. The formulas are defined as follows:

$$\varphi ::= p \mid i \mid \top \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \Diamond_h\varphi \mid \Diamond_v\varphi \mid \Diamond_h^{-1}\varphi \mid \Diamond_v^{-1}\varphi \mid \blacklozenge_h\varphi \mid \blacklozenge_v\varphi.$$

The definition of a frame for this language is the same as definition 4.2, but the definition of a model is slightly different from definition 4.3.

Definition 5.2 A *hybrid model* is a pair $\mathcal{M} = (\mathcal{F}, \mathbf{V})$, where $\mathcal{F}$ is a frame and $\mathbf{V}$ is a valuation function mapping proposition symbols into subsets of V, i.e., $\mathbf{V} : \Phi \mapsto \mathcal{P}(V)$ and mapping nominals into singleton subsets of V, i.e, if i is a nominal then $\mathbf{V}(i) = \{v\}$ for some $v \in V$. We call this unique state that belongs to $\mathbf{V}(i)$ the *denotation* of i under $\mathbf{V}$. We can also say that i denotes the single state belonging to $\mathbf{V}(i)$.

The notion of satisfaction is defined as follows:

Definition 5.3 The notion of satisfaction is defined adding the following extra clause to definition 4.4: $\mathcal{M}, v \Vdash i$ iff $v \in \mathbf{V}(i)$.

5.2   Hybrid Definability

Using this hybrid language, we can now express H-V intransitivity.

Theorem 5.4 *A graph G respects H-V intransitivity if and only if $\mathcal{F} \Vdash \phi$, where $\mathcal{F}$ is the frame that represents G and ϕ is the formula*

$$\phi = (i \wedge \neg j \wedge \blacklozenge_v(j \wedge \neg k \wedge \blacklozenge_h k)) \rightarrow \neg k.$$

Proof. ($\Leftarrow$) Suppose that $\mathcal{F} \Vdash \phi$ but G does not respect H-V intransitivity. Then, there are at least three vertices x, y, z, $x \neq y$ and $y \neq z$, in G such that $xU_v y$, $yU_h z$ and $x = z$. We evaluate ϕ in a model with a valuation $\mathbf{V}$ such that $\mathbf{V}(i) = \{x\}$, $\mathbf{V}(j) = \{y\}$ and $\mathbf{V}(k) = \{z\}$. Then, it is straightforward to see that $(\mathcal{F}, \mathbf{V}), x \not\Vdash \phi$, which contradicts the fact that ϕ is valid in $\mathcal{F}$.

($\Rightarrow$) Suppose that G respects H-V intransitivity but $\mathcal{F} \not\Vdash \phi$. Then, there is a valuation $\mathbf{V}$ and a vertex u such that $(\mathcal{F}, \mathbf{V}), u \not\Vdash \phi$. Let $\mathbf{V}(i) = \{x\}$, $\mathbf{V}(j) = \{y\}$ and $\mathbf{V}(k) = \{z\}$. Then, we must have that $u = x$, $x \neq y$, $y \neq z$, $xU_v y$, $yU_h z$ and $(\mathcal{F}, \mathbf{V}), u \Vdash k$, which means that $u = x = z$. This contradicts the fact that G respects H-V intransitivity. $\qquad\square$

6   Conclusion

It is known that left and right commutativity and the Church-Rosser and reverse Church-Rosser properties are necessary conditions for a graph (frame) to be a non-trivial (cartesian) product of two other graphs, but their conjunction is not a sufficient condition. We introduce a new property called H-V intransitivity, that, together with the former ones, form a necessary and sufficient condition for a finite and connected graph to be a product. The proof of the necessity of these properties is fairly simple and is done directly, without the need to assume that the graph is finite or connected. On the other hand, the proof of the sufficiency is done in two steps. First, we prove that if a finite and connected graph satisfies the five properties stated above, then its horizontal and vertical components must satisfy a

particular isomorphism. Then, we show that if a finite and connected graph satisfies H-V intransitivity and its components satisfy this particular isomorphism, then the graph is a product.

The limits to the expressive power of basic modal languages are well known. We show that the property of H-V intransitivity is not modally definable. We also show that no condition that is necessary and sufficient for a graph to be a product can be modally definable. Extending our language to a hybrid language, we are then able to express H-V intransitivity.

A natural continuation of the present work would be to carefully study the proofs of the theorems in section 3 and analyze whether it is possible to weaken or even to drop the hypotheses of connectivity and finiteness that we currently enforce.

As another future work, we would like to investigate hybrid axiomatizations of products of logics. There are sound and complete axiomatizations for products of logics in the literature, but their completeness proofs are very specific and usually cannot be easily generalized to products of other logics. As an example, the completeness proof for $S5 \times S5$ (the set of formulas that are valid in the class of frames that are products of two reflexive, transitive and symmetric frames) is not a direct and straightforward generalization of the proof for $K \times K$ (the set of formulas that are valid in the class of frames that are products of two frames). This is very different from the standard case, where the completeness proof for K (the set of formulas that are valid in the class of all frames) easily generalizes to $S5$ (the set of formulas that are valid in the class of reflexive, transitive and symmetric frames) using the fact that the extra axioms of $S5$ force the frames to be reflexive, transitive and symmetric. We hope that, by including the H-V intransitivity in the axiomatic systems, we may be able to obtain axiomatizations with more well-behaved completeness proofs, based on the standard technique of canonical models ([3]), so that they may be easier to generalize.

It is known that in hybrid logics with satisfaction operators ($@_i$, for each nominal i), a completeness proof for an axiomatic system $\mathcal{A}$ is easily generalized to a completeness proof for any axiomatic system $\mathcal{A} + L$, where L is a set of *pure* axioms, that is, axioms that contain only nominals and no proposition symbols (for details on satisfaction operators and the technique for hybrid completeness proofs, [4] and [3] can be consulted). Since many graph properties, such as reflexivity, irreflexivity, transitivity, intransitivity, symmetry, asymmetry and antisymmetry ([3] presents other examples), can be expressed by pure formulas, if we can extend this result on hybrid completeness proofs to products of logics, then we would be able to automatically prove completeness for a large number of products of logics through a single completeness proof. Some preliminary work on hybrid axiomatization of products of logics is presented in the last chapter of [8].

Acknowledgement

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