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Electronic Notes in Theoretical Computer Science

Electronic Notes in Theoretical Computer Science 221 (2008) 3–15

www.elsevier.com/locate/entcs

Stability for Effective Algebras

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Abstract

We give a general method for showing that all numberings of certain effective algebras are recursively equivalent. The method is based on *computable approximation-limit pairs*. The approximations are elements of a finitely generated subalgebra, and obtained by *computable (non-deterministic) selection*. The results are a continuation of the work by Mal'cev, who, for example, showed that finitely generated semicomputable algebras are computably stable. In particular, we generalise the result that the recursive reals are computably stable, if the limit operator is assumed to be computable, to spaces constructed by inverse limits.

Keywords: Recursively equivalent, computably stable, effective partial algebras.

1 Introduction

It is known, [4,1] and others, that the recursive reals have exactly one numbering, up to recursive equivalence, in the presence of a limit algorithm. Given the setting of effective partial algebras in [7] this result is reinvestigated. Many algebras have partial operations, in particular, this is the case for the reals, where the inverse is a partial operation.

The process of obtaining a sequence of approximations of elements is identified as an important step in the existing proofs. We introduce the concept of approximation-limit pairs, where the approximation and the limit processes are linked. For the reals, an approximation is a rational number that is sufficiently

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doi:10.1016/j.entcs.2008.12.002

close to the real, such a rational number can be found up to any desired accuracy by use of the ordering to search for approximations. In general, approximation is a relation between elements of the algebra, a natural number specifying the level of the approximation, and a finitely generated subalgebra. We would like to computably select approximations from a code of an element, but this is not a well-defined function on the level of the algebra, as the selection often depends on the code. We use a weaker notion of computable (non-deterministic) selection.

The main result is that an algebra with an approximation-limit pair which has computable non-deterministic selection and a computable limit process has exactly one numbering, up to recursive equivalence. This result is exemplified by general spaces constructed from inverse limits. Inverse limits are common in computer science as they are often used to lift algebras from finite data to infinite data, for example, from finite sequences to streams, and from finite processes to non-terminating processes.

1.1 Background on recursion theory

We assume very basic knowledge of recursion theory as can be found in any basic text. Our terminology and notation is standard. In particular, we let $(\varphi_e)_{e\in\mathbb{N}}$ be a standard numbering of the recursive functions, and $(W_e)_{e\in\mathbb{N}}$ the corresponding numbering of the recursively enumerable (r.e.) sets. We use $\downarrow$ and $\uparrow$ to denote convergence and divergence of a computation respectively. The strong (Kleene) equality is denoted by $\simeq$.

Let A be a set. A *numbering* of A is a surjective function $\alpha : \Omega_\alpha \to A$, where $\Omega_\alpha \subseteq \mathbb{N}$. It should be thought of as a coding of A by natural numbers. A *numbered set* is a pair (A, α) such that α is a numbering of A. For convenience, we repeat the definitions of computability for partial functions from [7].

Definition 1.1 Let (A, α) and (B, β) be numbered sets and let $f : A \to B$ be a partial function.

(i) f is (α, β)-*computable* if there exists a partial recursive function $\bar{f} : \mathbb{N} \to \mathbb{N}$ such that for each $n \in \Omega_\alpha$,
 (a) $\bar{f}(n)\downarrow \implies \bar{f}(n) \in \Omega_\beta$; and
 (b) $f(\alpha(n)) \simeq \beta(\bar{f}(n))$.

(ii) f is *weakly* (α, β)-*computable* if there exists a partial recursive function $\bar{f} : \mathbb{N} \to \mathbb{N}$ such that

$$f(\alpha(n))\downarrow \implies f(\alpha(n)) \simeq \beta(\bar{f}(n))\,.$$

In either case, we say that $\bar{f}$ *tracks* f.

Definition 1.2 Let α and β be numberings of a set A.

(i) α *recursively reduces to* β, denoted $\alpha \leq \beta$, if there is a partial recursive function f such that for each $n \in \Omega_\alpha$, $\alpha(n) \simeq \beta(f(n))$. That is, the identity function on A is (α, β)-computable.

(ii) α is *recursively equivalent* to β, denoted $\alpha \sim \beta$, if $\alpha \leq \beta$ and $\beta \leq \alpha$.

2 Computable non-deterministic selection

We say that a relation $R \subseteq A \times B$ is *left-total* if each $x \in A$ is related to some $y \in B$, or equivalently, if the projection to the first coordinate is surjective. For a left-total relation we would like a function $s : A \to B$, called a selection function, such that for each $x \in A$, $x \ R \ s(x)$. When formalising effectivity of such a function there are situation where the selection by necessity is dependent on the representation of x. These considerations lead to the following weaker definition of computable selection.

Definition 2.1 Let (A, α) and (B, β) be numbered sets, and let $R \subseteq A \times B$ be a left-total relation. The relation R has (α, β)-*computable (non-deterministic) selection* if there exists a partial recursive function f s.t. $f(\Omega_\alpha) \subseteq \Omega_\beta$, and for all $n \in \Omega_\alpha$

(i) $f(n)\downarrow$, and

(ii) $(\alpha(n), \beta(f(n))) \in R$.

We say that f tracks the selection for R.

Note that there is no requirement that $m \equiv_\alpha n$ implies $f(m) \equiv_\beta f(n)$, hence f need not induce a well-defined function from A to B. On the level of the sets A and B the selection is non-deterministic. Other authors have chosen to model this behaviour as multi-valued functions [8].

Also note that f may not give codes for all possible second coordinates for a given first coordinate x, even if f is applied to all possible codes of x, i.e., we only have

$$\beta(f(\alpha^{-1}(x))) \subseteq \{y : (x, y) \in R\} .$$

Definition 2.2 Let (A_i, α_i) be a numbered sets for $i = 1, \ldots, n$. Let α be the product numbering $\alpha_1 \times \cdots \times \alpha_n$ of $\prod_{i=1}^n A_i$. An n-ary relation $R \subseteq \prod_{i=1}^n A_i$ is α-*semicomputable* if there is a recursively enumerable n-ary relation W such that

$$(\alpha)^{-1}(R) = \Omega_\alpha \cap W .$$

Proposition 2.3 *Let (A, α) and (B, β) be numbered sets, where Ω_β is r.e. An (α, β)-semicomputable left-total relation $R \subseteq A \times B$ has computable selection.*

3 Effective partial Σ-algebras

In this section we review some of the results in [7]. By an effective partial Σ-algebra we mean a Σ-algebra where the partial operations are effective.

Definition 3.1 Let A be a partial Σ-algebra and let α be a numbering of A.

(i) (A, α) is an *effective* partial Σ-algebra if each k-ary partial operation σ of A is (α^k, α)-computable, where α^k is the product numbering of A^k obtained from α.

(ii) (A, α) is a *weakly effective* partial Σ-algebra if each k-ary partial operation σ of A is weakly (α^k, α)-computable.

In the sequel we assume that there is a computable enumeration of the operation symbols in Σ along with their arities. In particular this is true when Σ is finite. Then the total term algebra $T(\Sigma, V)$, where $V = \{v_0, v_1, v_2, \ldots\}$ is a countable set of variables, has a *standard numbering* which we denote by γ (see [6]).

Let A be a partial Σ-algebra and let $e\colon \mathbb{N} \to A$ be a partial sequence in A. Then we let $\mathrm{TE}_e\colon T(\Sigma, V) \to A$ be the corresponding partial term evaluation map (sending v_i to $e(i)$). Define the partial function $\gamma_e\colon \mathbb{N} \to A$ by

$$\gamma_e(n) \simeq \mathrm{TE}_e(\gamma(n)).$$

We denote $\mathrm{dom}(\gamma_e)$ by Ω_e.

Let $\langle e\rangle$ be the partial Σ-subalgebra generated by the partial sequence e, i.e., $\langle e\rangle$ is the image of γ_e. It is shown in [7] that $(\langle e\rangle, \gamma_e)$ is a weakly effective partial Σ-algebra.

We say that a partial sequence $e\colon \mathbb{N} \to A$, where A is a set numbered by α, is (weakly) α-computable if it is (weakly) (id, α)-computable. Note that every partial sequence e is weakly γ_e-computable, tracked by the partial function $\ulcorner v_i \urcorner \mapsto e(i)$ (where $\ulcorner v_i \urcorner$ is a γ-code for v_i), and e is γ_e-computable if, and only if, $\mathrm{dom}(e)$ is r.e. Normally, but not always, our sequences e will be total and hence γ_e-computable.

Theorem 3.2 ([7]) *Let (A, α) be a weakly effective partial Σ-algebra and let $e\colon \mathbb{N} \to A$ be a partial sequence that is weakly α-computable.*

- *(i) Then $TE_e\colon T(\Sigma, V) \to A$ is weakly (γ, α)-computable and the inclusion $\iota\colon \langle e\rangle \to A$ is (γ_e, α)-computable.*
- *(ii) If (A, α) is effective and e is α-computable then Ω_e is r.e. and $(\langle e\rangle, \gamma_e)$ is effective.*

We say that an effective partial Σ-algebra (A, α) is (*semi-*)*computable* if the equality relation is α-(semi-)computable.

Proposition 3.3 *Let (A, α) be a semicomputable partial Σ-algebra. If e is an α-computable sequence such that $\langle e\rangle = A$ then $\gamma_e \sim \alpha$.*

Corollary 3.4 *If A is a finitely generated partial Σ-algebra then A has at most one semicomputable numbering up to equivalence.*

4 Approximation-limit pair

We denote the set of all sequences on a set A by $A^{\mathbb{N}}$. For a numbered set (A, α) we denote the set of all α-computable sequences in A by $A^{\mathbb{N}}_k$. A natural numbering α^* of $A^{\mathbb{N}}_k$ obtained from α is defined by

$$\alpha^*(n) = \lambda k.\alpha(\varphi_n(k))\,,$$

for all n such that $\alpha \circ \varphi_n$ is total.

We now define what we mean by an approximation-limit pair and by effectivity of such a pair.

Definition 4.1 Let A be a partial Σ-algebra and let $e\colon \mathbb{N} \to A$ be a partial sequence. Assume that $\mathrm{aprx} \subseteq (A \times \mathbb{N}) \times \langle e \rangle$ is a left-total relation, i.e., for each $x \in A$ and $n \in \mathbb{N}$ there exists $y \in B$ satisfying the relation; and that $\lim\colon \langle e \rangle^{\mathbb{N}} \to A$ is a partial function.

(i) An *approximation sequence* for x is a sequence $(y_n)_n$ satisfying $(x, n, y_n) \in \mathrm{aprx}$ for all $n \in \mathbb{N}$.

(ii) The pair $(\mathrm{aprx}, \lim)$ is an *approximation-limit* pair for A and e if for each approximation sequence $(a_n)_n$ for x,

$$\lim((a_n)_n) \simeq x\,.$$

Consider a partial Σ-algebra A and a partial sequence $e\colon \mathbb{N} \to A$. Recall that $(\langle e \rangle, \gamma_e)$ is a weakly effective partial Σ-algebra (effective if the operations are total) and e is weakly γ_e-computable. We consider the numbered set $(\langle e \rangle_k^{\mathbb{N}}, \gamma_e^*)$.

Definition 4.2 Let A be a partial Σ-algebra, $e\colon \mathbb{N} \to A$ a partial sequence, and let α and β be numberings of A (as a set). An approximation-limit pair $(\mathrm{aprx}, \lim)$ for A and e is *computable with respect to* α and β if

(i) aprx has (α, γ_e)-computable non-deterministic selection, and

(ii) $\lim\colon \langle e \rangle_k^{\mathbb{N}} \to A$ is weakly (γ_e^*, β)-computable.

When $\alpha = \beta$ in the definition we say that $(\mathrm{aprx}, \lim)$ is computable with respect to α.

We now define a new numbering $\bar{\gamma}_e$ depending on $(\mathrm{aprx}, \lim)$ and e. Let

$$\Omega_{\bar{\gamma}_e} = \{n \in \Omega_{\gamma_e^*} : \lambda k.\gamma_e \varphi_n(k) \text{ is an approximation sequence}\},$$

and define $\bar{\gamma}_e\colon \Omega_{\bar{\gamma}_e} \to A$ by

$$\bar{\gamma}_e(n) = \lim(\lambda k.\gamma_e \varphi_n(k)).$$

Definition 4.3 The partial Σ-algebra A is *recursively constructible* with respect to $(\mathrm{aprx}, \lim)$ and e if for each $x \in A$ there is $n \in \Omega_{\bar{\gamma}_e}$ such that $\bar{\gamma}_e(n) \simeq x$.

Thus A is recursively constructible if every element of A is the limit of a γ_e-computable approximation sequence in $\langle e \rangle$, i.e., if $(A, \bar{\gamma}_e)$ is a numbered set.

Proposition 4.4 *If there is a numbering α of A such that* aprx *has* $(\alpha \times \mathrm{id}, \gamma_e)$*-computable non-deterministic selection then A is recursively constructible.*

Theorem 4.5 *Let A be a partial Σ-algebra with numberings α and β and let e be a partial sequence in A. Let* $(\mathrm{aprx}, \lim)$ *be an approximation-limit pair for A and e.*

(i) If aprx *has* $(\alpha \times \mathrm{id}, \gamma_e)$*-computable non-deterministic selection then $\bar{\gamma}_e$ is a numbering of A and $\alpha \leq \bar{\gamma}_e$.*

(ii) If A is recursively constructible and lim *is weakly* (γ_e^*, β)*-computable then $\bar{\gamma}_e \leq \beta$.*

(iii) If $(\mathrm{aprx}, \lim)$ *is computable with respect to* α *and* β *then* $\alpha \leq \bar{\gamma}_e \leq \beta$.

It is often the case in applications that every *effective* numbering α of a partial Σ-algebra A makes it possible to define aprx with $(\alpha \times \mathrm{id}, \gamma_e)$-computable selection. Put differently, Σ includes operations sufficient for computably tracking selection for aprx for all effective numberings of A.

Corollary 4.6 *If* $(\mathrm{aprx}, \lim)$ *is computable with respect to* α *then* $\bar{\gamma}_e \sim \alpha$.

Thus if there is one numbering making $(\mathrm{aprx}, \lim)$ computable then it is (up to recursive equivalence) $\bar{\gamma}_e$.

The numbering $\bar{\gamma}_e$ has some further nice properties. First note that if A is recursively constructible then aprx has $(\bar{\gamma}_e \times \mathrm{id}, \gamma_e)$-computable non-deterministic selection. For if $n \in \Omega_{\bar{\gamma}_e}$ then the sequence $\lambda k.\gamma_e\varphi_n(k)$ is an approximation sequence so the selection for aprx is tracked by $\overline{\mathrm{aprx}}(n, k) \simeq \varphi_n(k)$.

We now consider sufficient conditions for $\bar{\gamma}_e$ being an effective numbering of A. In case there is an effective numbering α making $(\mathrm{aprx}, \lim)$ computable then $\bar{\gamma}_e \sim \alpha$ and hence $\bar{\gamma}_e$ is effective. Without a priori access to such a numbering we need a notion of effective continuity with respect to an approximation-limit pair.

Definition 4.7 Let σ be a partial n-ary operation of A. Then σ is *effectively continuous* with respect to $(\mathrm{aprx}, \lim)$ if there is an $(n + 1)$-ary partial recursive function t such that whenever $\sigma(\bar{\gamma}_e(m_1), \ldots, \bar{\gamma}_e(m_n))\downarrow$ then

(i) $t(m_1, \ldots, m_n, k)\downarrow$ for each k, and

(ii) if $(\bar{\gamma}_e(m_i), t(m_1, \ldots, m_n, k), a_{ik}) \in \mathrm{aprx}$ for $i = 1, \ldots, n$ then
$$(\sigma(\bar{\gamma}_e(m_1), \ldots, \bar{\gamma}_e(m_n)), k, \sigma(a_{1k}, \ldots, a_{nk})) \in \mathrm{aprx}\ .$$

Theorem 4.8 *Let* A *be a partial* Σ*-algebra and let* $e\colon \mathbb{N} \to A$ *be a partial sequence. Assume that there is an approximation-limit pair* $(\mathrm{aprx}, \lim)$ *for* A *and* e *such that* A *is recursively constructible and every operation of* A *is effectively continuous with respect to* $(\mathrm{aprx}, \lim)$*. Then* $(A, \bar{\gamma}_e)$ *is a weakly effective partial* Σ*-algebra.*

In case each operation of A is total then the conclusion of the theorem states that $(A, \bar{\gamma}_e)$ is effective.

The use of γ_e is not restrictive. Each weakly effective numbering α is equivalent to γ_e for some e.

Proposition 4.9 *Let* (A, α) *be a weakly effective partial* Σ*-algebra. Then* $\alpha \sim \gamma_e$ *for some partial sequence* e.

In closing we show how it is often possible to construct an effective recursive completion within a partial Σ-algebra for natural approximation-limit pairs.

Let A be a partial Σ-algebra, possibly uncountable, $e\colon \mathbb{N} \to A$ a partial sequence, and $(\mathrm{aprx}, \lim)$ an approximation-limit pair for A and e. We say that $x \in A$ is *computable* with respect to $(\mathrm{aprx}, \lim)$ and e if x is the limit of a γ_e-computable approximation sequence, i.e.,
$$x = \lim(\lambda k.\gamma_e\varphi_n(k))$$

for some n such that $\lambda k.\gamma_e \varphi_n(k)$ is an approximation sequence with respect to aprx. We define $\bar{\gamma}_e$ as before.

Definition 4.10 (i) The set $A_k = \bar{\gamma}_e(\Omega_{\bar{\gamma}_e})$ of computable elements is the *recursive completion* of $\langle e \rangle$ in A with respect to $(\mathrm{aprx}, \lim)$.

(ii) The recursive completion A_k is *proper* if $\langle e \rangle \subseteq A_k$ and the inclusion $\iota\colon \langle e \rangle \to A_k$ is $(\gamma_e, \bar{\gamma}_e)$-computable.

Note that the recursive completion is proper if $(a, n, a) \in \mathrm{aprx}$ for each $a \in \langle e \rangle$ and $n \in \mathbb{N}$, since then each constant sequence in $\langle e \rangle$ is an approximation sequence. Furthermore, $(\mathrm{aprx}, \lim)$ restricted to A_k is $\bar{\gamma}_e$-computable and A_k is recursively constructible in the sense of Definition 4.3.

Theorem 4.11 *Let A be a partial Σ-algebra, $e\colon \mathbb{N} \to A$ a partial sequence, and* $(\mathrm{aprx}, \lim)$ *an approximation-limit pair for A and e. Let A_k be the recursive completion of $\langle e \rangle$ in A with respect to* $(\mathrm{aprx}, \lim)$ *and assume it is proper.*

(i) $(A_k, \bar{\gamma}_e)$ is a numbered set and if α is a numbering of A_k making $(\mathrm{aprx}, \lim)$ *computable then $\alpha \sim \bar{\gamma}_e$.*

(ii) If each operation of A restricted to A_k is effectively continuous with respect to $(\mathrm{aprx}, \lim)$ *then $(A_k, \bar{\gamma}_e)$ is a weakly effective partial Σ-algebra and $\iota\colon \langle e \rangle \to A_k$ is a $(\gamma_e, \bar{\gamma}_e)$-computable embedding.*

5 Applications

5.1 Real numbers

In this section we briefly indicate how the notion of approximation-limit pair fits in with the construction of the ordered field of recursive real numbers. The result by Moschovakis [4], Hertling [1] and others that there is exactly one effective numbering of the recursive reals (up to recursive equivalence) having a limit algorithm then easily follows.

Let $\mathbb{R}$ be the partial algebra $\mathbb{R} = (\mathbb{R}; 0, 1, +, -, \times, (\cdot)^{-1}, \chi_<)$, let Σ be its signature, and let $e\colon \mathbb{N} \to \mathbb{R}$ be the constant function with value 0. Thus $\langle e \rangle = \mathbb{Q}$. It is well-known how to construct an effective numbering α of $\mathbb{Q}$ such that $=$ and $<$ are α-decidable. Of course, e is α-computable for any numbering of $\mathbb{Q}$ and hence $\alpha \sim \gamma_e$ by Proposition 3.3 and γ_e is the only such numbering up to recursive equivalence making equality decidable.

We now define a natural approximation-limit pair $(\mathrm{aprx}, \lim)$ for $\mathbb{R}$ and e. Define $\mathrm{aprx} \subseteq \mathbb{R} \times \mathbb{N} \times \langle e \rangle$ by

$$(x, k, a) \in \mathrm{aprx} \iff |x - a| < 2^{-k} .$$

An approximation sequence for aprx in the sense of Section 4 is what is known in the literature as a *fast Cauchy sequence*. The limit of a fast Cauchy sequence exists in $\mathbb{R}$ and every element in $\mathbb{R}$ is the limit of a fast Cauchy sequence in $\mathbb{Q}$. It follows that $(\mathrm{aprx}, \lim)$ is an approximation-limit pair for $\mathbb{R}$ and e, where lim is the limit operator for fast Cauchy sequences.

Let $\mathbb{R}_k$ be the recursive completion of $\langle e\rangle$ in $\mathbb{R}$ with respect to $(\mathrm{aprx}, \lim)$. The completion is proper since $(a, k, a) \in \mathrm{aprx}$ for all $a \in \langle e\rangle$ and $k \in \mathbb{N}$. It is routine to verify that each operation of $\mathbb{R}$ restricted to $\mathbb{R}_k$ is effectively continuous with respect to $(\mathrm{aprx}, \lim)$ and hence $(\mathbb{R}_k, \bar{\gamma}_e)$ is a weakly effective partial Σ-algebra by Theorem 4.11.

Proposition 5.1 *The ordered field* $(\mathbb{R}_k, \bar{\gamma}_e)$ *is an effective partial* Σ*-algebra.*

Now suppose α is an effective numbering of the Σ-algebra $\mathbb{R}_k$.

Proposition 5.2 *If* $(\mathbb{R}_k, \alpha)$ *is effective then* aprx *has* $(\alpha \times \mathrm{id}, \gamma_e)$*-computable selection.*

Let LIM be the function taking an α-computable fast Cauchy sequence in $\mathbb{R}_k$ to its limit. We say that $(\mathbb{R}_k, \alpha)$ has a *limit algorithm* if LIM is weakly (α^*, α)-computable.

Proposition 5.3 $(\mathbb{R}_k, \bar{\gamma}_e)$ *has a limit algorithm.*

Thus we see that $(\mathbb{R}_k, \bar{\gamma}_e)$ is recursively constructible in the sense that $\mathbb{R}_k$ is closed under taking limits of $\bar{\gamma}_e$-computable fast Cauchy sequences.

Theorem 5.4 *Let* $\mathbb{R}_k = (\mathbb{R}_k; 0, 1, +, -, \times, (\cdot)^{-1}, \chi_<)$ *be the recursive completion with respect to* $(\mathrm{aprx}, \lim)$ *and* e *and let* α *be an effective numbering of* $\mathbb{R}_k$*. Then the following hold.*

(i) $\alpha \leq \bar{\gamma}_e$.

(ii) If $(\mathrm{aprx}, \lim)$ *is computable with respect to* α *then* $\alpha \sim \bar{\gamma}_e$.

(iii) $(\mathbb{R}_k, \alpha)$ *has a limit algorithm if, and only if,* $\alpha \sim \bar{\gamma}_e$.

We see that in the class of effective numberings of $\mathbb{R}_k$ there is a largest numbering (up to recursive equivalence) and it is characterised by having a limit algorithm. Such a numbering is said to be a *standard numbering* of the recursive reals. Thus a standard numbering is determined from the indexing of fast Cauchy sequences in $\mathbb{Q}$.

If Corollary 4.6 is applied directly to the unordered recursive reals, then the numbering is still unique. This is not really a strengthening of the above result since the ordering can be recovered from our chosen approximation relation aprx.

The algebraic formulation of Theorem 4.5 makes it easy to translate results to slightly different algebras. An easy exercise is to modify Theorem 5.4 to $\mathbb{R}_k$ built from fast Cauchy sequences over the dyadic numbers $\mathbb{D}$. As above, it is easy to give a computable numbering of $\mathbb{D}$ for the (total) algebra $\mathbb{R}' = (\mathbb{R}; 0, \frac{1}{2}, 1, +, -, \times, \chi_<)$. Note that the algebra does not use the partial operation $(\cdot)^{-1}$ but add a new constant, $\frac{1}{2}$. Thus, the results in Theorem 5.4 for $\mathbb{R}_k$ also hold for $\mathbb{R}'_k$.

5.2 Inverse limits

Taking the inverse limit of an inverse system of algebras is an important "completion" process in mathematics and in modelling computations. A natural approximation-limit pair for this construction is obtained using the projection functions, providing a tool for analysing its effective content

We consider inverse limits obtained from an algebra via a family of separating congruences. The more general situation of the inverse limit of an inverse system can be handled similarly only with a little more notational complexity.

Let A be a Σ-algebra. In this section we assume the operations of A to be total. Then $\{\equiv_n\}_n$ is a *family of separating congruences* on A if each $\equiv_n$ is a congruence on A and the following hold:

(i) $x \equiv_{n+1} y \implies x \equiv_n y$, and

(ii) $\bigcap_{n\in\mathbb{N}} \equiv_n = \{(x,x) : x \in A\}$.

We set $A_n = A/{\equiv_n}$ and let $v_n: A \to A_n$ be the factoring epimorphism, i.e., $v_n(a) = [a]_n = \{b \in A : a \equiv_n b\}$. For $n \geq m$ we let $\phi^n_m: A_n \to A_m$ be the epimorphism given by $[a]_n \mapsto [a]_m$. Then $(A_n, \phi^n_m)_{n\geq m}$ is an inverse system. Its inverse limit is denoted by $\lim_{\leftarrow}(A_n, \phi^n_m)$, or simply $\lim_{\leftarrow} A_n$ or $\bar{A}$, along with the epimorphisms $\bar{\phi}_n: \bar{A} \to A_n$ satisfying $\phi^n_m \circ \bar{\phi}_n = \bar{\phi}_m$ for each $n \geq m$.

Recall that $\bar{A}$ is, up to isomorphism, the Σ-algebra

$$\bar{A} = \{(a_n)_n \in \prod_{n=0}^{\infty} A_n : \forall n.\phi^{n+1}_n(a_{n+1}) = a_n\}$$

along with $\bar{\phi}_n: \bar{A} \to A_n$ given by $\bar{\phi}_n((a_k)_k) = a_n$, and the operations on $\bar{A}$ act pointwise. By the universal property of inverse limits there is a unique embedding $\theta: A \to \bar{A}$. It is given by $\theta(a) = ([a]_n)_n$.

Throughout we now let (A, α) be an effective Σ-algebra with a family $\{\equiv_n\}_n$ of separating congruences on A. Note that no assumptions are made on the effectivity of $\equiv_n$. We define a numbering $\alpha_n: \Omega_\alpha \to A_n$ by $\alpha_n(k) = [\alpha(k)]_n$. Then (A_n, α_n) is an effective Σ-algebra since $\equiv_n$ is a congruence, and v_n is (α, α_n)-computable, tracked by the identity.

By Proposition 4.9 there is a weakly α-computable partial sequence e in A such that α is recursively equivalent to γ_e. Thus we may use the results from Section 4 with α playing the role of γ_e and $\bar{\alpha}$ of $\bar{\gamma}_e$. Furthermore, for notational simplicity, we will consider an approximation function taking values in A rather than in its isomorphic image $\theta(A)$.

Define a relation $\mathrm{aprx} \subseteq \bar{A} \times \mathbb{N} \times A$ by

$$(x, n, a) \in \mathrm{aprx} \iff \bar{\phi}_n(x) = v_n(a).$$

Then define the partial function $\lim: A^{\mathbb{N}} \to \bar{A}$ as follows. Suppose $(a_n)_n$ is an approximation sequence in A with respect to aprx and induced by x. Then

$$\lim((a_n)_n) \simeq ([a_n]_n)_n \simeq x.$$

It follows that (aprx, lim) is an approximation-limit pair for A and α.

Let $\bar{A}_{k,\alpha}$ be the recursive completion of A and α with respect to (aprx, lim). Then $(\bar{A}_{k,\alpha}, \bar{\alpha})$ is a numbered set (where $\bar{\alpha}$ plays the role of $\bar{\gamma}_e$, i.e., $\bar{\alpha}(n) \simeq \lambda k.v_k \alpha\varphi_n(k)$ when $\lambda k.\alpha\varphi_n(k)$ is an approximation sequence). Let ϕ_n be the restriction of $\bar{\phi}_n$ to $\bar{A}_{k,\alpha}$.

Proposition 5.5 *(i)* $\theta\colon A \to \bar{A}_{k,\alpha}$ *is* $(\alpha, \bar{\alpha})$*-computable.*

(ii) $\phi_n\colon \bar{A}_{k,\alpha} \to A_n$ *is* $(\bar{\alpha}, \alpha_n)$*-computable, uniformly in* n*.*

(iii) $(\bar{A}_{k,\alpha}, \bar{\alpha})$ *is an effective* Σ*-algebra.*

A sequence $(a_n)_n$ in A is said to be a *Cauchy sequence* if $a_{n+1} \equiv_n a_n$ for each n. Note that this corresponds to being an approximation sequence with respect to aprx. Similarly, a sequence $(x_n)_n$ in $\bar{A}$ is said to be a *Cauchy sequence* if $\bar{\phi}_n(x_{n+1}) = \bar{\phi}_n(x_n)$ for each n. Thus if $x_n = ([a_{nk}]_k)_k$, where $a_{nk} \in A$, then it is required that $a_{n+1,n} \equiv_n a_{nn}$ for each n. Using the above notation let the partial function $\mathrm{LIM}\colon \bar{A}^{\mathbb{N}} \to \bar{A}$ be defined on each Cauchy sequence $(x_n)_n$ by

$$\mathrm{LIM}((x_n)_n) \simeq \lim((a_{nn})_n).$$

Note that $(a_{nn})_n$ is a Cauchy sequence in A since

$$a_{n+1,n+1} \equiv_n a_{n+1,n} \equiv_n a_{nn},$$

and therefore $\mathrm{LIM}((x_n)_n)$ is defined. LIM is the limit operator with respect to the natural ultrametric on $\bar{A}$ induced by $\{\equiv_n\}_n$. This will be further discussed in Section 5.3.

Let β be a numbering of $\bar{A}_{k,\alpha}$. We say that $(\bar{A}_{k,\alpha}, \beta)$ *has a limit algorithm* if LIM is weakly (β^*, β)-computable.

Proposition 5.6 $(\bar{A}_{k,\alpha}, \bar{\alpha})$ *has a limit algorithm.*

Theorem 5.7 *Let* (A, α) *be an effective* Σ*-algebra and let* $\{\equiv_n\}_n$ *be a family of separating congruences on* A*. Let* β *be a numbering of* $\bar{A}_{k,\alpha}$ *such that* $\phi_n\colon \bar{A}_{k,\alpha} \to A_n$ *is* (β, α_n)*-computable, uniformly in* n*. Then*

(i) $\beta \leq \bar{\alpha}$.

(ii) If the embedding $\theta\colon A \to \bar{A}_{k,\alpha}$ *is* (α, β)*-computable then* $\beta \sim \bar{\alpha}$ *if, and only if,* $(\bar{A}_{k,\alpha}, \beta)$ *has a limit algorithm.*

We conclude with some remarks on the complexity of $\equiv_n$ on A. The congruence relation $\equiv_n$ on A is extended to $\bar{A}$ by, for $x, y \in A$,

$$x \equiv_n y \Longleftrightarrow \bar{\phi}_n(x) = \bar{\phi}_n(y).$$

Proposition 5.8 *Let* (A, α) *be an effective* Σ*-algebra with a family* $\{\equiv_n\}_n$ *of separating congruences, assume* Ω_α *is r.e., and let* β *be a numbering of* $\bar{A}_{k,\alpha}$*.*

(i) Assume $\theta\colon A \to \bar{A}_{k,\alpha}$ *is* (α, β)*-computable. If* $\{\equiv_n\}_n$ *on* $\bar{A}_{k,\alpha}$ *is* β*-semicomputable, uniformly in* n*, then* ϕ_n *is* (β, α_n)*-computable, uniformly in* n*.*

(ii) If ϕ_n is (β, α_n)-computable and $\{\equiv_n\}_n$ on A is α-semicomputable, uniformly in n, then $\{\equiv_n\}_n$ on $\bar{A}_{k,\alpha}$ is β-semicomputable, uniformly in n.

Thus, in the event that $\{\equiv_n\}_n$ on A is α-computable, uniformly in n, and Ω_α is r.e., then ϕ_n is (β, α_n)-computable, uniformly in n, if, and only if, $\{\equiv_n\}_n$ is β-semicomputable, uniformly in n.

5.3 Effective metric partial Σ-algebras

A metric space M with metric d is *effective* with respect to a numbering α of M if d is $(\alpha \times \alpha, \rho)$-computable, where ρ is a standard numbering of the computable reals $\mathbb{R}_k$ as in Section 5.1.

A metric partial Σ-algebra is a partial Σ-algebra A equipped with a metric $d: A \times A \to \mathbb{R}$ such that each partial operation of A is continuous. We say that A is (*weakly*) *effective* with respect to a numbering α if (A, α) is an effective metric space and each operation of A is (weakly) effective.

Now let (A, α) be an effective metric partial Σ-algebra and suppose e is an α-computable sequence such that $\langle e \rangle$ is dense in A. Then, by Theorem 3.2, $\langle e \rangle$ is a dense partial Σ-subalgebra effective under the numbering γ_e, the inclusion $\iota: \langle e \rangle \to A$ is (γ_e, α)-computable, and Ω_e is r.e.

Define a relation aprx $\subseteq A \times \mathbb{N} \times \langle e \rangle$ by

$$(x, n, a) \in \text{aprx} \iff d(x, a) < 2^{-n}.$$

An approximation sequence with respect to aprx is thus a fast Cauchy sequence with a limit in A. Let $\lim: \langle e \rangle^{\mathbb{N}} \to A$ be the corresponding limit function defined by $\lim((a_n)_n \simeq x$ whenever $(a_n)_n$ is an approximation sequence with respect to aprx and generated by $x \in A$. Thus lim is well-defined.

The relation aprx is $(\alpha \times \text{id} \times \gamma_e)$-semicomputable since $<$ is ρ-semicomputable, using the computability of d and ι, and it is left-total since $\langle e \rangle$ is dense in A. Hence aprx has computable selection by Proposition 2.3.

It now follows from Theorem 4.5 that $\bar{\gamma}_e$ is a numbering of A and that $\alpha \leq \bar{\gamma}_e$. Furthermore, if lim is weakly $(\bar{\gamma}_e, \alpha)$-computable then $\alpha \sim \bar{\gamma}_e$.

We say that (A, α) has a *limit algorithm* if the function LIM taking fast α-computable Cauchy sequences in A to its limit in A, if it exists, is weakly (α^*, α)-computable. Using the (γ_e, α)-computable inclusion $\iota: \langle e \rangle \to A$ it follows that lim is $(\bar{\gamma}_e, \alpha)$-computable if (A, α) has a limit algorithm.

Using the above observations we obtain the following theorem with a proof completely analogous to that of Theorem 5.4.

Theorem 5.9 *Let (A, α) be an effective metric partial Σ-algebra and let e be an α-computable sequence such that $\langle e \rangle$ is dense in A.*

(i) $\bar{\gamma}_e$ is a numbering of A and $\alpha \leq \bar{\gamma}_e$.

(ii) If lim *is weakly $(\bar{\gamma}_e, \alpha)$-computable then $\alpha \sim \bar{\gamma}_e$.*

(iii) (A, α) has a limit algorithm if, and only if, $\alpha \sim \bar{\gamma}_e$.

An important role of α in the above was to guarantee that the metric d was computable on $\langle e\rangle$ with respect to γ_e. Suppose A is a metric partial Σ-algebra and e is a sequence in A such that the metric d restricted to $\langle e\rangle$ is (γ_e,ρ)-computable. Then we may consider the recursive closure $\overline{\langle e\rangle}^A_{k,\gamma_e}$ of $\langle e\rangle$ in A with respect to aprx defined above. It is straight-forward to verify that d restricted to $\overline{\langle e\rangle}^A_{k,\gamma_e}$ is $(\bar\gamma_e,\rho)$-computable, using the triangle inequality of the metric. If the operations are effectively continuous with respect to $(\mathrm{aprx},\lim)$ then we obtain the recursive completion $\overline{\langle e\rangle}^A_{k,\gamma_e}$ as a weakly effective metric partial Σ-algebra and its numbering $\bar\gamma_e$ is unique up to recursive equivalence having a limit algorithm. As an example, the ordered field $\mathbb{R}_k$ of recursive reals with its standard numbering was obtained in this way. For further results in this direction we refer to [7].

To conclude this section we return to the inverse limit construction of Section 5.2. Let A be a Σ-algebra with a family of separating congruences $\{\equiv_n\}_n$. There is a natural metric d on A defined by

$$d(x,y)=\begin{cases}0 & \text{if } x=y\\ 2^{-n_0} & \text{where } n_0 \text{ least such that } x\not\equiv_n y.\end{cases}$$

In fact, d is an ultrametric. Furthermore, each ultrametric Σ-algebra with non-expansive operations can be construed as a Σ-algebra with a family of separating congruences $\{\equiv_n\}_n$ up to topological equivalence (i.e., homeomorphic algebras).

It is natural to ask when the recursive completion for the inverse limit as in Section 5.2 corresponds to the recursive completion as a metric algebra. That is, are there reasonable conditions for when the metric d is computable?

Theorem 5.10 *Let (A,α) be an effective Σ-algebra such that Ω_α is r.e., and let $\{\equiv_n\}_n$ be a family of separating congruences on A such that $\equiv_n$ is α-decidable uniformly in n. Let β be a numbering of $\bar A_{k,\alpha}$ such that ϕ_n is (β,α_n)-computable, uniformly in n. Then the metric d is $(\beta\times\beta,\rho)$-computable.*

Note that $\bar\alpha$ is an example of such a β.

An interesting example is the completion of a local ring. Let R be a commutative local Noetherian ring in the signature for rings and let $\mathbf{m}$ be its unique maximal ideal. Define $\equiv_n$ on R by

$$a\equiv_n b \Longleftrightarrow a-b\in\mathbf{m}^n.$$

Then, by a theorem of Krull, $\{\equiv_n\}_n$ is a family of separating congruences on R. Assume that (R,α) is semicomputable, i.e., it is effective, Ω_α is r.e., and equality is α-semicomputable. Then it is straight forward to see that $\{\equiv_n\}_n$ is α-semicomputable, uniformly in n. However, it is shown in [5] that if (R,α) is computable as a ring then $\{\equiv_n\}_n$ is in fact α-decidable, uniformly in n, and hence the associated metric on $\bar R_{k,\alpha}$ is effective. This fact allowed us to construct an effective domain representation of the inverse limit of R.

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