Deciding Whether a Grid is a Topological Subgraph of a Planar Graph is NP-Complete

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Abstract

The Topological Subgraph Containment (TSC) Problem is to decide, for two given graphs G and H, whether H is a topological subgraph of G. It is known that the TSC Problem is NP-complete when H is part of the input, that it can be solved in polynomial time when H is fixed, and that it is fixed-parameter tractable by the order of H.

Motivated by the great significance of grids in graph theory and algorithms due to the Grid-Minor Theorem by Robertson and Seymour, we investigate the computational complexity of the Grid TSC Problem in planar graphs. More precisely, we study the following decision problem: given a positive integer k and a planar graph G, is the $k \times k$ grid a topological subgraph of G? We prove that this problem is NP-complete, even when restricted to planar graphs of maximum degree six, via a novel reduction from the Planar Monotone 3-SAT Problem.

Keywords: topological subgraph, subgraph homeomorphism, subdivision, grids, planar graph, NP-complete

1 Introduction

Given a graph G, the *subdivision* of an edge uv of G consists of its deletion and the addition of a new path of length two with ends u and v. For two graphs G and H, we say that H is a *topological subgraph* of G, or that G *contains a subdivision* of H, if G has a subgraph isomorphic to a graph obtained from H by repeatedly subdividing

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edges. This notion appears for example in the classical characterization of planar graphs by Kuratowski.

Our work concerns the computational complexity of the GRID TOPOLOGICAL SUBGRAPH CONTAINMENT (GRID TSC) PROBLEM in planar graphs. The general question, known as the TOPOLOGICAL SUBGRAPH CONTAINMENT (TSC) PROBLEM or as the SUBGRAPH HOMEOMORPHISM PROBLEM, is to determine for two given graphs G and H, whether H is a topological subgraph of G. To the best of our knowledge, investigations on this problem started with the work of LaPaugh and Rivest [6], who observed that, when H is part of the input, the TSC PROBLEM is NP-complete. Indeed, when G is a graph on n vertices and H is a cycle on n vertices, then solving the TSC PROBLEM means to decide whether G contains a Hamilton cycle. Since the HAMILTON CYCLE PROBLEM remains NP-complete when restricted to planar graphs [4], it also follows that the TSC PROBLEM restricted to planar graphs is NP-complete. This is different when the graph H is fixed. An algorithmic result within the famous Graph Minor Theorem of Robertson and Seymour [7] is that, when H is fixed, the TSC PROBLEM can be solved in time polynomial in the order of the input graph G. However, the constants involved in their result are enormous, which implies that their algorithm for the TOPOLOGICAL SUBGRAPH PROBLEM is not practical. More efficient algorithms are known for certain graphs H, including the complete bipartite graph $K_{3,3}$ [1] and wheels with up to seven spokes [3,8,9]. More recently, Grohe et al. [5] showed that the TSC PROBLEM is fixed-parameter tractable by the order of H. Their algorithm solves the TSC PROBLEM in time proportional to $f(n_H) \cdot n_G^3$, where n_G and n_H denote the number of vertices of the given graphs G and H, respectively, and $f(n_H)$ does not depend on n_G.

In this work, we prove that the TSC PROBLEM remains NP-complete when G is a planar graph and H is a grid. More precisely, we study the GRID TSC PROBLEM, which is to decide whether a given graph G contains the $k \times k$ grid as a topological subgraph, where k is part of the input. We show the following.

Theorem 1.1 *The* GRID TSC PROBLEM *in planar graphs is NP-complete, even when restricted to planar graphs with maximum degree 6.*

In other words, Theorem 1.1 says that finding largest topological grid minors in planar graphs with maximum degree 6 is NP-hard. Our proof of the previous theorem is a novel reduction, sketched in Section 1.4, from the PLANAR MONOTONE 3-SAT PROBLEM, which is NP-complete [2].

A concept related to topological subgraphs is the concept of minors. A graph H is a *minor* of a graph G if a graph isomorphic to H can be obtained from G by a series of edge contractions, and vertex and edge deletions. If G contains H as a topological subgraph, then G also contains H as a minor, and the reverse holds when $\Delta(H) \leq 3$. Moreover, for every graph H, there is a finite list of graphs $H_1, \ldots, H_\ell$ such that G contains H as a minor if and only if G contains one of the graphs $H_1, \ldots, H_\ell$ as a topological subgraph.

1.1   Organization of the paper

In Sections 1.2 and 1.3, we introduce basic concepts and terminology. In Section 1.4, we sketch the main idea of the reduction for the proof of Theorem 1.1. In Section 2, we describe the gadgets of the reduction. In Section 3, these gadgets are used to construct the graph G_ϕ, which depends on a monotone rectilinear drawing of a 3-SAT formula ϕ. Finally, in Section 4 we present the reduction, i.e., we prove that G_ϕ contains a subdivision of a grid of certain size if and only if ϕ is satisfiable.

Due to space limitations, along some work some definitions and technicalities are not formally established but are presented in a rather simplified and hopefully intuitive manner.

1.2   The Planar Monotone 3-SAT Problem

Let $\mathcal{U} = \{x_1, x_2, \ldots, x_n\}$ be a set of boolean variables and $\mathcal{C} = \{C_1, \ldots, C_m\}$ be a set of clauses over $\mathcal{U}$, where each clause C_i with $i \in [m]$ is a disjunction of at most 3 literals, that is, variables from $\mathcal{U}$ or their negation. Then, $\phi = \phi(\mathcal{U}, \mathcal{C}) = C_1 \wedge C_2 \wedge \ldots \wedge C_m$ is a *3-SAT formula* over $\mathcal{U}$ and is called *satisfiable* if there exists an assignment of TRUE and FALSE to the variables in $\mathcal{U}$ such that ϕ evaluates to TRUE. If a clause C contains only positive or only negative literals, then C is called *positive* or *negative*, respectively. A 3-SAT formula ϕ is called *monotone* if each clause in ϕ is either positive or negative, and it is called *planar* if the following bipartite graph G is planar: vertex set of G is $\{x_1, \ldots, x_n\} \cup \{C_1, \ldots, C_m\}$ and $\{x, C\}$ is an edge of G if and only if the clause C uses x or its negation $\bar{x}$.

Assume that $\phi = \phi(\mathcal{U}, \mathcal{C})$ is a monotone and planar 3-SAT formula. Consider an orthogonal coordinate system in the plane consisting of a horizontal and a vertical axis. A *monotone rectilinear representation* of ϕ is a drawing in the plane with the following properties, see Figure 1a):

$C_3 = (\bar{x}_1 \vee \bar{x}_1 \vee \bar{x}_1)$ $C_2 = (x_2 \vee x_2 \vee x_3)$
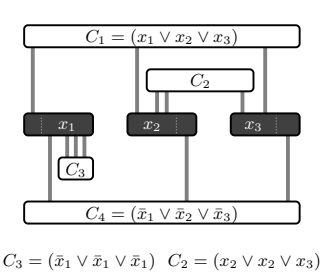
a) A monotone rectilinear representation of $\phi = \bigwedge C_i$.
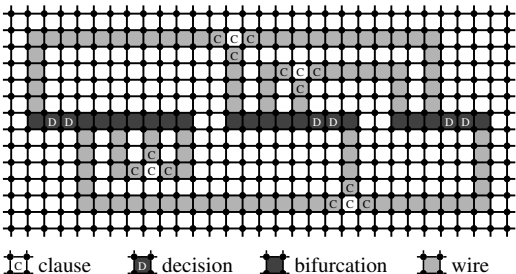
b) Plan for adding gadgets to the grid $\tilde{G}$.

Fig. 1.

- Variable in $\mathcal{U}$ and clauses in $\mathcal{C}$ are represented by pairwise disjoint rectangles in the plane, each of whose sides is parallel to the horizontal or the vertical axis.

- The horizontal axis intersects each rectangle representing a variable in $\mathcal{U}$ and no rectangle representing a clause in $\mathcal{C}$. Further, each rectangle representing

a positive clause in $\mathcal{C}$ is drawn above the horizontal axis and each rectangle representing a negative clause in $\mathcal{C}$ is drawn below the horizontal axis.

- For each variable $x \in \mathcal{U}$ and for each clause $C \in \mathcal{C}$ such that C contains x or $\bar{x}$ there is a vertical line segment that joins the rectangles representing x and C and that does neither intersect other vertical line segments nor other rectangles.

Given a monotone rectilinear representation of a 3-SAT formula ϕ, the PLANAR MONOTONE 3-SAT PROBLEM is to decide whether ϕ is satisfiable. This problem is NP-complete [2].

Consider a planar monotone 3-SAT formula $\phi = \phi(\mathcal{U}, \mathcal{C})$. Throughout this paper, we make no distintion between variables and clauses of ϕ and their representations as rectangles in $\mathcal{R}$. Furthermore, without loss of generality, we assume that each variable in $\mathcal{U}$ appears in at least one positive clause and at least one negative clause, i.e., ϕ uses both literals x and $\bar{x}$ for each $x \in \mathcal{U}$. Also, we assume that each clause contains exactly three literals and, in $\mathcal{R}$, each clause is incident to exactly three vertical lines.

1.3 Basic definitions and terminology

Throughout this paper, we use the notation $[n] := \{1, 2, \ldots, n\}$ for $n \in \mathbb{N}$. For each $k \geq 3$, the $k \times k$ grid $\tilde{G}$ is the graph with vertex set $\{(i, j) : i, j \in [k]\}$ and edge set

$$\left\{ \{(i, j), (i', j')\} : |i - i'| + |j - j'| = 1 \right\}.$$

The *canonical embedding* of $\tilde{G}$ refers to a drawing of $\tilde{G}$ in the plane, where the vertex (i, j) for $i \in [k]$ and $j \in [k]$ is embedded at the point (i, j) in a coordinate system whose horizontal axis refers to the first coordinate and whose vertical axis refers to the second coordinate and each edge of $\tilde{G}$ is represented by a line segment. For $k \geq 3$, the unique infinite face of the canonical embedding of $\tilde{G}$ is called the *outer face* of $\tilde{G}$ and each other face of $\tilde{G}$ is referred to as an *inner face* of $\tilde{G}$.

For $i \in [k]$, the path that is induced in $\tilde{G}$ by the vertices in $\{(i, j) : j \in [k]\}$ is called the ith *vertical grid-path* of $\tilde{G}$ and, for $j \in [k]$, the path that is induced in $\tilde{G}$ by the vertices in $\{(i, j) : i \in [k]\}$ is called the jth *horizontal grid-path* of $\tilde{G}$. Edges of $\tilde{G}$ are called either *vertical* or *horizontal*, accordingly.

1.4 Reduction Idea

Consider an instance $\phi = \phi(\mathcal{U}, \mathcal{C})$ of the PLANAR MONOTONE 3-SAT PROBLEM, i.e., a drawing $\mathcal{R}$ as in Figure 1a). To prove that the GRID TSC PROBLEM in planar graphs is NP-hard (Theorem 1.1), we construct a planar graph G_ϕ and define an appropriate value k such that the $k \times k$ grid is a topological subgraph of G_ϕ if and only if ϕ is satisfiable.

The construction of G_ϕ starts with the $k \times k$ grid $\tilde{G}$. We present gadgets for clauses and variables as well as wire gadgets that are used to connect variable gadgets to clause gadgets. Each variable gadget consists of one decision gadget and several bifurcation gadgets. The purpose of the decision gadget is to encode whether

a variable of ϕ is set to TRUE or FALSE and the purpose of the clause gadget is to ensure that at most two of the three literals of the clause are set to FALSE. The bifurcation and wire gadgets duplicate and propagate the information through the graph G_ϕ. Using the drawing $\mathcal{R}$, these gadgets are placed into some *inner* faces of $\tilde{G}$, i.e., faces of $\tilde{G}$ that are bounded by a cycle of length 4, see Figure 1.

We show that, if G_ϕ contains a subdivision H of the $k \times k$ grid, then, roughly speaking, H is $\tilde{G}$ except for a few local differences. In the construction of G_ϕ, for each variable gadget, an edge of $\tilde{G}$ is deleted. Such deletion forces the corresponding grid-path of H to bend and, as a result, all grid-paths along some wire gadgets connecting that variable to positive or negative clauses have to bend. Here, a bend of a grid-path can be interpreted as a variable sending the value FALSE towards a clause. The clause gadget, placed into an inner face f of $\tilde{G}$, is designed in such a way that at most two grid-paths can bend into the face f. As a consequence, if the three variables connected to the clause gadget in f are set to FALSE, one of the grid-paths cannot bend into the face f and G_ϕ does not contain a subdivision of the $k \times k$ grid.

2   Gadgets

In this section, we consider the $k \times k$ grid $\tilde{G}$ along with its canonical embedding. We refer to deleting an edge, subdividing an edge, adding a new vertex, and adding a new edge as *modifications*. Let G be a plane graph obtained from $\tilde{G}$ by modifications. In order to construct the gadgets we apply modifications to certain inner faces of $\tilde{G}$. These modifications often split an inner face of $\tilde{G}$ into several faces, or simply change the boundary of an inner face. Throughout this work, for a graph G obtained from $\tilde{G}$ by modifications which create a gadget, we refer to an inner face of G as the inner face of $\tilde{G}$ with the updated boundary if necessary.

In the figures of the gadgets, new edges are drawn thicker than edges in $\tilde{G}$ and edges arising from subdivisions.

2.1   Variable Gadgets

The variable gadget consists of one decision gadget and several bifurcation gadgets. The decision gadget is the part of the variable gadgets that encodes the TRUE or FALSE assignment to the variable. Then, the bifurcation gadget replicates the information encoded by the decision gadget as many times as needed.

The following definition is used throughout the construction of the gadgets. Let f be an inner face of $\tilde{G}$ and let $i, j \in [k]$ be such that f is bounded by the cycle $((i, j), (i+1, j), (i+1, j+1), (i, j+1))$. To add a *left arrow* in the face f means to replace the edge $\{(i, j), (i, j+1)\}$ by the path $((i, j), s_1, s_2, (i, j+1))$, where s_1 and s_2 are new vertices, and to add the edges $\{(i+1, j), s_1\}$ and $\{(i+1, j+1), s_2\}$. Similarly, we define *right arrow*, *up arrow*, and *bottom arrow*.

Decision gadget

Consider a vertical edge e of $\tilde{G}$. Denote by f^+ and f^- the two inner faces of $\tilde{G}$ whose edge at the right is e and whose edge at the left is e, respectively. To add a decision gadget for $x \in \mathcal{U}$ means to add a left arrow in f^+, to add a right arrow in f^-, and to delete the edge $e_x = e$, see Figure 2.

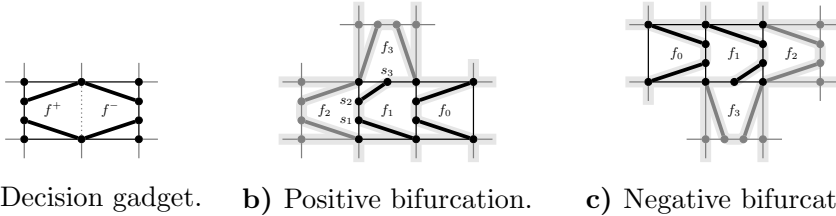**a)** Decision gadget.   **b)** Positive bifurcation.   **c)** Negative bifurcation.

Fig. 2. Decision and bifurcation gadgets. In Part b) and c), the highlighted subgraph visualizes, how the bend in f_0 is split into two bends in the connection faces f_2 and f_3.

Faces f^+ and f^-, are called the *positive* and the *negative face* of the variable x, respectively and the new edges in f^+ and f^- are called the *positive* and the *negative edges* of the variable x, respectively.

Bifurcation gadget

Roughly speaking, a *positive bifurcation gadget* consists of two left arrows, where one of them is slightly twisted, see Figure 2. More precisely, consider two inner faces f_0 and f_1 of $\tilde{G}$ such that f_1 is directly at the left of f_0. Denote by e_1 the unique edge of $\tilde{G}$ that is on the boundary of f_0 and f_1. Let e_2 and e_3 be the edges at the left and at the right of f_1, respectively. The following modifications are applied to add a positive bifurcation gadget in f_0 and f_1. Add a left arrow in f_0. Subdivide e_2 with two vertices s_1 and s_2 and subdivide e_3 once, say with the vertex s_3. Without loss of generality, assume that s_2 and s_3 have a common neighbor. Insert the new edge $\{s_2, s_3\}$ and a new edge joining s_1 to the common vertex of the edges at the bottom and at the right of the face f_1. Faces f_1 and f_0 are called *bifurcation faces*, and f_0 is also called the *right connection* face of the positive bifurcation gadget. This finishes the description of the positive bifurcation gadget. We add some more notation. Denote by f_2 and f_3 the inner faces of $\tilde{G}$ that are directly at the left and at the top of f_1, respectively. The faces f_2 and f_3 are called the *left* and the *top connection faces* of the positive bifurcation gadget; but they do not belong to the positive bifurcation gadget.

The *negative bifurcation gadget* is obtained by rotating the positive bifurcation gadget around 180 degrees, see Figure 2 and the terminology for the negative bifurcation gadget is naturally adapted.

2.1.1   Assembling the Variable Gadget

In what follows, we consider an instance $\phi = \phi(\mathcal{U}, \mathcal{C})$ of the PLANAR MONOTONE 3-SAT. Consider a variable $x \in \mathcal{U}$. We denote by $\deg^+(x)$ (resp. $\deg^-(x)$) the number of appearances of x (resp. $\bar{x}$) in clauses of $\mathcal{C}$. A *variable gadget* for x consists of $\deg^+(x)$ positive bifurcation gadgets, $\deg^-(x)$ negative bifurcation gadgets, all

top and bottom connection faces of the bifurcation gadgets, a left connection face of a positive bifurcation gadget and a right connection face of a negative bifurcation gadget. See Figure 3. More precisely, let $(f_1, e_2, \ldots, e_d, f_d)$ be a face-sequence
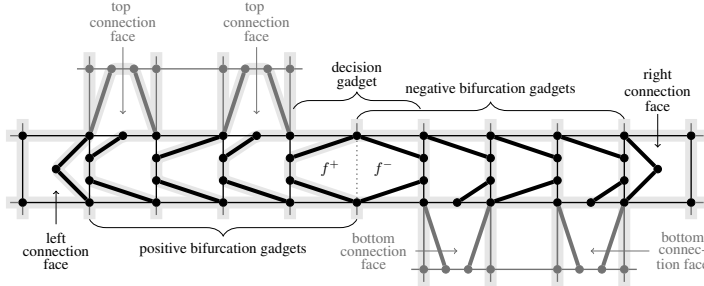

Fig. 3. The variable gadget. In the highlighted subgraph, the edge e_x bends into the positive face f^+, which results in bends in all top connection faces.

with $d := 2(\deg^+(x) + \deg^-(x))$ such that f_{h+1} is the face at the right of f_h for all $h \in [d-1]$. For each $h \in [\deg^+(x)]$, the faces f_{2h-1} and f_{2h} are modified according to the positive bifurcation gadget, the edge $e_{2\deg^+(x)+1}$ is removed, and, for each integer h with $\deg^+(x) + 1 \leq h \leq \deg^+(x) + \deg^-(x)$, the faces f_{2h-1} and f_{2h} are modified according to the negative bifurcation gadget. Note that the faces $f_{2\deg^+(x)}$ and $f_{2\deg^+(x)+1}$ are automatically modified according to a decision gadget, which we set as the decision gadget for x. Further, an up arrow is added to each top connection face and a down arrow is added to each bottom connection face, top and bottom connection faces of the bifurcation gadgets are now called *top and bottom connection faces* of the variable gadget for x. Finally, the following modifications are applied to the left connection face of the "first" positive bifurcation gadget and the right connection face of the "last" negative bifurcation gadget: a new vertex v_0 is inserted in the left connection face, say f_0, and two new edges are added so that v_0 is incident to both ends of the right edge of f_0, and a new vertex v_{d+1} is inserted in the right connection face f_{d+1} and two new edges are added so that v_{d+1} is incident to both ends of the left edge of f_{d+1}. Naturally, those faces are now called the *left and the right connection face* of the variable gadget for x, respectively.

2.2 Clause Gadgets

Consider a positive clause $C \in \mathcal{C}$. A *positive clause gadget* for C consists of one inner face f of $\tilde{G}$, called *clause face*, and faces f_l, f_b, f_r, which are the faces directly at the left, below, and at the right of f, respectively, called *connection faces*. In Figure 4, the positive clause gadget is depicted.

The *negative clause gadget* is obtained from the positive clause gadget by rotating it around 180 degrees. Its clause face and its *connection faces*: faces directly at the left, above, and at the right of the clause face, are defined accordingly.
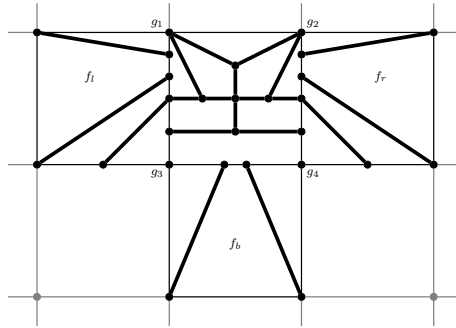
Fig. 4. A positive clause gadget. Define $X_C = \{g_1, g_2, g_3, g_4\}$.

2.3   Wire Gadgets

The task of the wire gadgets is to transfer the information from the variable gadgets to the clause gadgets.

Let $F = (f_0, e_1, f_1, \ldots, e_d, f_d)$ be a face-sequence for some integer d. Then, F is called *straight* if $d \leq 1$ or, $e_h \cap e_{h+1} = \emptyset$ for all $h \in [d-1]$. Moreover, for an integer $h \in [d-1]$, the face-sequence F *is almost straight (and it turns at f_h)* if F is not straight but $F' := (f_0, e_1, \ldots, f_h)$ and $F'' := (f_h, e_{h+1}, \ldots, f_d)$ are straight. Let $F = (f_0, e_1, f_1, \ldots, e_d, f_d)$ be a face-sequence that is straight or almost straight. To add a *wire gadget* along F consists of the following modifications: add an arrow to each face of F in such a way that the arrows point along F, see Figure 5. If F turns at f_h, then only one edge of the arrow is added to f_h, as shown in Figure 5.
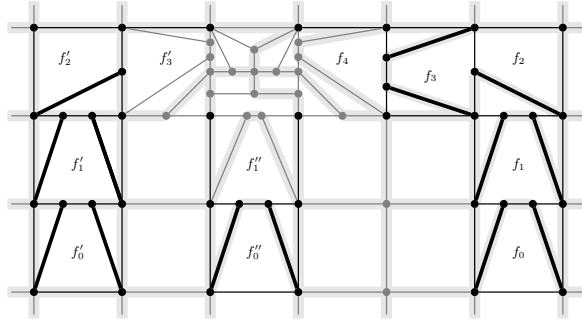

Fig. 5.   Wire gadgets along the face-sequences $F' = (f'_0, e'_1, f'_1, e'_2, f'_2, e'_3, f'_3)$, $F'' = (f''_0, e''_1, f''_1)$, and $F = (f_0, e_1, f_1, e_2, f_2, e_3, f_3, e_4, f_4)$. The faces f'_3, f''_1, and f_4 are faces of the clause gadget, which causes the modifications in these faces. The highlighted subdivision bends along F'' and F.

3   Construction of G_ϕ

Consider an instance $\phi = \phi(\mathcal{U}, \mathcal{C})$ of the PLANAR MONOTONE 3-SAT PROBLEM with its monotone rectilinear representation $\mathcal{R}$. Define $n := |\mathcal{U}|$ and $m := |\mathcal{C}|$. Set $k := 8m + 2n + 5$. Denote by $\tilde{G}$ the $k \times k$ grid and consider $\tilde{G}$ together with its canonical embedding. In the following, we describe the way that gadgets are added to $\tilde{G}$ in order to construct G_ϕ.

First, the variable gadgets for the variables in $\mathcal{U}$ are added one after another in the middle row of $\tilde{G}$. More precisely, let $j_v = \left\lfloor \frac{1}{2}k \right\rfloor$ and let $F_{\mathcal{U}} := (f_1, e_2, f_2, \ldots, f_{k-1})$ be the straight face-sequence, where f_1 is the inner face of $\tilde{G}$ whose boundary contains the grid vertices $(1, j_v)$ and $(1, j_v + 1)$ and f_{k-1} is the inner face of $\tilde{G}$ whose boundary contains the grid vertices (k, j_v) and $(k, j_v + 1)$. Let $\mathcal{U} = \{x_1, \ldots, x_n\}$ and assume that $x_1, \ldots, x_n$ is the order in which the variables appear in the drawing $\mathcal{R}$. The variable gadgets for $x_1, \ldots, x_n$ are placed, one after another, along $F_{\mathcal{U}}$ so that the left connection face of the variable gadget for x_1 is f_{m+3} and the right connection face of the variable gadget for x_n is $f_{7m+2n+2}$. In order to see that this is possible let $d_i = 2(\deg^+(x_i) + \deg^-(x_i))$ and note that $\sum_{x \in \mathcal{U}} \frac{1}{2} d_h$ counts each vertical line of $\mathcal{R}$ once and each clause in $\mathcal{R}$ touches exactly three vertical lines. Hence, $d_1 + \ldots + d_n = 6m$ and thus, $m + 2 + 2n + d_1 + \ldots + d_n = 7m + 2n + 2$. Denote by G_1 the graph obtained from $\tilde{G}$ by adding the variable gadgets in the described way.

In the following, more faces of $\tilde{G}$ in G_1 are chosen for placing clause and wire gadgets. Using the rectilinear drawing $\mathcal{R}$, we can choose distinct inner faces f_C of $\tilde{G}$ for each clause $C \in \mathcal{C}$ and a straight or almost straight face-sequence F_L in $\tilde{G}$ for each vertical line L in $\mathcal{R}$ such that the following two properties hold (see Figure 1).

(1) If L is a vertical line in $\mathcal{R}$ that joins a variable $x \in \mathcal{U}$ to a positive (resp. negative) clause $C \in \mathcal{C}$, then the face-sequence $F_L = (f_0, e_1, f_1 \ldots, f_d)$ satisfies: face f_0 is a top (resp. bottom) connection face of the variable gadget corresponding to x, face f_1 is the face directly above (resp. below) f_0, and face f_d is the clause face f_C. Face-sequences F_L are disjoint, possibly except for the last face, which is always a clause face and, for each clause $C \in \mathcal{C}$ there are exactly three face-sequences with last face f_C.

(2) Each face in $F_L = (f_0, v_{e_1}, f_1, \ldots, f_d)$ has boundary distance at least $m + 2$; the *boundary distance* of an inner face f of $\tilde{G}$ is defined as the length of a shortest v, w-path such that v is on the boundary of f and w is on the boundary of the outer face of $\tilde{G}$. If F_L is an almost straight face-sequence, then F_L turns at f_h with $h \leq d - 2$.

We now finalize the description of G_ϕ. Starting with the graph G_1, for each face-sequence $F_L = (f_0, e_1, \ldots, f_{d-1}, e_d, f_d)$ where L is a vertical line in $\mathcal{R}$, add a wire gadget along F_L. Observe that, for each clause $C \in \mathcal{C}$, the faces directly above, below, at the left and at the right of f_C as well as f_C itself have not been modified so far. Next, for each positive (negative resp.) clause $C \in \mathcal{C}$, add a positive (negative resp.) clause gadget such that its clause face is f_C. Due to the construction of each gadget, the planarity of G_1 and property (1), the graph G_ϕ is planar. Also, observe that, in each face f of $\tilde{G}$ at most a constant number of vertices and edges have been added to construct the graph G_ϕ. Consequently, G_ϕ has size polynomial in k and also polynomial in the size of ϕ.

Finally, let $\mathcal{F}_x^+$ be the set of all face-sequences F_L for each vertical line L in $\mathcal{R}$ that joins x to a positive clause in $\mathcal{C}$. Analogously, define $\mathcal{F}_x^-$.

4   Reduction

Here, we argue that the graph G_ϕ constructed in Section 3 contains a subdivision of a $k \times k$ grid if and only if the formula ϕ is satisfiable.

4.1   If ϕ is Satisfiable

Assume that ϕ is satisfiable. We prove that G_ϕ contains a subgraph H, which is isomorphic to a subdivision of the $k \times k$ grid. Fix a satisfying assignment $T : \mathcal{U} \rightarrow \{\text{TRUE}, \text{FALSE}\}$ of ϕ. Let e_x denote the unique edge of $\tilde{G}$ that was deleted due to the decision part of the variable gadget of x and define $E_\mathcal{U} = \{e_x \colon x \in \mathcal{U}\}$. Moreover, for each $e = \{u, v\} \in E(\tilde{G}) \setminus E_\mathcal{U}$, let the path P_e denote the u,v-path of G_ϕ that replaces e in the construction of G_ϕ. Let $\tilde{G}_s$ be the graph obtained from $\tilde{G}$ by replacing each edge $e \in E(\tilde{G}) \setminus E_\mathcal{U}$ by P_e. Clearly, $\tilde{G}_s$ is a subdivision of $\tilde{G}$ and it is a subgraph of $G_\phi + E_\mathcal{U}$. We argue now that it is possible to remove each edge $e \in E_\mathcal{U}$ from $\tilde{G}_s$ while maintaining a subdivision of a $k \times k$ grid by introducing some bends.

Let H be the subgraph of G_ϕ obtained from $\tilde{G}_s$ by the following modifications. For each variable $x \in \mathcal{U}$ with $T(x) = \text{FALSE}$, the edge e_x is replaced by the positive edges of x and the edge $\tilde{e}$ joining them in G_ϕ. In other words, instead of using e_x, in H, the edge e_x bends into the positive face of the variable gadget for x. Next, the grid-path of $\tilde{G}_s$ that uses the edge $\tilde{e}$ is repaired by bending the left edge of the positive face into the face on its left side. We repeat the repairing process sequentially until all repaired grid-paths are vertex-disjoint paths of G_ϕ, as indicated by the subgraphs in Figure 3 and Figure 5. The final graph H bends along each face-sequence in $\mathcal{F}_x^+$ with $T(x) = \text{FALSE}$. Similarly, for each variable $x \in \mathcal{U}$ with $T(x) = \text{TRUE}$, bend e_x into the negative face of x and repair the grid-paths so that H bends along each face-sequence in $\mathcal{F}_x^-$ with $T(x) = \text{TRUE}$.
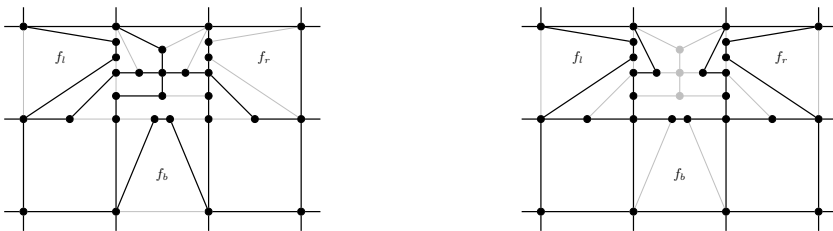
a) The edges e_l and e_b bend into f_l and f_b, respectively.

b) The edges e_l and e_r bend into f_l and f_r, respectively.

Fig. 6. local grid subdivisions in a positive clause gadget.

To see that the construction of H is feasible, it suffices to check that, in H, the edges on the boundary of the clause faces can bend into the clause faces. Consider a positive clause $C \in \mathcal{C}$; the following argument is easy to adjust for a negative clause. As in Section 2.2, denote by f_l, f_b, and f_r the connection faces of the clause C and, for each $h \in \{l, b, r\}$, let e_h be the unique edge on the boundary of f_h that contains

no vertex on the boundary of the clause face f_C. Due to the examples in Figure 6 and symmetry, it is easy to see that locally there are grid subdivisions where up to two edges e_h with $h \in \{l, b, r\}$ bend into f_h simultaneously. Since T is a satisfying assignment, there is a variable x that is used by C and satisfies $T(x) = \text{TRUE}$. Denote by L a line segment of $\mathcal{R}$ that joins x to C. Since the edge e_x bends into the negative face of x in H, the construction of H does not require any modifications along F_L. Consequently, there is an $h \in \{l, b, r\}$ such that e_h does not bend into f_h in H and the construction of H is feasible.

4.2 If G_ϕ Contains a Subdivision of a $k \times k$ Grid

Now, assume that G_ϕ contains a subdivision of a $k \times k$ grid. We argue that ϕ is satisfiable. The main part of the proof is the next lemma. Define $X_\phi = \bigcup_{C \in \mathcal{C}} X_C$, where $X_C \subset V(G_\phi)$ is defined in Figure 4, for each $C \in \mathcal{C}$.

Lemma 4.1 *If G_ϕ contains a subgraph H that is isomorphic to a subdivision of a $k \times k$ grid, then (i) the boundary of the outer face of $\tilde{G}$ is exactly the boundary of the outer face of H and (ii) all the vertices in $V(\tilde{G}) \setminus X_\phi$ are in $V(H)$.*

Sketch of Proof of Lemma 4.1
Note that few new vertices of degree at least 4 are created when constructing G_ϕ from $\tilde{G}$ and these vertices are in the clause faces. Further, each clause face can contain at most 5 vertices with degree 4 in H. Thus, most of the intersection vertices of H with degree 4 must be in $V(\tilde{G})$. Hence, there must be vertices of the outer face of H, which are vertices of the outer face of $\tilde{G}$, and no vertex of the outer face of H belongs to a clause face. The previous fact allows us to prove that each clause face in fact contains at most 4 vertices with degree 4 in H. We can conclude that the outer face of H coincides with the outer face of G_ϕ, i.e., (i) is satisfied. Moreover, a separation argument implies that (ii) is satisfied, which completes the proof of the lemma.

Lemma 4.1 is used to argue that ϕ is satisfiable. Assume that G_ϕ contains a subgraph H that is isomorphic to a subdivision of the $k \times k$ grid. In order to define a truth assignment $T : \mathcal{U} \to \{\text{TRUE}, \text{FALSE}\}$ for ϕ, consider a variable $x \in \mathcal{U}$. In H, the edge e_x of x, bends into the positive or into the negative face of x. Set $T(x) = \text{TRUE}$ if and only if e_x bends into the negative face of x. For a contradiction, assume that there is a positive clause $C \in \mathcal{C}$ that is not satisfied by the assignment T; the following is easy to adjust for a negative clause. Let x be an arbitrary variable used in C and denote by L a line segment in $\mathcal{R}$ that joins x to C. Then $T(x) = \text{FALSE}$ and, hence, e_x bends into the positive face of x. Then, H bends along F_L. As, in $\mathcal{R}$, there are exactly three line segments that touch the clause C, there are three distinct face-sequences F_L ending in a connection face of C and H bends along each of these face-sequences. However, the last is not possible and, hence, C must be satisfied. Consequently, ϕ has a satisfying assignment.

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