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# Domain Equations Based on Sets with Families of Pre-orders [1]

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**Abstract**

In this paper we are interested in finding solutions of domain equations based on posets with families of pre-orders. Let $(P, \sqsubseteq)$ be a poset and let $(\omega, \leq)$ be the natural number set. If $\mathcal{R} = (\sqsubseteq_n)_{n\in\omega}$ is a family of pre-order relations on $P$, where $\sqsubseteq_0 = P \times P$, such that (i)$\forall n, m \in \omega, m \leq n$ implies $\sqsubseteq_n \subseteq \sqsubseteq_m$, and (ii)$\bigcap_{n\in\omega} \sqsubseteq_n = \sqsubseteq$, then we call $(P, \sqsubseteq)$ a poset with pre-order family $\mathcal{R}$. We write it $\mathcal{R}$-poset or rpos for short and denote it briefly by $(P, \sqsubseteq; \mathcal{R})$ [13, L.Fan]. $\mathcal{R}$-posets are a particular case of quasi-metric spaces (qms) [6] and generalized ultrametric spaces (gums) [3]. $\mathcal{R}$-poset is a 'nonsymmetric' version of sfe [1, L.Monteiro]. We propose a fixed points theorem that can be used for solving domain equations. The paper ends in a final coalgebra theorem.

*Keywords:* $\mathcal{R}$-posets, fixed points, domain equations, final coalgebra

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# 1 Introduction and Preliminary

Solving domain equations is a fundamentally important theme in domain theory. As a fundamental mathematical problem in semantics, finding solutions of some operators's fixed points is the essence of finding solutions of a domain equation. The traditional mathematical frameworks for study of such solutions [7,8] are those of complete partial orders with continuous (monotone) functions (e.g.Scott fixed points theorem and Tarski fixed points theorem), and of complete metric spaces with nonexpanding and contracting functions (e.g.Banach fixed points theorem). In this paper, we will restrict our attention to the posets with a decreasing sequence of pre-order relations whose intersections in the cases of intersect is the given order relations on them. We propose a fixed point theorem based on $\mathcal{R}$-complete rpos's(crpos, Definition 1.2) with $\mathcal{R}$-continuous and $\mathcal{R}$-approximating functions (Definition 1.5). Roughly speaking, $\mathcal{R}$-continuous and $\mathcal{R}$-approximating functions on crpos's correspond to both continuous (monotone) functions on complete partial orders and (locally) $\mathcal{R}$-approximating and $\mathcal{R}$-continuous endofunctors on crpos-categories. In [15], we obtain a kind of $\mathcal{R}$-complete categories of $\mathcal{R}$-posets. Hence, a category-version of fixed point theorem based on crpos may be used to find the solutions of domain equations of an $\mathcal{R}$-approximating and $\mathcal{R}$-continuous endofunctor on crpos-categories.

The structure we are interested in is that of a set where a pre-order relation $\sqsubseteq_n$ is defined for every $n \in \omega$. We interpret $x \sqsubseteq_n y$ as indicating the extent $\frac{1}{n}$ (the smaller the better) to which the transitions of $x$ can be simulated by $y$. Thus, it is not surprising that we assume that any two elements are in relation $\sqsubseteq_0$ and that $\sqsubseteq_n \subseteq \sqsubseteq_m$ for $m \leq n$ in $\omega$. A typical application of this notion is to objects that can be structured or evaluated in stepwise manner, where it makes sense to state that an object can be simulated by another object up to level $n$. Another situation is when we have a battery of 'ordered observations' [11] and an object can be simulated by another object in first $n$ steps. Examples of both situations are given in the sequel.

The motivation for studying sfe's is that they embody the type of metric reasoning used in semantics, and that they are a kind of simpler mathematical setting. $\mathcal{R}$-poset is an order-version of sfe. The generalization from sfe to $\mathcal{R}$-poset is motivated by the desire to have a better world of reconciling metric spaces with domains. $\mathcal{R}$-posets are a particular case of quasi-metric spaces (qms) [6] and generalized ultrametric spaces (gums) [3]. We define a distance $d(x, y) = 2^{-n}$ when $x$ can be simulated by $y$ up to the greatest (level) $n$ (if $x$ can be simulated by $y$ up to every level $n$ then $d(x, y) = 0$). The paper generalizes part of the theory developed for both sfe's and metric spaces, ending with a final coalgebra theorem. The theory is based on a simple idea, that is, replacing the arbitrary $\epsilon \geq 0$ with a particular sequence such as $(1/n)_{n\in\omega}$. The advantage of restriction to the $\mathcal{R}$-poset is that it simplifies definitions and allows a larger set of constructions. But when we trade the pre-order relation for the more general notion of metric we loose some advantages that come with the particular structure we started with.

In the sequel we partly develop a domain theory of these sets with families of pre-order relations, by mimicking part of theory developed for the posets and the metric spaces. The main references are [1,2,3,5,6,8]. Many results in [1] are special cases of our results. Most main results in this paper are adaptations of results in those sources, but it is worth pointing out that the methods of their proofs are an "organic combination" of proofs of results in those sources. The notion of set with a family of pre-orders was borrowed from both [1] and [3], and the development of the theory in the present paper is similar to [1] and [3].

Let $(D, \sqsubseteq)$ be be a poset, let $x, y \in D$ be two elements, then $x \sqsubseteq y$ can be interpreted as that not only are they different but that $y$ is a better approximation of $y$. In other words, $x \sqsubseteq y$ means intuitively that $y$ is consistent with $x$ and is (possibly) more accurate than $x$. For short, $x$ "approximates" $y$. Many efforts (e.g.[6, qms by M.B. Smyth], and [3, gums by J.J.M.M.Rutten]) have been made to reconcile the theory of metric spaces with domain theory. The question remains as to how quantities can be introduced to domain theory in a simple and elegant way. This paper was intended as an attempt to do this. The $\mathcal{R}$-poset structure, with denumerable pre-orders on a set, was proposed by the second author of this paper in [13].

**Example 1.1** (i) Let $S$ be a set and let $(\mathcal{U}_n)_{n\in\omega}$ be a family of sets $\mathcal{U}_n$ of subsets of $S$ where $\mathcal{U}_0 = \{S\}$. Define $s \sqsubseteq_n t$ by requiring that $s \in U$ imply $t \in U$ for every $U \in \mathcal{U}_0 \bigcup \cdots \bigcup \mathcal{U}_n$. This gives an rpos.

(ii) Consider a transition system $\langle S, A, \rightarrow\rangle$. For $a \in A$ and $U \subseteq S$, let $p_a U = \{s | (\exists t \in U) s \xrightarrow{a} t\}$ be the set of $a$-predecessors of $U$. Extend this to traces by $p_\varepsilon U = U$ and $p_{av} = p_a p_v U$. Then $s \in p_v S$ if and only if $s$ has trace $v$. Let $\mathcal{U}_n = \{p_v S | v \text{ has length } n\}$, so that $\mathcal{U}_0 \bigcup \cdots \bigcup \mathcal{U}_n = \{p_v S | v$ has length $\leq n\}$. We define $s \sqsubseteq_n t$ if and only if, for every trace $v$ of length $\leq n, s \in p_v S$ implies $t \in p_v S$, that is to say, if $s$ has traces of length $\leq n$ then $t$ has the same traces of length $\leq n$. Explicitly, $s \sqsubseteq_n t$ means that at least any first $n$ consecutive transition steps that can be taken starting in state $s$ can all be simulated by steps from $t$.

(iii) Now we want to evaluate e (Euler number). Not very exactly, we know that e is greater than 2 and less than 3. If there are no further accuracy demands, any value between 2 and 3 can be viewed as e. If we are required to improve the accuracy of evaluation of e then one may obtain a more accurate value of e between 2.7 and 2.8. This process can be formalized as follows. The real interval $[2, 3]$ is made into an rpos by stipulating that $x \sqsubseteq_n y$ if and only if the decimal expansions of $x$ and $y$ agree on the first $n$ digits after the decimal point (all expansions are assumed to be infinite by adding 0 to the right if necessary, and 3 is represented by 2.999...). Then we have a sequence $2 \sqsubseteq_0 2.7 \sqsubseteq_1 2.71 \sqsubseteq_2 \cdots \sqsubseteq_{n-1} x_n \sqsubseteq_n \cdots$, where there exists some $m$ such that first $n$ digits of

$$1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{m!},$$

after radix point give $x_n$ for every $n$.

The above examples show that how computations evolve in transition systems. Formally we have the following definitions.

**Definition 1.2** [**$\mathcal{R}$-chain, $\mathcal{R}$-complete**] Let $(A, \sqsubseteq; (\sqsubseteq_n)_{n\in\omega})$ be an rpos and let $(x_n)_{n\in\omega}$ be a sequence in $A$. $(x_n)_{n\in\omega}$ is called an $\mathcal{R}$-chain if $x_n \sqsubseteq_n x_{n+1}$ for every $n \in \omega$. If $x \in A$ satisfies (i)$\forall n \in \omega, x_n \sqsubseteq_n x$, (ii) we have $x \sqsubseteq y$ for every $y \in A$ satisfying (i), then $x$ is called the least $\mathcal{R}$-upper bound or $\mathcal{R}$-limit of $(x_n)_{n\in\omega}$. We denote it by $x = \bigsqcup^{\mathcal{R}}_{n\in\omega} x_n$. In this case the sequence is said to be convergent (to $x$). $A$ is an $\mathcal{R}$-complete $\mathcal{R}$-poset (crpos for short) if every $\mathcal{R}$-chain in $A$ is convergent.

**Remark 1.3** The following assumption will be needed throughout the paper. It is required that $\mathcal{R}$-limits preserve every order in $\mathcal{R}$, i.e., $\bigsqcup^{\mathcal{R}}_{n\in\omega} x_n \sqsubseteq_m \bigsqcup^{\mathcal{R}}_{n\in\omega} y_n$ holds if $x_n \sqsubseteq_m y_n$ for some $m \in \omega$ and all $n \geq n_0$ where $n_0 \in \omega$. In particular, if $x \sqsubseteq_n y$ then obviously, we have $x \sqsubseteq_m x \sqsubseteq_{m+1} x \sqsubseteq_{m+2} \cdots$ and $y \sqsubseteq_m y \sqsubseteq_{m+1} y \sqsubseteq_{m+2} \cdots$. It is not surprising that the assumption only means that $\bigsqcup^{\mathcal{R}}_{n\in\omega} x = x \sqsubseteq_n y = \bigsqcup^{\mathcal{R}}_{n\in\omega} y$ holds. Let us mention an important consequence [14] of the assumption.

**Lemma 1.4** *If $(x_{mn})_{m,n\in\omega}$ is an $\mathcal{R}$-chain in a* crpos *for both $m$ and $n$, then $\bigsqcup^{\mathcal{R}}_{m\in\omega}(\bigsqcup^{\mathcal{R}}_{n\in\omega} x_{mn}) = \bigsqcup^{\mathcal{R}}_{n\in\omega}(\bigsqcup^{\mathcal{R}}_{m\in\omega} x_{mn}) = \bigsqcup^{\mathcal{R}}_{n\in\omega} x_{nn}$.*

**Proof.** Because $(x_{mn})_{m,n\in\omega}$ is an $\mathcal{R}$-chain for both $m$ and $n$, so $x_{mn} \sqsubseteq_m x_{m+1\ n}$ for $m, n \in \omega$. It implies that $\bigsqcup^{\mathcal{R}}_{n\in\omega} x_{mn} \sqsubseteq_m \bigsqcup^{\mathcal{R}}_{n\in\omega} x_{m+1\ n}$ by assumption in the remark above. Consequently, $(\bigsqcup^{\mathcal{R}}_{n\in\omega} x_{mn})_{m\in\omega}$ is an $\mathcal{R}$-chain and $\bigsqcup^{\mathcal{R}}_{m\in\omega}(\bigsqcup^{\mathcal{R}}_{n\in\omega} x_{mn})$ is well-defined. Let $\bigsqcup^{\mathcal{R}}_{m\in\omega}(\bigsqcup^{\mathcal{R}}_{n\in\omega} x_{mn}) = x$ and $\bigsqcup^{\mathcal{R}}_{n\in\omega} x_{nn} = y$. By the definition of $\mathcal{R}$-chains and $\mathcal{R}$-limits, $x_{mm} \sqsubseteq_m \bigsqcup^{\mathcal{R}}_{n\in\omega} x_{mn} \sqsubseteq_m x$ for all $m \in \omega$, hence $y \sqsubseteq x$. On the other hand, $x_{mn} \sqsubseteq_m y$, because $\forall n \geq m, x_{mn} \sqsubseteq_m x_{nn} \sqsubseteq_n y$. Therefore $\bigsqcup^{\mathcal{R}}_{n\in\omega} x_{mn} \sqsubseteq_m y$. Notice that $m$ is arbitrary, $x = \bigsqcup^{\mathcal{R}}_{m\in\omega}(\bigsqcup^{\mathcal{R}}_{n\in\omega} x_{mn}) \sqsubseteq y$. It follows that $x = y$. $\square$

Functions between sets with pre-order families are the fundamental mechanism relating one $\mathcal{R}$-poset to another [13].

**Definition 1.5** [**$\mathcal{R}$-monotone, $\mathcal{R}$-approximating, $\mathcal{R}$-continuous**] Let $f : S \to T$ be a mapping between rpos's $S$ and $T$. $f$ is $\mathcal{R}$-monotone if $f$ is monotone with respect to (w.r.t for short) every $\sqsubseteq_n$ in $\mathcal{R}$. $f$ is $\mathcal{R}$-approximating if $x \sqsubseteq_n y$ implies $f(x) \sqsubseteq_{n+1} f(y)$ for all $x, y \in S$ and every $n \in \omega$. $f$ is $\mathcal{R}$-continuous if $f$ is $\mathcal{R}$-monotone and $f(\bigsqcup^{\mathcal{R}}_{n\in\omega} x_n) = \bigsqcup^{\mathcal{R}}_{n\in\omega} f(x_n)$ for every $\mathcal{R}$-chain $(x_n)_{n\in\omega}$ in $S$.

**Lemma 1.6** *If $f : S \to T$ and $g : T \to U$ are $\mathcal{R}$-monotone functions such that one of them is $\mathcal{R}$-approximating, then the composition $g \circ f$ is $\mathcal{R}$-approximating.*

**Proof.** Suppose $f$ is $\mathcal{R}$-approximating. If $s \sqsubseteq_n t$ in $S$ then $f(s) \sqsubseteq_{n+1} f(t)$ in $T$, hence $g(f(s)) \sqsubseteq_{n+1} g(f(t))$ in $U$. If $g$ is $\mathcal{R}$-approximating the proof is similar. □

**Proposition 1.7** *A function $f : S \to T$ between crpos's is $\mathcal{R}$-continuous if and only if it maps convergent sequences in $S$ to convergent sequences in $T$, preserving the $\mathcal{R}$-limits.*

**Proof.** If $f$ is $\mathcal{R}$-continuous and the $\mathcal{R}$-chain $(s_n)_{n\in\omega}$ in $S$ converges to $s$, it is immediate that $(f(s_n))_n$ converges to $f(s)$ since

$$\bigsqcup^{\mathcal{R}}_{n\in\omega} f(s_n) = f(\bigsqcup^{\mathcal{R}}_{n\in\omega} s_n) = f(s).$$

Conversely, it remains to prove that $f$ is $\mathcal{R}$-monotone. Suppose $s \sqsubseteq_n t$ and consider the $\mathcal{R}$-chain $(s_k)_{m\in\omega}$ in $S$ where $s_k = s$ for $k \le n$ and $s_k = t$ for $k > n$. This sequence converges to $t$, hence $(f(s_k))_k$ converges to $f(t)$. In particular, $f(s) = f(s_n) \sqsubseteq_n f(t)$. Therefore $f$ is $\mathcal{R}$-monotone. □

We see at once that the identity functions are $\mathcal{R}$-continuous ($\mathcal{R}$-monotone) and the composition of $\mathcal{R}$-continuous ($\mathcal{R}$-monotone) functions is an $\mathcal{R}$-continuous ($\mathcal{R}$-monotone) function. The category **CRPOS** has $\mathcal{R}$-complete $\mathcal{R}$-posets as objects and $\mathcal{R}$-continuous functions as morphisms. The category **CRPOS**$_{\mathcal{R}}$ has $\mathcal{R}$-complete $\mathcal{R}$-posets that $\mathcal{R}$-limits preserve pre-orders in $\mathcal{R}$ as objects and $\mathcal{R}$-continuous functions as morphisms.

## 2 Some Basic Constructions on $\mathcal{R}$-posets

**Definition 2.1** [**Loosening**] The loosening of $(S, \sqsubseteq; (\sqsubseteq_n)_{n\in\omega})$, denoted by $(S, \sqsubseteq^o; \mathcal{R}^o)$ ($S^\circ$ for short), is the rpos with the same underlying set and orders $\sqsubseteq^o_n$ defined by $\sqsubseteq^o_0 = \sqsubseteq_0$ and $\sqsubseteq^o_{n+1} = \sqsubseteq_n$ for all $n \ge 0$. Evidently, $\sqsubseteq^o = \sqsubseteq$.

The sequence of orders of $S$ and $S^o$ are illustrated in the following table:

$$\begin{array}{ll} S: & \sqsubseteq_0 \quad \sqsubseteq_1 \quad \sqsubseteq_2 \quad \sqsubseteq_3 \cdots \\ S^o: & \sqsubseteq_0 \quad \sqsubseteq_0 \quad \sqsubseteq_1 \quad \sqsubseteq_2 \cdots . \end{array}$$

**Proposition 2.2** *Let $S$ be an* rpos*. Then $S^o$ is an* rpos*, $\mathcal{R}$-complete if $S$ is $\mathcal{R}$-complete. The identity function on $S$ is $\mathcal{R}$-approximating as a function $S \to S^o$. An $\mathcal{R}$-monotone ($\mathcal{R}$-approximating, $\mathcal{R}$-continuous) function $f : S \to T$ is also $\mathcal{R}$-monotone ($\mathcal{R}$-approximating, $\mathcal{R}$-continuous) as a function from $S^o \to T^o$.*

**Proof.** Suppose $S$ is $\mathcal{R}$-complete and $(s_n)_{n\in\omega}$ is an $\mathcal{R}$-chain in $S^o$, then $s_0 \sqsubseteq_0 s_1 \sqsubseteq_0 s_2 \sqsubseteq_1 s_3 \sqsubseteq_2 \cdots$. As $S$ is $\mathcal{R}$-complete we have $\bigsqcup^{\mathcal{R}}_{n\in\omega} s_n = s$ for some $s \in S$. By the definition of loosening, $s_n \sqsubseteq^o_n s$ if $s_n \sqsubseteq_{n-1} s$. If $t \in S$ such that $s_n \sqsubseteq^o_n t$ for all $n > 0$, then $s_n \sqsubseteq_{n-1} t$. By the definition of $\mathcal{R}$-limits, $s \sqsubseteq t$. Again by the definition of $\mathcal{R}$-limits, $s = \bigsqcup^{\mathcal{R}^o}_{n\in\omega} s_n$. If $f$ is $\mathcal{R}$-approximating and $s \sqsubseteq^o_0 t$ then $f(s) \sqsubseteq^o_1 f(t)$ since $\sqsubseteq^o_1 = \sqsubseteq_0$ and $\sqsubseteq^o_2 \subseteq \sqsubseteq^o_1$. If $s \sqsubseteq^o_{n+1} t$ then $s \sqsubseteq_n t$. Hence $f(s) \sqsubseteq_{n+1} f(t)$. Thus $f(s) \sqsubseteq^o_{n+2} f(t)$. If $f$ is $\mathcal{R}$-continuous then

$$f(\textstyle\bigsqcup^{\mathcal{R}^o}_{n\in\omega} s_n) = f(\bigsqcup^{\mathcal{R}}_{n\in\omega} s_n) = \bigsqcup^{\mathcal{R}}_{n\in\omega} f(s_n) = \bigsqcup^{\mathcal{R}^o}_{n\in\omega} f(s_n).$$

The more details are left to the reader. □

**Definition 2.3** [**Function spaces**] Let $S$ and $T$ be rpos's, let $[S \to T]$ be the set of all $\mathcal{R}$-continuous functions from $S$ to $T$, where $f \sqsubseteq_n g$ if and only if $f(s) \sqsubseteq_n g(s)$ for all $s \in S$. The subset of $[S \to T]$ of all $\mathcal{R}$-approximating and $\mathcal{R}$-continuous functions, with induced orders, is written as $[S \Rightarrow T]$.

**Definition 2.4** [**Powerdomains**] Let $S$ be an rpos, let $X$ and $Y$ be subsets of $S$. Put $X \sqsubseteq_0 Y$ and, for $n > 0$, $X \sqsubseteq_n Y$ if every $x \in X$ has some $y \in Y$ such that $x \sqsubseteq_n y$. We denote by $\mathcal{P}(S)$ the set of all subsets of $S$, by $\mathcal{P}_{ne}$ the set of all nonempty subsets of $S$. Note that if $X \subseteq Y$, then one sees immediately that $X \sqsubseteq_n Y$ for every $n$.

**Proposition 2.5** *If $S$ is an* rpos *then $\mathcal{P}(S)$ and $\mathcal{P}_{ne}(S)$ are* rpos*'s. If $f : S \to T$ is $\mathcal{R}$-monotone, then the function $\mathcal{P}(f) : \mathcal{P}(S) \to \mathcal{P}(T)$ defined by $\mathcal{P}(f)(X) = \{f(s)|s \in X\}$ and its restriction $\mathcal{P}_{ne}(f) : \mathcal{P}_{ne}(S) \to \mathcal{P}_{ne}(T)$ are $\mathcal{R}$-monotone. If furthermore $f$ is $\mathcal{R}$-approximating then $\mathcal{P}_{ne}(f)$ is also $\mathcal{R}$-approximating.*

**Proof.** It is easily seen that $\mathcal{P}(S)$ and $\mathcal{P}_{ne}(S)$ are rpos's. To prove $\mathcal{P}(f)$ is $\mathcal{R}$-monotone, suppose that $X \sqsubseteq_n Y$ in $\mathcal{P}(S)$. If $Y$ is empty then $X$ is empty. In particular they are equal, so the conclusion is immediate. If neither is empty, to prove that $\mathcal{P}(f)(X) \sqsubseteq_n \mathcal{P}(f)(Y)$ pick some element in $\mathcal{P}(f)(X)$, which has necessarily the form $f(x)$ for some $y \in Y$ such that $x \sqsubseteq_n y$. But

$f$ is $\mathcal{R}$-monotone, so $f(x) \sqsubseteq_n f(y)$. The proof of $\mathcal{P}(f)(X) \sqsubseteq_{n+1} \mathcal{P}(f)(Y)$ is similar. $\square$

Next we give some results [14] that will be applied in the proofs of the following propositions and theorems. The following basic fact provide a natural and intrinsic way to define the $\mathcal{R}$-limit of an $\mathcal{R}$-chain of $\mathcal{R}$-monotone ($\mathcal{R}$-continuous) functions.

**Lemma 2.6** *Let $S$ and $T$ be* crpos*'s. If $(f_n)_{n\in\omega}$ is an $\mathcal{R}$-chain of $\mathcal{R}$-monotone mappings from $S$ to $T$ then for some $\mathcal{R}$-monotone mapping $f : S \to T$, $\bigsqcup^{\mathcal{R}}_{n\in\omega} f_n = f$ if and only if $\bigsqcup^{\mathcal{R}}_{n\in\omega} f_n(s) = f(s)$ for every $s \in S$. Moreover, if $(f_n)_{n\in\omega}$ are $\mathcal{R}$-continuous then $f$ is also $\mathcal{R}$-continuous.*

**Proof.** If $(f_n)_{n\in\omega}$ is an $\mathcal{R}$-chain and $\bigsqcup^{\mathcal{R}}_{n\in\omega} f_n = f$ then $(f_n(s))_{n\in\omega}$ is an $\mathcal{R}$-chain and $f_n(s) \sqsubseteq_n f(s)$ for every $s \in S$ and $n \in \omega$. Let $\bigsqcup^{\mathcal{R}}_{n\in\omega} f_n(s) = g(s)$. If $x \sqsubseteq_m y$ then $g(x) = \bigsqcup^{\mathcal{R}}_{n\in\omega} f_n(x) \sqsubseteq_m \bigsqcup^{\mathcal{R}}_{n\in\omega} f_n(y) = g(y)$. It follows that $g$ is $\mathcal{R}$-monotone. Obviously, $g \sqsubseteq f$. But $f \sqsubseteq g$ since $f_n \sqsubseteq_n g$ for every $n \in \omega$. Hence $f = g$. Conversely, $\bigsqcup^{\mathcal{R}}_{n\in\omega} f_n(s) = f(s)$ implies that $f_n(s) \sqsubseteq_n f(s)$ holds for all $s \in S$ and $n \in \omega$. For every $s \in S$ and $n \in \omega$, if $g \in [S \to T]$ and $f_n(s) \sqsubseteq_n g(s)$ then $f(s) \sqsubseteq g(s)$. Thus $f \sqsubseteq g$. Consequently, $\bigsqcup^{\mathcal{R}}_{n\in\omega} f_n = f$.

If $(f_n)_{n\in\omega}$ are $\mathcal{R}$-continuous then we only need to show that

$$f(\textstyle\bigsqcup^{\mathcal{R}}_{m\in\omega} x_m) = \bigsqcup^{\mathcal{R}}_{m\in\omega} f(x_m)$$

whenever $(x_m)_{m\in\omega}$ is an $\mathcal{R}$-chain in $S$. Indeed

$$f(\textstyle\bigsqcup^{\mathcal{R}}_{m\in\omega} x_m) = \bigsqcup^{\mathcal{R}}_{n\in\omega} f_n(\bigsqcup^{\mathcal{R}}_{m\in\omega} x_m) = \bigsqcup^{\mathcal{R}}_{n\in\omega} \bigsqcup^{\mathcal{R}}_{m\in\omega} f_n(x_m)$$

and

$$\textstyle\bigsqcup^{\mathcal{R}}_{m\in\omega} f(x_m) = \bigsqcup^{\mathcal{R}}_{m\in\omega} \bigsqcup^{\mathcal{R}}_{n\in\omega} f_n(x_m).$$

We are reduced to proving

$$\textstyle\bigsqcup^{\mathcal{R}}_{n\in\omega} \bigsqcup^{\mathcal{R}}_{m\in\omega} f_n(x_m) = \bigsqcup^{\mathcal{R}}_{m\in\omega} \bigsqcup^{\mathcal{R}}_{n\in\omega} f_n(x_m).$$

For $k \in \omega$, $f_n(x_k) \sqsubseteq_k \bigsqcup^{\mathcal{R}}_{m\in\omega} f_n(x_m)$ for all $n \in \omega$. Therefore

$$\textstyle\bigsqcup^{\mathcal{R}}_{n\in\omega} f_n(x_k) \sqsubseteq_k \bigsqcup^{\mathcal{R}}_{n\in\omega} \bigsqcup^{\mathcal{R}}_{m\in\omega} f_n(x_m).$$

Notice that $k$ is arbitrary,

$$\textstyle\bigsqcup^{\mathcal{R}}_{k\in\omega} \bigsqcup^{\mathcal{R}}_{n\in\omega} f_n(x_k) \sqsubseteq \bigsqcup^{\mathcal{R}}_{n\in\omega} \bigsqcup^{\mathcal{R}}_{m\in\omega} f_n(x_m).$$

Similarly,

$$\textstyle\bigsqcup^{\mathcal{R}}_{n\in\omega} \bigsqcup^{\mathcal{R}}_{m\in\omega} f_n(x_m) \sqsubseteq \bigsqcup^{\mathcal{R}}_{k\in\omega} \bigsqcup^{\mathcal{R}}_{n\in\omega} f_n(x_k).$$

We also have a direct proof as follows by Lemma 1.4

$$\begin{aligned} & \bigsqcup\nolimits^{\mathcal{R}}_{n\in\omega} \bigsqcup\nolimits^{\mathcal{R}}_{m\in\omega} f_n(x_m) \\ = \;& \bigsqcup\nolimits^{\mathcal{R}}_{n\in\omega} f_n(x_n) \\ = \;& \bigsqcup\nolimits^{\mathcal{R}}_{m\in\omega} \bigsqcup\nolimits^{\mathcal{R}}_{n\in\omega} f_n(x_m) \end{aligned}$$

because $(f_n(x_m))_{m,n\in\omega}$ is an $\mathcal{R}$-chain for both $m$ and $n$ in $\mathsf{crpos}$ $T$. □

**Proposition 2.7** *If $S$ and $T$ are* crpos*'s then so is $[S \to T]$ where $[S \to T]$ is the set of $\mathcal{R}$-continuous functions from $S$ to $T$.*

**Proof.** Immediate from Lemma 1.4 and Lemma 2.6. □

**Theorem 2.8 ([14]) $\mathbf{CRPOS}_{\mathcal{R}}$** *is a cartesian closed full subcategory of* **CRPOS**.

**Lemma 2.9** *If $\bigsqcup^{\mathcal{R}}_{n\in\omega} f_n = f$ in $[S \to T]$ and $\bigsqcup^{\mathcal{R}}_{n\in\omega} g_n = g$ in $[T \to U]$ then $\bigsqcup^{\mathcal{R}}_{n\in\omega} g_n \circ f_n = g \circ f$ in $[S \to U]$.*

**Proof.** From Lemma 2.6, it suffices to show that $\bigsqcup^{\mathcal{R}}_{n\in\omega} g_n \circ f_n(s) = g \circ f(s)$ for every $s \in S$. Indeed

$$\begin{aligned} g \circ f(s) = \;& g(\bigsqcup\nolimits^{\mathcal{R}}_{n\in\omega} f_n(s)) \\ = \;& \bigsqcup\nolimits^{\mathcal{R}}_{m\in\omega} g_m(\bigsqcup\nolimits^{\mathcal{R}}_{n\in\omega} f_n(s)) \\ = \;& \bigsqcup\nolimits^{\mathcal{R}}_{m\in\omega} \bigsqcup\nolimits^{\mathcal{R}}_{n\in\omega} (g_m \circ f_n(s)) \\ = \;& \bigsqcup\nolimits^{\mathcal{R}}_{n\in\omega} g_n \circ f_n(s) \text{ (Lemma 1.4).} \end{aligned}$$

□

We return to the topic of finding solutions of domain equations based on $\mathcal{R}$-posets. Let's see some natural constructions of $\mathcal{R}$-continuous ($\mathcal{R}$-approximating) functions.

**Proposition 2.10** *Let $S, S', T$ and $T'$ be* crpos*'s.*

(i) *The function $f \mapsto f^o$ from $[S \to S']$ to $[S^o \to S'^o]$ is $\mathcal{R}$-approximating and $\mathcal{R}$-continuous.*

(ii) *The function $(f, g) \mapsto f \times g$ from $[S \to S'] \times [T \to T']$ to $[S \times T \to S' \times T']$ is $\mathcal{R}$-continuous.*

(iii) *The function $(f, g) \mapsto f + g$ from $[S \to S'] \times [T \to T']$ to $[S + T \to S' + T']$ is $\mathcal{R}$-continuous.*

(iv) *The function $(f, g) \mapsto [f \to g]$ from $[S' \to S] \times [T \to T']$ to $[[S \to T] \to$*

$[S' \to T']]$ *is* $\mathcal{R}$*-continuous, where* $[f \to g]h = g \circ h \circ f$.

$$\begin{tikzcd} S' \arrow[r, dotted, "g\circ h\circ f"] \arrow[d, "f"'] & T' \\ S \arrow[r, "h"'] & T \arrow[u, "g"'] \end{tikzcd}$$

(v) *The function* $f \mapsto \mathcal{P}(f)$ *from* $[S \to S']$ *to* $[\mathcal{P}(S) \to \mathcal{P}(S')]$ *is* $\mathcal{R}$*-monotone, as is the corresponding function for the functor* $\mathcal{P}_{ne}$.

**Proof.**

(i) It is evident that $f \mapsto f^o$ is $\mathcal{R}$-approximating. That $f \mapsto f^o$ is $\mathcal{R}$-continuous follows from $(\bigsqcup^{\mathcal{R}}_{n\in\omega} f_n)^o = \bigsqcup^{\mathcal{R}}_{n\in\omega} f^o_n$.

(ii) If $f$ and $g$ are $\mathcal{R}$-continuous then

$$\begin{aligned} & f \times g(\textstyle\bigsqcup^{\mathcal{R}}_{n\in\omega}(s_n, t_n)) \\ = \;& (f \times g)(\textstyle\bigsqcup^{\mathcal{R}}_{n\in\omega} s_n, \bigsqcup^{\mathcal{R}}_{n\in\omega} t_n) \\ = \;& \textstyle\bigsqcup^{\mathcal{R}}_{n\in\omega}(f(s_n), g(t_n)) \\ = \;& \textstyle\bigsqcup^{\mathcal{R}}_{n\in\omega}(f \times g)(s_n, t_n). \end{aligned}$$

Hence $f \times g$ is $\mathcal{R}$-continuous. Because $\bigsqcup^{\mathcal{R}}_{n\in\omega}(f_n, g_n) = (\bigsqcup^{\mathcal{R}}_{n\in\omega} f_n, \bigsqcup^{\mathcal{R}}_{n\in\omega} g_n)$ for every $\mathcal{R}$-chain $(f_n, g_n)_{n\in\omega}$ and

$$\begin{aligned} & (\textstyle\bigsqcup^{\mathcal{R}}_{n\in\omega} f_n \times \bigsqcup^{\mathcal{R}}_{n\in\omega} g_n)(s, t) \\ = \;& (\textstyle\bigsqcup^{\mathcal{R}}_{n\in\omega} f_n(s), \bigsqcup^{\mathcal{R}}_{n\in\omega} g_n(t)) \\ = \;& \textstyle\bigsqcup^{\mathcal{R}}_{n\in\omega}(f_n(s), g_n(t)) \\ = \;& \textstyle\bigsqcup^{\mathcal{R}}_{n\in\omega}(f_n \times g_n)(s, t), \end{aligned}$$

$(f, g) \mapsto f \times g$ is $\mathcal{R}$-continuous.

(iii) For an arbitrary $\mathcal{R}$-chain $(1, x_n)_{n\in\omega}$, we have

$$\begin{aligned} & (f + g)(\textstyle\bigsqcup^{\mathcal{R}}_{n\in\omega}(1, x_n)) \\ = \;& (f + g)(1, \textstyle\bigsqcup^{\mathcal{R}}_{n\in\omega} x_n) \\ = \;& (1, \textstyle\bigsqcup^{\mathcal{R}}_{n\in\omega} f(x_n)) \\ = \;& \textstyle\bigsqcup^{\mathcal{R}}_{n\in\omega}(f + g)(1, x_n) \end{aligned}$$

In the same manner we can see that $(f + g)(\bigsqcup^{\mathcal{R}}_{n\in\omega}(2, y_n)) = \bigsqcup^{\mathcal{R}}_{n\in\omega}(f + g)(2, y_n)$ for an arbitrary $\mathcal{R}$-chain $(2, y_n)$. It is easy to check that $(f, g) \mapsto f + g$ is $\mathcal{R}$-monotone. The verification of $\bigsqcup^{\mathcal{R}}_{n\in\omega} f_n + \bigsqcup^{\mathcal{R}}_{n\in\omega} g_n = \bigsqcup^{\mathcal{R}}_{n\in\omega}(f_n +$

$g_n$) is similar to (ii).

(iv) $g \circ h \circ f : S' \to T'$ is $\mathcal{R}$-continuous because the composite of $\mathcal{R}$-continuous functions is an $\mathcal{R}$-continuous function. From

$$\begin{aligned} & [f \to g](\textstyle\bigsqcup^{\mathcal{R}}_{n\in\omega} h_n) \\ = \; & g \circ (\textstyle\bigsqcup^{\mathcal{R}}_{n\in\omega} h_n) \circ f \\ = \; & \textstyle\bigsqcup^{\mathcal{R}}_{n\in\omega} g \circ h_n \circ f \\ = \; & \textstyle\bigsqcup^{\mathcal{R}}_{n\in\omega} [f \to g] h_n, \end{aligned}$$

we deduce that $[f \to g]$ is $\mathcal{R}$-continuous. If $f \sqsubseteq_n f'$ and $g \sqsubseteq_n g'$ then

$$[f \to g](h) = g \circ h \circ f \sqsubseteq_n g' \circ h \circ f' = [f' \to g'](h).$$

Notice that $h$ is arbitrary, $[f \to g] \sqsubseteq_n [f' \to g']$. From

$$\begin{aligned} & [\textstyle\bigsqcup^{\mathcal{R}}_{n\in\omega} f_n \to \bigsqcup^{\mathcal{R}}_{n\in\omega} g_n] h \\ = \; & (\textstyle\bigsqcup^{\mathcal{R}}_{n\in\omega} g_n) \circ h \circ (\bigsqcup^{\mathcal{R}}_{n\in\omega} f_n) \\ = \; & \textstyle\bigsqcup^{\mathcal{R}}_{n\in\omega} g_n \circ h \circ f_n \\ = \; & \textstyle\bigsqcup^{\mathcal{R}}_{n\in\omega} [f_n \to g_n](h), \end{aligned}$$

it follows that $(f, g) \mapsto [f \to g]$ is $\mathcal{R}$-continuous.

(v) We only consider the case of $\mathcal{P}_{ne}$. Suppose $f \sqsubseteq_n f'$ and $X \in \mathcal{P}_{ne}(S)$. An element in $\mathcal{P}_{ne}(f)(X)$ has the form $f(s)$ for some $s \in X$. The element $f'(s)$ is in $\mathcal{P}_{ne}(f')(X)$ and $f(s) \sqsubseteq_n f'(s)$. This shows that $\mathcal{P}_{ne}(f)(X) \sqsubseteq_n \mathcal{P}_{ne}(f')(X)$. As this is true for every $X$ we conclude that $\mathcal{P}_{ne}(f) \sqsubseteq_n \mathcal{P}_{ne}(f')$.

□

## 3 A Fixed Point Theorem

Domain theory's development strongly depends on fixed point theorems. Up till now, many kinds of fixed point theorem have been proposed and applied to different fields. The Scott fixed point theorem is one of the order versions of fixed point theorems. A typical case of the metric versions of fixed point theorems is the Banach fixed point theorem (also known as the contraction mapping theorem or contraction mapping principle). Combining order with metric we propose an rpos version based on $\mathcal{R}$-continuous and $\mathcal{R}$-approximating self mapping on $\mathcal{R}$-complete $\mathcal{R}$-posets.

**Theorem 3.1 (A Fixed Point Theorem Based on crpos)** *An $\mathcal{R}$- approximating and $\mathcal{R}$ -continuous function $f : S \to S$ on* *a non-empty* crpos *$S$ has a unique fixed point.*

**Proof.** Choose $x_0 \in S$ since $S$ is not empty. Of course, $x_0 \sqsubseteq_0 f(x_0)$, hence $f(x_0) \sqsubseteq_1 f^2(x_0)$. Thus $f^n(x_0) \sqsubseteq_n f^{n+1}(x_0)$ for all $n$. It follows that $(f^n(x_0))_{n\in\omega}$ is an $\mathcal{R}$-chain in $S$. Notice that $S$ is $\mathcal{R}$-complete, $\bigsqcup^{\mathcal{R}}_{n\in\omega} f^n(x_0)$ exists. From

$$f(\textstyle\bigsqcup^{\mathcal{R}}_{n\in\omega} f^n(x_0)) = \bigsqcup^{\mathcal{R}}_{n\in\omega} f^{n+1}(x_0) = \bigsqcup^{\mathcal{R}}_{n\in\omega} f^n(x_0),$$

$f$ has a fixed point $\bigsqcup^{\mathcal{R}}_{n\in\omega} f^n(x_0)$. Suppose $y$ is another fixed point of $f$, i.e., $y = f(y)$. It is easily seen that $x_0 \sqsubseteq_0 y$, hence $f(x_0) \sqsubseteq_1 f(y) = y$. Thus $f^n(x_0) \sqsubseteq_n y$ and $\bigsqcup^{\mathcal{R}}_{n\in\omega} f^n(x_0) \sqsubseteq y$. On the other hand, from $y \sqsubseteq_0 x_0$ and $y = f(y) \sqsubseteq_1 f(x_0)$ we conclude that $y \sqsubseteq_n f^n(x_0)$ for all $n$. Therefore $y \sqsubseteq \bigsqcup^{\mathcal{R}}_{n\in\omega} f^n(x_0)$. Consequently, $y = \bigsqcup^{\mathcal{R}}_{n\in\omega} f^n(x_0)$, which proves the theorem. □

In order to have a category-version of this fixed point theorem, we need the following preparations.

**Definition 3.2** [**$\omega$-chains, cocones, colimits**] Let $\mathbf{C}$ be a category. An $\omega$-chain in $\mathbf{C}$ is a sequence $\Sigma = (S_n, \alpha_n)_{n\in\omega}$ of objects and morphisms

$$S_0 \xrightarrow{\alpha_0} S_1 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} S_{n+1} \cdots .$$

A cocone $\sigma : \Sigma \to S$ over $\Sigma$ is an object $S$ together with a sequence $\sigma = (\sigma_n)_{n\in\omega}$ of morphisms $\sigma_n : S_n \to S$ such that, $\sigma_n = \sigma_{n+1} \circ \alpha_n$ for every $n \in \omega$, i.e., the diagram

$$\begin{tikzcd}
S_0 \arrow[r, "\alpha_0"] \arrow[drrr, "\sigma_0"'] & S_1 \arrow[r, "\alpha_1"] \arrow[drr, "\sigma_1"] & \cdots \arrow[r, "\alpha_{n-1}"] & S_n \arrow[r, "\alpha_n"] \arrow[d, "\sigma_n"] & S_{n+1} \cdots \arrow[dl, "\sigma_{n+1}"] \\
 & & & S &
\end{tikzcd}$$

commutes.

A colimit of $\Sigma$ (written $\mathrm{S} = \mathrm{Colim}\Sigma$) is a cocone $\sigma : \Sigma \to S$ such that for every other cocone $\tau : \Sigma \to T$, there is a unique morphism $\iota : S \to T$ satisfying $\tau_n = \iota \circ \sigma_n$ for every $n \in \omega$, i.e., the diagram

$$\begin{tikzcd}
S_0 \arrow[r, "\alpha_0"] \arrow[drrr, "\sigma_0"'] & S_1 \arrow[r, "\alpha_1"] \arrow[drr, "\sigma_1"] & \cdots \arrow[r, "\alpha_{n-1}"] & S_n \arrow[r, "\alpha_n"] \arrow[d, "\sigma_n"] \arrow[dr, "\tau_n"] & S_{n+1} \cdots \arrow[dl, "\sigma_{n+1}"'] \arrow[d, "\tau_{n+1}"] \\
 & & & \mathrm{S} = \mathrm{Colim}\Sigma \arrow[r, dotted, "\iota"] & T
\end{tikzcd}$$

commutes. If $\iota : S \to T$ exists but not necessarily unique, then $S$ is a weak colimit of $\Sigma$.

The following theorem is well known and plays an important role in the sequel. But it is not difficult to prove this theorem. The details are left to the reader.

**Theorem 3.3** *Let* $\mathbf{C}$ *be a category and let* $F : \mathbf{C} \to \mathbf{C}$ *be a functor. Let* $\alpha_0 : S_0 \to FS_0$ *be a morphism in* $\mathbf{C}$*. Let the* $\omega$*-chain* $\Sigma = (S_n, \alpha_n)_n$ *be defined by* $S_{n+1} = FS_n$ *and* $\alpha_{n+1} = F\alpha_n$ *for all* $n \geq 0$*. If this* $\omega$*-chain has a colimit* $\sigma : \Sigma \to S$ *and if* $F\Sigma = (F\Sigma, F\alpha_n)_n$ *has a colimit* $F\sigma : F\Sigma \to FS$ *with* $F\sigma = (F\sigma_n)_n$ *then* $FS \cong S$*.*

**Definition 3.4** [crpos-categories] A category $\mathbf{C}$ is crpos-enriched, or a crpos-category, if every hom-set $\mathbf{C}(S,T)$ is a crpos and composition of morphisms is $\mathcal{R}$-continuous.

**Example 3.5** The category $\mathbf{CRPOS}_{\mathcal{R}}$ is crpos-enriched, where each $\mathbf{CRPOS}_{\mathcal{R}}(S,T)$ is the function space $[S \to T]$. The composite of $\mathcal{R}$-continuous functions is $\mathcal{R}$-continuous because $\bigsqcup^{\mathcal{R}}_{n\in\omega} g_n \circ \bigsqcup^{\mathcal{R}}_{n\in\omega} f_n = \bigsqcup^{\mathcal{R}}_{n\in\omega} g_n \circ f_n$. If $\mathbf{C}$ and $\mathbf{C}'$ are crpos-categories, the product category $\mathbf{C} \times \mathbf{C}'$ is also a crpos-category.

The following propositions and definitions from Definition 3.6 to Theorem 3.13 are from [15]. For more details we refer the reader to [15].

**Definition 3.6** [**Embedding-projection pairs**]An embedding-projection pair $\alpha : D \to E$ between crpos'*s* $D$ and $E$ is a pair of $\mathcal{R}$-continuous maps $\alpha^e : D \to E$ and $\alpha^p : E \to D$ satisfying $\alpha^p \circ \alpha^e = 1_D$. We denote it briefly by $\alpha = \langle \alpha^e, \alpha^p \rangle$. If $\beta : E \to U$ is another e-p pair, the composition $\beta \circ \alpha$ is the e-p pair $\langle \beta^e \circ \alpha^e, \alpha^p \circ \beta^p \rangle$. We write $\mathbf{CRPOS}^{ep}_{\mathcal{R}}$ to express the category with the same objects as $\mathbf{CRPOS}_{\mathcal{R}}$ and e-p pairs as morphisms.

**Remark 3.7** For $\alpha^e : D \to E$, we have $\alpha^e(s) \sqsubseteq_n \alpha^e(t)$ implies $s = \alpha^p(\alpha^e(s)) \sqsubseteq_n \alpha^p(\alpha^e(t)) = t$. Hence $\alpha^e$ is an embedding w.r.t $\sqsubseteq_n$ and $\sqsubseteq$.

**Definition 3.8** [**Pre-orders on** $\mathbf{CRPOS}^{ep}_{\mathcal{R}}$] Let $D$ and $E$ be rpos's. We write $D \sqsubseteq_n E$ if and only if there exists an embedding-projection pair $\alpha_n : D \to E$ such that $\alpha^e_n \circ \alpha^p_n \sqsubseteq_n 1_E$.

**Lemma 3.9** $(\sqsubseteq_n)_{n\in\omega}$ *is a pre-order family on the class of* $\mathcal{R}$*-posets.*

**Lemma 3.10** *Let* $\Sigma = (S_n, \alpha_n)_n$ *be an* $\omega$*-chain in* $\mathbf{CRPOS}^{ep}_{\mathcal{R}}$*. Let* $\sigma : \Sigma \to S$ *and* $\tau : \Sigma \to T$ *be cocones. Then* $\Sigma$ *is an* $\mathcal{R}$*-chain if and only if the sequence* $(\tau^e_n \circ \sigma^p_n)_n$ *is an* $\mathcal{R}$*-chain in* $[S \to T]$*.*

**Theorem 3.11** *Let* $\Sigma = (S_n, \alpha_n)_{n\in\omega}$ *be an* $\mathcal{R}$*-chain in* $\mathbf{CRPOS}^{ep}_{\mathcal{R}}$*. A cocone* $\sigma : \Sigma \to S$ *is an* $\mathcal{R}$*-limit of* $\Sigma$ *if and only if* $\bigsqcup^{\mathcal{R}}_{n\in\omega} \sigma^e_n \circ \sigma^p_n = 1_S$*.*

**Remark 3.12** Lemma 3.10 and Theorem 3.11 can be easily generalized to crpos-categories with decomposable projections. To avoid repetitions the detailed proof is omitted. A projection in a category $\mathbf{C}$ is a morphism $p : S \to S$

such that $p \circ p = p$. We say $\mathbf{C}$ has decomposable projections if every projection $p : S \to S$ has an e-p pair $\mu : T \to S$ such that $p = \mu^e \circ \mu^p$.

**Theorem 3.13** $\mathbf{CRPOS}^{ep}_{\mathcal{R}}$ *is $\mathcal{R}$-complete, i.e., every $\mathcal{R}$-chain in* $\mathbf{CRPOS}^{ep}_{\mathcal{R}}$ *is convergent.*

**Notation 3.1** *The category with the same objects as* $\mathbf{C}$ *and e-p pairs as morphisms is denoted by* $\mathbf{C}^{ep}$*. A functor $F$ from* $\mathbf{C}$ *to* $\mathbf{D}$ *gives a functor $F^{ep}$ from* $\mathbf{C}^{ep}$ *to* $\mathbf{D}^{ep}$ *by putting $F^{ep}S = FS$ for any object $S$ and $F^{ep}\mu = \langle F\mu^e, F\mu^p\rangle$ for any e-p pair $\mu : S \to T$. We simply write $F$ instead of $F^{ep}$ when no confusion can arise. Unless otherwise stated the following functors between* $\mathbf{C}^{ep}$ *and* $\mathbf{D}^{ep}$ *are as given above.*

**Definition 3.14** [**$\mathcal{R}$-monotone ($\mathcal{R}$-approximating, $\mathcal{R}$-continuous) functors**] Let $\mathbf{C}$ and $\mathbf{D}$ be crpos-categories. A functor $F$ from $\mathbf{C}^{ep}$ to $\mathbf{D}^{ep}$ is $\mathcal{R}$-monotone if for every e-p pair $\mu : S \to T$ in $\mathbf{C}^{ep}$ and every $n \geq 0$, if $\mu^e \circ \mu^p \sqsubseteq_n 1_T$ then $(F\mu)^e \circ (F\mu)^p \sqsubseteq_n 1_{FT}$. If, in the same conditions, $(F\mu)^e \circ (F\mu)^p \sqsubseteq_{n+1} 1_{FT}$, the functor $F$ is said to be $\mathcal{R}$-approximating. If $F$ is $\mathcal{R}$-monotone and $F(\bigsqcup^{\mathcal{R}}_{n\in\omega} \sigma^e_n \circ \sigma^p_n) = \bigsqcup^{\mathcal{R}}_{n\in\omega}(F\sigma^e_n \circ F\sigma^p_n)$, then $F$ is said to be an $\mathcal{R}$-continuous functor.

**Proposition 3.15** *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be* crpos*-categories. If $F$ is an $\mathcal{R}$-continuous functor from* $\mathbf{C}^{ep}$ *to* $\mathbf{D}^{ep}$ *and if $\Sigma = (S_n, \alpha_n)_{n\in\omega}$ is an $\mathcal{R}$-chain in* $\mathbf{C}^{ep}$ *with colimit $\sigma : \Sigma \to S$ then $F\Sigma = (FS_n, F\alpha_n)_{n\in\omega}$ is an $\mathcal{R}$-chain in* $\mathbf{D}^{ep}$ *with colimit $F\sigma : F\Sigma \to FS$ with $F\sigma = (F\sigma_n)_{n\in\omega}$.*

**Proof.** As $F$ is $\mathcal{R}$-monotone and $\Sigma$ is an $\mathcal{R}$-chain, from $\alpha^e_n \circ \alpha^p_n \sqsubseteq_n 1_{S_{n+1}}$ we conclude that $(F\alpha_n)^e \circ (F\alpha_n)^p \sqsubseteq_n 1_{FS_{n+1}}$, so $F\Sigma$ is an $\mathcal{R}$-chain. Since $F$ is a functor, $F\sigma$ is a cocone of $F\Sigma$. By hypothesis , $\sigma : \Sigma \to S$ is an $\mathcal{R}$-limit of $\Sigma$, so by Theorem 3.11, $\bigsqcup^{\mathcal{R}}_{n\in\omega} \sigma^e_n \circ \sigma^p_n = 1_S$. As $F$ is $\mathcal{R}$-continuous,

$$F(\textstyle\bigsqcup^{\mathcal{R}}_{n\in\omega} \sigma^e_n \circ \sigma^p_n) = \bigsqcup^{\mathcal{R}}_{n\in\omega}(F\sigma^e_n) \circ (F\sigma^p_n) = 1_{FS},$$

hence $F\sigma$ is an $\mathcal{R}$-limit of $F\Sigma$ by Theorem 3.13. □

Here a category-version of the fixed point theorem (Theorem 3.1) will be proposed to find the solutions of domain equations of an $\mathcal{R}$-approximating and $\mathcal{R}$-continuous endofunctor on crpos-categories.

**Theorem 3.16** *Let* $\mathbf{C}$ *be a* crpos*-category with decomposable projections and let $F$ be an $\mathcal{R}$-approximating and $\mathcal{R}$-continuous endofunctor on* $\mathbf{C}^{ep}$*. Suppose there is an e-p pair $\alpha_0 : S_0 \to FS_0$. If* $\mathbf{C}^{ep}$ *is $\mathcal{R}$-complete then there is an $S$ with $S \cong FS$.*

**Proof.** We define inductively $S_{n+1} = FS_n$ and $\alpha_{n+1} = F\alpha_n$. This gives an $\omega$-chain $\Sigma = (S_n, \alpha_n)_n$. Let's assume inductively that $\alpha_n^e \circ \alpha_n^p \sqsubseteq_n 1_{S_{n+1}}$, which is certainly true for $n = 0$. Since $F$ is $\mathcal{R}$-approximating, $\alpha_{n+1}^e = (F\alpha_n)^e$ and $\alpha_{n+1}^p = (F\alpha_n)^p$, it follows that $\alpha_{n+1}^e \circ \alpha_{n+1}^p \sqsubseteq_{n+1} 1_{S_{n+2}}$. So $\Sigma$ is an $\mathcal{R}$-chain. Let $\sigma : \Sigma \to S$ be a colimit of $\Sigma$, which exists because $\mathbf{C}^{ep}$ is $\mathcal{R}$-complete. As $F$ is $\mathcal{R}$-continuous and $\mathcal{R}$-approximating, hence $F\Sigma = (FS_n, F\alpha_n)_n$ is an $\mathcal{R}$-chain with $\mathcal{R}$-limit $F\sigma : F\Sigma \to FS$ by Proposition 3.15. By Theorem 3.1, $S \cong FS$. □

**Definition 3.17** [**Locally $\mathcal{R}$-monotone, $\mathcal{R}$-approximating, $\mathcal{R}$-continuous**] A functor $F$ between crpos-categories $\mathbf{C}$ and $\mathbf{D}$ is locally $\mathcal{R}$-monotone (locally $\mathcal{R}$-approximating) if the function $\alpha \mapsto F\alpha$ from $\mathbf{C}(S,T)$ to $\mathbf{D}(FS, FT)$ is $\mathcal{R}$-monotone ($\mathcal{R}$-approximating) for all objects $S$ and $T$ in $\mathbf{C}$. If $F$ is locally $\mathcal{R}$-monotone and $F(\bigsqcup_{n\in\omega}^{\mathcal{R}} \alpha_n) = \bigsqcup_{n\in\omega}^{\mathcal{R}} F(\alpha_n)$ for every $\mathcal{R}$-chain $(\alpha_n)_{n\in\omega}$ then $F$ is said to be locally $\mathcal{R}$-continuous.

**Proposition 3.18** *Let $\mathbf{C}$ and $\mathbf{D}$ be crpos-categories. If $F : \mathbf{C}^{ep} \to \mathbf{D}^{ep}$ is locally $\mathcal{R}$-monotone (locally $\mathcal{R}$-approximating, locally $\mathcal{R}$-continuous) then $F$ is $\mathcal{R}$-monotone ($\mathcal{R}$-approximating, $\mathcal{R}$-continuous).*

**Proof.** If $\mu : S \to T$ is an e-p pair with $\mu^e \circ \mu^p \sqsubseteq_n 1_T$ then $(F\mu)^e \circ (F\mu)^p = F(\mu^e \circ \mu^p) \sqsubseteq_n 1_{FT}$. Because $F$ is locally $\mathcal{R}$-continuous,

$$F(\textstyle\bigsqcup_{n\in\omega}^{\mathcal{R}} \sigma_n^e \circ \sigma_n^p) = \bigsqcup_{n\in\omega}^{\mathcal{R}} (F\sigma_n^e \circ F\sigma_n^p).$$

Therefore $F$ is $\mathcal{R}$-continuous. The similar conclusion can be drawn for the $\mathcal{R}$-approximating case. □

By Proposition 2.10, the loosening functor is locally $\mathcal{R}$-approximating and $\mathcal{R}$-continuous, and the product and sum functors are locally $\mathcal{R}$-continuous. Several other useful functors includes the identity functor and the constant functors.

## 4 A Final Coalgebra Theorem in $\mathbf{CRPOS}_{\mathcal{R}}$

In this part it is shown that every locally $\mathcal{R}$-approximating and locally $\mathcal{R}$-continuous functor on $\mathbf{CRPOS}_{\mathcal{R}}$ has a final $F$-coalgebra.

**Definition 4.1** [**Coalgebra**] Let $\mathbf{C}$ be a category and $F : \mathbf{C} \to \mathbf{C}$ a functor. An $F$-coalgebra is a pair $(S, \alpha)$ where $S$ is an object and $\alpha : S \to FS$ is a morphism. An $F$-coalgebra morphism from $(S, \alpha)$ to $(T, \beta)$ is a morphism $\sigma : S \to T$ in $\mathbf{C}$ such that $\beta \circ \sigma = (F\sigma) \circ \alpha$, i.e., the following diagram commutes

$$\begin{tikzcd} S \arrow[r, "\sigma"] \arrow[d, "\alpha"'] & T \arrow[d, "\beta"] \\ FS \arrow[r, "F\sigma"'] & FT \end{tikzcd}$$

An $F$-coalgebra is final if there is a unique morphism to it from any $F$-coalgebra.

**Theorem 4.2** *Let $F : \mathbf{C} \to \mathbf{C}$ be a locally $\mathcal{R}$-approximating and locally $\mathcal{R}$-continuous functor on a* crpos*-category $\mathbf{C}$. If $\beta : T \cong FT$ is an isomorphism then $(T, \beta)$ is a final $F$-coalgebra.*

**Proof.** Let $(S, \alpha)$ be an $F$-coalgebra. Define $\Psi : \mathbf{C}(S,T) \to \mathbf{C}(S,T)$ by $\Psi(\gamma) = \beta^{-1} \circ (F\gamma) \circ \alpha$ for all $\gamma$.

$$\begin{tikzcd} S \arrow[r, dotted, "\Psi(\gamma)"] \arrow[d, "\alpha"'] & T \\ FS \arrow[r, "F\gamma"'] & FT \arrow[u, "\beta^{-1}"'] \end{tikzcd}$$

As $F$ is locally $\mathcal{R}$-continuous, we have

$$\begin{aligned} & \Psi(\textstyle\bigsqcup^{\mathcal{R}}_{n\in\omega} \gamma_n) \\ = \;& \beta^{-1} \circ F(\textstyle\bigsqcup^{\mathcal{R}}_{n\in\omega} \gamma_n) \circ \alpha \\ = \;& \textstyle\bigsqcup^{\mathcal{R}}_{n\in\omega} \beta^{-1} \circ F(\gamma_n) \circ \alpha \\ = \;& \textstyle\bigsqcup^{\mathcal{R}}_{n\in\omega} \Psi(\gamma_n). \end{aligned}$$

Hence $\Psi$ is $\mathcal{R}$-continuous. The function $\Psi$ is $\mathcal{R}$-approximating because $F$ is locally $\mathcal{R}$-approximating. By Theorem 3.1, $\Psi$ has a unique fixed point $\sigma$. $\sigma$ is a morphism because $\beta \circ \sigma = \beta \circ \Psi(\sigma) = (F\sigma) \circ \alpha$. It is immediate that any morphism is a fixed point of $\Psi$. Therefore $\sigma$ is the only morphism from $(S, \alpha)$ to $(T, \beta)$. □

**Theorem 4.3 (A Final Coalgebra Theorem)** *Every locally $\mathcal{R}$-approximating and locally $\mathcal{R}$-continuous functor $F : \mathbf{CRPOS}_{\mathcal{R}} \to \mathbf{CRPOS}_{\mathcal{R}}$ has a final $F$-coalgebra.*

**Proof.** Because $F$ is a locally $\mathcal{R}$-approximating and locally $\mathcal{R}$-continuous functor, so $F$ is $\mathcal{R}$-approximating and $\mathcal{R}$-continuous. Let $\alpha : S_0 \to FS_0$ be a morphism in $\mathbf{CRPOS}^{ep}_{\mathcal{R}}$ (e.g. take $S_0$ to be a singleton and $\alpha^e$ arbitrary). By Theorem 3.16, $F^{ep}$ has a fixed point. That fixed point is also a fixed point of $F$, and so by the previous theorem 4.2 it is a final $F$-coalgebra. □

**Corollary 4.4** *Every locally $\mathcal{R}$-approximating and locally $\mathcal{R}$-continuous functor $F : \mathbf{CRPOS}_{\mathcal{R}} \to \mathbf{CRPOS}_{\mathcal{R}}$ has a unique fixed point up to isomorphism.*

**Proof.** This is because final coalgebras are unique up to isomorphism. □

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## References

[1] L.Monteiro. Semantic domains based on sets with families of equivalences. *Electronic Notes in Theoretical Computer Science*, 11, pp1-34. 1998.

[2] D.Turi and J.J.M.M.Rutten. On the foundations of final coalgebra semantics: non-well-founded sets, partial orders, metric spaces. *Mathematical Structures in Computer Science*,8(5): 481-540. Augest 1998.

[3] J.J.M.M.Rutten. Elements of Generalized Ultrametric Domain Theory. Technical Report CS-R9507, CWI, Amsterdam, 1995.

[4] M. W . Mislove: Topology, domain theory and theoretical computer science. *Topology and its Applications*, Vol. 89, pp3-59. 1998.

[5] M.B.Smyth and G.D.Plotkin. The category-theoretic solution of recursive domain equations(extended abstract). *Proceedings of 18th Annual Symposium on Foundations of Computer Science*, pp13-17. OCT. 31-NOV. 2, 1977.

[6] M.B.Smyth. Quasi uniformities: reconciling domains with metric spaces. *Proceedings of the 3rd Workshop on Mathematical Foundations of Programming Language Semantics*, APR.8-10,1987. Lecture Notes In Computer Science, Vol. 298, pp236-253. Springer-Verlag, Berlin, 1988.

[7] V.Stoltenberg-Hansen, I.Lindström and E.R.Griffor. *Mathematical Theory of Domains.* Cambridge University Press, 1994.

[8] S.Abramsky and A.Jung. Domain theory, In S.Abramsky, D.Gabay and T.S.E.Maibaum, editors. *Handbook of Logic in Computer Science*, Vol.3: 1-168. Oxford University Press, 1995.

[9] S.MacLane. *Categories for the Working Mathematicians*, Springer-Verglag, Berlin. New York. Heidelberg, 1971.

[10] B.A.Davey and H.A.Priestley. *Introduction To Lattices and Order*. Cambridge Press, 2002.

[11] W.L.Wang and L.Fan. Ordered observation systems. *Fuzzy Systems and Mathematics*, Vol.20 (Supplement), pp157-160. September,2005.(In Chinese).

[12] L.Fan. *Some Questions in Domain Theory*, Ph.D Thesis, Capital Normal University 2001. (In Chinese).

[13] L.Fan,W.Ji and W.L.Wang. *The Information Order Approximation and Generalized Chains'Completion*, Beijing: Capital Normal University, Preprint, 2005, (in Chinese).

[14] L.G.Wu and L.Fan: Cartesian closed category of $\mathcal{R}$-Posets, *Proceedings of the 2009 International Conference on Foundations of Computer Science (FCS2009)*, Las Vegas, Nevada, July 13-16, 2009, USA CSREA Press 2009 (to be published).

[15] L.G.Wu, L.Fan and Y.Gao. A kind of $\mathcal{R}$-complete categories for $\mathcal{R}$-posets. *Proceedings of the Second International Joint Conference on Computational Sciences and Optimization (CSO2009)*, Vol.1, pp639-643. IEEE Computer Society Press, 2009.