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Rates of Convergence of Recursively Defined Sequences

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Abstract

This paper gives a generalization of a result by Matiyasevich which gives explicit rates of convergence for monotone recursively defined sequences. The generalization is motivated by recent developments in fixed point theory and the search for applications of proof mining to the field. It relaxes the requirement for monotonicity to the form $x_{n+1} \leq (1 + a_n)x_n + b_n$ where the parameter sequences have to be bounded in sum, and also provides means to treat computational errors.
The paper also gives an example result, an application of proof mining to fixed point theory, that can be achieved by the means discussed in the paper.

Keywords: monotone sequences, proof mining, computable limits, fixed point theory

1 Introduction

In classical mathematics many interesting results are based on the fact that

(1) every bounded monotone sequence of real numbers converges to a finite limit.

Unfortunately, this fact is not reflected constructively, i.e. there is no theorem letting us compute the speed of the convergence of the sequence which would be needed to compute the limit. Moreover, as shown by Specker in [8], it is possible to construct

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computable monotone sequences whose limit is non-computable, thus finding the speed is even impossible in certain cases.

If we are interested in extracting effective data from a proof that uses this fact, this imposes a significant problem. Since we do not have a constructive analog for it, we would generally be unable to continue past an application of it. The quantitative information hidden within the proof may thus seem inaccessible.

Sometimes it is possible to bypass this problem using (constructively) weakened versions of the statement of convergence like its Herbrand Normal Form (HNF):

$$\forall k \in \mathbb{N}\, \forall g : \mathbb{N} \to \mathbb{N}\, \exists n \,(g(n) \geq n \to |x_n - x_{g(n)}| \leq 2^{-k}). \tag{2}$$

This modification is not strong enough to allow the Specker examples, and we even have a solution for the general theorem in the form of a functional (for $u \leq x_i \leq x_{i+1} \leq v$ for all i)

$$H(u, v, k, g, x) \equiv \mu i \leq (v-u)2^k \left(g^i(0) < g^{i-1}(0) \vee |x_{g^{i-1}(0)} - x_{g^i(0)}| \leq 2^{-k} \right). \tag{3}$$

By certain proof-theoretic techniques (a combination of negative translation and an appropriate so-called monotone version of Gödel's functional interpretation, see [2]) one can show that in extracting bounds from proofs based on (1) it is sufficient to maintain bounds for (2) throughout the proof if the conclusion has a sufficiently simple logical form or is weakened accordingly. This is due to the fact that having a bound on (2) is nothing else than having a realizer for the monotone functional interpretation of the negative translation of (1).

The weakened form may be sufficient to get a full rate of convergence. E.g. [4] treats a proof in fixed point theory that uses the fact (1), but does not require its full power as even an instance of its HNF with $g(n) = n + 1$ suffices to yield the final result. In other cases (e.g. [5]) the weakened form appears in the final result, but may be sufficient to extract valuable information.

However, this approach does not always give the needed answer. It is thus interesting to investigate whether other approaches can resolve the ineffectivity caused by the use of this fact without weakening it. Naturally, these would rely on additional information about the converging sequence.

An approach to handling this problem was taken by Matiyasevich in [6]. He addressed the question of the convergence of a bounded monotone sequence which is defined recursively. It is known that if a diagonalization of the function that defines the recursion has a unique root in the interval where the sequence lies, then the sequence converges to that root. Matiyasevich proved the following theorem:

Theorem 1.1 *Let (x_n) be a non-decreasing sequence of real numbers from the segment $[u, v]$ and let F be a uniformly continuous function defined on all r real numbers $\langle y_1, y_2, \ldots, y_r \rangle$ such that $u \leq y_1 \leq y_2 \leq \ldots \leq y_r \leq v$ with modulus of continuity ω, i.e.*

$$\forall k\, \forall x_0 \leq x_1 \leq \ldots \leq x_{r-1} \in [u, v]\, \forall y_0 \leq y_1 \leq \ldots \leq y_{r-1} \in [u, v]$$

$$(4) \quad \left(\bigwedge_{i=0}^{r-1} |x_i - y_i| < 2^{-\omega(k)} \to |F(x_0, x_1, \dots x_{r-1}) - F(y_0, y_1, \dots y_{r-1})| < 2^{-k} \right).$$

If $F(x_k, x_{k+1}, \dots, x_{k+r-1}) = 0$ *for every* k*, and moreover, the equation* $F(x, x, \dots, x) = 0$ *has a unique root with a "modulus of uniqueness"*[3] η*, i.e.*

$$(5) \quad \begin{gathered} \forall k \, \forall x, y \in [u, v] \\ \big((|F(x, x, \dots x)| < 2^{-\eta(k)} \wedge |F(y, y, \dots y)| < 2^{-\eta(k)}) \to |x - y| < 2^{-k} \big), \end{gathered}$$

then

$$\forall k \, \forall m > \phi(k) \, (|x_m - x_{\phi(k)}| \leq 2^{-k}),$$

where $\phi(k) = 2(r-1)(v-u)2^{\omega(\eta(k))}$.

In Section 2 we give an example for an application of this result to a theorem in fixed point theory.

The main subject of this paper is to treat a generalization of this result of Matiyasevich where the sequences do not need to be monotone. They need to satisfy the inequality $x_{n+1} \leq (1 + a_n)x_n + b_n$, where (a_n) and (b_n) are non-negative and bounded in sum. We would call such sequences *almost monotone*.

The convergence of sequences satisfying this inequality, introduced by Qihou in [9], finds wide use in fixed point theory in recent papers ([5,9,10] among others) to treat Krasnoselski-Mann iterations of asymptotically quasi-nonexpansive mappings with error terms. The form of the inequality reflects the most current iterative schemes used to treat asymptotically non-expansive functions (introduced in [7]) via the (a_n) term and also allows for computational errors in the evaluation of the schemes ([11]) via the (b_n) term.

Error terms are also interesting for recursively-defined sequences, as e.g. computations with computers often introduce errors which can be arbitrarily reduced by increasing the computational precision but never completely eliminated. One would be interested whether a computation of a recursive sequence can start at a low precision and be subsequently refined to achieve the correct final result.

It turns out that it can, provided that the condition on the sequence being almost monotone can be preserved.

The main theorem to be proved in Section 3 gives an explicit rate for the convergence of almost monotone recursively-defined sequences, for which moduli of the kind (4) and (5) can be found. The result also allows computational error, and is uniform in the sense that it only reflects the sequence and recursive definition through a selection of parameters, and is thus applicable to the full range of functions that satisfy the same parameters.

[3] this term was introduced in [3] in a much more general context. Special forms such moduli also occur in numerical analysis (notably in approximation theory) under the name of strong unicity

2 Application of Matiyasevich's result to fixed point theory

In [1] Hillam proved the following generalization of Krasnoselski's Theorem on the real line:

Theorem 2.1 *Let $f : [u, v] \to [u, v]$ be a function that satisfies a Lipschitz condition with constant L. Let x_0 in $[u, v]$ be arbitrary and define $x_{n+1} = (1-\lambda)x_n + \lambda f(x_n)$ where $\lambda = 1/(L+1)$. If (x_n) denotes the resulting sequence, then (x_n) converges monotonically to a point z in $[u, v]$ where $f(z) = z$.*

Hillam proves this statement using three cases:

- $\exists n. f(x_n) = x_n$*:* since $\forall m > n(x_m = x_n = f(x_n))$, the sequence converges to x_n. In the treatment that follows we will allocate this case to one of the others;
- $f(x_0) > x_0$*:* by the continuity of f there exists a fixed point between x_0 and v and then the sequence increases monotonically and is bounded from above by that fixed point [4], therefore by (1) it converges;
- $f(x_0) < x_0$ is analogous to the previous.

After that with a simple triangle inequality he proves that the point to which the sequence converges is a fixed point of f.

Suppose that in addition to the conditions in Theorem 2.1 we know that the mapping has a unique fixed point with modulus of uniqueness η. Then, using Matiyasevich's result, we can formulate the following theorem:

Theorem 2.2 *Let $f : [u, v] \to [u, v]$ be a function that satisfies a Lipschitz condition with constant L. Let f have a unique fixed point within $[u, v]$ with a modulus of uniqueness η. Let x_0 in $[u, v]$ be arbitrary and define $x_{n+1} = (1-\lambda)x_n + \lambda f(x_n)$ where $\lambda = 1/(L+1)$. Then the following is true:*

$$\forall k \left(|x_{\phi(k)} - f(x_{\phi(k)})| \leq (L+1)2^{-k} \wedge \forall m > \phi(k)(|x_m - x_{\phi(k)}| \leq 2^{-k})\right),$$

(i.e. the sequence converges with rate of convergence ϕ and its limit is a fixed point of f) where $\phi(k) = 2(v-u)2^{\eta(k+\lceil \log_2(L+1)\rceil)}$.

Proof. Let $F(y_1, y_2) = (1-\lambda)y_1 + \lambda f(y_1) - y_2$. Since $\lambda L \leq 1$ this function has a Lipschitz constant 1 and thus a modulus of continuity $\omega_F(k) = k$.

$F(y, y) = \lambda(f(y) - y)$ and thus we can infer that the solution to $F(y, y) = 0$ has a modulus of uniqueness $\eta_F(k) = \eta\left(k + \lceil \log_2(L+1)\rceil\right)$.

By Matiyasevich's theorem any monotone sequence within $[u, v]$ defined recursively by F converges with rate $\phi(k)$.

Suppose $f(x_0) \geq x_0$. By Hillam's proof either $x_0 < x_1 < \ldots < p$ where p is a the least fixed point of f greater than x_0, or there exists an n, such that $x_0 < \ldots < x_n = \ldots = p$. In

[4] see the details in [1]

either case (x_n) is monotonically increasing and bounded from above by p. Alternatively, if $f(x_0) \le x_0$, by the same reasoning $(u+v-x_n)$ is monotonically increasing and bounded from above by $u+v-p$.

In both cases the sequence is monotonically increasing and bounded within $[u, v]$, hence Matiyasevich's result applies and therefore

$$\forall k\, \forall m > \phi(k)\ \left(|x_m - x_{\phi(k)}| \le 2^{-k}\right).$$

It remains to show that the point it converges to is a fixed point. Let k be arbitrary and $n = \phi(k)$:

$$|x_n - f(x_n)| \le |x_n - x_{n+1}| + |(1-\lambda)(x_n - f(x_n))|,$$

thus

$$|x_n - f(x_n)| \le \frac{1}{\lambda}|x_n - x_{n+1}| \le (L+1)2^{-k}.$$

□

3 Almost monotone recursive sequences

We will start with an investigation into some of the properties of almost monotone sequences:

Lemma 3.1 *Let (x_n) be a sequence of non-negative real numbers such that*

$$x_{n+1} \le (1 + a_n)x_n + b_n \tag{6}$$

where $0 \le a_n, b_n$.

Then for any m and n

$$x_{n+m} \le (x_n + \sum_{j=0}^{m-1} b_{n+j}) \cdot e^{\sum_{j=0}^{m-1} a_{n+j}}, \tag{7}$$

and if additionally $\sum_{i=0}^{\infty} a_i \le A \in \mathbb{R}$ and $\sum_{i=0}^{\infty} b_i \le B \in \mathbb{R}$, then

$$\forall n (0 \le x_n \le (x_0 + B)e^A). \tag{8}$$

Proof. By induction we can show for any $n, m \in \mathbb{N}$

$$x_{n+m} \le x_n \cdot \prod_{j=0}^{m-1}(1 + a_{n+j}) + \sum_{i=0}^{m-1} b_{n+i} \cdot \prod_{j=i+1}^{m-1}(1 + a_{n+j})$$

and also (by the arithmetic-geometric mean inequality)

$$\prod_{j=0}^{m-1}(1 + a_{n+j}) \le (1 + \frac{\sum_{j=0}^{m-1} a_{n+j}}{m})^m \le e^{\sum_{j=0}^{m-1} a_{n+j}}.$$

Combining them yields (7), and (8) is an instance of this inequality with $n \equiv 0$. □

In this main result, we generalize Matiyasevich's result by extending the class of sequences to almost monotone ones and introducing computational error. Note that, even though we define the error sequence (c_n) separately, it will usually be reflected in the parameters (a_n) and (b_n) as well.

Theorem 3.2 *Let (x_n) be a sequence of non-negative real numbers such that*

$$x_{n+1} \leq (1 + a_n)x_n + b_n$$

and

$$|F(x_n, x_{n+1}, \ldots, x_{n+r-1})| \leq c_n,$$

where $(a_n), (b_n), (c_n)$ are sequences that satisfy

$$0 \leq a_n, \sum_{i=0}^{\infty} a_i \leq A \in \mathbb{R}, \forall k\, \forall m > \alpha(k)\, (a_m < 2^{-k}),$$

$$0 \leq b_n, \sum_{i=0}^{\infty} b_i \leq B \in \mathbb{R}, \forall k\, \forall m > \beta(k)\, (b_m < 2^{-k}).$$

$$0 \leq c_n, \forall k\, \forall m > \gamma(k)\, (c_m < 2^{-k})$$

for some A, α, B, β and γ.

Let $d = (x_0 + B)e^A$ and F be uniformly continuous on $[0, d]$ with modulus ω, i.e.

$$\forall k\, \forall x_0, x_1, \ldots, x_{r-1} \in [0, d]\, \forall y_0, y_1, \ldots y_{r-1} \in [0, d]$$

$$\left(\bigwedge_{i=0}^{r-1} |x_i - y_i| < 2^{-\omega(k)} \to |F(x_0, x_1, \ldots x_{r-1}) - F(y_0, y_1, \ldots y_{r-1})| < 2^{-k} \right) \tag{9}$$

and have a unique solution of $F(x, x, \ldots, x) = 0$ within $[0, d]$ with uniform modulus of uniqueness η, i.e.

$$\begin{gathered} \forall k\, \forall x, y \in [0, d] \\ \left((|F(x, x, \ldots x)| < 2^{-\eta(k)} \wedge |F(y, y, \ldots y)| < 2^{-\eta(k)}) \to |x - y| < 2^{-k} \right). \end{gathered} \tag{10}$$

Then

$$\forall k\, \forall m \geq \phi(k)\ \left(|x_{\phi(k)} - x_m| < 2^{-k} \right), \tag{11}$$

where

$$\begin{gathered} \phi(k) = p(r - 1) + \max(\alpha(q), \beta(q), \gamma(\eta(k + 1) + 1)) \\ q = \theta(k) + 3 + \lceil \log_2 (d + 1)r \rceil \\ p = \lfloor d \cdot 2^{\theta(k)} \rfloor + 1 \\ \theta(k) = \max(k, \omega(\eta(k + 1) + 1)) \end{gathered}$$

Proof. Lemma 3.1 ensures that all members of the sequence (x_n) lie within $[0, d]$, thus we can safely use the moduli ω and η and freely substitute d as an upper bound for any x_n.

Using α and β we can make sure that, from a certain point onwards, any growth of the sequence (x_n) in a group of r consecutive elements is sufficiently restricted. For any $i \geq \max(\alpha(q), \beta(q))$ (using Lemma 3.1, $r < 2^q$, and $e^x \leq 1 + 2x$ for $x \leq 1$) we have:

$$\begin{aligned} x_{i+j} - x_i &\leq (x_i + j2^{-q})e^{j2^{-q}} - x_i \leq (x_i + j2^{-q})(1 + j2^{-q+1}) - x_i \\ &\leq j2^{-q}(2x_i + 1 + j2^{-q+1}) < (d+1)r2^{-q+2} \leq 2^{-\theta(k)-1} \end{aligned} \tag{12}$$

for any $j < r$.

We will now show that a significant distance between elements of (x_n) sufficiently far in the sequence has to be repeated in the distance between another pair of elements. Let n and m be natural numbers, $m, n \geq \max(\alpha(q), \beta(q), \gamma(\eta(k+1)+1))$, and suppose $x_{m+(r-1)} \geq x_n + 2^{-k}$. From its definition we know that $\theta(k) \geq k$ and thus (12) yields

$$\begin{aligned} x_m - x_n &= x_{m+(r-1)} - x_n - (x_{m+(r-1)} - x_m) \\ &\geq 2^{-k} - 2^{-\theta(k)-1} \geq 2^{-k-1}. \end{aligned}$$

By the uniqueness (10) of the root of $F(x, x, \ldots, x) = 0$ we can infer $|F(x_i, x_i, \ldots, x_i)| \geq 2^{-\eta(k+1)}$ where i is either n or m. By the continuity (9) of F applied to $F(x_i, x_i, \ldots, x_i)$ and $F(x_i, x_{i+1}, \ldots, x_{i+r-1})$, where

$$\begin{aligned} |F(x_i, x_i, \ldots, x_i) - F(x_i, x_{i+1}, \ldots, x_{i+r-1})| &\geq 2^{-\eta(k+1)} - c_i \\ &\geq 2^{-\eta(k+1)} - 2^{-(\eta(k+1)+1)} \\ &\geq 2^{-(\eta(k+1)+1)} \end{aligned}$$

we know there must exist $j \in \{1, 2, \ldots, r-1\}$ such that $|x_i - x_{i+j}| \geq 2^{-\omega(\eta(k+1)+1)} \geq 2^{-\theta(k)}$. Because of (12) the sequence cannot be growing that much between x_i and x_{i+j}, therefore

$$x_i - x_{i+j} \geq 2^{-\theta(k)}. \tag{13}$$

Now the distance between the pair $x_m, x_{n+(r-1)}$ has to be at least 2^{-k} (for simplicity we will only write the case $i = n$, the case $i = m$ yields an identical result):

$$\begin{aligned} x_m - x_{n+(r-1)} &\geq (x_m - x_{m+(r-1)}) + (x_{m+(r-1)} - x_n) + (x_n - x_{n+r-1}) \\ &> -2^{-\theta(k)-1} + 2^{-k} + (x_i - x_{i+j} + x_{i+j} - x_{i+r-1}) \\ &> -2^{-\theta(k)-1} + 2^{-k} + 2^{-\theta(k)} - 2^{-\theta(k)-1} \\ &= 2^{-k}. \end{aligned} \tag{14}$$

The same distance is maintained. Provided $m - (r-1)$ continues to be greater than or equal to $\max(\alpha(q), \beta(q), \gamma(\eta(k+1)+1))$, this argument can be applied again.

To continue with the main part of the proof, fix an arbitrary k and let $n_0 = \phi(k)$ and $m_0 \in \mathbb{N}$. Suppose $|x_{m_0+n_0} - x_{n_0}| \geq 2^{-k}$. Consider the following cases:

Case 1. $x_{m_0+n_0} \geq x_{n_0} + 2^{-k}$**.**

Let $n_{i+1} = n_i + (r-1)$ and $m_{i+1} = m_i - 2(r-1)$. By induction, using (14) with $n = n_i, m = n_i + m_i - (r-1)$ as the induction step, we know that at least for $i \leq \lfloor \frac{m_0}{(r-1)} \rfloor$

(since $n_i + m_i$ remains greater than or equal to $\max(\alpha(q), \beta(q), \gamma(\eta(k+1)+1))$)

$$x_{m_i+n_i} \geq x_{n_i} + 2^{-k}.$$

In particular, for $s = \lfloor \frac{m_0}{2(r-1)} \rfloor$, we have $0 \leq m_s < r$ and $x_{n_s+m_s} - x_{n_s} \geq 2^{-k}$, which is a contradiction with (12).

Case 2. $x_{m_0+n_0} \leq x_{n_0} - 2^{-k}$**.**

Let $n_{i+1} := n_i - (r-1)$ and $m_{i+1} := m_i + 2(r-1)$. Since $n_0 = \max(\alpha(q), \beta(q), \gamma(\eta(k+1)+1)) + p(r-1)$, we can apply (14) p times, taking $n = m_i + n_i$ and $m = n_i - (r-1)$. Therefore for any $i \leq p$ we have

$$x_{n_i} \geq x_{m_i+n_i} + 2^{-k},$$

and moreover (using (13)), for each iteration there is a distinct index $l_i \in \{n_i - (r-1), n_i + m_i\}$ where we have a significant drop in the values of the sequence, i.e. where

$$x_{l_i} - x_{l_i+j} \geq 2^{-\theta(k)}$$

for some $j < r$. (note that the drops cannot coincide because the points are at least $(r-1)$-apart)

We will prove that these drops accumulate and our choice of p makes this impossible. We will define two additional sequences to measure how big (x_n) could grow, and what difference there is between that and the real (x_n).

Let $y_0 = x_0, y_{n+1} = (1+a_n)y_n + b_n, z_n = y_n - x_n$. We can easily see that $x_n \leq y_n \leq y_{n+1}$ and $z_{n+1} \geq (1+a_n)y_n + b_n - (1+a_n)x_n - b_n \geq z_n$ for all n. Lemma 3.1 can also be used for (y_n) as an instance of a sequence that satisfies (6) with the same constants, thus (y_n) (and thereby (z_n)) also lies within $[0, d]$.

For each $i < p$, there exists $j < r$, such that

$$z_{l_i+j} = y_{l_i+j} - x_{l_i+j} \geq y_{l_i} - x_{l_i+j} = z_{l_i} + x_{l_i} - x_{l_i+j} \geq z_{l_i} + 2^{-\theta(k)},$$

and because of the monotonicity of (z_n) and $l_i \in \{n_i - (r-1), n_i + m_i\}$ also

$$\begin{aligned} z_{m_{i+1}+n_{i+1}} - z_{n_{i+1}} &= z_{m_i+n_i+(r-1)} - z_{n_i-(r-1)} \\ &= z_{m_i+n_i} - z_{n_i} + \left(z_{m_i+n_i+(r-1)} - z_{m_i+n_i}\right) + \left(z_{n_i} - z_{n_i-(r-1)}\right) \\ &\geq z_{m_i+n_i} - z_{n_i} + 2^{-\theta(k)}. \end{aligned}$$

Iterating this argument we arrive at

$$z_{m_p+n_p} \geq z_{m_p+n_p} - z_{n_p} \geq z_{m_0+n_0} - z_{n_0} + p2^{-\theta(k)} \geq p2^{-\theta(k)} > d,$$

which is a contradiction.

In either case the assumption $|x_{n_0} - x_{m_0+n_0}| \geq 2^{-k}$ causes a contradiction with our choice of n_0, therefore (since k and m_0 were arbitrary)

$$\forall k\, \forall m \geq \phi(k) \left(|x_{\phi(k)} - x_m| < 2^{-k}\right).$$

□

4 Conclusions and future work

In this paper we have approached the problem of recovering effective information from ineffective mathematical proofs by using an approach by Matiyasevich. We have given an application of it and a generalization motivated by recent developments in fixed point theory.

As future work within the topic, we are interested in non-trivial applications of the generalized result. On the other hand, by a result of Kohlenbach in [3], the main prerequisite of the treatments presented here, a modulus of uniqueness, can be extracted under very general conditions even from highly ineffective proofs of the uniqueness of the root. We are interested in finding applications of either the original theorem of Matiyasevich or the generalized result presented here, where finding the modulus is non-trivial, but can be achieved using the theorem from [3].

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