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Counting Sparse k-edge-connected Hypergraphs with Given Number of Vertices and Edges

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Abstract

In this paper, we provide an asymptotic formula for the number of k-edge-connected r-uniform hypergraphs with n vertices and $m = O(n \log n)$ edges, where $r \geq 3$ and $k \geq 2$ are fixed constants.

Keywords: hypergraphs, enumeration, probabilistic method, connectivity

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1 Introduction

Finding closed formulæ for the number of combinatorial structures with some desired properties is a fundamental problem in Combinatorics. It has a long history of interesting results and a great variety of techniques has been applied to it. For example, the problem of counting graphs with some desired property (e.g. connectedness) has been approached with generating functions, algebraic tools, probabilistic methods, among others (see [9,6,8,7]).

Probabilistic methods have been used with great success to obtain asymptotic formulæ since many enumeration problems are related to computing probabilities in a suitable random model. For example, the problem of counting some classes graphs has been investigated by a number of authors using probabilistic methods [14,15,11]. Pittel and Wormald [14] studied the problem of counting graphs with minimum degree at least k with given number of vertices and edges. In the random model they used, when the degree sequences restricted to a set of "relevant" degree sequences, the graphs had high probability of being k-connected for $k \geq 3$. This can be used to show that the formula obtained was also an asymptotic formula for the number of k-connected graphs with given number of vertices and edges. Interestingly, for $k = 2$, this was not the case since in a sparse regimen even with all degrees at least 2, the random graph has isolated cycles asymptotically almost surely (which we abbreviate by a.a.s. from now on). This case has been addressed in a paper by the fourth author with Kemkes and Wormald [10].

It is then quite natural to address the same enumeration problems for hypergraphs. Intuitively, one would expect that the problems would get much more difficult as this is the case with many graphs problems when generalized to hypergraphs. It is certainly the case for the problem of finding an asymptotic formula for the number of connected r-uniform hypergraphs with given number of vertices and edges (see [5,1,4,3]).

In this paper, we investigate the problem of counting k-edge-connected r-uniform hypergraphs with given number of vertices and edges where $k \geq 2$ and $r \geq 3$ are fixed constants. We will also include the proofs of the case $r = 2$ and $k \geq 3$ for completeness. We use similar techniques used by Pittel and Wormald [14] and Łuczak [12]. We make use of many facts proved by Pittel and Wormald [14].

Results

An r-uniform hypergraph is a pair $H = (V, E)$, where V is a finite set and $E \subseteq \binom{V}{r}$ (where $\binom{V}{r}$ is the set of subsets of V of size r).

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An r-uniform hypergraph $H = (V, E)$ is *connected* if there is no proper nonempty $S \subset V$ with no edge intersecting both S and $V \setminus S$. We say that H is *k-edge-connected* if there is no set $F \subset E$ with $|F| < k$ such that $(V, E \backslash F)$ is not connected. Let $C_{r,k}(n, m)$ denote the number of k-edge-connected r-uniform hypergraphs with vertex-set $[n] = \{1, \ldots, n\}$ and m edges. The main contribution of this paper is an asymptotic formula for $C_{r,k}(n, m)$ in the range as follows.

Theorem 1.1 *Let r and k be constants such that $r \geq 2$ and $k \geq 2$. For $m = O(n \log n)$, if $r \geq 3$ and $k \geq 2$,*

$$C_{r,k}(n, m) \sim \frac{(rm)! f_k(\lambda_c)^n}{\lambda_c^{rm} r!^m} \exp\left(-\frac{(r-1)\eta_c}{r}\right) \cdot \sigma,$$

if $r = 2$ and $k \geq 3$,

$$C_{r,k}(n, m) \sim \frac{(rm)! f_k(\lambda_c)^n}{\lambda_c^{rm} r!^m} \exp\left(-\frac{\eta_c}{2} - \frac{\eta_c^2}{4}\right) \cdot \sigma,$$

where λ_c is defined as the unique root of the equation

$$\frac{\lambda f_{k-1}(\lambda)}{f_k(\lambda)} = c,$$

where $c := \dfrac{rm}{n}$, $f_\ell(\lambda) = e^\lambda - \sum_{i=0}^{\ell-1} \dfrac{\lambda^j}{j!}$, $\eta_c = \lambda_c f_{k-2}(\lambda_c)/f_{k-1}(\lambda_c)$ and, for $R := rm - kn$,

$$\sigma = \begin{cases} \dfrac{1}{\sqrt{2\pi nc(1 + \eta_c - c)}}, & \text{if } R \to \infty; \\ \dfrac{e^{-R} R^R}{R!}, & \text{if } R = o(n^{2/5}). \end{cases}$$

Let $H(\mathbf{d})$ be the random multi-hypergraph with degree sequence $\mathbf{d}$ presented in Section 2. The *degree* of a vertex v is the number of edges containing v. A crucial step in our proof is to show that $H(\mathbf{d})$ is k-edge-connected a.a.s., for $\mathbf{d} \in \mathcal{D}'_{r,k}(n, m)$, where $\mathcal{D}'_{r,k}(n, m)$ is defined in (6).

Theorem 1.2 *Let $r \geq 2$ and $k \geq 2$ be constants such that $k \geq 3$ if $r = 2$. Uniformly for $\mathbf{d} \in \mathcal{D}'_{r,k}(n, m)$, we have that $H(\mathbf{d})$ is k-edge-connected a.a.s.*

We only consider the range $m = O(n \log n)$ since above this range random r-uniform hypergraphs generated uniformly at random with m edges are k-edge-connected a.a.s. and so an asymptotic formula would simply be $\binom{\binom{n}{r}}{m}$, which counts all r-uniform hypergraphs with vertex-set $[n]$ and m edges.

Organization of the paper

The rest of the paper is organized as follows. In Section 2, we introduce the random model used in the proof and show some of its properties. In Section 3, we present an outline with the main steps of the proof of Theorem 1.1. In Section 4, we present an overview of the proof of Theorem 1.2.

> In the rest of the paper, let $r \geq 2$ and $k \geq 2$ be constants such that $k \geq 3$ if $r = 2$. We will work with r-uniform hypergraphs with vertex-set $[n] = \{1, \ldots, n\}$ and $m \leq (C/r)n \log n$ edges, where C is a fixed constant.

2 Random model

In this section, we introduce the random model for r-uniform hypergraphs that we will use throughout the paper.

Definition 2.1 Let $\mathcal{D}_{r,k}(n, m) = \{\mathbf{d} \in \mathbb{N}^n : d_i \geq k \ \forall i \sum_i d_i = rm\}$. Let $\mathbf{d} \in \mathcal{D}_{r,k}(n, m)$ and let $H(\mathbf{d})$ denote the random multi-hypergraph obtained from the following procedure.

- Step 1: Let $V_1, V_2, \ldots, V_n$ be disjoint sets such that $|V_i| = d_i$ for every $i \in [n]$. Let $\mathcal{V} = \bigcup_{i \in [n]} V_i$. The sets $V_1, \ldots, V_n$ are called *vertex-bins*. The elements in $\mathcal{V}$ are called *points*.
- Step 2: Let $E_1, \ldots, E_m$ be disjoint sets (also disjoint from $\mathcal{V}$) each of size r. Let $\mathcal{E} = \bigcup_{j \in [m]} E_j$. The sets $E_1, \ldots, E_m$ are called *edge-bins*. The elements in $\mathcal{E}$ are also called *points*.
- Step 3: Let M be a perfect matching obtained uniformly at random from the perfect matchings in the bipartite graph with classes $(\mathcal{V}, \mathcal{E})$.
- Step 4: Let $H(\mathbf{d})$ denote the multi-hypergraph with vertex-set $[n]$ such that, for each E_j, we include an edge e_j so that for each uw with $u \in E_j$ and $w \in V_i$, we include $i \in e_j$ (with multiplicities).

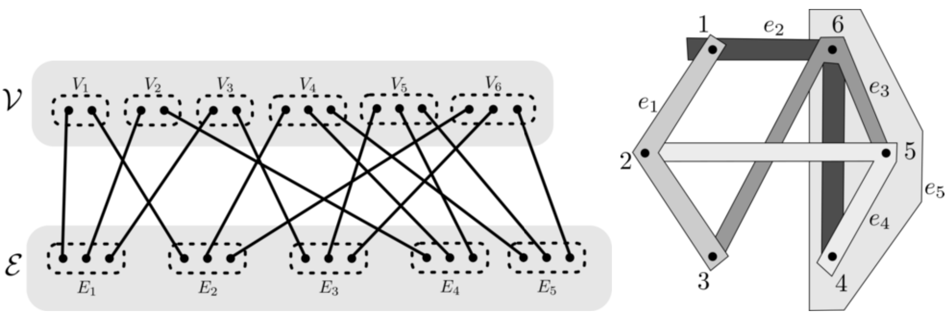

Fig. 1. A hypergraph generated by the model described in Definition 2.1

Remark: The model above can generate hypergraphs with *loops* (edges containing the same vertex more than once) and *multiple edges* (repeated edges), which is why we use the term *multi-hypergraph*. But our interest in this paper is restricted to *simple* hypergraphs.

This model is a simple extension of the configuration model (also known as pairing model) for generating multigraphs with a given degree sequence. One important property of $H(\mathbf{d})$ is that simple hypergraphs are generated by the same number of matchings.

Proposition 2.2 *Let H be a simple hypergraph with degree sequence $\mathbf{d}$, there are exactly $m!r!^m \prod_{i=1}^n d_i!$ choices for matchings in Step 3 that generate H in Step 4. This implies that by conditioning $H(\mathbf{d})$ to simple hypergraphs we obtain a uniform probability space on the r-uniform hypergraphs with degree sequence $\mathbf{d}$.*

It is then important to have an estimate for the probability of obtaining a simple hypergraph. We define $\mathcal{D}'_{r,k}(n,m)$ in (6) so that we can easily apply the following result by Blinovsky and Greenhill [2] for hypergraphs and by McKay [13] for graphs, which we re-state in terms of probabilities (instead of the original enumeration statements).

Let $\eta(\mathbf{d}) = (1/m)\sum_{i=1}^n \binom{d_i}{2}$.

Theorem 2.3 (Blinovsky and Greenhill [2]) *Let $r \geq 3$ be a fixed integer. Suppose that $\mathbf{d} \in \mathbb{N}^n$ is such that $\sum_i d_i = rm$ for $m \in \mathbb{N}$, $2 \leq \max_i d_i = o(m^{1/3})$ and $m \to \infty$. Then*

$$\mathbb{P}[H(\mathbf{d}) \text{ is simple }] \sim \exp\left(-\frac{(r-1)\eta(\mathbf{d})}{r}\right)$$

Theorem 2.4 (McKay [13]) *Suppose that $\mathbf{d} \in \mathbb{N}^n$ is such that $\sum_i d_i = 2m$ for $m \in \mathbb{N}$, $2 \leq \max_i d_i = o(m^{1/4})$ and $m \to \infty$. Then*

$$\mathbb{P}[H(\mathbf{d}) \text{ is simple }] \sim \exp\left(-\frac{\eta(\mathbf{d})}{2} - \frac{\eta(\mathbf{d})^2}{4}\right)$$

3 Outline of the proof

As we mentioned in the introduction, our approach is similar to Pittel and Wormald [14]'s approach to obtain an asymptotic formula for the number of graphs with minimum degree at least k. First they compute the probability that a graph generated by the configuration model is simple for a restricted set of degree sequences. Here we use $\mathcal{D}'_{r,k}(n,m)$ to denote such set and we call is the set of *critical degree sequences* (in the sense, that we will be able to ignore the degree sequences not in $\mathcal{D}'_{r,k}(n,m)$).

Pittel and Wormald use the truncated Poisson distribution Poisson(λ, k) defined by

$$\mathbb{P}[Y = j] = \begin{cases} \dfrac{\lambda^j}{j! f_k(\lambda)}, & \text{if } j \geq k; \\ 0, & \text{otherwise;} \end{cases} \tag{1}$$

where

$$f_k(\lambda) = e^\lambda - \sum_{i=0}^{k-1} \frac{\lambda^j}{j!}. \tag{2}$$

Let $\mathbf{Y} = (Y_1, \ldots, Y_n)$ be such that $Y_1, \ldots, Y_n$ are independent random variables with distribution Poisson(λ, k). Note that, given $\mathbf{d} \in \mathcal{D}_{r,k}(n,m)$,

$$\mathbb{P}[\mathbf{Y} = \mathbf{d}] = \prod_{i=1}^{n} \frac{\lambda^{d_i}}{d_i! f_k(\lambda)} = \frac{\lambda^{rm}}{f_k(\lambda)^n \prod_{i=1}^{n} d_i!}, \tag{3}$$

which has the term $\prod_{i=1}^{n} d_i!$ in common with the number of matchings generating a single hypergraph in Proposition 2.2.

Let Σ denote the event that $\sum_{Y_1,\ldots,Y_m} Y_i = rm$. Let

$$Q_{r,k}(n,m) = \frac{f_k(\lambda)^n}{\lambda^{rm}} \mathbb{P}[\Sigma].$$

Then, similarly as in Pittel and Wormald [14, Equation (13)], let $U(\mathbf{d})$ denote the probability that $H(\mathbf{d})$ is k-edge-connected and simple. We have that

$$\begin{aligned}
C_{r,k}(n,m) &= \sum_{\mathbf{d} \in \mathcal{D}_{r,k}(n,m)} \frac{(rm)!}{m! r!^m \prod_{i=1}^{n} d_i!} U(\mathbf{d}) \\
&= \frac{rm! f_k(\lambda)^n}{\lambda^{rm} r!^m} \sum_{\mathbf{d} \in \mathcal{D}_{r,k}(n,m)} \mathbb{P}[\mathbf{Y} = \mathbf{d}]\; U(\mathbf{d}) \\
&= \frac{rm!}{r!^m} Q_{r,k}(n,m) \sum_{\mathbf{d} \in \mathcal{D}_{r,k}(n,m)} \mathbb{P}[\mathbf{Y} = \mathbf{d} \mid \Sigma]\; U(\mathbf{d}) \\
&= \frac{rm!}{r!^m} Q_{r,k}(n,m) \mathbb{E}\left[U(\mathbf{Y}) \mid \Sigma\right].
\end{aligned} \tag{4}$$

Hence, in order to obtain an asymptotic formula for $C_{r,k}(n,m)$ it suffices to precisely estimate $\mathbb{E}[U(\mathbf{Y}) \mid \Sigma]$ and $\mathbb{P}[\Sigma]$. This simple "trick" allows us to ignore some degree sequences that would be very rare when considering the distribution of $\mathbf{Y}$ since they would have negligible effect on the expectation. We defined $\mathcal{D}'_{r,k}(n,m)$ so that the degree sequences outside it are those rare degree sequences.

Note that Equation (4) works for any $\lambda > 0$, but we choose λ so that it maximizes $\mathbb{P}[\Sigma]$. Also note that if Y has distribution Poisson(λ, k), then

$$\mathbb{E}Y = \frac{\lambda f_{k-1}(\lambda)}{f_k(\lambda)}.$$

Let $c = rm/n$ and define λ_c as the root for the equation

$$\frac{\lambda f_{k-1}(\lambda)}{f_k(\lambda)} = c.$$

Its existence and many properties have been proved by Pittel and Wormald [14]. Let

$$\eta_c = \frac{\lambda_c f_{k-2}(\lambda_c)}{f_{k-1}(\lambda_c)}. \tag{5}$$

Recall that $\eta(\mathbf{d}) = (1/m)\sum_{i=1}^{n}\binom{d_i}{2}$. It is not hard to show that $\mathbb{E}[\eta(Y)] = \eta_c$. We can then define $\mathcal{D}'_{r,k}(n,m)$ so that $\eta(Y) \sim \eta_c$ very sharply. For $\varepsilon \in (0, 1/2)$, define

$$\mathcal{D}'_{r,k}(n,m) = \left\{\mathbf{d} \in \mathcal{D}_{r,k}(n,m) : \max_i d_i \le 6C\log n,\ |\eta(\mathbf{d}) - \eta_c| \le n^{-1/2+\varepsilon}\right\}. \quad (6)$$

Then it is not hard to show that

$$\mathbb{P}[\mathbf{Y} \notin \mathcal{D}'_{r,k}(n,m)] \le \frac{\exp(-\Theta(\log n)^3)}{\mathbb{P}[\Sigma]}. \quad (7)$$

Thus, for $\mathbf{d} \in \mathcal{D}'_{r,k}(n,m)$, assuming Theorem 1.2, we have that $U(\mathbf{d}) \sim \exp(-(r-1)\eta_c/r - \mathbf{1}_{r=2}\eta_c^2/4)$, where $\mathbf{1}_{r=2} = 1$ if $r = 2$, and $\mathbf{1}_{r=2} = 0$ otherwise. For $\mathbf{d} \notin \mathcal{D}'_{r,k}(n,m)$, we use $U(\mathbf{d}) \le 1$ since it is a probability. Thus,

$$\mathbb{E}\left[U(\mathbf{Y}) \mid \Sigma\right] = (1+o(1))\exp\left(-\frac{(r-1)\eta_c}{r} - \mathbf{1}_{r=2}\frac{\eta_c^2}{4}\right)\left(1 - \frac{\exp(-\Theta(\log n)^3)}{\mathbb{P}[\Sigma]}\right) + \frac{\exp(-\Theta(\log n)^3)}{\mathbb{P}[\Sigma]}.$$

To finish the proof, we use the following result by Pittel and Wormald that computes $\mathbb{P}[\Sigma]$ (stated in our scenario):

Theorem 3.1 (Pittel and Wormald [14], Theorem 4) *Let $R = rm - kn$. If $R \to \infty$, then*

$$\mathbb{P}[\Sigma] = \frac{1 + O(R^{-1})}{\sqrt{2\pi nc(1 + \eta_c - c)}}.$$

If $R = O(n^{2/5})$, then

$$\mathbb{P}[\Sigma] = (1 + O(R^{5/2}n^{-1}))\frac{e^{-R}R^R}{R!}.$$

It follows that $\mathbb{P}[\Sigma] = \Omega(1/\sqrt{R})$ and so

$$\begin{aligned}\mathbb{E}\left[U(\mathbf{Y}) \mid \Sigma\right] &= (1+o(1))\exp\left(-\frac{(r-1)\eta_c}{r} - \mathbf{1}_{r=2}\frac{\eta_c^2}{4}\right) + \exp(-\Theta(\log n)^3)\\ &= (1+o(1))\exp\left(-\frac{(r-1)\eta_c}{r} - \mathbf{1}_{r=2}\frac{\eta_c^2}{4}\right).\end{aligned}$$

Therefore

$$C_{r,k}(n,m) \sim \frac{rm!}{r!^m}Q_{r,k}(n,m)\exp\left(-\frac{(r-1)\eta_c}{r} - \mathbf{1}_{r=2}\frac{\eta_c^2}{4}\right),$$

which gives the desired asymptotic formula by subbing in the values for $\mathbb{P}[\Sigma]$ (obtained in Theorem 3.1) in $Q_{r,k}(n,m)$.

4 Overview of the proof of Theorem 1.2

We will follow the same strategy used by Łuczak [12] for the graph case. We will show that, a.a.s., there is no set of edges of size $k-1$ that disconnects the random multigraph. The two main lemmas are the following: the first one bounds the number of induced edges in small sets of vertices and the second one bounds the probability that a large set has less than k edges crossing to its complement.

Lemma 4.1 *Let $\varepsilon > 0$ and let $D > 0$ be fixed constants. A.a.s., there is no set $S \subseteq [n]$ of size $r \leq s \leq D \log n$ with at least $(1+\varepsilon)s/(r-1)$ induced edges.*

Lemma 4.2 *Let $D' > 0$ be a fixed constant. A.a.s., there is no set $S \subseteq [n]$ of size $D' \log n \leq s \leq n - D' \log n$ with less than k edges intersecting both S and $[n] \setminus S$.*

Now we show how to prove Theorem 1.2. It suffices to show that, a.a.s., there is no nonempty proper subset $S \subseteq [n]$ such that there exists a set of edges R of size $|R| < k$ such that R is a minimal edge-cut for S (that is, R is a minimal set such that by removing R the set S becomes disconnected from $[n] \setminus S$). Note that, for $|S| < r$, this follows immediately from the fact that the minimum degree of a vertex is at least k. Thus, each edge with vertices in $|S|$ has at least one vertex outside S and so there are at least k such edges.

If $|S| > D \log n$, our claim follows immediately from Lemma 4.2. Note that, for the case $r = 2$ and $|S| = 2$, since the minimum degree is at least k, we have that there are at least $2k - 1 \geq k$ edges in the edge-cut of S.

Now assume $r \leq |S| \leq D \log n$ for $r \geq 3$ and assume $r + 1 \leq |S| \leq D \log n$ for $r = 2$. Let $S' = S \cup \bigcup_{e \in R} e$. Suppose R is a minimal edge-cut for S. Then each edge in R must intersect both S and its complement $[n]$. Let T denote the number of points in R matched to points outside S. Then,

$$|R| \leq |T| \leq (r-1)(k-1) \quad \text{and} \quad |S'| \leq |S| + |T|. \tag{8}$$

The number of edges induced by S' is then at least $(k|S| + |T|)/r$. It is straightforward, by computing derivatives, to prove that, for $\varepsilon = 1/(5r)$,

$$\frac{k|S| + |T|}{r} \geq \frac{(1+\varepsilon)|S'|}{r-1}, \tag{9}$$

which contradicts Lemma 4.1. Thus, there is no such R a.a.s., and so, Theorem 1.2 holds.

We present the proof for Lemma 4.1 and omit the proof of Lemma 4.2 since their proofs follow the same strategy.

4.1 Proof of Lemma 4.1

Let $S \subseteq [n]$ with $s := |S|$. Let $\mathcal{V}_S$ be the set of points in vertex-bins corresponding to vertices in S and recall that $\mathcal{V}$ is the set of all points in vertex-bins. Let $P_S = |\mathcal{V}_S|$ and $P = |\mathcal{V}|$. Note that $P \geq kn$ since each vertex-bin has at least k points and $P_S \leq \Delta s$.

Using the union bound, the probability that it induces at least $\ell = \ell(s) := (1+\varepsilon)s/(r-1)$ edges is at most

$$\binom{m}{\ell}\binom{P_S}{r\ell}\frac{(r\ell)!(P-r\ell)!}{P!} = \binom{m}{\ell}\prod_{i=0}^{r\ell-1}\frac{P_S-i}{P-i} \sim \binom{m}{\ell}\left(\frac{P_S}{P}\right)^{r\ell} \leq \binom{m}{\ell}\left(\frac{\Delta s}{rm}\right)^{r\ell},$$

where $\binom{m}{\ell}$ is the number of ways of choosing the induced edges, $\binom{P_S}{r\ell}$ is the number of ways of choosing points in S to be matched to these induced edges, $(r\ell)!$ is the number of matchings between these chosen points, $(P-r\ell)!$ is the number of matchings between the remaining points and $P!$ is the total number of matchings; for the last inequality we use $P = rm$ and $P_S \leq \Delta s$.

Summing this value over all subsets S with size between r and $D\log n$, the probability that there is a set of size s inducing $\ell(s)$ edges is at most

$$\begin{aligned}
\sum_{r\leq s\leq D\log n}\binom{n}{s}\binom{m}{\ell}\left(\frac{\Delta s}{rm}\right)^{r\ell} &\leq \sum_{r\leq s\leq D\log n}\left(\frac{en}{s}\right)^{s}\left(\frac{em}{\ell}\right)^{\ell}\left(\frac{\Delta s}{rm}\right)^{r\ell}\\
&\leq \sum_{r\leq s\leq D\log n}\left(\frac{en}{s}\right)^{s}\left(\frac{e}{\ell}\right)^{\ell}\frac{(\Delta s/r)^{r\ell}}{m^{r\ell-\ell}}\\
&\leq \sum_{r\leq s\leq D\log n}\left(\frac{en}{s}\right)^{s}\left(\frac{e}{\ell}\right)^{\ell}\frac{(\Delta s/r)^{r\ell}}{(kn/r)^{r\ell-\ell}}, \quad \text{since } m\geq kn/r\\
&= \sum_{r\leq s\leq D\log n}\frac{e^{s+\ell}s^{\ell}\Delta^{s+\ell}}{k^{r\ell-\ell}r^{\ell}}\left(\frac{\Delta s}{n}\right)^{r\ell-s-\ell}\\
&\leq \sum_{r\leq s\leq D\log n}(\Theta(1)(\log n)^2)^{s+\ell}\left(\frac{\Delta s}{kn}\right)^{r\ell-s-\ell},\\
&\qquad \text{using } s=O(\log n) \text{ and } \Delta = O(\log n)\\
&= \sum_{r\leq s\leq D\log n}(\Theta(1)(\log n)^2)^{s+\ell}\left(\frac{\Theta(1)(\log n)^2}{n}\right)^{r\ell-s-\ell}\\
&= \sum_{r\leq s\leq D\log n}(\Theta(1)(\log n)^2)^{(r+\varepsilon)s/(r-1)}\left(\frac{\Theta(1)(\log n)^2}{n}\right)^{\varepsilon s}\\
&\leq \sum_{r\leq s\leq D\log n}(\Theta(1)(\log n)^4)^{s}\left(\frac{\Theta(1)(\log n)^2}{n}\right)^{\varepsilon s}\\
&\leq (D\log n)\left(\frac{\Theta(1)(\log n)^{2+4/\varepsilon}}{n}\right)^{\varepsilon r} = o(1),
\end{aligned}$$

which finishes the proof of Lemma 4.1.

References

[1] M. Berherisch, A. Coja-Oghlan, and M. Kang. The asymptotic number of connected d-uniform hypergraphs. *Combinatorics, Probability and Computing*, 23(3):367–385, 2014.

[2] V. Blinovsky and C. Greenhill. Asymptotic enumeration of sparse uniform hypergraphs with given degrees. *European Journal of Combinatorics*, 51, 09 2014.

[3] B. Bollobás and O. Riordan. Counting connected hypergraphs via the probabilistic method. *Combinatorics, Probability and Computing*, 25(1):21–75, 2016.

[4] B. Bollobás and O. Riordan. Counting dense connected hypergraphs via the probabilistic method. *Random Structures & Algorithms*, 53(2):185–220, 2018.

[5] A. Coja-Oghlan, C. Moore, and V. Sanwalani. Counting connected graphs and hypergraphs via the probabilistic method. *Random Structures & Algorithms*, 31(3):288–329, 2007.

[6] P. Flajolet and R. Sedgewick. *Analytic Combinatorics.* Cambridge University Press, 2009.

[7] O. Gimenez and M. Noy. Asymptotic enumeration and limit laws of planar graphs. *J Amer Math Soc*, 22, 02 2005.

[8] F. Harary and E. M. Palmer. *Graphical Enumeration.* Elsevier Science, 2014.

[9] M. Isaev and B. D. McKay. Complex martingales and asymptotic enumeration. *Random Structures & Algorithms*, 52(4):617–661, 2017.

[10] G. Kemkes, C. M. Sato, and N. Wormald. Asymptotic enumeration of sparse 2-connected graphs. *Random Structures & Algorithms*, 43(3):354–376, 2013.

[11] L. Lu and L. Székely. Using lovász local lemma in the space of random injections. *Electr. J. Comb.*, 14, 09 2007.

[12] T. Łuczak. Sparse random graphs with a given degree sequence. In *Proceedings of the Symposium on Random Graphs, Poznan*, pages 165–182, 1989.

[13] B.D. McKay. Asymptotics for symmetric 0-1 matrices with prescribed row sums. *Ars Combinatoria*, 19A:15–25, 1985.

[14] B. Pittel and N. C. Wormald. Asymptotic enumeration of sparse graphs with a minimum degree constraint. *Journal of Combinatorial Theory, Series A*, 101(2):249 – 263, 2003.

[15] B. Pittel and N. C. Wormald. Counting connected graphs inside-out. *Journal of Combinatorial Theory, Series B*, 93(2):127 – 172, 2005.