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C^1 rational quadratic trigonometric spline

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Abstract A Bézier like C^1 rational quadratic trigonometric polynomial spline is developed. It defines two shape parameters in each subinterval. The approximation and geometric properties are investigated. The curvature continuity is established. The developed rational quadratic trigonometric polynomial spline is extended to C^1 piecewise rational bi-quadratic function with four shape parameters in each rectangular patch. Data dependent constraints are developed on the shape parameters in the description of piecewise rational quadratic and bi-quadratic trigonometric polynomial spline for shape preservation of curve and regular surface data. The developed shape preserving schemes provide tangent continuity in quadratic form and does not restrict interval length, derivatives or data.

1. Introduction

Rational trigonometric interpolating splines with tension(shape parameters) are preferred because these can interpolate the same data in the form of straight line and curve. These tension parameters do not affect the order of continuity of spline. Moreover, the rational structure of these splines allows coping with singularities. Data gathered, whether, physically or experimentally has at least one of the shape properties, positivity, monotonicity and convexity. The amount of rainfall, gas discharge and exponential functions are a few positive data producing sources. The path inscribed by reboots arm movement and designing of its circuits are few examples of monotone and convex data.

Bao et al. [2] presented a rational blended interpolant with free parameters. The developed interpolant was used for value control and shape control, and the range of parameters was determined by minimizing the bending energy. Brodlie et al. [3] extended the cubic Hermite to bi-cubic Hermite for positive and constrained data interpolation. The developed scheme was C^1. Delgado and Peña in [6] investigated the rational Bézier surfaces for shape preservation of data and established that rational Bézier surfaces are not monotonicity preserving. Duan et al. [7] revisited the rational spline [8] which was C^1 only if slope of secant line coincided with slope of tangent line. The boundedness, value control, inflection point control and convexity at a point of the interpolant were studied for uniform data. Duan et al. [9] developed a bivariate rational interpolant with four shape parameters in each rectangular patch. The developed interpolant was C^1 for equally spaced data with a suitable choice of shape parameters. The sufficient restrictions were developed on shape parameters for constrained

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interpolation of data. Duan et al. [10] discussed the rate of convergence of a rational spline with two shape parameters.

Han [14] presented the cubic trigonometric polynomial curves with shape parameters. The partition of knot vector and value of free parameter affected the order of continuity. Han et al. [16] presented a cubic trigonometric Bézier curve with two shape parameters and compared it to the cubic Bézier curve. The effect of shape parameters on the shape of the curve was analyzed found that it was closer to control polygon than the cubic Bézier curves. Hussain et al. [17] used rational quadratic function with a shape parameter to preserve the shape of curve data. The developed scheme of this paper only assured position continuity. Lamberti and Manni [18] utilized the cubic Hermite in its parametric form to preserve the shape of data. The authors constrained the knot vector to introduce the tension effect and shape preservation. C^2-continuity was established by a set of restrictions on first order derivatives at the knots. Manni and Sablonnière [19] developed a C^1 parametric Hermite interpolation technique for comonotone (monotone w.r.t. x and/or y) regular data using piecewise quadratic and bi-quadratic components. The shape preserving scheme imposed a restriction on knot vector and derivatives. Zhang et al. [21] constructed a bivariate rational function with shape parameters. The developed bivariate rational interpolant was C^1 if the derivative was equal to slope of tangent line. A positive convex discriminant function was constructed for the suitable choice of shape parameters to assure convex surface through convex data.

The study in this research paper is motivated by Bao et al. [2], Delgado and Peña [6] and Hussain et al. [17]. The aim is to develop a rational interpolating spline to ensure local control on a single interval and C^1 continuity in quadratic structure.

The remainder of the paper is organized as follows. Section 2 presents a new rational quadratic trigonometric function with two tension parameters in each subinterval. Section 3 transforms this rational quadratic trigonometric function to trigonometric spline and discusses its approximation properties and shape properties. Section 4 extends trigonometric spline to C^1 bivariate rational quadratic trigonometric function. Section 5 is of numerical examples and Section 6 concludes the paper.

2. Bézier like rational quadratic trigonometric function

Let the data under consideration be $\{(x_i, f_i), i = 0, 1, 2, \ldots, n\}$. The usual increasing partition of domain is adopted. The Bézier like rational quadratic trigonometric function $S(x)$ is defined over the interval $I_i = [x_i, x_{i+1}]$, $i = 0, 1, 2, \ldots, n-1$ as:

$$S(x) = \sum_{k=0}^{3} R_k(x) P_k, \tag{1}$$

$R_k(x)$, $k = 0, 1, 2, 3$ are the rational quadratic trigonometric basis functions defined as

$$R_0(x) = \frac{B_0(x)}{R(x)}, \quad R_1(x) = \frac{\mu_i B_1(x)}{R(x)},$$
$$R_2(x) = \frac{\eta_i B_2(x)}{R(x)}, \quad R_3(x) = \frac{B_3(x)}{R(x)},$$

$$B_0(x) = (1 - \sin(\delta_i))^2, \quad B_1(x) = (1 - \sin(\delta_i))\sin(\delta_i),$$
$$B_2(x) = (1 - \cos(\delta_i))\cos(\delta_i),$$

$$B_3(x) = (1 - \cos(\delta_i))^2,$$
$$R(x) = B_0(x) + \mu_i B_1(x) + \eta_i B_2(x) + B_3(x),$$

$$\delta_i = \frac{\pi(x - x_i)}{2h_i}, \quad h_i = x_{i+1} - x_i, \quad i = 0, 1, 2, \ldots, n-1.$$

P_k, $k = 0, 1, 2, 3$ are the control points so the Bézier like rational quadratic trigonometric function (1) is termed as control point form. The parameters μ_i and η_i are the shape design parameters or tension parameters in each subinterval $I_i = [x_i, x_{i+1}]$, $i = 0, 1, 2, \ldots, n-1$.

Theorem 1. *The rational quadratic trigonometric function (1) satisfies the following properties:*

1. *Endpoint interpolation property: $S(\delta_i = 0) = P_0$ and $S(\delta_i = \frac{\pi}{2}) = P_3$.*
2. *Convex Hull property: The curve always lies in the convex hull of control points P_k, $k = 0, 1, 2, 3$.*
3. *The rational quadratic trigonometric function is invariant under the affine transformations.*
4. *The rational quadratic trigonometric function represents conic section under certain restrictions on shape design parameters.*

Proof.

1. It can be easily established by substituting $\delta_i = 0$ and $\delta_i = \frac{\pi}{2}$ in Eq. (1).
2. Since $B_k(x) \geqslant 0$, $k = 0, 1, 2, 3$ for $\delta_i \in [0, \frac{\pi}{2}]$ and shape design parameters (μ_i, η_i) are assumed positive real numbers, so $R_k(x)$, $k = 0, 1, 2, 3$ are non-negative. The simple summation establishes that $\sum_{k=0}^{3} R_k(x) = 1$. It asserts convex hull property, i.e. the curve will always lie in the convex hull of control points.
3. Let T be an affine transformation defined as $T(X) = AX + T_1$, X is the vector to be transformed, A is transformation matrix and T_1 is the translation vector. Applying the affine transformation T to the rational quadratic trigonometric function (1) we have

$$T(S(x)) = T\left(\sum_{k=0}^{3} R_k(x) P_k\right) = A\sum_{k=0}^{3} R_k(x) P_k + T_1. \tag{2}$$

Since $\sum_{k=0}^{3} R_k(x) = 1$, so the expression (2) takes the form:

$$T(S(x)) = A\sum_{k=0}^{3} R_k(x) P_k + \sum_{k=0}^{3} R_k(x) T_1 = \sum_{k=0}^{3} R_k(x)(AP_k + T_1)$$
$$= \sum_{k=0}^{3} R_k(x) T(P_k).$$

Ellipse is finite over whole domain, parabola tends to infinity at most at a single point of domain and hyperbola tends to infinity at most two points of the domain. While interpolating a conic segment (hyperbola, parabola and ellipse) by the rational quadratic trigonometric function $S(x)$ defined in (1) it is necessary to choose those values of shape design parameters (μ_i η_i) which fulfill the aforesaid restrictions. The values of shape design parameters (μ_i η_i) are computed as follows:

Substituting the values of rational quadratic trigonometric basis functions $R_k(x)$, $k = 0, 1, 2, 3$ in Eq. (1) it takes the form

$$S(x) = \frac{B_0(x) + \mu_i B_1(x) + \eta_i B_2(x) + B_3(x)}{R(x)}. \tag{3}$$

Hence $S(x)$ is infinite if $R(x) = 0$. To simplify the computation we shall restrict to special case that is $\mu_i = \eta_i$. Substituting the values of $B_k(x)$, $k = 0, 1, 2, 3$; $\mu_i = \eta_i$in $R(x) = B_0(x) + \mu_i B_1(x) + \eta_i B_2(x) + B_3(x) = 0$ and after some simplification it reduces to

$$R(x) = 2(\eta_i - 2)^2 \sin^2(\delta_i) + 2(\eta_i - 2)(3 - \eta_i)\sin(\delta_i) + (5 - 2\eta_i) = 0. \tag{4}$$

Eq. (4) is quadratic in $\sin(\delta_i)$. The discriminant of $R(x) = 0$ w.r.t $\sin(\delta_i)$ is $\Delta = 4(\eta_i - 2)^2\ \{(3 - \eta_i)^2 - 2(5 - 2\eta_i)\}$. It can be easily computed from Δ that roots of $R(x) = 0$ will be

(a) real and distinct if $\eta_i \in (-\infty, 1 - \sqrt{2}) \cup (1 + \sqrt{2}, \infty)$;
(b) single real root if $\eta_i \in \{1 - \sqrt{2}, 1 + \sqrt{2}\}$;
(c) no real root(roots are imaginary) if $\eta_i \in (1 - \sqrt{2}, 1 + \sqrt{2})$.

$S(x)$ represents hyperbola if $\eta_i \in (-\infty, 1 - \sqrt{2}) \cup (1 + \sqrt{2}, \infty)$, parabola if $\eta_i \in \{1 - \sqrt{2}, 1 + \sqrt{2}\}$ and ellipse if $\eta_i \in (1 - \sqrt{2}, 1 + \sqrt{2})$. $\square$

3. Rational quadratic trigonometric spline

The rational quadratic trigonometric function (3) is transformed to spline by applying the following C^1-continuity conditions at the end points of the interval $I_i = [x_i, x_{i+1}]$

$$S(x_i) = f_i, \quad S(x_{i+1}) = f_{i+1}, \quad S'(x_i) = d_i, \quad S'(x_{i+1}) = d_{i+1}.$$

The rational quadratic trigonometric spline over the subinterval $I_i = [x_i, x_{i+1}]$ is defined as

$$S(x) = \frac{B_0(x)A_0 + B_1(x)A_1 + B_2(x)A_2 + B_3(x)A_3}{R(x)}, \tag{5}$$

$$A_0 = f_i, \quad A_1 = \mu_i f_i + \frac{2h_i d_i}{\pi}, \quad A_2 = \eta_i f_{i+1} - \frac{2h_i d_{i+1}}{\pi}, A_3 = f_{i+1}.$$

Theorem 2. *For $g(x) \in C^3[x_0, x_n]$, let $S(x)$ be a rational quadratic trigonometric spline (5) interpolating $g(x)$ in $[x_i, x_{i+1}]$, then for the positive parameters μ_i and η_i, the error of interpolating function satisfies $|g(x) - S(x)| \leqslant \|g^{(3)}(\lambda)\| h_i^3 \smile{c}_i$, with $\smile{c}_i = \max_{0 \leqslant \theta \leqslant 1} \varpi(\mu_i, \eta_i, \theta)$, where*

$$\varpi(\mu_i, \eta_i, \theta) = \begin{cases} \max\ \varpi_1(\mu_i, \eta_i, \theta), & 0 \leqslant \eta_i \leqslant 1, \quad 0 \leqslant \theta \leqslant 1, \\ \max\ \varpi_2(\mu_i, \eta_i, \theta), & \eta_i > 1 + \frac{2}{\pi}, \quad 0 \leqslant \theta \leqslant \theta^*, \\ \max\ \varpi_3(\mu_i, \eta_i, \theta), & \eta_i > 1 + \frac{2}{\pi}, \quad \theta^* \leqslant \theta \leqslant 1. \end{cases}$$

$$\begin{aligned}\varpi_1 = \frac{h_i^3}{R(x)}\Bigg\{ & \frac{2B_2\theta^2 - \pi(1-\theta)(\eta_i B_2 + B_3)\theta^2}{\pi} \\ & + \frac{4B_2(1-\theta)\theta - \pi(\eta_i B_2 + B_3)(1-\theta)^2\theta}{\pi} + \frac{(B_0 + \mu_i B_1)\theta^3}{3} \\ & + \frac{(1-\theta)^2(6B_2 - \pi(\eta_i B_2 + B_3)(1-\theta))}{3\pi}\Bigg\},\end{aligned}$$

$$\begin{aligned}\varpi_2 = \frac{h_i^3}{R(x)}\Bigg\{ & -\frac{2(B_0 + \mu_i B_1)}{3}\left(\frac{E-D}{C}\right)^3 \\ & + \frac{(4B_2 - 2\pi(1-\theta)(\eta_i B_2 + B_3))}{\pi}\left(\frac{E-D}{C}\right)^2 \\ & - \left(\frac{8B_2(1-\theta) - 2\pi(\eta_i B_2 + B_3)(1-\theta)^2}{\pi}\right)\left(\frac{E-D}{C}\right) \\ & - \frac{(B_0 + \mu_i B_1)\theta^3}{3} - \left(\frac{2B_2 - (1-\theta)(\eta_i B_2 + B_3)}{\pi}\right)\theta^2 \\ & - \left(\frac{4B_2(1-\theta) - (\eta_i B_2 + B_3)(1-\theta)^2}{\pi}\right)\theta \\ & - \left(\frac{2B_2 - (1-\theta)(\eta_i B_2 + B_3)}{\pi}\right)\theta^2 + \frac{64B_2^3}{3\pi^3(\eta_i B_2 + B_3)^2} \\ & - \frac{2B_2(1-\theta)^2}{\pi} + \frac{(1-\theta)^3(\eta_i B_2 + B_3)}{3}\Bigg\},\end{aligned}$$

$$\begin{aligned}\varpi_3 = \frac{h_i^3}{R(x)}\Bigg\{ & \frac{2(B_0 + \mu_i B_1)}{3}\left(\frac{E-D}{C}\right)^3 \\ & - \left(\frac{4B_2 - 2\pi(1-\theta)(\eta_i B_2 + B_3)}{\pi}\right)\left(\frac{E-D}{C}\right)^2 \\ & + \left(\frac{8B_2(1-\theta) - 2\pi(\eta_i B_2 + B_3)(1-\theta)^2}{\pi}\right)\left(\frac{E-D}{C}\right) \\ & - \frac{2(B_0 + \eta_i B_1)}{3}\left(\frac{E+D}{C}\right)^3 \\ & + \left(\frac{4B_2 - 2\pi(1-\theta)(\eta_i B_2 + B_3)}{\pi}\right)\left(\frac{E+D}{C}\right)^2 \\ & + \left(\frac{8B_2(1-\theta) - 2\pi(\eta_i B_2 + B_3)(1-\theta)^2}{\pi}\right)\left(\frac{E+D}{C}\right) \\ & + \frac{(B_0 + \mu_i B_1)\theta^3}{3} + \left(\frac{4B_2(1-\theta) - \pi(\eta_i B_2 + B_3)(1-\theta)^2}{\pi}\right)\theta \\ & + \left(\frac{6B_2 - \pi(1-\theta)(\eta_i B_2 + B_3)}{3\pi}\right)(1-\theta)^2 \\ & + \left(\frac{2B_2 - \pi(1-\theta)(\eta_i B_2 + B_3)}{\pi}\right)\theta^2\Bigg\},\end{aligned}$$

$$C = 2(B_0 + \mu_i B_1),$$

$$D = \sqrt{\left(\frac{4B_2 - 2(\eta_i B_2 + B_3)(1-\theta)}{\pi}\right)^2 - 4(B_0 + \mu_i B_1)(1-\theta)(4B_2 - (\eta_i B_2 + B_3)(1-\theta))},$$

$$E = \frac{4B_2 - 2(\eta_i B_2 + B_3)(1-\theta)}{\pi},$$

$$\theta = \frac{x - x_i}{h_i}, \quad \theta^* = 1 - \frac{2}{\pi(\eta_i - 1)}.$$

Proof. Let $\{(x_i, f_i), i = 0,1,2,\ldots,n\}$ be the plane data defined over the interval $[a, b]$. If we interpolate this data by rational quadratic trigonometric spline (5) then $S(x_i) = f_i$, $S'(x_i) = d_i$, $i = 0,1,2,\ldots,n$ but $S(x)$ approximates the data in between the knots. To measure the accuracy of this approximation suppose that the data are generated from a function $g(x) \in C^3[x_0, x_n]$. The error of approximation over an arbitrary subinterval $I_i = [x_i, x_{i+1}]$ is defined by Peano Kernel Theorem [20] as:

$$F[g] = g(x) - S(x) = \frac{1}{2}\int_{x_i}^{x_{i+1}} g^{(3)}(\lambda)F_x(\psi)d\lambda,$$

where F_x is known as Peano Kernel and $\psi = (x-\lambda)_+^2$ is the truncated power function. The absolute error of approximation over the subinterval $I_i = [x_i, x_{i+1}]$ is defined as:

$$|g(x) - S(x)| \leqslant \frac{1}{2}\|g^{(3)}(\lambda)\| \int_{x_i}^{x_{i+1}} |F_x(\psi)|d\lambda. \tag{6}$$

Due to truncated power function, $F_x(\psi)$ is partitioned into two subintervals as follows:

$F_x(\psi) = f_1(x, \lambda)$, for $\lambda \in (x_i, x)$ and $F_x(\psi) = f_2(x, \lambda)$, for $\lambda \in (x, x_{i+1})$, where

$$f_1(x,\lambda) = (x-\lambda)^2 - \frac{(\eta_i B_2 + B_3)(x_{i+1}-\lambda)^2 - \frac{4h_i}{\pi}B_2(x_{i+1}-\lambda)}{R(x)},$$

$$f_2(x,\lambda) = -\frac{(\eta_i B_2 + B_3)(x_{i+1}-\lambda)^2 - \frac{4h_i}{\pi}B_2(x_{i+1}-\lambda)}{R(x)}.$$

B_k are the $B_k(x)$, $k = 0, 1, 2, 3$ are already defined in Section 2.

The integral involved in Eq. (6) is expressed $\int_{x_i}^{x_{i+1}} |F_x(x)|d\lambda = \int_{x_i}^{x} |f_1(x,\lambda)|d\lambda + \int_{x}^{x_{i+1}} |f_2(x,\lambda)|d\lambda$.

It is observed by simple computation that if $\eta_i < \min\left(1, 1+\frac{2}{\pi}\right) = 1$, then the roots of f_1 $(x, x)$and $f_2(x, x)$ in $[0,1]$ are $\theta = 0$, $\theta = 1$. If $\eta_i > \max\left(1, 1+\frac{2}{\pi}\right) = 1+\frac{2}{\pi}$, then the roots of $f_1(x, x)$ and $f_2(x, x)$ in $[0,1]$ are $\theta = 0$, $\theta = 1$ and $\theta = \theta^*$, where $\theta^* = 1 - \frac{2}{\pi(\eta_i - 1)}$.

To locate the roots of $f_1(x, \lambda)$, it is rearranged as:

$$f_1(x,\lambda) = \frac{1}{R(x)}\Big\{(x-\lambda)^2(B_0+\mu_i B_1) + (x-\lambda)h_i\left(\frac{4}{\pi}B_2 - 2(\eta_i B_2 + B_3)(1-\theta)\right) + h_i^2\left(\frac{4}{\pi}B_2(1-\theta) - (\eta_i B_2 + B_3)(1-\theta)^2\right)\Big\}.$$

The roots of $f_1(x, \lambda)$ are $\lambda_1^* = x + h_i\left(\frac{E-D}{C}\right)$ and $\lambda_2^* = x + h_i\left(\frac{E+D}{C}\right)$, where

$$C = 2(B_0 + \mu_i B_1),$$
$$D = \sqrt{\left(\frac{(4B_2 - 2\pi(\eta_i B_2 + B_3)(1-\theta))}{\pi}\right)^2 - 4(B_0+\mu_i B_1)(4B_2(1-\theta) - (\eta_i B_2 + B_3)(1-\theta)^2)},$$
$$E = \frac{(4B_2 - 2\pi(\eta_i B_2 + B_3)(1-\theta))}{\pi}.$$

The roots of $f_2(x, \lambda)$ are $\lambda^* = x_{i+1}$ and $\lambda^* = x_{i+1} - \frac{4h_i B_2}{\pi(\eta_i B_2 + B_3)}$.

1. For $\theta \in [0,1]$, $\theta^* \notin [0,1]$ and $\eta_i < 1$, $|g(x) - S(x)| \leqslant \frac{1}{2}\|g^{(3)}(\lambda)\|h_i^3\varpi_1(\mu_i, \eta_i, \theta)$,

$$\varpi_1 = \int_{x_i}^{x} |f_1(x,\lambda)|d\lambda + \int_{x}^{x_{i+1}} |f_2(x,\lambda)|d\lambda = \int_{x_i}^{x} f_1(x,\lambda)d\lambda + \int_{x}^{x_{i+1}} f_2(x,\lambda)d\lambda = \frac{h_i^3}{R(x)}\Big\{\frac{2B_2^2 - \pi(1-\theta)(\eta_i B_2 + B_3)\theta^2}{\pi} + \frac{4B_2(1-\theta)\theta - \pi(\eta_i B_2 + B_3)(1-\theta)^2\theta}{\pi} + \frac{(B_0+\mu_i B_1)\theta^3}{3} + \frac{(1-\theta)^2(6B_2 - \pi(\eta_i B_2 + B_3)(1-\theta))}{3\pi}\Big\}.$$

2. For $\theta \leqslant \theta^*$, $\theta^* \in [0,1]$ and $\eta_i > 1+\frac{2}{\pi}$, $|g(x) - S(x)| \leqslant \frac{1}{2}\|g^{(3)}(\lambda)\|h_i^3\varpi_2(\mu_i, \eta_i, \theta)$,

$$\varpi_2 = \int_{x_i}^{x} |f_1(x,\lambda)|d\lambda + \int_{x}^{x_{i+1}} |f_2(x,\lambda)|d\lambda = -\int_{x_i}^{\lambda_1^*} f_1(x,\lambda)d\lambda + \int_{\lambda_1^*}^{x} f_1(x,\lambda)d\lambda - \int_{x}^{\lambda^*} f_2(x,\lambda)d\lambda + \int_{\lambda^*}^{x_{i+1}} f_2(x,\lambda)d\lambda$$
$$= \frac{h_i^3}{R(x)}\Big\{-\frac{2(B_0+\mu_i B_1)}{3}\left(\frac{E-D}{C}\right)^3 + \frac{(4B_2 - 2\pi(1-\theta)(\eta_i B_2 + B_3))}{\pi}\left(\frac{E-D}{C}\right)^2 - \left(\frac{8B_2(1-\theta) - 2\pi(\eta_i B_2 + B_3)(1-\theta)^2}{\pi}\right)\left(\frac{E-D}{C}\right) - \frac{(B_0+\mu_i B_1)\theta^3}{3} - \left(\frac{2B_2 - (1-\theta)(\eta_i B_2 + B_3)}{\pi}\right)\theta^2 - \left(\frac{4B_2(1-\theta) - (\eta_i B_2 + B_3)(1-\theta)^2}{\pi}\right)\theta + \frac{64B_2^3}{3\pi^3(\eta_i B_2 + B_3)^2} - \frac{2B_2(1-\theta)^2}{\pi} + \frac{(1-\theta)^3(\eta_i B_2 + B_3)}{3}\Big\}.$$

3. For $\theta \geqslant \theta^*$, $\theta^* \in [0,1]$ and $\eta_i > 1+\frac{2}{\pi}$, $|g(x) - S(x)| \leqslant \frac{1}{2}\|g^{(3)}(\lambda)\|h_i^3\varpi_3(\mu_i, \eta_i, \theta)$,

$$\varpi_3 = \int_{x_i}^{x} |f_1(x,\lambda)|d\lambda + \int_{x}^{x_{i+1}} |f_2(x,\lambda)|d\lambda = \int_{x_i}^{\lambda_1^*} f_1(x,\lambda)d\lambda - \int_{\lambda_1^*}^{\lambda_2^*} f_2(x,\lambda)d\lambda + \int_{\lambda_2^*}^{x} f_1(x,\lambda)d\lambda + \int_{x}^{x_{i+1}} f_2(x,\lambda)d\lambda$$
$$= \frac{h_i^3}{R(x)}\Big\{\frac{2(B_0+\mu_i B_1)}{3}\left(\frac{E-D}{C}\right)^3 - \left(\frac{4B_2 - 2\pi(1-\theta)(\eta_i B_2 + B_3)}{\pi}\right)\left(\frac{E-D}{C}\right)^2 + \left(\frac{8B_2(1-\theta) - 2\pi(\eta_i B_2 + B_3)(1-\theta)^2}{\pi}\right)\left(\frac{E-D}{C}\right) - \frac{2(B_0+\eta_i B_1)}{3}\left(\frac{E+D}{C}\right)^3 + \left(\frac{4B_2 - 2\pi(1-\theta)(\eta_i B_2 + B_3)}{\pi}\right)\left(\frac{E+D}{C}\right)^2 + \left(\frac{8B_2(1-\theta) - 2\pi(\eta_i B_2 + B_3)(1-\theta)^2}{\pi}\right) \times \left(\frac{E+D}{C}\right) + \frac{(B_0+\mu_i B_1)\theta^3}{3} + \left(\frac{4B_2(1-\theta) - \pi(\eta_i B_2 + B_3)(1-\theta)^2}{\pi}\right)\theta + \left(\frac{6B_2 - \pi(1-\theta)(\eta_i B_2 + B_3)}{3\pi}\right)(1-\theta)^2 + \left(\frac{2B_2 - \pi(1-\theta)(\eta_i B_2 + B_3)}{\pi}\right)\theta^2\Big\}. \qquad \square$$

Remark 1. It is of great interest that for suitably selected parameters μ_i and η_i, the piecewise rational quadratic trigonometric spline $S(x)$ is curvature continuous at each break point if $S^{(2)}(x_i-) = S^{(2)}(x_i+)$ or $S^{(2)}_{i-1}(x_i-)\Big|_{\delta_i=\frac{\pi}{2}} = S^{(2)}_i(x_i+)|_{\delta_i=0}$.

The above condition gives the following system of equations:

$$2h_i d_{i-1} + \{(4\eta_{i-1} - 4)h_i + (4\mu_i - 4)h_{i-1}\}d_i + 2h_{i-1}d_{i+1} = \pi\{\mu_{i-1}h_i\Delta_{i-1} + \eta_i h_{i-1}\Delta_i\}, \quad i = 1, 2, 3, \ldots, n-1. \tag{7}$$

Eq. (7) represents system of $(n-1)$ equations. If the derivative parameters d_i, $i = 0,1,2,\ldots,n$ are unknowns and shape design parameters (μ_i, η_i, $i = 0,1,2,\ldots,n$) are known then it represents system of $(n-1)$ equations in $(n+1)$ unknowns d_i, $i = 0,1,2,\ldots,n$. If the shape design parameters are also unknowns, then there are $(3n+1)$ unknowns. The unique solution can be determined by applying the end conditions.

Theorem 3. *For a positive data, the rational quadratic trigonometric spline defined in* Eq. (5) *preserves positivity if the shape parameters* μ_i *and* η_i *in each subinterval* $[x_i, x_{i+1}]$ *satisfy the following conditions:*

$$\mu_i > \max\left\{0, -\frac{2h_i d_i}{\pi f_i}\right\} \quad \text{and} \quad \eta_i > \max\left\{0, \frac{2h_i d_{i+1}}{\pi f_{i+1}}\right\}.$$

Proof. Assume that the given set of positive data be $\{(x_0, f_0), (x_1, f_1), \ldots, (x_n, f_n)\}$, $x_i < x_{i+1}$, $i = 0,1,2,\ldots,n-1$ and $f_i > 0$, $\forall i$. The curve produced by the interpolation of positive data will be positive if $S(x) > 0$, for all $x \in [x_0, x_n]$. Since each of the $B_i(x)$, $k = 0, 1, 2, 3$ is positive for $\delta_i \in \left[0, \frac{\pi}{2}\right]$ and for all $\mu_i > 0$, $\eta_i > 0$, so, $R(x)$ is positive over the whole domain. Now the problem of positivity of $S(x)$ has been reduced to the positivity of

$$B_0(x)f_i + B_1(x)\left(\mu_i f_i + \frac{2h_i d_i}{\pi}\right) + B_2(x)\left(\eta_i f_{i+1} - \frac{2h_i d_{i+1}}{\pi}\right) + B_3(x)f_{i+1},$$

$$B_0(x)f_i + B_1(x)\left(\mu_i f_i + \frac{2h_i d_i}{\pi}\right) + B_2(x)\left(\eta_i f_{i+1} - \frac{2h_i d_{i+1}}{\pi}\right) + B_3(x)f_{i+1} > 0 \text{ only if } \mu_i > -\frac{2h_i d_i}{\pi f_i} \quad \text{and} \quad \eta_i > \frac{2h_i d_{i+1}}{\pi f_{i+1}}.$$

Combining $\mu_i > 0$, $\eta_i > 0$, $\mu_i > -\frac{2h_i d_i}{\pi f_i}$ and $\eta_i > \frac{2h_i d_{i+1}}{\pi f_{i+1}}$, we have the required result.

4. Bivariate rational quadratic trigonometric function

Let $\{(x_i, y_j, F_{i,j}), i = 0,1,2,\ldots,n_1; j = 0,1,2,\ldots,n_2\}$ be the given set of regular data arranged over the rectangular grid $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$, $i = 0,1,2,\ldots,n_1-1$; $j = 0,1,2,\ldots,n_2-1$. We wish to interpolate these data by using rational quadratic trigonometric spline (5). Since each rectangle is bounded by four boundary curves. The final surface patch is obtained by blending these boundary curves. Interpolation and blending of boundary curves by rational quadratic trigonometric spline defined in Eq. (5) give birth to the following bivariate rational quadratic trigonometric function over each rectangular patch.

$$U(x,y) = \frac{(1-\sin(\varphi_j))^2 D_0 + (1-\sin(\varphi_j))\sin(\varphi_j)D_1 + (1-\cos(\varphi_j))\cos(\varphi_j)D_2 + (1-\cos(\varphi_j))^2 D_3}{\widehat{Q}_{i,j}(\varphi_j)}, \tag{8}$$

$$\begin{aligned}
&D_0 = U(x, y_j), \quad D_1 = \hat{\mu}_{i,j}U(x,y_j) + \hat{h}_j U_x(x,y_j), \quad D_2 = \hat{\eta}_{i,j}U(x,y_{j+1}) - \hat{h}_j U_x(x,y_{j+1}),\\
&D_3 = U(x,y_{j+1}), \quad \varphi_j = \frac{\pi(y-y_j)}{2\hat{h}_j}, \quad \hat{h}_j = y_{j+1} - y_j,\\
&\widehat{Q}_{i,j}(\varphi_j) = \left(1-\sin(\varphi_j)\right)^2 + \hat{\mu}_{i,j}\left(1-\sin(\varphi_j)\right)\sin(\varphi_j) + \hat{\eta}_{i,j}(1-\cos(\varphi_j))\cos(\varphi_j) + (1-\cos(\varphi_j))^2,
\end{aligned} \tag{9}$$

$$\begin{aligned}
&U(x,y_j) = \frac{(1-\sin(\delta_i))^2 E_0 + (1-\sin(\delta_i))\sin(\delta_i)E_1 + (1-\cos(\delta_i))\cos(\delta_i)E_2 + (1-\cos(\delta_i))^2 E_3}{Q_{i,j}(\delta_i)},\\
&E_0 = F_{i,j}, \quad E_1 = \mu_{i,j}F_{i,j} + h_i F^x_{i,j}, \quad E_2 = \eta_{i,j}F_{i+1,j} - h_i F^x_{i+1,j}, \quad E_3 = F_{i+1,j},\\
&Q_{i,j}(\delta_i) = (1-\sin(\delta_i))^2 + \mu_{i,j}(1-\sin(\delta_i))\sin(\delta_i) + \eta_{i,j}(1-\cos(\delta_i))\cos(\delta_i) + (1-\cos(\delta_i))^2,\\
&\delta_i = \frac{\pi(x-x_i)}{2h_i} \text{ and } h_i = x_{i+1} - x_i.\\
&U_y(x,y_j) = \frac{(1-\sin(\delta_i))^2 G_0 + (1-\sin(\delta_i))\sin(\delta_i)G_1 + (1-\cos(\delta_i))\cos(\delta_i)G_2 + (1-\cos(\delta_i))^2 G_3}{Q_{i,j}(\delta_i)},\\
&G_0 = F^y_{i,j}, G_1 = \mu_{i,j}F^y_{i,j} + h_i F^{xy}_{i,j}, G_2 = \eta_{i,j}F^y_{i+1,j} - h_i F^{xy}_{i+1,j}, G_3 = F^y_{i+1,j}.
\end{aligned} \tag{10}$$

$U(x, y_{j+1})$ and $U_y(x, y_{j+1})$ are obtained by replacing j by $j+1$ in Eqs. (8) and (10).

$\mu_{i,j}$ and $\eta_{i,j}$ are free parameters along x-axis and $\hat{\mu}_{i,j}$ and $\hat{\eta}_{i,j}$ are free parameters along y-axis. $\left\{F^x_{k,l}, F^y_{k,l}, F^{xy}_{k,l} : k = i, i+1; l = j, j+1\right\}$ are the partial derivatives at four corners of (i, j)th-patch. The bivariate rational quadratic trigonometric function (8)) has the following properties:

$$U(x_i, y_j) = F_{i,j}, \quad \frac{\partial U(x_i, y_j)}{\partial x} = F^x_{i,j},$$
$$\frac{\partial U(x_i, y_j)}{\partial y} = F^y_{i,j}, \quad \frac{\partial^2 U(x_i, y_j)}{\partial x \partial y} = F^{xy}_{i,j}.$$

Theorem 4. *The piecewise bivariate rational quadratic trigonometric function $U(x, y)$ is C^1 over the whole domain if the shape design parameters satisfy the following relation:*

(i) $\mu_{i,j} = \mu_i$ *and* $\eta_{i,j} = \eta_i$, $i = 0, 1, 2, \ldots, n_1 - 1$ *and for all values of* j.
(ii) $\hat{\mu}_{i,j} = \hat{\mu}_j$ *and* $\hat{\eta}_{i,j} = \hat{\eta}_j$, $j = 0, 1, 2, \ldots, n_2 - 1$ *and for all values of* i.

Proof. The rational quadratic trigonometric function (8) interpolates the data values $F_{i,j}$ and partial derivatives $F^x_{i,j}, F^y_{i,j}, F^{xy}_{i,j}$ defined at four corners of rectangular patch, i.e.

$$U(x_i, y_j) = F_{i,j}, \quad \frac{\partial U(x_i, y_j)}{\partial x} = F^x_{i,j}, \quad \frac{\partial U(x_i, y_j)}{\partial y} = F^y_{i,j},$$
$$\frac{\partial^2 U(x_i, y_j)}{\partial x \partial y} = F^{xy}_{i,j}.$$

Since each rectangular patch is bounded by four boundary curves so to blend the rectangular patches to generate a C^1 continuous surface following sufficient conditions must be satisfied along the four boundaries of each rectangular patch:

$$\left.\frac{\partial U_{i,j}(x_{i+1}, y)}{\partial x}\right|_{\delta_i=\frac{\pi}{2}} - \left.\frac{\partial U_{i+1,j}(x_{i+1}, y)}{\partial x}\right|_{\delta_{i+1}=0} = 0,$$
$$\left.\frac{\partial U_{i-1,j}(x_i, y)}{\partial x}\right|_{\delta_{i-1}=\frac{\pi}{2}} - \left.\frac{\partial U_{i,j}(x_i, y)}{\partial x}\right|_{\delta_i=0} = 0,$$
$$\left.\frac{\partial U_{i,j}(x, y_{j+1})}{\partial y}\right|_{\varphi_j=\frac{\pi}{2}} - \left.\frac{\partial U_{i,j+1}(x, y_{j+1})}{\partial y}\right|_{\varphi_{j+1}=0} = 0,$$

$$\left.\frac{\partial U_{i,j-1}(x, y_j)}{\partial y}\right|_{\varphi_{j-1}=\frac{\pi}{2}} - \left.\frac{\partial U_{i,j}(x, y_j)}{\partial y}\right|_{\varphi_j=0} = 0.$$

After some simple computation it is observed that $\left.\frac{\partial U_{i,j}(x_{i+1}, y)}{\partial x}\right|_{\delta_i=\frac{\pi}{2}} - \left.\frac{\partial U_{i+1,j}(x_{i+1}, y)}{\partial x}\right|_{\delta_{i+1}=0} = 0,$

$$\begin{aligned} \text{if } \hat{\mu}_{i+1,j} = \hat{\mu}_{i,j}, \hat{\eta}_{i+1,j} &= \hat{\eta}_{i,j}, F^x_{i+1,j}\left(\hat{\mu}_{i+1,j}\hat{\eta}_{i,j} - \hat{\eta}_{i+1,j}\hat{\mu}_{i,j}\right) \\ &\quad + \frac{2\hat{h}_j}{\pi} F^{xy}_{i+1,j}\left(\hat{\mu}_{i,j} - \hat{\mu}_{i+1,j}\right) \\ &= 0. \end{aligned} \tag{11}$$

$$\left.\frac{\partial U_{i-1,j}(x_i, y)}{\partial x}\right|_{\delta_{i-1}=\frac{\pi}{2}} - \left.\frac{\partial U_{i,j}(x_i, y)}{\partial x}\right|_{\delta_i=0} = 0$$
$$\begin{aligned} &\text{if } \hat{\mu}_{i,j} = \hat{\mu}_{i-1,j}, \quad \hat{\eta}_{i,j} = \hat{\eta}_{i-1,j}, \quad F^x_{i,j}\left(\hat{\mu}_{i,j}\hat{\eta}_{i-1,j} - \hat{\eta}_{i,j}\hat{\mu}_{i-1,j}\right) \\ &\quad + \frac{2\hat{h}_j}{\pi} F^{xy}_{i,j}\left(\hat{\mu}_{i-1,j} - \hat{\mu}_{i,j}\right) = 0. \end{aligned} \tag{12}$$

$$\left.\frac{\partial U_{i,j}(x, y_{j+1})}{\partial y}\right|_{\varphi_j=\frac{\pi}{2}} - \left.\frac{\partial U_{i,j+1}(x, y_{j+1})}{\partial y}\right|_{\varphi_{j+1}=0} = 0$$
$$\begin{aligned} &\text{if } \mu_{i,j+1} = \mu_{i,j}, \quad \eta_{i,j+1} = \eta_{i,j}, \quad F^y_{i,j+1}\left(\mu_{i,j+1}\eta_{i,j} - \eta_{i,j+1}\mu_{i,j}\right) \\ &\quad + \frac{2h_i}{\pi} F^{xy}_{i,j+1}\left(\mu_{i,j} - \mu_{i,j+1}\right) = 0, \end{aligned} \tag{13}$$
$$F^y_{i+1,j+1}(\mu_{i,j+1}\eta_{i,j} - \eta_{i,j+1}\mu_{i,j}) + \frac{2h_i}{\pi} F^{xy}_{i+1,j+1}(\mu_{i,j} - \mu_{i,j+1}) = 0.$$

$$\left.\frac{\partial U_{i,j-1}(x, y_j)}{\partial y}\right|_{\varphi_{j-1}=\frac{\pi}{2}} - \left.\frac{\partial U_{i,j}(x, y_j)}{\partial y}\right|_{\varphi_j=0} = 0 \text{ if}$$
$$\begin{aligned} &\mu_{i,j} = \mu_{i,j-1}, \quad \eta_{i,j} = \eta_{i,j-1}, \quad F^y_{i,j}(\mu_{i,j}\eta_{i,j-1} - \eta_{i,j}\mu_{i,j-1}) \\ &\quad + \frac{2h_i}{\pi} F^{xy}_{i,j}(\mu_{i,j-1} - \mu_{i,j}) = 0, \end{aligned} \tag{14}$$
$$F^y_{i+1,j}\left(\mu_{i,j}\eta_{i,j-1} - \eta_{i,j}\mu_{i,j-1}\right) + \frac{2h_i}{\pi} F^{xy}_{i+1,j}(\mu_{i,j-1} - \mu_{i,j})0.$$

The system of Eqs. (11)–(14) is satisfied only if $\mu_{i,j} = \mu_i$ and $\eta_{i,j} = \eta_i$, $i = 0, 1, 2, \ldots, n_1 - 1$ and for all values of j. $\hat{\mu}_{i,j} = \hat{\mu}_j$ and $\hat{\eta}_{i,j} = \hat{\eta}_j$, $j = 0, 1, 2, \ldots, n_2 - 1$ and for all values of i.

Theorem 5. *The C^1 bivariate rational quadratic trigonometric function $U(x, y)$ is positive over the whole domain if the following sufficient conditions are satisfied:*

$$U(x_i, y_j) = F_{i,j}, \quad \forall i = 0, 1, 2, \ldots, n_1; \quad j = 0, 1, 2, \ldots, n_2;$$

$$\mu_i > \max\left\{0, -\frac{2h_i F^x_{i,j}}{\pi F_{i,j}}, -\frac{2h_i F^x_{i,j+1}}{\pi F_{i,j+1}}\right\},$$
$$\eta_i > \max\left\{0, \frac{2h_i F^x_{i+1,j}}{\pi F_{i+1,j}}, \frac{2h_i F^x_{i+1,j+1}}{\pi F_{i+1,j+1}}\right\}$$

$$\hat{\mu}_j > \max\left\{0, -\frac{2\hat{h}_j F^y_{i,j}}{\pi F_{i,j}}, -\frac{2\hat{h}_j F^y_{i+1,j}}{\pi F_{i+1,j}}, -\frac{2\hat{h}_j\left(\pi\mu_i F^y_{i,j} + 2h_i F^{xy}_{i,j}\right)}{\pi\left(\mu_i F_{i,j} + 2h_i F^x_{i,j}\right)}, -\frac{2\hat{h}_j\left(\pi\eta_i F^y_{i+1,j} - 2h_i F^{xy}_{i+1,j}\right)}{\pi\left(\pi\eta_i F_{i+1,j} - 2h_i F^x_{i+1,j}\right)}\right\},$$

$$\hat{\eta}_j > \max\left\{0, \frac{2\hat{h}_j F^y_{i,j+1}}{\pi F_{i,j+1}}, \frac{2\hat{h}_j F^y_{i+1,j+1}}{\pi F_{i+1,j+1}}, \frac{2\hat{h}_j\left(\pi\mu_i F^y_{i,j+1} + 2h_i F^{xy}_{i,j+1}\right)}{\pi\left(\pi\mu_i F_{i,j+1} + 2h_i F^x_{i,j+1}\right)}, \frac{2\hat{h}_j\left(\pi\eta_i F^y_{i+1,j+1} - 2h_i F^{xy}_{i+1,j+1}\right)}{\pi\left(\pi\eta_i F_{i+1,j+1} - 2h_i F^x_{i+1,j+1}\right)}\right\}.$$

Proof. Let $\{(x_i, y_j, F_{i,j}): i = 0, 1, 2, \ldots, n_1; j = 0, 1, 2, \ldots, n_2\}$ be the given set of positive regular data defined over the domain $D = [c, d] \times [e, f]$. The requirement is to develop an interpolating C^1 bivariate positive rational quadratic trigonometric function, i.e. $U(x_i, y_j) = F_{i,j}$, $i = 0, 1, 2, \ldots, n_1$; $j = 0, 1, 2, \ldots, n_2$, and $U(x, y) > 0$, for all $(x, y) \in D$. The C^1 bivariate rational quadratic trigonometric function (8) is rearranged as:

$$U(x,y) = \times \frac{(1-\sin(\delta_i))^2 P_0 + (1-\sin(\delta_i))\sin(\delta_i)P_1 + (1-\cos(\delta_i))\cos(\delta_i)P_2 + (1-\cos(\delta_i))^2 P_3}{(1-\sin(\delta_i))^2 + \mu_i(1-\sin(\delta_i))\sin(\delta_i) + \eta_i(1-\cos(\delta_i))\cos(\delta_i) + (1-\cos(\delta_i))^2},$$

$$P_0 = (1-\sin(\varphi_j))^2 T_{0,0} + (1-\sin(\varphi_j))\sin(\varphi_j)T_{0,1} + (1-\cos(\varphi_j))\cos(\varphi_j)T_{0,2} + (1-\cos(\varphi_j))^2 T_{0,3},$$

$$P_1 = \left(1-\sin(\varphi_j)\right)^2 T_{1,0} + \left(1-\sin(\varphi_j)\right)\sin(\varphi_j)T_{1,1} + (1-\cos(\varphi_j))\cos(\varphi_j)T_{1,2} + (1-\cos(\varphi_j))^2 T_{1,3},$$

$$P_2 = (1-\sin(\varphi_j))^2 T_{2,0} + (1-\sin(\varphi_j))\sin(\varphi_j)$$

$$T_{2,1} + (1-\cos(\varphi_j))\cos(\varphi_j)T_{2,2} + (1-\cos(\varphi_j))^2 T_{2,3}$$

$$P_3 = (1-\sin(\varphi_j))^2 T_{3,0} + (1-\sin(\varphi_j))\sin(\varphi_j)$$

$$T_{3,1} + (1-\cos(\varphi_j))\cos(\varphi_j)T_{3,2} + (1-\cos(\varphi_j))^2 T_{3,3}$$

$$T_{0,0} = F_{i,j}, \quad T_{0,1} = \hat{\mu}_j F_{i,j} + \frac{2\hat{h}_j}{\pi} F^y_{i,j},$$

$$T_{0,2} = \hat{\eta}_j F_{i,j+1} - \frac{2\hat{h}_j}{\pi} F^y_{i,j+1}, \quad T_{0,3} = F_{i,j+1},$$

$$T_{1,0} = \mu_i F_{i,j} + \frac{2h_i}{\pi} F^x_{i,j},$$

$$T_{1,1} = \hat{\mu}_j\left(\mu_i F_{i,j} + \frac{2h_i}{\pi} F^x_{i,j}\right) + \frac{2\hat{h}_j}{\pi}\left(\mu_i F^y_{i,j} + \frac{2h_i}{\pi} F^{xy}_{i,j}\right),$$

$$T_{1,2} = \hat{\eta}_j\left(\mu_i F_{i,j+1} + \frac{2h_i}{\pi} F^x_{i,j+1}\right) - \frac{2\hat{h}_j}{\pi}\left(\mu_i F^y_{i,j+1} + \frac{2h_i}{\pi} F^{xy}_{i,j+1}\right),$$

Table 1 A 2D positive data set.

x	1.0	2.0	3.0	8.0	10.0	11.0	12.0	14.0
y	14.0	8.0	2.0	0.8	0.5	0.25	0.4	0.37

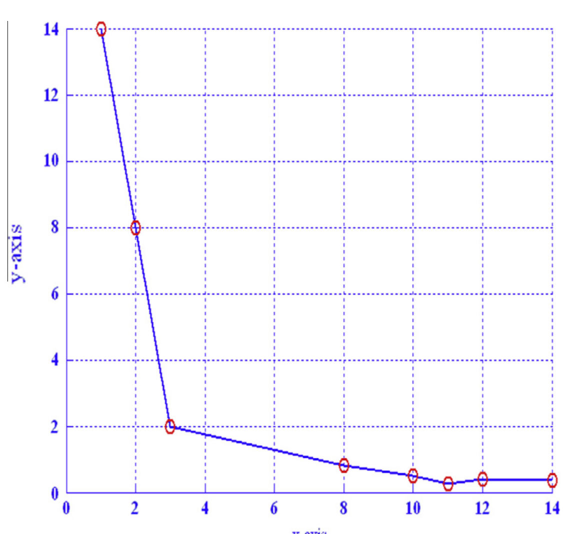

Figure 1 Linear interpolation of data.

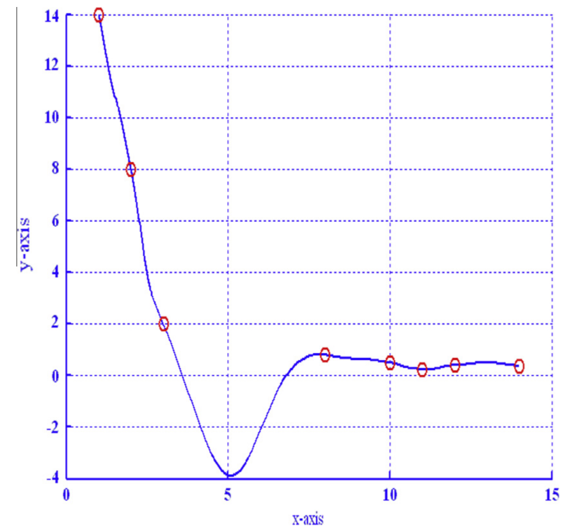

Figure 2 Rational quadratic trigonometric spline with $\mu_i = 0.4$ and $\eta_i = 0.6$.

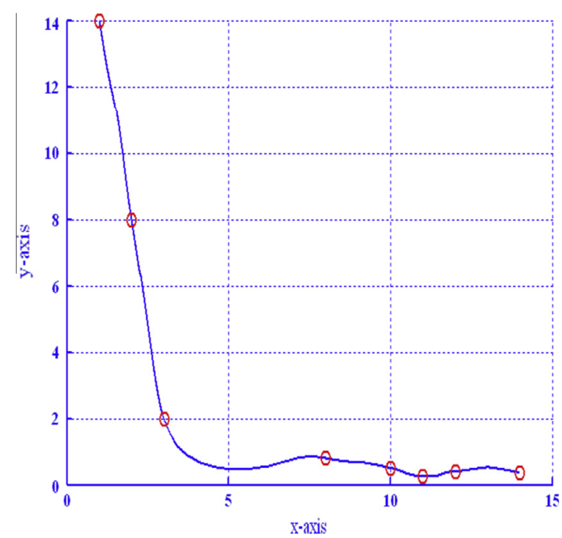

Figure 3 Positive rational quadratic trigonometric spline.

$$T_{1,3} = \mu_i F_{i,j+1} + \frac{2h_i}{\pi} F^x_{i,j+1}, T_{2,0} = \eta_i F_{i+1,j} - \frac{2h_i}{\pi} F^x_{i+1,j},$$

$$T_{2,1} = \hat{\mu}_j\left(\eta_i F_{i+1,j} - \frac{2h_i}{\pi} F^x_{i+1,j}\right) + \frac{2\hat{h}_j}{\pi}\left(\eta_i F^y_{i+1,j} - \frac{2h_i}{\pi} F^{xy}_{i+1,j}\right),$$

$$T_{2,2} = \hat{\eta}_j\left(\eta_i F_{i+1,j+1} - \frac{2h_i}{\pi} F^x_{i+1,j+1}\right) - \frac{2\hat{h}_j}{\pi}\left(\eta_i F^y_{i+1,j+1} - \frac{2h_i}{\pi} F^{xy}_{i+1,j+1}\right),$$

$$T_{2,3} = \eta_i F_{i+1,j+1} - \frac{2h_i}{\pi} F^x_{i+1,j+1},$$

$$T_{3,0} = F_{i+1,j}, \quad T_{3,1} = \hat{\mu}_j F_{i+1,j} + \frac{2\hat{h}_j}{\pi} F^y_{i+1,j},$$

$$T_{3,2} = \hat{\eta}_j F_{i+1,j+1} - \frac{2\hat{h}_j}{\pi} F^y_{i+1,j+1}, \quad T_{3,3} = F_{i+1,j+1}.$$

Table 2 A 3D positive data set.

y/x	-3	-2	-1	0	1	2	3
-3	0.0123	0.0236	0.0399	0.0493	0.0399	0.0236	0.0123
-2	0.0236	0.0624	0.1599	0.2499	0.1599	0.0624	0.0236
-1	0.0399	0.1599	0.9999	3.9996	0.9999	0.1599	0.0399
0	0.4444	0.2499	3.9996	40,000	3.9996	0.2499	0.4444
1	0.0399	0.1599	0.9999	3.9996	0.9999	0.1599	0.0399
2	0.0236	0.0624	0.1599	0.2499	0.1599	0.0624	0.0236
3	0.0123	0.0236	0.0399	0.0493	0.03999	0.0236	0.0123

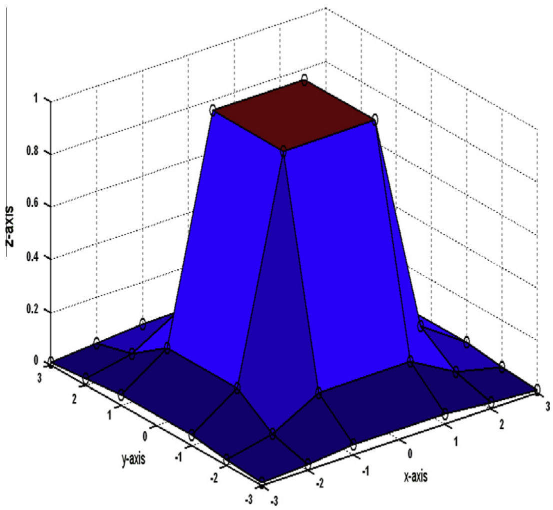

Figure 4 Linear interpolation of positive data.

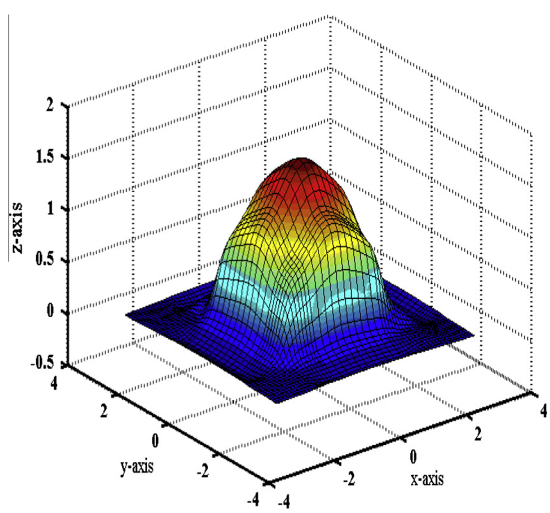

Figure 5 C^1 bivariate rational quadratic trigonometric function with ($\mu_i = 0.6$, $\eta_i = 2$, $\hat{\mu}_j = 0.8$ and $\hat{\eta}_j = 2.5$).

Since the shape design parameters μ_i and η_i are assumed as positive real numbers. Moreover, the parameters δ_i and φ_j are restricted to subinterval $\left[0, \frac{\pi}{2}\right]$. The denominator of $U(x, y)$ is always positive. Thus, positivity of $U(x, y)$ depends upon the positivity of P_k, $k = 0, 1, 2, 3$.

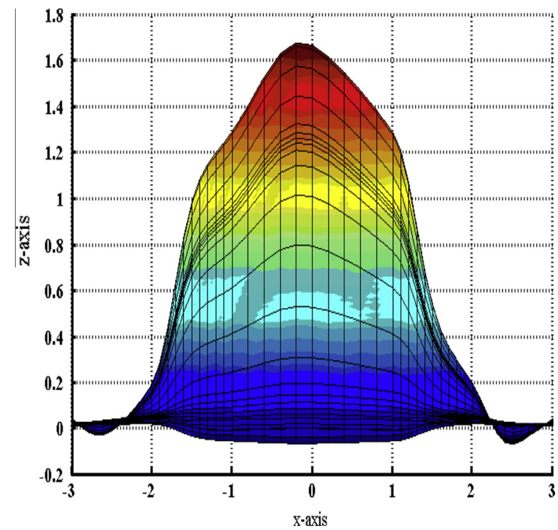

Figure 6 *xz*-view of Fig. 5.

$P_0 > 0$ is positive if $T_{0,k} > 0$, $k = 0, 1, 2, 3$.

$T_{0,k} > 0$, $k = 0, 1, 2, 3$ if $\hat{\mu}_j > -\frac{2\hat{h}_j F^y_{i,j}}{\pi F_{i,j}}$ and $\hat{\eta}_j > \frac{2\hat{h}_j F^y_{i,j+1}}{\pi F_{i,j+1}}$.

Similarly, $P_1 > 0$ if

$$\mu_i > -\frac{2h_i F^x_{i,j}}{\pi F_{i,j}}, \quad \mu_i > -\frac{2h_i F^x_{i,j+1}}{\pi F_{i,j+1}},$$

$$\hat{\mu}_j > -\frac{2\hat{h}_j\left(\pi\mu_i F^y_{i,j} + 2h_i F^{xy}_{i,j}\right)}{\pi\left(\pi\mu_i F_{i,j} + 2h_i F^x_{i,j}\right)}, \quad \hat{\eta}_j > \frac{2\hat{h}_j\left(\pi\mu_i F^y_{i,j+1} + 2h_i F^{xy}_{i,j+1}\right)}{\pi\left(\pi\mu_i F_{i,j+1} + 2h_i F^x_{i,j+1}\right)}.$$

Similarly for P_2 and P_3 are positive if

$$\eta_i > \frac{2h_i F^x_{i+1,j}}{\pi F_{i+1,j}}, \quad \eta_i > \frac{2h_i F^x_{i+1,j+1}}{\pi F_{i+1,j+1}}, \quad \hat{\eta}_j > \frac{2\hat{h}_j F^y_{i+1,j+1}}{\pi F_{i+1,j+1}},$$

$$\hat{\mu}_j > -\frac{2\hat{h}_j F^y_{i+1,j}}{\pi F_{i+1,j}}, \quad \hat{\mu}_j > -\frac{2\hat{h}_j\left(\pi\eta_i F^y_{i+1,j} - 2h_i F^{xy}_{i+1,j}\right)}{\pi\left(\pi\eta_i F_{i+1,j} - 2h_i F^x_{i+1,j}\right)},$$

$$\hat{\eta}_j > \frac{2\hat{h}_j\left(\pi\eta_i F^y_{i+1,j+1} - 2h_i F^{xy}_{i+1,j+1}\right)}{\pi\left(\pi\eta_i F_{i+1,j+1} - 2h_i F^x_{i+1,j+1}\right)}.$$

5. Numerical examples

Example 1. Here rational quadratic trigonometric spline interpolation of positive data set $\{(x, y)$: $(1.0, 14.0)$, $(2.0, 8.0)$,

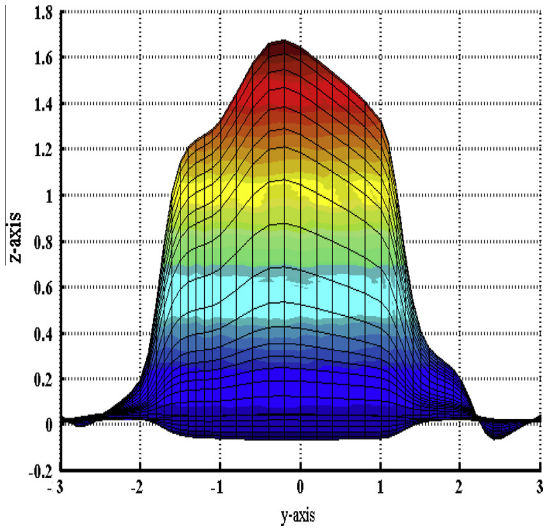

Figure 7 yz-view of Fig. 5.

(3.0, 2.0), (8.0, 0.8), (10.0, 0.5), (11.0, 0.25), (12.0, 0.4), (14.0, 0.37)} is discussed. The tabular form of this data set is (see Table 1)

This data is linearly interpolated by MATLAB build in function **plot** in Fig. 1 which shows that its shape is positive over whole domain. In Fig. 2, the same data are interpolated by rational quadratic trigonometric spline with $\mu_i = 0.4$ and $\eta_i = 0.6$. This figure clearly indicates that rational quadratic trigonometric spline is unable to preserve the positive shape of data for arbitrary values of shape design parameters. The strength of Theorem 3 is checked by implementing it on these data shown in Fig. 3 which reveals that rational quadratic trigonometric spline interpolates positive shape of data positively if the shape design parameters obey the restrictions developed in Theorem 4.

Example 2. The regular positive surface data are generated from the square function $h(x,y) = \frac{4}{(x^2+y^2)^2+0.0001}$. Although this function is positive over every domain but for the ease of computation the domain is restricted to $S_1 \times S_2$, where $S_1 = S_2 = \{-3,-2,-1,0,1,2,3\}$. The tabular form of this positive data set is (see Table 2)

The positive shape of the data is produced in Fig. 4 by its linear interpolation. C^1 bivariate rational quadratic trigonometric function for arbitrary values of shape design parameters, is unable to preserve the positive shape of the data. It is exposed in Fig. 5 ($\mu_i = 0.6$, $\eta_i = 2$, $\hat{\mu}_j = 0.8$ and $\hat{\eta}_j = 2.5$), Fig. 6(xz-view) and Fig. 7(yz-view). Fig. 8 is the test of Theorem 7 on this data. The surface produced in Fig. 8 is positive over the whole domain.

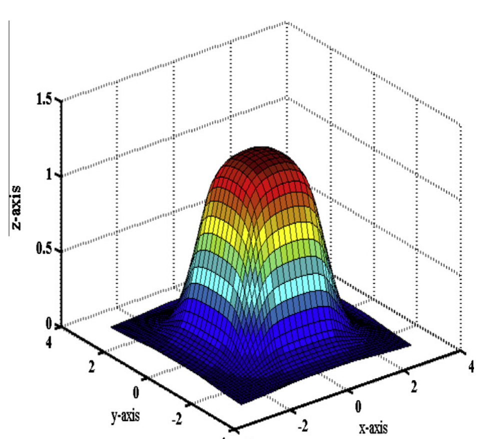

Figure 8 Positive C^1 bivariate rational quadratic trigonometric function.

6. Conclusion

In this study, a Bézier like rational quadratic trigonometric spline is developed to assure C^1-continuity in rational quadratic structure. The shape preserving schemes proposed in [13] was C^1 for uniform knot, [17] was C^0. In [14,15] order of continuity was dependent on multiplicity of knot and shape parameters, in [19] C^1-continuity was dependent on knot vectors and choice of derivatives, [20] restricted the derivatives to Δ_i for tangent continuity. The data arising in most of the applications are non-uniform and does not restrict the derivatives; hence, these schemes are not applicable to a wide range of functions where derivative preservation is also mandatory. The order of continuity of rational quadratic trigonometric spline of this paper is independent of knot spacing, slope of secant line and shape parameters.

The developed curve and regular surface data interpolants are likely to preserve the shape of data, unlike [1,4,5,11,12,18,19,21], without constraining the interval and derivatives.

The order of approximation of the developed scheme is $O(h_i^3)$ in quadratic structure, whereas the order of approximation of rational interpolant used in [17] was $O(h_i^2)$.

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