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Monadic Σ^1_1 and Modal Logic with Quantified Binary Relations

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Abstract

We investigate the expressive power of a range of modal logics extended with second-order prenex quantification of binary and unary relations. Our principal result is that $\Sigma^1_1(BML^=)$, i.e., Boolean modal logic extended with the identity modality and existential prenex quantification of binary and unary relations, translates into *monadic* Σ^1_1. We also briefly discuss a variety of decidability results in multimodal logic implied by our result.

Keywords: Monadic Σ^1_1, Boolean modal logic, expressive power, decidability.

1 Introduction

Modal correspondence theory concerns itself with the classification of formulae of modal logic according to whether they define elementary classes of Kripke frames. On the level of frames, modal logic can be regarded as a fragment of monadic Π^1_1, also known as $\forall MSO$. Hence correspondence theory studies a special fragment of $\forall MSO$. When inspecting a modal formula from the point of view of frames, one universally quantifies the proposition symbols occurring in the formula. It is therefore rather natural to ask what happens if one also quantifies binary relation symbols occurring in (the standard translation of) a modal formula. This question is studied in [10], where the focus is on the expressive power of multimodal logic with universal prenex quantification of (not necessarily all of the) binary and unary relation symbols occurring in a formula. It is natural to ask whether there exists

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any class of multimodal frames definable in this logic, let us call it $\Pi^1_1(ML)$, but not definable in monadic second-order logic (MSO). This can be regarded as a question of modal correspondence theory. In this case, however, the correspondence language is MSO rather than FO. In addition to [10], modal logic with quantification of binary relations is investigated for example in [2,11,15].

In the current paper we study two systems of multimodal logic with existential second-order prenex quantification of binary and unary relation symbols. We warm up by showing that formulae of $\Sigma^1_1(ML)$, i.e., ordinary multimodal logic with existential second-order prenex quantification of binary and unary relation symbols, translate into $\exists MSO(MLE)$, i.e., multimodal logic with the universal modality and existential second-order prenex quantification of *only unary* predicate symbols. The method of proof is based on the definition of the accessibility relation in a *largest filtration* (see [1] for the definition). We then push the method and establish that $\Sigma^1_1(BML^=)$, i.e., Boolean modal logic with the identity modality and existential second-order prenex quantification of binary and unary relation symbols, translates into monadic Σ^1_1, also known as $\exists MSO$. Note that both of these results directly imply that $\Pi^1_1(ML)$ translates into $\forall MSO$, and therefore show that MSO would be a dull correspondence language for correspondence theory of $\Pi^1_1(ML)$.

It could be argued that $\{\neg, \cup, \cap, \circ, ^*, ^\smile, E, \neq\}$ is, more or less, the core collection of operations on binary relations used for defining extensions of modal logic for the purposes of applications. Here $\neg$, $\cup$, $\cap$, $\circ$, *, and $^\smile$ denote the complement, union, intersection, composition, transitive reflexive closure, and converse operations, respectively. E and $\neq$ denote the constant operations universal modality and difference modality. Logics where these operations are used (possibly together with other operations) include for example PDL [3,7], Boolean modal logic [4,12], description logics [8,14], modal logic with the universal modality [5], and modal logic with the difference modality [17]. The fact that $BML^=$ subsumes a large number of typical extensions of modal logic is one of the motivations for our study.

We describe a possible application of our result concerning $\Sigma^1_1(BML^=)$ (cf. Theorem 4.11). Let $\mathcal{D}$ be a class of Kripke frames (W, R_1), and consider the class $\mathcal{C} = \{\ (W, R_1, R_2, ...) \mid R_i \subseteq W \times W,\ (W, R_1) \in \mathcal{D}\ \}$ of multimodal Kripke frames. Assume that the $\forall MSO$-theory of $\mathcal{D}$, that is, the $\forall MSO$-theory of the class of R_1-reducts of structures in $\mathcal{C}$, is decidable. For example, we could assume that $\mathcal{C}$ is the class of countably infinite frames $(W, R_1, R_2...)$, where R_1 is a dense linear ordering without endpoints (see [16]). Assume we wish to know whether the satisfiability problem of multimodal logic (perhaps extended with, say, the difference modality) with respect to $\mathcal{C}$ is decidable. By Theorem 4.11 below, we immediately see that, indeed, it is. Theorem 4.11 implies a whole range of decidability results for multimodal logic. We note that there exists a large body of knowledge concerning structures and classes of structures with a decidable MSO- (and therefore $\forall MSO$-) theory, see [18] for example.

Another motivation for our study is related to descriptive complexity theory [9]. FO^2 is the fragment of first-order logic, where the use of only two variables is allowed. $\Sigma^1_1(FO^2)$ is the extension of this logic with existential second-order prenex quantification. In [6], Grädel and Rosen pose the question whether there exists any class of finite directed graphs definable in $\Sigma^1_1(FO^2)$, but not definable in $\exists MSO$. Lutz, Sattler, and Wolter show in [13] that $BML^=$ extended with the converse operator is expressively complete for FO^2. Therefore, in order to show that $\Sigma^1_1(FO^2)$ translates into $\exists MSO$, one would have to modify our proof such that it takes into account the possibility of using the converse operation. We have succeeded neither in this, nor in finding a $\Sigma^1_1(FO^2)$ definable class of directed graphs that is not definable in $\exists MSO$. However, we find modal logic a promising framework for working on this problem.

2 Preliminaries

In this section we introduce technical notions that occupy a central role in the rest of the discourse.

2.1 Syntax and Semantics

With a model we mean a model of predicate logic. A *pointed model* is a pair (M, w), where M is a model and $w \in Dom(M)$. We only consider models associated with a relational vocabulary containing unary and binary relation symbols. If V is a vocabulary, we let V_1 and V_2 denote the sets of unary and binary relation symbols in V, respectively. The following *BNF* determines the set $MP(V_2)$ of *modal parameters* over V_2:

$$\mathcal{M} ::= id \mid R \mid \neg\mathcal{M} \mid (\mathcal{M} \cap \mathcal{M})$$

Here $R \in V_2$ and $id \notin V$ is a constant relation symbol. The following BNF determines the set of formulae of $BML^=$ over vocabulary V:

$$\varphi ::= P \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \langle\mathcal{M}\rangle\varphi$$

Here $P \in V_1$ and $\mathcal{M} \in MP(V_2)$. Operators $\langle\mathcal{M}\rangle$ are called *diamonds*. The *modal depth* of a formula φ, or $Md(\varphi)$, is the maximum number of nested diamonds in φ.

Let M be a model. The *extension* $\mathcal{M}^M$ of a modal parameter $\mathcal{M}$ over M is a binary relation over $Dom(M)$. The extension of $R \in V_2$ over M is simply the interpretation R^M of the symbol R. The extension id^M of the symbol id is $\{(w, w) \mid w \in Dom(M)\}$. Other modal parameters are interpreted recursively such that $\neg\mathcal{M}^M = (Dom(M) \times Dom(M)) \setminus \mathcal{M}^M$ and $(\mathcal{M} \cap \mathcal{N})^M = \mathcal{M}^M \cap \mathcal{N}^M$. The satisfaction relation $\Vdash$ of $BML^=$ is defined as follows:

$$(M, w) \Vdash P \quad \Leftrightarrow \quad w \in P^M,$$

$$(M, w) \Vdash \neg\varphi \quad \Leftrightarrow \quad (M, w) \nVdash \varphi,$$

$$(M, w), \Vdash \varphi \wedge \psi \quad \Leftrightarrow \quad (M, w) \Vdash \varphi \text{ and } (M, w) \Vdash \psi,$$

$$(M, w) \Vdash \langle\mathcal{M}\rangle\, \varphi \quad \Leftrightarrow \quad \exists u \in Dom(M) \text{ such that } (w, u) \in \mathcal{M}^M \text{ and } (M, u) \Vdash \varphi.$$

For each formula φ, we let $||\varphi||^M$ denote the set $\{w \in Dom(M) \mid (M,w) \Vdash \varphi\}$. The set $||\varphi||^M$ is called the *extension* of φ over M. We write $\varphi \Vdash \psi$ if $(M,w) \Vdash \varphi \Rightarrow (M,w) \Vdash \psi$ for all pointed models (M,w).

A formula φ of $\Sigma^1_1(BML^=)$ of vocabulary V is a formula of type $\exists S_1 ... \exists S_n \exists P_1 ... \exists P_m(\psi)$, where the variables S_i are binary and P_i unary relation symbols, and ψ is a $BML^=$ formula of vocabulary $V \cup \{S_1, ..., S_n, P_1, ..., P_m\}$. We define $(M,w) \Vdash \varphi$ if there exists an expansion $M' = (M, S_1^{M'}, ..., S_n^{M'}, P_1^{M'}, P_m^{M'})$ of the model M such that $(M', w) \Vdash \psi$. We define the logic $\Pi^1_1(BML^=)$ similarly, but with universal second-order quantifiers instead of existential ones. ML is the fragment of $BML^=$, where the modal parameters are required to be atomic binary relation symbols in the vocabulary under discourse. MLE is the extension of ML with the universal diamond $\langle E \rangle$, i.e., the diamond $\langle \neg(id \cap \neg id) \rangle$. The logics $\Sigma^1_1(ML)$ and $\Sigma^1_1(MLE)$ are defined in the natural way. $\exists MSO(MLE)$ is the fragment of $\Sigma^1_1(MLE)$, where we only allow second-order quantifiers quantifying *unary* relation symbols.

We assume the reader is familiar with the systems $\exists MSO$ (i.e., monadic Σ^1_1) and $\forall MSO$ (i.e., monadic Π^1_1) of predicate logic. We write $M, \frac{u}{x}\frac{v}{y} \models \varphi(x,y)$, if M satisfies the formula $\varphi(x,y)$ of predicate logic under the assignment $x \mapsto u \in Dom(M)$, $y \mapsto v \in Dom(M)$. A modal formula φ and a formula $\psi(x)$ of predicate logic are called *equivalent* if for all pointed models (M,w), $(M,w) \Vdash \varphi \Leftrightarrow M, \frac{w}{x} \models \psi(x)$.

2.2 Types

Let V be a finite vocabulary. Let U be a set of size $|V_2| + 1$ such that for all $T \in V_2$, exactly one of T and $\neg T$ is in U, and exactly one of id and $\neg id$ is in U. Let $\mathcal{M} \in MP(V_2)$ be an intersection consisting of all the members of U. Note that if $|V_2| \neq 0$, there are several such intersections corresponding to U. Therefore, for each U, we always assume some standard ordering and bracketing of the related modal parameter $\mathcal{M}$, so that there is a one-to-one correspondence between the sets U and the related modal parameters. We call such modal parameters *access types* (over V). We let ATP_V denote the set of access types over V.

Let $\mathcal{M}$ be an access type over V, and let $T \in V_2 \cup \{id\}$. We write $T \leq \mathcal{M}$ if T occurs in $\mathcal{M}$, i.e., $\neg T$ does *not* occur in $\mathcal{M}$. Let $U \subseteq V$ be a finite vocabulary and $\mathcal{N}$ an access type over U. We say that $\mathcal{M}$ is *consistent* with $\mathcal{N}$ (or alternatively, $\mathcal{N}$ is consistent with $\mathcal{M}$), if for all symbols $T \in U_2 \cup \{id\}$, $T \leq \mathcal{M}$ iff $T \leq \mathcal{N}$.

Let (M,w) be a pointed model of vocabulary V. We define

$$\tau^0_{(M,w)} := \bigwedge_{\substack{P \in V_1, \\ (M,w) \Vdash P}} P \wedge \bigwedge_{\substack{Q \in V_1, \\ (M,w) \nVdash Q}} \neg Q.$$

Formula $\tau^0_{(M,w)}$ is the *type* of modal depth 0 of (M,w). We choose formulae $\tau^0_{(M,w)}$ such that if $\tau^0_{(M,w)}$ and $\tau^0_{(N,v)}$ are equivalent for some pointed models (M,w) and (N,v), then actually $\tau^0_{(M,w)} = \tau^0_{(N,v)}$. We let TP^0_V be the set containing exactly the formulae τ such that for some pointed model (M,w) of vocabulary V, τ is the type of modal depth 0 of (M, w). Clearly the set TP_V^0 is finite. Now assume we have defined the formulae $\tau^n_{(M,w)}$ for all pointed models (M, w), and that the set TP_V^n is a finite set containing exactly all these formulae. We then define

$$\begin{aligned}\tau^{n+1}_{(M,w)} \;:=\; & \tau^{n}_{(M,w)} \\ & \wedge \bigwedge\{\ \langle\mathcal{M}\rangle\sigma \mid \mathcal{M} \in ATP_V,\ \sigma \in TP_V^n,\ (M,w) \Vdash \langle\mathcal{M}\rangle\sigma\} \\ & \wedge \bigwedge\{\ \neg\langle\mathcal{M}\rangle\sigma \mid \mathcal{M} \in ATP_V,\ \sigma \in TP_V^n,\ (M,w) \nVdash \langle\mathcal{M}\rangle\sigma\}.\end{aligned}$$

Again we assume some standard ordering of the conjuncts and some standard bracketing, so that there is exactly one formula $\tau^{n+1}_{(M,w)}$. Also, we choose these formulae such that if $\tau^{n+1}_{(M,w)}$ and $\tau^{n+1}_{(N,v)}$ are equivalent, then in fact $\tau^{n+1}_{(M,w)} = \tau^{n+1}_{(N,v)}$. Formula $\tau^{n+1}_{(M,w)}$ is the *type* of modal depth $n+1$ of (M, w). We let TP_V^{n+1} be the set containing exactly the formulae τ such that for some pointed model (M, w) of vocabulary V, τ is the type of modal depth $n + 1$ of (M, w). We see that TP_V^{n+1} is finite.

We then list a number of properties of types that are straightforward to prove. Let (M, w) be a pointed model of vocabulary V and let $U \subseteq V$ be a finite vocabulary. Let $n \in \mathbb{N}$. Firstly, (M, w) satisfies exactly one type in TP_U^n. Also, for all $\tau \in TP_U^n$ and all $m \leq n$, there exists exactly one type $\sigma \in TP_U^m$ such that $\tau \Vdash \sigma$. Let $\alpha \in TP_U^n$ and let ψ be an arbitrary formula of vocabulary U and of modal depth $m \leq n$. Then either $\alpha \Vdash \psi$ or $\alpha \Vdash \neg\psi$, and also, for all points $u, v \in Dom(M) \cap ||\alpha||^M$, we have $(M, u) \Vdash \psi$ iff $(M, v) \Vdash \psi$. Finally, ψ is equivalent to $\bigvee\{\tau \in TP_U^n \mid \tau \Vdash \psi\}$.

3 $\Sigma^1_1(ML) \leq\ \exists MSO(MLE)$

In this subsection we show how to translate $\Sigma^1_1(ML)$ formulae to equivalent formulae of $\exists MSO(MLE)$. We begin by fixing a $\Sigma^1_1(ML)$ formula φ and show how to translate it to an equivalent formula $\varphi^*(x)$ of $\exists MSO$. We then show that the first-order part of $\varphi^*(x)$ translates to an equivalent MLE formula.

Let $\varphi = \mathcal{Q}(\psi)$, where $\mathcal{Q}$ is a vector of existential second-order quantifiers and ψ a formula of ML. We let V_1^ψ and V_2^ψ denote the sets of unary and binary relation symbols, respectively, that occur in ψ. We define $V^\psi = V_1^\psi \cup V_2^\psi$. We let Q_1^ψ and Q_2^ψ denote the sets of unary and binary relation symbols, respectively, that occur in $\mathcal{Q}$. We define $Q^\psi = Q_1^\psi \cup Q_2^\psi$. We let SUB_ψ denote the set of subformulae of ψ.

We then fix a fresh (i.e., not occuring in φ) symbol P_α for each $\alpha \in SUB_\psi$. Before fixing the translated formula $\varphi^*(x)$, we define a number of auxiliary formulae. We begin by defining the following formulae for all $P, \neg\alpha, (\beta \wedge \gamma), \langle R\rangle\rho, \langle S\rangle\sigma \in SUB_\psi$, where $P \in V_1^{\psi}$, $R \in V_2^{\psi} \setminus Q_2^{\psi}$ and $S \in Q_2^{\psi}$:

$$\begin{aligned}
&\psi_P := \forall x\Big(P_P(x) \leftrightarrow P(x)\Big),\\
&\psi_{\neg\alpha} := \forall x\Big(P_{\neg\alpha}(x) \leftrightarrow \neg P_{\alpha}(x)\Big),\\
&\psi_{(\beta\wedge\gamma)} := \forall x\Big(P_{(\beta\wedge\gamma)}(x) \leftrightarrow (P_{\beta}(x) \wedge P_{\gamma}(x))\Big),\\
&\psi_{\langle R\rangle\rho} := \forall x\Big(P_{\langle R\rangle\rho}(x) \leftrightarrow \exists y(R(x,y) \wedge P_{\rho}(y))\Big),\\
&\psi_{\langle S\rangle\sigma} := \forall x\Big(P_{\langle S\rangle\sigma}(x) \leftrightarrow \exists y(Access_S(x,y) \wedge P_{\sigma}(y))\Big),
\end{aligned}$$

where

$$Access_S(x,y) := \bigwedge_{\langle S\rangle\chi \,\in\, SUB_{\psi}} \Big(P_{\chi}(y) \rightarrow P_{\langle S\rangle\chi}(x)\Big).$$

Finally, we define

$$\delta_{\psi} := \bigwedge_{\alpha \,\in\, SUB_{\psi}} \psi_{\alpha}, \quad \text{and} \quad \varphi^*(x) := Q^*(\delta_{\psi} \wedge P_{\psi}(x)),$$

where Q^* is a vector of existential quantifiers quantifying all the predicate symbols $P \in Q_1^{\psi}$ and also all the symbols P_{α}, where $\alpha \in SUB_{\psi}$.

We then establish that $(M,w) \Vdash \varphi$ implies $M, \frac{w}{x} \models \varphi^*(x)$. We therefore assume that $(M,w) \Vdash \varphi$, whence there exists some expansion M_2 of M by interpretations of the binary and unary symbols in Q^{ψ} such that $(M_2, w) \Vdash \psi$. We then define an expansion M_1 of M by interpretations of the unary predicate symbols occurring in $\mathcal{Q}^*$. We let $P^{M_1} = P^{M_2}$ for all $P \in Q_1^{\psi}$. For all P_{α}, where $\alpha \in SUB_{\psi}$, we define $P_{\alpha}^{M_1} = ||\alpha||^{M_2}$.

Lemma 3.1 *Let $\langle S\rangle\sigma \in SUB_{\psi}$, where $S \in Q_2^{\psi}$. Let $v \in Dom(M)$. We have $(M_2, v) \Vdash \langle S\rangle\sigma$ iff $M_1, \frac{v}{x} \models \exists y(Access_S(x,y) \wedge P_{\sigma}(y))$.*

Proof. Assume that $(M_2, v) \Vdash \langle S\rangle\sigma$. Therefore $(v,u) \in S^{M_2}$ for some $u \in ||\sigma||^{M_2} = P_{\sigma}^{M_1}$. In order to establish that $M_1, \frac{v}{x} \models \exists y(Access_S(x,y) \wedge P_{\sigma}(y))$, it therefore suffices to show that for all $\langle S\rangle\chi \in SUB_{\psi}$, if $u \in P_{\chi}^{M_1}$, then $v \in P_{\langle S\rangle\chi}^{M_1}$. Therefore assume that $u \in P_{\chi}^{M_1}$ for some $\langle S\rangle\chi \in SUB_{\psi}$. As $||\chi||^{M_2} = P_{\chi}^{M_1}$, we conclude that $u \in ||\chi||^{M_2}$. As $(v,u) \in S^{M_2}$, we have $(M_2, v) \Vdash \langle S\rangle\chi$. As $||\langle S\rangle\chi||^{M_2} = P_{\langle S\rangle\chi}^{M_1}$, we have $v \in P_{\langle S\rangle\chi}^{M_1}$, as desired.

Assume then that $M_1, \frac{v}{x} \models \exists y(Access_S(x,y) \wedge P_{\sigma}(y))$. Thus $M_1, \frac{v}{x}\frac{u}{y} \models Access_S(x,y)$ for some $u \in P_{\sigma}^{M_1} = ||\sigma||^{M_2}$. By the definition of the formula $Access_S(x,y)$, we see immediately that $v \in P_{\langle S\rangle\sigma}^{M_1}$. As $||\langle S\rangle\sigma||^{M_2} = P_{\langle S\rangle\sigma}^{M_1}$, we conclude that $v \in ||\langle S\rangle\sigma||^{M_2}$. Thus $(M_2, v) \Vdash \langle S\rangle\sigma$, as desired. $\square$

Lemma 3.2 *$(M,w) \Vdash \varphi$ implies $M, \frac{w}{x} \models \varphi^*(x)$.*

Proof. We prove the claim by establishing that $M_1, \frac{w}{x} \models \delta_\psi \wedge P_\psi(x)$. As $(M_2, w) \Vdash \psi$ and $||\psi||^{M_2} = P_\psi^{M_1}$, we have $M_1, \frac{w}{x} \models P_\psi(x)$. Hence we only need to establish that $M_1 \models \delta_\psi$. The non-trivial part of this is showing that $M_1 \models \psi_{\langle S \rangle \sigma}$ for an arbitrary $\langle S \rangle \sigma \in SUB_\psi$, where $S \in Q_2^\psi$. This follows directly from Lemma 3.1, as $P_{\langle S \rangle \sigma}^{M_1} = ||\langle S \rangle \sigma||^{M_2}$. □

We then show that $M, \frac{w}{x} \models \varphi^*(x)$ implies $(M, w) \Vdash \varphi$. Therefore we assume that $M, \frac{w}{x} \models \varphi^*(x)$, whence there exists an expansion M_1' of M by interpretations of the unary predicate symbols occurring in $\mathcal{Q}^*$ such that $M_1, \frac{w}{x} \models \delta_\psi \wedge P_\psi(x)$. We then define an expansion M_2' of M by interpretations of the binary and unary symbols occurring in $\mathcal{Q}$. For all $P \in Q_1^\psi$, we let $P^{M_2'} = P^{M_1'}$. For all $S \in Q_2^\psi$, we let $(v, u) \in S^{M_2'}$ if and only if $M_1', \frac{v}{x} \frac{u}{y} \models Access_S(x, y)$.

Lemma 3.3 *Let $\alpha \in SUB_\psi$ and $v \in Dom(M)$. We have $(M_2', v) \Vdash \alpha$ if and only if $M_1', \frac{v}{x} \models P_\alpha(x)$.*

Proof. We prove the claim by induction on the structure of α. As $M_1' \models \delta_\psi$, the claim holds trivially for all atomic formulae $P \in V_1^\psi$. Also, the cases where α is of type $\neg\beta$, $(\beta \wedge \gamma)$ or $\langle R \rangle \beta$, where $R \in V_2^\psi \setminus Q_2^\psi$, are straightforward to deal with, as $M_1' \models \delta_\psi$.

We assume that $(M_2', v) \Vdash \langle S \rangle \sigma$, where $S \in Q_2^\psi$ and $\langle S \rangle \sigma \in SUB_\psi$. Thus $(v, u) \in S^{M_2'}$ for some $u \in ||\sigma||^{M_2'}$. Hence $M_1, \frac{v}{x} \frac{u}{y} \models Access_S(x, y)$ by our definition of $S^{M_2'}$. By the induction hypothesis, we have $P_\sigma^{M_1'} = ||\sigma||^{M_2'}$. Therefore $u \in P_\sigma^{M_1'}$, whence $M_1', \frac{v}{x} \models \exists y(Access_S(x, y) \wedge P_\sigma(y))$. Therefore, as $M_1' \models \psi_{\langle S \rangle \sigma}$, we conclude that $v \in P_{\langle S \rangle \sigma}^{M_1'}$.

For the converse, assume that $M_1', \frac{v}{x} \models P_{\langle S \rangle \sigma}(x)$. As $M_1' \models \psi_{\langle S \rangle \sigma}$, we conclude that $M_1', \frac{v}{x} \models \exists y(Access_S(x, y) \wedge P_\sigma(y))$. Therefore there exists some $u \in P_\sigma^{M_1'}$ such that $M_1', \frac{v}{x} \frac{u}{y} \models Access_S(x, y)$. We now have $(v, u) \in S^{M_2'}$ by our definition of $S^{M_2'}$. As $u \in P_\sigma^{M_1'}$ and as $||\sigma||^{M_2'} = P_\sigma^{M_1'}$ by the induction hypothesis, we therefore conclude that $(M_2', v) \Vdash \langle S \rangle \sigma$, as desired. □

By Lemma 3.3 we immediately see that $M, \frac{w}{x} \models \varphi^*(x)$ implies $(M, w) \Vdash \varphi$. This, combined with Lemma 3.2, justifies the following conclusion:

Theorem 3.4 *Each formula of $\Sigma_1^1(ML)$ is expressible in $\exists MSO$.*

It is easy to see that $\varphi^*(x)$ is expressible in $\exists MSO(MLE)$: Let $S \in Q_2^\psi$ and let A be the subset of SUB_ψ of formulae of type $\langle S \rangle \alpha$. Formula $\exists y(Access_S(x, y) \wedge P_\sigma(y))$ is equivalent to the following formula of MLE:

$$\bigvee_{B \subseteq A} \Big(\bigwedge_{\langle S \rangle \chi \in B} P_{\langle S \rangle \chi} \wedge \langle E \rangle \Big(P_\sigma \wedge \bigwedge_{\langle S \rangle \chi \in B} P_\chi \wedge \bigwedge_{\langle S \rangle \chi \in A \setminus B} \neg P_\chi \Big) \Big)$$

Thus we see that each formula ψ_α, where $\alpha \in SUB_\psi$, can be expressed in MLE. We may therefore draw the following conclusion:

Theorem 3.5 *Each formula of $\Sigma^1_1(ML)$ is expressible in $\exists MSO(MLE)$.*

4 $\Sigma^1_1(BML^=) \leq \exists MSO$

In this section we prove that each formula of $\Sigma^1_1(BML^=)$ can be translated to an equivalent formula of $\exists MSO$.

4.1 A translation from $\Sigma^1_1(BML^=)$ into $\exists MSO$

In this subsection we define a translation of $\Sigma^1_1(BML^=)$ formulae to equivalent formulae of $\exists MSO$. For this aim, we fix a $\Sigma^1_1(BML^=)$ formula φ and show how it is translated. Let $\varphi = \mathcal{Q}(\psi)$, where $\mathcal{Q}$ is vector of existential second-order quantifiers and ψ a formula of $BML^=$. For technical reasons, we assume w.l.o.g. that $Md(\psi) \geq 2$. We let V_1^ψ and V_2^ψ denote the sets of unary and binary relation symbols, respectively, that occur in ψ. We define $V^\psi = V_1^\psi \cup V_2^\psi$. We let Q_1^ψ and Q_2^ψ denote the sets of unary and binary relation symbols, respectively, that occur in $\mathcal{Q}$. We define $Q^\psi = Q_1^\psi \cup Q_2^\psi$. We let ATP_ψ denote the set of access types over V^ψ. Let $n \in \mathbb{N}$. We let TP_ψ^n denote the set of types of modal depth n over V^ψ, and define $TP_\psi = \bigcup_{i \leq Md(\psi)} TP_\psi^i$.

We then define fresh (i.e., not occuring in φ) unary predicate symbols. We fix a unique symbol P_τ for each $\tau \in TP_\psi$. We also fix a symbol $P_{(\alpha,\, \mathcal{M},\, \beta)}$ for all $\alpha \in TP_\psi^{Md(\psi)}$, $\mathcal{M} \in ATP_\psi$, $\beta \in TP_\psi^{Md(\psi)-1}$.

The translation $\varphi^*(x)$ of φ will be the formula

$$\Big(\exists P\Big)_{P\, \in\, Q_1^\psi}\Big(\exists P_\tau\Big)_{\tau\, \in\, TP_\psi}\Big(\exists P_{(\alpha,\, \mathcal{M},\, \beta)}\Big)_{\substack{\alpha\, \in\, TP_\psi^{Md(\psi)},\\ \mathcal{M}\, \in\, ATP_\psi,\\ \beta\, \in\, TP_\psi^{Md(\psi)-1}}}\Big(\ \psi^*(x)\ \Big),$$

where $\psi^*(x)$ is a first-order formula in one free variable, x. We let $\mathcal{Q}^*$ denote the above vector of monadic existential second-order quantifiers.

The main idea in the translation is that we can use the symbols P_τ in order to encode information about the extensions of types τ over models of vocabulary V^ψ. The symbols $P_{(\alpha,\, \mathcal{M},\, \beta)}$, in turn, will help us fix formulae that encode information about the extensions of access types $\mathcal{M} \in ATP_\psi$. Before fixing the translation $\varphi^*(x)$ of φ, we define a number of auxiliary formulae. The first formula we define ensures that for all $n \in \{0, ..., Md(\psi)\}$, the extensions of the predicate symbols P_τ, where $\tau \in TP_\psi^n$, always cover all of the domain of a model and never overlap. We define

$$\psi_{uniq} := \forall x\Big(\bigwedge_{0\, \leq\, i\, \leq\, Md(\psi)}\Big(\bigvee_{\tau\, \in\, TP_\psi^i}\Big(P_\tau(x)\ \wedge \bigwedge_{\substack{\sigma\, \in\, TP_\psi^i,\\ \sigma\, \neq\, \tau}}\neg P_\sigma(x)\Big)\Big)\Big).$$

The next formula asserts that each predicate symbol P_β, where $\beta \in TP_\psi^{Md(\psi)-1}$, must be interpreted such that for all symbols P_τ, where $Md(\tau) < Md(\beta)$, the extension of P_β is either wholly included in the extension of P_τ, or does not overlap with it. We let

$$\psi_{pack} := \forall x \forall y \bigwedge_{\beta \in TP_\psi^{Md(\psi)-1}} \Big((P_\beta(x) \wedge P_\beta(y)) \rightarrow \bigwedge_{\tau \in TP_\psi^{<Md(\psi)-1}} (P_\tau(x) \leftrightarrow P_\tau(y)) \Big).$$

We then define formulae that encode information about access types connecting points in models of vocabulary V^ψ:

$$Access_{\mathcal{M}}(x, y) := \bigvee_{\substack{\alpha \in TP_\psi^{Md(\psi)}, \\ \beta \in TP_\psi^{Md(\psi)-1}}} \Big(P_\alpha(x) \wedge P_\beta(y) \wedge P_{(\alpha,\ \mathcal{M},\ \beta)}(y) \Big).$$

Next we define formulae for all $\tau \in TP_\psi^n$, $0 \leq n \leq Md(\psi)$ that recursively force the interpretations of P_τ to match the extensions of τ over models of vocabulary V^ψ. First, let $\tau \in TP_\psi^0$. We define

$$\chi_\tau(x) := \bigwedge_{\substack{P \in V_1^\psi, \\ \tau \Vdash P}} P(x) \qquad \wedge \bigwedge_{\substack{Q \in V_1^\psi, \\ \tau \nVdash Q}} \neg Q(x).$$

Now let $\tau \in TP_\psi^{n+1}$, where $0 \leq n \leq Md(\psi) - 1$. We define

$$\chi_\tau^+(x) := \bigwedge_{\substack{\mathcal{M} \in ATP(\psi), \\ \sigma \in TP_\psi^n, \\ \tau \Vdash \langle \mathcal{M} \rangle \sigma}} \exists y (Access_{\mathcal{M}}(x, y) \wedge P_\sigma(y)),$$

$$\chi_\tau^-(x) := \bigwedge_{\substack{\mathcal{M} \in ATP_\psi, \\ \sigma \in TP_\psi^n, \\ \tau \Vdash \neg \langle \mathcal{M} \rangle \sigma}} \neg \exists y (Access_{\mathcal{M}}(x, y) \wedge P_\sigma(y)),$$

and

$$\chi_\tau(x) := P_{\tau'}(x) \wedge \chi_\tau^+(x) \wedge \chi_\tau^-(x),$$

where τ' is the unique type in TP_ψ^n such that $\tau \Vdash \tau'$.

Let $A \subseteq ATP_\psi$, $A \neq \emptyset$. Let $\alpha \in TP_\psi^{Md(\psi)}$ and $\beta \in TP_\psi^{Md(\psi)-1}$. The next formula encodes information about the *sets* of access types connecting points in extensions of α to points in extensions of β in models of vocabulary V^ψ. We define

$$\psi_{(\alpha,\ A,\ \beta)}(x) := P_\alpha(x) \wedge \bigwedge_{\mathcal{M}\ \in\ A} \exists y(Access_{\mathcal{M}}(x,y) \wedge P_\beta(y)).$$

The next two formulae ensure that information about the access types realized in models of vocabulary V^ψ is consistent with the interpretation of the access types $\mathcal{R} \in ATP_{V^\psi \setminus Q^\psi}$, i.e., the access types describing non-quantified binary relations. We define a linear ordering on ATP_ψ. Let $A(k)$ denote the k^{th} member of a set $A \subseteq ATP_\psi$ with respect to this ordering, and let $\chi_{A(k)}(x, y_k)$ denote a first-order formula stating that x and y_k are connected according to the unique access type in $ATP_{V^\psi \setminus Q^\psi}$ consistent with the access type $A(k) \in A$. We define

$$\psi_{cons} :=\ \forall x\Big(\bigwedge_{\substack{A\ \subseteq\ ATP_\psi,\ A\ \neq\ \emptyset,\\ \alpha\ \in\ TP_\psi^{Md(\psi)},\\ \beta\ \in\ TP_\psi^{Md(\psi)-1}}} \Big(\ \ \psi_{(\alpha,\ A,\ \beta)}(x) \rightarrow$$
$$\exists y_1, ..., y_{|A|}\Big(\bigwedge_{\substack{i,\ j\ \in\ \{1\ ,...,\ |A|\},\\ i\ \neq\ j}} y_i \neq y_j\ \wedge$$
$$\bigwedge_{k\ \in\ \{1\ ,...,\ |A|\}} (\chi_{A(k)}(x, y_k) \wedge P_\beta(y_k))\Big)\Big)\Big).$$

For each $\mathcal{R} \in ATP_{V^\psi \setminus Q^\psi}$, we let $c(\mathcal{R})$ denote the set $A \subseteq ATP_\psi$ of access types that are consistent with $\mathcal{R}$. We define

$$\psi'_{cons} := \forall x\Big(\bigwedge_{\substack{\mathcal{R}\ \in\ ATP_{V^\psi \setminus Q^\psi},\\ \beta\ \in\ TP_\psi^{Md(\psi)-1}}} \Big(\exists y(\psi_{\mathcal{R}}(x,y) \wedge P_\beta(y)) \rightarrow$$
$$\bigvee_{\mathcal{M}\ \in\ c(\mathcal{R})} \exists y(Access_{\mathcal{M}}(x,y) \wedge P_\beta(y))\Big)\Big),$$

where $\psi_{\mathcal{R}}(x,y)$ denotes a first-order formula stating that x and y are connected according to the access type $\mathcal{R}$.

Finally, we define

$$\delta_\psi := \psi_{uniq} \wedge \psi_{pack} \wedge \psi_{cons} \wedge \psi'_{cons}\ \ \wedge \bigwedge_{\tau\ \in\ TP_\psi} \forall x\Big(P_\tau(x) \leftrightarrow \chi_\tau(x)\Big)$$

and

$$\varphi^*(x) := \mathcal{Q}^*\Big(\quad \delta_\psi \quad \wedge \bigvee_{\substack{\alpha \,\in\, TP_\psi^{Md(\psi)},\\ \alpha \,\Vdash\, \psi}} P_\alpha(x) \quad \Big).$$

We then fix an arbitrary pointed model (M, w) of vocabulary $V^\psi \setminus Q^\psi$. In the next subsection we establish that $(M, w) \Vdash \varphi$ if and only if $M, \frac{w}{x} \models \varphi^*(x)$.

4.2 $\Sigma_1^1(BML^=) \leq \exists MSO$

We first show that $(M, w) \Vdash \varphi$ implies $M, \frac{w}{x} \models \varphi^*(x)$. Thus we assume that $(M, w) \Vdash \varphi$. Therefore there exists some expansion M_2 of M by interpretations of the binary and unary symbols in Q^ψ such that $(M_2, w) \Vdash \psi$.

We then define an expansion M_1 of M by interpreting the unary variable symbols in Q_1^ψ, and also the symbols P_τ and $P_{(\alpha,\mathcal{M},\beta)}$ for all $\tau \in TP_\psi$, $\alpha \in TP_\psi^{Md(\psi)}$, $\mathcal{M} \in ATP_\psi$, $\beta \in TP_\psi^{Md(\psi)-1}$. For all $P \in Q_1^\psi$, we define $P^{M_1} = P^{M_2}$. For all $\tau \in TP_\psi$, we let $P_\tau^{M_1} = ||\tau||^{M_2}$. We choose exactly one point from each set $||\alpha||^{M_2} \subseteq Dom(M)$, where $\alpha \in TP_\psi^{Md(\psi)}$ and $||\alpha||^{M_2} \neq \emptyset$. We call such a point the *selector* of $||\alpha||^{M_2}$ and denote it by v_α. We use selectors in order to fix extensions of the predicate symbols $P_{(\alpha,\ \mathcal{M},\ \beta)}$. For each $\alpha \in TP_\psi^{Md(\psi)}$, $\mathcal{M} \in ATP_\psi$, and $\beta \in TP_\psi^{Md(\psi)-1}$, where $||\alpha||^{M_2} \neq \emptyset$, we define

$$P_{(\alpha,\ \mathcal{M},\ \beta)}^{M_1} = \{u \in Dom(M) \mid (v_\alpha, u) \in \mathcal{M}^{M_2},\ u \in P_\beta^{M_1}\}.$$

If $||\alpha||^{M_2} = \emptyset$, we define $P_{(\alpha,\ \mathcal{M},\ \beta)}^{M_1} = \emptyset$.

Next we prove a number of auxiliary lemmata, and then establish that $M_1, \frac{w}{x} \models \psi^*(x)$. The first two lemmata follow directly from the above definitions.

Lemma 4.1 *Let $\alpha \in TP_\psi^{Md(\psi)}$ and $\mathcal{M} \in ATP_\psi$. Let $u \in Dom(M)$. Then $(v_\alpha, u) \in \mathcal{M}^{M_2}$ if and only if $M_1, \frac{v_\alpha}{x}\frac{u}{y} \models Access_{\mathcal{M}}(x, y)$.*

Lemma 4.2 *Let $\alpha \in TP_\psi^{Md(\psi)}$ and $\mathcal{M} \in ATP_\psi$. Let $v \in P_\alpha^{M_1}$. Then, for all $u \in Dom(M)$, $M_1, \frac{v}{x}\frac{u}{y} \models Access_{\mathcal{M}}(x, y)$ if and only if $M_1, \frac{v_\alpha}{x}\frac{u}{y} \models Access_{\mathcal{M}}(x, y)$.*

We then show that the formula $Access_{\mathcal{M}}(x, y)$ captures all the relevant information about the action of the diamond $\langle\mathcal{M}\rangle$ on M_2:

Lemma 4.3 *Let $\tau \in TP_\psi^{<Md(\psi)}$ and $\mathcal{M} \in ATP_\psi$. Let $v \in Dom(M)$. Then $(M_2, v) \Vdash \langle\mathcal{M}\rangle\tau$ if and only if $M_1, \frac{v}{x} \models \exists y(Access_{\mathcal{M}}(x, y) \wedge P_\tau(y))$.*

Proof. Assume that $(M_2, v) \Vdash \langle\mathcal{M}\rangle\tau$. Let α be the type in $TP_\psi^{Md(\psi)}$ such that $v \in ||\alpha||^{M_2}$. As $(M_2, v) \Vdash \langle\mathcal{M}\rangle\tau$, also $(M_2, v_\alpha) \Vdash \langle\mathcal{M}\rangle\tau$. Therefore there exists some $u \in ||\tau||^{M_2}$ such that $(v_\alpha, u) \in \mathcal{M}^{M_2}$. We conclude that $M_1, \frac{v_\alpha}{x}\frac{u}{y} \models Access_{\mathcal{M}}(x, y)$ by Lemma 4.1, and therefore $M_1, \frac{v}{x}\frac{u}{y} \models Access_{\mathcal{M}}(x, y)$ by Lemma

4.2. As $u \in ||\tau||^{M_2} = P_\tau^{M_1}$, we have $M_1, \frac{v}{x}\frac{u}{y} \models Access_{\mathcal{M}}(x,y) \wedge P_\tau(y)$, whence $M_1, \frac{v}{x} \models \exists y(Access_{\mathcal{M}}(x,y) \wedge P_\tau(y))$, as desired.

Assume $M_1, \frac{v}{x} \models \exists y(Access_{\mathcal{M}}(x,y) \wedge P_\tau(y))$. We conclude, using Lemma 4.2, that $M_1, \frac{v_\alpha}{x}\frac{u}{y} \models Access_{\mathcal{M}}(x,y) \wedge P_\tau(y)$ for some $u \in P_\tau^{M_1}$. Therefore $(v_\alpha, u) \in \mathcal{M}^{M_2}$ by Lemma 4.1. As $P_\tau^{M_1} = ||\tau||^{M_2}$, we conclude that $(M_2, v_\alpha) \Vdash \langle \mathcal{M} \rangle \tau$, and therefore $(M_2, v) \Vdash \langle \mathcal{M} \rangle \tau$. □

Interpretations of the formulae $\chi_\tau(x)$ and the predicate symbols P_τ coincide:

Lemma 4.4 *Let $v \in Dom(M)$ and $\tau \in TP_\psi$. Then $M_1, \frac{v}{x} \models P_\tau(x)$ iff $M_1, \frac{v}{x} \models \chi_\tau(x)$.*

Proof. As $||P||^{M_2} = P^{M_1}$ for all $P \in V_1^\psi$, the claim follows directly for all $\tau \in TP_\psi^0$. Therefore we may assume that $\tau \in TP_\psi^{\geq 1}$. Throughout the proof, we let τ' denote the unique type in $TP_\psi^{Md(\tau)-1}$ such that $\tau \Vdash \tau'$.

Assume that $M_1, \frac{v}{x} \models P_\tau(x)$. As $P_\tau^{M_1} = ||\tau||^{M_2}$, we have $(M_2, v) \Vdash \tau$. As $\tau \Vdash \tau'$, we have $(M_2, v) \Vdash \tau'$. Since $P_{\tau'}^{M_1} = ||\tau'||^{M_2}$, we conclude that $M_1, \frac{v}{x} \models P_{\tau'}(x)$. We still need to establish that $M_1, \frac{v}{x} \models \chi_\tau^+(x) \wedge \chi_\tau^-(x)$. Therefore assume that $\tau \Vdash \langle \mathcal{M} \rangle \sigma$, where $\mathcal{M} \in ATP_\psi$ and $\sigma \in TP_\psi^{Md(\tau)-1}$. As $(M_2, v) \Vdash \tau$, we have $(M_2, v) \Vdash \langle \mathcal{M} \rangle \sigma$. Therefore $M_1, \frac{v}{x} \models \exists y(Access_{\mathcal{M}}(x,y) \wedge P_\sigma(y))$ by Lemma 4.3. Similarly, if $\tau \Vdash \neg\langle \mathcal{M} \rangle \sigma$, we conclude that $M_1, \frac{v}{x} \models \neg\exists y(Access_{\mathcal{M}}(x,y) \wedge P_\sigma(y))$ by Lemma 4.3. Thus $M_1, \frac{v}{x} \models \chi_\tau^+(x) \wedge \chi_\tau^-(x)$, as desired.

Assume then that $M_1, \frac{v}{x} \models \chi_\tau(x)$. In order to show that $M_1, \frac{v}{x} \models P_\tau(x)$, we will establish that $(M_2, v) \Vdash \tau$. This suffices, as $P_\tau^{M_1} = ||\tau||^{M_2}$. We immediately see that $(M_2, v) \Vdash \tau'$, as $M_1, \frac{v}{x} \models P_{\tau'}(x)$ and $P_{\tau'}^{M_1} = ||\tau'||^{M_2}$. Assume then that $\tau \Vdash \langle \mathcal{M} \rangle \sigma$, where $\mathcal{M} \in ATP_\psi$ and $\sigma \in TP_\psi^{Md(\tau)-1}$. As $M_1, \frac{v}{x} \models \chi_\tau^+(x)$, we have $M_1, \frac{v}{x} \models \exists y(Access_{\mathcal{M}}(x,y) \wedge P_\sigma(y))$, and therefore $(M_2, v) \Vdash \langle \mathcal{M} \rangle \sigma$ by Lemma 4.3. Similarly, if $\tau \Vdash \neg\langle \mathcal{M} \rangle \sigma$, then, as $M_1, \frac{v}{x} \models \chi_\tau^-(x)$, we conclude that $M_1, \frac{v}{x} \models \neg\exists y(Access_{\mathcal{M}}(x,y) \wedge P_\sigma(y))$, and therefore $(M_2, v) \Vdash \neg\langle \mathcal{M} \rangle \sigma$ by Lemma 4.3. Thus $(M_2, v) \Vdash \tau$, and hence $M_1, \frac{v}{x} \models P_\tau(x)$, as desired. □

We then conclude the first direction of the proof:

Lemma 4.5 *If $(M, w) \Vdash \varphi$, then $M, \frac{w}{x} \models \varphi^*(x)$.*

Proof. We establish the claim by showing that $M_1, \frac{w}{x} \models \psi^*(x)$.

Let ψ' denote a disjunction of all types $\alpha \in TP_\psi^{Md(\psi)}$ such that $\alpha \Vdash \psi$. As ψ and ψ' are equivalent, we have $(M_2, w) \Vdash \psi'$. Therefore $(M_2, w) \Vdash \alpha$ for some $\alpha \in TP_\psi^{Md(\psi)}$ occurring in the disjunction. Hence, as $||\alpha||^{M_2} = P_\alpha^{M_1}$, we have $M_1, \frac{w}{x} \models P_\alpha(x)$.

We then show that $M_1 \models \psi_{cons}$. Let $v \in Dom(M)$ and assume $M_1, \frac{v}{x} \models \psi_{(\alpha,\ A,\ \beta)}(x)$ for some $\alpha \in TP_\psi^{Md(\psi)}$, $A \subseteq ATP_\psi$, $\beta \in TP_\psi^{Md(\psi)-1}$. Recall that we may write $A = \{A(1), ..., A(|A|)\}$, where $A(k)$ refers to the k^{th} member of the set A w.r.t. the ordering of ATP_ψ we fixed. As $M_1, \frac{v}{x} \models \psi_{(\alpha,\ A,\ \beta)}(x)$, we see by Lemma 4.3 that $(M_2, v) \Vdash \langle A(k) \rangle \beta$ for all $k \in \{1, ..., |A|\}$. Thus there must exist distinct points $u_1, ..., u_{|A|} \in ||\beta||^{M_2} = P_\beta^{M_1}$ such that $(v, u_k) \in A(k)^{M_2}$ for each k. Let $\mathcal{R}_k$ be the type in $ATP_{V^\psi \setminus Q^\psi}$ consistent with $A(k)$. Recall that $\chi_{A(k)}(x, y_k)$ is a first-order formula stating that x is connected to y_k by $\mathcal{R}_k$. We have $(v, u_k) \in \mathcal{R}_k^{M_2} = \mathcal{R}_k^{M_1}$ for each k, and thus $M_1, \frac{v}{x}\frac{u_k}{y} \models \chi_{A(k)}(x, y_k) \wedge P_\beta(y_k)$ for each k.

We then show that $M_1 \models \psi'_{cons}$. Let $v \in Dom(M)$ and assume $M_1, \frac{v}{x}\frac{u}{y} \models \psi_{\mathcal{R}}(x, y)$ for some $u \in P_\beta^{M_1}$ with $\beta \in TP_\psi^{Md(\psi)-1}$ and some $\mathcal{R} \in ATP_{V^\psi \setminus Q^\psi}$. Let $\mathcal{M}$ be the access type in ATP_ψ such that $(v, u) \in \mathcal{M}^{M_2}$. Thus $(M_2, v) \Vdash \langle \mathcal{M} \rangle \beta$, whence $M_1, \frac{v}{x} \models \exists y(Access_{\mathcal{M}}(x, y) \wedge P_\beta(y))$ by Lemma 4.3. Clearly $\mathcal{M}$ is consistent with $\mathcal{R}$. Therefore $M_1 \models \psi'_{cons}$.

We have $M_1 \models \psi_{uniq} \wedge \psi_{pack}$ directly by properties of types. Therefore, in order to conclude the proof, we only need to establish that for all $\tau \in TP_\psi^{Md(\psi)}$ and all $v \in Dom(M)$, $M_1, \frac{v}{x} \models P_\tau(x) \leftrightarrow \chi_\tau(x)$. This follows directly from Lemma 4.4. □

We then show that $M, \frac{w}{x} \models \varphi^*(x)$ implies $(M, w) \Vdash \varphi$. Thus we assume that $M, \frac{w}{x} \models \varphi^*(x)$. Therefore there exists an expansion M'_1 of M by interpretations of the unary symbols P_τ and $P_{(\alpha,\ \mathcal{M},\ \beta)}$ for all $\tau \in TP_\psi$, $\alpha \in TP_\psi^{Md(\psi)}$, $\mathcal{M} \in ATP_\psi$, $\beta \in TP_\psi^{Md(\psi)-1}$, and also the symbols $P \in Q_1^\psi$, such that $M'_1, \frac{w}{x} \models \psi^*(x)$.

We then define an expansion of M by interpreting all the relation symbols, unary and binary, in Q^ψ. We call the resulting expansion M'_2. For all $P \in Q_1^\psi$, we define $P^{M'_2} = P^{M'_1}$. Let $v \in P_\alpha^{M'_1}$ and $\beta \in TP_\psi^{Md(\psi)-1}$. Let $A \subseteq ATP_\psi$ be the set of access types $\mathcal{M}$ such that $M'_1, \frac{v}{x}\frac{u}{y} \models Access_{\mathcal{M}}(x, y)$ for some $u \in P_\beta^{M'_1}$. As M'_1 satisfies the formula ψ_{cons}, we see that there exists a bijection f from A to a set $B \subseteq P_\beta^{M'_1}$ such that for all $\mathcal{M} \in A$, we have $(v, f(\mathcal{M})) \in \mathcal{R}_{\mathcal{M}}^{M'_1}$, where $\mathcal{R}_{\mathcal{M}}$ is the access type in $ATP_{V^\psi \setminus Q^\psi}$ consistent with $\mathcal{M}$. Let $S \in Q_2^\psi$. We define $(v, f(\mathcal{M})) \in S^{M'_2}$ if $S \leq \mathcal{M}$. We then consider the points in $P_\beta^{M'_1} \setminus B$. Thus let $u \in P_\beta^{M'_1} \setminus B$. Let $\mathcal{T}$ be the access type in $ATP_{V^\psi \setminus Q^\psi}$ such that $(v, u) \in \mathcal{T}^{M'_1}$. As M'_1 satisfies ψ'_{cons}, we see that there exists some $\mathcal{M} \in ATP_\psi$ consistent with $\mathcal{T}$ and some $u' \in P_\beta^{M'_1}$ such that $M'_1, \frac{v}{x}\frac{u'}{y} \models Access_{\mathcal{M}}(x, y)$. We define, for all $S \in Q_2^\psi$, $(v, u) \in S^{M'_2}$ if $S \leq \mathcal{M}$. We go through each $\alpha \in TP_\psi^{Md(\psi)}$ and $\beta \in TP_\psi^{Md(\psi)-1}$, and construct the extensions $S^{M'_2}$ of the symbols $S \in Q_2^\psi$ in the described way. As M'_1 satisfies ψ_{uniq}, each pair in $Dom(M) \times Dom(M)$ becomes associated with exactly one access type in ATP_ψ. Therefore M'_2 is well defined.

We first prove a number of auxiliary lemmata, and then establish that $(M'_2, w) \Vdash \psi$. The following lemma is a direct consequence of the way we define the extensions $S^{M'_2}$ of the relation symbols $S \in Q_2^\psi$.

Lemma 4.6 *Let $\beta \in TP_\psi^{Md(\psi)-1}$ and $\mathcal{M} \in ATP_\psi$. Let $v \in Dom(M)$. Then $(v, u) \in \mathcal{M}^{M'_2}$ for some $u \in P_\beta^{M'_1}$ iff $M'_1, \frac{v}{x}\frac{u'}{y} \models Access_{\mathcal{M}}(x, y)$ for some $u' \in P_\beta^{M'_1}$.*

We then show that the diamond $\langle \mathcal{M} \rangle$ captures relevant information about the relation that the formula $Access_{\mathcal{M}}(x, y)$ defines over M'_1:

Lemma 4.7 *Let $\tau \in TP_\psi^{<Md(\psi)}$ and $\mathcal{M} \in ATP_\psi$. Assume that $||\tau||^{M_2'} = P_\tau^{M_1'}$ and let $v \in Dom(M)$. Then $(M_2', v) \Vdash \langle\mathcal{M}\rangle\tau$ iff $M_1', \frac{v}{x} \models \exists y(Access_{\mathcal{M}}(x,y) \wedge P_\tau(y))$.*

Proof. Assume $(M_2', v) \Vdash \langle\mathcal{M}\rangle\tau$. Thus $(v,u) \in \mathcal{M}^{M_2'}$ for some $u \in ||\tau||^{M_2'} = P_\tau^{M_1'}$. As $M_1' \models \psi_{uniq}$, there is a unique $\beta \in TP_\psi^{Md(\psi)-1}$ such that $u \in P_\beta^{M_1'}$. Therefore $M_1', \frac{v}{x}\frac{u'}{y} \models Access_{\mathcal{M}}(x,y)$ for some $u' \in P_\beta^{M_1'}$ by Lemma 4.6. Since $M_1' \models \psi_{pack}$ and as $u \in P_\tau^{M_1'} \cap P_\beta^{M_1'}$ and $u' \in P_\beta^{M_1'}$, we have $u' \in P_\tau^{M_1'}$. Therefore $M_1', \frac{v}{x} \models \exists y(Access_{\mathcal{M}}(x,y) \wedge P_\tau(y))$.

For the converse, assume $M_1', \frac{v}{x} \models \exists y(Access_{\mathcal{M}}(x,y) \wedge P_\tau(y))$. Thus $M_1', \frac{v}{x}\frac{u}{y} \models Access_{\mathcal{M}}(x,y)$ for some $u \in P_\tau^{M_1'} = ||\tau||^{M_2'}$. As $M_1' \models \psi_{uniq}$, there is a unique $\beta \in TP_\psi^{Md(\psi)-1}$ such that $u \in P_\beta^{M_1'}$. By Lemma 4.6, we therefore have $(v,u') \in \mathcal{M}^{M_2'}$ for some $u' \in P_\beta^{M_1'}$. Since $M_1' \models \psi_{pack}$ and as $u \in P_\tau^{M_1'} \cap P_\beta^{M_1'}$ and $u' \in P_\beta^{M_1'}$, we have $u' \in P_\tau^{M_1'}$. As $P_\tau^{M_1'} = ||\tau||^{M_2'}$, we conclude that $(M_2', v) \models \langle\mathcal{M}\rangle\tau$. □

Next we show that extensions of the types $\tau \in TP_\psi$ and interpretations of the predicate symbols P_τ coincide, and then conclude the section.

Lemma 4.8 *Let $\tau \in TP_\psi$ and $v \in Dom(M)$. Then $(M_2', v) \Vdash \tau$ iff $M_1', \frac{v}{x} \models P_\tau(x)$.*

Proof. We prove the claim by induction on the modal depth of τ. If $\tau \in TP_\psi^0$, then, as $M_1' \models \forall x(P_\tau(x) \leftrightarrow \chi_\tau(x))$, the claim follows immediately.

Assume that $(M_2', v) \Vdash \tau$ for some $\tau \in TP_\psi^{n+1}$ with $n \geq 0$. We will show that $M_1', \frac{v}{x} \models P_{\tau'}(x) \wedge \chi_\tau^+(x) \wedge \chi_\tau^-(x)$, where τ' is the type of modal depth n such that $\tau \Vdash \tau'$. This directly implies that $M_1', \frac{v}{x} \models P_\tau(x)$, since $M_1' \models \forall x(P_\tau(x) \leftrightarrow \chi_\tau(x))$.

As $\tau \Vdash \tau'$, we have $(M_2', v) \Vdash \tau'$. Therefore $M_1, \frac{v}{x} \models P_{\tau'}(x)$ by the induction hypothesis. In order to establish that $M_1', \frac{v}{x} \models \chi_\tau^+(x) \wedge \chi_\tau^-(x)$, let $\tau \Vdash \langle\mathcal{M}\rangle\sigma$, where $\sigma \in TP_\psi^n$ and $\mathcal{M} \in ATP_\psi$. Therefore $(M_2', v) \Vdash \langle\mathcal{M}\rangle\sigma$. Since $||\sigma||^{M_2'} = P_\sigma^{M_1'}$ by the induction hypothesis, we conclude that $M_1', \frac{v}{x} \models \exists y(Access_{\mathcal{M}}(x,y) \wedge P_\sigma(y))$ by Lemma 4.7. Similarly, if $\tau \Vdash \neg\langle\mathcal{M}\rangle\sigma$, then $M_1', \frac{v}{x} \models \neg\exists y(Access_{\mathcal{M}}(x,y) \wedge P_\sigma(y))$ by Lemma 4.7 and the induction hypothesis. Thus $M_1', \frac{v}{x} \models \chi_\tau^+(x) \wedge \chi_\tau^+(x)$.

Assume then that $M_1', \frac{v}{x} \models P_\tau(x)$, where $\tau \in TP_\psi^{n+1}$. Now, since $M_1' \models \delta_\psi$, we conclude that $M_1', \frac{v}{x} \models \chi_\tau(x)$. Therefore $M_1', \frac{v}{x} \models P_{\tau'}(x)$, where τ' is the type of modal depth n such that $\tau \Vdash \tau'$. Thus $(M_2, v) \Vdash \tau'$ by the induction hypothesis. Assume then that $\tau \Vdash \langle\mathcal{M}\rangle\sigma$, where $\sigma \in TP_\psi^n$ and $\mathcal{M} \in ATP_\psi$. As $M_1', \frac{v}{x} \models \chi_\tau(x)$, we have $M_1', \frac{v}{x} \models \chi_\tau^+(x)$, and therefore $M_1', \frac{v}{x} \models \exists y(Access_{\mathcal{M}}(x,y) \wedge P_\sigma(y))$. Hence, as $||\sigma||^{M_2'} = P_\sigma^{M_1'}$ by the induction hypothesis, we conclude that $(M_2', v) \Vdash \langle\mathcal{M}\rangle\sigma$ by Lemma 4.7. Similarly, if $\tau \Vdash \neg\langle\mathcal{M}\rangle\sigma$, we conclude that $(M_2', v) \Vdash \neg\langle\mathcal{M}\rangle\sigma$ using Lemma 4.7 and the induction hypothesis. We have therefore established that $(M_2', v) \Vdash \tau$, as required. □

Lemma 4.9 *If $M, \frac{w}{x} \models \varphi^*(x)$, then $(M, w) \Vdash \varphi$.*

Proof. We prove the claim by showing that $(M_2', w) \Vdash \psi$. As $M_1', \frac{w}{x} \models \psi^*(x)$, we have $M_1', \frac{w}{x} \models P_\alpha(x)$ for some type $\alpha \in TP^{Md(\psi)}$ such that $\alpha \Vdash \psi$. Therefore

$(M_2', w) \Vdash \alpha$ by Lemma 4.8. As $\alpha \Vdash \psi$, we have $(M_2', w) \Vdash \psi$, as desired. □

The following theorem is a direct consequence of Lemmata 4.5 and 4.9:

Theorem 4.10 *Each formula of $\Sigma_1^1(BML^=)$ translates to an equivalent formula of $\exists MSO$.*

Theorem 4.10 implies a range of decidability results:

Theorem 4.11 *Let V and $U \subseteq V$ be sets of indices. Let $\mathcal{D}$ be a class of Kripke frames $(W, \{R_j\}_{j\in U})$. Consider the class $\mathcal{C} = \{\ (W, \{R_i\}_{i\in V}) \mid (W, \{R_j\}_{j\in U}) \in \mathcal{D}\ \}$ of Kripke frames. Now, if the $\forall MSO$-theory of $\mathcal{D}$ is decidable, then the satisfiability problem for $BML^=$ w.r.t. $\mathcal{C}$ is decidable.*

Proof. Given a formula φ of $BML^=$, we existentially quantify all the relation symbols occurring φ, except for those in $\{R_j\}_{j\in U}$. We end up with a $\Sigma_1^1(BML^=)$ formula, which we then effectively translate to an equivalent $\exists MSO$ formula $\varphi^*(x)$, applying our result. We then modify this formula to an $\exists MSO$ *sentence* χ equivalent to $\exists x(\varphi^*(x))$. Using the decision procedure for the $\forall MSO$-theory of $\mathcal{D}$, we then check whether the sentence χ is satisfiable w.r.t. $\mathcal{D}$. If yes, then φ is satisfiable w.r.t. $\mathcal{C}$, and if not, then φ is not satisfiable w.r.t. $\mathcal{C}$. □

5 Conclusions

We have investigated the expressive power of modal logics with prenex quantification of binary relations. We have shown that $\Sigma_1^1(BML^=)$ translates into $\exists MSO$, and also that $\Sigma_1^1(ML)$ translates into $\exists MSO(MLE)$. We have briefly discussed how these results can be used in order to prove decidability results for (extensions of) multimodal logic.

It remains to be seen whether our investigations provide a stepping stone towards settling the question about the existence of any class of directed graphs definable in $\Sigma_1^1(FO^2)$ but not definable in $\exists MSO$.

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