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Giry and the Machine

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Abstract

We present a general method - the Machine - to analyse and characterise in finitary terms natural transformations between well-known functors in the category **Pol** of Polish spaces. The method relies on a detailed analysis of the structure of **Pol** and a small set of categorical conditions on the domain and codomain functors. We apply the Machine to transformations from the Giry and positive measures functors to combinations of the Vietoris, multiset, Giry and positive measures functors. The multiset functor is shown to be defined in **Pol** and its properties established. We also show that for some combinations of these functors, there cannot exist more than one natural transformation between the functors, in particular the Giry monad has no natural transformations to itself apart from the identity. Finally we show how the Dirichlet and Poisson processes can be constructed with the Machine.

Keywords: probability, topology, category theory, monads

1 Introduction

Classical tools of probability theory are not geared towards compositionality, and especially not compositional approximation (Kozen, [13]). This has not prevented authors from developing powerful techniques (Chaput et al. [5], Kozen et al. [14]) based on structural approaches to probability theory (Giry, [9]). Here, we adopt a slightly different standpoint: we propose to tackle this tooling problem *globally*, by combining structural insights of **Pol** together with some classical tools of probability theory and topology put in functorial form. The outcome is the Machine, an axiomatic reconstruction in category-theoretic terms of developments carried out in [7]. Thus, we get a simpler and more conceptual proof of our previous results. We also obtain a much more comprehensive picture and prove that natural transformations between Giry-like functors are entirely characterised by their components on

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finite spaces. For instance, the monadic data of the Giry functor are easily obtained from the finite case (which is completely elementary) and applying the Machine. But the construction is not limited to probability functors: we deal similarly with the multiset and the Vietoris (the topological powerdomain of compact subsets) functors. This allows one to consider transformations mixing probabilistic and ordinary non-determinism, in a way which is reminiscent of (Keimel et al., [12]). Another byproduct of our Machine is that we reconstruct from finitary data classical objects of probability theory and statistics, namely the Poisson and Dirichlet processes. It is worth noting that Poisson, Dirichlet (and many other similar constructions obtained by recombining the basic ingredients differently) are obtained as natural and continuous maps: naturality expresses the stability of the "behaviour" in a change of granularity, and as such is a fundamental property of consistency, but continuity (which to our knowledge is proved here for the first time) expresses a no less important property, namely the robustness of the behaviour in changes in "parameters". This has potential implications in Bayesian learning.

The structure of the paper is as follows. In Sec. 3, we show that **Pol** is stratified into the subcategories $\mathbf{Pol}_f$, $\mathbf{Pol}_{cz}$ $\mathbf{Pol}_z$ of finite, compact zero-dimensional and zero-dimensional Polish spaces respectively and show how these subcategories are related. In Sec. 4, the Machine is introduced: we identify a small set of categorical conditions on functors F, G that guarantee that any natural transformation from F to G in $\mathbf{Pol}_f$ can be extended step-by-step through the subcategories to a natural transformation on **Pol**. In Sec. 5, we illustrate the workings of the Machine on natural transformations connecting the Giry and positive measure functors to combinations of the Vietoris, multiset, Giry and positive measure functors. As far as we know, the multiset functor is defined in **Pol** for the first time and its properties are established. As a first application of the Machine, we develop in Sec. 6 general criteria under which there can exist at most one natural transformation from a functor F to the Giry functor. In particular, we show that there exists at most one natural transformation between the Vietoris, multiset, positive measure and Giry functor to the Giry functor. Lastly, we show in Sec. 7 how transformations of the type $\mathsf{M}^+ \Rightarrow \mathsf{G}H$ where M^+ is the finite measure functor and H is either the multiset or the finite measure functor can be built in $\mathbf{Pol}_f$ from a single generating morphism $\mathsf{M}^+(1) \to \mathsf{G}H(1)$ and give criteria for this transformation to be natural. In particular, we show that the Dirichlet and Poisson distributions satisfy these criteria and use the Machine to build Dirichlet and Poisson processes.

2 Notations

Most of our developments take place in the category **Pol** of Polish spaces and continuous maps. **Pol** is a full subcategory of the category **Top** of topological spaces and continuous maps. **Pol** has all countable limits and all countable coproducts (Bourbaki [4], IX). The functor mapping any space to the measurable space having the same underlying set and the Borel σ-algebra and interpreting continuous maps as measurable ones will be denoted by $\mathcal{B} : \mathbf{Pol} \to \mathbf{Meas}$, where **Meas** is the category of measurable spaces and measurable maps. A *countable codirected diagram* (*ccd* for short) is given by a countable directed partial order $\mathcal{I}$ and a contravariant functor $D : \mathcal{I}^{op} \to \mathbf{Pol}$ such that for all $i \leq_{\mathcal{I}^{op}} j$, $D(i \leq_{\mathcal{I}^{op}} j)$ is surjective. We moreover assume that *ccd*s range over non-empty spaces. With that assumption, the categorical limit of a *ccd* D, which we denote by $\lim D$, is always non-empty.

3 The structure of Pol

Pol can be decomposed according to the following diagram of inclusions:

$$\begin{tikzcd} \mathbf{Pol}_f \arrow[r, hook, "I_{cz}"] & \mathbf{Pol}_{cz} \arrow[r, hook, "I_z"] & \mathbf{Pol}_z \arrow[r, hook, "I_p"] & \mathbf{Pol} \end{tikzcd} \tag{1}$$

Here, $\mathbf{Pol}_f$ is the full subcategory of finite (hence discrete) spaces, $\mathbf{Pol}_{cz}$ is the full subcategory of compact zero-dimensional spaces and $\mathbf{Pol}_z$ is the full subcategory of zero-dimensional spaces while I_{cz}, I_z and I_p are the obvious inclusion functors. To this picture, we add categories of *based spaces* and *base-preserving maps*.

Definition 3.1 (Categories of based spaces) A *based space* is a pair $(X, \mathcal{F})$ of $X \in Obj(\mathbf{Pol})$ and of a countable base $\mathcal{F}$ of the topology of X. A *base-preserving map* from $(X, \mathcal{F})$ to $(Y, \mathcal{G})$ is a function $f : X \to Y$ such that $f^{-1}(\mathcal{G}) \subseteq \mathcal{F}$ (it is therefore continuous). One easily checks that this defines a category having based spaces as objects and base-preserving maps as morphisms. We denote this category by $\mathbf{Pol}^\flat$. Similarly, a *based zero-dimensional space* is a pair $(Z, \mathcal{F})$ where $Z \in Obj(\mathbf{Pol}_z)$ and $\mathcal{F}$ is a *countable base of clopen sets which is also a boolean algebra*. We denote by $\mathbf{Pol}_z^\flat$ the category of based zero-dimensional spaces and base-preserving maps.

Of course, there exists for each such based category a (faithful, but not full!) forgetful functor, that we will denote by resp. U_z and U_p. The situation is summed up in the following commutative diagram in **Cat**:

$$\begin{tikzcd} & & \mathbf{Pol}_z^\flat \arrow[r, hook, "I_p^\flat"] \arrow[d, "U_z"] & \mathbf{Pol}^\flat \arrow[d, "U_p"] \\ \mathbf{Pol}_f \arrow[r, hook, "I_{cz}"] & \mathbf{Pol}_{cz} \arrow[ur, hook, "I_z^\flat"] \arrow[dr, hook, "I_z"'] & & \\ & & \mathbf{Pol}_z \arrow[r, hook, "I_p"'] & \mathbf{Pol} \end{tikzcd}$$

In the remainder of this section, we will unravel further relationships between these categories.

$\mathbf{Pol}_f$ is a codense subcategory of $\mathbf{Pol}_{cz}$. Objects of $\mathbf{Pol}_f$ are finite discrete spaces. Note that every subset of a discrete space is clopen; as a consequence, any map between two finite spaces is continuous. We will denote objects of $\mathbf{Pol}_f$ by their cardinality m, n. The objects of $\mathbf{Pol}_{cz}$ are the compact zero-dimensional (or *profinite*) spaces, a prime example being the Cantor space $2^{\mathbb{N}}$. These spaces are homeomorphic to limits of countable codirected diagrams (*ccd*s for short) taking values in $\mathbf{Pol}_f$. This is exactly captured by the concept of codensity (see [15], X.6).

Proposition 3.2 $\mathbf{Pol}_f$ *is codense in* $\mathbf{Pol}_{cz}$.

Proof. Let X be a compact zero-dimensional space, and consider the comma category $X \downarrow I_{cz}$. We denote by $D_X : (X \downarrow I_{cz}) \to \mathbf{Pol}_f$ the diagram corresponding to the base of this cone. It is enough to prove that for all $X \in Obj(\mathbf{Pol}_{cz})$, $X \cong \lim D_X$. Following (Mac Lane [15], IX.3), it is in turn enough to exhibit a diagram $D : \mathcal{I}^{op} \to \mathbf{Pol}_f$ verifying $X \cong \lim D$ and a cofinal ("initial" in [15]) functor $c : \mathcal{I}^{op} \to (X \downarrow I_{cz})$. Proposition 3.1 of [7] yields the existence of such a diagram D where $\mathcal{I}$ is the set of finite partitions of X taken in the boolean algebra of clopen sets of X (that we denote by $Clo(X)$), partially ordered by partition refinement and directed by partition intersection. Observe that any continuous map $f : X \to n$ induces a finite clopen partition of X by considering its fibres. Let us denote this partition by X/f. Let c be the functor mapping any finite partition $n \in \mathcal{I}^{op}$ seen as an object of $\mathbf{Pol}_f$ to the quotient map $q_n : X \twoheadrightarrow n$, and any refinement $m \leq_{\mathcal{I}^{op}} n$ to to the obvious map π_{mn} such that $q_m = \pi_{mn} \circ q_n$. For any $f : X \to n$ the partition X/f is mapped to $c(X/f) : X \to X/f$, and there trivially exists a map $\pi : c(X/f) \to f$. For any two $f, f' \in Obj(X \downarrow I_{cz})$, one can easily exhibit a partition $i \in \mathcal{I}$ of X such that there exists $\pi : c(i) \to f$ and $\pi' : c(i) \to f'$. $\square$

$\mathbf{Pol}_{cz}$ is a reflective subcategory of $\mathbf{Pol}_z^\flat$. Objects of $\mathbf{Pol}_z$ are *zero-dimensional* spaces, i.e. spaces whose topology admits a (countable) base of clopen sets. Discrete spaces (such as $\mathbb{N}$) are always zero-dimensional. A less trivial example is the Baire space $\mathbb{N}^\mathbb{N}$. The bridge between $\mathbf{Pol}_{cz}$ and $\mathbf{Pol}_z$ is provided by *compactifiying* zero-dimensional spaces, as explained in full length in ([7], Sec. 3). Let us recall the underpinnings of this compactification. Let Z be some zero-dimensional space and $\mathcal{F}$ be a countable base of clopens of Z. One easily verifies that the boolean algebra generated by $\mathcal{F}$, that we denote by $Bool(\mathcal{F})$, still generates the same topology and is still countable. Therefore, one can witout loss of generality assume that the base $\mathcal{F}$ of Z is a countable Boolean algebra of clopen sets. Let $\mathcal{I}_\mathcal{F}$ be the directed partial order of finite partitions of Z taken in $\mathcal{F}$ and let $D_\mathcal{F} : \mathcal{I}_\mathcal{F}^{op} \to \mathbf{Pol}_f$ be the diagram defined by $D_\mathcal{F}(i \in \mathcal{I}_\mathcal{F}^{op}) \triangleq i$ on objects (seeing finite partitions of Z as finite discrete spaces) and $D_\mathcal{F}(j \leq_{\mathcal{I}_\mathcal{F}^{op}} i) = q_{ij}$ where $q_{ij} : j \to i$ is the obvious quotient map.

Proposition 3.3 (Wallman compactification ([7], Prop. 3.12)) $\lim D_\mathcal{F}$ *is a zero-dimensional compactification of* Z *that we denote by* $\omega_\mathcal{F}(Z)$. *We denote by* $\eta_\mathcal{F} : Z \hookrightarrow \omega_\mathcal{F}(Z)$ *the canonical embedding of* Z *into its compactification.*

Note that this compactification is not universal, in the sense that $\mathbf{Pol}_{cz}$ is not a reflective subcategory of $\mathbf{Pol}_z$ (see [15], IV.3 for a definition of reflective subcategory). However, we will show that $\mathbf{Pol}_{cz}$ *is* a reflective subcategory of $\mathbf{Pol}_z^\flat$. In the following, recall that $Clo(X)$ is the boolean algebra of clopen sets of a compact zero-dimensional space X.

Proposition 3.4 *Let* $I_z^\flat$ *be the operation that maps any compact zero-dimensional space* X *to the pair* $(X, Clo(X))$ *and which acts identically on maps between such spaces.* $I_z^\flat$ *is a full and faithful functor from* $\mathbf{Pol}_{cz}$ *to* $\mathbf{Pol}_z^\flat$.

Proof. For any space $X \in Obj(\mathbf{Pol}_{cz})$, its boolean algebra of clopen sets $Clo(X)$ is countable and therefore, $(X, Clo(X))$ is a based zero-dimensional space. By continuity, maps between such spaces are base-preserving. Functoriality, fullness and faithfulness are trivial. □

Our compactification naturally lives in $\mathbf{Pol}_z^\flat$:

Proposition 3.5 *For any $(Z, \mathcal{F}) \in Obj(\mathbf{Pol}_z^\flat)$, the embedding $\eta_\mathcal{F} : (Z, \mathcal{F}) \to I_z^\flat(\omega_\mathcal{F}(Z))$ is base preserving.*

Proof. By construction of $\omega_\mathcal{F}(Z)$, any finite clopen partition of this space will induce through $\eta_\mathcal{F}$ a finite partition of Z taken in $\mathcal{F}$. Therefore, $\eta_\mathcal{F}$ is base preserving. □

The following proposition states the functoriality of compactification in this new setting, and the fact that $\mathbf{Pol}_{cz}$ is a reflective subcategory of $\mathbf{Pol}_z^\flat$.

Proposition 3.6 (ω as a reflector) *(i) Let $f : (Z, \mathcal{F}) \to (Z', \mathcal{F}')$ be a base-preserving map. There exists a unique $\omega_{\mathcal{F}\mathcal{F}'}(f) : \omega_\mathcal{F}(Z) \to \omega_{\mathcal{F}'}(Z')$ such that $\omega_{\mathcal{F}\mathcal{F}'}(f) \circ \eta_\mathcal{F} = \eta_{\mathcal{F}'} \circ f$. (ii) $\omega : \mathbf{Pol}_z^\flat \to \mathbf{Pol}_{cz}$ is a functor defined on objects by $\omega(Z, \mathcal{F}) \triangleq \omega_\mathcal{F}(Z)$ and on base-preserving maps $f : (Z, \mathcal{F}) \to (Z', \mathcal{F}')$ by $\omega(f) \triangleq \omega_{\mathcal{F}\mathcal{F}'}(f)$, and it is left adjoint to the inclusion functor $I_z^\flat$ (the unit being given by η).*

Proof. (i) This is Prop. 3.13 and Corollary 3.14 of [7]. Let us sketch the argument. As f is base-preserving, any finite clopen partition of Z' taken in $\mathcal{F}'$ will induce a unique finite clopen partition of Z taken in $\mathcal{F}$. Using the notations of Prop. 3.3, we deduce that $D_{\mathcal{F}'}$ is a sub-diagram of $D_\mathcal{F}$. Therefore, there exists a unique mediating map (that we denote $\omega_{\mathcal{F}\mathcal{F}'}(f)$) from $\lim D_\mathcal{F}$ to $\lim D_{\mathcal{F}'}$, i.e. from $\omega_\mathcal{F}(Z)$ to $\omega_{\mathcal{F}'}(Z')$, such that $\eta_{\mathcal{F}'} \circ f = \omega_{\mathcal{F}\mathcal{F}'}(f) \circ \eta_\mathcal{F}$. (ii) ω trivially preserves identities. For all f, f', the equality $W(f' \circ f) = W(f') \circ W(f)$ is a consequence of the uniqueness of factorisations in (i). According to (Mac Lane [15], IV.3), left adjointness of ω is a direct consequence of (i), as any map $f : (Z, \mathcal{F}) \to I_z^\flat(X)$ will factor uniquely through $\eta_\mathcal{F} : (Z, \mathcal{F}) \to I_z^\flat(\omega_\mathcal{F}(Z))$. □

This reflection is summarised in the following diagram:

$$\begin{tikzcd} \mathbf{Pol}_z^\flat \arrow[r, "\omega"'] \arrow[rr, bend left, "Id_{\mathbf{Pol}_z^\flat}"] & \mathbf{Pol}_{cz} \arrow[r, "I_z^\flat"'] \arrow[u, phantom, "\Downarrow\eta"] & \mathbf{Pol}_z^\flat \end{tikzcd} \tag{2}$$

$\mathbf{Pol}_z^\flat$ is a coreflective subcategory of $\mathbf{Pol}^\flat$. The penultimate step in our structural analysis of **Pol** is to relate $\mathbf{Pol}_z^\flat$ and $\mathbf{Pol}^\flat$. This is accomplished by associating zero-dimensional refinements to arbitrary spaces, in an operation called *zero-dimensionalisation*. Let us define this operation.

Proposition 3.7 (Zero-dimensionalisation ([7], Prop. 3.2)) *Let X be a space with underlying set $U(X)$ and let $\mathcal{F}$ be a countable base of X. The topological space $z_{\mathcal{F}}(X) \triangleq (U(X), \langle Bool(\mathcal{F})\rangle)$ having as underlying set $U(X)$ and whose topology is generated by the boolean algebra $Bool(\mathcal{F})$ verifies the following properties:*

- (i) *$z_{\mathcal{F}}(X)$ is Polish;*
- (ii) *$z_{\mathcal{F}}(X)$ is zero-dimensional.*
- (iii) *measurable sets are preserved: $\mathcal{B}(X) = \mathcal{B}(z_{\mathcal{F}}(X))$.*

In a similar fashion to compactifications, this operation is better typed as a functor from $\mathbf{Pol}^\flat$ to $\mathbf{Pol}^\flat_z$. Let us make zero-dimensionalisation into a functor:

Proposition 3.8 *Let $f : (X, \mathcal{F}) \to (Y, \mathcal{G})$ be a base-preserving map in $\mathbf{Pol}^\flat$. Then $f : (z_{\mathcal{F}}(X), Bool(\mathcal{F})) \to (z_{\mathcal{G}}(Y), Bool(\mathcal{G}))$ is base-preserving in $\mathbf{Pol}^\flat_z$. We denote by $z : \mathbf{Pol}^\flat \to \mathbf{Pol}^\flat_z$ the functor defined by $z(X, \mathcal{F}) = (z_{\mathcal{F}}(X), Bool(\mathcal{F}))$ on objects and acting identically on arrows.*

Proof. It is sufficient to consider the case of an arbitrary finite union of literals $L = A_1^{\epsilon_1} \cup \ldots \cup A_n^{\epsilon_n} \in Bool(\mathcal{G})$, where $A_i \in \mathcal{G}$ and $A_i^{\epsilon_i}$ denotes either A_i^c or A_i. We have $f^{-1}(L) = \cup_{i=1}^n f^{-1}(A_i)^{\epsilon_i}$, since f is base-preserving in $\mathbf{Pol}^\flat$ we deduce that $f^{-1}(L) \in Bool(\mathcal{F})$. Continuity of f in $\mathbf{Pol}^\flat_z$ is a direct consequence of base preservation. The fact that z is a functor is now trivial. □

The following result now follows easily:

Proposition 3.9 (z as a coreflector) *z is right adjoint to the inclusion functor $I^\flat_p$, i.e. $\mathbf{Pol}^\flat_z$ is a coreflective subcategory of $\mathbf{Pol}^\flat$.*

Proof. Observe that for all $(X, \mathcal{F}) \in Obj(\mathbf{Pol}^\flat)$, the identity function $\epsilon_{\mathcal{F}} \triangleq id : I^\flat_p z(X, \mathcal{F}) \to (X, \mathcal{F})$ is base-preserving. This indeed constitutes the counit of the coreflection: one easily verifies that for all $f : I^\flat_p(Z, \mathcal{F}) \to (X, \mathcal{G})$ there exists a unique $f' : I^\flat_p(Z, \mathcal{F}) \to I^\flat_p z(X, \mathcal{G})$ such that $f = \epsilon_{\mathcal{G}} \circ f'$ (and f' is equal to f as a function). □

This coreflection is summarised in the following diagram:

$$\begin{tikzcd} \mathbf{Pol}^\flat \arrow[r, "z"'] \arrow[rr, bend left=50, "Id_{\mathbf{Pol}^\flat}"] & \mathbf{Pol}^\flat_z \arrow[r, "I^\flat_p"'] \arrow[u, phantom, "\Uparrow\epsilon"] & \mathbf{Pol}^\flat \end{tikzcd} \tag{3}$$

Relating $\mathbf{Pol}^\flat$ and $\mathbf{Pol}$. For all space $X \in Obj(\mathbf{Pol})$, let us denote the set of countable bases of X, partially ordered by inclusion, by $Bases(X)$. Observe that $Bases(X)$ is directed by taking the union of the bases and closing under finite intersections. Accordingly, if $\mathcal{F} \subseteq \mathcal{G}$ are two countable bases of X, the identity function $id : (X, \mathcal{G}) \to (Y, \mathcal{F})$ is trivially base-preserving. This defines a codirected diagram $B_X : Bases(X)^{op} \to \mathbf{Pol}^\flat$ mapping any base $\mathcal{F}$ to $(X, \mathcal{F})$ and any pair

$\mathcal{F} \subseteq \mathcal{G}$ to the identity function. Recall that $U_p : \mathbf{Pol}^\flat \to \mathbf{Pol}$ is the base-forgetting functor. The next definition and proposition provide a universal characterisation of Polish spaces in terms of their zero-dimensionalisation.

Definition 3.10 (Diagram of zero-dimensionals) We define the *diagram of zero-dimensionals of* X:

$$Z_X \triangleq U_p I_p^\flat z B_X : Bases(X)^{op} \to \mathbf{Pol}$$

that maps bases $\mathcal{F} \in Bases(X)$ to $Z_X(\mathcal{F}) \triangleq z_{\mathcal{F}}(X)$.

We state without proof the following result, which is a category-theoretic reformulation of ([7], Theorem 3.5):

Proposition 3.11 *For all space* $X \in Obj(\mathbf{Pol})$*,* $X \cong \operatorname{colim} Z_X$*.*

In more concrete terms, any space X has the final topology for the family of identity functions $\{id : z_{\mathcal{F}}(X) \to X\}_{\mathcal{F}}$ where $\mathcal{F}$ ranges over $Bases(X)$. Let us conclude this section by summarising our structural decomposition of $\mathbf{Pol}$ in the following diagram:

$$\begin{tikzcd}
\mathbf{Pol}_f \arrow[r, hook, "I_{cz}"] & \mathbf{Pol}_{cz} \arrow[r, phantom, "\perp"] \arrow[r, hook, bend right, "I_z^\flat"'] \arrow[dr, hook, bend right, "I_z"'] & \mathbf{Pol}_z^\flat \arrow[l, bend right, "\omega"'] \arrow[d, "U_z"] \arrow[r, phantom, "\top"] \arrow[r, hook, bend right, "I_p^\flat"'] & \mathbf{Pol}^\flat \arrow[l, bend right, "z"'] \arrow[d, "U_p"] \\
& & \mathbf{Pol}_z \arrow[r, hook, "I_p"'] & \mathbf{Pol}
\end{tikzcd} \tag{4}$$

4 The Machine

We will leverage the structural decomposition of $\mathbf{Pol}$ given in the previous section to characterise some "profinite" natural transformations, in the sense that their behaviour on arbitrary spaces is entirely determined by their behaviour on finite spaces. We proceed in a stepwise and modular fashion: the Machine is presented as a series of extension theorems giving sufficient conditions for a natural transformation to be uniquely extended from a subcategory to the ambient one (Theorems 4.2-4.11). These results are combined in Theorem 4.12.

I. From $\mathbf{Pol}_f$ to $\mathbf{Pol}_{cz}$. One can completely characterise the subcategory of the functor category $[\mathbf{Pol}_{cz}; \mathbf{Pol}]$ consisting of functors commuting with certain codirected limits in terms of $[\mathbf{Pol}_f; \mathbf{Pol}]$. These functors are defined below.

Definition 4.1 ($\mathbf{Pol}_f$-continuous functors) A functor $F : \mathbf{Pol} \to \mathbf{Pol}$ is $\mathbf{Pol}_f$-*continuous* if for all *ccd* $D : \mathcal{I}^{op} \to \mathbf{Pol}_f$, $F(\lim D) \cong \lim FD$.

The key result is the following:

Theorem 4.2 *Let* $F, G : \mathbf{Pol}_{cz} \Rightarrow \mathbf{Pol}$ *be two functors. If* G *is* $\mathbf{Pol}_f$*-continuous, then* $Nat(F|_{\mathbf{Pol}_f}, G|_{\mathbf{Pol}_f}) \cong Nat(F, G)$*.*

This isomorphism arises from the existence of a functor computing right Kan extension along I_{cz} (see [15], X), denoted by $\mathrm{Ran}_{I_{cz}}$ in the following:

Proposition 4.3 *The functor* $\mathrm{Ran}_{I_{cz}} : [\mathbf{Pol}_f; \mathbf{Pol}] \to [\mathbf{Pol}_{cz}; \mathbf{Pol}]$ *is full and faithful.*

Proof. In the following, for any $X \in Obj(\mathbf{Pol}_{cz})$, $D_X : (X \downarrow I_{cz}) \to \mathbf{Pol}_f$ stands for the diagram verifying $X \cong \lim D_X$ (see proof of Prop. 3.2). We first prove that any functor $F : \mathbf{Pol}_f \to \mathbf{Pol}$ admits a right Kan extension $\mathrm{Ran}_{I_{cz}} F$ along I_{cz}. Following (Mac Lane [15], X.3, Corollary 4) it is sufficient to prove that for all $X \in Obj(\mathbf{Pol}_{cz})$, the diagram $F \circ D_X : (X \downarrow I_{cz}) \to \mathbf{Pol}$ has a limit. By a cofinality argument similar to that used in the proof of Prop. 3.2, one can show that $\lim F \circ D_X \cong \lim F \circ D$ for a countable diagram D and since $\mathbf{Pol}$ is countably complete this limit exists, therefore F admits a right Kan extension. Let us prove that the extension is full and faithful. Since I_{cz} is full and faithful, the universal arrow $\epsilon_F : (\mathrm{Ran}_{I_{cz}} F)I_{cz} \Rightarrow F$ is an iso. Given $F, G : \mathbf{Pol}_f \to \mathbf{Pol}_{cz}$ and $\alpha : F \Rightarrow G$, there exists a unique $\sigma : \mathrm{Ran}_{I_{cz}} F \Rightarrow \mathrm{Ran}_{I_{cz}} G$ such that $\alpha \circ \epsilon_F : (\mathrm{Ran}_{I_{cz}} F)I_{cz} \to G$ factors as $\alpha \circ \epsilon_F = \epsilon_G \circ \sigma I_{cz}$. Therefore, $\mathrm{Ran}_{I_{cz}}$ defines a functor from $[\mathbf{Pol}_f; \mathbf{Pol}]$ to $[\mathbf{Pol}_{cz}; \mathbf{Pol}]$ which is full and faithful by the bijection $Nat(\mathrm{Ran}_{I_{cz}} F, \mathrm{Ran}_{I_{cz}} G) \cong Nat(F, G)$. □

Proof. [Theorem 4.2] Prop. 4.3 and the universal property of Ran yields an isomorphism $Nat(F|_{\mathbf{Pol}_f}, G|_{\mathbf{Pol}_f}) \cong Nat(F, \mathrm{Ran}_{I_{cz}} G|_{\mathbf{Pol}_f})$. Recall that $\mathrm{Ran}_{I_{cz}} G|_{\mathbf{Pol}_f}(X) = \lim G \circ D_X \cong \lim G \circ D$ where D_X and D are as in the proof of Prop. 4.3. By $\mathbf{Pol}_f$-continuity of G, $\mathrm{Ran}_{I_{cz}} G|_{\mathbf{Pol}_f}(X) \cong G(\lim D) = G(X)$. □

II. From $\mathbf{Pol}_{cz}$ to $\mathbf{Pol}^\flat_z$. As seen in Prop. 3.6, the Wallman compactification makes $\mathbf{Pol}_{cz}$ into a reflective subcategory of $\mathbf{Pol}^\flat_z$. The extension of a natural transformation from $\mathbf{Pol}_{cz}$ to $\mathbf{Pol}^\flat_z$ can be framed componentwise as a *restriction* of the natural transformation to a space embedded into its compactification, that we construct using intersections.

Definition 4.4 (Intersections, preservation of intersections) If $j_1 : X \hookrightarrow Z, j_2 : Y \hookrightarrow Z$ are two embeddings, we define the intersection $X \cap Y \to Z$ as the pullback of j_1 and j_2 (Eq. 5). We say that an endofunctor $G : \mathbf{Pol} \to \mathbf{Pol}$ preserves intersections if the diagram in Eq. 6 is an intersection.

$$\begin{tikzcd} X \cap Y \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, hook, "j_1"] \\ Y \arrow[r, hook, "j_2"'] & Z \end{tikzcd} \qquad (5)$$

$$\begin{tikzcd} G(X \cap Y) \arrow[r, "G(p_1)"] \arrow[d, "G(p_2)"'] \arrow[dr, phantom, "\lrcorner", very near start] & G(X) \arrow[d, hook, "G(j_1)"] \\ G(Y) \arrow[r, hook, "G(j_2)"'] & G(Z) \end{tikzcd} \qquad (6)$$

The following Lemma characterises the topology of intersections in $\mathbf{Pol}$.

Lemma 4.5 *$X \cap Y$ is the* **Set***-theoretic intersection of X, Y together with the subspace topology induced by Z.*

Recall that if $f : X \to Y$ is a morphism in a category $\mathbb{C}$, its *cokernel pair* (if it exists) is the pushout of f with itself (Mac Lane [15], III.3). In **Top**, there is a well-known characterisation of embeddings as limits of their cokernel pair (see e.g. (Adamek et al. [1], 7.56-7.58)). In $\mathbf{Pol}$, we have the following:

$$\begin{tikzcd} X \arrow[r, hook, "f"] \arrow[d, hook, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, hook, "j_1"] \\ Y \arrow[r, hook, "j_2"'] & Y +_X Y \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Proposition 4.6 *Let X, Y be Polish and $f : X \hookrightarrow Y$ be an embedding. Then (i) the pushout object $Y +_X Y$ is Polish, (ii) the cokernel arrows $j_1, j_2 : Y \to Y +_X Y$ are embeddings and (iii) the intersection of j_1 and j_2 is homeomorphic to X.*

The following Lemma ensures that the pushout object of an embedding with range in $\mathbf{Pol}_{cz}$ is still compact zero-dimensional.

Lemma 4.7 *Let $f : X \hookrightarrow Y$ be an embedding in* $\mathbf{Pol}$ *such that $Y \in Obj(\mathbf{Pol}_{cz})$. Then $Y +_X Y \in Obj(\mathbf{Pol}_{cz})$.*

Proof. The proof that $Y +_X Y$ is Polish is routine. It thus remains to see that it is compact and zero-dimensional. Since finite unions of compacts are compact, the coproduct $Y + Y$ is compact. By universality of coproducts, the cokernel maps $j_1, j_2 : Y \to Y +_X Y$ define a unique continuous map $j_1 + j_2 : Y + Y \to Y +_X Y$, which is easily seen to be surjective, and it follows that $Y +_X Y$ is the continuous image of a compact, i.e. is compact. To see that it is zero-dimensional, we use the fact that on compact Hausdorff spaces zero-dimensionality coincides with being totally disconnected. Let $x \in Y +_X Y$ and let U_x be a subset such that $x \in U_x$. We can assume w.l.o.g. that x is in the first copy of Y and that U_x is included in this copy. Since Y is totally disconnected, if $U_x \neq \{x\}$ it can be written as the union of two disjoint opens V_1, V_2 in the subspace topology induced by Y and and thus also by $Y +_X Y$. It follows that if $U_x \neq \{x\}$ it cannot be connected in $Y +_X Y$. □

Theorem 4.8 *Let $F, G : \mathbf{Pol}_z^\flat \to \mathbf{Pol}$ be a pair of functors such that G preserves embeddings and intersections. Then $Nat(F, G) \cong Nat(F|_{\mathbf{Pol}_{cz}}, G|_{\mathbf{Pol}_{cz}})$.*

Proof. In the interest of readability, we will elude the inclusion $I_z^\flat : \mathbf{Pol}_{cz} \to \mathbf{Pol}_z^\flat$. Let $\alpha : F|_{\mathbf{Pol}_{cz}} \Rightarrow G|_{\mathbf{Pol}_{cz}}$ be a natural transformation. We prove that (i) for all $X \in Obj(\mathbf{Pol}_z^\flat)$, $\alpha_{\omega(X)} : F(\omega(X)) \to G(\omega(X))$ restricts uniquely to a morphism $\alpha_X : F(X) \to G(X)$ such that $\alpha_{\omega(X)} \circ (F\eta)_X = (G\eta)_X \circ \alpha_X$, and (ii) this restriction uniquely extends α to a natural transformation from F to G.

(i) Consider, given $X \in Obj(\mathbf{Pol}_z^\flat)$, the embedding $\eta_X : X \hookrightarrow \omega(X)$. By Prop. 4.6, X is the intersection of the cokernel maps $j_1, j_2 : \omega(X) \hookrightarrow \omega(X) +_X \omega(X)$. Moreover by Lemma 4.7, there exists a component $\alpha_{\omega X +_X \omega X}$. By functoriality and naturality of η, the diagram in Fig. 1 (ignoring α_X) commutes. Since G preserves embeddings and intersections, there exists a unique mediating map $\alpha_X : F(X) \to G(X)$ making the whole diagram commute.

(ii) Finally, we need to check that extending α to $F|_{\mathbf{Pol}_z^\flat} \to G|_{\mathbf{Pol}_z^\flat}$ in this way is natural. Let $f : X \to Y$ in $\mathbf{Pol}_z^\flat$ and let η_X, η_Y denote the embeddings of X and Y in their respective zero-dimensional compactifications. The corresponding diagram is depicted in Fig. 2. The top, bottom, front, back and right-hand square commute, and it follows that $(G\eta)_Y \circ G(f) \circ \alpha_X = (G\eta)_Y \circ \alpha_Y \circ F(f)$. Since η_Y is an embedding and since G preserves embeddings, $(G\eta_Y)$ is an embedding and in particular is injective, and it follows that i.e. $G(f) \circ \alpha_X = \alpha_Y \circ F(f)$ as desired.

$$\begin{tikzcd}
 & F(\omega(X)) \arrow[rr, "F(j_2)"] \arrow[dd, "\alpha_{\omega(X)}"'] & & F(\omega(X) +_X \omega(X)) \arrow[dd, "\alpha_{\omega(X)+_X\omega(X)}"] \\
F(X) \arrow[ur, "(F\eta)_X"] \arrow[rr, "(F\eta)_X"] \arrow[dd, dashed, "\alpha_X"'] & & F(\omega(X)) \arrow[ur, "F(j_1)"'] \arrow[dd, "\alpha_{\omega(X)}"'] & \\
 & G(\omega(X)) \arrow[rr, "G(j_2)"] & & G(\omega(X) +_X \omega(X)) \\
G(X) \arrow[ur, "(G\eta)_X"] \arrow[rr, "(G\eta)_X"] & & G(\omega(X)) \arrow[ur, "G(j_1)"'] &
\end{tikzcd}$$

Fig. 1.

$$\begin{tikzcd}
 & F(Y) \arrow[rr, "(F\eta)_Y"] \arrow[dd, "\alpha_Y"'] & & F(\omega(Y)) \arrow[dd, "\alpha_{\omega(Y)}"] \\
F(X) \arrow[ur, "F(f)"] \arrow[rr, "(F\eta)_X"] \arrow[dd, "\alpha_X"'] & & F(\omega(X)) \arrow[ur, "F(\omega(f))"'] \arrow[dd, "\alpha_{\omega(X)}"'] & \\
 & G(Y) \arrow[rr, hook, "(G\eta)_Y"] & & G(\omega(Y)) \\
G(X) \arrow[ur, "G(f)"] \arrow[rr, hook, "(G\eta)_X"] & & G(\omega(X)) \arrow[ur, "G(\omega(f))"'] &
\end{tikzcd}$$

Fig. 2.

□

III. From $\mathbf{Pol}_z^\flat$ to Pol. The last part of the Machine is a procedure to extend natural transformations from $\mathbf{Pol}_z^\flat$ to **Pol**. We have seen in Prop. 3.11 that Polish spaces are the colimits of their "diagrams of zero-dimensionals". We will require functors in the domain of natural transformations to commute with these colimits.

Definition 4.9 (Z-cocontinuous functors) A functor $F : \mathbf{Pol} \to \mathbf{Pol}$ is *Z-cocontinuous* if for all $X \in Obj(\mathbf{Pol})$, $F(X) \cong \operatorname{colim} FZ_X$ where Z_X is defined in Def. 3.10.

Moreover, we will require these functors to be *Z-stable*, which means that the underlying sets of the spaces in the range of the considered functors are invariant by zero-dimensionalisation. As we will prove later, this is for instance the case of the Giry, multiset and list functors.

Definition 4.10 (Z-stable functor) A functor $F : \mathbf{Pol} \to \mathbf{Pol}$ is *Z-stable* if $UFX = UFZ_X(\mathcal{F})$ for all $\mathcal{F} \in Bases(X)$.

Theorem 4.11 *Let $F, G : \mathbf{Pol} \to \mathbf{Pol}$ be a pair of functors such that F is Z-cocontinuous and Z-stable. Then $Nat(F, G) \cong Nat(FU_pI_p^\flat, GU_pI_p^\flat)$.*

Proof. Let $\alpha : FU_pI_p^\flat \Rightarrow GU_pI_p^\flat$ and $X \in Obj(\mathbf{Pol})$ be given. By Z-cocontinuity, $F(X)$ is the colimiting object of the diagram $FZ_X = FU_pI_p^\flat zB_X : Bases(X)^{op} \to \mathbf{Pol}$ (Def. 3.10). Applying α, we get a natural transformation $\alpha zB_X : FZ_X \Rightarrow GZ_X$. Composing with the counit $\epsilon : I_p^\flat z \to Id_{\mathbf{Pol}^\flat}$ yields a natural transformation $(GU_p\epsilon)(\alpha zB_X) : FZ_X \Rightarrow GU_pId_{\mathbf{Pol}^\flat}B_X$. Note that $GU_pId_{\mathbf{Pol}^\flat}B_X$ is equal to the constant functor with value $G(X)$. Therefore, we have constructed a cocone from FZ_X to $G(X)$. The situation above is summed up in the following diagram:

$$\begin{tikzcd}
 & & & \mathbf{Pol}^\flat \arrow[r, "U_p"] \arrow[dd, phantom, "\Downarrow\alpha"] & \mathbf{Pol} \arrow[dr, "F"] & \\
Bases(X)^{op} \arrow[r, "B_X"] & \mathbf{Pol}^\flat \arrow[r, "z"] \arrow[drr, bend right, "Id_{\mathbf{Pol}^\flat}"'] & \mathbf{Pol}_z^\flat \arrow[ur, "I_p^\flat"] \arrow[dr, "I_p^\flat"] \arrow[d, phantom, "\Downarrow\epsilon"] & & & \mathbf{Pol} \\
 & & {} & \mathbf{Pol}^\flat \arrow[r, "U_p"'] & \mathbf{Pol} \arrow[ur, "G"'] &
\end{tikzcd}$$

By universality, there exists a unique map $u_X : F(X) \to G(X)$ such that $u_X \circ (FU_p\epsilon B_X)_{\mathcal{F}} = (GU_p\epsilon B_X)_{\mathcal{F}} \circ (\alpha z B_X)_{\mathcal{F}}$. Let us prove naturality of $\{u_X\}_{X\in Obj(\mathbf{Pol})}$. For all $f : X \to Y$ and for all base $\mathcal{G}$ of Y, there exists a base $\mathcal{F}$ of X such that $f : (X, \mathcal{F}) \to (Y, \mathcal{G})$, is base-preserving, and by functoriality, so is $z(f) : Z_X(\mathcal{F}) \to Z_Y(\mathcal{G})$. We get the following diagram:

$$\begin{tikzcd}
FZ_X(\mathcal{F}) \arrow[rrr, "(\alpha z B_X)_{\mathcal{F}}"] \arrow[dr, "(FU_p\epsilon B_X)_{\mathcal{F}}"] \arrow[ddd, "Fz(f)"'] & & & GZ_X(\mathcal{F}) \arrow[dl, "(GU_p\epsilon B_X)_{\mathcal{F}}"'] \arrow[ddd, "Gz(f)"] \\
& FX \arrow[r, "u_X"] \arrow[d, "F(f)"'] & GX \arrow[d, "G(f)"] & \\
& FY \arrow[r, "u_Y"'] & GY & \\
FZ_Y(\mathcal{G}) \arrow[ur] \arrow[rrr, "(\alpha z B_Y)_{\mathcal{G}}"'] & & & GZ_Y(\mathcal{G}) \arrow[ul]
\end{tikzcd}$$

In the above diagram, the left and right cells commute by naturality of ϵ while the top and bottom cells commute by construction of the arrows u_X, u_Y. Note that the arrow $(FU_p\epsilon B_X)_{\mathcal{F}}$ is the image through F of the identity function $\epsilon_{\mathcal{F}} = id : Z_X(\mathcal{F}) \to X$. Since F is Z-stable, this arrow is surjective. We conclude that the central square commute, and we extend α by setting for all X $\alpha_X = u_X$ as constructed above. □

IV. The Machine. Bringing the parts of the Machine together, we obtain:

Theorem 4.12 *Let $F, G : \mathbf{Pol} \to \mathbf{Pol}$ be a pair of functors such that:*

(i) *F is Z-cocontinuous and Z-stable,*

(ii) *G is $\mathbf{Pol}_f$-continuous, preserves embeddings and intersections.*

Then one has $Nat(F, G) \cong Nat(F|_{\mathbf{Pol}_f}, G|_{\mathbf{Pol}_f})$.

5 Feeding the Machine

We now investigate the properties of some functors, with an eye on applying the Machine.

The Giry functor. For any space X, we denote by $\mathsf{G}(X)$ the space of Borel probability measures over X, endowed with the weak topology (Giry, [9]). This operation can be extended to a functor $\mathsf{G} : \mathbf{Pol} \to \mathbf{Pol}$ which admits the Giry monad structure (G, δ, μ) (Giry, [9]). The action of G on maps $f : X \to Y$ is defined by $\mathsf{G}(f)(P) \triangleq P \circ f^{-1}$. The unit is given by the Dirac delta: $\delta_X : X \to G(X)$ while the multiplication is defined by averaging: $\mu_X : G^2(X) \to G(X) \triangleq P \mapsto \int_{G(X)} p\, dP(p)$. G is a rather well-behaved functor:

Proposition 5.1 *(i) For all* ccd *D, $\mathsf{G}(\lim D) \cong \lim \mathsf{G} \circ D$; (ii) G is Z-cocontinuous and Z-stable; (iii) G preserves injections and embeddings; (iv) G preserves intersections.*

Proof. (i) is the Bochner extension theorem in functorial form ([7], Theorem 2.5). (ii) Z-cocontinuity is in ([7], Theorem 3.7); Z-stability stems from Prop. 3.7, (iii). For (iii), see e.g. ([7], Lemma 2.1). Now for (iv): let $j_1, j_2 : A, B \rightarrowtail X$ be two embeddings, let $p_1 : A \cap B \to A$ and $p_2 : A \cap B \rightarrowtail B$ be the corresponding embeddings and consider $\mu \in \mathsf{G}(A), \nu \in \mathsf{G}(B)$ such that $\mathsf{G}(j_1)(\mu) = \mathsf{G}(j_2)(\nu)$. It follows from (Kechris [11], Theorem 15.1) and the fact that p_1 is injective that whenever U is a Borel set of $A \cap B$, $p_1[U]$ is a Borel set of A, and similarly for p_2. We can therefore define $\lambda \in \mathsf{G}(A \cap B)$ by $\lambda(U) = \mu(p_1[U]) = \nu(p_2[U])$. To see that the equality on the right holds, note that since j_1 in injective $p_1[U] = j_1^{-1}(j_1[p_1[U]])$, and thus

$$\begin{aligned}\mu(p_1[U]) &= \mu(j_1^{-1}(j_1[p_1[U]])) = \mathsf{G}(j_1)(\mu)(j_1[p_1[U]]) = \mathsf{G}(j_2)(\nu)(j_1[p_1[U]]) \\ &= \mathsf{G}(j_2)(\nu)(j_2[p_2[U]]) = \nu(p_2[U])\end{aligned}$$

This assignment from pairs (μ, ν) such that $\mathsf{G}j_1(\mu) = \mathsf{G}j_2(\nu)$ to $\lambda \in \mathsf{G}(A \cap B)$ is clearly injective, and it follows that $\mathsf{G}(A \cap B) \cong \mathsf{G}A \cap \mathsf{G}B$ as sets. Since G preserves embeddings, $\mathsf{G}(j_1 \circ p_1) = \mathsf{G}(j_2 \circ p_2)$ is an embedding, and it follows that $\mathsf{G}(A \cap B)$ and $\mathsf{G}A \cap \mathsf{G}B$ are in fact homeomorphic. □

Example 5.2 Theorem 4.12 implies that the unit $\delta : \mathsf{Id} \to \mathsf{G}$ of the Giry monad is entirely determined by its finite components. We do not yet know whether G^2 is Z-cocontinuous, and thus whether the multiplication $\mu : \mathsf{G}^2 \to \mathsf{G}$ is determined by its finite components. However it follows from Theorem 4.8 that the restriction of μ to $\mathbf{Pol}_z$ *is* determined by its finite components. We conjecture that this result extends to the entire category **Pol**.

The non-zero finite measures functors. We will also consider functors closely related to G: we let M^+ be the functor mapping any space X to the space of non-zero positive finite measures over X with the weak topology, and acting on maps similarly as G. The following is trivial (consider the normalisation of a finite non-zero measure):

Proposition 5.3 *For all space X, we have the isomorphism* $\mathsf{M}^+(X) \cong \mathsf{G}(X) \times \mathbb{R}_{>0}$.

As a consequence, M^+ verifies all the properties listed in Prop. 5.1. Note that for all finite space n, $\mathsf{M}^+(n)$ is also homeomorphic to $\mathbb{R}^n_{\geq 0} \setminus \{\mathbf{0}\}$.

The multiset functor. We consider the multiset functor $\mathsf{B} : \mathbf{Pol} \to \mathbf{Pol}$. It is given explicitly by

$$\mathsf{B}(X) \triangleq \coprod_{n \in \mathbb{N}} X^n / S_n$$

where X^n/S_n is the quotient of X^n under the obvious action of S_n – the permutation group on n elements – on tuples together with the quotient topology, i.e. the final topology for the quotient map $q : X^n \twoheadrightarrow X^n/S_n$. See Appendix A for a proof that $\mathsf{B}(X)$ is Polish. Its action on maps is given by setting for any $f : X \to Y$ and $\mu \in \mathsf{B}(X)$, $\mathsf{B}(f)(\mu) = y \mapsto \sum_{x \in f^{-1}(y)} \mu(x)$. This is easily shown to be continuous. Observe also that for X finite, $\mathsf{B}(X) \cong \mathbb{N}^X$. The multiset functor verifies the following properties:

Proposition 5.4 *(i)* B *is* $\mathbf{Pol}_f$*-continuous; (ii)* B *preserves injections and embeddings; (iii)* B *preserves intersections.*

Proof. See Appendix B. □

The Vietoris functor. As a non-probabilistic example, we will consider the Vietoris functor. We recall its definition.

Definition 5.5 We denote by $\mathsf{V} : \mathbf{Pol} \to \mathbf{Pol}$ the functor mapping any space X to the space of compact subsets of X topologised with the Hausdorff distance, and mapping any continuous function $f : X \to Y$ to $\mathsf{V}(f) \triangleq K \in \mathsf{V}(X) \mapsto f(K)$.

See (Kechris [11], 4.F) for a proof that $\mathsf{V}(X)$ is indeed Polish. V has the following properties:

Proposition 5.6 *(i)* V *is* $\mathbf{Pol}_f$*-continuous; (ii)* V *preserves injections and embeddings; (iii)* V *preserves intersections.*

Proof. (i) is in Appendix B. (ii) and (iii) are in Appendix B. □

Example 5.7 An interesting example of transformation which is not natural in **Pol**, due to Michael Mislove, is provided by the *support* of a measure. Usually, the support of $p \in \mathsf{G}(X)$ is defined to be the smallest closed subset of measure 1. On finite spaces, for $p \in \mathsf{G}(n)$, we define $\mathrm{supp}_n(p) \triangleq \{x \in n \mid p(x) > 0\}$. Let us check that this is natural: for $f : m \to n$, we have that $\mathrm{supp}(G(f)(p)) = \mathrm{supp}(p \circ f^{-1}) = \{x \in n \mid f^{-1}(x) \cap \mathrm{supp}(p) \neq \emptyset\}$, i.e. $\mathrm{supp}(G(f)(p)) = f(\mathrm{supp}(p)) = \mathsf{V}(f)(\mathrm{supp}(p))$. However, this does not define a natural transformation in **Pol**: consider the sequence of measures $(p_n)_{n\in\mathbb{N}}$ on $X = \{0, 1\}$ defined by $p_n = \frac{n-1}{n}\delta_0 + \frac{1}{n}\delta_1$. p_n weakly converges to δ_0 as $n \to \infty$ and for all n, $\mathrm{supp}(p_n) = \{0, 1\}$ but $\mathrm{supp}(\delta_0) = \{0\}$. Therefore, supp is not continuous!

6 Rigidity

The results presented in Sec. 4 allow to construct natural transformations from finitary specifications. In this section, we apply these results to exhibit striking *rigidity* properties of G and related functors.

Definition 6.1 A pair of functors $F, G : \mathbb{C} \to \mathbb{D}$ is called *rigid*, if there exists at most one natural transformation $\eta : F \Rightarrow G$. In particular, we will say that an functor $F : \mathbb{C} \to \mathbb{D}$ is **rigid** if the identity natural transformation $id : F \Rightarrow F$ is the only natural transformation that exists from F to itself.

For each finite space k and functor $T : \mathbf{Pol} \to \mathbf{Pol}$, there exists a canonical action of S_k, the permutation group over k elements, given by:

$$\alpha : S_k \times T(k) \to T(k), (\pi, x) \mapsto T\pi(x)$$

We will call this action the *canonical action*. We will call an element $x \in T(k)$ stabilised by the entire group S_k under the canonical action an *isotropic element*. Isotropic elements will play a crucial role in our theorem.

Theorem 6.2 (Rigidity Theorem) *Let $H : \mathbf{Pol} \to \mathbf{Pol}$ be a subfunctor of the Giry monad G satisfying the following conditions: (i) $H(k) = \mathsf{G}(k)$ for every finite Polish space k; (ii) H is $\mathbf{Pol}_f$-continuous;(iii) H preserves injections. Let also $T : \mathbf{Pol} \to \mathbf{Pol}$ be a functor such that (iv) for each finite Polish space k there exists a dense subset $Q_k \subseteq T(k)$ with the property that if $x \in Q_k$ there exists a finite Polish space k', a morphism $f : k' \to k$ and an isotropic element $x' \in T(k')$ such that $T(f)(x') = x$. In these circumstances the pair (T, H) is rigid.*

We prove this theorem in steps. But let us first show some example of functors satisfying the property above.

Example 6.3 Let us show that the Vietoris functor V satisfies the condition (iv). Note first that for every k, the full set $k \in \mathsf{V}(k)$ is isotropic: for any $\pi \in S_k$ $\alpha(\pi, k) = \mathsf{V}\pi(k) = k$ since π is bijective. Now take $Q_k = \mathsf{V}(k)$ (which is trivially dense) and $x = \{x_1, \ldots, x_n\} \in \mathsf{V}(k)$. Consider the full set $n \in \mathsf{V}(n)$ along with the map $f : n \to k, i \mapsto x_i$, it is clear that $\mathsf{V}(f(n)) = x$, and n is isotropic.

Example 6.4 The Giry monad G satisfies all conditions of Theorem 6.2: it satisfies (i) trivially, it satisfies (ii) and (iii) by Prop. 5.1. Let us show that it satisfies (iv) as well. Note first that the uniform probabilities are the isotropic elements: if $\left(\frac{1}{k}, \ldots, \frac{1}{k}\right)$ denotes the uniform distribution on k elements, then

$$\alpha\left(\pi, \left(\frac{1}{k}, \ldots, \frac{1}{k}\right)\right) = \mathsf{G}(\pi)\left(\frac{1}{k}, \ldots, \frac{1}{k}\right) = \left(\frac{1}{k}, \ldots, \frac{1}{k}\right) \circ \pi^{-1} = \left(\frac{1}{k}, \ldots, \frac{1}{k}\right)$$

Consider now $Q_k = \Delta_k \cap \mathbb{Q}^k$, the rational probabilities on k elements. It is clearly dense in $\mathsf{G}(k)$. Any $x \in Q_k$, can without loss of generality be written as $\left(\frac{p_1}{n}, \ldots, \frac{p_m}{n}\right)$ for a common denominator n. Now consider the projection map defined by

$$p : n \to k, i \mapsto \begin{cases} 1 & \text{if } 1 \leq i \leq p_1 \\ 2 & \text{if } p_1 + 1 \leq i \leq p_1 + p_2 \\ \ldots & \\ k & \text{if } \sum_{i=1}^{k-1} p_i + 1 \leq i \leq \sum_{i=1}^{k} p_i \end{cases}$$

It is easy to check from this definition that $\left(\frac{p_1}{n}, \ldots, \frac{p_m}{n}\right) = \mathsf{G}(p)\left(\frac{1}{n}, \ldots, \frac{1}{n}\right)$, where $\left(\frac{1}{n}, \ldots, \frac{1}{n}\right)$ is isotropic.

Example 6.5 Let $\mathsf{M}^+ : \mathbf{Pol} \to \mathbf{Pol}$ be the finite non-zero positive measure functor. It follows easily from Prop. 5.3 that this functor satisfies condition (iv): the isotropic elements are those of the shape $((1/k, \ldots, 1/k), \lambda)$ for $\lambda \in \mathbb{R}_{>0}$. A dense subset is provided by $(\mathbb{Q}^k \cap \mathsf{G}(k)) \times \mathbb{R}_{>0}$ and the same argument as in Example 6.4 shows that every element $((p_1/n, \ldots, p_k/n), \lambda)$ is the image of $((1/n, \ldots, 1/n), \lambda)$ by $\mathsf{G}(p) \times id$ with p defined as in Example 6.4.

Example 6.6 The multiset functor B also has the property (iv). $\mathsf{B}(k)$ has one isotropic element: the unordered list $[(1,\dots,k)]$, and any $[(x_1,\dots,x_k)] \in \mathsf{B}(k)$ is the image of $[(1,\dots,k)]$ under $\mathsf{B}(f)$ for the map $f : k \to k, i \mapsto x_i$ (which might very well not be injective).

Let us proceed to the proof of Theorem 6.2. The following settles the finite case:

Lemma 6.7 *Let (T,H) be a pair of functors satisfying the conditions of Theorem 6.2, then (T,H) is rigid on $\mathbf{Pol}_f$.*

Proof. Let $\nu : T \Rightarrow H$ be a natural transformation. We first show that if $x \in T(k)$ is isotropic then

$$\nu_k(x) = \left(\frac{1}{k},\dots,\frac{1}{k}\right) \tag{7}$$

where $\left(\frac{1}{k},\dots,\frac{1}{k}\right)$ denotes the uniform probability distribution on k. Fix $i \in \{1,\dots,k\}$, and consider the permutations $(ij) \in S_k, 1 \leq j \leq k$ sending i to j, j to i and leaving all other elements of k unchanged. We have

$$\begin{aligned}
\nu_k(x)(i) &= \nu_k(T(ij)(x))(i) && (x \text{ isotropic})\\
&= H(ij)(\nu_k(x))(i) && (\text{By naturality})\\
&= \mathsf{G}(ij)(\nu_k(x))(i) && (H = \mathsf{G} \text{ on } \mathbf{Pol}_f)\\
&= \nu_k(x)(ij)^{-1}(i) && (\text{By def. of } \mathsf{G})\\
&= \nu_k(x)(j) && (\text{By def. of } (ij))
\end{aligned}$$

Since this holds for every $1 \leq j \leq k$ we have $\sum_{j=1}^{k} \nu_k(x)(j) = \sum_{j=1}^{k} \nu_k(x)(i) = k\nu_k(x)(i) = 1$ and thus $\nu_k(x)(i) = \frac{1}{k}$ for every $1 \leq i \leq k$, i.e. $\nu_k(x) = \left(\frac{1}{k},\dots,\frac{1}{k}\right)$. Let us now consider an arbitrary $x \in Q_k$, by assumption there exist $f : k' \to k$ and an isotropic element $x' \in T(k')$ such that $T(f)(x') = x$. It follows that

$$\begin{aligned}
\nu_k(x) &= \nu_k(T(f)(x')) && (\text{By assumption on } T)\\
&= H(f)(\nu_k(x')) && (\text{By naturality})\\
&= \mathsf{G}(f)(\nu_k(x')) && (H = \mathsf{G} \text{ on } \mathbf{Pol}_f)\\
&= \mathsf{G}(f)\left(\frac{1}{k'},\dots,\frac{1}{k'}\right) && (x' \text{ is isotropic and (7)})
\end{aligned}$$

Clearly, the same reasoning applies to any other natural transformation $\rho : T \Rightarrow H$. We have thus shown that for each finite Polish set k, ν_k is unique on a dense subset Q_k of $T(k)$. Since ν_k is a morphism in **Pol** it is continuous, and since Polish spaces are complete, it is in fact Cauchy-continuous. It follows that the restriction of ν_k to Q_k has a unique extension to $T(k)$. Since the restriction of ν_k to Q_k is unique, it follows that ν_k is also unique. □

Note that the entire group S_k was necessary to show Lemma 6.7, i.e. a weaker notion of isotropic element would not be sufficient.

Lemma 6.8 *Let (T, H) be a pair of functors satisfying the conditions of Theorem 6.2, then (T, H) is rigid on* $\mathbf{Pol}_{cz}$.

Proof. Assume $\nu : T|_{\mathbf{Pol}_f} \Rightarrow H|_{\mathbf{Pol}_f}$ is given. By Lemma 6.7, ν is unique. Since H is $\mathbf{Pol}_f$-continuous, Theorem 4.2 applies and the proof is complete. □

Lemma 6.9 *Let (T, H) be a pair of functors satisfying the conditions of Theorem 6.2, then (T, H) is rigid on* $\mathbf{Pol}_z^\flat$.

Proof. It is enough to reuse the uniqueness part of the proof of Theorem 4.8. □

We can finally prove Theorem 6.2.

Proof. (**Theorem** 6.2) Let $\alpha : T|_{\mathbf{Pol}_z^\flat} \Rightarrow H|_{\mathbf{Pol}_z^\flat}$ be given. By Lemma 6.9, α is the unique such transformation. Let $\beta, \beta' : T \Rightarrow H$ be given, extending α. For all X and $\mathcal{F} \in Bases(X)$, the identity function $id : z_{\mathcal{F}}(X) \to X$ is continuous. By the rigidity assumption, $\beta_{z_{\mathcal{F}}(X)} = \beta'_{z_{\mathcal{F}}(X)}$. Using this equation and naturality,

$$\beta_X \circ T(id) = H(id) \circ \beta_{z_{\mathcal{F}}(X)} = H(id) \circ \beta'_{z_{\mathcal{F}}(X)} = \beta'_X \circ T(id)$$

Therefore $\beta = \beta'$. □

Example 6.10 We have shown earlier that G satisfies all the conditions of Theorem 6.2. It follows that there can only exist a single natural transformation $\mathsf{G} \Rightarrow \mathsf{G}$, and since the identity transformation is natural, it follows that G is rigid.

Example 6.11 Let $\mathsf{M}^+ : \mathbf{Pol} \to \mathbf{Pol}$ be the finite positive measure functor. We can check that the following transformation is natural: define $\nu : \mathsf{M}^+ \to \mathsf{G}$ at a Polish space X by $\nu_X(Q) \triangleq A \mapsto \frac{Q(A)}{Q(X)}$ for A a Borel set of X. This is well defined since $0 < Q(X) < \infty$. It is also natural: if $f : X \to Y$ is a map in **Pol**, then for each Q in $\mathsf{M}^+(X)$ and Borel set B of Y we have:

$$\begin{aligned}\mathsf{G}(f)(\nu_X(Q))(B) = \nu_X(Q)(f^{-1}(B)) &= \frac{Q(f^{-1}(B))}{Q(X)} = \frac{Q(f^{-1}(B))}{Q(f^{-1}(Y))} \\ &= \nu_Y(\mathsf{M}^+(f)(Q))(B)\end{aligned}$$

Since M^+ satisfies (iv), it follows from Theorem 6.2, that the normalisation transformation ν we have just defined is the only natural transformation $\mathsf{M}^+ \Rightarrow \mathsf{G}$.

7 Applications

In previous work [7], we showed that a cornerstone of nonparametric Bayesian statistics, the Dirichlet process [8,10], is in fact a natural transformation from M^+ to G^2. This result hinged on a non-axiomatic version of the Machine of Sec. 4. In order to validate our new developments we first give a short construction of the Dirichlet process in axiomatic form. The value of our general framework is then illustrated by constructing the Poisson process as a natural transformation. At the heart of these constructions are families of distributions which are stable by convolution (mistakenly taken to be infinitely divisible in [7]). Common examples include: the Γ distribution, the Gaussian distribution, the Poisson distribution, etc. What examples such as Dirichlet or Poisson processes have in common is that they can all be represented by natural transformations of the shape $\mathsf{M}^+ \Rightarrow \mathsf{G}H$ where the functor H can be either B or M^+. Since M^+ is Z-cocontinuous, since G and H are $\mathbf{Pol}_f$-continuous, preserve injections, embeddings and intersections (see Appendix B) we can define a natural transformation of this type by restricting ourselves to $\mathbf{Pol}_f$ and running the Machine.

In the cases which we have mentioned above, the natural transformation in $\mathbf{Pol}_f$ can in fact be defined by a single map! The fundamental property which makes this possible is that both M^+ and B turn coproducts into products. When this is the case it is sometimes possible to define $\phi : \mathsf{M}^+ \Rightarrow \mathsf{G}H$ on $\mathbf{Pol}_f$ from a map $\phi_1 : \mathsf{M}^+(1) \to \mathsf{G}H(1)$. For this we need a fundamental result which holds very generally in the category **Meas** of measurable spaces and measurable maps. We define the *product measure* natural transformation between the bifunctors $\pi : \mathsf{G} - \times \mathsf{G} - \to \mathsf{G}(- \times -)$ at each pair of measurable spaces $((X, \Sigma_X), (Y, \Sigma_Y))$ by $\pi_{(X,Y)}(p, q) \mapsto p \times q$ where $p \times q$ is the product measure defined on the product σ-algebra $(\Sigma_X \otimes \Sigma_Y)$.

Theorem 7.1 *The transformation $\pi : \mathsf{G} - \times \mathsf{G} - \to \mathsf{G}(- \times -)$ is natural in both its arguments.*

Let us now fix a continuous map $\phi_1 : \mathsf{M}^+(1) \to \mathsf{G}H(1)$. For any n in $\mathbf{Pol}_f$ we use the fact that $n = \coprod_{i=1}^n 1$ and the fact that M^+ and H turn coproducts into products to define $\phi_n : \mathsf{M}^+(n) \to \mathsf{G}H(n)$ by

$$\begin{tikzcd} \mathsf{M}^+(n) \cong \mathsf{M}^+(1)^n \arrow[r, "\phi_1^n"] \arrow[rr, bend left, "\phi_n"] & (\mathsf{G}H(1))^n \arrow[r, "\bigotimes^n_{H1}"] & \mathsf{G}(H1)^n \cong \mathsf{G}H(n) \end{tikzcd}$$

where $\bigotimes^n_{H(1)}$ is the n-fold measure product at $H(1)$. The maps ϕ_n define the component of a transformation $\mathsf{M}^+ \Rightarrow \mathsf{G}H$. But when is it natural? A simple criterion is given in the following result.

Theorem 7.2 *A transformation $\phi : \mathsf{M}^+ \to \mathsf{G}H$ built as above is natural in $\mathbf{Pol}_f$* iff *the following diagrams commute:*

$$\begin{tikzcd} \mathsf{M}^+(2) \arrow[r, "\phi_2"] \arrow[d, "\mathsf{M}^+(e)"'] & \mathsf{G}H(2) \arrow[d, "\mathsf{G}H(e)"] \\ \mathsf{M}^+(1) \arrow[r, "\phi_1"'] & \mathsf{G}H(1) \end{tikzcd} \quad (8) \qquad \begin{tikzcd} \mathsf{M}^+(n) \arrow[r, "\phi_n"] \arrow[d, "\mathsf{M}^+(ij)"'] & \mathsf{G}H(n) \arrow[d, "\mathsf{G}H(ij)"] \\ \mathsf{M}^+(n) \arrow[r, "\phi_n"'] & \mathsf{G}H(n) \end{tikzcd} \quad (9)$$

$$\begin{tikzcd} \mathsf{M}^+(1) \arrow[r, "\phi_1"] \arrow[d, "\mathsf{M}^+(i_1)"'] & \mathsf{G}H(1) \arrow[d, "\mathsf{G}H(i_1)"] \\ \mathsf{M}^+(2) \arrow[r, "\phi_2"'] & \mathsf{G}H(2) \end{tikzcd} \quad (10) \qquad \begin{tikzcd} \mathsf{M}^+(1) \arrow[r, "\phi_2"] \arrow[d, "\mathsf{M}^+(i_2)"'] & \mathsf{G}H(2) \arrow[d, "\mathsf{G}H(i_2)"] \\ \mathsf{M}^+(2) \arrow[r, "\phi_1"'] & \mathsf{G}H(2) \end{tikzcd} \quad (11)$$

where $e : 2 \to 1$ is the obvious unique epimorphism, $(ij) : n \to n$ is any permutation of two elements of n, and $i_1, i_2 : 1 \to 2$ are the two injections of 1 into $2 = 1 + 1$.

Proof. Any map $f : m \to n$ between finite sets can be written as a permutation $\pi : n \to n$ followed by a monotone surjection $q : n \twoheadrightarrow k$ followed by a monotone injection $i : k \rightarrowtail n$. Since every permutation of n can be written as a composition of permutation of two elements, repeated usage of Diagram (9) shows that $\mathsf{G}H\pi \circ \phi_n = \phi_n \circ \mathsf{M}^+\pi$. Monotone surjections $q : m \twoheadrightarrow n$ can be written as a composition of maps of the shape

$$id_1 + id_1 + \ldots + e + id_1 + \ldots + id : k \to k - 1$$

For notational clarity let us consider the case $e + id_1 : 3 \twoheadrightarrow 2$. The following square commutes:

$$\begin{tikzcd} \mathsf{M}^+(3) \cong \mathsf{M}^+(2) \times \mathsf{M}^+(1) \arrow[r, "\phi_2 \times \phi_1"] \arrow[d, "\mathsf{M}^+(e) \times id_1"'] & \mathsf{G}H(2) \times \mathsf{G}H(1) \arrow[r, "\otimes"] \arrow[d, "\mathsf{G}H(e) \times id_1"] & \mathsf{G}(H(2) \times H(1)) \cong \mathsf{G}H(3) \arrow[d, "\mathsf{G}(H(e) \times id_1)"] \\ \mathsf{M}^+(2) \cong \mathsf{M}^+(1) \times \mathsf{M}^+(1) \arrow[r, "\phi_1 \times \phi_1"'] & \mathsf{G}H(1) \times \mathsf{G}H(1) \arrow[r, "\otimes"'] & \mathsf{G}(H(1) \times H(1)) \cong \mathsf{G}H(2) \end{tikzcd}$$

Indeed, the right-hand side square commutes by Theorem 7.1, whilst the left-hand side square commutes by assumption that Diagram 8 commutes. Monotone injections are treated in a similar way. □

We will call a family of probability distributions $\phi_n : \mathsf{M}^+(n) \to \mathsf{G}H(n)$ *additive* if (8) holds, *exchangeable* if (9) holds, and say that it *admits zero parameters* if (10) and (11) hold.

The Γ distribution $\Gamma_1 : \mathsf{M}^+(1) \to G\mathsf{M}^+(1)$ maps any parameter $\lambda \in \mathsf{M}^+(1)$ to a probability with density $x \mapsto \frac{x^{\lambda-1}e^{-x}}{\Gamma(\lambda)}$ w.r.t. Lebesgue [3]. The family of probability distributions Γ_n generated by Γ_1 is clearly exchangeable; it is also additive [7] and one can easily adapt the definition so that it admits zero parameters. It follows from Theorem 7.2 that $\Gamma_n : \mathsf{M}^+(n) \to \mathsf{G}\mathsf{M}^+(n)$ is a natural transformation on $\mathbf{Pol}_f$ which extends to **Pol**. The Dirichlet process is then simply defined as $\mathscr{D} : \mathsf{M}^+ \Rightarrow G^2 \triangleq$

$(\mathsf{G}\nu)\Gamma$, where $\nu : \mathsf{M}^+ \Rightarrow \mathsf{G}$ is the normalisation natural transformation (unique, by rigidity!).

Similarly, if we define $\Pi_1 : \mathsf{M}^+(1) \to \mathsf{GB}(1) \cong \mathsf{G}(\mathbb{N})$ by $\Pi_1(\lambda)(k) = \frac{\lambda^k e^{-\lambda}}{k!}$, then it is well-known that the family Π_n generated by Π_1 (similarly to the previous case) is additive. It is also clearly exchangeable. Finally to allow for zero parameters, we extend $\Pi_1 : \mathsf{M}_{\geq 0}(1) \to \mathsf{G}(\mathbb{N})$ by putting $\Pi_1(0) = \delta_0$, the Dirac delta at 0. It is clear that for any test function $f : \mathbb{N} \to \mathbb{R}$

$$\sum_{k=0} f(k)\frac{\lambda^k e^{-\lambda}}{k!} = f(0)e^{-\lambda} + \sum_{k=1} f(k)\frac{\lambda^k e^{-\lambda}}{k!} \quad \xrightarrow{\lambda \to 0} \quad f(0) = \sum_k f(k)\delta_0$$

i.e. our extension is continuous for the weak topology. This fact is the exact analogue of Proposition 4.2 in [7]. The family $\Pi_n : \mathsf{M}_{\geq 0}(n) \to \mathsf{G}\mathbb{N}^n$ thus defines a natural transformation in $\mathbf{Pol}_f$ by Theorem 7.2, and by applying the Machine we produce a natural transformation on $\mathbf{Pol}$. The processes $\Pi_X : \mathsf{M}^+_{\geq 0}(X) \to \mathsf{GB}(X)$ (for X in $\mathbf{Pol}$) defined by this natural transformation are very well-known in probability theory, they are the (inhomogeneous) *Poisson point processes* on X parameterised by a measure on X.

8 Outlook

Our results allow the compositional and finitary approximation of a class of parameterised "stochastic" processes seen as natural transformations between probability-like functors satisfying some general axioms. It is worth noting that all the conditions on endofunctors that we require for the codomain of natural transformatins are preserved by composition (if we strengthen $\mathbf{Pol}_f$-continuity to commutation with all limits of *ccd*s). Indeed, we are confident that compositionality can be pushed further: following coalgebraic practice, we will investigate whether functors in e.g. the polynomial closure of Giry can be fed to the Machine. For this to happen, parts of the Machine have yet to be better understood, in particular the special role played by the requirement of Z-cocontinuity (commutation with diagrams of zero-dimensional refinements). For instance, we ignore whether the Vietoris functor and the multiset functors are Z-cocontinuous, or whether Z-cocontinuity is preserved by composition.

Rigidity is an unexpected mathematical outcome of our structural decomposition of $\mathbf{Pol}$. Where the Machine allows to prove existence of natural transformations, rigidity allows to prove *unicity* and is somewhat dual to the former. We expect that the notion of isotropic element will find applications beyond the scope of these developments.

On the applications side, we are confident that many processes beside Dirichlet and Poisson can be subject to the same treatment. Poisson-Dirichlet, Cox processes and some form of Gaussian processes seem to be easy targets. In the case of Dirichlet, we already know that the Machine allows to prove an asymptotic "learning" property. The work of (Culbertson et al, [6]) will provide a convenient setting where we will study how topological properties of Bayesian models such as continuity relate to asymptotic properties of Bayesian update. The finitary handle provided by the Machine might also be useful in deriving new computability or complexity results in the field of probability.

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A Construction of the multiset functor B

Proposition A.1 *For X Polish, let $\mathsf{B}(X) \triangleq \coprod_{n\in\mathbb{N}} X^n/S_n$, where X^n/S_n is the quotient of X^n under the obvious action of S_n on tuples with the quotient topology, i.e. the final topology for the quotient map $q : X^n \twoheadrightarrow X^n/S_n$. $\mathsf{B}(X)$ is Polish.*

Proof. We first shown that if Q is dense in X, then Q^n/S_n is dense in X^n/S_n: let U be an open set of X^n/S_n, then $q^{-1}(U)$ is open in X^n and intersects Q^n, i.e. there exists $(r_1,\dots,r_n) \in Q^n$ with $(r_1,\dots,r_n) \in q^{-1}(U)$, but this means that $q(r_1,\dots,r_n) \in U$ and $q(r_1,\dots,r_n) \in Q^n/S_n$. To see that it is completely metrisable, let d be a complete metric for X,and consider the metric on X^n/S_n given by:

$$d_q([x],[y]) = \min_{\pi\in S_n} d^n(x,\pi(y))$$

where $[x],[y]$ represent the orbits of $x,y \in X^n$ respectively, and d^n is the product metric given by

$$d^n((x_1,\dots,x_n),(y_1,\dots,y_n)) = \left(\sum_i d(x_i,y_i)^p\right)^{\frac{1}{p}} \tag{A.1}$$

for some $0 < p < \infty$ (any choice of p generates an equivalent topology on X^n). Note that d^n is invariant under permutations of S_n, i.e. for any permutation $\pi \in S_n$, $d^n(x,y) = d^n(\pi(x),\pi(y))$ since this simply amounts to re-arranging the summands in Eq. (A.1). It is not immediately clear that d_q is well-defined or that it defines a metric. To see that it is well defined let x' be another representative of $[x]$, then by definition there exists $\rho \in S_n$ such that $\rho(x) = x'$, and it follows that

$$\min_{\pi\in S_n} d^n(x',\pi(y)) = \min_{\pi\in S_n} d^n(\rho(x),\pi(y)) = \min_{\pi\in S_n} d^n(x,\rho^{-1}\pi(y)) = \min_{\pi\in S_n} d^n(x,\pi(y))$$

It follows that d_q is well-defined. Let us now check that it is a metric. For any x,y we clearly have $d_q([x],[y]) \geq 0$ and $d_q([x],[y]) = 0$ means that there exists $\pi \in S_n$ such that $d^n(x,\pi(y)) = 0$ i.e. $x = \pi(y)$ since d^n is a metric, and it follows that $[x] = [y]$. It is straightforward to verify symmetry condition:

$$\begin{aligned} d_q([x],[y]) &= \min_{\pi\in S_n} d^n(x,\pi(y)) \\ &= \min_{\pi\in S_n} d^n(\pi^{-1}(x),y) && d^n \text{ invariant under } \pi^{-1} \\ &= \min_{\pi\in S_n} d^n(y,\pi^{-1}(x)) && d^n \text{ is symmetric} \\ &= d_q([y],[x]) \end{aligned}$$

Finally, we need to check the triangular inequality. Since d^n satisfies the triangular inequality we have for any choice $\pi_1,\pi_2 \in S_n$ that:

$$\begin{aligned} d^n(x,\pi_1(y)) &\leq d^n(x,\pi_2(z)) + d^n(\pi_2(z),\pi_1(x)) \\ &\leq d^n(x,\pi_2(z)) + d^n(z,\pi_2^{-1}\pi_1(x)) && d^n \text{ invariant under } \pi_2^{-1} \end{aligned}$$

and it follows that $d_q([x],[y]) \leq d_q([x],[z]) + d_q([z],[y])$ since going through all the combinations $\pi_2^{-1}\pi_1$ will exhaust the entire group S_n. The fact that $(X^n/S_n, d_q)$ is complete follows from the fact that (X^n, d^n) is. Let us prove that d_q induces the topology of X^n/S_n. Let us take an open set U in X^n/S_n. By definition $q^{-1}(U)$ is open in X^n, and can therefore be written as a union of open balls (for the metric d^n) $U = \cup_i B_{d^n}(x_i, \epsilon_i)$. By definition $q^{-1}(U)$ is invariant under permutation, so

$$\begin{aligned} q^{-1}(U) &= \bigcup_{\pi\in S_n} \pi(q^{-1}(U)) = \bigcup_{\pi\in S_n} \pi\left(\bigcup_i B_{d^n}(x_i,\epsilon_i)\right) = \bigcup_{\pi\in S_n}\bigcup_i \pi\left(B_{d^n}(x_i,\epsilon_i)\right) \\ &= \bigcup_i \bigcup_{\pi\in S_n} \pi\left(B_{d^n}(x_i,\epsilon_i)\right) \end{aligned}$$

since direct images commute with unions. It follows from the fact that each π is an homeomorphism that $\bigcup_{\pi\in S_n} \pi(B_{d^n}(x_i,\epsilon_i))$ is open in X^n. Moreover, $\bigcup_{\pi\in S_n} \pi(B_{d^n}(x_i,\epsilon_i))$ is by construction invariant under permutation, so

$$q^{-1}(q(\bigcup_{\pi\in S_n} \pi(B_{d^n}(x_i,\epsilon_i)))) = \bigcup_{\pi\in S_n} \pi(B_{d^n}(x_i,\epsilon_i))$$

and therefore each $q(\bigcup_{\pi\in S_n} \pi(B_{d^n}(x_i,\epsilon_i))$ is an open in X^n/S_n. We conclude by observing that $q(\bigcup_{\pi\in S_n} \pi(B_{d^n}(x_i,\epsilon_i)) = B_{d_q}(q(x_i),\epsilon_i)$ and that

$$q^{-1}(B_{d_q}(q(x_i),\epsilon_i)) = q^{-1}(q(\bigcup_{\pi\in S_n} \pi(B_{d^n}(x_i,\epsilon_i))) = \bigcup_{\pi\in S_n} \pi(B_{d^n}(x_i,\epsilon_i))$$

is open in X^n. Therefore, the balls $B_{d_q}(q(x_i),\epsilon_i)$ are open in X^n/S_n, and since direct images commute with unions it is not difficult to see that $U = \bigcup_i B_{d_q}(q(x_i),\epsilon_i)$ is a union of opens from the basis generated by the metric. Since each X^n/S_n is Polish and since **Pol** has countable coproducts, $\mathsf{B}(X)$ is Polish. $\square$

B Properties of the functors B and V

Proposition B.1 *The multiset functor* B *preserves injections, embeddings and intersections.*

Proof. Let $i : B \rightarrowtail X$ be a mono, i.e. an injective continuous map. Note first that $\mathsf{B}(i)$ is defined component-wise i.e. via $\mathsf{B}_n(i) : B^n/S_n \rightarrowtail X^n/S_n$ injecting an equivalence class of n-tuples of element of B in X^n/S_n. The fact that $\mathsf{B}(i)$ is injective follows from the fact that every component $\mathsf{B}_n(i)$ is. Similarly, to show that if i is an embedding so is $\mathsf{B}(i)$, it is enough to show that each $\mathsf{B}_n(i)$ is an embedding. To see that this is the case we need to show that for every open U of B^n/S_n there exists an open V of X^n/S_n such that $U = V \cap B^n/S_n$ and conversely that every subset of this shape is open in B^n/S_n. We write $p_n : B^n \twoheadrightarrow B^n/S_n$ and $q_n : X^n \twoheadrightarrow X^n/S_n$. For the first direction, let U be open in B^n/S_n, it follows that $p_n^{-1}(U)$ is open in B^n, and thus that there exists an open V of X^n such that $p_n^{-1}(U) = B^n \cap V$. If we can choose V to be closed under permutation we are done. Every permutation is a bijective isometry and thus a homeomorphism, and thus an open map, i.e. $\pi(V)$ is open for every $\pi \in S_n$. It follows that

$$V^* = \bigcup_{\pi \in S_n} \pi(V)$$

is open and closed under permutations (this procedure amounts to taking all the reflections of tuples along the diagonal). It follows that $q_n^{-1}(q_n[V^*]) = V^*$ and $q_n(V^*)$ is thus open in X^n/S_n. Moreover since $B^n \cap V$ is already closed under permutations $B^n \cap V = B^n \cap V^*$, and therefore $U = B^n/S_n \cap q_n(V^*)$. For the opposite direction, let U be open in X^n/S_n and consider $U \cap B^n/S_n$, it is clear that

$$p_n^{-1}(U \cap B^n/S_n) = p_n^{-1}(U) \cap p_n^{-1}(B^n/S_n) = (q_n^{-1}(U) \cap B) \cap B = q_n^{-1}(U) \cap B$$

which is open in B^n since $q_n(U)$ is open in X^n.

For intersections, we proceed as in Proposition 5.1. Let $j_1, j_2 : A, B \rightarrowtail X$ be two embeddings, let $p_1 : A \cap B \to A$ and $p_2 : A \cap B \rightarrowtail B$ be the corresponding embeddings and consider $\mu \in \mathsf{B}A, \nu \in \mathsf{B}B$ such that $\mathsf{B}j_1(\mu) = \mathsf{B}j_2(\nu)$. We define $\lambda \in \mathsf{B}(A \cap B)$

$$\lambda(x) = \mu(p_1(x)) = \nu(p_2(x))$$

We check that the last equality holds in exactly the same way as in the proof of Proposition 5.1, and the rest of the proof also follows identically. □

Proposition B.2 *The Vietoris functor* V *preserves monomorphisms, embeddings and intersections.*

Proof. It is clear that V preserves injective maps. To see that it preserves embeddings, consider an element of the basis of the topology on $\mathsf{V}(X)$, i.e. an element of the form (Kechris [11] I, 4.F)

$$W = \{K \in \mathsf{V}(X) \mid K \subseteq U_0 \& K \cap U_1 \neq \emptyset \& \ldots \& K \cap U_n \neq \emptyset\}$$

for $U_0, \ldots, U_n$ opens in X. It follows that

$$\begin{aligned} &W \cap \mathsf{V}(B) \\ &= \{K \in \mathsf{V}(B) \mid K \subseteq (U_0 \cap B) \& K \cap (U_1 \cap B) \neq \emptyset \& \ldots \& K \cap (U_n \cap B) \neq \emptyset\} \end{aligned}$$

which is an element of the basis of the topology of $\mathsf{V}(B)$, since elements of the shape $U_i \cap B$ are precisely the opens of B. Conversely therefore, starting from an element W' of this shape it is clear that by removing all the intersections with B we get an element W of the basis of the topology on $\mathsf{V}(X)$ such that $W \cap \mathsf{V}(B) = W'$, and V thus preserves embeddings.

For intersections, let $j_1, j_2 : A, B \rightarrowtail X$ be two embeddings, let $p_1 : A \cap B \to A$ and $p_2 : A \cap B \rightarrowtail B$ be the corresponding embeddings and consider $K_A \in \mathsf{V}A, K_B \in \mathsf{V}B$ such that $\mathsf{V}j_1(K_A) = \mathsf{V}j_2(K_B)$, i.e. such that $j_1[K_A] = j_2[K_B]$. This means that $K = K_A = K_B$ is a subset of $A \cap B$. To see that it is compact in $A \cap B$, let $\bigcup_i U_i \supseteq K$

be an open cover: for each i either U_i is of the form $p_1^{-1}(V_i)$ for some V_i open in A, or it is of the form $p_2^{-1}(V_i)$ for some V_i open in B. In the latter case, since j_2 is an embedding, there exists W_i open in C such that $U_i = p_2^{-1}(j^{-1}(W_i))$, but then $U_i = p_1^{-1}(j_1^{-1}(W_i))$, which means that we can assume without loss of generality that for each i the element U_i of the cover is of the form $p_1^{-1}(V_i)$ for some V_i open in A. It is easy to see that V_i is an open cover of K in A, from which we can extract a finite sub-cover, whose inverse image under p_1 will be an finite sub-cover of K in $A \cap B$. It follows that $\mathsf{V}A \cap \mathsf{V}B \simeq \mathsf{V}(A \cap B)$ as sets, and since V preserves embeddings, they are also homeomorphic. □

Proposition B.3 B *is* $\mathbf{Pol}_f$*-continuous.*

Proof. Let $X_i, i \in I$ be a ccd of $\mathbf{Pol}_f$ objects. We show $\lim \mathsf{B}X_i = \mathsf{B}(\lim X_i)$. For this we need to show that the unique continuous map $u : \mathsf{B}(\lim X_i) \to \lim \mathsf{B}X_i$ is a homeomorphism. To show this will show that it is bijective and open. We start by defining an inverse $\phi : \lim \mathsf{B}X_i \rightarrowtail \mathsf{B}(\lim X_i)$. Since the set underlying the limits are computed in **Set**, showing that ϕ exists and is an inverse as a function will be enough to prove that u is bijective. We can assume w.l.o.g. that the morphisms between the finite Polish spaces X_i are surjective.

Given a 'thread' $(\mu_i)_{i\in I} \in \lim \mathsf{B}X_i$ we need to define a finitely supported multiset on the threads $(x_i)_{i\in I} \in \lim X_i$. For the thread (μ_i) consider the projective system of supports $(supp(\mu_i))_{i\in I}$ together with the obvious restrictions $f_{ij} \restriction supp(\mu_i)$ of the connecting maps $f_{ij} : X_i \to X_j$ which are also surjective. We claim that $\lim supp(\mu_i)$ is finite and forms the support of the multiset $\phi((\mu_i)_{i\in I})$ on $\lim X_i$. We make the following observation:

- (i) Each support is finite
- (ii) The size of the support cannot increase by following the connecting arrows, since they are surjective.
- (iii) The total mass k of $\mu_i, i \in I$ is constant throughout the thread because B applied to a connecting morphism preserves the total mass of a multiset.
- (iv) The cardinality of the set $supp(\mu_i)$ is bounded by k since we cannot assign a weight less than one to any element in the support.
- (v) There exists an $i \in I$ after which the cardinality of $supp(\mu_i)$ remains constant, i.e. such that $|supp(\mu_k)| = |supp(\mu_j)|$ for each $j > k$. If this wasn't the case it would contradict the previous points.

Thus let k be such that $|supp(\mu_k)| = |supp(\mu_j)|$ for each $j > k$, we claim that $p_k : \lim supp(\mu_i) \to supp(\mu_k)$ is a bijection. It is surjective since the connecting morphisms in the diagram are surjective. If $(x_i)_{i\in I}, (y_i)_{i\in I}$ are two threads such that $p_k(x_i) = p_k(y_i)$ then $x_k = y_k$. Now take any $k' \in I$, by co-directedness there exists $j > k, k'$ and by assumption on k, $supp(\mu_k)$ and $supp(\mu_j)$ have the same cardinality, i.e. the connecting morphism p_{jk} is bijective. There therefore exists a unique $x_j \in supp(\mu_j)$ such that $p_{jk}(x_j) = x_k = y_k$, and it follows that both thread must go through the same element at k' too, for any k', which shows that p_k is injective. We define $\phi((\mu_i)_{i\in I})$ as the multiset on $\lim X_i$ defined by:

$$(x_i)_{i\in I} \mapsto \begin{cases} 0 & \text{if } (x_i)_{i\in I} \notin \lim supp(\mu_i) \\ \mu_k(x_k) & \text{else (where } k \text{ is defined as above)} \end{cases}$$

We need to show that the definition is independent of the choice of k. Consider another index k' such that $|supp(\mu_{k'})| = |supp(\mu_j)|$ for each $j > k'$. Again by co-directedness there exists $j > k, k'$. We now calculate:

$$\begin{aligned} \mu_k(x_k) &= \mu_k(f_{jk}(x_j)) = \mathsf{B}f_{jk}(\mu_j)(f_{jk}(x_j)) = \mu_j(x_j) = \mathsf{B}f_{jk'}(\mu_j)(f_{jk'}(x_j)) \\ &= \mu_{k'}(x_{k'}) \end{aligned}$$

Let us now show that ϕ thus defined is a left and right inverse to u. Given a multiset $\mu \in \mathsf{B}(\lim X_i)$ on threads of $\lim X_i$, $u(\mu)$ is the thread of multisets ν_i on X_i defined by $\nu_i(x) = \mu[p_i^{-1}(\{x\})]$, i.e. the mass given by μ to the set of threads going through $x \in X_i$. This family forms a thread since for every $f_{ij} : X_i \to X_j$ and $y \in X_i$,

$$\nu_j(y) = \mu[p_j^{-1}(\{y\})] = \mu[p_i^{-1}(f_{ij}^{-1}(y)] = \nu_i(f_{ij}^{-1}(y)) = \mathsf{B}f_{ij}(\nu_i)(y)$$

For $\mu \in \mathsf{B}(\lim X_i)$, let $u(\mu) = (\nu_i)_{i\in I}$. The support $supp(\nu_i)$ is given by the set $Y_i \subseteq X_i$ of points traversed by a thread in the support of μ, and it is therefore not hard to see that $\lim supp(\nu_i)$ with the multiplicities defined by ϕ is precisely μ, i.e. $\phi \circ u = id_{\mathsf{B}(\lim X_i)}$. Conversely $u \circ \phi = id_{\lim X_i}$ by universality of $\lim X_i$.

Finally, we show that the unique $u : \mathsf{B}(\lim X_i) \to \lim \mathsf{B}X_i$ is a homeomorphism. We already know that it is continuous and bijective, so it remains to be shown that it is open. For this we must look at the topologies on $\mathsf{B}(\lim X_i)$ and $\lim \mathsf{B}X_i$. In the former U is an open exactly when $q_n^{-1}(U)$ is open in $(\lim X_i)^n$ for each n where $q_n : (\lim X_i)^n \twoheadrightarrow (\lim X_i)^n/S_n$. Any subset U of $\mathsf{B}(\lim X_i)$ can be written as an union of sets U_n in $(\lim X_i)^n/S_n$, so it is sufficient to show that u maps opens of $(\lim X_i)^n/S_n$ (corresponding to sets of multisets of total mass n) to opens in $\lim \mathsf{B}X_i$. It is not hard to check that U is open in $(\lim X_i)^n/S_n$ iff there exists V open in $(\lim X_i)^n$ such that $q_n[V^*] = U$ where $V^* = \bigcup_{\pi\in S_n} \pi[V]$. To check that $u(U)$ is open it is therefore enough to check that $u \circ q_n \circ \pi[V]$ is open for any open V in $(\lim X_i)^n$ and any $\pi \in S_n$; and since π is a homeomorphism this really means checking that $u \circ q_n[V]$ is open when V is. By the definition of the product topology and of the topology on $\lim X_i$ it is enough to check that $u \circ q_n[Y_k^j]$ is open for Y_k^j the set of n-tuples of threads of $\lim X_i$ whose j^{th} component goes through $Y_k \subseteq X_k$. The morphism q_n collapses such an n-tuple to a multiset on threads and $q_n[Y_k^j]$ is the set of multisets of total mass n which assigns mass at least one to threads going through Y_k.

To check that u sends these open sets to open sets we need to describe the topology on the codomain. Fortunately is it much simpler. Since each X_i is finite, X_i^n/S_n is finite, and must therefore have the discrete topology. Since $\mathsf{B}(X_i) = \coprod_n X_i^n/S_n$ is given the final topology for all the injections its topology must also be discrete. The topology on $\lim \mathsf{B}(X_i)$ is thus generated by the opens of the shape $p_i^{-1}(U_i)$ where U_i is any subset of $\mathsf{B}(X_i)$ and p_i is the canonical projection.

We can now check that u is open. Let us denote $q_n[Y_k^j] = Y_k^n$ the set of multisets of total mass n which assigns mass at least one to threads going through Y_k. It gets mapped to a set $\mathsf{B}(p_k(Y_n^k))$ of multisets on X_k, which in turns defines $u(Y_k^n) = p_k^{-1}(\mathsf{B}(p_k(Y_n^k)))$ which is indeed open. $\square$

Proposition B.4 *Let* $(X_i)_{i\in I}$ *be a* ccd *of compact spaces. Then* $\mathsf{V}(\lim X_i) \cong \lim \mathsf{V} X_i$

Proof. Let $(X_i)_{i\in I}$ be a *ccd* of compact Polish space; we must show that $\mathsf{V}\lim X_i = \lim \mathsf{V} X_i$. Let us first show that there exists a bijection between these sets. We write $p_i : \lim X_i \to X_i$ for the canonical projections. We know that there exists a unique continuous map $u : \mathsf{V}\lim X_i \to \lim \mathsf{V} X_i$; it takes a compact K of $\lim X_i$ and maps it to the thread $(p_i[K])_{i\in I}$ of $\lim \mathsf{V} X_i$ (since the continuous image of a compact is compact). Let K, K' be two compacts of $\lim X_i$ such that $p_i[K] = p_i[K']$ for every $i \in I$, for every thread (x_i) in $\lim X_i$ it is clear that $(x_i) \in K$ iff $p_i(x_i) \in p_i[K]$ iff $p_i(x_i) \in p_i[K']$ iff $(x_i) \in K'$ and thus u is injective.

We now define an inverse map $\phi : \lim \mathsf{V} X_i \to \mathsf{V}\lim X_i$ as follows. For each thread of compacts $(K_i)_{i\in I}$ in $\lim \mathsf{V} X_i$, since each X_i is Hausdorff, each K_i is closed and thus $p_i^{-1}(K_i)$ is a closed subset of $\lim X_i$. We define

$$\phi((K_i)_{i\in I}) = \bigcap_i p_i^{-1}(K_i)$$

To see that this is well-defined, we need to show that $\phi((K_i)_{i\in I})$ is compact. Since each $p_i^{-1}(K_i)$ is closed, their intersection $\phi((K_i)_{i\in I})$ is closed. We also know that since each X_i is compact $\lim X_i$ is a closed subspace of the product $\prod X_i$ which is compact by Tychonoff's theorem. It follows that $\lim X_i$ is compact, and since V sends compacts to compacts (Kechris Theorem 4.26), $\mathsf{V}\lim X_i$ is compact. Finally since $\phi((K_i)_{i\in I})$ is closed in a compact it is itself compact.

To see that ϕ is a left inverse of u, start with $K \in \mathsf{V}\lim X_i$, $u(K) = (p_i[K])_{i\in I}$ and

$$\phi(u(K)) = \bigcap_i p_i^{-1}(p_i[K])$$

Let (x_i) be a thread in K, then clearly $p_i((x_i)) = x_i \in p_i[K]$ for all i, and thus $(x_i) \in \phi(u(K))$. Conversely, let (x_i) be a thread in $\phi(u(K))$ then by definition of ϕ, $p_i((x_i)) \in p_i[K]$ for every i, i.e. $x_i \in p_i[K]$ for every i, i.e. $(x_i) \in K$, and it follows that $\phi \circ u = id_{\mathsf{V}\lim X_i}$. Conversely, ϕ is a right inverse since $u \circ \phi = id_{\lim \mathsf{V} X_i}$ by universality of u. We have thus established that u is bijective.

Finally, since $u : \mathsf{V}\lim X_i \to \lim \mathsf{V} X_i$ is a continuous bijection with a compact domain and a Hausdorff codomain, it is a homeomorphism, which concludes the proof. $\square$