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Quadratically adjustable robust linear optimization with inexact data via generalized S-lemma: Exact second-order cone program reformulations☆

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ABSTRACT

Adjustable robust optimization allows for some variables to depend upon the uncertain data after its realization. However, the uncertainty is often not revealed exactly. Incorporating inexactness of the revealed data in the construction of ellipsoidal uncertainty sets, we present an exact second-order cone program reformulation for robust linear optimization problems with inexact data and quadratically adjustable variables. This is achieved by establishing a generalization of the celebrated S-lemma for a separable quadratic inequality system with at most one non-homogeneous function. It allows us to reformulate the resulting separable quadratic constraints over an intersection of two ellipsoids in terms of second-order cone constraints. We illustrate our results via numerical experiments on adjustable robust lot-sizing problems with demand uncertainty, showing improvements over corresponding problems with affinely adjustable variables as well as with exactly revealed data.

1. Introduction

Adjustable robust optimization (ARO) has proved to be a powerful deterministic methodology to handle dynamic decision-making problems, where the decision-maker is able to adjust her strategy to information revealed in stages (see Ben-Tal et al., 2009; Delage and Iancu, 2015). It allows for some decision variables to depend upon the uncertain data after realization. However, in many applications, the uncertainty is not revealed exactly, due to, for example, measurement error. The construction of uncertainty sets,that incorporates inexactness of revealed data, has been shown to give high quality solutions for practical dynamic decision-making problems. For example, de Ruiter et al. (2017) demonstrated that for the multi-stage inventory production problem, ignoring inexactness of revealed demand can lead to violation of the robust constraints in up to 80% of simulations.

Such a scheme has recently been developed for the case when the adjustable decisions take on an Affine Decision Rule (ADR), and the uncertain data together with the inexactly revealed data and the estimation error lie in closed and convex sets (de Ruiter et al., 2017). Applications of the methods have included multi-stage-inventory problems (Ben-Tal et al., 2009), planning and scheduling problems (Ning and You, 2017) and treatment-length optimization problems in radiation therapy (Eikelder et al., 2019). In particular, robust optimization techniques that employ inexactly revealed data are of great interest in dynamic decision-making problems of medicine and health sciences, such as the two-stage robust optimization models with time-dependent uncertainty sets that appear in radiation therapy planning problems (Nohadani and Roy, 2017). Here, inexactly revealed data can be incorporated in the construction of the uncertainty sets, because a patient's condition or health care needs can change during the course of a treatment (Eikelder et al., 2019; Nohadani and Roy, 2017), but estimates of their condition can be taken and considered mid-treatment.

In this paper, we consider the two-stage robust linear optimization problem with adjustable variables and inexactly revealed data:

$$(IP) \qquad \min_{x,y(\cdot)} \quad c^T x$$
$$\text{subject to} \quad A(z)x + By(\hat{z}) \le d(z), \text{ for all } (z,\hat{z}) \in \mathcal{Z} \subset \mathbb{R}^q \times \mathbb{R}^q,$$

where $c \in \mathbb{R}^n$, $B \in \mathbb{R}^{p\times s}$,

$$A(z) = \begin{bmatrix} (a_1 + A_1 z)^T \\ \vdots \\ (a_p + A_p z)^T \end{bmatrix} \in \mathbb{R}^{p\times n} \text{ and } d(z) = \begin{bmatrix} v_1 + v_1^T z \\ \vdots \\ v_p + v_p^T z \end{bmatrix} \in \mathbb{R}^p \tag{1}$$

for some $a_1,\dots,a_p \in \mathbb{R}^n$, $A_1,\dots,A_p \in \mathbb{R}^{n\times q}$, $v_1,\dots,v_p \in \mathbb{R}$ and $v_1,\dots,v_p \in \mathbb{R}^q$. The vector $x \in \mathbb{R}^n$ is the first-stage "here and now" decision vector and $y(\cdot)$ is the second-stage "wait and see" adjustable decision function $y : \mathbb{R}^q \to \mathbb{R}^s$. The uncertainty set of (IP) is defined as

$$\mathcal{Z} = \left\{(z,\hat{z}) \in \mathbb{R}^q \times \mathbb{R}^q : \|\hat{z}-u\|^2 \le \gamma_1^2,\ \|z-\hat{z}\|^2 \le \gamma_2^2\right\}, \tag{2}$$

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where $\gamma_1, \gamma_2 > 0$ and $u \in \mathbb{R}^q$ are given. That is, the adjustable decisions $y(\cdot)$ depend on an estimate $\hat{z} \in \mathbb{R}^q$ of the actual uncertain data $z \in \mathbb{R}^q$, with the *estimation error* $w := z - \hat{z}$ bounded in $\mathcal{W} = \{w \in \mathbb{R}^q : \|w\|^2 \le \gamma_2^2\}$, and $\hat{z}$ bounded in the *estimation range* $\mathcal{V} = \{\hat{z} \in \mathbb{R}^q : \|\hat{z} - u\|^2 \le \gamma_1^2\}$, with given scalars $\gamma_1, \gamma_2 > 0$, and known vector $u \in \mathbb{R}^q$, see de Ruiter et al. (2017).

As shown in Section 3, the uncertainty set $\mathcal{Z}$ can equivalently be described as an intersection of two ellipsoids in the higher-dimensional space, $\mathbb{R}^{2q}$. It is indeed a closed, bounded and convex uncertainty set. Note that our definition of $\mathcal{Z}$ differs from that used in [8] since we do not specify $z \in \mathcal{V}$. Our uncertainty set is therefore more general, and is useful in cases where the estimation range is known, but the true uncertainty set is not known. An example of such a case is discussed in Section 4.

We allow the adjustable decisions $y(\cdot)$ to admit a parameterized separable quadratic decision rule (Woolnough et al., 2021) of the form

$$y(\hat{z}) = \theta(y_0 + P\hat{z}) + (1-\theta)\begin{pmatrix} \hat{z}^T Q_1 \hat{z} \\ \vdots \\ \hat{z}^T Q_s \hat{z} \end{pmatrix}, \tag{3}$$

where $\theta \in (0,1]$ is a user-specified parameter, $y_0 \in \mathbb{R}^s$, $P \in \mathbb{R}^{s\times q}$ and $Q_r \in S^q$, $r = 1,\dots,s$, are diagonal matrices with $Q_r = \mathrm{diag}(\xi_r^{(1)},\dots,\xi_r^{(q)})$. This covers the commonly used affine decision rule where $\theta = 1$ (see Avraamidou and Pistikopoulos, 2020; Ben-Tal et al., 2009; Ben-Tal et al., 2004; Chen and Zhang, 2009; de Ruiter et al., 2017; Yanikoglu et al., 2019) and non-homogeneous separable quadratic decision rule where $0 < \theta < 1$ (see Ben-Tal et al., 2009; Xu and Hanasusanto, 2018).

The quadratic decision rules, including a separable quadratic decision rule, were first introduced by Ben-Tal et al. (2009) and these rules were used to examine numerically tractable safe approximations to various classes of intractable adjustable robust optimization problems. They include classes of robust problems where an uncertainty set is allowed to be an intersection of ellipsoids. The interplay between the quadratic optimization and adjustable robust optimization in the case of quadratic decision rules was elegantly discussed in Bomze and Gabi (2021). These quadratic decision rules were also employed to study the links between copositive optimization and multi-stage robust optimization in Xu and Hanasusanto (2018). Recently, the parameterized separable quadratic rule (3) has been shown by the authors (Woolnough et al., 2021) to admit exact second-order cone programming reformulations for adjustable robust linear optimization problems under single ellipsoidal uncertainty based on exact revealed data. More general nonlinear decision rules have also been examined for adjustable robust optimization problems in Ben-Tal et al. (2009) and Yanikoglu et al. (2019).

We make the following technical contributions, in this paper, to two-stage robust optimization.

(i) We show that an equivalent second-order cone program reformulation holds for (IP) with the parameterized quadratic decision rule (QDR) and inexactly revealed data. In contrast to the conic program reformulation of adjustable robust linear optimization with the parameterized QDR and *exact* revealed data, called exact QDR, established by the authors recently in Woolnough et al. (2021) with the aid of S-lemma (Ben-Tal and Nemirovski, 2001), the present reformulation for (IP) with the QDR and inexactly revealed data, called inexact QDR, requires a generalization of S-lemma to handle uncertainty sets involving the intersection of two ellipsoids and a non-convex and nonhomogeneous quadratic function. It is known that generalizations of S-lemma for non-convex quadratic systems involving more than two quadratic functions are often not possible unless the system enjoys some hidden convexity (see Example 2.1). For a survey of S-lemma and its generalizations, see Polik (2007), Jeyakumar et al. (2009a) and Derinkuyu and Pinar (2006).

(ii) Using the standard hyperplane separation arguments and the convexity of the associated epigraphical set, we first show that a general form of S-lemma holds for a non-homogeneous quadratic inequality system that admits a convex epigraphical set (see Section 3 for details). We then prove that the convex epigraphical set, which is hidden in many known classes of fully homogeneous quadratic systems (Ben-Tal and Hertog, 2014; Dines, 1941; Polyak, 1998), is also hidden in a separable quadratic inequality systems involving at most one non-homogeneous function. This results in a form of S-lemma for the corresponding separable quadratic inequality system, allowing us to handle the, not necessarily convex, separable quadratic constraints over the intersection of ellipsoidal uncertainty sets that appear in the reformulation of (IP). Consequently, we also deduce a form of S-lemma for a simultaneously diagonalizable quadratic inequality system involving at most one non-homogeneous function.
We also note that sufficient conditions which guarantee epigraphical convexity have been given for specially structured quadratic systems, such as the ones that appear in extended trust-region problems (see Jeyakumar et al., 2009b; Jeyakumar and Li, 2013). Related convexifiability conditions in terms of epigraphical set have been used in Chieu et al. (2019, 2020) for studying duality properties of various classes of non-convex quadratic optimization problems.

(iii) Finally, we illustrate our results via numerical experiments on adjustable robust lot-sizing problems with demand uncertainty. Our results show that the current inexact QDR approach improves over the recent exact QDR scheme by the authors (Woolnough et al., 2021) as well as the commonly used affine decision rule method with inexactly revealed data, where $\theta = 1$ in (3) and other references therein. Related results with affine decision rules and exactly revealed data may also be found in Chuong and Jeyakumar (2020).

The outline of the paper is as follows. Section 2 presents generalizations of S-lemma. Section 3 establishes a second-order cone program reformulation of (IP). Section 4 describes the numerical experiments and their outcomes for robust lot-sizing problems with demand uncertainty. Section 5 concludes with a discussion on possible future work.

2. Generalized S-lemma

In this Section, we present generalizations of S-Lemma for systems involving more than two quadratic functions. We begin by fixing some preliminaries. The notation $\mathbb{R}^n$ signifies the Euclidean space for each $n \in \mathbb{N} := \{1,2,\dots\}$ and S^l is the space of all real $l \times l$ symmetric matrices. As usual, the symbol I_n stands for the identity $(n\times n)$ matrix, while $\mathbb{R}_+ := [0,+\infty) \subset \mathbb{R}$. The inner product in $\mathbb{R}^n$ is defined by $\langle x, y\rangle := x^T y$ for all $x, y \in \mathbb{R}^n$. We use $\|x\|$ to denote the standard Euclidean norm of x. A symmetric $(n\times n)$ matrix A is said to be positive semi-definite, denoted by $A \succeq 0$, whenever $x^T A x \ge 0$ for all $x \in \mathbb{R}^n$. The notation $0_{m\times n}$ denotes the matrix of all zeros of dimensions $m \times n$.

For $z \in \mathbb{R}^q$, consider the following quadratic functions:

$$f(z) = z^T W z + w^T z + \alpha \text{ and } g_i(z) = z^T W_i z + u_i^T z + \beta_i,\ i = 1,\dots,m,$$

where W and W_i are symmetric $(q\times q)$ matrices, $w, u_i \in \mathbb{R}^q$ and $\alpha, \beta_i \in \mathbb{R}$, $i = 1,\dots,m$. The epigraphical set $U(f, g_1,\dots,g_m)$ associated with the functions f and $g_1,\dots,g_m$, is given by

$$U(f, g_1,\dots,g_m) = \{(v_1,\dots,v_m,r) : v_i \ge g_i(z) \text{ and } r \ge f(z) \text{ for some } z \in \mathbb{R}^q\}.$$

Recall that the epigraphical set $U(f, g_1,\dots,g_m)$, which plays a key role later in establishing generalizations of the S-lemma for systems involving more than two quadratic functions, is related to the epigraph of the value function of the quadratic optimization problem (QP_v):

$$(QP_v)\ \min_{z\in\mathbb{R}^q} f(z) \quad \text{s.t.} \quad g_i(z) \le v_i,\ i = 1,\dots,m,$$

where $v = (v_1,\dots,v_m) \in \mathbb{R}^m$. The set $U(f, g_1,\dots,g_m)$ is intrinsically related to the epigraph of the optimal value function of (QP_v). Indeed, if the optimal value function of (QP_v) is ϕ, that is, $\phi(v) =$

$\inf(QP_v)$, then a direct verification shows that $U(f, g_1, \ldots, g_m) \subseteq \text{epi}\phi \subseteq \text{cl}\, U(f, g_1, \ldots, g_m)$. Note that, for a subset A of $\mathbb{R}^n$, $\text{cl}\, A$ denotes the closure of the set A.

Moreover, the epigraphical set can be written as the Minkowski sum of the non-negative orthant $\mathbb{R}^{m+1}_+$ and the joint range set

$$R(f, g_1, \ldots, g_m) = \{(v_1, \ldots, v_m, r) : v_i = g_i(z) \text{ and } r = f(z) \text{ for some } z \in \mathbb{R}^q\}$$

Therefore, it is easy to see that if the joint range set $R(f, g_1, \ldots, g_m)$ is a convex set, then the epigraphical set $U(f, g_1, \ldots, g_m)$ is also a convex set. The convexity of the joint range set is known for various special classes of quadratic systems. The case where $m = 1$ and both f and g_1 are homogeneous quadratic functions is given in Dine's theorem (Dines, 1941), whereas the case where $m = 2$, and f, g_1, g_2 are homogeneous quadratic functions which satisfy a positive definite condition is given in Polyak's convexity theorem (Polyak, 1998). An extension of Dine's theorem may be found in Jeyakumar et al. (2009b).

In the case where f, g_i are all homogeneous and the matrices $W, W_1, \ldots, W_m$ are pairwise commutative (Polik, 2007) (for example, when they are diagonal matrices), the joint range set reduces to a polyhedral set which is, in particular, convex.

On the other hand, if f, g_i are nonhomogeneous quadratic functions, then the epigraphical set, in general, is a nonconvex set. However, the convexity of the epigraphical set can be satisfied by many specially structured systems of non-homogeneous quadratic functions. For instance, as shown in Jeyakumar and Li (2013), if $f(z) = z^T W z + w^T z + \alpha$, $g_0(z) = \|z - z_0\|^2 - \beta_0$ and $g_i(z) = b_i^T z - \beta_i$, $i = 1, 2, \ldots, m$ and if $\dim \text{Ker}(W - \lambda_{\min}(W) I_n) \geq s + 1$ with $\dim \text{span}\{b_1, \ldots, b_m\} = s$, then the epigraphical set $U(f, g_0, g_1, \ldots, g_m) = \{(v_0, v_1, \ldots, v_m, r) : \exists z \in \mathbb{R}^q, v_i \geq g_i(z), i = 0, 1, 2, \ldots, m,\ r \geq f(z)\}$ is a convex set, where $A \in S^q$, $u, z_0, b_i \in \mathbb{R}^q$ and $\alpha, \beta_i \in \mathbb{R}$. An extensive discussion on the relationship between the epigraphical convexity and S-lemma can be found in the recent paper (Bomze and Gabi, 2021). For related results, see Jeyakumar and Li (2018) and Jeyakumar et al. (2009b).

As we see later in this Section, the epigraphical convexity is also hidden in certain systems of non-homogeneous separable quadratic functions. We first establish an abstract generalization of S-lemma under epigraphical convexity of a general non-homogeneous quadratic system.

Lemma 2.1. ***(Generalized S-Lemma for nonconvex quadratic inequality systems)*** *Let $W, W_i \in S^q$, $i = 1, \ldots, m$, be symmetric matrices. Let $f(z) = z^T W z + w^T z + \alpha$ and $g_i(z) = z^T W_i z + u_i^T z + \beta_i, i = 1, \ldots, m$, where $\alpha, \beta_i \in \mathbb{R}$, $w, u_i \in \mathbb{R}^q$. Suppose that the epigraphical set $U(f, g_1, \ldots, g_m) = \{(v_1, \ldots, v_m, r) : \exists z \in \mathbb{R}^q, v_i \geq g_i(z), i = 1, 2, \ldots, m,\ r \geq f(z)\}$ is a convex set. If there exists $\bar{z} \in \mathbb{R}^n$ such that $g_i(\bar{z}) < 0$, $i = 1, \ldots, m$, then, the following statements are equivalent*

(a) $z^T W_i z + u_i^T z + \beta_i \leq 0,\ i = 1, \ldots, m \Rightarrow z^T W z + w^T z + \alpha \geq 0;$

(b) *There exist $\lambda_i \geq 0$, $i = 1, \ldots, m$, such that, for all $z \in \mathbb{R}^q$,*

$$z^T(W + \sum_{i=1}^m \lambda_i W_i) z + (w + \sum_{i=1}^m \lambda_i u_i)^T z + \alpha + \sum_{i=1}^m \lambda_i \beta_i \geq 0.$$

(c) *There exist $\lambda_i \geq 0$, $i = 1, \ldots, m$ such that*

$$\begin{pmatrix} W + \sum_{i=1}^m \lambda_i W_i & (w + \sum_{i=1}^m \lambda_i u_i)/2 \\ (w + \sum_{i=1}^m \lambda_i u_i)^T/2 & \alpha + \sum_{i=1}^m \lambda_i \beta_i \end{pmatrix} \succeq 0. \tag{4}$$

Proof. Clearly, (b) and (c) are equivalent. Moreover, by construction, (b) implies (a). We show that (a) implies (b). Suppose that (a) holds. Let $C = \{(\gamma_1, \ldots, \gamma_m, r) : \gamma_k \leq 0 \text{ and } r \leq 0\}$ and let $U := U(f, g_1, \ldots, g_m)$, for simplicity. Then, from (a), $\text{int}C \cap U = \emptyset$ (Otherwise, there exists $(\gamma_1, \ldots, \gamma_m, r) \in U$ with $\gamma_k < 0$ and $r < 0$. This shows that there exists $z \in \mathbb{R}^q$ such that $0 > \gamma_k \geq g_k(z)$ and $0 > r \geq f(z)$. This contradicts (a).) Since the epigraphical set U is a convex set, it follows by the convex separation theorem (Rockafellar, 1970, Theorem 11.3)there exists $(\mu_1, \ldots, \mu_m, t) \in \mathbb{R}^{m+1} \backslash \{0\}$ such that $\sum_{k=1}^m \mu_k \gamma_k + t \cdot r \geq 0$ for all $(\gamma_1, \ldots, \gamma_m, r) \in U$. Note that $U + \mathbb{R}^{m+1}_+ = U$. It follows that $(\mu_1, \ldots, \mu_m, t) \in \mathbb{R}^{m+1}_+ \backslash \{0\}$. Moreover, as $(g_1(z), \ldots, g_m(z), f(z)) \in U$ for all $z \in \mathbb{R}^q$, $\sum_{k=1}^m \mu_k g_k(z) + t f(z) \geq 0$ for all $z \in \mathbb{R}^q$. If $t = 0$, then, $(\mu_1, \ldots, \mu_m) \in \mathbb{R}^m_+ \backslash \{0\}$ and $\sum_{k=1}^m \mu_i g_k(z) \geq 0$ for all $z \in \mathbb{R}^q$. This is impossible as $g_k(\bar{z}) < 0$ for all $k = 1, \ldots, m$. So, $t > 0$. Thus, dividing by t on both sides, we have $f(z) + \sum_{k=1}^m \lambda_k g_k(z) \geq 0$ for all $z \in \mathbb{R}^q$, where $\lambda_k = \mu_k / t$. Then the statement (b) holds. □

In passing observe that Lemma 2.1 includes the celebrated homogeneous S-lemma and Farkas' lemma for convex quadratic inequality systems. Indeed, if f and g_1 are homogeneous quadratic functions, then it is known by Dine's theorem that the joint range $R(f, g_1)$ is convex, and so, the epigraphical set $U(f, g_1)$ is also convex in this case. Thus, Lemma 2.1 reduces to the homogeneous S-lemma. Moreover, if f, g_i, $i = 1, \ldots, m$, are all convex quadratic functions, then clearly the epigraphical set $U(f, g_1, \ldots, g_m)$ is also a convex set.

It is also worth noting that a robust separable quadratic inequality over an intersection of ellipsoids of the form (2), that appears in the formulations of AROs with inexact data, requires a form of S-lemma for a separable quadratic inequality system involving both homogeneous and non-homogeneous functions with a total of at least three inequalities. Although the epigraphical convexity is known to be hidden in several structured quadratic inequality systems, the following simple one-dimensional example shows that the epigraphical convexity assumption (and also Lemma 2.1) may fail, in general, for a separable system of three inequalities with two non-homogeneous quadratic functions.

Example 2.1. (Failure of epigraphical convexity & Lemma 2.1 with two non-homogeneous inequalities) Let $g_1(z) = z$, $g_2(z) = 1 - z^2$ and $f(z) = z^2 - z - 2$. Then, g_1 and f are two non-homogeneous separable quadratic functions, and

$$g_1(z) \leq 0, g_2(z) \leq 0 \ \Rightarrow z \leq -1 \ \Rightarrow \ f(z) = z^2 - z - 2 \geq 0.$$

So, statement (a) of Lemma 2.1 holds, but statement (b) of Lemma 2.1 fails because there does not exist $\lambda_1, \lambda_2 \geq 0$ such that

$$f_1(z) + \lambda_1 g_1(z) + \lambda_2 g_2(z) \geq 0 \text{ for all } z \in \mathbb{R}.$$

To see this, suppose on the contrary that, for some $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$,

$$f_1(z) + \lambda_1 g_1(z) + \lambda_2 g_2(z) = (1 - \lambda_2) z^2 + (\lambda_1 - 1) z + (\lambda_2 - 2) \geq 0 \quad \text{for all } z \in \mathbb{R}.$$

This gives us that $1 - \lambda_2 \geq 0$ and $\lambda_2 - 2 \geq 0$, which is not possible.

Note that the epigraphical set $U(f, g_1, g_2) = \{(s_1, s_2, r) : \exists z \in \mathbb{R}, g_i(z) \leq s_i, f(z) \leq r, i = 1, 2\}$ is not a convex set. To see this, take $x_0 = (-1, 0, 0)$ and $y_0 = (0, 1, -2)$. Then, $x_0, y_0 \in U(f, g_1, g_2)$, because $g_1(-1) = -1$, $g_2(-1) = 0$ and $f(-1) = 0$; $g_1(0) = 0$, $g_2(0) = 1$ and $f(0) = -2$. We now see that their mid point $\frac{1}{2}(x_0 + y_0) = (-1/2, 1/2, -1) \notin U(f, g_1, g_2)$. Otherwise, we have $(-1/2, 1/2, -1) \in U(f, g_1, g_2)$, and so, there exists $z \in \mathbb{R}$ such that

$$z \leq -1/2,\ 1 - z^2 \leq 1/2 \text{ and } z^2 - z - 2 \leq -1.$$

The first two relations imply that $z \leq -\frac{\sqrt{2}}{2}$. The last relation implies that $\frac{-\sqrt{5}+1}{2} \leq z \leq \frac{\sqrt{5}+1}{2}$, which is impossible. So, $U(f, g_1, g_2)$ is not a convex set.

The following theorem shows that a form of S-lemma holds for a system of separable quadratic functions with at most one non-homogeneous function. It is proved by first establishing the convexity of the epigraphical set of the associated quadratic system.

Theorem 2.1. ***(Generalized S-Lemma with at most one non-homogeneous function)*** *Let $W, W_i \in S^q$, $i = 1, 2 \ldots, m$ be diagonal matrices, $\alpha, r_i \in \mathbb{R}, i = 1, 2, \ldots, m$ and $w \in \mathbb{R}^q$. Suppose that there exists $\bar{z} \in \mathbb{R}^q$ such that $\bar{z}^T W_i \bar{z} < r_i$, $i = 1, \ldots, m$. Then, the following statements are equivalent*

(a) $z^T W_i z \le r_i,\ i = 1, 2, \ldots, m\ \Rightarrow z^T W z + w^T z + \alpha \ge 0$;
(b) *There exist* $\lambda_i \ge 0,\ i = 1, \ldots, m$ *such that, for all* $z \in \mathbb{R}^q$,

$$z^T W z + w^T z + \alpha + \sum_{i=1}^m \lambda_i (z^T W_i z - r_i) \ge 0;$$

(c) *There exist* $\lambda_i \ge 0,\ i = 1, \ldots, m$ *and* $s_j \ge 0,\ j = 1, \ldots, q$ *such that*

$$\begin{cases} W_{jj} + \sum_{i=1}^m \lambda_i (W_i)_{jj} \ge 0,\ j = 1, \ldots, q \\ \alpha - \sum_{i=1}^m \lambda_i r_i - \sum_{j=1}^q s_j \ge 0 \\ \left\| \begin{pmatrix} w_j \\ s_j - W_{jj} - \sum_{i=1}^m \lambda_i (W_i)_{jj} \end{pmatrix} \right\| \le s_j + W_{jj} + \sum_{i=1}^m \lambda_i (W_i)_{jj},\ j = 1, \ldots, q. \end{cases}$$

where, for $j = 1, \ldots, q$, W_{jj} *and* $(W_i)_{jj}$ *are the* (j, j)*th element of the diagonal matrix* W *and* W_i, $i = 1, \ldots, m$, *respectively.*

Proof. Let $g_i(z) = z^T W_i z - r_i = \sum_{j=1}^q (W_i)_{jj} z_j^2 - r_i,\quad i = 1, \ldots, m,$ and $f(z) = z^T W z + w^T z + \alpha$.The equivalence between (a) and (b) follows from Lemma 2.1 if we show that the epigraphical set

$$U(f, g_1, \ldots, g_m) = \{(v_1, \ldots, v_m, r) : v_i \ge g_i(z) \text{ and } r \ge f(z) \text{ for some } z \in \mathbb{R}^q\}$$

is a convex set. Although this equivalence can be proved by verifying the convexity from the definition of a convex set, as done in many known cases (c.f Dines, 1941; Jeyakumar et al., 2009b; Jeyakumar and Li, 2018; Polik, 2007; Polyak, 1998), we give the details of the arguments here for the purpose of completeness.

If $w = 0$, then $g_i(z) = z^T W_i z_i - r_i,\ i = 1, \ldots, m$, and $f(z) = z^T W z + \alpha$. In this case, it is known that the joint range set $R(f, g_1, \ldots, g_m) = \{(g_1(z), \ldots, g_m(z), f(z)) : z \in \mathbb{R}^q\}$ is a convex set (Polik, 2007). Thus, $U(f, g_1, \ldots, g_m) = R(f, g_1, \ldots, g_m) + \mathbb{R}_+^{m+1}$ is also convex.

Therefore, without loss of generality, we assume that $w \ne 0$. Let $\lambda \in [0, 1]$, $(v_1^{(1)}, \ldots, v_m^{(1)}, \gamma^{(1)}) \in U(f, g_1, \ldots, g_m)$ and $(v_1^{(2)}, \ldots, v_m^{(2)}, \gamma^{(2)}) \in U(f, g_1, \ldots, g_m)$. Then, there exist $z^{(l)} \in \mathbb{R}^q$, $l = 1, 2$ such that

$$v_i^{(l)} \ge g_i(z^{(l)}) = \sum_{j=1}^q (W_i)_{jj} (z_j^{(l)})^2 - r_i,\ i = 1, \ldots, m \tag{5}$$

and

$$\gamma^{(l)} \ge f(z^{(l)}) = (z^{(l)})^T W (z^{(l)}) + w^T z^{(l)} + \alpha. \tag{6}$$

Consider the function $\hat{g}_i : \mathbb{R}_+^q \to \mathbb{R}$, $i = 1, \ldots, m$ and $\hat{f} : \mathbb{R}_+^q \to \mathbb{R}$ given by $\hat{g}_i(y) = \sum_{j=1}^q (W_i)_{jj} y_j - r_i$, for all $y \in \mathbb{R}_+^q$, and $\hat{f}(y) = \sum_{j=1}^q W_{jj} y_j - \sum_{j=1}^q |w_j| \sqrt{y_j} + \alpha$, for all $y \in \mathbb{R}_+^q$. Direct verification shows that $\hat{g}_i$ and $\hat{f}$ are all convex functions on $\mathbb{R}_+^q$. Let

$$y^{(l)} = \left((z_1^{(l)})^2, \ldots, (z_q^{(l)})^2 \right)^T \in \mathbb{R}_+^q.$$

It then follows from (5) and (6) that $\hat{g}_i(y^{(l)}) = g_i(z^{(l)}) \le v_i^{(l)}$ and

$$\hat{f}(y^{(l)}) = \sum_{j=1}^q W_{jj} (z^{(l)})_j^2 - \sum_{j=1}^q |w_j| \cdot |(z^{(l)})_j| + \alpha \le f(z^{(l)}) \le \gamma^{(l)}.$$

Then, by the convexity of $\hat{g}_i$ and $\hat{f}$ on $\mathbb{R}_+^q$,

$$\hat{g}_i(\lambda y^{(1)} + (1 - \lambda) y^{(2)}) \le \lambda v_i^{(1)} + (1 - \lambda) v_i^{(2)}$$

and

$$\hat{f}(\lambda y^{(1)} + (1 - \lambda) y^{(2)}) \le \lambda \gamma^{(1)} + (1 - \lambda) \gamma^{(2)}.$$

Now, define $z = (z_1, \ldots, z_q)^T$ where $z_j = -\mathrm{sign}(w_j) \sqrt{\lambda y_j^{(1)} + (1 - \lambda) y_j^{(2)}}$. Then, one has, for all $i = 1, \ldots, m$,

$$\begin{aligned} g_i(z) &= \sum_{j=1}^q (W_i)_{jj} z_j^2 - r_i = \sum_{j=1}^q (W_i)_{jj} (\lambda y_j^{(1)} + (1 - \lambda) y_j^{(2)}) - r_i \\ &= \hat{g}_i(\lambda y^{(1)} + (1 - \lambda) y^{(2)}) \le \lambda v_i^{(1)} + (1 - \lambda) v_i^{(2)}, \end{aligned}$$

and

$$\begin{aligned} f(z) &= \sum_{j=1}^q W_{jj} z_j^2 + \sum_{j=1}^q w_j z_j + \alpha \\ &= \sum_{j=1}^q W_{jj} (\lambda y_j^{(1)} + (1 - \lambda) y_j^{(2)}) - \sum_{j=1}^q |w_j| \cdot \sqrt{\lambda y_j^{(1)} + (1 - \lambda) y_j^{(2)}} + \alpha \\ &= \hat{f}(\lambda y^{(1)} + (1 - \lambda) y^{(2)}) \le \lambda \gamma^{(1)} + (1 - \lambda) \gamma^{(2)}. \end{aligned}$$

This shows that

$$\lambda (v_1^{(1)}, \ldots, v_m^{(1)}, \gamma^{(1)}) + (1 - \lambda)(v_1^{(2)}, \ldots, v_m^{(2)}, \gamma^{(2)}) \in U(f, g_1, \ldots, g_m).$$

So, $U(f, g_1, \ldots, g_m)$ is a convex set.

The equivalence between (b) and (c) follows if we show that (4) is equivalent to (c). This follows by the standard matrix algebra arguments together with the fact that $W + \sum_{i=1}^m W_i$ is a diagonal matrix and the following elementary equivalence,

$$\begin{pmatrix} \alpha & t/2 \\ t/2 & \beta \end{pmatrix} \succeq 0 \ \Leftrightarrow t^2 \le 4\alpha\beta,\ \alpha, \beta \ge 0 \ \Leftrightarrow\ \|(t, \alpha - \beta)\| \le \alpha + \beta.$$

□

Now, we show that the generalised S-lemma continues to hold for quadratic inequality system with a simultaneously diagonalizablity structure. The simultaneous diagonalizability property can be satisfied for quadratic systems that appear in several important quadratic optimization problems such as the classical trust region problems and some of their variants (Ben-Tal and Hertog, 2014; Jeyakumar and Li, 2018).

Corollary 2.1. ***(Generalized S-Lemma for simultaneously diagonalizable systems)*** *Let* $W, W_i \in S^q$, $i = 1, 2 \ldots, m$, $\alpha, r_i \in \mathbb{R}, i = 1, 2, \ldots, m$ *and* $w \in \mathbb{R}^q$. *Suppose that* W *and* $W_i, i = 1, \ldots, m$ *are simultaneously diagonalizable, that is, there exists an invertible matrix* $U \in \mathbb{R}^{q \times q}$ *such that* $U^T W U$ *and* $U^T W_i U$, $i = 1, \ldots, m$, *are all diagonal matrices. Suppose further that there exists* $\bar{z} \in \mathbb{R}^q$ *such that* $\bar{z}^T W_i \bar{z} < r_i$, $i = 1, \ldots, m$. *Then, the following statements are equivalent*

(a) $z^T W_i z \le r_i,\ i = 1, 2, \ldots, m\ \Rightarrow z^T W z + w^T z + \alpha \ge 0$;
(b) *There exist* $\lambda_i \ge 0,\ i = 1, \ldots, m$ *such that, for all* $z \in \mathbb{R}^q$,

$$z^T W z + w^T z + \alpha + \sum_{i=1}^m \lambda_i (z^T W_i z - r_i) \ge 0;$$

(c) *There exist* $\lambda_i \ge 0,\ i = 1, \ldots, m$ *and* $s_j \ge 0,\ j = 1, \ldots, q$ *such that*

$$\begin{cases} (U^T W U)_{jj} + \sum_{i=1}^m \lambda_i (U^T W_i U)_{jj} \ge 0,\ j = 1, \ldots, q \\ \alpha - \sum_{i=1}^m \lambda_i r_i - \sum_{j=1}^q s_j \ge 0 \\ \left\| \begin{pmatrix} (U^T w)_j \\ s_j - (U^T W U)_{jj} - \sum_{i=1}^m \lambda_i (U^T W_i U)_{jj} \end{pmatrix} \right\| \\ \le s_j + (U^T W U)_{jj} + \sum_{i=1}^m \lambda_i (U^T W_i U)_{jj},\ j = 1, \ldots, q. \end{cases}$$

where, for $j = 1, \ldots, q$, $(U^T W U)_{jj}$ *and* $(U^T W_i U)_{jj}$ *are the* (j, j)*th element of the diagonal matrix* $U^T W U$ *and* $U^T W_i U$, $i = 1, \ldots, m$, *respectively.*

Proof. [(a) ⇔ (b)] It suffices to show (a) implies (b) as (b) always implies (a). Let U be the invertible matrix such that $U^T W U = D$ and $U^T W_i U = D_i$, $i = 1, \ldots, m$, where D and D_i are all diagonal matrices. Letting $y = U^{-1} z$, we see that (a) is equivalent to

$$y^T D_i y \le r_i,\ i = 1, 2, \ldots, m\ \Rightarrow y^T D y + (U^T w)^T y + \alpha \ge 0.$$

Note also that $\bar{y}^T D_i \bar{y} = \bar{z}^T W_i \bar{z} < r_i$, $i = 1, \ldots, m$, where $\bar{y} = U^{-1}\bar{z}$. Applying Theorem 2.1, there exist $\lambda_i \geq 0$, $i = 1, \ldots, m$ such that for all $y \in \mathbb{R}^q$,

$$y^T D y + (U^T w)^T y + \alpha + \sum_{i=1}^{m} \lambda_i \left(y^T D_i y - r_i \right) \geq 0.$$

This shows that for all $z \in \mathbb{R}^q$,

$$z^T W z + w^T z + \alpha + \sum_{i=1}^{m} \lambda_i \left(z^T W_i z - r_i \right) \geq 0.$$

Thus, (b) holds. So, (a) implies (b), and hence, (a) is equivalent to (b).

[(b) ⇔ (c)] As U is invertible, we first note that (b) is equivalent to

$$y^T (U^T W U) y + (U^T w)^T y + \alpha + \sum_{i=1}^{m} \lambda_i \left(y^T (U^T W_i U) y - r_i \right) \geq 0,$$

for all $y \in \mathbb{R}^q$.

Then, the conclusion follows by Theorem 2.1. □

Remark 2.1. It is worth remarking that the convexity theorems of the epigraphical set involving non-homogeneous functions with Z-matrices, established in Jeyakumar et al. (2009b), do not yield corresponding S-lemma type results for diagonal matrices due to homogenization of the given non-homogeneous system. Simplified S-lemma results are known for homogeneous systems involving diagonal matrices in Ben-Tal et al. (2009); Polik (2007)

3. Exact SOCP relaxations for AROs with inexact data

In this section we present an exact SOCP relaxation problem for a robust linear program with adjustable decisions that admit a Separable Parameterized Quadratic Decision Rule (Woolnough et al., 2021), with inexact data as well as the estimation error contained within ellipsoidal uncertainty sets.

The two-stage robust linear optimization problem (IP) with separable parameterized quadratic decision rule given as in (3) can be written in the following compact form:

$$\begin{aligned}
(IP-QDR) \quad & \min_{x, y_0, P, \xi_r^j} \quad c^T x \\
& \text{subject to} \quad A(z)x + By(\hat{z}) \leq d(z), \\
& \qquad \text{for all } (z, \hat{z}) \in \mathcal{Z} \subset \mathbb{R}^q \times \mathbb{R}^q, \\
& \qquad y(\hat{z}) = \theta(y_0 + P\hat{z}) + (1-\theta) \begin{pmatrix} \hat{z}^T Q_1 \hat{z} \\ \vdots \\ \hat{z}^T Q_s \hat{z} \end{pmatrix},
\end{aligned}$$

where $Q_r = \text{diag}(\xi_r^{(1)}, \ldots, \xi_r^{(q)})$. Recall that a_l, A_l and v_l, v_l are given as in (1). We also use b_l^T to denote the lth row of the matrix B in (IP) and let $(b_l)_r$ be the rth element of b_l (that is, the (l, r)th entry of B). In passing, note that the constraint system of $(IP-QDR)$ is a non-convex, semi-infinite constraint. We now associate with $(IP-QDR)$ the following second order cone programming problem:

$$\begin{aligned}
(SOCP_{IP}) \quad & \min_{\substack{x \in \mathbb{R}^n, \tilde{y}_0 \in \mathbb{R}^s \\ \tilde{P} \in \mathbb{R}^{s \times q}, \tilde{\xi}_r^{(j)} \in \mathbb{R} \\ \lambda_1^{(l)} \in \mathbb{R}, \lambda_2^{(l)} \in \mathbb{R}, \\ \mu_j^{(l)} \in \mathbb{R}, \sigma_j^{(l)} \in \mathbb{R}}} \quad c^T x \\
\text{s.t.} \quad & -(1-\theta) \sum_{r=1}^{s} b_{ir} \tilde{\xi}_r^{(j)} + \lambda_1^{(l)} \geq 0, \; j = 1, \ldots, q, \; l = 1, \ldots, p, \\
& (a_l^T x + \theta b_l^T \tilde{y}_0 - v_l) + (A_l^T x - v_l)^T u + \lambda_1^{(l)} \gamma_1^2 + \lambda_2^{(l)} \gamma_2^2 \\
& + \sum_{j=1}^{q} (\mu_j^{(l)} + \sigma_j^{(l)}) \leq 0, \; l = 1, \ldots, p, \\
& \left\| \begin{pmatrix} -(A_l^T x + \theta \tilde{P}^T b_l - v_l)_j \\ \mu_j^{(l)} + (1-\theta) \sum_{r=1}^{s} b_{ir} \tilde{\xi}_r^{(j)} - \lambda_1^{(l)} \end{pmatrix} \right\| \leq \mu_j^{(l)} \\
& -(1-\theta) \sum_{r=1}^{s} b_{ir} \tilde{\xi}_r^{(j)} + \lambda_1^{(l)}, \; j = 1, \ldots, q, \; l = 1, \ldots, p, \\
& \left\| \begin{pmatrix} -(A_l^T x - v_l)_j \\ \sigma_j^{(l)} - \lambda_2^{(l)} \end{pmatrix} \right\| \leq \sigma_j^{(l)} + \lambda_2^{(l)}, \; j = 1, \ldots, q, \; l = 1, \ldots, p, \\
& \lambda_1^{(l)}, \lambda_2^{(l)}, \mu_j^{(l)}, \sigma_j^{(l)} \geq 0, \; j = 1, \ldots, q, \; l = 1, \ldots, p,
\end{aligned}$$

where $\theta \in (0, 1]$ is a parameter given in (3), and $u \in \mathbb{R}^q$ is given in the description of the uncertainty set $\mathcal{Z}$.

Next, we show that $(IP-QDR)$ and its associated second order cone programming problem are equivalent in the sense that an optimal solution to $(IP-QDR)$ can be uniquely determined from an optimal solution to $(SOCP_{IP})$. We derive this exact second order cone program relaxation by using the new S-lemma for a partially homogeneous separable quadratic system involving a non-homogeneous quadratic function and more than two homogeneous quadratic inequalities. This extends the approach used in Woolnough et al. (2021) where a single ellipsoid and the classical S-lemma were used.

Theorem 3.1. *Consider the adjustable robust linear program with separable parameterized quadratic decision rule and inexact ellipsoidal data $(IP-QDR)$, and the second order cone programming problem $(SOCP_{IP})$.*

(a) *It holds that* $\min(IP-QDR) = \min(SOCP_{IP})$, *where* $\min(IP-QDR)$ *(resp.* $\min(SOCP_{IP})$ *denotes the optimal value of the problem (IP-QDR) (resp.* $(SOCP_{IP})$*).*
(b) *Moreover,* $(x^*, y_0^*, P^*, \xi_1^{(1)^*}, \ldots, \xi_s^{(q)^*})$ *is a solution to* $(IP-QDR)$ *if and only if there exist* $\lambda_1^{(l)}, \lambda_2^{(l)}, \mu_j^{(l)}, \sigma_j^{(l)}$, $l = 1, \ldots, p$, $j = 1, \ldots, q$ *such that*

$$(x^*, \tilde{y}_0^*, \tilde{P}^*, \tilde{\xi}_1^{(1)^*}, \ldots, \tilde{\xi}_s^{(q)^*}, \lambda_1^{(l)}, \lambda_2^{(l)}, \mu_1^{(1)}, \ldots, \mu_q^{(p)}, \sigma_1^{(1)}, \ldots, \sigma_q^{(p)}),$$

is a solution to $(SOCP_{IP})$, *where* $(\tilde{y}_0^*, \tilde{P}^*, \tilde{\xi}_1^{(1)^*}, \ldots, \tilde{\xi}_s^{(q)^*})$ *is given by the one-to-one correspondence*

$$(\tilde{y}_0^*, \tilde{P}^*, \tilde{\xi}_1^{(1)^*}, \ldots, \tilde{\xi}_s^{(q)^*}) = \left(y_0^* + P^* u + \frac{1-\theta}{\theta} \begin{pmatrix} u^T Q_1^* u \\ \vdots \\ u^T Q_s^* u \end{pmatrix}, P^* + \frac{1-\theta}{\theta} \begin{pmatrix} 2u^T Q_1^* \\ \vdots \\ 2u^T Q_s^* \end{pmatrix}, \xi_1^{(1)^*}, \ldots, \xi_s^{(q)^*} \right)$$

where $Q_r^* = diag(\xi_r^{(1)^*}, \ldots, \xi_r^{(q)^*})$, $r = 1, \ldots, s$.

Proof. Fix $l \in \{1, \ldots, p\}$. Notice that the l^{th} constraint of $(IP-QDR)$ can be equivalently rewritten as

$$(a_l + A_l z)^T x + b_l^T \left(\theta(y_0 + P\hat{z}) + (1-\theta) \begin{pmatrix} \hat{z}^T Q_1 \hat{z} \\ \vdots \\ \hat{z}^T Q_s \hat{z} \end{pmatrix} \right) \leq v_l + v_l^T z, \forall (z, \hat{z}) \in \mathcal{Z}. \tag{7}$$

We first apply a linear transformation to our uncertainty set

$$\mathcal{Z} = \{(z, \hat{z}) \in \mathbb{R}^q \times \mathbb{R}^q : \|\hat{z} - u\|^2 \leq \gamma_1^2, \|z - \hat{z}\|^2 \leq \gamma_2^2\}.$$

Let

$$\zeta^{(1)} = \hat{z} - u, \; \zeta^{(2)} = z - \hat{z}, \; \zeta = \begin{pmatrix} \zeta^{(1)} \\ \zeta^{(2)} \end{pmatrix} \in \mathbb{R}^{2q}.$$

Then

$$\begin{aligned}
(z, \hat{z}) \in \mathcal{Z} \iff \zeta \in \mathcal{U} &= \left\{ \zeta = \begin{pmatrix} \zeta^{(1)} \\ \zeta^{(2)} \end{pmatrix} \in \mathbb{R}^{2q} : \|\zeta^{(1)}\|^2 \leq \gamma_1^2, \|\zeta^{(2)}\|^2 \leq \gamma_2^2 \right\} \\
\iff \zeta \in \mathcal{U} &= \left\{ \zeta \in \mathbb{R}^{2q} : \sum_{j=1}^{2q} w_{1j} \zeta_j^2 \leq \gamma_1^2, \; \sum_{j=1}^{2q} w_{2j} \zeta_j^2 \leq \gamma_2^2 \right\},
\end{aligned}$$

where

$$w_{1j} = \begin{cases} 1, & 1 \leq j \leq q \\ 0, & q+1 \leq j \leq 2q \end{cases}, \quad w_{2j} = \begin{cases} 0, & 1 \leq j \leq q \\ 1, & q+1 \leq j \leq 2q. \end{cases} \tag{8}$$

That is, $\mathcal{Z}$ can be represented as an intersection of two ellipsoids of the form consistent with Theorem 2.1, in a higher dimension. Thus, we can rewrite (7) under the uncertainty set $\mathcal{U}$:

$$\begin{aligned}&\left(a_l + A_l(\zeta^{(1)}+\zeta^{(2)}+u)\right)^T x\\ &+b_l^T\left(\theta(y_0+P(\zeta^{(1)}+u))+(1-\theta)\begin{pmatrix}(\zeta^{(1)}+u)^TQ_1(\zeta^{(1)}+u)\\ \vdots\\ (\zeta^{(1)}+u)^TQ_s(\zeta^{(1)}+u)\end{pmatrix}\right)\\ &\le v_l + v_l^T(\zeta^{(1)}+\zeta^{(2)}+u), \forall(\zeta^{(1)},\zeta^{(2)})\in\mathcal{U}.\end{aligned} \tag{9}$$

Define the one-to-one correspondence between variables $(y_0, P, \xi_1^{(1)}, \dots, \xi_s^{(q)})$ and new variables $(\tilde{y}_0, \tilde{P}, \tilde{\xi}_1^{(1)}, \dots, \tilde{\xi}_s^{(q)})$ by

$$\tilde{y}_0 = y_0 + Pu + \frac{1-\theta}{\theta}\begin{pmatrix}u^TQ_1u\\ \vdots\\ u^TQ_su\end{pmatrix}\in\mathbb{R}^s,\ \tilde{P} = P + \frac{1-\theta}{\theta}\begin{pmatrix}2u^TQ_1\\ \vdots\\ 2u^TQ_s\end{pmatrix}\in\mathbb{R}^{s\times q}$$

and

$$\tilde{\xi}_r^{(j)} = \xi_r^{(j)},\ r=1,\dots,s,\ j=1,\dots,q.$$

To accommodate the transformation of $(z,\hat{z})$ into ζ in the higher dimensional space $\mathbb{R}^{2q}$, we also define

$$\begin{aligned}\tilde{a}_l &= a_l + A_l u \in\mathbb{R}^n,\ \tilde{A}_l = \begin{pmatrix}A_l & A_l\end{pmatrix}\in\mathbb{R}^{n\times 2q},\\ \tilde{v}_l &= v_l + v_l^T u\in\mathbb{R},\ \tilde{v}_l = \begin{pmatrix}v_l\\ v_l\end{pmatrix}\in\mathbb{R}^{2q}\end{aligned} \tag{10}$$

Then (9) is equivalent to Theorem 2.1 (a) with the settings

$$\begin{aligned}&m=2,\quad z=\zeta\in\mathbb{R}^{2q},\quad r_1=\gamma_1^2,\quad r_2=\gamma_2^2,\quad \alpha=-\tilde{a}_l^Tx-\theta b_l^T\tilde{y}_0+\tilde{v}_l\in\mathbb{R},\\ &w = -\tilde{A}_l^Tx-\theta\begin{pmatrix}\tilde{P}^T\\ 0_{q\times s}\end{pmatrix}b_l+\tilde{v}_l\in\mathbb{R}^{2q},\\ &W=\begin{pmatrix}-(1-\theta)\sum_{r=1}^s(b_l)_rQ_r & \\ & 0_{q\times q}\end{pmatrix},\quad W_1=\begin{pmatrix}w_{11} & & & \\ & w_{12} & & \\ & & \ddots & \\ & & & w_{1(2q)}\end{pmatrix},\\ &W_2=\begin{pmatrix}w_{21} & & & \\ & w_{22} & & \\ & & \ddots & \\ & & & w_{2(2q)}\end{pmatrix}\end{aligned} \tag{11}$$

Note that $W_{jj} = -(1-\theta)\sum_{r=1}^s(b_l)_r\tilde{\xi}_r^{(j)}$ for $j=1,\dots,q$. Applying Theorem 2.1, and transforming $(\tilde{a}_l,\tilde{A}_l,\tilde{v}_l,\tilde{v}_l)$ back to (a_l,A_l,v_l,v_l), give us that (9) is equivalent to the condition that there exist $\lambda_1^{(l)},\lambda_2^{(l)},\mu_j^{(l)},\sigma_j^{(l)}\ge 0, j=1,\dots,q$, such that

$$\begin{cases}-(1-\theta)\sum_{r=1}^s(b_l)_r\tilde{\xi}_r^{(j)}+\lambda_1^{(l)}\ge 0,\ j=1,\dots,q\\ (-a_l^Tx-\theta b_l^T\tilde{y}_0+v_l)+(-A_l^Tx+v_l)^Tu-\lambda_1\gamma_1^2-\lambda_2\gamma_2^2-\sum_{j=1}^q\mu_j^{(l)}\\ \quad -\sum_{j=1}^q\sigma_j^{(l)}\le 0\\ \left\|\begin{pmatrix}(-A_l^Tx-\theta\tilde{P}^Tb_l+v_l)_j\\ \mu_j^{(l)}+(1-\theta)\sum_{r=1}^s(b_l)_r\tilde{\xi}_r^{(j)}-\lambda_1^{(l)}\end{pmatrix}\right\|\le\mu_j^{(l)}-(1-\theta)\sum_{r=1}^s(b_l)_r\tilde{\xi}_r^{(j)}+\lambda_1^{(l)},\\ j=1,\dots,q\\ \left\|\begin{pmatrix}(-A_l^Tx+v_l)_j\\ \sigma_j^{(l)}-\lambda_2^{(l)}\end{pmatrix}\right\|\le\sigma_j^{(l)}+\lambda_2^{(l)},\ j=1,\dots,q,\end{cases} \tag{12}$$

where $\mu_j^{(l)}=s_j$, $\sigma_j^{(l)}=s_{j+q}$, $j=1,\dots,q$. The result then follows by substitution of (12) for each of the l^{th} constraints of $(IP-QDR)$, $l=1\dots,p$. $\square$

Remark 3.1. It is worth noting that the authors in Woolnough et al. (2021) established exact conic programming reformulations for two-stage robust optimization problems with exactly revealed data, where $\hat{z}=z$ in (2). In this case, our two-staged robust problem is equivalent to the robust optimization model examined in Woolnough et al. (2021).

4. Adjustable robust lot-sizing problems with inexact data

In the lot-sizing problem, given a network of n stores one needs to determine the stock allocation x_i (in units) at each store $i=1,\dots,n$, (denoted by $x:=(x_1,\dots,x_n)$), and the stock y_{ij} to be transported from store i to j (with y_{ii} necessarily set to 0), in order to meet the demand d_i for each store over a set period of time (for example, one day). We assume that the storage cost in store i is c_i with respect to the stock allocation x_i, and that the cost of transporting one unit of stock from store i to j is t_{ij}, $i,j=1,\dots,n$, where $t_{ii}=0$ for $i=1,\dots,n$. This problem can be mathematically described as

$$\begin{aligned}(\mathrm{LS}_0)\quad &\min_{\substack{x\in\mathbb{R}^n\\ y_{ij}\in\mathbb{R}}}\ \sum_{i=1}^n c_ix_i+\sum_{i,j=1}^n t_{ij}y_{ij}\\ &\text{s.t.}\quad x_i+\sum_{j=1}^n y_{ji}-\sum_{j=1}^n y_{ij}\ge d_i, i=1,\dots,n\\ &\qquad\ x_i\ge 0,\ y_{ii}=0,\ y_{ij}\ge 0,\ i=1,\dots,n,\ j=1,\dots,n,\ i\ne j\end{aligned}$$

In practice the true demand $d=(d_1,d_2,\dots,d_n)^T\in\mathbb{R}^n$ is uncertain, and partially revealed at some point after the initial allocation of stock; for example, by using a projection of current demand until end of day.

4.1. Experimental design

Problem Set-up: Suppose a nominal demand $d_0\in\mathbb{R}^n$ is chosen for the model (using, e.g. historical data). The true demand is known to lie within some neighbourhood of this nominal demand. At the beginning of the period, an initial stock allocation x_i is decided for each store in the network. Then, at a pre-specified point in time later in the period, an estimate $\hat{d}\in\mathbb{R}^n$ is taken of the true demand for the entire period (using, for example, satisfied demand up until that time point). We will assume that $\hat{d}$ lies within a ball centred at the nominal demand (the *estimation range*), and that this is expressed in the form

$$\|\hat{d}-d_0\|\le\frac{\alpha}{100}\|d_0\|.$$

At this point in time we make a decision to transport some stock y_{ij} between stores, based on our estimate $\hat{d}$. In reality the estimate $\hat{d}$ will deviate from the true demand d with some *estimation error* δ, such that $\delta=d-\hat{d}$. We assume also that this deviation lies in its own ball: $\|\delta\|\le\beta$ with $\beta>0$.

By also choosing to assign a separable parameterized QDR to each $y_{ij}(\hat{d})$, we can express the problem as a two-stage adjustable robust linear optimization problem with inexactly-revealed ellipsoidal data:

$$\begin{aligned}(\mathrm{LS})\quad &\min_{\substack{x\in\mathbb{R}^n,y_{ij}\in\mathbb{R}\\ w_{ij}\in\mathbb{R}^n,q_{ij}\in\mathbb{R}^n}}\ \sum_{i=1}^n c_ix_i+\max_{\hat{d}\in D}\left\{\sum_{i,j=1}^n t_{ij}y_{ij}(\hat{d})\right\}\\ &\text{s.t.}\quad x_i+\sum_{j=1}^n y_{ji}(\hat{d})-\sum_{j=1}^n y_{ij}(\hat{d})\ge d_i, i=1,\dots,n\\ &\qquad x_i\ge 0,\ y_{ii}(\hat{d})=0,\ y_{ij}(\hat{d})\ge 0,\ i=1,\dots,n,\ j=1,\dots,n,\ i\ne j\\ &\qquad y_{ij}(\hat{d})=\theta\left(y_{ij}^0+w_{ij}^T\hat{d}\right)+(1-\theta)(\hat{d}^T\mathrm{diag}(q_{ij})\hat{d}),\quad\text{for all }(\hat{d},d)\in\mathcal{D},\end{aligned}$$

where $\mathcal{D}=\left\{(\hat{d},d)\in\mathbb{R}^n\times\mathbb{R}^n:\|d-\hat{d}\|^2\le\beta^2,\ \|\hat{d}-d_0\|^2\le\frac{\alpha^2}{10\,000}\|d_0\|^2\right\}$. Notice that our uncertainty set $\mathcal{D}$ takes on the same form as $\mathcal{Z}$ in (IP). The formulation aims to satisfy the uncertain true demand, whilst making decisions based on the estimated uncertainty, with these decisions functionally determined before estimates are taken.

Numerical Experiment Design. We will compare the solution to problem (LS) (referred to here on as the "inexact approach") to its juxtaposition where we ignore the inexactness of revealed data and instead assume that $d = \hat{d}$ (here-on referred to as the "exact" approach).

For $n = 2, 4, 8, 10, 12, 15$, we will choose our parameters as follows:

$$c_i = 1,\ t_{ij} = \begin{cases} 2, & i \neq j \\ 0, & i = j \end{cases},\ d_0 = (5, \ldots, 5)^T \in \mathbb{R}^n, \alpha = 50,$$

$$\beta = \frac{1}{2}\left(\frac{\alpha}{100}\|d_0\|\right),\ i, j = 1, \ldots, n.$$

Our experiments analyse the solutions for each of the approaches by considering the solution after realization of the true uncertain parameters d and $\hat{d}$, which we simulate (see below). Recall that after this point, our adjustable variables $y_{ij}(\hat{d}) \in \mathbb{R}$ reduce to the scalar $\theta(y_{ij}^{0\,*} + w_{ij}^{*\,T}\hat{d}) + (1-\theta)\left(\hat{d}^T \mathrm{diag}(q_{ij}^*)\hat{d}\right)$ where $(y_{ij}^{0\,*}, w_{ij}^*, q_{ij}^*)$ are the optimal solution found by solving (LS) with the given approach.

Our simulations require us to simulate first the demand estimate $\hat{d}$ and then, based on the outcome, simulate the true demand d. In our case this is done by first uniformly generating 200 simulated values for $\hat{d}$ within the ellipsoid $\|\hat{d} - d_0\| \leq \frac{\alpha}{100}\|d_0\|$, and then, for each of these, generating 200 simulated values for the true demand d within the ellipsoid $\|d - \hat{d}\| \leq \beta$.

For proper discussion of the performance of our adjustable robust models, we need to investigate how the solution behaves under realization of some true demand d, by examining its realized cost:

$$\sum_{i=1}^{n} c_i x_i + \sum_{i,j=1}^{n} t_{ij} y_{ij}(\hat{d})$$

Price of Robustness. We define the price of robustness POR as the percentage difference between the realized cost achieved by the adjustable solution (which is the optimal value c produced by solving the problem (LS)), and the optimal value c_{LS_0} of the nominal problem (LS_0), relative to c_{LS_0}:

$$POR = \frac{c - c_{\mathrm{LS}_0}}{c_{\mathrm{LS}_0}}.$$

We emphasise that both values are dependent on the same estimated true demand parameter $\hat{d}$ and its associated simulations of the true demand d. Note that this realized cost for the adjustable solution is the cost that would be incurred in practice by utilizing the adjustable model, and likewise the optimal cost of the nominal problem is the lowest possible cost achievable by having all uncertain information available before making any decisions.

Our definition of the price of robustness is equivalent to that employed by Ben-Tal et al. (2004) in discussions of ADR solutions to multi-stage adjustable robust inventory production problems. We have chosen this definition due to its relevance to our experiments, as opposed to, for example, the definition provided by Bertsimas and Sim (2004) where it is defined as a trade-off between the probability of violating robust constraints and the effect on the resulting objective function, and only within the scope of Robust Optimization.

Our experiments consist of two parts:

1. Firstly, we compare the price of robustness of the two approaches by comparing their (realised) solutions to the nominal solution to (LS_0). For this case we choose $\theta = \frac{1}{2}$ for the exact approach, and both $\theta = \frac{1}{2}$ and $\theta = 1$ (i.e. ADR) for the inexact approach. Results for this experiment are presented in Tables 1 and 2 and Fig. 1. We will not compare to the exact approach with an ADR, as this result has been examined in Woolnough et al. (2021).
2. Secondly, we compare the exact and inexact approaches (both $\theta = \frac{1}{2}$ and $\theta = 1$) in terms of the feasibility of their solutions against the (inexactly-revealed) true demand. A solution given by one of our adjustable models is considered feasible for a realization d if the constraints of (LS_0) hold for this solution and realized d. Results for this experiment are presented in Table 3.

Table 1
Price of Robustness, over all d in the neighbourhood of $\hat{d}_{\max}$.

n	Approach	Mean	Max	Min	Med.
2	Inexact QDR	0.82	1.31	0.49	0.79
	Inexact ADR	0.84	1.33	0.51	0.81
	Exact QDR	0.50	0.90	0.23	0.48
4	Inexact QDR	0.70	1.03	0.43	0.69
	Inexact ADR	0.86	1.21	0.56	0.85
	Exact QDR	0.33	0.58	0.12	0.32
8	Inexact QDR	1.59	2.22	1.14	1.55
	Inexact ADR	2.18	2.96	1.63	2.13
	Exact QDR	0.86	1.32	0.54	0.84
10	Inexact QDR	1.83	2.55	1.43	1.81
	Inexact ADR	2.63	3.56	2.12	2.61
	Exact QDR	0.98	1.48	0.70	0.97
12	Inexact QDR	1.27	1.63	0.97	1.26
	Inexact ADR	2.01	2.50	1.62	2.01
	Exact QDR	0.54	0.79	0.34	0.54
15	Inexact QDR	1.95	2.40	1.57	1.94
	Inexact ADR	3.09	3.72	2.57	3.07
	Exact QDR	0.94	1.23	0.69	0.93

Table 2
Price of Robustness, over all d in the neighbourhood of $\hat{d}_{\min}$.

n	Approach	Mean	Max	Min	Med.
2	Inexact QDR	0.93	1.44	0.55	0.89
	Inexact ADR	0.93	1.44	0.56	0.90
	Exact QDR	0.59	1.02	0.28	0.57
4	Inexact QDR	1.97	3.08	1.29	1.89
	Inexact ADR	2.25	3.46	1.50	2.16
	Exact QDR	1.32	2.19	0.79	1.26
8	Inexact QDR	1.28	1.65	0.91	1.27
	Inexact ADR	1.81	2.27	1.35	1.80
	Exact QDR	0.64	0.91	0.38	0.64
10	Inexact QDR	1.52	1.97	1.10	1.51
	Inexact ADR	2.23	2.81	1.70	2.22
	Exact QDR	0.76	1.08	0.47	0.76
12	Inexact QDR	1.92	2.42	1.45	1.93
	Inexact ADR	2.88	3.55	2.26	2.89
	Exact QDR	0.99	1.33	0.67	0.99
15	Inexact QDR	2.01	2.51	1.59	2.01
	Inexact ADR	3.17	3.86	2.59	3.17
	Exact QDR	0.97	1.30	0.70	0.97

Table 3
Infeasibility results for the inexact and exact approach. Columns defined above. Note that the inexact approach never delivers an infeasible solution.

n	Approach	Total	Max	Min	Med.
2	Inexact QDR	0	0	0	0
	Exact QDR	4024	104	0	0
	Exact ADR	15,000	200	0	0
4	Inexact QDR	0	0	0	0
	Exact QDR	13,111	166	0	60
	Exact ADR	33,600	200	0	200
8	Inexact QDR	0	0	0	0
	Exact QDR	25,369	195	1	136
	Exact ADR	38,000	200	0	200
10	Inexact QDR	0	0	0	0
	Exact QDR	29,158	192	4	155
	Exact ADR	40,000	200	200	200
12	Inexact QDR	0	0	0	0
	Exact QDR	31,291	200	6	164
	Exact ADR	40,000	200	200	200
15	Inexact QDR	0	0	0	0
	Exact QDR	34,436	198	86	179
	Exact ADR	40,000	200	200	200

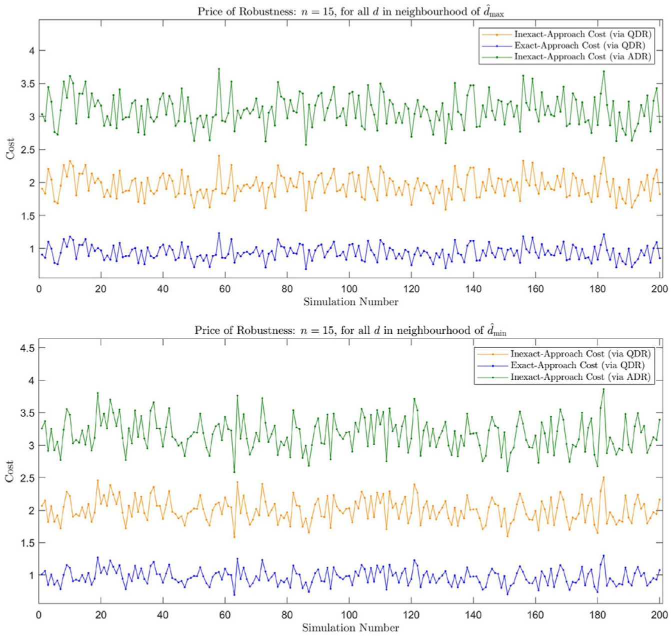

Fig. 1. Price of Robustness, for the case $n = 15$. Top: the price of robustness for the three approaches, over all simulated d in the neighbourhood of $\hat{d}_{\max}$. Bottom: likewise, but over all simulated d in the neighbourhood of $\hat{d}_{\min}$.

For the purposes of analysis we cannot present the price of robustness results for all 40,000 simulations; therefore we have selected from all 200 simulated $\hat{d}$ two realizations: $\hat{d}_{\min}$, the estimate that produced the *least* number of infeasible solutions when applying the exact QDR approach to all 200 simulated d in its neighbourhood; and $\hat{d}_{\max}$, the estimate that produced the *most* number of infeasible solutions when applying the exact QDR approach to all 200 simulated d in its neighbourhood. These two choices allow for coverage of both extreme cases

All computations were performed using a 3.2GHz Intel(R) Core(TM) i7-8700 and 16GB of RAM, equipped with MATLAB R2019A. All optimization problems were solved via the MOSEK software (ApS, 2019), handled through the YALMIP interface (Lofberg, 2004).

4.2. Results

Price of Robustness Comparison. The price of robustness was calculated for each n and each generated demand d in the neighbourhoods of $\hat{d}_{\min}$ and $\hat{d}_{\max}$. Tables 1 and 2 provide statistics for the two cases. The columns are given by:

- ***n***: the number of stores
- **Approach**: method used: either the inexact QDR approach, inexact ADR approach, or the exact QDR approach
- **Mean**: the mean price of robustness over all (200) relevant d
- **Max**: the maximum price of robustness over all (200) relevant d
- **Min**: the minimum price of robustness over all (200) relevant d
- **Med.**: the median price of robustness over all (200) relevant d.

We first note that the QDR (both approaches) outperforms the ADR in all simulations, achieving a lower price of robustness. It can also be seen that the counterpart to infeasible solutions (see below, Infeasibility Comparison) is a lower price of robustness, as would be expected. Interestingly, the infeasible solutions (also included in the below plots) never find a lower cost than the nominal solution (which is a possible consequence of infeasibility). The price of robustness for each individual simulation, for all three methods and $n = 15$, are illustrated in Fig. 1.

Infeasibility Comparison. The results are summarised in Table 3 for which the columns are given by:

- ***n***: the number of stores
- **Approach**: either the exact or inexact QDR, or the exact ADR.
- **Total**: the total number of infeasible solutions found across all 40 000 generated values for d
- **Max**: the maximum number of infeasible solutions found for a single generated $\hat{d}$, out of 200 (i.e. instance $\hat{d}_{\max}$)
- **Min**: the minimum number of infeasible solutions found for a single generated $\hat{d}$, out of 200 (i.e. instance $\hat{d}_{\min}$)
- **Med.**: the median number of infeasible solutions found for a single generated $\hat{d}$, out of 200.

As expected, the inexact approach never returns infeasible solutions, by design of its uncertainty set to accommodate all possible realisations of the uncertainty. The exact approach, however, ignores inexactness and, in assuming that $d = \hat{d}$, frequently generates adjustable decisions that are infeasible for the problem. Also notice that the number of infeasible solutions increases as the network increases to a larger, more realistic size. This shows a clear advantage for real-world applications, which often contain varying degrees of uncertainty.

Our experiments also reported that there were some simulated problem instances wherein the true demand d lay within the estimation range, and yet the exact approaches returned realized solutions which were infeasible for this choice of d, despite satisfying all conditions to be an optimal solution. This illustrates the true benefit of an inexact approach, which is that it can guarantee that a returned solution will always be feasible for any realized true demand within the estimation error. While this result may appear counter-intuitive to the robustness of the exact approaches, it is worth noticing that the exact approach is immune to uncertainty realization as long as its core assumption - that the realized value of the uncertainty, $\hat{d}$, is *exact* - holds. This is not the case for our experiments as we are specifically analysing the case $\hat{d} \neq d$. More explicitly, for solution $(x, y(\cdot))$ returned by the exact approach, there is no reason that the constraints

$$x_i + \sum_{j=1}^{n} y_{ji}(\hat{d}) + \sum_{j=1}^{n} y_{ij}(\hat{d}) \geq d_i, \ \forall i = 1, \dots, n$$

hold in general for all d satisfying $\|d - \hat{d}\| \leq \beta$. This is in stark contrast to the inexact approach, which understands that $\hat{d}$ can be different to d with deviation up to the estimation error.

5. Conclusion and outlook

We have shown that a second-order cone program reformulation holds for adjustable robust linear optimization problems with a parameterized quadratic decision rule (QDR) and inexactly revealed data. We achieved this by first establishing a generalization of S-lemma which allowed us to transform the semi-infinite non-convex separable quadratic inequality constraint into a second-order cone constraint. We illustrated our results via numerical experiments on adjustable robust lot-sizing problems with demand uncertainty and showed that the inexactly revealed data approach with QDRs improves over the infeasibility of the exact QDR approach, whilst maintaining its better performance over an ADR approach in achieving a low price of robustness.

Our adjustable robust optimization approach with inexact data is likely to lead to applications in dynamic decision-making problems of medicine and health sciences, where the conventional static decisions that do not include adjustable (recourse) decisions are often practically meaningless because a patient's condition or health care needs can change during the course of a treatment (Eikelder et al., 2019; Nohadani and Roy, 2017), but estimates of their condition can be taken and considered mid-treatment. In particular, our approach may be extended to two-stage robust optimization models with time-dependent uncertainty sets that appear in radiation therapy planning problems (Nohadani and Roy, 2017).

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