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An Inductive-style Procedure for Counting Monochromatic Simplexes of Symmetric Subdivisions with Applications to Distributed Computing

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Abstract

In the *weak symmetry breaking* (WSB) task, each of $n+1$ processes has to decide either 0 or 1 such that not all processes decide the same value. A WSB algorithm should be *wait-free*, namely, processes are *asynchronous* (there is no bound on their relative speeds), and potentially *faulty* (any proper subset may halt without warning). Also, it should be *comparison based*, namely, processes can only use comparison operations ($<, >, =$) on the values they read from the memory.
Symmetric chromatic subdivisions of an n-simplex have been used to represent the executions of a distributed algorithm solving the WSB task. Informally, each simplex of such a subdivision corresponds to an execution of the algorithm. Each vertex is labeled with the local state of a process in the execution; it is colored with the process ID, and also with the binary value the process decides in the execution. The symmetry properties of such a complex come from the comparison based requirement of a WSB algorithm.
Let $\mathcal{C}$ denote the number of monochromatic n-simplexes of such a subdivision, counted by orientation. Previous work has shown that $\mathcal{C} = 1 + \sum_{i=0}^{n-1} \binom{n+1}{i+1} k_i$, for some coefficients $k_i \in \mathbb{Z}$. This characterization of $\mathcal{C}$ implies that the WSB task on $n+1$ processes is solvable if and only if n is such that the binomial coefficients are relatively prime, or equivalently, if and only if n is not a primer power.
This paper presents an inductive style procedure that yields an alternative proof of the characterization of $\mathcal{C}$. Roughly speaking, the proof consists in a procedure for modifying gradually the binary coloring of a symmetric chromatic subdivision, and computing the degree of the maps produced during the procedure.

Keywords: Distributed computing, Weak symmetry breaking, Renaming, Combinatorial topology.

1 Introduction

A *task* $\mathcal{T}$ on $n+1$ processes is specified by an *input complex*, $\mathcal{I}$, an *output complex*, $\mathcal{O}$, and an input-output relation Δ. The input complex specifies the possible inputs

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to the processes. Each simplex in $\mathcal{I}$ specifies input values for the processes. For each input simplex, Δ specifies a set of output simplexes, containing the values the processes are allowed to decide in an execution. Topological methods have been used in Distributed Computing to study which tasks can be solved, and with what complexity (see for example [3,9,10,11,12,13]). Roughly speaking, executions of a distributed algorithm for a task $\mathcal{T}$ starting in an input simplex, can be represented by an *algorithm complex*. Each simplex of such a complex corresponds to an execution of the algorithm. The vertices of a simplex are labeled with the local state of the process in the execution; each one is colored with a distinct process *id*, and hence the complex is called *chromatic*. The values decided by the processes induce a simplicial map from the algorithm complex to the output complex, that respects the specification given by Δ. Using topological invariants of the algorithm complex one can prove that $\mathcal{T}$ is not solvable, or derive an algorithm for solving $\mathcal{T}$. For the wait-free model considered in this paper, the topological invariant is that the algorithm complex always induces a subdivision of the input complex. In a *wait-free* model, processes are *asynchronous* (there is no bound on their relative speeds), and potentially *faulty* (any proper subset may halt without warning). Also, the processes communicate with each other using a read/write shared memory.

In this paper we are interested in the *weak symmetry breaking* (WSB) task [8]: each of $n+1$ processes has a unique ID in $[n] = \{0, \dots, n\}$, and after communicating with each other, processes decide either 0 or 1, such that not all processes decide the same value. An algorithm for the WSB task is required to be *comparison based*: processes can only use comparison operations $(<, >, =)$ on the values read from the memory. Recall that the WSB task is equivalent to the celebrated M*-renaming* task [1], when $M = 2n$. In this task each process is issued a unique name taken from a large namespace, and after coordinating with one another, processes choose unique names taken from a (much smaller) namespace of size M.

Prior research [2,5,10,11] has shown that the executions of a WSB algorithm can be represented as a chromatic subdivision $\mathcal{K}$ of an n-simplex, with a binary coloring on its vertexes that is "symmetric" on the boundary. The binary coloring represents the values decided by the processes, and the monochromatic n-simplexes of $\mathcal{K}$ correspond to the executions in which all processes decide the same output value. Let $\mathcal{C}$ denote the number of monochromatic n-simplexes in $\mathcal{K}$ counted by orientation. In [4] it is proved that $\mathcal{C} = 1 + \sum_{i=0}^{n-1} \binom{n+1}{i+1} k_i$, for some integers $k_0, \dots, k_{n-1}$. This characterization of $\mathcal{C}$ has implications on the solvability of the WSB task: if $\binom{n+1}{1}, \dots, \binom{n+1}{n}$ are not relatively prime, then there are no integers $k_0, \dots, k_{n-1}$ such that $\mathcal{C} = 0$, which implies that there is no WSB algorithm for $n+1$ processes. Otherwise there is a WSB algorithm. In [6] it is observed that n is a prime power if and only if $\binom{n+1}{1}, \dots, \binom{n+1}{n}$ are not relatively prime. Therefore, there is a WSB algorithm for $n+1$ processes if and only if n is not a prime power.

Two combinatorial approaches for proving the characterization of $\mathcal{C}$ were introduced in [4]. Both are based on the Index Lemma 5.2, which is a restatement of Corollary 2 in [7]. One approach consists of replacing the inside of $\mathcal{K}$ with a very simple complex, where the number of monochromatic simplexes can be counted, and observing that this number is not modified so long as the boundary of the resulting complex is the same as the boundary of $\mathcal{K}$, by the Index Lemma. This approach is described in detail in [5].

It is presented in [6] a third approach which is algebraic. Roughly speaking, it consists on, first, noticing that the colorings of $\mathcal{K}$ induce a chain map ϕ from the chain complex of $\mathcal{K}$ to the chain complex of an annulus of dimension n, that is equivariant with respect to the symmetric group over $[n]$; then, exploiting the symmetric properties of ϕ, it computes the degree of ϕ, which implies the characterization of $\mathcal{C}$.

In this paper we describe in detail the other approach in [4]. We present an inductive style procedure that yields an alternative proof of the characterization of $\mathcal{C}$. The procedure gradually modifies the binary coloring of the subdivision, making sure that each time the binary coloring of a vertex is changed, the binary coloring of other vertexes on the boundary also changes, to preserve the symmetry of the binary coloring. The proof consists in computing how each one of these color changes affects $\mathcal{C}$. In algebraic topology language, the proof consists on computing the degree of the chain maps induced by the simplicial maps produced during the procedure.

The rest of the paper is organized as follows. Section 2 present a brief review of some combinatorial topology concepts and Section 3 presents the notion of divided images that is used in the formal statement of the characterization of $\mathcal{C}$ in Theorem 6.1. Sections 4 and 5 contain some lemmas used in the proof of Theorem 6.1 in Section 6.

2 Combinatorial topology preliminaries

We assume the reader is familiar with concepts such as (combinatorial) simplexes, (combinatorial) complexes, simplicial maps and orientability.

Let $\mathcal{K}$ be an n-complex. The *i-graph*, $0 \leq i \leq n$, of $\mathcal{K}$ has a node for every i-simplex of $\mathcal{K}$ and an edge between two vertexes if they share an $(i-1)$-face. We say that $\mathcal{K}$ is *i-connected* [4] if its i-graph is connected, or if it consists of a single vertex when $i = 0$. If we say that an n-complex is *connected*, we mean to the highest dimension n, i.e., the complex is n-connected.

A complex $\mathcal{K}^n$ is a *n-pseudomanifold* if each of its i-simplexes, $i \leq n$, is a face of at least one of its n-simplexes, and each of its $(n-1)$-simplex is face of either one or two n-simplexes. The *boundary* of a pseudomanifold $\mathcal{K}^n$, $bd(\mathcal{K}^n)$, is the subcomplex induced by its $(n-1)$-simplexes that are contained in exactly one n-simplex.

A *coloring* of a complex $\mathcal{K}$ is a function f from its vertexes to a set of *colors*. The set of colors of the vertexes of a simplex $\tau \in \mathcal{K}$, is denoted $f(\tau)$. A *binary coloring* of $\mathcal{K}$ is a coloring with colors $\{0, 1\}$. A coloring of a simplex is *proper* if it gives different values to different vertexes. If a coloring of a simplex gives the same value

[4] This definition is not equivalent to the usual definition of i-connected. Roughly speaking, the usual definition of i-connected means that the complex does not have "holes" of dimension less or equal than i (the homology group of dimension $k \leq i$ is trivial).

b to every vertex, we say the simplex is b-*monochromatic* or just *monochromatic*. An n-complex is *chromatic* if it has a coloring that uses $n+1$ colors and each one of its simplexes is properly colored.

Let σ^n be a simplex that has proper coloring id with $[n]$. Then, $d = +1$ denotes the *positive* orientation that contains the sequence $\langle 0, 1, \ldots, n \rangle$, i.e., the sequence of vertexes $\langle v_0, v_1, \ldots, v_n \rangle$ of σ^n such that $id(v_i) = i$, $0 \leq i \leq n$, and $d = -1$ denotes *negative* orientation. If σ^n is oriented d, its $(n-1)$-face without color i, receives the induced orientation $(-1)^i d$.

A pseudomanifold $\mathcal{K}^n$ is *orientable* if it is possible to give an orientation to each of its n-simplexes such that if $\sigma, \sigma' \in \mathcal{K}^n$ share an $(n-1)$-face τ then τ gets opposite induced orientations from σ and σ'. Such an orientation is a *coherent orientation* of $\mathcal{K}^n$.

3 Divided images

Divided images are introduced and studied in [2], to model the structure of the complex associated to a distributed algorithm. This section briefly reviews the main properties of these combinatorial topology objects.

Definition 3.1 [2, Definition 4.1] Let $\mathcal{K}^n$, $\mathcal{L}^n$ be complexes and ψ be a function that maps every simplex of $\mathcal{L}^n$ to a finite subcomplex of $\mathcal{K}^n$. The complex $\mathcal{K}^n$ is a *divided image* of $\mathcal{L}^n$ under ψ if:

(1) $\psi(\emptyset) = \emptyset$

(2) $\forall \tau \in \mathcal{K}^n$, there is a simplex $\sigma \in \mathcal{L}^n$ such that $\tau \in \psi(\sigma)$

(3) $\forall \sigma^0 \in \mathcal{L}^n$, $\psi(\sigma^0)$ is a single vertex

(4) $\forall \sigma_1, \sigma_2 \in \mathcal{L}^n$, $\psi(\sigma_1 \cap \sigma_2) = \psi(\sigma_1) \cap \psi(\sigma_2)$

(5) $\forall \sigma \in \mathcal{L}^n$, $\psi(\sigma)$ is a $dim(\sigma)$-pseudomanifold with $bd(\psi(\sigma)) = \psi(bd(\sigma))$

The complex $\mathcal{K}^n$ is a *divided image* of $\mathcal{L}^n$ if there exists ψ such that $\mathcal{K}^n$ is a divided image of $\mathcal{L}^n$ under ψ.

Figure 1 depicts a divided image of dimension 2 where $\mathcal{L}^2$ is the complex consisting of a 2-simplex and all its faces, and the arrows show how ψ maps the vertexes of $\mathcal{L}^2$. It is worth noticing that a divided image is not necessarily a subdivision, even if it is connected. For example, a torus $\mathcal{L}$ of dimension 2 with a 2-simplex τ removed from it, is a divided image of a 2-simplex σ: $bd(\sigma)$ is mapped to $bd(\tau)$ and σ is mapped to $\mathcal{L}$.

Let $\mathcal{K}^n$ be a divided image of $\mathcal{L}^n$ under ψ. $\mathcal{K}^n$ is *connected* if for every i-simplex $\sigma \in \mathcal{L}^n$, if $i \geq 1$ then $\psi(\sigma)$ is i-connected, and if $i \geq 2$ then $bd(\psi(\sigma))$ is $(i-1)$-connected. Similarly, $\mathcal{K}^n$ is *orientable* if for every $\sigma \in \mathcal{L}^n$, $\psi(\sigma)$ is orientable. And $\mathcal{K}^n$ is *coherently oriented* if for every n-simplex $\sigma \in \mathcal{L}^n$, the n-pseudomanifold $\psi(\sigma^n)$ is coherently oriented.

The *carrier* of a simplex $\tau \in \mathcal{K}^n$, $carr(\tau)$, is the simplex $\sigma \in \mathcal{L}^n$ of smallest dimension such that $\tau \in \psi(\sigma)$. Assume $\mathcal{L}^n$ is chromatic. The set colors of a simplex $\sigma \in \mathcal{L}^n$ is denoted $id(\sigma)$. The divided image $\mathcal{K}^n$ is *chromatic* if every simplex $\tau \in \mathcal{K}^n$

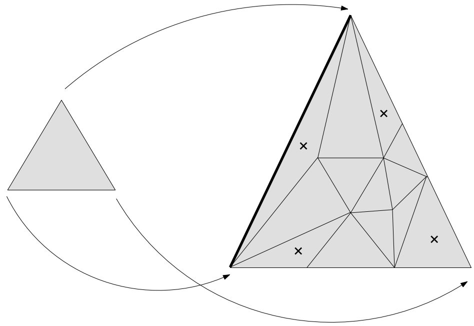

Fig. 1. A divided image of dimension 2.

with $dim(\tau) = dim(carr(\tau))$, is properly colored with $id(carr(\tau))$. Figure 2 depicts a chromatic divided image of a 2-simplex.

Let $\mathcal{K}^n$ be a divided image of σ^n under ψ. A *cross edge* of $\mathcal{K}^n$ is a 1-simplex $\{u, v\} \in bd(\mathcal{K}^n)$ such that there exist distinct i-faces σ, σ' of σ^n, $0 \le i \le n-2$, such that $u \in \psi(\sigma)$, $u \notin \psi(\sigma')$, $v \notin \psi(\sigma)$ and $v \in \psi(\sigma')$. This implies that if $\mathcal{K}^n$ has no cross edges then for every $\sigma \subset \sigma^n$, there exists $v \in \psi(\sigma)$ such that $carr(v) = \sigma$. The bold edge in Figure 1 is a cross edge.

Roughly speaking, the *j-corners* of a divided image are the j-simplexes that have an i-face in the boundary, for every $i \le j$. The 2-simplexes marked with a small cross are the 2-corners of the divided image in Figure 1.

Definition 3.2 Let $\mathcal{K}^n$ be a divided image of σ^n under ψ, and σ^j be a j-face of σ^n. The set of *j-corners* of $\psi(\sigma^j)$ is:

$$
\begin{aligned}
j\text{-}corners(\psi(\sigma^j)) = \{\, & \tau^j \in \psi(\sigma^j) | \forall\, 0 \le k \le j, \exists\, \sigma^k, \rho^k, \text{ such that} \\
& \sigma^k \subseteq \sigma^j, \rho^k \in \psi(\sigma^k) \text{ and } \rho^0 \subset \rho^1 \subset \ldots \subset \rho^j = \tau^j \}
\end{aligned}
$$

4 Some lemmas about orientability

This section presents some lemmas concerning orientability of chromatic divided images. For the rest of the paper, for a simplex σ^n that is properly colored with $[n]$, let σ_i^{n-1} denote the $(n-1)$-face of σ^n without color $i \in [n]$. The proof of Lemma 4.1 is easy and is left to the reader.

Lemma 4.1 *Let $\mathcal{K}^n$ be a chromatic, connected and orientable divided image of σ^n under ψ. In any coherent orientation of $\mathcal{K}^n$, $\psi(\sigma_i^{n-1})$ has a coherent induced orientation.*

Lemma 4.2 *Let $\mathcal{K}^n$ be a chromatic, connected and orientable divided image of σ^n under ψ. In any coherent orientation of $\mathcal{K}^n$, all simplexes of n-corners$(\mathcal{K}^n)$ have the same orientation.*

Proof. Consider faces $\sigma^1, \sigma^2, \ldots, \sigma^{n-1}$ of σ^n such that $\sigma^1 \subset \sigma^2 \subset \ldots \subset \sigma^{n-1}$. For $1 \le i \le n-1$, let $\mathcal{K}^i$ denote the complex $\psi(\sigma^i)$. Assume $\mathcal{K}^i$ has the induced orientation by $\mathcal{K}^{i+1}$. By Lemma 4.1, $\mathcal{K}^i$ is coherently oriented. We proceed by induction on n. For the base, we have that $\mathcal{K}^1$ has an odd number of 1-simplexes because $\mathcal{K}^1$ is chromatic and connected. Also, the 1-corners of $\mathcal{K}^1$ are the 1-simplexes containing the two vertexes of its boundary. It is not hard to see that these simplexes have the same orientation. Suppose the lemma holds for $i-1$. We prove that it holds for i.

By Definition 3.2 of n-corners, every simplex of $(i-1)$-*corners*$(\mathcal{K}^{i-1})$ is contained in some simplex of i-*corners*$(\mathcal{K}^i)$. However, it is not necessary true that every simplex of i-*corners*$(\mathcal{K}^i)$ contains a simplex of $(i-1)$-*corners*$(\mathcal{K}^{i-1})$. Let τ^{i-1} and ρ^{i-1} be simplexes of $(i-1)$-*corners*$(\mathcal{K}^{i-1})$ and τ^i and ρ^i be the simplexes of i-*corners*$(\mathcal{K}^i)$ such that $\rho^{i-1} \subset \rho^i$ and $\tau^{i-1} \subset \tau^i$. By definition of orientability, τ^i induces its orientation multiplied by $(-1)^k$ to τ^{i-1}, and ρ^i induces its orientation multiplied by $(-1)^k$ to ρ^{i-1}, for some k. Also, by induction hypothesis, the simplexes of $(i-1)$-*corners*$(\mathcal{K}^{i-1})$ have the same orientation, and thus τ^i and ρ^i have the same orientation. Consider now a face λ^{i-1} of σ^i such that $\lambda^{i-1} \neq \sigma^{i-1}$. Let $\mathcal{L}^{i-1}$ denote the complex $\psi(\lambda^{i-1})$ and $\mathcal{L}^{i-2}$ denote the complex $\psi(\sigma^{i-1} \cap \lambda^{i-1})$. Consider a simplex γ^{i-2} of $(i-2)$-*corners*$(\mathcal{L}^{i-2})$. Let $\tau^{i-1} \in (i-1)$-*corners*$(\mathcal{K}^{i-1})$, $\rho^{i-1} \in (i-1)$-*corners*$(\mathcal{L}^{i-1})$ and $\tau^i, \rho^i \in i$-*corners*$(\mathcal{K}^i)$ be the simplexes such that $\gamma^{i-2} \subset \tau^{i-1} \subset \tau^i$ and $\gamma^{i-2} \subset \rho^{i-1} \subset \rho^i$. Using the fact that $\mathcal{K}^i$ is a connected and chromatic divided image, one can prove that τ^i and ρ^i have the same orientation. By the previous case, this one holds. This complete the proof. □

For a simplex σ^n, in what follows let σ^n denote the complex containing all faces of σ^n. Consider a chromatic divided image $\mathcal{K}^n$ of σ^n under ψ. Let σ and σ' be i-faces of σ^n. A simplicial bijection $\mu : \psi(\sigma) \to \psi(\sigma')$ is *id-preserving* if for every $u, v \in \psi(\sigma)$, if $id(u) = id(v)$ then $id(\mu(u)) = id(\mu(v))$. If in addition, for every $u \in \psi(\sigma)$, $rk(id(u)) = id(\mu(u))$, where $rk : id(\sigma) \to id(\sigma')$ is the bijection such that if $x < y$ then $rk(x) < rk(y)$, then μ is *id-rank-preserving*. Notice that there can be only one id-rank-preserving bijection. $\mathcal{K}^n$ has *structural-symmetry* if for every two i-faces σ and σ' of σ^n, there is an id-preserving simplicial bijection between $\psi(\sigma)$ and $\psi(\sigma')$. Similarly, $\mathcal{K}^n$ has *structural-rank-symmetry* if for every two i-faces σ and σ' of σ^n, there is an *id-rank-preserving* simplicial bijection between $\psi(\sigma)$ and $\psi(\sigma')$. Clearly, if $\mathcal{K}^n$ has structural-rank-symmetry, it has structural-symmetry.

Assume $\mathcal{K}^n$ has structural-symmetry. For every i-faces σ and σ' of σ^n, fix an id-preserving simplicial bijection $\mu_{\sigma\sigma'} : \psi(\sigma) \to \psi(\sigma')$ such that $\mu^{-1}_{\sigma\sigma'} = \mu_{\sigma'\sigma}$. Let $\mathcal{F}$ be the family of simplicial bijections $\mu_{\sigma\sigma'}$. Then $\mathcal{K}^n$ has *structural-symmetry with respect to* $\mathcal{F}$. For each $\mu_{\sigma\sigma'}$, $u \in \psi(\sigma)$ and $v \in \psi(\sigma')$ are *isomorphic with respect to* μ if $\mu(u) = v$. Isomorphic simplexes with respect to μ are defined similarly. Observe that isomorphic simplexes between $\psi(\sigma)$ and $\psi(\sigma')$ are well defined since $\mu^{-1}_{\sigma\sigma'} = \mu_{\sigma'\sigma}$.

Let $\mathcal{K}^n$ be a chromatic divided image of σ^n under ψ with structural-symmetry with respect to a family $\mathcal{F}$, and with a binary coloring b. The coloring b is *symmetric with respect to* $\mathcal{F}$ if every $\mu_{\sigma\sigma'} \in \mathcal{F}$ is color-preserving, i.e., for every $v \in \psi(\sigma)$, $b(v) = b(\mu(v))$. If there is a family of simplicial bijections such that b is symmetric with respect to it, then b is *symmetric*. Also, b is *rank-symmetric* if it is symmetric with respect to the family of id-rank-preserving simplicial bijections. Therefore a divided image with a (rank-)symmetric binary coloring, has structural-(rank-

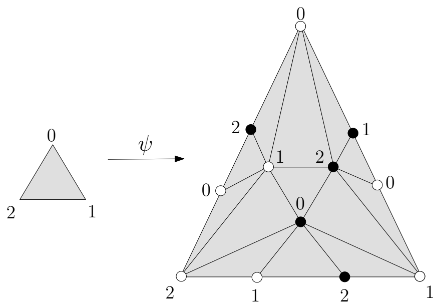

Fig. 2. A chromatic divided image with a symmetric binary coloring.

)symmetry. Figure 2 presents a chromatic divided image with a rank-symmetric binary coloring (white and black circles represent binary colors 0 and 1).

Lemma 4.3 *Let $\mathcal{K}^n$ be a chromatic, connected and orientable divided image of σ^n, with structural symmetry with respect to $\mathcal{F}$. In any coherent orientation of $\mathcal{K}^n$, the n-simplexes of $\mathcal{K}^n$ that contain isomorphic $(n-1)$-simplexes of $bd(\mathcal{K}^n)$, have the same orientation.*

Proof. Let $\mathcal{K}_i^{n-1}$ and $\mathcal{K}_j^{n-1}$ denote $\psi(\sigma_i^{n-1})$ and $\psi(\sigma_j^{n-1})$. Consider isomorphic simplexes $\rho^{n-1} \in \mathcal{K}_i^{n-1}$ and $\tau^{n-1} \in \mathcal{K}_j^{n-1}$, i.e., for $f_{\sigma_i^{n-1}\sigma_j^{n-1}} \in \mathcal{F}$, $f_{\sigma_i^{n-1}\sigma_j^{n-1}}(\rho^{n-1}) = \tau^{n-1}$. Let ρ^n and τ^n be the unique simplexes of $\mathcal{K}^n$ such that $\rho^{n-1} \subset \rho^n$ and $\tau^{n-1} \subset \tau^n$. The induced orientation of ρ^{n-1} is the orientation of ρ^n multiplied by $(-1)^i$ and the induced orientation of τ^{n-1} is the orientation of τ^n multiplied by $(-1)^j$. Thus, it is sufficient to prove that ρ^{n-1} and τ^{n-1} have the same induced orientation, multiplied by $(-1)^i$ and $(-1)^j$, respectively.

By Definition 3.2, every simplex of $(n-1)$-*corners*$(\mathcal{K}_i^{n-1})$ or $(n-1)$-*corners*$(\mathcal{K}_j^{n-1})$ is face of a simplex in n-*corners*$(\mathcal{K}^n)$. Consider simplexes $\gamma^{n-1} \in (n-1)$-*corners*$(\mathcal{K}_i^{n-1})$ and $\lambda^{n-1} \in (n-1)$-*corners*$(\mathcal{K}_j^{n-1})$. By Lemma 4.2, γ^{n-1} and λ^{n-1} have the same induced orientation, multiplied by $(-1)^i$ and $(-1)^j$. Since $\mathcal{K}_i^{n-1}$ is connected, $\mathcal{K}_i^{n-1}$ has only two possible coherent orientations. Therefore, an orientation of an $(n-1)$-simplex of $\mathcal{K}_i^{n-1}$ induces the orientation of the other $(n-1)$-simplexes in a coherent orientation. Something similar happens with $\mathcal{K}_j^{n-1}$. It can be easily proved by induction on n, that any *ids*-preserving simplicial bijection $f : \mathcal{K}_i^{n-1} \to \mathcal{K}_j^{n-1}$, maps $(n-1)$-corners to $(n-1)$-corners. Thus, an $(n-1)$-simplex of $\mathcal{K}_i^{n-1}$ is isomorphic to an $(n-1)$-simplex of $\mathcal{K}_j^{n-1}$ with the same orientation, multiplied by $(-1)^i$ and $(-1)^j$. □

5 Counting monochromatic simplexes

Consider a chromatic n-pseudomanifold $\mathcal{K}$ that has a binary coloring on its vertexes. This section presents a lemma that can be used to count the monochromatic n-simplexes (with respect to the binary coloring) of $\mathcal{K}$, by counting on $bd(\mathcal{K})$. This lemma is the basis for proving Theorem 6.1 in Section 6.

Consider an oriented simplex σ with a proper coloring c. Let $\langle c_0, \ldots, c_{dim(\sigma)} \rangle$ be the sequence of the c colors of σ in ascending order. The simplex σ is *counted* *by orientation* with respect to c in the following way. It is counted as $+1$ if the sequence $\langle c_0, \ldots, c_{dim(\sigma)} \rangle$ belongs to its orientation, i.e., the sequence of vertexes $\langle v_0, v_1, \ldots, v_{dim(\sigma)} \rangle$ such that $c(v_i) = c_i$, $0 \leq i \leq dim(n)$, belongs to the orientation of σ. Otherwise it is counted as -1.

Definition 5.1 [Index and Content] Consider a coherently oriented pseudomanifold $\mathcal{K}^n$ with the induced orientation on its boundary. Let c be a coloring, not necessarily proper, of $\mathcal{K}^n$ with $[n]$.

(1) The *content* of $\mathcal{K}^n$, $\mathcal{C}(\mathcal{K}^n)$, is the number of the properly colored n-simplexes of $\mathcal{K}^n$ counted by orientation.

(2) The *index i* of $\mathcal{K}^n$, $\mathcal{I}_i(\mathcal{K}^n)$, is the number of properly colored $(n-1)$-simplexes of $bd(\mathcal{K}^n)$ with $[n] - \{i\}$ counted by orientation.

If there is no ambiguity, we just write $\mathcal{C}$ or $\mathcal{I}_i$. Lemma 5.2 below is the restatement of Corollary 2 in [7] using our notation.

Lemma 5.2 (Index Lemma) *Let $\mathcal{K}^n$ be a coherently oriented, connected and colored pseudomanifold with $[n]$. Then $\mathcal{C} = (-1)^i \mathcal{I}_i$.*

Figure 3 shows a pseudomanifold with its 2-simplexes counterclockwise oriented. Notice that colors 0, 1 and 2, in this order, of the unique properly colored 2-simplex, denoted by the circular arrow, follow the counterclockwise direction, and thus $\mathcal{C} = +1$. An edge in the boundary with colors 0 and 1, is counted $+1$ or -1 according to its induced orientation and the direction followed by 0 and 1, in this order. Hence $\mathcal{I}_2 = +1$. It can be easily verified that $(-1)^2\mathcal{I}_2 = (-1)^1\mathcal{I}_1 = (-1)^0\mathcal{I}_0$. Notice that the coloring c induces a simplicial map from $\mathcal{K}^n$ to a properly colored simplex σ^n. Thus we can think of the index of $\mathcal{K}^n$ as the number of times that $bd(\mathcal{K}^n)$ is "wrapped around" $bd(\sigma^n)$, i.e., a combinatorial version of the notion of degree.

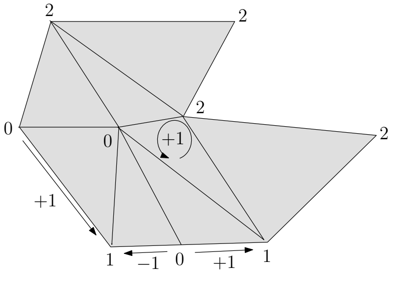

Fig. 3. The Index Lemma.

For a chromatic pseudomanifold with a binary coloring, we define the coloring c, Definition 5.3, that uses colors $[n]$.

Definition 5.3 Let $\mathcal{K}^n$ be a chromatic pseudomanifold with a binary coloring. For every $v \in \mathcal{K}^n$, the coloring c is defined as $c(v) = (id(v) + b(v)) \bmod (n+1)$, where id and b are the chromatic and binary coloring of $\mathcal{K}^n$, respectively.

Lemma 5.4 proves that the number of monochromatic n-simplexes under b and the properly colored n-simplexes under c are related. Thus the boundary induces the number of monochromatic n-simplexes, by the Index Lemma.

Lemma 5.4 ([5, Lemma 3.5]) *Let $\mathcal{K}^n$ be chromatic pseudomanifold with a binary coloring b and a coloring c as in Definition 5.3. An n-simplex of $\mathcal{K}^n$ is monochromatic under b if and only if it is properly colored under c.*

6 An inductive style proof

This section proves Theorem 6.1, which formalizes the symmetric chromatic subdivisions described in the Introduction. The theorem considers that the content $\mathcal{C}$ of $\mathcal{K}^n$ is computed with respect to the coloring c defined in Definition 5.3. By Lemma 5.4, $\mathcal{C}$ counts the monochromatic n-simplexes of $\mathcal{K}^n$. Also observe that $\mathcal{K}^n$ is not necessarily a chromatic subdivision of σ^n; for example, for dimension 2, it can be a torus without a 2-simplex.

Theorem 6.1 *Let $\mathcal{K}^n$ be a chromatic, connected and orientable divided image of σ^n, with a rank-symmetric binary coloring and no cross edges. Then, for some integers $k_0, \ldots, k_{n-1}$, $\mathcal{C} = 1 + \sum_{i=0}^{n-1} \binom{n+1}{i+1} k_i$.*

First Section 6.1 describes the idea used in the proof of Theorem 6.1 and then Section 6.2 presents the proof.

6.1 An Example

The strategy is to start with a binary coloring equal to 0 on the boundary, $\forall v \in bd(\mathcal{K}^n), b(v) = 0$, and then *process* groups of isomorphic vertexes (change their binary color to 1) with carriers of dimension ℓ, until $bd(\mathcal{K}^n)$ gets its original binary coloring. This action is called *ℓ-step* and it may be done more than once in each dimension ℓ. A step guarantees that after executing it, the coloring b of $\mathcal{K}^n$ remains rank-symmetric. Moreover, steps are done by dimension: a vertex with carrier of dimension $\ell + 1$ is processed if and only if every vertex with carrier of dimension ℓ has its correct binary color. For example, for dimension 3, first, if necessary, the corners are processed, then the vertexes inside the divided images of the edges, and finally the vertexes inside the divided images of the triangles. The vertexes inside the divided image of the tetrahedron are not modified and actually their coloring does not matter. The main part of the proof is to analyze how all these steps affect the index of $\mathcal{K}^n$. It will be proved that all changes in a step affect the index in the same way.

Figure 4 presents an example of the inductive procedure. The vertexes have colorings b and c. Assume the 2-simplexes are counterclockwise oriented. For a properly colored 1-simplex on the boundary, the arrow shows the direction followed by c colors 1 and 2, and -1 or $+1$ denotes how this simplex is counted by $\mathcal{I}_0$. The procedure begins with a binary coloring equal to 0 on the boundary, Figure 4 (a). The index at the beginning of the procedure always is equal to ± 1, according to

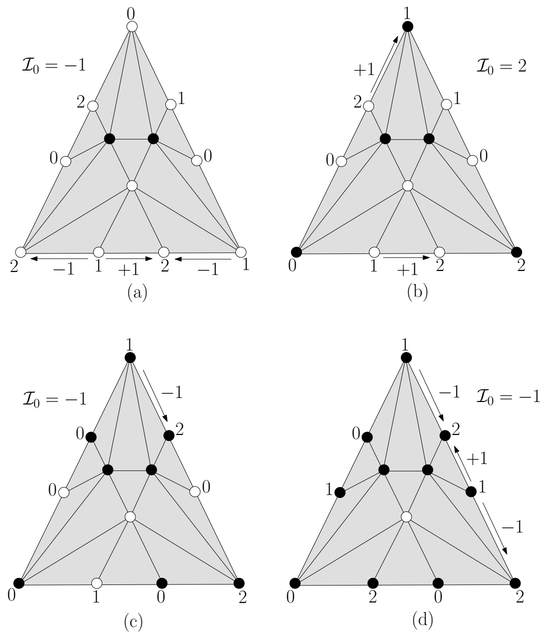

Fig. 4. An example of the inductive procedure.

the orientation. The procedure has a 0-step, Figure 4 (b), that adds a multiple of three to the index because a 2-simplex has three 0-faces. Figure 4 (c) shows a 1-step which adds a multiple of three to the index because a 2-simplex has three 1-faces. The procedure ends with the 1-step in Figure 4 (d).

6.2 The proof

Recall that the index and content of $\mathcal{K}^n$ are computed with respect to the coloring c defined in Definition 5.3. Assume $\mathcal{K}^n$ has a coherent orientation. The Lemma 6.2 computes the value of the index at the beginning of the procedure.

Lemma 6.2 *If for every $v \in bd(\mathcal{K}^n)$, $b(v) = 0$, then $\mathcal{I}_i = \pm 1$.*

Proof. Consider the faces $\sigma^0, \sigma^1, \ldots, \sigma^{n-1}$ of σ^n such that $id(\sigma^i) = [i]$. Let $\mathcal{K}^i$ denote $\psi(\sigma^i)$. It is clear that $\mathcal{K}^i$ is a chromatic, connected and orientable divided image of σ^i under $\psi|_{\sigma^i}$. Assume $\mathcal{K}^i$ has the the induced orientation by $\mathcal{K}^{i+1}$. By Lemma 4.1, $\mathcal{K}^i$ has a coherent induced orientation. By the definition of c, Definition 5.3, for every $v \in bd(\mathcal{K}^n)$, $c(v) = id(v)$. Notice that $\mathcal{K}^{n-1}$ contains all the properly colored $(n-1)$-simplexes of $bd(\mathcal{K}^n)$ with $[n-1]$. Actually, every $(n-1)$-simplex of $\mathcal{K}^{n-1}$ is properly colored with $[n-1]$. Therefore, we can recursively use Index Lemma 5.2. That is, $\mathcal{I}_n(\mathcal{K}^n) = \mathcal{C}(\mathcal{K}^{n-1})$ and, by Index Lemma, $\mathcal{I}_n(\mathcal{K}^n) = (-1)^{n-1}\mathcal{I}_{n-1}(\mathcal{K}^{n-1})$. We can do the same with $\mathcal{K}^{n-1}$ and $\mathcal{K}^{n-2}$, i.e., $\mathcal{I}_{n-1}(\mathcal{K}^{n-1}) = (-1)^{n-2}\mathcal{I}_{n-2}(\mathcal{K}^{n-2})$, and so on. Thus, $\mathcal{I}_n(\mathcal{K}^n) = (-1)^{1+2+\ldots+n-1}\mathcal{I}_1(\mathcal{K}^1)$. Observe that $\mathcal{I}_1(\mathcal{K}^1) = \pm 1$. And by Index Lemma, $(-1)^n\mathcal{I}_n(\mathcal{K}^n) = (-1)^i\mathcal{I}_i(\mathcal{K}^n)$. □

Consider $v \in bd(\mathcal{K}^n)$ such that $b(v) = 0$. The vertex v is *processed* when its binary color is changed from 0 to 1. Colorings, simplexes and values after processing

v, are marked with a dot ($\dot{}$). Thus, $\mathcal{I}_i$ and $\dot{\mathcal{I}}_i$ denote the index of $\mathcal{K}^n$ before and after processing v, and $c(v)$ and $\dot{c}(v)$ are its coloring c before and after processing it.

Let k denote $\binom{n+1}{\ell+1}$. Let $\sigma_1, \sigma_2 \dots \sigma_k$ be the ℓ-faces of σ^n. For $1 \leq j \leq k$, consider the id-rank-preserving bijection $f_j : \psi(\sigma_1) \to \psi(\sigma_j)$. Let us assume that theres exists $v \in \psi(\sigma_1)$ such that $carr(v) = \sigma_1$ and $b(v) = 0$. For $1 \leq j \leq k$, let v_j denote $f_j(v)$, the isomorphic vertex of v in $\psi(\sigma_j)$. Thus $v = v_1$. Also $b(v_j) = 0$ and $carr(v_j) = \sigma_j$. An *ℓ-step* consists of processing one by one the vertexes $v_1, v_2 \dots v_k$. The vertexes of $bd(\mathcal{K}^n)$ are processed by dimension, i.e., the procedure applies an ℓ-step if and only if all necessary $(\ell-1)$-steps have been done. Therefore, when an ℓ-step is the next step in the procedure, each vertex with carrier of dimension smaller than ℓ, has its correct binary color, and each vertex with carrier of dimension greater than ℓ, has binary color 0. It is clear that b is rank-symmetric at the beginning of the procedure, however b is not symmetric in the middle of a step. Colorings, simplexes and values after a step are denoted with a circumflex ($\hat{}$). For the rest of the proof, fix the ℓ-step associated with the vertexes $\mathcal{V} = \{v_1 \dots v_k\}$ and assume that none of the vertexes of $\mathcal{V}$ has been processed. The assumption that the binary coloring b of $\mathcal{K}^n$ is rank-symmetric helps in proving that b remains rank symmetric after an ℓ-step.

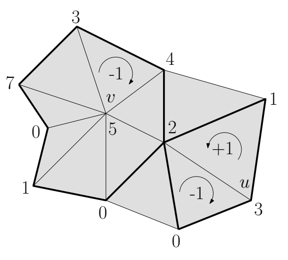

Fig. 5. The extended definition of content.

The core of the proof is computing how the index of $\mathcal{K}^n$ changes when a vertex of $\mathcal{V}$ is processed. To do that, the definition of content is extended for colored pseudomanifolds with an arbitrary number of colors. For a colored and oriented pseudomanifold $\mathcal{L}^n$ (possibly colored with more that $n+1$ colors) and a set of $n+1$ colors $\mathcal{H}$, $\mathcal{C}(\mathcal{L}^n, \mathcal{H})$ denotes the number of properly colored n-simplexes in $\mathcal{L}^n$ with $\mathcal{H}$, counted by orientation. $\mathcal{C}(\mathcal{L}^n, \mathcal{H})$ is the *content of $\mathcal{L}^n$ with $\mathcal{H}$*. For $st(v, \mathcal{L}^n)$, we write $\mathcal{C}(v, \mathcal{L}^n, \mathcal{H})$ instead of $\mathcal{C}(st(v, \mathcal{L}^n), \mathcal{H})$. Figure 5 presents a colored pseudomanifold $\mathcal{L}^2$ in which $st(u, \mathcal{L}^2)$ and $st(v, \mathcal{L}^2)$ are the regions bounded by bold lines. The reader can check that $\mathcal{C}(u, \mathcal{L}^2, \{3,4,5\}) = -1$, $\mathcal{C}(v, \mathcal{L}^2, \{1,2,3\}) = 1$ and $\mathcal{C}(u, v, \mathcal{L}^2, \{0,2,3\}) = -1$, assuming each 2-simplex is counterclockwise oriented.

Lemma 6.3 below describes how the index $\mathcal{I}_i$ changes when any vertex in $bd(\mathcal{K}^n)$ is processed. For the rest of the section, assume $bd(\mathcal{K}^n)$ has the induced orientation by $\mathcal{K}^n$.

Lemma 6.3 *Consider a vertex $v \in bd(\mathcal{K}^n)$ such that $b(v) = 0$. If v is processed then $\dot{\mathcal{I}}_i = \mathcal{I}_i + \dot{\mathcal{C}}(v, bd(\mathcal{K}^n), [n] - \{i\}) - \mathcal{C}(v, bd(\mathcal{K}^n), [n] - \{i\})$.*

Proof. First, observe $c(v) \neq \dot{c}(v)$. Consider an $(n-1)$-simplex $\tau \in st(v, bd(\mathcal{K}^n))$. We have two cases. If $c(\tau) \neq [n] - \{i\}$ then it is possible $\dot{c}(\tau) = [n] - \{i\}$. In the other case, if $c(\tau) = [n] - \{i\}$ then $\dot{c}(\tau) \neq [n] - \{i\}$. Thus, $\dot{\mathcal{I}}_i$ is $\mathcal{I}_i$ plus all those $(n-1)$-simplexes of $bd(\mathcal{K}^n)$ that will be properly colored with $[n] - \{i\}$ after v is processed, minus all those properly colored $(n-1)$-simplexes of $bd(\mathcal{K}^n)$ with $[n] - \{i\}$ before v is processed. Also, notice that $st(v, bd(\mathcal{K}^n))$ contains all those $(n-1)$-simplexes that change their coloring c when v is processed. Therefore, $\dot{\mathcal{I}}_i = \mathcal{I}_i + \dot{\mathcal{C}}(v, bd(\mathcal{K}^n), [n] - \{i\}) - \mathcal{C}(v, bd(\mathcal{K}^n), [n] - \{i\})$. □

The following two lemmas and corollary intuitively say that when a vertex of $\mathcal{V}$ is processed, $\dot{\mathcal{I}}_i$ can be computed by counting in a specific "region" of $bd(\mathcal{K}^n)$.

Lemma 6.4 *Consider an $(n-1)$-simplex $\tau \in st(v, bd(\mathcal{K}^n))$. If $c(\tau) = [n] - \{i\}$ then $\tau \in st(v, \psi(\sigma_i^{n-1}))$.*

Proof. First, by Definition 3.1 of a divided image and because $\sigma_i^{n-1} \in bd(\sigma^n)$, we have that $\psi(\sigma_i^{n-1}) \subset bd(\mathcal{K}^n)$. We have two cases. If $\ell = n-1$, $\mathcal{V}$ is an $n-1$ step, then v has a carrier of dimension $n-1$ and hence $v \notin bd(\psi(\sigma_i^{n-1}))$ (notice if $v \in bd(\psi(\sigma_i^{n-1}))$ then it cannot have a carrier of dimension $n-1$). Thus $st(v, bd(\mathcal{K}^n)) = st(v, \psi(\sigma_i^{n-1}))$.

The second case is $\ell < n-1$. Let σ be the carrier of v. Consider a face σ_j^{n-1} of σ^n such that $\sigma \subset \sigma_j^{n-1}$ and $\sigma_i^{n-1} \neq \sigma_j^{n-1}$. We have $v \in \psi(\sigma_j^{n-1})$ and $i \in id(\sigma_j^{n-1})$. Consider an ℓ-simplex $\rho \in st(v, \psi(\sigma))$. Let γ be a simplex of $st(v, \psi(\sigma_j^{n-1}))$ such that $\rho \subset \gamma$. Let u be the vertex of γ such that $id(u) = i$. Observe that $u \notin \rho$ and hence $u \notin \psi(\sigma)$. Since $\mathcal{K}^n$ does not have cross edges, then u has a carrier of dimension greater than ℓ. Thus $b(w) = 0$ and hence $c(w) = i$ and $c(\gamma) \neq [n] - \{i\}$. This implies $st(v, \psi(\sigma_i^{n-1}))$ contains every properly colored $(n-1)$-simplexes of $st(v, bd(\mathcal{K}^n))$ with $[n] - \{i\}$. □

Corollary 6.5 *Let v be a vertex of $\mathcal{V}$ such that $v \in \psi(\sigma_i^{n-1})$. Then $\mathcal{C}(v, bd(\mathcal{K}^n), [n] - \{i\}) = \mathcal{C}(v, \psi(\sigma_i^{n-1}), [n] - \{i\})$.*

Lemma 6.6 *Let v be a vertex of $\mathcal{V}$ such that $v \in \psi(\sigma_i^{n-1})$. If v is processed then $\dot{\mathcal{I}}_i = \mathcal{I}_i - \mathcal{C}(v, \psi(\sigma_i^{n-1}), [n] - \{i\})$.*

Proof. By Lemma 6.3, if v is processed, $\dot{\mathcal{I}}_i = \mathcal{I}_i + \dot{\mathcal{C}}(v, bd(\mathcal{K}^n), [n] - \{i\}) - \mathcal{C}(v, bd(\mathcal{K}^n), [n] - \{i\})$. And by Corollary 6.5, $\mathcal{C}(v, bd(\mathcal{K}^n), [n] - \{i\}) = \mathcal{C}(v, \psi(\sigma_i^{n-1}), [n] - \{i\})$. Consider an $(n-1)$-simplex $\tau \in st(v_i, \psi(\sigma_i^{n-1}))$. Recall that $id(\tau) = [n] - \{i\}$. By the definition of coloring c, Definition 5.3, one can conclude $c(\tau) = [n] - \{i\}$ if and only if τ is 0-monochromatic. Also observe $\dot{\tau}$ is not 0-monochromatic and hence $\dot{c}(\tau) \neq [n] - \{i\}$. Thus, $\dot{\mathcal{C}}(v_i, bd(\mathcal{K}^n), [n] - \{i\}) = 0$ and so $\dot{\mathcal{I}}_i = \mathcal{I}_i - \mathcal{C}(v, bd(\mathcal{K}^n), [n] - \{i\})$. □

Lemma 6.7 below shows that the content of vertexes $u, v \in \mathcal{V}$, are essentially the same, assuming none of them have been processed. This property will imply that all the modifications in a step affect the index in the same way.

Lemma 6.7 *Let $u, v \in \mathcal{V}$ be vertexes of $\psi(\sigma_i^{n-1})$ and $\psi(\sigma_j^{n-1})$, respectively. Then $(-1)^i \mathcal{C}(u, \psi(\sigma_i^{n-1}), [n] - \{i\}) = (-1)^j \mathcal{C}(v, \psi(\sigma_j^{n-1}), [n] - \{j\})$.*

Proof. Consider the faces $\sigma^{\ell+1}, \ldots, \sigma^{n-1}, \sigma^n$ such that $id(\sigma^m) = [m]$, $\ell < m \leq n$. For $m < n$, assume $\psi(\sigma^m)$ has the induced orientation by $\psi(\sigma^{m+1})$. By Lemma 4.1, $\psi(\sigma^m)$ has a coherent induced orientation. It is clear that $\psi(\sigma^m)$ is a chromatic, connected and orientable divided image of σ^m under $\psi|\sigma^m$ with a rank symmetric binary coloring and no cross edges. By induction on m, we prove that the lemma holds for the vertexes of $\mathcal{V}$ that belong to $\psi(\sigma^m)$.

For the base of the induction, $\ell + 1$, consider faces σ_i^ℓ and σ_j^ℓ of $\sigma^{\ell+1}$, without id color i and j. Let $\mathcal{L}_i^\ell$ and $\mathcal{L}_j^\ell$ denote $\psi(\sigma_i^\ell)$ and $\psi(\sigma_j^\ell)$. Notice that $\mathcal{L}_i^\ell$ and $\mathcal{L}_j^\ell$ only contain one vertex of $\mathcal{V}$, respectively. Consider $u, v \in \mathcal{V}$ such that $u \in \mathcal{L}_i^\ell$ and $v \in \mathcal{L}_j^\ell$. By the definition of coloring c, Definition 5.3, one can conclude that, for $\tau \in st(u, \mathcal{L}_i^\ell)$, $c(\tau) = id(\tau)$ if and only if τ is 0-monochromatic. Consider an ℓ-simplex $\tau \in st(u, \mathcal{L}_i^\ell)$. Recall that $id(\tau) = [\ell + 1] - \{i\}$. Suppose τ is 0-monochromatic. Thus, for each $w \in \tau$, $c(w) = id(w)$. Notice that if τ has induced orientation d then $\mathcal{C}(u, \mathcal{L}_i^\ell, [\ell + 1] - \{i\})$ counts τ as d. Something similar happens with ℓ-simplexes in $st(v, \mathcal{L}_j^\ell)$. Consider the isomorphic ℓ-simplex ρ of τ in $st(v, \mathcal{L}_j^\ell)$. Notice that ρ is 0-monochromatic. By Lemma 4.3, τ and ρ have the same orientation, multiplied by $(-1)^i$ and $(-1)^j$, respectively. Therefore, $(-1)^i\mathcal{C}(u, \mathcal{L}_i^\ell, [\ell+1]-\{i\}) = (-1)^j\mathcal{C}(v, \mathcal{L}_j^\ell, [\ell + 1] - \{j\})$.

Suppose the lemma is true for $m - 1$. We prove it is true for m. Consider the faces σ^{m-1} and σ_k^{m-1} of σ^m. Thus, $id(\sigma^{m-1}) = [m - 1]$ and $id(\sigma_k^{m-1}) = [m] - \{k\}$. Let $\mathcal{L}^{m-1}$ and $\mathcal{L}_k^{m-1}$ denote $\psi(\sigma^{m-1})$ and $\psi(\sigma_k^{m-1})$. We have that $\mathcal{L}^{m-1}$ and $\mathcal{L}_k^{m-1}$ contain more than one vertex of $\mathcal{V}$. Consider $u, v \in \mathcal{V}$ such that $u \in \mathcal{L}^{m-1}$, $v \in \mathcal{L}_k^{m-1}$ and they are isomorphic. As in the base of the induction, it can be easily proved that $(-1)^m\mathcal{C}(u, \mathcal{L}^{m-1}, [m - 1]) = (-1)^k\mathcal{C}(v, \mathcal{L}_k^{m-1}, [m] - \{k\})$. Consider a vertex $w \in \mathcal{V}$ such that $w \neq u$ and $w \in \mathcal{L}^{m-1}$. Observe that if we prove $\mathcal{C}(u, \mathcal{L}^{m-1}, [m - 1]) = \mathcal{C}(w, \mathcal{L}^{m-1}, [m - 1])$, the lemma follows.

Let σ_i^{m-2} and σ_j^{m-2} be faces of σ^{m-1} such that $u \in \psi(\sigma_i^{m-2})$ and $w \in \psi(\sigma_j^{m-2})$. Assume, w.l.o.g., faces σ_i^{m-2} and σ_j^{m-2} do not have colors i and j. Let $\mathcal{L}_i^{m-2}$ and $\mathcal{L}_j^{m-2}$ denote $\psi(\sigma_i^{m-2})$ and $\psi(\sigma_j^{m-2})$. The idea is to prove that $\mathcal{C}(u, \mathcal{L}^{m-1}, [m-1]) = (-i)^i\mathcal{C}(u, \mathcal{L}_i^{m-2}, [m - 1] - \{i\})$ and $\mathcal{C}(w, \mathcal{L}^{m-1}, [m - 1]) = (-i)^j\mathcal{C}(w, \mathcal{L}_j^{m-2}, [m - 1] - \{j\})$, by using Index Lemma on complexes $st(u, \mathcal{L}^{m-1})$ and $st(w, \mathcal{L}^{m-1})$. Then, $\mathcal{C}(u, \mathcal{L}^{m-1}, [m - 1]) = \mathcal{C}(w, \mathcal{L}^{m-1}, [m - 1])$, by induction hypothesis. However, it is possible c colors $st(u, \mathcal{L}^{m-1})$ and $st(w, \mathcal{L}^{m-1})$ with more than m colors, and thus Index Lemma cannot be used on them. So it is defined an extra coloring c' for these two complexes that uses m colors.

Consider $st(u, \mathcal{L}^{m-1})$. The coloring c' is defined as follows. For each $x \in st(u, \mathcal{L}^{m-1})$, if $b(x) = 0$ then $c'(x) = c(x)$, otherwise $c'(x) = c(u)$. Since $b(u) = 0$ and for every $x \in st(u, \mathcal{L}^{m-1})$, $id(x) \in [m - 1]$, we have that c' uses colors $[m - 1]$. Therefore, Index Lemma can be applied on $st(u, \mathcal{L}^{m-1})$. Also, as noticed above, for each $\tau \in st(u, \mathcal{L}^{m-1})$, $c(\tau) = id(\tau)$ if and only if τ is 0-monochromatic, and thus $c'(\tau) = id(\tau)$ if and only if $c(\tau) = id(\tau)$. Thus, $\mathcal{C}(st(u, \mathcal{L}^{m-1}))$ and $\mathcal{C}(st(u, \mathcal{L}_i^{m-2}))$ with respect to c', are equal to $\mathcal{C}(u, \mathcal{L}^{m-1}, [m - 1])$ and $\mathcal{C}(u, \mathcal{L}_i^{m-2}, [m - 1] - \{i\})$ with respect to c, respectively. Now, for an $(m - 2)$-simplex $\tau \in bd(st(u, \mathcal{L}^{m-1}))$, if $c'(\tau) = [m-1] - \{i\}$ then $\tau \in st(u, \mathcal{L}_i^{m-2})$. In other words, $\mathcal{L}_i^{m-2}$ is the only "region" of $bd(st(u, \mathcal{L}^{m-1}))$ containing properly colored simplexes with $[m-1]-\{i\}$. First observe that $\sigma_i^{m-2} \in bd(\sigma^{m-1})$ and hence $\mathcal{L}_i^{m-2} \subset \mathcal{L}^{m-1}$. Also if $\tau \notin \mathcal{L}_i^{m-2}$ then there must exists $x \in \tau$ such that $id(x) = i$. Since $x \in st(u, \mathcal{L}^{m-1})$, there exists a 1-simplex connecting x and u. Because there are no cross edges, x has a carrier of dimension greater than ℓ and hence $b(x) = 0$. Therefore $c'(x) = i$ and so $c'(\tau) \neq [m-1] - \{i\}$.

By Index Lemma 5.2, $\mathcal{C}(st(u, \mathcal{L}^{m-1})) = (-i)^i \mathcal{I}_i(st(u, \mathcal{L}^{m-1}))$. Also, we get $\mathcal{I}_i(st(u, \mathcal{L}^{m-1})) = \mathcal{C}(st(u, \mathcal{L}_i^{m-2}))$ because for every $\tau^{m-2} \in bd(st(u, \mathcal{L}^{m-1}))$ such that $c'(\tau^{m-2}) = [m-1] - \{i\}$, we have that $\tau^{m-2} \in \mathcal{L}_i^{m-2}$. Therefore, $\mathcal{C}(st(u, \mathcal{L}^{m-1})) = (-i)^i \mathcal{C}(st(u, \mathcal{L}_i^{m-2}))$. Similarly, by adding the appropriate c' coloring to $st(w, \mathcal{L}^{m-1})$, we get $\mathcal{C}(st(w, \mathcal{L}^{m-1})) = (-i)^j \mathcal{C}(st(w, \mathcal{L}_j^{m-2}))$. By induction hypothesis, $(-i)^i \mathcal{C}(st(u, \mathcal{L}_i^{m-2})) = (-i)^j \mathcal{C}(st(w, \mathcal{L}_j^{m-2}))$, thus $\mathcal{C}(st(u, \mathcal{L}^{m-1})) = \mathcal{C}(st(w, \mathcal{L}^{m-1}))$. The lemma follows. □

For Lemma 6.8, recall that a dot ($\dot{}$) denotes a value after a vertex is processed and a circumflex ($\widehat{}$) denotes a value after a step is done. Lemma 6.8 directly implies Lemma 6.9.

Lemma 6.8 *After the ℓ-step associated to $\mathcal{V}$ is done, we have that $\widehat{\mathcal{I}}_i = \mathcal{I}_i + \binom{n+1}{i+1} k$, for some $k \in \mathbb{Z}$.*

Proof. Consider vertexes $v_i, v_j \in \mathcal{V}$ such that $v_i \in \psi(\sigma_i^{n-1})$ and $v_j \in \psi(\sigma_j^{n-1})$. By Index Lemma 5.2, $(-1)^{i-j} \mathcal{I}_i = \mathcal{I}_j$ and $(-1)^{i-j} \dot{\mathcal{I}}_i = \dot{\mathcal{I}}_j$. And by Lemma 6.6, $\dot{\mathcal{I}}_j = \mathcal{I}_j - \mathcal{C}(v_j, \psi(\sigma_j^{n-1}), [n] - \{j\})$, when v_j is processed. Combining these three equations, we get $\dot{\mathcal{I}}_i = \mathcal{I}_i - (-1)^{j-i} \mathcal{C}(v_j, \psi(\sigma_j^{n-1}), [n] - \{j\})$. Using Lemma 6.7, we get $\mathcal{C}(v_i, \psi(\sigma_i^{n-1}), [n] - \{i\}) = (-1)^{j-i} \mathcal{C}(v_j, \psi(\sigma_j^{n-1}), [n] - \{j\})$, and hence $\dot{\mathcal{I}}_i = \mathcal{I}_i - \mathcal{C}(v_i, \psi(\sigma_i^{n-1}), [n] - \{i\})$. In other words, when the vertex v_j is processed, the index $\mathcal{I}_i$ changes as if v_i is processed. Since $\mathcal{K}^n$ does not have cross edges, there is not a 1-simplex connecting v_i and v_j, and so $v_j \notin st(v_i, bd(\mathcal{K}^n))$. Therefore, the c coloring of the $(n-1)$-simplexes in $st(v_i, \psi(\sigma_i^{n-1}))$ do not change when v_j is processed, and hence $\mathcal{C}(v_i, \psi(\sigma_i^{n-1}), [n] - \{i\}) = \dot{\mathcal{C}}(v_i, \psi(\sigma_i^{n-1}), [n] - \{i\})$. Thus, we have $\widehat{\mathcal{I}}_i = \mathcal{I}_i - \mathcal{C}(v_i, \psi(\sigma_i^{n-1}), [n] - \{i\}) \binom{n+1}{i+1}$ at the end of the step, because $|\mathcal{V}| = \binom{n+1}{i+1}$. □

Lemma 6.9 *Let $\mathcal{I}_i$ and $\widehat{\mathcal{I}}_i$ be the indexes of $\mathcal{K}^n$ before and after all the ℓ-steps in the procedure are done. Then $\widehat{\mathcal{I}}_i = \mathcal{I}_i - \binom{n+1}{i+1} k$, for some $k \in \mathbb{Z}$.*

By Lemma 6.2, $\mathcal{I}_i = \pm 1$ at the beginning of the procedure, according to the orientation. And by Lemma 6.9, after all ℓ-steps in the procedure, $\widehat{\mathcal{I}}_i = \mathcal{I}_i - \binom{n+1}{i+1} k$, Therefore, at the end of the procedure, $\mathcal{I}_i = 1 + \sum_{i=0}^{n-1} \binom{n+1}{i+1} k_\ell$, for some $k_\ell \in \mathbb{Z}$. By Index Lemma 5.2, Theorem 6.1 follows.

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