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Coalgebras for Fuzzy Transition Systems

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Abstract

This paper studies a coalgebraic theory of fuzzy transition systems. Main conclusions include: the functor $\mathcal{F}^A$ for deterministic fuzzy transition systems and the functor $(\mathcal{P} \circ \mathcal{F})^A$ for nondeterministic fuzzy transition systems preserve weak pullbacks, and the functor $\mathcal{F}^A$ has a final coalgebra under some restricted conditions. Moreover, we show how to get a concrete (fuzzy) bisimulation from a coalgebraic bisimulation.

Keywords: Fuzzy transition system; Fuzzy bisimulation; Coalgebra; Weak pullback; Final coalgebra

1 Introduction

The notion of (deterministic) fuzzy transition system (**FLTS**, for short) was first proposed by Errico and Loreti in [7], which is a triple (S, A, α). In this triple, S is a set of states, A is a set of actions and α, the fuzzy transition function, is a mapping from $S \times A$ to $\mathcal{L}(S)$, where $\mathcal{L}(S) = \{\mu \mid \mu : S \to L\}$ is the set of all lattice-valued sets with a complete lattice L as a codomain. Hence, this is a lattice-valued transition system. The same notion has appeared in [11] too. Recently, Cao et al. [2] also proposed this notion using the unit interval $[0, 1]$ instead of the complete lattice L, and with an initial state. In this case, for any $s, s' \in S$ and $a \in A$, $\alpha(s, a)(s')$ means the possibility that s performs an a action to enter a successor state s'. Errico and Loreti [7] give the notion of (fuzzy) bisimulation and applied it to fuzzy reasoning.

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Cao et al. [2] define a (fuzzy) bisimulation relation between two different **FLTS**s by a correlational pair based on some relation; Ćirić and his colleagues in [4] introduce two types of simulations (forward and backward) and four types of bisimulations (forward, backward, forward-backward, and backward-forward) for fuzzy automata.

Coalgebra is an abstract theory that provides a uniform framework for various kinds of transition systems, arising as a rather recent theory within, or closely connected to category theory. Many researches on coalgebras for the classical transition systems and probabilistic transition systems have been done (see for example, [5,6,13,14,15]). However, as far as we know, there are no researches on coalgebras for fuzzy transition systems.

Two issues are often concerned in investigating coalgebras. That is, a final coalgebra [8] and the preservation of weak pullback of the functor [9]. In system-theoretic terms, final coalgebras are of interest because they form so-called minimal representations: they are canonical realisations containing all the possible behaviours of a system. The preservation of weak pullback of the functor is helpful devoted to a coalgebraic theory from a coalgebraic bisimulation. An important consequence of this property is that T-bisimulation in any T-coalgebra coincides with equality in the final coalgebra of the T.

This paper is devoted to a coalgebraic theory of fuzzy transition systems, including deterministic and nondeterministic fuzzy transition systems. We prove that the functor $\mathcal{F}^A$ for deterministic fuzzy transition systems and the functor $(\mathcal{P} \circ \mathcal{F})^A$ for nondeterministic fuzzy transition systems preserve weak pullbacks, and the functor $\mathcal{F}^A$ has a final coalgebra under some restricted conditions. Moreover, we show how to get a concrete (fuzzy) bisimulation from coalgebraic bisimulation.

2 Preliminaries

In this section, we introduce fuzzy transition systems and coalgebras.

2.1 Fuzzy Transition Systems

In this subsection, we recall some basic facts on fuzzy set theory and the notion of fuzzy transition systems.

Let X be a universal set. A fuzzy set on X is a mapping from X to the unit interval. We denote by $\mathcal{F}(X)$ the set of all fuzzy sets of X. The *support* of a fuzzy set μ is a set defined as $\mathrm{supp}(\mu) = \{x \in X : \mu(x) > 0\}$. For any $\mu \in \mathcal{F}(X)$ and $U \subseteq X$, the notation $\mu(U)$ stands for $\sup_{x \in U} \mu(x)$; for any $e \in \mathcal{F}(X \times Y)$, $x \in X$ and $V \subseteq Y$, the notation $e(x, V)$ stands for $\sup_{y \in V} e(x, y)$. Let $(\sqcup_{i \in I} \mu_i)(x) = \sup_i \mu_i(x)$ for any family $\mu_i (i \in I)$ of $\mathcal{F}(X)$ and $x \in X$. In addition, a fuzzy set μ of X is called *normal* if $\mu(X) = 1$.

Definition 2.1 [2] A deterministic fuzzy labelled transition system (**FLTS**, for short) is a triple $\mathcal{S} = (S, A, \alpha)$, where

(1) S is a finite or infinite set of states;

(2) A is a finite or infinite set of actions;

(3) α, the fuzzy transition function, is a mapping from $S \times A$ to $\mathcal{F}(S)$.

Determinism here means for each state s and label a, at most one fuzzy set $\alpha(s,a)$ is returned by α. The symbols $s \xrightarrow{a} \mu$ and $s \xrightarrow{a[\lambda]} s'$ denote $\alpha(s,a) = \mu$ and $\alpha(s,a)(s') = \lambda$, respectively.

Definition 2.2 [3] A nondeterministic fuzzy labelled transition system (**NFLTS**, for short) is a triple $\mathcal{S} = (S, A, \alpha)$ where S and A as Definition 2.1 and the transition function α is a mapping from $S \times A$ to $\mathcal{P}(\mathcal{F}(\mathcal{S}))$, the powerset of $\mathcal{F}(S)$.

Nondeterminism means more than one fuzzy set may be returned by α for each state s and label a. For $s \in S$ and $a \in A$, if $\mu \in \alpha(s,a)$, we write $s \overset{a}{\rightsquigarrow} \mu$.

2.2 Category and Coalgebra

In this subsection, we introduce some notions of category theory and coalgebra. The following three functors in **Set** category are often used below.

(1)The powerset functor $\mathcal{P}$ maps any set to the set of its subset

$$\mathcal{P}(X) = \{Z \mid Z \subseteq X\}$$

and for a function $f : X \to Y$, gives

$$\mathcal{P}(f)(U) = f(U) = \{f(x) \mid x \in U\} \text{ for } U \subseteq X.$$

(2) Let A be a fixed set. The constant exponent functor $\mathcal{I}d^A$ maps a set X to the set of all functions from A to X, i.e.

$$\mathcal{I}d^A(X) = X^A = \{\xi \mid \xi : A \to X\},$$

and it maps a function $f : X \to Y$ to the function $\mathcal{I}d^A(f) : X^A \to Y^A$ defined by

$$\mathcal{I}d^A(f)(\xi) = f \circ \xi \text{ for } \xi : A \to X.$$

(3) The fuzzy functor $\mathcal{F}$ maps a set X to $\mathcal{F}(X)$, and it maps a function $f : X \to Y$ to the function $\mathcal{F}(f) : \mathcal{F}(X) \to \mathcal{F}(Y)$ such that

$$\mathcal{F}(f)(\mu)(y) = \mu(f^{-1}(y)) \text{ for } \mu \in \mathcal{F}(X) \text{ and } y \in Y.$$

We will use the notation T^A for the composition of $\mathcal{I}d^A$ with a functor T, i.e. $T^A = \mathcal{I}d^A \circ T$. Thus, T^A is a functor since the composition of functors is still a functor.

Let $\mathbf{C}$ be a category and $T : \mathbf{C} \to \mathbf{C}$ an endofunctor. A T-coalgebra is a tuple (X, α), where X is an object in $\mathbf{C}$ and α is an arrow in $\mathbf{C}$, i.e. $\alpha : X \to T(X)$. Sometimes, we simply call T-coalgebra α. A homomorphisms between two T-coalgebras (X, α) and (Y, β) is $f : X \to Y$ in $\mathbf{C}$ such that $T(f) \circ \alpha = \beta \circ f$. The class of T-coalgebras with their T-homomorphisms form a category $\mathbf{Coalg}_T$.

A final object in $\mathbf{Coalg}_T$ is a T-coalgebra γ such that for each T-coalgebra α there is exactly one homomorphism from α to γ.

Now, we return to **FLTS**. An **FLTS** is just a mapping $\alpha : S \to (A \to \mathcal{F}(S))$, or equivalently, $\alpha : S \to \mathcal{F}(S)^A$. Thus, an **FLTS** is precisely a coalgebra (S, α) of the functor $\mathcal{F}^A$, which maps a function $f : X \to Y$ to the function $\mathcal{F}^A(f) : \mathcal{F}^A(X) \to \mathcal{F}^A(Y)$ such that for any $t \in \mathcal{F}^A(X)$, $y \in Y$ and $a \in A$, $\mathcal{F}^A(f)(t(a))(y) = (t(a))(f^{-1}(y))$, where $t(a)$ is a fuzzy set on X. Particularly, if the fuzzy transition system is unlabelled, then it is a coalgebra $(S, \alpha : S \to \mathcal{F}(S))$ of the fuzzy functor $\mathcal{F}$. Similarly, an **NFLTS** is a $(\mathcal{P} \circ \mathcal{F})^A$-coalgebra.

Definition 2.3 A T-bisimulation between two T-coalgebras (X, α) and (Y, β) is a relation $R \subseteq X \times Y$ such that there exists a coalgebra structure $\gamma : R \to T(R)$ making the projections $\pi_1 : R \to X$ and $\pi_2 : R \to Y$ coalgebra homomorphisms, satisfying $\alpha \circ \pi_1 = T(\pi_1) \circ \gamma$ and $\beta \circ \pi_2 = T(\pi_2) \circ \gamma$. That is, the following diagram commutes:

$$\begin{array}{ccccc} X & \xleftarrow{\pi_1} & R & \xrightarrow{\pi_2} & Y \\ {\scriptstyle\alpha}\downarrow & & {\scriptstyle\exists\gamma}\downarrow & & \downarrow{\scriptstyle\beta} \\ T(X) & \xleftarrow{T(\pi_1)} & T(R) & \xrightarrow{T(\pi_2)} & T(Y) \end{array}$$

The arrow γ is called mediating morphism between α and β. We say that two states $x \in X$ and $y \in Y$ are bisimilar, and write $x \sim_{\alpha\beta} y$, if they are related by some bisimulations between (X, α) and (Y, β).

This definition of bisimulation was introduced in [1]. It gives a categorical formulation of a notion that has various manifestations in different kinds of state-transition system.

A bisimulation relation on a coalgebra (X, α) is any bisimulation between (X, α) and itself. A bisimulation equivalence is a bisimulation on a coalgebra that is also an equivalence.

We next list some properties of bisimulation. The proofs and more details can be found in [13].

Proposition 2.4 *The following properties hold:*

(i) The diagonal $\Delta_S = \{(s, s) \mid s \in S\}$ *is a bisimulation equivalence on any coalgebra with the state set* S.

(ii) If R *is a bisimulation between between two coalgebras* (X, α) *and* (Y, β)*, then* R^{-1} *is a bisimulation between* (Y, β) *and* (X, α).

(iii) A function $f : X \to Y$ *is a homomorphism between two coalgebras* (X, α) *and* (Y, β) *if and only if its graph* $Graph(f)$ *is a bisimulation between* (X, α) *and* (Y, β)*, where* $Graph(f) = \{(x, y) \in X \times Y \mid f(x) = y\}$.

The bisimilarity relation between (X, α) and (Y, β) is the union of all bisimulations which is the greatest bisimulation. Proposition 2.4(i) shows that the bisimilarity relation $\sim_\alpha$ on any coalgebra α is reflexive since $\Delta_S \subseteq \sim_\alpha$ (i.e. $\sim_{\alpha\alpha}$), whereas (ii) shows $\sim_\alpha$ is also symmetric. However, it need not be transitive, thus $\sim_\alpha$ is not necessarily an equivalence relation.

Naturally, instantiating coalgebraic bisimulation with fuzzy functors $\mathcal{F}^A$ and

$(\mathcal{P} \circ \mathcal{F})^A$, we can get the corresponding coalgebraic bisimulations for **FLTS** and **NFLTS**. Moreover, $\mathcal{F}^A$ and $(\mathcal{P} \circ \mathcal{F})^A$ -bisimulations have all properties in Proposition 2.4.

A pullback of functions $f : X \to Z$ and $g : Y \to Z$ is a triple $(P, k : P \to X, l : P \to Y)$ with $f \circ k = g \circ l$ such that for any set Q and functions $i : Q \to X$ and $j : Q \to Y$ with $f \circ i = g \circ j$ there exists a unique (so-called mediating) function $h : Q \to P$ with $k \circ h = i$ and $l \circ h = j$, where $(Z, f : X \to Z, g : Y \to Z)$ and $(P, k : P \to X, l : P \to Y)$ are called a cospan and span of objects Z and P between X and Y, respectively.

In **Set**, a pullback of functions $f : X \to Z$ and $g : Y \to Z$ always exists: the set

$$P = \{(x, y) \in X \times Y \mid f(x) = g(y)\}$$

with projections $\pi_1 : P \to X$ and $\pi_2 : P \to Y$, is a pullback of f and g.

A weak pullback is defined in the same way as a pullback, but without the requirement that the mediating function be unique. A functor T : **Set** $\to$ **Set** preserves weak pullbacks if applying T to a weak pullback $(P, k : P \to X, l : P \to Y)$, of functions $f : X \to Z$ and $g : Y \to Z$ yields again a weak pullback: $(T(P), T(k) : T(P) \to T(X), T(l) : T(P) \to T(Y))$, now of the functions $T(f) : T(X) \to T(Z)$ and $T(g) : T(Y) \to T(Z)$. The functor T weakly preserves a pullback of a diagram if it transforms it into a weak pullback of the transformed diagram.

If the functor preserves weak pullback, then the following proposition holds [13].

Proposition 2.5 *Assume that the functor T preserves weak pullbacks. Then:*

(i) The relational composition of two bisimulations is again a bisimulation, where the composition of relation $R \subseteq X \times Y$ and $Q \subseteq Y \times Z$ is $R \circ Q = \{(x, z) \mid \exists y \in Y \text{ such that } (x, y) \in R \text{ and } (y, z) \in Q\}$.

(ii) Bisimilarity on a coalgebra is an equivalence.

Proposition 2.5 gives a sufficient condition that $\sim_\alpha$ on any coalgebra α is an equivalence. Therefore, in order to show that $\mathcal{F}^A$-bisimulation and $(\mathcal{P} \circ \mathcal{F})^A$-bisimulation have also properties given by Proposition 2.5, it suffices to prove that these two functors preserve weak pullbacks.

3 Weak Pullback Preservations

The following three lemmas are helpful for proving that $\mathcal{F}^A$ and $(\mathcal{P} \circ \mathcal{F})^A$ preserve weak pullbacks. The first two are taken from [10].

Lemma 3.1 *In* **Set**, *a functor T preserves weak pullbacks if and only if it weakly preserves pullbacks.*

Lemma 3.2 *A* **Set** *endofunctor T preserves weak pullbacks if and only if for any cospan $(Z, c_1 : X \to Z, c_2 : Y \to Z)$ we have: Given u and v with $T(c_1)(u) = T(c_2)(v)$ then there exists a $w \in T\{(x, y) \mid c_1(x) = c_2(y)\}$ with $T(\pi_1)(w) = u$ and $T(\pi_2)(w) = v$.*

The following lemma is also necessary when we prove Theorem 3.4.

Lemma 3.3 *Let μ and ν, respectively, be fuzzy sets on X and Y such that $\mu(X) = \nu(Y)$. Then there exists a fuzzy set e on the product space $X \times Y$ such that for any $x \in X$ and $y \in Y$, $e(x, Y) = \mu(x)$ and $e(X, y) = \nu(y)$.*

Proof. For any $x \in X$ and $y \in Y$, let $e(x, y) = \mu(x) \wedge \nu(y)$. Then

$$\begin{aligned} e(x, Y) &= \sup_{y \in Y} e(x, y) \\ &= \sup_{y \in Y} \mu(x) \wedge \nu(y) \\ &= \mu(x) \wedge \sup_{y \in Y} \nu(y) \\ &= \mu(x) \wedge \nu(Y) \\ &= \mu(x) \wedge \mu(X) \\ &= \mu(x). \end{aligned}$$

Similarly, one can prove that $e(X, y) = \nu(y)$. □

It should be pointed out that Lemma 3.3 also holds in the probabilistic setting. However, its proof is rather complicated, we refer the reader to [14] for details.

Theorem 3.4 *The fuzzy functor $\mathcal{F}$ preserves weak pullback.*

Proof. It can be proven by Lemmas 3.1, 3.2 and 3.3, we skip. □

Corollary 3.5 *Functors $\mathcal{F}^A$ and $(\mathcal{P} \circ \mathcal{F})^A$ preserve weak pullbacks.*

Proof. By the following facts (see Lemma 3.5.4 and Lemma 3.5.7 in [14]):

(1) The powerset functor $\mathcal{P}$ and the constant exponent functor $\mathcal{I}d^A$ preserve weak pullbacks;

(2) the composition $T \circ G$ of functors T and G preserves weak pullback if T and G preserve weak pullbacks.

It is not hard to get that $\mathcal{F}^A$ and $(\mathcal{P} \circ \mathcal{F})^A$ preserve weak pullbacks since $\mathcal{F}^A = \mathcal{I}d^A \circ \mathcal{F}$ and $(\mathcal{P} \circ \mathcal{F})^A = \mathcal{I}d^A \circ (\mathcal{P} \circ \mathcal{F})$. □

4 Final Coalgebra

The final coalgebra is important, its members can be thought of as representing all possible "behaviours" of process, because members x and y of coalgebras α and β, respectively, are typically "behaviourally indistinguishable" precisely when they are identified by the unique homomorphisms from α and β to the final coalgebra γ. This section will investigate final $\mathcal{F}^A$(or $\mathcal{F}$)-coalgebra.

It is well known that if (X, α) is a final T-coalgebra, then $\alpha : X \to T(X)$ is an isomorphism in **Set** (see Theorem 9.1 in [13]), i.e. $X \cong T(X)$. However $X \ncong \mathcal{F}(X) = [0, 1]^X$ since the cardinal number of X is strictly smaller than the one of 2^X by Cantor theorem which shows there has not an injection from 2^X to

X, where 2^X is the powerset of X, and the cardinal number of 2^X is smaller than or equal to the one of $[0,1]^X$. Further, $X \ncong \mathcal{F}^A(X)$. Thus, in order to get a final $\mathcal{F}^A$(or $\mathcal{F}$)-coalgebra we need to consider some restricted classes. We define

$$\begin{aligned}
\mathcal{F}_n(X) &= \{\mu \mid \mu : X \to [0,1], \mu(X) = 1\} \\
\mathcal{F}_c(X) &= \{\mu \mid \mu : X \to [0,1], \mathrm{supp}(\mu) \text{ is countable}\} \\
\mathcal{F}_n^A(X) &= \{t \mid t : A \to (X \to [0,1]), t(a)(X) = 1 \ \text{ for any } a \in A\} \\
\mathcal{F}_c^A(X) &= \{t \mid t : A \to (X \to [0,1]),\ \mathrm{supp}(t(a)) \text{ is countable for any } a \in A\}
\end{aligned}$$

Theorem 4.1 *The normal functors $\mathcal{F}_n$, $\mathcal{F}_n^A$ and countable support functor $\mathcal{F}_c$ have final coalgebras. Moreover, if A is countable, then the functor $\mathcal{F}_c^A$ has also a final coalgebra.*

Proof. The final coalgebras for $\mathcal{F}_n$ and $\mathcal{F}_n^A$ are trivial, they equal the one element set, i.e. $(\{s\}, \alpha)$. In this case there is only an element μ in $\mathcal{F}_n(\{s\})$ such that $\mu(s) = 1$ and only an element t in $\mathcal{F}_n^A(\{s\})$ such that $t(a)(s) = 1$ for any $a \in A$. The final coalgebras for $\mathcal{F}_c$ and $\mathcal{F}_c^A$ are nontrivial. They can be proven by a basic fact (Theorem* 10.4, [13]): any bounded functor has a final coalgebra. The functors $\mathcal{F}_c$ and $\mathcal{F}_c^A$ are bounded (one can refer to the proof of Theorem 4.6 in [5] for detail). Hence, this conclusion holds. $\square$

Likewise, a final coalgebra does not exist for functor $(\mathcal{P} \circ \mathcal{F})^A$. However, we do not know whether a final coalgebra exists for some restricted classes of functor $(\mathcal{P} \circ \mathcal{F})^A$, for example, the functor $(\mathcal{P}_f \circ \mathcal{F}_c)^A$, where $\mathcal{P}_f$ is the finite powerset functor, i.e. $\mathcal{P}_f(X) = \{U \subseteq X \mid U \text{ is finite}\}$ for any set X.

Theorem 4.2 *(Full abstract) Let A be a countable set and (S, α) be an $\mathcal{F}_c^A$-coalgebra, (D, β) a final $\mathcal{F}_c^A$-coalgebra and $f : S \to D$ the unique homomorphism from (S, α) to (D, β). For any $x, y \in S$, $f(x) = f(y)$ if and only if $x \sim_\alpha y$.*

Proof. Let $x \sim_\alpha y$. Then (x, y) belongs to some bisimulation R from α to itself. Hence, there exists a coalgebra γ on R such that the projections give homomorphisms $\pi_1 : R \to S$ and $\pi_2 : R \to S$. Then $f \circ \pi_1 = f \circ \pi_2 =$ the unique homomorphism $R \to D$. So $f(x) = f \circ \pi_1(x, y) = f \circ \pi_2(x, y) = f(y)$.

For another direction, $f(x) = f(y)$ implies $x \sim_\alpha y$. First, by Proposition 2.4(iii), we have $x \sim_{\alpha\beta} f(x)$. Again, by Proposition 2.4 (ii), $Graph^{-1}(f)$ is a bisimulation from (D, β) to (S, α), so $f(y) \sim_{\beta\alpha} y$. Further, by Proposition 2.5(i) and $f(x) = f(y)$ we have $x \sim_\alpha y$. $\square$

This conclusion also holds for functors $\mathcal{F}_n, \mathcal{F}_n^A$ (A is countable) and $\mathcal{F}_c$. The element $f(s)$ in the final coalgebra can be viewed as the 'observable behaviour' of s.

5 Concrete Bisimulation

In this section, we present a way of relation lifting transforming a coalgebraic bisimulation to a concrete (fuzzy) bisimulation, where the notion of concrete bisimulation is due to Sokolova [14]. This approach of transforming a coalgebraic bisimulation to a concrete bisimulation also can be found in [12].

Definition 5.1 Let $R \subseteq S \times T$ be a relation. The relation R can be lifted to a relation $Rel(\mathcal{F}^{\mathcal{A}})(R) \subseteq \mathcal{F}^{\mathcal{A}}(S) \times \mathcal{F}^{\mathcal{A}}(T)$ defined by

$$(f, g) \in Rel(\mathcal{F}^{\mathcal{A}})(R) \Longleftrightarrow \exists k \in \mathcal{F}^{\mathcal{A}}(R),\ \mathcal{F}^{\mathcal{A}}(\pi_1)(k) = f \text{ and } \mathcal{F}^{\mathcal{A}}(\pi_2)(k) = g.$$

where $\pi_1 : R \to S$ and $\pi_2 : R \to T$.

Proposition 5.2 *A relation $R \subseteq S \times T$ is a bisimulation between the $\mathcal{F}^A$ coalgebras (S, α) and (T, β) if and only if*

$$(s, t) \in R \Longrightarrow (\alpha(s), \beta(t)) \in Rel(\mathcal{F}^{\mathcal{A}})(R). \tag{1}$$

Proof. Let R be a bisimulation between the $\mathcal{F}^{\mathcal{A}}$ coalgebras (S, α) and (T, β) and let $(s, t) \in R$. Let γ be the mediating coalgebra structure for R. Then $\gamma((s, t))$ satisfies $\mathcal{F}^{\mathcal{A}}(\pi_1)(\gamma((s, t))) = \alpha(s)$ and $\mathcal{F}^{\mathcal{A}}(\pi_2)(\gamma((s, t))) = \beta(t)$. Hence, by Definition 5.1 $(\alpha(s), \beta(t)) \in Rel(\mathcal{F}^{\mathcal{A}})(R)$.

For the opposite, assume R satisfies condition (3), For $(s, t) \in R$, since $(\alpha(s), \beta(t)) \in Rel(\mathcal{F}^{\mathcal{A}})(R)$, by Definition 5.1 we can find $k \in \mathcal{F}^{\mathcal{A}}(R)$ such that $\mathcal{F}^{\mathcal{A}}(\pi_1)(k) = \alpha(s), \mathcal{F}^{\mathcal{A}}(\pi_2)(k) = \beta(t)$. Let $\gamma((s, t)) = k$. Then R is a bisimulation with mediating coalgebra structure γ. □

By Proposition 5.2 and Definition 5.1 a relation $R \subseteq S \times T$ is a bisimulation between the $\mathcal{F}^{\mathcal{A}}$-coalgebras (S, α) and (T, β) if and only if for any $a \in A$ and $(s, t) \in R$, there exists $k \in \mathcal{F}^{\mathcal{A}}(R)$ such that

$$\begin{aligned} \sup_{\{t' \mid (s', t') \in R\}} k(a)(s', t') &= \alpha(s)(a)(s') \\ \sup_{\{s' \mid (s', t') \in R\}} k(a)(s', t') &= \beta(t)(a)(t'). \end{aligned}$$

We know for any $a \in A$, $k(a) \in \mathcal{F}(R)$, $\alpha(s)(a) \in \mathcal{F}(S)$ and $\beta(t)(a) \in \mathcal{F}(T)$. $k(a)$ can be extended to a fuzzy set e in $\mathcal{F}(S \times T)$ defined by $e(x, y) = k(a)(x, y)$ if $(x, y) \in R$ otherwise 0. Thus, we can get a definition of relation lifting as follows:

Definition 5.3 (Relation lifting) The relation $R \subseteq S \times T$ can be lifted to a relation $Rel(\mathcal{F})(R) \subseteq \mathcal{F}(S) \times \mathcal{F}(T)$ such that for any $\mu \in \mathcal{F}(S), \nu \in \mathcal{F}(T)$, $(\mu, \nu) \in Rel(\mathcal{F})(R)$ iff there exists a fuzzy set $e \in \mathcal{F}(S \times T)$ satisfying the following conditions.

(1) $e(s, T) = \mu(s)$, for any $s \in S$;
(2) $e(S, t) = \nu(t)$, for any $t \in T$;
(3) $e(s, t) = 0$, if $(s, t) \notin R$.

Since R^{-1} is also a bisimulation when R is a bisimulation and $Rel(\mathcal{F})(R^{-1}) = [Rel(\mathcal{F})(R)]^{-1}$, we can get the definition of concrete (fuzzy) bisimulations.

Definition 5.4 (Concrete bisimulation) Let (S, A, α) and (T, A, β) be two **FLTS**s. A relation $R \subseteq S \times T$ is called a (fuzzy) bisimulation between (S, A, α) and (T, A, β) if and only if for all $(s, t) \in R$ and $a \in A$,

(1) $s \xrightarrow{a} \mu$ implies $t \xrightarrow{a} \nu$ such that $(\mu, \nu) \in Rel(\mathcal{F})(R)$;

(2) $t \xrightarrow{a} \nu$ implies $s \xrightarrow{a} \mu$ such that $(\nu, \mu) \in Rel(\mathcal{F})(R^{-1})$, i.e. $(\mu, \nu) \in Rel(\mathcal{F})(R)$.

Similarly, applying functor $(\mathcal{P} \circ \mathcal{F})^A$ to the relation lifting, we can get the concrete (fuzzy) bisimulation between two **NFLTS**s.

Definition 5.5 (Concrete bisimulation) Let (S, A, α) and (T, A, β) be two **NFLTS**s. A relation $R \subseteq S \times T$ is called (fuzzy) bisimulation between (S, A, α) and (T, A, β) if and only if for all $(s, t) \in R$ and $a \in A$,

(1) if $s \overset{a}{\rightsquigarrow} \mu$, then there exists a ν with $t \overset{a}{\rightsquigarrow} \nu$ and $(\mu, \nu) \in Rel(\mathcal{F})(R)$;

(2) if $t \overset{a}{\rightsquigarrow} \nu$, then there exists a μ with $s \overset{a}{\rightsquigarrow} \mu$ and $(\mu, \nu) \in Rel(\mathcal{F})(R)$.

The following theorem shows the relation between **FLTS** and **NFLTS**.

Theorem 5.6 *For a $(\mathcal{P} \circ \mathcal{F})^A$-coalgebra, there exists an $\mathcal{F}^A$-coalgebra such that if s and t are bisimilar in $(\mathcal{P} \circ \mathcal{F})^A$-coalgebra then they are bisimilar in $\mathcal{F}^A$-coalgebra.*

Proof. Let (S, α) be a $(\mathcal{P} \circ \mathcal{F})^A$-coalgebra. Define $\beta(s)(a) = \sqcup_{\mu \in \alpha(s)(a)} \mu$ for any $s \in S$ and $a \in A$. Then it is easy to verify that (S, β) is an $\mathcal{F}^A$-coalgebra. Let $s \sim_\alpha t$. Then there exists a relation R containing (s, t) such that for any $\mu \in \alpha(s)(a)$, there exists a $\nu \in \alpha(t)(a)$ with $(\mu, \nu) \in Rel(\mathcal{F})(R)$. Further, one can prove that $(\sqcup_{i \in I} \mu_i, \sqcup_{i \in I} \nu_i) \in Rel(\mathcal{F})(R)$ provided that $(\mu_i, \nu_i) \in Rel(\mathcal{F})(R)$ for any $i \in I$. Thus, $s \sim_\beta t$ as desired. $\square$

A possible application of concrete bisimulation is the following.

Example 5.7 Assume that there is an unknown classical bacterial infect. The physicians by their experience think that three drugs, i.e. u_1, u_2, u_3, may be useful to this disease. Three possible negative symptoms, i.e. v_1, v_2, v_3, have also been taken into account. Further, the physicians consider the patient's condition roughly to be four cases, i.e. "poor", "fair", "good" and "excellent", which are denoted as q_1, q_2, q_3 and q_4, respectively. A treatment (or a negative symptom) may lead a state to multi-states with respective degrees. For example, $q_2 \xrightarrow{u_1[0.6]} q_3$ $(\alpha(q_2, u_1)(q_3) = 0.6)$ means that the patient's condition is changed from "fair" to "good" with possibility 0.6 after the drug u_1 is used, whereas $q_2 \xrightarrow{v_1[0.3]} q_1$ means that the patient's condition is changed from "fair" to "poor" with possibility 0.3 if the patient has the negative symptom v_1. The transition possibilities of these events among states are estimated by physicians. Let $S = \{q_1, q_2, q_3, q_4\}$ and $A = \{u_1, u_2, u_3, v_1, v_2, v_3\}$. Then we get an **FLTS** (S, A, α). A patient's initial condition may be "poor" and should become "fair", or "good", even "excellent" after certain treatment. When a patient's health becomes "fair", we naturally hope it to be better and better, say "excellent", instead of deteriorating. Analogously, if the patient's condition has been "excellent", it is desired to keep the good health and thus a supervisor is necessary to disable the events v_1, v_2, v_3 in case they are controllable. Xing et al. [16] developed a theory by using bisimulation to solve this problem better. We refer the reader to [16] for details. We would like to use concrete bisimulation in this paper to determine whether a supervisor exists between the specification system and the controlled system. Another possible application is that one can use concrete bisimulation to determine whether the behaviour of two fuzzy automata is identical.

6 Conclusion and Future Work

This paper uses the approach of coalgebra to investigate fuzzy transition systems, some important properties are obtained. Indeed these conclusions also appeared some literature about probabilistic transition systems. From the research point of view, it seems necessary to point out these conclusions. Of course, more attentions should focus on the differences between fuzzy transition systems and probabilistic transition systems. Hence, further work is necessary.

Further work includes: (1) consider whether main conclusions of this paper can be generalized into more general lattice-valued transition systems and whether there is a general result based on max-plus semirings that brings theses results under a common denominator? (2) consider fuzzy coalgebraic logic following the work of Doberkat on stochastic coalgebraic logic [6]. We know that main conclusions of [6] are based on the **ANL** the category of analytic spaces with surjective Borel maps as morphisms. Probably, these conclusions in the fuzzy case can hold under the general **Set** category.

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