Available online at www.sciencedirect.com
SCIENCE DIRECT®
Electronic Notes in Theoretical Computer Science
ELSEVIER Electronic Notes in Theoretical Computer Science 120 (2005) 111–123
www.elsevier.com/locate/entcs

Some Examples of Non-Metrizable Spaces Allowing a Simple Type-2 Complexity Theory

Daren Kunkle[1]

Theoretische Informatik I
FernUniversität Hagen
Hagen, Germany

Matthias Schröder[2]

LFCS, School of Informatics
University of Edinburgh
Edinburgh, UK

Abstract

Representations of spaces are the key device in Type-2 Theory of Effectivity (TTE) for defining computability on non-countable spaces. *Almost-compact* representations permit a simple measurement of the time complexity of functions using *discrete* parameters, namely the desired output precision together with "size" information about the argument, rather than continuous ones.
We present some interesting examples of non-metrizable topological vector spaces that have almost-compact admissible representations, including spaces of real polynomial functions and of distributions with compact support.

Keywords: Topological Vector Spaces, Distributions, Type-2 Theory of Effectivity, Complexity Theory

1 Introduction

Up to now, most investigations in complexity theory deal with discrete spaces. In this article, however, we consider computational complexity of functions on non-discrete spaces. For studying time complexity of real functions there exist

[1] Email: dareng@web.de
[2] Email: mschrode@inf.ed.ac.uk

1571-0661 © 2005 Elsevier B.V. Open access under CC BY-NC-ND license.
doi:10.1016/j.entcs.2004.06.038

already several approaches. They can be divided into two classes, depending on whether or not they are "bit-oriented". Bit-oriented models take into account the infinitesimal and approximative nature of real numbers and the finitary aspects of computations on digital computers, whereas non bit-oriented ones assume that an arithmetic operation can be performed on a real number in one step. An example of the latter is the real-RAM model by L. Blum, M. Shub, S. Smale (cf. [1]), examples of the former are the approach by K. Ko (cf. [7]) and Type-2 theory of effectivity (TTE) developed by K. Weihrauch (cf. [14,9]).

In this paper we use Type-2 theory of effectivity. TTE provides a computational model for functions on sets with cardinality of the continuum. The basic idea is to equip a given set X with a *representation*, which provides the objects of X with names and is formally a surjective partial function from the Baire space $\mathbb{N}^\omega$ onto X. On these names the actual computation is performed by a Type-2 machine. This kind of computability is called *relative computability*. Details can be found in Section 2 or in [14,10]. The computation by a Type-2 machine is potentially infinite and produces increasingly better approximations of the result. As a mathematical model to describe approximations, we use topological spaces (cf. [4]).

Since the computation by a Type-2 machine does not terminate, we have to define time complexity of functions on non-discrete spaces different to the discrete case. For every "precision" $m \in \mathbb{N}$, we count the finite number of steps which the realizing Type-2 machine needs to produce an approximation of the result with precision m. As we use infinite words for names rather than finite ones, there is, unlike the discrete case, no natural notion of a "size" of an input. So in general the time complexity of a relatively computable function has to be a function from $\mathbb{N}^\omega \times \mathbb{N}$ to $\mathbb{N}$. However, for the sake of simplicity we are interested in time complexity functions of the type $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$. *Almost-compact* representations are defined in such a way that indeed the time complexity of functions which are relatively computable w.r.t. almost-compact representations can be described by functions of the type $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$ (cf. Subsection 2.3). In [12], several nice characterizations of the class of Hausdorff spaces equipped with an almost-compact admissible representation are shown.

In Section 3 we repeat the definitions of inductive limit spaces and of Silva spaces. We show conditions under which these spaces have almost-compact admissible representations. As examples, we prove in Section 4 that the space of real polynomials, the ℓ_p-spaces, the space of analytic functions on the interval $[0, 1]$ and the space of distributions with compact support have, under suitable topologies, almost-compact admissible representations. The considered almost-compact admissible representation of the real polynomials admits computation of the evaluation operator in polynomial time.

1.1 Notation and Terminology

By $\overline{\mathbb{N}}$ we denote the set $\mathbb{N} \cup \{\infty\} = \{0, 1, \dots\} \cup \{\infty\}$, by $\mathbb{N}^*$ the set of finite words over $\mathbb{N}$ and by $\mathbb{N}^\omega := \{p \,|\, p : \mathbb{N} \to \mathbb{N}\}$ the set of infinite words over $\mathbb{N}$. For $p \in \mathbb{N}^\omega$, $n \in \mathbb{N}$ and $w \in \mathbb{N}^*$ let $p^{<n} := p(0) \dots p(n-1) \in \mathbb{N}^*$, $p^{>n} := p(n+1)p(n+2) \dots \in \mathbb{N}^\omega$ and $w\mathbb{N}^\omega := \{p \in \mathbb{N}^\omega \,|\, (\exists n \in \mathbb{N})\, p^{<n} = w\} \subseteq \mathbb{N}^\omega$. On $\mathbb{N}^\omega$ and $\overline{\mathbb{N}}$ we use the usual metrizable topologies $\tau_{\mathbb{N}^\omega} := \big\{ \bigcup_{w \in W} w\mathbb{N}^\omega \,\big|\, W \subseteq \mathbb{N}^* \big\}$ and $\tau_{\overline{\mathbb{N}}} := \big\{ U \subseteq \overline{\mathbb{N}} \,\big|\, \infty \in U \Longrightarrow (\forall^\infty n \in \mathbb{N})\, n \in U \big\}$, where the quantifier $(\forall^\infty n)$ means *for almost all* n, i.e. $(\exists n_0 \in \mathbb{N})(\forall n \geq n_0)$. The convergence relation of a topological space $\mathfrak{X} = (X, \tau_{\mathfrak{X}})$ is denoted by $\to_{\mathfrak{X}}$, i.e., we write $(x_n)_n \to_{\mathfrak{X}} x_\infty$ to express that $(x_n)_n$ converges to x_∞ in $\mathfrak{X}$, which is defined by $(\forall U \in \tau_{\mathfrak{X}})(x_\infty \in U \Longrightarrow (\forall^\infty n)\, x_n \in U)$, cf. [4]. The closure of a subset M in $\mathfrak{X}$ is denoted by $\mathrm{Cls}_{\mathfrak{X}}(M)$, and $\mathrm{dom}(f)$ denotes the domain of a partial function $\phi :\subseteq A \to B$.

2 Basics of Type Two Theory

We repeat in this section the notions of relative computability and complexity with respect to representations and motivate the notion of an almost-compact representation. Details can be found e.g. in [14,11,12].

2.1 Computability

Type-2 Theory of Effectivity defines computability for functions between sets with cardinality of the continuum by introducing computability for functions on the Baire space $\mathbb{N}^\omega$ via Type-2 machines and by transferring this computability notion via representations. Briefly, a k-ary Type-2 machine M is a usual Turing machine with changed semantics. It has k input tapes, several work tapes, and an *one-way* output tape and is controlled by a finite flowchart. In each cell of these tapes, one symbol from our alphabet $\mathbb{N}$ is stored. The domain of the function $\Gamma_M :\subseteq (\mathbb{N}^\omega)^k \to \mathbb{N}^\omega$ computed by M consists of those tuples $\bar{p} \in (\mathbb{N}^\omega)^k$ for which M with input $\bar{p}$ writes step by step infinitely many symbols onto the output tape, the corresponding sequence q is defined to be $\Gamma_M(\bar{p})$. Since M cannot change a symbol already written onto the output tape, each prefix of the output only depends on some prefixes of the inputs. This *finiteness property* implies that Γ_M is continuous w.r.t. the Baire space topology $\tau_{\mathbb{N}^\omega}$.

Given representations $\delta_i :\subseteq \mathbb{N}^\omega \to X_i$, a function $f : X_1 \times \ldots \times X_k \to$

X_{k+1} is called $(\delta_1, \ldots, \delta_{k+1})$-*computable* iff there exists a Type-2 machine M such that Γ_M *realizes* f with respect to these representations, meaning that $\gamma\big(\Gamma_M(p_1, \ldots, p_k)\big) = f\big(\delta_1(p_1), \ldots, \delta_1(p_k)\big)$ holds for all $p_1 \in \mathrm{dom}(\delta_1), \ldots, p_k \in \mathrm{dom}(\delta_k)$. Moreover, f is called $(\delta_1, \ldots, \delta_{k+1})$-*continuous* iff there is a continuous function g realizing f w.r.t. $\delta_1, \ldots, \delta_{k+1}$. As computable functions on the Baire space are continuous, relative computability implies relative continuity.

The property of admissibility is defined to reconcile relative continuity with mathematical continuity. We call $\delta :\subseteq \mathbb{N}^\omega \to X$ an *admissible* representation of a topological space $\mathfrak{X} = (X, \tau_{\mathfrak{X}})$ iff δ is continuous and for every continuous representation $\phi :\subseteq \mathbb{N}^\omega \to X$ there is some continuous function $g :\subseteq \mathbb{N}^\omega \to \mathbb{N}^\omega$ with $\phi = \delta \circ g$. From [9,10] we know

Proposition 2.1 *Let δ_i be an admissible representation of a topological space $\mathfrak{X}_i = (X_i, \tau_{\mathfrak{X}_i})$ for $i = 1 \ldots k+1$. Then a function $f : X_1 \times \ldots \times X_k \to X_{k+1}$ is $(\delta_1, \ldots, \delta_{k+1})$-continuous if and only if f is sequentially continuous (i.e., f maps convergent sequences to convergent sequences).*

2.2 Time complexity of Type-2 machines

We assign to a k-ary Type-2 machine M a time complexity function $\mathrm{Time}_M : \mathrm{dom}(\Gamma_M) \times \mathbb{N} \to \mathbb{N}$. For $\bar{p} \in \mathrm{dom}(\Gamma_M)$ and $n \in \mathbb{N}$, $\mathrm{Time}_M(\bar{p}, n)$ is defined to be the number of steps which M on input $\bar{p}$ executes until the prefix $\Gamma_M(\bar{p})(0) \ldots \Gamma_M(\bar{p})(n)$ is written onto the output tape [3]. We extend Time_M to a function of the type $2^{\mathrm{dom}(\Gamma_M)} \times \mathbb{N} \to \mathbb{N} \cup \{\infty\}$ defined by [4]

$$\mathrm{Time}_M(S, n) := \sup \big\{\mathrm{Time}_M(\bar{p}, n) \,\big|\, \bar{p} \in S\big\}$$

for $S \subseteq \mathrm{dom}(\Gamma_M)$ and $n \in \mathbb{N}$. For a function $t : \mathbb{N} \to \mathbb{N}$, we say that M *works on S in time t* iff $\mathrm{Time}_M(S, m) \le t(m)$ holds for all m. For arbitrary subsets S such a time bound might not exist. However, compact subsets $K \subseteq \mathrm{dom}(\Gamma_M)$ satisfy $\mathrm{Time}_M(K, n) < \infty$, since the function $\bar{p} \mapsto \mathrm{Time}_M(\bar{p}, n)$ is continuous by the finiteness property and since continuous functions map compact sets to compacts sets (cf. [4]). Hence elements of a compact subset of $\mathrm{dom}(\Gamma_M)$ share a common time bound $t : \mathbb{N} \to \mathbb{N}$.

[3] To make good sense of it, reading as well as writing a symbol a of the infinite alphabet $\mathbb{N}$ has to cost $\lg(a)$ steps rather than one step, where $\lg(a)$ denotes the length of the binary notation of the number a.

[4] For unbounded sets $B \subseteq \mathbb{N}$ let $\sup B := \infty$.

2.3 Complexity w.r.t. proper and almost-compact representations

Let $\delta :\subseteq \mathbb{N}^\omega \to X$ and $\gamma :\subseteq \mathbb{N}^\omega \to Y$ be admissible representations of topological spaces $\mathfrak{X}$ and $\mathfrak{Y}$, let $f : X \to Y$, $t : \mathbb{N} \to \mathbb{N}$ be functions, and let $A \subseteq X$. We say that f is (δ, γ)-*computable in time t on A* iff there is a Type-2 machine M such that Γ_M realizes f w.r.t. δ and γ and M works on $\delta^{-1}[A]$ in time t.

By the previous subsection, a time bound for A exists, if $\delta^{-1}[A]$ is compact. By continuity of δ, compactness of $\delta^{-1}[A]$ implies compactness of A (cf. [11,12]). A continuous representation such that the preimages of all compacts sets are compact is called *proper*. If δ is proper, then the time complexity of f can be estimated by a function $T : \mathcal{K}(\mathfrak{X}) \times \mathbb{N} \to \mathbb{N}$, where $\mathcal{K}(\mathfrak{X})$ denotes the set of compact subsets of $\mathfrak{X}$.

The signed-digit representation is an example of a proper admissible representation of the Euclidean space $\mathfrak{R} = (\mathbb{R}, \tau_\mathbb{R})$, cf. [14]. It may be defined by $\varrho_\mathbb{R}(p) := \sum_{i\in\mathbb{N}} \nu_\mathbb{Z}(p(i)) \cdot 2^{-i}$ for all $p \in \mathbb{N}^\omega$ such that $(\forall i \geq 1)\, \nu_\mathbb{Z}(p(i)) \in \{-1, 0, 1\}$, where $\nu_\mathbb{Z} : \mathbb{N} \to \mathbb{Z}$ is given by $\nu_\mathbb{Z}(2n) = -n$ and $\nu_\mathbb{Z}(2n+1) = n+1$.

We are now interested in representations δ which allow to estimate complexity by natural number functions. This means that complexity is measured by a *discrete* parameter on the input (and, of course, by the output precision). Since the existence of a time bound is only guaranteed on compact name sets $S \subseteq \mathrm{dom}(\delta)$, we have to require the domain of δ to be a countable union of compact sets. Moreover, it is reasonable to demand that it is possible to compute the index of (one of) the set(s) in which a given name $p \in \mathrm{dom}(\delta)$ lies. Note that the situation in discrete complexity theory is similar: the set Σ^l of words of length l over a finite alphabet Σ is a compact subset of the set Σ^* of all words, which is the countable union of the sets Σ^i. Furthermore, the length of a word can be computed.

These considerations motivates the following definition. We call δ an *almost-compact* representation iff there exists a computable [5] *size function* $\kappa_\delta : \mathrm{dom}(\delta) \to \mathbb{N}^k$ such that $\kappa_\delta^{-1}\{(a_1, \ldots, a_k)\}$ is compact for every $(a_1, \ldots, a_k) \in \mathbb{N}^k$. In the presence of such a size function κ_δ, we say that f is (δ, γ)-*computable in time $T : \mathbb{N}^{k+1} \to \mathbb{N}$ in κ_δ* iff there is a Type-2 machine M such that Γ_M realizes f w.r.t. δ and γ and

$$\mathrm{Time}_M\big(\kappa_\delta^{-1}\{a_1, \ldots, a_k\}, n\big) \leq T(a_1, \ldots, a_k,\, n)$$

holds for all $a_1, \ldots, a_k,\, n \in \mathbb{N}$. The signed-digit representation is an example of an almost-compact admissible representation: the corresponding size function $\kappa_{\varrho_\mathbb{R}} : \mathrm{dom}(\varrho_\mathbb{R}) \to \mathbb{N}$ can simply be defined by $\kappa_{\varrho_\mathbb{R}}(p) := \big|\nu_\mathbb{Z}(p(0))\big|$.

From [12] we obtain the following characterization theorem.

[5] w.r.t. canonical representations of $\mathrm{dom}(\delta)$ and $\mathbb{N}^k$

Theorem 2.2 *Let $\mathfrak{X}$ be a sequential Hausdorff space.*

(i) *The space $\mathfrak{X}$ has a proper admissible representation if and only if $\mathfrak{X}$ is separable metrizable.*

(ii) *The space $\mathfrak{X}$ has an almost-compact admissible representation if and only if $\mathfrak{X}$ is a direct limit (cf. Section 3) of compact metrizable spaces.*

3 Inductive limits and Silva spaces

The spaces we will deal with in Section 4 are topologized by suitable *inductive limit topologies*. Given a sequence $(\mathfrak{X}_m)_m = (X_m, \tau_m)_m$ of Hausdorff spaces, its *inductive limit* $\underrightarrow{\mathrm{Lim}}\,(\mathfrak{X}_m)_m$ is defined to be the topological space having $\bigcup_{m\in\mathbb{N}} X_m$ as its underlying set and

$$\underrightarrow{\mathrm{Lim}}\,(\tau_m)_m := \Big\{O \subseteq \bigcup_{m\in\mathbb{N}} X_m \;\Big|\; (\forall m \in \mathbb{N})\; O \cap X_m \in \tau_m\Big\} \tag{1}$$

as its topology.

Remark 3.1 Note that the inductive limit topology may equivalently be defined by the property of being the finest topology τ on the set $\bigcup_{m\in\mathbb{N}} X_m$ such that all the inclusion mappings $\mathfrak{X}_m \hookrightarrow \big(\bigcup_{m\in\mathbb{N}} X_m, \tau\big)$ are continuous.

If additionally for every a, b there is some c such that the spaces $\mathfrak{X}_a$ and $\mathfrak{X}_b$ are subspaces of $\mathfrak{X}_c$, then the inductive limit is called *directed*. From [9, Theorem 19] we obtain the following simple construction of an admissible representation of $\underrightarrow{\mathrm{Lim}}\,(\mathfrak{X}_m)_m$.

Proposition 3.2 *Let $\underrightarrow{\mathrm{Lim}}\,(\mathfrak{X}_m)_m$ be a directed limit of sequential Hausdorff spaces $\mathfrak{X}_m$, and let δ_m be an admissible representation of $\mathfrak{X}_m$ for $m \in \mathbb{N}$. Then the function $\delta :\subseteq \mathbb{N}^\omega \to \bigcup_{m\in\mathbb{N}} X_m$ defined by*

$$\delta(p) := x \;:\Longleftrightarrow\; x \in X_{p(0)} \;\wedge\; \delta_{p(0)}(p^{>0}) = x$$

is an admissible representation of $\underrightarrow{\mathrm{Lim}}\,(\mathfrak{X}_m)_m$.

In general, for non-directed inductive limits, the representation constructed in Proposition 3.2 fails to be admissible. For Silva spaces, however, one can show admissibility to hold even in the non-directed case. A *Silva space* (cf. [13], [3, Chapter 4.2.3]) is defined to be an inductive limit of a sequence of Banach spaces (complete normed vector spaces) $(\mathfrak{X}_m)_m$ such that for every m there exists a continuous inclusion $\iota_m^{m+1} : \mathfrak{X}_m \to \mathfrak{X}_{m+1}$ which is *compact*, meaning that $\mathrm{Cls}_{\mathfrak{X}_{m+1}}\big(\iota_m^{m+1}(B_m)\big)$ is compact in $\mathfrak{X}_{m+1}$, where $B_m := \big\{x \in X_m \;\big|\; \|x\|_m < 1\big\}$ denotes the unit ball in $\mathfrak{X}_m$.

Compactness of the mappings ι_m^{m+1} allows us to interpret the inductive limit topology of a Silva space in a different way suitable for our purposes:

Proposition 3.3 *Let $\mathfrak{X} = (X, \tau_{\mathfrak{X}})$ be a Silva space which is the inductive limit of Banach spaces $\mathfrak{X}_m = (X_m, \|.\|_m)$, with compact inclusions $\iota_m^{m+1} : \mathfrak{X}_m \to \mathfrak{X}_{m+1}$, $m \in \mathbb{N}$. For every $m, n \in \mathbb{N}$ define $K_{m,n} := \mathrm{Cls}_{\mathfrak{X}_{m+1}}\big(\iota_m^{m+1}(n \cdot B_m)\big)$.*

(i) *The topological spaces $\mathfrak{K}_{m,n} := (K_{m,n}, \tau_{\mathfrak{X}}|_{K_{m,n}})$, $m, n \in \mathbb{N}$, are compact, separable and metrizable.*

(ii) *The inductive limit topology of the $\mathfrak{K}_{m,n}$, $m, n \in \mathbb{N}$, coincides with the topology $\tau_{\mathfrak{X}}$, i.e., $\mathfrak{X} = \varinjlim \mathfrak{K}_{m,n}$.*

Proof. (i) is a well-known consequence from the theory of Silva spaces, it also follows from e.g. [6, Proposition 8.5.3].

To show (ii), denote the topology of $\varinjlim \mathfrak{K}_{m,n}$ by $\tau_{\mathfrak{K}}$. For every $O \in \tau_{\mathfrak{X}}$ and every $m, n \in \mathbb{N}$, the set $O \cap K_{m,n}$ is open in $\mathfrak{K}_{m,n}$, thus according to Remark 3.1, O is open in $\tau_{\mathfrak{K}}$. This shows $\tau_{\mathfrak{X}} \subseteq \tau_{\mathfrak{K}}$.

For the opposite inclusion note that for every $m, n \in \mathbb{N}$ the inclusion $(n \cdot B_m, \tau_{\mathfrak{X}_m}|_{n \cdot B_m}) \hookrightarrow \mathfrak{K}_{m,n}$ is continuous because so does the inclusion $\mathfrak{X}_m \hookrightarrow \mathfrak{X}$. If then $O \in \tau_{\mathfrak{K}}$, then the sets $O \cap K_{m,n}$, $m, n \in \mathbb{N}$ are open in $\mathfrak{K}_{m,n}$, therefore the $O \cap n \cdot B_m$ are open in $\tau_{\mathfrak{X}_m}$. Thus $O \cap X_m = \bigcup_{n \in \mathbb{N}} O \cap n \cdot B_m$ is open in $\mathfrak{X}_m$ for every $m \in \mathbb{N}$. With Remark 3.1 we get $O \in \tau_{\mathfrak{X}}$. Thus $\tau_{\mathfrak{K}} \subseteq \tau_{\mathfrak{X}}$. $\square$

From Proposition 3.3 and Theorem 2.2 we immediately get the

Theorem 3.4 *Every Silva space has an almost-compact admissible representation.*

Examples of Silva spaces relevant to analysis are presented in Section 4. Note that an infinite dimensional Silva space can never be metrizable (cf. [3, Proposition 4.2.3.5]).

4 Examples

4.1 Polynomials

As a first simple example, we consider the set $\mathcal{P}$ of polynomials on the reals. A straightforward representation $\psi_{\mathcal{P}}$ of $\mathcal{P}$ can be constructed by using an admissible representation of $\mathbb{R}^{\mathbb{N}}$ like[6] $\boxtimes_{i=0}^{\infty} \varrho_{\mathbb{R}}$. We define $\psi_{\mathcal{P}}$ by

$$\psi_{\mathcal{P}}(q)(x) := \sum_{i \in \mathbb{N}} a_i \cdot x^i$$

[6] cf. [10, Section 4.1.4]; in [14, Definition 3.3.3] this representation is denoted by $[\varrho_{\mathbb{R}}]^{\omega}$.

for all $q \in \mathrm{dom}(\boxtimes_{i=0}^{\infty} \varrho_{\mathbb{R}})$ such that $(a_i)_i := (\boxtimes_{i=0}^{\infty} \varrho_{\mathbb{R}})(q)$ is an eventually vanishing sequence. From [10, Proposition 4.1.6] it follows that $\psi_{\mathcal{P}}$ is an admissible representation of a topological space $\mathfrak{X}_{\mathcal{P}}$ which is isomorphic (via the obvious isomorphism) to the subspace of $\mathbb{R}^{\mathbb{N}}$ consisting of all eventually vanishing sequences.

It is easy to see that the evaluation function $\mathrm{eval} : \mathcal{P} \times \mathbb{R} \to \mathbb{R}$ is not even $(\psi_{\mathcal{P}}, \varrho_{\mathbb{R}}, \varrho_{\mathbb{R}})$-continuous, because the names of $\psi_{\mathcal{P}}$ do not yield continuously accessible information about the degree of the encoded polynomial. This gives rise to the following representation $\varrho_{\mathcal{P}}$ which explicitly carries an upper bound of the degree of the polynomial. It is defined by

$$\varrho_{\mathcal{P}}(q) := P \;:\Longleftrightarrow\; \psi_{\mathcal{P}}(q^{>0}) = P \;\wedge\; \big(\forall i > q(0)\big)\, a_i = 0\,,$$

where $(a_i)_i := (\boxtimes_{i=0}^{\infty} \varrho_{\mathbb{R}})(q^{>0})$. Clearly, eval is $(\varrho_{\mathcal{P}}, \varrho_{\mathbb{R}}, \varrho_{\mathbb{R}})$-computable. We define $\kappa_{\varrho_{\mathcal{P}}} : \mathrm{dom}(\varrho_{\mathcal{P}}) \to \mathbb{N}^2$ by $\kappa_{\varrho_{\mathcal{P}}}(q) := \big(q(0), \max_{i \le q(0)} |z_i|\big)$, where z_i is the integer part of the i-th coefficient encoded in $q^{>0}$. Since $\varrho_{\mathbb{R}}$ and thus $(\boxtimes_{i=0}^{\infty} \varrho_{\mathbb{R}})$ are proper, $\kappa_{\varrho_{\mathcal{P}}}^{-1}\{(d,e)\}$ is compact for all $d, e \in \mathbb{N}$. Let $\mathfrak{P}_m$ be the subspace of $\mathfrak{X}_{\mathcal{P}}$ consisting of all polynomials of degree at most m. The restriction δ_m of $\psi_{\mathcal{P}}$ to $\mathfrak{P}_m$ is an admissible representation of $\mathfrak{P}_m$. Hence $\varrho_{\mathcal{P}}$ is an admissible representation of $\mathfrak{P} := \varinjlim \mathfrak{P}_m$ by being constructed as in Proposition 3.2.

In order to prove that $\mathfrak{P}$ is not metrizable, we use the well-known and easily provable fact that the convergence relation $\to_{\mathfrak{Y}}$ of a first-countable topological space $\mathfrak{Y}$ has the following property, which is often denoted by "(L4)":

(L4) if $(y_{i,j})_j \to_{\mathfrak{Y}} z_i$ for every $i \in \mathbb{N}$ and $(z_k)_k \to_{\mathfrak{Y}} z_\infty$, then there are functions $\varphi, \psi : \mathbb{N} \to \mathbb{N}$ with $(y_{\varphi(n),\psi(n)})_n \to_{\mathfrak{Y}} z_\infty$,

cf. [4, Ex. 1.7.18]. An example proving $\mathfrak{P}$ to fail (L4) is provided by the sequences of polynomials $(f_{i,j})_{i,j}$ and $(g_k)_{k \le \infty}$ defined by

$$f_{i,j}(x) := 1/2^j \cdot x^i + 1/2^i\,,\;\; g_i(x) := 1/2^i \text{ and } g_\infty(x) := 0\,.$$

By Proposition 3.2, we have $(\forall i \in \mathbb{N})\,(f_{i,j})_j \to_{\mathfrak{P}} g_i$ and $(g_k)_k \to_{\mathfrak{P}} g_\infty$. Assume that there are functions $\varphi, \psi : \mathbb{N} \to \mathbb{N}$ with $(f_{\varphi(n),\psi(n)})_n \to_{\mathfrak{P}} g_\infty$. Proposition 3.2 implies that on the one hand $(\varphi(n))_n$ is bounded and on the other hand the sequence $(1/2^{\varphi(n)})_n = (f_{\varphi(n),\psi(n)}(0))_n$ converges to $0 = g_\infty(0)$, a contradiction. Therefore $\mathfrak{P}$ does not satisfy Axiom (L4). Hence $\mathfrak{P}$ is neither first-countable nor metrizable. We summarize these results:

Theorem 4.1 *The space $\mathfrak{P}$ of real polynomials has an almost-compact admissible representation and is not metrizable.*

It is well-known that integer multiplication can be done in polynomial time. From this fact one can deduce that the evaluation function $\mathrm{eval} : \mathcal{P} \times \mathbb{R} \to$

$\mathbb{R}$ is $(\varrho_{\mathcal{P}}, \varrho_{\mathbb{R}}, \varrho_{\mathbb{R}})$-computable in polynomial time in $\kappa_{\varrho_{\mathcal{P}}}$, $\kappa_{\varrho_{\mathbb{R}}}$ and the output precision. More precisely, there is a Type-2 machine M and a polynomial $T : \mathbb{N}^2 \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that Γ_M realizes eval w.r.t. $\varrho_{\mathcal{P}}$ and $\varrho_{\mathbb{R}}$ and

$$\mathrm{Time}_M(q, r, n) \leq T\big(\kappa_{\varrho_{\mathcal{P}}}(q), \kappa_{\varrho_{\mathbb{R}}}(r), n\big)$$

holds for all $q \in \mathrm{dom}(\varrho_{\mathcal{P}})$, $r \in \mathrm{dom}(\varrho_{\mathbb{R}})$ and $n \in \mathbb{N}$.

4.2 ℓ_p-spaces

For $p \geq 1$, the vector space ℓ_p consists of all real sequences $(a_i)_i$ with $\|(a_i)_i\|_p := \sqrt[p]{\sum_{i\in\mathbb{N}} |a_i|^p} < \infty$. An almost-compact admissible representation $\varrho_{\ell_p} :\subseteq \mathbb{N}^\omega \to \ell_p$ can be constructed similar to $\varrho_{\mathcal{P}}$ by

$$\varrho_{\ell_p}(q) = x \;:\Longleftrightarrow\; (\boxtimes_{i=0}^{\infty} \varrho_{\mathbb{R}})(q^{>0}) = x \;\wedge\; q(0) \geq \|x\|_p \,.$$

The size function $\kappa_{\varrho_{\ell_p}} : \mathrm{dom}(\varrho_{\ell_p}) \to \mathbb{N}$ can be chosen as $\kappa_{\varrho_{\ell_p}}(q) := q(0)$. An analogue representation has been investigated by V. Brattka in [2, Section 15]. We omit the proof that the final topology of ϱ_{ℓ_p} is a vector space topology.

4.3 Real analytic functions on the unit interval

An important space of functions considered in functional analysis and numerical mathematics is the vector space $A([0, 1])$ of real analytic functions on the unit interval. It may be defined as those functions $f : [0, 1] \to \mathbb{R}$ that for every $x \in [0, 1]$ may be expanded into a Taylor series which is convergent in some complex neighbourhood of x.

The classical way to topologize the vector space $A([0, 1])$ is to identify its elements with the Silva space of those functions on the unit interval that have a unique holomorphic extension into some complex neighbourhood of $[0, 1]$ (cf. [8]). We shortly recall this construction:

Consider some bounded domain $U \subseteq \mathbb{C}$ and denote by $H_\infty(U)$ the vector space of continuous functions on the closure $\mathrm{Cls}_{\mathbb{C}}(U)$ which are holomorphic on U. By elementary facts from complex analysis, one can see that $H_\infty(U)$ is a closed subspace of the space of continuous functions on $\mathrm{Cls}_{\mathbb{C}}(U)$, in particular it is a separable Banach space with respect to the norm $\|f\|_{H_\infty(U)} := \sup_{z\in U} |f(z)|$.

Take now an arbitrary sequence $(U_m)_{m\in\mathbb{N}}$ of bounded domains in $\mathbb{C}$ such that we have $U_m \supseteq \mathrm{Cls}_{\mathbb{C}}(U_{m+1})$, $m \in \mathbb{N}$, and additionally $\bigcap_{m\in\mathbb{N}} U_m = [0, 1]$ and define for every m the Banach space $\mathcal{H}_m := (H_\infty(U_m), \tau_m)$, where τ_m is the norm topology on $H_\infty(U_m)$. This gives a sequence of linear, continuous inclusions $\mathcal{H}_1 \overset{\iota_1^2}{\hookrightarrow} \mathcal{H}_2 \overset{\iota_2^3}{\hookrightarrow} \mathcal{H}_3 \overset{\iota_3^4}{\hookrightarrow} \ldots$, where the $\iota_m^{m+1} : \mathcal{H}_m \to \mathcal{H}_{m+1}$, $m \in \mathbb{N}$, are defined as restrictions, i.e. $f \mapsto f|_{\mathrm{Cls}_{\mathbb{C}}(U_{m+1})}$. (Note that we can interpret the ι_m^{m+1} as inclusions as their the injectivity follows from the identity theorem for holomorphic functions).

Also with the identity theorem we can for every $m \in \mathbb{N}$ interpret the mapping $\iota_m : H_\infty(U_m) \to A([0,1])$, $f \mapsto f|_{[0,1]}$ as an inclusion, which gives us the canonical identification $A([0,1]) = \bigcup_{m \in \mathbb{N}} H_\infty(U_m)$. This also yields a standard way to topologize the space $A([0,1])$:

Definition 4.2 Define the topological space $\mathcal{A}$ as the inductive limit

$$(A([0,1]), \tau_{\mathcal{A}}) := \varinjlim (\mathcal{H}_m)_m.$$

Every real analytic function is thus interpreted as a holomorphic function in some neighbourhood of $[0,1]$. If the neighbourhoods $(U_m)_m$ are chosen in a suitable way, the index $m \in \mathbb{N}$ for which $f \in H_\infty(U_m)$ gives information on the radius of convergence of the Taylor series expansions of f. The open (resp. closed) sets in $\mathcal{A}$ are those subsets $O \subseteq A([0,1])$ for which the intersection with every space $\mathcal{H}_m$, $m \in \mathbb{N}$, is open (resp. closed).

Note that as the spaces $\mathcal{H}_m$ are closed subspaces of spaces of continuous functions on compact sets, the theorem of Arzela-Ascoli is applicable and immediately yields the fact that the closed unit ball in $\mathcal{H}_m$ is relatively compact when restricted to the smaller set U_{m+1}. Using Theorem 3.4 we have the

Theorem 4.3 *The space $\mathcal{A}$ is a Silva space and thus has an almost-compact admissible representation.*

We wish to describe an almost-compact admissible representation of $\mathcal{A}$ in some more detail. Define analogously to Proposition 3.3 for every $m, n \in \mathbb{N}$ the set $K_{m,n} := \mathrm{Cls}_{\mathcal{A}}(n \cdot B_m)$, where B_m is the unit ball in $\mathcal{H}_m$, and the compact metrizable space $\mathfrak{K}_{m,n} := (K_{m,n}, \tau_{\mathcal{A}}|_{K_{m,n}})$. By Theorem 2.2(i), for every $m, n \in \mathbb{N}$ there is an admissible representation $\delta_{m,n} :\subseteq \mathbb{N}^\omega \to K_{m,n}$ with compact domain. Using Proposition 3.2, we get the

Proposition 4.4 *Define $\delta_{\mathcal{A}} :\subseteq \mathbb{N}^\omega \to \mathcal{A}$ by*

$$\delta_{\mathcal{A}}(p) := f \;:\Longleftrightarrow\; f \in K_{p(0),p(1)} \;\wedge\; \delta_{p(0),p(1)}(p^{>1}) = f.$$

Then $\delta_{\mathcal{A}}$ is an admissible almost-compact representation of $\mathcal{A}$.

Proof. From Proposition 3.2 we get admissibility and for the computable function $\kappa_{\delta_{\mathcal{A}}} : \mathrm{dom}(\delta_{\mathcal{A}}) \to \mathbb{N}^2$ defined by $\kappa_{\delta_{\mathcal{A}}}(p) := (p(0), p(1))$ the preimages $\kappa_{\delta_{\mathcal{A}}}^{-1}(m,n) = \delta_{m,n}^{-1}(K_{m,n})$, $m, n \in \mathbb{N}$, are compact. □

Remark 4.5 We do not exactly specify the neighbourhoods U_m and the representations $\delta_{m,n}$, as their choice will depend heavily on the application. For the $\delta_{m,n}$ one can e.g. take a proper admissible representation δ of the Banach space $C([0,1])$ of continuous functions on $[0,1]$ and take as $\delta_{m,n}$ the restriction of δ to the sets $K_{m,n}$ (note that the inclusion of $\mathcal{A}$ into $C([0,1])$ with its standard topology is continuous, thus the $K_{m,n}$, $m,n \in \mathbb{N}$, are compact in this space and have compact preimages under δ).

4.4 Distributions with compact support

A very important space in distribution theory is the space of distributions over $\mathbb{R}$ with compact support. Recall that the support of a distribution T over $\mathbb{R}$ is the set of those $x \in \mathbb{R}$ such that for every neighbourhood U of x there exists a test function φ with $supp(\varphi) \subseteq U$ and $T(\varphi) \neq 0$. [7]

A very classical fact is that this space may be identified with the dual space[8] $\mathcal{E}'$ of the space $\mathcal{E}$ of infinitely differentiable functions on $\mathbb{R}$ (see e.g. [5, Theorem 2.3.1]).

We shortly describe the spaces $\mathcal{E}$ and $\mathcal{E}'$ and show the existence of an almost-compact admissible representation of $\mathcal{E}'$ under a suitable topology.

Consider the vector space $C^\infty(\mathbb{R})$ of infinitely differentiable functions on $\mathbb{R}$ with the semi-norms $\|f\|_{k,m} := \sup\big\{|f^{(j)}(x)| \;\big|\; |x| \leq m\,,\, j \leq k\big\}$, $m,k \in \mathbb{N}$. With the metric defined by $d(f,g) := \sum_{k,m=0}^{\infty} 2^{-(k+m)} \cdot \frac{\|f-g\|_{k,m}}{1+\|f-g\|_{k,m}}$, this space is a complete and separable metric space, classically denoted by $\mathcal{E}$. A basis of the neighbourhood filter of zero in $\mathcal{E}$ is given by the sets

$$U_{k,m,n} := \big\{f \in C^\infty(\mathbb{R}) \;\big|\; \|f\|_{k,m} \leq 1/(n+1)\big\}, \quad k,m,n \in \mathbb{N}. \tag{2}$$

The standard vector space topology on the dual $\mathcal{E}'$ is given by the topology τ_{pc} of "precompact convergence" for which a basis of the neighbourhood filter of zero is given by the sets

$$V_{K,\varepsilon} := \big\{y \in \mathcal{E}' \;\big|\; \sup_{x \in K} |y(x)| \leq \varepsilon\big\}\,, \quad \varepsilon > 0,\ K \text{ relatively compact in } \mathcal{E}.$$

We will denote by $\mathcal{E}'$ also the dual of $\mathcal{E}$ equipped with this topology.

With the zero neighbourhoods $U_{k,m,n}$ of $\mathcal{E}$ as in (2), we define the polar sets

$$U^o_{k,m,n} := \big\{T \in \mathcal{E}' \;\big|\; \sup_{f \in U_{k,m,n}} |T(f)| \leq 1\big\}\,.$$

[7] The space of test functions is defined as those $\varphi \in C^\infty(\mathbb{R})$ such that its support, defined as $supp(\varphi) := \mathrm{Cls}_{\mathbb{R}}\left(\{y|\varphi(y) \neq 0\}\right)$, is compact.

[8] The dual space X' of a topological vector space X is the vector space of continuous linear functionals on X.

The properties of the $U^o_{k,m,n}$ needed in our context are well-known facts from the theory of locally convex spaces. They are summarized in the following

Proposition 4.6

(i) *For every $k, m, n \in \mathbb{N}$ the set $U^o_{k,m,n}$ is a compact, separable and metrizable subset of $\mathcal{E}'$.*

(ii) *For every $k, m, n \in \mathbb{N}$ the linear span $[U^o_{k,m,n}]$ of $U^o_{k,m,n}$ is a Banach space with respect to the norm $\|T\|_{U^o_{k,m,n}} := \inf\{\lambda > 0 \mid T \in \lambda \cdot U^o_{k,m,n}\}$ and the embeddings $([U^o_{k,m,n}], \|\cdot\|_{U^o_{k,m,n}}) \hookrightarrow \mathcal{E}'$ are continuous.*

(iii) *With $\mathfrak{U}_k := ([U^o_{k,k,k}], \|\cdot\|_{U^o_{k,k,k}})$, $k \in \mathbb{N}$, we have $\mathcal{E}' = \varinjlim \mathfrak{U}_k$ as a Silva space.*

Proof. Compactness is the well-known theorem of Alaoglou-Bourbaki (see e.g. [6, Theorem 8.5.2]) and the other assertions in (i) follow from standard duality theory (e.g. [6, Chapter 8.5]). (ii) and (iii) are folklore from the theory of locally convex spaces. □

Theorem 3.4 and Proposition 4.6 yield

Theorem 4.7 *The space $\mathcal{E}$ has an almost-compact admissible representation.*

Using the $U^o_{k,m,n}$, $k, m, n \in \mathbb{N}$, we can construct an almost-compact admissible representation $\delta_{\mathcal{E}'}$ as in Proposition 4.4:

Proposition 4.8 *Let $\delta_{k,m,n}$ be an admissible representation with compact domain of the compact metrizable space $(U^o_{k,m,n}, \tau_{pc}|_{U^o_{k,m,n}})$. Then the representation $\delta_{\mathcal{E}'} :\subseteq \mathbb{N}^\omega \to \mathcal{E}'$ defined by*

$$\delta_{\mathcal{E}'}(p) = T \;:\Longleftrightarrow\; T \in U^o_{p(0),p(1),p(2)} \;\wedge\; \delta_{p(0),p(1),p(2)}(p^{>2}) = T$$

is an almost-compact admissible representation of $\mathcal{E}'$.

The representation $\delta_{\mathcal{E}'}$ carries as the prefixes of a name of a distribution T the indices k, m, n of the corresponding compact set $U^o_{k,m,n}$. We can interpret this prefix as follows: If a distribution T is contained in the set $U^o_{k,m,n}$, then we have $|T(\varphi)| \leq n \cdot \sup\left\{|\varphi^{(j)}(x)| \;\middle|\; j \leq k, |x| \leq m\right\}$ for all test functions φ, thus T is of order [9] at most k with support contained in the interval $[-m, m]$.

From [5, Theorem 2.3.10] we get that a distribution with order k and $supp(T) \subseteq [-m, m]$ is contained in some $U^o_{k,m,n}$ and can be extended to a

[9] The order of a distribution T is the smallest $k \in \mathbb{N} \cup \{\infty\}$ such that there exist a compact K and a constant $C > 0$ such that $|T(\varphi)| \leq C \cdot \sup\left\{|\varphi^{(j)}(x)| \;\middle|\; j \leq k, x \in K\right\}$ for all test functions φ.

continuous linear functional on the space $C^k(\mathbb{R})$ of k-times differentiable functions. Then n gives a bound for the operator norm of T in that dual space.

Thus our almost-compact admissible representation $\delta_{\mathcal{E}'}$ has as prefixes of a name of a distribution T bounds for the order of T, for the support of T and for the norm of T viewed as a continuous linear functional on $C^k(\mathbb{R})$.

References

[1] L. Blum, M. Shub, S. Smale: *On a theory of computations and complexity over the real numbers: NP-completeness, recursive functions and universal machines*, Bulletin of the American Mathematical Society 21(1) (1989), pp. 1–46

[2] V. Brattka: *Computability of Banach space principles*, Informatik-Berichte 286, FernUniversität Hagen (2001)

[3] R. Cristescu: *Topological Vector Spaces*, Noordhoff International Publishing, Leyden (1977)

[4] R. Engelking: *General Topology*, Heldermann, Berlin (1989)

[5] L. Hörmander: *The Analysis of Linear Partial Differential Operators I*, Springer, Berlin (1983)

[6] H. Jarchow: *Locally convex spaces*, B.G. Teubner, Stuttgart (1981)

[7] K-I Ko: *Complexity Theory of Real Functions*, Birkhäuser, Boston (1991)

[8] A. Martineau: *Sur la topologie des espaces de fonctions holomorphes*, Math. Ann. **163** (1966), pp. 62–88

[9] M. Schröder: *Extended Admissibility*, Theoretical Computer Science, Vol. 284 (2002), pp. 519–538

[10] M. Schröder: *Admissible Representations for Continuous Computations*, PhD-thesis, Informatik-Berichte 299, FernUniversität Hagen (2002)

[11] M. Schröder: *Spaces Allowing Type-2 Complexity Theory Revisited*, In: Workshop on Computability and Complexity in Analysis, Proceedings of the international CCA conference 2003, Informatik-Berichte 302, FernUniversität Hagen (2003)

[12] M. Schröder: *Spaces Allowing Type-2 Complexity Theory Revisited*, To appear in Mathematical Logic Quarterly (2004)

[13] J.S. Silva: *Su certi classi di spazi localmente convessi importanti per le applicazioni*, Universita di Roma, Rendiconti di Matematico, 14 (1955), pp. 388–410

[14] K. Weihrauch: *Computable Analysis*, Springer, Berlin (2000)