Available online at www.sciencedirect.com
ScienceDirect
Electronic Notes in Theoretical Computer Science 175 (2007) 51–61
Electronic Notes in Theoretical Computer Science
www.elsevier.com/locate/entcs

Adhesive DPO Parallelism for Monic Matches

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Abstract

This paper presents indispensable technical results of a general theory that will allow to systematically derive from a given reduction system a behavioral congruence that respects concurrency. The theory is developed in the setting of adhesive categories and is based on the work by Ehrig and König on borrowed contexts; the latter are an instance of relative pushouts, which have been proposed by Leifer and Milner. In order to lift the concurrency theory of DPO rewriting to borrowed contexts we will study the special case of DPO rewriting with monic matches in adhesive categories: more specifically we provide a generalized Butterfly Lemma together with a Local Church Rosser and Parallelism theorem.

Keywords: Adhesive category, behavioral congruence, borrowed context, DPO

1 Introduction and Motivation

Process calculi are a well established tool to describe interactive systems. The progression of a process, if it is interpreted as a *closed system*, is described by a *reduction system* (RS); moreover each process is a state of a *labeled transition system* (LTS), which describes how the process may interact with its environment: in this case the process is thought of as an *open system*. Also the the double pushout approach (DPO) can be used to model closed and open systems: a reduction step corresponds to a DPO *rewrite* while interaction with the environment is described as a transition that is labeled by a *borrowed context*, which is a part of the environment. One of the advantages of the DPO approach is that one can distinguish between concurrent and necessarily interleaved events of a closed system. Now the main motivation of this paper to lift this advantage to the setting of open systems, i.e. to provide LTSs with labels that describe *concurrent interaction* with the environment.

One of the first approaches to derive a LTS from a given RS, was presented in [13]. The transitions of the generated LTS are labeled by the "minimal" contexts that allow a reduction (as a consequence all the internal actions of a system correspond

[1] This work is partially supported by the SegraVis research training network, the IST 2004-16004 SEnSOria, and the MIUR PRIN 2005015824 ART.

doi:10.1016/j.entcs.2007.04.016

to transitions which have the "empty" environment as label). For example in CCS, which has the reduction rule $\bar{x}.P \mid x.Q \to P \mid Q$, the process $\bar{a}.0$ cannot perform any reduction by itself but can only be reduced in a context of the form $[-] \mid a.P$: hence the derived LTS contains a transition $\bar{a}.0 \xrightarrow{[-]|a.P} P$. The main property of the derived LTS is that its associated bisimulation relation is a congruence, i.e. it relates two processes that exhibit the same behavior w.r.t. to every environment. However to check bisimilarity one does *not* need to check *all* contexts but it is enough to consider the "minimal" ones, which are given as the labels.

Leifer and Milner's work [13] has been extended to an enriched category context by Sassone and Sobocinski in [14], while Ehrig and König developed a similar framework for DPO rewriting (on **Graphs**) in [6], called *borrowed context rewriting* (DPOBC). Finally [15] introduces an encompassing theory (following the bicategorical approach of DPO rewriting of [7,8]). The results of this last most general work apply to every adhesive category. This means that given a system specification by an adhesive rewriting system [12] one can generate a LTS with an associated bisimulation congruence.

Whereas RSs and LTSs are (families of) relations between *states* of a system, the concurrency theory for DPO rewriting is concerned with relations between the *transitions*, i.e. the rewrites (see e.g. [10,1]). For example two consecutive applications of the rule $\circ\ \circ \leftarrow \circ\ \circ \rightarrow$  may result in the graph . The two rewrites are sequential independent, i.e. one can swap them without any further complications; moreover one can even apply them "at the same time", that is *concurrently*: the concurrent application corresponds to a single application of the *parallel rule* $\circ\ \circ \leftarrow \circ\ \circ \rightarrow$ . In contrast, consider a coffee vending machine: it can sell a coffee and then a latte macchiato or do this in the reversed order but not at the same time (unless you operate a buggy machine which produces a puddle of cappuccino as the result of the concurrent execution). The latter example explains the difference between the two CCS processes $\bar{c} \mid \bar{m}$ and $\bar{c}.\bar{m} + \bar{m}.\bar{c}$, which nevertheless are equivalent according to the standard bisimulation of CCS. Also the generated LTSs discussed before do *not* take into account these finer differences in behavior.

This paper is aimed towards the generation of bisimulation congruences that do respect concurrency. Here we report about the first steps of research in this direction. The main idea is to saturate a given set of productions with *all parallel productions* and then apply the borrowed context method to generate a bisimulation congruence that respects concurrency. More specifically, given an initial set of rules P, we will construct a saturation $\bar{P}$ that will be used to synthesize a LTS using the results of [15]; the set $\bar{P}$ contains for every (finite) subset $P' \subseteq P$ and every way in which the members of P' might be applied concurrently the corresponding parallel production.

One central issue is the apropriate notion of parallel rule. Parallel rules are usually defined as coproducts in DPO; but this construction cannot be used in DPOBC since there, matching morphisms are required to be monic. The required notion of parallel rule is given in [10], which studies DPO rewriting with monic matches ($\text{DPO}^{\text{a/i}}$), for the case of **Graphs**. However this work cannot be directly adapted to the adhesive setting since the proofs of its results depend on coproducts, which adhesive cateogries do not have in general.

2 Local Church Rosser and Parallelism for DPO$^{\mathrm{a/i}}$

We first recall the essential definitions of DPO rewriting in adhesive categories as presented in [12], to which we refer the reader for more details. For the remainder of this section we fix an adhesive category $\mathbb{C}$, to which all mentioned objects and arrows belong.

Definition 2.1 (Productions and rewriting) *A* production p *is a span of arrows* $p = L \overset{l}{\leftarrowtail} K \overset{r}{\to} R$ *with monic* l*. Given an arrow* $f\colon L \to C$ *we say that* p rewrites C to D at match f*, and we write* $C \overset{\langle f,p\rangle}{\Longrightarrow} D$ *if there exists a diagram containing two pushouts as shown on the right.*

$$\begin{tikzcd} L \arrow[d, "f"'] & K \arrow[l, tail, "l"'] \arrow[r, "r"] \arrow[d, "g"] & R \arrow[d, "h"] \\ C & E \arrow[l, tail, "v"] \arrow[r, tail, "w"'] & D \end{tikzcd}$$

In the theory of borrowed contexts in adhesive categories, one only encounters the special case where the matching morphism f is monic, and hence from now on we will assume all matches to be monic. This fragment of DPO rewriting in the category of **Graphs** has been studied in [10] by the name DPO$^{\mathrm{a/i}}$. Their results involve the strong versions of sequential and parallel independence.

Definition 2.2 ((Strong) parallel and sequential independence) *Let be given productions* $p_i = L_i \overset{l_i}{\leftarrowtail} K_i \overset{r_i}{\to} R_i$ *for* $i \in \{1,2\}$ *and let there be given the rewrites* $D_1 \overset{\langle f_1,p_1\rangle}{\Longleftarrow} C \overset{\langle f_2,p_2\rangle}{\Longrightarrow} D_2$ *(*$C \overset{\langle f_1,p_1\rangle}{\Longrightarrow} D_1 \overset{\langle f_2',p_2\rangle}{\Longrightarrow} D$*). They are* parallel (sequential) independent*, if there exist morphisms* s *and* t *(*s' *and* t'*) such that they commute in the composed diagram of the rewrites below.*

$$\begin{tikzcd} R_1 \arrow[d] & K_1 \arrow[l] \arrow[r, tail] \arrow[d] & L_1 \arrow[dr, tail, "f_1"'] \arrow[rrd, bend left=60, "s"] & L_2 \arrow[d, tail, "f_2"] \arrow[dll, bend left=60, "t"] & K_2 \arrow[l, tail] \arrow[r, tail] \arrow[d] & R_2 \arrow[d] \\ D_1 & E_1 \arrow[l, tail, "w_1"] \arrow[rr, tail] & & C & E_2 \arrow[l, tail] \arrow[r, tail, "w_2"'] & D_2 \end{tikzcd}
\qquad
\left(\begin{tikzcd} L_1 \arrow[d, tail, "f_1"] & K_1 \arrow[l] \arrow[r, tail] \arrow[d] & R_1 \arrow[dr, tail] \arrow[rrd, bend left=60, "s'"] & L_2 \arrow[d, tail, "f_2'"] \arrow[dll, bend left=60, "t"] & K_2 \arrow[l, tail] \arrow[r, tail] \arrow[d] & R_2 \arrow[d] \\ C & E_1 \arrow[l, tail, "v_1"] \arrow[rr, tail] & & D_1 & E_3 \arrow[l, tail] \arrow[r, tail, "w_3"'] & D \end{tikzcd}\right)$$

They are strongly *parallel (sequential) independent if* $w_1 \circ t$ *and* $w_2 \circ s$ *(*$v_1 \circ t'$ *and* $w_3 \circ s'$*) are monic.*

In [10] the Parallelism theorem for DPO$^{\mathrm{a/i}}$ for the case of **Graphs** has been proven. However the proof cannot be lifted directly to adhesive categories since it depends on the existence of coproducts. Moreover the Parallelism theorem for adhesive categories with coproducts presented in [12], does not transfer to DPO$^{\mathrm{a/i}}$.

Technical contribution

The main idea is to replace coproducts, which are just pushouts from the empty graph in **Graphs**, by pushouts. This will allow us to make the DPO$^{\mathrm{a/i}}$ theory of [10] available for adhesive categories. How coproducts can be replaced by pushouts will be explained in terms of the next definition.

Definition 2.3 *Let the following squares be pushouts:*

$$\begin{tikzcd}[column sep=small] & Q \arrow[dl, "x_2"'] \arrow[dr, "x_1"] & \\ A_2 \arrow[dr, "i_2"'] & \ulcorner & A_1 \arrow[dl, "i_1"] \\ & A & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & Q \arrow[dl, "y_2"'] \arrow[dr, "y_1"] & \\ B_2 \arrow[dr, "j_2"'] & \ulcorner & B_1 \arrow[dl, "j_1"] \\ & B & \end{tikzcd} .$$

Then we will denote A by $A_1 +_Q A_2$ and B by $B_1 +_Q B_2$.

Let f_1 and f_2 be two morphisms satisfying $y_1 = f_1 \circ x_1$ and $y_2 = f_2 \circ x_2$, and let $f : A \to B$ be the unique morphism which satisfies $f \circ i_1 = j_1 \circ f_1$ and $f \circ i_2 = j_2 \circ f_2$; then f will be denoted by $f_1 +_Q f_2$.

For the initial object 0 the expression $A_1 +_0 A_2$ is equivalent to $A_1 + A_2$, and similarly for $f_1 +_0 f_2$ and $f_1 + f_2$. This "generalized coproduct" is used to describe the parallel composition of two rules that rewrite an object in a parallel independent way: a combined rule is constructed that allows to apply the two rules "at the same time", i.e. concurrently. More specifically the two rules need to be glued together at the intersection of their read-only parts.

Definition 2.4 (Parallel productions) *Let $p_1 = L_1 \xleftarrow{l_1} K_1 \xrightarrow{r_1} R_1$ and $p_2 = L_2 \xleftarrow{l_2} K_2 \xrightarrow{r_2} R_2$ be productions, and let $K_1 \xleftarrow{x_1} Q \xrightarrow{x_2} K_2$ be a span of morphisms. If the pushouts for all the pairs $(l_1 \circ x_1, l_2 \circ x_2), (x_1, x_2)$ and $(r_1 \circ x_1, r_2 \circ x_2)$ exist, then the* parallel composition of p_1 and p_2 over Q *is*

$$p_1 +_Q p_2 = L_1 +_Q L_2 \xleftarrow{l_1 +_Q l_2} K_1 +_Q K_2 \xrightarrow{r_1 +_Q r_2} R_1 +_Q R_2.$$

The production $p_1 +_Q p_2$ is called proper *if all the morphisms of the three involved pushout diagrams are monic.*[2]

Now we are ready to formulate the main theorem, which might be of interest whenever one uses DPO$^{\mathrm{a/i}}$ rewriting in adhesive categories. The proof relies on an adapted version of the Butterfly Lemma of [11] for "generalized" coproducts (see Appendix[3]).

Theorem 2.5 (Parallelism and Local Church Rosser in DPO$^{\mathrm{a/i}}$) *Let $p_1 = L_1 \xleftarrow{l_1} K_1 \xrightarrow{r_1} R_1$ and $p_2 = L_2 \xleftarrow{l_2} K_2 \xrightarrow{r_2} R_2$ be productions,*[4] *and let $L_1 \xrightarrowtail{f_1} C$ and $L_2 \xrightarrowtail{f_2} C$ be morphisms. Then the following are equivalent.*

(i) *There are strongly parallel independent rewrites $D_1 \xLeftarrow{\langle f_1, p_1 \rangle} C \xRightarrow{\langle f_2, p_2 \rangle} D_2$.*

(ii) *There are strongly sequential independent rewrites $C \xRightarrow{\langle f_1, p_1 \rangle} D_1 \xRightarrow{\langle f_2', p_2 \rangle} D$.*

(iii) *There are strongly sequential independent rewrites $C \xRightarrow{\langle f_2, p_2 \rangle} D_2 \xRightarrow{\langle f_1', p_1 \rangle} D$.*

(iv) *There is a rewrite $C \xRightarrow{\langle f_1 +_Q f_2, p_1 +_Q p_2 \rangle} D$ with a proper parallel production $p_1 +_Q$*

2 This construction is equivalent to the one given in Definition 9.5 of [10].

3 As [11] is a rather inaccessible source, we chose to give the whole proof.

4 These are not required to be linear, as is assumed in [12].

p_2 *where* Q *is constructed as the pullback* $Q \begin{smallmatrix} \nearrow K_1 \rightarrow C \\ \rightarrow K_2 \nearrow \end{smallmatrix}$, *i.e.* $Q = K_1 \cap K_2$.

3 Conclusion and work in progress

Motivated by extending the existing concurrency theory of DPO rewriting to the interactive setting of DPO with borrowed contexts (DPOBC), we have defined the required kind of parallel productions and proved the Local Church Rosser and Parallelism theorem for DPO$^{a/i}$ in adhesive categories. Besides filling this gap in the literature, these theorems might prove useful for future research concerned with DPO$^{a/i}$ rewriting in adhesive categories. This is not unlikely since the DPO$^{a/i}$ approach is more intuitive and more expressive than DPO as shown in [10]. In fact, DPOBC is not the only application where the requirement of monic matches arises naturally: consider e.g. the work on processes of adhesive rewriting systems [2] and encondig of nominal calculi [9].

We will use the presented results for the generation of a concurrency respecting bisimulation congruence from a given set of rules. More specifically the construction of parallel rules will be used to generate a closure of all given productions as follows: given a set of productions P we construct the closure $\bar{P}$ via the two rules

$$\frac{p \in P}{p \in \bar{P}} \qquad \frac{p, p' \in \bar{P} \ \&\ K_p \xleftarrow{i} Q \xrightarrow{j} K_{p'}}{p +_Q p' \in \bar{P}}$$

where K_p denotes the interface of a rule p, i.e. given a rule $p = X \leftarrow Y \rightarrow Z$ we write K_p for Y.

Usually in borrowed context rewriting and in the more general setting of the theory of reactive systems, the LTS is derived using the set of rules P, while here we propose to use $\bar{P}$. Reconsider the CCS example from the introduction where we hinted at the difference between the two processes $\bar{c}|\bar{m}$ and $\bar{c}.\bar{m}+\bar{m}.\bar{c}$. This now can be made formal, since the LTS generated from $\bar{P}$ using the borrowed context technique of [15] allows the former to communicate with the environment concurrently at the channels m and c (this corresponds to the transition $\bar{c} \mid \bar{m} \xrightarrow{[-]|c.P|m.Q} P \mid Q$) while the latter cannot (in signs $\bar{c}.\bar{m} + \bar{m}.\bar{c} \stackrel{[-]|c.P|m.Q}{\not\longrightarrow} P \mid Q$).

There are several other proposals of bisimulations that respect concurrency [4,3,5] however they are based on the notion of causality. Our proposal conceptually differs from these since it does not allow the environment to observe causality but just the possible ways in which a system could interact with the environment concurrently. In other words, we consider systems as black boxes, while intuitively the existing equivalences seem to open the black box by observing causal dependencies. Reconsidering our CCS example, our proposed bisimilarity distinguishes $\bar{c} \mid \bar{m}$ and $\bar{c}.\bar{m} + \bar{m}.\bar{c}$ because an external observer can parallely communicate with the former but not with the latter, while the bisimilarities of the cited works distinguish the processes because the former can perform its transitions independently and the latter cannot. The subtle interplay between causality and concurrency especially in the context of borrowed context rewriting is the main interest of ongoing research.

Acknowledgement

We would like to thank Fabio Gadducci, Ugo Montanari and the anonymous referees for their comments on the paper.

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A The extended Butterfly Lemma

Lemma A.1 (General Butterfly Lemma)

$$
\begin{tikzcd}
 & & Q \arrow[dll, "y_1"'] \arrow[dl, "x_1"'] \arrow[dr, "x_2"] \arrow[drr, "y_2"] & & \\
B_1 \arrow[ddrr, "j_1"'] & A_1 \arrow[l, "f_1"'] \arrow[dr, "i_1"'] & & A_2 \arrow[r, "f_2"] \arrow[dl, "i_2"] & B_2 \arrow[ddll, "j_2"] \\
 & & A \arrow[d, dotted, bend right, "f"'] & & \\
 & & B & &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & Q \arrow[dl, "x_1"'] \arrow[dr, "x_2"] & \\
A_1 \arrow[dr, "i_1"'] \arrow[ddr, bend right, "a_1"'] & & A_2 \arrow[dl, "i_2"] \arrow[ddl, bend left, "a_2"] \\
 & A \arrow[d, dotted, "a"] & \\
 & C &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & Q \arrow[dl, "y_1"'] \arrow[dr, "y_2"] & \\
B_1 \arrow[dr, "j_1"'] \arrow[ddr, bend right, "e_1"'] & & B_2 \arrow[dl, "j_2"] \arrow[ddl, bend left, "e_2"] \\
 & B \arrow[d, dotted, "e"] & \\
 & E &
\end{tikzcd}
\tag{A.1}
$$

Let the above be commuting diagrams where all interior squares and the boundary of the left one are pushouts, and $f\colon A \to B, a\colon A \to C$ *and* $e\colon B \to E$ *are the unique mediating morphisms, such that*

$$j_1 \circ f_1 = f \circ i_1 \qquad j_2 \circ f_2 = f \circ i_2 \tag{A.2}$$

$$a_1 = a \circ i_1 \qquad a_2 = a \circ i_2 \tag{A.3}$$

$$e_1 = e \circ j_1 \qquad e_2 = e \circ j_2 \tag{A.4}$$

Finally let $\mathbb{C}$ *have pushouts of the diagrams* $B_1 \xleftarrow{f_1} A_1 \xrightarrow{a_1} C$ *and* $B_2 \xleftarrow{f_2} A_2 \xrightarrow{a_2} C$. *Then for any morphism* $c\colon C \to E$ *the following are equivalent.*

(i) *There exists a commuting diagram*

$$
\begin{tikzcd}
 & A_1 \arrow[dl, "f_1"'] \arrow[dr, "a_1"] & & A_2 \arrow[dl, "a_2"'] \arrow[dr, "f_2"] & \\
B_1 \arrow[dr, "b_1"'] \arrow[ddrr, bend right, "e_1"'] \arrow[rr, phantom, "(*)"] & & C \arrow[dl, "c_1"] \arrow[dr, "c_2"'] \arrow[dd, "c"] \arrow[rr, phantom, "(\dagger)"] & & B_2 \arrow[dl, "b_2"] \arrow[ddll, bend left, "e_2"] \\
 & D_1 \arrow[dr, "d_1"'] & (\ddagger) & D_2 \arrow[dl, "d_2"] & \\
 & & E & &
\end{tikzcd}
$$

where the squares $(*), (\dagger)$ *and* $(\ddagger)$ *are pushouts.*

(ii) *The diagram*

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "(\S)"] & B \arrow[d, "e"] \\
C \arrow[r, "c"'] & E
\end{tikzcd}
$$

is a pushout.

Proof. First we show that the implication (i) $\Rightarrow$ (ii) holds. For this assemble the given diagrams into one (see Diagram (A.5)).

Next we need to check that $e \circ f = c \circ a$; for this we will use that i_1 and i_2 are jointly epic, i.e. it is enough to show that both $e \circ f \circ i_1 = c \circ a \circ i_1$ and $e \circ f \circ i_2 = c \circ a \circ i_2$ hold; these two equations are derivable using Item (i), Equation (A.4), Diagram (A.5), Square $(*)$, and Equation (A.3). In other words Square (§) of Item *(ii)* commutes; it remains to show that it satisfies the universal property of pushouts.

$$\begin{tikzcd}
 & & Q \arrow[dl, "x_1"'] \arrow[dr, "x_2"] \arrow[dll, bend right, "y_1"'] \arrow[drr, bend left, "y_2"] \arrow[ddddd, bend left=90, "f"] & & \\
 & A_1 \arrow[r, "i_1"] \arrow[dl, "f_1"'] \arrow[dr, "a_1"] \arrow[dd, phantom, "(*)"] & A \arrow[d, "a"] & A_2 \arrow[l, "i_2"'] \arrow[dl, "a_2"'] \arrow[dr, "f_2"] \arrow[dd, phantom, "(\dagger)"] & \\
B_1 \arrow[dr, "b_1"'] \arrow[ddrr, bend right, "e_1"'] \arrow[dddrr, bend right, "j_1"'] & & C \arrow[dl, "c_1"] \arrow[dr, "c_2"'] \arrow[dd, bend right, "c"'] \arrow[dd, phantom, "(\ddagger)"] & & B_2 \arrow[dl, "b_2"] \arrow[ddll, bend left, "e_2"] \arrow[dddll, bend left, "j_2"] \\
 & D_1 \arrow[dr, "d_1"'] & & D_2 \arrow[dl, "d_2"] & \\
 & & E & & \\
 & & B \arrow[u, dotted, "e"'] & &
\end{tikzcd} \tag{A.5}$$

Hence assume there is a commuting diagram as

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "(\S)"] & B \arrow[d, "e"] \arrow[dr, bend left, "k"] & \\
C \arrow[r, "c"] \arrow[rr, bend right, "h"'] & E & X
\end{tikzcd} . \tag{A.6}$$

Now we derive $h \circ a_1 = k \circ j_1 \circ f_1$ and $h \circ a_2 = k \circ j_2 \circ f_2$ using Equation (A.3), Diagram (A.6) and Diagram (A.1). Because the Squares $(*)$ and $(\dagger)$ are pushouts there are uniquely determined morphisms $z_1 \colon D_1 \to X$ and $z_2 \colon D_2 \to X$ which satisfy

$$z_1 \circ c_1 = h \qquad\qquad z_2 \circ c_2 = h \tag{A.7}$$

$$z_1 \circ b_1 = k \circ j_1 \qquad\qquad z_2 \circ b_2 = k \circ j_2. \tag{A.8}$$

Using Equation (A.7) and the fact that Square $(\ddagger)$ is a pushout we derive that there is exactly one morphism $z \colon E \to X$ such that

$$z \circ d_1 = z_1 \qquad \text{and} \qquad z \circ d_2 = z_2 \tag{A.9}$$

hold. This z is a candidate for the mediating morphism we are are looking for (see Diagram (A.6)).

Further we derive $z \circ e \circ j_1 = k \circ j_1$ and $z \circ e \circ j_2 = k \circ j_2$ using Equation (A.4), Diagram (A.1), Equation (A.9) and Equation (A.8). However j_1 and j_2 are jointly epic, which yields $z \circ e = k$. Moreover one can show $z \circ c = h$ using Square $(\ddagger)$, Equation (A.9) and Equation (A.7), i.e. we have the equalities

$$z \circ e = k \qquad \text{and} \qquad z \circ c = h. \tag{A.10}$$

It remains to show that z is the unique mediating morphism, i.e. that every other morphism $\zeta\colon E \to X$ satisfying

$$\zeta \circ e = k \qquad \text{and} \qquad \zeta \circ c = h \tag{A.11}$$

is equal to z. So assume that some morphism ζ satisfying Equation (A.11) is given. We put

$$\zeta_1 := \zeta \circ d_1 \qquad \text{and} \qquad \zeta_2 := \zeta \circ d_2. \tag{A.12}$$

Now we derive

$$\zeta_1 \circ b_1 = z_1 \circ b_1 \qquad \text{and} \qquad \zeta_2 \circ b_2 = z_2 \circ b_2 \tag{A.13}$$

using Equation (A.12), Item (i), Equation (A.4), Equation (A.11), Equation (A.10), Item (i), Equation (A.12) and Equation (A.9).

Further

$$\zeta_1 \circ c_1 = z_1 \circ c_1 \qquad \text{and} \qquad \zeta_2 \circ c_2 = z_2 \circ c_2 \tag{A.14}$$

follow using Equation (A.12), Square (‡), Equation (A.11), Equation (A.10), Square (‡) and Equation (A.9).

Using Equation (A.13) and Equation (A.14) we may conclude that $\zeta_1 = z_1$ and $\zeta_2 = z_2$ since the pairs (b_1, c_1) and (b_2, c_2) are jointly epic, and because the squares $(*)$ and $(\dagger)$ are pushouts. However using Equation (A.12) and Equation (A.9) we can also derive $\zeta \circ d_1 = z \circ d_1$ and $\zeta \circ d_2 = z \circ d_2$ from which $z = \zeta$ follows since d_1 and d_2 are jointly epic.

Second we show the implication (ii) $\Rightarrow$ (i). By assumption we have the following commuting diagrams.

$$\begin{tikzcd} A_1 \arrow[r, "i_1"] \arrow[dr, "a_1"'] & A \arrow[d, "a"] & A_2 \arrow[l, "i_2"'] \arrow[dl, "a_2"] \\ & C & \end{tikzcd} \tag{A.15a}$$

$$\begin{tikzcd} B_1 \arrow[r, "j_1"] \arrow[dr, "e_1"'] & B \arrow[d, "e"] & B_2 \arrow[l, "j_2"'] \arrow[dl, "e_2"] \\ & E & \end{tikzcd} \tag{A.15b}$$

Further we construct the pushouts for the pairs (f_1, a_1) and (f_2, a_2), and assemble them into the following diagram

$$\begin{tikzcd}
& & Q \arrow[dl, "x_1"'] \arrow[dr, "x_2"] & & \\
& A_1 \arrow[r, "i_1"'] \arrow[dl, "f_1"'] \arrow[dr, "a_1"'] & A \arrow[d, "a"] & A_2 \arrow[l, "i_2"] \arrow[dl, "a_2"] \arrow[dr, "f_2"] & \\
B_1 \arrow[dr, "b_1"'] & (*) & C \arrow[dl, "c_1"] \arrow[dr, "c_2"'] & (\dagger) & B_2 \arrow[dl, "b_2"] \\
& D_1 & & D_2 &
\end{tikzcd}$$

where the upper triangle commutes by assumption. Now we derive $c \circ a_1 = e_1 \circ f_1$ using Diagram (A.15a), Square (§), Diagram (A.1) and Diagram (A.15b); the equation $c \circ a_2 = e_2 \circ f_2$ follows similarly. Hence there are unique morphisms $d_1\colon D_1 \to E$ and $d_2\colon D_2 \to E$ such that the following hold.

$$d_1 \circ b_1 = e_1 \quad \text{and} \quad d_1 \circ c_1 = c \tag{A.16a}$$

$$d_2 \circ b_2 = e_2 \quad \text{and} \quad d_2 \circ c_2 = c \tag{A.16b}$$

It remains to show that the square $C \xrightarrow{c_1} D_1 \xrightarrow{d_1} E$, $C \xrightarrow{c_2} D_2 \xrightarrow{d_2} E$ is a pushout.

For this let there be two morphisms $h_1\colon D_1 \to X$ and $h_2\colon D_2 \to X$ such that

$$h_1 \circ c_1 = h_2 \circ c_2. \tag{A.17}$$

Hence after defining $k_1 := h_1 \circ b_1$ and $k_2 := h_2 \circ b_2$, and because Diagram (A.1) commutes we arrive at the following commuting Diagram (A.18).

$$\begin{tikzcd}
 & & Q \arrow[dl, "x_1"'] \arrow[dr, "x_2"] \arrow[dll, bend right, "y_1"'] \arrow[drr, bend left, "y_2"] & & \\
 & A_1 \arrow[r, "i_1"'] \arrow[dl, "f_1"'] \arrow[dr, "a_1"] & A \arrow[d, "a"] & A_2 \arrow[l, "i_2"] \arrow[dr, "f_2"] \arrow[dl, "a_2"'] & \\
B_1 \arrow[dr, "b_1"'] \arrow[dddrr, bend right, "k_1"'] \arrow[rr, phantom, "(*)"] & & C \arrow[dl, "c_1"'] \arrow[dr, "c_2"] \arrow[dd, bend right, "c"'] \arrow[rr, phantom, "(\dagger)"] & & B_2 \arrow[dl, "b_2"] \arrow[dddll, bend left, "k_2"] \\
 & D_1 \arrow[dr, "d_1"] \arrow[ddr, bend right, "h_1"'] \arrow[rr, phantom, "(\ddagger)"] & & D_2 \arrow[dl, "d_2"'] \arrow[ddl, bend left, "h_2"] & \\
 & & E & & \\
 & & X & &
\end{tikzcd} \tag{A.18}$$

Using its commutativity and Equation (A.17) we derive $k_1 \circ y_1 = k_2 \circ y_2$; therefore there exists a unique morphism $k\colon B \to X$ such that

$$k_1 = k \circ j_1 \qquad \text{and} \qquad k_2 = k \circ j_2. \tag{A.19}$$

Moreover $k_1 \circ y_1 = k_2 \circ y_2$ implies $k_1 \circ f_1 \circ x_1 = k_2 \circ f_2 \circ x_2$ by “expansion” of y_1 and y_2, which provides us with a uniquely determined morphism $u\colon A \to X$ such that

$$k_1 \circ f_1 = u \circ i_1 \qquad \text{and} \qquad k_2 \circ f_2 = u \circ i_2. \tag{A.20}$$

Now inspecting

$$\begin{tikzcd}
 & & Q \arrow[dll, "y_1"'] \arrow[dl, "x_1"] \arrow[dr, "x_2"'] \arrow[drr, "y_2"] & & \\
B_1 \arrow[ddrr, "j_1"] \arrow[dddrr, bend right, "k_1"'] & A_1 \arrow[l, "f_1"'] \arrow[dr, "i_1"] & & A_2 \arrow[r, "f_2"] \arrow[dl, "i_2"'] & B_2 \arrow[ddll, "j_2"'] \arrow[dddll, bend left, "k_2"] \\
 & & A \arrow[d, dotted, bend right, "f"'] & & \\
 & & B \arrow[d, "k"] & & \\
 & & X & &
\end{tikzcd}$$

we see that $k_1 \circ f_1 = k \circ f \circ i_1$ and $k_2 \circ f_2 = k \circ f \circ i_2$; hence

$$k \circ f = u \tag{A.21}$$

follows from the characterization of u in (A.20). However we can also derive $k_1 \circ f_1 = h_1 \circ c_1 \circ a \circ i_1$ and $k_2 \circ f_2 = h_2 \circ c_2 \circ a \circ i_2$ from it, whence $h_1 \circ c_1 \circ a = u \overset{(A.21)}{=} k \circ f$ where we used uniqueness of the mediating morphism u to derive the first equality (see Equation (A.20)). Since $h_1 \circ c_1 = h_2 \circ c_2$ we also get $h_2 \circ c_2 \circ a = k \circ f$.

Now since Square (§) is a pushout we know that there is a unique morphism $z \colon E \to X$ such that

$$z \circ c = h \quad \text{and} \quad z \circ e = k \qquad\qquad \text{where } h = h_1 \circ c_1 = h_2 \circ c_2. \tag{A.22}$$

This morphism z is the candidate for the mediating morphism we are looking for.

To show that it is the unique one we will use that b_1 and c_1 are jointly epic to derive that $z \circ d_1 = h_1$. Now using Square (§) and (A.22) we derive $z \circ d_1 \circ c_1 = h_1 \circ c_1$; further using (A.16a), (A.15b), (A.19) and (A.18) we derive $z \circ d_1 \circ b_1 = h_1 \circ b_1$. This shows that $z \circ d_1 = h_1$ and mutatis mutandis $z \circ d_2 = h_2$; hence z is a mediating morphism from $E \to X$. It remains to show that it is the only one.

Let $\zeta \colon E \to X$ be a morphism such that $\zeta \circ d_1 = h_1$ and $\zeta \circ d_2 = h_2$ hold; we have to show that $\zeta = z$. Using Square (‡), the assumption and Equation (A.22) we derive $\zeta \circ c = z \circ c$. If also $\zeta \circ e = k$ then $z = \zeta$ holds because e and c are jointly epic; thus it remains to show that $\zeta \circ e = k$.

Since j_1 and j_2 are jointly epic it is enough to show that $\zeta \circ e \circ j_1 = k \circ j_1$ and $\zeta \circ e \circ j_2 = k \circ j_2$. However we can derive (see Diagram (A.18), the assumption and Diagram (A.1)) that $k_1 \circ y_1 = \zeta \circ e \circ j_1 \circ y_1$, and mutatis mutandis also $k_2 \circ y_2 = \zeta \circ e \circ j_2 \circ y_2$. This yields that $\zeta \circ e$ is the unique arrow such that $\zeta \circ e \circ j_1 = k_1$ and $\zeta \circ e \circ j_2 = k_2$. Expanding the definition of k_1 and k_2 we arrive at $\zeta \circ e \circ j_1 = k \circ j_1$ and $\zeta \circ e \circ j_2 = k \circ j_2$ and the proof is finished.