Available online at www.sciencedirect.com
ScienceDirect
Electronic Notes in Theoretical Computer Science 273 (2011) 51–67
Electronic Notes in Theoretical Computer Science
www.elsevier.com/locate/entcs

Axiomatizing Hybrid Products of Monotone Neighborhood Frames

Katsuhiko Sano

JSPS Research Fellow
Department of Humanistic Informatics,
Graduate School of Letters, Kyoto University
Yoshida Hommachi, Sakyo-ku
Kyoto, 606-8501, JAPAN

Abstract

The main aim of this paper is to propose a robust way to combine two monotone hybrid logics. This work can be regarded as a further extension of both topological semantics for hybrid logic (Ten Cate and Litak 2007) and bi-hybrid logic of *products of Kripke frames* (Sano 2010). First, we generalize the notion of *product of topologies* (Van Benthem, et al 2006) to the monotone neighborhood frames and introduce two kinds of *nominals*: i (e.g. for a moment of time) and a (e.g. for a spatial point), and the corresponding *satisfaction operators*: $@_i$ and $@_a$ to describe a product of monotone neighborhood frames. Second, we give five interaction axioms and establish a general completeness result called *pure completeness* of *bi*-hybrid logic of monotone neighborhood frames. By extending this, we also establish a pure completeness result of bi-hybrid logic of products of topologies.

Keywords: product of topologies, hybrid logic, product of neighborhood frames, monotone neighborhood frames, pure completeness.

1 Introduction

When we want to formalize the inference containing two dimensional information (e.g. space and time, time and the individual domain, etc.), we encounter with the following problem: how can we deal with two kinds of information in one setting? In other words, we need to know how to combine two modal logics, provided we deal with each of two dimensional (e.g. spatial and temporal) information in terms of modal logic. *Product of modal logics* should be counted as one answer to this question. It have been studied comprehensively since [5] (see also [4]), based on the notion of *product of Kripke frames*. Both dimension of a given two-dimensional structure, however, are not always relational. They might be topological. So, it would be desirable to combine not only two relational structures (i.e. Kripke frames) but also two topological structures. Van Benthem, et al. [18] generalized the notion of product of Kripke frames to the notion of product of topologies (see

doi:10.1016/j.entcs.2011.06.012

also [15]). They proposed a way of combining two topological modal logics, based on topological semantics for modal logic studied by McKinsey and Tarski [11]. Moreover, they showed that the fusion of two **S4**s is complete with respect to the product of two rational lines, while Kremer [8] showed that this fusion is incomplete with respect to the product of two real lines. Up to now, however, there is no study of the weaker notion of *product of neighborhood frames* (for neighborhood semantics of modal logic, the reader can refer to [3]).

Hybrid logic is an extended modal logic, which overcomes a weakness of expressive power of the basic modal logic over Kripke frames. For example, we can define the class of partial orders (the reflexive, antisymmeric and transitive relations) by nominals i and satisfaction operators $@_i$, while this class is not definable by any set of ordinary modal formulas. Moreover, we can define the class of partial orders by *pure formulas*, i.e., formulas not containing any ordinary proposition letters. For example, the conjunction of $@_i\Diamond i$, $@_i\Diamond j \wedge @_j\Diamond i \to @_i j$ and $@_i\Diamond j \wedge @_j\Diamond k \to @_i\Diamond k$ defines the class of partial orders. It is known that if a class F of Kripke frames is definable by a set of pure formulas, we can *always* obtain a strong completeness of the logic of F with respect to F [1]. Such result is called *pure completeness.* The author [13] extended the notion of product of modal logics to hybrid logics over Kripke semantics and established a pure completeness result for products of Kripke frames. He also showed that the product of any two pure complete logics enjoys a completeness result. In this sense, he expanded the range of combining logics without losing a completeness result.

The most significant feature of [13] is to propose the idea of *naming lines* by two kinds of nominals, instead of the ordinary idea of naming points in hybrid logic. This idea allows us to extend pure completeness result to two-dimensional Kripke semantics. In this paper, we keep this key feature also for two-dimensional neighborhood or topological semantics and see if it gives rise to the corresponding general completeness result. As a result, we will demonstrate that the idea of naming lines is robust for obtaining the completeness result.

To be more precise, this paper generalizes the method of [13] to both the product of monotone neighborhood frames and the product of topological spaces [18]. This study can also be regarded as a further extension of topological and monotone neighborhood semantics for hybrid logic studied by Ten Cate and Litak [17] and the author [14]. A main theorem of this paper is a pure completeness result for products of monotone neighborhood frames (Theorem 5.11). By extending it, we will also establish a pure completeness result for products of topological spaces (Theorem 5.14). As a corollary, we will show that any topo-product of two pure topo-complete logics enjoys a completeness result (Corollary 6.3). For example, this corollary tells us that we can provide a complete axiomatization of the logic (in two-dimensional hybrid language) of all products of dense-in-itself T_1-spaces, since the condition of being of T_1 are definable by a pure formula [6]. Finally, as a limitative result, we also show that this pure axiomatizable logic of all products of dense-in-itself T_1-spaces is incomplete with respect to the product of real lines (Theorem 6.7).

2 Product of Monotone Neighborhood Frames

A topological space is a pair $\langle T, \mathcal{O} \rangle$ such that a set $\mathcal{O} \subseteq \mathcal{P}(T)$ of *open sets* is closed under arbitrary unions and arbitrary finite intersections. Unlike this ordinary definition, we adopt the definition of topological spaces in terms of *local neighborhood basis* at a point, since this formulation allows us to regard the notion of topological space as a special case of the notion of monotone neighborhood frames as follows.

Definition 2.1 We say that $\langle T, \tau \rangle$ is a *neighborhood frame* if $T \neq \emptyset$ and $\tau : T \to \mathcal{PP}(T)$. A neighborhood frame $\langle T, \tau \rangle$ is *monotone* if, for any $x \in T$,

(supplementedness) $X \cap Y \in \tau(x)$ implies $X, Y \in \tau(x)$.

$\langle T, \tau \rangle$ is *normal* if it is monotone and it satisfies: for any $x \in T$,

(non-emptiness) $\tau(x) \neq \emptyset$.

(intersection) $X, Y \in \tau(x)$ implies $X \cap Y \in \tau(x)$.

$\langle T, \tau \rangle$ is a *topological space* if it is normal and it satisfies: for any $x \in T$,

(T) For all $X \in \tau(x)$, $x \in X$.

(4) For all $X \in \tau(x)$, $\{ y \in T \mid X \in \tau(y) \} \in \tau(x)$.

Definition 2.2 Let $\mathfrak{T}_1 = \langle T_1, \tau_1 \rangle$ and $\mathfrak{T}_2 = \langle T_2, \tau_2 \rangle$ be monotone. We define the *product* $\mathfrak{T}_1 \otimes \mathfrak{T}_2 = \langle T_1 \times T_2, \tau_h, \tau_v \rangle$ of $\mathfrak{T}_1$ and $\mathfrak{T}_2$ by:

$$\begin{aligned} \tau_h(x, y) &= \{ P \subseteq T_1 \times T_2 \mid \exists X \in \tau_1(x).\, X \times \{ y \} \subseteq P \}, \\ \tau_v(x, y) &= \{ P \subseteq T_1 \times T_2 \mid \exists Y \in \tau_2(y).\, \{ x \} \times Y \subseteq P \}. \end{aligned}$$

We say that τ_h is a *horizontal neighborhood structure* on $T_1 \times T_2$ and τ_v is a *vertical neighborhood structure* on $T_1 \times T_2$. If $\mathfrak{T}_1$ and $\mathfrak{T}_2$ are topological spaces, we say that $\mathfrak{T}_1 \otimes \mathfrak{T}_2$ is the *product of topologies*.

It is easy to see that τ_h and τ_v are monotone. Given two topological spaces $\mathfrak{T}_1$ and $\mathfrak{T}_2$, let us remark that $\mathfrak{T}_1 \otimes \mathfrak{T}_2 = \langle T_1 \times T_2, \tau_h, \tau_v \rangle$ does not coincide with the *product topology* $\langle T_1 \times T_2, \tau_{1,2} \rangle$ of $\mathfrak{T}_1$ and $\mathfrak{T}_2$, where $\tau_{1,2}(x, y)$ is the $\supseteq$-closure of $\{ X \times Y \mid X \in \tau_1(x) \text{ and } Y \in \tau_2(y) \}$. However, we can regard, e.g., τ_h as the product topology of the topology determined by τ_1 and the *discrete topology* on T_2. By this view and the following proposition, we can state that our definition of product of topologies in terms of local neighborhood basis and Van Benthem, et al. [18]'s definition in terms of open sets, are the same.

Proposition 2.3 *Let* $\mathfrak{T}_1 = \langle T_1, \tau_1 \rangle$ *and* $\mathfrak{T}_2 = \langle T_2, \tau_2 \rangle$ *be monotone.* (i) *If* τ_1 *and* τ_2 *are normal, then* τ_h *and* τ_v *are normal.* (ii) *If* τ_1 *and* τ_2 *are topological spaces, then* τ_h *and* τ_v *are also topological spaces.*

Proof. (i) is easy. Let us show (ii). It suffices to check that (T) of τ_1 implies (T) of τ_h and that (4) of τ_1 implies (4) of τ_h. We only show (4), since (T) is easy to show. Assume that τ_1 satisfies (4) and that $P \in \tau_h(x, y)$. We need to establish that $\{ \langle x', y' \rangle \in T_1 \times T_2 \mid P \in \tau_h(x', y') \} \in \tau_h(x, y)$, i.e., there exists $Z \in$

$\tau_1(x)$ such that $Z \times \{ y \} \subseteq \{ \langle x', y' \rangle \in T_1 \times T_2 \,|\, P \in \tau_h(x', y') \}$. By assumption, there exists $X \in \tau_1(x)$ such that $X \times \{ y \} \subseteq P$. Since τ_1 satisfies (4) and $X \in \tau_1(x)$, $\{ x' \in T_1 \,|\, X \in \tau_1(x') \} \in \tau_1(x)$. Since $X \times \{ y \} \subseteq P$, $\{ x' \in T_1 \,|\, X \in \tau_1(x') \} \times \{ y \} \subseteq \{ \langle x', y' \rangle \in T_1 \times T_2 \,|\, P \in \tau_h(x', y') \}$. □

3 Hybrid Semantics on Product on Monotone Neighborhood Frames

Let us introduce the syntax. First of all, it is worth noting that our syntax has two disjoint sets NOM_1 and NOM_2 of nominals. E.g., one can consider that an element of NOM_1 represents an instant of time and an element of NOM_2 represents a coordinate of space. So, our vocabulary consists of:

(i) two countable but disjoint sets of nominals $\mathsf{NOM}_1 = \{ i, j, k, \dots \}$ and $\mathsf{NOM}_2 = \{ a, b, c, \dots \}$,

(ii) a countable set PROP of propositional variables, where we assume that PROP is disjoint from $\mathsf{NOM}_1 \cup \mathsf{NOM}_2$,

(iii) Boolean connectives: $\neg$, $\wedge$,

(iv) two modal operators: $\Box_1$ (e.g. for time) and $\Box_2$ (e.g. for space) ($\Diamond_\alpha$ is the defined dual of $\Box_\alpha$, where $\alpha = 1$ or 2),

(v) two kinds of satisfaction operators: $@_i$ ($i \in \mathsf{NOM}_1$), $@_a$ ($a \in \mathsf{NOM}_2$).

Then, the set of formulas is defined inductively by:

$$\varphi ::= i \,|\, a \,|\, p \,|\, \neg\varphi \,|\, \varphi \wedge \psi \,|\, \Box_1\varphi \,|\, \Box_2\varphi \,|\, @_i\varphi \,|\, @_a\varphi.$$

We say that φ is *pure* if φ does not contain any propositional variables. For example, $@_i @_a \Diamond_1(i \wedge \Diamond_2 a)$ is pure. If φ is constructed only from the vocabulary (ii), (iii) and (iv) above, we say that φ is a *two-dimensional modal formula*. Moreover, we define the following two sublanguages: $\mathcal{L}_1 := \{ \neg, \wedge, \Box_1 \} \cup \mathsf{PROP} \cup \mathsf{NOM}_1 \cup \{ @_i \,|\, i \in \mathsf{NOM}_1 \}$ and $\mathcal{L}_2 := \{ \neg, \wedge, \Box_2 \} \cup \mathsf{PROP} \cup \mathsf{NOM}_2 \cup \{ @_a \,|\, a \in \mathsf{NOM}_2 \}$. We say that φ is a $\mathcal{L}_\alpha$*-formula* if it is constructed from the vocabulary of $\mathcal{L}_\alpha$ ($\alpha = 1$, 2).

Let us provide the semantics to our syntax. Intuitively, we define our valuation so that the denotation of $i \in \mathsf{NOM}_1$ is a vertical line $\{ x \} \times T_2$ and the denotation of $a \in \mathsf{NOM}_2$ is a horizontal line $T_1 \times \{ y \}$ over $T_1 \times T_2$. In this sense, we call $i, j, k, \cdots$ *vertical nominals* and $a, b, c, \cdots$ *horizontal nominals* below in this paper. So, let us define a *valuation* as follows. Given any product $\mathfrak{T}_1 \otimes \mathfrak{T}_2$ of monotone neighborhood frames, we say that a mapping $V : \mathsf{PROP} \cup \mathsf{NOM}_1 \cup \mathsf{NOM}_2 \to \mathcal{P}(T_1 \times T_2)$ is a *valuation* if (i) for any $i \in \mathsf{NOM}_1$, $|\pi_1[V(i)]| = 1$ and $\pi_2[V(i)] = T_2$; (ii) for any $a \in \mathsf{NOM}_2$, $|\pi_2[V(a)]| = 1$ and $\pi_1[V(a)] = T_1$, where $\pi_\alpha : T_1 \times T_2 \to T_\alpha$ is the projection onto T_α [1] [2]. Note that the denotation of p is a subset of $T_1 \times T_2$. Let

[1] As for $V(i)$, the condition $\pi_2[V(i)] = T_2$ excludes the possibility that $V(i)$ is a *non-empty proper subset* of $\{ x \} \times T_2$.

[2] Alternatively, we can also define $V(i) := \pi_1^{-1}[\{ x \}]$ for some $x \in T_1$ and $V(a) := \pi_2^{-1}[\{ y \}]$ for some $y \in T_2$.

us denote a unique element of $\pi_1[V(i)]$ by i^V and a unique element of $\pi_2[V(a)]$ by a^V. Then, we can derive that $V(i) = \{ i^V \} \times T_2$ and $V(a) = T_1 \times \{ a^V \}$. We call a pair $\langle \mathfrak{T}_1 \otimes \mathfrak{T}_2, V \rangle$ a *monotone neighborhood product model* (simply *product model*, when it causes no confusion).

Then, for any pair $\mathfrak{M} = \langle \mathfrak{T}_1 \otimes \mathfrak{T}_2, V \rangle$, any $\langle x, y \rangle \in T_1 \times T_2$ and any φ, the *satisfaction relation* $\Vdash$ is defined inductively as follows:

$$\begin{array}{lll}
\mathfrak{M}, \langle x, y \rangle \Vdash p & \text{iff} & \langle x, y \rangle \in V(p) \\
\mathfrak{M}, \langle x, y \rangle \Vdash i & \text{iff} & x = i^V \\
\mathfrak{M}, \langle x, y \rangle \Vdash a & \text{iff} & y = a^V \\
\mathfrak{M}, \langle x, y \rangle \Vdash \neg\varphi & \text{iff} & \mathfrak{M}, \langle x, y \rangle \nVdash \varphi \\
\mathfrak{M}, \langle x, y \rangle \Vdash \varphi \wedge \psi & \text{iff} & \mathfrak{M}, \langle x, y \rangle \Vdash \varphi \text{ and } \mathfrak{M}, \langle x, y \rangle \Vdash \psi \\
\mathfrak{M}, \langle x, y \rangle \Vdash \Box_1\varphi & \text{iff} & [\![\varphi]\!] \in \tau_h(x, y) \\
\mathfrak{M}, \langle x, y \rangle \Vdash \Box_2\varphi & \text{iff} & [\![\varphi]\!] \in \tau_v(x, y) \\
\mathfrak{M}, \langle x, y \rangle \Vdash @_i\varphi & \text{iff} & \mathfrak{M}, \langle i^V, y \rangle \Vdash \varphi \\
\mathfrak{M}, \langle x, y \rangle \Vdash @_a\varphi & \text{iff} & \mathfrak{M}, \langle x, a^V \rangle \Vdash \varphi,
\end{array}$$

where $[\![\varphi]\!] = \{ \langle x, y \rangle \,|\, \mathfrak{M}, \langle x, y \rangle \Vdash \varphi \}$. We usually write $\langle x, y \rangle \Vdash \varphi$, when the underlying model $\mathfrak{M}$ is clear from the context. By monotonicity of τ_h and τ_v, we can simplify the satisfactions of $\Box_1\varphi$ and $\Box_2\varphi$ as:

$$\begin{array}{lll}
\mathfrak{M}, \langle x, y \rangle \Vdash \Box_1\varphi & \text{iff} & \exists X \in \tau_1(x).\, \forall x' \in X.\, \mathfrak{M}, \langle x', y \rangle \Vdash \varphi, \\
\mathfrak{M}, \langle x, y \rangle \Vdash \Box_2\varphi & \text{iff} & \exists Y \in \tau_2(y).\, \forall y' \in Y.\, \mathfrak{M}, \langle x', y' \rangle \Vdash \varphi.
\end{array}$$

Remark that the behavior of $@_i\varphi$ is different from one-dimensional hybrid logic. In one-dimensional semantics, if φ holds at the state named by i, then $@_i\varphi$ holds at *all* states. In our two-dimensional semantics, $(i^V, y) \in [\![\varphi]\!]$ does not imply $[\![@_i\varphi]\!] = T_1 \times T_2$ in general.

We need more semantic definitions. A formula φ is *valid on a product model* $\mathfrak{M}$ (notation: $\mathfrak{M} \Vdash \varphi$) if $\mathfrak{M}, \langle x, y \rangle \Vdash \varphi$ for any pair $\langle x, y \rangle$ in $\mathfrak{M}$. We say that φ is *valid on* $\mathfrak{T}_1 \otimes \mathfrak{T}_2$ (notation: $\mathfrak{T}_1 \otimes \mathfrak{T}_2 \Vdash \varphi$) if $\langle \mathfrak{T}_1 \otimes \mathfrak{T}_2, V \rangle \Vdash \varphi$ for any valuation V. We also say that a set Λ of formulas is *valid* on $\mathfrak{T}_1 \otimes \mathfrak{T}_2$ (notation: $\mathfrak{T}_1 \otimes \mathfrak{T}_2 \Vdash \Lambda$) if $\mathfrak{T}_1 \otimes \mathfrak{T}_2 \Vdash \varphi$ for any $\varphi \in \Lambda$. A set Λ of formulas *defines* a class F of product of monotone neighborhood frames if, for any $\mathfrak{T}_1 \otimes \mathfrak{T}_2$, $\mathfrak{T}_1 \otimes \mathfrak{T}_2 \Vdash \Lambda$ iff $\mathfrak{T}_1 \otimes \mathfrak{T}_2 \in \mathsf{F}$. A set Λ of formulas is *satisfiable* in a class F of product of monotone neighborhood frames if there exists some $\mathfrak{T}_1 \otimes \mathfrak{T}_2 \in \mathsf{F}$ and some valuation V on it and some pair $\langle x, y \rangle$ from $\mathfrak{T}_1 \otimes \mathfrak{T}_2$ such that all formulas of Λ are true at $\langle x, y \rangle$ of $\langle \mathfrak{T}_1 \otimes \mathfrak{T}_2, V \rangle$.

Proposition 3.1 *All the formulas in Table 1 are valid on any product of monotone neighborhood frames.*

Table 1
Interaction Axioms for Product of Monotone Neighborhood Frames

Com@	$@_a @_i p \leftrightarrow @_i @_a p$
Com$\Box_1$**@**$_2$	$\Box_1 @_a p \leftrightarrow @_a \Box_1 p$
Com$\Box_2$**@**$_1$	$\Box_2 @_i p \leftrightarrow @_i \Box_2 p$
Red@$_1$	$@_i a \leftrightarrow a$
Red@$_2$	$@_a i \leftrightarrow i$

Proof. We only show the validity of **Com**$\Box_1$**@**$_2$. $\langle x, y \rangle \Vdash \Box_1 @_a p$ iff :

$$\exists X \in \tau_1(x).\, \forall x' \in X.\, \langle x', y \rangle \Vdash @_a p \text{ iff } \exists X \in \tau_1(x).\, \forall x' \in X.\, \langle x', a^V \rangle \Vdash p$$

iff $\langle x, a^V \rangle \Vdash \Box_1 p$ iff $\langle x, y \rangle \Vdash @_a \Box_1 p$, as desired. $\Box$

The one-dimensional nature of the horizontal and vertical neighborhood frames is emphasized by the following proposition (cf. [18, Proposition 3.10]).

Proposition 3.2 (i) *A $\mathcal{L}_1$-formula φ is valid on $\mathfrak{T}_1 \otimes \mathfrak{T}_2$* iff *$\varphi$ is valid on $\mathfrak{T}_1$.*
(ii) *A $\mathcal{L}_2$-formula φ is valid on $\mathfrak{T}_1 \otimes \mathfrak{T}_2$* iff *$\varphi$ is valid on $\mathfrak{T}_2$.*

Proof. We can show these two items by the similar argument for product of Kripke frames [13, Proposition 2.2]. $\Box$

4 Monotone Hybrid Product Logic

Definition 4.1 A set Λ of formulas is a **Name**-*logic* if Λ contains all tautologies and Λ is closed under **MP** and **Name** in Table 2. Λ is a *monotone bi-hybrid logic* if Λ is a **Name**-logic and Λ contains **K@**, **Selfdual**, **Ref**, **Intro**, **BMon**, **Agree** in Table 2 and Λ is closed under **Mon**, **Nec@**, **Sub** in Table 2. A monotone bi-hybrid logic Λ is *normal* if it contains **N** and **R**.

One difference from the notion of bi-hybrid logic in [13] consists in the axiom **BMon** in Table 2. In [13], the author used the axiom **Back**: $@_n p \to \Box_\alpha @_n p$, where $n \in \mathsf{NOM}_\alpha$ ($\alpha = 1, 2$). However, we cannot use it in this context, because $@_i p \to \Box_1 @_i p$ defines (non-emptiness) of τ_1 [14]. On the other hand, it is easy to see that $@_i p \wedge \Box_1 q \to \Box_1(@_i p \wedge q)$ is valid on all monotone neighborhood frames $\langle T_1, \tau_1 \rangle$. Another difference consists in whether we include the inference rule **BG** from Table 2 in the definition. We will discuss this below.

In order to capture the interaction between two dimensions, however, we also need the five interaction axioms in Table 1.

Definition 4.2 A monotone bi-hybrid logic Λ is a *monotone hybrid product logic* if Λ contains all formulas: **Com@**, **Com**$\Diamond_1$**@**$_2$, **Com**$\Diamond_2$**@**$_1$, **Red@**$_1$ and **Red@**$_2$ in Table 1. We denote the smallest monotone hybrid product logic by $(\mathbf{M}^{\mathbf{Name}}_{\mathcal{H}(@)}, \mathbf{M}^{\mathbf{Name}}_{\mathcal{H}(@)})$.

Table 2
A List of Axioms and Rules

Axioms for Monotone Hybrid Product Logic	
All the interaction axioms in Table 1.	
K@	$@_n(p \to q) \to (@_n p \to @_n q)$, where $n = i$ or a.
Selfdual	$\neg @_n p \leftrightarrow @_n \neg p$, where $n = i$ or a.
Ref	$@_n n$, where $n = i$ or a.
Intro	$n \wedge p \to @_n p$, where $n = i$ or a.
BMon	$@_n p \wedge \Box_\alpha q \to \Box_\alpha(@_n p \wedge q)$, where $n \in \mathsf{NOM}_\alpha$ $(\alpha = 1, 2)$.
Agree	$@_n @_m p \to @_m p$, where $\langle n, m \rangle = \langle i, j \rangle$ or $\langle a, b \rangle$.
Rules for Monotone Hybrid Product Logic	
MP	From $\varphi \to \psi$ and φ, we may infer ψ
Mon	From $\varphi \to \psi$, we may infer $\Box\varphi \to \Box\psi$, where $\Box \in \{ \Box_1, \Box_2 \}$.
Nec@	From φ, we may infer $@_n \varphi$, where $n \in \mathsf{NOM}_1 \cup \mathsf{NOM}_2$.
Sub	From φ, we may infer $\sigma(\varphi)$, where σ denotes a substitution that uniformly replaces proposition letters by formulas and nominals from NOM_α by nominals from NOM_α $(\alpha = 1, 2)$.
Name	From $n \to \varphi$, we may infer φ, where $n \in \mathsf{NOM}_1 \cup \mathsf{NOM}_2$ does not occur in φ.
Additional Axioms	
R	$(\Box p \wedge \Box q) \to \Box(p \wedge q)$ where $\Box \in \{ \Box_1, \Box_2 \}$.
N	$\Box\top$ where $\Box \in \{ \Box_1, \Box_2 \}$.
T	$\Box p \to p$ where $\Box \in \{ \Box_1, \Box_2 \}$.
4	$\Box p \to \Box\Box p$ where $\Box \in \{ \Box_1, \Box_2 \}$.
Sep$_0$	$@_n \Diamond_\alpha m \vee @_m \Diamond_\alpha n \to @_n m$ where $n, m \in \mathsf{NOM}_\alpha$ $(\alpha = 1, 2)$.
Sep$_1$	$\Diamond_\alpha n \to n$ where $n \in \mathsf{NOM}_\alpha$ $(\alpha = 1, 2)$.
Di	$\neg \Box_\alpha n$ where $n \in \mathsf{NOM}_\alpha$ $(\alpha = 1, 2)$.
com$_\to$	$\Box_1 \Box_2 p \to \Box_2 \Box_1 p$
com$_\leftarrow$	$\Box_2 \Box_1 p \to \Box_1 \Box_2 p$
chr	$\Diamond_1 \Box_2 p \to \Box_2 \Diamond_1 p$
Additional Rules	
BG	From $@_n \Diamond_\alpha m \to @_m \varphi$, we may infer $@_n \Box_\alpha \varphi$, where $n, m \in \mathsf{NOM}_\alpha$ and $m \neq n$ does not appear in φ $(\alpha = 1, 2)$.

We also say that a monotone hybrid product logic Λ is *normal* if Λ contains **N** and **R**. Λ is a *topological hybrid product logic* if Λ is a normal hybrid product logic and it contains **T** and **4** in Table 2. We denote the smallest topological hybrid product logic by $(\mathbf{S4}^{\mathbf{Name}}_{\mathcal{H}(@)}, \mathbf{S4}^{\mathbf{Name}}_{\mathcal{H}(@)})$.

Let us go back to the inference rule **BG**.

Definition 4.3 A monotone $\langle T, \tau \rangle$ is *augmented* if $\bigcap \tau(x) \in \tau(x)$ for any $x \in T$, i.e., $\tau(x)$ has a smallest element. A topological space $\langle T, \tau \rangle$ is *Alexandrov* if $\langle T, \tau \rangle$ is augmented.

For one-dimensional hybrid logic, Ten Cate and Litak [17] showed that **BG**-rule characterizes the class of all Alexandrov spaces, and they also generalized it to a characterization of the class of all augmented monotone neighborhood frames by **BG**. We will discuss now how to extend their results in the context of the present paper (for a similar kind of characterization by **BG** in a different context, see [12]).

Given two valuations V and V' and a horizontal or vertical nominal m, we say that V' is an *m-variant* of V if V and V' agree on all elements from the domain except possibly for m. Let us say that $\mathfrak{T}_1 \otimes \mathfrak{T}_2$ *admits* **BG** for $\Box_\alpha$ if any valuation V on $\mathfrak{T}_1 \otimes \mathfrak{T}_2$ falsifying the consequent $@_n \Box_\alpha \varphi$ can be changed to some valuation V' such that it falsifies the antecedent $@_n \Diamond_\alpha m \to @_m \varphi$ and V' is an m-variant of V.

Proposition 4.4 *If a monotone $\mathfrak{T}_1$ is augmented, then $\mathfrak{T}_1 \otimes \mathfrak{T}_2$ admits* **BG** *for $\Box_1$ for any monotone $\mathfrak{T}_2$.*

Proof. Assume that a monotone $\mathfrak{T}_1$ is augmented and that V on $\mathfrak{T}_1 \otimes \mathfrak{T}_2$ falsifies $@_i \Box_1 \varphi$. Then, we can find some $\langle x, y \rangle$ such that $x = i^V$ and $\langle x, y \rangle \not\Vdash \Box_1 \varphi$. By augmentation of τ_1, it follows that we can choose $x' \in \bigcap \tau_1(x)$ such that $\langle x', y \rangle \notin [\![\varphi]\!]_V$. Let us consider some j-variant V' of V such that $V'(j) := \{ x' \} \times T_2$ (recall that j is fresh in $@_i \Box_1 \varphi$). Then, it is easy to see that $@_i \Diamond_1 j \to @_j \varphi$ is false at $\langle x, y \rangle$ under V'. $\Box$

Proposition 4.5 *If a monotone $\mathfrak{T}_1$ is not augmented, $\mathfrak{T}_1 \otimes \mathfrak{T}_1$ fails to admit* **BG** *for $\Box_1$.*

Proof. The proof is similar to [17, Theorem 3.4]. It suffices to care about the vertical dimension. Assume that a monotone $\mathfrak{T}_1$ is not augmented. Thus, $\tau_1(x)$ has no smallest element for some $x \in T_1$. Fix such x. Then, it follows that $\{ x' \in X \mid \exists X' \in \tau_1(x). x' \notin X' \} \neq \emptyset$ for any $X \in \tau_1(x)$. By the axiom of choice, we can find a sequence $(g(X))_{X \in \tau_1(x)}$ such that (0) $g(X) \in X$, (1) $\forall X \in \tau_1(x). X \cap \{ g(X) \mid X \in \tau_1(x) \} \neq \emptyset$, and (2) $\forall X \in \tau_1(x). \exists X' \in \tau_1(x). g(X) \notin X'$. Define $V(i) = \{ x \} \times T_1$ and $V(p) = (T_1 \setminus \{ g(X) \mid X \in \tau_1(x) \}) \times T_1$. By (1), we can establish $\langle x, x \rangle \notin [\![@_i \Box_1 p]\!]_V$. Consider any j-variant V' of V. We show that $@_i \Diamond_1 j \to @_j p$ is true at all points $\langle x_1, x_2 \rangle$ from $T_1 \times T_1$ under V'. By definition of $(g(X))_{X \in \tau(x)}$ and (2), it is easy to see that $\langle x_1, x_2 \rangle \notin [\![@_j p]\!]_{V'}$ implies $\langle x_1, x_2 \rangle \notin [\![@_i \Diamond_1 j]\!]_{V'}$. $\Box$

Corollary 4.6 *Let $\mathfrak{T}_1$ be monotone. Then, the following are equivalent:*

(i) $\mathfrak{T}_1$ *is augmented.*

(ii) $\mathfrak{T}_1 \otimes \mathfrak{T}_2$ *admits* **BG** *for* $\Box_1$ *for any monotone* $\mathfrak{T}_2$*.*

(iii) $\mathfrak{T}_1 \otimes \mathfrak{T}_1$ *admits* **BG** *for* $\Box_1$*.*

Proof. By Propositions 4.4 and 4.5. $\Box$

If we restrict our attention to the product of topologies, Van Benthem, et al [18] showed equivalence of several characterizations of Alexandrovness in terms of $\mathbf{com}_{\leftarrow}$, $\mathbf{com}_{\rightarrow}$ and **chr** from Table 2. So, we can combine these characterizations with ours and obtain the following.

Corollary 4.7 *Let* $\mathfrak{T}_1$ *be a topological space. Then, the following are equivalent:*

(i) $\mathfrak{T}_1$ *is Alexandrov.*

(ii) $\mathfrak{T}_1 \otimes \mathfrak{T}_2$ *admits* **BG** *for* $\Box_1$ *for any topological space* $\mathfrak{T}_2$*.*

(iii) $\mathfrak{T}_1 \otimes \mathfrak{T}_1$ *admits* **BG** *for* $\Box_1$*.*

(iv) $\mathfrak{T}_1 \otimes \mathfrak{T}_1 \Vdash \mathbf{com}_{\leftarrow} \wedge \mathbf{com}_{\rightarrow}$*.*

(v) $\mathfrak{T}_1 \otimes \mathfrak{T}_2 \Vdash \mathbf{com}_{\leftarrow}$ *for any topological space* $\mathfrak{T}_2$*.*

(vi) $\mathfrak{T}_2 \otimes \mathfrak{T}_1 \Vdash \mathbf{com}_{\rightarrow}$ *for any topological space* $\mathfrak{T}_2$*.*

(vii) $\mathfrak{T}_1 \otimes \mathfrak{T}_1 \Vdash \mathbf{chr}$*.*

Proof. By Corollary 4.6 and [18, Corollary 4.19 and Corollary 4.22]. $\Box$

Remark 4.8 A natural question to ask is whether we can generalize the above characterization taken from [18] to *monotone neighborhood frames*. This is not the case. The proof for (i) $\Rightarrow$ ((v) & (vi)) and (i) $\Rightarrow$ (vii) in Corollary 4.7 can go through even for monotone neighborhood frames (cf. [18, Propositions 4.15 and 4.20]). Their proof of (iv) $\Rightarrow$ (i), however, requires (non-emptiness) and (intersection) of $\langle T, \tau \rangle$ (cf. [18, Propositions 4.18]). Moreover, the following tells us that the above generalization is impossible.

Proposition 4.9 *There is a non-augmented monotone* $\mathfrak{T} = \langle T, \tau \rangle$ *such that* $\mathfrak{T} \otimes \mathfrak{T} \models \mathbf{com}_{\leftarrow} \wedge \mathbf{com}_{\rightarrow}$*.*

Proof. Fix some non-empty T. Define $\tau : T \to \mathcal{PP}(T)$ by $\tau(x) = \emptyset$ for any $x \in T$. Then, (T, τ) is not augmented, since $\bigcap \tau(x) = T \notin \tau(x)$. Fix any $\langle x, y \rangle \in T \times T$ and any valuation V. Since both $\tau(x)$ and $\tau(y)$ are empty, we trivially have $\langle x, y \rangle \notin [\![\Box_1 \Box_2 p]\!]_V$ and $\langle x, y \rangle \notin [\![\Box_2 \Box_1 p]\!]_V$, as desired. $\Box$

5 Pure Completeness for Product of Monotone Frames

Definition 5.1 Let Λ be a **Name**-logic. φ is *deducible* in Λ from Γ if there is a finite subset Γ' of Γ such that $\bigwedge \Gamma' \to \varphi \in \Lambda$, where $\bigwedge \Gamma'$ is the conjunction of all finite elements of Γ' (if $\Gamma' = \emptyset$, we define $\bigwedge \Gamma' := \top$). Γ is Λ-*consistent* if $\bot$ is not deducible from Γ in Λ.

Lemma 5.2 *The following derivation rules are admissible in all monotone bi-hybrid logic:*

(i) If $\vdash @_j(\varphi \to \psi)$ *and* j *is fresh in* $\varphi \to \psi$*, then* $\vdash @_i(\Box_1\varphi \to \Box_1\psi)$.

(ii) If $\vdash \alpha \to @_j(\varphi \to \psi)$ *and* j *is fresh in* α *and* $\varphi \to \psi$*, then* $\vdash \alpha \to (@_i\Box_1\varphi \to @_i\Box_1\psi)$.

(iii) If $\vdash @_b(\varphi \to \psi)$ *and* b *is fresh in* $\varphi \to \psi$*, then* $\vdash @_a(\Box_2\varphi \to \Box_2\psi)$.

(iv) If $\vdash \alpha \to @_b(\varphi \to \psi)$ *and* b *is fresh in* α *and* $\varphi \to \psi$*, then* $\vdash \alpha \to (@_a\Box_2\varphi \to @_a\Box_2\psi)$.

Proof. It suffices to show (i) and (ii). Let us show (i). Assume that $\vdash @_j(\varphi \to \psi)$ and that j is fresh in $\varphi \to \psi$. First of all, remark that we can always use distributivity of @ over Boolean connectives from our axioms for @. Then, we can derive from **Intro** that $@_j(\varphi \to \psi) \to (j \to (\varphi \to \psi))$. By this and our assumption, we obtain $\vdash j \to (\varphi \to \psi)$. We deduce from **Name** and the freshness assumption of j that $\vdash \varphi \to \psi$. It follows from **Mon** that $\vdash \Box_1\varphi \to \Box_1\psi$. By $\mathbf{Nec}_{@}$, we obtain $@_i(\Box_1\varphi \to \Box_1\psi)$, as required. Next, we show (ii). Assume that j is fresh in α and $\varphi \to \psi$ and that $\vdash \alpha \to @_j(\varphi \to \psi)$, i.e., $\vdash (\alpha \wedge @_j\varphi) \to @_j\psi$. Let us choose k such that k does not occur in $\alpha \to (@_i\Box_1\varphi \to @_i\Box_1\psi)$ and $k \neq j$. By $\mathbf{Nec}_{@}$, $\vdash @_k((\alpha \wedge @_j\varphi) \to @_j\psi)$. We obtain $\vdash @_j((@_k\alpha \wedge \varphi) \to \psi)$ by **Agree** and distributivity of @ over Boolean connectives. It follows from (i) that $\vdash @_i(\Box_1(@_k\alpha \wedge \varphi) \to \Box_1\psi)$, since j does not occur in $(@_k\alpha \wedge \varphi) \to \psi$. By **BMon**, $\vdash @_i((@_k\alpha \wedge \Box_1\varphi) \to \Box_1\psi)$. It follows from the similar argument by **Agree** to the above that $\vdash @_k((\alpha \wedge @_i\Box\varphi) \to @_i\Box\psi)$, i.e., $\vdash @_k(\alpha \to (@_i\Box_1\varphi \to @_i\Box_1\psi))$. By **Name**, we conclude $\vdash \alpha \to (@_i\Box_1\varphi \to @_i\Box_1\psi)$. $\Box$

Definition 5.3 Let Δ be any set of formulas.

- Δ is *labelled* if $i \wedge a \in \Delta$ for some $\langle i, a \rangle$.
- Δ is *monotonically* $\Box_1$*-saturated* if, for every $\neg(@_i\Box_1\varphi \to @_i\Box_1@_a\psi) \in \Delta$, there is a vertical nominal j which does not appear in φ and ψ such that $\neg @_j(\varphi \to @_a\psi) \in \Delta$.
- Δ is *monotonically* $\Box_2$*-saturated* if, for every $\neg(@_a\Box_2\varphi \to @_a\Box_2@_i\psi) \in \Delta$, there is a horizontal nominal b which does not appear in φ and ψ such that $\neg @_b(\varphi \to @_i\psi) \in \Delta$.

The following is immediate from Definition 5.3.

Lemma 5.4 *Let* Λ *be a monotone hybrid product logic. Suppose that* Δ *is monotonically* $\Box_\alpha$*-saturated* Λ*-MCS* (α = *1, 2*).

(i) If $@_j(\varphi \to @_a\psi) \in \Delta$ *for all vertical nominals* j*, then* $@_j\Box_1\varphi \to @_j\Box_1@_a\psi \in \Delta$ *for all vertical nominals* j.

(ii) If $@_b(\varphi \to @_i\psi) \in \Delta$ *for all horizontal nominals* b*, then* $@_b\Box_2\varphi \to @_b\Box_2@_i\psi \in \Delta$ *for all horizontal nominals* b.

Lemma 5.5 (Lindenbaum Lemma) *Let* Λ *be a monotone hybrid product logic.*

Every Λ-consistent set of formulas can be extended to a labelled, monotonically $\Box_1$-saturated and monotonically $\Box_2$-saturated Λ-MCS, by adding both countably many new horizontal nominals and countably many new vertical nominals to the language.

Proof. Suppose that Σ is Λ-consistent (henceforth 'consistent'). Let $(i_n)_{n\in\omega}$ and $(a_n)_{n\in\omega}$ be two disjoint sets of countable fresh nominals. Let also $(\varphi_n)_{n\in\omega}$ be an enumeration of all formulas in this expanded syntax. We are going to construct a sequence of consistent extensions $(\Sigma^n)_{n\in\omega}$ of Σ by induction on n.

(Basis) Define $\Sigma^0 := \Sigma\cup\{\, i_0 \wedge a_0 \,\}$. By two kinds of **Name**-rule, we easily establish that Σ^0 is consistent.

(Inductive Step) Suppose that Σ^n is consistent. Let us define Σ^{n+1} as follows: If $\Sigma^n \cup \{\, \varphi_n \,\}$ is inconsistent, $\Sigma^{n+1} := \Sigma^n$. Otherwise, Σ^{n+1} is defined by:

$$\Sigma^{n+1} := \begin{cases} \Sigma^n \cup \{\, \varphi_n, \neg @_j(\varphi \to @_a\psi) \,\} & \text{if } \varphi_n \equiv \neg(@_i\Box_1\varphi \to @_i\Box_1 @_a\psi) \\ \Sigma^n \cup \{\, \varphi_n, \neg @_b(\varphi \to @_i\psi) \,\} & \text{if } \varphi_n \equiv \neg(@_a\Box_2\varphi \to @_a\Box_2 @_i\psi) \\ \Sigma^n \cup \{\, \varphi_n \,\} & \text{o.w.} \end{cases}$$

where $b \in \{\, a_n \,\}_{n\in\omega}$ and $j \in \{\, i_n \,\}_{n\in\omega}$ are *first unused* vertical and horizontal nominals in $\Sigma^n \cup \{\, \varphi_n \,\}$, respectively.

Claim 5.6 Σ^{n+1} *is consistent.*

Proof of Claim. It suffices to check the case where $\Sigma^n \cup \{\, \varphi_n \,\}$ is consistent and $\varphi_n \equiv \neg(@_i\Box_1\varphi \to @_i\Box_1 @_a\psi)$. Recall that j is fresh in $\Sigma^n\cup\{\, \varphi_n \,\}$. Assume for the purpose of *reductio* that Σ^{n+1} is inconsistent. Then there exist $\gamma_1, \ldots, \gamma_n \in \Sigma^n$ such that $\vdash \varphi_n \wedge \neg @_j(\varphi \to @_a\psi) \to \neg \bigwedge_l \gamma_l$. Let us put $\eta{:=}\bigwedge_l \gamma_l$. It follows from propositional logic that $\vdash \varphi_n \wedge \eta \to @_j(\varphi \to @_a\psi)$. By Lemma 5.2 (ii) and the choice of j, we obtain $\vdash \varphi_n \wedge \eta \to (@_i\Box_1\varphi \to @_i\Box_1 @_a\psi)$, which is equivalent to $\vdash \neg(\varphi_n \wedge \eta)$ by $\varphi_n \equiv \neg(@_i\Box_1\varphi \to @_i\Box_1 @_a\psi)$. This tells us the inconsistency of $\Sigma^n \cup \{\, \varphi_n \,\}$. A contradiction. $\dashv$

Finally, we put $\Sigma^\omega {:=} \bigcup_{n\in\omega} \Sigma^n$. Then, by construction we can easily establish that Σ^ω is a labelled, monotonically $\Box_1$- and $\Box_2$-saturated MCS. $\Box$

Let us now define the notion of a *Henkin-style product model.*

Definition 5.7 Let Λ be a monotone hybrid product logic. Given any Λ-MCS Δ, we define a *Henkin-style product model* $\mathfrak{M}_\Delta = \langle\, \mathfrak{T}_1 \otimes \mathfrak{T}_2, V_\Delta \,\rangle$ where $\mathfrak{T}_\alpha := \langle\, T_\alpha, \tau_\alpha \,\rangle$ ($\alpha = 1, 2$), as follows:

- For any vertical nominal i and any horizontal nominal a, let us define: $[i] := \{\, j \mid @_i j \in \Delta \,\}$, $|a| := \{\, b \mid @_a b \in \Delta \,\}$.
- Define $T_1 := \{\, [i] \mid i : \text{vertical nominal} \,\}$ and $T_2 := \{\, |a| \mid a : \text{horizontal nominal} \,\}$.
- We also define $\tau_1 : T_1 \to \mathcal{PP}(T_1)$ and $\tau_1 : T_2 \to \mathcal{PP}(T_2)$ as follows:

$$X \in \tau_1([i]) \text{ iff } \exists\, \theta.\, (@_i\Box_1\theta \in \Delta \text{ and } \forall\, k.\, (@_k\theta \in \Delta \text{ implies } [k] \in X)),$$

$$Y \in \tau_2(|a|) \text{ iff } \exists\, \theta.\, (@_a\Box_2\theta \in \Delta \text{ and } \forall\, c.\, (@_c\theta \in \Delta \text{ implies } |c| \in Y)).$$

- Define the mapping V_Δ by: $V_\Delta(l) = \{ \langle [j], |b| \rangle \mid @_j @_b l \in \Delta \}$ for any $l \in \mathsf{PROP} \cup \mathsf{NOM}_1 \cup \mathsf{NOM}_2$.

It is clear that τ_1 and τ_2 are monotone: Let us check that τ_1 is monotone. Assume that $X \in \tau_1([i])$ and $X \subseteq Y$. By definition, there exists some θ such that $@_i \Box_1 \theta \in \Delta$ and $\forall k. (@_k \theta \in \Delta \text{ implies } [k] \in X))$. Since $X \subseteq Y$, we have $\forall k. (@_k \theta \in \Delta \text{ implies } [k] \in Y))$. Then, we can conclude that $Y \in \tau_1([i])$.

Lemma 5.8 (Truth Lemma) *Let Λ be a monotone hybrid product logic. For all monotonically $\Box_1$- and $\Box_2$-saturated Λ-MCSs Δ, all pairs $\langle i, a \rangle$ and all formulas φ, $\mathfrak{M}_\Delta, \langle [i], |a| \rangle \Vdash \varphi$ iff $@_i @_a \varphi \in \Delta$.*

Proof. First, we check that V_Δ is really a valuation. In order to show that, it suffices to show that $V_\Delta(i) := \{ [j] \mid @_i j \in \Delta \} \times T_2$ and that $\{ [j] \mid @_i j \in \Delta \}$ is a singleton. We can establish the first clause, since $V_\Delta(i) = \{ \langle [j], |b| \rangle \mid @_j @_b i \in \Gamma \}$ and $\vdash @_j @_b i \leftrightarrow @_j i$ (by **Nec**@ and **Red**$@_2$) and $\vdash @_j i \leftrightarrow @_i j$. As for the second clause, it suffices to note that we have $\vdash @_i i$ and $\vdash @_i j \wedge @_j k \to @_i k$.

Second, we prove our main statement by induction on φ. We only demonstrate it for the following case: φ is of the form $\Box_2 \psi$. The proofs for the cases: (a) φ is of the form j and (b) φ is of the form $@_j \varphi$, are the same as in the proof of [13, Lemma 3.11].

We can demonstrate the case where φ is of the form $\Box_2 \psi$ as follows:

$$
\begin{array}{ll}
& \mathfrak{M}_\Delta, \langle [i], |a| \rangle \Vdash \Box_2 \psi \\
\text{iff} & \exists Y \in \tau_2(|a|).\ \forall y \in Y.\, \langle [i], y \rangle \Vdash \psi \\
\text{iff} & \exists \theta. (@_a \Box_2 \theta \in \Delta \text{ and } \forall c. (@_c \theta \in \Delta \text{ implies } |c| \in Y) \text{ and } \forall y \in Y.\, \langle [i], y \rangle \Vdash \psi) \\
\text{iff} & \exists \theta. (@_a \Box_2 \theta \in \Delta \text{ and } \forall c. (@_c \theta \in \Delta \text{ implies } \langle [i], |c| \rangle \in [\![\psi]\!])) \\
\text{iff} & \exists \theta. (@_a \Box_2 \theta \in \Delta \text{ and } \forall c. (@_c \theta \in \Delta \text{ implies } @_i @_c \psi \in \Delta)) \\
\text{iff} & @_i @_a \Box_2 \psi \in \Delta
\end{array}
$$

As for the right-to-left direction of the last equivalence, take $@_i \psi$ as θ. Then, by **Com**@ and **Com**$\Box_2 @_1$, we can establish the desired statement. As for the left-to-right direction, let us fix our witness θ. By Lemma 5.4 (monotone $\Box_2$-saturation), we obtain $@_a \Box_2 @_i \psi \in \Delta$. By **Com**@ and **Com**$\Box_2 @_1$, we establish $@_i @_a \Box_2 \psi \in \Delta$, as required. $\Box$

Definition 5.9 $\mathfrak{M} = \langle \mathfrak{T}_1 \otimes \mathfrak{T}_2, V \rangle$ is *named* if, for any $\langle x, y \rangle$ in $\mathfrak{M}$, there exists $\langle i, a \rangle$ such that $x = i^V$ and $y = a^V$.

Then, we can easily establish the following (cf. [1, Lemma 7.22]).

Lemma 5.10 *Given any named product model $\mathfrak{M} = \langle \mathfrak{T}_1 \otimes \mathfrak{T}_2, V \rangle$ and any pure formula φ, if $\mathfrak{M} \Vdash \sigma(\varphi)$ for all uniform substitutions σ, then $\mathfrak{T}_1 \otimes \mathfrak{T}_2 \Vdash \varphi$.*

This lemma tells us that the notion of uniform substitution (**Sub** in Table 2) fits well with a named model also in monotone hybrid product logic.

Theorem 5.11 *Let Γ be a set of pure formulas and Λ the smallest monotone hybrid product logic containing Γ. Then, Λ is sound and strongly complete for the class of all products of monotone neighborhood frames defined by Γ.*

Proof. Soundness is straightforward. In order to establish the strong completeness, assume that Δ is Λ-consistent. By Lemma 5.5, there exists a labelled, monotonically $\Box_1$- and $\Box_2$-saturated MCS Δ^+ such that $\Delta \subseteq \Delta^+$. Construct the Henkin-style product model $\mathfrak{M}_{\Delta^+} = \langle \mathfrak{T}_1 \otimes \mathfrak{T}_2, V_{\Delta^+} \rangle$. Since Δ^+ is labelled, $i \wedge a \in \Delta^+$ for some pair $\langle i, a \rangle$. By **Intro** and $i \wedge a \in \Delta^+$, $@_i @_a \varphi \in \Delta^+$ holds for any $\varphi \in \Delta^+$. So, we derive from Lemma 5.8 that Δ is satisfiable in $\mathfrak{M}_{\Delta^+}$. Finally, we show that $\mathfrak{T}_1 \otimes \mathfrak{T}_2$ belongs to the class F of product frames defined by Γ. For any $\gamma \in \Gamma$, we have $\mathfrak{M}_{\Delta^+} \Vdash \sigma(\gamma)$ for all uniform substitutions σ. So, $\mathfrak{T}_1 \otimes \mathfrak{T}_2 \Vdash \Gamma$ by Lemma 5.10. $\Box$

Corollary 5.12 $(\mathbf{M}^{\mathbf{Name}}_{\mathcal{H}(@)}, \mathbf{M}^{\mathbf{Name}}_{\mathcal{H}(@)})$ *is sound and strongly complete for the class of all product of monotone neighborhood frames.*

Theorem 5.13 *Let Γ be a set of pure formulas. The smallest normal hybrid product logic Λ containing Γ is sound and strongly complete for the class of all product of normal neighborhood frames defined by Γ.*

Proof. Soundness is straightforward. As for strong completeness, it suffices to show that $\mathfrak{T}_\alpha$ ($\alpha = 1, 2$) satisfies (non-emptiness) and (intersection) in the Henkin-style product model $\mathfrak{M}_{\Delta^+} = \langle \mathfrak{T}_1 \otimes \mathfrak{T}_2, V_{\Delta^+} \rangle$ (recall the proof of Theorem 5.11). We can easily establish (non-emptiness), because the axiom **N** is pure and **N**: $\Box_1 \top$ and $\Box_2 \top$ define (non-emptiness) of τ_1 and τ_2, respectively. Let us establish that τ_1 and τ_2 satisfy (intersection) by the axiom **R**. Consider $X, X' \in \tau_1([i])$. We demonstrate $X \cap X' \in \tau_1([i])$. Thus,

$$\exists\, \theta_X.\, (@_i \Box_1 \theta_X \in \Delta \text{ and } \forall\, k.\, (@_k \theta_X \in \Delta \text{ implies } [k] \in X)),$$

$$\exists\, \theta_{X'}.\, (@_i \Box_1 \theta_{X'} \in \Delta \text{ and } \forall\, k.\, (@_k \theta_{X'} \in \Delta \text{ implies } [k] \in X')).$$

Then, we have $@_i \Box_1 \theta_X \wedge @_i \Box_1 \theta_{X'} \in \Delta$. By **R** for $\Box_1$, $@_i \Box_1 (\theta_X \wedge \theta_{X'}) \in \Delta$. It is easy to see that $\forall\, k.\, (@_k (\theta_X \wedge \theta_{X'}) \in \Delta \text{ implies } [k] \in X \cap X'))$. $\Box$

Theorem 5.14 *Let Γ be a set of pure formulas. The smallest topological hybrid product logic Λ containing Γ is sound and strongly complete for the class of all product of topological spaces defined by Γ.*

Proof. We only show the strong completeness. It suffices to show that $\mathfrak{T}_\alpha$ ($\alpha = 1$, 2) satisfies the conditions: (T) and (4) in the Henkin-style product model $\mathfrak{M}_{\Delta^+} = \langle \mathfrak{T}_1 \otimes \mathfrak{T}_2, V_{\Delta^+} \rangle$ (recall the proof of Theorem 5.11 and Theorem 5.13). First, let us establish (T) of τ_1. Assume $X \in \tau_1([i])$. We show that $[i] \in X$. This means that:

$$@_i \Box_1 \theta_X \in \Delta \text{ and } \forall\, k.\, (@_k \theta_X \in \Delta \text{ implies } [k] \in X).$$

for some θ_X. By **T**-axiom, we have $@_i \theta_X \in \Delta$. Then, we obtain $[i] \in X$, as required. Second, let us establish (4) of τ_1. Assume $X \in \tau_1([i])$. We show $\{\, [j] \mid X \in \tau_1([j]) \,\} \in$

$\tau_1([i])$. Similarly to the argument for (T), we can find θ_X such that $@_i\Box_1\theta_X \in \Delta$ and $\forall\, k.\ (@_k\theta_X \in \Delta \text{ implies } [k] \in X)$. By **4**-axiom, we have $@_i\Box_1\Box_1\theta_X \in \Delta$. For the witness of $\{\, [j] \,|\, X \in \tau_1([j]) \,\} \in \tau_1([i])$, let us consider $\Box_1\theta_X$. Then, it suffices to check that: $\forall\, k.\, (@_k\Box_1\theta_X \in \Delta \text{ implies } X \in \tau_1([k]))$. Consider any k with $@_k\Box_1\theta_X \in \Delta$. Our witness for $X \in \tau_1([k])$ should be θ_X. Then, it suffices to check that:

$$\forall\, k'.\, @_{k'}\theta_X \in \Delta \text{ implies } [k'] \in X.$$

However, this is trivial. □

Corollary 5.15 $(\mathbf{S4}^{\mathbf{Name}}_{\mathcal{H}(@)}, \mathbf{S4}^{\mathbf{Name}}_{\mathcal{H}(@)})$ *is sound and strongly complete for the class of all product of topological spaces.*

6 An Application of Topological Pure Completeness

Definition 6.1 Let $\mathfrak{T} = \langle T, \tau \rangle$ be a topological space.

(i) $\mathfrak{T}$ is a T_0-*space* if, for any $x, y \in T$, $x \neq y$ implies that there exists $X \subseteq T$ such that $(y \notin X$ and $X \in \tau(x))$ or $(x \notin X$ and $X \in \tau(y))$.

(ii) $\mathfrak{T}$ is a T_1-*space* if for any $x, y \in T$, $x \neq y$ implies that there exist $X \in \tau(x)$ and $Y \in \tau(y)$ such that $y \notin X$ and $x \notin Y$.

(iii) $\mathfrak{T}$ is *dense-in-itself* if $\{\, x \,\} \notin \tau(x)$ for any $x \in T$.

Fact 6.2 (Gabelaia [6] (cf. [16])) *With respect the class of all topological spaces, we have:*

(i) $\mathbf{Sep}_0$ *for* $\Box_1$ *in Table 2 defines the class of all* T_0*-spaces.*

(ii) $\mathbf{Sep}_1$ *for* $\Box_1$ *in Table 2 defines the class of all* T_1*-spaces.*

(iii) **Di** *for* $\Box_1$ *in Table 2 defines the class of all dense-in-itself spaces.*

Remark that all the formulas in Fact 6.2 are *pure*.

Let us consider the one-dimensional hybrid language $\mathcal{L}_\alpha$. We say that a set Λ of $\mathcal{L}_\alpha$-formulas is a (*one-dimensional*) *topological hybrid logic* if it contains all axioms *of* $\mathcal{L}_\alpha$ in monotone bi-hybrid logic as well as **N**, **R**, **T** and **4** and is closed under **MP**, **Mon**, **Nec@**, the uniform substitution **Sub**, **Name** for $\mathcal{L}_\alpha$ (remark that we do not require the closure under **BG**). A topological hybrid logic Λ of $\mathcal{L}_\alpha$-formulas is *topologically complete* if there exists a class F of topological spaces such that Λ is the logic of F, i.e., $\Lambda = \{\, \varphi \text{ of } \mathcal{L}_\alpha \,|\, \varphi \text{ is valid on } \mathsf{F} \,\}$. We also say that Λ is *pure topo-complete* if there exists a class F of topological spaces such that Λ is the logic of F and F is definable by some set of pure formulas in $\mathcal{L}_\alpha$. Let Λ_α be a topologically complete logic in $\mathcal{L}_\alpha$ ($\alpha = 1, 2$). The *topo-product logic* $\Lambda_1 \times_t \Lambda_2$ is defined as the set of all valid formulas (of two-dimensional hybrid language) on any product $\mathfrak{T}_1 \otimes \mathfrak{T}_2$ such that Λ_α is valid on $\mathfrak{T}_\alpha$ ($\alpha = 1, 2$). We define (Λ_1, Λ_2) as the smallest topological hybrid product logic containing both Λ_1 and Λ_2.

Corollary 6.3 *Let* Λ_α *be a pure topo-complete logic of* $\mathcal{L}_\alpha$ ($\alpha = 1, 2$). *Then:*

$$(\Lambda_1, \Lambda_2) = \Lambda_1 \times_t \Lambda_2.$$

Proof. By Theorem 5.14 and Proposition 3.2. □

Let denote the smallest topological hybrid product logic containing $\mathbf{Sep}_1$ and $\mathbf{Di}$ by $(\mathbf{S4T_1Di^{Name}_{\mathcal{H}(@)}}, \mathbf{S4T_1Di^{Name}_{\mathcal{H}(@)}})$. Then, this corollary and Fact 6.2 assure us that $(\mathbf{S4T_1Di^{Name}_{\mathcal{H}(@)}}, \mathbf{S4T_1Di^{Name}_{\mathcal{H}(@)}})$ is strongly complete with respect to all products of two dense-in-itself T_1-spaces.

Since $\mathbb{R}$ with the Euclidean topology (i.e., the real line) satisfies T_1 and density-in-itself, it is tempting to think that $(\mathbf{S4T_1Di^{Name}_{\mathcal{H}(@)}}, \mathbf{S4T_1Di^{Name}_{\mathcal{H}(@)}})$ is (weakly) complete with respect to $\mathbb{R} \otimes \mathbb{R}$, in the sense that $\varphi \in (\mathbf{S4T_1Di^{Name}_{\mathcal{H}(@)}}, \mathbf{S4T_1Di^{Name}_{\mathcal{H}(@)}})$ iff $\mathbb{R} \otimes \mathbb{R} \Vdash \varphi$, for any φ. In what follows in this section, we show that this is not the case. Let us recall that a *two-dimensional modal formula* is a formula does not contain any hybrid vocabulary. We define the set $\mathbf{S4} \oplus \mathbf{S4}$ (usually called *fusion*) of two-dimensional modal formulas as the smallest normal bimodal logic containing $\mathbf{S4}$-axioms for both $\Box_1$ and $\Box_2$. Below, let $\mathbb{Q}$ be the rational line.

Fact 6.4 (Van Benthem, et al [18]) *Then,* $\mathbf{S4} \oplus \mathbf{S4}$ *is complete with respect to* $\mathbb{Q} \otimes \mathbb{Q}$, *i.e.,* $\varphi \in \mathbf{S4} \oplus \mathbf{S4}$ iff $\mathbb{Q} \otimes \mathbb{Q} \Vdash \varphi$ *for any two dimensional modal formula* φ.

Lemma 6.5 $(\mathbf{S4T_1Di^{Name}_{\mathcal{H}(@)}}, \mathbf{S4T_1Di^{Name}_{\mathcal{H}(@)}})$ *is conservative over* $\mathbf{S4} \oplus \mathbf{S4}$, *i.e., for any two dimensional modal formula* φ, $\varphi \in (\mathbf{S4T_1Di^{Name}_{\mathcal{H}(@)}}, \mathbf{S4T_1Di^{Name}_{\mathcal{H}(@)}})$ *implies* $\varphi \in \mathbf{S4} \oplus \mathbf{S4}$.

Proof. We show the contrapositive implication. Suppose that $\varphi \notin \mathbf{S4} \otimes \mathbf{S4}$. We deduce from Fact 6.4 that $\mathbb{Q} \otimes \mathbb{Q} \nVdash \varphi$. Remark that the rational line $\mathbb{Q}$ satisfies T_1 and density-in-itself. Then, it is easy to see that $(\mathbf{S4T_1Di^{Name}_{\mathcal{H}(@)}}, \mathbf{S4T_1Di^{Name}_{\mathcal{H}(@)}})$ is sound with respect to $\mathbb{Q} \otimes \mathbb{Q}$, i.e.,

$$\psi \in (\mathbf{S4T_1Di^{Name}_{\mathcal{H}(@)}}, \mathbf{S4T_1Di^{Name}_{\mathcal{H}(@)}}) \text{ implies } \mathbb{Q} \otimes \mathbb{Q} \Vdash \psi$$

for any (two dimensional hybrid) formula ψ. Therefore, it follows from $\mathbb{Q} \otimes \mathbb{Q} \nVdash \varphi$ that $\varphi \notin (\mathbf{S4T_1Di^{Name}_{\mathcal{H}(@)}}, \mathbf{S4T_1Di^{Name}_{\mathcal{H}(@)}})$. □

Fact 6.6 (Kremer [8]) $\mathbf{S4} \oplus \mathbf{S4}$ *is incomplete with respect to* $\mathbb{R} \otimes \mathbb{R}$, *i.e., there exists some two dimensional modal formula* φ *such that* $\varphi \notin \mathbf{S4} \oplus \mathbf{S4}$ *and* $\mathbb{R} \otimes \mathbb{R} \Vdash \varphi$.

Theorem 6.7 $(\mathbf{S4T_1Di^{Name}_{\mathcal{H}(@)}}, \mathbf{S4T_1Di^{Name}_{\mathcal{H}(@)}})$ *is incomplete with respect to* $\mathbb{R} \otimes \mathbb{R}$, *i.e., there exists some formula* φ *such that* $\varphi \notin (\mathbf{S4T_1Di^{Name}_{\mathcal{H}(@)}}, \mathbf{S4T_1Di^{Name}_{\mathcal{H}(@)}})$ *and* $\mathbb{R} \otimes \mathbb{R} \Vdash \varphi$.

Proof. By Fact 6.6 and Lemma 6.5. □

7 Further Directions and Open Problems

7.1 Local Definability

In the same sprit as did in [17], we can also include the *downarrow binders* in our syntax. The downarrow binder $\downarrow i$ (or $\downarrow a$) binds a nominal i (or a) to the first (or, second, respectively) argument of the current state. Given any $\langle T_1 \times T_2, \tau_h, \tau_v, V \rangle$, we can define

$$\langle T_1 \times T_2, \tau_h, \tau_v, V \rangle, \langle x, y \rangle \Vdash \downarrow i. \varphi \text{ iff } \langle T_1 \times T_2, \tau_h, \tau_v, V[i \mapsto x] \rangle, \langle x, y \rangle \Vdash \varphi,$$

where $V[i \mapsto x]$ is the i-variant of V such that it sends i to $\{x\} \times T_2$. We can also give the similar clause to $\downarrow a. \varphi$. A technique of *local definability* allows us to capture this semantics by the axiom $\mathbf{DA}_1$: $@_j(\downarrow i. \varphi \leftrightarrow \varphi[i/j])$, where $\varphi[i/j]$ is the result of replacing all free instances of i by j in φ. We can also consider the corresponding axiom $\mathbf{DA}_2$ for $\downarrow a. \varphi$. By the similar argument to [2, Theorem 5], we can immediately transfer Theorems 5.11, 5.13, and 5.14 to the syntax extended with the downarrow binders $\downarrow i$ and/or $\downarrow a$.

7.2 Dependant Product of Monotone Neighborhood Frames

In [13], the author considered the dependence of the horizontal dimension to the vertical dimension by the notion of *dependent product of Kripke frames* and revealed that we still retain pure completeness result. It would be interesting to see if we can obtain the corresponding result for the notion of *dependant product of monotone neighborhood frames.*

7.3 Hybrid Product Logic over Product of Euclidean spaces

Theorem 6.7 established the incompleteness of $(\mathbf{S4T_1Di}^{\mathbf{Name}}_{\mathcal{H}(@)}, \mathbf{S4T_1Di}^{\mathbf{Name}}_{\mathcal{H}(@)})$ with respect to $\mathbb{R} \otimes \mathbb{R}$. In this stage, the author does not know if we can obtain a complete axiomatization of the logic of $\mathbb{R} \otimes \mathbb{R}$ in our two-dimensional hybrid syntax. If the syntax for one-dimensional hybrid logic is expanded with the *global modality* $\mathrm{E}\varphi$ (read: 'φ holds at some states'), the logic of the real line $\mathbb{R}$ in this syntax is not *finite* axiomatizable [9] (via Gargov-Goranko translation [7], see also [10]). However, Kudinov [9] also showed that the logic of $\mathbb{R}^n$ ($n \geq 2$ is fixed) in the above syntax with the global modality is axiomatizable. Therefore, it would be interesting to study whether the hybrid product logic of $\mathbb{R}^n \otimes \mathbb{R}^n$ ($n \geq 2$ is fixed) is axiomatizable or not.

Acknowledgement

I would like to thank João Marcos for his encouragement to submit this paper to Hylo 2010. I would like to thank Tadeusz Litak for his extensive comments to my dissertation [14]. His comments are useful also in writing up this paper. Finally, all the comments and suggestions from three reviewers are helpful and useful to revise this paper. All errors, however, are mine.

References

[1] Blackburn, P., M. de Rijke and Y. Venema, "Modal Logic," Cambridge Tracts in Theoretical Computer Science, Cambridge University Press, Cambridge, 2001.

[2] Blackburn, P. and B. ten Cate, *Pure extensions, proof rules, and hybrid axiomatics*, Studia Logica **84** (2006), pp. 277–322.

[3] Chellas, F., Brian, "Modal Logic," Cambridge University Press, Cambridge, Massachusetts, 1980.

[4] Gabbay, D. M., A. Kurucz, F. Wolter and M. Zakharyaschev, "Many-Dimensional Modal Logics : Theory and Applications," Studies in Logic and the Foundations of Mathematics, Elsevier, North-Holland, 2003.

[5] Gabbay, D. M. and V. B. Shehtman, *Products of modal logics, part 1*, Logic Journal of IGPL **6** (1998), pp. 73–146.

[6] Gabelaia, D., "Modal Definability in Topology," Master's thesis, University of Amsterdam (2001).
URL http://www.illc.uva.nl/Publications/ResearchReports/MoL-2001-11.text.pdf

[7] Gargov, G. and V. Goranko, *Modal logic with names*, Journal of Philosophical Logic **22** (1993), pp. 607–36.
URL http://www.springerlink.com/content/w887321858708085/

[8] Kremer, P., *The incompleteness of* **S4** $\oplus$ **S4** *for the product space* $\mathbb{R} \times \mathbb{R}$, forthcoming in *Studia Logica*.
URL http://individual.utoronto.ca/philipkremer/onlinepapers/RxR.pdf

[9] Kudinov, A. V., *Topological modal logic of* $\mathbb{R}$ *with inequality*, Russian Mathematical Surveys **63** (2008), pp. 163–165.

[10] Litak, T., *Isomorphism via translation*, , **6** (2006), pp. 333–351.
URL http://www.aiml.net/volumes/volume6/Litak.ps

[11] McKinsey, J. and A. Tarski, *The algebra of topology.*, Annals of Mathematics **45** (1944), pp. 141–191.

[12] Sano, K., *Hybrid counterfactual logics: David Lewis meets Arthur Prior again*, Journal of Logic, Language and Information **18** (2009), pp. 515–539.
URL http://www.springerlink.com/content/r168848r1216m08k/

[13] Sano, K., *Axiomatizing hybrid products: How can we reason many-dimensionally in hybrid logic?*, Journal of Applied Logic **8** (2010), pp. 459–474, special Issue on Hybrid Logics.

[14] Sano, K., "Semantical Investigation into Extended Modal Languages," Ph.D. thesis, Graduate School of Letters, Kyoto University (2010).

[15] Sarenac, D., "Products of topological modal logics," Ph.D. thesis, Stanford University, Ph.D. Philosophy (2006).

[16] Ten Cate, B., D. Gabelaia and D. Sustretov, *Modal languages for topology: expressivity and definability*, Annals of Pure and Applied Logic **159** (2008), pp. 146–170.
URL http://staff.science.uva.nl/~bcate/papers/tengasus.pdf

[17] Ten Cate, B. and T. Litak, *Topological perspective on the hybrid proof rules*, Electronic Notes in Theoretical Computer Science **174** (2007), pp. 79–94.

[18] Van Benthem, J., G. Bezhanishvili, B. Ten Cate and D. Sarenac, *Multimodal logics of products of topologies*, Studia Logica **84** (2006), pp. 369–392.