Extending Algebraic Operations to D-Completions

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Abstract

In this article we show how separately continuous algebraic operations on T_0-spaces and the laws that they satisfy, both identities and inequalities, can be extended to the D-completion, that is, the universal monotone convergence space completion. Indeed we show that the operations can be extended to the lattice of closed sets, but in this case it is only the linear identities that admit extension. Via the Scott topology, the theory is shown to be applicable to dcpo-completions of posets. We also explore connections with the construction of free algebras in the context of monotone convergence spaces.

Keywords: monotone convergence space completion, extension of algebraic operations, extension of identities, dcpo-completeion, free algebra constructions.

1 Introduction

In the theory of lattices and partially ordered sets completions of various types play a basic role. For the theory of posets as they are used for modeling in theoretical computer science, the appropriate type of completeness is directed completeness, the existence of suprema for directed subsets. A directed complete partial ordered set is called a dcpo, for short. A topological analogue occurs in the theory of T_0-spaces, where one requires that directed sets in the order of specialization have suprema, and that a directed set converges to its supremum. Such spaces have been called *monotone convergence spaces* or *d-spaces* and they yield an appropriate notion of

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completion for T_0-spaces, the D-completion, which has been discussed by the authors recently [11]. In this paper we consider algebraic operations defined on a T_0-space that are separately continuous and the identities, equalities and inequalities that they satisfy, and show how these may be extended to the D-completion, which is the universal monotone convergence space completion.

As an intermediary step we first choose a very large kind of completion, the space of all nonempty closed subsets with the weak topology, and we extend the algebraic operations to this space. We then restrict these algebraic operations to the sobrification of the original space; the sobrification can indeed be viewed as a subspace of the space of all nonempty closed subsets by restricting to the irreducible ones. We finally restrict to the D-completion, a subspace of the sobrification.

Throughout the paper we assume that all topological spaces under consideration are T_0 and let TOP_0 denote the category of T_0-spaces and continuous maps. Besides facilitating ease of presentation, this assumption is equivalent to assuming that the order of specialization is a partial order, which is the case of interest to us. Since TOP_0 is a full reflective subcategory of the category TOP of all topological spaces and continuous maps, a number of the results of this paper extend to TOP by composing with the T_0-reflection functor, as we occasionally point out.

Some Notation. In a topological space the closure of a subset A is denoted by $\overline{A}$ and also by A^-. The order of specialization is defined by $x \leq y$ iff $x \in \overline{\{y\}}$. In a partially ordered set we denote by $\downarrow x$ the set of all $y \leq x$. Note that $\downarrow x = \overline{\{x\}}$ for the specialization order in a topological space.

2   Scott and separate continuity

We recall some basic notions concerning the Scott topology, as they may be found, for example, in [1] or [9]. By definition the closed sets in the Scott topology of a poset are precisely those that are lower sets and are closed with respect to taking any existing directed sups. A function between partially ordered sets is *Scott-continuous* if it is continuous with respect to the Scott topologies. The following lemma is standard, although not usually stated in this generality (see [9, Proposition II-2.1]).

Lemma 2.1 *A function $f: X \to Y$ between posets is Scott-continuous if and only if it preserves all existing directed sups. In this case f is order preserving.*

Recall that a function $f: \Pi_{i=1}^{n} X_i \to Y$ of topological spaces is *separately continuous* if f restricted to every slice $\{x_1\} \times \cdots \times X_i \times \cdots \times \{x_n\}$ is continuous.

Lemma 2.2 *Let $P_1, \ldots, P_n, Q$ be posets and let $f: \Pi_{i=1}^{n} P_i \to Q$ be a function. Then f is Scott-continuous (with respect to the Scott topology of the coordinatewise order on $\prod_{i=1}^{n} P_i$) if and only if f is separately Scott-continuous.*

Proof. Extend the proof of [9, Lemma II-2.8] from dcpos to posets and from two variables to n-variables (see [9, Exercise II-2.27]).   □

Remark 2.3 Note in Lemma 2.2 that the Scott topology on the product is always finer and may be strictly finer than the product of the Scott topologies.

For T_0-spaces, the Scott topology and Scott continuity are defined in terms of the order of specialization.

Lemma 2.4 *Let* $f: \Pi_{i=1}^{n} X_i \to Y$ *be a separately continuous function, where each* X_i *is a monotone convergence space. Then* f *is separately Scott-continuous, Scott-continuous, and hence order preserving.*

Proof. Since each slice is a monotone convergence space, f is separately Scott-continuous by [9, Lemma II-3.13]. By Lemma 2.2 it is Scott-continuous, and by Lemma 2.1 it is order-preserving. □

3 Spaces of closed subsets and extensions of functions

For a topological space X, we let ΓX denote the sup-semilattice (with respect to the inclusion order) of all nonempty closed subsets, whereas in the antecedent paper [11] ΓX denoted the lattice of all closed subsets.

Recall that the *weak upper topology*, or more simply the *weak topology*, on a poset P has as a subbase for the closed sets all principal lower sets $\downarrow x$, $x \in X$. The weak upper topology is a T_0-topology, and it is the weakest topology for which the order of specialization agrees with the original order on P. For the case of a topological space X, the subbase of closed sets for the weak topology on $\Gamma(X)$ is given by

$$\downarrow A = \{B \in \Gamma X : B \subseteq A\}$$

as A varies over the nonempty closed subsets of X. There is a canonical map $\eta_X: X \to \Gamma X$ assigning the singleton closure $\eta_X(x) = \overline{\{x\}}$ to every $x \in X$. This map is easily seen to be a topological embedding (for T_0-spaces); see [9, Exercises V-4.9, V-5.34].

Lemma 3.1 *Any dcpo is a monotone convergence space with respect to the weak upper topology. This applies in particular to the set ΓX ordered by inclusion arising from a topological space X, since ΓX is a complete sup-semilattice, and hence a dcpo.*

Proof. For the first assertion, see [9, Exercise II-1.31(i)].

We observe that ΓX is closed under arbitrary nonempty sups (just take the closure of the union for any nonempty family of members of ΓX), hence in particular suprema of directed families exist, so ΓX is a dcpo (indeed a complete sup-semilattice). □

Remark 3.2 The assignment X to ΓX extends to an endofunctor on the category of topological spaces. For a continuous function $f: X \to Y$, define $\Gamma f: \Gamma X \to \Gamma Y$ by $\Gamma f(A) = \overline{f(A)}$. Then Γf is continuous since the inverse image of a subbasic closed set is again subbasic closed:

$$(\Gamma f)^{-1}(\downarrow A) = \downarrow f^{-1}(A).$$

It is straightforward to verify the functorial properties of Γ.

Definition 3.3 We consider a variant of Γf. Let $f \colon \prod_{i=1}^n X_i \to Y$ be a (not necessarily continuous) function defined on a product of topological spaces. We define $f^\gamma \colon \prod_{i=1}^n \Gamma X_i \to \Gamma Y$ by:

$$f^\gamma(A_1, \ldots, A_n) = (f(\prod_{i=1}^n A_i))^-$$
$$= \{f(x_1, \ldots, x_n) \mid x_i \in A_i \text{ for } 1 \le i \le n\}^-.$$

Lemma 3.4 Let $f \colon \prod_{i=1}^n X_i \to Y$ be separately continuous. Then for arbitrary nonempty subsets $A_i \subseteq X_i$ for $1 \le i \le n$ and $B \subseteq Y$:

(i) $f(\prod_{i=1}^n \overline{A_i}) \subseteq \overline{B}$ whenever $f(\prod_{i=1}^n A_i) \subseteq B$;

(ii) $(f(\prod_{i=1}^n \overline{A_i}))^- = (f(\prod_{i=1}^n A_i))^-$;

(iii) $f^\gamma \colon \prod_{i=1}^n \Gamma X_i \to \Gamma Y$ is separately continuous, hence Scott-continuous.

Proof. (i) If $f(\prod_{i=1}^n A_i) \subseteq B$, then for any $(a_2, \ldots, a_n) \in \prod_{i=2}^n A_i$, we have $f(\overline{A_1} \times \Pi_{i=2}^n \{a_i\}) \subseteq \overline{B}$ by separate continuity, and thus $f(\overline{A_1} \times \Pi_{i=2}^n A_i) \subseteq \overline{B}$. Applying this argument next in the second coordinate, we conclude that $f(\overline{A_1} \times \overline{A_2} \times \Pi_{i=3}^n A_i) \subseteq \overline{B}$, and by induction the conclusion of the lemma follows.

(ii) The inclusion from left to right follows from part (i) and the other direction is trivial.

(iii) Let B be a closed subset of Y, and fix $(A_2, \ldots, A_n) \in \prod_{i=2}^n \Gamma X_i$. Set

$$A_1 = \{x \in X_1 : f(\{x\} \times \prod_{i=2}^n A_i) \subseteq B\}$$

It follows from part (i) that A_1 is closed. One sees directly from this that for $(A_2, \ldots, A_n)$ fixed, the inverse image under f^γ of $\downarrow B$ in ΓX_1 is $\downarrow A_1$, so f^γ is continuous in the first variable. Applying the same argument to the other variables, we conclude that f^γ is separately continuous.

It follows from the fact that each ΓX_i is a monotone convergence space (Lemma 3.1) and f^γ is separately continuous that f^γ is Scott-continuous (Lemma 2.4).   □

Let S be a topological space endowed with a separately continuous binary operation $\mu : S \times S \to S$. Write $x * y$ for $\mu(x, y)$. Then for $(A, B) \in \Gamma S \times \Gamma S$, $\mu^\gamma(A, B) = (A * B)^-$, where $A * B = \{a * b : a \in A, \ b \in B\}$ is the set product. We alternatively write $A *_\gamma B$ for $\mu^\gamma(A, B)$.

Proposition 3.5 Let S be a topological space endowed with a separately continuous binary operation $\mu : S \times S \to S$. Then $\mu^\gamma : \Gamma S \times \Gamma S \to \Gamma S$ is separately continuous, associative if μ is and commutative if μ is.

Proof. The separate continuity follows from Lemma 3.4(iii). Assume now that $*$

is associative. Then we have

$$A *_\gamma (B *_\gamma C) = (A * (B * C)^-)^- = (A * (B * C))^-,$$

where the second equality follows from Lemma 3.4(ii). Similarly $(A *_\gamma B) *_\gamma C = ((A * B) * C)^-$. Thus the operation $*_\gamma$ is associative, since $A * (B * C) = (A * B) * C$ by associativity of $*$. If $*$ is commutative, then

$$A *_\gamma B = (A * B)^- = (B * A)^- = B *_\gamma A.$$

$\square$

Thus, if S is a semitopological semigroup, that is, a topological space with a separately continuous associative operation $*$, then ΓS is also a separately continuous semigroup, which is commutative iff S is.

4 General semitopological algebras

In this section we generalize Proposition 3.5 from semigroups to general algebraic structures.

A *signature* Σ is understood to be a set of operation symbols μ each being assigned a finite arity $n_\mu \in \mathbb{N}$. A *(general) algebra of signature* Σ, a *Σ-algebra*, for short, will be a set A together with a collection of operations $\mu^A : A^n \to A$, one for every operation symbol μ of arity $n = n_\mu$. In most cases we will simply write μ instead of μ^A, when there is no need to distinguish the operation symbol from the concrete operation. If the algebra A carries a partial order such that all operations are order preserving, then we say that we have a *(general) partially ordered algebra*. If A carries a topology such that all operations $\mu^A : A^n \to A$ are separately continuous, then we talk about a *(general) semitopological algebra*. If all of these operations μ^A are jointly continuous, we say that A is a *(general) topological algebra*.

With respect to the specialization order every semitopological algebra is a partially ordered algebra; indeed, continuous functions preserve the specialization order, the specialization order on a product of spaces is the product of the specialization orderings on the factors, and a function defined on a direct product of partially ordered sets is monotone if and only if it is separately monotone. Conversely, every partially ordered algebra A can be viewed as a topological algebra: just endow A with the *Alexandroff* or *A-discrete topology*, the open sets of which are all upper sets. The specialization order is just the original order.

We establish now that Γ lifts to a functor (see 3.2) in the context of semitopological algebras.

By Lemma 3.4(iii) each separately continuous operation $\mu : A^n \to A$ of a semitopological algebra A induces a separately continuous operation $\mu^\gamma : (\Gamma A)^n \to \Gamma A$ by defining

$$\mu^\gamma(A_1, \ldots, A_n) = \mu(A_1 \times \ldots \times A_n)^-$$

$$= \{\mu(a_1, \ldots, a_n) \mid a_i \in A_i, i = 1, \ldots, n\}^-.$$

Thus we obtain a semitopological algebra ΓA of the same signature as A. For lifting algebra homomorphisms we use the following:

Lemma 4.1 *Let* $\mu \colon X^n \to X$ *and* $\nu \colon Y^n \to Y$ *be separately continuous and let* $f \colon X \to Y$ *be continuous and satisfy*

$$f(\mu(x_1, \ldots, x_n)) = \nu(f(x_1), \ldots, f(x_n))$$

for all $(x_1, \ldots, x_n) \in X^n$. *Then*

$$\Gamma f(\mu^\gamma(A_1, \ldots, A_n)) = \nu^\gamma(\Gamma f(A_1), \ldots, \Gamma f(A_n))$$

for all $(A_1, \ldots, A_n) \in (\Gamma X)^n$.

Proof. For nonempty subsets $A_1, \ldots, A_n$ of X, we have $f(\mu(\prod_{i=1}^n A_i)) = \nu(\prod_{i=1}^n f(A_i))$ by hypothesis. It follows from this and Lemma 3.4(ii) that

$$
\begin{aligned}
\nu^\gamma(\Gamma f(A_1), \ldots, \Gamma f(A_n)) &= (\nu(f(A_1)^- \times \ldots \times f(A_n)^-))^- \\
&= (\nu(f(A_1) \times \ldots \times f(A_n)))^- \\
&= (f(\mu(\prod_{i=1}^n A_i)))^- \\
&= (f(\mu(\prod_{i=1}^n A_i)^-))^- \\
&= \Gamma f(\mu(\prod_{i=1}^n A_i)^-) \\
&= \Gamma f(\mu^\gamma(A_1, \ldots, A_n)).
\end{aligned}
$$

$\square$

It follows directly from the preceding lemma that if $f \colon A \to A'$ is a continuous homomorphism of semitopological Σ-algebras then $\Gamma f \colon \Gamma A \to \Gamma A'$ is also. We summarize:

Proposition 4.2 *The functor* Γ *applied to a semitopological* Σ-*algebra* A *in the manner described above gives rise to a semitopological* Σ-*algebra* ΓA, *and furthermore, if* $f \colon A \to A'$ *is a continuous homomorphism of semitopological* Σ-*algebras, then* $\Gamma f \colon \Gamma A \to \Gamma A'$ *is also. There results an endofunctor, again called* Γ, *on the category* $\mathcal{S}(\Sigma)$ *of semitopological* Σ-*algebras for any fixed signature* Σ *and continuous* Σ-*algebra homomorphisms.*

In order to talk about the equational and inequational theory of an algebra A of signature Σ we need the notion of a term. For this we choose variables $x_1, x_2, \ldots$. Terms are defined inductively: variables x_i are terms to begin with, and if $\tau_1(x_{11}, \ldots, x_{1n_1}), \ldots, \tau_m(x_{m1}, \ldots, x_{mn_m})$ are terms and μ an operation symbol in

Σ of arity m, then

$$\mu(\tau_1(x_{11}, \ldots, x_{1n_1}), \ldots, \tau_m(x_{m1}, \ldots, x_{mn_m}))$$

is also a term. We briefly write $\tau(x_1, \ldots, x_n)$ for a term, where $x_1, \ldots, x_n$ are n different variables among which appear all the variables occurring in the term. In an algebra A of signature Σ every term $\tau(x_1, \ldots, x_n)$ induces a *term function* $(a_1, \ldots, a_n) \mapsto \tau^A(a_1, \ldots, a_n) : A^n \to A$ by assigning values $a_1, \ldots, a_n \in A$ to the variables $x_1, \ldots, x_n$.

In a topological algebra, all the term functions are continuous. But in a semi-topological algebra term functions need not be separately continuous; for example, the term function $x \mapsto x \cdot x$ need not be (separately) continuous in a semitopological semigroup, as one sees from simple examples such as the one-point compactification S of an infinite discrete set T with distinguished element 0, and with multiplication defined in S by $x^2 = x$ for $x \in T$ and $xy = 0$ otherwise. The problem is that in the term $x \cdot x$ the variable x occurs twice. If in the term τ each variable occurs at most once — such terms are called *linear* — then one easily proves by induction on the structure of the term:

Lemma 4.3 *In a semitopological Σ-algebra the linear term functions are separately continuous.*

Linear term functions have another noteworthy property (see [8, Lemma 2]):

Lemma 4.4 *For a linear term $\tau(x_1, \ldots, x_n)$ one has for arbitrary subsets $A_1, \ldots, A_n$ of an algebra A:*

$$\tau(A_1, \ldots, A_n) = \{\tau(a_1, \ldots, a_n) \mid a_i \in A_i, i = 1, \ldots, n\}$$

Again this property is not true for the nonlinear term $\tau(x) = x \cdot x$ over a semigroup as $\tau(A_1) = A_1 \cdot A_1 = \{a \cdot a' \mid a, a' \in A_1\}$ properly contains $\{\tau(a) = a \cdot a \mid a \in A_1\}$, in general.

We say that in an algebra A the *equational law*

$$\tau(x_1, \ldots, x_n) = \rho(x_1, \ldots, x_n)$$

holds, where τ and ρ are terms, if

$$\tau^A(a_1, \ldots, a_n) = \rho^A(a_1, \ldots, a_n) \text{ for all } a_1, \ldots, a_n \in A,$$

i.e., if τ and ρ induce the same term function on A. In an analogous way we say that in a partially ordered algebra A the *inequational law* $\tau \le \rho$ holds, if

$$\tau^A(a_1, \ldots, a_n) \le \rho^A(a_1, \ldots, a_n) \text{ for all } a_1, \ldots, a_n \in A.$$

In a semitopological algebra an inequational law always refers to the specialization order. An (in)equational law is called linear if both τ and ρ are linear terms. It

is trivial to note that, in a semitopological algebra, a term function τ is separately continuous, whenever there is a linear term function ρ such that the equational law $\tau = \rho$ holds in A.

Example 4.5 A *cone* is an algebra with a binary operation $+$, a constant ($=$ nullary operation) 0 and a unary operation $a \mapsto r \cdot a$ for every real number $r \geq 0$ satisfying the usual equational laws:

$$x + (y + z) = (x + y) + z$$

$$x + y = y + x$$

$$x + 0 = x$$

$$r \cdot (x + y) = r \cdot x + r \cdot y$$

$$(r + s) \cdot x = r \cdot x + s \cdot x$$

$$(rs) \cdot x = r \cdot (s \cdot x)$$

$$1 \cdot x = x$$

$$0 \cdot x = 0$$

From these laws it follows that every term function equals a linear one of the form $\sum_{i=1}^{n} r_i x_i$. Thus, in a semitopological cone, every term function is separately continuous. Notice that all the axioms for cones are linear with one exception: $(r + s)x = rx + sx$.

In general, distributive laws are nonlinear, whereas associativity and commutativity are typical for linear laws. Idempotency is nonlinear.

Proposition 4.6 *When A is a semitopological algebra, then A and ΓA satisfy the same linear equational and inequational laws.*

Proof. Let A be a semitopological algebra. As A is topologically and algebraically embedded in ΓA, all inequational laws satisfied in ΓA are also satisfied in A. Conversely, let $\tau \leq \rho$ be a linear inequational law holding in A, that is,

$$\tau(a_1, \ldots, a_n) \leq \rho(a_1, \ldots, a_n) \text{ for all } (a_1, \ldots, a_n) \in A^n$$

Let $A_1, \ldots, A_n$ be closed subsets of A. By Lemma 4.4 we have for the linear terms τ and ρ:

$$\tau(A_1, \ldots, A_n) = \{\tau(a_1, \ldots, a_n) \mid a_i \in A_i, i = 1, \ldots, n\}$$

$$\rho(A_1, \ldots, A_n) = \{\rho(a_1, \ldots, a_n) \mid a_i \in A_i, i = 1, \ldots, n\}$$

As closed sets are lower sets,

$$\tau^\gamma(A_1, \ldots, A_n) = \{\tau(a_1, \ldots, a_n) \mid a_i \in A_i, i = 1, \ldots, n\}^-$$
$$\subseteq \{\rho(a_1, \ldots, a_n) \mid a_i \in A_i, i = 1, \ldots, n\}^-$$
$$= \rho^\gamma(A_1, \ldots, A_n)$$

This shows that the inequational law $\tau \leq \rho$ holds in ΓA. The statement for equational laws is now immediate, since an equational law is satisfied if and only if the two corresponding inequality laws are satisfied. □

This proposition subsumes Proposition 3.5, as the laws for associativity and commutativity are linear.

The special case of Proposition 4.6 where A is a discrete algebra, and hence $\Gamma A = \mathfrak{P}A$, the set of all nonempty subsets, has been previously established by Gautam [7] and Grätzer and Lakser [8] for equational laws. Indeed the result extends to equational laws derived from linear equational laws by identification of variables, provided the equational law holds for a whole variety containing A.

Since a partially ordered algebra A is a topological algebra if we endow it with the A-discrete topology, the collection $\mathfrak{L}A$ of nonempty lower sets (the nonempty A-discrete closed sets) ordered by inclusion becomes an ordered algebra if every basic operation $\mu: A^n \to A$ is lifted to an operation $\mu^\gamma: (\mathfrak{L}A)^n \to \mathfrak{L}A$ by defining

$$\mu^\gamma(A_1, \ldots, A_n) = \ \downarrow\!\mu(A_1 \times \cdots \times A_n)$$
$$= \ \downarrow\!\{\mu(a_1, \ldots, a_n) \mid a_i \in A_i, i = 1, \ldots, n\}.$$

Proposition 4.6 yields:

Corollary 4.7 *For a partially ordered algebra A, the linear equational resp. inequational laws holding in $\mathfrak{L}A$ are exactly the linear equational resp. inequational laws holding in A.*

Remark 4.8 The space ΓX of nonempty closed subsets of a space X always carries a semilattice operation, namely binary union $A_1 \cup A_2$, which is (jointly) continuous with respect to the weak topology for obvious reasons. Besides the equational laws of idempotence, associativity and commutativity this semilattice operation is *inflationary* in the sense that $A_1 \subseteq A_1 \cup A_2$. As subset inclusion is the specialization order for the weak topology on ΓX, this means that the semilattice operation union satisfies the inequational law

$$\text{(I)} \qquad\qquad\qquad\qquad x_1 \leq x_1 \vee x_2$$

A. Schalk [15, Theorem 6.9] has shown that $(\Gamma X, \cup)$ is the free sober topological inflationary semilattice over X, i.e., for every sober topological inflationary semilattice S and every continuous map $f: X \to S$, there is a unique continuous semilattice homomorphism $\widetilde{f}: \Gamma X \to S$ such that $\widetilde{f} \circ \eta_X = f$. Of course, the Hoare power domain (Scott-closed subsets of a dcpo) was considered long before Schalk's thesis, where one can find appropriate references, and the generalization to the space of closed subsets of a topological space was considered by M. Smyth in [16].

For a semitopological algebra A, we may endow the algebra ΓA with the additional semilattice operation $\cup$. For every basic operation $\mu: A^n \to A$ one has

$$\text{(D)} \qquad \mu^\gamma(A_1 \cup A_1', A_2, \ldots, A_n) = \mu^\gamma(A_1, A_2, \ldots, A_n) \cup \mu^\gamma(A_1', A_2, \ldots, A_n)$$

and similarly for the coordinates $i = 2, \ldots, n$ as one easily verifies. Thus, the

semilattice operation satisfies the following distributivity laws

$$\mu(x_1 \vee x_1', x_2, \ldots, x_n) = \mu(x_1, x_2, \ldots, x_n) \vee \mu(x_1', x_2, \ldots, x_n)$$

and similarly for the coordinates $i = 2, \ldots, n$ for every $\mu \in \Sigma$. We conjecture that $(\Gamma A, \cup)$ is a kind of free sober semitopological inflationary semilattice algebra satisfying the distributivity laws (D). But this observation is outside the main thrust of this paper.

It is a well-known fact that a continuous function from a dense subset of a topological space to a Hausdorff space has at most one continuous extension. The following is a variant for separately continuous functions.

Lemma 4.9 *Let $X_1, \ldots, X_n$ be topological spaces with dense subspaces $D_1, \ldots D_n$. Then any function $f : \prod_{i=1}^{n} D_i \to Y$, Y Hausdorff, has at most one separately continuous extension to $\prod_{i=1}^{n} X_i$.*

Proof. Let $g, h : \prod_{i=1}^{n} X_i \to Y$ be separately continuous extensions of f. For any $(x_2, \ldots, x_n) \in \prod_{i=2}^{n} D_i$, g and h restricted to $X_1 \times \{x_2\} \times \cdots \times \{x_n\}$ are continuous and agree with f, and hence each other, on the dense subset $D_1 \times \{x_2\} \times \cdots \times \{x_n\}$. Hence they agree on $X_1 \times \{x_2\} \times \cdots \times \{x_n\}$. It follows that they agree on $X_1 \times \prod_{i=2}^{n} D_i$. Proceeding one coordinate at a time (as in the proof of Lemma 3.4(i)), we eventually conclude that f and g agree on $\prod_{i=1}^{n} X_i$. This shows the uniqueness of any possible continuous extension.   □

The *dual A-discrete topology* has all lower sets for its set of open sets. Note that any order-preserving map between posets is continuous for both the A-discrete and the dual A-discrete topologies.

We can convert a T_0-space into a bitopological space by assigning it the the dual A-discrete topology as a second topology, where the dual A-discrete topology is defined with respect to the order of specialization. We call the join (or patch) of these two topologies the *strong topology*. In the case of a topological space X, we denote ΓX equipped with the strong topology by $\Gamma_s X$.

Remark 4.10 If a function $f : X \to Y$ between T_0-topological spaces is continuous, then it is order-preserving with respect to the specialization order (see e.g. Section 0.5 of [9]). Hence it is continuous with respect to the dual A-topology, and thus continuous with respect to the strong topology. Thus Γ_s extends to an endofunctor on the category of topological spaces.

Lemma 4.11 *Let X be a T_0-space.*

  (i) *The specialization order is a closed order with respect to the strong topology.*

  (ii) *The space X equipped with the strong topology is a totally disconnected Hausdorff space.*

Proof. Let $x \nleq y$. The $\downarrow y$ is open in the dual A-topology and $X \setminus \downarrow y$ is open in the given T_0-topology, since $\overline{\{y\}} = \downarrow y$. Thus $X \setminus \downarrow y \times \downarrow y$ is an open set containing

(x, y) that misses $\leq= \{(u, v) : u \leq v\}$, which establishes (i). It follows also that $\downarrow y$ is a clopen set missing x and similarly $\downarrow x$ is a clopen set missing y if $y \not\leq x$. Thus (ii) follows. $\qquad\square$

Remark 4.12 We recall a common construction for the sobrification X^s of a T_0-space X as the subspace of ΓX consisting of all irreducible closed sets. We call this the *standard sobrification*. There is a homeomorphic embedding of $\eta_X : X \to X^s$, the sobrification map, sending x to $\overline{\{x\}} = \downarrow x$, the lower set of x with respect to the order of specialization. The space X^s may be alternatively characterized as the strong closure in ΓX of the embedded image of X. In particular, X is strongly dense in X^s. For $f : X \to Y$ continuous, we have seen (Remark 4.10) that $\Gamma f : \Gamma X \to \Gamma Y$ is strongly continuous, and hence must carry the strong closure X^s of X into the strong closure Y^s of Y. This restriction and corestriction of Γ gives the sobrification functor.

Corollary 4.13 *A separately continuous function* $f : \prod_{i=1}^n X_i \to Y$ *extends uniquely to a separately continuous function* $f^s : \prod_{i=1}^n X_i^s \to Y^s$. *If* $g : \prod_{i=1}^n X_i \to Y$ *is a second separately continuous function such that* $f(x_1 \ldots, x_n) \leq g(x_1 \ldots, x_n)$ *for all* $(x_1 \ldots, x_n) \in \prod_{i=1}^n X_i$, *then* $f^s \leq g^s$ *holds for the extension on all of* $\prod_{i=1}^n X_i^s$.

Proof. The extension $f^\gamma : \prod_{i=1}^n \Gamma X_i \to \Gamma Y$ defined by $f^\gamma(A_1, \ldots, A_n) = (f(\prod_{i=1}^n A_i))^-$ is separately continuous by Lemma 3.4(iii). It follows that f^γ is strongly separately continuous (see Remark 4.10). Since the closure of the product is the product of the closures, the strong closure of $\prod_{i=1}^n X_i$, as an embedded subspace of $\prod_{i=1}^n \Gamma X_i$, is $\prod_{i=1}^n X_i^s$, which by Lemma 3.4(i) is carried into Y^s, the strong closure of Y. The uniqueness of the extension follows from Lemma 4.9. The claim on the preservation of inequalities follows from Lemma 4.11(i). $\qquad\square$

Remark 4.14 The function f^s in the previous corollary may be viewed as an extension of the morphism component of the sobrification functor, since it is a standard result that the sobrification of the product is the product of the sobrifications.

Corollary 4.15 *Let* $f, g : \prod_{i=1}^n X_i^s \to Y^s$ *be two separately continuous functions. If* f *and* g *agree on* $\prod_{i=1}^n X_i$, *then they are equal. If* $f \leq g$ *when restricted to* $\prod_{i=1}^n X_i$, *then* $f \leq g$ *on all of* $\prod_{i=1}^n X_i^s$

Proof. In the first case, both are separately continuous extensions of their restriction to $\prod_{i=1}^n X_i$ and hence are equal by the uniqueness in Corollary 4.13. The second case is essentially a restatement from Corollary 4.13. $\qquad\square$

We apply the preceding results to the sobrification of a semitopological Σ-algebra and we obtain that the sobrification functor induces an endofunctor on the category of semitopological Σ-algebras and continuous Σ-algebra homomorphisms.

Proposition 4.16 *The basic operations μ of a semitopological algebra A extend in a unique way to separately continuous operations μ^s on the sobrification A^s which in this way becomes a semitopological algebra of the same signature as A.*

If $f: A \to B$ is a continuous homomorphism of semitopological Σ-algebras, then $f^s : A^s \to B^s$ is also a continuous homomorphism.

Proof. We may apply Corollary 4.13 to the basic operations μ of A in order to see that they extend in a unique way to separately continuous operations μ^s on the sobrification A^s which in this way becomes a semitopological algebra of the same signature as A.

By Proposition 4.2 a continuous homomorphism $f: A \to B$ of semitopological algebras induces a continuous homomorphism $\Gamma f : \Gamma A \to \Gamma B$. As remarked in the proof of Corollary 4.13 the restriction of Γf to A^s carries A^s into B^s. Hence f continuously extends to a homomorphism from A^s to B^s. Since the continuous extension is unique, it must agree with f^s, and thus f^s is a homomorphism.   □

As for ΓA, the linear equational and inequational laws satisfied by a semitopological algebra A are also satisfied in the sobrification A^s. This is an immediate consequence of Corollary 4.15, as linear term functions are separately continuous. Conversely, the linear equational and inequational laws satisfied by A^s are also satisfied by A, as A may be considered to be a subalgebra of A^s via the embedding $\eta_A = (x \mapsto \overline{\{x\}})$. We have not investigated the question whether there may be more than just the linear equational laws inherited by A^s from A. The problem is that nonlinear term functions need not be separately continuous in semitopological algebras. Since in topological algebras all term functions are continuous, we have:

Proposition 4.17 *The basic operations, as well as the term functions, of a topological Σ-algebra A extend uniquely to its sobrification A^s. The topological Σ-algebra A^s so obtained satisfies the same equational and inequational laws as A.*

Proof. The sobrification of a (continuous) term function is again continuous from the sobrification of the product, which is the product of the sobrifications, to the sobrification A^s. Hence A^s is a topological Σ-algebra and by Corollary 4.15 satisfies any equational resp. inequational law that A does.   □

The next examples illustrate limitations that one encounters in trying to extend equational laws.

Example 4.18 Let $A = \{0, 1, 2, 3, \ldots\}$ equipped with the cofinite topology (the nonempty open sets are the cofinite sets), the nullary operation sending every element to 0, and the binary operation $\mu(x, y) = |x - y|$. The last operation is separately, but not jointly, continuous. The sobrification is $A^\top = A \cup \{\top\}$, where the nonempty open sets are the cofinite sets containing $\top$. The semicontinuous function μ extends to $\mu^s : A^s \times A^s \to A^s$ given by $\mu^s(x, \top) = \mu^s(\top, x) = \top$. The original algebra A satisfies the equality $\mu(x, x) = 0$, but this is is not true for A^s, since $\mu(\top, \top) = \top$. Thus equational laws in semitopological algebras need not extend to the sobrification.

Example 4.19 We consider again the example preceding Lemma 4.3, the one-point compactification S of an infinite discrete set T with distinguished element 0, and with multiplication defined in S by $x^2 = x$ for $x \in T$ and $xy = 0$ otherwise. Then T

is dense in the compact Hausdorff semitopological semigroup S, and the equational law $x^2 = x$ holds in T, but not in S. This shows that in the setting of Hausdorff separately continuous algebras, equational laws in an algebra need not extend to a semigroup compactification, even when the operation(s) extend semicontinuously.

5   The D-completion and extension of functions

We define a subset A of a poset P to be *d-closed* if for every directed subset $D \subseteq A$ that possesses a supremum $\vee D$, it is the case that $\vee D \in A$. It is immediate that an arbitrary intersection of d-closed sets is again d-closed and almost immediate that the same is true for finite unions (see [11]). Hence the d-closed sets form the closed sets for a topology, called the *d-topology*. We define the *d-topology of a T_0-space* to be the d-topology of the associated order of specialization.

The next lemma is an immediate consequence of the definition of the d-closed sets.

Lemma 5.1 *Let A be a subset of a monotone convergence space X. Then the closure of A in the d-topology is the smallest sub-dcpo, i. e., the smallest subset closed with respect to directed sups, containing A.*

There are other quickly derived elementary facts about the d-topology (see [11, Lemmas 5.1,5.3]).

Lemma 5.2 (1) *A lower set is Scott-closed if and only if it is d-closed.*
(2) *Any upper set is d-closed, and hence any lower set is d-open.*
(3) *If $\vee D$ exists for a directed set D, then the directed set converges to $\vee D$ in the d-topology.*
(4) *A function $f : P \to Q$ between posets is Scott-continuous if and only if it preserves all existing directed sups if and only if it is d-continuous and order preserving.*

The theory of D-completions was the topic of [11].

Definition 5.3 For a topological space X, the D-completion $\xi_X : X \to X^d$ is defined (up to categorical equivalence) by the universal property that given a continuous $f : X \to Y$ into a monotone convergence space Y, there exists a unique continuous function $f^d : X^d \to Y$ such that $f^d \circ \xi_X = f$, i.e., such that the following diagram commutes:

$$X \xrightarrow{\ \xi_X\ } X^d$$
$$f \searrow \quad \downarrow f^d$$
$$Y$$

Equivalently, and more directly, we may define X^d to be the subspace of the standard sobrification X^s obtained by taking the d-closure of the topologically embedded image $\eta_X(X)$ in X^s, where the d-topology is that associated with the order of specialization of X^s, that is, X^d as a topological space is equal to $(cl_d(\eta_X(X)), \tau)$,

where τ is the subspace topology from X^s. As a set X^d is also the smallest sub-dcpo of X^s (or ΓX) containing $\eta_X(X)$. In this setting we take the D-*completion* to be the corestriction of η_X from X into X^s, denoted $\xi_X : X \to X^d$. We often speak simply (and loosely) of X^d as the d-completion of X. We refer to this construction as the *standard* D-*completion*

It follows readily from the preceding considerations and is worked out in detail in [11] that the D-completion defines a functor that is a reflector (left adjoint to the inclusion) from the category of T_0-spaces to the category of monotone convergence spaces (morphisms in both cases being continuous maps).

The next theorem, Theorem 6.7 of [11], asserts that up to isomorphism the D-completion of X is the unique monotone convergence space completion in which X is d-dense.

Theorem 5.4 *Let* $j : X \to Y$ *be a topological embedding of* X *into a monotone convergence space* Y, *and let* $\tilde{X}$ *be the d-closure of* $j(X)$ *in* Y. *Then* $j : X \to \tilde{X}$ *is a* D-*completion.*

Lemma 5.5 *For finitely many spaces* $X_1, \ldots, X_n$, *the topological embeddings* $\xi_i : X_i \to X_i^d$ *yield a product embedding*

$$\xi : X_1 \times \cdots \times X_n \to X_1^d \times \cdots \times X_n^d$$

for which the image of ξ *is d-dense in* $X_1^d \times \cdots \times X_n^d$. *Hence* ξ *is a* D-*completion.*

Proof. A product of topological embeddings is a topological embedding. It follows from Lemma 5.4 of [11] (and induction) that the image of $\prod_{i=1}^n X_i$ is d-dense in $\prod_{i=1}^n X_i^d$, and thus by Theorem 5.4 that the map ξ is a D-completion. $\quad\square$

Proposition 5.6 *A separately continuous function* $f : \prod_{i=1}^n X_i \to Y$ *extends uniquely to a separately continuous function* $f^d : \prod_{i=1}^n X_i^d \to Y^d$. *The function* f^d *is Scott-continuous.*

Proof. The proof follows along the lines of that of Corollary 4.13, except that the d-topology plays the role of the strong topology. The last assertion follows from Lemma 2.4. $\quad\square$

We label the extension of a continuous or separately continuous function f to the D-completion by f^d, or alternatively by $\mathsf{D}f$.

We apply the previous proposition to the basic operations $\mu : A^n \to A$ of a semitopological Σ-algebra A which are are separately continuous and we obtain:

Corollary 5.7 *The basic operations of a semitopological* Σ-*algebra* A *extend uniquely to separately continuous operations on the* D-*completion* A^d *which in addition are Scott-continuous. In this way the* D-*completion* A^d *becomes a semitopological algebra of the same signature as* A.

In an analogous way the following corollary follows from Proposition 4.17. In the next section we show that the equational and inequational laws extend to D-completions in the semitopological setting as well.

Corollary 5.8 *The basic operations of a topological Σ-algebra A extend uniquely to continuous operations on the* D*-completion A^d. In this way the* D*-completion A^d becomes a topological algebra of the same signature as A, which obeys the same equational and inequational laws as A.*

The functoriality of the D-completion on the level of semitopological Σ-algebras is established through the following:

Corollary 5.9 *A continuous homomorphism $f : A \to B$ between semitopological Σ-algebras extends uniquely to a continuous homomorphism $f^d : A^d \to B^d$.*

Proof. If $f : A \to B$ is a continuous homorphism of semitopological algebras, then by Proposition 4.16 f extends to a continuous homomorphism $f^s : A^s \to B^s$. The restriction of f^s to A^d will be d-continuous, hence have image contained in B^d. Thus f extends to a continuous homomorphism from A^d to B^d. Since the continuous extension is unique the homomorphic extension is f^d. □

The universal property of the D-completion carries over to the algebraic setting. The proof follows easily from Corollary 5.9 since $B = B^d$:

Proposition 5.10 *Let A, B be semitopological algebras of the same signature, and let $f : A \to B$ be a continuous homomorphism. If B is additionally a monotone convergence space, then the map f extends uniquely to a continuous homomorphism $f^d : A^d \to B$.*

In the next section we show that A^d satisfies the same equational and inequational laws as A.

6 Directed induction

As mathematical induction is fundamental for reasoning involving the natural numbers, so what we call "directed induction" is fundamental for reasoning about D-completions.

Proposition 6.1 (*Principle of Directed Induction*) *If a property holds for all members of a space X considered as a subspace of X^d and if whenever it holds for all members of a directed subset D of X^d, it holds for $\sup D$, then the property holds for all members of X^d.*

Proof. Let us call the property, property P. We set

$$A := \{x \in X^d : x \text{ satisfies Property } P\}.$$

Then by hypothesis $X \subset A$, and A is closed under directed sups, i.e., A is d-closed. By definition X^d is the d-closure of X, and hence $A = X^d$. □

We remark that the principle of directed induction is reminiscent of Scott induction (see, e.g., [6]): to prove a property P of the least fixed point of a functional

$Y(F)$, it is enough to prove $P(\perp)$ and that P is closed under F (that is, $P(x)$ implies $P(F(x))$), provided P is closed under directed sups.

If a space X is equipped with the structure of a semitopological algebra, then the Principle of Directed Induction can be used to show that a wide range of algebraic identities and inequalities extend from X to X^d, even though they may fail to extend to ΓX, or even the sobrification. We illustrate with two examples.

Proposition 6.2 *If $*: X \times X \to X$ is a separately continuous idempotent operation on X, the same is true of its extension to X^d.*

Proof. The function $F: X^d \to X^d \times X^d \to X^d$ defined by $x \mapsto (x, x) \mapsto x * x$ is a composition of Scott-continuous maps, hence Scott-continuous. By hypothesis F agrees with the identity map 1_{X_d} on X. If these two Scott-continuous maps agree on directed set D, then by Scott continuity they agree at $d = \sup D$. By the principle of directed induction they agree on X^d, which establishes the proposition.   □

We remark that Corollary 4.15 cannot be used to extend the conclusion to the sobrification, since the extended composition need not be separately continuous (equal continuous in this case).

Since the D-completion $\xi_X : X \to X^d$ is a homeomorphic embedding, the order of specialization on X^d restricted to X agrees with the order of specialization on X. This suggests the consideration of the extension of inequalities from X to X^d.

Lemma 6.3 *If $f, g: Y^d \to X^d$ are Scott-continuous functions such that $f(y) \leq g(y)$ for all $y \in Y$, then $f(y) \leq g(y)$ for all $y \in Y^d$.*

Proof. A straightforward application of proof by directed induction.   □

As separately continuous functions are Scott-continuous, we infer:

Corollary 6.4 *If $f, g : \prod_{k=1}^{n} X_i \to X$ are separately continuous and $f(x_1, \ldots, x_n) \leq g(x_1, \ldots, x_n)$ for all $(x_1, \ldots, x_n) \in \prod_{k=1}^{n} X_i$, then $f^d(x_1, \ldots, x_n) \leq g^d(x_1, \ldots, x_n)$ for all $(x_1, \ldots, x_n) \in \prod_{k=1}^{n} X_i^d$.*

Consider now a semitopological algebra A. By 5.7 we know that the D-completion A^d is a semitopological algebra, too. The basic operations μ of A extend uniquely to separately continuous operations on A^d. The same holds for linear term functions. For arbitrary term functions we have:

Lemma 6.5 *For any term $\tau(x_1, \ldots, x_n)$ the term function $\tau^A: A^n \to A$ is Scott-continuous and extends uniquely to a Scott-continuous function $\tau^{A^d}: (A^d)^n \to A^d$.*

Proof. The term $\tau(x_1, \ldots, x_n)$ need not be linear; there may be multiple occurrences of the variables as in the law of idempotency. The straightforward Lemma 1 in [8] tells us that there are an integer $m \geq n$, a linear term $\check{\tau}(x_1, \ldots, x_m)$ and a surjection $\varphi: \{1, \ldots, m\} \to \{1, \ldots, n\}$ such that $\tau(x_1, \ldots, x_n) = \check{\tau}(x_{\varphi(1)}, \ldots, x_{\varphi(m)})$. (One just has to introduce new variables for multiple occurrences of variables.) Now consider the map:

$$F: (a_1, \ldots, a_n) \mapsto (a_{\varphi(1)}, \ldots, a_{\varphi(m)}): (A^d)^n \to (A^d)^m$$

which is continuous. The term function from $\check{\tau} \colon A^m \to A$ induced by the linear term $\check{\tau}$ is separately continuous by Lemma 4.3. Thus it has a unique separately continuous extension $\check{\tau}^d \colon (A^d)^m \to A^d$ by Proposition 5.6. The maps $F \colon (A^d)^n \to (A^d)^m$ and $\check{\tau}^d \colon (A^d)^n \to A^d$ being Scott-continuous, their composition $\check{\tau}^d \circ F$ is Scott-continuous, too. As $(\check{\tau}^d \circ F)(a_1, \ldots, a_n) = \tau(a_1, \ldots, a_n)$ for all $(a_1, \ldots, a_n) \in A^n$, the function $(\check{\tau}^d \circ F)$ extends the term function τ on A^n. The uniqueness follows from Proposition 5.6.   □

Suppose now that A satisfies an inequational law

$$\tau(x_1, \ldots, x_n) \leq \rho(x_1, \ldots, x_n)$$

that is, $\tau(a_1, \ldots, a_n) \leq \rho(a_1, \ldots, a_n)$ for all $(a_1, \ldots, a_n) \in A^n$. By Lemma 6.5 the functions τ and ρ have Scott-continuous extensions $\check{\tau}, \check{\rho}$ to A^d. By Lemma 6.3 we infer that the inequality holds for all $(a_1, \ldots, a_n) \in (A^d)^n$. As the equational law $\tau = \rho$ is equivalent to the conjunction of the inequational laws $\tau \leq \rho$ and $\rho \leq \tau$, we have:

Theorem 6.6 *A semitopological Σ-algebra and its D-completion satisfy the same equational and inequational laws.*

We can apply this theorem to Example 4.5 and conclude that

Corollary 6.7 *The D-completion of a semitopological cone is again a semitopological cone.*

Recall that a topological space is *conditionally up-complete* if every directed subset that is bounded above has a supremum to which it converges. In Section 8 of [11] it is shown that every topological space has a conditional D-completion, a strongly dense embedding that is universal among continuous maps into conditionally up-complete spaces. Indeed for a space X this conditional D-completion may be obtained as the lower set of the image of X in X^d, where X^d is the D-completion. It follows directly from the fact that the basic operations of a semitopological algebra A and their extensions to A^d are order-preserving that this lower set is a subalgebra of A^d.

Corollary 6.8 *The basic operations of a semitopological algebra A extend to the conditional D-completion, making it a semitopological algebra obeying all the equational and inequational laws of A.*

7   Presenting dcpos and dcpo-algebras

In this section we show that the construction of freely generated dcpos by Jung, Moshier and Vickers [10] fits perfectly into the topological framework of D-completions as considered in [11] and recalled in Section 5 of this paper and that their results on extending algebraic operations under preservation of equational and inequational laws to the freely generated dcpos are subsumed by our results above. The methods applied are surprisingly similar.

Jung, Moshier and Vickers [10] define a *dcpo presentation* to consist of a set P of generators equipped with a preorder $\lesssim$ and a relation $a \lhd U$, called the *covering relation*, between elements a and directed subsets U of P. We suppose that $a \lhd \{b\}$ whenever $a \lesssim b$ in P, thus coding the preorder into the relation $\lhd$. A map $f: P \to Q$ between dcpo presentations P and Q is said to *preserve covers* if

$$a \lhd U \Rightarrow f(a) \lhd f(U)$$

Note that such a map preserves the preorder and hence maps directed sets to directed sets. We denote by PRES the category of dcpo presentations and covering preserving maps.

A dcpo D carries a canonical structure of a dcpo presentation

$$a \lhd U \iff a \leq \bigsqcup{}^{\uparrow} U.$$

A Scott-continuous map between dcpos is the same as a cover preserving map for these canonical covers. In this way we get a full forgetful functor from the category DCPO of dcpos and Scott-continuous maps to the category PRES of dcpo presentations. In [10, Theorem 2.7] it is shown:

Theorem 7.1 *The forgetful functor from* DCPO *to* PRES *has a left adjoint, that is, for every dcpo presentation P there is a dcpo $\overline{P}$ and a cover preserving map $\eta_P: P \to \overline{P}$ with the universal property that, for every cover preserving map f from P into a dcpo D, there is a unique Scott-continuous map $\overline{f}: \overline{P} \to D$ such that $\overline{f} \circ \eta_P = f$.*

One also says that $\overline{P}$ is the dcpo freely generated by the dcpo presentation P.

Let P be a dcpo presentation. We define a subset A of P to be $\lhd$-*closed* if, whenever U is a directed set contained in A and $a \lhd U$, then $a \in A$; we denote by $\Gamma_0 P$ the collection of all $\lhd$-closed sets and by ΓP the nonempty ones. It is almost straightforward to verify that arbitrary intersections and finite unions of $\lhd$-closed subsets are $\lhd$-closed. Thus, the $\lhd$-closed sets are the closed sets of a topology that we denote by $\tau_\lhd$. The $\lhd$-closed sets have been called C-ideals in [10] and the lattice of $\lhd$-closed sets is their lattice $C\text{-}\mathsf{Idl}(P)$ of C-ideals. Observe that a map between two dcpo presentations is cover preserving if and only if it is continuous for the respective $\tau_\lhd$-topologies. Further, the $\lhd$-closed sets in a dcpo D are precisely the Scott-closed sets; thus, the topology $\tau_\lhd$ coincides with the Scott topology.

We denote by P^d the (standard) D-completion of the space P with the topology $\tau_\lhd$ and by $\eta_P: P \to P^d$ the map assigning to every $a \in P$ the smallest $\lhd$-closed set, i.e., the smallest C-ideal, containing a, which is the same as $\downarrow a = \{x \in P \mid x \lesssim a\}$. Comparing the standard D-completion with the definition of the dcpo $\overline{P}$ freely generated by P in [10] we see that these two constructions are identical. Thus Theorem 7.1 can be viewed as a special case of Theorem 5.4 above. It even gives a slightly stronger result, as it asserts that the universality property holds with respect to continuous maps f from P into arbitrary monotone convergence spaces D instead of dcpos.

After presenting dcpos, Jung, Moshier and Vickers proceed to presentations of dcpo-algebras of some signature Σ. A dcpo-algebra is an algebra D equipped with a directed complete partial order in such a way that all the basic operations μ are Scott-continuous. The latter is equivalent to saying that the operations μ are separately continuous with respect to the Scott topology. In this way a dcpo-algebra may be considered to be a semitopological algebra with the Scott topology.

In [10] an n-nary operation $\mu \colon P^n \to P$ on a dcpo presentation P is said to be *stable* if it is cover preserving separately in each coordinate. This is equivalent to saying that μ is separately continuous for the topology $\tau_\triangleleft$.

Now consider a dcpo presentation P which is also a Σ-algebra such that all the basic operations μ are stable. This is equivalent to saying that P with the topology $\tau_\triangleleft$ is a semitopological Σ-algebra. The algebraic structure can be lifted to the set of nonempty $\triangleleft$-closed subsets $\Gamma P = C\text{-}\mathsf{Idl}(P)$, which then becomes a semitopological Σ-algebra with respect to the weak topology satisfying the same linear equational and inequational laws as P by Proposition 4.6. Further, the algebra structure on P extends in a unique way to a Scott-continuous algebra structure on the D-completion $P^d = \overline{P}$ in such a way that it becomes a dcpo algebra satisfying the same equational and inequational laws as P by Theorem 6.6. In this way, the results in this paper subsume the results in section 3 and 4 of [10].

Remark 7.2 The considerations of this section generalize in a straightforward manner from preordered sets to topological spaces. Let (X, τ) be a (not necessarily T_0) topological space and suppose that the corresponding (pre)order of specialization is equipped with a dcpo presentation. We define a subset A of X to be $\triangleleft$-*closed* if it is τ-closed and if, whenever U is a directed set contained in A and $a \triangleleft U$, then $a \in A$. (We observe that in this setting it is not necessary to code the (pre)order into the dcpo presentation, since the closed sets will automatically be lower sets.) The collection $\Gamma_0 P$ of all $\triangleleft$-closed sets form the closed sets of a topology and the desired completion is the D-completion of this topological space. In this context we might call the dcpo presentation for the space X a D-*presentation* and the resulting $\overline{X}$ the monotone convergence space *freely generated* by the D-presentation. One then obtains a topological version of Theorem 7.1. The dcpo case follows as a special case by endowing a preordered space with the A-discrete topology.

8 Directed completions and ideal completions

An important special case for D-completions are posets with the Scott topology. The Scott topology on posets and some of its properties were already addressed at the beginning of Section 2. The D-completion of a poset P with the Scott topology agrees with the *dcpo-completion* which is a dcpo $\widetilde{P}$ together with a map ξ which is an embedding for the respective Scott topologies of P onto a d-dense subset of $\widetilde{P}$ as explained in [11, Section 7].

The dcpo-completion of a poset P just mentioned coincides with the dcpo freely generated by the following dcpo presentation in the sense of [10] as discussed in the previous section: we define the covering relation $a \triangleleft U$ for a directed set U to hold

iff U has a supremum in P and $a \leq \bigvee^{\uparrow} U$. The associated $\lhd$-closed sets are just the Scott-closed ones.

Accordingly, we can consider partially ordered algebras A for which the basic operations $\mu \colon A^n \to A$ are Scott-continuous. The D-completion or equivalently the dcpo freely generated by the dcpo presentation of A will yield a dcpo-algebra. As noted in Remark 2.3, the Scott topology on A^n may be strictly finer than the product of the Scott topologies on A. Thus, the basic operations μ need not be jointly continuous, but they are separately continuous and we may apply our results from the previous sections or alternatively the results in [10]. In particular, Corollary 5.7 and Theorem 6.6 yield the following:

Proposition 8.1 *Let A be a partially ordered algebra with Scott-continuous basic operations μ. These basic operations on A extend uniquely to Scott-continuous operations on the dcpo-completion $\tilde{A}$. The dcpo-algebra $\tilde{A}$ obtained in this way satisfies the same equational and inequational laws as A.*

As a second special case we consider C-spaces as investigated by Erné [3] and Ershov [4], by the latter under the name of α-spaces. A topological space X is a *C-space* if each of its points y has a neighborhood basis of sets of the form $\uparrow x = \{z \in X \mid x \leq z$ for the specialization order$\}$. We write $x \prec y$ iff $\uparrow x$ is a neighborhood of y. In [11, Proposition 9.1] we stated that the D-completion of a C-space X agrees with its sobrification and also with its *round ideal completion* $\mathsf{RI}(X)$ equipped with its Scott topology: A *round ideal* is a directed lower set $I \subseteq X$ with the property that for every $x \in I$ there is a $y \in I$ with $x \prec y$. The set $\mathsf{RI}(X)$ of all round ideals ordered by inclusion is a dcpo and, for C-spaces, even a continuous domain in the sense of [9]. The C-space X is embedded in its round ideal completion $\mathsf{RI}(X)$ via the map $y \mapsto \{x \mid x \prec y\}$. Every separately continuous map on a product of C-spaces is jointly continuous (see [5, Proposition 2]). Thus, every semitopological algebra structure on a C-space is a topological algebra. Using 4.17 or alternatively 6.6 we can summarize:

Proposition 8.2 *Every semitopological algebra A on an underlying C-space is a topological algebra. Its D-completion A^d coincides with its sobrification and also with its round ideal completion $\mathsf{RI}(X)$; hence A^d is a continuous domain with its Scott topology. The operations of the algebra A extend in a unique way to continuous operations on A^d, and the topological algebra A^d satisfies the same equational and inequational laws as A.*

A particular case of a C-space is a continuous poset in the sense of [9] with the Scott topology. We say that A is a *continuous partially ordered algebra* if A is an algebra and a continuous poset such that all basic operations of the algebra are Scott-continuous. If in addition A is a dcpo, that is, a continuous domain, then we say that A is a continuous dcpo-algebra. For the Scott topology, a continuous partially ordered algebra is a topological algebra. From the previous proposition we infer:

Corollary 8.3 *For a continuous partially ordered algebra A the round ideal comple-*

tion $\mathsf{RI}(A)$ *is the sobrification as well as the* D-*completion with respect to the Scott topology. The operations of* A *can be extended in a unique way to Scott-continuous operations on the round ideal completion* $\mathsf{RI}(A)$. *The continuous dcpo-algebra* $\mathsf{RI}(A)$ *obtained in this way satisfies the same equational and inequational laws as* A.

Explicitly the extension of a Scott-continuous operation $\mu\colon A^n \to A$ to the round ideal completion is given by

$$\mu^d(I_1, \ldots, I_n) = \{x \mid x \prec \mu(y_1, \ldots, y_n) \text{ for some } y_i \in I_i, i = 1, \ldots, n\}$$
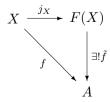
As a third special case we consider partially ordered algebras A with the A-discrete topology. The basic algebra operations are now only supposed to be order preserving, not Scott-continuous. But they are continuous for the A-discrete topology. With respect to the A-discrete topology a poset is a C-space in which the relation $\prec$ coincides with the partial order $\leq$. The round ideals are just the ideals, i.e., the directed lower sets. The round ideal completion $\mathsf{RI}(A)$ coincides with the *ideal completion* $I(A)$, the set of all ideal ordered by containment, which is an algebraic domain. We conclude:

Corollary 8.4 *For a partially ordered algebra* A *the ideal completion* $I(A)$ *with the Scott topology is the sobrification as well as the* D-*completion of* A *with the A-discrete topology. The operations of* A *can be extended in a unique way to Scott-continuous operations on* $I(A)$. *The algebraic dcpo-algebra* $I(A)$ *obtained in this way satisfies the same equational and inequational laws as* A.

9 Free algebras

In this section we develop some fairly standard and familiar categorical constructions and considerations in our context, and hence proceed in a somewhat informal fashion.
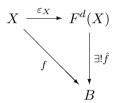
For a given signature Σ and family $\mathcal{E}$ of equational and inequational laws, we denote by $\mathcal{S}(\Sigma, \mathcal{E})$ the category of semitopological Σ-algebras (equipped with the order of specialization) that satisfy all the equational and inequational laws in $\mathcal{E}$ and continuous homomorphisms. There is an inclusion of $\mathcal{S}(\Sigma, \mathcal{E})$ into the category of T_0-spaces which "forgets" the algebraic structure. A straightforward application of the adjoint functor theorem yields for each T_0-space X, a *free algebra* $(F(X), j_X)$ over $\mathcal{S}(\Sigma, \mathcal{E})$ consisting of an algebra $F(X)$ in $\mathcal{S}(\Sigma, \mathcal{E})$ and a map $j_X : X \to F(X)$, such that for any continuous $f : X \to A$, where A is a semitopological algebra in $\mathcal{S}(\Sigma, \mathcal{E})$, there exists a unique continuous homomorphism $\tilde{f} : F(X) \to A$ such that the following diagram commutes:

$$
\begin{array}{ccc}
X & \xrightarrow{\;j_X\;} & F(X) \\
 & \searrow{\scriptstyle f} & \big\downarrow{\scriptstyle \exists! \tilde{f}} \\
 & & A
\end{array}
$$

The canonical map $j_X : X \to F(X)$ need not be a topological embedding or an injection. The equational laws may impose restrictions on the underlying topology; for topological groups, for example, the T_0-axiom implies the Hausdorff separation axiom.

We denote by $F^d(X)$ the D-completion of $F(X)$. By Theorem 6.6 $F^d(X)$ is a semitopological algebra in $\mathcal{S}(\Sigma, \mathcal{E})$.

Proposition 9.1 *Let X be a T_0 space, and let $\varepsilon_X : X \to F^d(X)$ be the composition of the canonical maps $j_X : X \to F(X)$ and $\xi_{F(X)} : F(X) \to F^d(X)$. If $f : X \to B$ is a continuous map from X into a semitopological algebra B in $\mathcal{S}(\Sigma, \mathcal{E})$ that is also a monotone convergence space, then there exists a unique continuous homomorphism $\hat{f} : F^d(X) \to B$ such that $\tilde{f} \circ \varepsilon_X = f$:*

$$
\begin{array}{ccc}
X & \xrightarrow{\ \varepsilon_X\ } & F^d(X) \\
 & {\scriptstyle f}\searrow & \ \Big\downarrow{\scriptstyle \exists! \hat{f}} \\
 & & B
\end{array}
$$

Proof. By the construction of $F(X)$ we have a unique continuous homomorphism $\tilde{f} : F(X) \to B$ such that $\tilde{f} \circ j_X = f$. By Proposition 5.10 there exists a unique continuous homomorphism $\hat{f} : F^d(X) \to B$ extending $\tilde{f}$. Combining these results, obtain the conclusion of the proposition.   □

Of course one could obtain the free algebra that is also a monotone convergence space directly as the adjoint to the inclusion of the category of semitopological algebras of signature Σ satisfying $\mathcal{E}$ that are also monotone convergence spaces into the category of T_0-spaces. One point of the preceding construction is that the free algebra $F(X)$ is algebraically generated by X (since the subalgebra of $F(X)$ generated by the image of X also satisfies the freeness property and the free object is unique). Indeed one can obtain $F(X)$ by first taking the free algebra in the algebraic setting over the set X (which is a quotient of the term algebra), giving it the finest topology for which the basic operations are separately continuous and the inclusion map from X remains continuous, and then taking the T_0-reflection. In specific cases this free object can sometimes be given a fairly concrete representation, and then one has a fairly direct two-step road to the study of $F^d(X)$.

In recent years Alex Simpson has advocated a domain theory based on monotone convergence spaces that are QCB-spaces (quotients of countably based spaces). In [2] he, Battenfeld, and Schröder have shown that the D-completion of a QCB-space is again a QCB-space. Thus if one can show for a class of semitopological algebras $\mathcal{S}(\Sigma, \mathcal{E})$ that the free algebra $F(X)$ is a QCB-space whenever X is, then one obtains by Proposition 9.1 that the free QCB-domain algebra is $F^d(X)$.

We now turn to the category PS of posets and Scott-continuous maps and the category $\mathcal{OA}(\Sigma, \mathcal{E})$ of patially ordered algebras with Scott-continuous operations satisfying the equational and inequational laws in $\mathcal{E}$ and Scott-continuous homomorphisms. Again there is an obvious forgetful functor from the latter category to

the former. Again the adjoint functor theorem yields a free ordered algebra algebra $F(P)$ in $\mathcal{OA}(\Sigma, \mathcal{E})$ over each poset P. Forming the dcpo-completion $F^d(P)$ we obtain again an algebra in $\mathcal{OA}(\Sigma, \mathcal{E})$ by Proposition 8.1 which is the free dcpo-algebra over P satisfying the equational and inequational laws prescribed in $\mathcal{E}$. This free dcpo-algebra over a poset P satisfying the appropriate laws has also been exhibited by Jung, Moshier and Vickers [10] via a covering relation approach, as discussed in Section 8.

10   Concluding remarks and questions

There are several lines of investigation suggested by the developments of the preceding section that remain unresolved. Since an infinite product of C-spaces or of continuous posets need not be a C-space or a continuous poset, it seems that the adjoint functor theorem is not applicable in these cases. Thus one would like to know general sufficient conditions for the free algebra $F(X)$ for $\mathcal{S}(\Sigma, \mathcal{E})$ of the previous section to be a C-space resp. a continuous poset resp. a QCB-space whenever X is. As Abramsky and Jung [1] have shown that the free dcpo-algebra over a continuous dcpo is a continuous dcpo-algebra, one conjectures that an analogous result holds for C-spaces and continuous posets.

There is also the following question: Consider a poset P as a T_0-space with its Scott topology and form the free algebra over P in $\mathcal{S}(\Sigma, \mathcal{E})$. Is it the same as the free algebra over P in $\mathcal{OA}(\Sigma, \mathcal{E})$?

One of the principal motivations of the authors for the study of D-completions has been an interest in their application to the study of cones, particularly T_0-cones, which arise as power domains in probabilistic semantics and in certain approaches to potential theory in mathematics. In Corollary 6.7 we have seen that the cone operations extend to the D-completion. We intend to do a more focused study of cones and the D-completion in future work.

Acknowledgement

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