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Eilenberg-Kelly Reloaded

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Abstract

The Eilenberg-Kelly theorem states that a category $\mathbb{C}$ with an object I and two functors $\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ and $\multimap\colon \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{C}$ related by an adjunction $- \otimes B \dashv B \multimap -$ natural in B is monoidal iff it is closed and moreover the adjunction holds internally. We dissect the proof of this theorem and observe that the necessity for a side condition on closedness arises because the standard definition of closed category is left-skew in regards to associativity. We analyze Street's observation that left-skew monoidality is equivalent to left-skew closedness and establish that monoidality is equivalent to closedness unconditionally under an adjusted definition of closedness that requires normal associativity. We also work out a definition of right-skew closedness equivalent to right-skew monoidality. We give examples of each type of structure; in particular, we look at the Kleisli category of a left-strong monad on a left-skew closed category and the Kleisli category of a lax closed monad on a right-skew closed category. We also view skew and normal monoidal and closed categories as special cases of skew and normal promonoidal categories and take a brief look at left-skew prounital-closed categories.

Keywords: skew and normal monoidal, closed, monoidal closed and bi-closed categories, Eilenberg-Kelly theorem, promonoidal categories, Kleisli construction

1 Introduction

The closed categories of Eilenberg and Kelly [13] are categories with a unit object I and an internal function space $A \multimap B$ between objects A and B. Examples include various categories of structured sets, such as normal bands and posets, where $A \multimap B$

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is the set of normal band morphisms (resp. monotone functions) between A and B and its normal band (resp. poset) structure is obtained from that of B by pointwise lifting. The singleton set with the trivial structure plays the role of the unit I in this case. Categories underlying various deductive systems, such as simply-typed λ-calculus, form another class of examples. The internal function space $A \multimap B$ is the type of functions between A and B, while the external function space consists of well-typed terms $x : A \vdash t : B$ with one free variable. In this case, the object I is the unit type.

In a large number of examples, the closed structure $(\mathsf{I}, \multimap)$ arises from an adjunction with the tensor product in a monoidal category $(\mathbb{C}, \mathsf{I}, \otimes)$ [2,23], but the original definition of closed category of Eilenberg and Kelly does not require the presence of a monoidal structure in the category. However, it is true that, in case the category $\mathbb{C}$ comes equipped with a unit I and two functors $\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ and $\multimap : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{C}$ related by an adjunction $- \otimes B \dashv B \multimap -$ natural in B, then $(\mathbb{C}, \mathsf{I}, \otimes)$ is monoidal if and only if $(\mathbb{C}, \mathsf{I}, \multimap)$ is closed and moreover the adjunction holds *internally*. This reveals that closedness of $\mathbb{C}$ is for some reason weaker than monoidality. On a closer look, the internal adjunction requirement is necessary for the associator α to be an isomorphism, as demanded in the definition of monoidal category, a condition which is not matched up by anything in the definition of closed category.

For properly understanding the reason behind this imbalance in the connection between monoidal and closed structures, it is helpful to start considering weak variants of monoidal and closed categories: the left-skew monoidal categories of Szlachányi [30] and the left-skew closed categories of Street [29].

Left-skew monoidal categories are a weakening of monoidal categories where the unitors and associators are not required to be isomorphisms but merely transformations in a particular direction. Skew monoidal categories arise naturally in many settings, for example in the study of relative monads [1] and of quantum categories [17], and have been thoroughly investigated by Street, Lack and colleagues [18,7,4,5]. In previous work, we contributed to the study of coherence for left-skew monoidal categories, first taking a rewriting approach [31] and later using proof-theoretic techniques [35,32].

Left-skew closed categories are analogous weak variants of closed categories. Street [29] proved a variant of the Eilenberg-Kelly theorem, stating that, given an adjunction $- \otimes B \dashv B \multimap -$ natural in B, $(\mathbb{C}, \mathsf{I}, \otimes)$ is left-skew monoidal if and only if $(\mathbb{C}, \mathsf{I}, \multimap)$ is left-skew closed. Crucially, the internal adjunction requirement from the original Eilenberg-Kelly result is no longer present.

In this paper, we continue on the line of work initiated by Street.

Left-skew monoidal categories differ from normal monoidal categories in that the two unitors and the associator are not invertible. When one or more of the structural laws is invertible, the resulting category is "more normal", in the sense that its monoidal structure is less skew and it looks more like a usual monoidal category. Similar degrees of normality sit in between left-skew closed and properly closed categories as well. We prove a refined version of Street's left-skew variant of

Eilenberg-Kelly theorem: not only there exists an isomorphism between left-skew monoidal $(\mathsf{I}, \otimes)$ and left-skew closed $(\mathsf{I}, \multimap)$ structures on $\mathbb{C}$ in the presence of an adjunction $- \otimes B \dashv B \multimap -$, but each normality condition on $(\mathsf{I}, \otimes)$ corresponds to a normality condition on $(\mathsf{I}, \multimap)$. This allows us to extrapolate a requirement on the closed structure corresponding to the invertibility of the associator α, which is equivalent to the existence of the internal adjunction but it is entirely expressible using the closed structure and can be formulated also when the category $\mathbb{C}$ is lacking a monoidal structure. This associative-normality condition for left-skew closed categories was first discovered by Day and Laplaza [10,12], but it has gone unnoticed in later literature.

Normality for left-skew closed categories corresponds to the invertibility of its structural laws. By dropping these laws while keeping their inverses (so the new laws point in the opposite direction compared to the original ones), we obtain the new notion of right-skew closed category. We prove a right-skew variant of Street's theorem connecting adjoint right-skew monoidal and right-skew closed structures on a category, and similar relationships between their normality conditions. More interestingly, we observe that left-skew closed $(\mathsf{I}, \multimap^{\mathsf{L}})$ and right-skew closed $(\mathsf{I}, \multimap^{\mathsf{R}})$ structures on a category $\mathbb{C}$ are in a bijective correspondence in case $\mathbb{C}$ is equipped with an isomorphism $\sigma_{A,B,C} : \mathbb{C}(A, B \multimap^{\mathsf{R}} C) \to \mathbb{C}(B, A \multimap^{\mathsf{L}} C)$ natural in A,B and C. We call the latter requirement the (external) Lambek condition since it resembles a characteristic feature of Lambek's syntactic calculus [19].

Throughout the paper we discuss a large number of examples, in particular for motivating the different normality conditions and the new notion of right-skew closed category. We show that the Kleisli category of a left-strong monad on a left-skew closed category inherits the left-skew closed structure. We also see how the right-skew closed structure on a category $\mathbb{C}$ can be lifted to the Kleisli category of a lax closed monad on $\mathbb{C}$. We discuss possible ways of skewing a skew closed structure further: using a lax closed comonad, we can skew a left-closed structure further to the left (an example also discussed by Street [29]) while using an oplax closed monad, we can skew a right-closed structure further to the right. Another example of a left-skew closed category, in fact a presentation of the free left-skew closed category on a set of generators, is given by the non-commutative linear typed λ-calculus with the unit type.

Building on an old observation by Day [10], Street noticed that both left-skew monoidal and left-skew closed categories arise as specific instances of the general notion of left-skew promonoidal category [29], in which the unit I and functor $\multimap$ are respectively replaced by a functor $J : \mathbb{C} \to \mathbf{Set}$ and a distributor $P : \mathbb{C}^{\mathsf{op}} \times \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbf{Set}$. By choosing $P(A, B; C) = \mathbb{C}(A, B \multimap C)$ but dropping the representability condition on J, one obtains another natural variation of left-skew closed category, a left-skew version of Shulman's prounital-closed categories [27, Rev. 49]. A canonical example of such a category results from the non-commutative linear typed λ-calculus without the unit type.

2 Left-Skew Monoidal and Closed Categories

Szlachányi's [30] left-skew monoidal categories are a variation of monoidal categories.

Definition 2.1 *A* left-skew monoidal category *is a category* $\mathbb{C}$ *together with a distinguished object* I*, a functor* $\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ *and three natural transformations* λ*,* ρ*,* α *typed*

$$\lambda_A : \mathsf{I} \otimes A \to A \qquad \rho_A : A \to A \otimes \mathsf{I} \qquad \alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$$

satisfying the equations

(m1)
$$\begin{tikzcd} & \mathsf{I} \otimes \mathsf{I} \arrow[dr, "\lambda_{\mathsf{I}}"] & \\ \mathsf{I} \arrow[ur, "\rho_{\mathsf{I}}"] \arrow[rr, equal] & & \mathsf{I} \end{tikzcd}$$

(m2)
$$\begin{tikzcd} (A \otimes \mathsf{I}) \otimes B \arrow[r, "\alpha_{A,\mathsf{I},B}"] & A \otimes (\mathsf{I} \otimes B) \arrow[d, "A \otimes \lambda_B"] \\ A \otimes B \arrow[u, "\rho_A \otimes B"] \arrow[r, equal] & A \otimes B \end{tikzcd}$$

(m3)
$$\begin{tikzcd} (\mathsf{I} \otimes A) \otimes B \arrow[rr, "\alpha_{\mathsf{I},A,B}"] \arrow[dr, "\lambda_A \otimes B"'] & & \mathsf{I} \otimes (A \otimes B) \arrow[dl, "\lambda_{A \otimes B}"] \\ & A \otimes B & \end{tikzcd}$$

(m4)
$$\begin{tikzcd} (A \otimes B) \otimes \mathsf{I} \arrow[rr, "\alpha_{A,B,\mathsf{I}}"] & & A \otimes (B \otimes \mathsf{I}) \\ & A \otimes B \arrow[ul, "\rho_{A \otimes B}"] \arrow[ur, "A \otimes \rho_B"'] & \end{tikzcd}$$

(m5)
$$\begin{tikzcd} (A \otimes (B \otimes C)) \otimes D \arrow[rr, "\alpha_{A,B \otimes C,D}"] & & A \otimes ((B \otimes C) \otimes D) \arrow[d, "A \otimes \alpha_{B,C,D}"] \\ ((A \otimes B) \otimes C) \otimes D \arrow[u, "\alpha_{A,B,C} \otimes D"] \arrow[r, "\alpha_{A \otimes B,C,D}"] & (A \otimes B) \otimes (C \otimes D) \arrow[r, "\alpha_{A,B,C \otimes D}"] & A \otimes (B \otimes (C \otimes D)) \end{tikzcd}$$

Notice that (m1)–(m5) are directed versions of the original axioms of Mac Lane [23]. Later, Kelly [15] observed that when α, ρ and λ are natural isomorphisms, laws (m1), (m3), and (m4) can be derived from (m2) and (m5), but for left-skew monoidal categories this is not the case.

Here we present a selection of examples of left-skew monoidal categories, which arise naturally in various different contexts [30,17,7,31].

Example 2.2 The category **Ptd** of pointed sets and point-preserving functions has the following left-skew monoidal structure.

Take $\mathsf{I} = (1, *)$ and $(X, p) \otimes (Y, q) = (X + Y, \mathsf{inl}\ p)$ (notice the "skew" in choosing the point). We define $\lambda_X : (1, *) \otimes (X, p) = (1 + X, \mathsf{inl}\ *) \to (X, p)$ by $\lambda_X\ (\mathsf{inl}\ *) = p$, $\lambda_X\ (\mathsf{inr}\ x) = x$ (this is not injective). We let $\rho_X : (X, p) \to (X + 1, \mathsf{inl}\ p) = (X, p) \otimes (1, *)$ by $\rho_X\ x = \mathsf{inl}\ x$ (this is not surjective). Finally, we let $\alpha_{X,Y,Z} : ((X, p) \otimes (Y, q)) \otimes (Z, r) = ((X + Y) + Z, \mathsf{inl}\ (\mathsf{inl}\ p)) \to (X + (Y + Z), \mathsf{inl}\ p) = (X, p) \otimes ((Y, q) \otimes (Z, r))$ be the obvious isomorphism.

Example 2.3 Given a left-skew monoidal category $(\mathbb{C}, \mathsf{I}, \otimes)$ and a comonad D on $\mathbb{C}$. Suppose D is *lax monoidal*, i.e., comes with a map $\mathsf{e} : \mathsf{I} \to D\,\mathsf{I}$ and a natural transformation $\mathsf{m} : D\,A \otimes D\,B \to D\,(A \otimes B)$ agreeing suitably with λ, ρ, α, ε, δ. Then the category $\mathbb{C}$ has also a left-skew monoidal structure $(\mathsf{I}, \otimes^D)$ where

$A \otimes^D B = A \otimes D\, B$. The unitors and associator are the following:

$$\lambda^D_A = \mathsf{I} \otimes D\, A \xrightarrow{\mathsf{I}\otimes\varepsilon_A} \mathsf{I} \otimes A \xrightarrow{\lambda_A} A \qquad \rho^D_A = A \xrightarrow{\rho_A} A \otimes \mathsf{I} \xrightarrow{A\otimes \mathsf{e}} A \otimes D\, \mathsf{I}$$

$$\begin{aligned} \alpha^D_{A,B,C} = {} & (A \otimes D\, B) \otimes D\, C \xrightarrow{(A\otimes DB)\otimes\delta_C} (A \otimes D\, B) \otimes D\,(D\, C) \\ & \xrightarrow{\alpha_{A,DB,D(DC)}} A \otimes (D\, B \otimes D\,(D\, C)) \xrightarrow{A\otimes \mathsf{m}_{B,DC}} A \otimes D\,(B \otimes D\, C) \end{aligned}$$

A similar left-skew monoidal category is obtained from an oplax monoidal monad $(T, \mathsf{e}, \mathsf{m})$ on $\mathbb{C}$ by taking $A\, {}^T\!\otimes B = T\, A \otimes B$. (Later, in Example 4.10, we will take $A \otimes^T B = A \otimes T\, B$ to produce a new right-skew monoidal structure on a given right-skew monoidal category $(\mathbb{C}, \mathsf{I}, \otimes)$.)

Example 2.4 Given a left-skew monoidal category $(\mathbb{C}, \mathsf{I}, \otimes)$ and a monad T on $\mathbb{C}$. Suppose that T is *lax monoidal*, i.e., comes with a map $\mathsf{e} : \mathsf{I} \to T\, \mathsf{I}$ and a natural transformation $m_{A,B} : T\, A \otimes T\, B \to T\,(A \otimes B)$ cohering with λ, ρ, α, η, μ (in fact, agreement with η makes e redundant as it forces that $\mathsf{e} = \eta_{\mathsf{I}}$). Then the Kleisli category $\mathbf{Kl}(T)$ has a left-skew monoidal structure $(\mathsf{I}, \otimes^T)$ where $A \otimes^T B = A \otimes B$ and

$$(f : A \to T\, A') \otimes^T (g : B \to T\, B') = A \otimes B \xrightarrow{f\otimes g} T\, A' \otimes T\, B' \xrightarrow{\mathsf{m}_{A',B'}} T\,(A' \otimes B')$$

The unitors and associator are pure and taken directly from the base category: $\lambda^T_A = J\, \lambda_A$, $\rho^T_A = J\, \rho_A$, $\alpha^T_{A,B,C} = J\, \alpha_{A,B,C}$ where $J : \mathbb{C} \to \mathbf{Kl}(T)$ is the left adjoint of the Kleisli adjunction, $J\, A = A$, $J\,(f : A \to B) = \eta_B \circ f$.

Instead of lax monoidality, one may require T to be both *left-strong* and *right-strong*, i.e., to come with natural transformations $\mathsf{mst}_{A,B} : A \otimes T\, B \to T\,(A \otimes B)$ and $\mathsf{mst}^{\mathsf{R}}_{A,B} : T\, A \otimes B \to T\,(A \otimes B)$ cohering individually with λ resp. ρ and α, η, μ. These natural transformations must additionally agree with each other via α (the *bi-strength* equation) and via μ (the *commutativity* equation):

$$\begin{tikzcd}
(A \otimes T\, B) \otimes C \arrow[r, "\mathsf{mst}_{A,B}\otimes C"] \arrow[d, "\alpha_{A,TB,C}"'] & T\,(A \otimes B) \otimes C \arrow[r, "\mathsf{mst}^{\mathsf{R}}_{A\otimes B,C}"] & T\,((A \otimes B) \otimes C) \arrow[d, "T\alpha_{A,B,C}"] \\
A \otimes (T\, B \otimes C) \arrow[r, "A\otimes\mathsf{mst}^{\mathsf{R}}_{B,C}"] & A \otimes T\,(B \otimes C) \arrow[r, "\mathsf{mst}_{A,B\otimes C}"] & T\,(A \otimes (B \otimes C))
\end{tikzcd}$$

$$\begin{tikzcd}
T\, A \otimes T\, B \arrow[rr, "\mathsf{mst}_{TA,B}"] \arrow[dd, "\mathsf{mst}^{\mathsf{R}}_{A,TB}"'] & & T\,(T\, A \otimes B) \arrow[d, "T\mathsf{mst}^{\mathsf{R}}_{A,B}"] \\
& & T\, T\,(A \otimes B) \arrow[d, "\mu_{A\otimes B}"] \\
T\,(A \otimes T\, B) \arrow[r, "T\mathsf{mst}_{A,B}"] & T\, T\,(A \otimes B) \arrow[r, "\mu_{A\otimes B}"] & T\,(A \otimes B)
\end{tikzcd}$$

The pairs $(\mathsf{mst}, \mathsf{mst}^{\mathsf{R}})$ of a left-strength and a right-strength of T agreeing with each other in this way are in a bijective correspondence with the lax monoidality witnesses m. Without commutativity, $(\mathbf{Kl}(T), \mathsf{I}, \otimes^T)$ is not left-skew monoidal, in fact, $\otimes^T$ is not even a bifunctor; we only get a structure that could be called left-skew premonoidal by extending the terminology of Power and Robinson [26].

Example 2.5 Consider two categories $\mathbb{J}$ and $\mathbb{C}$ with a functor $J : \mathbb{J} \to \mathbb{C}$, and assume that the left Kan extension $\mathrm{Lan}_J\, F : \mathbb{C} \to \mathbb{C}$ exists for every $F : \mathbb{J} \to \mathbb{C}$. Then the functor category $[\mathbb{J}, \mathbb{C}]$ has a skew monoidal structure given by $\mathsf{I} = J$, $F \otimes G = F \cdot^J G = \mathrm{Lan}_J F \cdot G$. The unitors and associator are the canonical natural transformations $\lambda_F : \mathrm{Lan}_J\ J \cdot F \to F$, $\rho_F : F \to \mathrm{Lan}_J\ F \cdot J$, $\alpha_{F,G,H} : \mathrm{Lan}_J\ (\mathrm{Lan}_J\ F \cdot G) \cdot H \to \mathrm{Lan}_J\ F \cdot \mathrm{Lan}_J\ G \cdot H$. This example is from the relative monads work of Altenkirch et al. [1]. Relative monads on J are monoids in the left-skew monoidal category $[\mathbb{J}, \mathbb{C}]$.

Left-skew closed categories were introduced by Street [29] as a variation of the closed categories of Eilenberg and Kelly [13]. Zeilberger considered a posetal variant of left-skew closed categories, which he called *imploids* [34].

Definition 2.6 *A* left-skew closed category *is a category* $\mathbb{C}$ *together with a distinguished object* I*, a functor* $\multimap : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{C}$*, and three natural transformations* j*,* i*, and* L [4] *typed*

$$j_A : \mathsf{I} \to A \multimap A \qquad i_A : \mathsf{I} \multimap A \to A \qquad L_{A,B,C} : B \multimap C \to (A \multimap B) \multimap (A \multimap C)$$

satisfying the laws

$$\text{(c1)}\quad \begin{tikzcd} & \mathsf{I} \multimap \mathsf{I} \arrow[dr, "i_{\mathsf{I}}"] & \\ \mathsf{I} \arrow[ur, "j_{\mathsf{I}}"] \arrow[rr, equal] & & \mathsf{I} \end{tikzcd} \qquad \text{(c2)}\quad \begin{tikzcd} A \multimap C \arrow[r, "L_{A,A,C}"] \arrow[d, equal] & (A \multimap A) \multimap (A \multimap C) \arrow[d, "j_A \multimap (A \multimap C)"] \\ A \multimap C & \mathsf{I} \multimap (A \multimap C) \arrow[l, "i_{A \multimap C}"'] \end{tikzcd}$$

$$\text{(c3)}\quad \begin{tikzcd} B \multimap B \arrow[rr, "L_{A,B,B}"] & & (A \multimap B) \multimap (A \multimap B) \\ & \mathsf{I} \arrow[ul, "j_B"] \arrow[ur, "j_{A \multimap B}"'] & \end{tikzcd} \qquad \text{(c4)}\quad \begin{tikzcd} B \multimap C \arrow[rr, "L_{\mathsf{I},B,C}"] \arrow[dr, "i_B \multimap C"'] & & (\mathsf{I} \multimap B) \multimap (\mathsf{I} \multimap C) \arrow[dl, "(\mathsf{I} \multimap B) \multimap i_C"] \\ & (\mathsf{I} \multimap B) \multimap C & \end{tikzcd}$$

$$\text{(c5)}\quad \begin{tikzcd} C \multimap D \arrow[rr, "L_{A,C,D}"] \arrow[dd, "L_{B,C,D}"'] & & (A \multimap C) \multimap (A \multimap D) \arrow[d, "L_{A \multimap B, A \multimap C, A \multimap D}"] \\ & & ((A \multimap B) \multimap (A \multimap C)) \multimap ((A \multimap B) \multimap (A \multimap D)) \arrow[d, "L_{A,B,C} \multimap ((A \multimap B) \multimap (A \multimap D))"] \\ (B \multimap C) \multimap (B \multimap D) \arrow[rr, "(B \multimap C) \multimap L_{A,B,D}"] & & (B \multimap C) \multimap ((A \multimap B) \multimap (A \multimap D)) \end{tikzcd}$$

Given a category $\mathbb{C}$ with I, $\multimap$, the natural transformations j and L are interdefinable with natural transformations

$$\widehat{\jmath}_{A,B} : \mathbb{C}(A, B) \to \mathbb{C}(\mathsf{I}, A \multimap B)$$
$$\widehat{L}_{A,B,C,D} : \int^X \mathbb{C}(A, X \multimap D) \times \mathbb{C}(B, C \multimap X) \to \mathbb{C}(A, B \multimap (C \multimap D))$$

[4] In fact, j and L are "extranatural" transformations in the terminology of Eilenberg and Kelly: the codomains of j_A and $L_{A,B,C}$ depend on A both covariantly and contravariantly. Following Street [29], we omit the "extra" adjective and refer to these simply as natural transformations.

where the integral notation denotes a coend. This is because of the following sequences of bijections: [5]

$$\frac{\mathbb{C}(A,B) \overset{\widehat{\jmath}_{A,B}}{\to} \mathbb{C}(\mathsf{I}, A \multimap B)}{\mathsf{I} \overset{j_B}{\to} B \multimap B}
\qquad
\cfrac{\cfrac{\cfrac{\cfrac{\int^E \mathbb{C}(A, E \multimap D) \times \mathbb{C}(B, C \multimap E) \overset{\widehat{L}_{A,B,C,D}}{\to} \mathbb{C}(A, B \multimap (C \multimap D))}{\mathbb{C}(A, E \multimap D) \times \mathbb{C}(B, C \multimap E) \to \mathbb{C}(A, B \multimap (C \multimap D))}}{\mathbb{C}(A, E \multimap D) \to \int_B \mathbb{C}(B, C \multimap E) \Rightarrow \mathbb{C}(A, B \multimap (C \multimap D)))}}{\mathbb{C}(A, E \multimap D) \to \mathbb{C}(A, (C \multimap E) \multimap (C \multimap D))}}{E \multimap D \overset{L_{C,E,D}}{\to} (C \multimap E) \multimap (C \multimap D)}$$

Explicitly, $\widehat{\jmath}_{A,B}\, f = A \multimap f \circ j_A$ and $\widehat{L}_{A,B,C,D}\,(E, f, g) = g \multimap (C \multimap D) \circ L_{C,E,D} \circ f$.

Examples of left-skew closed categories are not as common in the literature as left skew-monoidal categories. Nevertheless, they arise also quite naturally, as demonstrated by the following examples. The first example is from Street [29, Proposition 3].

Example 2.7 Given a left-skew closed category $(\mathbb{C}, \mathsf{I}, \multimap)$ and a comonad D on it. Suppose D is *lax closed*, i.e., comes with a map $\mathsf{e} : \mathsf{I} \to D\,\mathsf{I}$ and a natural transformation $\mathsf{c}_{B,C} : D\,(B \multimap C) \to D\,B \multimap D\,C$ cohering as appropriate with j, i, L, ε, δ. Then the category $\mathbb{C}$ also has a left-skew closed structure $(\mathsf{I}, {}^D\!\!\multimap)$ where $B \;{}^D\!\!\multimap C = D\,B \multimap C$. The laws of a left-skew closed category are defined by

$$ {}^D j_A = \mathsf{I} \xrightarrow{j_A} A \multimap A \xrightarrow{\varepsilon_A \multimap A} D\,A \multimap A \quad {}^D i_A = D\,\mathsf{I} \multimap A \xrightarrow{\mathsf{e} \multimap A} \mathsf{I} \multimap A \xrightarrow{i_A} A$$

$$\begin{aligned}
{}^D L_{A,B,C} = \; & D\,B \multimap C \xrightarrow{L_{D\,A,D\,B,C}} (D\,A \multimap D\,B) \multimap (D\,A \multimap C) \\
& \xrightarrow{(\delta_A \multimap D B) \multimap (D\,A \multimap C)} (D\,(D\,A) \multimap D\,B) \multimap (D\,A \multimap C) \\
& \xrightarrow{\mathsf{c}_{D A,B} \multimap (D A \multimap C)} D\,(D\,A \multimap B) \multimap (D\,A \multimap C)
\end{aligned}$$

Example 2.8 Given a left-skew closed category $(\mathbb{C}, \mathsf{I}, \multimap)$ and a monad T on it. Suppose that T is *left-strong* (or *internally functorial*), i.e., endowed with a natural transformation $\mathsf{cst}_{A,B} : B \multimap C \to T\,B \multimap T\,C$ cohering appropriately with j, L, η, μ.

The monad T being internally functorial is the same as it being *internally Kleisli*, i.e., equipped with a natural transformation $\mathsf{iext}_{B,C} : B \multimap T\,C \to T\,B \multimap T\,C$ satisfying internal versions (in terms of j, L, η, μ) of the equations of a Kleisli extension system à la Manes [25] (a.k.a. a monad in the no-iteration form): the left-strengths cst and the internal Kleisli extensions iext of T are in a bijective correspondence.

Then $\mathbf{Kl}(T)$ has a left-skew closed structure $(\mathsf{I}, \multimap^T)$ where $B \multimap^T C = B \multimap T\,C$

[5] All lines in the sequences of bijections here in the rest of the paper stand for families of maps natural in all free object variables.

and

$$(g : B' \to T\,B) \multimap^T (h : C \to T\,C') = \\ J\,(\,B \multimap T\,C \xrightarrow{\mathsf{iext}_{B,C}} T\,B \multimap T\,C \xrightarrow{g \multimap T h} B' \multimap T\,T\,C' \xrightarrow{B' \multimap \mu_{C'}} B' \multimap T\,C'\,)$$

Its structural laws are defined by

$$j^T_A = J\,(\,\mathsf{I} \xrightarrow{j_A} A \multimap A \xrightarrow{A \multimap \eta_A} A \multimap T A\,) \qquad i^T_A = \mathsf{I} \multimap T\,A \xrightarrow{i_{T A}} T\,A$$

$$L^T_{A,B,C} = J\,(\,B \multimap T\,C \xrightarrow{\mathsf{iext}_{B,C}} T\,B \multimap T\,C \xrightarrow{L_{A,TB,TC}} (A \multimap T\,B) \multimap (A \multimap T\,C) \\ \xrightarrow{(A \multimap TB) \multimap \eta_{A \multimap TC}} (A \multimap T\,B) \multimap T\,(A \multimap T\,C)\,)$$

Notably, all of $g \multimap^T h$, j^T_A, $L^T_{A,B,C}$ are pure maps (whereby the map $g \multimap^T h$ is pure even if g and h are not), i.e., in the image of the left adjoint $J : \mathbb{C} \to \mathbf{Kl}(T)$ of the Kleisli adjunction. Only the map i^T_A is impure.

Example 2.9 Another example of left-skew closed category is given by non-commutative linear typed λ-calculus with the unit type. Types are generated by the grammar $A, B ::= X \mid \mathsf{I} \mid A \multimap B$, where X is an atomic type. Contexts are lists of types. Well-formed terms are constructed as follows:

$$\frac{}{x : A \vdash x : A} \qquad \frac{}{\vdash \star : \mathsf{I}} \qquad \frac{\Gamma \vdash t : \mathsf{I} \quad \Delta \vdash u : A}{\Gamma, \Delta \vdash \mathsf{let}\ \star = t\ \mathsf{in}\ u : A} \qquad \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.\, t : A \multimap B} \qquad \frac{\Gamma \vdash t : A \multimap B \quad \Delta \vdash u : A}{\Gamma, \Delta \vdash t\ u : B}$$

Definitional equality of terms is given by $\beta\eta$-equality (a non-commutative variant of the term equality of Hyland and de Paiva [14]). The natural transformations j, i and L are derivable as follows:

$$\frac{\dfrac{\dfrac{}{x : \mathsf{I} \vdash x : \mathsf{I}} \quad \dfrac{}{y : A \vdash y : A}}{x : \mathsf{I}, y : A \vdash \mathsf{let}\ \star = x\ \mathsf{in}\ y : A}}{x : \mathsf{I} \vdash j_A = \lambda y.\ \mathsf{let}\ \star = x\ \mathsf{in}\ y : A \multimap A} \qquad \frac{\dfrac{}{x : \mathsf{I} \multimap A \vdash x : \mathsf{I} \multimap A} \quad \dfrac{}{\vdash \star : \mathsf{I}}}{x : \mathsf{I} \multimap A \vdash i_A = x\ \star : A}$$

$$\frac{\dfrac{\dfrac{}{x : B \multimap C \vdash x : B \multimap C} \quad \dfrac{\dfrac{}{y : A \multimap B \vdash y : A \multimap B} \quad \dfrac{}{z : A \vdash z : A}}{y : A \multimap B, z : A \vdash y\ z : B}}{x : B \multimap C, y : A \multimap B, z : A \vdash x\ (y\ z) : C}}{x : B \multimap C \vdash L_{A,B,C} = \lambda y.\ \lambda z.\ x\ (y\ z) : (A \multimap B) \multimap (A \multimap C)}$$

The non-commutative linear typed λ-calculus with unit type over a set of atomic types At is a concrete presentation of the *free* left-skew closed category generated by At.

The following theorem is from Street [29] (modulo the observation about the internal right transpose, which was not explicit).

Theorem 2.10 *Let $\mathbb{C}$ be a category with an object I and functors $\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ and $\multimap : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{C}$, with an adjunction $- \otimes B \dashv B \multimap -$ natural in B.*

Then $(\mathbb{C}, \mathsf{I}, \otimes)$ is left-skew monoidal if and only if $(\mathbb{C}, \mathsf{I}, \multimap)$ is left-skew closed. More precisely, there is a bijection between the (λ, ρ, α) left-skew monoidal and the (j, i, L) left-skew closed structures on $(\mathbb{C}, \mathsf{I}, \otimes)$ resp. $(\mathbb{C}, \mathsf{I}, \multimap)$.

Moreover, both α and L are also interdefinable with a natural transformation

$$\mathsf{p}_{A,B,C} : (A \otimes B) \multimap C \to A \multimap (B \multimap C)$$

(an internal version of the right transpose $\pi_{A,B,C} : \mathbb{C}(A \otimes B, C) \to \mathbb{C}(A, B \multimap C)$).

The bijection of structures is actually finer. First, there are three independent bijections between the λ and j data, the ρ and i data and the α and L data (as well as p). Second, the equations (m1–m5) and (c1–c5) also entail each other pairwise. So the overall bijection is there even between partial structures.

Proof. The requisite bijections between the above pairs of data follow from the the sequences of bijections below. These hold both for natural transformations in the forward direction only (relevant here), for natural transformations in both directions, in particular natural isomorphisms (relevant for Theorems 3.8, 4.11 below) and also for natural transformations in the backward direction only (relevant for Theorem 4.8).

$$\begin{array}{c} \mathsf{I} \otimes A \overset{\lambda_A}{\rightleftarrows} A \\ \hline\hline \mathbb{C}(A, B) \rightleftarrows \mathbb{C}(\mathsf{I} \otimes A, B) \\ \hline\hline \mathbb{C}(A, B) \overset{\widehat{\jmath}_{A,B}}{\rightleftarrows} \mathbb{C}(\mathsf{I}, A \multimap B) \end{array} \qquad \begin{array}{c} A \overset{\rho_A}{\rightleftarrows} A \otimes \mathsf{I} \\ \hline\hline \mathbb{C}(A \otimes \mathsf{I}, B) \rightleftarrows \mathbb{C}(A, B) \\ \hline\hline \mathbb{C}(A, \mathsf{I} \multimap B) \rightleftarrows \mathbb{C}(A, B) \\ \hline\hline I \multimap B \overset{i_B}{\rightleftarrows} B \end{array}$$

$$\begin{array}{c} (A \otimes B) \otimes C \overset{\alpha_{A,B,C}}{\rightleftarrows} A \otimes (B \otimes C) \\ \hline\hline \mathbb{C}(A \otimes (B \otimes C), D) \rightleftarrows \mathbb{C}((A \otimes B) \otimes C, D) \\ \hline\hline \mathbb{C}(A, (B \otimes C) \multimap D) \rightleftarrows \mathbb{C}(A \otimes B, C \multimap D) \\ \hline\hline \mathbb{C}(A, (B \otimes C) \multimap D) \rightleftarrows \mathbb{C}(A, B \multimap (C \multimap D)) \quad \| \quad (B \otimes C) \multimap D \overset{\mathsf{p}_{B,C,D}}{\rightleftarrows} B \multimap (C \multimap D) \\ \hline\hline \int^X \mathbb{C}(A, X \multimap D) \times \mathbb{C}(B \otimes C, X) \rightleftarrows \mathbb{C}(A, B \multimap (C \multimap D)) \\ \hline\hline \int^X \mathbb{C}(A, X \multimap D) \times \mathbb{C}(B, C \multimap X) \overset{\widehat{L}_{A,B,C,D}}{\rightleftarrows} \mathbb{C}(A, B \multimap (C \multimap D)) \end{array}$$

The mutual entailments between the respective pairs of equations need separate verification; we are not showing these calculations here. □

Definition 2.11 Under the assumptions of Theorem 2.10, if $(\mathbb{C}, \mathsf{I}, \otimes)$ is left-skew monoidal or, equivalently, $(\mathbb{C}, \mathsf{I}, \multimap)$ is left-skew closed, the category $(\mathbb{C}, \mathsf{I}, \otimes, \multimap)$ is said to be *left-skew monoidal closed.*

Example 2.12 Given a left-skew monoidal closed category $(\mathbb{C}, \mathsf{I}, \otimes, \multimap)$ and comonad D on it. The comonad D is lax monoidal if and only if it is lax closed. Let $A \otimes^D B = A \otimes D\, B$ and $B \,{}^D\!\!\multimap C = D\, B \multimap C$ as in Examples 2.3 and 2.7. The adjunction $- \otimes D\, B \dashv D\, B \multimap -$ immediately yields an adjunction $- \otimes^D B \dashv B \,{}^D\!\!\multimap -$. Hence, as soon D is lax monoidal or, equivalently, lax closed, then $(\mathbb{C}, \mathsf{I}, \otimes^D, {}^D\!\!\multimap)$ is a left-skew monoidal closed category.

Example 2.13 Given a left-skew monoidal closed category $(\mathbb{C}, \mathsf{I}, \otimes, \multimap)$ and a monad T on it. The monad T is lax monoidal if and only if it is lax closed: its lax monoidality witnesses m and lax closedness witnesses c are in a bijection. Likewise, T is left-strong wrt. $\otimes$ if and only if it is left-strong wrt. $\multimap$, i.e., internally functorial: the left-strengths mst and cst are in a bijection. The same can be said about T being right-strong wrt. $\otimes$ or $\multimap$: the right-strengths $\mathsf{mst}^{\mathsf{R}}$ and $\mathsf{cst}^{\mathsf{R}}$ are in a bijection. Indeed, look at the sequences of bijections

$$\dfrac{\dfrac{\dfrac{A \otimes T\,B \overset{\mathsf{mst}_{A,B}}{\to} T\,(A \otimes B)}{\mathbb{C}(A \otimes B, C) \to \mathbb{C}(A \otimes T\,B, T\,C)}}{\mathbb{C}(A, B \multimap C) \to \mathbb{C}(A, T\,B \multimap T\,C)}}{B \multimap C \to T\,B \overset{\mathsf{cst}_{B,C}}{\multimap} T\,C} \qquad \dfrac{\dfrac{\dfrac{T\,A \otimes B \overset{\mathsf{mst}^{\mathsf{R}}_{A,B}}{\to} T\,(A \otimes B)}{\mathbb{C}(A \otimes B, C) \to \mathbb{C}(T\,A \otimes B, T\,C)}}{\mathbb{C}(A, B \multimap C) \to \mathbb{C}(T\,A, B \multimap T\,C)}}{T\,(B \multimap C) \to B \overset{\mathsf{cst}^{\mathsf{R}}_{B,C}}{\multimap} T\,C}$$

The corresponding pairs of equations are also interderivable.

Let $A \otimes^T B = A \otimes B$ and $B \multimap^T C = B \multimap T\,C$ as in Examples 2.4 and 2.8. As a consequence of the above observations, $(\mathbf{Kl}(T), \mathsf{I}, \otimes^T)$ is a left-skew monoidal category as soon as the monad T is lax monoidal or, equivalently, lax closed. Also, $(\mathbf{Kl}(T), \mathsf{I}, \multimap^T)$ is left-skew monoidal as soon as T is left-strong wrt. $\otimes$ or, equivalently, left-strong wrt. $\multimap$ (which is weaker than lax monoidality/lax closedness of T).

In contrast, for $(\mathbf{Kl}(T), \mathsf{I}, \otimes^T, \multimap^T)$ to be left-skew monoidal closed, lax monoidality/lax closedness of T does not suffice. This is because, in general, we have no adjunction

$$\begin{tikzcd} \mathbf{Kl}(T) \arrow[r, bend left, "- \otimes^T B"] \arrow[r, phantom, "\perp"] & \mathbf{Kl}(T) \arrow[l, bend left, "B \multimap^T -"] \end{tikzcd}$$

natural in $B \in \mathbf{Kl}(T)$. We only have an adjunction

$$\begin{tikzcd} \mathbb{C} \arrow[r, bend left, "J - \otimes J B"] \arrow[r, phantom, "\perp"] & \mathbf{Kl}(T) \arrow[l, bend left, "B \multimap R -"] \end{tikzcd}$$

natural in $B \in \mathbb{C}$ where $R : \mathbf{Kl}(T) \to \mathbb{C}$ is the right adjoint of the Kleisli adjunction defined by $R\,A = T\,A$, $R\,(f : A \to TB) = \mu_B \circ T\,f$.

3 Normality Conditions

Left-skew monoidal categories differ from monoidal categories in that the unitors and the associator are not invertible. When one of the structural laws is invertible, the resulting category is said to be satisfying a *normality condition* [17].

Definition 3.1 *A left-skew monoidal category is said to be*

(i) left-normal *if the left unitor λ is a natural isomorphism;*

(ii) right-normal *if the right unitor ρ is a natural isomorphism;*

(iii) associative-normal *(or* Hopf*) if the associator α is a natural isomorphism.*

Monoidal categories can be defined as left-skew monoidal categories satisfying all three normality conditions.

Example 3.2 Given a left-skew monoidal category $(\mathbb{C}, \mathsf{I}, \otimes)$ and a comonad D on $\mathbb{C}$ that is *monoidal*, i.e., lax monoidal with e, m natural isomorphisms (in fact, just e being invertible suffices for our purpose). Let $A \otimes^D B = A \otimes D\,B$ as in Example 2.3. Then the left-skew monoidal category $(\mathbb{C}, \mathsf{I}, \otimes^D)$ is right-normal if $(\mathbb{C}, \mathsf{I}, \otimes)$ is right-normal with $(\rho^D_A)^{-1} = \rho^{-1}_A \circ (A \otimes \mathsf{e}^{-1})$.

Left-skew closed categories also admit degrees of normality similar to the monoidal case. It does not make sense though to require the invertibility of the structural laws j and L. We can specify the corresponding normality conditions by asking for the invertibility of $\widehat{\jmath}$ and $\widehat{L}$ instead.

Definition 3.3 *A left-skew closed category is called*

(i) left-normal *if $\widehat{\jmath}$ is a natural isomorphism;*

(ii) right-normal *if i is a natural isomorphism;*

(iii) associative-normal *if $\widehat{L}$ is a natural isomorphism.*

By the commonly adopted terminology of [13], a left-skew closed category is called *closed* if it is left-normal and right normal. Associative-normality is not required in this definition. As a consequence, the Eilenberg-Kelly theorem [13] (which we adjust in Theorem 3.8 below) is not perfectly balanced. We note that Day [10, Example 3.2] did consider associative-normality, but this went unnoticed in the later literature.

Example 3.4 Given a left-skew closed category $(\mathbb{C}, \mathsf{I}, \multimap)$ and a comonad D on $\mathbb{C}$ that is *closed*, i.e., lax closed with e, c natural isomorphisms (just e being invertible suffices for our purposes). Let $B \;{}^D\!\!\multimap C = D\,B \multimap C$ as in Example 2.7. Then the left-skew closed category $(\mathbb{C}, \mathsf{I}, {}^D\!\!\multimap)$ is right-normal if $(\mathbb{C}, \mathsf{I}, \multimap)$ is right-normal with ${}^D i^{-1}_A = (\mathsf{e}^{-1} \multimap A) \circ i^{-1}_A$. Notice the parallel to Example 3.2.

Example 3.5 Given a left-skew closed category $(\mathbb{C}, \mathsf{I}, \multimap)$ and a monad T on $\mathbb{C}$ that is *lax closed*. This is the same as T being left-strong and right-strong with the the left-strength cst and the right-strength $\mathsf{cst}^{\mathsf{R}}$ making a bi-strength and commuting. Let $B \multimap^T C = B \multimap T\,C$ as in Example 2.8. Because T is left-strong, we have that $(\mathbf{Kl}(T), \mathsf{I}, \multimap^T)$ is left-skew closed. Now, if $(\mathbb{C}, \mathsf{I}, \multimap)$ happens to be left-normal, i.e., if $\widehat{\jmath}$ is invertible, then with the help of the right-strength $\mathsf{cst}^{\mathsf{R}}$, we can define a natural transformation ${j^{\mathsf{R}}}^T_{A,B} : \mathbf{Kl}(T)(\mathsf{I}, A \otimes^T B) \to \mathbf{Kl}(T)(A, B)$ by

$$ {j^{\mathsf{R}}}^T_{A,B}\,(\,\mathsf{I} \xrightarrow{g} T\,(A \multimap T\,B)\,) = A \xrightarrow{\widehat{\jmath}^{-1}_{A,TB}\,g'} T\,B $$

where $g' = \mathsf{I} \xrightarrow{g} T\,(A \multimap T\,B) \xrightarrow{\mathsf{cst}^{\mathsf{R}}_{A,TB}} A \multimap T\,T\,B \xrightarrow{A \multimap \mu_B} A \multimap T\,B$. For naturality of ${j^{\mathsf{R}}}^T$, commutativity is needed. Now, by its type, ${j^{\mathsf{R}}}^T$ is a good candidate for the inverse of $\widehat{\jmath}^T$, but it is only a retraction. It is generally the case that ${j^{\mathsf{R}}}^T\,(\widehat{\jmath}^T\,f)) = f$ but the section equation $\widehat{\jmath}^T\,({j^{\mathsf{R}}}^T\,g)) = g$ fails already for T the reader monad $T\,A = S \multimap A$. So $(\mathbb{C}, \mathsf{I}, \multimap^T)$ is not left-normal.

Similarly, if $(\mathbb{C}, \mathsf{I}, \multimap)$ is right-normal, i.e., has i invertible, then a natural transformation ${i^{\mathsf{R}}}^T_A : A \to \mathsf{I} \multimap^T A$ in $\mathbf{Kl}(T)$ is readily defined by

$$ {i^{\mathsf{R}}}^T_A = J\,(\,A \xrightarrow{i_A^{-1}} \mathsf{I} \multimap A \xrightarrow{\mathsf{I}\multimap\eta_A} \mathsf{I} \multimap T\,A\,) $$

${i^{\mathsf{R}}}^T$ is a section of i^T, but fails to be a retraction already for the reader monad. Hence, $(\mathbb{C}, \mathsf{I}, \multimap^T)$ is not right-normal.

Finally, if $(\mathbb{C}, \mathsf{I}, \multimap)$ is associative-normal, i.e., has $\widehat{L}$ invertible, then, using the right-strength $\mathsf{cst}^{\mathsf{R}}$, we can define a candidate for the inverse of $\widehat{L}^T$ by

$$ \begin{aligned} &{L^{\mathsf{R}}}^T_{A,B,C,D}\,(\,A \xrightarrow{h} T\,(B \multimap T\,(C \multimap T\,D))\,) = \\ &\qquad (E, J\,(\,A \xrightarrow{f} E \multimap T\,D\,), J\,(\,B \xrightarrow{g} C \multimap E \xrightarrow{C\multimap\eta_E} C \multimap T\,E\,)) \end{aligned} $$

where

$$ \begin{aligned} h' = \; & A \xrightarrow{h} T\,(B \multimap T\,(C \multimap T\,D)) \xrightarrow{\mathsf{cst}^{\mathsf{R}}_{B,T(C\multimap T\,D)}} B \\ & \multimap T\,T\,(C \multimap T\,D) \xrightarrow{B\multimap\mu_{C\multimap T\,D}} B \multimap T\,(C \multimap T\,D) \\ & \xrightarrow{B\multimap\mathsf{cst}^{\mathsf{R}}_{C,T\,D}} B \multimap (C \multimap T\,T\,D) \xrightarrow{B\multimap(C\multimap\mu_D)} B \multimap (C \multimap T\,D) \end{aligned} $$

and

$$ (E,\, A \xrightarrow{f} E \multimap T\,D\,,\, B \xrightarrow{g} C \multimap E\,) = \widehat{L}^{-1}_{A,B,C,T\,D}\, h'. $$

But ${L^{\mathsf{R}}}^T$ is not a section of $\widehat{L}^T$.

Example 3.6 The left-skew monoidal category $([\mathbb{J}, \mathbb{C}], J, \cdot^J)$ from Example 2.5 is right-normal if and only if the functor J is fully-faithful. It is left-normal if and only if J is dense (i.e., the nerve of J is fully-faithful). Its associative-normality condition is equivalent to a more involved condition on the functor J [1].

Example 3.7 The left-skew closed structure on the non-commutative linear typed λ-calculus introduced in Example 2.9 is left-normal. The inverse of $\widehat{\jmath}$ is derivable as follows:

$$ \dfrac{\dfrac{}{\vdash \star : \mathsf{I}} \quad \dfrac{y : \mathsf{I} \vdash t : A \multimap B \quad \overline{x : A \vdash x : A}}{y : \mathsf{I}, x : A \vdash t\;x : B}}{x : A \vdash \widehat{\jmath}^{-1}_{A,B}\, t = (t\;x)[\star/y] : B} $$

This derivation uses substitution

$$ \dfrac{\Gamma \vdash t : A \quad \Delta_0, x : A, \Delta_1 \vdash u : C}{\Delta_0, \Gamma, \Delta_1 \vdash u[t/x] : C} $$

which is an admissible rule in this calculus—by induction on derivations we can prove that if the premises are derivable, the conclusion is as well.

Inverses of i and $\widehat{L}$ are not derivable. We can impose right-normality by replacing the elimination rule for I in Example 2.9 with the following more permissive rule:

$$\frac{\Gamma \vdash t : \mathsf{I} \quad \Delta_0, \Delta_1 \vdash u : A}{\Delta_0, \Gamma, \Delta_1 \vdash \mathsf{let}\ \star = t\ \mathsf{in}\ u : A}$$

With this rule in place we can derive the inverse of i:

$$\cfrac{\cfrac{\cfrac{}{y : \mathsf{I} \vdash y : \mathsf{I}} \quad \cfrac{}{x : A \vdash x : A}}{x : A, y : \mathsf{I} \vdash \mathsf{let}\ \star = y\ \mathsf{in}\ x : A}}{x : A \vdash i_A^{-1} = \lambda y.\, \mathsf{let}\ \star = y\ \mathsf{in}\ x : \mathsf{I} \multimap A}$$

Associative-normality can be imposed by extending the grammar for types with a tensor product $\otimes$ and well-formed terms:

$$\frac{\Gamma \vdash t : A \quad \Delta \vdash u : B}{\Gamma, \Delta \vdash (t, u) : A \otimes B} \qquad \frac{\Gamma \vdash t : A \otimes B \quad \Delta_0, x : A, y : B, \Delta_1 \vdash u : C}{\Delta_0, \Gamma, \Delta_1 \vdash \mathsf{let}\ (x, y) = t\ \mathsf{in}\ u : C}$$

In this way we obtain an associative-normal (also left-normal) left-skew monoidal closed category. We do not currently know how to impose associative-normality on the non-commutative linear typed λ-calculus without simultaneously enforcing left-normal and associative-normal monoidality as well.

In a left-skew monoidal closed category, each normality condition on the monoidal structure exactly corresponds to a normality condition on the closed structure. Notice that associative-normality of $(\mathbb{C}, \mathsf{I}, \multimap)$, i.e., invertibility of $\widehat{L}$, is equivalent to invertibility of the internal right transpose p, but the latter is a condition not in terms of $\multimap$ and L (through which $\widehat{L}$ is defined) but in terms of $\otimes$, $\multimap$ and α (through which p is defined).

Theorem 3.8 *Let $(\mathbb{C}, \mathsf{I}, \otimes, \multimap)$ be a left-skew monoidal closed category. Then*

(i) $(\mathbb{C}, \mathsf{I}, \otimes)$ *is left-normal iff* $(\mathbb{C}, \mathsf{I}, \multimap)$ *is left-normal.*

(ii) $(\mathbb{C}, \mathsf{I}, \otimes)$ *is right-normal iff* $(\mathbb{C}, \mathsf{I}, \multimap)$ *is right-normal.*

(iii) $(\mathbb{C}, \mathsf{I}, \otimes)$ *is associative-normal iff* p *is a natural isomorphism iff* $(\mathbb{C}, \mathsf{I}, \multimap)$ *is associative-normal.*

Proof. The sequences of bijections shown in the proof of Theorem 2.10 above hold not only for natural transformations in the forward direction but also natural isomorphisms. □

4 Skewing to the Right

There is nothing indicating that it should not make sense to consider skewing normal monoidality and closedness to the right instead of to the left. The definition of right-skew monoidal category is obvious.

Definition 4.1 *A* right-skew monoidal category *is a category $\mathbb{C}$ together with a distinguished object I, a bifunctor $\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ and three natural transformations*

ρ^{R}, λ^{R}, α^{R} *typed*

$$\rho^{\mathsf{R}}_A : A \otimes \mathsf{I} \to A \qquad \lambda^{\mathsf{R}}_A : A \to \mathsf{I} \otimes A \qquad \alpha^{\mathsf{R}}_{A,B,C} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C$$

satisfying five laws analogous to (m1)–(m5) in Definition 2.1.

This category is called right-normal, left-normal *resp.* associative-normal *if* ρ^{R}, λ^{R}, *resp.* α^{R} *is a natural isomorphism.*

Results analogous to Theorem 2.10 and 3.8 relate left-skew and right-skew monoidal structures on a given category.

Theorem 4.2 *Given a category* $\mathbb{C}$ *with an object* I *and two bifunctors* $\otimes^{\mathsf{L}}$, $\otimes^{\mathsf{R}}$ *together with a natural isomorphism* $\gamma_{A,B} : A \otimes^{\mathsf{L}} B \to B \otimes^{\mathsf{R}} A$. *Then* $(\mathbb{C}, \mathsf{I}, \otimes^{\mathsf{L}})$ *is left-skew monoidal iff* $(\mathbb{C}, \mathsf{I}, \otimes^{\mathsf{R}})$ *is right-skew monoidal; more precisely, the* (λ, ρ, α) *and* $(\rho^{\mathsf{R}}, \lambda^{\mathsf{R}}, \alpha^{\mathsf{R}})$ *structures on them are in a bijection.*

Definition 4.3 *Under the assumptions of Theorem 4.2, if* $(\mathbb{C}, \mathsf{I}, \otimes^{\mathsf{L}})$ *is left-skew monoidal or, equivalently,* $(\mathbb{C}, \mathsf{I}, \otimes^{\mathsf{R}})$ *is right-skew monoidal, we say that* $(\mathbb{C}, \mathsf{I}, \otimes^{\mathsf{L}}, \otimes^{\mathsf{R}})$ *is* skew bi-monoidal.

Theorem 4.4 *Given a skew bi-monoidal category* $(\mathbb{C}, \mathsf{I}, \otimes^{\mathsf{L}}, \otimes^{\mathsf{R}})$. *Then*

(i) $(\mathbb{C}, \mathsf{I}, \otimes^{\mathsf{L}})$ *is left-normal iff* $(\mathbb{C}, \mathsf{I}, \otimes^{\mathsf{R}})$ *is right-normal;*

(ii) $(\mathbb{C}, \mathsf{I}, \otimes^{\mathsf{L}})$ *is right-normal iff* $(\mathbb{C}, \mathsf{I}, \otimes^{\mathsf{R}})$ *is left-normal;*

(iii) $(\mathbb{C}, \mathsf{I}, \otimes^{\mathsf{L}})$ *is associative-normal iff* $(\mathbb{C}, \mathsf{I}, \otimes^{\mathsf{R}})$ *is associative-normal.*

As a specification of the correct concept of right-skew closed category, we can take that an analogue of Street's version of the Eilenberg-Kelly theorem must also hold in the right-skew case (Theorem 4.8 below).

Definition 4.5 *A* right-skew closed category *is a category* $\mathbb{C}$ *together with a distinguished object* I, *a functor* $\multimap : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{C}$ *and three natural transformations* i^{R}, j^{R}, L^{R} [6] *typed*

$$i^{\mathsf{R}}{}_A : A \to \mathsf{I} \multimap A \qquad j^{\mathsf{R}}{}_{A,B} : \mathbb{C}(\mathsf{I}, A \multimap B) \to \mathbb{C}(A, B)$$

$$L^{\mathsf{R}}{}_{A,B,C,D} : \mathbb{C}(A, B \multimap C \multimap D) \to \int^X \mathbb{C}(A, X \multimap D) \times \mathbb{C}(B, C \multimap X)$$

satisfying the equations

(cR1)
$$\begin{tikzcd}
& \mathbb{C}(\mathsf{I}, \mathsf{I} \multimap B) \arrow[dr, "{j^{\mathsf{R}}}_{\mathsf{I},B}"] & \\
\mathbb{C}(\mathsf{I}, B) \arrow[ur, "{\mathbb{C}(\mathsf{I}, {i^{\mathsf{R}}}_B)}"] \arrow[rr, equal] & & \mathbb{C}(\mathsf{I}, B)
\end{tikzcd}$$

[6] One might well at first think that a good type for an equivalent version for the structural law L of a left-skew closed category or for the structural law L^{R} of a right-skew closed category could be $A \multimap B \to (B \multimap C) \multimap (A \multimap C)$. But neither is true. The closest that we can get to is the existence of a canonical natural transformation typed $A \multimap^{\mathsf{L}} B \to (B \multimap^{\mathsf{L}} C) \multimap^{\mathsf{R}} (A \multimap^{\mathsf{L}} C)$ in the situation of Theorem 4.12. There we have two internal homs $\multimap^{\mathsf{L}}$, $\multimap^{\mathsf{R}}$; one is part of a left-skew, the other of a right-skew closed structure on the same category. Cf. the question of Fritz and the discussion contribution of Kataoka on the ncatlab page [27].

(cR2)
$$\begin{tikzcd}
\mathbb{C}(A, \mathsf{I} \multimap C \multimap D) \arrow[r, "{L^{\mathsf{R}}{}_{A,\mathsf{I},C,D}}"] & \int^X \mathbb{C}(A, X \multimap D) \times \mathbb{C}(\mathsf{I}, C \multimap X) \arrow[d, "{\int^X \mathbb{C}(A,X \multimap D) \times j^{\mathsf{R}}{}_{C,X}}"] \\
\mathbb{C}(A, C \multimap D) \arrow[u, "{\mathbb{C}(A, i^{\mathsf{R}}{}_{C \multimap D})}"] & \int^X \mathbb{C}(A, X \multimap D) \times \mathbb{C}(C, X) \arrow[l, "\cong"']
\end{tikzcd}$$

(cR3)
$$\begin{tikzcd}
\mathbb{C}(A, B \multimap \mathsf{I} \multimap D) \arrow[r, "{L^{\mathsf{R}}{}_{A,B,\mathsf{I},D}}"] & \int^X \mathbb{C}(A, X \multimap D) \times \mathbb{C}(B, \mathsf{I} \multimap X) \\
\mathbb{C}(A, B \multimap D) \arrow[u, "{\mathbb{C}(A, B \multimap i^{\mathsf{R}}{}_{D})}"] & \int^X \mathbb{C}(A, X \multimap D) \times \mathbb{C}(B, X) \arrow[l, "\cong"'] \arrow[u, "{\int^X \mathbb{C}(A,X \multimap D) \times \mathbb{C}(B, i^{\mathsf{R}}{}_{X})}"']
\end{tikzcd}$$

(cR4)
$$\begin{tikzcd}
\mathbb{C}(\mathsf{I}, B \multimap C \multimap D) \arrow[r, "{L^{\mathsf{R}}{}_{\mathsf{I},B,C,D}}"] \arrow[d, "{j^{\mathsf{R}}{}_{B,C \multimap D}}"'] & \int^X \mathbb{C}(\mathsf{I}, X \multimap D) \times \mathbb{C}(B, C \multimap X) \arrow[d, "{\int^X j^{\mathsf{R}}{}_{X,D} \times \mathbb{C}(B, C \multimap X)}"] \\
\mathbb{C}(B, C \multimap D) & \int^X \mathbb{C}(X, D) \times \mathbb{C}(B, C \multimap X) \arrow[l, "\cong"']
\end{tikzcd}$$

(cR5)
$$\begin{tikzcd}
\mathbb{C}(A, B \multimap C \multimap D \multimap E) \arrow[r, "{L^{\mathsf{R}}{}_{A,B,C,D \multimap E}}"] & \int^X \mathbb{C}(A, X \multimap D \multimap E) \times \mathbb{C}(B, C \multimap X) \arrow[d, "{\int^X L^{\mathsf{R}}{}_{A,X,D,E} \times \mathbb{C}(B,C \multimap X)}"] \\
\int^Y \mathbb{C}(Y, C \multimap D \multimap E) \times \mathbb{C}(A, B \multimap Y) \arrow[u, "\cong"] \arrow[dd, "{\int^Y L^{\mathsf{R}}{}_{Y,C,D,E} \times \mathbb{C}(A,B \multimap Y)}"'] & \int^X (\int^Y \mathbb{C}(A, Y \multimap E) \times \mathbb{C}(X, D \multimap Y)) \times \mathbb{C}(B, C \multimap X) \arrow[d, "\cong"] \\
 & \int^Y \mathbb{C}(A, Y \multimap E) \times \int^X \mathbb{C}(X, D \multimap Y) \times \mathbb{C}(B, C \multimap X) \arrow[d, "\cong"] \\
\int^Y (\int^X \mathbb{C}(Y, X \multimap E) \times \mathbb{C}(C, D \multimap X)) \times \mathbb{C}(A, B \multimap Y) \arrow[d, "\cong"'] & \int^Y \mathbb{C}(A, Y \multimap E) \times \mathbb{C}(B, C \multimap D \multimap Y) \arrow[d, "{\int^Y \mathbb{C}(A,Y \multimap E) \times L^{\mathsf{R}}{}_{B,C,D,Y}}"] \\
\int^X (\int^Y \mathbb{C}(Y, X \multimap E) \times \mathbb{C}(A, B \multimap Y)) \times \mathbb{C}(C, D \multimap X) \arrow[d, "\cong"'] & \int^Y \mathbb{C}(A, Y \multimap E) \times \int^X \mathbb{C}(B, X \multimap Y) \times \mathbb{C}(C, D \multimap X) \arrow[d, "\cong"] \\
\int^X \mathbb{C}(A, B \multimap X \multimap E) \times \mathbb{C}(C, D \multimap X) \arrow[r, "{\int^X L^{\mathsf{R}}{}_{A,B,X,E} \times \mathbb{C}(C,D \multimap X)}"] & \int^X (\int^Y \mathbb{C}(A, Y \multimap E) \times \mathbb{C}(B, X \multimap Y)) \times \mathbb{C}(C, D \multimap X)
\end{tikzcd}$$

This category is called right-normal, left-normal *resp.* associative-normal *if* i^{R}, j^{R} *resp.* L^{R} *is a natural isomorphism.*

The equations (cR1)–(cR5) admit various interderivable versions. For example, (cR1) is interderivable with $j^{\mathsf{R}}{}_{\mathsf{I},\mathsf{I}}\, i^{\mathsf{R}}{}_{\mathsf{I}} = \mathsf{id}_{\mathsf{I}}$. But similar interderivable versions of (cR2)–(cR5) are all clumsier because of the coends involved.

Here is a particular construction of right-skew closed categories. Thanks to the implication

$$\begin{array}{c}
[(B \boxtimes_{D,D} C, \boxtimes\mathsf{E}_{D,B,C}, \boxtimes\mathsf{I}_{D,B,C})] \in \int^X \mathbb{C}(B \multimap (C \multimap D), X \multimap D) \times \mathbb{C}(B, C \multimap X) \\
\hline
1 \to \int^X \mathbb{C}(B \multimap (C \multimap D), X \multimap D) \times \mathbb{C}(B, C \multimap X) \\
\hline\hline
\mathbb{C}(A, B \multimap C \multimap D) \overset{L^{\mathsf{R}}{}_{A,B,C,D}}{\to} \int^X \mathbb{C}(A, X \multimap D) \times \mathbb{C}(B, C \multimap X)
\end{array}$$

the natural transformation L^{R} can obtained from a functor $\boxtimes : (\mathbb{C}^{\mathsf{op}} \times \mathbb{C}) \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ (kind of a weak tensor) and two natural (in fact, extranatural) transformations $\boxtimes\mathsf{E}$ and $\boxtimes\mathsf{I}$ typed

$$\boxtimes\mathsf{E}_{D,B,C} : B \multimap (C \multimap D) \to (B \boxtimes_{D,D} C) \multimap D \qquad \boxtimes\mathsf{I}_{D,B,C} : B \to C \multimap (B \boxtimes_{D,D} C)$$

(the elimination and introduction rules for this weak tensor; notice that elimination is only available for motif D) satisfying certain equations deriving (cR2–cR5). We note that $\int^X \mathbb{C}(B \multimap (C \multimap D), X \multimap D) \times \mathbb{C}(B, C \multimap X)$ consists of triples of an object X and maps $e : B \multimap (C \multimap D) \to X \multimap D$ and $i : B \to C \multimap X$ identified up to the least equivalence relation $\sim_{D,B,C}$ closed under the rule

$$\frac{\xi : X \to X' \quad e = (\xi \multimap D) \circ e' \quad (C \multimap \xi) \circ i = i'}{(X, e, i) \sim_{D,B,C} (X', e', i')}$$

Example 4.6 Given a right-skew closed category $(\mathbb{C}, \mathsf{I}, \multimap)$ and a monad T on $\mathbb{C}$ that is *oplax closed*, i.e., endowed with a map $\mathsf{e}^{\mathsf{R}} : T\,\mathsf{I} \to \mathsf{I}$ and a natural transformation $\mathsf{c}^{\mathsf{R}}_{A,B,C} : \mathbb{C}(T\,A, T\,B \multimap C) \to \int^X \mathbb{C}(T\,X, C) \times \mathbb{C}(A, B \multimap X)$ cohering suitably with $i^{\mathsf{R}}, j^{\mathsf{R}}, L^{\mathsf{R}}, \eta, \mu$. Then the category $\mathbb{C}$ also has a right-skew closed structure $(\mathsf{I}, {}^{T}\!\!\multimap)$ where $B\,{}^{T}\!\!\multimap C = T\,B \multimap C$. The structural laws are defined by

$$
\begin{aligned}
{}^{T}i^{\mathsf{R}}{}_{A} &= A \xrightarrow{i^{\mathsf{R}}{}_{A}} \mathsf{I} \multimap A \xrightarrow{\mathsf{e}^{\mathsf{R}} \multimap A} T\,\mathsf{I} \multimap A \\
{}^{T}j^{\mathsf{R}}{}_{A,B} &= \mathbb{C}(\mathsf{I}, T\,A \multimap B) \xrightarrow{\mathbb{C}(\mathsf{I}, \eta_A \multimap B)} \mathbb{C}(\mathsf{I}, A \multimap B) \xrightarrow{j^{\mathsf{R}}{}_{A,B}} \mathbb{C}(A, B) \\
{}^{T}L^{\mathsf{R}}{}_{A,B,C,D} &= \mathbb{C}(A, T\,B \multimap T\,C \multimap D) \\
&\xrightarrow{L^{\mathsf{R}}{}_{A,TB,TC,D}} \textstyle\int^X \mathbb{C}(A, X \multimap D) \times \mathbb{C}(T\,B, T\,C \multimap X) \\
&\xrightarrow{\int^X \mathsf{id} \times \mathbb{C}(TB, \mu_X \multimap X)} \textstyle\int^X \mathbb{C}(A, X \multimap D) \times \mathbb{C}(T\,B, T\,T\,C \multimap X) \\
&\xrightarrow{\int^X \mathsf{id} \times \mathsf{c}^{\mathsf{R}}_{B,TC,X}} \textstyle\int^X \mathbb{C}(A, X \multimap D) \times \int^Y \mathbb{C}(T\,Y, X) \times \mathbb{C}(B, T\,C \multimap Y) \\
&\xrightarrow{\cong} \textstyle\int^Y (\int^X \mathbb{C}(A, X \multimap D) \times \mathbb{C}(T\,Y, X)) \times \mathbb{C}(B, T\,C \multimap Y) \\
&\xrightarrow{\cong} \textstyle\int^Y \mathbb{C}(A, T\,Y \multimap D) \times \mathbb{C}(B, T\,C \multimap Y)
\end{aligned}
$$

Example 4.7 In Example 3.5, we assumed a closed category $(\mathbb{C}, \mathsf{I}, \multimap)$ with a lax closed monad (T, c) and tried to establish that the left-skew closed category $(\mathbf{Kl}(T), \mathsf{I}, \multimap^T)$ where $B \multimap^T C = B \multimap T\,C$ was normal. This did not work out. We had candidates for the inverses of $\widehat{\jmath}^T$, i^T, $\widehat{L}^T$ that we built from $\widehat{\jmath}^{-1}$, i^{-1}, $\widehat{L}^{-1}$ using the right-strength $\mathsf{cst}^{\mathsf{R}}$ of T, but none of them were both a section and a retraction.

The constructions of those candidates turn out to give us a right-skew closed structure on $(\mathbf{Kl}(T), \mathsf{I}, \multimap^T)$ if we build them from j^{R}, i^{R}, L^{R} when $(\mathbb{C}, \mathsf{I}, \multimap)$ is a right-skew closed category with a monad T on it that is lax closed (the left-strength cst is needed to define $\multimap^T$ on maps, the right-strength $\mathsf{cst}^{\mathsf{R}}$ is needed to define the structural laws, commutativity is needed to ensure that they are natural).

Now, because we arranged it so, results analogous to Theorems 2.10, 3.8 are obtained, with left-skewness replaced with right-skewness.

Theorem 4.8 *Let $\mathbb{C}$ be a category with an object I and functors $\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ and $\multimap : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{C}$, with an adjunction $- \otimes B \dashv B \multimap -$ natural in B.*

Then $(\mathbb{C}, \mathsf{I}, \otimes)$ is right-skew monoidal if and only if $(\mathbb{C}, \mathsf{I}, \multimap)$ is right-skew closed. More precisely, there is a bijection between the $(\rho^{\mathsf{R}}, \lambda^{\mathsf{R}}, \alpha^{\mathsf{R}})$ right-skew monoidal and the $(i^{\mathsf{R}}, j^{\mathsf{R}}, L^{\mathsf{R}})$ right-skew closed structures on $(\mathbb{C}, \mathsf{I}, \otimes)$ resp. $(\mathbb{C}, \mathsf{I}, \multimap)$.

Moreover, both α^{R} *and* L^{R} *are interdefinable with a natural transformation*

$$\mathsf{p}^{\mathsf{R}}_{A,B,C} : A \multimap (B \multimap C) \to (A \otimes B) \multimap C$$

(an internal version of the left transpose of the adjunction).

Proof. The required bijections between ρ^{R} and i^{R}; λ^{R} and j^{R} and α^{R}, L^{R} and p^{R} follow from the sequences of bijections of natural transformations in the backward direction in the proof of Theorem 2.10.

For this, the bijections in that proof need to be read under the following translation: $\lambda^{-1} := \lambda^{\mathsf{R}}$, $\rho^{-1} := \rho^{\mathsf{R}}$, $\alpha^{-1} := \alpha^{\mathsf{R}}$, $\widehat{\jmath}^{-1} := j^{\mathsf{R}}$, $i^{-1} := i^{\mathsf{R}}$, $\widehat{L}^{-1} := L^{\mathsf{R}}$, $\mathsf{p}^{-1} := \mathsf{p}^{\mathsf{R}}$.

Verification of the mutual entailments between the respective equations needs to be done separately. But of course the equations (cR1–cR5) were designed for this theorem to hold. □

The bijection in the proof of Theorem 4.8 interdefines $(\boxtimes, \boxtimes\mathsf{E}, \boxtimes\mathsf{I})$ with p^{R} by

$$B \boxtimes_{D^-,D^+} C = B \otimes C$$

$$B \multimap (C \multimap D) \overset{\substack{\mathsf{p}^{\mathsf{R}}_{B,C,D} \\ = \boxtimes\mathsf{E}_{D,B,C}}}{\to} (B \otimes C) \multimap D \qquad \dfrac{\overline{B \otimes C \to B \otimes C}}{B \overset{\boxtimes\mathsf{I}_{D,B,C}}{\to} C \multimap (B \otimes C)}$$

Definition 4.9 *Under the assumptions of Theorem 4.8, if* $(\mathbb{C}, \mathsf{I}, \otimes)$ *is right-skew monoidal or, equivalently,* $(\mathbb{C}, \mathsf{I}, \multimap)$ *is right-skew closed, we say that* $(\mathbb{C}, \mathsf{I}, \otimes, \multimap)$ *is right-skew monoidal closed.*

Example 4.10 Given a right-skew monoidal closed category $(\mathbb{C}, \mathsf{I}, \otimes, \multimap)$ and a monad T on $\mathbb{C}$. The monad T is oplax monoidal if and only if it is oplax closed: there is a bijection between the oplax monoidality witnesses m^{R} and the oplax closedness witnesses c^{R} of T:

$$\dfrac{\dfrac{\dfrac{T(A \otimes B) \overset{\mathsf{m}^{\mathsf{R}}_{A,B}}{\to} TA \otimes TB}{\mathbb{C}(TA \otimes TB, C) \to \mathbb{C}(T(A \otimes B), C)}}{\mathbb{C}(TA, TB \multimap C) \to \int^X \mathbb{C}(TX, C) \times \mathbb{C}(A \otimes B, X)}}{\mathbb{C}(TA, TB \multimap C) \overset{\mathsf{c}^{\mathsf{R}}_{A,B,C}}{\to} \int^X \mathbb{C}(TX, C) \times \mathbb{C}(A, B \multimap X)}$$

Suppose that T is oplax monoidal or, equivalently, oplax closed. Let $A \otimes^T B = A \otimes TB$ and $B \;{}^T\!\!\multimap C = TB \multimap C$ as in Examples 2.3 and 4.6. The adjunction $- \otimes TB \dashv TB \multimap -$ immediately gives an adjunction $- \otimes^T B \dashv B \;{}^T\!\!\multimap -$. Hence, $(\mathbb{C}, \mathsf{I}, \otimes^T, {}^T\!\!\multimap)$ is right-skew monoidal closed.

Theorem 4.11 *Given a right-skew monoidal closed category* $(\mathbb{C}, \mathsf{I}, \otimes, \multimap)$*. Then*

(i) $(\mathbb{C}, \mathsf{I}, \otimes)$ *is right-normal iff* $(\mathbb{C}, \mathsf{I}, \multimap)$ *is right-normal;*

(ii) $(\mathbb{C}, \mathsf{I}, \otimes)$ *is left-normal iff* $(\mathbb{C}, \mathsf{I}, \multimap)$ *is left-normal;*

(iii) $(\mathbb{C}, \mathsf{I}, \otimes)$ *is associative-normal iff* p^{R} *is a natural isomorphism iff* $(\mathbb{C}, \mathsf{I}, \multimap)$ *is associative-normal.*

Proof. The required bijections follow from the sequences of bijections of natural isomorphisms in the proof of Theorem 2.10 under the translation described in the proof of Theorem 4.8. □

The following two theorems are considerably more interesting. Left-skew closed $(\mathsf{I}, \multimap^{\mathsf{L}})$ and right-skew closed $(\mathsf{I}, \multimap^{\mathsf{R}})$ structures on a category $\mathbb{C}$ are in a bijective correspondence in case $\mathbb{C}$ satisfies an additional condition resembling one of the requirements imposed by Lambek in his syntactic calculus [19].

Theorem 4.12 *Let* $\mathbb{C}$ *be a category with an object* I *and functors* $\multimap^{\mathsf{L}}, \multimap^{\mathsf{R}} : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{C}$ *together with what we call the* external Lambek condition, *viz., a bijection*

$$\sigma_{A,B,C} : \mathbb{C}(A, B \multimap^{\mathsf{R}} C) \to \mathbb{C}(B, A \multimap^{\mathsf{L}} C)$$

natural in A, B *and* C.

Then $(\mathbb{C}, \mathsf{I}, \multimap^{\mathsf{L}})$ *is left-skew closed if and only if* $(\mathbb{C}, \mathsf{I}, \multimap^{\mathsf{R}})$ *is right-skew closed: there is a bijection between the* (j, i, L) *left-skew closed and the* $(i^{\mathsf{R}}, j^{\mathsf{R}}, L^{\mathsf{R}})$ *right-skew closed structures on* $(\mathbb{C}, \mathsf{I}, \multimap^{\mathsf{L}})$ *resp.* $(\mathbb{C}, \mathsf{I}, \multimap^{\mathsf{R}})$.

Moreover, both L *and* L^{R} *are also interdefinable with a natural transformation*

$$\mathsf{s}_{A,B,C} : A \multimap^{\mathsf{L}} (B \multimap^{\mathsf{R}} C) \to B \multimap^{\mathsf{R}} (A \multimap^{\mathsf{L}} C)$$

(an internal version of σ*).*

Proof. The requisite bijections between the relevant pairs of data follow from the following sequences of bijections. These hold both for natural transformations in the forward direction only (relevant here) and for natural transformations in both directions, in particular natural isomorphisms (relevant for Theorem 4.15).

$$\begin{array}{c}
\mathbb{C}(A,B) \overset{\widehat{\jmath}_A}{\rightleftarrows} \mathbb{C}(\mathsf{I}, A \multimap^{\mathsf{L}} B) \\ \hline\hline
\mathbb{C}(A,B) \rightleftarrows \mathbb{C}(A, \mathsf{I} \multimap^{\mathsf{R}} A) \\ \hline\hline
B \overset{{i^{\mathsf{R}}}_B}{\rightleftarrows} \mathsf{I} \multimap^{\mathsf{R}} B
\end{array}
\qquad\qquad
\begin{array}{c}
\mathsf{I} \multimap^{\mathsf{L}} B \overset{i_B}{\rightleftarrows} B \\ \hline\hline
\mathbb{C}(A, \mathsf{I} \multimap^{\mathsf{L}} B) \rightleftarrows \mathbb{C}(A,B) \\ \hline\hline
\mathbb{C}(\mathsf{I}, A \multimap^{\mathsf{R}} B) \overset{{j^{\mathsf{R}}}_{A,B}}{\rightleftarrows} \mathbb{C}(A,B)
\end{array}$$

$$\begin{array}{c}
\int^X \mathbb{C}(A, X \multimap^{\mathsf{L}} D) \times \mathbb{C}(B, C \multimap^{\mathsf{L}} X) \overset{\widehat{L}_{A,B,C,D}}{\rightleftarrows} \mathbb{C}(A, B \multimap^{\mathsf{L}} (C \multimap^{\mathsf{L}} D)) \\ \hline\hline
\int^X \mathbb{C}(X, A \multimap^{\mathsf{R}} D) \times \mathbb{C}(B, C \multimap^{\mathsf{L}} X) \rightleftarrows \mathbb{C}(A, B \multimap^{\mathsf{L}} (C \multimap^{\mathsf{L}} D)) \\ \hline\hline
\mathbb{C}(B, C \multimap^{\mathsf{L}} (A \multimap^{\mathsf{R}} D)) \rightleftarrows \mathbb{C}(B, A \multimap^{\mathsf{R}} (C \multimap^{\mathsf{L}} D)) \\ \hline\hline
\mathbb{C}(C, B \multimap^{\mathsf{R}} (A \multimap^{\mathsf{R}} D)) \rightleftarrows \int^X \mathbb{C}(X, C \multimap^{\mathsf{L}} D) \times \mathbb{C}(B, A \multimap^{\mathsf{R}} X) \\ \hline\hline
\mathbb{C}(C, B \multimap^{\mathsf{R}} (A \multimap^{\mathsf{R}} D)) \overset{{L^{\mathsf{R}}}_{C,B,A,D}}{\rightleftarrows} \int^X \mathbb{C}(C, X \multimap^{\mathsf{R}} D) \times \mathbb{C}(B, A \multimap^{\mathsf{R}} X)
\end{array}
\quad \| \quad
C \multimap^{\mathsf{L}} (A \multimap^{\mathsf{R}} D) \overset{\mathsf{s}_{C,A,D}}{\rightleftarrows} A \multimap^{\mathsf{R}} (C \multimap D)$$

□

The bijection in the proof of Theorem 4.12 interdefines $(\boxtimes, \boxtimes\mathsf{E}, \boxtimes\mathsf{I})$ with s:

$$B \boxtimes_{D^-, D^+} C = (B \multimap^{\mathsf{R}} (C \multimap^{\mathsf{R}} D^-)) \multimap^{\mathsf{L}} D^+$$

$$
\begin{array}{c}
\hline
B \multimap^{\mathsf{R}} (C \multimap^{\mathsf{R}} D)) \multimap^{\mathsf{L}} D \to (B \multimap^{\mathsf{R}} (C \multimap^{\mathsf{R}} D)) \multimap^{\mathsf{L}} D \\
\hline\hline
B \multimap^{\mathsf{R}} (C \multimap^{\mathsf{R}} D) \overset{\boxtimes\mathsf{E}_{D,B,C}}{\to} ((B \multimap^{\mathsf{R}} (C \multimap^{\mathsf{R}} D)) \multimap^{\mathsf{L}} D) \multimap^{\mathsf{R}} D
\end{array}
\qquad
\begin{array}{c}
A \multimap^{\mathsf{L}} (C \multimap^{\mathsf{R}} D) \overset{\mathsf{s}_{A,C,D}}{\to} C \multimap^{\mathsf{R}} (A \multimap^{\mathsf{L}} D) \\
\hline\hline
\mathbb{C}(B, A \multimap^{\mathsf{L}} (C \multimap^{\mathsf{R}} D)) \to \mathbb{C}(B, A \multimap^{\mathsf{L}} (C \multimap^{\mathsf{R}} D)) \\
\hline\hline
\mathbb{C}(A, B \multimap^{\mathsf{R}} (C \multimap^{\mathsf{R}} D)) \to \mathbb{C}(B, A \multimap^{\mathsf{L}} (C \multimap^{\mathsf{R}} D)) \\
\hline\hline
B \overset{\boxtimes\mathsf{I}_{D,B,C}}{\to} C \multimap^{\mathsf{R}} ((B \multimap^{\mathsf{R}} (C \multimap^{\mathsf{R}} D)) \multimap^{\mathsf{L}} D)
\end{array}
$$

Notice that in this case $B \boxtimes_{D,D} C$ looks like a monomorphic Church encoding of a (non-existent) right-skew tensor $B \otimes^{\mathsf{R}} C$ for the specific motif D.

Definition 4.13 *Under the assumptions of Theorem 4.12, if* $(\mathbb{C}, \mathsf{I}, \multimap^{\mathsf{L}})$ *is left-skew closed or, equivalently,* $(\mathbb{C}, \mathsf{I}, \multimap^{\mathsf{R}})$ *is right-skew closed, we say that* $(\mathbb{C}, \mathsf{I}, \multimap^{\mathsf{L}}, \multimap^{\mathsf{R}})$ *is* skew bi-closed*.*

Example 4.14 Given a skew bi-closed category $(\mathbb{C}, \mathsf{I}, \multimap^{\mathsf{L}}, \multimap^{\mathsf{R}})$ and a monad T on it. Then T is left-strong wrt. $\multimap^{\mathsf{L}}$ if and only if it is right-strong wrt. $\multimap^{\mathsf{R}}$: there is a bijection between the left-strengths cst and the right-strengths $\mathsf{cst}^{\mathsf{R}}$:

$$
\begin{array}{c}
A \multimap^{\mathsf{L}} C \overset{\mathsf{cst}_{A,C}}{\to} T\,A \multimap^{\mathsf{L}} T\,C \\
\hline\hline
\mathbb{C}(B, A \multimap^{\mathsf{L}} C) \to \mathbb{C}(B, T\,A \multimap^{\mathsf{L}} T\,C) \\
\hline\hline
\mathbb{C}(A, B \multimap^{\mathsf{R}} C) \to \mathbb{C}(B, T\,A \multimap^{\mathsf{L}} T\,C) \\
\hline\hline
B \to T\,(B \multimap^{\mathsf{R}} C) \multimap^{\mathsf{L}} T\,C \\
\hline\hline
T\,(B \multimap^{\mathsf{R}} C) \overset{\mathsf{cst}^{\mathsf{R}}_{B,C}}{\to} B \multimap^{\mathsf{R}} T\,C
\end{array}
$$

Suppose that T is lax closed wrt. $\multimap^{\mathsf{R}}$, which entails that T is right-strong wrt. $\multimap^{\mathsf{R}}$ and hence left-strong wrt. $\multimap^{\mathsf{L}}$. Let $B \multimap^{\mathsf{L}^T} C = B \multimap^{\mathsf{L}} T\,C$ and $B \multimap^{\mathsf{R}^T} C = B \multimap^{\mathsf{R}} T\,C$ as in Examples 3.5 and 4.7. By what we have learned before, $(\mathbf{Kl}(T), \mathsf{I}, \multimap^{\mathsf{L}^T})$ is left-skew closed and $(\mathbf{Kl}(T), \mathsf{I}, \multimap^{\mathsf{R}^T})$ is right-skew closed. But $(\mathbf{Kl}(T), \mathsf{I}, \multimap^{\mathsf{L}^T}, \multimap^{\mathsf{R}^T})$ is not skew bi-closed as we do not have $\mathbb{C}(A, T\,(B \multimap^{\mathsf{R}} T\,C)) \cong \mathbb{C}(B, T\,(A \multimap^{\mathsf{L}} T\,C))$ and hence cannot have $\mathbf{Kl}(T)(A, B \multimap^{\mathsf{R}^T} C) \cong \mathbf{Kl}(T)(B, A \multimap^{\mathsf{L}^T} C)$.

The normality conditions on $\multimap^{\mathsf{L}}$ and $\multimap^{\mathsf{R}}$ also correspond in the Lambek situation.

Theorem 4.15 *Given a skew bi-closed category* $(\mathbb{C}, \mathsf{I}, \multimap^{\mathsf{L}}, \multimap^{\mathsf{R}})$*. Then*

(i) $(\mathbb{C}, \mathsf{I}, \multimap^{\mathsf{L}})$ *is left-normal iff* $(\mathbb{C}, \mathsf{I}, \multimap^{\mathsf{R}})$ *is right-normal;*

(ii) $(\mathbb{C}, \mathsf{I}, \multimap^{\mathsf{L}})$ *is right-normal iff* $(\mathbb{C}, \mathsf{I}, \multimap^{\mathsf{R}})$ *is left-normal;*

(iii) $(\mathbb{C}, \mathsf{I}, \multimap^{\mathsf{L}})$ *is associative-normal iff* s *is a natural isomorphism iff* $(\mathbb{C}, \mathsf{I}, \multimap^{\mathsf{R}})$ *is associative-normal.*

Proof. The sequences of bijections shown in the proof of Theorem 4.12 above hold not only for natural transformations in the forward direction but also natural isomorphisms. □

Theorems 4.12, 4.15, connecting $\multimap^{\mathsf{L}}$ and $\multimap^{\mathsf{R}}$, are shortcuts past $\otimes^{\mathsf{L}}$ and $\otimes^{\mathsf{R}}$ in their absence. If $(\mathbb{C}, \mathsf{I}, \otimes^{\mathsf{L}}, \multimap^{\mathsf{L}})$ is left-skew closed monoidal and $(\mathbb{C}, \mathsf{I}, \otimes^{\mathsf{R}}, \multimap^{\mathsf{R}})$ is right-skew closed monoidal and there is an isomorphism $A \otimes^{\mathsf{L}} B \cong B \otimes^{\mathsf{R}} A$, then the "external Lambek" condition is a consequence of the sequence of bijections

$$\dfrac{\dfrac{\dfrac{A \to B \multimap^{\mathsf{R}} C}{A \otimes^{\mathsf{R}} B \to C}}{B \otimes^{\mathsf{L}} A \to C}}{B \to A \multimap^{\mathsf{L}} C}$$

Theorem 4.12 is in that case a corollary of Theorems 2.10, 4.2, 4.8 whereas Theorem 4.15 follows from Theorems 3.8, 4.4, 4.11.

5 Left-Skew Promonoidal and Prounital-Closed Categories

As remarked by Street [29] building on an old observation of Day [10], it is possible to view both left-skew monoidal and left-skew closed categories as instances of a common pattern, namely that of a *left-skew promonoidal category*. This is defined in [29] as a category $\mathbb{C}$ together with a pair of functors $J : \mathbb{C} \to \mathbf{Set}$ and $P : \mathbb{C}^{\mathsf{op}} \times \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbf{Set}$ and three natural transformations λ, ρ, α typed

$$\lambda_{A,B} : \mathbb{C}(A,B) \to \int^X J(X) \times P(X,A;B) \quad \rho_{A,B} : \int^X P(A,X;B) \times J(X) \to \mathbb{C}(A,B)$$
$$\alpha_{A,B,C,D} : \int^X P(A,X;D) \times P(B,C;X) \to \int^X P(X,C;D) \times P(A,B;X)$$

satisfying five equations omitted here. To understand where the definition comes from, it is helpful to invoke the more abstract notion of *left-skew monoidale in a monoidal bicategory* introduced by Lack and Street, namely an object C equipped with 1-cells $p : C \otimes C \to C$ and $j : \mathsf{I} \to C$ and a triple of 2-cells (λ, ρ, α) satisfying five equations, see [17] for details. A left-skew monoidal category is the same thing as a left-skew monoidale in **Cat**, the bicategory of categories, functors, and natural transformations, with the monoidal product given by the cartesian product of categories, whereas a left-skew promonoidal category in the above sense is just a left-skew monoidale in $\mathbf{Dist}^{\mathsf{op}(1,2)}$, the bicategory whose objects are categories, 1-cells $\mathbb{C} \nrightarrow \mathbb{D}$ are *distributors* (a.k.a. profunctors) $\mathbb{C}^{\mathsf{op}} \times \mathbb{D} \to \mathbf{Set}$, and 2-cells between distributors F and G are reverse natural transformations $\theta_{A,B} : G(A,B) \to F(A,B)$, and with the tensor again given on objects by the cartesian product of categories.

As the notation suggests, $\mathbf{Dist}^{\mathsf{op}(1,2)}$ is the result of reversing both 1-cells and 2-cells in **Dist**, the bicategory of categories and distributors in its more common orientation [3]. Recall that there are pseudofunctors $(-)^{\oplus} : \mathbf{Cat} \to \mathbf{Dist}$ and $(-)^{\ominus} : \mathbf{Cat} \to \mathbf{Dist}^{\mathsf{op}(1,2)}$, which act as the identity on objects, and which send a functor $F : \mathbb{C} \to \mathbb{D}$ to the *representable* distributors $F^{\oplus} = \mathbb{D}(-, F-)$ and $F^{\ominus} = \mathbb{D}(F-, -)$, respectively. Since both of these pseudofunctors are monoidal, they transport a left-skew monoidale in **Cat** to a pair of left-skew monoidales, one in **Dist** and one in $\mathbf{Dist}^{\mathsf{op}(1,2)}$. On the other hand, a left-skew closed category only induces a left-skew monoidale in $\mathbf{Dist}^{\mathsf{op}(1,2)}$, hence the choice of orientation in Street's definition of "left-skew promonoidal category". Explicitly, the left-skew promonoidal structure induced by a left-skew monoidal category is given by $J(A) = \mathbb{C}(\mathsf{I}, A)$ and $P(A, B; C) = \mathbb{C}(A \otimes B, C)$, while for a left-skew closed category it is given by $J(A) = \mathbb{C}(\mathsf{I}, A)$ and $P(A, B; C) = \mathbb{C}(A, B \multimap C)$. Conversely, both left-skew monoidal categories and left-skew closed categories can be defined as left-skew promonoidal categories with different representability conditions [29, Proposition 22].

A benefit of analyzing left-skew monoidal and left-skew closed categories in this way is that one may consider selectively relaxing the representability conditions. In particular, by dropping the representability condition on J one obtains a natural notion of left-skew *prounital-closed* category.[7]

Definition 5.1 *A* left-skew prounital-closed category *is a category* $\mathbb{C}$ *equipped with functors* $J : \mathbb{C} \to \mathbf{Set}$ *and* $\multimap : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{C}$ *and natural transformations* j, i, L *typed*

$$j_A \in J\,(A \multimap A) \quad i_{A,B} : J\,A \to \mathbb{C}(A \multimap B, B)$$

$$L_{A,B,C} \in \mathbb{C}(B \multimap C, (A \multimap B) \multimap (A \multimap C))$$

satisfying the following equations as well as equation (c5) *where we write* $\circ_0 : \mathbb{C}(A, B) \times J\,A \to J\,B$ *for* J *as a left action, i.e.,* $f \circ_0 e = J\,f\,e$*:*

- (p1) $e = i_{A,A}\,e \circ_0 j_A \in J\,A$ *for* $e \in J\,A$*;*
- (p2) $i_{A\multimap A, A\multimap C}\,j_A \circ L_{A,A,C} = \mathsf{id}_{A\multimap C} \in \mathbb{C}(A \multimap C, A \multimap C)$*;*
- (p3) $L_{A,B,B} \circ_0 j_B = j_{A\multimap B} \in J\,((A \multimap B) \multimap (A \multimap B))$*;*
- (p4) $i_{A,B}\,e \multimap C = ((A \multimap B) \multimap i_{A,C}\,e) \circ L_{A,B,C} \in \mathbb{C}(B \multimap C, (A \multimap B) \multimap C)$ *for* $e \in J\,A$.

The natural transformations j, i *and* L *are interdefinable with natural transformations* $\widehat{\jmath}$, $\widehat{\imath}$ *resp.* $\widehat{L}$ *typed*

$$\widehat{\jmath}_{A,B} : \mathbb{C}(A, B) \to J(A \multimap B) \qquad \widehat{\imath}_{A,B} : \textstyle\int^X \mathbb{C}(A, X \multimap B) \times J\,X \to \mathbb{C}(A, B)$$

$$\widehat{L}_{A,B,C,D} : \textstyle\int^X \mathbb{C}(A, X \multimap D) \times \mathbb{C}(B, C \multimap X) \to \mathbb{C}(A, B \multimap (C \multimap D))$$

If $\widehat{\jmath}$, $\widehat{\imath}$ *or* $\widehat{L}$ *is a natural isomorphism, then we say that the left-skew prounital-closed category* $(\mathbb{C}, J, \multimap)$ *is* left-normal, right-normal *resp.* associative-normal.

The set $J\,A$ plays the role of the set of global elements of A, replacing $\mathbb{C}(\mathsf{I}, A)$ in the absence of I. The operation $\circ_0$ is application of a map to an element, as opposed to the operation $\circ$ of composition of two maps.

Example 5.2 With the unit I and the associating term formers dropped, the non-commutative linear typed λ-calculus from Example 2.9 is a presentation of the free left-skew prounital-closed category. Just as the free left-skew closed category, it happens to be left-normal.

[7] The definition was suggested but not fully worked out by Shulman [27, Rev. 49]. The original term was 'non-unital', but this was somewhat misleading since the unit is "morally there", albeit not represented.

6 Conclusion and Future Work

We demonstrated that it pays off to match the data, equations and normality conditions of left-skew monoidal categories vs. left-skew closed categories one by one and to consider partial normality. This gives a clearer and finer picture of what is going on both in different general situations as well as in examples. In particular, closed categories in the sense of the standard terminology correspond to monoidal categories that are left-skew in regards to associativity. We showed that it makes sense to adjust the definition of closed category to include associative-normality so that closed categories then correspond to normal monoidal categories. We also demonstrated that there are well-justified notions of right-skew closed category and left-skew prounital-closed category with nontrivial examples.

This note reports only the first findings of a project. We plan to continue it along several avenues.

We intend to find more examples, in particular examples relevant to mathematical semantics of programming (based on monads, comonads, lax monoidal/closed functors etc.). We plan to determine which examples of nonmonoidal closed categories from the mathematical literature (especially the catalog of de Schipper [28, Ch. 5]) are left-skew and which are normal in the associativity respect. We intend to investigate the Yoneda embedding of skew/partially-normal monoidal/closed categories into presheaf categories as was done for monoidal closed and closed categories by Day and Laplaza [10,12,22]. Finally, in continuation to our prior work [35,32] and relating to Bourke and Lack's skew multicategories work [5], we plan to work out in detail the proof theory (sequent calculus, natural deduction) for left-skew/partially-normal monoidal, closed, prounital-closed, monoidal closed and bi-closed categories. In the left-skew monoidal sequent calculus, which we already explored [33], left-normality corresponds to the addition of an extra $\otimes$ right-introduction rule (which makes passivation invertible and renders the stoup unnecessary), right-normality corresponds to allowing the I left-introduction rule not only in the stoup but also in the passive context (generalized to all formulae made of I and $\otimes$ only) and associative-normality corresponds to similar liberalization of the $\otimes$ left-introduction rule.

In a longer term, we would also like to extend the scope of this project beyond the structures we considered here, to skew and normal versions of symmetry and braidedness for skew/partially-normal monoidal/closed categories, above all their proof theory. This is related to recent work by Bourke and Lack [6].

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