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Continuous Monads

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Abstract

Continuous monads are an axiomatic class of submonads of the double power set monad. ρ-sets are an axiomatic generalization of directed sets. The ρ-generalization of continuous lattices arises as the algebras of a continuous monad and conversely. Each ρ-continuous poset has two topologies which respectively generalize the Scott and Lawson topologies. Each ρ-contnuous lattice is compact in the canonical topology if and only if the corresponding continuous monad contains the ultrafilter monad.

Keywords: continuous monad, conditional suprema, continuous lattice, completely distributive lattice, Scott monad

1 Introduction

The vitality of the 1980 six-author *Compendium of Continuous Lattices* [3] stems, in part, by invoking two distinct traditions. With the Scott topology, the partial order $x \leq y$ in a continuous lattice is interpreted to mean that "y has at least as much information as x". A "set of finite approximations of f" forms a directed set whose supremum f is the semantics of the computation. This is the forerunner of "domain theory" (cf [1], [4] and others). On the other hand, a continuous lattice with the Lawson topology is a particular type of compact topological semilattice, part of the very different tradition of topological algebra.

Recall that a *continuous lattice* is a complete lattice satisfying $x = \bigvee\{y : y \ll x\}$ for all x, where the *way below relation* $y \ll x$ means that for all directed D, if $x \leq \bigvee D$ then there exists $d \in D$ with $y \leq d$. If X is a continuous lattice then $U \subset X$ is *Scott open* if U is an upper set (i.e. $x \geq u \in U \Rightarrow x \in U$) and if whenever $D \subset X$ is directed with $\bigvee D \in U$ then $U \cap D \neq \emptyset$. These form the open sets of the *Scott topology*.

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In any category, given a subcategory $\mathcal{M}$, an object I is $\mathcal{M}$-*injective* if given $I \xleftarrow{f} X \xrightarrow{m} Y$ with $m \in \mathcal{M}$ there exists $g : Y \to I$ with $gm = f$. Let $\mathbf{CL}_\sigma$ be the category of continuous lattices and morphisms which preserve directed suprema. In the seminal paper [10], Dana Scott proved that $\mathbf{CL}_\sigma$ is isomorphic (via the Scott topology) to the full subcategory of T_o-spaces and continuous maps of all $\mathcal{M}$-injective objects where $\mathcal{M}$ is the subcatgory of subspace inclusions. That the Scott topology determines the partial order is seen from $x \le y \Leftrightarrow x \in \overline{\{y\}}$.

Given a continuous lattice, the topology of open sets generated by the Scott open sets and the complements $(\uparrow x)'$ of the principal upper sets is called the *Lawson topology*. The Lawson topology is compact Hausdorff. On a continuous lattice, the Lawson topology determines the Scott topology since the Scott open sets are the Lawson open upper sets. The partial order is not determined by the Lawson topology since the 4-element Boolean algebra and the 4-element chain are different continuous lattices with the same Lawson topology. Let $\mathbf{CL}_\lambda$ be the subcategory of $\mathbf{CL}_\sigma$ again with all continuous lattices as objects, but with morphisms that also preserve arbitrary infima. Via the Lawson topology, $\mathbf{CL}_\lambda$ is a full subcategory of the category of compact Hausdorff topological semilattices with continuous semilattice homomorphisms as morphisms.

This preliminary report is based on the following idea. Alan Day [2] and Oswald Wyler [11] have independently shown that $\mathbf{CL}_\lambda$ is isomorphic to the category of algebras of the filter monad. The filter monad is a member of the broader class of *continuous monads* whose algebras are cousins to continuous lattices. There, directed sets are replaced by ρ-sets with ρ characteristic of the particular monad. These new posets have two topologies, the *Sierpiński topology* and the *canonical topology* which respectively recover the Scott and Lawson topologies when the continuous monad is the filter monad.

We give specific examples (see Table 5) and shall develop tools to find other examples.

As a rule, deep results for topological semigroups require compactness (see [7]). Continuous monads whose algebras have compact Hausdorff canonical topology are characterized in Theorem 8.12.

We thank the referees for helpful suggestions.

2 Continuous Monads

We begin by reminding the reader of fundamental definitions.

Definition 2.1 A **monad T** in a category $\mathcal{K}$ is $\mathbf{T} = (T, \eta, \mu)$ with $T : \mathcal{K} \to \mathcal{K}$ a functor and $\eta : \mathrm{id} \to T$, $\mu : TT \to T$ natural transformations subject to the equations $\mu(\eta T) = \mathrm{id}_T = \mu(T\eta)$ and $\mu(T\mu) = \mu(\mu T)$.

Definition 2.2 If $\mathbf{T} = (T, \eta, \mu)$ is a monad in $\mathcal{K}$, a **T-algebra** is (X, ξ) with $\xi : TX \to X$ satisfying $\xi\, \eta_X = \mathrm{id}_X$ and $\xi\, \mu_X = \xi(T\xi)$. Here, ξ is the **structure map** of the algebra. A $\mathbf{T}$-homomorphism $f : (X, \xi) \to (Y, \theta)$ is a morphism $f : X \to Y$ satisfying $\theta(Tf) = f\xi$. This gives rise to the category $\mathcal{K}^{\mathbf{T}}$ of $\mathbf{T}$-algebras with underlying functor $\mathcal{K}^{\mathbf{T}} \to \mathcal{K}$.

Let T be an object function $\mathrm{ob}(\mathcal{K}) \to \mathrm{ob}(\mathcal{K})$ and let $\eta_X : X \to TX$ be a morphism for each X. Given morphisms $\mu_X : TTX \to TX$, to establish that (T, η, μ) is a monad, one must define $Tf : TX \to TY$ for each f and prove two axioms to show that T is a functor; there are two more axioms to show η and T are natural; then, in verifying the remaining three axioms, one must chase elements of $TTTX$ (if $\mathcal{K}$ is the category of sets) which is horrendous, say, if TX is the set of filters on X.

There is a well-known equivalent definition of a monad, $\mathbf{T} = (T, \eta, (\cdot)^{\#})$ where again T is an object function $\mathrm{ob}(K) \to \mathrm{ob}(K)$ and η_X is a morphism $X \to TX$ together with a new operator $f : X \to TY \mapsto f^{\#} : TX \to TY$ subject to a total of three axioms in which T is never iterated. The axioms are $f^{\#}\eta_X = f$, $(\eta_X)^{\#} = \mathrm{id}_{TX}$ and for $X \xrightarrow{f} Y \xrightarrow{g} Z$, $(g^{\#}f)^{\#} = g^{\#}f^{\#}$. Here, (X, ξ) is an algebra if $\xi\,\eta_X = \mathrm{id}_X$ and if given $f, g : W \to TX$ with $\xi f = \xi g$ then also $\xi f^{\#} = \xi g^{\#}$.

The correspondences between the definitions are as follows. $f^{\#} = TX \xrightarrow{Tf} TTY \xrightarrow{\mu_Y} TY$, $Tf = (X \xrightarrow{f} Y \xrightarrow{\eta_Y} TY)^{\#}$, $\mu_X = (\mathrm{id}_{TX})^{\#}$. (X, ξ) is an algebra for (T, η, μ) if and only if it is an algebra for $(T, \eta, (\cdot)^{\#})$. We will often use both viewpoints and think of a monad as $(T, \eta, \mu, (\cdot)^{\#})$.

In this paper we will be interested only in monads in the category **Set** of sets and (total) functions.

Example 2.3 The monad $\mathbf{B} = (B, \eta, \mu, (\cdot)^{\#})$ is defined by

$$\begin{aligned} BX &= 2^{2^X} \\ \eta_X\, x &= \mathrm{prin}(x) \;=\; \{A \subset X : x \in A\} \\ \text{For } X \xrightarrow{f} Y,\ (Bf)\mathcal{A} &= \{D \subset Y : f^{-1}D \in \mathcal{A}\} \\ \mu_X(\mathcal{H}) &= \{A \subset X : \Box A \in \mathcal{H}\} \text{ where } \Box A = \{\mathcal{A} \in BX : A \in \mathcal{A}\} \\ \text{For } X \xrightarrow{f} BY,\ f^{\#}(\mathcal{A}) &= \{B \subset Y : \{x : B \in fx\} \in \mathcal{A}\} \end{aligned}$$

Definition 2.4 Let $\mathbf{T}$ be a monad in **Set**. If for each set X, we are given $SX \subset TX$, then S is a **submonad of T** if S is closed under the monad operations, that is, if $x \in X$ then $\eta_X\, x \in SX$ and, if $f : X \to SY$, then $(X \xrightarrow{f} SY \subset TY)^{\#}$ maps SX into SY. In that case, $(S, \eta, (\cdot)^{\#})$ is a monad in its own right.

Given a group, a subset is or is not a subgroup accordingly as it is closed under the group operations. The situation with submonads is exactly the same. For $\mathbf{T}$ a monad in **Set**, given $SX \subset TX$ for every set X, S either is or is not a submonad $\mathbf{S}$ of $\mathbf{T}$.

It is obvious that any intersection of submonads is a submonad.

Definition 2.5 For $\mathcal{A} \in BX$, let $\mathcal{A}^c = \{D \subset X : \exists A \in \mathcal{A}\ \ D \supset A\}$. Then $\mathcal{A}$ is closed under supersets if and only if $\mathcal{A} = \mathcal{A}^c$. For $\mathcal{F} \in BX$, $\mathcal{F}$ is a **filter** on X if

$\emptyset \neq \mathcal{F} = \mathcal{F}^c$ and if the intersection of two elements of $\mathcal{F}$ is again in $\mathcal{F}$. The unique filter with $\emptyset \in \mathcal{F}$ is 2^X and it is called the **improper filter**. All other filters are **proper**.

Example 2.6 $FX = \{\mathcal{F} \in BX : \mathcal{F}$ is a filter on $X\}$ is a submonad of $\mathbf{B}$, the **filter monad**.

The papers of Day and Wyler already cited established that an $\mathbf{F}$-algebra is a continuous lattice. Here, the structure map $\xi : FX \to X$ is $\xi(\mathcal{F}) = \bigvee_{A\in\mathcal{F}} \bigwedge A$. This type of "lim-inf" operator was central to Scott's motivation in [10] to relate the convergence to f of finite approximations of f as a topological limit (in the Scott topology). We would hope that continuous monads will have this sort of structure.

For more insight, let us recall that the algebras of a monad are a model of universal algebra [6]. We think of (TX, μ_X) as the free algebra generated by X with inclusion of the generators η_X. If (Y, θ) is an algebra and $f : X \to Y$ is a function there exists a unique $\mathbf{T}$-homomorphism $\psi : (TX, \mu_X) \to (Y, \theta)$ with $\psi\, \eta_X = f$, namely $\psi = TX \xrightarrow{Tf} TY \xrightarrow{\theta} Y$. The elements of free algebras are built up recursively from variables given operations by applying operations to expressions already built. Now notice that a filter $\mathcal{F}$ on X satisfies

$$(1) \quad \mathcal{F} = \bigcup_{A\in\mathcal{F}} \bigcap_{x\in A} \mathrm{prin}(x)$$

This shows how a filter is a lim-inf expression (noting that $\mathrm{prin}(x)$ is the typical variable) and suggests that the algebras should at least be complete lattices.

Not every submonad of $\mathbf{B}$ will be appropriate. In fact, $\mathbf{B}$ itself is far off. The $\mathbf{B}$-algebras are complete atomic Boolean algebras, with structure map $\xi : BX \to X$, $\xi(\mathcal{A}) = \bigvee\{x : x$ is an atom and $\uparrow x \in \mathcal{A}\}$ (this is proved on pages 116-118 in [6]) and here there is no lim-inf in sight.

Now notice that $\mathcal{F} \in BX$ satisfies (1) if and only if $\mathcal{F} = \mathcal{F}^c$. This leads us to the definition of a pre-continuous monad.

Example 2.7 $B^cX = \{\mathcal{A} \in BX : \mathcal{A} = \mathcal{A}^c\}$ is a submonad of $\mathbf{B}$.

We now define the central structure of this paper.

Definition 2.8 A **continuous monad** is a submonad of $\mathbf{B^c}$ satisfying the following three axioms.

(CM.1) If $\emptyset \neq A \subset X$, $\mathrm{Prin}(A) \in TX$ where $\mathrm{Prin}(A) = \{A\}^c$.

(CM.2) If $\mathcal{A} \in TX$ then $\{2^A : A \in \mathcal{A}\}^c \in T(2^X)$.

(CM.3) If $\mathcal{A}_i \in TX_i$ $(i \in I)$ then $\{\prod A_i : A_i \in \mathcal{A}_i\}^c \in T(\prod X_i)$.

A **pre-continuous monad** is a submonad of $\mathbf{B^c}$ satisfying (CM.1). For a submonad $\mathbf{T}$ of $\mathbf{B^c}$, T_o is a submonad of $\mathbf{T}$ if $T_oX = \{\mathcal{A} \in TX : \mathcal{A} \neq 2^X\}$. The proof is in Lemma 7.2. It is routine to check that $\mathbf{T}_o$ is pre-continuous if $\mathbf{T}$ is and is continuous if $\mathbf{T}$ is.

We will postpone the roles of (CM.2, CM.3) to later sections.

Example 2.9 $\mathbf{B^c}$ is a continuous monad.

Example 2.10 The filter monad $\mathbf{F}$ is a continuous monad.

Example 2.11 The **neighborhood monad** first defined in [5] is the submonad of $\mathbf{F}_o$ defined by $NX = \{\mathcal{F} \in F_oX : \bigcap \mathcal{F} \neq \emptyset\}$. For $f : X \to NY$, for each $x \in X$ let $y_x \in A$ for each $A \in fx$. Given $\mathcal{F} \in NX$, let $x_o \in W$ for all $W \in \mathcal{F}$ and set $y = y_{x_o}$. Consider $W \in f^{\#}\mathcal{F}$. As $\{x : W \in fx\} \in \mathcal{F}$, $W \in fx_o$, so $y \in W$. This monad is continuous.

Example 2.12 Let βX be the set of ultrafilters on X. This is a submonad of the filter monad but it is not pre-continuous.

Example 2.13 $IX = \{\mathcal{A} \in B^cX : \mathcal{A}$ has the finite intersection property$\}$ is a submonad of $\mathbf{B^c}$. To see this, let $\mathcal{A} \in IX$ and let $W_1, \ldots, W_k \in f^{\#}(\mathcal{A})$. Then $A_i = \{x : W_i \in fx\} \in \mathcal{A}$ so there exists $x \in A_1 \cap \cdots \cap A_k$. For that x, all $W_i \in fx$ so $W_1 \cap \cdots \cap W_k \neq \emptyset$ as needed. This monad is continuous.

We note that the empty family is a member of each IX. This is unavoidable as follows. Let $\mathcal{P}$ be a partition of a set X in such a way that there is a subset A of X which is not a union of blocks of $\mathcal{P}$. Let $f : X \to X/\mathcal{P}$ be the canonical projection. Then $\{A\} \in IX$ and $(If)\{A\} = \{W \subset X/\mathcal{P} : f^{-1}W = A\} = \emptyset$.

Routinely, every intersection of continuous monads is a continuous monad. This is an important general tool to construct examples. For example, given a particular ultrafilter there exists a smallest continuous monad containing that ultrafilter.

3 Nonempty Infima

In this section, we establish that every algebra of a pre-continuous monad is a partially ordered set with non-empty imfima.

The **power set monad** $\mathbf{P} = (P, \eta, \mu)$ is well known, $PX = 2^X$, $\eta_X x = \{x\}$, $\mu_X \mathcal{A} = \bigcup \mathcal{A}$. Here for $f : X \to PY$, $f^{\#}A = \bigcup_{a\in A} fa$. For example, $(g^{\#}f)^{\#} = g^{\#}f^{\#}$ because both sides map A to $\bigcup_{a\in A}\bigcup_{b\in fa} gb$.

Notice that $P_oX = \{A \in PX : A \neq \emptyset\}$ is a submonad of $\mathbf{P}$. The following proposition is unusual in that we are able to characterize that algebras of all submonads at once.

Proposition 3.1 *Let* $\mathbf{T}$ *be a submonad of* $\mathbf{P}$*. Then the category of* $\mathbf{T}$*-algebras is isomorphic over* **Set** *to the category of partially-ordered sets* $(X, \leq)$ *in which* $A \subset X$ *has an infimum whenever* $A \in TX$*. The morphisms are those functions which preserve such infima.*

Proof. Let $\mathcal{D}$ be the category of all posets $(X, \leq)$ in which $\bigwedge A$ exists whenever $A \in TX$. A morphism $f : (X, \leq) \to (Y, \leq)$ in $\mathcal{D}$ must satisfy $f(\bigwedge A) = \bigwedge(fA)$ whenever $A \in TX$; notice that $fA = (Tf)A$ indeed is in TY. Given $(X, \leq)$ in $\mathcal{D}$, define $\xi : TX \to X$ by $\xi A = \bigwedge A$. The $\mathbf{T}$-algebra axioms on ξ are

$$\xi(\{x\}) = x \text{ for all } x \in X$$

$$\xi(\bigcup \mathcal{A}) = \xi\{\xi A : A \in \mathcal{A}\} \text{ for all } \mathcal{A} \in TTX$$

To see these axioms hold, $\xi(\{x\}) = \wedge\{x\} = x$ and, for $\mathcal{A} \in TTX$, $\xi(\{\xi A : A \in \mathcal{A}\}) = \xi(\{\bigwedge A : A \in \mathcal{A}\}) = \bigwedge\{\bigwedge A : A \in \mathcal{A}\} = \bigwedge(\bigcup \mathcal{A}) = \xi(\bigcup \mathcal{A})$.

A **T**-homomorphism $f : (X, \xi) \to (Y, \theta)$ must satisfy $\theta(Tf) = f\xi$. This is precisely the statement that $\bigwedge(fA) = f(\bigwedge A)$ for all $A \in TX$. It remains to show that for an arbitrary **T**-algebra (X, ξ) there exists a partial order $\leq$ for which $(X, \leq) \in \mathcal{D}$ with $\xi(A) = \bigwedge A$ for all $A \in TX$. To that end, define $x \wedge y = \xi\{x, y\}$. Then $x \wedge y = y \wedge x$ trivially and $x \wedge x = \xi(\{x\}) = x$. To see this operation is associative,

$$x \wedge (y \wedge z) = \xi\{\xi\{x\}, \xi\{y, z\}\} = \xi\{\{x\} \cup \{y, z\}\} = \xi\{x, y, z\}$$

and $(x \wedge y) \wedge z = \xi\{x, y, z\}$ similarly. Thus $(X, \leq)$ ia a poset with $x \leq y$ if $x \wedge y = x$. To complete the proof we must show that $\xi A = \bigwedge A$ for arbitrary $A \in TX$. If $\emptyset \in TX$ then for all $x \in X$, $x \wedge \xi\emptyset = \xi\{\xi\{x\}, \xi\emptyset\} = \xi\{\{x\} \cup \emptyset\} = x$ so $\xi\emptyset$ is indeed the greatest element, the empty infimum. Now let $\emptyset \neq A \in TX$. For $a \in A$, $\xi\{a, \xi A\} = \xi(A \cup \{a\}) = \xi(A)$ so $\xi A \leq a$ for all $a \in A$. Finally, suppose $y \leq a$ for all $a \in A$. Then we have $\xi\{y, \xi A\} = \xi\{A \cup \{y\}\} = \xi(\bigcup(\{a, y\} : a \in A) = \xi\{\xi\{a, y\} : a \in A\} = \xi\{y\} = y$ so $\xi A \leq y$ and $\xi A = \bigwedge A$. □

In the previous theorem, passing to the opposite order $(X, \leq) \mapsto (X, \geq)$ is an isomorphism of the category $\mathcal{D}$ with the category of posets $(X, \geq)$ in which every $A \in TX$ has a supremum, $\xi A = \bigvee A$. It is only a matter of taste as to whether ξA should be an infimum rather than a supremum. Since the structure map $\mu_X : TTX \to TX$ is the union map, our choice would seem in bad taste. The justification for the choice lies in the fact that we wish to represent submonads of **P** as submonads of $\mathbf{B^c}$, identifying the subset $A \subset X$ with its principal filter $\text{Prin}(A) = \{A\}^c \in B^cX$, and Prin is order reversing, $A \subset W \Leftrightarrow \text{Prin}(A) \supset \text{Prin}(W)$.

Proposition 3.2 *$\tau : P \to B^c$, $\tau_X(A) = Prin(A)$ is a monad map, representing* **P** *as a submonad of* $\mathbf{B^c}$.

Proof. For basic facts about monad maps we refer the reader to [6, Definition 2.2, Proposition 2.15, Theorem 3.39]. We must prove that $X \xrightarrow{\eta_X} PX \xrightarrow{\tau_X} B^cX = X \xrightarrow{\text{prin}_X} B^cX$ and that $PPX \xrightarrow{\bigcup_X} PX \xrightarrow{\tau_X} B^cX = PPX \xrightarrow{P\tau_X} PB^cX \xrightarrow{\tau_{B^cX}} B^cB^cX \xrightarrow{\mu_X} B^cX$. The first equality is obvious. For the second,

$$\begin{aligned}
\mu_X\, \tau_{B^cX}\, (P\tau_X)\, \mathcal{A} &= \mu_X\, \tau_{B^cX}\{\text{Prin}(A) : A \in \mathcal{A}\} \\
&= \mu_X\{\mathcal{B} \subset B^cX : \{\text{Prin}(A) : A \in \mathcal{A}\} \subset \mathcal{B}\} \\
&= \{W \subset X : \{\text{Prin}(A) : A \in \mathcal{A}\} \subset \Box W\} \\
&= \{W \subset X : \forall\, A \in \mathcal{A}\ \ A \subset W\} \\
&= \{W \subset X : \bigcup \mathcal{A} \subset W\} \ = \ \tau_X(\bigcup \mathcal{A}).
\end{aligned}$$

□

In general, if $\lambda : \mathbf{S} \to \mathbf{T}$ is a monad map and $\xi : TX \to X$ is a **T**-algebra, then $SX \xrightarrow{\lambda_X} TX \xrightarrow{\xi} X$ is an **S**-algebra. If **T** is a continuous monad, $\mathbf{P_o}$ is a submonad of $\mathbf{T}$ by (CM.1). Thus every $\mathbf{T}$-algebra (X,ξ) is a poset with non-empty infima $\bigwedge A = \xi(\mathrm{Prin}(A))$.

Proposition 3.3 *If $\mathbf{T}$ is a continuous monad and X is any set, the poset $(TX, \subset)$ is closed under non-empty intersections.*

Proof. The non-empty infimum operation of (TX, μ_X) is given by $P_oTX \xrightarrow{\tau_{TX}} TTX \xrightarrow{\mu_X} TX$. This operation is

$$\begin{aligned}\bigwedge \mathcal{A}_i &= \mu_X\{\mathrm{Prin}(\mathcal{A}_i)\} \;=\; \mu_X\{\mathcal{B}\subset TX : \{\mathcal{A}_i\}\subset \mathcal{B}\}\\ &= \{W\subset X : \{\mathcal{A}_i\}\subset \Box W\} \;=\; \{W\subset X : W\in\mathcal{A}_i \text{ for all } i\}\\ &= \bigcap \mathcal{A}_i\end{aligned}$$

□

Let $\mathcal{L}$ denote the category of $\mathbf{P_o}$-algebras, that is, the category of posets with non-empty infima and morphisms which preserve these. We have the following "$\mathcal{L}$-splitting lemma" which will find use below in Lemma 6.3, Proposition 6.4, Theorem 7.4, Lemma 8.5 and Corollary 7.6.

Lemma 3.4 *Every surjective morphism in $\mathcal{L}$ splits.*

Proof. Let $f : X \to Y$ be a surjective $\mathcal{L}$-morphism and define $g : Y \to X$ by the non-empty infimum

$$gy \;=\; \bigwedge\{x : fx = y\}$$

It is routine to check that g is a morphism with $fg = \mathrm{id}_Y$. □

4 Conditionals

The set FX of filters on X is $\{\mathcal{A}\in B^cX : \mathcal{A} \text{ is directed in } (2^X, \supset)\}$. New monads result by generalizing "directed set" to "ρ-set" according to the next definition. There are two definitions with two sets of axioms depending on whether or not a greatest element is desired in the semantics.

Definition 4.1 A **pre-conditional for suprema** is an assignment ρ to each poset $(X, \leq)$ of a collection of subsets of X called **ρ-sets** in such a way that axioms $(\rho.1, \rho.2, \rho.3)$ hold.

(ρ.1) Every subset with a greatest element is a ρ-set.

(ρ.2) The image of a ρ-set under an order-preserving map is a ρ-set.

(ρ.3) If A_i is a ρ-set in $(X_i, \leq_i)$ then $\prod A_i$ is a ρ-set in $\prod(X_i, \leq_i)$.

For a pre-conditional ρ, define

(2) $T_\rho X = \{\mathcal{A}\in B^cX : \mathcal{A} \text{ is a } \rho\text{-set in } (2^X\backslash\{\emptyset\}, \supset)\}$

(3) $\overline{T}_\rho X = \{\mathcal{A}\in B^cX : \mathcal{A} \text{ is a } \rho\text{-set in } (2^X, \supset)\}$

With further axioms, these will be seen to be submonads of $\mathbf{B^c}$ which are continuous. We note that if (ρ.2) holds then $T_\rho X \subset \overline{T}_\rho X$. We say that a pre-conditional

ρ is a **proper conditional** if axioms (ρ.4, ρ.5) hold whereas ρ is an **improper conditional** if axioms ($\overline{\rho}$.4, $\overline{\rho}$.5) hold.

($\rho.4 \mid \overline{\rho}.4$) If $\{\mathcal{A}_i : i \in I\}$ is a ρ-set in $(T_\rho X, \subset) \mid (\overline{T}_\rho X, \subset)$ then $\bigcup \mathcal{A}_i \in T_\rho X \mid \overline{T}_\rho X$.

($\rho.5 \mid \overline{\rho}.5$) If $\mathcal{A} \in T_\rho X \mid \overline{T}_\rho X$ and $\mathcal{B}_x \in T_\rho Y \mid \overline{T}_\rho Y$ for each $x \in X$ then $\{D \subset Y : \{x : D \in \mathcal{B}_x\} \in \mathcal{A}\} \in T_\rho Y \mid \overline{T}_\rho Y$.

Pre-conditionals, improper conditionals and proper conditionals each form a complete lattice with pointwise intersection as infimum. We name the following examples all of which are simultaneously improper conditionals and proper conditionals. Verification is routine.

Example 4.2 The least conditional is ρ_g where a ρ_g-set is a set with a greatest element.

The greatest conditional is ρ_a where all subsets are ρ_a-sets.

A ρ_c-set is a consistent set, that is, every finite subset has an upper bound.

A ρ_b-set is a bounded set, that is, the whole set has an upper bound.

A ρ_d set is a directed set.

$\rho_{db} = \rho_d \cap \rho_b$.

5 ρ-Continuous Posets

Let ρ be a an proper conditional. A **ρ-poset** is a poset in which every non-empty subset has an infimum and every ρ-set has a supremum. Morphisms of ρ-posets must preserve non-empty infima and ρ-suprema. Let ρ be an improper conditional. The subcategory of **improper ρ-posets** has as objects all ρ-posets with a greatest element and whose morphisms also preserve the greatest element.

In a ρ-poset, define the **ρ-below relation**

$$x \ll_\rho y \;\Leftrightarrow\; \text{for } D \text{ a } \rho\text{-set with } \bigvee D \leq y \;\exists\, d \in D \text{ with } x \leq d$$

A **ρ-continuous poset** is a ρ-poset such that for all x there exists a ρ-set D with $D \subset \{y : y \ll_\rho x\}$ such that $x = \bigvee D$. Morphisms preserve non-empty infima and ρ-suprema. An **improper** ρ-continuous poset is a ρ-continuous poset with a greatest element. Morphisms must additionally preserve the greatest element.

Thus an improper ρ_d-continuous poset is but a continuous lattice and a ρ_d-continuous poset is but a dcpo with non-empty infima.

Regarding the following definition, note that we assume the axiom of choice.

Definition 5.1 A ρ-poset is **completely ρ-distributive** if it satisfies the equation

$$\bigwedge_{j \in J} \bigvee_{k \in K(j)} x_{jk} \;=\; \bigvee_{g} \bigwedge_{j \in J} x_{j,gj} \qquad (\mathrm{CD}_\rho)$$

whenever $J \neq \emptyset$, $K(j) \neq \emptyset$ $(j \in J)$, $g \in \prod_{j \in J} K(j)$ and all suprema are ρ-suprema.

In (CD_ρ), the left hand side is always $\geq$ the right hand side, so the equation holds if $\leq$ can be shown.

Before going further, we note that, in (CD_ρ), if the supremum on the left hand side is a ρ-supremum then the supremum on the right hand side necessarily also is as follows. We assume $Q_j = \{x_{jk} : k \in K(j)\}$ is a ρ-set for each j. By axiom (ρ.3), $Q = \prod Q_j$ is a ρ-set in X^J. Notice that $(q_j) \in Q \Leftrightarrow \exists f \in \prod K(j)$ with $q_j = x_{j,fj}$. As $J \neq \emptyset$, $\bigwedge : X^J \to X$ exists and is order preserving so, by (ρ.2), the supremum on the right hand side is a ρ-supremum.

Theorem 5.2 *A ρ-poset is ρ-continuous if and only if it is completely ρ-distributive.*

Proof. The proof is very like that of [3, Theorem I.2.3] or [1, Theorem 7.1.1]. □

The proof of the next lemma is obvious.

Lemma 5.3 *Let ρ be a conditional for suprema as in Definition 4.1 and let $\mathcal{X} \subset 2^Y$, a poset under inclusion. Suppose that $(\mathcal{X}, \subset)$ has non-empty intersections and ρ-suprema which are unions. Then $(\mathcal{X}, \subset)$ is a completely ρ-distributive poset.*

Example 5.4 Improper ρ_a-continuous posets are completely distributive lattices.

6 Universal-Algebraic Properties of ρ-Posets

Any category monadic over **Set** has the properties we study in this section which concern products, subalgebras and homomorphic images. We establish some of these properties for ρ-posets.

Proposition 6.1 *Let ρ be a proper or improper conditional and let $(X_i, \leq_i)$ be ρ-posets ($i \in I$). Consider the product poset $(X, \leq) = \prod (X_i, \leq_i)$ (with the coordinatewise ordering) with projections $\pi_i : X \to X_i$. Then $(X, \leq)$ is a ρ-poset if each $(X_i, \leq_i)$ is and then $\pi_i : (X, \leq) \to (X_i, \leq_i)$ is a product in the category of ρ-posets. If each $(X_i, \leq_i)$ is ρ-continuous then $(X, \leq)$ also is.*

Proof. $\pi_i : (X, \leq) \to (X_i, \leq_i)$ is a product in $\mathcal{L}$. If A is a ρ-set in $(X, \leq)$ then $A_i = \pi_i A$ is a ρ-set in $(X_i, \leq_i)$ by (ρ.2) so $\alpha_i = \bigvee A_i$ exists. As the partial order is coordinatewise, $\alpha = (\alpha_i) = \bigvee A$. Using similar reasoning, (CD_ρ) holds if it holds coordinatewise. The remaining details are routine. □

Definition 6.2 Let ρ be a proper or improper conditional and let $(X, \leq)$ be a ρ-poset, $A \subset X$. Say that A is a **sub ρ-poset** if it is closed under non-empty infima, if it contains the greatest element if ρ is improper, and if every ρ-set in A has its supremum in A. (In more detail: if B is a ρ-set in A it is a ρ-set in X so has a supremum in X which we require to be in A).

It is evident that if A is a sub ρ-poset then it is a ρ-poset in its owns right and that the inclusion of $(A, \leq)$ in $(X, \leq)$ is a morphism of ρ-posets. It is further clear that an instance of (CD_ρ) in A is also an instance of (CD_ρ) in X, so $(A, \leq)$ is ρ-continuous if $(X, \leq)$ is.

Lemma 6.3 *Let ρ be a proper or improper conditional and let $f : (X, \leq) \to (Y, \leq)$ be a surjective morphism in $\mathcal{L}$. Let C be a ρ-set of $(Y, \leq)$. Then there exists a ρ-set $A \subset X$ with $fA = C$.*

Proof. By Lemma 3.4, there exists $\gamma : (Y, \leq) \to (X, \leq)$ in $\mathcal{L}$ with $f\gamma = \mathrm{id}_Y$. Then $A = \gamma C$ is a ρ-set and $fA = f\gamma C = C$. □

Proposition 6.4 *Let ρ be a proper or improper conditional and let $f : (X, \leq) \to (Y, \leq)$ be a morphism of ρ-posets with image factorization $f = (X, \leq) \xrightarrow{p} (fX, \leq) \xrightarrow{i} (Y, \leq)$ (i an inclusion) in $\mathcal{L}$. Then $(fX, \leq)$ is a sub ρ-poset of $(Y, \leq)$ and $p : (X, \leq) \to (fX, \leq)$ is a morphism of ρ-posets. Moreover, if $(X, \leq)$ and $(Y, \leq)$ are ρ-continuous, so too is $(fX, \leq)$.*

Proof. The result holds in $\mathcal{L}$. As p is a surjective morphism in $\mathcal{L}$, if B is a ρ-set in $(fX, \leq)$ there exists a ρ-set A in $(X, \leq)$ with $fA = B$ by Lemma 6.3. Thus $f(\bigvee A) = \bigvee B$ in $(Y, \leq)$. As $f(\bigvee A) \subset fX$, $(fX, \leq)$ is closed under ρ-suprema. To see $(fX, \leq)$ is ρ-continuous, starting with (y_{jk}) in (CD_ρ) for fX, use Lemma 6.3 to choose (x_{jk}) with $f(x_{jk}) = y_{jk}$ and each $\{x_{ik} : k \in K(j)\}$ a ρ-set in $(X, \leq)$. In this way, (CD_ρ) in X gives (CD_ρ) in fX. □

Proposition 6.5 *Let $(X, \leq)$ be a ρ-poset and let $R \subset X \times X$ be an equivalence relation which is also a sub-ρ-poset. Let $\theta : X \to X\backslash R$ be the canonical projection. Then there exists a unique ρ-poset structure on $X\backslash R$ such that $\theta : (X, \leq) \to (X\backslash R, \leq)$ is a morphism of ρ-posets.*

Proof. There exists unique $\leq$ such that $\theta : (X, \leq) \to (X\backslash R, \leq)$ is a morphism in $\mathcal{L}$. Let $A \subset X$ be a ρ-set. There exists $x \in X$ wih $\theta a \leq \theta x$ for all $a \in A$ (for example, let $x = \bigvee A$). For any such x, the map $X \to X$, $y \mapsto y \wedge x$ is order preserving, so $\{a \wedge x : a \in A\}$ is a ρ-set in $(X, \leq)$. For $a \in A$, $\theta(a \wedge x) = \theta a \wedge \theta x = \theta a$, so $(a \wedge x, a) \in R$ for all $a \in A$. It follows that $\{(a \wedge x, a) : a \in A\}$ is a ρ-set in R with supremum $(\bigvee(a \wedge x : a \in A), \bigvee A) \in R$ (noting that R is assumed closed under ρ-suprema). Thus $\theta(\bigvee A) = \theta(\bigvee(a \wedge x : a \in A) \leq \theta x$. This shows that $\theta(\bigvee A) = \bigvee(\theta A)$ so $X\backslash R$ has and θ preserves ρ-suprema. □

7 The Main Theorems

Definition 7.1 A submonad T of $\mathbf{B^c}$ is **improper** if $2^X \in TX$ for all sets X and is otherwise **proper**.

Evidently, T is improper $\Leftrightarrow \{\emptyset\} \in T\emptyset$.

Lemma 7.2 *Let T be an improper submonad of $\mathbf{B^c}$. Then $T_oX = TX\backslash\{2^X\}$ is a proper submonad.*

Proof. Let $f : X \to T_oY$, $\mathcal{A} \in T_oX$. Suppose $2^X \in f^\#\mathcal{A}$. Then $\emptyset = \{x : 2^X \in fx\} \in \mathcal{A}$, the desired contradiction, so T_o is a submonad. □

Theorem 7.3 *If ρ is a proper conditional then T_ρ is a proper continuous monad. If ρ is an improper conditional then $\overline{T}_\rho$ is an improper continuous monad. Conversely, if T is a continuous monad then, accordingly as T is proper or improper there exists a largest proper conditional, respectively improper conditional ρ with $T = T_\rho$,* *respectively $T = \overline{T}_\rho$; for this ρ, $A \subset (X, \leq)$ is a ρ-set if and only if $\{\uparrow a : a \in A\}^c \in TX$.*

Proof. First assume that T is a continuous monad. For (ρ.1), If $A \subset (X, \leq)$ with greatest element a_o, $\{\uparrow a : a \in A\}^c = \mathrm{Prin}((\uparrow a_o)) \in TX$ by (CM.1).

For (ρ.2), let $f : (X, \leq) \to (Y, \leq)$ be monotone, $A \subset X$ a ρ-set. We must show $\mathcal{B} = \{\uparrow fa : a \in A\}^c \in TY$ given that $\mathcal{A} = \{\uparrow a : a \in A\}^c \in TX$. Define $g : X \to TY$ by $gx = \mathrm{Prin}(\uparrow(fx))$. This is well-defined by (CM.1). Then $b \geq a \Rightarrow \uparrow(fb) \subset \uparrow(fa)$ so

$$\begin{aligned}\mathcal{B} &= \{D \subset Y : \exists\, a \in A \;\; \uparrow(fa) \subset D\}\\ &= \{D \subset Y : \exists\, a \in A\, \forall\, b \geq a \;\; \uparrow(fb) \subset D\}\\ &= \{D \subset Y : \exists a \in A \; \{x : \uparrow(fx) \subset D\} \supset \uparrow a\}\\ &= \{D \subset Y : \{x : D \in gx\} \in \mathcal{A}\\ &= g^{\#}(\mathcal{A}) \in TY\end{aligned}$$

For (ρ.3), let A_i be a ρ-set in $(X_i, \leq_i)$ so that $\{\uparrow a : a \in A_i\}^c \in TX_i$. We must show $\prod_i (\uparrow a : a \in A_i)^c \in T(\prod X_i)$. We have

$$\begin{aligned}\prod_i (\uparrow a : a \in A_i)^c &= \{\uparrow a : a \in \prod A_i\}^c\\ &= \{A \subset \prod X_i : \exists\, a \in \prod A_i \;\; \uparrow a \subset A\}\\ &= \{A \subset \prod X_i : \forall\, i\; \exists a_i \in A_i \prod \uparrow a_i \subset A\}\\ &= \prod \{(\uparrow a_i)^c : a_i \in A_i\} \;\in\; T(\prod X_i) \quad \text{(by CM.3)}\end{aligned}$$

We next show that $T = \overline{T}_\rho$ given that $\{\emptyset\} \in T\emptyset$. The case that $T = T_\rho$ if $\{\emptyset\} \notin T\emptyset$ is similar.

$$\begin{aligned}\overline{T}_\rho X &= \{\mathcal{A} \in B^c X : \mathcal{A} \text{ is a } \rho\text{-set in } (2^X, \supset)\}\\ &= \{\mathcal{A} \in B^c X : \{\uparrow A : A \in \mathcal{A}\}^c \in T(2^X)\}\\ &= \{\mathcal{A} \in B^c X : \{2^A : A \in \mathcal{A}\}^c \in T(2^X)\}\end{aligned}$$

so $TX \subset \overline{T}_\rho X$ by (CM.2). Conversely, if $\mathcal{A} \in \overline{T}_\rho X$, consider $\mathrm{Prin}_X : 2^X \to TX$ mapping A to $\mathrm{Prin}(A)$. We have

$$\begin{aligned}(\mathrm{Prin}_X)^{\#}(\uparrow A : A \in \mathcal{A}\}^c &= \{D \subset X : \{E \subset X : D \in \mathrm{Prin}(D)\} \in \{\uparrow A : a \in \mathcal{A}\}^c\\ &= \{D \subset X : \exists\, A \in \mathcal{A} \;\; 2^A \subset 2^D\}\\ &= \{D \subset X : \exists\, A \in \mathcal{A} \;\; D \supset A\}\\ &= \mathcal{A}^c \;=\; \mathcal{A}\end{aligned}$$

showing that $\overline{T}_\rho X \subset TX$.

In particular, T_ρ or $\overline{T}_\rho$ are submonads which gives (ρ.5) or ($\overline{\rho}$.5). To complete the proof of one direction, we'll show ($\overline{\rho}$.4). The proof of (ρ.4) is similar. Let $\{\mathcal{A}_i : i \in I\}$ be a ρ-set in $(T_\rho X, \subset) = (TX, \subset)$ so that $\{\uparrow \mathcal{A}_i : i \in I\}^c \in TTX$. Then

$$\begin{aligned}\bigcup \mathcal{A}_i &= \{D \subset X : \exists\, i \; D \in \mathcal{A}_i\}\\ &= \{D \subset X : \exists\, i \; \Box D \supset \uparrow \mathcal{A}_i\}\end{aligned}$$

$= \mu_X\{\uparrow\mathcal{A}_i : i \in I\}^c \in TX = T_\rho X$

Now the converse statement. Let ρ be a proper conditional and show that T_ρ is a continuous monad. The improper case is similar. Thus $T_\rho X = \{\mathcal{A} \in B^c X : \mathcal{A}$ is a ρ-set in $(2^X\backslash\{\emptyset\}, \supset)\}$.

To show (CM.1), let $\emptyset \neq A \subset X$. Then $\mathrm{Prin}(A) \in T_\rho X$ by (ρ.1) because A is the greatest elememt of $\mathrm{Prin}(A)$.

In particular, $\mathrm{prin}(x) \in T_\rho X$ for $x \in X$. Together with (ρ.5) this shows that T_ρ is a submonad of $\mathbf{B^c}$.

For (CM.2), the map $f : (2^X, \supset) \to (2^{2^X}, \supset)$, $fA = \uparrow 2^A = \{\mathcal{D} : \mathcal{D} \supset 2^A\}$ is order-preserving so by (ρ.2) maps a ρ-set $\mathcal{A} \in TX$ to a ρ-set $\{2^A : A \in \mathcal{A}\}^c \in T(2^X)$.

For (CM.3) let $\mathcal{A}_i \in TX_i\,(i \in I)$ and show $\{\prod A_i : A_i \in \mathcal{A}_i\}^c \in T(\prod X_i)$. By ($\rho$.3), $\prod \mathcal{A}_i$ is a ρ-set in $\prod(2^{X_i}\backslash\{\emptyset\}, \supset)$. Using the axiom of choice, we have an order-preserving map

$$(\prod(2^{X_i}\backslash\{\emptyset\}), \supset) \xrightarrow{f} ((2^{\prod X_i})\backslash\{\emptyset\}, \supset)$$

defined by $f(A_i) = \prod A_i$. Thus $\{\prod A_i : A_i \in \mathcal{A}_i\}$ is a ρ-set in $(2^{\prod X_i}, \supset)$. As $\mathcal{B} \mapsto \mathcal{B}^c$ is order-preserving, $\{\prod A_i : A_i \in \mathcal{A}_i\}^c \in T(\prod X_i)$.

To complete the proof, we must show that if $\widehat{\rho}$ is a proper conditional with $T_{\widehat{\rho}} = T$ then $\widehat{\rho} \subset \rho$. Let $A \subset (X, \leq)$ be a $\widehat{\rho}$-set. Then the map $f : (X, \leq) \to (2^X\backslash\{\emptyset\}, \supset)$, $fx = \{A \subset X : A \supset\uparrow x\}$ is order preserving so $fA = \{\uparrow a : a \in A\}^c \in T_{\widehat{\rho}}X = TX$ and so A is a ρ-set. □

Theorem 7.4 *Let* $\mathbf{T}$ *be a continuous submonad of* $\mathbf{B^c}$ *and let* ρ *be the corresponding largest conditional of Theorem 7.3. If* $\mathbf{T}$ *is proper, its category of algebras* $\mathbf{Set^T}$ *is the category of* ρ*-continuous posets. If* $\mathbf{T}$ *is improper,* $\mathbf{Set^T}$ *is the category of improper* ρ*-continuous posets.*

Proof. We know from Proposition 3.3 that TX is closed under non-empty intersections and by (ρ.4, $\overline{\rho}$.4), ρ-suprema exist. Thus, by Lemma 5.3, $(TX, \subset)$ is a completely ρ-distributive ρ-poset. For $A \subset X$ with inclusion $i : A \to X$, $Ti : TA \to TX$ maps $\mathcal{A}$ to $\{B \subset X : A \cap B \in \mathcal{A}\}$. Applying this to $A = \emptyset$, if $\emptyset \in T\emptyset$ then $(Ti)\emptyset = \emptyset \in TX$ so $(TX, \subset)$ has $\emptyset$ as least element. Also, if $\{\emptyset\} \in T\emptyset$, $(Ti)\{\emptyset\} = 2^X$ provides $(TX, \subset)$ with greatest element 2^X. Thus $(TX, \subset)$ is an object of $\mathcal{C}(\mathbf{T}, \rho)$, here defined to be the category of ρ-continuous posets if $\mathbf{T}$ is proper or the category of improper ρ-continuous posets if $\mathbf{T}$ is improper. Now let $(Y, \leq)$ be an object of $\mathcal{C}(\mathbf{T}, \rho)$ and let $f : X \to Y$ be a function. Claim that $(TX, \subset)$ is freely generated by X. For this, we must prove that there exists a unique morphism $\psi : (TX, \subset) \to (Y, \leq)$ in $\mathcal{C}(\mathbf{T}, \rho)$ with $\psi\, prin_X = f$. Define such ψ by

(4) $\psi(\mathcal{A}) = \bigvee_{A \in \mathcal{A}} \bigwedge_{x \in A} fx$

This map is well defined as follows. Define $(P_{\mathbf{T}}X, \supset)$ to be $(2^X\backslash\{\emptyset\}, \supset)$ or $(2^X, \supset)$ accordingly as $\mathbf{T}$ is proper or improper. The map $(P_{\mathbf{T}}X, \supset) \to (Y, \leq)$, $B \mapsto \bigwedge B$ is order-preserving and so, for each $\mathcal{B} \in TY$, maps the ρ-set B to the ρ-set $\{\wedge B : B \in \mathcal{B}\}$. For $\mathcal{A} \in TX$, $(Tf)\mathcal{A} = \{B \subset Y : f^{-1}B \in \mathcal{A}\} \in TY$. Thus

$\bigvee_{A\in\mathcal{A}}\bigwedge_{x\in A} fx = \bigvee_{f^{-1}B\in\mathcal{A}}\bigwedge B$ exists as desired. That ψ extends f is verified as follows: $\psi(\mathrm{prin}(x)) = \bigvee_{x\in A}\bigwedge fA = \mathrm{prin}(fx)$. Since $\mathcal{A} = \bigcup_{A\in\mathcal{A}}\bigcap_{x\in A}\mathrm{prin}(x)$, any morphism extending f must agree with ψ on nonempty $\mathcal{A}$. Since $\psi\emptyset$ is the empty supremum, ψ is indeed unique. To complete this part of the proof we must show that ψ preserves non-empty infima and ρ-suprema. The case of ρ-suprema is easy to verify:

$$\psi(\bigcup\mathcal{A}_i) = \bigvee_i \bigvee_{A\in\mathcal{A}_i}\bigwedge_{x\in A} fx = \bigvee_i \psi\mathcal{A}_i$$

For infima, the calculation is as follows.

$$\begin{aligned}\bigwedge_j \psi\mathcal{A}_j &= \bigwedge_j \bigvee_{A\in\mathcal{A}_j}\bigwedge_{x\in A} fx = \bigvee_{g\in\prod\mathcal{A}_j}\bigwedge_j\bigwedge_{x\in gj} fx \qquad (\mathrm{CD}_\rho)\\ \psi(\bigcap_j\mathcal{A}_j) &= \bigvee_{A\in\bigcap_j\mathcal{A}_j}\bigwedge_{x\in A} fx\end{aligned}$$

If $A\in\bigcap_j\mathcal{A}_j$, define $g\in\prod\mathcal{A}_j$ as the constant function $gj = A$. Then $\bigwedge_{x\in A} fx = \bigwedge_j\bigwedge_{x\in gj} fx \le \bigwedge_j \psi\mathcal{A}_j$ so $\psi(\bigcap_j\mathcal{A}_j) \le \bigwedge_j\psi\mathcal{A}_j$. Conversely, let $g\in\prod_j\mathcal{A}_j$. Then $\bigwedge_j\bigwedge_{x\in gj} fx = \bigwedge_{x\in\cup gj} fx \le \bigvee_{A\in\cap\mathcal{A}_j}\bigwedge_{x\in A} fx$ since $\cup gj\in\cap\mathcal{A}_j^c = \cap\mathcal{A}_j$. Thus $\bigwedge_j\psi\mathcal{A}_j \le \psi(\bigcap\mathcal{A}_j)$.

To finish the proof we must establish the "Beck coequalizer condition" which, here, means we must show that if (X,ξ) is a $\mathbf{T}$-algebra then there exists a unique partial order $\le$ such that $(X,\le)$ is an object of $\mathcal{C}(\mathbf{T},\rho)$ with $\xi : (TX,\subset)\to(X,\le)$ a morphism in $\mathcal{C}(\mathbf{T},\rho)$. To that end, let R be the equivalence relation of ξ, $R = \{(\mathcal{A},\mathcal{B})\in TX\times TX : \xi\mathcal{A} = \xi\mathcal{B}\}$. Let $p,q : R\to X$ be the two projections and consider $p^\#, q^\# : TR\to TX$. Since $\xi p^\#$ and $\xi q^\#$ are $\mathbf{T}$-homomorphisms $(TR,\mu_R)\to(X,\xi)$ which agree when preceded by prin_R, $\xi p^\# = \xi q^\#$. Thus there exists a unique function $\theta : TR\to R$ with $p\theta = p^\#$, $q\theta = q^\#$. As p,q are jointly monic, $\theta\,\mathrm{prin}_R = \mathrm{id}_R$. (In fact, one easily goes on to prove that (R,θ) is a $\mathbf{T}$-algebra, but we do not need this here). As a result, θ is surjective so that R is the image of $[p^\#, q^\#] : TR\to TX\times TX$ and this map is a morphism in $\mathcal{C}(\mathbf{T},\rho)$ by Proposition 6.4. Now use Propositon 6.5. □

One easily computes that the free continuous lattice on three elements has seven elements. In general, TX is finite if X is, so we have

Corollary 7.5 *A finitely-generated ρ-continuous poset is finite.*

The next result is well known for continuous lattices [3, Lemma I.1.12].

Corollary 7.6 *Each ρ-continuous poset is ρ-**meet continuous**, that is, the law*

$$(\vee x_i)\wedge x = \vee(x_i\wedge x) \qquad (MC_\rho)$$

holds whenever $\{x_i\}$ is a ρ-set.

Proof. The law trivially holds in $(TX,\subset)$ and such a law is preserved by quotients using Lemma 6.3. □

Lemma 7.7 *Let $\mathbf{T}$ be a continuous monad with largest conditional ρ with $T = T_\rho$. Let $A\subset(X,\le)$ be a ρ-set and let $B\subset X$ be such that A, B are mutually cofinal,* *that is, for $a \in A$ there is $d \in D$ with $a \leq d$ and for $d \in D$ there is $a \in A$ with $d \leq a$. Then B is a ρ-set.*

Proof. $\{\uparrow a : a \in A\}^c = \{\uparrow b : b \in B\}^c$. $\square$

Corollary 7.8 *Let ρ be the largest conditional with $T = T_\rho$ for continuous $\mathbf{T}$. If $(X, \leq)$ is a ρ-continuous poset and $x \in X$ then $\{y : y \ll_\rho x\}$ is a ρ-set. Thus for all x, $x = \bigvee\{y : y \ll_\rho x\}$.*

In view of the theory of this section, it is easy to establish the following table which identifies the conditionals for specific continuous monads. We have filled in the third column only in cases where there is an established name for the corresponding ρ-continuous poset in the literature.

(5)

ρ	monad $\mathbf{T}$	ρ-continuous posets
ρ_a	$\mathbf{B^c}$	completely distributive lattices
ρ_b	$\mathbf{B^c_o}$	
ρ_d	$\mathbf{F}$	continuous lattices
ρ_d	$\mathbf{F_o}$	dcpos with non-empty infima
ρ_{db}	$\mathbf{N}$	
ρ_{fb}	$\mathbf{I}$	
ρ_g	$\mathbf{P}$	complete inf-semilattices
ρ_g	$\mathbf{P_o}$	

8 The Sierpiński and Canonical Topologies

For a continuous monad $\mathbf{T}$, its inclusion in $\mathbf{B^c}$ is a monad map which then induces a forgetful functor over **Set** from the category of completely distributive lattices to ρ-continuous posets. Such functors always preserve limits. It follows that 2 is canonically a ρ-continuous poset and that the power 2^X is a product in three categories, completely distributive lattices, ρ-continuous posets for $T = T_\rho$ and posets with non-empty infima. All three are the same as posets because the restriction to P_o determines the infimum. Now via ξ, (X, ξ) is a quotient of the free algebra (TX, μ_X) which is in turn a subalgebra of the product algebra 2^{2^X}. It follows that the two-element algebra generates the variety of ρ-continuous posets in that every algebra is a quotient of a subalgebra of a power of 2. In the same way, each topology on 2 induces a topology on a $\mathbf{T}$-algebra (X, ξ), namely TX has the subspace topology of the product topology and X then has the quotient topology.

Definition 8.1 Let (X, ξ) be a $\mathbf{T}$-algebra. The **Sierpiński topology** on (X, ξ) is induced by the topology on $2 = \{0, 1\}$ in which $\{1\}$ is open $\{2\}$ is not. The **canonical topology** on (X, ξ) is induced by the discrete topology on 2.

We now explore some properties of these topologies.

Lemma 8.2 *Let 2 have any topology and let $TX \to 2^{2^X}$ have the subspace topology of the product topology where T is any submonad of* $\mathbf{B}$*. Let $f : X \to Y$ be a function. Then $Tf : TX \to TY$ is continuous.*

Proof. The monad inclusion $\iota_X : TX \to BX$ is a natural transformation giving rise, for $A \subset Y$, to the diagram

$$\begin{tikzcd} TX \arrow[r, "i_X"] \arrow[d, "Tf"'] & BX \arrow[d, "Bf"] \arrow[dr, "\pi_{f^{-1}A}"] & \\ TY \arrow[r, "i_Y"'] & BY \arrow[r, "\pi_A"'] & Y \end{tikzcd}$$

The triangle shows Bf is continuous. As ι_Y is a subspace and $(Bf)\iota_X$ is continuous, Tf is continuous. □

Lemma 8.3 *Let* $\mathbf{T}$ *be a continuous monad. In both the Sierpiński and canonical topologies, TX is a subspace of 2^{2^X}.*

Proof. Consider the diagram

$$\begin{tikzcd} TTX \arrow[rr, "\mu_X"] \arrow[d, "\iota_{TX}"'] & (\iota \text{ monad map}) & TX \arrow[d, "\iota_X"] \\ BTX \arrow[r, "B(\iota_X)"] & BBX \arrow[r, "\nu_X"] & BX = 2^{2^X} \end{tikzcd}$$

Here, ν_X is continuous since it is continuous followed by each projection, $\pi_B\,\nu_X = \pi_{\Box B}$. $B(\iota_X)$ is continuous by Lemma 8.2. Now let ι_{TX} be a subspace and let μ_X be a quotient. We must show ι_X is a subspace. Equivalently, instead let ι_X be a subspace and prove that μ_X is a quotient. As $\iota_X\mu_X$ is continuous, μ_X is continuous. For $\mathrm{prin}_X : X \to TX$, it is a monad law that $\mu_X T(\mathrm{prin}_X) = \mathrm{id}_{TX}$. As $T(\mathrm{prin}_X)$ is continuous by Lemma 8.2, μ_X is split epic in **Top**, hence is a quotient map. □

Proposition 8.4 *Let* $\mathbf{S}$ *be a continuous monad which is a submonad of the continuous monad* $\mathbf{T}$*. The following hold for a* $\mathbf{T}$*-algebra (X, ξ) which is also, then, an* $\mathbf{S}$*-algebra $SX \xrightarrow{\iota_X} TX \xrightarrow{\xi} X$ where ι is the inclusion monad map.*

(i) *SX is a subspace of TX in both the Sierpiński and canonical topologies.*

(ii) *If $U \subset X$ is Sierpiński-open in (X, ξ) it is again Sierpiński-open in $(X, \xi\,\iota_X)$. Similarly for the canonical topology.*

Proof. We have ι_X is a subspace because j_X and $j_X\,\iota_X$ are. For the second statement, if $U \subset (X, \xi)$ is open (in either topology) then $\xi^{-1}U$ is open in TX so $(\xi\,\iota_X)^{-1}U = SX \cap \xi^{-1}U$ is open in SX since SX is a subspace of TX. □

Let $\mathbf{T}$ be a continuous monad. For $A_1, \ldots, A_m$, $B_1, \ldots, B_n \subset X$, define

$$\square(A_1,\ldots,A_m) = \{\mathcal{A} \in TX : \text{all } A_i \in \mathcal{A}\}, \quad \square'(B_1,\ldots,B_n) = \{\mathcal{A} \in TX : \text{all } B_j \notin \mathcal{A}\}$$

By the definition of the cartesian product topology, a base for the Sierpiński topology on TX is all $\square(A_1,\ldots,A_m)$ whereas a base for the canonical topology on TX is all $\square(A_1,\ldots,A_m) \bigcap \square'(B_1,\ldots,B_n)$.

Lemma 8.5 *For a ρ-continuous poset (X,ξ), a Sierpiński-open set $U \subset X$ is an upper set.*

Proof. This is true in (TX,μ_X) because $\square(A_1,\ldots,A_m)$ is an upper set and any union of upper sets is upper. Thus $\xi^{-1}U$ is an upper set. By Lemma 3.4 there exists order-preserving $g : X \to TX$ with $\xi g = \mathrm{id}_X$. If $u \in U$ and $u \leq v$ then $gu \leq gv$ with $gu \in \xi^{-1}U$ so $gv \in \xi^{-1}U$ and $v = \xi gv \in U$. □

Proposition 8.6 *In a ρ-continuous poset, the order and the Sierpiński topology are related by $\overline{\{y\}} = \downarrow y$, that is, the order is the specialization order of its Sierpińsky topology which is then necessarily T_o.*

Proof. Let $\mathbf{T}$ be a continuous monad with algebra (X,ξ). For $y \in X$,

$$\xi^{-1}(\downarrow y) = \{\mathcal{A} \in TX : \xi(\mathcal{A}) \leq y\} \;=\; \{\mathcal{A} : \bigvee_{A\in\mathcal{A}} \bigwedge A \;\leq y\}$$

If $(\mathcal{A}_i)$ is a net in $\xi^{-1}(\downarrow y)$ which converges to $\mathcal{A} \in TX$ then, for $A \in \mathcal{A}$, $\mathcal{A} \in \square A$ so $\mathcal{A}_i$ is eventually in $\square A$, that is, A is eventually in $\mathcal{A}_i$. This shows $\bigwedge A \leq y$ and, so, $\xi^{-1}(\downarrow y)$ is Sierpiński-closed in TX. By Lemma 8.5, closed sets are lower sets. Thus $\downarrow y$ is the smallest closed set containing y as needed. □

In this paper, a compact space is not required to be Hausdorff. The constructions in the next proof mirror the approach of [11].

Theorem 8.7 *Let $\mathbf{T}$ be a continuous monad. Then every algebra (X,ξ) is compact in its canonical topology if and only if every ultrafilter on X belongs to TX.*

Proof. The usual beta-compactification of (discrete) X is realized as the set βX of all ultrafilters on X which is a subspace of the Cantor space 2^{2^X}. First suppose that TX is compact. Then TX is a closed subspace of 2^{2^X}. Let $\mathcal{U} \in \beta X$. As X is dense in βX, there exists a net $\mathrm{prin}(x_i)$ converging to $\mathcal{U}$ in 2^{2^X}. As $\mathrm{prin}(x_i) \in TX$ and TX is closed, this shows $\mathcal{U} \in TX$. Conversely, we assume that β is a submonad of $\mathbf{T}$ inducing a forgetful functor $\Phi : \mathbf{Set}^{\mathbf{T}} \to \mathbf{Set}^{\beta}$. (It is well known that β is a submonad of $\mathbf{B^c}$ and that $\mathbf{Set}^{\beta}$ is the category of compact Hausdorff spaces). Φ maps the $\mathbf{T}$-subalgebra TX of 2^{2^X} to the closed subspace TX of the Cantor space so TX is compact in its canonical topology. As (X,ξ) is a quotient, it too is compact.□

Definition 8.8 A continuous monad $\mathbf{T}$ is a **Scott monad** if $\beta X \subset TX$ for all sets X.

Proposition 8.9 *Let $\mathbf{S}$ be a Scott monad which is a submonad of the Scott monad $\mathbf{T}$. The following hold for a $\mathbf{T}$-algebra (X,ξ) and the resulting $\mathbf{S}$-algebra $SX \xrightarrow{\iota_X}$*

$TX \xrightarrow{\xi} X$ *where* ι *is the inclusion monad map.*

(i) *The canonical topologies of* (X, ξ) *and* $(X, \xi\,\iota_X)$ *coincide and are compact Hausdorff.*

(ii) *If* $\mathbf{S}$ *is an improper Scott monad,* (X, ξ) *and* $(X, \xi\,\iota_X)$ *are continuous lattices and the compact Hausdorff topology of (i) is the Lawson topology.*

Proof. For (i), a continuous identity function from a compact space to a Hausdorff space must be a homeomorphism. For (ii), $FX \subset SX$ by Proposition 3.3 because every proper filter is an intersection of ultrafilters. □

It may seem puzzling that the $\mathbf{F_o}$ algebras, "continuous lattices without greatest element" remain compact in the canonical topology. This is explained by the fact that, in a continuous lattice, the greatest element is isolated in the Lawson topology.

Proposition 8.10 *For a Scott monad* $\mathbf{T}$ *with largest conditional* ρ*. the following hold.*

(i) *Every directed set is a* ρ*-set.*

(ii) *If* (X, ξ) *is a* $\mathbf{T}$*-algebra, every subalgebra* $Q \subset X$ *is closed in the canonical topology.*

Proof. For (i), if $A \subset (X, \leq)$ is directed then $\{\uparrow x : x \in A\}^c$ is a filter and hence is in TX. For (ii), ξ^{-1} maps subalgebras to subalgebras so it suffices to show that each subalgebra $Q \subset (TX, \mu_X)$ is closed in the canonical topology. Let $(\mathcal{A}_i)$ be a net in A which converges to $\mathcal{A}$ in TX. We have

$$\begin{aligned} A \in \mathcal{A} &\Leftrightarrow A \text{ is eventually in } \mathcal{A}_i \\ &\Leftrightarrow A \in \bigcup_i \bigcap_{j \geq i} \mathcal{A}_j \end{aligned}$$

But this union is a directed union hence is a ρ-supremum of infima which is again in the subalgebra Q. □

The proof of our final result is left to the reader.

Proposition 8.11 *Let* $\mathbf{T}$ *be a Scott monad,* (X, ξ) *an algebra,* $U \subset X$*. The following are equivalent.*

(i) $(\,(\mathcal{A} \in TX) \wedge (\xi\mathcal{A} \in U)\,) \;\Rightarrow\; U \in \mathcal{A}$.

(ii) $U = \uparrow U$ and, for all ρ-sets D, $\bigvee D \in U \Rightarrow U \cap D \neq \emptyset$.

(iii) U is open in the Sierpiński topology.

In particular, a Sierpiński-open set is Scott-open.

Theorem 8.12 *Let* $\mathbf{T}$ *be a continuous monad. Then every algebra* (X, ξ) *is compact in its canonical topology if and only if every ultrafilter on* X *belongs to* TX*.*

Proof. The usual beta-compactification of (discrete) X is realized as the set βX of all ultrafilters on X which is a subspace of the Cantor space 2^{2^X}. First suppose that TX is compact. Then TX is a closed subspace of 2^{2^X}. Let $\mathcal{U} \in \beta X$. As X is dense in βX, there exists a net $\mathrm{prin}(x_i)$ converging to $\mathcal{U}$ in 2^{2^X}. As $\mathrm{prin}(x_i) \in TX$ and TX is closed, this shows $\mathcal{U} \in TX$. Conversely, we assume that β is a submonad of $\mathbf{T}$ inducing a forgetful functor $\Phi : \mathbf{Set}^{\mathbf{T}} \to \mathbf{Set}^{\beta}$. (It is well known that β is a submonad of $\mathbf{B^c}$ and that $\mathbf{Set}^{\beta}$ is the category of compact Hausdorff spaces). Φ maps the $\mathbf{T}$-subalgebra TX of 2^{2^X} to the closed subspace TX of the Cantor space so TX is compact in its canonical topology. As (X, ξ) is a quotient, it too is compact.$\square$

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