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Domain Theory its Ramifications and Interactions

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Abstract

At a conference that is devoted to a specific research area – Domain Theory in this case – there should be room for reflecting its rôle within the concert of scientific fields that sometimes cooperate and sometimes compete for recognition. It is the intention of this paper to stimulate interactions of Domain Theory with other research areas. Thus, this is not a research paper. It is the intention, firstly, to recall that Domain Theory has its historical roots in quite different fields, secondly, to indicate its interactions with other areas and not only with Computer Science, and thirdly. to put forward a new development in the theory of C*-algebras that opens new perspectives.

Keywords: Domain Theory and its history, Interactions with other areas, Cuntz semigroup, C*-algebra

1 Interaction versus Application

Competing for positions and grants, mathematical disciplines have to justify themselves. From the outside the impact on APPLICATIONS is often put forward as the relevant criterion. For this reason, titles of Journals, of Conferences, etc., like to take the form

[Something] and its Applications.

Whenever you see such a title, you can be sure that this is a Journal or a Conference on Pure Mathematics.

What is meant by 'applications' is quite subtle. Are these applications in fields outside of mathematics, like Engineering, Medicine, Computer Science (Physics)? Or applications in Statistics, Optimization, Numerical Methods? Or applications to other mathematical theories, or as background theories to applied mathematics?

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In this paper we put forward another criterion for judging the liveliness of a field of mathematical research: Ramifications into and Interactions with other fields inside or outside of mathematics, no matter whether they are theoretical or applied.

Here, my question is: What are the ramifications of Domain Theory into other fields and what are the interactions with other fields? While ramifications can go just one way from Domain Theory into other fields, interactions are meant to go both ways.

Domain Theory cannot be compared to a field like Number Theory which has a long history, problems that one can explain to non-mathematicians and that is extremely rich with respect to the variety of methods that are used. Domain Theory can be compared to General Topology. General Topology provides a conceptual framework for phenomena that occur allover in Mathematics like *space, convergence, nearness, connectedness, compactness, etc.*, but it has not been developed in order to solve specific problems. Every theory has its internal life and creates its internal problems, but General Topology is mainly perceived as a background theory, and its latest internal developments may not matter to the outside world.

At the birth of Domain Theory, specific problems were waiting to be solved. Domain Theory provides a conceptual framework for phenomena occurring in various fields combining order and topology. It cannot be justified entirely by its internal problems. Thus, Domain Theory can only remain a lively research field if it interacts with other research areas inside or outside of Mathematics.

It is the purpose of this paper, firstly, to remind that historically the roots of Domain Theory have grown out of quite different origins, secondly, to indicate ramifications into other fields and interactions of Domain Theory with other fields that have occurred in the past and, thirdly, to draw the attention to a new development in a classical field of Mathematics, where Domain Theory appears as a relevant ingredient.

Because of the large variety of subjects to be touched, the paper will be far from being self-contained. It is hoped that the descriptions will be clear enough to give the flavor of things, so that one may be tempted to look at the original literature that is listed at the end of each of the three sections.

I apologize that my list of interactions will be incomplete and I would appreciate hints for making this paper more complete. What I write may be biased and can be criticized. And I hope to get lots of reactions.

The language of Domain Theory will be used freely. In particular, *poset* stands for partially ordered set, *dcpo* for directed complete poset, *domain* for continuous dcpo, $\ll$ denotes the way-below relation.

References are inserted at the end of each section, since they are of a different nature according to the very different contents of the three sections.

2 The roots

Domain Theory has emerged for very different reasons and from various origins. I can see five quite different roots not all of the same impact. Although I have to talk about them in some order, they should be seen as interleaving.

Computability theory (computable functionals of higher type),
models for computation and the lambda calculus,
topological algebra,
universal algebra,
general topology.

Computability theory

The notion of computability was first investigated for functions from the natural numbers into themselves, that is, at ground type. But of course the question came up when a real function is computable or when an operator (that transforms functions into new functions) is computable, and so on. This is the question of computability of functionals of higher type. In 1959, Kleene [2.10] and Kreisel [2.11] were proposing notions of computability for functionals of higher type. Kreisel singled out a class of computable functionals from within a wider class of functionals that were continuous with respect to neighborhood systems based on the idea of approximation via finite information. These topological ideas were simplified by Yu. L. Ershov who arrived in 1972 [2.4] at his f-spaces and a simplified topological characterization of the Kleene-Kreisel computable functionals of higher type [2.5]. Ershov mentions that he became aware of Scott's 1972 paper [2.15] on continuous lattices and models for the untyped λ-calculus just after writing the two papers mentioned above. In 1973 he then published his paper on A-spaces that generalize f-spaces and relate closely to continuous lattices. Ershov did not require directed completeness for his f-spaces and A-spaces. Thus, he worked with what are called abstract bases in [2.1], an aspect that – in the author's opinion – has not attracted enough attention in the past.

Models for computation and λ-calculus

In the late 1960ies, D. S. Scott was fascinated by the problem of sound foundations for the emerging variety of programming languages. He wrote an extremely influential manuscript [2.13] that at the time was circulated within the interested community only, published retroactively in 1993. In this manuscript, which is very worth of being read also nowadays, one finds the roots of denotational semantics. The models are partially ordered sets with a bottom element that are ω-complete (that is, increasing sequences have a least upper bound). Computable functionals are modeled by monotone functions preserving suprema of increasing sequences. Thus, Scott started with an order theoretical point of view, adding a topological ingredient in his seminal 1972 paper [2.15] where he introduced the notion of a continuous lattice with the aim of building the first models for the untyped λ-calculus. Let me restate Scott's definition:

On every partially ordered set P, a topology σ is defined (later called the *Scott topology*): a subset U is open if, for every directed subset D of P that has a lub $\sup D \in U$, there is a $d \in D \cap U$. For elements in a complete lattice L, Scott defines $x \prec y$ if y is in the σ-interior of the upper set $\uparrow x = \{z \in L \mid x \leq z\}$. A complete lattice he said to be *continuous* if every element y is the lub of the set of elements $x \prec y$.

Topological algebra

Animated by A. D. Wallace in the 1950ies, a rather active research community dealt with a theory of compact semigroups in the 1960ies (the Hausdorff separation axiom was always subsumed). When a problem about compact semigroups was reduced to a problem about compact groups, it was considered to be solved. But there are semigroups far away from groups. The most distant situation is that in which all elements are idempotent so that all subgroups are singletons. In the commutative case, these are just the compact semilattices. In the commutative case, one would have liked to mimic Pontryagin duality for compact Abelian groups. Thus, for compact semilattices the first question was whether the continuous semilattice homomorphisms into the unit interval $\mathbb{I}$ – considered as a compact semilattice with the usual topology and the operation $\min(x, y)$ – were separating the points. In 1969, J. D. Lawson [2.12] characterized the compact semilattices having this separation property as those having a basis of open subsemilattices.

K. H. Hofmann and A. R. Stralka [2.9] then searched for a purely order theoretical characterization of the semilattices identified by Lawson. On p. 27 of their Memoir one finds the statement that a complete lattice admits a compact Hausdorff topology such that the meet-operation is continuous and satisfies Lawson's condition if, and only if, it is *relatively algebraic* according in the following sense:

> "It is now notationally convenient to call an element x in a lattice L *relatively compact under* y iff it is contained in any ideal I of L with $\sup I \geq y$. This is ostensibly equivalent to saying that *for all subsets* $D \subseteq L$ *with* $\sup D \geq y$ *there is a finite subset subset* $F \subseteq Y$ *with* $x \leq \sup F$. Let us call a lattice *relatively algebraic* if it is complete and every element in it is the l.u.b. of all relatively compact elements under it."

This is the precisely the definition of continuous lattices used until today (see, e. g., [2.1,2.8]). Long before the Memoir [2.9] finally appeared in print, it was discovered that these 'relatively algebraic lattices' where the same as the 'continuous lattices' introduced Scott in his 1972 paper.

Universal Algebra

In the 1960ies and 1970ies, Universal Algebra was developing rapidly. A category theoretical version of universal algebra was developed, first under the name of a *triple*, and now known under the name of a *monad*. A well known monad beyond universal algebra was the monad of ultrafilters over sets. The algebras of this monad are the compact Hausdorff spaces. It was then natural to consider monad of all filters over sets, but a characterization of the algebras of the filter monad had been an open question for some years. In a paper submitted in 1973 [2.2], Alan Day then proved that the algebras of the filter monad are precisely Scott's continuous lattices and he gave an equational characterization as suggested by universal algebra.

General Topology

In 1970, Day and Kelly [2.3] answered the question for which topological spaces Z it is true that the function $f \times id_Z : X \times Z \to Y \times Z$ is a quotient map for every quotient map $f: X \to Y$. Their answer was that these are the spaces Z for which the lattice of open subsets is continuous. They did not phrase their answer in these terms. Their formulation was that for every open neighborhood W of a point $z \in Z$, there was a Scott-open neighborhood $\mathcal{W}$ of W in the lattice of open subsets of Z such that $\bigcap\{U \mid U \in \mathcal{W}\}$ is a neighborhood of z. It was discovered quite later that this condition was equivalent to the continuity of the lattice of open sets.

Note

Without any doubt, the work of Scott and its inspiration for denotational semantics has been the decisive moment for the development of Domain Theory which rapidly expanded from continuous lattices to continuous directed complete posets. Its simultaneous appearance in various fields led to a rapid development of the mathematical theory [2.8]. But what feature of continuous dcpos is it that made it appear in so different contexts? Undoubtedly the fact that the idea of approximation by relatively finite approximands is optimally coded by the way-below relation and the notion of a continuous domain.

The historical account above can also be seen as a warning. Some of the fields that were present at the birth of Domain Theory are no longer in the focus of the mathematical community. The theory of compact semigroups has almost disappeared. Universal algebra attracts much less attention now except for some specific orientations. And hasn't Domain Theory its best years behind itself?

Domain Theory will continue to be used in semantics. But will there be new developments required for semantics, or is the existing bulk of methods sufficient?

Remaining introverted and generating problems and new notions just inside the theory will not lead to high recognition by the scientific community outside. In order to remain a lively theory, ramifications into other fields and interactions with other areas are essential.

In the following I will list such ramifications and interactions that have occurred in the past as far as I am aware of them.

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3 Ramifications and interactions

I will not talk about the applications of Domain Theory in Semantics, as these a widely known and well documented. For the relation to Computability Theory, I refer to the books by Stoltenberg-Hansen, Lindström and Griffor [3.13] and the excellent recent book by Longley and Normann [3.7]. The work on exact real number computation can be seen as a combination of both aspects.

Dynamical systems, Differentials

The work of Abbas Edalat in Domain Theory is characterized by his orientation to domain theoretical models for approximating objects in all kinds of fields in a computable way. His earlier work up to 1997 is summarized in the survey [3.3]. Since then it has largely extended and developed further. It is impossible to describe all of its facets. Let us just pick out three of them.

ITERATED FUNCTION SYSTEMS An iterated function system on a compact metric space X is given by finitely many continuous selfmaps $f_1, \ldots, f_n$ of X. Such a system is said to be *weakly hyperbolic* [3.3] if, for every infinite sequence $i_1, i_2, i_3, \ldots$, the diameter of the sets $f_{i_1} f_{i_2} \cdots f_{i_k}(X)$ tends to 0 with $k \to \infty$. The set $\mathcal{H}(X)$

of nonempty closed subsets C of X ordered by reverse inclusion is a bounded complete domain. A weakly hyperbolic iterated function system on X induces a Scott-continuous selfmap $F(C) = f_1(C)\cup\cdots\cup f_n(C)$ of $\mathcal{H}(X)$. Using fixed point theorems of Domain Theory one can show that there is a unique nonempty closed subset C such that $F(C) = C$, the unique attractor of the weakly hyperbolic system. Algorithms for calculating the attractor are given. Such iterated functions systems are used in image processing. Using domain theoretical versions of measure theory, these results an be extended from black-and-white to gray scale images.

The Clarke derivative A variant of Scott's discovery [2.15] that continuous lattices, when endowed with there Scott topology, are injective in the category of topological spaces, says: Every continuous function from a dense subspace Y of a topological space X into a bounded complete domain has a greatest continuous extension to all of X. Edalat [3.4] uses a topology on the set of all Lipschitz maps $f\colon \mathbb{R}^n \to \mathbb{R}$ with the property that the continuously differentiable maps are dense. Since mapping a continuously differentiable f to its classical derivative Df is continuous, this map has a greatest continuous extension to the space of all Lipschitz maps. Edalat proves the surprising result that, for a Lipschitz map f, the extension yields the Clarke derivative Df.

The Clarke derivative, a standard tool in optimization, assigns to every point $x \in \mathbb{R}^n$ a non-empty compact convex subset $Df(x)$ of $\mathbb{R}^n$ which is continuous with respect the Scott topology on the bounded complete domain $\mathrm{Conv}(\mathbb{R}^n)$ of non-empty compact convex subsets of $\mathbb{R}^n$. The continuous functions from $\mathbb{R}^n$ to $\mathrm{Conv}(\mathbb{R}^n)$ form again a bounded complete domain; and this is the domain which is used for the extension result quoted above.

Thus, Domain Theory allows to introduce the Clarke derivative in a smooth way compared to the original quite technical definition.

Differential Equations The initial value problem $y' = v(y),\ y(0) = 0$, for a Lipschitz continuous vector field v defined on a neighborhood of $0 \in \mathbb{R}^n$ can be solved by Picard iteration. Edalat and Pattinson [3.5] provide a directly implementable algorithm that guarantees an approximation of the exact solution for any given precision given in advance within a short runtime. (Of course, one has to restrict to a finite interval.) For this they use a canonical extension of the problem to interval domains. On $\mathbb{R}$ one uses the domain of closed intervals $[a, b]$, $a \leq b$, and on $\mathbb{R}^n$ rectangles, that is, direct products $[a_1, b_1] \times \cdots \times [a_n, b_n]$. Ordered by reverse inclusion, these intervals form bounded complete domains $\mathbb{IR}^n$. A continuous function $f\colon \mathbb{R} \to \mathbb{R}$ is extended to a Scott-continuous function $\mathbb{I}f\colon \mathbb{IR} \to \mathbb{IR}$ by $\mathbb{I}f([a, b]) = [\inf_{x\in[a,b]} f(x), \sup_{x\in[a,b]} f(x)]$, and similarly for the vector fields v. The Picard iteration, which uses integration, can be extended to Scott-continuous maps between the interval domains. The Lipshitz condition on the vector field v assures the convergence of the successive approximations to the solution. The key idea now is to use intervals with rational endpoints as a countable basis for the interval domains. The exact solution y is approximated (on finite intervals) by finitely many basic overlapping intervals covering (the graph of) y.

For such applications, Domain Theory is an appropriate setting which provides an environment for finitary approximation and, in the presence of a countable explicit basis, a natural notion of computability. But the proofs need additional tools depending of the specific situation.

Causal structures and General Relativity

Already in 1967, Kronheimer and Penrose [3.6] in their approach to spacetime for general relativity introduced an abstract notion of a *causal structure*. This is a triple $(P, \leq, \ll)$ where $\leq$ is a partial order and, in the language of Domain Theory, $\ll$ is an auxiliary relation (I do not use Penrose's notation who writes $\prec$ for the partial order). In addition, they require that $x \ll x$ never holds. The partial order is read as x causally precedes y, and $x \ll y$ as x chronologically precedes y. The antireflexivity of the chronological order means that there are no jumps in time. The chronological order $\ll$ which is always supposed to be transitive is said to be *full* if, firstly, for every y there is an x such that $x \ll y$ and if, secondly, whenever $x_i \ll y$ for $i = 1, 2$, one can interpolate a z such that $x_i \ll z \ll y$ for $i = 1, 2$. Structures $(P, \ll)$ with these properties have been called *abstract bases* in Domain Theory (see, e.g., [2.1]). Since domains are the round ideal completions of abstract bases, the germs of Domain Theory are already contained this 1967 paper [3.6].

The domain theoretical approach to these aspects of general relativity was fully adopted by Martin and Panangaden [3.8,3.9]. Under natural hypotheses, a causal structure on a manifolds M leads to a partial order $\leq$ and a relation $\ll$ such that not only $(M, \leq)$ is bounded directed complete and continuous with $\ll$ being the way-below relation, but also for the opposite order $\geq$ the same properties hold. One may call such posets *bounded bidirected complete bidomains*. Martin and Panangaden have shown that domain theoretical methods are useful in this context.

It seems to me that in more general contexts of causal structures on manifolds there is a subjacent domain structure. Methods from Domain Theory may be useful and should be explored. The aspect of 'bicontinuity' has not been investigated a lot. But one should not expect spectacular results from domain theoretical methods alone.

Abstract potential theory

Interesting examples of continuous lattices arise in potential theory. A function u defined on an open subset U of the complex plane with values in $\mathbb{R} \cup \{+\infty\}$ is *superharmonic* if it is lower semiconntinuous and satisfies

$$u(x) \geq \int_{-\pi}^{\pi} u(x + re^{i\theta}) \frac{d\theta}{2\pi}$$

that is, for every $x \in U$, the average value of the function u on a circle with radius r around the center x is greater or equal to the value of u in the center x, provided the closed disc with radius r around x is contained in U. (The harmonic functions are those for which equality holds instead of $\geq$.) The nonnegative superharmonic functions form a lattice ordered cone which is a continuous lattice; addition is Scott-continuous and preserves the way-below relation.

An abstract approach to potential theory and superharmonic functions has been developed by Boboc, Bucur and Cornea [3.2]. Their notion of a *standard H-cone* isolates some of properties of the cone of superharmonic functions. Their standard H-cones have the properties of the cone of superharmonic functions listed above. Their notion of an *integral* on a standard H-cone is a that of a Scott-continuous linear functional into the nonnegative reals extended by $+\infty$. Their *natural topology* is the Lawson topology. These connections to Domain Theory – not known to the community of abstract potential theory – have been described by M. Rauch [3.12]. Unfortunately it seems that abstract potential theory is no longer an active field of research. But it leads to non-trivial examples for Domain Theory.

Philosophy

In 1998, Th. Mormann [3.11]published a philosophical paper using basic concepts of Domain Theory. He presents a solution to a problem in the Whiteheadian theory of regions: Starting with a purely mereological system of regions a topological space is constructed such that the class of regions is isomorphic to the Boolean lattice of regular open sets of that space. This construction is based on a round ideal completion yielding a continuous Heyting algebra the spectrum of which gives the desired topological space. Thus, the argument of the paper relies on the theories of continuous lattices and pointless topology.

Notes

Set-valued analysis seems to be an area where Domain theory can be used profitably, not as a tool for solving specific problems but as an appropriate conceptual framework in which problems can be formulated conveniently. Another promising area one should look at is Idempotent Analysis. One could add quite some literature. To begin with, [3.1,3.10] are useful sources of information.

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4 The Cuntz semigroup of a C*-algebra

In recent years, domains and domain theoretical concepts are used in the theory of C*-algebras. Remarkably, this happened without being noticed by the Domain Theory community for several years. This section is devoted to report about these new developments. Domain Theory is a substantial ingredient and new domain theoretical results and questions arise.

Recall shortly that a C*-algebra is a Banach algebra with an involution $a \mapsto a^*$ satisfying $\|aa^*\| = \|a\|^2$. One should think of the algebra of complex $n \times n$-matrices and, more generally, of closed *-subalgebras of the algebra of bounded operators on a Hilbert space, the involution being given by the adjoint of an operator.

Already in 1978, J. Cuntz [4.4] had associated a commutative monoid which each C*-algebra. In 2008, searching for an invariant, more powerful than the mainly K-theoretical invariants used before, Coward, Elliott, and Ivanescu [4.3] enriched this semigroup with the structure of a domain. They use a variant of domain theory: partial orders are only required to be ω-complete in the sense that every increasing sequence has a least upper bound. They also use a countable version of the way-below relation: $a \ll_\omega b$ if, for every increasing sequence $x_1 \leq x_2 \leq \ldots$ such that $b \leq \sup_n x_n$, there is an n such that $a \leq x_n$. They then require that, for every element b, there is a sequence $a_1 \ll_\omega a_2 \ll_\omega \ldots$ such that $b = \sup_n a_n$. Let us call such structures *ω-domains*. Scott-continuity of maps is understood to be preservation of suprema of increasing sequences. The *Cuntz semigroup*[3] $Cu(A)$ associated with any given C*-algebra A is a commutative monoid enriched with the structure of an ω-domain such that 0 is the smallest element and such that addition is not only Scott-continuous but also $\ll_\omega$-preserving, that is:

$$a \ll_\omega b, a' \ll_\omega b' \Longrightarrow a + a' \ll_\omega b + b'$$

A monoid with the structure of an ω-domain having these properties is called an *abstract Cuntz semigroup*.

A main difference to domain theory is that morphisms f between abstract Cuntz semigroups are not only required to be Scott-continuous monoid homomorphisms;

[3] The construction of these concrete Cuntz semigroups is quite involved and beyond the scope of this paper. The properties defining an abstract Cuntz semigroup are not exhaustive. We refer the interested reader to the original sources.

they are also required to preserve the way-below relation, that is:

$$a \ll_\omega b \Longrightarrow f(a) \ll_\omega f(b)$$

The category of abstract Cuntz semigroups with these morphisms is denoted by CuSgr. In Domain Theory preservation of the way-below relation has only be considered in connection with adjoints. This preservation property is close to the statement that f is an open map: The upper set generated by the image of a Scott-open set is Scott-open.

In the last years, a considerable number of publications has been devoted to this subject. And domain theoretical concepts play an important role there. From the background of C*-algebras new aspects and new problems arise. But also the existing Domain Theory is helpful. Here a selection of results will be presented.

Already Coward, Elliott, and Ivanescu [4.3] had shown that every $*$-homomorphism f between C*-algebras induces a Scott-continuous semigroup homomorphism $Cu(f)$ preserving $\ll_\omega$ between the corresponding Cuntz semigroup so that Cu becomes a functor from the category of C*-algebras and $*$-homomorphisms to the category CuSgr. This functor has quite some preservation properties. Since the Cuntz semigroup is difficult to describe concretely, these preservation properties help to determine new Cuntz semigroups from known ones.

Direct limits

It is not difficult to see that colimits of sequences $S_1 \to S_2 \to \ldots$ exist in the category CuSgr. One first takes the algebraic semigroup colimit S_∞. One then defines a partial order and a relation $\ll_\omega$ on S_∞ as the (set theoretical) colimit of the corresponding relations on the S_n. For this uses that the bonding maps preserve the order and the relation $\ll_\omega$. This set theoretical colimit is not yet complete for suprema of increasing sequences. Thus one executes a completion process, the completion by countably generated round $\ll_\omega$-ideals. Addition of round ideals is well-defined, since addition preserves the relation $\ll_\omega$. Already in [4.3] it has been shown:

Theorem 4.1 *The functor Cu from the category of C*-algebras and $*$-homomorphisms to the category* CuSgr *preserves direct limits of sequences $A_1 \to A_2 \to \cdots$.*

Closed ideals and quotients

Antoine, Perera and Thiel [4.2, Section 5.1] have shown that the Cuntz semigroup of a closed ideal J of a C*-algebra A is a Scott-closed sub-monoid of the Cuntz-semigroup of A and conversely:

Theorem 4.2 *$J \mapsto Cu(J)$ is an order isomorphism between the collection of closed ideals of the C*-algebra A and the Scott-closed sub-monoids of the Cuntz semigroup $Cu(A)$ of A.*

For a Scott-closed sub-monoid I of an abstract Cuntz semigroup S one can define a preorder $\prec_I$ on S compatible with the monoid structure by $a \prec_I b$ if $a \leq b + x$ for some $x \in I$. The quotient semigroup $S/\sim_I$ modulo the congruence relation $a \sim_I b$ if $a \prec_I b$ and $b \prec_I a$ is again an abstract Cuntz semigroup for the partial order $\prec_I$. The quotient map is a morphism. According to [4.2, Section 5.1] the functor Cu preserves quotients:

Theorem 4.3 *If J is a closed ideal of the C*-algebra A, then the Cuntz semigroup of the quotient algebra A/J is the quotient of the Cuntz semigroup of A modulo the congruence relation $\sim_{Cu(J)}$ defined as above by the Scott-closed submonoid $Cu(J)$.*

Altogether, one can say that the functor Cu preserves short exact sequences: For a short exact sequence $0 \to J \to A \to B \to 0$ of C*-algebras, $0 \to Cu(J) \to Cu(A) \to Cu(B) \to 0$ is a short exact sequence of Cuntz semigroups.

Traces and quasi-traces

The positive cone A_+ of a C*-algebra A consists of the elements of the form xx^*, $x \in A$, equivalently, of the elements the spectrum of which consists of non-negative real numbers only.

Adopting a general point of view, a *trace* is a map $t\colon A_+ \to \overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{\infty\}$ which is lower semicontinuous and satisfies $t(xx*) = t(x * x)$ for all $x \in A$, $t(ra) = rt(a)$ for $0 < r \in \mathbb{R}$ and $a \in A_+$, and $t(a+b) = t(a) + t(b)$ for all $a, b \in A_+$. Quasi-traces are defined in the same way except that additivity is only required when a and b generate a commutative *-subalgebra of A; one requires in addition that t estends to a quasi-trace on the the C*-algebra of 2×2-matrices $\begin{pmatrix} x & y \\ z & w \end{pmatrix}$ with entries in A. Both traces and quasi-traces form cones $T(A)$ and $QT(A)$, respectively, since the defining properties are preserved under addition and multiplication by scalars $r \geq 0$.

In [4.3] it has been shown that the cones $T(A)$ and $QT(A)$ can be endowed with a compact Hausdorff topology such that addition and multiplication with scalars $r > 0$ are continuous. For the case of traces, a direct proof is presented. For the case of quasi-traces , the authors use that quasi-traces can be identified with Scott-continuous monoid homomorphisms from the Cuntz semigroup $Cu(A)$ to $\overline{\mathbb{R}}_+$, and then they proceed with a similar proof as for traces. These results are closely related to Plotkin's Banach-Alaoglu Theorem [4.8] for continuous dcpo-cones with an additive way-below relation. In fact, both results can be seen special cases of a general domain theoretical result in [4.6] as shown in [4.7]:

Theorem 4.4 *Let S be an abstract Cuntz semigroup and S^* the cone of all Scott-continuous monoid homomorphisms $t\colon S \to \overline{\mathbb{R}}_+$. Endowed with the weakest topology τ making all point evaluations $t \mapsto t(x)$ continuous, S becomes a stably compact topological cone. Endowed with the patch topology associated with τ, S^* is a compact Hausdorff topological cone with a closed order.*

Function spaces

Antoine, Perera and Santiago [4.1, Theorem 5.15] have obtained a new result on function spaces, that one might not have conjectured in Domain Theory:

Theorem 4.5 *Let X be a separable compact metric space of finite (covering) dimension and S an abstract Cuntz semigroup (endowed with its Scott topology). Then the continuous functions from X to S form again an abstract Cuntz semigroup.*

For this has to prove that the space of continuous functions from X to S is a continuous domain which is done by induction over the dimension. It seems to me that this is a purely domain theoretical result, that one can extract by close inspection of the proof: The space of continuous functions from a separable, compact, finite dimensional, metrizable space to a domain with a bottom element is a domain.

Questions and remarks

The Cuntz semigroup $Cu(A)$ of a C*-algebra A as defined by Coward, Elliott, and Ivanescu [4.3] is in fact a kind of completion of Cuntz' original semigroup $W(A)$; indeed a completion by countably generated round ideals (see [4.2]). Alternatively, $Cu(A)$ is isomorphic to $W(A \otimes K)$ where K is the C*-algebra of compact operators on a separable Hilbert space.

Instead of $Cu(A)$ as defined by Coward, Elliott, and Ivanescu, consider Cuntz' original semigroup $W(A)$ endowed with the relation $\ll$ as its domain theoretical enrichment, where $\ll$ is the restriction of the way-below relation on $Cu(A)$.

Then $(W(A), \ll)$ is an object of the following category: The objects are structures $(S, +, 0, \ll)$ where $(S, +, 0)$ is a commutative monoid and $\ll$ a transitive relation with $0 \ll x$ for all x and with the interpolation property (thus, $(S, \ll)$ is an abstract basis) such that addition is continuous and $\ll$-preserving. Morphisms are continuous $\ll$-preserving monoid homomorphisms. Still we have a functor from C*-algebras into this new category preserving, e.g., inductive limits and short exact sequences.

The invariant $(W(A), \ll)$ is finer than $Cu(A)$, since there is no reason why it should be possible to recover $W(A)$ from $Cu(A)$. Isn't it a better invariant than its countable completion preferred by [4.3]?

For some purposes it might still be useful to consider completions of $(W(A), \ll)$, but for non-separable C*-algebras the general round ideal completion looks to be more useful than the completion by countably generated round ideals.

In the context of the Cuntz semigroup, maps between domains that are not only continuous but also preserve the way-below relation are of major interest. These maps do not play a rôle in domain theory. But here the question for Cartesian closed categories of domains with this restricted class of morphisms arises. It is interesting to observe that Z. Q. Yang [4.9] has proved that the category of domains and Scott-continuous $\ll$-preserving maps is Cartesian closed, not knowing that this result would become of interest for Cuntz semigroups. It is a step towards proving that the category of (abstract) Cuntz semigroups is symmetric monoidal closed.

One may have noticed that at several places in this paper abstract bases are mentioned which are extensively used in [2.1]. I feel that this notion deserves more attention and, for this, another name may be helpful; I propose to call them *predomains.* Thus a *predomain* is a set P with a relation $\ll$ that is transitive and has the interpolation property: For every finite subset F and every element z, if $x \ll z$ for every $x \in F$, then there is a y such that $x \ll y \ll z$ for every $x \in F$. A predomain carries a natural topology, a basis of which are the sets $\{y \mid x \ll y\}$. The topological spaces arising in this way have been called *core spaces* by Erné [4.5]. Core spaces are the topological variant of predomains. Domains are obtained as round ideal completions of predomains, equivalently, as sobrifications of core spaces.

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