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Transition Systems over Continuous Time-Space

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Abstract

This paper gives a transition system over continuous time-space, which is a system of functions of continuous time-space into descrete states. This system is situated between the cellular automata and the partial differential equations. This paper shows the reasonable sufficient condition of the uniquness of the solution.

Keywords: dynamics, time development, cellular automaton, partial defferential equation, general topology

1 Introduction

1.1 Motivation

The time development is described as a function $\phi : X \times T \to A$ in general, where the set X represents the space, or the degree of freedom, T is the set of the time, and A is the set of attributes. The laws of time development is described with some constraint of such functions $\phi : X \times T \to A$.

According to the purpose of each application, the set X is a discrete set of a few elements which represents a small degree of freedom of the system, a set of lots of elements which represents a grid of the space, or a continuous space $\mathbb{R}$. Similarly, the set T is sometimes $\mathbb{Z}$ or sometimes $\mathbb{R}$. The set of attributes A is

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sometimes a discrete set of states $Q = \{q_1, q_2, .., q_n\}$, or sometimes continuous $\mathbb{R}$ which represents the displacement of a wave.

As for discrete time $T = \mathbb{Z}$, a symbolic dynamics is an example of $\{\cdot\} \times \mathbb{Z} \to Q$. A cellular automaton [1] is for $\mathbb{Z} \times \mathbb{Z} \to Q$, a map dynamics is for $\{\cdot\} \times \mathbb{Z} \to \mathbb{R}$, and a map dynamics of higher dimension is for $X \times \mathbb{Z} \to \mathbb{R}$ for some $X \subset \mathbb{Z}$. One of the examples of $\mathbb{R} \times \mathbb{Z} \to \mathbb{R}$ is a map dynamics over a functional space [5,6].

As for continuous time $T = \mathbb{R}$, an ordinary differential equation describes a system of $\{\cdot\} \times \mathbb{R} \to \mathbb{R}$, an ordinary differential equation with higher dimension describes a system of $X \times \mathbb{R} \to \mathbb{R}$ for some $X \subset \mathbb{Z}$, and a partial differential equation describes a system of $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$. It is known that a non-linear partial differential equation simulates a kind of cellular automata [2,3,4].

This work studies transition systems over continuous time-space, which are systems of $\mathbb{R}^n \times \mathbb{R} \to Q$. This system is situated between the cellular automata and the partial differential equations.

1.2 Determinism and Zeno's paradox

We are interested in a deterministic system of $\mathbb{R}^n \times \mathbb{R} \to Q$. That is because, in usual, non-deterministic systems include deterministic systems as the special case and deterministic systems are considered to be simpler than non-deterministic systems. A definition of non-deterministic systems is not intersting unless the definition can describe both determinism and proper non-determinism.

The simplest example of dynamics with higher degree of freedom is a progressive wave. As for a cellular automaton, we define the transition rules for $\phi : \mathbb{Z} \times \{0, 1, 2, ...\} \to \{0, 1\}$ as:

- $\phi(x, t) = 1$ if either $\phi(x-1, t-1) = 1$, $\phi(x, t-1) = 1$ or $\phi(x+1, t-1) = 1$,
- $\phi(x, t) = 0$, otherwise

and the initial condition is:

$$\phi(0, 0) = 1, \text{ and } \phi(x, 0) = 0 \text{ for } x \neq 0.$$

Then the solution is:

$$\phi(x, t) = 1 \text{ if } -t \leq x \leq t, \text{ and } \phi(x, t) = 0 \text{ otherwise.}$$

This ϕ denotes the waves progressing both to left and to right.

As for partial differential equation, put the equation of the wave function $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ as:

$$\frac{\partial^2 \phi}{\partial t^2} = -\frac{\partial^2 \phi}{\partial x^2}.$$

Then the solution is $\phi(x, t) = \phi^+(x - t) + \phi^-(x + t)$, which is a summation of two wave; one progresses to left and the other progresses to right.

The naïve translation of that cellular automaton above into the system of $\mathbb{R} \times \mathbb{R} \to \{0,1\}$ would be as the following. The naïve transition rules for $\phi : [0,\infty) \times \mathbb{R} \to \{0,1\}$ is written as:

- $\phi(x,t) = 1$ if, for any $\epsilon > 0$, there exists (x',t') such that $t - \epsilon < t' < t$, $|x' - x| < t - t'$ and $\phi(x',t') = 1$,
- $\phi(x,t) = 0$, otherwise

and the initial condition is:

$\phi(0,0) = 1$, and $\phi(x,0) = 0$ for $x \neq 0$.

Then one of its solutions is:

$\phi_1(x,t) = 1$ if $-t \leq x \leq t$, and $\phi_1(x,t) = 0$ otherwise.

This is the solution which we expect. However, unfortunately, any functions of the following form for $0 < v < 1$ also satisfy the rules and the initial condition above:

$\phi_v(x,t) = 1$ if $-vt \leq x \leq vt$, and $\phi_v(x,t) = 0$ otherwise.

That is because the rules above cannot write the lower bound of the speed of the wave.

This difficulty looks similar to Zeno's paradox of the flying arrow. Why $\phi_v(1,1) \neq 1$? Because $\phi_v(\xi,\xi) \neq 1$ for $1/2 \leq \xi < 1$. Why $\phi_v(1/2,1/2) \neq 1$? Because $\phi_v(\xi,\xi) \neq 1$ for $1/4 \leq \xi < 1/2$. Why $\phi_v(1/2^n,1/2^n) \neq 1$? Because $\phi_v(\xi,\xi) \neq 1$ for $1/2^{n+1} \leq \xi < 1/2^n$. And so forth. In order to escape this difficulty, we write the lower bound v of the speed of the wave in the rule. Actually, we write the rule as:

- If (x,t) satisfies the condition p, then there exists $\epsilon > 0$ such that, for any points (x',t') such that $t \leq t' < t + \epsilon$ and $|x' - x| \leq v(t' - t)$, it holds that $\phi(x',t') = q$.

instead of the naïve form:

- If (x,t) satisfies the condition p, then $\phi(x,t) = q$.

Therefore, we write a transition system for progressing waves as follows.

Example 1.1 The transition rules and the initial condition of $\phi : \mathbb{R} \times [0,\infty) \to \{0,1\}$ are the followings.

Transition rules:

- If $\forall\delta > 0.\ \exists(x',t').\ t - \delta < t' < t,\ |x' - x| < t - t',\ \phi(x',t') = 1$, then $\exists\epsilon > 0.\ \forall(x',t').\ t \leq t' < t + \epsilon,\ |x' - x| < t' - t \Rightarrow \phi(x',t') = 1$.
- Otherwise, $\exists\epsilon > 0.\ \forall(x',t').\ t \leq t' < t + \epsilon,\ |x' - x| < t' - t \Rightarrow \phi(x',t') = 0$.

Initial condition:

$\phi(0,0) = 1$, and $\phi(x,0) = 0$ for $x \neq 0$.

Actually, the function ϕ_1 above satisfies these conditions, and the function ϕ_v above for $v < 1$ does not.

There is another difficulty on define a deterministic system. We have to keep the development form starting the wave without any cause. Unfortunately, the following ϕ' also satisfies Example 1.1:

$\phi'(x,t) = 1$ if $0 < t$ or $x = 0$, and $\phi'(x,0) = 0$ for $x \neq 0$.

We regard the function as a wave, that is, the wave exists where $\phi'(x,t) = 1$. At the beginning time $t = 0$, the wave exists only at $x = 0$. But the wave exists everywhere after the beginning time, that is, $t > 0$. This difficulty seems the inverse of Zeno's paradox. Why does the wave exists without the cause? That is because the wave at $t = 1$ has the cause at time $t = 1/2$, the wave at $t = 1/2$ has the cause at time $t = 1/4$, the wave at $t = 1/2^n$ has the cause at time $t = 1/2^{n+1}$, and so forth.

In order to prevent this difficulty, we define the notions of fundamental functions and speed condition. The intuitive meaning of fundamental function is the follows. In the development described by a fundamental function, one can always trace back the cause of each phenomenon to the initial condition. The formal definition of it is given in Definition 2.10 and 4.2. The speed condition is a condition for the transition rules. The intuitive meaning of speed condition is that a wave progressing at speed v does not make the effect faster than its speed v. The formal definition of it is given in Definition 4.5.

Through these devices, we prove the uniqness of the solution under a certain kind of restriction. That is Theorem 4.16, which is the main theorem of this paper.

As for Example 1.1, the set of transition rules actually satisfies speed condition. The function ϕ' is not a fundamental function, and the function ϕ_1 is the unique fundamental function which is the solution of the transition rules.

2 Topological Operators over Time-Space

Definition 2.1 (Time-Space) Let X be a complete metric space. Then the set $U = X \times \mathbb{R}$ is called a *time-space*, where the left part X is regarded as the space and the right part $\mathbb{R}$ is regarded as the time.

Example 2.2 Let X be a closed subset of $\mathbb{R}^n$. Then $X \times \mathbb{R}$ is a time-space.

Definition 2.3 (Accumulation operator) Let U be a time-space. We define operators A_v^+ and A_v^- over $E \subset U$ for a real number $v \geq 0$.

- $A_v^+(E)$

$= \{(x,t) \mid \forall\epsilon > 0.\exists(x',t').\, t-\epsilon < t' < t,\ d(x,x') \le v(t-t'),\ (x',t') \in E\}$

- $A_v^-(E) = \{(x,t) \mid (x,-t) \in A_v^+(\{(x',t') \mid (x',-t') \in E\})\}$
  $= \{(x,t) \mid \forall\epsilon > 0.\exists(x',t').\, t < t' < t+\epsilon,\ d(x,x') \le v(t-t'),\ (x',t') \in E\}$

The operator A_v^+ is called the *forward accumulation operator at speed limit* v, and A_v^- is called the *backward accumulation operator at speed limit* v.

Remark 2.4 The operators A_v^+ and A_v^- are appropriate for the phenomena progressing at a constant speed. When we deal with the phenomena with the speed getting slower, the following operators are appropriate.

$$A_{>v}^+(E) = \bigcap_{w>v} A_w^+(E), \qquad A_{<v}^-(E) = \bigcup_{w<v} A_w^-(E).$$

Proposition 2.5 *1.* $U = A_v^+(U)$ *2.* $\emptyset = A_v^+(\emptyset)$
3. $E \subset E' \Rightarrow A_v^+(E) \subset A_v^+(E')$ *4.* $A_v^+(A_v^+(E)) \subset A_v^+(E)$
5. $A_v^+(E \cup E') = A_v^+(E) \cup A_v^+(E')$ *6.* $v \le w \Rightarrow A_v^+(E) \subset A_w^+(E)$
The similar properties hold for A_v^-.

Definition 2.6 (Closure, interior, boundary) The *closure operators* F_v^+ and F_v^-, the *interior operators* G_v^+ and G_v^-, and the *boundary operators* ∂_v^+ and ∂_v^- are defined as corresponding to A_v^+ and A_v^-. For $E \subset U$ and a real number $v \ge 0$, they are defined as:

$$F_v^-(E) = E \cup A_v^+(E), \qquad F_v^-(E) = E \cup A_v^-(E),$$
$$G_v^+(E) = U - F_V^+(U - E), \qquad G_v^-(E) = U - F_v^-(U - E),$$
$$\partial_v^+(E) = F_v^+(E) - G_v^+(E), \qquad \partial_v^-(E) = F_v^-(E) - G_v^-(E).$$

Remark 2.7 The properties similar to Prop. 2.5 hold for these operators F_v^+, F_v^-, G_v^+, G_v^-, ∂_v^+ and ∂_v^-.

Definition 2.8 (Closed set, open set) The families of *closed sets* and *open sets* are defined for each real number $v \ge 0$ as follows:

$$\mathcal{F}_v^+ = \{E \subset U \mid E = F_v^+(E)\}, \qquad \mathcal{F}_v^- = \{E \subset U \mid E = F_v^-(E)\},$$
$$\mathcal{G}_v^+ = \{E \subset U \mid E = G_v^+(E)\}, \qquad \mathcal{G}_v^- = \{E \subset U \mid E = G_v^-(E)\}.$$

Remark 2.9 If $v \le w$, then the following hold: $\mathcal{F}_w^+ \subset \mathcal{F}_v^+$, $\mathcal{F}_w^- \subset \mathcal{F}_v^-$, $\mathcal{G}_w^+ \subset \mathcal{G}_v^+$, and $\mathcal{G}_w^- \subset \mathcal{G}_v^-$.

Definition 2.10 (Fundamental set) For each real number $v \ge 0$, we define a family of sets $\mathcal{D}_v$ as:

$$\mathcal{D}_v = \{E \subset U \mid E \in \mathcal{F}_0^-,\ F_v^-(\partial_0^-(E)) \subset E\}.$$

A set $E \in \mathcal{D}_v$ is called a *fundamental set at speed limit* v.

Remark 2.11 If $v \le w$ then $\mathcal{D}_w \subset \mathcal{D}_v$.

Remark 2.12 This $\mathcal{D}_v$ is closed under finite union.

3 Modal Logic

Definition 3.1 (State) A finite set $Q = \{q_1, q_2, ..., q_n\}$ is called the *set of states*, and an element $q_i \in Q$ is called a *state*. States play the rôle of atomic formulae.

Definition 3.2 (Formula) *Formulae* are defined as the following syntax.

$$P ::= q \mid \neg P \mid P \wedge P \mid \Diamond_v P \qquad (q \in Q, v \geq 0)$$

We write P for the set of all the formulae. We write P_v for the set of all the formulae p which do not have any occurrences of modal operators $\Diamond_w$ such that $v < w$. Note that $Q \subset P_v \subset P$.

Notation 3.3 $p \vee p' = \neg(\neg p \wedge \neg p')$, $\Box_v p = \neg \Diamond_v \neg p$.

Notation 3.4 The powers of connection of $\neg$, $\Diamond$ and $\Box$ are strongest. The next is that of $\wedge$, and that of $\vee$ is the weakest.

Definition 3.5 (Totally Modalised formula) A formula $p \in P$ is *totally modalised* if any occurrences of any $q \in Q$ in p are in scopes of $\Diamond$'s. We write P^M for the set of all the totally modalised formulae, and write P_v^M as $P_v^M = P^M \cap P_v$.

Definition 3.6 (Interpretation of formulae) For a partial function ϕ of U into Q and a formula $p \in P$, the *interpretation* $[\![p]\!]_\phi \subset U$ is defined as follows.

$$[\![q]\!]_\phi = \phi^{-1}(q), \qquad [\![\neg p]\!]_\phi = U - [\![p]\!]_\phi, \qquad [\![p \wedge p']\!]_\phi = [\![p]\!]_\phi \cap [\![p']\!]_\phi,$$
$$[\![\Diamond_v p]\!]_\phi = A_v^+([\![p]\!]_\phi).$$

Notation 3.7 For a function $\phi : U \to Q$, we write $\phi|_E$ for the function made by restricting the domain into E.

Proposition 3.8 *For any $v \geq 0$, any subset $E \in \mathcal{G}_v^+$, any functions $\phi, \phi' : U \to Q$ such that $\phi|_E = \phi'|_E$ and any formula $p \in P_v$, it holds that $[\![p]\!]_\phi \cap E = [\![p]\!]_{\phi'} \cap E$.*

Proposition 3.9 *For any $v \geq 0$, any subset $E \in \mathcal{G}_v^+$, any functions $\phi, \phi' : U \to Q$ such that $\phi|_E = \phi'|_E$ and any formula $p \in P_v^M$, it holds that $[\![p]\!]_\phi \cap F_v^+(E) = [\![p]\!]_{\phi'} \cap F_v^+(E)$.*

Definition 3.10 (Positive occurrence, negative occurrence) The notions of *positive occurrence* and *negative occurrence* of a state $q \in Q$ in a formula $p \in P$ are defined in the usual way.

Remark 3.11 For each formula p, there is a formula p' which is logically equivalent to p and has only positive occurrences of states q's. The logical equivalence of two formula p and p' is defined as: $[\![p]\!]_\phi = [\![p']\!]_\phi$ for any total functions $\phi : U \to Q$. Actually, such p' is made from p by replacing each negative occurrence of each state q with the formula $\neg \bigvee_{q' \neq q} q'$.

Definition 3.12 (Propagating speed in a formula) The *propagating speed* of a state q in a formula $p \in P^M$ is the greatest number v such that the formula p has a positive occurrence of q, or a negative occurrence of some q' other than q, in the scope of $\Diamond_v$. If there is no such v, then the propagating speed is defined as $-\infty$.

We write $Pr(q, p)$ for the propagating speed of q in p.

Example 3.13 It holds that $Pr(q, \Diamond_v q) = v$ and $Pr(q, \Diamond_v \neg q) = -\infty$.

The formula $\neg\Diamond_v q$ has a negative occurrence of q in the scope of $\Diamond_v$. Indeed q occurs positively in $\Diamond_v$, but it occurs negatively in $\neg\Diamond_v q$. Thus $Pr(q, \neg\Diamond_v q) = -\infty$ and $Pr(q', \neg\Diamond_v q) = v$ for a state $q' \neq q$.

Remark 3.14 The definition of propagating speed of q mentions the negative occurrences of the state other than q. That is because the negative occurrences of the state other than q can be turned into the positive occurrence of q by the replacement as in Remark 3.11.

Lemma 3.15 *Put $\phi, \phi' : U \to Q$ and $p \in P^M$. Suppose that $(x, t) \in [\![p]\!]_\phi$ and $(x, t) \notin [\![p]\!]_{\phi'}$. Then, for any $\epsilon > 0$, there is a point (x', t') such that:*

$$0 < t - t' < \epsilon, \ \frac{d(x, x')}{t - t'} \leq Pr(\phi(x', t'), p), \ \textit{and} \ \frac{d(x, x')}{t - t'} \leq Pr(\phi'(x', t'), \neg p).$$

4 Transition System

Notation 4.1 $U_{\geq t_0} = \{(x, t) \in U_| t \geq t_0\}$, $U_{>t_0} = \{(x, t) \in U_| t > t_0\}$, $U_{(t_1,t_2)} = \{(x, t) \in U_| t_1 < t < t_2\}$

Definition 4.2 (Fundamental function) For $v \geq 0$, a function $\phi : U \to Q$ or $\phi : U_{\geq t_0} \to Q$ is a *fundamental function at speed limit v* iff $\phi^{-1}(q) \in \mathcal{D}_v$ for each $q \in Q$.

We write Φ_v for the set of all the fundamental functions $\phi : U \to Q$ at speed limit v. We write $\Phi_{v,\geq t_0}$ for the set of all the fundamental functions $\phi : U_{\geq t_0} \to Q$ at speed limit v.

Remark 4.3 The notion of fundamental functions is a generalisation of the notion of half-open intervals. Actually, for a fundamental function $\phi \in \Phi_v$ and a state $q \in Q$, the set $\{t \in T \mid \phi(x, t) = q\}$ is a countable union of half-open intervals $[t, t')$.

Remark 4.4 If $\phi \in \Phi_v$, then $\phi|_{U_{\geq t_0}} \in \Phi_{v,\geq t_0}$.

The inverse also holds. For each function $\phi' \in \Phi_{v,\geq t_0}$, there is a total function $\phi \in \Phi_v$ such that $\phi' = \phi|_{U_{\geq t_0}}$. This ϕ is constructed as:

$\phi(x,t) = \phi'(x,t)$ if $t \geq t_0$, and $\phi(x,t) = \phi'(x,t_0)$ if $t < t_0$.

Definition 4.5 (Transition rule) A triple $\langle q, v, p\rangle \in Q \times [0,\infty) \times P^M$ is called a *transition rule.* The rule $\langle q, v, p\rangle$ means that, if p holds, then the state is changed into q at expanding speed v.

Definition 4.6 (Satisfaction of rules at a point) A partial function ϕ of U into Q *satisfies* a transition rule $\langle q, v, p\rangle$ at a point $(x,t) \in U$ iff they satisfy the relation $\phi \models_{(x,t)} \langle q, v, p\rangle$ defined as:

$$\phi \models_{(x,t)} \langle q, v, p\rangle \iff (x,t) \in G^-_v(\phi^{-1}(q)) \cup [\![\neg p]\!]_\phi.$$

Definition 4.7 (Satisfaction of rules in an area) A partial function ϕ of U into Q *satisfies* a transition rule $\langle q, v, p\rangle$ at an area $E \subset U$ iff they satisfy the relation $\phi \models_E \langle q, v, p\rangle$ defined as:

$$\phi \models_E \langle q, v, p\rangle \iff \forall (x,t) \in E.\ \phi \models_{(x,t)} \langle q, v, p\rangle$$
$$\iff E \cap [\![p]\!]_\phi(x,t) \subset G^-_v(\phi^{-1}(q)).$$

Definition 4.8 (Transition system) A *transition system* is a finite set S of transition rules. A *transition system at speed limit* v is a finite subset $S \subset Q \times [0, v] \times P^M_v$.

Definition 4.9 (Completeness) A transition system $S = \{\langle q_1, v_1, p_1\rangle, \langle q_2, v_2, p_2\rangle, ..., \langle q_k, v_k, p_k\rangle\}$ is *complete at speed limit* v iff, for each function $\phi \in \Phi_v$, it holds that $U = \bigcup_{1\leq i\leq k} [\![p_i]\!]_\phi$, which is equivalent to: $\phi \models_U p_1 \vee p_2 \vee ... \vee p_k$.

Conjecture 4.10 *For a transition system* $S = \{\langle q_1, v_1, p_1\rangle, \langle q_2, v_2, p_2\rangle, ..., \langle q_k, v_k, p_k\rangle\}$*, if* S *is complete at some speed limit* v*, then for each function* $\phi : U \to Q$*, it holds that* $U = \bigcup_{1\leq i\leq k} [\![p_i]\!]_\phi$.

Definition 4.11 (Propagating speed in a system) The *propagating speed* of a state q in a system S is the maximum of $Pr(q, p)$ for p's such that $\langle q', v, p\rangle \in S$. Note that the state q' in $\langle q', v, p\rangle$ may be different to q.

We write $Pr(q, S)$ for the propagating speed of q in S.

Definition 4.12 (Speed condition) A system S satisfies *speed condition* iff there is $q_0 \in Q$ such that, for each $\langle q, v, p\rangle \in S$, it holds either

$Pr(q, S) \leq v$,

or

$q = q_0$ and $Pr(q_0, p) = 0$, and moreover, for any other $\langle q', v', p'\rangle \in S$,

if $q' \neq q_0$ and $Pr(q', p) \geq 0$, then $Pr(q, p') < 0$.
This q_0 is called the *exception state*.

Remark 4.13 If it holds $Pr(q, S) \leq v$ for each $\langle q, v, p\rangle \in S$, then the system S satisfies speed condition.

Definition 4.14 (Solution at the initial condition with positive thickness) For a transition system S and a function $\phi_0 : U_{(-\epsilon,0)} \to Q$, a function $\phi : U_{\geq -\epsilon} \to Q$ is a *solution* of S at the initial condition ϕ_0 iff the followings hold:

$$\phi|_{U(-\epsilon,0)} = \phi_0, \text{ and } \phi \models_{U_{\geq 0}} R \text{ for each } R \in S.$$

Definition 4.15 (Solution at Hagiya's initial condition) For a transition system S and a function $\phi_0 : X \to Q$, a function $\phi : U_{\geq 0} \to Q$ is a *solution* of S at the initial condition ϕ_0 iff the followings hold:

$$\phi(x, 0) = \phi_0(x), \text{ and } \phi \models_{U_{>0}} R \text{ for each } R \in S.$$

Theorem 4.16 (Main theorem) *Let S be a transition system at speed limit v. Suppose that S is complete at speed limit v and S satisfies speed condition. Let ϕ_0 be a function $X \to Q$. Then, the solution of S at the initial condition ϕ_0 in $\Phi_{v,\geq 0}$ is unique if it exists. On the other words, if both of functions $\phi, \phi' \in \Phi_{v,\geq 0}$ are the solution of S at the initial condition ϕ_0, then $\phi = \phi'$.*

We will show the proof of this theorem latter. First, we put the corollary of this theorem.

Corollary 4.17 *Let S be a transition system at speed limit v. Suppose that s is complete at speed limit v and S satisfies speed condition. Let ϕ_0 be a function in $U_{(-\epsilon,0)} \to Q$. Then, the solution of S at the initial condition ϕ_0 in $\Phi_{v,\geq -\epsilon}$ is unique if it exists.*

Outline of the proof. The function $\phi|_{X\times\{0\}}$, which is the solution at $t = 0$, is uniquely determined from the initial condition ϕ_0, because of Proposition 3.9. The function $\phi|_{U_{\geq 0}}$ is the solution of S at Hagiya's initial condition $\phi|_{X\times\{0\}}$. Therefore, the whole function ϕ is uniquely determined because of the main theorem (Thm. 4.16). $\square$

Hereafter we define several notions in order to prove the main theorem (Thm. 4.16).

Definition 4.18 (Difference set) Let S be a transition system at speed limit v which is complete at speed limit v, and satisfies speed condition. For two functions $\phi, \phi' \in \Phi_{v,\geq 0}$ such that both $\phi \models_{U_{>0}} S$ and $\phi' \models_{U_{>0}} S$. Then, the *difference set* Δ is the set $\{(x, t) \in U_{\geq 0} \mid \phi(x, t) \neq \phi'(x, t)\}$.

Definition 4.19 (Cause-effect relation) For two point $(x,t),(x',t') \in \Delta$, the point (x,t) is a *cause* of (x',t') iff the followings hold:

$$1.\ t < t', \quad 2.\ \frac{d(x,x')}{t'-t} \leq Pr(\phi(x,t),S), \quad 3.\ \frac{d(x,x')}{t'-t} \leq Pr(\phi'(x,t),S),$$

4. Either $(x,t) \in \partial_0^-(\phi^{-1}(\phi(x,t)))$ or $(x,t) \in \partial_0^-(\phi'^{-1}(\phi'(x,t)))$.

The relation of these (x,t) and (x',t') are called the *cause-effect relation.*

Lemma 4.20 *For each $(x,t) \in \Delta$, there exists $(x',t') \in \Delta$ which is a cause of (x,t).*

Remark 4.21 The cause-effect relation is not transitive.

Definition 4.22 (Cause-effect chain) Let $(D,<)$ be an ordered set. A function $f : D \to \Delta$ is a *cause-effect chain* iff the followings hold:
– f is injective.
– For any $e,e' \in D$, if $e < e'$, then there is a finite sequence $e = e_0 < e_1 < e_2 < ... < e_n = e'$ in D such that $f(e_{i+1})$ is a cause of $f(e_i)$ for each $i \leq n-1$.

Notation 4.23 We write $\aleph_1$ for the least uncountable ordinal, as is usual. We write $Ord(\alpha)$ for the set of all the ordinals $\beta < \alpha$, which is sometimes identified to α itself in the set theory.

Lemma 4.24 *If $\{(x_1,t_1),(x_2,t_2),(x_3,t_3),...\}$ which is regarded as a function of $Ord(\omega) \to \Delta$ is a cause-effect chain, then $\lim_{n\to\infty}(x_n,t_n) \in \Delta$. Moreover, there is $(x_\omega,t_\omega) \in \Delta$ such that $\{(x_1,t_1),(x_2,t_2),(x_3,t_3),...,(x_\omega,t_\omega)\}$ which is regarded as a function of $Ord(\omega+1) \to U$ is a cause-effect chain.*

Proof of the main theorem (Thm. 4.16). Suppose that there are two distinct solutions $\phi,\phi' \in \Phi_{v,\geq 0}$ of S at the initial condition ϕ_0, and we will derive the contradiction from this assumption. Note that $\Delta \cap (X \times \{0\}) = \emptyset$, because both ϕ and ϕ' follow the same initial condition ϕ_0.

We will construct a cause-effect chain $f : Ord(\aleph_1) \to U_{>0}$.

For zero: Because ϕ and ϕ' are distinct, the difference set Δ is not empty. Thus there is a point $(x,t) \in \Delta$. We put $f(0) = (x,t)$ for this (x,t).

For successors: We will define $f(\alpha+1)$ for a successor $\alpha+1$. As the induction hypothesis, we have already defined $f(\alpha)$. By Lemma 4.20, there is $(x,t) \in \Delta \subset U_{>0}$ which is a cause of $f(\alpha)$. We put $f(\alpha+1) = (x,t)$ for this (x,t).

We have to check that $f|_{\{\beta|\beta\leq\alpha+1\}}$ is a cause-effect chain. As the induction hypothesis, we already have that $f|_{\{\beta|\beta\leq\alpha\}}$ is a cause-effect chain. Hence it suffices to show that there is a finite cause-effect chain from $f(\beta)$ to $f(\alpha+1)$ for each $\beta \leq \alpha$. That is shown because we already have a finite cause-effect

chain from $f(\beta)$ to $f(\alpha)$.
For limits: We will define $f(\alpha)$ for a limit ordinal α. There is a increasing sequence $\alpha_1, \alpha_2, \alpha_3, ...$ which converges into α. As the induction hypothesis, we have already defined $f(\beta)$ for $\beta < \alpha$ and $f|_{\{\beta|\beta<\alpha\}}$ is a cause-effect chain. Thus, we can choose the sequence $\{f(\alpha_1), f(\alpha_2), f(\alpha_3), ...\}$ as a cause-effect chain. By Lemma 4.24, there is $(x,t) \in \Delta$ such that the chain $\{f(\alpha_1), f(\alpha_2), f(\alpha_3), ..., (x,t)\}$ is a cause-effect chain. we put $f(\alpha) = (x,t)$ for this (x,t).

We have to check that $f|_{\{\beta|\beta\leq\alpha\}}$ is a cause-effect chain. It suffices to show that there is a finite cause-effect chain from $f(\beta)$ to $f(\alpha)$ for each $\beta \leq \alpha$. That is shown because we already have a finite cause-effect chain from $f(\beta)$ to $f(\alpha_n)$, for some $\alpha_n > \beta$.

Thus we have constructed a cause-effect chain $f : \mathit{Ord}(\aleph_1) \to U_{>0}$. For $\alpha < \aleph_1$, put (x_α, t_α) as $(x_\alpha, t_\alpha) = f(\alpha)$. Then $0 < t_\alpha < t_\beta$ for any $\beta < \alpha < \aleph_1$. Therefore, the chain $\{t_\alpha\}_{\alpha<\aleph_1}$ is an uncountable bounded monotone chain. However, there is no uncountable bounded monotone chain in real numbers. That is contradiction. $\Box$

Remark 4.25 The main theorem (Thm. 4.16) gives a reasonable sufficient condition which ensures the uniqueness of the solution. It is similar to the theorem of Cauchy and Kovalevskaja for differential equations. Unfortunately, it seems very much difficult to give a reasonable sufficient condition which ensures the existence of the solution, although the condition in the theorem of Cauchy and Kovalevskaja also ensures the existence of the solution.

5 Examples

Example 5.1 Example 1.1 is formalised as below. The space is a line $X = \mathbb{R}$, and the set of states is $Q = \{0, 1\}$. The transition system S and Hagiya's initial condition ϕ_0 are defined as follows:

$$S = \{\langle 1, 1, \Diamond_1 1\rangle, \langle 0, 1, \neg\Diamond_1 1\rangle\},$$
$$\phi_0(0) = 1, \text{ and } \phi_0(x) = 0 \text{ for } x \neq 0.$$

This S is a transition system at speed limit 1, is complete at speed limit 1, and satisfies speed condition. The function ϕ_1 in Section 1 is the unique solution of S at the initial condition ϕ_0 in $\Phi_{1,\geq 0}$.

Example 5.2 (Hagiya's Example) The space is a plane $X = \mathbb{R}^2$, and the set of states is $Q = \{\mathsf{white}, \mathsf{red}, \mathsf{blue}, \mathsf{black}\}$, which consists of four colours. The transition system S and Hagiya's initial condition ϕ_0 are defined as follows.

$$R_1 = \langle \mathsf{red}, 1, p_1\rangle,\ p_1 = \Diamond_1 \mathsf{red} \wedge \Box_1(\mathsf{white} \vee \mathsf{red}),$$
$$R_2 = \langle \mathsf{blue}, 1, p_2\rangle,\ p_2 = \Diamond_1 \mathsf{blue} \wedge \Box_1(\mathsf{white} \vee \mathsf{blue}),$$

$R_3 = \langle \mathsf{black}, 0, p_3 \rangle,\ p_3 = \Diamond_1 \mathsf{red} \wedge \Diamond_1 \mathsf{blue},$
$R_4 = \langle \mathsf{black}, 0, p_4 \rangle,\ p_4 = \Diamond_0 \mathsf{black},$
$R_5 = \langle \mathsf{white}, 1, p_5 \rangle,\ p_5 = \Box_1 \mathsf{white},$
$R_6 = \langle \mathsf{white}, 1, \neg(p_1 \vee p_2 \vee p_3 \vee p_4 \vee p_5) \rangle,$
$S = \{R_1, R_2, R_3, R_4, R_5, R_6\}$

$$\phi_0 : X \to Q,\ \phi_0(x, y) = \begin{cases} \mathsf{red}, & (x, y) = (1, 0) \\ \mathsf{blue}, & (x, y) = (-1, 0) \\ \mathsf{white}, & \text{otherwise} \end{cases}$$

This S is a transition system at speed limit 1, is complete at speed limit 1, and satisfies speed condition. The function $\phi \in \Phi_{1,\geq 0}$ defined as follows is the solution of S at the initial condition ϕ_0.

$$\phi : U_{\geq 0} \to Q,\ \phi(x, y, t) = \begin{cases} \mathsf{red}, & x > 0,\ t \geq \sqrt{(x-1)^2 + y^2} \\ \mathsf{blue}, & x < 0,\ t \geq \sqrt{(x+1)^2 + y^2} \\ \mathsf{black}, & x = 0,\ t \geq \sqrt{y^2 + 1} \\ \mathsf{white}, & \text{otherwise} \end{cases}$$

The rule R_6 is a dummy rule; this rule is not applied at any points actually, that is, $[\![\neg(p_1 \vee p_2 \vee p_3 \vee p_4 \vee p_5)]\!]_\phi = \emptyset$ for this ϕ. The system S has the rule R_6 because it makes the system complete.

Example 5.3 The space is a line $X = \mathbb{R}$, and the set of states is $Q = \mathbb{Z}/5\mathbb{Z} = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}\}$ which is the cyclic group of period 5. It holds in Q that $\overline{-1} = \bar{4} = \bar{9} = ...$, $\overline{-2} = \bar{3} = \bar{8} = ...$ and so forth. The transition system S is defined as below.

For $\bar{i} \in Q$,

$R_{1,\bar{i}} = \langle \bar{i}, 1, P_{1,\bar{i}} \rangle,\ P_{1,\bar{i}} = \Diamond_1 \bar{i} \wedge \Box_1(\bar{i} \vee \overline{i-1}),$
$R_{2,\bar{i}} = \langle \bar{i}, 1, P_{2,\bar{i}} \rangle,\ P_{2,\bar{i}} = \Diamond_1 \bar{i} \wedge \Box_1(\bar{i} \vee \overline{i-2}),$
$R_{3,\bar{i}} = \langle \bar{i}, 1, P_{3,\bar{i}} \rangle,\ P_{3,\bar{i}} = \Diamond_1 \overline{i-1} \wedge \Diamond_1 \overline{i-2},$
$R_{4,\bar{i}} = \langle \bar{i}, 1, P_{4,\bar{i}} \rangle,\ P_{4,\bar{i}} = \Box_1 \bar{i},$

$R_5 = \langle \bar{0}, 1, \neg(P_{1,\bar{0}} \vee ... \vee P_{4,\bar{4}}) \rangle,$
$S = \{R_{1,\bar{0}}, ..., R_{4,\bar{4}}, R_5\}$

This S is a transition system at speed limit 1, is complete at speed limit 1, and satisfies speed condition.

The initial condition is defined as below. The functions $\phi \in \Phi_{1,\geq -\epsilon}$ and $\phi : U_{(-\epsilon,0)} \to Q$ are defined as:

$$m, n \in \mathbb{Z},\ n \le t + x < n + 1,\ m \le t - x < m + 1 \Rightarrow \phi(x,t) = \overline{n + 2m}.$$

The area $\{(x,t) \mid n \le t + x < n + 1,\ m \le t - x < m + 1\}$ is an oblique square whose lowest point is $({}^{m+n}/_2, {}^{n-m}/_2)$. The set of such squares fills the whole plain. Therefore this ϕ is well-defined. The initial condition ϕ_0 is defined as: $\phi_0 = \phi|_{U_{(-\epsilon,0)}}$.

Then, the function ϕ is the solution of S at the initial condition ϕ_0.

Remark 5.4 This ϕ in this example makes periodic intervals on the one-dimensional line. In general, on n-dimensional space $\mathbb{R}^n$, we can construct a transition system and an initial condition such that the solution makes periodic cells.

The set of states in this example represents only the phases of cells. If we put the set of states as the direct product of the set of colours and the set of phases, such as $Q = \{\mathsf{white},\ \mathsf{black}\} \times \mathbb{Z}/5\mathbb{Z}$, then the system can realise a cellular automaton. This can be generalised into the spaces of higher dimensions.

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