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Quantum Logic in Dagger Kernel Categories

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Abstract

This paper investigates quantum logic from the perspective of categorical logic, and starts from minimal assumptions, namely the existence of involutions/daggers and kernels. The resulting structures turn out to (1) encompass many examples of interest, such as categories of relations, partial injections, Hilbert spaces (also modulo phase), and Boolean algebras, and (2) have interesting categorical/logical properties, in terms of kernel fibrations, such as existence of pullbacks, factorisation, and orthomodularity. For instance, the Sasaki hook and and-then connectives are obtained, as adjoints, via the existential-pullback adjunction between fibres.

Keywords: Dagger kernel category, quantum logic, categorical logic

1 Introduction

Dagger categories $\mathbf{D}$ come equipped with a special functor $\dagger\colon \mathbf{D}^{\mathrm{op}} \to \mathbf{D}$ with $X^\dagger = X$ on objects and $f^{\dagger\dagger} = f$ on morphisms. A simple example is the category **Rel** of sets and relations, where $\dagger$ is reversal of relations. A less trivial example is the category **Hilb** of Hilbert spaces and continuous linear transformations, where $\dagger$ is induced by the inner product. The use of daggers, mostly with additional assumptions, dates back to [16,19]. Daggers are currently of interest in the context of quantum computation [1,23,6]. The dagger abstractly captures the reversal of a computation.

Mostly, dagger categories are used with fairly strong additional assumptions, like compact closure in [1]. Here we wish to follow a different approach and start from minimal assumptions. This paper is a first step to understand quantum logic, from the perspective of categorical logic (see *e.g.* [17,14,24,12]). It grew from the work of one of the authors [11]. Although that paper enjoys a satisfactory relation to traditional quantum logic [10], this one generalises it, by taking the notion of dagger category as starting point, and adding kernels, to be used as predicates. The interesting thing is that in the presence of a dagger functor $\dagger$ much else can be derived. As usual, it is quite subtle what precisely to take as primitive.

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Upon this structure of "dagger kernel categories" the paper constructs pullbacks of kernels and factorisation (both similar to [8]). It thus turns out that the kernels form a "bifibration" (both a fibration and an opfibration, see [12]). This structure can be used as a basis for categorical logic, which captures substitution in predicates by reindexing (pullback) f^{-1} and existential quantification by op-reindexing $\exists_f$, in such a way that $\exists_f \dashv f^{-1}$. From time to time we use fibred terminology in this paper, but familiarity with this fibred setting is not essential. We find that the posets of kernels (fibres) are automatically orthomodular lattices [13], and that the Sasaki hook and and-then connectives appear naturally from the existential-pullback adjunction. Additionally, a notion of Booleanness is identified for these dagger kernel categories. It gives rise to a generic construction that generalises how the category of partial injections can be obtained from the category of relations.

Apart from this general theory, the paper brings several important examples within the same setting—of dagger kernel categories. Examples are the categories **Rel** and **PInj** of relations and partial injections. Additionally, the category **Hilb** is an example—and, interestingly—also the category **PHilb** of Hilbert spaces modulo phase. The latter category provides the framework in which physicists typically work [5]. It has much weaker categorical structure than **Hilb**. Finally, we present a construction to turn an arbitrary Boolean algebra into a dagger kernel category. We suspect that there is a similar construction for orthomodular lattices, but to our regret, we have not been able to produce it.

The authors are acutely aware of the fact that several of the example categories have much richer structure, involving for instance a tensor sum $\oplus$ and a tensor product $\otimes$ with associated scalars and traced monoidal structure. But investigation of this additional structure is postponed to follow-up work. There are interesting differences between our main examples: for instance, **Rel** and **PInj** are Boolean, but **Hilb** is not; in **PInj** and **Hilb** "zero-epis" are epis, but not in **Rel**; **Rel** and **Hilb** have biproducts, but **PInj** does not.

The paper is organised as follows. After introducing the notion of dagger kernel category in Section 2, the main examples are described in Section 3. Factorisation and (co)images occur in Sections 4 and 5. Finally, Section 6 introduces the Sasaki hook and and-then connectives via adjunctions, and investigates Booleanness.

2 Daggers and kernels

To start we shall work with the following notion.

Definition 2.1 A *dagger kernel category* consists of:

(i) a dagger category $\mathbf{D}$, with dagger $\dagger\colon \mathbf{D}^{\mathrm{op}} \to \mathbf{D}$;

(ii) a zero object 0 in $\mathbf{D}$;

(iii) kernels $\ker(f)$ of arbitrary maps f in $\mathbf{D}$, which are $\dagger$-monos.

Definition 2.2 A dagger kernel category is called Boolean if $m \wedge n = 0$ implies $m^\dagger \circ n = 0$, for all kernels m, n.

The name Boolean will be explained in Theorem 6.2. We shall later rephrase the Booleanness condition as: kernels are disjoint if and only if they are orthogonal, see Lemma 2.3.

A category **DCK** is formed with these dagger categories with kernels as objects and functors F between them that preserve the relevant structure: dagger (*i.e.* $F(f^\dagger) = F(f)^\dagger$), zero object ($F(0)$ is again zero object), and kernels ($F(k)$ is kernel of $F(f)$ if k is kernel of f).

The dagger operation $\dagger$ satisfies $X^\dagger = X$ on objects and $f^{\dagger\dagger} = f$ on morphisms. It comes with a number of definitions. A map f in $\mathbf{D}$ is called a $\dagger$-mono(morphism) if $f^\dagger \circ f = \mathrm{id}$ and a $\dagger$-epi(morphism) if $f \circ f^\dagger = \mathrm{id}$. Hence f is a $\dagger$-mono if and only if $f^\dagger$ is a $\dagger$-epi. A map f is a $\dagger$-iso(morphism) when it is both a $\dagger$-mono and a $\dagger$-epi; in that case $f^{-1} = f^\dagger$ and f is sometimes called unitary (in analogy with Hilbert spaces). An endomap $p\colon X \to X$ is called self-adjoint if $p^\dagger = p$.

The zero object $0 \in \mathbf{D}$ is by definition both initial and final. Actually, in the presence of $\dagger$, initiality implies finality, and vice-versa. For an arbitrary object $X \in \mathbf{D}$, the unique map $X \to 0$ is then a $\dagger$-epi and the unique map $0 \to X$ is a $\dagger$-mono. The "zero" map $0 = 0_{X,Y} = (X \to 0 \to Y)$ satisfies $(0_{X,Y})^\dagger = 0_{Y,X}$. Notice that $f \circ 0 = 0 = 0 \circ g$. Usually there is no confusion between 0 as zero object and 0 as zero map. Two maps $f\colon X \to Z$ and $g\colon Y \to Z$ with common codomain are called orthogonal, written as $f \perp g$, if $g^\dagger \circ f = 0$—or, equivalently, $f^\dagger \circ g = 0$.

We recall that a kernel of a map $f\colon X \to Y$ is a universal map $k\colon \ker(f) \to X$ with $f \circ k = 0$. Universality means that for an arbitrary $g\colon Z \to X$ with $f \circ g = 0$ there is a unique map $g'\colon Z \to \ker(f)$ with $k \circ g' = g$. Kernels are automatically (ordinary) monos. Definition 2.1 requires that kernels are $\dagger$-monos.[1] We shall write $\mathrm{KSub}(X)$ for the poset of (equivalence classes) of kernels with codomain X. Sometimes we are a bit sloppy and confuse the kernel object $\ker(f)$ with the kernel map, for instance in defining the cokernel $\mathrm{coker}(f)$ as $\ker(f^\dagger)^\dagger$. This cokernel is a $\dagger$-epi. Finally, we define $m^\perp = \ker(m^\dagger)$, which we often write as $m^\perp : M^\perp \rightarrowtail X$ if $m : M \rightarrowtail X$. This notation is especially used when m is a mono. In diagrams we typically write a kernel as $\triangleright\!\!\longrightarrow$ and a cokernel as $\longrightarrow\!\!\!\triangleright$.

We start with some basic observations.

Lemma 2.3 *In a dagger kernel category,*

(i) $\ker(X \overset{0}{\to} Y) = (X \overset{\mathrm{id}}{\to} X)$ *and* $\ker(X \overset{\mathrm{id}}{\to} X) = (0 \overset{0}{\to} X)$*; they yield the top and bottom elements* $1, 0 \in \mathrm{KSub}(X)$*;*

(ii) $\ker(\ker(f)) = 0$*;*

(iii) $\ker(\mathrm{coker}(\ker(f))) = \ker(f)$*, as subobjects;*

(iv) $m^{\perp\perp} = m$ *if* m *is a kernel;*

(v) f *factors through* $g^\perp$ *iff* $f \perp g$ *iff* $g \perp f$ *iff* g *factors through* $f^\perp$*; in particular* $m \le n^\perp$ *iff* $n \le m^\perp$*, for monos* m, n*; hence* $(-)^\perp\colon \mathrm{KSub}(X) \xrightarrow{\cong} \mathrm{KSub}(X)^{\mathrm{op}}$*;*

[1] This requirement involves a subtlety: kernels are closed under arbitrary isomorphisms but $\dagger$-monos are only closed under $\dagger$-isomorphisms. Hence we should be more careful in this requirement. What we really mean is that for every kernel there is a $\dagger$-mono that is isomorphic to it. Hence we can always choose a kernel in such a way that it is a $\dagger$-mono.

(vi) *if $m \leq n$, for monos m, n, say via $m = n \circ \varphi$, then:*
 (a) *if m, n are †-monic, then so is φ;*
 (b) *if m is a kernel, then so is φ.*

(vii) *Booleanness amounts to $m \wedge n = 0 \Leftrightarrow m \perp n$, i.e. disjointness is orthogonality, for kernels.*

Proof. We skip the first two points because they are obvious and start with the third one. Consider for an arbitrary $f\colon X \to Y$ the diagram:

$$\begin{tikzcd}
\ker(f) \arrow[r, tail, "k"] \arrow[d, dotted, shift right, "k'"'] & X \arrow[r, "f"] \arrow[dr, two heads, "c"] & Y \\
\ker(\mathrm{coker}(\ker(f))) \arrow[u, dotted, shift right, "\ell'"'] \arrow[ur, tail, "\ell"'] & & \mathrm{coker}(\ker(f)) \arrow[u, dotted, "f'"']
\end{tikzcd}$$

By construction $f \circ k = 0$ and $c \circ k = 0$. Hence there are f' and k' as indicated. Since $f \circ \ell = f' \circ c \circ \ell = f' \circ 0 = 0$ one gets ℓ'. Hence the kernels ℓ and k are equal, as subobjects.

For the fourth point we now notice that if $m = \ker(f)$,

$$m^{\perp\perp} = \ker(\ker(m^\dagger)^\dagger) = \ker(\mathrm{coker}(\ker(f))) = \ker(f) = m.$$

Next,

$$\begin{aligned} f \text{ factors through } g^\perp &\iff g^\dagger \circ f = 0 \\ &\iff f^\dagger \circ g = 0 \iff g \text{ factors through } f^\perp. \end{aligned}$$

If, in the sixth point, $m = n \circ \varphi$ and m, n are †-monos, then $\varphi^\dagger \circ \varphi = (n^\dagger \circ m)^\dagger \circ \varphi = m^\dagger \circ n \circ \varphi = m^\dagger \circ m = \mathrm{id}$. And if $m = \ker(f)$, then $\varphi = \ker(f \circ n)$, since: (1) $f \circ n \circ \varphi = f \circ m = 0$, and (2) if $f \circ n \circ g = 0$, then there is a unique ψ with $m \circ \psi = n \circ g$; but then $\varphi \circ \psi = g$ since n is monic.

Finally, Booleanness means that $m \wedge n = 0$ implies $m^\dagger \circ n = 0$, which is equivalent to $n^\dagger \circ m = 0$, which is $m \perp n$ by definition. The reverse implication is easy: if $m \circ f = n \circ g$, then $f = m^\dagger \circ m \circ f = m^\dagger \circ n \circ g = 0 \circ g = 0$. Similarly, $g = 0$. Hence the zero object 0 is the pullback of m, n. □

Certain constructions from the theory of Abelian Categories [8] also work in the current setting. This applies to the pullback construction in the next result, but also, to a certain extend, to the factorisation of Section 4.

Lemma 2.4 *Pullbacks of kernels exist, and are kernels again. Explicitly, given a kernel n and map f one obtains a pullback:*

$$\begin{tikzcd}
M \arrow[r, "f'"] \arrow[d, tail, "f^{-1}(n)"'] \arrow[dr, phantom, "\lrcorner", very near start] & N \arrow[d, tail, "n"] \\
X \arrow[r, "f"] & Y
\end{tikzcd} \qquad \textit{as} \qquad f^{-1}(n) = \ker(\mathrm{coker}(n) \circ f).$$

In case this f is a $\dagger$-epi, then so is f'.

By duality there are of course similar results about pushouts of cokernels.

Proof. For convenience write $m = f^{-1}(n) = \ker(\mathrm{coker}(n) \circ f)$. By construction, $\mathrm{coker}(n) \circ f \circ m = 0$, so that $f \circ m$ factors through $\ker(\mathrm{coker}(n)) = n$, say via $f' \colon M \to N$ with $n \circ f' = f \circ m$, as in the diagram. This yields a pullback: if $a \colon Z \to X$ and $b \colon Z \to N$ satisfy $f \circ a = n \circ b$, then $\mathrm{coker}(n) \circ f \circ a = \mathrm{coker}(n) \circ n \circ f' = 0 \circ f' = 0$, so that there is a unique map $c \colon Z \to M$ with $m \circ c = a$. Then $f' \circ c = b$ because n is monic.

In case f is a $\dagger$-epi we have $f \circ f^\dagger = \mathrm{id}$. Hence there are two adjacent pullbacks:

$$\begin{tikzcd} N \arrow[r, "f''"] \arrow[d, tail, "n"'] \arrow[dr, phantom, very near start, "\lrcorner"] & M \arrow[r, "f'"] \arrow[d, tail, "m=f^{-1}(n)"] \arrow[dr, phantom, very near start, "\lrcorner"] & N \arrow[d, tail, "n"] \\ Y \arrow[r, "f^\dagger"] & X \arrow[r, "f"] & Y \end{tikzcd}$$

Then $f' \circ f'' = \mathrm{id}$ because n is monic. Further, $f'' = m^\dagger \circ m \circ f'' = m^\dagger \circ f^\dagger \circ n = f'^\dagger \circ n^\dagger \circ n = f'^\dagger$. Hence f' is $\dagger$-epi. $\square$

Corollary 2.5 *Given these pullbacks of kernels,*

(i) *the mapping $X \mapsto \mathrm{KSub}(X)$ yields an indexed category $\mathbf{D}^{\mathrm{op}} \to \mathbf{PoSets}$ and forms a setting in which one can develop categorical logic for dagger categories;*

(ii) *the following diagram is a pullback,*

$$\begin{tikzcd} \ker(f) \arrow[r] \arrow[d, tail] \arrow[dr, phantom, very near start, "\lrcorner"] & 0 \arrow[d, tail] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

showing that, logically speaking, falsum—i.e. the bottom element $0 \in \mathrm{KSub}(Y)$—is in general not preserved under substitution. Also, negation $(-)^\perp$ does not commute with substitution, because $1 = 0^\perp$ and $f^{-1}(1) = 1$. $\square$

One may also describe the indexed category KSub from (i) as a split fibration [12] $\left(\begin{smallmatrix}\mathrm{KSub}(\mathbf{D}) \\ \downarrow \\ \mathbf{D}\end{smallmatrix}\right)$ where the "total" category $\mathrm{KSub}(\mathbf{D})$ has (equivalence classes of) kernels $M \rightarrowtail X$ as objects, and morphisms $(M \overset{m}{\rightarrowtail} X) \longrightarrow (N \overset{n}{\rightarrowtail} Y)$ are maps $f \colon X \to Y$ in $\mathbf{D}$ with:

$$\begin{tikzcd} M \arrow[r, dashed] \arrow[d, tail, "m"'] & N \arrow[d, tail, "n"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \qquad \textit{i.e.}\text{ with} \qquad m \le f^{-1}(n).$$

We shall sometimes refer to this fibration as the "kernel fibration". Every functor $F \colon \mathbf{D} \to \mathbf{E}$ in $\mathbf{DCK}$ induces a map of fibrations:

$$\begin{tikzcd} \mathrm{KSub}(\mathbf{D}) \arrow[r] \arrow[d] & \mathrm{KSub}(\mathbf{E}) \arrow[d] \\ \mathbf{D} \arrow[r, "F"] & \mathbf{E} \end{tikzcd} \tag{1}$$

because F preserves kernels and pullbacks of kernels—the latter since pullbacks can be formulated in terms of constructions that are preserved by F, see Lemma 2.4. As we shall see, in some situations, this diagram (1) is a pullback—also called a change-of-base situation in this context, see [12]. It means that the map $\mathrm{KSub}(X) \to \mathrm{KSub}(FX)$ is an isomorphism.

Being able to take pullbacks of kernels has some important consequences.

Lemma 2.6 *Kernels are closed under composition—and hence cokernels are, too.*

Proof. We shall prove the result for cokernels, because it uses pullback results as we have just seen. So assume we have (composable) cokernels e, d; we wish to show $e \circ d = \mathrm{coker}(\ker(e \circ d))$. We first notice, using Lemma 2.4,

$$\ker(e \circ d) = \ker(\mathrm{coker}(\ker(e)) \circ d) = d^{-1}(\ker(e)),$$

yielding a pullback:

$$\begin{tikzcd}
 & A \arrow[r, "d'"] \arrow[d, tail, "m=\ker(e\circ d)"'] \arrow[dr, phantom, very near start, "\lrcorner"] & B \arrow[d, tail, "\ker(e)"] & \\
K \arrow[ur, dashed, "\varphi"] \arrow[r, tail, "\ker(d)"] & X \arrow[r, two heads, "d"] & D \arrow[r, two heads, "e"] & E
\end{tikzcd}$$

We intend to prove $e \circ d = \mathrm{coker}(m)$. Clearly, $e \circ d \circ m = e \circ \ker(e) \circ d' = 0 \circ d' = 0$. And if $f \colon X \to Y$ satisfies $f \circ m = 0$, then $f \circ \ker(d) = f \circ m \circ \varphi = 0$, so because $d = \mathrm{coker}(\ker(d))$ there is $f' \colon D \to Y$ with $f' \circ d = f$. But then: $f' \circ \ker(e) \circ d' = f' \circ d \circ m = f \circ m = 0$. Then $f' \circ \ker(e) = 0$, because d' is †-epi because d is, see Lemma 2.4. This finally yields $f'' \colon E \to Y$ with $f'' \circ e = f'$. Hence $f'' \circ e \circ d = f$. □

As a result, the logic of kernels has intersections, preserved by substitution. More precisely, the indexed category $\mathrm{KSub}(-)$ from Corollary 2.5 is actually a functor $\mathrm{KSub} \colon \mathbf{D}^{\mathrm{op}} \to \mathbf{MSL}$ to the category $\mathbf{MSL}$ of meet semi-lattices. Each poset $\mathrm{KSub}(X)$ also has disjunctions, by $m \vee n = (m^\perp \wedge n^\perp)^\perp$, but they are not preserved under substitution/pullback f^{-1}. But we do have $m \vee m^\perp = (m^\perp \wedge m^{\perp\perp})^\perp = (m^\perp \wedge m)^\perp = 0^\perp = 1$.

Proposition 2.7 *Orthomodularity holds: for kernels $m \leq n$, say via φ with $n \circ \varphi = m$, one has pullbacks:*

$$\begin{tikzcd}
M \arrow[r, tail, "\varphi"] \arrow[d, equal] \arrow[dr, phantom, very near start, "\lrcorner"] & N \arrow[d, tail, "n"] & P \arrow[l, tail, "\varphi^\perp"'] \arrow[d] \arrow[dl, phantom, very near start, "\llcorner"] \\
M \arrow[r, tail, "m"] & X & M^\perp \arrow[l, tail, "m^\perp"']
\end{tikzcd}$$

This means that $m \vee (m^\perp \wedge n) = n$.

Proof. The square on the left is obviously a pullback. For the one on the right we use a simple calculation, following Lemma 2.4:

$$\begin{aligned} n^{-1}(m^{\perp}) &= \ker(\mathrm{coker}(m^{\perp}) \circ n) \\ &= \ker(\mathrm{coker}(\ker(m^{\dagger})) \circ n) \\ &= \ker(m^{\dagger} \circ n) \qquad \text{since } m^{\dagger} \text{ is a cokernel} \\ &\overset{(*)}{=} \ker(\varphi^{\dagger}) \\ &= \varphi^{\perp}, \end{aligned}$$

where the marked equation holds because $n \circ \varphi = m$, so that $\varphi = n^{\dagger} \circ n \circ \varphi = n^{\dagger} \circ m$ and thus $\varphi^{\dagger} = m^{\dagger} \circ n$. Then:

$$m \vee (m^{\perp} \wedge n) = (n \circ \varphi) \vee (n \circ \varphi^{\perp}) \overset{(*)}{=} n \circ (\varphi \vee \varphi^{\perp}) = n \circ \mathrm{id} = n.$$

The marked equation holds because $n \circ (-)$ preserves joins, since it is a left adjoint: $n \circ k \leq m$ iff $k \leq n^{-1}(m)$, for kernels k, m. □

The following notion does not seem to have an established terminology. Hence we introduce our own.

Definition 2.8 In a category with a zero object, a map m is called a zero-mono if $m \circ f = 0$ implies $f = 0$, for each map f. Dually, e is zero-epi if $f \circ e = 0$ implies $f = 0$. In diagrams we write $\rightarrowtail\!\circ\!\rightarrow$ for zero-monos and $\rightarrow\!\circ\!\twoheadrightarrow$ for zero-epis.

Clearly, a mono is zero-mono, since $m \circ f = 0 = m \circ 0$ implies $f = 0$ if m is monic. The following points are worth making explicit.

Lemma 2.9 *In a dagger kernel category,*

(i) *m is a zero-mono iff* $\ker(m) = 0$ *and e is a zero-epi iff* $\mathrm{coker}(e) = 0$*;*

(ii) $\ker(m \circ f) = \ker(f)$ *if m is a zero-mono, and similarly,* $\mathrm{coker}(f \circ e) = \mathrm{coker}(f)$ *if e is a zero-epi.*

(iii) *a kernel which is zero-epic is an isomorphism.* □

We shall mostly be interested in zero-epis (instead of zero-monos), because they arise in the factorisation of Section 4. In the presence of dagger equalisers, zero-epis are ordinary epis. This applies to **Hilb** and **PInj**. This fact is not really used, but is included because it gives a better understanding of the situation. A *dagger equaliser category* is a dagger category that has equalisers which are dagger monic.

Lemma 2.10 *In a dagger equaliser category* **D** *where every dagger mono is a kernel, zero-epis in* **D** *are ordinary epis.*

Proof. Assume a zero-epi $e \colon E \to X$ with two maps $f, g \colon X \to Y$ satisfying $f \circ e = g \circ e$. We need to prove $f = g$. Let $m \colon M \rightarrowtail X$ be the equaliser of f, g, with

$h = \mathrm{coker}(m)$, as in:

$$\begin{tikzcd} E \arrow[r, "e"] \arrow[d, dashed, "\varphi"'] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, "h=\mathrm{coker}(m)"] & Y \\ M \arrow[ur, "m"'] & Z \end{tikzcd}$$

This e factors through the equaliser m, as indicated, since $f \circ e = g \circ e$. Then: $h \circ e = h \circ m \circ \varphi = 0 \circ \varphi = 0$. Hence $h = 0$ because e is zero-epi. But m, being a dagger mono, is a dagger kernel. Hence $m = \ker(\mathrm{coker}(m)) = \ker(h) = \ker(0) = \mathrm{id}$, so that $f = g$. □

3 Main examples

This section will describe our four main examples, namely **Rel**, **PInj**, **Hilb** and **PHilb**, and additionally a general construction to turn a Boolean algebra into a dagger kernel category.

3.1 *The category* **Rel** *of sets and relations*

Sets and binary relations $R \subseteq X \times Y$ between them can be organised in the familiar category **Rel**, using relational composition. Alternatively, such a relation may be described as a Kleisli map $X \to \mathcal{P}(Y)$ for the powerset monad $\mathcal{P}$; in line with this representation we sometimes write $R(x) = \{y \in Y \mid R(x, y)\}$. A third way is to represent such a morphism in **Rel** as (an equivalence class of) a pair of maps $(X \overset{r_1}{\leftarrow} R \overset{r_2}{\to} Y)$ whose tuple $\langle r_1, r_2\rangle \colon R \to X \times Y$ of legs is injective.

There is a simple dagger operation on **Rel** by reversal of relations: $R^\dagger(y, x) = R(x, y)$. A map $R\colon X \to Y$ is a †-mono in **Rel** if $R^\dagger \circ R = \mathrm{id}$, which amounts to the equivalence:

$$\exists_{y \in Y}.\, R(x, y) \wedge R(x', y) \quad \Longleftrightarrow \quad x = x',$$

for all $x, x' \in X$. It can be split into two statements:

$$\forall_{x\in X}.\, \exists_{y\in Y}.\, R(x, y) \quad \text{and} \quad \forall_{x,x'\in X}.\, \forall_{y\in Y}.\, R(x, y) \wedge R(x', y) \Rightarrow x = x'.$$

Hence such a †-mono R is given by a span of the form:

$$\left(\begin{tikzcd} & R \arrow[dl, two heads, "r_1"'] \arrow[dr, tail, "r_2"] & \\ X & & Y \end{tikzcd}\right) \tag{2}$$

with surjection as first leg and injection as second leg. A †-epi has the same shape, but with legs exchanged.

The empty set 0 is a zero object in **Rel**, and the resulting zero map $0\colon X \to Y$ is the empty relation $\emptyset \subseteq X \times Y$.

The category **Rel** also has kernels. For an arbitrary map $R\colon X \to Y$ one takes $\ker(R) = \{x \in X \mid \neg\exists_{y\in Y}.\, R(x, y)\}$ with map $k\colon \ker(R) \to X$ in **Rel** given by $k(x, x') \Leftrightarrow x = x'$. Clearly, $R \circ k = 0$. And if $S\colon Z \to X$ satisfies $R \circ S = 0$, then

$\neg\exists_{x\in X}.\, R(x,y) \wedge S(z,x)$, for all $z \in Z$ and $y \in Y$. This means that $S(z,x)$ implies there is no y with $R(x,y)$. Hence S factors through the kernel k. Kernels are thus of the following form:

$$\left(\begin{array}{ccc} & K & \\ K \nearrow\!\!\!\!/\!/ & & \searrow X \end{array}\right) \quad \text{with} \quad K = \{x \in X \mid R(x) = \emptyset\}.$$

Kernels are thus essentially given by subsets: $\mathrm{KSub}(X) = \mathcal{P}(X)$. Indeed, **Rel** is Boolean, in the sense of Definition 2.1. A cokernel has the reversed shape.

Finally, a relation R is zero-mono if its kernel is 0, see Lemma 2.9. This means that $R(x) \neq \emptyset$, for each $x \in X$, so that R's left leg is a surjection.

Proposition 3.1 *In* **Rel** *there are proper inclusions:*

$$\textit{kernel} \subsetneq \dagger\textit{-mono} \subsetneq \textit{mono} \subsetneq \textit{zero-mono}.$$

Subsets of a set X correspond to kernels in **Rel** *with codomain X.*

There is of course a dual version of this result, for cokernels and epis.

Proof. We still need to produce (1) a zero-mono which is not a mono, and (2) a mono which is not a $\dagger$-mono. As to (1), consider $R \subseteq \{0,1\} \times \{a,b\}$ given by $R = \{(0,a),(1,a)\}$. Its first leg is surjective, so R is a zero-mono. But it is not a mono: there are two different relations $\{(*,0)\}, \{(*,1)\} \subseteq \{*\} \times \{0,1\}$ with $R \circ \{(*,0)\} = \{(*,a)\} = R \circ \{(*,1)\}$.

As to (2), consider the relation $R \subseteq \{0,1\} \times \{a,b,c\}$ given by $R = \{(0,a),(0,b),(1,b),(1,c)\}$. Clearly, the first leg of R is a surjection, and the second one is neither an injection nor a surjection. We check that R is monic. Suppose $S,T\colon X \to \{0,1\}$ satisfy $R \circ S = R \circ T$. If $S(x,0)$, then $(R \circ S)(x,a) = (R \circ T)(x,a)$, so that $T(x,0)$. Similarly, $S(x,1) \Rightarrow T(x,1)$. □

We add that the pullback $R^{-1}(n)$ of a kernel $n = (N = N \rightarrowtail Y)$ along a relation $R \subseteq X \times Y$, as described in Lemma 2.4 is the subset of X given by the modal formula $\Box_R(n)(x) = R^{-1}(n)(x) \Leftrightarrow (\forall_y.\, R(x,y) \Rightarrow N(y))$. As is well-known in modal logic $\Box_R$ preserves conjunctions, but no disjunctions. Interestingly, the familiar "graph" functor $\mathcal{G}\colon \mathbf{Sets} \to \mathbf{Rel}$ yields a map of fibrations:

$$\begin{array}{ccc} \mathrm{Sub}(\mathbf{Sets}) & \longrightarrow & \mathrm{KSub}(\mathbf{Rel}) \\ \downarrow & & \downarrow \\ \mathbf{Sets} & \xrightarrow{\mathcal{G}} & \mathbf{Rel} \end{array} \tag{3}$$

which forms actually a pullback (or a "change-of-base" situation, see [12]). This means that the familiar logic of sets can be obtained from this kernel logic on relations. In this diagram we use that inverse image is preserved: for a function

$f\colon X \to Y$ and predicate $N \subseteq Y$ one has:

$$\begin{aligned}\mathcal{G}(f)^{-1}(N) \;=\; \Box_{\mathcal{G}(f)}(N) &= \{x \in X \mid \forall_y.\, \mathcal{G}(f)(x,y) \Rightarrow N(y)\}\\ &= \{x \in X \mid \forall_y.\, f(x) = y \Rightarrow N(y)\}\\ &= \{x \in X \mid N(f(x))\}\\ &= f^{-1}(N).\end{aligned}$$

3.2 The category **PInj** *of sets and partial injections*

There is a subcategory **PInj** of **Rel** also with sets as objects but with "partial injections" as morphisms. These are special relations $F \subseteq X \times Y$ satisfying $F(x,y) \wedge F(x,y') \Rightarrow y = y'$ and $F(x,y) \wedge F(x',y) \Rightarrow x = x'$. We shall therefore often write morphisms $f\colon X \to Y$ in **PInj** as spans with the notational convention

$$\left(X \xrightarrow{f} Y\right) = \left(\begin{tikzcd}[column sep=small] & F \arrow[dl, tail, "f_1"'] \arrow[dr, tail, "f_2"] & \\ X & & Y\end{tikzcd}\right),$$

where spans $(X \overset{f_1}{\leftarrowtail} F \overset{f_2}{\rightarrowtail} Y)$ and $(X \overset{g_1}{\leftarrowtail} G \overset{g_2}{\rightarrowtail} Y)$ are equivalent if there is an isomorphism $\varphi\colon F \to G$ with $g_i \circ \varphi = f_i$, for $i = 1, 2$—like for relations.

Composition of $X \overset{f}{\to} Y \overset{g}{\to} Z$ can be described as relational composition, but also via pullbacks of spans. The identity map $X \to X$ is given by the span of identities $X \leftarrowtail X \rightarrowtail X$. The involution is inherited from **Rel** and can be described as $\left(X \overset{f_1}{\leftarrowtail} F \overset{f_2}{\rightarrowtail} Y\right)^\dagger = \left(Y \overset{f_2}{\leftarrowtail} F \overset{f_1}{\rightarrowtail} X\right)$.

It is not hard to see that $f = \left(X \overset{f_1}{\leftarrowtail} F \overset{f_2}{\rightarrowtail} Y\right)$ is a $\dagger$-mono—*i.e.* satisfies $f^\dagger \circ f = \mathrm{id}$—if and only if its first leg $f_1\colon F \rightarrowtail X$ is an isomorphism. For convenience we therefore identify a mono/injection $m\colon M \rightarrowtail X$ in **Sets** with the corresponding $\dagger$-mono $\left(M \overset{\mathrm{id}}{\leftarrowtail} M \overset{m}{\rightarrowtail} X\right)$ in **PInj**.

By duality: f is $\dagger$-epi iff $f^\dagger$ is $\dagger$-mono iff the second leg f_2 of f is an isomorphism. Further, f is a $\dagger$-iso iff f is both $\dagger$-mono and $\dagger$-epi iff both legs f_1 and f_2 of f are isomorphisms.

Like in **Rel**, the empty set is a zero object, with corresponding zero map given by the empty relation, and $0^\dagger = 0$.

For the description of the kernel of an arbitrary map $f = \left(X \overset{f_1}{\leftarrowtail} F \overset{f_2}{\rightarrowtail} Y\right)$ in **PInj** we shall use the *ad hoc* notation $\neg_1 F \overset{\neg f_1}{\rightarrowtail} X$ for the negation of the first leg $f_1\colon F \rightarrowtail X$, as subobject/subset. It yields a map:

$$\ker(f) = \left(\begin{tikzcd}[column sep=small] & \neg_1 F \arrow[dl, equal] \arrow[dr, tail, "\neg f_1"] & \\ \neg_1 F & & X\end{tikzcd}\right)$$

It satisfies $f \circ \ker(f) = 0$. It is a $\dagger$-mono by construction. Notice that kernels are the same as $\dagger$-monos, and are also the same as zero-monos. They all correspond to subsets, so that $\mathrm{KSub}(X) = \mathcal{P}(X)$ and **PInj** is Boolean, like **Rel**.

The next result summarises what we have seen so far and shows that **PInj** is very different from **Rel** (see Proposition 3.1).

Proposition 3.2 *In* **PInj** *there are proper identities:*

$$\textit{kernel} \;=\; \dagger\textit{-mono} \;=\; \textit{mono} \;=\; \textit{zero-mono}.$$

These all correspond to subsets.

3.3 The category **Hilb** *of Hilbert spaces*

Our third example is the category **Hilb** of (complex) Hilbert spaces and continuous linear maps. Recall that a Hilbert space is a vector space X equipped with an inner product, *i.e.* a function $\langle - \,|\, - \rangle : X \times X \to \mathbb{C}$ that is linear in the first and anti-linear in the second variable, satisfies $\langle x \,|\, x \rangle \geq 0$ with equality if and only if $x = 0$, and $\langle x \,|\, y \rangle = \overline{\langle y \,|\, x \rangle}$. Moreover, a Hilbert space must be complete in the metric induced by the inner product by $d(x, y) = \sqrt{\langle x - y \,|\, x - y \rangle}$.

The Riesz representation theorem provides this category with a dagger functor. Explicitly, for $f\colon X \to Y$ a given morphism, $f^\dagger\colon Y \to X$ is the unique morphism satisfying

$$\langle f(x) \,|\, y \rangle_Y = \langle x \,|\, f^\dagger(y) \rangle_X$$

for all $x \in X$ and $y \in Y$. The zero object is inherited from the category of (complex) vector spaces: it is the zero-dimensional Hilbert space $\{0\}$, with unique inner product $\langle 0 \,|\, 0 \rangle = 0$.

In the category **Hilb**, †-mono's are usually called isometries, because they preserve the metric: $f^\dagger \circ f = \mathrm{id}$ if and only if

$$d(fx, fy) = \langle f(x - y) \,|\, f(x - y) \rangle^{\frac{1}{2}} = \langle x - y \,|\, (f^\dagger \circ f)(x - y) \rangle^{\frac{1}{2}} = d(x, y).$$

Kernels are inherited from the category of vector spaces. For $f : X \to Y$, we can choose $\ker(f)$ to be (the inclusion of) $\{x \in X \mid f(x) = 0\}$, as this is complete with respect to the restricted inner product of X. Hence kernels correspond to (inclusions of) closed subspaces. Being inclusions, kernels are obviously †-monos. Hence **Hilb** is indeed an example of a dagger kernel category. However, **Hilb** is not Boolean. The following proposition shows that it is indeed different, categorically, from **Rel** and **PInj**.

Proposition 3.3 *In* **Hilb** *one has:*

$$\textit{kernel} \;=\; \dagger\textit{-mono} \;\subsetneq\; \textit{mono} \;=\; \textit{zero-mono}.$$

Proof. For the left equality, notice that both kernels and isometries correspond to closed subspaces. It is not hard to show that the monos in **Hilb** are precisely the injective continuous linear functions, establishing the middle proper inclusion. Finally, **Hilb** has equalisers by $\mathrm{eq}(f, g) = \ker(g - f)$, which takes care of the right equality. □

As is well-known, the ℓ^2 construction forms a functor $\ell^2 \colon \mathbf{PInj} \to \mathbf{Hilb}$ (but not a functor $\mathbf{Sets} \to \mathbf{Hilb}$), see *e.g.* [2,9]. Since it preserves daggers, zero object and kernels it is a map in the category **DCK**, and thus yields a map of kernel fibrations like in (1). It does not form a pullback (change-of-base) between these fibrations, since the map $\mathrm{KSub}_{\mathbf{PInj}}(X) = \mathcal{P}(X) \to \mathrm{KSub}_{\mathbf{Hilb}}(\ell^2(X))$ is not an isomorphism.

3.4 The category **PHilb***: Hilbert spaces modulo phase*

The category **PHilb** of *projective Hilbert spaces* has the same objects as **Hilb**, but its homsets are quotiented by the action of the circle group $U(1) = \{z \in \mathbb{C} \mid |z| = 1\}$. That is, continuous linear transformations $f, g : X \to Y$ are identified when $x = z \cdot y$ for some phase $z \in U(1)$.

Equivalently, we could write $PX = X_1/U(1)$ for an object of **PHilb**, where $X \in \mathbf{Hilb}$ and $X_1 = \{x \in X \mid \|x\| = 1\}$. Two vectors $x, y \in X_1$ are therefore identified when $x = z \cdot y$ for some $z \in U(1)$. Continuous linear transformations $f, g : X \to Y$ then descend to the same function $PX \to PY$ precisely when they are equivalent under the action of $U(1)$. This gives a full functor $P : \mathbf{Hilb} \to \mathbf{PHilb}$.

The dagger of **Hilb** descends to **PHilb**, because if $f = z \cdot g$ for some $z \in U(1)$, then

$$\langle f(x) \,|\, y\rangle = \bar{z} \cdot \langle g(x) \,|\, y\rangle = \bar{z} \cdot \langle x \,|\, g^\dagger(y)\rangle = \langle x \,|\, \bar{z} \cdot g^\dagger(y)\rangle,$$

whence also $f^\dagger = \bar{z} \cdot g^\dagger$, making the dagger well-defined.

Also dagger kernels in **Hilb** descend to **PHilb**. More precisely, the kernel $\ker(f) = \{x \in X \mid f(x) = 0\}$ of a morphism $f : X \to Y$ is well-defined, for if $f = z \cdot f'$ for some $z \in U(1)$, then

$$\ker(f) = \{x \in X \mid z \cdot f'(x) = 0\} = \{x \in X \mid f'(x) = 0\} = \ker(f').$$

Proposition 3.4 *In* **PHilb** *one has:*

$$\textit{kernel} \;=\; \dagger\textit{-mono} \;\subsetneq\; \textit{mono} \;=\; \textit{zero-mono}.$$

Proof. It remains to be shown that every zero-mono is a mono. So let $m : Y \to Z$ be a zero-mono, and $f, g : X \to Y$ arbitrary morphisms in **PHilb**. More precisely, let m, f and g be morphisms in **Hilb** representing the equivalence classes $[m], [f]$ and $[g]$ that are morphisms in **PHilb**. Suppose that $[m \circ f] = [m \circ g]$. Then $m \circ f \sim m \circ g$, say $m \circ f = z \cdot (m \circ g)$ for $z \in U(1)$. So $m \circ (f - z \cdot g) = 0$, and $f - z \cdot g = 0$ since m is zero-mono. Then $f = z \cdot g$ and hence $f \sim g$, *i.e.* $[f] = [g]$. Thus m is mono. □

The full functor $P : \mathbf{Hilb} \to \mathbf{PHilb}$ preserves daggers, the zero object and kernels. Hence it is a map in the category **DCK**. In fact it yields a pullback (change-of-base) between the corresponding kernel fibrations.

$$\begin{tikzcd}
\mathrm{KSub}(\mathbf{Hilb}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{KSub}(\mathbf{PHilb}) \arrow[d] \\
\mathbf{Hilb} \arrow[r, "P"] & \mathbf{PHilb}
\end{tikzcd} \tag{4}$$

3.5 From Boolean algebras to dagger kernel categories

The previous four examples were concrete categories. At the end we add a generic construction, which turns an arbitrary Boolean algebra into a (Boolean) dagger kernel category.

To start, let B with $(1, \wedge)$ be a meet semi-lattice. We can turn it into a category, for which we use the notation $\widehat{B}$. The objects of $\widehat{B}$ are elements $x \in B$, and its morphisms $x \to y$ are elements $f \in B$ with $f \leq x, y$, *i.e.* $f \leq x \wedge y$. There is an identity $x\colon x \to x$, and composition of $f\colon x \to y$ and $g\colon y \to z$ is simply $f \wedge g\colon x \to z$. This $\widehat{B}$ is a dagger category with $f^\dagger = f$. A map $f\colon x \to y$ is a †-mono if $f^\dagger \circ f = f \wedge f = x$. Hence a †-mono is of the form $x\colon x \to y$ where $x \leq y$.

It is not hard to see that the construction $B \mapsto \widehat{B}$ is functorial: a morphism $h\colon B \to C$ of meet semi-lattices yields a functor $\widehat{h}\colon \widehat{B} \to \widehat{C}$ by $x \mapsto h(x)$. It clearly preserves †.

Proposition 3.5 *If B is a Boolean algebra, then $\widehat{B}$ is a Boolean dagger kernel category. This yields a functor* $\mathbf{BA} \to \mathbf{DCK}$.

Proof. The bottom element $0 \in B$ yields a zero object $0 \in \widehat{B}$, and also a zero map $0\colon x \to y$. For an arbitrary map $f\colon x \to y$ there is a kernel $\ker(f) = \neg f \wedge x$, which is a †-mono $\ker(f)\colon \ker(f) \to x$ in $\widehat{B}$. Clearly, $f \circ \ker(f) = f \wedge \neg f \wedge x = 0 \wedge x = 0$. If also $g\colon z \to x$ satisfies $f \circ g = 0$, then $g \leq x, z$ and $f \wedge g = 0$. The latter yields $g \leq \neg f$ and thus $g \leq \neg f \wedge x = \ker(f)$. Hence g forms the required mediating map $g\colon z \to \ker(f)$ with $\ker(f) \circ g = g$.

Notice that each †-mono $m\colon m \to x$, where $m \leq x$, is a kernel, namely of its cokernel $\neg m \wedge x\colon x \to (\neg m \wedge x)$. For two kernels $m\colon m \to x$ and $n\colon n \to x$, where $m, n \leq x$, one has $m \leq n$ as kernels iff $m \leq n$ in B. Thus $\mathrm{KSub}(x) = {\downarrow} x$, which is again a Boolean algebra (with negation $\neg_x m = \neg m \wedge x$). The intersection $m \wedge n$ as subobjects is the meet $m \wedge n$ in B. This allows us to show that $\widehat{B}$ is Boolean: if $m \wedge n = 0$, them $m^\dagger \circ n = m \circ n = m \wedge n = 0$. □

It remains an open question whether a similar construction can be performed for orthomodular lattices (see [13]), instead of Boolean algebras. The straightforward extension of the above construction does not work: in order to get kernels one needs to use the and-then connective (&, see Proposition 6.1) for composition; but & is neither associative nor commutative, unless the lattice is Boolean [15].

4 Factorisation

In this section we assume that $\mathbf{D}$ is an arbitrary dagger kernel category. We will show that each map in $\mathbf{D}$ can be factored as a zero-epi followed by a kernel, in an essentially unique way. This factorisation leads to existential quantifiers $\exists$, as usual.

The "image" $\mathrm{Im}(f)$ of $f\colon X \to Y$ is defined as $\mathrm{Im}(f) = \ker(\mathrm{coker}(f))$ with kernel map (and hence †-mono) $i_f\colon \mathrm{Im}(f) \rightarrowtriangle Y$ obtained as follows. First take the kernel:

$$\ker(f^\dagger) \overset{k}{\rightarrowtriangle} Y \xrightarrow{f^\dagger} X$$

and define i_f as the kernel of $k^\dagger$ as in:

$$\begin{tikzcd} \mathrm{Im}(f) = \ker(k^\dagger) \arrow[r, tail, "i_f"] & Y \arrow[r, "k^\dagger"] & \ker(f^\dagger) \\ X \arrow[u, dashed, "e_f"] \arrow[ur, "f"'] & & \end{tikzcd} \tag{5}$$

The map $e_f \colon X \to \mathrm{Im}(f)$ is obtained from the universal property of kernels, since $k^\dagger \circ f = (f^\dagger \circ k)^\dagger = 0^\dagger = 0$. Since i_f is a †-mono, this e_f is determined as $e_f = \mathrm{id} \circ e_f = (i_f)^\dagger \circ i_f \circ e_f = (i_f)^\dagger \circ f$.

The image of a map f is therefore defined as kernel $\ker(\mathrm{coker}(f))$. Conversely, every kernel $m = \ker(f)$ arises as an image, since $\ker(\mathrm{coker}(m)) = m$ by Lemma 2.3.

The maps that arise as e_f in (5) can be characterised.

Proposition 4.1 *The maps in* **D** *that arise of the form e_f, as in diagram (5), are precisely the zero-epis.*

Proof. We first show that e_f is a zero-epi. Assume therefor a map $h \colon \ker(k^\dagger) \to Z$ satisfying $h \circ e_f = 0$. Recall $e_f = (i_f)^\dagger \circ f$, so that:

$$f^\dagger \circ (i_f \circ h^\dagger) = (h \circ (i_f)^\dagger \circ f)^\dagger = (h \circ e_f)^\dagger = 0^\dagger = 0.$$

This means that $i_f \circ h^\dagger$ factors through the kernel of $f^\dagger$, say via $a \colon Z \to \ker(f^\dagger)$ with $k \circ a = i_f \circ h^\dagger$. Since k is a †-mono we now get:

$$a = k^\dagger \circ k \circ a = k^\dagger \circ i_f \circ h^\dagger = 0 \circ h^\dagger = 0.$$

But then $i_f \circ h^\dagger = k \circ a = k \circ 0 = 0 = i_f \circ 0$, so that $h^\dagger = 0$, because i_f is mono, and $h = 0$, as required.

Conversely, assume $g \colon X \to Y$ is a zero-epi, so that $\mathrm{coker}(g) = 0$ by Lemma 2.9. Trivially, $i_g = \ker(\mathrm{coker}(g)) = \ker(X \to 0) = \mathrm{id}_X$, so that $e_g = g$. □

The factorisation $f = i_f \circ e_f$ from (5) describes each map as a zero-epi followed by a kernel. In fact, these zero-epis and kernels also satisfy what is usually called the "diagonal fill-in" property.

Lemma 4.2 *In any commuting square of shape*

$$\begin{tikzcd} \cdot \arrow[r, two heads, "\circ"] \arrow[d] & \cdot \arrow[d] \\ \cdot \arrow[r, tail] & \cdot \end{tikzcd} \quad \textit{there is a (unique) diagonal} \quad \begin{tikzcd} \cdot \arrow[r, two heads, "\circ"] \arrow[d] & \cdot \arrow[d] \arrow[dl, dashed] \\ \cdot \arrow[r, tail] & \cdot \end{tikzcd}$$

making both triangles commute.

As a result, the factorisation (5) is unique up to isomorphism. Indeed, kernels and zero-epis form a factorisation system (see [3]).

Proof. Assume the zero-epi $e\colon E \to Y$ and kernel $m = \ker(h)\colon M \rightarrowtail X$ satisfy $m \circ f = g \circ e$, as below,

$$\begin{tikzcd} E \arrow[r, two heads, "e"] \arrow[d, "f"'] & Y \arrow[d, "g"] & \\ M \arrow[r, "m"'] & X \arrow[r, "h"'] & Z \end{tikzcd}$$

Then: $h \circ g \circ e = h \circ m \circ f = 0 \circ f = 0$ and $h \circ g = 0$ because e is zero-epi. This yields the required diagonal $d\colon Y \to M$ with $m \circ d = g$ because m is the kernel of h. Using that m is monic we get $d \circ e = f$. □

Factorisation standardly gives a left adjoint to inverse image (pullback), corresponding to existential quantification in logic. In this self-dual situation there are alternative descriptions.

Proposition 4.3 *For $f\colon X \to Y$, the pullback functor $f^{-1}\colon \mathrm{KSub}(Y) \to \mathrm{KSub}(X)$ from Lemma 2.4 has a left adjoint $\exists_f$ given as image:*

$$\Big(M \xrightarrow{m} X\Big) \longmapsto \Big(\mathrm{Im}(f \circ m) \xrightarrow{\exists_f(m) = i_{f\circ m}} Y\Big)$$

Alternatively, $\exists_f(m) = \Big((f^\dagger)^{-1}(m^\perp)\Big)^\perp$.

Proof. It is standard/straightforward that $m \le f^{-1}(n)$ iff there is a $\varphi\colon M \to N$ with $n \circ \varphi = f \circ m$ iff $\exists_f(m) \le n$. For the alternative description:

$$\begin{aligned} \Big((f^\dagger)^{-1}(m^\perp)\Big)^\perp \le n &\iff n^\perp \le (f^\dagger)^{-1}(m^\perp) \\ &\iff \text{there is a } \psi\colon N^\perp \to M^\perp \text{ with } m^\perp \circ \psi = f^\dagger \circ n^\perp \\ &\overset{(*)}{\iff} \text{there is a } \varphi\colon M \to N \text{ with } n \circ \psi = f \circ m \\ &\iff m \le f^{-1}(n). \end{aligned}$$

For the direction ($\Rightarrow$) of the marked equivalence, recall that $n = \ker(\mathrm{coker}(n))$, so we show: $\mathrm{coker}(n) \circ f \circ m = (f^\dagger \circ n^\perp)^\dagger \circ m = (m^\perp \circ \psi)^\dagger \circ m = \psi^\dagger \circ \mathrm{coker}(m) \circ m = \psi^\dagger \circ 0 = 0$. The reverse direction works similarly: given φ one gets: $m^\dagger \circ f^\dagger \circ n^\perp = (f \circ m)^\dagger \circ n^\perp = (n \circ \varphi)^\dagger \circ n^\perp = \varphi^\dagger \circ n^\dagger \circ n^\perp = \varphi^\dagger \circ 0 = 0$. □

This adjunction $\exists_f \dashv f^{-1}$ makes the kernel fibration $\left(\begin{smallmatrix}\mathrm{KSub}(\mathbf{D})\\ \downarrow \\ \mathbf{D}\end{smallmatrix}\right)$ an opfibration, and thus a bifibration, see [12].

Recall the *Beck-Chevalley condition*: if the left square below is a pullback, then the right one must commute.

$$\begin{tikzcd} P \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} \mathrm{KSub}(P) \arrow[d, "\exists_p"'] & \mathrm{KSub}(Y) \arrow[l, "q^{-1}"'] \arrow[d, "\exists_g"] \\ \mathrm{KSub}(X) & \mathrm{KSub}(Z) \arrow[l, "f^{-1}"] \end{tikzcd} \qquad \text{(BC)}$$

This condition ensures that $\exists$ commutes with substitution. Beck-Chevalley holds for the pullbacks from Lemma 2.4 that are known to exist. In the notation from Lemma 2.4, for kernels $k\colon K \rightarrowtail Y$ and $g\colon Y \rightarrowtail Z$,

$$\begin{aligned} f^{-1}(\exists_g(k)) &= f^{-1}(g \circ k) && \text{because both } g, k \text{ are kernels} \\ &= p^{-1}(k) \circ q && \text{by composition of pullbacks} \\ &= \exists_p(q^{-1}(k)). \end{aligned}$$

In **Hilb** all pullbacks exist and Beck-Chevalley holds for all of them by [4, II, Proposition 1.7.6] using **Hilb**'s biproducts and equalisers.

The final result in this section brings more clarity; it underlies the relations between the various maps in the propositions in the previous section.

Lemma 4.4 *If zero-epis are (ordinary) epis, then* $\dagger$*-monos are kernels.*

Recall that Lemma 2.10 tells that zero-epis are epis in the presence of equalisers.

Proof. Suppose $m\colon M \rightarrowtail X$ is a $\dagger$-mono, with factorisation $m = i \circ e$ as in (5), where i is a kernel and a $\dagger$-mono, and e is a zero-epi and hence an epi by assumption. We are done if we can show that e is an isomorphism. Since $m = i \circ e$ and i is $\dagger$-monic we get $i^\dagger \circ m = i^\dagger \circ i \circ e = e$. Hence $e^\dagger \circ e = (i^\dagger \circ m)^\dagger \circ e = m^\dagger \circ i \circ e = m^\dagger \circ m = \mathrm{id}$ because m is $\dagger$-mono. But then also $e \circ e^\dagger = \mathrm{id}$ because e is epi and $e \circ e^\dagger \circ e = e$. $\square$

Example 4.5 In the category **Rel** the image of a morphism $(X \overset{r_1}{\leftarrow} R \overset{r_2}{\rightarrow} Y)$ is the relation $i_R = (Y' \overset{=}{\leftarrow} Y' \rightarrowtail Y)$ where $Y' = \{y \in Y \mid \exists_x.\, R(x,y)\}$ is the image of the second leg r_2 in **Sets**. The associated zero-epi is $e_R = (X \overset{r_1}{\leftarrow} R \overset{r_2}{\twoheadrightarrow} Y')$. Existential quantification $\exists_R(M)$ from Proposition 4.3 corresponds to the modal diamond operator (for the reversed relation $R^\dagger$):

$$\exists_R(M) = \{y \in Y \mid \exists_{x \in M}.\, R(x,y)\} = \Diamond_{R^\dagger}(M) = \neg\Box_{R^\dagger}(\neg M).$$

It is worth mentioning that the "graph" map of fibrations (3) between sets and relations is also a map of opfibrations: for a function $f\colon X \to Y$ and a predicate $M \subseteq X$ one has:

$$\begin{aligned} \exists_{\mathcal{G}(f)}(M) &= \{y \mid \exists_x.\, \mathcal{G}(f)(x,y) \wedge M(x)\} \\ &= \{y \mid \exists_x.\, f(x) = y \wedge M(x)\} \\ &= \{f(x) \mid M(x)\} \\ &= \exists_f(M), \end{aligned}$$

where $\exists_f$ in the last line is the left adjoint to pullback f^{-1} in the category **Sets**.

In **PInj** the image of a map $f = (X \overset{f_1}{\hookleftarrow} F \overset{f_2}{\rightarrowtail} Y)$ is given as $i_f = (F \overset{\mathrm{id}}{\hookleftarrow} F \overset{f_2}{\rightarrowtail} Y)$. The associated map e_f is $(X \overset{f_1}{\hookleftarrow} F \overset{\mathrm{id}}{\rightarrowtail} F)$, so that indeed $f = i_f \circ e_f$. Notice that this e_f is a $\dagger$-epi in **PInj**.

In **Hilb**, the image of a map $f : X \to Y$ is (the inclusion of) the closure of the set-theoretic image $\{y \in Y \mid \exists_{x \in X} . y = f(x)\}$. This descends to **PHilb**: the image of a morphism is the equivalence class represented by the inclusion of the closure of the set-theoretic image of a representative.

The functor $\ell^2 \colon \mathbf{PInj} \to \mathbf{Hilb}$ is a map of opfibrations: for a partial injection $f = (X \overset{f_1}{\hookleftarrow} F \overset{f_2}{\rightarrowtail} Y)$ and a kernel $m : M \rightarrowtail X$ in **PInj** one has:

$$\begin{aligned}
\exists_{\ell^2(f)}(\ell^2(m)) &= \mathrm{Im}_{\mathbf{Hilb}}(\ell^2(f \circ m)) \\
&= \mathrm{Im}_{\mathbf{Hilb}}(\lambda\varphi : \ell^2(M).\lambda y : Y. \sum_{x \in (f \circ m)^{-1}(y)} \varphi(x))) \\
&\cong \overline{\{\varphi \in \ell^2(X) \mid \mathrm{supp}(\varphi) \subseteq F \cap M\}} \\
&= \{\varphi \in \ell^2(X) \mid \mathrm{supp}(\varphi) \subseteq F \cap M\} \\
&\cong \ell^2(f_2 \circ f_1^{-1}(m)) \\
&= \ell^2(\exists_f(m)).
\end{aligned}$$

Also the full functor $P \colon \mathbf{Hilb} \to \mathbf{PHilb}$ is a map of opfibrations: for $f \colon X \to Y$ and a kernel $m : M \rightarrowtail X$ in **Hilb** one has:

$$\begin{aligned}
\exists_{Pf}(Pm) &= \mathrm{Im}_{\mathbf{PHilb}}(P(f \circ m)) \\
&= \overline{\{f(x) \mid x \in M\}} \\
&= P(\overline{\{f(x) \mid x \in M\}}) \\
&= P(\mathrm{Im}_{\mathbf{Hilb}}(f \circ m)) \\
&= P(\exists_f(m)).
\end{aligned}$$

In the category $\widehat{B}$ obtained from a Boolean algebra the factorisation of $f \colon x \to y$ is the composite $x \xrightarrow{f} f \xrightarrow{f} y$. In particular, for $m \leq x$, considered as kernel $m \colon m \to x$ one has $\exists_f(m) = (m \wedge f \colon (m \wedge f) \to x)$.

Example 4.6 In [18] the domain $\mathrm{Dom}(f)$ of a map $f \colon X \to Y$ is the negation of its kernel, so $\mathrm{Dom}(f) = \ker(f)^\perp$, and hence a kernel itself. It can be described as an image, namely of $f^\dagger$, since:

$$\mathrm{Dom}(f) = \ker(f)^\perp = \ker(\ker(f)^\dagger) = \ker(\mathrm{coker}(f^\dagger)) = i_{f^\dagger}.$$

It is shown in [18] that the composition $f \circ \mathrm{Dom}(f)$ is zero-monic—or “total”, as it is called there. This also holds in the present setting, since:

$$f \circ \mathrm{Dom}(f) = f^{\dagger\dagger} \circ i_{f^\dagger} = (i_{f^\dagger} \circ e_{f^\dagger})^\dagger \circ i_{f^\dagger} = (e_{f^\dagger})^\dagger \circ (i_{f^\dagger})^\dagger \circ i_{f^\dagger} = (e_{f^\dagger})^\dagger.$$

This $e_{f^\dagger}$ is zero-epic, by Proposition 4.1, so that $(e_{f^\dagger})^\dagger$ is indeed zero-monic.

There is one further property that is worth making explicit, if only in examples. In the kernel fibration over **Rel** one finds the following correspondences.

$$\mathrm{KSub}(X) \cong \mathcal{P}(X) \cong \mathbf{Sets}(X, 2) \cong \mathbf{Sets}(X, \mathcal{P}(1)) \cong \mathbf{Rel}(X, 1).$$

This suggests that one has "kernel classifiers", comparable to "subobject classifiers" in a topos—or more abstractly, "generic objects", see [12]. But the naturality that one has in toposes via pullback functors f^{-1} exists here via their left adjoints $\exists_f$. Indeed, there are natural "characteristic" isomorphisms:

$$\begin{array}{rcl} \mathrm{KSub}(X) = \mathcal{P}(X) & \xrightarrow[\cong]{\text{char}} & \mathbf{Rel}(1, X) \\ (M \subseteq X) & \longmapsto & \{(*, x) \mid x \in M\}. \end{array}$$

Then, for $S\colon X \to Y$ in **Rel**,

$$\begin{aligned} S \circ \mathrm{char}(M) &= \{(*, y) \mid \exists_x.\, \mathrm{char}(M)(*, x) \wedge S(x, y)\} \\ &= \{(*, y) \mid \exists_x.\, M(x) \wedge S(x, y)\} \\ &= \{(*, y) \mid \exists_S(M)(y)\} \\ &= \mathrm{char}(\exists_S(M)). \end{aligned}$$

Hence one could say that **Rel** has a kernel "op-classifier".

The same thing happens in the dagger categories $\widehat{B}$ from Subsection 3.5. There one has, for $x \in B$,

$$\begin{array}{rcl} \mathrm{KSub}(x) = {\downarrow} x & \xrightarrow[\cong]{\text{char}} & \widehat{B}(1, x) \\ (m \leq x) & \longmapsto & (m\colon 1 \to x) \end{array}$$

As before, $f \circ \mathrm{char}(m) = f \wedge m = \exists_f(m) = \mathrm{char}(\exists_f(m))$.

5 Images and coimages

We continue to work in an arbitrary dagger kernel category **D**. In the previous section we have seen how each map $f\colon X \to Y$ in **D** can be factored as $f = i_f \circ e_f$ where the image $i_f = \ker(\mathrm{coker}(f))\colon \mathrm{Im}(f) \rightarrowtail Y$ is a kernel and e_f is a zero-epi. We can apply this same factorisation to the dual $f^\dagger$. The dual of its image, $(i_{f^\dagger})^\dagger = \mathrm{coker}(\ker(f))\colon X \twoheadrightarrow \mathrm{Im}(f^\dagger)$, is commonly called the coimage of f. It is a cokernel and $\dagger$-epi by construction. Thus we have:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "e_f"'] & & Y \\ & \mathrm{Im}(f) \arrow[ur, "i_f"'] & \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[rr, "f^\dagger"] \arrow[dr, "e_{f^\dagger}"'] & & X \\ & \mathrm{Im}(f^\dagger) \arrow[ur, "i_{f^\dagger}"'] & \end{tikzcd}$$

By combining these factorisations we get two mediating maps m by diagonal fill-in, as in:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "e_f"] \arrow[ddr, "(i_{f^\dagger})^\dagger"'] & & Y \\ & \mathrm{Im}(f) \arrow[ur, "i_f"'] & \\ & \mathrm{Im}(f^\dagger) \arrow[u, dotted, "m_f"] \arrow[uur, "(e_{f^\dagger})^\dagger"'] & \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[rr, "f^\dagger"] \arrow[dr, "e_{f^\dagger}"] \arrow[ddr, "(i_f)^\dagger"'] & & X \\ & \mathrm{Im}(f^\dagger) \arrow[ur, "i_{f^\dagger}"'] & \\ & \mathrm{Im}(f) \arrow[u, dotted, "m_{f^\dagger}"] \arrow[uur, "(e_f)^\dagger"'] & \end{tikzcd}$$

We claim that $(m_f)^\dagger = m_{f^\dagger}$. This follows easily from the fact that $(i_{f^\dagger})^\dagger$ is epi:

$$(m_{f^\dagger})^\dagger \circ (i_{f^\dagger})^\dagger = (i_{f^\dagger} \circ m_{f^\dagger})^\dagger = (e_f)^{\dagger\dagger} = e_f = m_f \circ (i_{f^\dagger})^\dagger.$$

Moreover, m_f is both a zero-epi and a zero-mono.

As a result we can factorise each map $f\colon X \to Y$ in $\mathbf{D}$ as:

$$\begin{tikzcd} X \arrow[r, "(i_{f^\dagger})^\dagger", "\text{coimage}"'] & \mathrm{Im}(f^\dagger) \arrow[r, "m_f", "\substack{\text{zero-epi}\\ \text{zero-mono}}"'] & \mathrm{Im}(f) \arrow[r, "i_f", "\text{image}"'] & Y\,. \end{tikzcd} \tag{6}$$

This coimage may also be reversed, so that a map in $\mathbf{D}$ can also be understood as a pair of kernels with a zero-mono/epi between them, as in:

$$\begin{tikzcd} X & \mathrm{Im}(f^\dagger) \arrow[l, "i_{f^\dagger}"'] \arrow[r] & \mathrm{Im}(f) \arrow[r, "i_f"] & Y \end{tikzcd}$$

The two outer kernel maps perform some “bookkeeping” to adjust the types; the real action takes place in the middle, see the examples below. The category **PInj** consists, in a sense, of only these bookkeeping maps, without any action. This will be described more systematically in Definition 6.4.

Example 5.1 We briefly describe the factorisation (6) in **Rel**, **PInj** and **Hilb**, using diagrammatic order for convenience (with notation $f; g = g \circ f$).

For a map $(X \overset{r_1}{\leftarrow} R \overset{r_2}{\rightarrow} Y)$ in **Rel** we take the images $X' \rightarrowtail X$ of r_1 and $Y' \rightarrowtail Y$ of r_2 in:

$$\left(\begin{tikzcd}[sep=small] & R \arrow[dl, "r_1"'] \arrow[dr, "r_2"] & \\ X & & Y \end{tikzcd}\right) = \left(\begin{tikzcd}[sep=small] & X' \arrow[dl] \arrow[dr, equal] & \\ X & & X' \end{tikzcd}\right); \left(\begin{tikzcd}[sep=small] & R \arrow[dl, "r_1"'] \arrow[dr, "r_2"] & \\ X' & & Y' \end{tikzcd}\right); \left(\begin{tikzcd}[sep=small] & Y' \arrow[dl, equal] \arrow[dr] & \\ Y' & & Y \end{tikzcd}\right)$$

In **PInj** the situation is simpler, because the middle part m in (6) is the identity, in:

$$\left(\begin{tikzcd}[sep=small] & F \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ X & & Y \end{tikzcd}\right) = \left(\begin{tikzcd}[sep=small] & F \arrow[dl, "f_1"'] \arrow[dr, equal] & \\ X & & F \end{tikzcd}\right); \left(\begin{tikzcd}[sep=small] & F \arrow[dl, equal] \arrow[dr, "f_2"] & \\ F & & Y \end{tikzcd}\right).$$

In **Hilb**, a morphism $f : X \to Y$ factors as $f = i \circ m \circ e$. The third part $i : I \to Y$ is given by $i(y) = y$, where I is the closure $\overline{\{f(x) : x \in X\}}$. The first part $e : X \to E$ is given by orthogonal projection on the closure $E = \overline{\{f^\dagger(y) : y \in Y\}}$; explicitly, $e(x)$ is the unique x' such that $x = x' + x''$ with $x' \in E$ and $\langle x'' \,|\, z\rangle = 0$ for all $z \in E$. Using the fact that the adjoint $e^\dagger : E \to X$ is given by $e^\dagger(x) = x$, we deduce that the middle part $m : E \to I$ is determined by $m(x) = (i \circ m)(x) = (f \circ e^\dagger)(x) = f(x)$. Explicitly,

$$\left(X \xrightarrow{f} Y\right) = \left(X \xrightarrow{e} E\right); \left(E \xrightarrow{m} I\right); \left(I \xrightarrow{i} Y\right).$$

6 Categorical logic

In this final section we further investigate the logic of dagger kernel categories. We shall first see how the so-called Sasaki hook [13] arises naturally in this setting, and then investigate Booleanness.

For a kernel $m\colon M \rightarrowtail X$ we shall write $P(m) = m \circ m^\dagger\colon X \to X$. This $P(m)$ is easily seen to be a self-adjoint idempotent [2]: one has $P(m)^\dagger = P(m)$ and $P(m) \circ P(m) = P(m)$. The endomap $P(m)\colon X \to X$ associated with a kernel/predicate m on X maps everything in X that is in m to itself, and what is perpendicular to m to 0, as expressed by the equations $P(m) \circ m = m$ and $P(m) \circ m^\perp = 0$. Of interest is the following result. It makes the dynamical aspects of quantum logic described in [7] explicit.

Proposition 6.1 *For kernels $m\colon M \rightarrowtail X$, $n\colon N \rightarrowtail X$ the pullback $P(m)^{-1}(n)$ is the Sasaki hook, written here as $\supset$:*

$$m \supset n \stackrel{def}{=} P(m)^{-1}(n) \;=\; m^\perp \vee (m \wedge n).$$

The associated left adjoint $\exists_{P(m)} \dashv P(m)^{-1}$ yields the "and then" operator:

$$k \mathbin{\&} m \stackrel{def}{=} \exists_{P(m)}(k) \;=\; m \wedge (m^\perp \vee k),$$

so that the "Sasaki adjunction" holds by construction:

$$k \mathbin{\&} m \leq n \Longleftrightarrow k \leq m \supset n.$$

Quantum logic based on this "and-then" & connective is developed in [15], see also [20,21]. This & connective is in general non-commutative and non-associative [3]. Some basic properties are: $m \mathbin{\&} m = m$, $1 \mathbin{\&} m = m \mathbin{\&} 1 = m$, $0 \mathbin{\&} m = m \mathbin{\&} 0 = 0$, and both $k \mathbin{\&} m \leq n$, $k^\perp \mathbin{\&} m \leq n$ imply $m \leq n$ (which easily follows from the Sasaki adjunction).

Proof. Consider the following pullbacks.

$$\begin{tikzcd} P \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & N \arrow[d, tail, "n"] \\ M \arrow[r, tail, "m"'] & X \end{tikzcd} \qquad\qquad \begin{tikzcd} Q \arrow[r, "s"] \arrow[d, "r"'] \arrow[dr, phantom, "\lrcorner", very near start] & P^\perp \arrow[d, tail, "(m\wedge n)^\perp = \ker(p^\dagger \circ m^\dagger)"] \\ M \arrow[r, tail, "m"'] & X \end{tikzcd}$$

---

[2] Sometimes these self-adjoint idempotents are called projections, but we shall use "projection" slightly differently, see a forthcoming paper, namely with additional requirement that it is less than or equal to the identity, for a suitably defined order on homsets.

[3] The "and-then" connective & should not be confused with the multiplication of a quantale [22], since the latter is always associative.

Then:

$$\begin{aligned}
m^{\perp} \vee (m \wedge n) &= \big(m \wedge (m \wedge n)^{\perp}\big)^{\perp} \\
&= \ker\big((m \wedge (m \wedge n)^{\perp})^{\dagger}\big) \\
&= \ker\big(r^{\dagger} \circ m^{\dagger}\big) \\
&= \ker\big(\ker(\mathrm{coker}((m \wedge n)^{\perp}) \circ m)^{\dagger} \circ m^{\dagger}\big) \\
&\qquad\qquad \text{by definition of } r \text{ as pullback, see Lemma 2.4} \\
&= \ker\big(\ker(\mathrm{coker}(\ker(p^{\dagger} \circ m^{\dagger})) \circ m)^{\dagger} \circ m^{\dagger}\big) \\
&= \ker\big(\ker(p^{\dagger} \circ m^{\dagger} \circ m)^{\dagger} \circ m^{\dagger}\big) \\
&\qquad\qquad \text{because } p^{\dagger} \circ m^{\dagger} \text{ is a cokernel, see Lemma 2.6} \\
&= \ker(\mathrm{coker}(p) \circ m^{\dagger}) \\
&= \big(m^{\dagger}\big)^{-1}(p) \\
&= \big(m^{\dagger}\big)^{-1}(m^{-1}(n)) \\
&= P(m)^{-1}(n). \qquad \square
\end{aligned}$$

We now turn to Booleanness. As we have seen, the categories **Rel**, **PInj** and $\widehat{B}$ (for a Boolean algebra B) are Boolean, but **Hilb** and **PHilb** are not. We start with a result that justifies the name "Boolean".

Theorem 6.2 *A dagger kernel category is Boolean if and only if each poset* $\mathrm{KSub}(X)$ *is a Boolean algebra.*

Proof. We already know that each poset $\mathrm{KSub}(X)$ is an orthomodular lattice, with bottom 0, top 1, negation $(-)^{\perp}$ (by Lemma 2.3), intersections $\wedge$ (by Lemma 2.6), and joins $m \vee n = (m^{\perp} \wedge n^{\perp})^{\perp}$. What is missing is distributivity $m \wedge (n \vee k) = (m \vee n) \wedge (m \vee k)$. We show that it is equivalent to the Booleanness requirement $m \wedge n = 0 \Rightarrow m \perp n$. Recall: $m \perp n$ iff $n^{\dagger} \circ m = 0$ iff $m \leq n^{\perp} = \ker(n^{\dagger})$.

First, assume Booleanness. To start,

$$(m \wedge (m \wedge n)^{\perp}) \wedge n = (m \wedge n) \wedge (m \wedge n)^{\perp} = 0$$

Hence $m \wedge (m \wedge n)^{\perp} \leq n^{\perp}$. Similarly, $m \wedge (m \wedge k)^{\perp} \leq k^{\perp}$. Therefore:

$$m \wedge (m \wedge n)^{\perp} \wedge (m \wedge k)^{\perp} \leq n^{\perp} \wedge k^{\perp} = (n \vee k)^{\perp},$$

and thus:

$$m \wedge (m \wedge n)^{\perp} \wedge (m \wedge k)^{\perp} \wedge (n \vee k) = 0.$$

But then we are done by using Booleanness again:

$$m \wedge (n \vee k) \leq ((m \wedge n)^{\perp} \wedge (m \wedge k)^{\perp})^{\perp} = (m \wedge n) \vee (m \wedge k).$$

The other direction is easier: if $m \wedge n = 0$, then:

$$\begin{aligned} m \;=\; m \wedge 1 = m \wedge (n \vee n^{\perp}) & \\ = (m \wedge n) \vee (m \wedge n^{\perp}) & \quad \text{by distributivity} \\ = 0 \vee (m \wedge n^{\perp}) \;=\; m \wedge n^{\perp}. & \end{aligned}$$

Hence $m \leq n^{\perp}$. □

The Booleanness property can be strengthened in the following way.

Proposition 6.3 *The Booleanness requirement $m \wedge n = 0 \Rightarrow m \leq n^{\perp}$, for all kernels m, n, is equivalent to the following: for each pullback of kernels:*

$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & N \arrow[d, tail, "n"] \\ M \arrow[r, tail, "m"] & X \end{tikzcd} \qquad \textit{one has} \qquad n^{\dagger} \circ m = p \circ q^{\dagger}.$$

Proof. It is easy to see that the definition of Booleanness is the special case $P = 0$. For the converse, we put another pullback on top of the one in the proposition:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & P^{\perp} \arrow[d, tail, "p^{\perp}"] \\ P \arrow[r, tail, "p"] \arrow[d, tail, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & N \arrow[d, tail, "n"] \\ M \arrow[r, tail, "m"] & X \end{tikzcd}$$

We use that p, q are kernels by Lemma 2.4. We see $m \wedge (n \circ p^{\perp}) = 0$, so by Booleanness we obtain:

$$\begin{aligned} m \leq (n \circ p^{\perp})^{\perp} &= \ker\left((n \circ \ker(p^{\dagger}))^{\dagger}\right) \\ &= \ker(\operatorname{coker}(p) \circ n^{\dagger}) \\ &= (n^{\dagger})^{-1}(p), \end{aligned}$$

where the pullback is as described in Lemma 2.4. Hence there is a map $\varphi \colon M \to P$ with $p \circ \varphi = n^{\dagger} \circ m$. This means that $\varphi = p^{\dagger} \circ p \circ \varphi = p^{\dagger} \circ n^{\dagger} \circ m = (n \circ p)^{\dagger} \circ m = (m \circ q)^{\dagger} \circ m = q^{\dagger} \circ m^{\dagger} \circ m = q^{\dagger}$. Hence we have obtained $p \circ q^{\dagger} = n^{\dagger} \circ m$, as required. □

Definition 6.4 Let $\mathbf{D}$ be a Boolean dagger kernel category. We write $\mathbf{D}_{KcK}$ for the category with the same objects as $\mathbf{D}$; morphisms $X \to Y$ in $\mathbf{D}_{KcK}$ are cokernel-kernel pairs (c, k) of the form $X \xrightarrow{c} \bullet \xrightarrow{k} Y$. The identity $X \to X$ is $X \xrightarrow{\text{id}} X \xrightarrow{\text{id}} X$, and composition of $X \xrightarrow{c} M \xrightarrow{k} Y$ and $Y \xrightarrow{d} N \xrightarrow{l} Z$ is the pair $(q^\dagger \circ c, l \circ p)$ obtained via the pullback:

$$\begin{tikzcd} & P \arrow[r, "p"] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & N \arrow[r, "l"] \arrow[d, "d^\dagger"] & Z \\ X \arrow[r, "c"] & M \arrow[r, "k"] & Y & \end{tikzcd} \tag{7}$$

To be precise, we identity (c, k) with $(\varphi \circ c, k \circ \varphi^{-1})$, for isomorphisms φ.

The reader may have noticed that this construction generalises the definition of **PInj**. Indeed, now we can say $\mathbf{PInj} = \mathbf{Rel}_{KcK}$.

Theorem 6.5 *The category $\mathbf{D}_{KcK}$ as described in Definition 6.4 is again a Boolean dagger kernel category, with a functor $D\colon \mathbf{D}_{KcK} \to \mathbf{D}$ in* **DCK** *and a change-of-base situation (pullback):*

$$\begin{tikzcd} \mathrm{KSub}(\mathbf{D}_{KcK}) \arrow[r] \arrow[d] & \mathrm{KSub}(\mathbf{D}) \arrow[d] \\ \mathbf{D}_{KcK} \arrow[r, "D"] & \mathbf{D} \end{tikzcd}$$

Moreover, in $\mathbf{D}_{KcK}$ one has:

$$\textit{kernel} \;=\; \dagger\textit{-mono} \;=\; \textit{mono} \;=\; \textit{zero-mono},$$

and $\mathbf{D}_{KcK}$ is universal among such categories.

Proof. The obvious definition $(c, k)^\dagger = (k^\dagger, c^\dagger)$ yields an involution on $\mathbf{D}_{KcK}$. The zero object $0 \in \mathbf{D}$ is also a zero object $0 \in \mathbf{D}_{KcK}$ with zero map $X \longrightarrow 0 \longrightarrow Y$ consisting of a cokernel-kernel pair. A map (c, k) is a $\dagger$-mono if and only if $(c, k)^\dagger \circ (c, k) = (k^\dagger, k)$ is the identity; this means that $k = \mathrm{id}$.

The kernel of a map $(d, l) = (Y \xrightarrow{d} N \xrightarrow{l} Z)$ is $\ker(d, l) = (N^\perp \xrightarrow{\mathrm{id}} N^\perp \xrightarrow{(d^\dagger)^\perp} Y)$, so that $\ker(d, l)$ is a $\dagger$-mono and $(d, l) \circ \ker(d, l) = 0$. If also $(d, l) \circ (c, k) = 0$, then $k \wedge d^\dagger = 0$ so that by Booleanness, $k \leq (d^\dagger)^\perp$, say via $\varphi\colon M \to N^\perp$ with $(d^\dagger)^\perp \circ \varphi = k$. Then we obtain a mediating map $(c, \varphi) = (X \xrightarrow{c} M \xrightarrow{\varphi} N^\perp)$ which satisfies $\ker(d, l) \circ (c, \varphi) = (\mathrm{id}, (d^\dagger)^\perp) \circ (c, \varphi) = (c, (d^\dagger)^\perp \circ \varphi) = (c, k)$. It is not hard to see that maps of the form (id, m) in $\mathbf{D}_{KcK}$ are kernels, namely of the cokernel $(m^\perp, \mathrm{id})$.

The intersection of two kernels $(\mathrm{id}, m) = (M = M \xrightarrow{m} X)$ and $(\mathrm{id}, n) = (N = N \xrightarrow{n} X)$ in $\mathbf{D}_{KcK}$ is the intersection $m \wedge n\colon P \rightarrowtail X$ in $\mathbf{D}$, with projections $(P = P \longrightarrow M)$ and $(P = P \longrightarrow N)$. Hence if the intersection of (id, m) and (id, n) in $\mathbf{D}_{KcK}$ is 0, then so is the intersection of m and n in $\mathbf{D}$, which yields $n^\dagger \circ m = 0$. But then in $\mathbf{D}_{KcK}$, $(\mathrm{id}, n)^\dagger \circ (\mathrm{id}, m) = (n^\dagger, \mathrm{id}) \circ (\mathrm{id}, m) = 0$. Hence $\mathbf{D}_{KcK}$ is also Boolean.

Finally, there is a functor $\mathbf{D}_{KcK} \to \mathbf{D}$ by $X \mapsto X$ and $(c, k) \mapsto k \circ c$. Composition is preserved by Proposition 6.3, since for maps as in Definition 6.4,

$$(d, l) \circ (c, k) \;=\; (q^\dagger \circ c, l \circ p) \longmapsto l \circ p \circ q^\dagger \circ c \;=\; (l \circ d) \circ (k \circ c).$$

We have already seen that $\mathrm{KSub}(X)$ in $\mathbf{D}_{KcK}$ is isomorphic to $\mathrm{KSub}(X)$ in $\mathbf{D}$. This yields the change-of-base situation.

We have already seen that kernels and †-monos coincide. We now show that they also coincide with zero-monos. So let $(d, l)\colon Y \to Z$ be a zero-mono. This means that $(d, l) \circ (c, k) = 0 \Rightarrow (c, k) = 0$, for each map (c, k). Using diagram (7), this means: $d^\dagger \wedge k = 0 \Rightarrow k = 0$. By Booleanness, the antecedent $d^\dagger \wedge k = 0$ is equivalent to $k \leq (d^\dagger)^\perp = \ker(d)$, which means $d \circ k = 0$. Hence we see that d is zero-monic in $\mathbf{D}$, and thus an isomorphism (because it is already a cokernel).

Finally, let $\mathbf{E}$ be a Boolean dagger kernel category in which zero-monos are kernels, with a functor $F\colon \mathbf{E} \to \mathbf{D}$ in $\mathbf{DCK}$. Every morphism f in $\mathbf{E}$ factors as $f = i_f \circ e_f$ for a kernel i_f and a cokernel e_f. Hence $G\colon \mathbf{E} \to \mathbf{D}_{KcK}$ defined by $G(X) = F(X)$ and $G(f) = (e_f, i_f)$ is the unique functor satisfying $F = D \circ G$. □

7 Conclusions and future work

The paper shows that a "dagger kernel category" forms a powerful notion that not only captures many examples of interest in quantum logic but also provides basic structure for categorical logic. Several research avenues are still open: construction of dagger kernel categories from orthomodular lattices (like in Subsection 3.5 for Boolean algebras), or further investigation of the relevance of "opfibred" structure in this setting (like for "op-classifiers" at the end of Section 4).

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