A Short Introduction to Clones

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Abstract

In universal algebra, clones are used to study algebras abstracted from their signature. The aim of this paper is to give a brief introduction to the theory thereof. We give basic definitions and examples, and we present several results and open problems, selected from almost one hundred years of ongoing research. We also discuss what is arguably the most important tool to study clones – the Galois connection between operations and relations built on the notion of preservation. We conclude the paper by explaining the connection between clones and the closely related category theoretic notion of Lawvere theory.

Keywords: clones, relational clones, composition closed classes, polymorphisms, invariant relations

1   Introduction

A set of functions is a clone if and only if it is the set of non-nullary term functions of some algebra. For this reason, clones can be thought of as a representation of algebras which abstracts from their signature.

This paper aims at being a brief introduction to their theory. After discussing the basic definitions, we present some of the most celebrated results from almost one hundred years of ongoing research in the field. Although there are far more significant results than we can mention in this short survey, there are even more open problems. Indeed, as soon as the cardinality of the set A exceeds two, very little is known about the structure of the lattice of all clones on A. We will present some of the most outstanding open problems, giving the reader an impression of

what we know and don't know about the seemingly incomprehensible variety of clones.

To study clones, people have used many different techniques, coming from fields such as combinatorics, set theory or topology. One technique, however, stands above all others and is arguably the most important item in a clone theorist's toolbox. It is the Galois connection Pol-Inv between operations and relations based on the notion of preservation. This connection, nicknamed the "most basic Galois connection in algebra" in [27], has literally been used thousands of times and is the heart and soul of many important contributions to the theory of clones. We explain this technique, illustrate its usefulness and point to some of its many specifications and generalizations.

As clones are composition-closed sets of functions that contain projections, it is perhaps not surprising that the concept was generalized to category theory. In his 1963 PhD thesis [22], Bill Lawvere introduced the notion of algebraic theory (nowadays called Lawvere theory), which can be thought of as a category theoretic abstraction of clones. Shortly after Lawvere's thesis was published, clone theorists captured the same level of abstraction in the notion of abstract clone [7,44,46]. We present these notions and explain their connection, also discussing how concrete questions from classical clone theory (even if asked for finite base sets) benefit from a more abstract view.

The paper is structured as follows: Following the introductory words, we start our short survey in Section 2, which contains basic definitions and motivating examples. We continue in Section 3 by giving some examples of typical objects of research in the field, including a selection of results and open problems. The fourth section explains the Galois connection Pol-Inv, and the last section presents the connection between clones and Lawvere theories.

It is important to note that this survey is just a brief introduction and contains only some (sometimes almost randomly chosen) examples of the research that has been going on for several decades. For a complete overview of the theory we refer to the monographs [32,43,21].

2 What is a clone?

Given a universal algebra (A, F), where F is a set of finitary operation on the set A, one is often interested in the term functions of the algebra rather than in the set F itself. In particular, if two algebras have the same set of term functions, then one might consider their difference as a mere question of representation. This motivates a notion that describes precisely those sets of functions that can arise as sets of term functions of an algebra – and that is exactly what a clone is:

Definition 2.1 Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{N}_+ = \{1, 2, \ldots\}$. For $n \in \mathbb{N}$, denote by $O_A^{(n)}$ the set of n-ary operations on A and set $O_A := \bigcup_{n \in \mathbb{N}_+} O_A^{(n)}$. A subset $C \subseteq O_A$ is a called a *clone* (or *clone of operations*) if it contains all the projection mappings $\mathrm{pr}_i^k \colon A^k \to A \colon (x_1, \ldots, x_k) \mapsto x_i$ and is closed with respect to superposition of

operations in the following sense: For $f \in O_A^{(n)} \cap C$ and $f_1, \ldots, f_n \in O_A^{(k)} \cap C$, the k-ary operation $f(f_1, \ldots, f_n)$, defined by setting

$$f(f_1, \ldots, f_n)(x_1, \ldots, x_k) := f(f_1(x_1, \ldots, x_k), \ldots, f_n(x_1, \ldots, x_k)),$$

is also in C.

In other words, clones describe the behaviour of algebras independent from their signature. Given the existence of nullary operational symbols in a signature, one might wonder why nullary operations are excluded from the definition of a clone. To explain this convention, one can argue that, for a given element $a \in A$, it is not desirable to distinguish between clones that contain the actual constant a and clones that contain all non-nullary constant functions with value a. Universal algebraists almost universally agree to follow this convention, but it should be mentioned that including nullary operations would only cause minor and rather obvious changes to the theory (see the detailed discussion in [2], which can also be found in this volume). Thus, ultimately, including or excluding nullary operations is a question of taste rather than of mathematical substance.

It should be noted that the name "clone" is not as old as the concept it denotes. Indeed, as recollected in [9], the term seems to have first appeared in 1965, namely in the monograph of Paul Cohn [7], who attributed the notion to Ph. Hall. The mathematical object, on the other hand, has been studied at least since the 1920s, when Emil Post described all "closed classes" of functions on a two-element set (published around 20 years later in [33]). Another name for a clone appearing in the literature is "Funktionenalgebra" (function algebra), which is for instance used in the monographs [32] and [21]. Nowadays, however, the term clone is widely accepted.

Here are some basic examples:

(i) The set O_A of all (finitary non-nullary) operations on A and the set J_A of all projections on A are clones, called the *full clone* and the *trivial clone*, respectively.

(ii) As explained in the first paragraph of this section, clones arise as sets of term functions. In fact, a set C of finitary operations on A is a clone if and only if there exists an algebra $\mathbf{A} = (A, F)$ such that C is the set of term functions of $\mathbf{A}$.

(iii) Given an algebra $\mathbf{A} := (A, F)$, the set of finitary homomorphisms $\bigcup_{n \in \mathbb{N}_+} \mathrm{Hom}(\mathbf{A}^n, \mathbf{A})$ is a clone on A, called the *centralizer clone* of $\mathbf{A}$. Although these clones are in a sense universal (it is shown in [49] that every clone is abstractly isomorphic to a centralizer clone), only finitely many of the infinitely many clones on a given finite set are centralizer clones of some algebra ([6]). In the literature, these clones have obtained special attention, see for instance [39,40,42,25].

(iv) Given a topological space $(X, \mathcal{T})$, all continuous operations on X form a clone, called the clone of $(X, \mathcal{T})$. Starting with the monograph [45], there is a series

of papers discussing the topological information that this clone contains about the space. See [48] for a recent survey on the results in this direction.

(v) All idempotent operations on a set A form a clone, where f is said to be *idempotent* whenever $f(x, \ldots, x) = x$ holds for all $x \in A$.

(vi) Given a partially ordered set $(A, \leq)$, all operations on A monotone in each variable with respect to $\leq$ form a clone, called the clone of that partial order.

3   What is studied?

On any given domain A, the set of all clones on A forms a complete lattice with respect to inclusion. This follows directly from the following observation: the intersection of clones is again a clone and thus, for each set of operations $F \subseteq O_A$, there exists a least clone C that contains F (this clone is denoted by $\mathrm{Clo}(F)$ and called the clone *generated* by F).

This lattice, denoted by $\mathcal{L}_A$, is clone theory's main object of study. While there is also ongoing research for $|A| \geq \aleph_0$ (see [12] and the references therein), the vast majority of clone theory deals with the case that A is finite.

For $|A| = 1$, the lattice contains only one element. For $|A| = 2$, it was completely described by Emil Post in [33]. It is infinite, but countable and of relatively easy structure. It contains 8 atoms and 5 coatoms, and it is infinite only because of the existence of 8 infinite chains. Figure 1 displays the lattice.
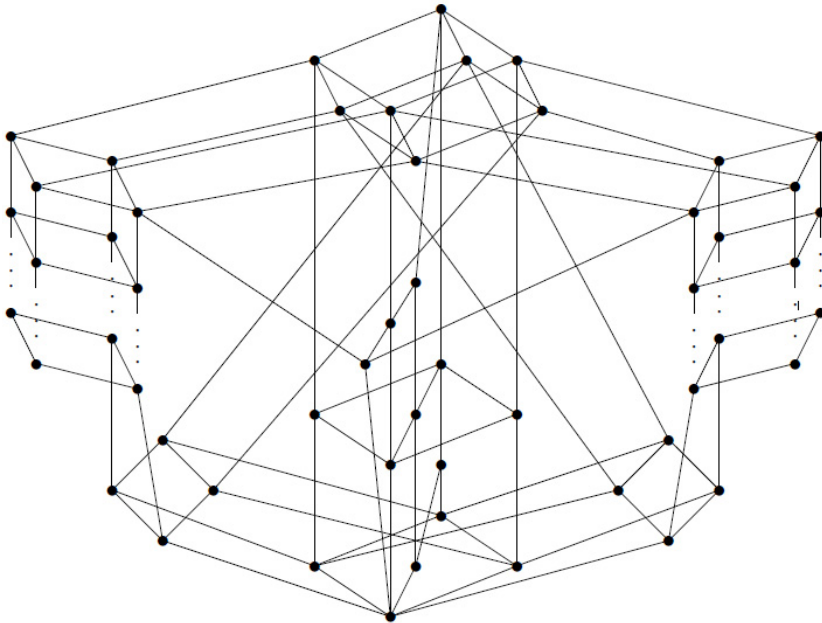

Fig. 1. The lattice of clones on a two-element set ([33])

However, as soon as $|A|$ exceeds two, things get more difficult. The lattice is

not countable any more, and very little is known about its form. Even for small cardinalities such as $|A| = 3$, it seems entirely hopeless to describe $\mathcal{L}_A$ completely. Besides the sheer sizes of these lattices, several results indicate that their structures are very complex. For instance, as soon as $|A| \geq 3$, there are no nontrivial lattice identities satisfied by $\mathcal{L}_A$ ([4]). Moreover, for $|A| \geq 4$, every countable product of finite lattices is a sublattice of $\mathcal{L}_A$ ([5]).

Hence, research aims at much more modest goals, like investigating particular parts or properties of these lattices. This work has been going on for several decades and is (to a large extent) summarized in the monographs [32,43,21]. Our goal in this section is to give the reader an impression of this research by choosing and discussing what we consider to be typical questions. Some of them have been solved, some of them are still open.

Definition 3.1 A clone on A is called *maximal* if it is a coatom in $\mathcal{L}_A$.

The interest in maximal clones is motivated by the completeness criterion that their knowledge provides. Indeed, a subset $F \subseteq O_A$ is complete (i.e., it generates the full clone O_A) if and only if, for every maximal clone M, there exists some $f \in F$ that is not contained in M.

Post's complete description of the lattice of clones on a two-element set gives us all maximal clones for $|A| = 2$. For $|A| = 3$, Jablonskiĭ characterized the maximal clones in 1958. It took another seven years before Ivo Rosenberg solved the problem for all finite A in his celebrated 1965 PhD thesis [36]. This result is often considered to be the greatest achievement in clone theory. We will state it in Section 4, where we get to know one of the major tools with which it was shown.

Knowing the maximal clones, an obvious question arises: can we also fully describe the minimal clones?

Definition 3.2 A clone on A is called *minimal* if it is an atom in $\mathcal{L}_A$.

Perhaps contrary to intuition, the answer to this question is no. While we know all minimal clones for $|A| \leq 3$ from [33,8], we have only partial results for $|A| \geq 4$. Indeed, the closest we come to a general description of all minimal clones is the following theorem, again found by Ivo Rosenberg.

Theorem 3.3 (Rosenberg's Classification Theorem (RCT) [37]) *A clone $C \subseteq O_A$ is minimal only if it is generated by an operation $f \in O_A \setminus J_A$ of one of the following types:*

(i) *a unary operation that is either a retraction (i.e., $f^2 = f$) or a permutation of prime order,*

(ii) *a binary idempotent operation (i.e., $f(x,x) = x$),*

(iii) *a majority operation (i.e., $f(x,x,y) = f(x,y,x) = f(y,x,x) = x$),*

(iv) *the (minority) operation $f(x,y,z) = x + y + z$ where $(A,+)$ is an elementary 2-group,*

(v) *a k-ary nontrivial semiprojection where $k \geq 3$ (i.e., there exists $i \in \{1,\ldots,k\}$ such that $f(x_1,\ldots,x_k) = x_i$ whenever $|\{x_1,\ldots,x_k\}| < k$).*

The cases (i) and (iv) ensure minimality, whereas the others do not. The problem of adding additional conditions to the other three cases such that this theorem becomes a characterization of minimal clones remains wide open.

Problem 3.4 *Characterize all minimal clones.*

For a summary of the research in this direction, we refer to the surveys [35,9].

However, not only the minimal clones are an open problem, we also do not know all clones that are directly below the maximal clones. These clones are called *submaximal* and are only known for $|A| \leq 3$ ([33,20]). Other than that, we know once again only partial results, see for instance [21,38,47].

Problem 3.5 *Find all submaximal clones on a given finite set A.*

All this shows how little we know about the structure of the lattice of clones. Going away from the hope of describing certain parts of the lattice precisely, one may also ask about the cardinality of a chosen interval. For instance, since we know all minimal clones on a three-element set, we may ask how many clones are above each atom. While this is easy for some of these clones, the last remaining bits of this question were only solved recently in [51].

Since the class of all clones on a given set is so complicated, it is also the ground for many questions concerning decidability. To give an impression of questions of this kind, let us state one (celebrated) result that we know and one that is still open.

Definition 3.6 A function $f \colon A^n \to A$ is said to be a *near-unanimity operation* if $f(x, \ldots, x, y) = f(x, \ldots, x, y, x) = \ldots = f(y, x, \ldots, x) = x$ for all $x, y \in A$.

Near-unanimity operations are of particular interest because clones that contain them have several special properties. Among other things, they are necessarily finitely generated (i.e., they are generated by a finite set of operations), and there are only finitely many clones that contain a given near-unanimity operation ([1]). They also play an important role in duality theory in the sense of [10]. Indeed, algebraically speaking, they could be considered the reason why dualities such as the Stone duality ([41]) or Priestley duality ([34]) work.

Theorem 3.7 ([24]) *Let A be a finite set and let C be a clone on A. It is decidable whether C contains a near-unanimity operation.*

One open problem reads as follows. Take a set of relations and consider the clone C of all functions that preserve these relations (what this means and why this is a clone will be explained in the next section).

Problem 3.8 *Is it decidable whether C is finitely generated?*

4   What is the main tool to study clones?

There are many different techniques that are used to study clones. In general, the study of clones on finite sets is mostly driven by combinatorial arguments, whereas

that of clones on infinite sets uses arguments from set theory and topology. In this section, we will describe one tool that has proven itself to be particularly useful for clones on both finite and infinite sets. It is a Galois connection between the set of finitary operations O_A and the set of finitary relations $R_A := \bigcup_{k \in \mathbb{N}_+} \mathcal{P}(A^k)$, based on the notion of preservation:

Definition 4.1 An operation $f \colon A^n \to A$ is said to *preserve* a k-ary relation σ on A, written $f \rhd \sigma$, if

$$\begin{pmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{1k} \end{pmatrix}, \ldots, \begin{pmatrix} a_{n1} \\ a_{n2} \\ \vdots \\ a_{nk} \end{pmatrix} \in \sigma \implies \begin{pmatrix} f(a_{11}, a_{21}, \ldots, a_{n1}) \\ f(a_{12}, a_{22}, \ldots, a_{n2}) \\ \vdots \\ f(a_{1k}, a_{2k}, \ldots, a_{nk}) \end{pmatrix} \in \sigma.$$

In that case, we also say that σ is *invariant* under f or that f is a *polymorphism* of σ.

If $r_1, \ldots, r_n \in A^k$, then one often simply writes $f(r_1, \ldots, r_n)$ to denote $\begin{pmatrix} f(r_{11}, r_{21}, \ldots, r_{n1}) \\ \vdots \\ f(r_{1k}, r_{2k}, \ldots, r_{nk}) \end{pmatrix}$ where $r_i = \begin{pmatrix} r_{i1} \\ \vdots \\ r_{ik} \end{pmatrix}$ for all $i \in \{1, \ldots, n\}$. Written in this way, we have $f \rhd \sigma$ if and only if $f(r_1, \ldots, r_n) \in \sigma$ for all $r_1, \ldots, r_n \in \sigma$. Alternatively, one can formulate the notion of preservation in terms of algebras or relational homomorphisms. That is, one has $f \rhd \sigma$ if and only if σ is a subuniverse of the algebra $(A, f)^k$, and this is also equivalent to f being a homomorphism from the relational structure $(A, \sigma)^n$ to (A, σ).

Definition 4.2 For $F \subseteq O_A$ and $R \subseteq R_A$, we define

$$\operatorname{Inv} F := \{\sigma \in R_A \mid \forall f \in F : f \rhd \sigma\},$$
$$\operatorname{Pol} R := \{f \in O_A \mid \forall \sigma \in R : f \rhd \sigma\}.$$

Evidently, Pol-Inv is a Galois connection between the set of finitary operations and relations on A. The concept of preservation is fundamental, but its history in clone theory is not entirely clear. To the best knowledge of the authors, it was first studied for unary operations on finite sets in [19]. The Galois connection as defined here seems to have first been studied in [11,3] for the case that A is finite. The case for arbitrary sets A is briefly discussed in [11] and intensively studied in [29,30]. However, as pointed out in [32, p. 20], preservation was also successfully used in the 1950s by A.V. Kuznecov.

Throughout the years, many specializations and generalizations of the connection were worked out. For an overview of these variants, we refer to the survey [31]. Recently, it was also generalized to a category-theoretic setting in [16].

Let us now take a look at the Galois closed classes of Pol-Inv. It is straightforward to check that $\operatorname{Pol} R$ is a clone for every set of relations (regardless of the

cardinality of A). Looking at the list of clones presented as examples in the first section, it is obvious that some of these clones can be described in exactly that way:

- the trivial clone on A is $\operatorname{Pol} R_A$,

- the full clone is $\operatorname{Pol} \emptyset$,

- the centralizer clone of an algebra (A, F) is $\operatorname{Pol} R$ where R is the set of graphs of the operations from F,

- the clone of idempotent operations is $\operatorname{Pol} R$ for $R := \{\{a\} \mid a \in A\}$,

- the clone of a partial order $\leq$ is $\operatorname{Pol}\{\leq\}$.

This raises the following question: is every clone the set of polymorphisms of some set of relations? On the most general level, the answer is a simple no. For instance, the clone of a topological space can almost never be written as $\operatorname{Pol} R$. We will understand the reason for this after characterizing the Galois closed classes of Pol-Inv. This requires us to present a few more definitions:

Definition 4.3 ([29,30]) A subset $R \subseteq R_A$ is called a *clone of relations* on A if and only if

(i) $\emptyset \in R$,

(ii) R is closed under *general superposition*, that is, the following holds: For an arbitrary index set I, let $\sigma_i \in R^{(k_i)}$ ($i \in I$) and let $\varphi\colon k \to \alpha$ and $\varphi_i\colon k_i \to \alpha$ be mappings where α is some cardinal number. Then the relation $\bigwedge^{\varphi}_{(\varphi_i)_{i \in I}}(\sigma_i)_{i \in I}$, defined by

$$\bigwedge^{\varphi}_{(\varphi_i)_{i \in I}}(\sigma_i)_{i \in I} := \{r \circ \varphi \mid \forall i \in I : r \circ \varphi_i \in \sigma_i,\, r \in A^\alpha\},$$

belongs to R.

For $R \subseteq R_A$, denote by $\operatorname{CLO}(R)$ the least clone of relations on A that contains R.

If one allows nullary operations in O_A (and hence in clones), then one has to omit condition (i) from this definition (indeed, we have $\emptyset \in \operatorname{Inv} F$ if and only if F contains no constants, see [2] for details).

Definition 4.4 Let $F \subseteq O_A$, $R \subseteq R_A$, $s \in \mathbb{N}_+$. We define the following local closure operators:

$$\text{s-Loc}\, F := \{f \in O_A^{(n)} \mid n \in \mathbb{N}_+, \forall r_1, \ldots, r_n \in A^s :$$
$$\exists f' \in F : f(r_1, \ldots, r_n) = f'(r_1, \ldots, r_n)\},$$
$$\text{s-LOC}\, R := \{\sigma \in R_A \mid \forall B \subseteq \sigma, |B| \leq s : \exists \sigma' \in R : B \subseteq \sigma' \subseteq \sigma\}.$$

Furthermore, let

$$\operatorname{Loc} F := \bigcap_{s \in \mathbb{N}_+} \text{s-Loc}\, F \text{ and } \operatorname{LOC} R := \bigcap_{s \in \mathbb{N}_+} \text{s-LOC}\, R.$$

Now we can present the main result for the Galois connection Pol-Inv.

Theorem 4.5 ([29,30]) *For $F \subseteq O_A$, $R \subseteq R_A$, we have*

(i) $\operatorname{Pol} \operatorname{Inv} F = \operatorname{Loc} \operatorname{Clo}(F)$,

(ii) $\operatorname{Inv} \operatorname{Pol} R = \operatorname{LOC} \operatorname{CLO}(R)$.

Thus, the Galois closed classes of Pol-Inv are precisely the locally closed clones of operations and the locally closed clones of relations, respectively. This explains why the clone of a topological space can almost never be described as the set of polymorphisms of some relations: it is almost never locally closed. As mentioned in the last section, however, most research in the field of clone theory assumes a finite base set A. Once we restrict ourselves to this framework, we can completely forget the local closure operators. Indeed, for a finite A, we have $\operatorname{Pol} \operatorname{Inv} F = \operatorname{Clo}(F)$ and $\operatorname{Inv} \operatorname{Pol} R = \operatorname{CLO}(R)$, that is, the Galois closed classes are precisely the clones of operations and the clones of relations, respectively. In particular, this means that every clone C on a finite set arises as the set of polymorphisms of some relations. Moreover, in case that A is finite, we can also simplify the definition of a clone insofar as we do not need to use the rather unappealing general superposition from Definition 4.3:

Definition 4.6 A relation $\sigma \in R_A^{(k)}$ is called *trivial* if it is empty or a *diagonal relation on A*, that is, there exists an equivalence relation $\theta \subseteq \{1, \ldots, k\} \times \{1, \ldots, k\}$ with

$$\sigma = \left\{ \begin{pmatrix} a_1 \\ \vdots \\ a_k \end{pmatrix} \in A^k \,\middle|\, \forall (i,j) \in \theta : a_i = a_j \right\}.$$

Proposition 4.7 *Let A be finite. A subset $R \subseteq R_A$ is a clone of relations if and only if it contains all trivial relations, is closed under direct (Cartesian) products, under intersection of (any family of) relations of the same arity, under permutation of coordinates, and under projections onto a set of coordinates.*

But why exactly is the Galois connection Pol-Inv such a powerful tool? There are several reasons. First, clone theory is often about the question whether a given operation f is generated by a given set of operations, that is, whether one has $f \in \operatorname{Clo}(F)$. By the Galois connection, $f \in \operatorname{Clo}(F)$ implies that every relation preserved by F is also preserved by f. Hence, one can show $f \notin \operatorname{Clo}(F)$ by finding a relation σ that is preserved by F but not by f. If A is finite, the existence of such relation is even equivalent to $f \notin \operatorname{Clo}(F)$ (on infinite A, it might happen that f is not generated by F but still preserves all $\sigma \in \operatorname{Inv} F$).

Example 4.8 Consider $f \in O_{\mathbb{N}}^{(2)}$ defined by $f(x, y) := x^y$. One might suspect that f is generated by addition and multiplication. However, we can show that this is not the case by observing that the binary relation $\sigma := \{ \binom{x}{y} \mid x - y \equiv 0 \pmod 3 \}$ is preserved by addition and multiplication but not by f.

This little trick has literally been used thousands of times and is arguably the most efficient technique to achieve this kind of result.

Also, every result that one might want to prove about the lattice of clones of operations can be proven by showing the dual result in the lattice of clones of

relations. In fact, the most celebrated result mentioned in this survey was obtained in exactly that way: Ivo Rosenberg did not directly determine the maximal clones of operations; he determined the minimal clones of relations. Having the Galois connection at his hand, this did not only give him the result he was after, but also a very elegant description of the maximal clones. He listed six types of relations on A such that a clone is maximal if and only if it is the set of polymorphisms for some relation from the list.

Definition 4.9 Let σ be a k-ary relation on A. We say that σ is

(i) *totally symmetric* if it is invariant under permutation of arguments,

(ii) *prime affine* if there exists some prime p and a binary operation $+\colon A^2 \to A$ such that $(A, +)$ is an abelian p-group and $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} \in \sigma$ if and only if $a_1 + a_2 = a_3 + a_4$,

(iii) *totally reflexive* if $\iota_k^A \subseteq \sigma$ where

$$\iota_k^A := \{(a_1, \ldots, a_k) \in A^k \mid \exists i, j \in \{1, \ldots, k\} : i \neq j \wedge a_i = a_j\},$$

(iv) *central* if it is totally symmetric, totally reflexive and

$$C(\sigma) := \{a \in A \mid \forall a_2, \ldots, a_k \in A : (a, a_2, \ldots, a_k) \in \sigma\}$$

(called the *center* of σ) is a proper nonempty subset of A, and

(v) *k-regularly generated* for $k \in \{1, \ldots, |A|\}$ if there exists $\lambda \in \mathbb{N}_+$ and a surjection $\varphi \colon A \to \{1, \ldots, k\}^\lambda$ such that $\sigma = \varphi^{-1}(\omega_\lambda)$, where is the k-ary relation on $\{1, \ldots, k\}^\lambda$ defined by

$$(\alpha_1, \ldots, \alpha_k) \in \omega_\lambda :\Longleftrightarrow \forall i \in \{1, \ldots, \lambda\} : (\mathrm{pr}_i^\lambda(\alpha_1), \ldots, \mathrm{pr}_i^\lambda(\alpha_k)) \in \iota_k^{\{1, \ldots, k\}}.$$

Theorem 4.10 ([36]) *Let A be a finite set. A clone $C \subseteq A$ is maximal if and only if it is of the form $\mathrm{Pol}\, \sigma$, where σ is a k-ary relation of one of the following six types:*

1. *a partial order with least and greatest element,*

2. *the graph of a permutation of prime order,*

3. *a non-trivial equivalence relation,*

4. *a prime-affine relation,*

5. *a central relation,*

6. *a k-regularly generated relation.*

The Galois connection also gives us an additional motivation for Problem 3.4. On a finite set A, finding all minimal clones of operations is equivalent to finding all maximal clones of relation. Hence, analogous to the completeness criterion for operations that was established by finding all maximal clones of operations, solving Problem 3.4 would give us a completeness criterion for relations.

5 What's the connection to Lawvere theories?

Since clones are sets of functions closed under composition, it is not surprising that they have been generalized to the world of category theory. Indeed, in 1963, Bill Lawvere gave a category theoretic formulation of a clone in his PhD thesis [22].

Definition 5.1 Consider $\aleph_0$ as a skeleton of the category of finite sets and all functions between them. A *Lawvere theory* is a small category $\mathcal{L}$ with (necessarily strictly associative) finite products such that there exists a strict finite-product preserving identity-on-objects functor $I : \aleph_0^{op} \longrightarrow \mathcal{L}$.

Thus the objects of any Lawvere theory $\mathcal{L}$ are exactly the objects of $\aleph_0$, which we may denote by the natural numbers in the obvious way. For most mathematical purposes, one understands a Lawvere theory by study of its models.

Definition 5.2 A *model* of a Lawvere theory $\mathcal{L}$ in a category $\mathcal{C}$ with finite products is a finite-product preserving functor $M : \mathcal{L} \longrightarrow \mathcal{C}$.

We can now formulate the connection between clones and Lawvere theories.

Proposition 5.3 *A subset $C \subseteq O_A$ is a clone on A if and only if there exists a model $M : \mathcal{L} \longrightarrow \mathcal{S}et$ of a Lawvere theory $\mathcal{L}$ in $\mathcal{S}et$ such that*

$$C = \bigcup_{n \in \mathbb{N}_+} \{M(f) \mid f \in \mathcal{L}(n,1)\}.$$

We again encounter the tradition of excluding nullary operations in clones that we have already mentioned after Definition 2.1. Since the category theoretic tradition is different insofar as Lawvere theories may include nullary operations, we have to require $M(f) \notin C$ for $f \in \mathcal{L}(0,1)$, which makes the connection between clones and models of Lawvere theories somewhat less natural.

Since it can sometimes be advantageous to treat clones abstractly, universal algebraists invented the definition of an abstract clone which turns out to be equivalent to the notion of Lawvere theory (again, of course, up to a caveat about nullary operations).

Definition 5.4 ([7,44,46]) An *abstract clone* consists of

- for each $n \in \mathbb{N}_+$, a set C_n, the elements of which are called n-ary operation symbols,
- for each $n \in \mathbb{N}_+$ and $i \in \{1, \dots, n\}$, an n-ary operation symbol pr_i^n,
- for each n-ary operation symbol g and m-ary operation symbols $f_1, \dots, f_n$, an m-ary operation symbol $g(f_1, \dots, f_n)$

such that, subject to the composites being defined,

- $(h(g_1, \dots, g_n))(f_1, \dots, f_m) = h(g_1(f_1, \dots, f_m), \dots, g_n(f_1, \dots, f_m))$,
- $\mathrm{pr}_i^n(f_1, \dots, f_n) = f_i$ for all $i \in \{1, \dots, n\}$,
- $f(\mathrm{pr}_1^n, \dots, \mathrm{pr}_n^n) = f$.

When abstract clones were introduced, the notion of Lawvere theory was acknowledged and the similarities were pointed out. In fact, Taylor already proved the equivalence of both notions in [44]. Since then, abstract clones have been used in the literature, for instance for the study of minimal clones ([28,23,50]). This is not surprising since, unlike maximality, minimality of a clone is an abstract property, that is, a clone is minimal if and only if its abstract clone has no proper (abstract) subclones. In the same way, a Lawvere theory $\mathcal{L}$ and all its models are minimal if and only if $\mathcal{L}$ has no nontrivial wide subcategories. Thus, we can reformulate Problem 3.4 as follows.

Problem 5.5 *Characterize all minimal Lawvere theories.*

Although universal algebraists are well aware of the equivalence between abstract clones and Lawvere theories as well as that of concrete clones and models of Lawvere theories in $\mathcal{S}et$, they almost never use Lawvere theories and their models for the study of the lattice of clones on a given set. Recently, however, there has been a step in this direction. Started in [25] and continued in [17], it was outlined how treating clones as models of Lawvere theories allows one to dualize them and to use the dualized notion to examine some of them in a more convenient way. Applications of this approach can be found in [26,18,15,14], for instance. It seems that this cannot be done in entirely universal algebraic terms, as the concept of duality is (in this generality) intrinsically tied to category theory. See also [13] in this volume.

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