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# Countably Sober Spaces

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**Abstract**

As a generalization of sober spaces, we introduce the concept of countably sober spaces and prove that some topological constructions preserve countable sobriety. In particularly, we prove that the category with countably sober spaces and continuous mappings is a complete category. We give some characterizations of countably sober spaces via countable filters and obtain the Hofmann-Mislove Theorem for countably sober spaces.

*Keywords:* countably sober space, countable filter, $\sigma$-Scott topology, $P$-space

## 1 Introduction

Sobriety is between $T_0$ and $T_2$ in topological spaces, and being $T_1$ and being sober are incomparable properties. Sober spaces have wonderful properties and play important roles in domain theory (see, e.g., [1,3,6,8,9,10,11]). For instance, a sober space is a directed complete poset (dcpo, for short) with respect to its *specialization order*, two sober spaces are homeomorphic iff their lattices of open subsets are order isomorphic, and the celebrated Hofmann-Mislove Theorem shows that there is an order isomorphism between the poset of compact saturated subsets and the poset of Scott open filters of open subsets in a sober space. There are many generalizations of sober spaces. In [10], the authors introduced the weaker notion of sobriety, which is called *bounded sobriety*, and proved that the subcategory of bounded sober spaces is reflective in the category of $T_0$ spaces. D. Zhao and W. K. Ho generalized

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bounded sober spaces to *$k$-bounded sober spaces* and obtained many interesting results in [11]. In this paper, we intend to generalize sober spaces to *countably sober spaces.*

Recall the definition of *sober spaces.* A subset $C$ of a topological space $X$ is *irreducible* if it is nonempty and if $C \subseteq A \cup B$, where $A$ and $B$ are closed, implies that $C \subseteq A$ or $C \subseteq B$. A topological space $X$ is sober if for every irreducible closed set $C$, there exists a unique $x \in X$ such that $C = \mathrm{cl}(\{x\})$. Equivalently, a nonempty set $C$ is irreducible if for finite closed set $B_1, B_2, ..., B_n$ in $X$, $C \subseteq \bigcup_{i=1}^{n} B_i$ implies that $C \subseteq B_i$ for some $i \in \{1, 2, ..., n\}$. Replacing finite closed sets in the definition of irreducible sets by countable closed sets, we define the notion of *countably irreducible sets* and introduce the concept of countably sober spaces via countably irreducible sets.

Countably approximating poset is a successful generalization of continuous poset (see [5,4]). To characterize countably approximating posets, the authors [4] introduced $\sigma$-Scott topology on a poset. We show that a countably approximating poset with its $\sigma$-Scott topology is a countably sober space, and prove that the topology generated by the $\sigma$-Scott topology and the lower topology on a bounded complete and countably directed complete poset is Lindelöf.

We investigate the properties of countably sober space. We prove that some topological constructions preserve countable sobriety. In particularly, we prove that the category with countably sober spaces and continuous mappings is a complete category. We give some characterizations of countably sober spaces via countable filters and obtain the Hofmann-Mislove Theorem for countably sober spaces.

## 2 Preliminaries

In this section, we recall some basic definitions and notations needed in this paper; more details can be founded in [1,3]. For a set $X$, the family of all finite sets (resp., countable sets) in $X$ is denoted by Fin $X$ (resp., Count $X$). For a poset $P$, $x \in P$, and $A \subseteq P$, let $\downarrow x = \{y \in P : y \leq x\}$, $\downarrow A = \bigcup\{\downarrow x : x \in A\}$; $\uparrow x$ and $\uparrow A$ are defined dually. A subset $D$ of $P$ is called *countably directed* if for any $E \in \mathrm{Count}D$, there exists $d \in D$ such that $E \subseteq \downarrow d$. $P$ is said to be a *countably directed complete poset* if every countably directed subset of $P$ has the least upper bound in $P$.

For a topological space $X$, let $\mathcal{O}(X)$ be the lattice of all open subsets in $X$. For $x \in X$, let $\mathcal{N}(x)$ and $\mathcal{O}(x)$ be the *neighbourhood* and *open neighbourhood* of point $x$ in $X$, respectively. That is, $\mathcal{N}(x) = \{U \subseteq X :$ there exists $O \in \mathcal{O}(X)$ such that $x \in O \subseteq U\}$, and $\mathcal{O}(x) = \{U \in \mathcal{O}(X) : x \in U\}$. For a $T_0$ space $(X, \tau)$, the *specialization order* $\leq$ on $X$ is defined by $x \leq y$ if and only if $x \in \mathrm{cl}(\{y\})$. Unless otherwise stated, throughout the paper, whenever an order-theoretic concept is mentioned, it is to be interpreted with respect to the specialization order on $X$.

**Definition 2.1** ([5,4]) Let $P$ be a countably directed complete poset. A binary relation $\ll_c$ on $P$ is defined as follows: $x \ll_c y$ iff for any countably directed set $D \subseteq P$, $y \leq \bigvee D$ implies that $x \leq d$ for some $d \in D$. Let $\downarrow_c x = \{y \in P : y \ll_c x\}$. $P$

is called a *countably approximating poset* if for all $x \in P$, $\downarrow_c x$ is countably directed and $x = \bigvee \downarrow_c x$.

Recall that the *$\sigma$-Scott topology* (see [4]) on a countably directed complete poset $P$. Let $\sigma_c(P)$ be a family subsets of $P$ and $U \in \sigma_c(P)$ iff $U$ satisfies the following two conditions: (i) $U = \uparrow U$; (ii) for any countably directed subset $D$ of $P$, $\bigvee D \in U$ implies $D \cap U \neq \emptyset$. Then $\sigma_c(P)$ is a topology on $P$, and we call $\sigma_c(P)$ the *$\sigma$-Scott topology* on $P$. It is easy to show that the intersection of countable open sets in $\sigma_c(P)$ is still open.

**Theorem 2.2** ([4,5]) *Let $P$ be a countably approximating poset.*

(1) *The relation $\ll_c$ satisfies interpolation property, i.e., $x \ll_c z$ implies $x \ll_c y \ll_c z$ for some $y \in P$.*

(2) *The family $\{\uparrow_c x : x \in P\}$ form a basis for the topology $\sigma_c(P)$, where $\uparrow_c x = \{y \in P : x \ll_c y\}$.*

## 3 Countably sober spaces

In this section we define the notion of countably irreducible sets and introduce the concept of countably sober spaces via countably irreducible closed sets. Some basic properties of countably sober spaces are proved.

**Definition 3.1** Let $X$ be a topological space and $C \subseteq X$.
(1) $C$ is called *countably irreducible* if $C$ is nonempty and if for any countable closed subsets $\{B_i : i \in \mathbb{Z}_+\}$, $C \subseteq \bigcup_{i \in \mathbb{Z}_+} B_i$ implies that $C \subseteq B_i$ for some $i \in \mathbb{Z}_+$.
(2) $X$ is called a *countably sober space* if for every countably irreducible closed set $C$, there exists a unique $x \in X$ such that $C = \mathrm{cl}(\{x\})$.

Note that countably irreducible closed sets are irreducible closed, sober spaces are countably sober spaces. If a space $X$ has only finite closed subsets (especially, $X$ is finite), then irreducible closed sets are countably irreducibly irreducible. Then $X$ is sober if $X$ is countably sober.

**Proposition 3.2** *Let $X$ be a set that has more than one element. If $X$ is endowed with the cofinite topology, then we have the following.*

(1) *$X$ is sober if and only if $X$ is a finite set;*

(2) *$X$ is countably sober if and only if $X$ is a countable set.*

**Remark 3.3** (1) If $X$ is a countable infinite set endowed with the cofinite topology. It follows from Proposition 3.2, $X$ is countably sober but not sober.
(2) Let $P$ be a finite poset with non-discrete partial order, then $(P, \sigma(P))$ is countably sober but not $T_1$.
(3) If an uncountable set $X$ is endowed with the cofinite topology, then $X$ is $T_1$ but not countably sober.

**Proposition 3.4** *Let $X$ be a space and $C$ a nonempty closed set of $X$. The following two conditions are equivalent:*

(1) *$C$ is countably irreducible;*

(2) *For any family of countable open sets $\{U_i : i \in \mathbb{Z}_+\}$ in $X$, if $C \cap U_i \neq \emptyset$ for every $i \in \mathbb{Z}_+$, then $C \cap (\bigcap_{i\in\mathbb{Z}_+} U_i) \neq \emptyset$.*

**Proposition 3.5** *For a countably approximating poset $P$, $(P, \sigma_c(P))$ is a countably sober space.*

**Proof.** Let $P$ be a countably approximating poset and $C$ a countably irreducible closed set in $(P, \sigma_c(P))$. Let

$$C^* = \{b \in C : \text{ there is an } a \in C \text{ with } b \ll_c a\} = \bigcup\{\downarrow_c a : a \in C\} = \downarrow_c C.$$

We claim that $C^*$ is countably directed. Let $\{b_i : i \in \mathbb{Z}_+\} \subseteq C^*$. Then there exists $a_i \in C$ such that $b_i \ll_c a_i$ for every $i \in \mathbb{Z}_+$. Since $C$ is a countably irreducible closed set and $\uparrow_c b_i \in \sigma_c(P)$ by Theorem 2.2, $C \cap (\bigcap_{i\in\mathbb{Z}_+} \uparrow_c b_i) \neq \emptyset$. Choose an $a \in C \cap (\bigcap_{i\in\mathbb{Z}_+} \uparrow_c b_i)$. Note that $\bigcap_{i\in\mathbb{Z}_+} \uparrow_c b_i \in \sigma_c(P)$, it follows from Theorem 2.2 that there exists $d \in \bigcap_{i\in\mathbb{Z}_+} \uparrow_c b_i$ such that $a \in \uparrow_c d \subseteq \bigcap_{i\in\mathbb{Z}_+} \uparrow_c b_i$. Thus $d \in C^*$ and $b_i \leq d$ for all $i \in \mathbb{Z}_+$. Since $P$ is a countably directed poset, $x = \bigvee C^*$ exists. Now we prove that $C = \mathrm{cl}_{\sigma_c(P)}(\{x\})$. Since $C$ is closed in $(P, \sigma_c(P))$ and $C^*$ is a countably directed subset of $C$, $x = \bigvee C^* \in C$. Thus $\downarrow x \subseteq C$. Let $y \in C$, then $y = \bigvee \downarrow_c y$ since $P$ is a countably approximating poset. But $\downarrow_c y \subseteq C^*$ implies $y = \bigvee \downarrow_c y \leq \bigvee C^* = x$. Thus $C \subseteq \downarrow x$. Note that $(P, \sigma_c(P))$ is $T_0$, $(P, \sigma_c(P))$ is countably sober. □

**Definition 3.6** Let $P$ be a countably directed complete poset. Then the common refinement $\sigma_c(P) \vee \omega(P)$ of the $\sigma$-Scott topology and the lower topology is called the $\sigma$-Lawson topology and is denoted by $\lambda_c(P)$.

**Theorem 3.7** *For a bounded complete and countably directed complete poset $P$, $(P, \lambda_c(P))$ is a Lindelöf space.*

**Proof.** Assume $\{U_i \in \sigma_c(P) : i \in I\}$ and $\{P \setminus \uparrow y_j : j \in J\}$ together form a cover of $P$, i.e., $(\bigcup_{i\in I} U_i) \cup (\bigcup_{j\in J} P \setminus \uparrow y_j) = (\bigcup_{i\in I} U_i) \cup (P \setminus \bigcap_{j\in J} \uparrow y_j) = P$. Then $\bigcap_{j\in J} \uparrow y_j \subseteq \bigcup_{i\in I} U_i$. We consider the following two cases.

**Case 1** $\bigcap_{j\in J} \uparrow y_j = \emptyset$.

We claim that there exists $J_0 \in \mathrm{Count}J$ such that $\bigcap_{j\in J_0} \uparrow y_j = \emptyset$. Suppose that $\bigcap_{j\in J_0} \uparrow y_j \neq \emptyset$ for all $J_0 \in \mathrm{Count}J$. Since $P$ is bounded complete, $\bigvee_{j\in J_0} y_j$ exists. Note that $\{\bigvee_{j\in J_0} y_j : J_0 \in \mathrm{Count}J\}$ is countably directed and $P$ is a countably directed complete poset, $\bigvee_{J_0\in \mathrm{Count}J} \bigvee_{j\in J_0} y_j$ exists and $\bigvee_{J_0\in \mathrm{Count}J} \bigvee_{j\in J_0} y_j = \bigvee_{j\in J} y_j$. Thus $\bigcap_{j\in J} \uparrow y_j = \uparrow \bigvee_{j\in J} y_j \neq \emptyset$. This is a contradiction. Thus there exists $J_0 \in \mathrm{Count}J$ such that $\bigcap_{j\in J_0} \uparrow y_j = \emptyset$. Therefore $\bigcup_{j\in J_0}(P \setminus \uparrow y_j) = P$.

**Case 2** $\bigcap_{j\in J} \uparrow y_j \neq \emptyset$.

Since $P$ is bounded complete, $\bigvee_{j\in J} y_j$ exists and $\bigcap_{j\in J} \uparrow y_j = \uparrow \bigvee_{j\in J} y_j$. Thus $\bigvee_{j\in J} y_j \in \bigcup_{i\in I} U_i$. Therefore there exits $i_0 \in I$ such that $\bigvee_{j\in J} y_j \in U_{i_0}$. Since $\{\bigvee_{j\in J_0} y_j : J_0 \in \mathrm{Count}J\}$ is countably directed and $\bigvee_{j\in J} y_j = \bigvee\{\bigvee_{j\in J_0} y_j : J_0 \in \mathrm{Count}J\} \in U_{i_0}$, there exists $J_0 \in \mathrm{Count}J$ such that $\bigvee_{j\in J_0} y_j \in U_{i_0}$. Thus $(P \setminus \uparrow \bigvee_{j\in J_0} y_j) \cup U_{i_0} = P$, i.e., $U_{i_0} \cup (\bigcup_{j\in J_0} P \setminus \uparrow y_j) = P$.

To summarize what has been mentioned above, $(P, \lambda_c(P))$ is a Lindelöf space.□

**Proposition 3.8** *Let $P$ be a countably approximating poset.*

(1) $(P, \lambda_c(P))$ *is regular and Hausdorff.*

(2) $(P, \lambda_c(P), \leq)$ *is a pospace, i.e., the relation $\leq$ is closed in the product space $(P \times P, \lambda_c(P) \times \lambda_c(P))$.*

The proofs of the following four propositions are straightforward, and we omit.

**Proposition 3.9** *If $f$ is a continuous map from topological space $X$ to topological space $Y$ and $C$ is a countably irreducible closed subset of $X$, then $\mathrm{cl}(f(C))$ is a countably irreducible closed subset of $Y$.*

**Proposition 3.10** *A closed subspace of a countably sober space is countably sober.*

**Proposition 3.11** *A saturated subspace of a countably sober space is countably sober.*

**Proposition 3.12** *A retract of a countably sober space is countably sober.*

**Theorem 3.13** *The product of countably sober spaces is countably sober.*

**Proof.** The statement has been proved for sober spaces in [3, Theorem 8.4.8], and the same proof carries over to this setting. □

**Lemma 3.14** *If $f, g : X \to Y$ are continuous, $X$ is countably sober, and $Y$ is $T_0$, then the equalizer $\{x \in X : f(x) = g(x)\}$ is countably sober.*

**Proof.** Let $Z = \{x \in X : f(x) = g(x)\}$, and let $C$ be a countably irreducible closed set in $Z$ (as a subspace of $X$). The the closure $\mathrm{cl}(C)$ of $C$ in $X$ is countably irreducible closed in $X$, and $C = \mathrm{cl}(C) \cap Z$. Since $X$ is countably sober, there is a unique $x \in X$ such that $\mathrm{cl}(C)$ is the closure of $x$ in $X$. We claim that $f(x) = g(x)$. Otherwise, $f(x) \neq g(x)$. Since $Y$ is $T_0$, we assume that there is an open set $U$ in $Y$ such that $f(x) \in U$ and $g(x) \notin U$ without loss of generality. Since $f(x) \in U$, $x \in f^{-1}(U)$. Since $x$ is also in the closure of $C$ in $X$, $f^{-1}(U) \cap C \neq \emptyset$. Let $y$ be any point in $f^{-1}(U) \cap C$. Since $y \in C \subseteq Z$, $g(y) = f(y) \in U$. Since $y \in C \subseteq \mathrm{cl}(C) = {\downarrow}x$, $y \leq x$, whence $g(y) \leq g(x)$. Since $g(y) \in U$, $g(x) \in U$, a contradiction. □

Let **Top** be the category of topological spaces and continuous maps. Recall that any limit can be obtained as an equalizer of two morphisms whose source and target objects are products. We can get the following.

**Theorem 3.15** *Any limit taken in* **Top** *of countably sober spaces is countably sober. The category of countably sober spaces and continuous maps is complete, and limits are computed as in* **Top**.

**Proposition 3.16** *Let $X$ be a topological space. If $Y$ is countably sober, then the set $Top(X, Y)$ of all continuous functions $f : X \to Y$ equipped with the topology of pointwise convergence is countably sober.*

**Proof.** For all $x \in X$ , let $p_x : Y^X \to Y$ be a projection. Let $A$ be a countably irreducible closed subset of $Top(X,Y)$ with the topology induced by the product topology on $Y^X$. Then $\mathrm{cl}_{Y^X}(A)$ is closed in $Y^X$, where $\mathrm{cl}_{Y^X}(A)$ is the closure of $A$ in $Y^X$. Thus $\mathrm{cl}_Y(p_x(\mathrm{cl}_{Y^X}(A))) = \mathrm{cl}_Y(p_x(A))$ is a countably irreducible closed set for each $x \in X$. As $Y$ is supposed to be countably sober, there is a unique $a_x \in Y$ such that $\mathrm{cl}_Y(p_x(\mathrm{cl}_{Y^X}(A))) = \mathrm{cl}_Y(p_x(A)) = \mathrm{cl}_Y(\{a_x\})$. Define $f : X \to Y$ as $f(x) = a_x$ for every $x \in X$. We now show that $f$ is continuous. Let $x \in X$ and $V \in \mathcal{O}(f(x))$, since $f(x) = a_x$ and $a_x \in \mathrm{cl}_Y(p_x(A))$, $p_x(A) \cap V \neq \emptyset$. That is, there exists $a \in A$ such that $a(x) \in V$. As $a$ is continuous, there is $U \in \mathcal{O}(X)$ such that $a(U) \subseteq V$. Since $a(z) \in p_z(A) \subseteq \mathrm{cl}_Y(\{a_z\})$ for every $z \in U$, $a(z) \leq a_z$. Note that $a_z = f(z)$ and $V = \uparrow V$, $f(z) \in V$. We conclude that $f(U) \subseteq V$. Thus $f$ is continuous and $A = \mathrm{cl}_{Top(X,Y)}(A) = (\mathrm{cl}_{Y^X}(A)) \cap Top(X,Y) = (\mathrm{cl}_{Y^X}(\{f\})) \cap Top(X,Y) = \mathrm{cl}_{Top(X,Y)}(\{f\})$.

□

# 4 Countable filters

In this section, we generalize filters to countable filters and give some characterizations of countably sober spaces via countable filters.

**Definition 4.1** Let $L$ be a complete lattice and $F \subseteq L$.

(1) $F$ is called a *countable filter* if $F = \uparrow F$ and $\bigwedge_{i \in \mathbb{Z}_+} x_i \in F$ for all $\{x_i : i \in \mathbb{Z}_+\} \subseteq F$.

(2) $F$ is called a *completely prime countable filter* if $F$ is a countable filter and for all $S \subseteq L$, $\bigvee S \in F$ implies $S \cap F \neq \emptyset$.

It is obvious that countable filters are filters in a complete lattice. Let $X$ be a set, $\mathcal{F} = \{X \setminus C : C \in \mathrm{Count}X\}$. Then $\mathcal{F}$ is a countable filter of the complete lattice $\mathcal{P}(X)$, where $\mathcal{P}(X)$ is the powerset lattice of $X$. Let $X$ be a topological space, $x \in X$. Then $\mathcal{N}(x)$ is a filter of $\mathcal{P}(X)$, but in general $\mathcal{N}(x)$ is not a countable filter of $\mathcal{P}(X)$. In the real number space $\mathbb{R}$, $\{(-1/n, 1/n) : n \in \mathbb{Z}_+\} \subseteq \mathcal{N}(0)$, but $\bigcap_{i \in \mathbb{Z}_+}(-1/n, 1/n) = \{0\} \notin \mathcal{N}(0)$. To our delight, the neighbourhood of each point in a $P$-space is a countable filter.

**Definition 4.2** ([2,7]) A point $p \in X$ is called a $P$-point if its filter of neighbourhoods is closed under countable intersection. A space $X$ is called a $P$-space if every point in $X$ is a $P$-point.

**Proposition 4.3** *Let $X$ be a topological space. The following conditions are equivalent:*

(1) *$X$ is a $P$-space;*

(2) *For all $x \in X$, $\mathcal{O}(x)$ is a countable filter of $\mathcal{O}(X)$;*

(3) *For all $x \in X$, $\mathcal{N}(x)$ is a countable filter of $\mathcal{P}(X)$;*

(4) *The intersection of any countable open sets in $X$ is an open set.*

**Proof.** (3) $\Rightarrow$ (4) $\Rightarrow$ (1): Trivial.
(1) $\Rightarrow$ (2): Let $x \in X$. Obviously, $\mathcal{O}(x)$ is an upper set in the lattice of open sets $\mathcal{O}(X)$. If $\{U_i : i \in \mathbb{Z}_+\} \subseteq \mathcal{O}(x)$, in order to show that $\bigwedge_{\mathcal{O}(X)}\{U_i : i \in \mathbb{Z}_+\} \in \mathcal{O}(x)$, we only need check $\bigcap_{i\in\mathbb{Z}_+} U_i \in \mathcal{O}(X)$. For all $y \in \bigcap_{i\in\mathbb{Z}_+} U_i$, $\{U_i : i \in \mathbb{Z}_+\} \subseteq \mathcal{N}(y)$. Since $X$ is a $P$-space, $\bigcap_{i\in\mathbb{Z}_+} U_i \in \mathcal{N}(y)$. Thus $\bigcap_{i\in\mathbb{Z}_+} U_i \in \mathcal{O}(X)$.
(2) $\Rightarrow$ (3): Let $x \in X$. Obviously, $\mathcal{N}(x)$ is an upper set in $\mathcal{P}(X)$. If $\{V_i : i \in \mathbb{Z}_+\} \subseteq \mathcal{N}(x)$, then there exists $U_i \in \mathcal{O}(X)$ such that $x \in U_i \subseteq V_i$ for all $i \in \mathbb{Z}_+$. Thus $\{U_i : i \in \mathbb{Z}_+\} \subseteq \mathcal{O}(X)$. Since $\mathcal{O}(x)$ is a countable filter of $\mathcal{O}(X)$ by (2), $\bigwedge_{\mathcal{O}(X)}\{U_i : i \in \mathbb{Z}_+\} = \mathrm{int}(\bigcap_{i\in\mathbb{Z}_+} U_i) \in \mathcal{O}(x)$. Thus $x \in \mathrm{int}(\bigcap_{i\in\mathbb{Z}_+} U_i) \subseteq \bigcap_{i\in\mathbb{Z}_+} U_i \subseteq \bigcap_{i\in\mathbb{Z}_+} V_i$. Therefore, $\bigcap_{i\in\mathbb{Z}_+} V_i \in \mathcal{N}(x)$. $\square$

**Example 4.4** (1) Recall that a topological space is called an Alexandroff-discrete space iff every intersection (even infinite) of opens is again open. All Alexandroff-discrete spaces are $P$-spaces.
(2) Let $P$ be countably directed complete poset. Then $(P, \sigma_c(P))$ is a $P$-space.

**Theorem 4.5** *Let $X$ be a topological space. Consider the following two conditions:*

(1) *$X$ is countably sober;*

(2) *For all completely prime countable filter $\mathcal{F}$ of $\mathcal{O}(X)$, there is a unique $x \in X$ such that $\mathcal{F} = \mathcal{O}(x)$.*

*Then* (1) $\Rightarrow$ (2)*; if $X$ is a $P$-space, then* (2) $\Rightarrow$ (1)*, and (1) and (2) are equivalent.*

**Proof.** (1) $\Rightarrow$ (2): Let $\mathcal{F}$ be a completely prime countable filter of $\mathcal{O}(X)$, and let $O = \bigcup\{U \in \mathcal{O}(X) : U \notin \mathcal{F}\}$. Then $O \in \mathcal{O}(X)$ and $O \notin \mathcal{F}$. Let $C = X \setminus O$. We claim that $C$ is a countably irreducible closed in $X$. Assume that $\{B_i : i \in \mathbb{Z}_+\}$ is a family closed sets in $X$ with $C \subseteq \bigcup_{i\in\mathbb{Z}_+} B_i$. Then $O = X \setminus C = X \setminus C \cap (\bigcup_{i\in\mathbb{Z}_+} B_i) = X \setminus \bigcup_{i\in\mathbb{Z}_+}(C \cap B_i) = \bigcap_{i\in\mathbb{Z}_+}(X \setminus (C \cap B_i)) = \bigwedge_{\mathcal{O}(X)}\{X \setminus (C \cap B_i) : i \in \mathbb{Z}_+\}$ and $\bigwedge_{\mathcal{O}(X)}\{X \setminus (C \cap B_i) : i \in \mathbb{Z}_+\} \notin \mathcal{F}$. Since $\mathcal{F}$ is a countable filter of $\mathcal{O}(X)$, there exists $i \in \mathbb{Z}_+$ such that $X \setminus (C \cap B_i) \notin \mathcal{F}$. Note that $X \setminus (C \cap B_i) \in \mathcal{O}(X)$, $X \setminus (C \cap B_i) \subseteq O$. Thus $C = X \setminus O \subseteq C \cap B_i \subseteq B_i$. Hence $C$ is a countably irreducible closed in $X$. Since $X$ is countably sober, there is a unique $x \in X$ such that $C = \mathrm{cl}(\{x\})$. We now prove that $\mathcal{F} = \mathcal{O}(x)$. For all $V \in \mathcal{O}(x)$, if $V \notin \mathcal{F}$, then $x \in V \subseteq O = X \setminus C$. Thus $x \notin C$, a contradiction. On the other hand, for all $V \in \mathcal{F}$, if $x \notin V$, then $V \cap \mathrm{cl}(\{x\}) = V \cap C = \emptyset$. Thus $V \subseteq X \setminus C = O$, hence $O \in \mathcal{F}$. This is a contradiction. Together, $\mathcal{F} = \mathcal{O}(x)$ and the uniqueness of $x$ follow.
(2) $\Rightarrow$ (1): Let $C$ be a countably irreducible closed set in $X$, and let $\mathcal{F} = \{U \in \mathcal{O}(X) : U \cap C \neq \emptyset\}$. We show $\mathcal{F}$ is a completely prime countable filter of $\mathcal{O}(X)$.
(i) If $U \in \mathcal{F}$, $V \in \mathcal{O}(X)$ with $U \subseteq V$, then $V \in \mathcal{F}$;
(ii) If $\{U_i : i \in \mathbb{Z}_+\} \subseteq \mathcal{F}$, then $U_i \cap C \neq \emptyset$ for all $i \in \mathbb{Z}_+$. Since $C$ is countably irreducible closed, $(\bigcap_{i\in\mathbb{Z}_+} U_i) \cap C \neq \emptyset$ by Proposition 4.3. Since $X$ is a $P$-space, $\bigcap_{i\in\mathbb{Z}_+} U_i \in \mathcal{O}(X)$ by Proposition 4.3. Hence $\bigwedge_{\mathcal{O}(X)}\{U_i : i \in \mathbb{Z}_+\} = \bigcap_{i\in\mathbb{Z}_+} U_i \in \mathcal{F}$.
(iii) For any $\{U_i : i \in I\} \subseteq \mathcal{O}(X)$, if $\bigcup_{i\in I} U_i \in \mathcal{F}$, then $(\bigcup_{i\in I} U_i) \cap C \neq \emptyset$. Thus $U_i \cap C \neq \emptyset$ for some $i \in I$. Hence $U_i \in \mathcal{F}$.

By precondition (2), there exists a unique $x \in X$ such that $\mathcal{F} = \mathcal{O}(x)$. We now prove $C = \mathrm{cl}(\{x\})$. For all $V \in \mathcal{O}(x) = \mathcal{F}$, $V \cap C \neq \emptyset$. Hence $x \in \mathrm{cl}(C) = C$. On the other hand, for all $y \in C$, if $U \in \mathcal{O}(y)$, then $y \in U \cap C \neq \emptyset$. Thus $U \in \mathcal{F} = \mathcal{O}(x)$. So $x \in U$ and $y \in \mathrm{cl}(\{x\})$. Thus $C \subseteq \mathrm{cl}(\{x\})$. Therefore $C = \mathrm{cl}(\{x\})$ and the uniqueness of $x$ is obvious. $\square$

**Theorem 4.6** *Let $X$ be a topological space. Consider the following conditions:*

(1) *$X$ is countably sober;*

(2) *For all Scott open countable filter $\mathcal{F}$ of $\mathcal{O}(X)$ and $U \in \mathcal{O}(X)$, $\bigcap \mathcal{F} \subseteq U$ implies $U \in \mathcal{F}$;*

(3) *For all completely prime countable filter $\mathcal{F}$ of $\mathcal{O}(X)$ and $U \in \mathcal{O}(X)$, $\bigcap \mathcal{F} \subseteq U$ implies $U \in \mathcal{F}$.*

*Then* $(1) \Rightarrow (2) \Rightarrow (3)$*; if $X$ is a $P$-space, and* $(3) \Rightarrow (1)$*, thus all three conditions are equivalent.*

**Proof.** $(1) \Rightarrow (2)$: Let $\mathcal{F}$ be a Scott open countable filter $\mathcal{F}$ of $\mathcal{O}(X)$ and $U \in \mathcal{O}(X)$ with $\bigcap \mathcal{F} \subseteq U$. Assume $U \notin \mathcal{F}$. Let $\mathcal{A} = \mathcal{O}(X) \setminus \mathcal{F}$, then $U \in \mathcal{A}$. Since $\mathcal{F}$ is a Scott open set of $\mathcal{O}(X)$, $(\mathcal{A}, \subseteq)$ is a dcpo. Thus there exists a maximal element $O$ in $\mathcal{A}$ such that $U \subseteq O$. Let $C = X \setminus O$. We now prove that $C$ is countably irreducible. Let $\{B_i : i \in \mathbb{Z}_+\}$ be a family of closed sets in $X$ with $C \subseteq \bigcup_{i\in\mathbb{Z}_+} B_i$. Then $O = X\setminus C = X\setminus\bigcup_{i\in\mathbb{Z}_+}(C\cap B_i) = \bigcap_{i\in\mathbb{Z}_+} X\setminus(C\cap B_i) = \bigwedge_{\mathcal{O}(X)}\{X\setminus(C\cap B_i) : i \in \mathbb{Z}_+\}$. Since $O \notin \mathcal{F}$ and $\mathcal{F}$ is a countable filter of $\mathcal{O}(X)$, there exists some $i \in \mathbb{Z}_+$ such that $X\setminus(C\cap B_i) = (X\setminus C)\cup(X\setminus B_i) = O\cup(X\setminus B_i) \notin \mathcal{F}$. Then $O\cup(X\setminus B_i) \in \mathcal{A}$. It follows from the maximality of $O$ that $X \setminus B_i \subseteq O \cup (X \setminus B_i) = O$. Then $C = X \setminus O \subseteq B_i$. Hence $C$ is a countably irreducible closed set in $X$. Since $X$ is countably sober, there is a unique $x \in X$ such that $C = \mathrm{cl}(\{x\})$. Since $U \subseteq O = X \setminus \mathrm{cl}(\{x\})$, $x \notin U$. Since $\bigcap \mathcal{F} \subseteq U$, $x \notin \bigcap \mathcal{F}$. Then there exists $V \in \mathcal{F}$ such that $x \notin V$. Thus $V \subseteq X \setminus \mathrm{cl}(\{x\}) = O$. Since $V \in \mathcal{F}$, $O \in \mathcal{F}$. But $O \in \mathcal{A} = \mathcal{O}(X) \setminus \mathcal{F}$. This is a contradiction. Therefore $U \in \mathcal{F}$.
$(2) \Rightarrow (3)$: Trivial.
$(3) \Rightarrow (1)$: Let $C$ be a countably irreducible closed set in $X$, and let $\mathcal{F} = \{U \in \mathcal{O}(X) : U \cap C \neq \emptyset\}$. Since $X$ is a $P$-space, $\mathcal{F}$ is a completely prime countable filter of $\mathcal{O}(X)$. We claim that $(\bigcap \mathcal{F}) \cap C \neq \emptyset$. In fact, if $(\bigcap \mathcal{F}) \cap C = \emptyset$, then $\bigcap \mathcal{F} \subseteq X \setminus C$. By precondition (3), $X \setminus C \in \mathcal{F}$, a contradiction. Let $x \in (\bigcap \mathcal{F}) \cap C$, then $\mathrm{cl}(\{x\}) \subseteq C$. On the other hand, if $C \nsubseteq \mathrm{cl}(\{x\})$, then $C \cap (X \setminus \mathrm{cl}(\{x\})) \neq \emptyset$. Thus $X \setminus \mathrm{cl}(\{x\}) \in \mathcal{F}$ and $x \in \bigcap \mathcal{F} \subseteq X \setminus \mathrm{cl}(\{x\})$. This is a contradiction. Hence $C \subseteq \mathrm{cl}(\{x\})$. Therefore $C = \mathrm{cl}(\{x\})$. $\square$

For a topological space $X$, the poset of all Scott open countable filters of $\mathcal{O}(X)$ is denoted by $\mathrm{OCFilt}(\mathcal{O}(X))$, and the poset of compact saturated subsets of $X$ with the order reverse to containment is denoted by $Q(X)$. For all $\mathcal{F} \in \mathrm{OCFilt}(\mathcal{O}(X))$, it follows from Theorem 4.6 that $\bigcap \mathcal{F} \in Q(X)$. For all $K \in Q(X)$, Let $\mathcal{U}(K) = \{O \in \mathcal{O}(X) : K \subseteq O\}$. Then by Proposition 4.3, $\mathcal{U}(K) \in \mathrm{OCFilt}(\mathcal{O}(X))$ if $X$ is a $P$-space. We now get the following theorem, which we call the Hofmann-Mislove Theorem for countably sober spaces.

**Theorem 4.7** *If $X$ is both a countably sober space and a $P$-space, then the mapping*

$$\Phi : Q(X) \to \mathrm{OCFilt}(\mathcal{O}(X)), \Phi(K) = \mathcal{U}(K)$$

*is an order isomorphism between $Q(X)$ and $\mathrm{OCFilt}(\mathcal{O}(X))$, and the inverse of $\Phi$ is:*

$$\Psi : \mathrm{OCFilt}(\mathcal{O}(X)) \to Q(X), \Psi(\mathcal{F}) = \bigcap \mathcal{F}.$$

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