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A Quantitative Analysis of Implicational Paradoxes in Classical Mathematical Logic

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Abstract

Classical mathematical logic includes a lot of "implicational paradoxes" as its logic theorems. This paper uses the property of strong relevance as the criterion to identify implicational paradoxes in logical theorems of classical mathematical logic, and enumerates logical theorem schemata of classical mathematical logic that do not satisfy the strong relevance. This quantitative analysis shows that classical mathematical logic is by far not a suitable logical basis for automated forward deduction.

Keywords: Knowledge representation and reasoning, Automated forward deduction, Relevant logics, Strong relevance

1 Introduction

A forward deduction engine is an indispensable component for any knowledge-based system to discover new knowledge or predict future incidents. Since any automated forward deduction for discovery or prediction has no explicitly specified proposition or theorem given previously as goal, intrinsically, to apply all inference rules to all given premises and previously deduced conclusions is the only way to deduce new knowledge or predictions. This naturally requires that a forward deduction engine deduces only conclusions that are certainly relevant to given premises.

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Within the framework of classical mathematical logic, CML for short, the conclusion of a valid deduction is not necessarily relevant to its premises, because CML is established based on the classical account of validity. It is well known that logical theorems of CML include a lot of "implicational paradoxes" [1,2]. On the other hand, relevant logics and strong relevant logics rejected those implicational paradoxes as their logical theorems and are adopted as the logical basis for automated forward deduction [1,2,3,4,5]. However, until now no one investigated quantitatively how CML is "bad" and/or how relevant logics are "good" for automated forward deduction. This paper presents the result of our quantitative analysis of implicational paradoxes in CML and explains its implications. We use the property of the strong relevance as the criterion to identify implicational paradoxes in logical theorems of CML, and enumerate logical theorem schemata of CML that do not satisfy the strong relevance. This quantitative analysis shows that CML is by far not a suitable logical basis for automated forward deduction.

The rest of this paper is organized as follows: Section 2 gives a very simple introduction to implicational paradoxes in CML and the mention of the strong relevance. Section 3 gives how to analyze and analysis. Some concluding remarks are given in Section 4.

2 Implicational paradoxes

2.1 Implicational paradoxes in Classical Mathematical Logic

In logic, a sentence in the form of "if ... then ..." is usually called a conditional proposition or simply conditional. A conditional must concern two parts which are connected by the connective "if ... then ..." and called the antecedent and the consequent of that conditional. The truth of a conditional depends not only on the truth of its antecedent and consequent but also, and more essentially, on a necessarily relevant and/or conditional relation between its antecedent and consequent.

In CML, the notion of conditional, which is intrinsically intensional but not truth-functional, is represented by the truth-functional extensional notion of material implication (denoted by $\rightarrow$ in this paper) that is defined as $A \rightarrow B =_{df} \neg(A \wedge \neg B)$ or $A \rightarrow B =_{df} \neg A \vee B$, where $\wedge$, $\vee$, and $\neg$ denote the connectives of conjunction, disjunction, and negation, respectively. However, the material implication is intrinsically different from the notion of conditional in meaning (semantics). It is no more than an extensional truth-function of its antecedent and consequent but does not require that there is a necessarily relevant and conditional relation between its antecedent and consequent, i.e. the truth-value of the formula $A \rightarrow B$ depends only on the truth-values of A and B, though there could exist no necessarily relevant and conditional relation between A and B. It is this intrinsic difference in meaning between the notion of material implication and the notion of conditional that leads to the well-known "implicational paradox problem" in CML. The problem is that if one regards the material implication as the notion of conditional and regards every logical theorem of CML as an entailment or valid reasoning form, then a great number of logical axioms and logical theorems of CML, such as $A \rightarrow (B \rightarrow B)$,

$B \to (\neg A \vee A)$, and so on, present some paradoxical properties and therefore they have been referred to in the literature as "implicational paradoxes" [1,2].

Note that any classical conservative extension or non-classical alternative of CML where the classical account of validity is adopted as the logical validity criterion and the notion of conditional is directly or indirectly represented by the material implication has the similar problems as the above problems in CML [3].

In the framework of CML and its various conservative extensions, even if a deduction is valid in the sense of CML, neither the necessary relevance between its premises and conclusion nor the truth of its conclusion in the sense of conditional can be guaranteed necessarily. Therefore, any forward deduction engine cannot deduce only conclusions that are certainly relevant to given premises based on CML or its various conservative extensions.

2.2 Strong Relevance

Relevant logics, RL for short, were constructed to obtain a notion of implication which is free from the implicational paradoxes [1,2]. However, although RL have rejected those paradoxes of implication, there still exist some "conjunction-implicational paradoxes" and "disjunction-implicational paradoxes" in logical theorems of RL [3,4,5]. Cheng has proposed some strong relevant logics, SRL for short, which do not include the paradoxes [4,5]. In the framework of SRL, if a reasoning is valid in the sense of SRL, then both the relevance between its premises and conclusions and the validity of its conclusions in the sense of conditional can be guaranteed in a certain sense of strong relevance.

Strong relevance is one of property in RL and SRL: every propositional variable (or pattern variable) in a formula (or formula schema) occurs at least once as an antecedent part and at least once as a consequent part [1]. The definition of an antecedent part and a consequent part is as follows, let A, B and C be well-formed formulas or their schemata,

(i) A is a consequent part of A,

(ii) if $\neg B$ is a consequent part (antecedent part) of A, then B is an antecedent part (consequent part) of A,

(iii) if $B \to C$ is a consequent part (antecedent part) of A, then B is an antecedent part (consequent part) of A, and C is consequent (antecedent part) of A,

(iv) if $B \wedge C$ or $B \vee C$ is a consequent part (antecedent part) of A, then both B and C are consequent parts (antecedent parts) of A.

The strong relevance is a property which formally guarantees the relationship between antecedent and consequent so that any theorem of SRL satisfies the strong relevance [4,5] and any theorem in the implication-negation fragments of RL also satisfies it [1]. On the other hand, implicational paradoxes in logical theorems of CML and conjunction-implicational paradoxes and disjunction-implicational paradoxes in logical theorems of RL do not satisfy the strong relevance. We therefore can distinguish implicational paradoxes from logical theorems of CML by checking whether a formula (schema) satisfies the strong relevance or not.

3 A Quantitative Analysis

3.1 Preparation

In this paper, we focus our discussion on the axiomatic system of CML with only the connectives of implication and negation on propositional calculus because it is possible to represent all logical theorems of CML with the connectives of implication and negation. We define well-formed formulas we investigate in this paper as follows.

Definition 3.1 (i) A propositional variable p is a well-formed formula, wff for short,

(ii) $\neg A$ is a wff, if A is a wff,

(iii) $A \to B$ is a wff, if both A and B are wffs,

(iv) Only i $\sim$ iii are wffs.

Definition 3.2 A sub-wff is a part of a wff and is itself a wff. Each wff is regarded as a sub-wff of itself.

Note that we are not concerned with the wffs which include double negation, such as $\neg(\neg A)$ whose A is a wff, as their sub-wff in our analysis.

By other hand, the number of propositional variables is infinite so that the set of logical theorems of CML is the infinite set. Hence we deal with the set of schemata of logical theorem.

Definition 3.3 A schema of a wff is a formula that replaced all propositional variable in the wff with pattern variables.

Pattern variables are variables for which it can substitute certain propositional variables. They are symbols with order relation and are not included the vocabulary of CML.

Definition 3.4 A logical theorem schema of CML is a schema of a logical theorem of CML.

Definition 3.5 Let $deg_{\to}(A) = k$ denote that k is the degree of nested the connective of implication ($\to$) of A. $deg_{\to}(A) = k$ is defined as follows.

(i) if there is no occurrence of $\to$ in A, then $deg_{\to}(A) = 0$,

(ii) $deg_{\to}(\neg A) = deg_{\to}(A)$,

(iii) $deg_{\to}(A \to B) = 1 + max\,(deg_{\to}(A), deg_{\to}(B))$.

Definition 3.6 A is a k^{th} degree logical theorem schema of CML iff A is a logical theorem schema of CML and $deg_{\to}(A) = k$.

Definition 3.7 k^{th} degree schemata fragment is the set of all j^{th} degree schemata $(1 \leq j \leq k)$, and denoted by F_k.

Definition 3.8 The k^{th} degree fragment of CML is the set of all j^{th} degree logical theorem schemata of CML ($1 \leq j \leq k$), and denoted by Th_k.

Th_k can be classified into two sets: the set of all implicational paradoxes in Th_k, denoted by IP_k, and the set of all paradox-free theorems, denoted by ThS_k. The relationship amount Th_k, IP_k, and ThS_k is as follows;

$$Th_k = IP_k \cup ThS_k. \tag{1}$$

We distinguish between implicational paradoxes and others by checking whether a logical theorem schema satisfies the strong relevance or not. We can therefore enumerate the paradoxes by 3 steps;

(i) get all schemata in F_k,

(ii) classify F_k into two sets: Th_k and others, by checking whether each schema in F_k is logical theorem or not.

(iii) classify Th_k into ThS_k and IP_k by using strong relevance as the criterion.

3.2 Analysis

At first, we produced all schemata in F_3. Any schema can be represented by rooted tree structure; the connective of implication is an internal node which can have one parent and must have two children; the connective of negation is other internal node which can have one parent and must have one child; pattern variables are labels put on leaves. For example $(\neg(\alpha \to \beta) \to \gamma)$ is can be represented by a rooted tree like figure 1, whose $\to$ and $\neg$ are the connectives of implication and negation respectively and α, β, γ are pattern variables. All schemata can be grouped by trunks of rooted trees. We define a trunk is a part of a tree which consists of only internal nodes without leaves, e.g., in figure 2. We can produce all schemata in F_k from all kinds of trunks which occur in $F(k)$ and all kinds of permutations of pattern variables whose length are from 1 to 2^k.

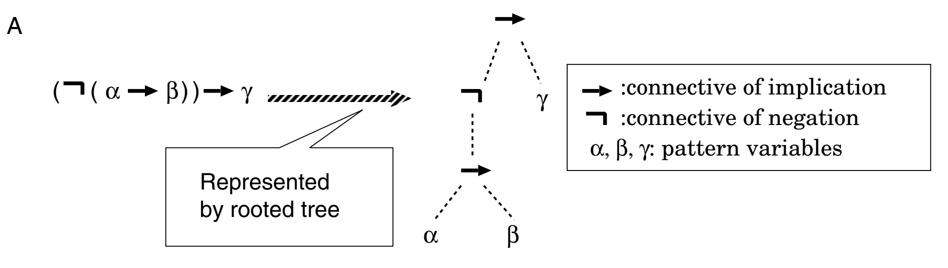

Fig. 1. An example of a rooted tree

Algorithm 1 *Produce all kinds of trunks which occur in $F(k)$.*

(i) $\tau_0 =_{df}$ {'$\neg$', ' ' (blank)}, *whose both '$\neg$' and ' ' (blank) are representations of trunks when no $\to$ occurs,*

(ii) $i \leftarrow 1$

(iii) *loop from iv to xviii,*

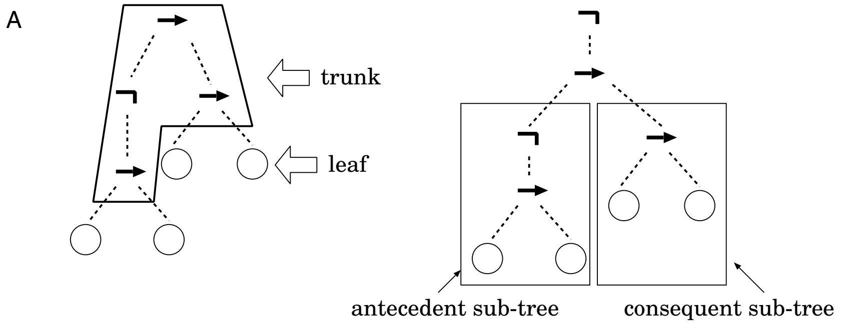

Fig. 2. Structure of rooted tree

(iv) $\tau_i \leftarrow \phi$,

(v) *loop from vi to xvii,*

(vi) *pick up a trunk* ζ *from* τ_{i-1},

(vii) *loop from viii to xi,*

(viii) *pick up a trunk* η *from* τ_j $(0 \leq j \leq i-2)$,

(ix) *create four new trunks:* $(\zeta \rightarrow \eta)$, $(\eta \rightarrow \zeta)$, $\neg(\zeta \rightarrow \eta)$, $\neg(\eta \rightarrow \zeta)$,

(x) *add those trunks into* τ_i,

(xi) *repeat from viii until picking up all elements of* τ_j *as* η,

(xii) *loop from xiii to xvi,*

(xiii) *pick up a trunk* θ *from* τ_{i-1},

(xiv) *create two new trunks:* $(\zeta \rightarrow \theta)$, $\neg(\zeta \rightarrow \theta)$,

(xv) *add those trunks into* τ_k,

(xvi) *repeat from xiii until picking up all elements of* τ_{i-1} *as* θ,

(xvii) *repeat from vi until picking up all elements of* τ_{i-1} *as* ζ,

(xviii) $i \leftarrow i+1$ *then repeat from iv until* $k = i$.

Algorithm 2 *Create all kinds of permutation of pattern variables whose length is from 1 to k.*

(i) $\rho_1 =_{df} \{$ 'a_0, ' $\}$ *whose* a_n $(1 \leq n \leq 2^k)$ *is a pattern variable,*

(ii) $i \leftarrow 2$

(iii) *loop from iv to xiii,*

(iv) $\rho_i \leftarrow \phi$,

(v) *loop from vi to xii,*

(vi) *pick up a permutation* λ *from* ρ_{i-1},

(vii) *loop from viii to xi,*

(viii) $j \leftarrow 1$

(ix) *create a new permutation by connecting* λ *with* a_j, *such as* 'λ, a_j',

(x) *add the permutation into* ρ_i,

(xi) $j \leftarrow j+1$ *then repeat from* viii *until* $j = m$, *whose* m *is the maximum number of the subscript of the pattern variable which occurs in* λ,

(xii) *repeat from* vi *until picking up all elements of* ρ_{i-1} *as* λ,

(xiii) $i \leftarrow i+1$ *then repeat from* iv *until* $k = i$.

We then distinguished logical theorem schemata and others in the set of produced schemata by checking whether a schema is universally true or not. After that, we judged whether these logical theorem schemata satisfy strong relevance or not. Table 1 shows the number of elements of F_k, IP_k and ThS_k $(1 \leq k \leq 3)$.

Table 1
The number of elements of F_k, IP_k, and ThS_k

k	F_k	IP_k	ThS_k	IP_k/ThS_k
1	1.60×10^1	0.00×10^0	2.00×10^0	0.00
2	2.26×10^3	2.16×10^2	9.80×10^1	2.20
3	1.67×10^8	3.94×10^7	2.44×10^6	16.13

Table 2
The number of elements of F_k

degree	F_k
4	2.92×10^{19}
5	1.63×10^{45}
6	4.29×10^{103}
7	1.02×10^{235}
8	8.15×10^{527}

In our enumeration method, we cannot deal with F_i when i is more than 3, because F_4 is too large amount. Table 2 shows the number of elements of F_k $(4 \leq k \leq 8)$. F_k becomes large exponentially against k becomes large linearly. Note it is possible to calculate the number of elements of F_k but difficult to get all elements of the set.

The number of elements of F_k can be calculated as follows. T_k^i denotes the kinds of trunks of rooted trees in F_k while the number of leaves of the all trees are i and the degree of nested implications is k. Let P_i be the kinds of permutations of pattern variables put on leaves of a certain rooted tree as labels while the number of the leaves is i. Then the number of elements of F_k is can be calculated from T_k^i and P_i.

T_k^i is defined as follows,

$$T_0^1 = 2,$$

$$T_1^2 = 8,$$
$$T_k^i = 0,\ (0 \leq k,\ i < k+1,\ 2^k < i),$$
$$(2)\quad T_k^i = 4\sum_{j=k}^{2^{(k-1)}}\sum_{e=0}^{k-2} T_{k-1}^j \cdot T_e^{i-j} + 2\sum_{j=k}^{2^{(k-1)}} T_{k-1}^j \cdot T_{k-1}^{i-j},\ (2 < k).$$

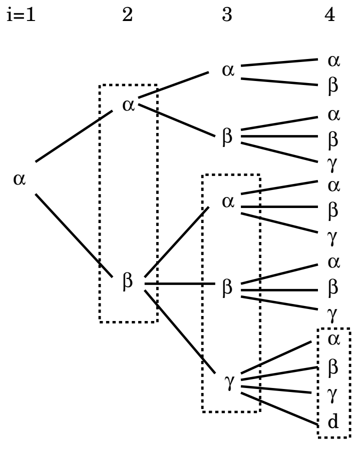

Fig. 3. The kinds of permutations of pattern variables

Figure 3 is the relationship between the number of leaves and the kinds of permutations of pattern variables. In the figure, pattern variables are represented by Greek alphabet. Let P_i be the kinds of permutations of pattern variables when the number of leaves is i. P_1 is 1 kind. P_2, P_3, and P_4 are 2 kinds, 5 kinds, and 15 kinds, respectively. We can see that same combinations that enclose them with the square in the figure appear many times. Hence P_i is defined as follows,

$$P_1 = 1,$$
$$(3)\quad P_i = \sum_{j=1}^{i-1}(j+1) \cdot S_{i-1}^j,\ (2 \leq i).$$

The coefficient S_m^n in equation (3) is the stirling numbers of the second kind. The stirling numbers of second kind is defined as follows,

$$S_m^1 = S_m^m = 1,\ (1 \leq m),$$
$$(4)\quad S_{m+1}^n = S_m^{n-1} + n \cdot S_m^n,\ (1 < m < n).$$

From equation (2) and equation (3), the number of elements of F_k is defined as follows,

$$(5)\qquad \sum_{j=1}^{k}\sum_{i=j+1}^{2^j} T_j^i \cdot P_i.$$

Table 1 shows that the IP_3 is 16.13 times as large as ThS_3. We think that this rate becomes larger as the degree of nested implications becomes larger. All logical theorem schemata in Th_k can be grouped by trunks of rooted trees. The number of elements in a set of logical theorem schemata which are grouped by a certain trunk of a rooted tree is as same as the kinds of permutations of pattern variables which can put on leaves of the tree as labels. The difference between amount of IP_k and that of ThS_k causes the kinds of permutations of pattern variables which can put on leaves of a rooted tree as labels.

Under a certain trunk of a rooted tree, the kinds of permutations of pattern variables which appear in elements of IP_k is by far more than that of ThS_k when k is large. In representation of rooted tree, we call antecedent (consequent) of a logical theorem schema antecedent (consequent) sub-tree of it. Note we call same rule if root of a tree is negation in this paper, e.g., in figure 2. $A \rightarrow B$ is a logical theorem schemata of CML iff the truth-value of A and B are follows,

case 1 the truth-value of A is universally false and the truth-value of B is universally true,

case 2 the truth-value of A is universally false and truth-value of B is not universally true,

case 3 the truth-value of A is not universally false and the truth-value of B is universally true,

case 4 the truth-value of A is not true if the truth-value of B is false, and both of them are contingent.

Similarly if $\neg(A \rightarrow B)$ is logical theorem schemata of CML then as follows,

case 5 the truth-value of A is universally true, and the truth-value of B is universally false,

If A is an element of IP_k then it can put any kind of permutations of pattern variables on leaves of its consequent sub-tree when the truth-value of its antecedent sub-tree is universally false. Similarly, if A is an element of IP_k then it can put any kind of the permutations of pattern variables on leaves of its antecedent sub-tree when the truth-value of its consequent sub-tree is universally true. However, if A is an element of ThS_k iff only kinds of permutations of pattern variables satisfying the strong relevance.

Let us discuss about the case that the truth-value of its consequent sub-tree is universally true. On the case, it is large that the difference between the kinds of permutations of pattern variables which appear in elements of IP_k and that of ThS_k under a certain trunk of a rooted tree. While the number of antecedent parts is a, and the number of consequent parts is c of antecedent sub-tree, the kinds of permutation of pattern variables which appear in an element of ThS_k is calculated as follows,

$$\sum_{j=1}^{min(a,c)} R(j,a) \cdot R(j,c) \cdot j!. \tag{6}$$

$R(m,n)$ is a function that calculates the kinds of permutation of pattern variables, whose n is the number of leaves and m is the kinds of pattern variables. $R(m,n)$ is defined as follows,

$$R(m,n) = 1,\ (n = m),$$

$$= \sum_{j=1}^{m} j, \ (n = m + 1),$$

$$= \sum_{j=1}^{m} j \cdot \delta_{n-m-1}^{j}, \ (m + 1 < n). \tag{7}$$

Let m be the kinds of pattern variables. δ_e^f is defined as follows,

$$\delta_1^f = \sum_{k=f}^{m} k, \ (1 \leq f \leq m),$$

$$\delta_e^f = \sum_{k=f}^{m} k \cdot \delta_{e-1}^{k}, \ (1 < e, \ 1 \leq f \leq m). \tag{8}$$

Under a trunk of a rooted tree, equation (3) gives the kinds of the permutations which appear in the antecedent sub-tree of elements of IP_k. In contrast, the kinds of permutations which appear in the antecedent sub-tree of elements of ThS_k is given by from equation (3) as maximum value to equation (6) as minimum value. Note equation (6) is minimum when $a + c = n, \ a = c$ in this assumption, whose a is the number of antecedent part, and n is the number of leaves. Table 3 shows the difference between the value of equation (3) and that of equation (6) if $a + c = n, \ a = c$. The value of equation (3) becomes larger than that of equation (6) as the number of leaves becomes large. The larger the number of leaves becomes, the larger the degree of nested implication becomes.

The same thing may be said of the other cases. We therefore consider that the difference between amount of IP_k and that of ThS_k becomes larger as the degree of nested implications becomes larger.

Table 3
The kinds of permutation of pattern variables

length	eq. (3)	eq. (6)	eq. (3) / eq. (6)
2	2.00×10^{0}	1.00×10^{0}	2.00×10^{0}
8	4.14×10^{3}	3.39×10^{2}	1.22×10^{1}
32	1.28×10^{26}	6.63×10^{23}	1.93×10^{2}
128	1.12×10^{158}	4.14×10^{153}	2.71×10^{4}

4 Concluding remarks

We enumerated implicational paradoxes in logical theorem of classical mathematical logic, CML for short. On the axiomatic system of CML with the connective of implication and negation, our analysis results showed that the number of implicational paradoxes in the set of $1^{st} \sim 3^{rd}$ degree logical theorem schemata of CML is 16.13 times as many as the number of paradox-free logical theorems in the set. The difference between amount of implicational paradoxes and that of paradox-free logical theorems of CML becomes larger as the degree of nested implications becomes larger.

With only connectives of entailment and negation, any logical theorem of relevant logics, RL for short, and strong relevant logics, SRL for short, is an element of paradox-free logical theorems of CML, if we regard material implication in CML and entailment in RL and SRL as a same connective to represent the notion of conditional. Hence, it is quite likely that a forward deduction engine based on CML deals with conclusions at least 16 times as many as conclusions which a forward deduction engine based on RL or SRL deal with. Therefore, as the logic systems underlying forward deduction, RL and SRL are quantitatively more suitable by far than CML. The same thing may be said of its various conservative extensions of CML which include all logical theorems of CML as their logical theorems.

In this paper, we investigated the implicational paradoxes in the axiomatic system of CML with only implication and negation. This is only a quantitative comparative study between CML and the intersection between SRL and RL. As future work, we should investigate whether SRL is quantitative suitable logic systems to underlie automated forward deduction than RL or not by quantitative comparative study between RL and SRL.

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