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The Better Bubbling Lemma

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Abstract

BCD [1] relies for its modeling of λ calculus in intersection type filters on a key theorem which I call BL (for the Bubbling Lemma, following someone). This lemma has been extended in [2,4] to encompass Boolean structure, including specifically union types; this extended lemma I call BBL (the Better Bubbling Lemma). There are resonances, explored in [5] and [4], between intersection and union type theories and the already existing minimal positive relevant logic **B+** of [10]. (Indeed [9] applies BL and BBL to get further results linking combinators to relevant theories and propositions.) On these resonances the *filters* of algebra become the *theories* of logic. The semantics of [8] yields here a new and short proof of BBL, which takes account of full Boolean structure by encompassing not only **B+** but also its conservative Boolean extension **CB** [10,7,8].

Keywords: semantics, subtyping, classical relevant logic, minimal relevant logic, CB, B+, type theory, bubbling.

1 Introduction

This paper is about the "complementarity" of the relevant logics of the philosophers and the type theories of computer scientists. [1] Differently motivated investigations have produced more or less *the same* formal systems. Relevant logics arose in the search for a better account of implication, the $\rightarrow$ connective which is at the heart of logic. Type theories seek to carve up an intuitive universe of discourse into manageable chunks, producing (it is hoped) more secure programming environments.

The paper is more specifically about a particular lemma, which I call the Better Bubbling Lemma (henceforth, BBL). I use here the semantics of relevant logics, as presented in [10] and [7], to prove BBL. BBL is itself an improvement of the

[1] Thanks to my colleagues in Logic and Computation, at the College of Engineering and Computer Science at ANU. I am particularly indebted to Professor John W Lloyd, whose travel grant made it possible for me to attend the Venice DCM workshop at which I presented this note in July 2006. Thanks also to my daughter Dorothy K Meyer, whose graduate study in Computer Science at the University of New England produced the LaTeX expertise that made possible the submission of this article for publication in the millennium in which it was written. I'm deeply grateful, Dorothy. Other debts will be noted as they arise.

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Bubbling Lemma (henceforth BL). I learned the Bubbling Lemma from Mariangiola Dezani, who thought it up. [2] At any rate, as Proposition 2.4, the Bubbling Lemma plays a crucial role in her paper [1] with Barendregt and Coppo, where it is used to nail down a "filter model" of λ calculus in intersection type theory. [3] This filter model is at the same time a logical model of λ in $\mathbf{B}\wedge\mathbf{T}$ theories, as we showed in [5].

Bubbling is what happens in the severely restricted $[\rightarrow, \wedge, \top]$ vocabulary to Better Bubbling. Relevant logicians and type theorists have a common goal in seeking to lift those restrictions. Logicians seek to add at least a disjunction $\vee$, and perhaps also a negation $\neg$; the type crowd sees the former as type union and the latter (also perhaps) as some sort of complement. On the logical side, a number of relevant logics were developed and given a semantical interpretation, including the natural minimal positive relevant logic $\mathbf{B}+$ of [10] and its conservative Boolean extension $\mathbf{CB}$ of [7]. [4] Type theorists also sought semantical underpinning of their work; the line that led to BBL was worked out in Paris by Frisch, Castagna and Benzaken [6,2,4].

2 Preliminaries

Our vocabulary here will be dual purpose (as in [5]). We begin with atoms p, etc., taken indifferently as propositional or type variables. There will be a constant $\top$ (a formula entailed by everything, or the whole space ω of [1]). Formulae (or types) A, B, etc., shall be built up from atoms and $\top$ under the binary operations $\wedge$ (conjunction, or intersection) and $\rightarrow$ (implication, or function space constructor). Statements are of the form $A \leq B$, where $\leq$ (logical entailment, or sub-type) is a binary relation symbol and A and B are formulae (or types). Thus our formal systems are, in the style of Curry [3], *relational* ones, more natural (as in [8,6,4]) for contact with *sub-typing* ideas.

We have just described the basic language L$\wedge\top$. [5] We extend it to the language L$+\top$ by adding the additional operation $\vee$ (disjunction, or union). Binary operations shall be ranked $\wedge$, $\vee$ (when present), $\rightarrow$ in order of *increasing* scope, with association otherwise to the *right*. We concentrate here on an additional language L$\neg$, which results when another unary connective $\neg$ (Boolean negation, or complement) is added. In this language L$\neg$, preferred henceforth, we get $\vee$ and $\top$ by the the following familiar Definitions:

D$\vee$. $A \vee B \overset{\text{def}}{=} \neg(\neg A \wedge \neg B)$

[2] According to me, Dezani also *named* it. (She has denied this, preferring Lemma 3.14159, or whatever it was called in [1].)

[3] [1] built on work by Coppo, Dezani and their Italian colleagues.

[4] $\mathbf{CB}$ was called $\mathbf{CB}+$ in [7].

[5] The care and feeding of $\top$ is interesting here. As a truth that is logically entailed by *every* formula, $\top$ seems a detour into *irrelevance*. Accordingly it is missing from $\mathbf{B}+$ and the kindred relevant logics of [10] and related work in the Anderson-Belnap tradition. $\top$ is equally dicey from a type-theoretic viewpoint; as the *universal type* ω , $\top$ violates Aristotle's dictum that "Being is not a genus." Among type theorists, Venneri has declared herself particularly suspicious of $\top$. On the other hand, $\top$ is useful, both type-theoretically and logically. And since it is trivial to *add* $\top$ conservatively on the semantics of [10] (just make $\top$ true at every state s), we treat it here as harmless. More than that, since $\top$ has *exactly* the properties imposed on ω in [1], it is in this paper a logical gift from a benevolent Creator.

D$\top$. $\top \stackrel{\text{def}}{=} p \vee \neg p$, where p is *first*.

We shall characterize the *theorems* of corresponding systems *semantically*. (For *syntactic* characterizations see the cited papers.) A Boolean *3-frame* **K** shall here be a pair $< K, R >$, where K is a set (of states) and R is a 3-place relation on K.[6]

Let **K** be a Boolean 3-frame, and let L be one of our languages above. Let $\mathbf{2} = \{0, 1\}$ be the set {false, true} of truth-values. A *possible interpretation* I of L in **K** shall be any function $I : L \times K \to \mathbf{2}$. That is, a possible interpretation is any function which assigns exactly one truth-value to each formula A in L at each state s in K.

Not all *possible* interpretations count as interpretations. This is *semantics*, and some attention to the *meanings* of the particles is in order. That attention is supplied by *truth-conditions* on the primitive particles.[7] Writing $[A]c$ for $I(A, c) = 1$ and $\neg[A]c$ for $I(A, c) = 0$ and using intuitive connectives and quantifiers in obvious ways, we have the following:

Truth-conditions:

Tω. $[\top]c$ always

T$\wedge$. $[A \wedge B]c = [A]c \wedge [B]c$

T$\vee$. $[A \vee B]c = [A]c \vee [B]c$

T$\neg$. $[\neg A]c = \neg[A]c$

T$\to$. $[A \to B]c = \forall a, b \in K\ (Rcab \Rightarrow [A]a \Rightarrow [B]b)$

A possible interpretation I is an *interpretation* provided that all applicable truth conditions hold for I. We have now

Verification condition on an interpretation I in a 3-frame **K**:

VI. $A \leq B$ is *verified* on I in **K** iff $\forall c \in K([A]c \Rightarrow [B]c)$

Validity condition in a 3-frame **K**:

VK. $A \leq B$ is *valid* in **K** iff $A \leq B$ is verified on all I in **K**

Basic validity condition:

VB. $A \leq B$ is *basically valid* iff $A \leq B$ is valid in all 3-frames **K**

3 Better Bubbling

This brings us to our main topic, the Bubbling and Better Bubbling Lemmas. I shall henceforth simply write '$A \leq B$' when that statement is basically valid.

[6] It is a pleasant consequence of various conservative extension results that we may restrict ourselves here to *Boolean* 3-frames, which were called b+ms on p. 70 of [8].

[7] The semantics of [10] also imposed a *heredity* condition H sensitive to a partial order $\leq$ on states, reminiscent of a similar condition in Kripke's semantics for intuitionist logic. It is still further evidence of divine benevolence that, since we can get by with *Boolean* 3-frames, the condition H is otiose in the situation here. The reason for this is that, without loss of generality, we may take actual equality as the partial order $\leq$. This also dispenses with the semantic postulates for **B**+ in [10].

3.1 Bubbling

Let I be a finite index set. Then the Bubbling Lemma says

BL. Suppose $\wedge_{i\in I}(A_i \to B_i) \leq A \to B$. Then there is a subset $J \subseteq I$ such that A $\leq \wedge_{j\in J}A_j$ and $\wedge_{j\in J}B_j \leq B$.

I put BL thus on the usual lattice-theoretic convention that, where Λ is the null set,

$\wedge_{j\in\Lambda}A_j = \top$.

BL is stated in [1] for the Intersection Type Discipline (henceforth, **ITD**); it comes to the same thing to say that it holds for the $\to \wedge\top$ modification **B**$\wedge$**T** of the minimal logic **B**+ of [10]; see [5]. The utility of BL in [1] and associated work is that it assures that the values on interpretation of terms of the form $\lambda x.M$ are indeed **ITD**-filters (= **B**$\wedge$**T**-theories).

3.2 Better Bubbling

On, now, to Better Bubbling! The Better Bubbling Lemma BBL (e.g., of [4,9]) has two parts.

BBL1. Suppose that $\wedge_{i\in I}(A_i \to B_i) \leq \vee_{j\in J}(C_j \to D_j)$. Then there is a particular j in J such that $\wedge_{i\in I}(A_i \to B_i) \leq C_j \to D_j$.

BBL2. Suppose that $\wedge_{i\in I}(A_i \to B_i) \leq A \to B$. Then for each subset $J \subseteq I$ we have

$$\text{(i)}\ \ A \leq \vee_{j\in J}A_j,\ \text{OR}$$
$$\text{(ii)}\ \ \wedge_{k\in I\setminus J}B_k \leq B$$

It is the case that Better Bubbling for **CB** entails Bubbling for **B**$\wedge$**T**. But I shall not go into all that now. Instead I will prove Better Bubbling SEMANTICALLY. [8] And I shall prove BBL for the full Boolean (conservative) extension **CB** of **B**+. **Proof** of BBL2 for **CB**. The proof is by *reductio*. The lemma claims that, if a conjunction over an index set I of $\to$ formulae $A_i \to B_i$ entails an $\to$ formula $A \to B$, then for every subset J of I we have either

$$\text{i.}\ \ A \leq \vee_{j\in J}A_j,\ \text{OR}$$
$$\text{ii.}\ \ \wedge_{k\in I\setminus J}B_k \leq B$$

So suppose, for some subset J of I, *both* i and ii are semantically *invalid*. We use this hypothesis to show that, in this case, the statement

iii. $\wedge_{i\in I}(A_i \to B_i) \leq A \to B$

is also semantically *invalid*. BBL2 then follows by contraposition.

Let then J be the subset of I for which *both* i and ii fail. By VI, T$\vee$, there is then by the failure of i, some interpretation I_a in a 3-frame $\mathbf{K_a} =< K_a, R_a >$ such that, on I_a, we have at a state a $\in K_a$ and for all $j \in J$,

[8] BBL is a *beautiful* theorem. I learned of it from Dezani, who found the idea in Frisch's Ph. D. thesis, done under the supervision of Giuseppe Castagna at ENS, Paris, investigating the *semantic* approach to sub-typing presented with Benzaken in [6] and [2]. It may have roots, according to Castagna and Frisch, in work by Hosoya.

$$(1)\ [A]a$$

$$(2)\ \neg[A_{\mathrm{j}}]a$$

Meanwhile, by the failure of ii, there is some interpretation I_b in a 3-frame $\mathbf{K_b} =< K_b, R_b >$ such that, on I_b, we have by VI, T$\wedge$ a state $b \in K_b$ such that, for all $k \in I\backslash J$,

$$(3)\ [B_k]b$$

$$(4)\ \neg[B]b$$

Let x be a new element foreign to both K_a and K_b. We construct a new 3-frame $\mathbf{K} =< K, R >$, where $K = \{x\} \cup K_a \cup K_b$. By defining R appropriately on K, we shall make the antecedent of iii true at x but its consequent false at x. This will suffice for the invalidity of iii, ending the argument.

We specify R as follows:

iv. *Rxab.*

v. For c, d, $e \in K_a$, $Rcde$ iff R_acde.

vi. For c, d, $e \in K_b$, $Rcde$ iff R_bcde.

vii. Otherwise $Rcde$ *fails*, for all c, d, $e \in K$.

The idea of this specification is that we are simply *pasting together* the two 3-frames that we already have, joining them at x via $Rxab$. We continue the pasting by defining an interpretation I in $\mathbf{K}$ that copies I_a on the K_a side and I_b on the K_b side. Specifically, for all $c \in K_a$ and $d \in K_b$, we lay down for each atom p that $I(p, c) = I_a(p, c)$ and $I(p, d) = I_b(p, d)$. As for the new element x, we simply set $I(p, x) = 0$ for all p. The imposition of the truth-conditions T$\rightarrow$, T$\wedge$, T$\neg$ (and, by definition, T$\vee$ and Tω as well) then assures that I is well-defined on all formulae E at each state e in K.

It is now an elementary structural induction, safely left to the reader, to show that I agrees with I_a on all formulae E at every state e in K_a, and with I_b on all E at every e in K_b. As for what it does at x, we must check that I makes the antecedent of iii true there but its consequent false. There is no problem with the latter. $I(A, a) = I_a(A, a) = 1$ and $I(B, b) = I_b(B, b) = 0$; whence, since $Rxab$, we have $I(A \rightarrow B, x) = 0$ by T$\rightarrow$. But we need also to check that each $A_i \rightarrow B_i$ is *true* on I at x (whence by T$\wedge$, so is the *whole* antecedent of iii). It all depends on whether i is in the special subset J from which we started.

Subcase α. $i \in J$. Then $I(A_i, a) = I_a(A_i, a) = 0$, by (2) above. But then, since $Rxab$ is the *only* triple involving x, we have $I(A_i \rightarrow B_i, x) = 1$ (by, so to speak, falsity of antecedent in T$\rightarrow$).

Subcase β. $i \in I\backslash J$. We then have $I(B_i, b) = I_b(B_i, b) = 1$, by (4) above. This also enforces $I(A_i \rightarrow B_i, x) = 1$ (by, so to speak, truth of consequent in T$\rightarrow$).

Thus all the $\rightarrow$ formulae in the antecedent of iii are true at x on I. But the consequent of iii was false at x on I. Thus iii is *not* basically valid, if any subset $J \subseteq I$ fails to satisfy one of (i), (ii). Contraposing, this ends the semantical proof of BBL2 for **CB**.

Proof of BBL1 for CB. By contraposition. Suppose that, for each $j \in J$, the statement

$$(5)\ \wedge_{i \in I}(A_i \to B_i) \leq C_j \to D_j$$

is *not* basically valid. We shall show that

$$(6)\ \wedge_{i \in I}(A_i \to B_i) \leq \vee_{j \in J}(C_j \to D_j)$$

is also invalid. Accordingly, since BBL assumes the validity of (6), there is a $j \in J$ for which (5) holds.

We proceed to construct, very carefully, for each $j \in J$ an interpretation I_j in a 3-frame $\mathbf{K_j} =< K_j, R_j >$. We might as well take the index j itself as the "state" at which the antecedent of (1) turns out true on I_j and its consequent false. That is, we have on I_j (using our abbreviated notation again), *new* states c_j, d_j such that, applying T$\wedge$, T$\to$, we get

$$(7)\ R_j j c_j d_j$$

$$(8)\ [C_j]c_j$$

$$(9)\ \neg[D_j]d_j$$

$$(10)\ \text{for each } i \in I,\ [A_i \to B_i]j$$

There is a very important point in this observation—namely, that we can *always* choose a fresh and new c_j and d_j when we are falsifying $\to$ statements at j.[9] In many logics, we do not have this luxury; for example one of the postulates of the logic **R** states that $Rxxx$ *always*, whence we must attend to repetitions of the arguments of the ternary relation. We note moreover that there is no reason to make any atom p true at any of the special states $j \in J$. So w.l.o.g., $I_j(p, j) = 0$ for all atoms p and $j \in J$.

Having *carefully* falsified each of the instances of (5), we now construct a countermodel to (6). We may assume, for $j \neq k(j, k \in J$, that $K_j \cap K_k$ is the empty set.

Let $K0 = \cup_{j \in J} K_j$. Let x be an element not in $K0$, and let $K = \{x\} \cup K0$. We define the ternary relation R on K as follows, for each $j \in J$:

(11) If a, b, $c \in K_j$ then $Rabc$ iff $R_j abc$.

(12) If a, $b \in K_j$ then $Rxab$ iff $R_j jab$.

(13) Otherwise *Rabc fails.*

This will make $\mathbf{K} =< K, R >$ a 3-frame. We go on to define an interpretation I in $\mathbf{K}$, thus:

(14) For all atoms p and $a_j \in K_j$, $I(p, a_j) = I_j(p, a_j)$.

(15) For all atoms p and $j \in J$, $I(p, x) = I_j(p, j) = 0$.

(16) For compound formulae C and all states $c \in K$, let $I(C, c)$ be determined in $\mathbf{K}$

[9] My former student and later ANU boss John Slaney e-mailed an elegant proof that this is the case. He and my ANU colleague Raje'ev Goré have my thanks, as do the graduate students Chunlai Zhou (Indiana University) and Koushik Pal (UC, Berkeley) for incisive insights.

by imposing the truth-conditions T→, T∧, T¬.

Lemma 3.1 *For all formulae A and all $a_j \in K_j$, $I(A, a_j) = I_j(A, a_j)$.*

Proof Obvious by structural induction, since truth-conditions are the same. □

Lemma 3.2 *For all $j \in J$ and consequents $C_j \to D_j$ of (6), $I(C_j \to D_j, x) = 0$.*

Proof By lemma 3.1, condition (12) above, and T→. □

Lemma 3.3 *For all conjoined antecedents $A_i \to B_i$ of (6), $I(A_i \to B_i, x) = 1$.*

Proof All of the I_j *agree* in making the $A_i \to B_i$ *true*. Suppose, for *reductio*, that $I(A_i \to B_i, x)$ were nonetheless false. Then there would be a, b such that $Rxab$ and $I(A_i, a) = 1$ and $I(B_i, b) = 0$. But then, by (12), there is a j such that both a, $b \in K_j$ and $R_j jab$. Whence, by lemma 1 and T→, $I_j(A_i \to B_i, j) = 0$, which is impossible. □

Theorem 3.4 *BBL1 holds.*

Proof As indicated. Suppose that (6) holds, but that (5) fails for all $j \in J$. Construct the interpretation I in the 3-frame $\mathbf{K} =< K, R >$. On I we have, on abbreviated notation,

$$(17)\ \ [\wedge_{i \in I}(A_i \to B_i)]x$$

$$(18)\ \ \neg[\vee_{j \in J}(C_j \to D_j)]x$$

We have (17) by T∧, because each $A_i \to B_i$ is true at x by lemma 3. And we have (18) by T∨, since each $C_j \to D_j$ is false at x by lemma 2. This shows that (6) is invalid after all, a contradiction, ending the semantic proof of BBL1. [10] □

4 Bibliographical references

References for The Better Bubbling Lemma The following abbreviations are used:

- AJL Australasian Journal of Logic
- JPL Journal of Philosophical Logic
- JSL The Journal of Symbolic Logic
- NDJFL Notre Dame Journal of Formal Logic

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[10] I have been chided by a referee, who correctly insists that the contributions of Castagna, Frisch and Benzaken in [6,2] and related research are at the heart of the topic of semantic subtyping. I am furthermore indebted to these authors, for entertaining me in Paris in July 2006 for conversations about their work and for sharing further projected papers of theirs.

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