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On Monotone Determined Spaces

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Abstract

In this paper, we investigate some basic properties, especially categorical properties, of monotone determined spaces. For a topology τ, we construct a monotone determined topology $md(\tau)$. The main results are: (1) for a space (X, τ), then $md(\tau)$ is the weakest monotone determined topology on X containing τ; (2) the category $\mathbf{Top_{md}}$ of monotone determined spaces with continuous maps is fully co-reflexive in the category $\mathbf{Top}$ of all topology spaces with continuous maps; (3) the category $\mathbf{Top_{md}}$ is cartesian closed.

Keywords: monotone determined space, weak Scott topology, co-reflective, cartesian closed

1 Introduction

For a space X, it is well known that a subset U of X is open iff every net that converges to a point in U is residually in U (cf. [2]). Foe certain order-defined topologies, it suffices to test that criterion for *monotone nets*. Spaces or topologies with that property is called *monotone determined* in [3]. Erné [3] has shown that all locally hypercompact spaces and all Scott spaces are monotone determined, compact open subsets of monotone determined spaces are hypercompact, and a space is

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hypercompactly based iff it is monotone determined and compactly based. It follows that the monotone determined monotone convergence spaces are exactly the Scott spaces of dcpos, the hypercompactly based sober spaces are exactly the Scott spaces of quasialgebraic domains, which gave a negative answer for the question posed by Priestley in [9]: whether there exists a non-quasicontinuous domain on which the Scott topology is spectral.

In this paper, we investigate some basic properties of monotone determined spaces, especially the categorical properties. For a topology τ, we construct a monotone determined topology $md(\tau)$. It is shown that $md(\tau)$ is the weakest monotone determined topology on X containing τ and $cl_\tau D = cl_{md(\tau)}D$ for any directed subset D. Let $\mathbf{Top_{md}}$ be the category of monotone determined spaces with continuous maps and $\mathbf{Top}$ the category of all topology spaces with continuous maps. We show that the product and the limit of monotone determined spaces are $md(\prod_{i\in I} X_i)$ and $md(\varprojlim_j X_j)$, respectively, where $\prod_{i\in I} X_i$ and $\varprojlim_j X_j$ are the product and the limit in $\mathbf{Top}$, respectively. It is proved that $\mathbf{Top_{md}}$ is fully co-reflexive in $\mathbf{Top}$ and $\mathbf{Top_{md}}$ is cartesian closed.

2 Preliminaries

In this section we recall some basic definitions and notations used in this note, more details can be found in [1,5,8]. Let P be a poset, $x \in P, A \subseteq P$. Let $\uparrow x = \{y \in P : x \leq y\}$ and $\uparrow A = \{y \in P : x \leq y$ for some $x \in A\}$, $\downarrow x$ and $\downarrow A$ are defined dually. A is said to be an *upper set* if $A = \uparrow A$. $A^\uparrow$ and $A^\downarrow$ denote the sets of all upper and lower bounds of A, respectively. Let $A^\delta = (A^\uparrow)^\downarrow$. P is said to be a *directed complete poset*, a dcpo for short, if every directed subset of P has the least upper bound in P. The *Alexandroff topology* $A(P)$ on P is the topology consisting of all its upper subsets. The topology generated by the collection of sets $P\backslash\downarrow x$(as subbasic open subsets) is called *upper topology* and denote by $\nu(P)$. A subset U of P is called *Scott open* if $U = \uparrow U$ and $D \cap U \neq \emptyset$ for all directed sets $D \subseteq P$ with $\vee D \in U$ whenever $\vee D$ exists. The topology formed by all the Scott open sets of P is called the *Scott topology*, written as $\sigma(P)$.

We order the collection of nonempty subsets of a poset P by $G \leq H$ if $\uparrow H \subseteq \uparrow G$. We say that a nonempty family of sets is directed if given F_1, F_2 in the family, there exists F in the family such that $F_1, F_2 \leq F$, i.e., $F \subseteq \uparrow F_1 \cap \uparrow F_2$. For nonempty subsets F and G of a dcpo L, we say F approximates G if whenever a directed subset D satisfies $\vee D \in \uparrow G$, then $d \in \uparrow F$ for some $d \in D$. A dcpo L is called a *quasicontinuous domain* if for all $x \in L$, $\uparrow x$ is the directed (with respect to reverse inclusion) intersection of sets of the forms $\uparrow F$, where F approximates $\{x\}$ and F is finite.

Give a topological space (X, τ), define an order $\leq_\tau$, called the *specialization order*, by $x \leq_\tau y$ if and only if $x \in cl_\tau\{y\}$. Clearly, each open set is an upper set and each closed set is a lower set with respect to the specialization order $\leq_\tau$. Denote the closure of subset $A \subseteq X$ by $cl_\tau A$ and interior of A by $int_\tau A$ in (X, τ).

Definition 2.1 ([3]) A space is called *locally hypercompact* if for any $x \in X$ and

$U \in \tau$ with $x \in U$, there exists a finite set F such that $x \in int_\tau \uparrow F \subseteq \uparrow F \subseteq U$.

Definition 2.2 ([6,10]) Let P be a poset.

(1) Given any two elements x and y in P, define a relation $\prec$ on P by $x \prec y$ iff $y \in int_{\nu(P)} \uparrow x$;

(2) P is called *hypercontinuous* if $x = \vee\{u \in P : u \prec x\}$ for all $x \in P$.

Theorem 2.3 ([10]) *A poset P is hypercontinuous if and only if for all $x \in P$ and $U \in \nu(P)$ with $x \in U$, there exists a $y \in P$ such that $x \in int_{\nu(P)} \uparrow y \subseteq \uparrow y \subseteq U$.*

Definition 2.4 ([10]) Let P be a poset. P is called *quasi-hypercontinuous* if for all $x \in P$ and $U \in \nu(P)$ with $x \in U$, there exists a finite set $F \subseteq P$ such that $x \in int_{\nu(P)} \uparrow F \subseteq \uparrow F \subseteq U$.

Theorem 2.5 ([11,12]) *Let P be a poset. Then the following conditions are equivalent:*

(1) *P is quasi-hypercontinuous;*

(2) *$(\nu(P), \subseteq)$ is a hypercontinuous lattice;*

(3) *$\nu(P)$ is locally hypercompact.*

Definition 2.6 ([13]) Let P be a poset.

(1) Given any two subsets G and H in P, we say that G approximates H and write $G \ll_2 H$, if for all directed sets $D \subseteq P$, $\uparrow H \cap D^\delta \neq \emptyset$ implies $\uparrow G \cap D \neq \emptyset$. Let $w(x) = \{F \subseteq P : F \text{ is finite and } F \ll_2 x\}$.

(2) P is called *s_2-quasicontinuous* if for each $x \in P$, $\uparrow x = \bigcap\{\uparrow F : F \in w(x)\}$ and $w(x)$ is directed.

Obviously, if P is a dcpo, then s_2-quasicontinuity is equivalent to the quasicontinuity.

Definition 2.7 ([3,4]) Let P be a poset. A subset $U \subseteq P$ is called *weak Scott open* if it satisfies

(1) $U = \uparrow U$;

(2) For all directed sets $D \subseteq P$, $D^\delta \cap U \neq \emptyset$ implies $D \cap U \neq \emptyset$.

The collection of all weak Scott open subsets of P forms a topology, it will be called the *weak Scott topology* of P and will be denoted by $\sigma_2(P)$. Clearly, $\nu(P) \subseteq \sigma_2(P) \subseteq \sigma(P)$ and $\sigma_2(P)$ coincide with $\sigma(P)$ on dcpos.

Theorem 2.8 ([13]) *For a poset P, the following statements are equivalent:*

(1) *P is an s_2-quasicontinuous poset;*

(2) *$\sigma_2(P)$ is locally hypercompact;*

(3) *$(\sigma_2(P), \subseteq)$ is a hypercontinuous lattice.*

3 Monotone determined spaces

In this section, we will give some properties of monotone determined spaces and construct a monotone determined topology $md(\tau)$ from any given topology τ.

Definition 3.1 ([3]) A topological space (X, τ) is called a *monotone determined space*, a *MD-space* for short, if any subset U meeting all directed sets whose closure meets U is open, that is, $U \in \tau$ iff $U \cap cl_\tau D \neq \emptyset$ implies $U \cap D \neq \emptyset$. The topology τ is called a *monotone determined topology*, a *MD-topology* for short,

Lemma 3.2 ([3]) (1) *The weakest monotone determined topology with a given specialization order is the weak Scott topology, the strongest is the Alexandorff topology;*

(2) *Every locally hypercompact space is a MD-space;*

(3) *A poset P associating with Scott topology $\sigma(P)$ is a MD-space.*

Definition 3.3 Let (X, τ) be a space, $U \subseteq X$, U is call *MD-open* if and only if for any directed subset $D \subseteq X$, $cl_\tau D \cap U \neq \emptyset$ implies $D \cap U \neq \emptyset$. The collection of all MD-open sets is denote by $md(\tau)$.

It is immediate that an arbitrary unions of MD-open sets is again MD-open and almost immediate that the same is true for finite intersection. Indeed, Let $U, V \in md(\tau)$ and $D \subseteq X$ be a directed subset which satisfies $cl_\tau D \cap (U \cap V) \neq \emptyset$, then $cl_\tau D \cap U \neq \emptyset$ and $cl_\tau D \cap V \neq \emptyset$. Since $U, V \in md(\tau)$, there exist $d_1, d_2 \in D$ such that $d_1 \in U$ and $d_2 \in V$, thus there is a $d \in D$ such that $d_1, d_2 \leq d$. Notice that U, V are upper sets, which implies $d \in U \cap V$, so $D \cap (U \cap V) \neq \emptyset$. This proves that $U \cap V \in md(\tau)$. Hence the MD-open sets form a topology. Obviously, we have $\tau \subseteq md(\tau)$.

Lemma 3.4 *Let (X, τ) be a space. Then for any directed subset D, $cl_\tau D = cl_{md(\tau)} D$ and $\leq_\tau = \leq_{md(\tau)}$;*

Proof. Clearly, $cl_\tau D \supseteq cl_{md(\tau)} D$ since $\tau \subseteq md(\tau)$. On the other side, for any $x \in cl_\tau D$ and $U \in md(\tau)$ with $x \in U$, we have $cl_\tau D \cap U \neq \emptyset$. From the definition of $md(\tau)$, it follows that $D \cap U \neq \emptyset$, thus $x \in cl_{md(\tau)} D$. Therefore, $cl_\tau D = cl_{md(\tau)} D$. For any $x, y \in X$, $x \leq_\tau y$ iff $x \in cl_\tau y = cl_{md(\tau)} y$ iff $x \leq_{md(\tau)} y$. □

Theorem 3.5 *Let (X, τ) be a space. Then $md(\tau)$=min$\{\alpha : \alpha$ is a MD-topology on X with $\tau \subseteq \alpha\}$, that is, $md(\tau)$ is the weakest MD-topology containing τ.*

Proof. Firstly, we show that $md(\tau)$ is a MD-topology. Let $D \subseteq X$ be a directed subset and $U \subseteq X$. Suppose that $cl_{md(\tau)} D \cap U \neq \emptyset$ implies $D \cap U \neq \emptyset$. Now we have to show $U \in md(\tau)$. If $cl_\tau D \cap U \neq \emptyset$, by Lemma 3.4, we have $cl_{md(\tau)} D \cap U \neq \emptyset$. Thus $D \cap U \neq \emptyset$. By Definition 3.3, $U \in md(\tau)$. Hence, $md(\tau)$ is a MD-topology.

Let α be a MD-topology on X with $\tau \subseteq \alpha$. For any $U \in md(\tau)$, if $cl_\alpha D \cap U \neq \emptyset$ for a directed subset D, note that $cl_\tau D \supseteq cl_\alpha D$, it follows that $cl_\tau D \cap U \neq \emptyset$. Since $U \in md(\tau)$ and α is a MD-topology, we have $D \cap U \neq \emptyset$ and $U \in \alpha$. Thus $md(\tau) \subseteq \alpha$. □

Follows from preceding theorem, it is easy to see that (X, τ) is a MD-space iff $\tau = md(\tau)$. Now question naturally arise: whether a poset P equipped with the upper topology $\nu(P)$ is a MD-space? If not, what is the $md(\nu(P))$? Next, we will give an example to show that even if P is a complete lattice, $(P, \nu(P))$ is not a MD-space, and it is proved that $md(\nu(P))$ is exactly the weak Scott topology $\sigma_2(P)$.

Example 3.6 Let $L = \{a\} \cup \{b_i : i \in N\} \cup \{c\}$, where N denotes the set of all natural numbers. The order $\leq$ on L is defined as follows: for any $i \in N$, $c \leq b_i \leq a$. Then $(L, \nu(L))$ is not a MD-space. In fact, we can conclude that $\uparrow b_1 \notin \nu(L)$. Suppose not, since $a \in \uparrow b_1$, there is a finite subset $F \subseteq L$ such that $a \in L \backslash \downarrow F \subseteq \uparrow b_1 = \{a, b_1\}$, we have $\downarrow F = \{b_i : i \in N \text{ and } i \geq 2\}$, which contradicts to F is finite. Hence $\uparrow b_1 \notin \nu(L)$. For any directed subset $D \subseteq L$, if $cl_{\nu(L)} D \cap \uparrow b_1 \neq \emptyset$, Note that $cl_{\nu(L)} D = \downarrow \vee D$ and $\uparrow b_1 \in \sigma(L)$, it follows that $D \cap \uparrow b_1 \neq \emptyset$. Thus $\uparrow b_1 \in md(\nu(L))$. Hence, $\nu(L) \neq md(\nu(L))$. Therefore, $(L, \nu(L))$ is not a MD-space, as desired.

Lemma 3.7 *Let P be a poset. Then for any directed subset $D \subseteq P$, $cl_{\sigma_2(P)} D = cl_{\nu(P)} D = D^\delta$;*

Proof. Clearly, $cl_{\sigma_2(P)} D = D^\delta$. Since $\nu(P) \subseteq \sigma_2(P)$, we have $D^\delta = cl_{\sigma_2(P)} D \subseteq cl_{\nu(P)} D$. Note that $D^\delta = \bigcap \{\downarrow x : D \subseteq \downarrow x\}$. Then it is easy to see that D^δ is a closed set in $\nu(P)$. So $cl_{\nu(P)} D \subseteq D^\delta$. Hence $cl_{\nu(P)} D = D^\delta$. □

Theorem 3.8 *Let P be a poset. Then $md(\nu(P)) = \sigma_2(P)$.*

Proof. Since $\nu(P) \subseteq \sigma_2(P)$, we have $md(\nu(P)) \subseteq md(\sigma_2(P)) = \sigma_2(P)$. Let $U \in \sigma_2(P)$ and $D \subseteq P$ be a directed set with $cl_{\nu(P)} D \cap U \neq \emptyset$. By Lemma 3.7, $cl_{\nu(P)} D = D^\delta$. Thus $D^\delta \cap U \neq \emptyset$, it follows that $D \cap U \neq \emptyset$. So $U \in md(\nu(P))$. Therefore, $md(\nu(P)) = \sigma_2(P)$. □

Let τ be an order compatible topology($\leq_\tau = \leq$) on P. Then $\nu(P) \subseteq \tau \subseteq A(P)$, by Theorem 3.8, $\sigma_2(P) = md(\nu(P)) \subseteq md(\tau) \subseteq md(A(P)) = A(P)$. Hence we get the following Corollary by this way which is different from the manner given by Erné.

Corollary 3.9 ([3]) *Let P be a poset and τ be an order compatible($\leq_\tau = \leq$) MD-topology on P. Then $\sigma_2(P) \subseteq \tau \subseteq A(P)$.*

The following Corollaries are easy to obtain and the proof is omitted.

Corollary 3.10 *Let P be a poset. $\nu(P)$ is the MD-topology if and only if $\nu(P) = \sigma_2(P)$.*

Corollary 3.11 *Let (X, τ) be a MD-space and τ be an order compatible topology. Then $cl_\tau D \subseteq D^\delta$ for any directed subset D.*

By Theorem 2.5, Theorem 2.8, Lemma 3.2 and Corollary 3.10, we immediately have:

Corollary 3.12 *Let P be a poset. Then the following conditions are equivalent:*

(1) $(\nu(P), \subseteq)$ *is a hypercontinuous lattice;*

(2) P *is s_2-quasicontinuous and* $\nu(P) = \sigma_2(P)$*;*

(3) P *is s_2-quasicontinuous and* $\nu(P)$ *is a MD-topology.*

Corollary 3.13 *Let L be a dcpo. Then the following conditions are equivalent:*

(1) P *is a quasi-hypercontinuous domain;*

(2) P *is a quasicontinuous domain and* $\nu(P) = \sigma(P)$*;*

(3) P *is a quasicontinuous domain and* $\nu(P)$ *is a MD-topology.*

Denote the set of all topologies on set X by $Top(X)$ and the set of all monotone determined topologies on X by $Top_{md}(X)$.

Proposition 3.14 *Let X be a set and* $\{\tau_i : i \in I\} \subseteq Top_{md}(X)$*,* $\bigvee\{\tau_i : i \in I\}$ *is the least upper bound of* $\{\tau_i : i \in I\}$ *in* $Top(X)$*. Then*

(1) $\bigcap\{\tau_i : i \in I\} \in Top_{md}(X)$*;*

(2) $md(\bigvee\{\tau_i : i \in I\}) = \bigvee_{Top_{md}(X)}\{md(\tau_i) : i \in I\}$.

Proof. (1) Let $\bigcap\{\tau_i : i \in I\} = \tau$. Since $\tau \subseteq \tau_i$, we have $x \leq_{\tau_i} y$, which implies $x \leq_\tau y$ for any $i \in I$. Thus, if D is a directed set in $(X, \leq_{\tau_i})$, then D is directed in $(X, \leq_\tau)$.

Let D be a directed subset of $(X, \leq_\tau)$. Suppose that $cl_\tau D \cap V \neq \emptyset$ implies $D \cap V \neq \emptyset$, we need to show $V \in \tau = \bigcap\{\tau_i : i \in I\}$, that is, we have to prove that $V \in \tau_i$ for each $i \in I$. For any $i \in I$, let D_i be a directed subset of $(X, \leq_{\tau_i})$, then D_i is directed in $(X, \leq_\tau)$. If $cl_{\tau_i} D_i \cap V \neq \emptyset$, then $cl_\tau D_i \cap V \neq \emptyset$ which implies $D_i \cap V \neq \emptyset$. Since τ_i is a MD-topology, we have $V \in \tau_i$. Hence $V \in \tau$. So $\bigcap\{\tau_i : i \in I\}$ is the MD-topology.

(2) Obviously, $md(\bigvee\{\tau_i : i \in I\}) \supseteq \bigvee_{Top_{md}(X)}\{md(\tau_i) : i \in I\}$. Suppose that τ is a MD-topology on X containing $md(\tau_i)$ for any $i \in I$. Next we have to show $\tau \supseteq md(\bigvee\{\tau_i : i \in I\})$. Since $\tau_i \subseteq md(\tau_i) \subseteq \tau$, we have $\bigvee\{\tau_i : i \in I\} \subseteq \tau$. Thus $md(\bigvee\{\tau_i : i \in I\}) \subseteq md(\tau) = \tau$. Hence, $md(\bigvee\{\tau_i : i \in I\})$ is the least upper bound containing $\bigvee\{\tau_i : i \in I\}$ in $Top_{md}(X)$. □

From Proposition 3.14, it can be easy to get the following theorem.

Theorem 3.15 *Let X be a set and for any* $\{\tau_i : i \in I\} \subseteq Top_{md}(X)$*,* $\bigvee\{\tau_i : i \in I\}$ *is the least upper bound of* $\{\tau_i : i \in I\}$ *in* $Top(X)$*. Then* $Top_{md}(X)$ *is a complete sublattice of* $Top(X)$*.*

4 The category of MD-spaces

Let **Top** denote the category of all topology spaces and continuous maps, and $\mathbf{Top_{md}}$ denote the category of all MD-spaces and continuous maps. In this section, we will discuss some category properties of MD-spaces, especially the cartesian closed property.

Lemma 4.1 *Let (X,τ) and (Y,σ) be two spaces. If the function $f:(X,\tau)\to(Y,\sigma)$ is continuous, then $f^*:(X,md(\tau))\to(Y,md(\sigma))$ which satisfies $f^*(x)=f(x)$ is continuous.*

Proof. For all $V\in md(\sigma)$, we need to show that $f^{-1}(V)\in md(\tau)$. For any directed subset $D\subseteq X$, suppose that $cl_\tau D\cap f^{-1}(V)\neq\emptyset$. Since $f:(X,\tau)\to(Y,\sigma)$ is continuous, $f(cl_\tau D)\subseteq cl_\sigma f(D)$, it follows that $cl_\sigma f(D)\cap V\neq\emptyset$. Since $V\in md(\sigma)$, we have $f(D)\cap V\neq\emptyset$, that is, $D\cap f^{-1}(V)\neq\emptyset$. Thus $f^{-1}(V)\in md(\tau)$. Therefore f^* is continuous. □

Lemma 4.2 *Let $(X,\tau),(Y,\sigma)$ be two MD-spaces and $f:(X,\tau)\to(Y,\sigma)$ be a function. Then the following statements are equivalent:*

(1) *f is continuous;*

(2) *For any directed subset $D\subseteq X$, $f(cl_\tau D)\subseteq cl_\sigma f(D)$.*

Proof. (1) $\Rightarrow$ (2) Obviously.

(2) $\Rightarrow$ (1) Firstly, we show f is order preserving. Let $x\leq_\tau y$ in X, then $x\in cl_\tau\{y\}$. So $f(x)\in f(cl_\tau\{y\})\subseteq cl_\sigma f(y)=\downarrow f(y)$. Thus $f(x)\leq_\sigma f(y)$. For any $U\in\sigma$, we will show that $f^{-1}(U)$ is an open set in X. Assume $cl_\tau D\cap f^{-1}(U)\neq\emptyset$ for a directed subset $D\subseteq X$, then there exists a $x\in cl_\tau D$ such that $f(x)\in U\cap f(cl_\tau D)$. Thus $f(x)\in U\cap cl_\sigma f(D)$. Since (Y,σ) is a MD-space, we have $f(D)\cap U\neq\emptyset$, that is $D\cap f^{-1}(U)\neq\emptyset$. Since (X,τ) is a MD-space, $f^{-1}(U)$ is an open set. Hence, f is continuous. □

Define $md:\mathbf{Top}\to\mathbf{Top_{md}}$ as following: for any $(X,\tau)\in Ob(\mathbf{Top})$ and $f\in Mor(\mathbf{Top})$, $md((X,\tau))=(X,md(\tau)), md(f)=f$. It is easy to see that md is a functor.

Theorem 4.3 *The category $\mathbf{Top_{md}}$ is fully co-reflexive in $\mathbf{Top}$.*

Proof. Given a space (X,τ), let $j:(X,md(\tau))\to(X,\tau)$ be an identity function($j(x)=x$). Suppose that $f:(Y,\alpha)\to(X,\tau)$ is a continuous map with (Y,α) a MD-space. By Lemma 4.1, $f^*:(Y,md(\alpha)=\alpha)\to(X,md(\tau))$ which satisfies $f^*(y)=f(y)$ for any $y\in Y$ is continuous, and $f=j\circ f^*$. Assume there exists a continuous function $g:(Y,\alpha)\to(X,md(\tau))$ such that $f=j\circ g$, then $f(y)=j(g(y))=g(y)$ for any $y\in Y$, thus $f^*=g$. Therefore, $\mathbf{Top_{md}}$ is fully co-reflexive in $\mathbf{Top}$. □

Let $(X,\tau),(Y,\sigma)$ be spaces, $X\times Y$ denoted the cartesian product. Define an order $\leq$ on $X\times Y$ as followings: $(x_1,y_1)\leq(x_2,y_2)\Leftrightarrow x_1\leq_\tau x_2, y_1\leq_\sigma y_2$, this order is said to be *pointwise order*. Obviously, the pointwise order coincide with the specialization order on $X\times Y$. For MD-spaces, it is natural to ask whether the cartesian product $X\times Y$ of MD-spaces is MD-space? Next, we will give a negative answer and construct the product in $\mathbf{Top_{md}}$.

Lemma 4.4 ([5]) *Let L be a complete lattice. If $\sigma(L\times L)=\sigma(L)\times\sigma(L)$, then $\sigma(L)$ is sober.*

Theorem 4.5 *Let L be a complete lattice. If $(L, \sigma(L))$ is not sober, then $(L, \sigma(L)) \times (L, \sigma(L))$ is not a MD-space.*

Proof. Clearly, $\nu(L \times L) = \nu(L) \times \nu(L) \subseteq \sigma(L) \times \sigma(L) \subseteq \sigma(L \times L)$. By Lemma 3.2(3), $\sigma(L \times L)$ is a MD-topology. Assume that $\sigma(L) \times \sigma(L)$ is a MD-topology. By Theorem 3.8 and $L \times L$ is a complete lattice, it follows that $\sigma(L \times L) = \sigma(L) \times \sigma(L)$. By Lemma 4.4, $(L, \sigma(L))$ is sober, a contradiction. □

In [7], Isbell constructed a complete lattice whose Scott topology is not sober. Hence we have conclusion that the product of MD-spaces is not a MD-space.

Theorem 4.6 *Let $\{X_i : i \in I\}$ be a family of MD-spaces. Then $md(\prod_{i \in I} X_i)$ is the product of $\{X_i : i \in I\}$ in $\mathbf{Top_{md}}$, that is $\prod_{Top_{md}} X_i = md(\prod_{i \in I} X_i)$, where $\prod_{i \in I} X_i$ is the product in $\mathbf{Top}$.*

Proof. For any $i \in I$, let $p_i : md(\prod_{i \in I} X_i) \to X_i$ be a project map. Suppose that (X, τ) is a MD-space and $f_i : X \to X_i$ is a continuous map. Since $\prod_{i \in I} X_i$ is the product in $\mathbf{Top}$, there exists a unique continuous maps $f : X \to \prod_{i \in I} X_i$ such that $p_i \circ f = f_i$ for any $i \in I$. By Lemma 4.1, $f^* : X \to md(\prod_{i \in I} X_i)$ satisfying $f^*(x) = f(x)$ is continuous and $p_i \circ f^* = f_i$. Hence, $md(\prod_{i \in I} X_i) = \prod_{Top_{md}} X_i$. □

In the remainder parts of this section, we denote the product of $\{(X_i, \tau_i) : i \in I\}$ in $\mathbf{Top_{md}}$ by $\prod_{md} X_i$ for convenience.

Theorem 4.7 *Let $D : \mathbf{J} \to \mathbf{Top_{md}}$ be a diagram. Then $md(\varprojlim_j X_j)$ is the limit of D in $\mathbf{Top_{md}}$, where $\varprojlim_j X_j$ is the limit in $\mathbf{Top}$.*

Proof. Denote the limiting cone in $\mathbf{Top}$ by $(\varprojlim_j X_j, p_j)$, where $p_j : \varprojlim_j X_j \to X_j$ is continuous. By Lemma 4.1, $p_j^* : md(\varprojlim_j X_j) \to X_j$ is continuous. Next we have to show that $(md(\varprojlim_j X_j), p_j^*)$ is the limiting cone in $\mathbf{Top_{md}}$. Let (X, τ) be a MD-space and given any cone $(X, , c_j)$ to D. Since $\varprojlim_j X_j$ is the limit in $\mathbf{Top}$, there is a unique continuous map $u : X \to \varprojlim_j X_j$ such that for all j, $p_j \circ u = c_j$, it follows that $u^* = u : X \to md(\varprojlim_j X_j)$ is continuous satisfying $p_j^* \circ u = c_j^*$. Hence, $md(\varprojlim_j X_j)$ is the limit of D in $\mathbf{Top_{md}}$. □

Using the similar proof, we can get the following results:

Theorem 4.8 *Let $\{X_i : i \in I\}$ be a family of MD-spaces, and $D : \mathbf{J} \to \mathbf{Top_{md}}$ be a diagram. Then*

(1) *$\coprod_{Top_{md}} X_i = md(\coprod_{i \in I} X_i)$, where $\coprod_{i \in I} X_i$ is the coproduct in $\mathbf{Top}$.*
(2) *$md(\varinjlim_j X_j)$ is the colimit of D in $\mathbf{Top_{md}}$, where $\varinjlim_j X_j$ is the colimit in $\mathbf{Top}$.*

From above theorems, we know that $\mathbf{Top_{md}}$ is a complete subcategory. So the functor $md : \mathbf{Top} \to \mathbf{Top_{md}}$ preserve limit and colimit.

At the end of this section, we will discuss the cartesian closed property of the category $\mathbf{Top_{md}}$. Let $(X, \tau), (Y, \sigma)$ be two spaces. Denote the set of all continuous maps from (X, τ) to (Y, σ) by Y^X, i.e., $Y^X = \{f : (X, \tau) \to (Y, \sigma) | f$ is continuous $\}$.

Definition 4.9 ([2,8]) Let $(X,\tau),(Y,\sigma)$ be two spaces. Given a point $x \in X$ and an open set $U \in \sigma$, let $S(x,U) = \{f : f \in Y^X \text{ and } f(x) \in U\}$, the sets $S(x,U)$ are a subbasis for topology on Y^X, which is called the topology of *pointwise convergence*. Denote the pointwise convergence topology by $\mathcal{S}$.

Define the usually pointwise order on Y^X: $f \leq g \Leftrightarrow f(x) \leq g(x)$ for any $x \in X$. It is easy to see that the specialization order $\leq_{\mathcal{S}}$ coincide with the pointwise order $\leq$ on Y^X. The set Y^X equipped with the pointwise convergence topology $\mathcal{S}$ denote by $\mathcal{S}(X,Y)$, that is, $\mathcal{S}(X,Y) = (Y^X, \mathcal{S})$. Hence, $md(\mathcal{S}(X,Y)) = (Y^X, md(\mathcal{S}))$, written as $S_{md}(X,Y)$. Obviously, $S_{md}(X,Y)$ is a MD-space.

Lemma 4.10 ([2,8]) *Let $\{f_d : d \in D\} \subseteq Y^X$ be a net. Then $\{f_d\}_{d\in D}$ converges to the function $f \in Y^X$ in the topology of pointwise convergence $\mathcal{S}$ if and only if for each $x \in X$, the net $\{f_d(x)\}_{d\in D}$ converges to $f(x)$.*

Lemma 4.11 *Let $(X,\tau),(Y,\sigma)$ and (Z,α) be MD-spaces, $(x,y) \in X \times Y$. If $f : X \times_{md} Y \to Z$ is continuous, then $\varphi(f) : X \to S_{md}(Y,Z)$ defined by $\varphi(f)(x)(y) = f(x,y)$ is continuous.*

Proof. Clearly, $\varphi(f)$ is order preserving. Since X and $S_{md}(Y,Z)$ are MD-spaces, we have only to show that $\varphi(f)(cl_\tau D) \subseteq cl_{md(\mathcal{S})}\varphi(f)(D)$ for any directed set D. Let $x \in cl_\tau D$. For any $y \in Y$, we have $(x,y) \in cl_\tau D \times cl_\sigma\{y\} = cl_{\tau\times\sigma} D \times \{y\} = cl_{md(\tau\times\sigma)} D \times \{y\}$. Suppose that $U \in \alpha$ with $\varphi(f)(x)(y) = f(x,y) \in U$, then $(x,y) \in f^{-1}(U)$ and $f^{-1}(U)$ is an open set in $X \times_{md} Y$. Thus $(D\times\{y\}) \cap f^{-1}(U) \neq \emptyset$, which implies that there is a $d \in D$ such that $(d,y) \in f^{-1}(U)$, that is ,$f(d,y) \in U$. Hence $\varphi(f)(x)(y) = f(x,y) \in cl_\alpha\{f(d,y) : d \in D\} = cl_\alpha \varphi(f)(D)(y)$. By arbitrary of y and Lemma 4.10, $\varphi(f)(x) \in cl_{\mathcal{S}}\varphi(f)(D) = cl_{md(\mathcal{S})}\varphi(f)(D)$. Thus $\varphi(f)(cl_\tau D) \subseteq cl_{md(\mathcal{S})}\varphi(f)(D)$. By Lemma 4.2, $\varphi(f)$ is continuous. □

Lemma 4.12 *Let $(X,\tau),(Y,\sigma)$ and (Z,α) be MD-spaces, $(x,y) \in X \times Y$. If $g : X \to S_{md}(Y,Z)$ is continuous, then $\psi(g) : X \times_{md} Y \to Z$ which satisfies $\psi(g)(x,y) = g(x)(y)$ is continuous.*

Proof. Firstly, we show that the evaluation map $e : S_{md}(Y,Z) \times_{md} Y \to Z$ which sends (f,y) to $f(y)$ is continuous. Clearly, e is order preserving. Let $D \subseteq S_{md}(Y,Z) \times_{md} Y$ be a directed subset and $(f,y) \in cl_{md(md(\mathcal{S})\times\sigma)} D = cl_{md(\mathcal{S})\times\sigma} D$. Then $f \in cl_{md(\mathcal{S})} p_1(D) = cl_{\mathcal{S}} p_1(D)$ and $y \in cl_\sigma p_2(D)$, where $p_1 : S_{md}(Y,Z) \times_{md} Y \to S_{md}(Y,Z)$ and $p_2 : S_{md}(Y,Z) \times_{md} Y \to Y$ are projection maps. Let $U \in \alpha$ with $e(f,y) = f(y) \in U$. Then $y \in f^{-1}(U) \in \sigma$. Note that $y \in cl_\sigma p_2(D)$. Thus $f^{-1}(U) \cap p_2(D) \neq \emptyset$, which implies there exists a $(g_1,y_1) \in D$ such that $f(y_1) \in U$. Since $f \in cl_{\mathcal{S}} p_1(D)$ and $\mathcal{S}$ is pointwise convergence topology, by Lemma 4.10, $f(y_1) \in cl_\alpha p_1(D)(y_1)$. Thus $U \cap p_1(D)(y_1) \neq \emptyset$, that is, there exists a $(g_2,y_2) \in D$ such that $g_2(y_1) \in U$. Since D is directed, it follows that there is a $(g_0,y_0) \in D$ such that $(g_1,y_1),(g_2,y_2) \leq (g_0,y_0)$, that is, $g_1,g_2 \leq g_0$ and $y_1,y_2 \leq y_0$, we know that $g_i (i = 0,1,2)$ are order preserving and U is an upper set, which implies that $g_2(y_1) \leq g_2(y_0) \leq g_0(y_0)$ and $g_0(y_0) \in U$. Hence $U \cap e(D) \neq \emptyset$, it follows that $e(f,y) \in cl_\alpha e(D)$. Thus $e(cl_{md(md(\mathcal{S})\times\sigma)} D) \subseteq cl_\alpha e(D)$. By Lemma 4.2, the evaluation map e is continuous.

Suppose $g : X \to S_{md}(Y, Z)$ is continuous. Since $\psi(g)$ equals the composite $g \times i_Y : X \times Y \to S_{md}(Y, Z) \times Y$ and $e : S_{md}(Y, Z) \times_{md} Y \to Z$, where i_Y is the identity map of Y, it follows that $\psi(g)$ is continuous. □

Theorem 4.13 *Let $(X, \tau), (Y, \sigma)$ and (Z, α) be MD-spaces. Then $S_{md}(X \times_{md} Y, Z)$ is isomorphism to $S_{md}(X, S_{md}(Y, Z))$.*

Proof. For any $(x, y) \in X \times Y$, $f \in Z^{X \times_{md} Y}$ and $g \in S_{md}(Y, Z)^X$, define $\varphi : S_{md}(X \times_{md} Y, Z) \to S_{md}(X, S_{md}(Y, Z))$ by $\varphi(f)(x)(y) = f(x, y)$ and $\psi : S_{md}(X, S_{md}(Y, Z)) \to S_{md}(X \times_{md} Y, Z)$ by $\psi(g)(x, y) = g(x)(y)$. By Lemma 4.11 and Lemma 4.12, φ and ψ are defined well. It is easy to see that $\varphi \circ \psi(g) = g$, $\psi \circ \varphi(f) = f$ and φ, ψ are order preserving.

Now we only have to show that φ and ψ are continuous. For any directed subset $D = \{f_i : i \in I\} \subseteq S_{md}(X \times_{md} Y, Z)$. Denote the pointwise convergence topology on $Z^{X \times_{md} Y}$ and $(S_{md}(Y, Z))^X$ by $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively. Let $f \in cl_{md(\mathcal{S}_1)} D = cl_{\mathcal{S}_1} D$, by Lemma 4.10, $f(x, y) \in cl_\alpha\{f_i(x, y) : i \in I\}$ for any $(x, y) \in X \times Y$. Then $\varphi(f)(x)(y) = f(x, y) \in cl_\alpha\{f_i(x, y) : i \in I\} = cl_\alpha\{\varphi(f_i)(x)(y) : i \in I\}$. Thus $\varphi(f) \in cl_{md(\mathcal{S}_2)}\{\varphi(f_i) : i \in I\}$, hence $\varphi(cl_{md(\mathcal{S}_1)} D) \subseteq cl_{md(\mathcal{S}_2)} \varphi(D)$. By Lemma 4.2, φ is continuous. Similarly, we can prove that ψ is continuous. All these show that $S_{md}(X \times_{md} Y, Z)$ is isomorphism to $S_{md}(X, S_{md}(Y, Z))$. □

Now we can immediately obtain the following theorem:

Theorem 4.14 *The category* $\mathbf{Top_{md}}$ *is cartesian closed.*

References

[1] Awodey, S., *Category theory*, Oxford University Press, 2010.

[2] Engelking, R., *General Topology*, Polish Scientific Publishers, Warszawa, 1977.

[3] Erné, M., *Infinite distributive laws versus local connectedness and compactness properties*. Topology and its Applications, 156: 2054–2069, 2009.

[4] Erné, M., *Scott convergence and Scott topology on partially ordered sets II*. In: B. Banaschewski and R.-E. Hoffman, eds., Continuous Lattices, Bremen 1979, Lecture Notes in Math. vol.871, Springer-Verlag, Berlin-Heidelberg-New York, 61–96, 1981.

[5] Gierz, G., Hofmann, K., Keimel, K., Lawson, J. D., Mislove, M., Scott, D., *Continuous Lattices and Domains*, Encyclopedia of Mathematics and its Applications 93. Cambridge University Press, 2003.

[6] Gierz, G., Lawson, J. D., *Generalized continuous and hypercontinuous lattices*. Rocky Mountain J. Math., 11: 271-296, 1981.

[7] Isbell, J., *Completion of a construction of Johnstone*. Proceedings of the American Mathematical Society, 85(3): 333-334, 1982.

[8] Munkres, J. R., *Topology*, Prentice Hall, 2000.

[9] Priestley, H. A., *Intrinsic spectral topologies*. Annals of the New York Academy of Sciences, 728(1): 78-95, 1994.

[10] Xu, X. Q., *Relational representations of complete lattices and their applications*, PhD thesis, Sichuan University, 2004.

[11] Yang, J. B., Luo, M. K., *Priestley spaces,quasi-hyperalgebraic lattices and Smyth powerdomains*. Acta Mathematics Sinica(English Series), 22: 951-958, 2006.

[12] Yang, J. B., *Quasi-hyperalgebraic Domains and quasi-hyperalgebraic lattices*, PhD thesis, Sichuan University, 2006.

[13] Zhang, W. F., Xu, X. Q., s_2-*Quasiconinuous posets*. Theoretical Computer Science, 574: 78-85, 2015.