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Paracomplete Logics Dual to the Genuine Paraconsistent Logics: The Three-valued Case

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Abstract

In 2016 Béziau, introduce a restricted notion of paraconsistency, the so-called *genuine paraconsistency*. A logic is *genuine paraconsistent* if it rejects the laws $\varphi, \neg\varphi \vdash \psi$ and $\vdash \neg(\varphi \wedge \neg\varphi)$. In that paper the author analyzes, among the three-valued logics, which of them satisfy this property. If we consider multiple-conclusion consequence relations, the dual properties of those above mentioned are $\vdash \varphi, \neg\varphi$ and $\neg(\psi \vee \neg\psi) \vdash$. We call *genuine paracomplete logics* those rejecting the mentioned properties. We present here an analysis of the three-valued genuine paracomplete logics.

Keywords: Many-valued logics, Paracomplete logics, Dual logic.

1 Introduction

Classically, a negation $\neg$ for a given logic **L** is semantically characterized by two properties: (1) for no sentence φ it is the case that φ and $\neg\varphi$ are simultaneously true; and (2) for no sentence φ it is the case that φ and $\neg\varphi$ are simultaneously false. Principle (1) is known as the *law of non-contradiction* (**NC**) (also known as the *law of explosion*), while (2) is usually called the *law of excluded middle* (**EM**). In terms of multiple-conclusion consequence relations,[4] both laws can be represented as follows:

$$(\mathbf{NC}) \quad \varphi, \neg\varphi \vdash \qquad \text{and} \qquad (\mathbf{EM}) \quad \vdash \varphi, \neg\varphi.$$

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[4] We can consider a multiple-conclusion consequence relation $\vdash$ as a binary relation between sets of formulas Γ and Δ, such that $\Gamma \vdash \Delta$ means that any model of every $\gamma \in \Gamma$ is also a model for some $\delta \in \Delta$ [8].

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This is why both laws are usually considered as being *dual* one from the other.[5] If $\mathcal{L}$ has a conjunction $\wedge$ (which corresponds to commas on the left-hand side of $\vdash$) and a disjunction $\vee$ (which corresponds to commas on the right-hand side of $\vdash$), then both laws can be written as

$$(\mathbf{NC})\ \ \varphi \wedge \neg\varphi \vdash \qquad \text{and} \qquad (\mathbf{EM})\ \ \vdash \varphi \vee \neg\varphi.$$

Let $\mathbf{L}$ be a logic with a negation $\neg$. If it satisfies $(\mathbf{NC})$, then the negation $\neg$ is said to be *explosive*, and $\mathbf{L}$ is *explosive* (w.r.t. $\neg$). On the other hand, $\mathbf{L}$ is said to be *paraconsistent* (w.r.t. $\neg$) if $(\mathbf{NC})$ does not hold in general, that is: $\varphi, \neg\varphi \nvdash$ in general. This means that there are formulas φ and ψ such that $\varphi, \neg\varphi \nvdash \psi$ (or $\varphi \wedge \neg\varphi \nvdash \psi$, if $\mathbf{L}$ has a conjunction). Dually, a logic $\mathbf{L}$ is *paracomplete* (w.r.t. $\neg$) if $(\mathbf{EM})$ does not hold in general, that is: $\nvdash \varphi, \neg\varphi$ in general. That is, there are formulas φ and ψ such that $\psi \nvdash \varphi, \neg\varphi$ (or $\psi \nvdash \varphi \vee \neg\varphi$, if $\mathbf{L}$ has a disjunction).

As observed in [3], $(\mathbf{NC})$ is sometimes expressed as follows:

$$(\mathbf{NC}')\ \ \vdash \neg(\varphi \wedge \neg\varphi).$$

However, as the authors have shown in [3], both principles are independent. Moreover, they show that several paraconsistent logics validate $(\mathbf{NC}')$, which is arguably counterintuitive or undesirable. This motivates the definition of a *strong paraconsistent logic* as being a logic in which both principles, $(\mathbf{NC})$ and $(\mathbf{NC}')$, are not valid in general. In subsequent papers (see, for instance, [2]) strong paraconsistent logics were rebaptized as *genuine paraconsistent logics*. Thus, a logic $\mathbf{L}$ with negation and conjunction is genuine paraconsistent if, for some formulas φ and ψ,

$$(\mathbf{GP}1)\ \ \varphi \wedge \neg\varphi \nvdash \qquad \text{and} \qquad (\mathbf{GP}2)\ \ \nvdash \neg(\psi \wedge \neg\psi).$$

Given the duality between $(\mathbf{NC})$ and $(\mathbf{EM})$, it makes sense to consider (in a logic with disjunction) the dual property of $(\mathbf{NC}')$, namely

$$(\mathbf{EM}')\ \ \neg(\varphi \vee \neg\varphi) \vdash .$$

This motivates the following definition:

Definition 1.1 A logic $\mathbf{L}$ with negation and disjunction is said to be a **genuine paracomplete logic** (or a *strong paracomplete logic*) if neither $(\mathbf{EM})$ nor $(\mathbf{EM}')$ is valid, that is: for some formulas φ and ψ,

$$(\mathbf{GP}1_D)\ \ \nvdash \varphi \vee \neg\varphi \qquad \text{and} \qquad (\mathbf{GP}2_D)\ \ \neg(\psi \vee \neg\psi) \nvdash .$$

Observe that, in terms of a tarskian (single-conclusion) consequence relation (see Definition 2.1), $(\mathbf{GP}2_D)$ is equivalent to the following:

$$(\mathbf{GP}2_D)\ \ \neg(\psi \vee \neg\psi) \nvdash \varphi \quad \text{for some formulas } \varphi, \psi.$$

[5] The readers should be advised that in [3] the authors use $\mathbf{NC}$ for representing $T \vdash \neg(\varphi \wedge \neg\varphi)$, where T is any set of formulas.

In semantical terms, if $(\mathbf{GP}2_D)$ holds for ψ then ψ is satisfiable, that is: it has some model.

Remark 1.2 If $\mathbf{L}$ is a logic with negation $\neg$ and conjunction $\wedge$ such that $\neg$ satisfies the right-introduction rule:

$$\Gamma, \varphi \vdash \Delta \quad \text{implies that} \quad \Gamma \vdash \neg\varphi, \Delta$$

(which implies that $(\mathbf{EM})$ is valid in $\mathbf{L}$, that is, $\mathbf{L}$ is not paracomplete) then $(\mathbf{NC})$ implies $(\mathbf{NC}')$. In this case, $\mathbf{L}$ is genuine paraconsistent if it satisfies $(\mathbf{GP}2)$ for some formula. Indeed, if $(\mathbf{GP}2)$ holds for some formula φ then $(\mathbf{GP}1)$ also holds for φ.

Dually, if $\mathbf{L}$ is a logic with negation $\neg$ and disjunction $\vee$ such that $\neg$ satisfies the left-introduction rule:

$$\Gamma \vdash \varphi, \Delta \quad \text{implies that} \quad \Gamma, \neg\varphi \vdash \Delta$$

(which implies that $(\mathbf{NC})$ is valid in $\mathbf{L}$, that is, $\mathbf{L}$ is not paraconsistent), then $(\mathbf{EM})$ implies $(\mathbf{EM}')$. In this case, $\mathbf{L}$ is genuine paracomplete, if it satisfies $(\mathbf{GP}2_D)$ for some formula. Indeed, if $(\mathbf{GP}2_D)$ holds for some φ, then $(\mathbf{GP}1_D)$ also holds for φ.

Example 1.3

(i) Propositional Intuitionistic logic **IPL** is paracomplete, but it is not genuine paracomplete: the formula $\neg(\varphi \vee \neg\varphi)$ is unsatisfiable.

(ii) The Belnap-Dunn logic $FOUR$ (with the truth ordering) is both genuine paraconsistent and genuine paracomplete.

(iii) Nelson logic **N4** is both genuine paraconsistent and genuine paracomplete.

(iv) The three-valued logic **MH**, introduced in [4], is genuine paracomplete and explosive. As we shall see, it is a three-valued genuine paracomplete logic which conservatively extends the 2-valued truth functions of classical logic **CPL**.

2 Basic concepts

We consider a formal propositional language $\mathcal{L} = \langle atom(\mathcal{L}), \mathcal{C}, \mathcal{A}\rangle$, where $atom(\mathcal{L})$ is an enumerable set, whose elements are called *atoms* and are denoted by lowercase letters; $\mathcal{C}$ is a set of connectives (the *signature* of $\mathcal{L}$) and $\mathcal{A}$ is a set of auxiliary symbols. Formulas are constructed as usual and will be denoted by lowercase Greek letters. The set of all formulas of $\mathcal{L}$ is denoted as $Form(\mathcal{L})$. Note that this is an algebra over $\mathcal{C}$ freely generated by $atom(\mathcal{L})$. Theories are sets of formulas and will be denoted by uppercase Greek letters.

Definition 2.1 A **(tarskian) consequence relation** $\vdash$ between theories and formulas is a relation satisfying the following properties, for every theory $\Gamma \cup \Delta \cup \{\varphi\}$:

(**Reflexivity**) if $\varphi \in \Gamma$, then $\Gamma \vdash \varphi$;

(**Monotonicity**) if $\Gamma \vdash \varphi$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash \varphi$;

(**Transitivity**) if $\Delta \vdash \varphi$ and $\Gamma \vdash \psi$ for every $\psi \in \Delta$, then $\Gamma \vdash \varphi$.

$\vdash$ is called structural if, in addition, it holds: $\Gamma \vdash \varphi$ implies that $\theta(\Gamma) \vdash \theta(\varphi)$, for every $\mathcal{L}$-substitution θ.[6] If there exists some non-empty theory Γ and some φ such that $\Gamma \nvdash \varphi$, $\vdash$ is called non-trivial.

Sometimes, in order to define a logic it is required that $\vdash$ be finitary.[7] However, here we consider a logic as it is established in Definition 2.2.

Definition 2.2 A **logic** is a pair $\mathbf{L} = \langle \mathcal{L}, \vdash_{\mathbf{L}} \rangle$, where $\vdash_{\mathbf{L}}$ is a structural and non-trivial consequence relation such that its language contains a binary connective $\rightarrow$ satisfying *Modus Ponens* (MP), that is: $\varphi \rightarrow \psi, \varphi \vdash_{\mathbf{L}} \psi$ for any formulas φ and ψ.

The notation $\Gamma \vdash_{\mathbf{L}} \varphi$ will be read as: φ can be inferred from Γ in $\mathbf{L}$. The subscript $\mathbf{L}$ will be dropped whenever the logic is clear from the context.

The usefulness of a logic depends on the available connectives in its language: thus, as we have pointed out in the introduction, for talking about paracompleteness we need a negation and a disjunction satisfying particular conditions. However, we are going to complete the language with an appropriate conjunction and an appropriate implication. In Definition 2.3 we establish some conditions on connectives so they can be considered as conjunction, disjunction, and implication.

Definition 2.3 [1] Let $\mathbf{L}$ be a logic in the language $\mathcal{L}$ with binary connectives $\wedge$, $\vee$ and $\rightarrow$. Then:

(i) $\wedge$ is said to be a **conjunction** for $\mathbf{L}$ when: $\Gamma \vdash \varphi \wedge \psi$ iff $\Gamma \vdash \varphi$ and $\Gamma \vdash \psi$.

(ii) $\vee$ is a **disjunction** for $\mathbf{L}$ when: $\Gamma, \varphi \vee \psi \vdash \sigma$ iff $\Gamma, \varphi \vdash \sigma$ and $\Gamma, \psi \vdash \sigma$.

(iii) $\rightarrow$ is an **implication** for $\mathbf{L}$ when: $\Gamma, \varphi \vdash \psi$ iff $\Gamma \vdash \varphi \rightarrow \psi$.

Observe that the notions of conjunction and disjunction are the usual ones considered in abstract logic, see for instance [9]. In order to find a suitable implication for the genuine paraconsistent logics **L3A** and **L3B** investigated in [6], the authors define the concept of classical implication as follows.

Definition 2.4 [6] Let $\mathbf{L}$ be a logic in the language $\mathcal{L}$ with a binary connective $\rightarrow$. Then $\rightarrow$ is said to be a **classical implication** if, for every $\Gamma \cup \{\varphi, \psi\} \subseteq \mathcal{L}$:

i) $\Gamma \vdash \varphi$ and $\Gamma \vdash \varphi \rightarrow \psi$ imply that $\Gamma \vdash \psi$;

ii) $\Gamma \vdash \varphi \rightarrow (\psi \rightarrow \varphi)$;

iii) $\Gamma \vdash \Big(\varphi \rightarrow (\psi \rightarrow \sigma)\Big) \rightarrow \Big((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \sigma)\Big)$.

It is not difficult to prove in the context of tarskian consequence relations that the notions of implication in Definition 2.3 and Definition 2.4 agree. The usual manner to define many-valued logics is by means of a matrix.

Definition 2.5 A **matrix** for a language $\mathcal{L}$, is a structure $M = \langle V, D, F \rangle$, where:

[6] That is, for every endomorphism θ over $\mathcal{L}$.

[7] Informally speaking, it means that every deduction can be obtained from a finite number of hypothesis.

V is a non-empty set of truth values (domain);

D is a subset of V (set of designated values);

$F := \{f_c \mid c \in \mathcal{C}\}$ is a set of truth functions, such that $f_c : D^n \to D$ if c is a logical connective in $\mathcal{L}$ with arity n.

Observe that $A = \langle V, F\rangle$ is an algebra for the signature $\mathcal{C}$ of $\mathcal{L}$.

Definition 2.6 Given a language $\mathcal{L}$, a function $v : atom(\mathcal{L}) \longrightarrow V$ that maps atoms into elements of the domain is called a **valuation** over M.

Any valuation v can be uniquely extended, as usual, to a homomorphism $v : Form(\mathcal{L}) \longrightarrow V$ such that $v(c(\alpha_1, \ldots, \alpha_n)) = f_c(v(\alpha_1), \ldots, v(\alpha_n))$. Now we can define the notion of model:

Definition 2.7 Given a matrix M, a valuation v over M is a **model** of the formula φ, denoted by $v \models_M \varphi$, if $v(\varphi) \in D$. A model of a set of formulas is a model of each of its elements. A formula φ is a **tautology** in M, denoted by $\models_M \varphi$, if every valuation is a model of φ.

Whenever the matrix is clear from the context, the subscript will be dropped. It is also possible to define a consequence relation by means of a matrix.

Definition 2.8 [1] Given a matrix M, its **induced consequence relation**, denoted by $\vdash_M$, is defined by: $\Gamma \vdash_M \varphi$ if every model of Γ is a model of φ. We denote by $\mathbf{L}_M = \langle \mathcal{L}, \vdash_M\rangle$ the logic obtained with this consequence relation.

In case a logic is defined via the induced consequence relation of a matrix M and the cardinality of set of truth values of M is $n < \omega$ then the logic is called an n-valued logic.

Now we define *neoclassical* connectives. This name can be easily understood if we identify the 'True' value with 'designated value' and the 'False' value with 'non-designated value'. These conditions are generalizations of those that satisfy and in some way define the nature of the connectives in Classical Logic.

Definition 2.9 [6] Let $M = \langle V, D, F\rangle$ be a matrix, $\overline{D}$ the set of non-designated values, and v any valuation over M. Then:

(i) $\wedge$ is a **neoclassical conjunction**, if it holds that:
$$v(\varphi \wedge \psi) \in D \text{ iff } v(\varphi) \in D \text{ and } v(\varphi) \in D.$$

(ii) $\vee$ is a **neoclassical disjunction**, if it holds that:
$$v(\varphi \vee \psi) \in \overline{D} \text{ iff } v(\varphi) \in \overline{D} \text{ and } v(\varphi) \in \overline{D}.$$

(iii) $\to$ is a **neoclassical implication**, if it holds that:
$$v(\varphi \to \psi) \in D \text{ iff } v(\varphi) \in \overline{D} \text{ or } v(\psi) \in D.$$

Observe that conditions of neoclassicality of Definition 2.9 are more restrictive than those on Definition 2.3. Specifically, we have that items (ii) and (iii) on Definition 2.9 imply items (ii) and (iii) on Definition 2.3. Moreover, item (i) on Definition 2.9 is equivalent to item (i) on Definition 2.3.

Definition 2.10 A n-valued function $\circledast$ of arity k ($\circledast : V^k \longrightarrow V$) is a:

Conservative extension of a m-valued function $\odot : V_1^k \longrightarrow V_1$ where $V_1 \subsetneq V$ and $|V_1| = m$, if the restriction of $\circledast$ to V_1 coincide with $\odot$ (i.e. $\circledast|_{V_1} = \odot$).

Molecular if the range of it is a proper subset of V. [3]

Definition 2.11 Let $\mathbf{L_1}$ be a n-valued logic whose set of truth values is V_1 and $\mathbf{L_2}$ a m-valued logic whose set of truth values is V_2, such that $V_1 \subset V_2$ and the set of connectives of $\mathbf{L_1}$ is a subset of the connectives of $\mathbf{L_2}$. $\mathbf{L_2}$ is called a **conservative extension** of $\mathbf{L_1}$ if all the truth functions in $\mathbf{L_1}$ are conservatively extended in $\mathbf{L_2}$.

The last definition can be recast in algebraic terms as follows: given $\mathbf{L_1}$ and $\mathbf{L_2}$ as above, let A_i be the algebra underlying the matrix M_i of $\mathbf{L_i}$, for $i = 1, 2$. Then, $\mathbf{L_2}$ is a conservative extension of $\mathbf{L_1}$ if and only if A_1 is a proper subalgebra of the reduct of A_2 to the signature of A_1.

3 Three-valued genuine paracomplete logics

In this section we study logics $L_M = \langle \mathcal{L}, \vdash_M \rangle$, where $M = \langle \{0, 1, 2\}, D, F \rangle$ and $0, 2$ are identified with *False* and *True* respectively. This implies that $2 \in D$, $0 \notin D$. We are looking for three-valued genuine paracomplete logics extending conservatively classical logic, apart from some extra conditions such as neoclassicality, non-molecularity, etc.

3.1 Independence of EM *and* EM′

In Definition 1.1, two conditions for a logic be genuine paracomplete were required. Let us see that these conditions are independent. If we define negation as $v(\neg\varphi) = 2 - v(\varphi)$ and disjunction as the maximum of the values of the disjuncts, then depending on the choice of the set of designated values we have one and just one principle satisfied. On the one hand, if the set of designated values is $\{1, 2\}$, we can see in Table 1 that the third column from left to right is composed only by designated values. Therefore **EM** is satisfied, meanwhile **EM′** is not, since in the fourth column there is a row which has one designated value. On the other hand, if we take the set of designated values as $\{2\}$, then the fourth column has not designated values and so **EM′** is satisfied, but since the third column has a not designated row **EM** does not hold. This shows that, in order to get a three-valued genuine paracomplete logic, it is necessary to use a different combination of truth tables for the connectives of negation and disjunction.

3.2 Genuine Paracomplete Negation

Let us start by analyzing the negation. Since we are considering connectives that are conservative extensions of the 2-valued truth functions of classical logic, we have already fixed some of the values of the truth table for negation, namely those that are boxed in Table 2. As a result of this, only the second row in the table should

φ	$\neg\varphi$	$\varphi \vee \neg\varphi$	$\neg(\varphi \vee \neg\varphi)$
0	2	2	0
1	1	1	1
2	0	2	0

Table 1
Independency of principles **EM** and **EM**′

φ	$\neg\varphi$
0	$\boxed{2}$
1	n
2	$\boxed{0}$

Table 2
Possible negations

be analyzed in order to fix the value of the variable n, that denotes the unknown value for negation.

Note that we cannot assign 2 to n. Otherwise, in Table 2, we would have that, in every row, either φ or $\neg\varphi$ are designated, validating **EM** in terms of multiple-conclusion consequence relations. Therefore, n must be in $\{0, 1\}$. Up to this point, we know that $2 \in D$ and $0 \notin D$ but, if we set 1 as designated, once again in every row either φ or $\neg\varphi$ will be designated, thus validating **EM**. Hence, in a three-valued genuine paracomplete logic we must have $D = \{2\}$.

3.3 Genuine Paracomplete Disjunction

As in the case of negation, due to the condition of being conservative extension, there are some fixed values in the truth table for the disjunction; they will be boxed in Table 3 in order to be identified. We want to obtain a neoclassical disjunction in order to keep the semantical behavior of the classical disjunction. This condition fixes two more values, which are circled in Table 3, and restricts the value of the three remaining ones as not designated. The symmetry condition reduces the number of variables to d_1, $d_2 \in \overline{D}$, since $0 \vee 1 = 1 \vee 0 = d_1$. Finally, the values for d_1 and d_2 depend on the choice of n in the truth table for negation, either $n = 0$ or $n = 1$. Let us analyze by cases.

Case $n = 0$

Considering the negation whose table takes the value 0 for n, we have the following sub-cases:

(i) If $d_1 = 0$, $\mathbf{DP1_D}$ and $\mathbf{DP2_D}$ hold and Definition 1.1 is satisfied, regardless of the value of d_2, as Table 4 shows. $\mathbf{DP1_D}$ holds since, in the third column,

$\vee$	0	1	2
0	$\boxed{0}$	d_1	$\boxed{2}$
1	d_1	d_2	2
2	$\boxed{2}$	2	$\boxed{2}$

Table 3
Possible disjunctions

φ	$\neg\varphi$	$\varphi \vee \neg\varphi$	$\neg(\varphi \vee \neg\varphi)$
0	2	2	0
1	0	0	2
2	0	2	0

Table 4
Truth tables for $\mathbf{DP1_D}$ and $\mathbf{DP2_D}$ case $n = 0$ and $d_1 = 0$

φ	$\neg\varphi$	$\varphi \vee \neg\varphi$	$\neg(\varphi \vee \neg\varphi)$
0	2	2	0
1	1	0	2
2	0	2	0

Table 5
Truth tables for $\mathbf{DP1_D}$ and $\mathbf{DP2_D}$ case $n = 1$ and $d_2 = 0$

$\varphi \vee \neg\varphi$ is not a tautology due to the value 0 in the second row. On the other hand, $\mathbf{DP2_D}$ holds since, in the fourth column, $\neg(\varphi \vee \neg\varphi)$ has a model due to the value 2 in the second row.

(ii) If $d_1 = 1$ then, for any valuation, $v(\neg(\varphi \vee \neg\varphi)) = 0$ and so $\mathbf{DP2_D}$ does not hold.

Therefore, the only acceptable value for d_1 is 0. Thus we have the combinations $d_1 = 0$, $d_2 = 0$ and $d_1 = 0$, $d_2 = 1$. However, if $d_1 = d_2 = 0$, the connective $\vee$ becomes molecular, which is not desirable. So, if $n = 0$ we have only one choice to get a genuine paracomplete disjunction, namely $d_1 = 0$ and $d_2 = 1$. See Table 8 left.

Case $n = 1$
When $n = 1$, we have the following sub-cases:

(i) If $d_2 = 0$, then $\mathbf{DP1_D}$ as well as $\mathbf{DP2_D}$ hold as desired, without considering d_2, as we can see in Table 5 analogously to Table 4.

(ii) If $d_2 = 1$, then $v(\neg(\varphi \vee \neg\varphi)) \in \overline{D}$ and $\mathbf{DP2_D}$ does not hold.

L3A → *dualizing* → → *reordering* → **L3A^D**

φ	$\neg\varphi$
0	2
1	2
2	0

$\xrightarrow{\textit{dualizing}}$

φ	$\neg^D\varphi$
2	0
1	0
0	2

$\xrightarrow{\textit{reordering}}$

φ	$\neg\varphi$
0	2
1	0
2	0

$\wedge$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	2

$\xrightarrow{\textit{dualizing}}$

$\wedge^D$	2	1	0
2	2	2	2
1	2	1	0
0	2	0	0

$\xrightarrow{\textit{reordering}}$

$\vee$	0	1	2
0	0	0	2
1	0	1	2
2	2	2	2

$\vee$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

$\xrightarrow{\textit{dualizing}}$

$\vee^D$	2	1	0
2	2	1	0
1	1	1	0
0	2	0	0

$\xrightarrow{\textit{reordering}}$

$\wedge$	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

Table 6
Duality between **L3A** and **L3A$^{\mathbf{D}}$**

Analogously to the case $n = 0$ we have only one choice to get a genuine paracomplete disjunction, namely $d_2 = 0$ and $d_1 = 1$. See Table 8 right.

The previous analysis leads us to two different three-valued genuine paracomplete logics in the language that include $\neg$ and $\vee$ as their unique connectives. An interesting fact is that we can get the same truth tables for negation and disjunction, up to reordering, just considering the negation and conjunction of the genuine paraconsistent logics **L3A** and **L3B**, and dualizing them.

By dualizing we mean switching the truth values 2 for 0 and 0 for 2, keeping 1 fixed. In Table 6 we show this process for **L3A**, in the first two rows the dualized connectives of negation and conjunction of **L3A** lead us to the negation and conjunction of **L3A**. Similar process is shown in Table 7 for **L3B**. This is the reason for naming these genuine paracomplete logics after their duals as **L3A$^{\mathbf{D}}$** and **L3B$^{\mathbf{D}}$** respectively.

Definition 3.1 The three-valued logic $\mathbf{L}_M = \langle \mathcal{L}, \vdash_M \rangle$, where M is the matrix with set of values $\{0, 1, 2\}$, 2 as the only designated value, and connectives taken from Table 8a, will be called **L3A$^{\mathbf{D}}$**. Otherwise, if we take the connectives on the Table 8b, we obtain a logic called **L3B$^{\mathbf{D}}$**.

L3B $\xrightarrow{dualizing}$ $\xrightarrow{reordering}$ $\mathbf{L3B^D}$

φ	$\neg\varphi$
0	2
1	1
2	0

$\xrightarrow{\textit{dualizing}}$

φ	$\neg^D\varphi$
2	0
1	1
0	2

$\xrightarrow{\textit{reordering}}$

φ	$\neg\varphi$
0	2
1	1
2	0

$\wedge$	0	1	2
0	0	0	0
1	0	2	1
2	0	1	2

$\xrightarrow{\textit{dualizing}}$

$\wedge^D$	2	1	0
2	2	2	2
1	2	0	1
0	2	1	0

$\xrightarrow{\textit{reordering}}$

$\vee$	0	1	2
0	0	1	2
1	1	0	2
2	2	2	2

$\vee$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

$\xrightarrow{\textit{dualizing}}$

$\vee^D$	2	1	0
2	2	1	0
1	1	1	0
0	2	0	0

$\xrightarrow{\textit{reordering}}$

$\wedge$	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

Table 7
Duality between **L3B** and $\mathbf{L3B^D}$

φ	$\neg\varphi$
0	2
1	0
2	0

$\vee$	0	1	2
0	0	0	2
1	0	1	2
2	2	2	2

(a) $\mathbf{L3A^D}$

φ	$\neg\varphi$
0	2
1	1
2	0

$\vee$	0	1	2
0	0	1	2
1	1	0	2
2	2	2	2

(b) $\mathbf{L3B^D}$

Table 8
Truth tables for $\neg$ and $\vee$ in a genuine paracomplete three-valued logic

3.4 Genuine Paracomplete Conjunction

Since the definition of genuine paracompleteness does not impose conditions over the conjunction connective we can choose any of the definable conjunctions in a three-valued logic. Considering, again, conservative extension, neoclassicality, symmetry and non-molecularity, we have a partial table for the conjunction as the one on Table 9 where c_1, c_2 and $c_3 \in \overline{D}$ and it is not the case that $c_1 = c_2 = c_3 = 0$. Then there are 7 different conjunctions satisfying all these restrictions. However, if we want to extend $\mathbf{L3A^D}$ and $\mathbf{L3B^D}$ with a conjunction keeping its duality with

$\wedge$	0	1	2
0	$\boxed{0}$	c_1	$\boxed{0}$
1	c_1	c_2	c_3
2	$\boxed{0}$	c_3	$\boxed{2}$

Table 9
Possible conjunctions

the paraconsistent logics **L3A** and **L3B**, we must dualize their disjunction i.e. the maximum function. The resulting conjunction for the genuine paracomplete logics $\mathbf{L3A^D}$ and $\mathbf{L3B^D}$ is the minimum function, see the third row in Tables 6 and 7 respectively.

3.5 Genuine Paracomplete Implication

In [6], a search for implications in **L3A** and **L3B** satisfying specific properties was done. Analogously, here we search for suitable implications in $\mathbf{L3A^D}$ and $\mathbf{L3B^D}$.

The condition of being a conservative extension fix four values, see boxes in Table 10a.

By the nature of the logics in this section and the fact $D = \{2\}$, see Section 3.2, the conditions in Definition 2.4 can be rewritten as follows:

For any valuation v:

If $v(\varphi) = 2$ and $v(\varphi \to \psi) = 2$, then $v(\psi) = 2$; **MP**

$v(\varphi \to (\psi \to \varphi)) = 2$; **A1**

$v(\ (\varphi \to (\psi \to \sigma)) \to ((\varphi \to \psi) \to (\varphi \to \sigma))) = 2.$ **A2**

Assume that $\to$ is a connective satisfying **MP, A1**, and **A2**. Then $i_5 \neq 2$ as a consequence of **MP**. Suppose that $v(\varphi) = 2$. As **A1** is satisfied then, by **MP**, we must have $v(\psi \to \varphi) = 2$, therefore $i_4 = 2$. If $i_3 = 1$ then $1 \to (1 \to 1) = 1$, a contradiction with **A1**. Hence $i_3 \neq 1$. This gives Table 10b.

Up to now we have two possibilities with respect to i_5: its value is 0 or 1. In the former case, if $i_5 = 0$ in Table 10b, then:

- $i_2 = 2$, otherwise **A1** would not hold when $v(\varphi) = 1$ and $v(\psi) = 2$;
- If $i_1 = 0$, then **A2** does not hold, for $v(\varphi) = 0$, $v(\psi) = 0$ and $v(\sigma) = 1$;
- If $i_1 = 1$, $i_3 = 0$, then **A1** does not hold for $v(\varphi) = 1$ and $v(\psi) = 0$;
- If $i_1 = 1$, $i_3 = 2$, then **A2** does not hold for $v(\varphi) = 0$, $v(\psi) = 2$ and $v(\sigma) = 1$;
- If $i_1 = 2$, $i_3 = 0$, then **A2** does not hold for $v(\varphi) = 1$, $v(\psi) = 0$ and $v(\sigma) = 1$.

This analysis for $i_5 = 0$ only leave us one option, namely $\to_0$ in Table 11. In the second case, when $i_5 = 1$ we have:

- $i_3 = 2$, otherwise **A1** would not hold when $v(\varphi) = 1$ and $v(\psi) = 2$;
- If $i_1 = 0$, then **A2** does not hold, for $v(\varphi) = 0$, $v(\psi) = 0$ and $v(\sigma) = 1$;
- If $i_1 = 1$, $i_2 = 1$, then **A1** does not hold for $v(\varphi) = 0$ and $v(\psi) = 0$;
- If $i_1 = 2$, $i_2 = 1$, then **A2** does not hold for $v(\varphi) = 1$, $v(\psi) = 1$ and $v(\sigma) = 0$.

This analysis for $i_5 = 1$ leave us with 4 options, namely $\rightarrow_1, \rightarrow_2, \rightarrow_3$ and $\rightarrow_4$ in Table 11.

The five connectives in Table 11 are conservative extensions of the classical implication and are implications according to 2.3. Now, if we ask for neoclassicality to be satisfied, see Definition 2.9, we only have $\rightarrow_0$ and $\rightarrow_1$. One nice additional feature of connectives $\rightarrow_0$ and $\rightarrow_1$ is that any of the logics obtained by extending $\mathbf{L3A^D}$ or $\mathbf{L3B^D}$ with any of the connectives $\rightarrow_0$ or $\rightarrow_1$, satisfies the positive fragment of classical logic.

There is a criterion for the construction of a 'paraconsistent' logic, but in terms of implication and negation. It is due to Jaśkowski and consists on the rejection of $\vdash \varphi \rightarrow (\neg\varphi \rightarrow \psi)$ (the law of Duns Scotus). Clearly, this definition is not equivalent to the definition of paraconsistency in terms of (**NC**) but provides another approach to study the implications found in [6]. By analogy, for the paracompleteness case there is also a criterion in terms of implication and negation. According to Karpenko and Tomova [7] a logic is 'paracomplete' iff $\vdash (\neg\varphi \rightarrow \varphi) \rightarrow \varphi$ (the law of Clavius), is not valid in it. This definition does not correspond with the definition of paracompletness in terms of (**EM**) but it gives an intuitionistic flavor to the implication connective. This kind of logics are called weakly-intuitionistic logics, see [5]. Observe that the logic $\mathbf{L3A^D}$ extended with the implications $\rightarrow_0$, $\rightarrow_1$ or $\rightarrow_3$ from Table 11 rejects the Clavius law. On the one hand, the logics obtained by extending $\mathbf{L3B^D}$ with any one of the implications from Table 11 also reject the Clavius law.

The logic obtained by extending $\mathbf{L3B^D}$ with $\rightarrow_0$, namely $\mathbf{L3B^D_{\rightarrow_0}}$, coincides with the logic **MH** introduced in [4], where a Hilbert system for it was presented. But, if we look for a non-molecular implication, just one option is left, namely $\rightarrow_1$. It is worth to mention that this implication corresponds to the implication of the three-valued logic of Kleene [1].

4 Conclusions

In this paper we introduce the notion of genuine paracomplete logic, which are logics rejecting the dual principles that define genuine paraconsistent logic. On the one hand, in a similar way to the analysis done in [3], we develop a study among three-valued logics in order to find all connectives defining genuine paracomplete logics. We found two unary connectives that can serve as negation. By fixing one of these

$\to$	0	1	2
0	$\boxed{0}$	i_1	$\boxed{2}$
1	i_2	i_3	i_4
2	$\boxed{2}$	i_5	$\boxed{2}$

(a)

$\to$	0	1	2
0	$\boxed{0}$	i_1	$\boxed{2}$
1	i_2	0/2	2
2	$\boxed{2}$	0/1	$\boxed{2}$

(b)

Table 10
Possible implications

$\to_0$	0	1	2
0	2	2	2
1	2	2	2
2	0	0	2

$\to_1$	0	1	2
0	2	2	2
1	2	2	2
2	0	1	2

$\to_2$	0	1	2
0	2	1	2
1	0	2	2
2	0	1	2

$\to_3$	0	1	2
0	2	2	2
1	0	2	2
2	0	1	2

$\to_4$	0	1	2
0	2	1	2
1	2	2	2
2	0	1	2

Table 11
Possible implications for $\mathbf{L3A^D}$ and $\mathbf{L3B^D}$

negations, we discover in each case just one disjunction that works accordingly to our requests established for this particular kind of paracompleteness. On the other hand, if we take the connectives defining genuine three-valued paraconsistent logics, **L3A** and **L3B**, and later perform a process of dualizing them, we obtain the same connectives as before. This process lead us to the logics $\mathbf{L3A^D}$ and $\mathbf{L3B^D}$, see Definition 3.1.

In a further step, trying to extend the language with a conjunction, we found seven different suitable connectives for conjunction. Among these, we select the minimum function in order to get **L3A** and **L3B** completely dualized.

Finally, by proceeding analogously to [6], we found implications for $\mathbf{L3A^D}$ and $\mathbf{L3B^D}$ which are neoclassical, namely $\to_0$ and $\to_1$. These implications give place to the logics $\mathbf{L3A^D_{\to_0}}, \mathbf{L3A^D_{\to_1}}, \mathbf{L3B^D_{\to_0}}$ and $\mathbf{L3B^D_{\to_1}}$. This completes our analysis, obtaining four genuine paracomplete three-valued logics that dualize **L3A** and **L3B**. In [4] a Hilbert system for one of these logics was presented. As a future work, we consider to find axiomatizations for the remaining ones in order to have a better understanding of the nature of these logics. For instance, the relations among the connectives is not evident from the truth tables, since they are defined individually. However, suitable axioms can help us to understand the way in which the connectives are related.

References

[1] Ofer Arieli and Arnon Avron. Three-valued paraconsistent propositional logics. In Jean-Yves Béziau, Mihir Chakraborty, and Soma Dutta, editors, *New Directions in Paraconsistent Logic*, pages 91–129, New Delhi, 2015. Springer India.

[2] Jean-Yves Béziau. Two genuine 3-valued paraconsistent logics. In Seiki Akama, editor, *Towards Paraconsistent Engineering*, pages 35–47. Springer International Publishing Switzerland, Cham, 2016.

[3] Jean-Yves Béziau and Anna Franceschetto. Strong three-valued paraconsistent logics. In Jean-Yves Béziau, Mihir Chakraborty, and Soma Dutta, editors, *New directions in paraconsistent logic*, pages 131–145, New Delhi, 2015. Springer India.

[4] Colin Caret. Hybridized paracomplete and paraconsistent logics. *The Australasian Journal of Logic*, 14(1):281–325, 2017.

[5] Janusz Ciuciura. A weakly-intuitionistic logic I1. *Logical Investigations*, 21(2):53–60, 2015.

[6] Alejandro Hernández-Tello, José Arrazola Ramírez, and Mauricio Osorio Galindo. The pursuit of an implication for the logics L3A and L3B. *Logica Universalis*, 11(4):507–524, 2017.

[7] Alexander Karpenko and Natalya Tomova. Bochvar's three-valued logic and literal paralogics: Their lattice and functional equivalence. *Logic and Logical Philosophy*, 26(2):207–235, 2017.

[8] David J. Shoesmith and Timothy John Smiley. *Multiple-Conclusion Logic*. Cambridge University Press, 1978.

[9] Ryszard Wójcicki. *Lectures on propositional calculi*. Ossolineum, Wroclaw, 1984.