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On the Largest Cartesian Closed Category of Stable Domains

Xiaoyong Xi [1,2]

College of Mathematics Sciences
Jiangsu Normal university
Xuzhou, China

Guohua Wu[3]

Division of Mathematical Sciences, School of Physical and Mathematical Sciences
Nanyang Technological University
Singapore

Abstract

Let **SABC** (resp., $\widetilde{\mathbf{SABC}}$) be the category of algebraic bounded complete domains with conditionally multiplicative mappings, that is, Scott-continuous mappings preserving meets of pairs of compatible elements (resp., stable mappings). Zhang showed that the category of dI-domains is the largest cartesian closed subcategory of ω-**SABC** and ω-$\widetilde{\mathbf{SABC}}$, with the exponential being the stable function space, where ω-**SABC** and ω-$\widetilde{\mathbf{SABC}}$ are full subcategories of **SABC** and $\widetilde{\mathbf{SABC}}$ respectively which contain countablly based algebraic bounded complete domains as objects. This paper shows that:

i) The exponentials of any full subcategory of **SABC** or $\widetilde{\mathbf{SABC}}$ are exactly function spaces;

ii) $\widetilde{\mathbf{SDABC}}$, the category of distributive algebraic bounded complete domains, is the largest cartesian closed subcategory of $\widetilde{\mathbf{SABC}}$;

The compact elements of function spaces in the category **SABC** are also studied.

Keywords: cartesian closed; stable mappings; distributive

1 Introduction and Preliminaries

Let **CONT, ALG** be the categories of continuous domains, algebraic domains with Scott-continuous mappings respectively. The cartesian closedness of full subcategories of **CONT** and **ALG** plays an important role in computer science [2,13]. It

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[1] Funded by NSFC (11361028) and Priority Academic Program Development of Jiangsu Higher Education Institutions of China

[2] Email: `littlebrook@sina.com`

[3] Email: `guohua@ntu.edu.sg`

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is investigated by Plotkin [12], Smyth [14], Jung [2] and others. Jung [2] systematically studied the cartesian closed maximal full subcategories of **CONT, ALG** and proved that there exist exactly four cartesian closed maximal full subcategories of **CONT** and **ALG**, respectively. On the other hand, motivated from the study of sequentiality, Berry [3] introduced stable mappings and the category of dI-domains which then was investigated by many authors [1, 5, 7, 10, 15,16, 17]. Similar to Jung's study direction, Zhang [17] studied the largest cartesian closed category of stable domains.

Before we introduce Zhang's results, we need to recall some basic definitions about stable mappings and the stable order which is due to Berry [3].

Let D be an ω-algebraic bounded complete domain and $a, b, k \in D$. We write $a^{\uparrow}b$ means that a and b are compatibale. Let $\uparrow k = \{x \in D \mid x \geq k\}$. D is called a dI-domain if D satisfies Axiom (d) $(\forall a, b, c \in D, b^{\uparrow}c \Rightarrow a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c))$ and Axiom I (for every compact element k of D, $\downarrow k$ is a finite set). A domain which satisfies Axiom I is also said to be finitary.

Let D, E be Scott domains. A mapping f from D to E is called a conditionally multiplicative mapping (CM for short) if it is continuous and it preserves meets of pairs of compatible elements, i.e.,

$$\forall x, y \in D, x^{\uparrow}y \Rightarrow f(x \wedge y) = f(x) \wedge f(y).$$

Note that the CM mapping is called a stable mapping in [17]. Let $[D \to_c E]$ be the set of CM mappings from D to E. Let $f, g \in [D \to_c E]$. Define

$$f \leq_s g \text{ if and only if } \forall x, y \in D, x \leq y \Rightarrow f(x) = f(y) \wedge g(x).$$

The order $\leq_c$ is called stable order in [17].

A scott-continuous mapping $f : D \to E$ is called stable if for every $y \leq f(x), m(f, x, y) = min\{t \mid y \leq f(t), t \leq x\}$ exists. The set of all stable mappings from D to E is denoted by $[D \to_s E]$. Define an order $\leq_s$ (called the stable order) on $[D \to_s E]$ by putting $f \leq_s g \Leftrightarrow, m(g, x, y)$ exists and $m(f, x, y) = m(g, x, y)(\forall x \in D, \forall y \leq f(x))$.

Zhang [17] obtained the following result.

Theorem 1.1 Let D be a Scott-domain and $[D \to_c D]$ its CM mappings space.

(i) If $[D \to_c D]$ has a countable basis, then D must be finitary, for any Scott-domain D;

(ii) If $[D \to_c D]$ is bounded complete, then D must be distributive, for any finitary Scott-domain D.

Note that CM mappings constructed by Zhang in [17] are also stable mappings which are introduced in the following. As an immediate consequence of above result, the category of dI-domains with CM mappings is the largest cartesian closed category within ω-algebraic bounded complete domains, with the exponential being the stable mapping space.

Zhang [17] mentioned that if the finiteness condition is dropped, Theorem 1.1 should still hold. But the proof may be complicated. This paper shows his conjecture is true. Let **SABC** (resp., $\widetilde{\mathbf{SABC}}$) be the category of algebraic bounded complete domains with CM mappings (resp., stable mappings). Furthermore, this paper shows that:

(i) The exponentials of any full subcategory of **SABC** or $\widetilde{\mathbf{SABC}}$ are exactly function spaces;

(ii) $\widetilde{\mathbf{SDABC}}$, the category of distributive algebraic bounded complete domains, is the largest cartesian closed subcategory of $\widetilde{\mathbf{SABC}}$.

Finally, compact elements of function spaces in the category $\widetilde{\mathbf{SABC}}$ are also studied.

We quickly recall some results about stable mappings. An L-domain is a cpo in which every principal ideal is a complete lattice.

Proposition 1.2 [6] Let D and E be L-domains and $f, g \in [D \to_s E]$. Then the followings are equivalent:

i)$f \leq_s g$;

ii)$f \leq g$ and $f(x) = f(x') \wedge g(x)$ for any $x, x' \in D$ with $x \leq x'$.

Lemma 1.3 [17] Let D, E be Scott-domains and f, g compatible in $[D \to_c E]$.

i) If x, y are compatible in D, then $f(x) \wedge g(y) = f(y) \wedge g(x)$.

ii) If $f(x) \leq g(x)$ for every $x \in D$, then $f \leq_c g$.

For a directed set of CM mappings, the least upper bound exists and it is determined coordinatewise.

Lemma 1.4 [17] If $\{f_i \mid i \in I\} \subseteq [D \to_c E]$ is directed with respect to the order $\leq_c$ then $(\bigsqcup_{i \in I} f_i)(x) = \bigvee_{i \in I} f_i(x)$.

The rest of this paper is organized as follows: In section 2 we show that the distributivity condition is necessary for preserving bounded completeness of spaces of stable mappings or CM mappings. In section 3 we show that exponentials of any full subcategory of **SABC** and $\widetilde{\mathbf{SABC}}$ are exactly function spaces. In section 4 we prove that $\widetilde{\mathbf{SDABC}}$ is the largest cartesian closed subcategory of $\widetilde{\mathbf{SABC}}$. In Section 5 we study compact elements of function spaces in the category **SABC**.

2 Distributivity

The following examples are useful for the discussion.

Example 2.1 Let $D = \{\bot, \top, a, b\}$ with $\bot, \top$ least and largest elements respectively, and a, b incomparable. $M_3 = \{p, q, u, v, w\}$ with p, q least and largest elements respectively, and u, v, w pairwise incomparable.

Define four mappings $f, g, h_1, h_2 \in [D \to_s M_3]$ as follows:

$$f(x) = \begin{cases} u, & x = \top, \\ p, & \text{otherwise,} \end{cases} \qquad g(x) = \begin{cases} v, & x = \top, \\ p, & \text{otherwise,} \end{cases}$$

$$h_1(x) = \begin{cases} q, & x = \top, \\ p, & \text{otherwise,} \end{cases} \qquad h_2(x) = \begin{cases} q, & x = \top, \\ p, & x = a \text{ or } \bot, \\ w, & x = b. \end{cases}$$

It is easy to see that h_1, h_2 are incomparable minimal upper bounds of f and g with respect to the stable order.

Example 2.2 Let D be the cpo of Example 2.1. Let $N_5 = \{p, q, u, v, w\}$ with p, q least and largest elements respectively, $v < w$, but u and v, w incomparable. Define four mappings $f, g, h_1, h_2 \in [D \to_s N_5]$ as follows:

$$f(x) = \begin{cases} u, & x \geq a, \\ p, & \text{otherwise,} \end{cases} \qquad g(x) = \begin{cases} v, & x \geq b, \\ p, & \text{otherwise,} \end{cases}$$

$$h_1(x) = \begin{cases} u, & x = a, \\ v, & x = b, \\ q, & x = \top, \\ p, & x = \bot, \end{cases} \qquad h_2(x) = \begin{cases} u, & x = a, \\ w, & x = b, \\ q, & x = \top, \\ p, & x = \bot. \end{cases}$$

It is easy to see that h_1, h_2 are incompatible minimal upper bounds of f and g with respect to the stable order.

We slightly change the concept of subdomain in [17].

Definition 2.3 Let D be a Scott-domain and let M be a non-empty subset of D. Then M is a subdomain of D if and only if M is a Scott-domain inheriting the orderings of D in such a way that, for any non-empty subset X of M, if $\vee X$ exists in D, then $\vee X \in M$, and if $\wedge X$ exists in D, then $\wedge X \in M$.

Note that we do not require that an element in M is compact with respect to M if and only if it is compact in D. So this definition is weaker than that of Zhang.

We write $M \hookrightarrow D$ to indicate that D has a subdomain isomorphic to M. It is easy to prove that $M \hookrightarrow D$ is stable.

Lemma 2.4 (17) *If the distributive law fails in a Scott-domain D, then it fails in a principle ideal $\downarrow p$ for some $p \in D$.*

Lemma 2.5 (17) *If D is a non-distributive Scott-domain, then either $M_3 \hookrightarrow D$ or $N_5 \hookrightarrow D$, where M_3 and N_5 are the cpos of Example 2.1 and Example 2.2 respectively.*

We are now in a position to prove our main result.

Theorem 2.6 *Let D be a Scott-domain such that $[D \to_s D]$ is consistently complete. Then D must be distributive.*

Note that a stable mapping is a CM mapping. We have the similar result for the case of CM mappings.

Theorem 2.7 *Let D be a Scott-domain such that $[D \to_c D]$ is consistently complete. Then D must be distributive.*

3 Exponentials of SABC and $\widetilde{\mathbf{SABC}}$

In this section, we prove that exponentials of any full subcategory of the category **SABC** or $\widetilde{\mathbf{SABC}}$ are exactly function spaces.

Definition 3.1 [2] Let $\mathcal{C}$ be any category. We say that $\mathcal{C}$ is cartesian closed if the following three conditions are satisfied:

(i) There is a terminal object T in $\mathcal{C}$ such that for any object $A \in \mathcal{C}$ there is exactly one morphism $\alpha : A \to T$.

(ii) For any two objects $A, B \in \mathcal{C}$ there exists an object $A \bigotimes B$ in $\mathcal{C}$ and morphisms $pr_1 : A \bigotimes B \to A, pr_2 : A \bigotimes B \to B$ such that for any object C and morphism $f : C \to A, g : C \to B$ there is a unique morphism $f \times g : C \to A \bigotimes B$ such that $pr_1 \circ (f \times g) = f$ and $pr_2 \circ (f \times g) = g$. The object $A \bigotimes B$ is called the product of A and B.

(iii) For any two objects $A, B \in \mathcal{C}$ there exists an object A^B in $\mathcal{C}$ and a morphism $ev : A^B \times B \to A$ such that for each $f : C \times B \to A$ there exists an unique morphism $\Lambda_f : C \to A^B$ such that $ev \circ (\Lambda_f \times id_B) = f$. The object A^B is called the exponential object for A and B.

Let **SAL** (resp., $\widetilde{\mathbf{SAL}}$) be the category of algebraic L-domains with CM mappings (resp., stable mappings).

Lemma 3.2 (5) $\widetilde{\mathbf{SAL}}$ *is cartesian closed in which the one-point domain, the cartesian product $A \times B$ and $[A \to_s B]$ are the terminal object, the categorical product and the exponential object respectively for algebraic L-domains A, B.*

We do not know if **SAL** is cartesian closed. The reason is that we do not know if the compact elements of CM function spaces can be captured by their traces which are studied in Section 5. However, this need not concern us since we are interested in the universal property of function spaces mentioned in (iii) of Definition 3.1.

Since the proof of following lemma is extremely similar to that of Lemma 3.2, we omit the proof.

Lemma 3.3 *Let A, B be algebraic L-domains.*

*(i) The one-point domain and the cartesian product $A \times B$ are the terminal object and the categorical product in **SAL** respectively.*

(ii) $[A \to_c B]$ is an L-domain.

(iii) Suppose $ev : [B \to_c A] \times B \to A$ defined by $\forall g \in [B \to_c A], \forall d \in B, ev(g, d) = g(d)$. Then for each $f : C \times B \to A$ there exists an unique morphism $\Lambda_f : C \to$

$[B \to_c A]$ *such that* $ev \circ (\Lambda_f \times id_B) = f$.

Let A, B be algebraic L-domains. By Lemma 3.2 the one-point domain, the cartesian product $A \times B$ and $[B \to_s A]$ are the terminal object, the categorical product and exponential object in $\widetilde{\mathbf{SAL}}$ respectively. Any full subcategory of $\mathcal{C}$ of $[B \to_c A]$, which is closed with respect to these three constructs is itself cartesian closed. On the other hand, there is not much choice for these constructs in a cartesian closed full subcategory of **SABC** or $\widetilde{\mathbf{SABC}}$. This can be seen from the following theorem which essentially appears in [14] already.

In [11] Liu and Li also studied the categorical product and exponential in the category **SAL** and obtained results which is similar to the following theorem. But their proof there has a gap and they only considered the case of stable mappings.

Theorem 3.4 *Let* $\mathcal{C}$ *be a cartesian closed full subcategory of* $\widetilde{\mathbf{SABC}}$ *(resp.,* **SABC***). Then the following holds for any two objects* $A, B \in \mathcal{C}$.

(i) The terminal object T *of* $\mathcal{C}$ *is isomorphic to the one-point domain.*

(ii) The category product of A *and* B *is isomorphic to the cartesian product* $A \times B$.

(iii) The exponential object A^B *in* $\mathcal{C}$ *is isomorphic to* $[B \to_s A]$ *(resp.,* $[B \to_c A]$*).*

4 The largest cartesian closed subcategory of SABC and $\widetilde{\mathbf{SABC}}$

By Theorem 3.4 the result of Zhang can be improved as follows.

Theorem 4.1 *The category of dI-domains is the largest cartesian closed subcategory of* ω-**SABC**.

Next we show that the category $\widetilde{\mathbf{SDABC}}$ is the largest cartesian closed subcategory of $\widetilde{\mathbf{SABC}}$. We need the following lemma.

Lemma 4.2 *Let* D *and* E *be distributive Scott-domains.* $f, g \in [D \to_s E]$. *If* f *and* g *are compatible, then* $f \vee_s g = f \vee g$ *and* $f \wedge_s g = f \wedge g$.

Theorem 4.3 *The category* $\widetilde{\mathbf{SDABC}}$ *is the largest cartesian closed subcategory of* $\widetilde{\mathbf{SABC}}$.

A natural question arises:

Question: Is **SDABC** the largest cartesian closed subcategory of **SABC**?

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