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An Algorithm for Computing Fundamental Solutions

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Abstract

For a partial differential operator $P = \sum_{|\alpha|\leq m} c_\alpha D^\alpha$ with constant coefficients c_α, a generalized function u is a fundamental solution, if $Pu = \delta$, where δ is the Dirac distribution. In this article we provide an algorithm which computes a fundamental solution for every such differential operator P on a Turing machine, if the input- and output-data are represented canonically.

Keywords: computable analysis, partial differential equations, fundamental solution

1 Introduction

In the theory of differential operators with constant coefficients, fundamental solutions have a central place. Let

$$P = P(D) = \sum_{|\alpha|\leq m} c_\alpha D^\alpha \tag{1}$$

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be a partial differential operator with constant coefficients c_α, where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index of order $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $D_j = (1/i)\partial/\partial x_j$ with $1 \leq j \leq n$ and $i = \sqrt{-1}$, $D = (D_1, \ldots, D_n)$ and $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n}$. A distribution is called a fundamental solution of the partial differential operator P if it is a solution of the point source problem

$$P(D)u = \sum_{|\alpha| \leq m} c_\alpha D^\alpha u = \delta \tag{2}$$

where δ is the Dirac measure at 0. In 1954 Malgrange and Ehrenpreis proved that every partial differential operator with constant coefficients has a fundamental solution. Fundamental solutions are very useful tools in the theory of partial differential equations, for instance, in solving inhomogeneous equations and in providing information about the regularity and growth of solutions. In the case of solving inhomogeneous equations, if E is a fundamental solution of the partial differential operator $P = P(D)$ and if f is a distribution, then, $E * f$ is a solution of the equation $P(D)u = f$ whenever the convolution is defined.

Many classical differential operators are known to have computable functions as fundamental solutions. For example, the Schwartz function

$$E_H(t, x) = (4\pi\nu t)^{-n/2} e^{-|x|^2/4\nu t}$$

which is a computable real function, is a fundamental solution of the heat equation

$$\frac{\partial u}{\partial t} = \nu \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}, \ (t, x) \in \mathbb{R} \times \mathbb{R}^n.$$

In general, however, a fundamental solution is a distribution. Has every partial differential operator $P(D)$ with computable coefficients a computable fundamental solution? In fact, this follows from the even more general result we prove in this note: There is a computable operator mapping every differential operator $P(D)$ (as in (1)) to a fundamental solution. We present an algorithm which in a well-defined realistic model of computation computes a fundamental solution of any given differential operator P from its coefficients, where abstract data are encoded canonically by, generally infinite, sequences of symbols and computations are (can be) performed by Turing machines.

2 Preliminaries

In this note, we consider the representation approach as a model of computation for analysis [5]. Computable functions on Σ^* and Σ^ω (the set of finite and infinite sequences of symbols, respectively, from the finite alphabet Σ) are defined by Turing machines which can read and write finite and infinite sequences. A multifunction $f :\subseteq X \rightrightarrows Y$ is a function which assigns to every $x \in X$ a set $f(x) \subseteq Y$, the set of "acceptable" values ($f(x) = \emptyset$, if $x \notin \mathrm{dom}(f)$).

Computability on other sets is defined by using Σ^* and Σ^ω as codes or names. A notation (representation) is a multifunction $\nu :\subseteq \Sigma^* \rightrightarrows M$ ($\delta :\subseteq \Sigma^\omega \rightrightarrows M$). For the natural numbers and the rational numbers we use canonical notations $\nu_{\mathbb{N}} :\subseteq \Sigma^* \to \mathbb{N}$ and $\nu_{\mathbb{Q}} :\subseteq \Sigma^* \to \mathbb{Q}$, respectively. For the real numbers we use the representation $\rho :\subseteq \Sigma^\omega \to \mathbb{R}$, where $\rho(p) = x$ if p encodes a sequence $(a_i)_i$ of rational numbers such that $|x - a_i| \leq 2^{-i}$. For a naming system $\gamma :\subseteq Y \to M$ ($Y \in \{\Sigma^*, \Sigma^\omega\}$) the representation $\gamma^k :\subseteq Y \to M^k$ is defined by $\gamma\langle y_1, \ldots, y_k\rangle := (\gamma(y_1), \ldots \gamma(y_k))$ where $\langle\ \rangle$ is a tupling function. For the complex numbers we use the representation ρ^2 of $\mathbb{R}^2$.

For naming systems $\gamma_i :\subseteq Y_i \rightrightarrows M_i$ a function $h :\subseteq Y_1 \to Y_2$ (on sequences of symbols) realizes a multifunction $f :\subseteq M_1 \rightrightarrows M_2$, if

$$\gamma_2 \circ h(y) \cap f(x) \neq \emptyset \quad \text{if} \ \ \gamma_1(y) \in \mathrm{dom}(f)\,,$$

that is, $h(y)$ is a name of some $z \in f(x)$ if y is a name of $x \in \mathrm{dom}(f)$. The multifunction f is (γ_1, γ_2)-computable, if it has a computable realization, and (γ_1, γ_2)-continuous, if it has a continuous realization.

Multifunctions occur naturally in Computable Analysis. As an example, there is an algorithm which maps every ρ-name p of $x \in \mathbb{R}$ (i.e., every sequence of rational numbers rapidly converging to x) to some $n \in \mathbb{N}$ such that $x < n$. This algorithm, however, might give another upper bound n' of x, if fed with another ρ-name p' of x. The algorithm is not "$(\rho, \nu_{\mathbb{N}})$-extensional". Nevertheless, the algorithm realizes the multifuntion $f : \mathbb{R} \rightrightarrows \mathbb{N}$, defined by $n \in f(x) \iff x < n$. There is no $(\rho, \nu_{\mathbb{N}})$-computable function $g : \mathbb{R} \to \mathbb{N}$ such that $x < f(x)$.

As generalizations of the "acceptable Gödel numbering $\varphi : \mathbb{N} \to P^{(1)}$" of the partial recursive functions, for any $a, b \in \{*, \omega\}$ there is a canonical representation $\eta^{ab} : \Sigma^\omega \to F^{ab}$ of continuous functions $f :\subseteq \Sigma^a \to \Sigma^b$ satisfying the "utm-theorem" and the "smn-theorem" [5]. For naming systems $\gamma_1 :\subseteq \Sigma^a \to M_1$ and $\gamma_2 :\subseteq \Sigma^b \to M_2$ a *representation* $[\gamma_1 \to \gamma_2]$ of the *total* (γ_1, γ_2)-continuous functions is defined by

$$[\gamma_1 \to \gamma_2](p) = f :\iff \eta^{ab}_p \ \text{ realizes } \ f \quad \text{w.r.t. } \ (\gamma_1, \gamma_2)\,,$$

and a *multirepresentation* $[\gamma_1 \to_p \gamma_2]$ of *all partial* (γ_1, γ_2)-continuous functions is defined by

$$f \in [\gamma_1 \to_p \gamma_2](p) :\iff \eta^{ab}_p \ \text{ realizes } \ f \quad \text{w.r.t. } \ (\gamma_1, \gamma_2)\,.$$

Notice that $[\gamma_1 \to \gamma_2]$ is the "weakest" representation δ of the set of total (γ_1, γ_2)-continuous functions such that the evaluation $(f, x) \mapsto f(x)$ becomes $(\delta, \gamma_1, \gamma_2)$-computable (Lemma 3.3.14 in [5]).

For definitions and mathematical properties of distributions see [1,2]. In [6] computability on distributions over the real line are studied. The definitions and theorems can be generalized straightforwardly to distributions over $\mathbb{R}^n$ ($n \geq 1$). For the space of continuous functions $f : \mathbb{R}^n \to \mathbb{C}$ we use the representation $[\rho^n \to \rho^2] :\subseteq \Sigma^\omega \to C(\mathbb{R}^n)$. For the space $C^\infty(\mathbb{R}^n)$ of infinitely differentiable functions $f : \mathbb{R}^n \to \mathbb{C}$ we use the representation $\delta_\infty :\subseteq \Sigma^\omega \to C^\infty(\mathbb{R}^n)$ defined by

$$\delta_\infty(\langle p_\alpha \rangle_{\alpha \in \mathbb{N}^n}) = f :\iff (\forall \alpha \in \mathbb{N}^n)\ D^\alpha f = [\rho^n \to \rho^2](p_\alpha)$$

(where $\langle p_\alpha \rangle_{\alpha \in \mathbb{N}^n} \in \Sigma^\omega$ is a canonical merging of the infinitely many infinite sequences $p_\alpha \in \Sigma^\omega$, $\alpha \in \mathbb{N}^n$). The topology of $C^\infty(\mathbb{R}^n)$ can be defined by the semi-norms $f \to |D^m f|_k = \sup_{|\alpha| \leq m}(\sup_{x \in K} |D^\alpha f(x)|)$, as m varies over the set of non-negative integers and K varies over the family of compact subsets of $\mathbb{R}^n$.

A test function is a function $f \in C^\infty(\mathbb{R}^n)$ with compact support $\mathrm{supp}(f) := \mathrm{cls}\{x \in \mathbb{R}^n \mid f(x) \neq 0\}$. The set of test functions is denoted by $\mathcal{D}(\mathbb{R}^n)$ and its topology is induced by $C^\infty(\mathbb{R}^n)$. We use the representation $\delta_D :\subseteq \Sigma^\omega \to \mathcal{D}(\mathbb{R}^n)$ defined by

$$\delta_D(0^k 1 p) = f :\iff \delta_\infty(p) = f \ \text{ and } \ \mathrm{supp}(f) \subseteq [-k; k]^n .$$

A Schwartz function is a function $f \in C^\infty(\mathbb{R}^n)$ such that $\sup_{x \in \mathbb{R}^n}(|x|^j |D^\alpha f|) \leq \infty$ for all $j \in \mathbb{N}$ and $\alpha \in \mathbb{N}^n$. The set of Schwartz functions is denoted by $\mathcal{S}(\mathbb{R}^n)$. We use the representation $\delta_S :\subseteq \Sigma^\omega \to \mathcal{S}(\mathbb{R}^n)$ defined by

$$\begin{aligned} \delta_D(\langle p, q \rangle) = f :\iff\ & \delta_\infty(p) = f \ \text{ and } q \text{ encodes a function } h : \mathbb{N}^{n+1} \to \mathbb{N} \\ & \text{such that } \ (\forall\, j, \alpha)\ \sup_{x \in \mathbb{R}^n}(|x|^j |D^\alpha f|) \leq h(j, \alpha). \end{aligned}$$

Of course, the above representations can be replaced by equivalent ones. Examples for the case $n = 1$ are discussed in [6].

A continuous linear map $T : \mathcal{D}(\mathbb{R}^n) \to \mathbb{C}$ ($T : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$) is called a distribution (tempered distribution), (where "continuous" means sequentially continuous w.r.t. the canonical convergence relations on $\mathcal{D}(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$, resp.). The set of all distributions (tempered distributions) is denoted as $\mathcal{D}'(\mathbb{R}^n)$ ($\mathcal{S}'(\mathbb{R}^n)$). For the set $\mathcal{D}'(\mathbb{R}^n)$ we use the representation $[\delta_D \to \rho^2]$ (restricted to the linear functions).

Remember that the evaluation function is computable, i.e., there is Type-2 Turing machine which computes a ρ^2-name of $T(\phi)$ whenever given a $[\delta_D \to \rho^2]$-name of T and a δ_D-name of ϕ as input. Usually $T(\phi)$ is written as $<T, \phi>$.

We conclude this section with recalling some further definitions and facts [2]. The Dirac measure δ on $\mathbb{R}^n$ at 0 is a tempered distribution defined by $\delta(\phi) = <\delta, \phi> = \phi(0)$ for any Schwartz function $\phi \in \mathcal{S}(\mathbb{R}^n)$. The Dirac measure can be viewed as a point source. The Fourier transform of a Schwartz function $\phi \in \mathcal{S}(\mathbb{R}^n)$, denoted as $\hat{\phi}$ or $\mathcal{F}(\phi)$, is defined as

$$\mathcal{F}(\phi)(\xi) = \hat{\phi}(\xi) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi} \phi(x) dx$$

where $x = (x_1, x_2, \ldots, x_n)$, $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ and $x\xi = x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n$. The Fourier transform is a linear bijection of $\mathcal{S}(\mathbb{R})$ to itself. For every partial derivative,

$$\mathcal{F}(D_j\phi) = \xi_j \mathcal{F}(\phi) \quad \text{and} \quad D_j\mathcal{F}(\phi) = \mathcal{F}(-\xi_j\phi) \tag{3}$$

Therefore, for any partial differential operator $P(D)$ with constant coefficients

$$<\mathcal{F}(P(D)), \phi> = P(\xi)\hat{\phi}, \quad \forall \phi \in \mathcal{S}(\mathbb{R}). \tag{4}$$

$D = -i\partial$.

Since $\mathcal{S}(\mathbb{R})$ is sequentially dense in $\mathcal{S}'(\mathbb{R})$, by a duality argument, $\mathcal{F}$ and D_j can be uniquely extended from $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}'(\mathbb{R})$ defined by the following formulae:

$$<\mathcal{F}T, \phi> := <T, \mathcal{F}\phi> , \tag{5}$$

$$<D_jT, \phi> := <T, -D_j\phi> \tag{6}$$

for any $T \in \mathcal{S}'(\mathbb{R})$ and $\phi \in \mathcal{S}(\mathbb{R})$. The Fourier transform $\hat{\delta}$ of the Dirac measure δ is a constant:

$$\mathcal{F}(\delta) = (2\pi)^{-n/2}. \tag{7}$$

Let $\mathcal{R}$ be the reflexion operator on $\mathcal{S}(\mathbb{R}^n)$ defined by $\mathcal{R}(u)(x) := u(-x)$.

Lemma 2.1 *For any $E \in \mathcal{D}'(\mathbb{R}^n)$ let $\langle \tilde{E}, \phi> := <E, \mathcal{R}\phi>$. Then E is a fundamental solution of $P(D)$, that is, $P(D)E = \delta$, iff*

$$<\tilde{E}, P(D)\phi> = \phi(0) \quad \textit{for all} \;\; \phi \in \mathcal{D}(\mathbb{R}^n)\,.$$

Proof By (6), $<E, \mathcal{R}D_j\phi> = <E, -D_j\mathcal{R}\phi> = <D_jE, \mathcal{R}\phi>$, and therefore,

$$<\tilde{E}, P(D)\phi> = <E, \mathcal{R}P(D)\phi> = <P(D)E, \mathcal{R}\phi> .$$

We obtain $P(D)E = \delta$, iff $(\forall\phi) <P(D)E, \mathcal{R}\phi> = \mathcal{R}\phi(0) = \phi(0)$, iff $(\forall\phi) <\tilde{E}, P(D)\phi> = \phi(0)$. □

3 Computing a Fundamental Solution

Formally, the problem of constructing a fundamental solution of a partial differential operator $P(D)$ is very easy. Indeed, suppose that we have

$$P(D)E = \delta\,.$$

Since $\mathcal{F}D_jT = \xi_j\mathcal{F}T$ (from (3), (5) and (6)), by taking the Fourier transform we obtain $P(\xi)\mathcal{F}E = (2\pi)^{-n/2}$, hence

$$\mathcal{F}E = \frac{(2\pi)^{-n/2}}{P(\xi)}\,,$$

and E should be defined as the inverse Fourier transform of $(2\pi)^{-n/2}/P(\xi)$. This is meaningful if $P(\xi)$ has no real zeros. In the general case, we will overcome the difficulty by selecting domains of integration which avoids the zeros of $P(\xi)$.

For $m \geq 1$ let $\mathcal{P}(m)$ be the linear space of all polynomials $P(\xi) = \sum_{|\alpha|\leq m} c_\alpha \xi^\alpha$, $c_\alpha \in \mathbb{C}$, in n variables of degree $\leq m$, where $\xi^\alpha = \xi_1^{\alpha_1} \cdot \ldots \cdot \xi_n^{\alpha_n}$ for $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $|\alpha| = \alpha_1 + \ldots + \alpha_n$. This space has dimension $N(m,n) := (m+n)!/m!n!$, the monomials ξ^α, $|\alpha| \leq m$, can be considered as a basis. On $\mathcal{P}(m)$ we consider the norm $\|\sum_{|\alpha|\leq m} c_\alpha \xi^\alpha\|_m := \sqrt{\sum_{|\alpha|\leq m} |c_\alpha|^2}$.

Lemma 3.1 *[3] From m a finite set $A_m = \{\nu_i \mid 1 \leq i \leq L(m,n)\} \subseteq \mathbb{Q}^n$ of vectors, $L(m,n) := (m+1)^{(n+1)}$, can be computed such that for all $P \in \mathcal{P}(m)$, $P \not\equiv 0$,*

$$(\exists\, \nu \in A_m)(\forall z \in \mathbb{C},\, |z| = 1)P(z\nu) \neq 0\,. \tag{8}$$

A_m can be chosen as follows: $A = \{\frac{k}{m}\nu : \nu \in A', k = 0,1,\ldots,m\}$, where $A' = \{\xi \in \mathbb{R}^n : \xi_i \in \{0,1,2,\ldots,m\}, 1 \leq i \leq n\}$. Observe that $|A_m| = (m+1)^{(n+1)} =: L(m,n)$.

Lemma 3.2 *There is a Type-2 Turing machine which, for every polynomial $P \in \mathcal{P}(m)$ of degree $m > 0$, computes a number $l \in \mathbb{N}$ such that*

$$2^{-l} < \sup_{\nu\in A_m} \inf_{|z|=1} |P(\xi + z\nu)| \qquad \text{for all} \quad \xi \in \mathbb{R}^n\,. \tag{9}$$

More precisely, from the degree m and ρ^2-names of the coefficients $c_\alpha \in \mathbb{C}$ ($|\alpha| \leq m$) of a polynomial $P(\xi) = \sum_{|\alpha|\leq m} c_\alpha \xi^\alpha$ of degree m the machine computes a number l such that (9).

Proof

By Lemma 3.1 there is some $\nu \in A_m$ such that for all $z \in \mathbb{C}$, $P(z\nu) \neq 0$ if $|z| = 1$. Since $z \mapsto |P(z\nu)|$ is continuous,

$$0 < \sup_{\nu \in A_m} \inf_{|z|=1} |P(z\nu)| \,. \tag{10}$$

For every $\nu \in A_m$ the function

$$(\tilde{c}, z) \mapsto |P(z\nu)| \quad \text{is } (\rho^{2N(m,n)}, \rho^2, \rho)\text{-computable}$$

($\tilde{c} := (c_\alpha)_{|\alpha| \le m} \in \mathbb{C}^{N(m,n)}$). By Theorem 3.3.15 in [5] on type conversion, the function

$$\tilde{c} \mapsto (z \mapsto |P(z\nu)|) \quad \text{is } (\rho^{2N(m,n)}, [\rho^2 \to \rho])\text{-computable.}$$

Since $\{z \in \mathbb{C} \mid |z| = 1\}$ is a κ^2-computable compact set (Def. 5.1.2 in [5]) the function

$$f \mapsto \inf_{|z|=1} f(z) \quad \text{is } ([\rho^2, \rho], \rho)\text{-computable}$$

by Cor. 6.2.5 in [5]. Therefore, for each $\nu \in A_m$, the function

$$\tilde{c} \mapsto \inf_{|z|=1} |P(z\nu)| \quad \text{is } (\rho^{2N(m,n)}, \rho)\text{-computable.} \tag{11}$$

By Thm. 6.1.2 in [5], the function

$$\tilde{c} \mapsto \sup_{\nu \in A_m} \inf_{|z|=1} |P(z\nu)| \quad \text{is } (\rho^{2N(m,n)}, \rho)\text{-computable.} \tag{12}$$

Since $B_m := \{\tilde{c} \mid \sqrt{\sum_{|\alpha| \le m} |c_\alpha|^2} = 1\}$ is compact and $\tilde{c} \mapsto \sup_{\nu \in A_m} \inf_{|z|=1} |P(z\nu)|$ is continuous by (12), by (10)

$$\inf_{\|\tilde{c}\|_m = 1} \sup_{\nu \in A_m} \inf_{|z|=1} |P(z\nu)| > 0 \,.$$

Since B_m is $\kappa^{2N(m,n)}$-computable, by (12) and Cor. 6.2.5 in [5],

$$\inf_{\|\tilde{c}\|_m = 1} \sup_{\nu \in A_m} \inf_{|z|=1} |P(z\nu)| > 0 \quad \text{is } \rho\text{-computable.}$$

Since all the above computations are uniform in the degree m, a rational number $C_m > 0$ can be computed from m such that

$$C_m < \sup_{\nu \in A_m} \inf_{|z|=1} |P(z\nu)| \qquad \text{if} \;\; \|P\|_m = 1 \,. \tag{13}$$

Now consider arbitrary polynomials $P(\xi) = \sum_{|\alpha| \le m} c_\alpha \xi^\alpha$ of degree m. There is a Type-2 machine which on input m and the coefficients $(c_\alpha)_{|\alpha| \le m}$ computes some rational number a_P such that $0 < a_P < |c_\beta|$ for some index β with $|\beta| = m$.

For fixed $\xi \in \mathbb{R}^n$, $P(\xi + \xi') = \sum c_\alpha (\xi + \xi')^\alpha = \sum d_\alpha(\xi)\, \xi'^\alpha$ where the coefficients of the polynomials $d_\alpha(\xi)$ can be determined by algebraic computation. By an easy observation, $d_\beta(\xi) = c_\beta$ since $|\beta| = m$. Let $Q(\xi') := P(\xi + \xi')$.

Then $a_P < |c_\beta| \le \sqrt{\sum |d_\alpha(\xi)|^2} = \|Q\|_m$. Since $Q/\|Q\|_m$ has norm 1, by (13), $C_m < \sup_{\nu\in A_m} \inf_{|z|=1} |Q(z\nu)|/\|Q\|_m$, therefore

$$a_P \cdot C_m < \|Q\|_m \cdot C_m \le \sup_{\nu\in A_m} \inf_{|z|=1} |Q(z\nu)| = \sup_{\nu\in A_m} \inf_{|z|=1} |P(\xi + z\nu)| \,.$$

Finally some $l \in \mathbb{N}$ can be computed such that $2^{-l} \le a_P C_m$. □

In the following we denote, for any $(\xi_1, \dots, \xi_n) \in \mathbb{R}^n$, $|(\xi_1, \dots, \xi_n)| := \sup_j |\xi_j|$. $\overline{X}$ will denote the closure and X° will denote the interior of a set X. For vectors in $\mathbb{R}^n$ let $(x_n, \dots, x_n) < (y_n, \dots, y_n)$, iff $x_i < y_i$ for all i. For $a, b \in \mathbb{R}^n$, $a < b$, let $(a; b]$ be the semi-open interval (box) $\{x \in \mathbb{R}^n \mid a < x \le b\}$. Let $\mathcal{B}_1 := \{(a; b] \mid a, b \in \mathbb{Q}^n \text{ and } a < b\}$ be the set of all semi-open rational intervals with canonical notation ν_{B1}. The set $\{I^\circ \mid I \in \mathcal{B}_1\}$ is a basis of the topology on $\mathbb{R}^n$. Let $\mathcal{B}$ be the set of all finite unions of elements from $\mathcal{B}_1$ with standard notation ν_B. Notice that $\emptyset \in \mathcal{B}$ and $\mathcal{B}$ is closed under union, intersection and difference $(I, J) \mapsto I \setminus J$ and that every $J \in \mathcal{B}$ is a finite union of pairwise disjoint semi-open intervals from $\mathcal{B}_1$. Union and difference are (ν_B, ν_B, ν_B)-computable.

Integration of total continuous functions over intervals is computable [5]. We need a generalization to partial functions. In [4] such multirepresentations have been used for partial continuous functions on computable metric spaces. The following lemma generalizes Thm. 6.4.1 in [5]. It can be proved by using standard techniques.

Lemma 3.3 (i) *The function $f \mapsto \int_{|z|=1} f(z)\, dz$ for continuous $f :\subseteq \mathbb{C} \to \mathbb{C}$ which is defined if $\{z \mid |z| = 1\} \subseteq \mathrm{dom}(f)$ is $([\rho^2 \to_p \rho^2], \rho^2)$-computable.*

(ii) *The function $(f, I) \mapsto \int_I f(\xi)\, d\xi$ for continuous $f :\subseteq \mathbb{R}^n \to \mathbb{C}$ and $I \in \mathcal{B}$ which is defined if $\overline{I} \subseteq \mathrm{dom}(f)$, is $([\rho^n \to_p \rho^2], \nu_B, \rho^2)$-computable.*

Since the boundary of every $I \in \mathcal{B}$ has measure 0, for every continuous function f,

$$\int_{\overline{I}} f(\xi)\, d\xi = \int_I f(\xi)\, d\xi = \int_{I^\circ} f(\xi)\, d\xi \,,$$

if $\overline{I} \subseteq \mathrm{dom}(f)$. After these preparations we can prove our main theorem.

Theorem 3.4 *There is a Type-2 Turing machine which for every differential operator $P(D) = \sum_{|\alpha|\le m} c_\alpha D^\alpha \not\equiv 0$ computes a fundamental solution $E \in \mathcal{D}'(\mathbb{R}^n)$. More precisely, from m and ρ^2-names of the $c_\alpha \in \mathbb{C}$ ($|\alpha| \le m$) it computes a $[\delta_D \to \rho^2]$-name of a fundamental solution E.*

Proof
First, we assume that the degree m of the polynomial $P(\xi) = \sum_{|\alpha|\le m} c_\alpha \xi^\alpha$ is fixed. Let $\tilde{c} := (c_\alpha)_{|\alpha|\le m} \in \mathbb{C}^{N(m,n)}$ be the vector of coefficients of $P(\xi)$. Let $A_m = \{\nu_i \mid 1 \le i \le L(m,n)\}$ be any set satisfying Lemma 3.1. Let $l \in \mathbb{N}$ be some constant such that (9). For $1 \le j \le L(m,n)$ Define

$$\Omega_j := \{\xi \in \mathbb{R}^n \mid 2^{-l} < \inf_{|z|=1} |P(\xi + z\nu_j)|\,\}\,. \tag{14}$$

Let $M_{-1} := M_0 := \emptyset \in \mathcal{B}$ and $M_k := ((-k,\dots,-k);(k,\dots,k)] \in \mathcal{B}$ for $k \ge 1$. For $k, j \in \mathbb{N}$, $1 \le j \le L(m,n)$, let $T_j^k \in \mathcal{B}$ be sets such that

$$T_j^k \subseteq \Omega_j, \quad T_i^k \cap T_j^k = \emptyset \ \text{ for } \ i \ne j \ \text{ and } \ \bigcup_j T_j^k = M_k \setminus M_{k-1}\,. \tag{15}$$

Since for each k the T_j^k $(1 \le j \le L(m,n))$ are a partition of $M_k \setminus M_{k-1}$ and the $M_k \setminus M_{k-1}$ are partition of $\mathbb{R}^n$, $T_j^k \cap T_{j'}^{k'} = \emptyset$ if $k \ne k'$ or $j \ne j'$. Define $T_j := \bigcup_k T_j^k$. Then

$$T_j \subseteq \Omega_j, \quad T_i \cap T_j = \emptyset \ \text{ for } \ i \ne j \ \text{ and } \ \bigcup_j T_j = \mathbb{R}^n\,. \tag{16}$$

For any $u \in \mathcal{D}(\mathbb{R}^n)$ and $k \in \mathbb{N}$ define

$$\tilde{E}_k(u) := (2\pi)^{-n/2} \sum_{j=1}^{L(m,n)} \int_{T_j^k} d\xi \frac{1}{2\pi i} \int_{|z|=1} \frac{\hat{u}(\xi + z\nu_j)}{P(\xi + z\nu_j)} \frac{1}{z} dz \tag{17}$$

and

$$\tilde{E}(u) := (2\pi)^{-n/2} \sum_{j=1}^{L(m,n)} \int_{T_j} d\xi \frac{1}{2\pi i} \int_{|z|=1} \frac{\hat{u}(\xi + z\nu_j)}{P(\xi + z\nu_j)} \frac{1}{z} dz\,. \tag{18}$$

If $P(\xi + z\nu_j)$ becomes 0 in the domain of integration, the integrals possibly do not exist. By (14), however, for any j, any $\xi \in \Omega_j$ and for $|z| = 1$,

$$\left| \frac{\hat{u}(\xi + z\nu_j)}{P(\xi + z\nu_j)} \frac{1}{z} \right| \le \frac{|\hat{u}(\xi + z\nu_j)|}{|P(\xi + z\nu_j)|} \frac{1}{|z|} \quad \le \quad 2^l \cdot |\hat{u}(\xi + z\nu_j)|\,. \tag{19}$$

The integrals exist, since $|z| = 1$ is considered and the function $\hat{u} \in \mathcal{S}(\mathbb{R}^n)$ is rapidly decaying.

Define E by $<E,\phi> \ := \ <\tilde{E},\mathcal{R}\phi>$ for all $\phi \in \mathcal{D}(\mathbb{R}^n)$. Then $<\tilde{E},\phi> \ = \ <\tilde{E},\mathcal{R}^2\phi> \ = \ <E,\mathcal{R}\phi>$ for all $\phi \in \mathcal{D}(\mathbb{R}^n)$. We show in the following that E is a fundamental solution of $P(D)$. By Lemma 2.1, E is a fundamental solution of $P(D)$ iff $\tilde{E}(P(D)v) = v(0)$ for any $v \in \mathcal{D}(\mathbb{R}^n)$. For any $u = P(D)v$, $v \in \mathcal{D}(\mathbb{R})$,

$$\begin{aligned}
\tilde{E}(u) &= \tilde{E}(P(D)v) \\
&= (2\pi)^{-n/2} \sum_{j=1}^{L(m,n)} \int_{T_j} d\xi \frac{1}{2\pi i} \int_{|z|=1} \frac{\mathcal{F}(P(D)v)(\xi + z\nu_j)}{P(\xi + z\nu_j)} \frac{1}{z} dz \\
&= (2\pi)^{-n/2} \sum_{j=1}^{L(m,n)} \int_{T_j} d\xi \frac{1}{2\pi i} \int_{|z|=1} \frac{P(\xi + z\nu_j)\hat{v}(\xi + z\nu_j)}{P(\xi + z\nu_j)} \frac{1}{z} dz \\
&= (2\pi)^{-n/2} \sum_{j=1}^{L(m,n)} \int_{T_j} d\xi \frac{1}{2\pi i} \int_{|z|=1} \hat{v}(\xi + z\nu_j) \frac{1}{z} dz \,.
\end{aligned}$$

We observe that

$$\begin{aligned}
&\frac{1}{2\pi i} \int_{|z|=1} \hat{v}(\xi + z\nu_j) \frac{1}{z} dz \\
&= \frac{1}{2\pi i} \int_{|z|=1} (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix(\xi + z\nu_j)} v(x) dx \frac{1}{z} dz \\
&= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi} v(x) dx \frac{1}{2\pi i} \int_{|z|=1} \frac{e^{-ixz\nu_j}}{z} dz \\
&= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi} v(x) e^{-ix0\nu_j} dx \qquad \text{(Cauchy integral)} \\
&= (2\pi)^{-n/2} \int_{\mathbb{R}} e^{-ix\xi} v(x) dx \\
&= \hat{v}(\xi) \,.
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
\tilde{E}(P(D)v) &= (2\pi)^{-n/2} \sum_{j=1}^{L(m,n)} \int_{T_j} \hat{v}(\xi) d\xi \\
&= (2\pi)^{-n/2} \int_{\mathbb{R}^n} \hat{v}(\xi) d\xi \\
&= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i0\xi} \hat{v}(\xi) d\xi \\
&= v(0) \,.
\end{aligned}$$

Next we determine an upper bound of $|\tilde{E}_k(u)|$. Since $\hat{u} \in \mathcal{S}(\mathbb{R}^n)$,

$$(\exists\, k' \in \mathbb{N}) \quad (|\hat{u}(\xi)|\, |\xi|^{n+2} \le 1 \,\text{if}\, |\xi| \ge k') \,. \tag{20}$$

Define $\bar{\nu} := \max_{\nu \in A_m} |\nu|$.
Let $k \ge k' + \bar{\nu} + 1$ and $k \ge 2\bar{\nu} + 2$. Then for $\xi \in \mathbb{R}^n$, $z \in \mathbb{C}$ and $1 \le j \le L(m,n)$,

$$|\xi + z\nu_j| \ge |\xi| - \bar{\nu} \ge k' \quad \text{if} \quad |\xi| \ge k - 1 \quad \text{and} \quad |z| = 1 \,. \tag{21}$$

We obtain

$$
\begin{aligned}
|\tilde{E}_k(u)| &\le (2\pi)^{-n/2} \sum_{j=1}^{L(m,n)} \int_{T_j^k} d\xi \frac{1}{2\pi} \int_{|z|=1} \frac{|\hat{u}(\xi + z\nu_j)|}{|P(\xi + z\nu_j)|} \frac{1}{|z|} |dz| \\
&\le 2^l (2\pi)^{-n/2} \sum_{j=1}^{L(m,n)} \int_{T_j^k} d\xi \frac{1}{2\pi} \int_{|z|=1} |\hat{u}(\xi + z\nu_j)||dz| \quad \text{(by (14), (15))} \\
&\le 2^l (2\pi)^{-n/2} \sum_{j=1}^{L(m,n)} \int_{T_j^k} d\xi \frac{1}{2\pi} \int_{|z|=1} \frac{|\hat{u}(\xi + z\nu_j)| \cdot |\xi + z\nu_j|^{n+2}}{|\xi + z\nu_j|^{n+2}} |dz| \\
&\le 2^l (2\pi)^{-n/2} \sum_{j=1}^{L(m,n)} \int_{T_j^k} d\xi \frac{1}{2\pi} \int_{|z|=1} \frac{1}{|\xi + z\nu_j|^{n+2}} |dz| \\
&\qquad \text{(by (20)and (21), since } |\xi| \ge k-1 \text{ for } \xi \in T_j^k) \\
&\le 2^l (2\pi)^{-n/2} \sum_{j=1}^{L(m,n)} \int_{T_j^k} d\xi \frac{1}{2\pi} \int_{|z|=1} \frac{1}{(k-1-\bar{\nu})^{n+2}} |dz| \\
&\le 2^l (2\pi)^{-n/2} \sum_{j=1}^{L(m,n)} \int_{T_j^k} d\xi \frac{1}{(k-1-\bar{\nu})^{n+2}} \\
&\le 2^l (2\pi)^{-n/2} \frac{(2k)^n}{(k-1-\bar{\nu})^{n+2}} \qquad \text{(by (15))} \\
&\le 2^l \cdot (2\pi)^{-n/2} \cdot 2^{2n+2} \frac{1}{k^2} \qquad (\text{since } k \ge 2\bar{\nu} + 2)
\end{aligned}
$$

Since $\sum_{k>K} \frac{1}{k^2} \le \frac{1}{K}$ we obtain for all $K \ge k' + 2\bar{\nu} + 2$,

$$
\sum_{k=K+1}^{\infty} |\tilde{E}_k(u)| \;\le\; 2^{l+2n+2} \cdot (2\pi)^{-n/2} \frac{1}{K},
$$

and by (15) - (18),

$$
\tilde{E}(u) \;=\; \sum_{k=0}^{\infty} \tilde{E}_k(u) \tag{22}
$$

and for all $K \ge k' + 2\bar{\nu} + 2$,

$$
|\tilde{E}(u) - \sum_{k=0}^{K} \tilde{E}_k(u)| \;\le\; 2^{l+2n+2} \cdot (2\pi)^{-n/2} \frac{1}{K}\,. \tag{23}
$$

It remains to prove that the fundamental solution E can be *computed* from the polynomial P. Concretely, a polynomial P of degree m is given by the list

$\tilde{c} = (c_\alpha)_{|\alpha|\le m}$, $c_\alpha \in \mathbb{C}$, of its coefficients represented by $\rho^{2N(m,n)}$. Let A_m be a set of vectors according to Lemma 3.1.

(a) By Lemma 3.2, the multifunction $\tilde{c} \rightrightarrows l$ such that (9) is $(\rho^{2N(m,n)}, \nu_{\mathbb{N}})$-computable.

(b) Next we compute the sets Ω_j defined in (14). For each number j the function $(l, \tilde{c}, \xi) \mapsto \inf_{|z|=1} |P(\xi + z\nu_j)| - 2^{-l}$ is $(\nu_{\mathbb{N}}, \rho^{2N(m,n)}, \rho^n, \rho)$-computable. The proof is essentially that of (11). By Thm. 3.3.12 in [5] on type conversion, for each j the function

$$(l, \tilde{c}) \mapsto (\xi \mapsto \inf_{|z|=1} |P(\xi + z\nu_j)| - 2^{-l})$$

is $(\nu_{\mathbb{N}}, \rho^{2N(m,n)}, [\rho^n \to \rho])$-computable. For the open subsets of $\mathbb{R}^n$ we use the representation $\theta_<$ defined by $\theta_<(p) = U$, iff $p \in \Sigma^\omega$ is (encodes) a list of all $I \in \mathcal{B}_1$ such that $\overline{I} \subseteq U$ (Def. 5.1.15 in [5]). Notice that $U = \bigcup_{\overline{I}\subseteq U} I^\circ = \bigcup_{\overline{I}\subseteq U} \overline{I}$. Since the set $\{y \in \mathbb{R} \mid y > 0\}$ is r.e. open, by Thm. 6.2.4.1 in [5] for each j the function

$$(l, \tilde{c}) \mapsto \Omega_j \ \text{ is } (\nu_{\mathbb{N}}, \rho^{2N(m,n)}, \theta_<)\text{-computable.}$$

(c) We describe how to compute sets T_j^k from sequences $p_j \in \Sigma^\omega$ such that $\theta_<(p_j) = \Omega_j$. Consider $k \in \mathbb{N}$. Simultaneously for all $j = 1, \ldots, L(m,n)$ produce the lists of intervals $I \in \mathcal{B}_1$ encoded by the p_j. Halt as soon as a finite set of intervals $C \subseteq \mathcal{B}_1$ has been found such that $\overline{M_k \setminus M_{k-1}} \subseteq \bigcup_{I\in C} I^\circ$ (finite covering of a compact set). For each j let $C_j \subseteq C$ be the set of intervals from C so far listed by p_j. Let $T_0^k := \emptyset$ and determine the $T_j^k \subseteq \mathcal{B}$ successively by the rule

$$T_j^k := (\bigcup_{I\in C_j} I \cap (M_k \setminus M_{k-1})) \setminus \bigcup_{j'<j} T_{j'}^k\,.$$

If (9) holds for the number l, then $\bigcup_j \Omega_j = \mathbb{R}^n$, a set C will be found by compactness of $\overline{M_k \setminus M_{k-1}}$, and (15) and (16) are true. As a summary, we have a multifunction

$$(l, \tilde{c}) \rightrightarrows ((k,j) \mapsto T_j^k) \quad \text{which is } (\nu_{\mathbb{N}}, \rho^{2N(m,n)}, [\nu_{\mathbb{N}}^2 \to \nu_B])\text{-computable.}$$

(d) Next we prove that $(\tilde{c}, k, u, ((k,j) \mapsto T_j^k)) \mapsto \tilde{E}_k(u)$ defined in (17) is $(\rho^{2N(m,n)}, \nu_{\mathbb{N}}, \delta_D, [\nu_{\mathbb{N}}^2 \to \nu_B], \rho^2)$-computable.

First let j be fixed. Since the identity from $\mathcal{D}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$ is (δ_D, δ_S)-computable, the Fourier transform is (δ_S, δ_S)-computable and the identity from $\mathcal{S}(\mathbb{R}^n)$ to $C(\mathbb{R}^n)$ is $(\delta_S, [\rho^n \to \rho^2])$-computable, the function

$$(u, \tilde{c}, \xi, z) \mapsto \frac{\hat{u}(\xi + z\nu_j)}{P(\xi + z\nu_j)} \frac{1}{z} \quad \text{is} \quad (\delta_D, \rho^{2N(m,n)}, \rho^n, \rho^2, \rho^2)\text{-computable}.$$

Therefore by type conversion,

$$(u, \tilde{c}, \xi) \mapsto (z \mapsto \frac{\hat{u}(\xi + z\nu_j)}{P(\xi + z\nu_j)} \frac{1}{z}) \;\text{ is }\; (\delta_D, \rho^{2N(m,n)}, \rho^n, [\rho^2 \to_p \rho^2])\text{-computable}.$$

By Lemma 3.3 the function

$$(u, \tilde{c}, \xi) \mapsto \int_{|z|=1} \frac{\hat{u}(\xi + z\nu_j)}{P(\xi + z\nu_j)} \frac{1}{z} dz \quad \text{is} \quad (\delta_D, \rho^{2N(m,n)}, \rho^n, \rho^2)\text{-computable}.$$

Again after type conversion by Lemma 3.3, the function

$$(I, u, \tilde{c}) \mapsto \int_I d\xi \int_{|z|=1} \frac{\hat{u}(\xi + z\nu_j)}{P(\xi + z\nu_j)} \frac{1}{z} dz \;\text{ is }\; (\nu_B, \delta_D, \rho^{2N(m,n)}, \rho^2)\text{-computable}.$$

Finally, since π is computable, we can conclude that

$$(\tilde{c}, k, u, ((k, j) \mapsto T_j^k)) \mapsto \tilde{E}_k(u)$$

is $(\rho^{2N(m,n)}, \nu_{\mathbb{N}}, \delta_D, [\nu_{\mathbb{N}}^2 \to \nu_B], \rho^2)$-computable. By Thm. 3.1.7 in [5] on primitive recursion the function

$$(\tilde{c}, K, u, ((k, j) \mapsto T_j^k)) \mapsto \sum_{k=0}^{K} \tilde{E}_k(u)$$

is $(\rho^{2N(m,n)}, \nu_{\mathbb{N}}, \delta_D, [\nu_{\mathbb{N}}^2 \to \nu_B], \rho^2)$-computable.

(e) As above, from $u \in \mathcal{D}(\mathbb{R}^n)$, $\hat{u} \in \mathcal{S}(\mathbb{R}^n)$ can be computed. By the definition of δ_S, a number k' can be computed from $\hat{u}$ such that $|v(\xi)||\xi|^{n+2} \le 1$ if $|\xi| \ge k'$. From l, k' and $L \in \mathbb{N}$, a number K can be computed such that $K \ge k' + 2\bar{\nu} + 2$ and $K \ge 2^{l+2n+2} \cdot (2\pi)^{-n/2} \cdot 2^{L+1}$. Then by (23),

$$|\tilde{E}(u) - \sum_{k=0}^{K} \tilde{E}_k(u)| \;\le 2^{-L-1}.$$

By (d) above, from $\tilde{c}, K, u, ((k, j) \mapsto T_j^k)$ and L a rational $b_L \in \mathbb{C}$ can be computed such that

$$|b_L - \sum_{k=0}^{K} \tilde{E}_k(u)| \leq 2^{-L-1}.$$

From the sequence $b_0, b_1, \ldots$ we can compute a ρ^2-name of $\tilde{E}(u)$. Therefore, after type conversion, the function

$$(\tilde{c}, l, ((k,j) \mapsto T^k_j)) \mapsto (u \mapsto \tilde{E}(u))$$

is $(\rho^{2N(m,n)}, \nu_{\mathbb{N}}, [\nu^2_{\mathbb{N}} \to \nu_B], [\delta_D \to \rho^2])$-computable.

From (a), (c) and (e), the multifunction $\tilde{c} \rightrightarrows \tilde{E}$ is $(\rho^{2N(m,n)}, [\delta_D \to \rho^2])$-computable. Since $<E, u> = <\tilde{E}, \mathcal{R}u> = <\mathcal{R}\tilde{E}, u>$ and $\mathcal{R}$ is computable on $\mathcal{D}'(\mathbb{R}^n)$ by Lemma 4.7 in [6], the operator $u \rightrightarrows E$ mapping a polynomial of degree m to some fundamental solution is $(\rho^{2N(m,n)}, [\delta_D \to \rho^2])$-computable.

So far we have assumed a fixed degree m of the polynomial. Since the set A_m can be computed from m (Lemma 3.1) and all the other computations are uniform also in m the proof is finished. □

In the proof multivalued functions occur several times. A critical part is the computation of the sets T^k_j which determines a partition of $\mathbb{R}^n$ into (at most) $L(m,n)$ sets. The resulting distribution depends essentially on this partition. However, the function from $\mathbb{C}^{N(m,n)}$ to these sets cannot be computable (hence continuous) and single-valued at the same time, provided it is not trivial (Lemma 4.3.15 in [5]).

Problem: Is there a *single-valued* computable function mapping any polynomial of degree m to a fundamental solution?

As usual in recursion theory, we have not merely proved the *existence* of a computable function (from names of input objects to names of output objects) but the proof is *constructive*, i.e. it explains how a concrete Turing machine or computer program can be constructed for computing this function. Of course the algorithm is not trivial since subroutines for integration, for Fourier transform on Schwartz space etc. have to be included. The presented algorithms can be improved, since, e.g., no derivatives of the Fourier transform $\hat{u}$ of the input test function $u \in \mathcal{S}(\mathbb{R}^n)$ are needed for computing the $\tilde{E}_k(u)$ by integration.

Problem: Can our informal algorithm be converted to a feasible numerical algorithm for computing fundamental solutions efficiently or is the problem inherently difficult?

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