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PSPACE-hardness of Two Graph Coloring Games

Eurinardo Costa[1] Victor Lage Pessoa[1] Rudini Sampaio[1] Ronan Soares[1]

Departamento de Computação
Departamento de Estatística e Matemática Aplicada
Universidade Federal do Ceará, Fortaleza, Brazil

Abstract

In this paper, we answer a long-standing open question proposed by Bodlaender in 1991: the game chromatic number is PSPACE-hard. We also prove that the game Grundy number is PSPACE-hard. In fact, we prove that both problems (the graph coloring game and the greedy coloring game) are PSPACE-Complete even if the number of colors is the chromatic number. Despite this, we prove that the game Grundy number is equal to the chromatic number for several superclasses of cographs, extending a result of Havet and Zhu in 2013.

Keywords: Coloring game, game chromatic number, greedy coloring, Grundy number, PSPACE-hardness.

1 Introduction

In the graph coloring game, given a graph G and a set C of integers, two players (Alice and Bob) alternate their turns (starting with Alice) in choosing an uncolored vertex to be colored by an integer of C not already assigned to one of its colored neighbors. In the greedy coloring game, the vertices must be colored by the least possible integer of C. Alice wins if all vertices are successfully colored. Otherwise, Bob wins the game. The game chromatic number $\chi_g(G)$ and the game Grundy number $\Gamma_g(G)$ are the least numbers of colors in the set C for which Alice has a winning strategy in the graph coloring game and the greedy coloring game, respectively.

Clearly, $\chi_g(G) \geq \chi(G)$ and $\chi(G) \leq \Gamma_g(G) \leq \Gamma(G)$, where $\chi(G)$ is the chromatic number of G and $\Gamma(G)$ is the Grundy number of G (the maximum number of colors that can be used by a greedy coloring of G).

[1] Emails: eurinardo@lia.ufc.br, victorlagepessoa@gmail.com, rudini@dc.ufc.br, ronan.soares@ufc.br

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The graph coloring game was first considered by Brams about 38 years ago in the context of coloring maps and was described by Gardner in 1981 in his "*Mathematical Games*" column of Scientific American [9]. It remained unnoticed until Bodlaender [2] reinvented it in 1991 as the "*Coloring Construction Game*". Bodlaender left the complexity of the problem as an open question. In his own words, "*the complexity of the Color Construction Game is an interesting open problem*".

Since then, the graph coloring game became a very active topic of research. In 1993, Faigle et al. [8] proved that $\chi_g(G) \leq 4$ for forests and, in 2007, Sidorowicz [18] proved that $\chi_g(G) \leq 5$ for cacti. In 1994, Kierstead and Trotter [13] proved that $\chi_g(G) \leq 7$ for outerplanar graphs. In 1999, Dinski and Zhu proved that $\chi_g(G) \leq k(k+1)$ for every graph with acyclic chromatic number k [6]. In 2000, Zhu proved that $\chi_g(G) \leq 3k+2$ for partial k-trees [20]. For planar graphs, Zhu [21] proved in 2008 that $\chi_g(G) \leq 17$, Sekiguchi [17] proved in 2014 that $\chi_g(G) \leq 13$ if the girth is at least 4 and Nakprasit et al. [15] proved in 2018 that $\chi_g(G) \leq 5$ if the girth is at least 7. In 2008, Bohman, Frieze and Sudakov [3] investigated the asymptotic behavior of $\chi_g(G_{n,p})$ for the random graph $G_{n,p}$.

Despite this, little progress was obtained in the complexity of the problem. In 2015, Dunn, Larsen, Lindke, Retter and Toci [7] investigated it and stated, in their own words, that "*more than two decades later, this question remains open*". They obtained partial progress on the complexity of the game chromatic number of a forest.

The greedy coloring game and the game Grundy number were introduced by Havet and Zhu [11] in 2013. They proved that $\Gamma_g(G) \leq 3$ in forests and $\Gamma_g(G) \leq 7$ in partial 2-trees. They also posed two questions regarding the game Grundy number:

- Problem 5 of [11]: $\chi_g(G)$ can be bounded by a function of $\Gamma_g(G)$?
- Problem 6 of [11]: Is it true that $\Gamma_g(G) \leq \chi_g(G)$ for every graph G?

In 2015, Krawczyk and Walczak [14] answered Problem 5 of [11] in the negative: $\chi_g(G)$ is not upper bounded by a function of $\Gamma_g(G)$. To the best of our knowledge, Problem 6 of [11] is still open.

In this paper, we prove that the graph coloring game and the greedy coloring game are PSPACE-Complete problems even if the number of colors is the chromatic number. That is, the game chromatic number and the game Grundy number are PSPACE-hard.

In 2013, Havet and Zhu showed that $\Gamma_g(G) = \chi(G)$ for any cograph (see Proposition 9 of [11]). In this paper, we extend this by proving that $\Gamma_g(G) = \chi(G)$ for several known superclasses of cographs, as P_4-sparse graphs, P_4-tidy graphs and P_4-laden graphs. For those graph classes, $\Gamma(G)$ can be larger than $\chi(G)$ as much as desired. Moreover, for those graph classes, Alice has a winning strategy with $\chi(G)$ colors in the greedy coloring game even if Bob starts the game and can pass any turn.

2 Game chromatic number is PSPACE-hard

As pointed out by Zhu [19], the graph coloring game "*exhibits some strange properties*" and the following "naive" question is still open (Question 1 of [19]): Does Alice have a winning strategy for the coloring game with $k+1$ colors if she has a winning strategy with k colors?

With this, we can define two decision problems for the coloring game: given a graph G and an integer k,

- (Game Coloring Problem 1) $\chi_g(G) \leq k$?
- (Game Coloring Problem 2) Does Alice have a winning strategy with k colors?

Game Coloring Problems 1 and 2 are equivalent if and only if Question 1 of [19] is true.

In this section, we prove that the following more restricted Game Coloring Problem 3 is PSPACE-Complete: given a graph G and its chromatic number $\chi(G)$,

- (Game Coloring Problem 3) $\chi_g(G) = \chi(G)$?

It is easy to see that Game Coloring Problems 1 and 2 are generalizations of Game Coloring Problem 3, since both problems are equivalent to it for $k = \chi(G)$. Notice that $\chi_g(G) \leq k = \chi(G)$ if and only if $\chi_g(G) = \chi(G)$, which is true if and only if Alice has a winning strategy with $k = \chi(G)$ colors. Then the PSPACE-hardness of Game Coloring Problem 3 implies the PSPACE-hardness of Game Coloring Problems 1 and 2.

We obtain a reduction from the POS-CNF problem, which is known to be log-complete in PSPACE [16]. In POS-CNF, we are given a set $\{X_1, \ldots, X_N\}$ of N variables and a CNF formula (conjunctive normal form) with M clauses $C_1, \ldots, C_M$, in which only positive variables appear (that is, no negations of variables). Two players (Alice and Bob) alternate turns setting a previously unset variable True or False. After all N variables are set, Alice wins if and only if the formula is True. Clearly, since there are only positive variables, we can assume that Alice and Bob always set variables True and False, respectively. We may assume that no clause contains all variables, since it is trivially satisfied.

One important ingredient of the reduction is the graph F_1 of Figure 2, which has a universal vertex s, two cliques $K^{(1)}$ and $K^{(2)}$ both with $\beta-1$ vertices and two independent sets $I^{(1)} = \{r_1, \ldots, r_{2\beta}\}$ and $I^{(2)} = \{t_1, \ldots, t_{2\beta}\}$ both with 2β vertices. Each vertex of $I^{(i)}$ is adjacent to all vertices in $K^{(i)}$ for $i \in \{1,2\}$. We start by proving that, with $2\beta-1$ colors, Alice wins the game in F_1 if and only if she is the first one to play and, in this case, she must color vertex s first.

Lemma 2.1 *Alice has a winning strategy for the graph F_1 of Figure 1 in the graph coloring game with $2\beta-1$ colors if and only if she is the first to play. In this case, she must color vertex s first. Moreover, a winning strategy also exists even if the opponnent can pass any turn.*

Proof. Without loss of generality, suppose that Alice in her first turn colors some vertex in $K^{(2)} \cup I^{(2)}$. Then, Bob has a winning strategy that consists, on his every

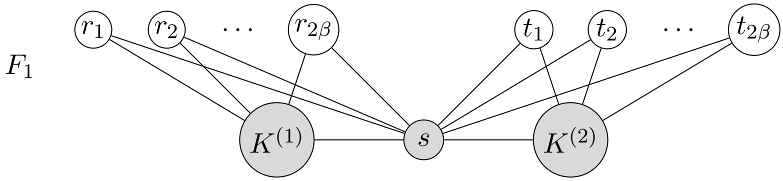

Fig. 1. The graph F_1. $K^{(1)}$ and $K^{(2)}$ are cliques of size $\beta - 1$. Vertices $r_1, \ldots, r_{2\beta}$ are adjacent to every vertex of $K^{(1)}$. Vertices $t_1, \ldots, t_{2\beta}$ are adjacent to every vertex of $K^{(2)}$. Vertex s is universal.

turn, to color one vertex of $I^{(1)}$ with a new color. If Bob uses this strategy, on the end of his β turn, we have that: $I^{(1)}$ has at least β colored vertices each with a different color. Since $K^{(1)} \cup \{s\}$ is a clique with β vertices adjacent to every vertex of $I^{(1)}$, then it is not possible to color F_1 with $2\beta - 1$ colors.

We now show that there is a winning strategy strategy for Alice, using $2\beta - 1$ colors. In her first turn Alice colors vertex s with color 1. For each subsequent turn Alice chooses the least available color to color a vertex following this order of priority: (a) a vertex of $K^{(i)}$, if Bob colored a vertex of $K^{(i)} \cup I^{(i)}$, (b) if all vertices of one of the cliques $K^{(i)}$ are colored, she colors an uncolored vertex of the other clique, and (c) any vertex of F_1, if all vertices of $K^{(1)}$ and $K^{(2)}$ are colored.

Without loss of generality suppose that both players play only on $K^{(1)} \cup I^{(1)} \cup \{s\}$. Consider some turn in which $K^{(1)}$ is completely colored. Note that, in this turn, Bob has played at most $\beta - 1$ times. Then, they have used β colors in $K^{(1)} \cup \{s\}$ and at most $\beta - 1$ colors in $I^{(1)}$. Since each vertex in $I^{(1)}$ can share the same colors of any vertex in $I^{(1)}$, they have used at most $\beta + \beta - 1$ colors and all uncolored vertices of $I^{(1)}$ can be colored with any color already used in $I^{(1)}$. □

Theorem 2.2 *Given a graph G, deciding whether $\chi_g(G) = \chi(G)$ is PSPACE-Complete. Consequently, if k is an integer, deciding whether $\chi_g(G) \leq k$ or deciding if Alice has a winning strategy with k colors are PSPACE-Complete problems.*

Proof. **[Sketch]** It is not difficult to see that the three decision problems are in PSPACE. Given a POS-CNF formula with N variables $X_1, \ldots, X_N$ and M clauses $C_1, \ldots, C_M$, let p_j (for $j = 1, \ldots, M$) be the size of clause C_j and let q_i (for $i = 1, \ldots, N$) be the number of clauses containing the variable X_i. Let $p = \max_{j=1,\ldots,M}\{p_j\}$ and $q = \max_{i=1,\ldots,N}\{q_i\}$. That is, every clause has at most p variables and every variable appears in at most q clauses. Let $\beta = \max\{p, q, 4N\} + 2$. We will construct a graph G such that $\chi(G) = 2\beta - 1$ and $\chi_g(G) = 2\beta - 1$ if and only if Alice has a winning strategy for the POS-CNF formula.

Initially, the constructed graph G is the graph F_1 of Figure 1. Add to G a new vertex y. For every variable X_i, create a vertex x_i in G. For every clause C_j, we will create a *clause clique* for it. First create a clique with vertices $\ell_{j,1}, \ldots, \ell_{j,p_j}$ and join $\ell_{j,k}$ to x_i with an edge if and only if both are associated to the same variable, for $k = 1, \ldots, p_j$. Also add the new vertex $\ell_{j,0}$ (which is not associated to variables) and join it with an edge to the vertex y. For every vertex $\ell_{j,k}$ ($j = 1, \ldots, M$ and $k = 0, \ldots, p_j$), replace it by two true-twin vertices $\ell'_{j,k}$ and $\ell''_{j,k}$, which are adjacent vertices with same neighborhood of $\ell_{j,k}$. Moreover, add to the clause clique of C_j a clique L_j with size $2(\beta - p_j) - 3 \geq 4N$ and join all vertices of L_j to s. With this,

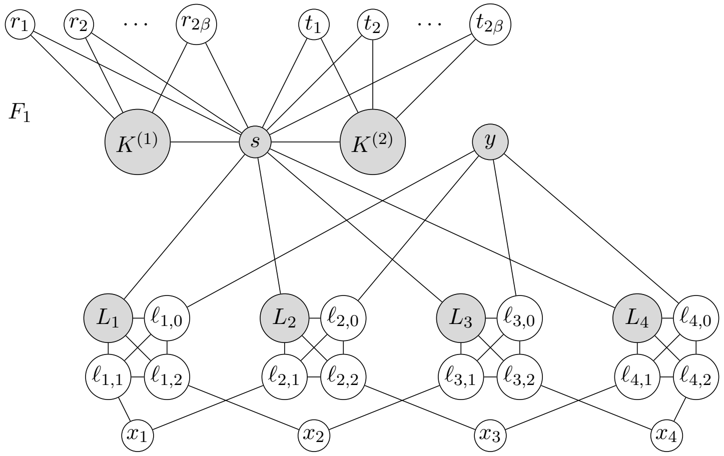

Fig. 2. Constructed graph G for the formula $(X_1 \vee X_2) \wedge (X_1 \vee X_3) \wedge (X_2 \vee X_4) \wedge (X_3 \vee X_4)$. Recall that each vertex $\ell_{j,k}$ represents two true-twins $\ell'_{j,k}$ and $\ell''_{j,k}$; L_1, L_2, L_3, L_4 are cliques with 29 vertices. Bob has a winning strategy with 35 colors in the graph coloring game.

all clause cliques have exactly $2\beta - 1$ vertices.

Figure 2 shows the constructed graph G for the formula $(X_1 \vee X_2) \wedge (X_1 \vee X_3) \wedge (X_2 \vee X_4) \wedge (X_3 \vee X_4)$. Notice that Bob has a winning strategy: if Alice sets X_1 True, Bob sets X_4 False; if Alice sets X_4 True, Bob sets X_1 False; if Alice sets X_2 True, Bob sets X_3 False; if Alice sets X_3 True, Bob sets X_2 False. In the reduction of this example, we have $N = 4$ variables, $M = 4$ clauses, $p = 2$, $q = 2$, $\beta = 4N + 2 = 18$ and the cliques L_1 to L_M have $2(\beta - p) - 3 = 29$ vertices each.

It is easy to verify that $\chi(G) = 2\beta - 1$. Color s with color 1, color the vertex y and every vertex x_i $(i = 1, \ldots, n)$ with color $2\beta - 1$. The vertices of each clique of F_1 can be colored with colors 2 to β and each independent set of F_1 can be colored using color $\beta + 1$. For every $j = 1, \ldots, M$, color the vertices $\ell'_{j,k}$ and $\ell''_{j,k}$ with colors $2k + 1$ and $2k + 2$ $(k = 0, \ldots, p_j)$. Finally, color the vertices of the clique L_j using the colors $2p_j + 3, \ldots, 2\beta - 1$. Since the cliques representing clauses contains $2\beta - 1$ vertices, then $\chi(G) = 2\beta - 1$.

In the following, we show that Alice has a winning strategy in the graph coloring game if and only if she has a winning strategy in POS-CNF. From Lemma 2.1, in her first move, Alice must color vertex s of F_1, since otherwise F_1 cannot be colored with $2\beta - 1$ colors. Also notice that all variable vertices have degree at most $2q < 2\beta - 1$ and then can always be colored with a color in $\{1, \ldots, 2\beta - 1\}$. We show that vertex y will be colored in the three first turns. With this, every vertex of a clause clique has degree exactly $2\beta - 1$. In order to be colored using the colors of $\{1, \ldots, 2\beta - 1\}$, Alice must guarantee that all colors appearing in the outside neighbors of a clause clique also appears inside the clique. On the other hand, we show that Bob's strategy is making all outside neighbors of a clause clique to be colored with the same color of s (which will represent False in POS-CNF) and thus impeding Alice of using this color inside the clause clique.

We first show that if Bob has a winning strategy in the POS-CNF game, then $\chi_g(G) > 2\beta - 1$. Assume that Bob wins in the POS-CNF game and Alice starts by coloring s with color 1 (recall Lemma 2.1). Bob can use the following strategy. In his first turn, he colors y with color 1, then at each subsequent turn: if Alice colors a vertex in $N[x_i]$ (the closed neighborhood of x_i) for some i, Bob considers that she marked X_i true in the POS-CNF game and colors with color 1 the vertex x_j representing the literal X_j chosen by him in his winning POS-CNF strategy; if Alice does not color any vertex in $N[x_i]$ for some i, then Bob plays as if Alice has passed her turn in the POS-CNF game. Since Bob has a winning strategy in the POS-CNF game, at some point all literals of some clause will be marked false. This means that all neighbors outside some clause clique will be colored with color 1. Since the clique has $2\beta - 1$ vertices and color 1 cannot be used, we have that $\chi_g(G) > 2\beta - 1$.

We now show that if Alice has a winning strategy in the POS-CNF game then $\chi_g(G) = 2\beta - 1$. Assume that Alice wins in the POS-CNF game and she colors vertex s of F_1 (with color 1).

At first, suppose that Bob does not color y with color 1 in his first turn. Then, in the next round, Alice can guarantee that y receives a color distinct from 1. After this, she can always color a vertex inside any clause clique with color 1 (for example, $\ell'_{i,0}$, $\ell''_{i,0}$ or other). Since each clique L_j has at least $4N$ vertices, then Alice can also color all variable vertices and guarantee that all colors of the variable vertices appear in L_j.

Thus assume that Bob colors vertex y with color 1. With this, Alice can play using the following strategy: (1) if Bob plays in a vertex of F_1 then she plays with her winning strategy in F_1 from Lemma 2.1; (2) if Bob plays on some twin obtained from vertex $\ell_{i,j}$ then Alice plays the least available color in the other twin; (3) if Bob plays on some L_j then Alice plays as if Bob has passed his turn on the POS-CNF game, meaning that she chooses a vertex x_j and color it with a color different from 1 where X_j is the literal chosen by her winning strategy in the POS-CNF game; (4) if Bob plays on x_i, then Alice plays as if Bob has chosen X_i to be false in the POS-CNF game; (5) if the other moves are not possible, then Alice chooses any vertex of G and color it with the least available color.

By following this strategy, when the POS-CNF game finishes, every clause clique has some outside neighbor with color different from 1 or a vertex colored with color 1. With this, Alice and Bob can finish coloring every clause cliques using colors $1, \ldots, 2\beta - 1$, since Alice can guarantee that every clique with $2\beta - 1$ vertices receive all colors of its ouside neighbors (recall again that each clique L_j has at least $4N$ vertices) □

3 Game Grundy number is PSPACE-hard

Similar results can also be obtained for the game Grundy number. Unlike in the game coloring problem, the greedy game coloring problem satisfies the following:

Proposition 3.1 *If Alice has a winning strategy with k colors in the greedy game coloring problem, then Alice has also a winning strategy with $k + 1$ colors.*

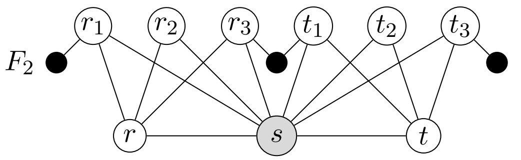

Fig. 3. Graph F_2: with 3 colors, the first to move wins the greedy game

Thus we can define the decision problem for the greedy coloring game: given a graph G and an integer k,

- (Greedy Game Coloring Problem 1) $\Gamma_g(G) \leq k$? That is, does Alice have a winning strategy using k colors in the greedy coloring game?

In this section, we prove that the following more restricted Greedy Game Coloring Problem 2 is PSPACE-Complete: given a graph G and its chromatic number $\chi(G)$,

- (Greedy Game Coloring Problem 2) $\Gamma_g(G) = \chi(G)$? That is, does Alice have a winning strategy using $\chi(G)$ colors in the greedy coloring game?

It is easy to see that Greedy Game Coloring Problem 1 is a generalization of Greedy Game Coloring Problem 2 (just set $k = \chi(G)$). Then the PSPACE-hardness of Greedy Game Coloring Problem 2 implies the PSPACE-hardness of Greedy Game Coloring Problem 1.

We obtain a very similar reduction from the POS-CNF problem (as shown in Section 2 for the game coloring problem). One important ingredient of the reduction is the graph F_2 of Figure 3. We start by proving that, with 3 colors, Alice wins the game in F_2 if and only she is the first one to play and, in this case, she must color vertex s first.

Lemma 3.2 *Alice has a winning strategy for the graph F_2 of Figure 3 in the greedy coloring game with 3 colors if and only if she is the first to play. In this case, she must color vertex s first.*

Proof. Suppose that Alice is the first to play and colors vertex r first (color 1 in the greedy game). Then Bob can color vertex t_2 (color 1). With this, s and t cannot be colored 1 and then Bob can force t_1 or t_3 be colored with 4 by coloring one of the two black vertices (color 1) in his next turn.

Now suppose that Alice is the first to play and colors vertex s first (color 1). Thus no vertex in $\{r_1, r_2, r_3, t_1, t_2, t_3\}$ can be colored 1 and then all black vertices will be colored 1 in the greedy game coloring. Therefore all vertices in $\{r_1, r_2, r_3, t_1, t_2, t_3\}$ can be colored with 2 or 3 in the game. Moreover, Alice can color r and t in her second and third turns using the colors $\{2, 3\}$ winning the game.

Finally suppose that Bob is the first to play. He can win the game by coloring vertex r_2 first (color 1). In his next turn, he can color the black neighbor of r_1 or r_3 (color 1). With this, r_1 or r_3 will be colored 4, and Bob wins the game. □

Theorem 3.3 *Given a graph G, deciding whether $\Gamma_g(G) = \chi(G)$ is PSPACE-Complete. Consequently, if k is an integer, deciding whether $\Gamma_g(G) \leq k$ is a*

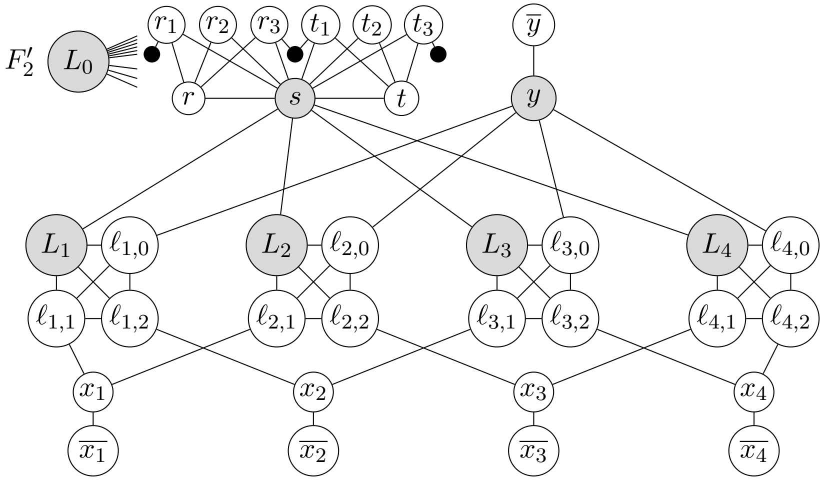

Fig. 4. Constructed graph G for the formula $(X_1 \vee X_2) \wedge (X_1 \vee X_3) \wedge (X_2 \vee X_4) \wedge (X_3 \vee X_4)$. Recall that each vertex $\ell_{j,k}$ represents two true-twins $\ell'_{j,k}$ and $\ell''_{j,k}$, L_0 is a clique with 32 vertices and L_1, L_2, L_3, L_4 are cliques with 29 vertices. Bob has a winning strategy with 35 colors in the greedy coloring game.

PSPACE-Complete problem.

Proof. [**Sketch**] We will follow almost the same reduction (from POS-CNF) of Theorem 2.2, including a neighbor $\overline{x_i}$ of degree 1 to each vertex x_i and replacing the graph F_1 by the graph F_2 with all vertices adjacent to a new clique L_0 with $2\beta - 4$ vertices. Also, in this case, we have that L_j is a clique with $2(\beta - p_j) - 3$ vertices, for $1 \leq j \leq M$.

Figure 3 shows the constructed graph G for the same formula $(X_1 \vee X_2) \wedge (X_1 \vee X_3) \wedge (X_2 \vee X_4) \wedge (X_3 \vee X_4)$. In the reduction of this example, we have the same values $N = 4$ variables, $M = 4$ clauses, $p = 2$, $q = 2$, $\beta = 4N + 2 = 18$, clique L_0 has $2\beta - 4 = 32$ vertices and cliques L_1 to L_M with $2(\beta - p) - 3 = 29$ vertices each.

As in Theorem 2.2, $\chi(G) = 2\beta - 1$. Using similar arguments as in the proof of Theorem 2.2, we obtain the result. The main difference is that, instead of coloring a vertex x_i with a color different from 1, Alice should color the vertex $\overline{x_i}$ with color 1. □

4 Game Grundy number of graphs with few P_4's

As mentioned in the introduction, Havet and Zhu showed in 2013 that $\Gamma_g(G) = \chi(G)$ for any cograph (see Proposition 9 of [11]), since $\chi(G) = \Gamma(G)$ in cographs. In this section, we prove that this also happens for known superclasses of cographs, as P_4-sparse graphs, P_4-tidy graphs and P_4-laden graphs, such that $\Gamma(G)$ can be larger than $\chi(G)$ as much as desired.

A *cograph* is a graph with no induced P_4 [5]. A graph G is P_4*-sparse* if every set of five vertices in G induces at most one P_4 [12]. P_4-sparse graphs have a nice structural decomposition in terms of unions, joins and *spiders*. Given graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the *union of* G_1 *and* G_2 is the graph $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$ and the *join of* G_1 *and* G_2 is the graph $G_1 \vee G_2 = (V_1 \cup V_2, E_1 \cup E_2 \cup \{uv : u \in$

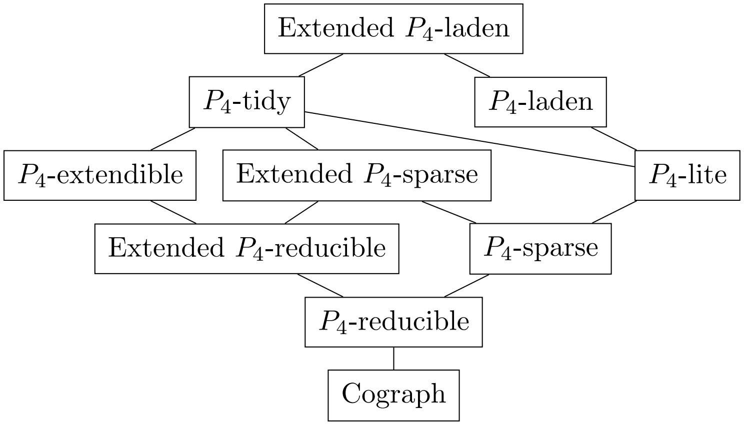

Fig. 5. Hierarchy of graphs with few P_4's.

$V_1, v \in V_2\}$). A *spider* is a graph whose vertex set has a partition (R, C, S), where $C = \{c_1, \ldots, c_k\}$ and $S = \{s_1, \ldots, s_k\}$ for $k \geq 2$ are respectively a clique and a stable set; s_i is adjacent to c_j if and only if $i = j$ (a thin spider), or s_i is adjacent to c_j if and only if $i \neq j$ (a thick spider); and every vertex of R is adjacent to each vertex of C and non-adjacent to each vertex of S. Notice that the complement of a thin spider is a thick spider, and vice-versa. Jamison and Olariu [12] proved that, if a graph G is P_4-sparse, then G is the disjoint union or the join of two P_4-sparse graphs, or G is a spider (R, C, S) such that $G[R]$ is a P_4-sparse graph, or G has at most one vertex.

As an example, the following sequence (G_k) of P_4-sparse graphs satisfies $\Gamma_g(G_k) = \chi(G_k) = 2k$ and $\Gamma(G_k) = 3k$. Let G_1 be a P_4 $a_1b_1c_1d_1$. For $k \geq 2$, let G_k be obtained from the join of G_{k-1} with a P_4 $a_kb_kc_kd_k$. Clearly, $\chi(G_k) = 2k$: color a_i and c_i with color $2i-1$ and color b_i and d_i with color $2i$ for every $1 \leq i \leq k$. Moreover, $\Gamma(G_k) = 3k$: color a_i and d_i with color $3i-2$ and color b_i and c_i with colors $3i-1$ and $3i$, respectively. Finally, $\Gamma_g(G_k) = 2k$: Alice can always avoid that both endpoints a_i and d_i of a P_4 $a_ib_ic_id_i$ receive the same color.

We say that a graph is *extended P_4-laden* if every induced subgraph with at most six vertices that contains more than two induced P_4's is $\{2K_2, C_4\}$-free. This graph class was introduced in [10], and a motivation to develop algorithms for extended P_4-laden graphs lies on the fact that they are on the top of a widely studied hierarchy of classes containing many graphs with few P_4's (see Figure 5), including cographs, P_4-sparse, P_4-lite, P_4-laden and P_4-tidy graphs. Therefore, solving interesting problems in an efficient way for extended P_4-laden graphs immediately imply efficient, generalized algorithms for all these classes. Another motivation is that extended P_4-laden graphs are not contained in perfect graphs; hence this work obtains coloring results not specifically related to perfection. There are recent papers dealing with coloring problems in extended P_4-laden graphs. For example, in the Grundy number [1] and in the cochromatic number [4].

Giakoumakis [10] proved an important structural characterization for extended P_4-laden graphs by special graphs, called *pseudo-splits* and *quasi-spiders*. Given a split graph G with vertex set partition (C, S), where C is a clique and S is an independent set, we say that G is *original* if every vertex in S has a non-neighbor in C and every vertex in C has a neighbor in S. We say that a graph G is a *pseudo-split* if its vertex set has a partition (R, C, S) such that S induces an independent set, C induces a clique, $C \cup S$ induces an original split graph and every vertex of R is adjacent to every vertex of C and non-adjacent to every vertex of S. Notice that the complement of a pseudo-split is also a pseudo-split and that spiders are pseudo-split graphs. A *quasi-spider* is a graph obtained from a spider (R, C, S) with at most one vertex from $C \cup S$ replaced by K_2 or $\overline{K_2}$ (keeping the neighborhood). Clearly, every spider is a quasi-spider.

Theorem 4.1 ([10]) *A graph G is extended P_4-laden if and only if exactly one of the following holds:*

(a) G is the disjoint union or the join of two non-empty extended P_4-laden graphs.

(b) G is a quasi-spider or a pseudo-split graph (R, C, S) such that $G[R]$ is an extended P_4-laden graph.

(c) G is isomorphic to C_5, P_5, $\overline{P_5}$, or has at most one vertex.

To determine $\Gamma_g(G)$ for extended P_4-laden graphs, we introduce the parameter $\Gamma'_g(G)$: the minimum number of colors such that Alice has a winning strategy in greedy coloring game even if Bob can start the game and also can pass any turn (that is, does not color a vertex in a turn). Notice that $\Gamma_g(G) \le \Gamma'_g(G)$. We will prove that $\Gamma'_g(G) = \chi(G)$ for extended P_4-laden graphs, implying that $\Gamma_g(G) = \chi(G)$ in this graph class. In the following, we obtain upper bounds on the union and join operations.

Lemma 4.2 *Given graphs G_1 and G_2,*

- $\Gamma'_g(G_1 \vee G_2) \le \Gamma'_g(G_1) + \Gamma'_g(G_2)$ *and*
- $\Gamma'_g(G_1 \cup G_2) \le \max\{\Gamma'_g(G_1), \Gamma'_g(G_2)\}$.

Proof. [**Sketch**] Alice only has to play in the same graph that Bob has played following her best strategy in this graph. If there is no uncolored vertex, Alice colors a vertex in the other graph (in this case, she considers that Bob passed his turn in this graph). □

Lemma 4.3 *If G is a quasi-spider or a pseudo-split graph (R, C, S), then $\Gamma'_g(G) = \chi(G)$. If G is isomorphic to C_5, P_5 or $\overline{P_5}$, then $\Gamma'_g(G) = \chi(G)$.*

Proof. [**Sketch**] There are some cases to analyse, which will be omitted because of space restrictions. In most cases, Alice only has to avoid that Bob colors all vertices of the independent set S with color 1. □

With this, we obtain our main theorem for extended P_4-laden graphs.

Theorem 4.4 *If G is an extended P_4-laden graph, then $\Gamma_g(G) = \Gamma'_g(G) = \chi(G)$, which can be computed in linear time. That is, Alice can win the greedy coloring game with $\chi(G)$ colors in extended P_4-laden graphs even if Bob starts the game and can also pass any turn.*

Proof. [**Sketch**] It follows by induction from Theorem 4.1 and Lemmas 4.2 and 4.3, since $\chi(G_1 \vee G_2) = \chi(G_1) + \chi(G_2)$ and $\chi(G_1 \cup G_2) = \max\{\chi(G_1), \chi(G_2)\}$ for any graphs G_1 and G_2. □

5 Final remarks

In this paper, we proved in Theorems 2.2 and 3.3 that, given a graph G and an integer k, deciding whether $\chi_g(G) \leq k$ or $\Gamma_g(G) \leq k$ are PSPACE-Complete problems. In the proofs of Theorems 2.2 and 3.3, the value of k obtained in the reductions (which is $k = \chi(G)$) can be very high. Considering only $k = 2$ colors in a connected graph G, it is known that $\chi_g(G) \leq 2$ if and only if G is a star $K_{1,n}$. For the greedy coloring game, we have the following characterization:

Lemma 5.1 *A connected graph G has $\Gamma_g(G) = 2$ if and only if G is bipartite with a vertex whose neighborhood is one of the parts of the bipartition.*

Proof. If $\Gamma_g(G) = 2$, then G must be bipartite since $\chi(G) \leq \Gamma_g(G) = 2$. Moreover, if G is not bipartite, then $\Gamma_g(G) \geq \chi(G) > 2$. Then assume that G is bipartite with bipartition (A, B).

Without loss of generality, suppose that G has a vertex v of A whose neighborhood is B. If Alice colors v in her first turn (color 1), then no vertex of B can be colored 1 and consequently all vertices of A will be colored 1. Thus all vertices of B will be colored 2.

Now suppose that every vertex of A has a non-neighbor in B and that every vertex of B has a non-neighbor in A. Let w be the first vertex colored by Alice. Assume without loss of generality that $w \in A$. Since G is connected, then w has a non-neighbor z in B at distance 3 (that is, w and z are the endpoints of a P_4). Thus, coloring z in his first turn (color 1), Bob forces 3 colors in this P_4. □

However, for $k = 3$ colors, the complexity of the greedy game coloring is not known.

Conjecture 5.2 *Deciding whether the Grundy number $\Gamma_g(G) \leq 3$ is PSPACE-Complete.*

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