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Clones with Nullary Operations [1]

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Abstract

This article discusses clones with nullary operations and the corresponding relational clones, both defined on arbitrary sets. By means of two pairs of kernel and closure operators, the relationship between such clones and clones in the traditional sense, i.e. without nullary operations, is investigated, and in particular the latter type of clones is located in the lattice of all clones. Finally, the fundamental theorem characterising Galois closed sets w.r.t. an adjusted version of Pol − Inv as local closures of clones and relational clones, respectively, is proven in the more comprehensive setting.

Keywords: clone, relational clone, nullary operation

1 Introduction

Clones (historically also known as *function algebras* or Post *classes*) are sets of finitary operations on a fixed carrier set that contain all projection operations and are closed under composition. They play an important role in modern universal algebra due to the fact that the set of all term operations of a universal algebra **A**, i.e. the union of all finitely generated free algebras in the equational class generated by **A**, always forms a clone. Moreover, important properties of algebras, like whether a subset forms a subuniverse, or a mapping has the homomorphism property, depend only on the clone of term operations of an algebra, not on its specific fundamental operations. In this way comparing clones of algebras is much more suitable for classifying algebras according to essentially different behaviour than comparing algebras themselves.

Clones can be seen as a higher arity generalisations of transformation monoids, and like transformation monoids give rise to the abstract notion of monoids,

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which forgets about the underlying carrier set, clones lead to the concept of *abstract clones* [25,4], which axiomatises composition of finitary functions and projections. This analogy even encloses CAYLEY's theorem in the sense that every abstract clone has a concrete representation as an isomorphic clone of finitary operations (cf. [4]).

Modulo a caveat about nullary operations, abstract clones are in turn just a slight reformulation of the concept of LAWVERE's *algebraic theories* [11,12] (see [2, Proposition 3.4, p. 10] for details). The latter constitute a common category theoretic means to capture equational theories invariantly of their presentation (i.e. of the chosen similarity type), and can be seen to be equivalent [13] to the notion of *finitary monad* [6,5] on the category of sets (see also [7] for a discussion of LAWVERE theories vs. monads in universal algebra).

This connection between clones and equational theories has also been recognised by the universal algebraic community: if $\mathbf{A}$ is an algebra generating an equational class $\mathcal{V}$, i.e. the class of all algebras of identical similarity type as $\mathbf{A}$ obeying all the equations satisfied by $\mathbf{A}$, then the clone of term operations of $\mathbf{A}$ is in one-to-one correspondence with $\mathcal{V}$ (up to term equivalence of equational classes and isomorphism of abstract clones; see e.g. [24]).

There remain, however, a few minor differences: some universal algebraists, e.g. [3], exclude the possibility of empty carrier sets for their algebras, and in traditional clone theory (see e.g. [18,23,10]) it is customary to study only operations of strictly positive arity (despite the fact that term operations of algebras are usually allowed to be nullary). On the contrary, for the category theoretic approach, having empty carrier sets as initial objects in the category of sets is central, as otherwise this category would fail to be co-complete, and arguments would become a lot messier. Likewise, it is fundamental to the notion of LAWVERE theory to include nullary operations, such that, from a category theorist's perspective, it is very natural to consider (abstract) clones possibly having nullary operations.

An important tool for the investigation of clones is the GALOIS connection $\mathrm{Pol}_A - \mathrm{Inv}_A$ induced by the binary *compatibility* (also *preservation*) relation between finitary operations and finitary relations on a common base set A. If A is finite, then all concrete clones on A can be described as GALOIS closures w.r.t. $\mathrm{Pol}_A - \mathrm{Inv}_A$. More generally, for an arbitrary set A the fixed points of $\mathrm{Pol}_A \mathrm{Inv}_A$ are exactly those clones that are *locally closed*. In this way the mentioned GALOIS correspondence links (locally closed) clones to certain sets of relations, the fixed points of $\mathrm{Inv}_A \mathrm{Pol}_A$, which are[3] so-called (locally closed) *relational clones*. Thus, at least for finite A, relational clones can be interpreted as an alternative, equivalent description of clones.

Like for clones, also their GALOIS theory ([17,16]) is mostly studied only for non-nullary operations (and non-nullary relations), this, again, mainly for historic reasons. It is the aim of this article to demonstrate that, without any problems, the definitions of clone and relational clone, as well as the GALOIS connection $\mathrm{Pol}_A - \mathrm{Inv}_A$, can be lifted to a setting with nullary operations (and relations), and that the cent-

[3] not by definition, but by proof

ral characterisation theorem for the closures of the Galois connection continues to hold in the more comprehensive framework (cf. Subsection 3.2).

Since every clone in the traditional sense (*old* clone) continues to fulfil the criteria of the more general *new* clones, conventional clone theory, as developed e.g. in [18,23,10,20], still remains a valuable piece of the general theory: it just focusses on the subclones of the clone $\mathrm{O}_A \setminus \mathrm{O}_A^{(0)}$ of all non-nullary operations. Moreover, it is interesting to ask about the connections between old and new (relational) clones, e.g. about the position of the old clones in the complete lattice of all new clones on a fixed carrier set. We shall address these questions in Subsection 3.1 and see that no dramatic changes of the lattice structure occur. We remark that, with minor technical modifications, the main results of this text are taken from the more elaborate report [1].

The author thinks that it is beneficial for an exchange of knowledge and methods if the notions of term operation, clone, abstract clone and Lawvere theory are compatible with each other, and thus clones may contain nullary operations. He is not alone regarding this: the monograph [14, p. 143, Definition 4.1] and some publications such as [26,22,15] define (and use) clones possibly including nullary operations. Recent work such as [8,9], generalising clone theory to arbitrary categories, at least strongly suggests the possibility to include them in the theory.

That this approach also contributes to a smoother theory concerning relational clones shall be demonstrated with at small example: in order to be in accordance with the classical notion of clone of operations and the preservation relation, relational clones always had to contain the empty relation, as it is preserved by any operation of positive arity. This should be seen as an artefact, rather than as intended: if nullary operations are part of $\mathrm{Pol}_A - \mathrm{Inv}_A$, then the least relational clone on a set A (w.r.t. set inclusion), which is the least fixed point of $\mathrm{Inv}_A \mathrm{Pol}_A$, solely consists of the so-called *diagonal relations*. These are all relations of the form

$$d_\theta := \{ (x_0, \dots, x_{m-1}) \in A^m \mid \forall 0 \leq i, j < m : (i,j) \in \theta \implies x_i = x_j \},$$

where $m \in \mathbb{N} \setminus \{0\}$ is a positive natural number and $\theta \in \mathrm{Eq}(m)$ is any equivalence relation on the set of indices $\{ i \mid 0 \leq i < m \}$. This is also the set of all relations that can be defined using *primitive positive first order formulæ* with an empty set of predicate symbols, so without any predicates. Such a description works, in point of fact, more generally, as on finite base sets A the least relational clone containing a given set Q of relations can be expressed as closure of Q w.r.t. primitive positively definable relations. If one excludes nullary functions on the side of operations, one has to artificially include the empty relation on the side of relations, which disturbs this characterisation of generated relational clones.

2 Basic definitions and observations

In this section we first make the reader familiar with some notation for sets, tuples, functions, and relations. Then we quickly recall the notions of kernel and closure system, the associated kernel and closure operators, and Galois connections. After that we present the definitions of clones, relational clones, the compatibility relation between functions and relations, the GALOIS connection $\mathrm{Pol}_A - \mathrm{Inv}_A$ and the local closure operators on sets of functions and relations, in each definition possibly allowing nullary operations. Subsequently, we sketch two basic facts from clone theory needed for Section 3.

2.1 Notation, functions and relations

Throughout the text $\mathbb{N}$ will denote the set of all natural numbers (including zero), and $\mathbb{N}_+$ will stand for $\mathbb{N} \setminus \{0\}$. We will employ the standard set theoretic representation of natural numbers by JOHN VON NEUMANN, i.e. $n = \{ i \in \mathbb{N} \mid i < n \}$ for $n \in \mathbb{N}$. Furthermore, we will write $\mathfrak{P}(S)$ for the *power set* of a set S.

In this text we will study finitary operations, relations, clones etc. on arbitrary sets, which are usually called A. Finiteness of the carrier set is not required unless explicitly mentioned, also $A = \emptyset$ is not excluded per se.

If A and B are sets, we use A^B to denote the set of all mappings from B to A. If $f \in A^B$ and $U \subseteq A$, then $f[U] := \{ f(u) \mid u \in U \}$ denotes the *image* of U under f. We call $\operatorname{im} f := f[A]$ simply the *image* of f. Composition of functions is written as $g \circ f \in C^A$ for $f \in B^A$ and $g \in C^B$, i.e. $g \circ f$ maps elements $a \in A$ to $g(f(a))$.

If $B = n \in \mathbb{N}$ is just a natural number, then $A^B = A^n$ is the set of all n-tuples $x = (x(i))_{i<n}$. We will also write x_i for the entry $x(i)$ $(i \in n)$, and, if convenient, we will also refer to the entries of tuples by different indexing, e.g. $x = (x_1, \ldots, x_n)$. Note that the only element of $A^0 = A^\emptyset$ is the empty mapping (tuple), whose graph is the empty set. It will consistently be denoted by $\emptyset$. As tuples are functions they can be composed with other functions, for instance, if $x \in A^n$ and $\alpha\colon m \longrightarrow n$, $(m, n \in \mathbb{N})$, then $x \circ \alpha$ is the tuple in A^m whose entries are $x_{\alpha(i)}$ $(i \in m)$. Similarly, if $g\colon A \longrightarrow B$, then $g \circ x = (g(x_i))_{i \in n}$ is an element of B^n.

Any mapping $f \in A^{A^n}$ $(n \in \mathbb{N})$ is called an *n-ary operation* on A. The set of all *finitary operations* on A is $\mathrm{O}_A := \bigcup_{k \in \mathbb{N}} A^{A^k}$. For a set of operations $F \subseteq \mathrm{O}_A$ we denote its *n-ary part* by $F^{(n)} := F \cap A^{A^n}$. One can extend this notation to operators yielding subsets of operations: if $\mathrm{OP}\colon S \longrightarrow \mathfrak{P}(\mathrm{O}_A)$ is an operator on a set S, then we define $\mathrm{OP}^{(n)}\colon A \longrightarrow \mathfrak{P}\left(\mathrm{O}_A^{(n)}\right)$ by $\mathrm{OP}^{(n)}(s) := (\mathrm{OP}(s))^{(n)}$ for $s \in S$. A function $f \in \mathrm{O}_A^{(n)}$ is called *constant* if it has a one-element image. Such functions are uniquely determined by the element $a \in A$ such that $\operatorname{im} f = \{a\}$, and we denote them by $c_a^{(n)}$. Furthermore, for a set $F \subseteq \mathrm{O}_A$ the set of its unary constant members will be written as $C_1(F) := \left\{ c_a^{(1)} \;\middle|\; a \in A \wedge c_a^{(1)} \in F \right\}$ (see also Definition 3.1).

For $m \in \mathbb{N}$ we call any subset $\varrho \subseteq A^m$ of m-tuples an *m-ary relation* on A. Thus $\mathfrak{P}(A^m)$ is the set of all *m-ary relations*, and the set of all *finitary relations* is defined by $\mathrm{R}_A := \bigcup_{\ell \in \mathbb{N}} \mathfrak{P}\left(A^\ell\right)$. If $Q \subseteq \mathrm{R}_A$, we use $Q^{(m)} := Q \cap \mathfrak{P}(A^m)$ to denote its *m-ary part*. Moreover, if $\mathrm{OP}\colon S \longrightarrow \mathfrak{P}(\mathrm{R}_A)$ is an operator on a set S, we put $\mathrm{OP}^{(m)}\colon S \longrightarrow \mathfrak{P}\left(\mathrm{R}_A^{(m)}\right)$, mapping $s \in S$ to $\mathrm{OP}^{(m)}(s) := (\mathrm{OP}(s))^{(m)}$.

Finally, if $(P, \leq)$ is a poset and $x \in P$, we write $\downarrow_{(P,\leq)} x$ for the *principal ideal* $\{ y \in P \mid y \leq x \}$, and $\uparrow_{(P,\leq)} x$ for the *principal filter* $\{ y \in P \mid x \leq y \}$. We allow ourselves to omit the order relation if it is clear from the context, i.e. we may write $\downarrow_P x$ or $\uparrow_P x$. In all use cases within this text the order relation will be set inclusion.

2.2 Kernel and closure systems, kernel and closure operators, Galois connections

We briefly recall the notions of closure and kernel operator, closure and kernel system, and Galois connections since they are principal tools for our investigations.

Let S be a set. A collection $\mathcal{C} \subseteq \mathfrak{P}(S)$ of subsets is a *closure system* on S, if it is closed under intersection of arbitrary subcollections, that is, if we have $\bigcap \mathcal{D} \in \mathcal{C}$ for any $\mathcal{D} \subseteq \mathcal{C}$. In particular, $S = \bigcap \emptyset \in \mathcal{C}$. Dually, $\mathcal{C} \subseteq \mathfrak{P}(S)$ is a *kernel system* on S, if it is closed under arbitrary unions, especially $\emptyset \in \mathcal{C}$.

There is a one-to-one correspondence between closure systems and closure operators. The latter are mappings $c\colon \mathfrak{P}(S) \longrightarrow \mathfrak{P}(S)$ being *extensive*, *monotone* and *idempotent*. That is to say, for $X_1, X_2 \subseteq S$ we have $X_1 \subseteq c(X_1)$, $c(X_1) \subseteq c(X_2)$ whenever $X_1 \subseteq X_2$, and $c(c(X_1)) = c(X_1)$. If $c\colon \mathfrak{P}(S) \longrightarrow \mathfrak{P}(S)$ is a closure operator, then $c[\mathfrak{P}(S)] \subseteq \mathfrak{P}(S)$ is the corresponding closure system on S, and conversely, if $\mathcal{C} \subseteq \mathfrak{P}(S)$ is a closure system, then $c(X) := \bigcap \{ C \in \mathcal{C} \mid X \subseteq C \}$ for $X \subseteq S$ defines a closure operator having this closure system.

The dual concept of a *kernel operator*, i.e. an *intensive*, *monotone* and *idempotent* operator $k\colon \mathfrak{P}(S) \longrightarrow \mathfrak{P}(S)$ is obtained by replacing extensivity by intensivity, i.e. by the requirement $X \supseteq k(X)$ for $X \subseteq S$. Kernel systems correspond to kernel operators in a similar way as closure systems and closure operators do.

A very rich source for closure operators are Galois *connections*, i.e. pairs of mappings $(\varphi\colon \mathfrak{P}(G) \longrightarrow \mathfrak{P}(M), \psi\colon \mathfrak{P}(M) \longrightarrow \mathfrak{P}(G))$ between power sets of sets G and M, that are both *antitone*, i.e. inclusion reversing, and yield extensive compositions $\psi \circ \varphi$ and $\varphi \circ \psi$. It is easy to see that these compositions are then closure operators on G and M, respectively.

2.3 Clones

The definition of a *clone* is easiest stated borrowing the notion of *tupling* of functions from category theory to state the composition closedness. Recall that a product P of objects $(A_i)_{i\in I}$ in a category is characterised by the property that any I-indexed family $\left(Q \xrightarrow{f_i} A_i\right)_{i\in I}$ gives rise to a unique comparison morphism $Q \xrightarrow{h} P$ simultaneously letting all f_i $(i \in I)$ factor via the projections of P. We call this comparison map *tupling of* $(f_i)_{i\in I}$ and denote it by $(f_i)_{i\in I}$. In the category of sets tuplings are given by $(f_i)_{i\in I}(q) := (f_i(q))_{i\in I}$ for $q \in Q$.

Furthermore, we introduce a special notation for the projection mappings belonging to finite Cartesian powers of sets A. For $n \in \mathbb{N}_+$ and any $i \in \{1, \dots, n\}$ we denote the *i-th n-ary projection* belonging to the product A^n by $e_i^{(n)}\colon A^n \longrightarrow A$ mapping $(a_1, \dots, a_n)$ to a_i. The *set of all projections over some set* A is written as $\mathrm{J}_A := \bigcup_{n\in\mathbb{N}_+} \left\{ e_i^{(n)} \;\middle|\; 1 \le i \le n \right\}$. Note that ${\mathrm{J}_A}^{(0)} = \emptyset$, i.e., there are no nullary projections.

Definition 2.1 A *(concrete) clone (of operations)* on a set A is a subset $F \subseteq \mathrm{O}_A$ of all finitary operations such that $\mathrm{J}_A \subseteq F$ and for all $m, n \in \mathbb{N}$ it is $f \circ (g_1, \ldots, g_n) \in F$ whenever $f \in F^{(n)}$, $(g_1, \ldots, g_n) \in \left(F^{(m)}\right)^n$.

In this definition the second condition, expressing closedness w.r.t. composition can be replaced by the following two more explicit ones:

(i) For all $n \in \mathbb{N}_+$, $m \in \mathbb{N}$ and all functions $f \in F^{(n)}$, $g_1, \ldots, g_n \in F^{(m)}$, we have $f \circ (g_1, \ldots, g_n) \in F$. Note that for $m = 0$ the resulting function is the nullary constant operation $c^{(0)}_{f(g_1(\emptyset),\ldots,g_n(\emptyset))}$.

(ii) For all $f \in F^{(0)}$ and every $m \in \mathbb{N}_+$, the set F contains the constant m-ary function $c^{(m)}_{f(\emptyset)}$, which is the composition of f with the empty tupling of m-ary operations.

One can easily extend the partial operations on O_A declared in Definition 2.1 to a set Φ of total ones in such a way that a subset $F \subseteq \mathrm{O}_A$ is a clone on A if and only if it is a subuniverse of the algebra $\mathrm{O}_{\mathbf{A}} = \langle \mathrm{O}_A; \Phi \rangle$.

From this it is clear that the *set* $\mathcal{L}_A$ *of all clones on* A forms a complete, algebraic lattice of subuniverses w.r.t. set inclusion, and thus also a closure system. The corresponding closure operator will be denoted by $\langle\ \rangle_{\mathrm{O}_A}$. It is not hard to see that for $F \subseteq \mathrm{O}_A$ the clone $\langle F \rangle_{\mathrm{O}_A}$ can be described as the set of all term operations of the algebra $\mathbf{A} = \langle A; F \rangle$ over the canonical signature given by the set F and the arities of the functions therein. The least element in $\mathcal{L}_A$ is the *clone* J_A *of all projections*, and the largest element is the *full clone* O_A.

In the following sections we will not make a sharp distinction between the set $\mathcal{L}_A$, the partially ordered set $(\mathcal{L}_A, \subseteq)$ and the complete lattice $(\mathcal{L}_A; \bigcap, \bigvee)$.

Let us note that clones in the traditional sense are simply clones $F \subseteq \mathrm{O}_A$ that consist of non-nullary operations, i.e. clones $F \subseteq \mathrm{O}_A \setminus \mathrm{O}_A^{(0)}$. Hence, these form a principal ideal in the lattice of all clones. For them condition (ii) in the definition above is void and can be ignored.

2.4 Relational clones

There is an analogous notion of a *clone of relations* or *relational clone*, which will be made precise in the following definition.

Definition 2.2 A *(concrete) clone of relations* on a set A is a subset $Q \subseteq \mathrm{R}_A$ of the set R_A of all finitary relations on A that is closed w.r.t. the so-called *general composition of relations*. For any index set I, any ordinal number μ (or any set), natural numbers $m, m_i \in \mathbb{N}$ $(i \in I)$, mappings $(\alpha_i \colon m_i \longrightarrow \mu)_{i \in I}$ and $\beta \colon m \longrightarrow \mu$, and relations $\varrho_i \in Q^{(m_i)}$ $(i \in I)$, we require Q to contain the m-ary relation defined by

$$\bigwedge_{(\alpha_i)_{i \in I}}^{\beta} (\varrho_i)_{i \in I} := \{ y \in A^m \mid \exists\, a \in A^{\mu} \colon y = a \circ \beta \wedge \forall\, i \in I \colon a \circ \alpha_i \in \varrho_i \}$$
$$= \{ a \circ \beta \mid a \in A^{\mu} \wedge \forall\, i \in I \colon a \circ \alpha_i \in \varrho_i \}\,.$$

It is easy to see that relational clones $Q \subseteq \mathrm{R}_A$ contain all diagonal relations (see page 3) by just using index mappings β and making $I = \emptyset$. Moreover, letting $\mu = m = m_i$ and $\beta = \alpha_i = \mathrm{id}_m$ for all $i \in I$ shows that relational clones are closed under arbitrary intersections of relations of the same arity.

On finite sets A it is not very hard to derive a set of finitary operations on R_A that suffice to describe the closedness condition for relational clones. Details can for instance be found in [17, Proposition 3.6, p. 28 et seq.].

For a fixed set A, it is also known that there is a bound on the ordinal μ and the cardinality of the set I needed in Definition 2.2. This means it suffices to ensure closedness w.r.t. only set-many partial operations (of possibly infinite arity) which can easily be extended in a conservative way to global ones on R_A^I. This argument shows that the *set* $\mathcal{L}_A^*$ *of all relational clones on* A is a complete lattice w.r.t. set inclusion, and as such a closure system, as well. The corresponding closure operator will be denoted by $[\ ]_{\mathrm{R}_A}$.

One can quickly check that the least relational clone in the lattice $\mathcal{L}_A^*$ is the *set* Diag_A *of all diagonal relations*, while the largest one is the *full clone* R_A of all finitary relations.

Furthermore, it is a small technical exercise that for a set $Q \subseteq \mathrm{R}_A$ the generated relational clone looks as follows:

$$[Q]_{\mathrm{R}_A} = \left\{ \bigwedge_{(\alpha_i)_{i\in I}}^{\beta} (\varrho_i)_{i\in I} \;\middle|\; \begin{array}{l} I \text{ a set}, \mu \text{ an ordinal}, m \in \mathbb{N}, (m_i)_{i\in I} \in (\mathbb{N})^I, \\ \beta \in \mu^m, (\alpha_i)_{i\in I} \in \prod\limits_{i\in I} \mu^{m_i}, (\varrho_i)_{i\in I} \in \prod\limits_{i\in I} Q^{(m_i)} \end{array} \right\}.$$

Throughout the paper we will adopt the same laxity w.r.t. the lattice $\mathcal{L}_A^*$ as for $\mathcal{L}_A$, that is, we will not sharply distinguish between the set $\mathcal{L}_A^*$, the poset $(\mathcal{L}_A^*, \subseteq)$ and the complete lattice structure $(\mathcal{L}_A^*; \bigcap, \bigvee)$.

As relational clones in the classical sense necessarily contain the empty relation, they are exactly the relational clones according to Definition 2.2 that additionally contain $\emptyset$. Thus, they form a principal filter above $[\{\emptyset\}]_{\mathrm{R}_A}$ in the lattice $\mathcal{L}_A^*$, and, as is well-known, also a closure system.

2.5 Compatibility (preservation) relation

In this subsection we introduce a binary relation between finitary operations and finitary relations on a set A, which gives rise to "[t]he most basic Galois connection in algebra" [14, p. 147, l. 20 et seq.]. This relation describes when a relation ϱ is *compatible with* (*invariant for*) an operation f, or, equivalently, the operation f *preserves* the relation ϱ. The relevance of preservation concerning clone theory lies in the fact that the closed sets w.r.t. the GALOIS connection it induces are so-called *locally closed* clones of operations and relations, respectively (we will prove in Subsection 3.2 that this well-known fact generalises to the more comprehensive setting including nullary operations). Since for trivial reasons on finite base sets A all clones are locally closed (see Subsection 2.6), the operators Pol_A and Inv_A of the GALOIS connection (see Definition 2.4) constitute order-antiisomorphisms between the lattices $\mathcal{L}_A$ and $\mathcal{L}_A^*$ in this case.

We will give the definition of the preservation relation in four different, but equivalent forms. That the conditions in Definition 2.3 are indeed equivalent is a straightforward calculation and will be omitted.

Definition 2.3 For an n-ary operation $f \in \mathrm{O}_A^{(n)}$ $(n \in \mathbb{N})$ and an m-ary relation $\varrho \in \mathrm{R}_A^{(m)}$ $(m \in \mathbb{N})$ on a set A, we say that f *preserves* ϱ and write $f \rhd \varrho$ if one of the following equivalent conditions is fulfilled:

(i) ϱ is a subuniverse of the m-th direct power $\langle A; f\rangle^m$. This motivates the alternative names *subpower* or *invariant* relation for ϱ.

(ii) $f\colon A^n \longrightarrow A$ induces a homomorphism between the n-th direct power $\langle A; \varrho\rangle^n$ and the relational structure $\langle A; \varrho\rangle$. This is why f is also called a *polymorphism* of ϱ.

(iii) For every $(m \times n)$-matrix $X \in A^{m\times n}$ the columns $X_{-,j} \in \varrho$ $(j \in n)$ of which are tuples in ϱ, the tuple $(f(X_{i,-}))_{i\in m}$ obtained by row-wise application of f to X is again a tuple of ϱ.

(iv) For every tuple $\mathbf{r} \in \varrho^n$, the composition of f with the tupling[4] $(\mathbf{r})$ of the tuples in $\mathbf{r}$ belongs again to the relation: $f \circ (\mathbf{r}) \in \varrho$.

As every binary relation the preservation relation gives rise to a Galois connection via the following two operators:

Definition 2.4 For sets $F \subseteq \mathrm{O}_A$ and $Q \subseteq \mathrm{R}_A$ we define the operators

$$\begin{aligned} \mathrm{Pol}_A\colon\ \mathfrak{P}(\mathrm{R}_A) &\longrightarrow \mathfrak{P}(\mathrm{O}_A) \\ Q &\longmapsto \mathrm{Pol}_A\, Q := \{ f \in \mathrm{O}_A \mid \forall \varrho \in Q\colon f \rhd \varrho \}, \\ \mathrm{Inv}_A\colon\ \mathfrak{P}(\mathrm{O}_A) &\longrightarrow \mathfrak{P}(\mathrm{R}_A) \\ F &\longmapsto \mathrm{Inv}_A\, F := \{ \varrho \in \mathrm{R}_A \mid \forall f \in F\colon f \rhd \varrho \}. \end{aligned}$$

The pair $(\mathrm{Pol}_A, \mathrm{Inv}_A)$ forms the Galois connection $\mathrm{Pol}_A - \mathrm{Inv}_A$ between finitary operations and relations.

We will conclude this subsection with a few basic facts about the Galois connection $\mathrm{Pol}_A - \mathrm{Inv}_A$. In the proof we require a fact about functions sometimes known as the *superassociativity law of composition*. Even though we just need this fact in the category of sets, it is easiest proven using category theoretic language and works in any category with products.

Lemma 2.5 *Let I and J be arbitrary index sets, $k, m, n \in \mathbb{N}$ natural numbers, and A, B, D, X and B_i $(i \in I)$, C_j $(j \in J)$ be objects in a category $\mathfrak{C}$. Furthermore, suppose that we are given morphisms $r\colon A \longrightarrow B$, $r_i\colon A \longrightarrow B_i$ $(i \in I)$, $g_j\colon B \longrightarrow C_j$ $(j \in J)$, and $f\colon \prod_{j\in J} C_j \longrightarrow D$, where $\prod_{j\in J} C_j$ together with projection morphisms $\left(\pi_j\colon \prod_{\nu\in J} C_\nu \longrightarrow C_j\right)_{j\in J}$ is a product of $(C_j)_{j\in J}$ in $\mathfrak{C}$. Assume moreover, that $\mathfrak{C}$*

[4] Recall that tuples are mappings.

contains a product $\prod_{i\in I} B_i$ *and the finite powers* X^k, X^m *and* X^n, *then the following equalities are valid:*

(i) $(g_j)_{j\in J} \circ r = (g_j \circ r)_{j\in J}$.

(ii) *If* $B = \prod_{i\in I} B_i$, *then* $(g_j)_{j\in J} \circ (r_i)_{i\in I} = \big(g_j \circ (r_i)_{i\in I}\big)_{j\in J}$, *and thus*

$$\Big(f \circ (g_j)_{j\in J}\Big) \circ (r_i)_{i\in I} = f \circ \big(g_j \circ (r_i)_{i\in I}\big)_{j\in J}.$$

(iii) *If* $B_i = C_j = D = X$ *for* $i \in I$ *and* $j \in J$, $A = X^k$ *and* $I = m$ *and* $J = n$, *then we have*

$$\begin{aligned}(f \circ (g_0, \ldots, g_{n-1})) &\circ (r_0, \ldots, r_{m-1}) \\ &= f \circ (g_0 \circ (r_0, \ldots, r_{m-1}), \ldots, g_{n-1} \circ (r_0, \ldots, r_{m-1})),\end{aligned}$$

the superassociativity law *for "finitary morphisms" over the object* X.

Here, the tupling of morphisms is denoted as above by $(\ldots)$, *and composition of morphisms is written using* $\circ$ *in the same way as defined for functions in Subsection 2.1.*

Proof The second fact follows from (i) by composition with f and associativity, the third fact is a special case of the second one. The equality in (i) follows from the uniqueness property of the tupling $(g_\iota \circ r)_{\iota\in J}$. □

Lemma 2.6 *For every set* $F \subseteq \mathrm{O}_A$ *of operations and every set* $Q \subseteq \mathrm{R}_A$ *of relations the following holds:*

$$\begin{aligned} &\mathrm{Pol}_A\, Q \in \mathcal{L}_A, && \mathrm{Inv}_A\, F \in \mathcal{L}^*_A, \\ &\langle F\rangle_{\mathrm{O}_A} \subseteq \mathrm{Pol}_A\, \mathrm{Inv}_A\, F, && [Q]_{\mathrm{R}_A} \subseteq \mathrm{Inv}_A\, \mathrm{Pol}_A\, Q, \\ &\mathrm{Pol}_A\, Q = \mathrm{Pol}_A\, [Q]_{\mathrm{R}_A}, && \mathrm{Inv}_A\, F = \mathrm{Inv}_A\, \langle F\rangle_{\mathrm{O}_A}. \end{aligned}$$

Proof Throughout the proof we consider fixed sets $F \subseteq \mathrm{O}_A$ and $Q \subseteq \mathrm{R}_A$.

Let us first show that $\mathrm{Pol}_A\, Q$ is a clone. Certainly, $\mathrm{J}_A \subseteq \mathrm{Pol}_A\, Q$ since for every $n \in \mathbb{N}_+$ and $1 \le i \le n$ the projection $e_i^{(n)}$ preserves any relation $\varrho \in Q$. Namely, if $(r_1, \ldots, r_n) \in \varrho^n$ is a tuple of tuples from ϱ, then $e_i^{(n)} \circ (r_1, \ldots, r_n) = r_i \in \varrho$. Furthermore, if $n, m \in \mathbb{N}$ and $f \in \mathrm{Pol}_A^{(n)} Q$ and $(g_0, \ldots, g_{n-1}) \in \Big(\mathrm{Pol}_A^{(m)} Q\Big)^n$, then the composition $h := f \circ (g_0, \ldots, g_{n-1}) \in \mathrm{Pol}_A\, Q$. This is true because h preserves any relation $\varrho \in Q$: if $\mathbf{r} \in \varrho^m$, then by the superassociativity law we have

$$h \circ (\mathbf{r}) = (f \circ (g_0, \ldots, g_{n-1})) \circ (\mathbf{r}) \overset{2.5(\mathrm{iii})}{=} f \circ (g_0 \circ (\mathbf{r}), \ldots, g_{n-1} \circ (\mathbf{r})),$$

the latter being a tuple in ϱ because $f \in \mathrm{Pol}_A\, Q$ and $(g_0 \circ (\mathbf{r}), \ldots, g_{n-1} \circ (\mathbf{r})) \in \varrho^n$ as every g_j belongs to $\mathrm{Pol}_A\, Q$ ($j \in \{0, 1, \ldots, n-1\}$).

To show that $\mathrm{Inv}_A\, F$ is a relational clone we consider any index set I, any ordinal number μ, natural numbers $m, m_i \in \mathbb{N}$ ($i \in I$), mappings $(\alpha_i \colon m_i \longrightarrow \mu)_{i\in I}$

and $\beta\colon m \longrightarrow \mu$, and relations $\varrho_i \in \operatorname{Inv}_A^{(m_i)} F$ $(i \in I)$. We have to show that $\operatorname{Inv}_A F$ contains the m-ary relation defined by

$$\varrho := \bigwedge_{(\alpha_i)_{i\in I}}^{\beta} (\varrho_i)_{i\in I} = \{\, a \circ \beta \mid a \in A^{\mu} \wedge \forall\, i \in I\colon a \circ \alpha_i \in \varrho_i \}\,.$$

For this consider any $n \in \mathbb{N}$ and $f \in F^{(n)}$. We have to show that f preserves ϱ; so, let any tuple $\mathbf{r} \in \varrho^n$ be given. By definition of ϱ there exists a tuple $\mathbf{a} \in (A^{\mu})^n$ such that $\mathbf{r}(j) = \mathbf{a}(j) \circ \beta$ and $\mathbf{a}(j) \circ \alpha_i \in \varrho_i$ for every $i \in I$ and $j \in \{0, 1, \ldots, n-1\}$. Clearly, the composition $b := f \circ (\mathbf{a})$ of f with the tupling $(\mathbf{a})$ is an element of A^{μ}. For any set x and any mapping $\gamma\colon x \longrightarrow \mu$, Lemma 2.5(i) yields

$$b \circ \gamma = f \circ (\mathbf{a}) \circ \gamma = f \circ (\mathbf{a}(j) \circ \gamma)_{j\in\{0,\ldots,n-1\}}\,.$$

In particular, for all $i \in I$, letting $\gamma = \alpha_i$, it is $b \circ \alpha_i = f \circ (\mathbf{a}(j) \circ \alpha_i)_{j\in\{0,\ldots,n-1\}} \in \varrho_i$ since $(\mathbf{a}(j) \circ \alpha_i)_{j\in\{0,\ldots,n-1\}} \in (\varrho_i)^n$ and $\varrho_i \in \operatorname{Inv}_A F \subseteq \operatorname{Inv}_A \{f\}$. Likewise, for $\gamma = \beta$ we obtain $b \circ \beta = f \circ (\mathbf{a}(j) \circ \beta)_{j\in\{0,\ldots,n-1\}} = f \circ (\mathbf{r}(j))_{j\in\{0,\ldots,n-1\}} = f \circ (\mathbf{r})$, and this is an element of ϱ by what was shown just before about the tuple $b \in A^m$.

The remaining facts are easy consequences of the already proven properties. By extensivity of $\operatorname{Pol}_A \operatorname{Inv}_A$, monotonicity of $\langle\ \rangle_{\mathrm{O}_A}$ and the fact that $\operatorname{Pol}_A \operatorname{Inv}_A F$ is a clone, we get $\langle F\rangle_{\mathrm{O}_A} \subseteq \langle \operatorname{Pol}_A \operatorname{Inv}_A F\rangle_{\mathrm{O}_A} = \operatorname{Pol}_A \operatorname{Inv}_A F$. By application of Inv_A on both sides of this inclusion, one obtains $\operatorname{Inv}_A \langle F\rangle_{\mathrm{O}_A} \supseteq \operatorname{Inv}_A \operatorname{Pol}_A \operatorname{Inv}_A F = \operatorname{Inv}_A F$. Extensivity yields $F \subseteq \langle F\rangle_{\mathrm{O}_A}$, hence we have $\operatorname{Inv}_A F \supseteq \operatorname{Inv}_A \langle F\rangle_{\mathrm{O}_A}$, i.e. equality $\operatorname{Inv}_A F = \operatorname{Inv}_A \langle F\rangle_{\mathrm{O}_A}$.

The outstanding assertions about Q follow by switching the roles of Inv_A and Pol_A, replacing $\langle\ \rangle_{\mathrm{O}_A}$ by $[\ ]_{\mathrm{R}_A}$, $\mathcal{L}_A$ by $\mathcal{L}_A^*$ and F by Q. $\square$

The previous lemma demonstrated that, as in the classical case, the closed sets of operations w.r.t. $\operatorname{Pol}_A - \operatorname{Inv}_A$ are certain clones, and, likewise, the closed sets of relations are relational clones. We shall introduce the ad-hoc terminology Galois *closed (relational) clone* for such (relational) clones. For clones without nullary operations it is well-known that the closure system of Galois closed clones can be characterised by being locally closed in the sense of Definition 2.7. Therefore, these clones are usually simply called locally closed clones. Until we have generalised the characterisation to the more comprehensive setting involving nullary operations in Subsection 3.2, we shall stay with the term “Galois closed”.

2.6 Local closure operators on operations and relations

On infinite carrier sets A it can happen for some sets $F \subseteq \mathrm{O}_A$ (see e.g. Example 3.8) that the inclusion $\langle F\rangle_{\mathrm{O}_A} \subseteq \operatorname{Pol}_A \operatorname{Inv}_A F$, proven in Lemma 2.6, is a proper one. Therefore, in general, the closure operator $\langle\ \rangle_{\mathrm{O}_A}$ is not strong enough to describe the Galois closure $\operatorname{Pol}_A \operatorname{Inv}_A$. Similar situations can arise with $[\ ]_{\mathrm{R}_A}$ and $\operatorname{Inv}_A \operatorname{Pol}_A$. Hence, additional closure operators are needed to close up (relational) clones to obtain Galois closed clones. From classical clone theory, it is known that these operators are given as so-called *local closures*. It is the purpose of this subsection to define them in our general framework and to verify that, as in the classical case, still all GALOIS closed clones are locally closed.

Definition 2.7 For $F \subseteq \mathrm{O}_A$ and $Q \subseteq \mathrm{R}_A$ we define

$$\operatorname{Loc}_A F := \bigcup_{n\in\mathbb{N}} \left\{ f \in \mathrm{O}_A^{(n)} \;\middle|\; \forall B \subseteq A^n, 0 \leq |B| < \aleph_0 \, \exists g \in F^{(n)} \colon g|_B = f|_B \right\},$$

$$\operatorname{LOC}_A Q := \bigcup_{m\in\mathbb{N}} \left\{ \sigma \in \mathrm{R}_A^{(m)} \;\middle|\; \forall B \subseteq \sigma, 0 \leq |B| < \aleph_0 \, \exists \varrho \in Q^{(m)} \colon B \subseteq \varrho \subseteq \sigma \right\}.$$

Without difficulty one shows that these operators are indeed closure operators on the respective sets of operations and relations. The closed sets of operations and relations, respectively, are called *locally closed*. The operators Loc_A and LOC_A add everything that can be interpolated *on any finite subset* B. Therefore, on finite sets A, any set $F \subseteq \mathrm{O}_A$ and $Q \subseteq \mathrm{R}_A$ is locally closed, since in this case relations and domains of operations are finite sets. In particular every (relational) clone on a finite set is locally closed. In general this is only true for GALOIS closed (relational) clones, a fact that will be shown in Theorems 3.17 and 3.20. The simpler statement, that all GALOIS closed clones are actually locally closed, is part of the following lemma.

Lemma 2.8 *For $n \in \mathbb{N}$ and $m \in \mathbb{N}$, any sets $F \subseteq \mathrm{O}_A$ and $Q \subseteq \mathrm{R}_A$ the following is true:*

$$\operatorname{Loc}_A^{(n)} F = \operatorname{Loc}_A \left(F^{(n)}\right), \qquad \operatorname{LOC}_A^{(m)} Q = \operatorname{LOC}_A \left(Q^{(m)}\right), \tag{1}$$

$$\operatorname{Loc}_A \operatorname{Pol}_A Q = \operatorname{Pol}_A Q, \qquad \operatorname{LOC}_A \operatorname{Inv}_A F = \operatorname{Inv}_A F, \tag{2}$$

in particular,

$$\operatorname{Loc}_A \operatorname{Pol}_A \operatorname{Inv}_A F = \operatorname{Pol}_A \operatorname{Inv}_A F, \qquad \operatorname{LOC}_A \operatorname{Inv}_A \operatorname{Pol}_A Q = \operatorname{Inv}_A \operatorname{Pol}_A Q. \tag{3}$$

Proof The equalities in (1) are evident from Definition 2.7 and (3) follows from (2). For both equalities in (2) it suffices to prove the inclusion "$\subseteq$". So let us first consider any l-ary operation $f \in \operatorname{Loc}_A^{(l)} \operatorname{Pol}_A Q$ ($l \in \mathbb{N}$). We have to show that f preserves any k-ary relation $\varrho \in Q^{(k)}$, for any $k \in \mathbb{N}$. To this end let $X \in A^{k\times l}$ be any matrix, all of whose columns $X_{-,j}$ ($j \in l$) are tuples in ϱ. The set $B := \{X_{i,-} \mid i \in k\}$ of rows is contained in A^l and has cardinality at most $k < \aleph_0$. Hence, by definition of Loc_A there is an operation $g \in \operatorname{Pol}_A^{(l)} Q$ interpolating f on B. This means $(f(X_{i,-}))_{i\in k} = (g(X_{i,-}))_{i\in k} \in \varrho$, because $g \rhd \varrho$, and we are done.

Second, we show that any h-ary relation $\sigma \in \operatorname{LOC}_A^{(h)} \operatorname{Inv}_A F$, $h \in \mathbb{N}$ is invariant for any $f \in F^{(k)}$, $k \in \mathbb{N}$. For this consider any $\mathbf{r} \in \sigma^k$ and set $B := \operatorname{im} \mathbf{r} \subseteq A^h$. Certainly, $|B| \leq k < \aleph_0$, so by definition of LOC_A we can find a subrelation $\varrho \in \operatorname{Inv}_A^{(h)} F$ satisfying $B \subseteq \varrho \subseteq \sigma$. The latter inclusion yields $\mathbf{r} \in \varrho^k$, and, since $\varrho \in \operatorname{Inv}_A F$, we know $f \rhd \varrho$, thus $f \circ (\mathbf{r}) \in \varrho \subseteq \sigma$. This shows that f preserves σ, finishing the proof. $\square$

3 Relating old and new clones

In order to explore some connections between clones in the usual sense (without nullary operations) and the new, general clones, in Subsection 3.1 we are going to define four operators on sets of finitary operations and relations. They will be used to locate the lattices of traditional clones in the lattices of new clones, and, more importantly, to determine the location of those clones that are strictly new, i.e. not part of the usual theory.

Subsequently, we demonstrate that the characterisation of Galois closed clones as locally closed clones continues to hold if nullary operations are admitted.

3.1 Two closure and kernel operator pairs

In this part, we shall introduce two closure operators $^{\circ}$, one acting on sets of finitary operations, the other one on finitary relations, and similarly two kernel operators $'$. We will see that these operators can be restricted to clones and that it turns out that the set of all kernels of clones of operations is precisely the old lattice of clones without nullary operations, while the closures of all new relational clones are exactly the old relational clones. In Subsection 3.1.3, the positions of clones in the traditional sense in the new general clone lattice are discussed.

The other subsections of this part deal with the interplay of the defined closure and kernel operators with familiar constructions from clone theory. A passage is devoted to each of the following: the operators of the Galois connection $\mathrm{Pol}_A - \mathrm{Inv}_A$, the local closure operators and the clone closures.

3.1.1 Definition of $^{\circ}$, $'$, and their closure and kernel system

Definition 3.1 For a constant k-ary ($k \in \mathbb{N}$) operation $f \in \mathrm{O}_A^{(k)}$ let $f^{\circ} \in \mathrm{O}_A^{(0)}$ be the constant nullary operation with the same value as f, i.e., $f^{\circ}(\emptyset) := x$, where $x \in A$ is uniquely determined by $\operatorname{im} f = \{x\}$. We define the following operations:

$$
\begin{aligned}
C_1\colon\ \mathfrak{P}(\mathrm{O}_A) &\longrightarrow \mathfrak{P}\left(\mathrm{O}_A^{(1)}\right)\\
F &\longmapsto C_1(F) := \left\{ f \in F^{(1)} \;\middle|\; |\operatorname{im} f| = 1 \right\},\\
{}'\colon\ \mathfrak{P}(\mathrm{O}_A) &\longrightarrow \mathfrak{P}(\mathrm{O}_A)\\
F &\longmapsto F' := F \cap \left(\mathrm{O}_A \setminus \mathrm{O}_A^{(0)}\right) = F \setminus \mathrm{O}_A^{(0)} = F \setminus F^{(0)},\\
{}^{\circ}\colon\ \mathfrak{P}(\mathrm{O}_A) &\longrightarrow \mathfrak{P}(\mathrm{O}_A)\\
F &\longmapsto F^{\circ} := F \cup \left\{ f^{\circ} \;\middle|\; f \in C_1(F) \right\},\\
{}'\colon\ \mathfrak{P}(\mathrm{R}_A) &\longrightarrow \mathfrak{P}(\mathrm{R}_A)\\
Q &\longmapsto Q' := \begin{cases} Q \setminus \{\emptyset\} & \text{if } \mathrm{Pol}_A^{(0)}(Q \setminus \{\emptyset\}) \neq \emptyset,\\ Q & \text{otherwise,} \end{cases}
\end{aligned}
$$

$$\begin{aligned} {}^{\circ}\colon \mathfrak{P}(\mathrm{R}_A) &\longrightarrow \mathfrak{P}(\mathrm{R}_A) \\ Q &\longmapsto Q^{\circ} := Q \cup \{\emptyset\}\,. \end{aligned}$$

The operator $'$ on sets of operations allows us to introduce a useful short notation for the clone of all non-nullary operations that we will apply from now on. Namely, by definition, we have $\mathrm{O}_A \setminus \mathrm{O}_A^{(0)} = \mathrm{O}'_A$.

In the next lemma, we relate the condition on nullary polymorphisms, occurring in the definition of the operator $'$ on sets of relations to the closure operators $[\ ]_{\mathrm{R}_A}$ and $\mathrm{Inv}_A \mathrm{Pol}_A$.

Lemma 3.2 *For subsets $Q \subseteq \mathrm{R}_A$ of relations, $F \subseteq \mathrm{O}_A$ of operations and a closure operator $\mathrm{Cl}\,()\colon \mathfrak{P}(\mathrm{R}_A) \longrightarrow \mathfrak{P}(\mathrm{R}_A)$ on all finitary relations such that the inclusion $[W]_{\mathrm{R}_A} \subseteq \mathrm{Cl}\,(W) \subseteq \mathrm{Inv}_A \mathrm{Pol}_A W$ is true for all $W \subseteq \mathrm{R}_A$, the following facts hold:*

(a) $\left\{ a \in A \,\middle|\, c_a^{(0)} \in \mathrm{Pol}_A^{(0)} Q \right\} \in [Q]_{\mathrm{R}_A}$.

(b) $\emptyset \in \mathrm{Inv}_A F \iff \mathrm{Pol}_A^{(0)} \mathrm{Inv}_A F = \emptyset \iff F^{(0)} = \emptyset$.

(c) $\emptyset \in \mathrm{Cl}\,(Q) \iff \mathrm{Pol}_A^{(0)} Q = \emptyset$.

Proof

(a) For every relation $\varrho \in Q$ of arity $\mathrm{ar}\,\varrho$ we let $\pi_\varrho\colon \mathrm{ar}\,\varrho \longrightarrow 1$ be the unique constant mapping with value 0. For any $a \in A$ and $\varrho \in Q$ the condition $c_a^{(0)} \rhd \varrho$ is certainly equivalent to $(a) \circ \pi_\varrho = (a, \ldots, a) \in \varrho$. Consequently, for every $a \in A$, we have $c_a^{(0)} \in \mathrm{Pol}_A^{(0)} Q$ if and only if $(a) \circ \pi_\varrho \in \varrho$ for all $\varrho \in Q$. Using the identity mapping $\mathrm{id}_1\colon 1 \longrightarrow 1$ in the general composition of relations (see Definition 2.2), we see that

$$\begin{aligned} [Q]_{\mathrm{R}_A} \ni \bigwedge_{(\pi_\varrho)_{\varrho \in Q}}^{\mathrm{id}_1} (\varrho)_{\varrho \in Q} &= \left\{ (a) = (a) \circ \mathrm{id}_1 \in A^1 \,\middle|\, \forall \varrho \in Q\colon\ (a) \circ \pi_\varrho \in \varrho \right\} \\ &= \left\{ a \in A \,\middle|\, c_a^{(0)} \in \mathrm{Pol}_A^{(0)} Q \right\}. \end{aligned}$$

(b) This is true since $\mathrm{Pol}_A \{\emptyset\} = \mathrm{O}_A \setminus \mathrm{O}_A^{(0)}$, i.e. nullary operations do not preserve the empty relation, and they are the only ones doing this.

(c) Whenever $\mathrm{Pol}_A^{(0)} Q = \emptyset$, statement (a) yields $\emptyset \in [Q]_{\mathrm{R}_A} \subseteq \mathrm{Cl}\,(Q) \subseteq \mathrm{Inv}_A \mathrm{Pol}_A Q$. The latter is equivalent to $\mathrm{Pol}_A^{(0)} Q = \emptyset$ by letting $F = \mathrm{Pol}_A Q$ in (b).

□

In the following lemma we will show that in Definition 3.1 we have declared certain kernel and closure operators, and we will characterise their kernel and closure systems, respectively.

Lemma 3.3 *The operators $'$ are kernel operators on the set of all finitary relations and operations, respectively, whereas the operators $^{\circ}$ are closure operators on these* *sets, respectively. The corresponding closure / kernel systems are the following:*

$$[\mathfrak{P}(\mathrm{O}_A)]' = \mathfrak{P}\left(\mathrm{O}'_A\right) = \left\{ F \subseteq \mathrm{O}_A \;\middle|\; F^{(0)} = \emptyset \right\} \tag{4}$$

$$[\mathfrak{P}(\mathrm{O}_A)]^{\circ} = \left\{ F \subseteq \mathrm{O}_A \;\middle|\; \forall a \in A\colon\; c_a^{(1)} \in F \implies c_a^{(0)} \in F \right\} \tag{5}$$

$$[\mathfrak{P}(\mathrm{R}_A)]' = \left\{ Q \subseteq \mathrm{R}_A \;\middle|\; \emptyset \notin Q \vee \mathrm{Pol}_A^{(0)}(Q \setminus \{\emptyset\}) = \emptyset \right\} \tag{6}$$

$$[\mathfrak{P}(\mathrm{R}_A)]^{\circ} = \{ Q \subseteq \mathrm{R}_A \mid \emptyset \in Q \}. \tag{7}$$

These operators restrict nicely to clones:

$$'\colon\; \mathcal{L}_A \longrightarrow {\downarrow_{\mathcal{L}_A}}\left(\mathrm{O}'_A\right) = \left\{ F \in \mathcal{L}_A \;\middle|\; F^{(0)} = \emptyset \right\} \tag{8}$$

$$\begin{aligned} {}^{\circ}\colon\; \mathcal{L}_A \longrightarrow\; & \left\{ F \in \mathcal{L}_A \;\middle|\; \forall a \in A\colon\; c_a^{(1)} \in F \implies c_a^{(0)} \in F \right\} \\ & = \left\{ F \in \mathcal{L}_A \;\middle|\; F^{(0)} \neq \emptyset \vee C_1(F) = \emptyset \right\}. \end{aligned} \tag{9}$$

If $\mathrm{Cl}()\colon \mathfrak{P}(\mathrm{R}_A) \longrightarrow \mathfrak{P}(\mathrm{R}_A)$ *is a closure operator on sets of relations such that* $[W]_{\mathrm{R}_A} \subseteq \mathrm{Cl}(W) \subseteq \mathrm{Inv}_A \mathrm{Pol}_A W$ *holds for all* $W \subseteq \mathrm{R}_A$*, and* $\mathcal{C} := \mathrm{Cl}(\mathfrak{P}(\mathrm{R}_A))$ *is the corresponding closure system, then we have*

$$Q' = \mathrm{Cl}(Q \setminus \{\emptyset\}) \tag{10}$$

for $Q \in \mathcal{C}$ *and the restriction*

$$\begin{aligned} '\colon \mathcal{C} \longrightarrow\; & \left\{ Q \in \mathcal{C} \;\middle|\; \emptyset \notin Q \vee \mathrm{Pol}_A^{(0)}(Q \setminus \{\emptyset\}) = \emptyset \right\} \\ & = \{ \mathrm{Cl}(Q \setminus \{\emptyset\}) \mid Q \in \mathcal{C} \}. \end{aligned} \tag{11}$$

If, furthermore, $Q^{\circ} \in \mathcal{C}$ *for* $Q \in \mathcal{C}$*, then it is*

$$^{\circ}\colon \mathcal{C} \longrightarrow {\uparrow_{\mathcal{C}}}(\mathrm{Cl}(\{\emptyset\})), \tag{12}$$

in particular, the assumptions of (12) *on* $\mathrm{Cl}()$ *are true for the closures* $\mathrm{Cl}() = [\;]_{\mathrm{R}_A}$ *and* $\mathrm{Cl}() = \mathrm{Inv}_A \mathrm{Pol}_A$.

Moreover, the restrictions mentioned in (8) *to* (12) *are surjective.*

Proof The only non-trivial part in proving that $^{\circ}$ and $'$ are closure and kernel operators is to show that $'$ on sets of relations is monotone and idempotent. For this let $P \subseteq Q \subseteq \mathrm{R}_A$. It has to be excluded that $P' = P \not\subseteq Q \setminus \{\emptyset\} = Q'$. We do this by deriving a contradiction from this assumption. As $P \subseteq Q$, the assumption yields that $\emptyset \in P \subseteq Q$. Since $P = P'$, one obtains $\mathrm{Pol}_A^{(0)}(P \setminus \{\emptyset\}) = \emptyset$, as otherwise one had $\emptyset \in P = P' = P \setminus \{\emptyset\}$. This implies $\mathrm{Pol}_A^{(0)}(Q \setminus \{\emptyset\}) \subseteq \mathrm{Pol}_A^{(0)}(P \setminus \{\emptyset\}) = \emptyset$ because $P \subseteq Q$. Hence, by definition of $'$ we have $Q \setminus \{\emptyset\} = Q' = Q$, so $\emptyset \notin Q$, contradicting $\emptyset \in P \subseteq Q$. Consequently, $'$ is monotone. To see that this operator is also idempotent, we note the equality $Q' \setminus \{\emptyset\} = Q \setminus \{\emptyset\}$. Thus, we have $\mathrm{Pol}_A^{(0)}(Q \setminus \{\emptyset\}) \neq \emptyset$ if and only if $\mathrm{Pol}_A^{(0)}(Q' \setminus \{\emptyset\}) \neq \emptyset$. If both conditions are false, we have $Q' = Q$, and so $Q'' = Q' = Q$. Otherwise, it is $Q' = Q \setminus \{\emptyset\} = Q' \setminus \{\emptyset\} = Q''$.

To discuss (4)–(12), let $F \subseteq \mathrm{O}_A$ and $Q \subseteq \mathrm{R}_A$ be arbitrary subsets.

Claims (4), (5) and (7) are easy consequences of Definition 3.1.

If $\mathrm{Pol}_A^{(0)}(Q \setminus \{\emptyset\}) \neq \emptyset$, then $\emptyset \notin Q \setminus \{\emptyset\} = Q'$. Otherwise, $\mathrm{Pol}_A^{(0)}(Q \setminus \{\emptyset\}) = \emptyset$ and so $Q' = Q$. Hence $\emptyset = \mathrm{Pol}_A^{(0)}(Q \setminus \{\emptyset\}) = \mathrm{Pol}_A^{(0)}(Q' \setminus \{\emptyset\})$ follows. Conversely, if $\emptyset \notin Q$, then in any case $Q' = Q$, and if $\mathrm{Pol}_A^{(0)}(Q \setminus \{\emptyset\}) = \emptyset$, then it is $Q' = Q$ by definition. This shows equality (6).

As $\mathrm{O}_A \setminus \mathrm{O}_A^{(0)}$ is a clone, for every $F \in \mathcal{L}_A$ we have $F' = F \cap \left(\mathrm{O}_A \setminus \mathrm{O}_A^{(0)}\right) \in \mathcal{L}_A$, since clones form a closure system, thus proving (8).

For (9) we first verify that for $F \in \mathcal{L}_A$ also $F^{\circ} \in \mathcal{L}_A$. Since F is a clone, we have $F^{\circ(0)} = \left\{ f^{\circ} \;\middle|\; f \in C_1(F) \right\}$. It follows that F° is closed under compositions involving nullary operations. Thus, $F^{\circ} \in \mathcal{L}_A$ for $F \in \mathcal{L}_A$.

For the equality stated in (9) we consider a clone F satisfying $C_1(F) \neq \emptyset$ and respecting all stated implications. Then for some $a \in A$ it is $c_a^{(1)} \in F$, and so $c_a^{(0)} \in F^{(0)} \neq \emptyset$. Conversely, let $F \in \mathcal{L}_A$ be such that $F^{(0)} \neq \emptyset$ or $C_1(F) = \emptyset$. So, if $c_a^{(1)} \in F$ for some $a \in A$, then $F^{(0)} \neq \emptyset$. This means $c_b^{(0)} \in F^{(0)}$ for some $b \in A$, and thus $c_a^{(0)} = c_a^{(1)} \circ \left(c_b^{(0)}\right) \in F$ as F is closed w.r.t. composition.

For the rest of the proof we assume that $\mathrm{Cl}()$ is a closure operator with corresponding closure system $\mathcal{C}$, satisfying $[W]_{\mathrm{R}_A} \subseteq \mathrm{Cl}(W) \subseteq \mathrm{Inv}_A \mathrm{Pol}_A W$ for $W \subseteq \mathrm{R}_A$.

First, we check equation (10), i.e. $Q' = \mathrm{Cl}(Q \setminus \{\emptyset\})$ for every $Q \in \mathcal{C}$: the inclusion $Q \setminus \{\emptyset\} \subseteq \mathrm{Cl}(Q \setminus \{\emptyset\}) \subseteq \mathrm{Cl}(Q) = Q$ is true for all closed sets $Q \in \mathcal{C}$. So, if $\mathrm{Pol}_A^{(0)}(Q \setminus \{\emptyset\}) \neq \emptyset$, then it is $\emptyset \notin \mathrm{Cl}(Q \setminus \{\emptyset\})$ by Lemma 3.2(c), and thus we have $\mathrm{Cl}(Q \setminus \{\emptyset\}) = Q \setminus \{\emptyset\} = Q'$. Otherwise, it is $Q' = Q$ and $Q \setminus \{\emptyset\} \subset \mathrm{Cl}(Q \setminus \{\emptyset\})$, wherefore $\mathrm{Cl}(Q \setminus \{\emptyset\}) = Q = Q'$. So in any case we have shown $Q' = \mathrm{Cl}(Q \setminus \{\emptyset\})$.

Using this we can see that $[\mathcal{C}]' = \{\mathrm{Cl}(Q \setminus \{\emptyset\}) \mid Q \in \mathcal{C}\}$, and because of (6) this set equals $\left\{ Q \in \mathcal{C} \;\middle|\; \emptyset \notin Q \vee \mathrm{Pol}_A^{(0)}(Q \setminus \{\emptyset\}) = \emptyset \right\}$.

The remaining statement (12) is clearly true, since $Q^{\circ} \in \mathcal{C}$ for $Q \in \mathcal{C}$.

For $\mathrm{Cl}() = [\;]_{\mathrm{R}_A}$ this condition is fulfilled, because for every relational clone $Q \in \mathcal{L}_A^*$, the set $Q^{\circ} = Q \cup \{\emptyset\}$ is again a relational clone: the general composition of relations from Q° is empty if at least one of the arguments is empty. Otherwise, all of them belong to the clone Q, and so does the resulting relation.

For $\mathrm{Cl}() = \mathrm{Inv}_A \mathrm{Pol}_A$ and $\mathcal{C} = \{\mathrm{Inv}_A Q \mid Q \subseteq \mathrm{R}_A\}$, we make use of Lemma 3.4, coming next, to show $\mathrm{Inv}_A \mathrm{Pol}_A \left(Q^{\circ}\right) \overset{3.4}{=} \mathrm{Inv}_A \left((\mathrm{Pol}_A Q)'\right) \overset{3.4}{=} (\mathrm{Inv}_A \mathrm{Pol}_A Q)^{\circ} = Q^{\circ}$ for all $Q \subseteq \mathrm{R}_A$. Here the second equality is true as $(\mathrm{Pol}_A Q)' \in \mathcal{L}_A$ due to Lemma 2.6 and (8). □

3.1.2 *Behaviour of* $^{\circ}$, $'$ *towards* $\mathrm{Pol}_A - \mathrm{Inv}_A$

In Lemma 3.3 we just saw that the operators $^{\circ}$ and $'$ can be restricted to operational and relational clones, and to Galois closed relational clones. It is our aim to establish this fact also for Galois closed clones of operations, which seems to be slightly more difficult. For this purpose, we study more generally, how the operators ${}^{\circ}$ and ${}'$ interact with the operators Pol_A and Inv_A.

Lemma 3.4 *For every subset $F \subseteq \mathrm{O}_A$ we have $\operatorname{Inv}_A(F') \supseteq (\operatorname{Inv}_A F)^{\circ}$. Equality, i.e. $\operatorname{Inv}_A(F') = (\operatorname{Inv}_A F)^{\circ}$, holds if and only if for every $a \in A$ it is $c_a^{(1)} \in \operatorname{Pol}_A \operatorname{Inv}_A(F')$ whenever $c_a^{(0)} \in F$. Clearly, this is the case for clones $F \in \mathcal{L}_A$.*

Dually, for every subset $Q \subseteq \mathrm{R}_A$ we have $\operatorname{Pol}_A\left(Q^{\circ}\right) = (\operatorname{Pol}_A Q)'$.

Proof Since $(F')^{(0)} = \emptyset$, it is $\emptyset \in \operatorname{Inv}_A(F')$, and thus we get $(\operatorname{Inv}_A F)^{\circ} \subseteq \operatorname{Inv}_A(F')$.

Now we assume $\operatorname{Inv}_A(F') = (\operatorname{Inv}_A F)^{\circ}$ and show that the condition in the lemma is necessary. For this we consider any $a \in A$ such that $c_a^{(0)} \in F$ and any relation $\varrho \in \operatorname{Inv}_A(F') \setminus \{\emptyset\} = (\operatorname{Inv}_A F)^{\circ} \setminus \{\emptyset\} = \operatorname{Inv}_A F \setminus \{\emptyset\} \subseteq \operatorname{Inv}_A F$. As $c_a^{(0)} \in F$, it is $c_a^{(0)} \rhd \varrho$, i.e. $(a, \ldots, a) \in \varrho$. Hence $c_a^{(1)} \rhd \varrho$, and so we have $c_a^{(1)} \in \operatorname{Pol}_A \operatorname{Inv}_A(F')$.

Let us now suppose that the condition on constant operations is true. To prove $\operatorname{Inv}_A(F') \subseteq (\operatorname{Inv}_A F)^{\circ}$, we consider any $\varrho \in \operatorname{Inv}_A(F') \setminus \{\emptyset\}$. It has to be shown that $\varrho \in \operatorname{Inv}_A F$. For every positive $n \in \mathbb{N}_+$ we have $F^{(n)} = F'^{(n)} \subseteq F'$, implying $\varrho \in \operatorname{Inv}_A(F') \subseteq \operatorname{Inv}_A\left(F'^{(n)}\right) = \operatorname{Inv}_A\left(F^{(n)}\right)$. For $n = 0$ we exploit the given condition to prove $c_a^{(0)} \rhd \varrho$ for any $a \in A$ where $c_a^{(0)} \in F$.

Certainly, any clone fulfils the condition regarding constants as with every nullary operation it also contains the corresponding unary one.

We finish the proof of this lemma by showing $\operatorname{Pol}_A\left(Q^{\circ}\right) = (\operatorname{Pol}_A Q)'$. Clearly, we have $\operatorname{Pol}_A\{\emptyset\} = \mathrm{O}_A \setminus \mathrm{O}_A^{(0)}$, and therefore, we obtain

$$\operatorname{Pol}_A\left(Q^{\circ}\right) = \operatorname{Pol}_A(\{\emptyset\} \cup Q) = \operatorname{Pol}_A\{\emptyset\} \cap \operatorname{Pol}_A Q = \mathrm{O}'_A \cap \operatorname{Pol}_A Q = (\operatorname{Pol}_A Q)'.$$

□

Lemma 3.5 *For every subset $Q \subseteq \mathrm{R}_A$ the following equalities hold:*

$$\operatorname{Pol}_A\left(Q'\right) = \operatorname{Pol}_A(Q \setminus \{\emptyset\}) = (\operatorname{Pol}_A Q)^{\circ}.$$

Proof We consider a fixed set of relations $Q \subseteq \mathrm{R}_A$. If $\operatorname{Pol}_A^{(0)}(Q \setminus \{\emptyset\}) \neq \emptyset$, then $Q' = Q \setminus \{\emptyset\}$, and the first equality is trivially true. Otherwise, we may assume $\operatorname{Pol}_A^{(0)}(Q \setminus \{\emptyset\}) = \emptyset$, i.e. $\emptyset \in \operatorname{Inv}_A \operatorname{Pol}_A(Q \setminus \{\emptyset\})$ by Lemma 3.2(c). So we get the inclusions $Q \subseteq Q \cup \{\emptyset\} = \{\emptyset\} \cup (Q \setminus \{\emptyset\}) \subseteq \operatorname{Inv}_A \operatorname{Pol}_A(Q \setminus \{\emptyset\})$. Thus, we obtain $\operatorname{Pol}_A Q \supseteq \operatorname{Pol}_A \operatorname{Inv}_A \operatorname{Pol}_A(Q \setminus \{\emptyset\}) = \operatorname{Pol}_A(Q \setminus \{\emptyset\}) \supseteq \operatorname{Pol}_A(Q') \supseteq \operatorname{Pol}_A Q$, where the last two inclusions follow from $Q \setminus \{\emptyset\} \subseteq Q' \subseteq Q$. Hence, from the previous chain of inclusions, we infer $\operatorname{Pol}_A(Q \setminus \{\emptyset\}) = \operatorname{Pol}_A(Q') = \operatorname{Pol}_A Q$.

For the second equality, recall that $(\operatorname{Pol}_A Q)^{\circ} = \operatorname{Pol}_A Q \cup \left\{ f^{\circ} \mid f \in C_1(\operatorname{Pol}_A Q)\right\}$. Let us now treat both inclusions of $\operatorname{Pol}_A(Q \setminus \{\emptyset\}) = (\operatorname{Pol}_A Q)^{\circ}$ separately.

"$\supseteq$" Clearly, it is $\operatorname{Pol}_A Q \subseteq \operatorname{Pol}_A(Q \setminus \{\emptyset\})$. Besides, for all $a \in A$ and all $\varrho \in \mathrm{R}_A \setminus \{\emptyset\}$, it is $c_a^{(1)} \rhd \varrho$ if and only if $c_a^{(0)} \rhd \varrho$. Thus, $\left\{ f^{\circ} \mid f \in C_1(\operatorname{Pol}_A Q)\right\} \subseteq \operatorname{Pol}_A(Q \setminus \{\emptyset\})$.

"$\subseteq$" For the converse inclusion let $g \in \operatorname{Pol}_A(Q \setminus \{\emptyset\})$ and assume $g \notin \operatorname{Pol}_A Q$. This implies $g \not\rhd \emptyset$, and so $g = c_a^{(0)}$ for some $a \in A$. Since $\operatorname{Pol}_A(Q \setminus \{\emptyset\})$ is a clone, it follows $c_a^{(1)} \in \operatorname{Pol}_A^{(1)} (Q \setminus \{\emptyset\})$, and thus $c_a^{(1)} \in C_1 (\operatorname{Pol}_A Q)$. Consequently, we have $g \in \left\{ f^{\circ} \;\middle|\; f \in C_1 (\operatorname{Pol}_A Q) \right\} \subseteq (\operatorname{Pol}_A Q)^{\circ}$.

$\square$

Lemma 3.6 *For every set $F \subseteq \mathrm{O}_A$ of operations the following equalities and inclusions hold:*

$$\operatorname{Inv}_A \left(F^{\circ}\right) = \begin{cases} \operatorname{Inv}_A F & \text{if } C_1 (F) = \emptyset, \\ (\operatorname{Inv}_A F) \setminus \{\emptyset\} & \text{otherwise.} \end{cases} \tag{13}$$

$$(\operatorname{Inv}_A F)' = \begin{cases} \operatorname{Inv}_A F & \text{if } C_1 (\operatorname{Pol}_A \operatorname{Inv}_A F) = \emptyset, \\ (\operatorname{Inv}_A F) \setminus \{\emptyset\} & \text{otherwise.} \end{cases} \tag{14}$$

$$(\operatorname{Inv}_A F)' \subseteq \operatorname{Inv}_A \left(F^{\circ}\right). \tag{15}$$

If, moreover, $C_1 (F) \neq \emptyset$, then $(\operatorname{Inv}_A F)' = (\operatorname{Inv}_A F) \setminus \{\emptyset\} = \operatorname{Inv}_A \left(F^{\circ}\right)$. Similarly, if $F^{(0)} \neq \emptyset$, then $(\operatorname{Inv}_A F)' = (\operatorname{Inv}_A F) \setminus \{\emptyset\} = \operatorname{Inv}_A F = \operatorname{Inv}_A \left(F^{\circ}\right)$. Furthermore, if $C_1 (\operatorname{Pol}_A \operatorname{Inv}_A F) = \emptyset$, then $(\operatorname{Inv}_A F)' = \operatorname{Inv}_A F = \operatorname{Inv}_A \left(F^{\circ}\right)$.

Proof To prove (13) recall that $F^{\circ} = F \cup \left\{ f^{\circ} \;\middle|\; f \in C_1 (F) \right\}$ by definition. Clearly, $F^{\circ} = F$ if $C_1 (F) = \emptyset$. So in this case $\operatorname{Inv}_A \left(F^{\circ}\right) = \operatorname{Inv}_A F$. Otherwise, we use that $c_a^{(1)} \rhd \varrho$ if and only if $c_a^{(0)} \rhd \varrho$ for $a \in A$ and $\varrho \in \mathrm{R}_A \setminus \{\emptyset\}$. Since $\emptyset \notin \operatorname{Inv}_A \mathrm{O}_A^{(0)}$, we get $\operatorname{Inv}_A \left\{ f^{\circ} \;\middle|\; f \in C_1 (F) \right\} = (\operatorname{Inv}_A C_1 (F)) \setminus \{\emptyset\}$. Consequently,

$$\begin{aligned} \operatorname{Inv}_A \left(F^{\circ}\right) &= \operatorname{Inv}_A \left(F \cup \left\{ f^{\circ} \;\middle|\; f \in C_1 (F) \right\}\right) = \operatorname{Inv}_A F \cap \operatorname{Inv}_A \left\{ f^{\circ} \;\middle|\; f \in C_1 (F) \right\} \\ &= \operatorname{Inv}_A F \cap ((\operatorname{Inv}_A C_1 (F)) \setminus \{\emptyset\}) = (\operatorname{Inv}_A F \cap \operatorname{Inv}_A C_1 (F)) \setminus \{\emptyset\} \\ &= \operatorname{Inv}_A (F \cup C_1 (F)) \setminus \{\emptyset\} = (\operatorname{Inv}_A F) \setminus \{\emptyset\}. \end{aligned}$$

Next, we verify that $\operatorname{Pol}_A^{(0)} ((\operatorname{Inv}_A F) \setminus \{\emptyset\}) = \emptyset$ if and only if $C_1 (\operatorname{Pol}_A \operatorname{Inv}_A F)$ is empty. By Lemma 3.5 for $Q = \operatorname{Inv}_A F$, we see $\operatorname{Pol}_A ((\operatorname{Inv}_A F) \setminus \{\emptyset\}) = (\operatorname{Pol}_A \operatorname{Inv}_A F)^{\circ}$, and so $\operatorname{Pol}_A^{(0)} ((\operatorname{Inv}_A F) \setminus \{\emptyset\}) = \emptyset$ implies $C_1 (\operatorname{Pol}_A \operatorname{Inv}_A F) = \emptyset$. Conversely, if this is the case, then applying equation (13) to the set $\operatorname{Pol}_A \operatorname{Inv}_A F$, we obtain that $\operatorname{Inv}_A \operatorname{Pol}_A ((\operatorname{Inv}_A F) \setminus \{\emptyset\}) = \operatorname{Inv}_A \left((\operatorname{Pol}_A \operatorname{Inv}_A F)^{\circ}\right) = \operatorname{Inv}_A \operatorname{Pol}_A \operatorname{Inv}_A F = \operatorname{Inv}_A F$, whence $\operatorname{Pol}_A^{(0)} ((\operatorname{Inv}_A F) \setminus \{\emptyset\}) = \operatorname{Pol}_A^{(0)} \operatorname{Inv}_A \operatorname{Pol}_A ((\operatorname{Inv}_A F) \setminus \{\emptyset\}) = \operatorname{Pol}_A^{(0)} \operatorname{Inv}_A F$. Since $\operatorname{Pol}_A \operatorname{Inv}_A F$ is a clone, the assumption $C_1 (\operatorname{Pol}_A \operatorname{Inv}_A F) = \emptyset$ now implies $\operatorname{Pol}_A^{(0)} ((\operatorname{Inv}_A F) \setminus \{\emptyset\}) = \operatorname{Pol}_A^{(0)} \operatorname{Inv}_A F = \emptyset$.

Combining this equivalence with the definition of $'$ directly yields (14).

Now, if $C_1 (F) = \emptyset$, then by (13), $\operatorname{Inv}_A \left(F^{\circ}\right) = \operatorname{Inv}_A F \supseteq (\operatorname{Inv}_A F)'$. If, otherwise, $C_1 (F) \neq \emptyset$, then we also have $\emptyset \neq C_1 (F) \subseteq C_1 (\operatorname{Pol}_A \operatorname{Inv}_A F)$. So from equations (13) and (14), one can infer $\operatorname{Inv}_A \left(F^{\circ}\right) = (\operatorname{Inv}_A F) \setminus \{\emptyset\} = (\operatorname{Inv}_A F)'$. This proves the first part of the last claim of the lemma and the inclusion (15) at the same time.

The following part, about $F^{(0)} \neq \emptyset$, is trivially true, since then $\emptyset \notin \operatorname{Inv}_A F$, so $(\operatorname{Inv}_A F) \setminus \{\emptyset\} = \operatorname{Inv}_A F$, and by equations (13) and (14) we are done.

If we suppose, for the remaining fact, that $C_1(\mathrm{Pol}_A \mathrm{Inv}_A F) = \emptyset$, then we also have $C_1(F) = \emptyset$. Hence, $(\mathrm{Inv}_A F)' \overset{(14)}{=} \mathrm{Inv}_A F \overset{(13)}{=} \mathrm{Inv}_A\left(F^{\circ}\right)$. □

Lemma 3.7 *For all sets $F \subseteq \mathrm{O}_A$ of operations we have*

$$\mathrm{Inv}_A\left(F^{\circ}\right) = (\mathrm{Inv}_A F)' \iff \left(C_1(\mathrm{Pol}_A \mathrm{Inv}_A F) = \emptyset \text{ or } C_1(F) \neq \emptyset \text{ or } F^{(0)} \neq \emptyset\right).$$

If $F \in \mathcal{L}_A$ is Galois *closed, i.e. $\mathrm{Pol}_A \mathrm{Inv}_A F = F$, then truly $\mathrm{Inv}_A\left(F^{\circ}\right) = (\mathrm{Inv}_A F)'$.*

Proof By Lemma 3.6, we only have to deal with the implication "$\implies$". To this end suppose that $\mathrm{Inv}_A\left(F^{\circ}\right) = (\mathrm{Inv}_A F)'$, $C_1(\mathrm{Pol}_A \mathrm{Inv}_A F) \neq \emptyset$ and $C_1(F) = \emptyset$. It has to be shown that $F^{(0)} \neq \emptyset$. Using the assumptions and Lemma 3.6, we can infer $\mathrm{Inv}_A F \overset{(13)}{=} \mathrm{Inv}_A\left(F^{\circ}\right) = (\mathrm{Inv}_A F)' \overset{(14)}{=} (\mathrm{Inv}_A F) \setminus \{\emptyset\}$. This is true if and only if $\emptyset \notin \mathrm{Inv}_A F$, which is equivalent to $F^{(0)} \neq \emptyset$ by Lemma 3.2(b). □

The following example shows that on infinite carrier sets there actually exist clones $F \in \mathcal{L}_A$ violating the condition from Lemma 3.7 characterising the equality $\mathrm{Inv}_A\left(F^{\circ}\right) = (\mathrm{Inv}_A F)'$. That is to say, the example exhibits a clone without nullary or unary constant operations, that is not Galois closed, and whose Galois closure $\mathrm{Pol}_A \mathrm{Inv}_A F$ contains a unary constant operation. We shall verify that in this case the inclusion (15) is a proper one.

Example 3.8 Let $A = \mathbb{N}$ and let us consider the unary function $f\colon \mathbb{N} \longrightarrow \mathbb{N}$ given by $k \mapsto f(k) := \max\{0, k-1\}$. Clearly, for $n \in \mathbb{N}$ iterates of this function have the form $f^n(k) = \max\{0, k-n\}$ for $k \in \mathbb{N}$, i.e. $f(k) = 0$ if $k \leq n$, and $f(k) = n-k$ otherwise. In particular, none of these functions is constant, so if we put $F := \langle\{f\}\rangle_{\mathrm{O}_A}$, then $F^{(1)} = \{ f^n \mid n \in \mathbb{N}\}$, $C_1(F) = \emptyset$ and hence $F^{\circ} = F$.

We will see that F is not Galois closed as $c_0^{(1)} \in \mathrm{Pol}_A \mathrm{Inv}_A \{f\}$, which by Lemma 2.6 equals $\mathrm{Pol}_A \mathrm{Inv}_A \langle\{f\}\rangle_{\mathrm{O}_A} = \mathrm{Pol}_A \mathrm{Inv}_A F$. This is true because every $\varrho \in (\mathrm{Inv}_A \{f\}) \setminus \{\emptyset\}$ contains some tuple $x \in \varrho$, and for $n := \max \mathrm{im}(x)$, we obtain $f^n \circ x = (0, \ldots, 0) \in \varrho$ since $f \rhd \varrho$. Thus, $c_0^{(1)} \rhd \varrho$, and so $c_0^{(1)} \in \mathrm{Pol}_A \mathrm{Inv}_A \{f\}$.

Hence, $C_1(\mathrm{Pol}_A \mathrm{Inv}_A F) \neq \emptyset$, and therefore, by equation (14) of Lemma 3.6, we obtain $(\mathrm{Inv}_A F)' = (\mathrm{Inv}_A F) \setminus \{\emptyset\} \subset \mathrm{Inv}_A F = \mathrm{Inv}_A\left(F^{\circ}\right)$.

Corollary 3.9 *Let $Q \subseteq \mathrm{R}_A$ and $F \subseteq \mathrm{O}_A$, then it is*

$$\begin{aligned} \mathrm{Inv}_A \mathrm{Pol}_A\left(Q'\right) &= (\mathrm{Inv}_A \mathrm{Pol}_A Q)', & \mathrm{Pol}_A \mathrm{Inv}_A\left(F'\right) &\subseteq (\mathrm{Pol}_A \mathrm{Inv}_A F)', \\ \mathrm{Inv}_A \mathrm{Pol}_A\left(Q^{\circ}\right) &= (\mathrm{Inv}_A \mathrm{Pol}_A Q)^{\circ}, & \mathrm{Pol}_A \mathrm{Inv}_A\left(F^{\circ}\right) &\subseteq (\mathrm{Pol}_A \mathrm{Inv}_A F)^{\circ}. \end{aligned}$$

Moreover, we have

$$\begin{aligned} \mathrm{Pol}_A \mathrm{Inv}_A\left(F'\right) = (\mathrm{Pol}_A \mathrm{Inv}_A F)' &\iff \mathrm{Inv}_A\left(F'\right) = (\mathrm{Inv}_A F)^{\circ} \\ &\iff \left(\forall a \in A\colon c_a^{(0)} \in F \Rightarrow c_a^{(1)} \in \mathrm{Pol}_A \mathrm{Inv}_A\left(F'\right)\right). \end{aligned}$$

In particular, this is the case if we consider a clone $F \in \mathcal{L}_A$.

Furthermore, we can characterise

$$\begin{aligned}\mathrm{Pol}_A \mathrm{Inv}_A \left(F^{\circ}\right) = \left(\mathrm{Pol}_A \mathrm{Inv}_A F\right)^{\circ} &\iff \mathrm{Inv}_A \left(F^{\circ}\right) = \left(\mathrm{Inv}_A F\right)' \\ &\iff \Big(C_1 \left(\mathrm{Pol}_A \mathrm{Inv}_A F\right) = \emptyset \text{ or } C_1 (F) \neq \emptyset \\ &\qquad \text{or } F^{(0)} \neq \emptyset\Big).\end{aligned}$$

The equalities above show that the operators ′ and ° map Galois *closed clones to* Galois *closed clones. Hence, the restrictions of these closure and kernel operators to the lattices of* Galois *closed clones are well-defined.*

Proof For the proof we fix subsets $Q \subseteq \mathrm{R}_A$ and $F \subseteq \mathrm{O}_A$. Using the lemmas established before, we can infer

$$\begin{aligned}\mathrm{Inv}_A \mathrm{Pol}_A \left(Q'\right) &\overset{3.5}{=} \mathrm{Inv}_A \left(\left(\mathrm{Pol}_A Q\right)^{\circ}\right) \overset{3.7}{=} \left(\mathrm{Inv}_A \mathrm{Pol}_A Q\right)', \\ \mathrm{Inv}_A \mathrm{Pol}_A \left(Q^{\circ}\right) &\overset{3.4}{=} \mathrm{Inv}_A \left(\left(\mathrm{Pol}_A Q\right)'\right) \overset{3.4}{=} \left(\mathrm{Inv}_A \mathrm{Pol}_A Q\right)^{\circ}.\end{aligned}$$

In the first line we have used that $\mathrm{Pol}_A Q$ is Galois closed and in the second that $\mathrm{Pol}_A Q \in \mathcal{L}_A$ (see Lemma 2.6). From Lemma 3.4 we read off $\mathrm{Inv}_A (F') \supseteq (\mathrm{Inv}_A F)^{\circ}$, so $\mathrm{Pol}_A \mathrm{Inv}_A (F') \subseteq \mathrm{Pol}_A \left((\mathrm{Inv}_A F)^{\circ}\right) \overset{3.4}{=} (\mathrm{Pol}_A \mathrm{Inv}_A F)'$. By formula (15) we obtain $(\mathrm{Inv}_A F)' \subseteq \mathrm{Inv}_A \left(F^{\circ}\right)$, thus $\mathrm{Pol}_A \mathrm{Inv}_A \left(F^{\circ}\right) \subseteq \mathrm{Pol}_A \left((\mathrm{Inv}_A F)'\right) \overset{3.5}{=} (\mathrm{Pol}_A \mathrm{Inv}_A F)^{\circ}$.

If we know that $\mathrm{Inv}_A \left(F^{\circ}\right) = (\mathrm{Inv}_A F)'$, then in the previous line we have equality. Conversely, if we suppose $\mathrm{Pol}_A \mathrm{Inv}_A \left(F^{\circ}\right) = (\mathrm{Pol}_A \mathrm{Inv}_A F)^{\circ}$, then we can derive

$$\begin{aligned}\mathrm{Inv}_A \left(F^{\circ}\right) = \mathrm{Inv}_A \mathrm{Pol}_A \mathrm{Inv}_A \left(F^{\circ}\right) = \mathrm{Inv}_A &\left(\left(\mathrm{Pol}_A \mathrm{Inv}_A F\right)^{\circ}\right) \\ &\overset{3.7}{=} \left(\mathrm{Inv}_A \mathrm{Pol}_A \mathrm{Inv}_A F\right)' = \left(\mathrm{Inv}_A F\right)',\end{aligned}$$

where the applicability of Lemma 3.7 is guaranteed by $\mathrm{Pol}_A \mathrm{Inv}_A F$ being a Galois closed clone. The second condition characterising $\mathrm{Pol}_A \mathrm{Inv}_A \left(F^{\circ}\right) = (\mathrm{Pol}_A \mathrm{Inv}_A F)^{\circ}$ is already proven in Lemma 3.7.

Likewise, for the equality $\mathrm{Pol}_A \mathrm{Inv}_A (F') = (\mathrm{Pol}_A \mathrm{Inv}_A F)'$ we only need to show that it is equivalent to $\mathrm{Inv}_A (F') = (\mathrm{Inv}_A F)^{\circ}$ since the second stated equivalence is already contained in Lemma 3.4. Evidently, the equality $\mathrm{Inv}_A (F') = (\mathrm{Inv}_A F)^{\circ}$ implies $\mathrm{Pol}_A \mathrm{Inv}_A (F') = \mathrm{Pol}_A \left((\mathrm{Inv}_A F)^{\circ}\right) \overset{3.4}{=} (\mathrm{Pol}_A \mathrm{Inv}_A F)'$. Conversely, if we know this, then we can conclude $\mathrm{Inv}_A (F') = \mathrm{Inv}_A \mathrm{Pol}_A \mathrm{Inv}_A (F') = \mathrm{Inv}_A \left((\mathrm{Pol}_A \mathrm{Inv}_A F)'\right)$, which equals $(\mathrm{Inv}_A \mathrm{Pol}_A \mathrm{Inv}_A F)^{\circ} = (\mathrm{Inv}_A F)^{\circ}$ by Lemma 3.4 and $\mathrm{Pol}_A \mathrm{Inv}_A F \in \mathcal{L}_A$ (see again Lemma 2.6). □

Since Example 3.8 above exhibits a proper inclusion in (15), it violates the condition in Lemma 3.7 and therefore, shows that the last equality in the previous lemma is not true for non-Galois closed clones without any constant operations. Explicitly, this is so because in Example 3.8 we had $C_1 (F) = \emptyset$, so $F = F^{\circ}$ and $\mathrm{Pol}_A \mathrm{Inv}_A \left(F^{\circ}\right) = \mathrm{Pol}_A \mathrm{Inv}_A F$. Furthermore, $F \subseteq \mathrm{O}_A \setminus \mathrm{O}_A^{(0)} = \mathrm{O}_A'$, so $\mathrm{Pol}_A \mathrm{Inv}_A F$

is a subclone of O'_A. Hence, $\mathrm{Pol}_A \mathrm{Inv}_A \left(F^{\circ}\right)$ does not contain nullary operations, but certainly there are nullary constants in $\left(\mathrm{Pol}_A \mathrm{Inv}_A F\right)^{\circ}$ because $C_1 \left(\mathrm{Pol}_A \mathrm{Inv}_A F\right) \neq \emptyset$. Consequently, 3.8 demonstrates a proper inclusion $\mathrm{Pol}_A \mathrm{Inv}_A \left(F^{\circ}\right) \subset \left(\mathrm{Pol}_A \mathrm{Inv}_A F\right)^{\circ}$.

In the following lemma we record how the operators $'$ and $^{\circ}$ interact with each other, when applied to any kind of clones. This is the last step needed to completely describe where the old clones of non-nullary operations lie in the general clone lattice, and in which places of this lattice those clones are situated that are strictly new, i.e. do not occur in the traditional theory.

Lemma 3.10 *For a clone $F \in \mathcal{L}_A$ and a relational clone $Q \in \mathcal{L}^*_A$ the following equalities are true:*

$$F'^{\circ} = F^{\circ}, \qquad (16) \qquad\qquad Q'^{\circ} = Q^{\circ}, \qquad (18)$$

$$F^{\circ\prime} = F', \qquad (17) \qquad\qquad Q^{\circ\prime} = Q'. \qquad (19)$$

Proof For equality (16) we only have to compare the nullary parts of F'° and F°, because the operations $'$ and $^{\circ}$ do not touch the higher arity part of the clone. Two cases can occur: if $F^{(0)} = \emptyset$, then $F' = F$, and (16) holds. Otherwise, if $F^{(0)} \neq \emptyset$, then $\left\{ c_a^{(1)} \;\middle|\; a \in A \wedge c_a^{(0)} \in F \right\} \subseteq F$ since $F \in \mathcal{L}_A$. This implies $F \subseteq F'^{\circ}$. As $F^{(0)} \neq \emptyset$, formula (9) yields $F = F^{\circ}$, so $F'^{\circ} \subseteq F^{\circ} = F \subseteq F'^{\circ}$, i.e. $F'^{\circ} = F^{\circ} = F$.

Equality (17) is clear as $F^{\circ\prime(0)} = \emptyset = F'^{(0)}$ and $F^{\circ\prime(n)} = F^{(n)} = F'^{(n)}$ for $n \in \mathbb{N}_+$.

In equality (18) the term Q'° equals $Q \cup \{\emptyset\}$ or $(Q \setminus \{\emptyset\}) \cup \{\emptyset\}$, depending on the result of Q'. In any case the final result will be $Q \cup \{\emptyset\} = Q^{\circ}$.

For equality (19) we note that $\left(Q^{\circ}\right) \setminus \{\emptyset\} = Q \setminus \{\emptyset\}$, and so by equation (10) we obtain $Q' \overset{(10)}{=} [Q \setminus \{\emptyset\}]_{\mathrm{R}_A} = \left[\left(Q^{\circ}\right) \setminus \{\emptyset\}\right]_{\mathrm{R}_A} \overset{(10)}{=} Q^{\circ\prime}$, having used $Q \in \mathcal{L}^*_A$ and also $Q^{\circ} \in \mathcal{L}^*_A$ (cf. (12) in Lemma 3.3). $\square$

3.1.3 Location of conventional clones in the new, general clone lattice

Now we try to put the information of the previous lemmas together, to find out more about *old* (Galois closed) clones, i.e. (Galois closed) subclones of $\mathrm{O}'_A = \mathrm{O}_A \setminus \mathrm{O}_A^{(0)}$, and *strictly new* (Galois closed) clones, i.e. those satisfying $F^{(0)} \neq \emptyset$. Mainly due to Corollary 3.9, this and the following two paragraphs can be read with or without the additional attribute "Galois closed". It therefore is always written in brackets to denote two alternative ways of reading.

Every strictly new (Galois closed) clone $F \nsubseteq \mathrm{O}'_A$ (meaning $F^{(0)} \neq \emptyset$) has got a distinguished lower cover, $F' \subseteq \mathrm{O}'_A$, which is clearly a (Galois closed) clone in the traditional sense. Certainly, the lower cover F' will contain constant operations of positive arity since F did. Hence, not all old (Galois closed) clones arise in this way, only those satisfying $C_1(F) \neq \emptyset$, i.e. lying above one minimal clone given by a constant unary operation. These are exactly the old (Galois closed) clones H that are not closed under $^{\circ}$ (see formula (9)). Thus they are mapped back by $^{\circ}$ to the upper cover among the strictly new (Galois closed) clones that induced them via $'$ (see equation (16) and recall that every strictly new clone is closed w.r.t. $^{\circ}$, (9)). For any other (Galois closed) clone we have $F^{\circ} = F$. Similarly, we have $F' = F$ for all

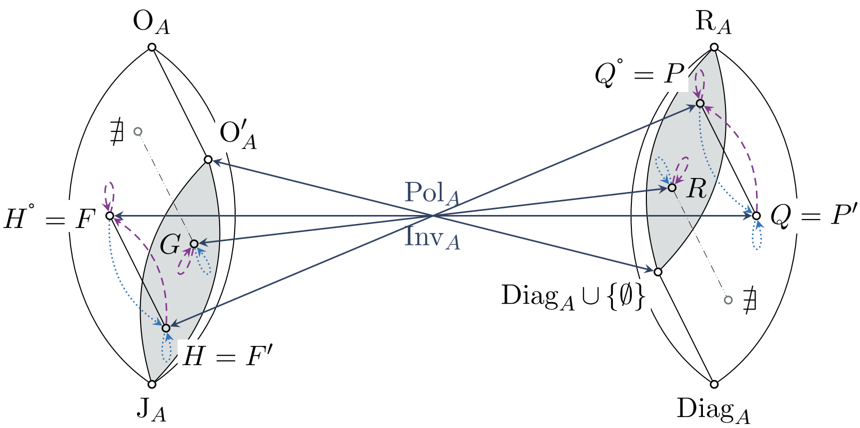

Figure 1. Relationship between old and new (Galois closed) clones and the operation of $'$ (⋯⋯→), $^{\circ}$ (- - →) and $\mathrm{Pol}_A - \mathrm{Inv}_A$. The depicted clones have the following properties: $F^{(0)} \neq \emptyset$, $C_1(H) \neq \emptyset$, $C_1(G) = \emptyset$, $\emptyset \notin Q$, $\emptyset \in P \cap R$ and $P \setminus \{\emptyset\}$ is a clone, whereas $R \setminus \{\emptyset\}$ is not.

old (Galois closed) clones, i.e. the traditional (Galois closed) clones are precisely the $'$-kernels of (Galois closed) clones.

The (Galois closed) relational clones have analogous properties. *Old* (Galois closed) relational clones are such that contain the empty relation, and *strictly new* (Galois closed) relational clones are those that do not. Every strictly new (Galois closed) relational clone Q has got an upper cover among the (Galois closed) old ones, namely Q°. Again, not all (Galois closed) old relational clones can arise in such a way, only those where $Q \setminus \{\emptyset\}$ is again a (Galois closed) relational clone. This is the case if and only if $Q' \overset{(10)}{=} [Q \setminus \{\emptyset\}]_{\mathrm{R}_A} = Q \setminus \{\emptyset\} \subset Q$. Clearly, Q' is the strictly new (Galois closed) relational clone that induced Q via $^{\circ}$. For any other (Galois closed) relational clone we have $Q' \overset{(10)}{=} [Q \setminus \{\emptyset\}]_{\mathrm{R}_A} = Q$, and $F^{\circ} = F$ for all old (Galois closed) clones.

The relationships explained above are visualised in Figure 1. The individual pictures of the clone lattices are correct for clones and Galois closed clones, however, the relating arrows labelled $\mathrm{Pol}_A - \mathrm{Inv}_A$ only make sense for Galois closed clones.

Figure 2 shows the location of the strictly new clones in the clone lattice as a copy of an order filter generated by minimal clones generated by constant operations. It is easy to see that this order filter is generally not a sublattice.

3.1.4 Behaviour of $^{\circ}$, $'$ towards local closures

So far we have looked at the interaction of the operators $^{\circ}$ and $'$ with Pol_A and Inv_A and with each other. It remains to study how they get along with the local closure operators Loc_A and LOC_A. We will see an answer to this question in Lemmas 3.11 and 3.12, the first one dealing with LOC_A, the second one with Loc_A.

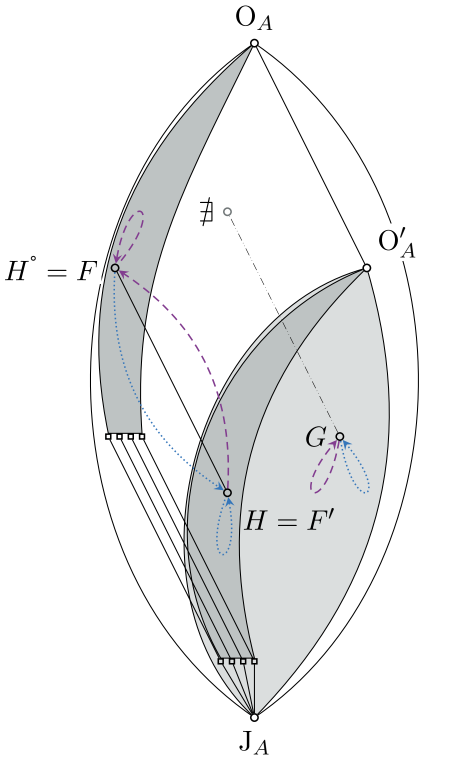

Figure 2. Location of (Galois closed) clones of operations in the traditional sense and of strictly new ones in the clone lattice. The strictly new clones, i.e. such satisfying $F^{(0)} \neq \emptyset$, are precisely the upper covers H° of (Galois closed) clones H in the order filter (dark grey) generated by the minimal clones given by constant unary operations. Other clones G where $C_1(G) = \emptyset$ do not have upper covers outside the classical clone lattice $\downarrow_{\mathcal{L}_A} (\mathrm{O}'_A)$.
The figure also depicts the action of the closure operator $^{\circ}$ ($\dashrightarrow$), whose closure system consists of the light grey and the upper, dark grey shaded part of the lattice. Likewise, one can see that the kernel system of the operator $'$ ($\cdots\rightarrow$) equals the light grey coloured part including the lower, dark grey filter.
A dual situation happens for the lattice of relational clones: there the (Galois closed) strictly new clones form a copy, consisting of lower covers, of an order ideal generated by certain maximal clones of relations.

Lemma 3.11 *For any subset $Q \subseteq \mathrm{R}_A$ of relations the following holds:*

$$\mathrm{LOC}_A (Q \setminus \{\emptyset\}) = (\mathrm{LOC}_A Q) \setminus \{\emptyset\},$$
$$\mathrm{LOC}_A \left(Q'\right) = (\mathrm{LOC}_A Q)',$$
$$\mathrm{LOC}_A \left(Q^{\circ}\right) = (\mathrm{LOC}_A Q)^{\circ}.$$

Proof Let us consider a fixed subset $Q \subseteq \mathrm{R}_A$. Certainly, $Q \setminus \{\emptyset\} \subseteq Q$ implies $\mathrm{LOC}_A (Q \setminus \{\emptyset\}) \subseteq \mathrm{LOC}_A Q$. Using interpolation on the empty subset, one readily checks that $\emptyset \in \mathrm{LOC}_A W$ always implies $\emptyset \in W$ (for any $W \subseteq \mathrm{R}_A$). Thus, it is $\emptyset \notin \mathrm{LOC}_A (Q \setminus \{\emptyset\})$, and so $\mathrm{LOC}_A (Q \setminus \{\emptyset\}) \subseteq (\mathrm{LOC}_A Q) \setminus \{\emptyset\}$. For the converse inclusion let us consider any $\sigma \in (\mathrm{LOC}_A Q) \setminus \{\emptyset\}$. For every finite subset $B \subseteq \sigma$, we can find some $\varrho \in Q$ such that $B \subseteq \varrho \subseteq \sigma$. If $B \neq \emptyset$, then also $\varrho \neq \emptyset$, so $\varrho \in Q \setminus \{\emptyset\}$. Since $\sigma \neq \emptyset$, there is at least one singleton subset $B \subseteq \sigma$, being interpolated by some $\varrho \in Q \setminus \{\emptyset\}$. This ϱ interpolates $B = \emptyset$, as well, whence $\sigma \in \mathrm{LOC}_A (Q \setminus \{\emptyset\})$.

We can use the previous result to show $\mathrm{LOC}_A(Q') = (\mathrm{LOC}_A Q)'$. Using the definition of LOC_A, it is straightforward to prove $\mathrm{Pol}_A \mathrm{LOC}_A W = \mathrm{Pol}_A W$ for all $W \subseteq \mathrm{R}_A$. Thus, the conditions $\emptyset \neq \mathrm{Pol}_A^{(0)} \mathrm{LOC}_A(Q \setminus \{\emptyset\}) = \mathrm{Pol}_A^{(0)}((\mathrm{LOC}_A Q) \setminus \{\emptyset\})$ and $\emptyset \neq \mathrm{Pol}_A^{(0)}(Q \setminus \{\emptyset\})$ are equivalent. Therefore, we have

$$\begin{aligned}
\mathrm{LOC}_A(Q') &= \begin{cases} \mathrm{LOC}_A(Q \setminus \{\emptyset\}) = (\mathrm{LOC}_A Q) \setminus \{\emptyset\} & \text{if } \emptyset \neq \mathrm{Pol}_A^{(0)}(Q \setminus \{\emptyset\}) \\ \mathrm{LOC}_A Q & \text{else,} \end{cases} \\
&= \begin{cases} (\mathrm{LOC}_A Q) \setminus \{\emptyset\} & \text{if } \emptyset \neq \mathrm{Pol}_A^{(0)}((\mathrm{LOC}_A Q) \setminus \{\emptyset\}) \\ \mathrm{LOC}_A Q & \text{else,} \end{cases} \\
&= (\mathrm{LOC}_A Q)'.
\end{aligned}$$

The remaining equality, $\mathrm{LOC}_A\left(Q^{\circ}\right) = (\mathrm{LOC}_A Q)^{\circ}$, will be proven via both set inclusions. Since $\emptyset \in (\mathrm{LOC}_A Q)^{\circ}$ it suffices to consider non-empty relations for "$\subseteq$". By the above, it is $\left(\mathrm{LOC}_A\left(Q^{\circ}\right)\right) \setminus \{\emptyset\} = \mathrm{LOC}_A\left(Q^{\circ} \setminus \{\emptyset\}\right) = \mathrm{LOC}_A(Q \setminus \{\emptyset\})$, being a subset of $\mathrm{LOC}_A Q \subseteq (\mathrm{LOC}_A Q)^{\circ}$ and thus demonstrating $\mathrm{LOC}_A\left(Q^{\circ}\right) \subseteq (\mathrm{LOC}_A Q)^{\circ}$. The inclusion "$\supseteq$" follows from $\mathrm{LOC}_A Q \subseteq \mathrm{LOC}_A\left(Q^{\circ}\right)$ and $\emptyset \in Q^{\circ} \subseteq \mathrm{LOC}_A\left(Q^{\circ}\right)$. □

The following lemma is the companion of Lemma 3.11. The restrictions we will have to make to achieve equality in formula (21) are not surprising regarding the conditions appearing in Corollary 3.9 for $^{\circ}$ defined on operations.

Lemma 3.12 *For any subset $F \subseteq \mathrm{O}_A$ of operations we have*

$$\mathrm{Loc}_A\left(F'\right) = (\mathrm{Loc}_A F)', \tag{20}$$

$$\mathrm{Loc}_A\left(F^{\circ}\right) \subseteq (\mathrm{Loc}_A F)^{\circ}. \tag{21}$$

Moreover, we have $\mathrm{Loc}_A\left(F^{\circ}\right) = (\mathrm{Loc}_A F)^{\circ}$ if and only if

$$\left\{ f^{\circ} \;\middle|\; f \in C_1(\mathrm{Loc}_A F)\right\} \subseteq F^{(0)} \cup \left\{ f^{\circ} \;\middle|\; f \in C_1(F)\right\}. \tag{22}$$

If $C_1(F) \neq \emptyset$, and $F^{(1)}$ is closed under substitution of unary constants from $C_1(F)$, e.g. if $\left(F^{(1)}, \circ\right)$ is a semigroup, then $C_1(\mathrm{Loc}_A F) = C_1(F)$, and this implies (22).

Likewise, if $F^{(0)} \neq \emptyset$, and F is closed under substitution of nullary constants from $F^{(0)}$ into unary operations, then $\left\{ f^{\circ} \;\middle|\; f \in C_1(\mathrm{Loc}_A F)\right\} \subseteq F^{(0)}$, and, again, this implies that (22) is fulfilled.

Consequently, (22) holds for clones $F \in \mathcal{L}_A$ where $C_1(F) \neq \emptyset$.

Proof It is clear that $F'^{(0)} = \emptyset$, $F'^{(n)} = F^{(n)}$ for $n \in \mathbb{N}_+$, and $\mathrm{Loc}_A \emptyset \subseteq \mathrm{Loc}_A F^{(1)}$.

We can use this to show (20):

$$\begin{aligned}
\mathrm{Loc}_A\left(F'\right) &= \bigcup_{n\in\mathbb{N}} \mathrm{Loc}_A^{(n)}\left(F'\right) \overset{(1)}{=} \bigcup_{n\in\mathbb{N}} \mathrm{Loc}_A\left(F'^{(n)}\right)\\
&= \mathrm{Loc}_A\left(F'^{(0)}\right) \cup \bigcup_{n\in\mathbb{N}_+} \mathrm{Loc}_A\left(F'^{(n)}\right) = \mathrm{Loc}_A\left(\emptyset\right) \cup \bigcup_{n\in\mathbb{N}_+} \mathrm{Loc}_A\left(F^{(n)}\right)\\
&\overset{(1)}{=} \bigcup_{n\in\mathbb{N}_+} \mathrm{Loc}_A^{(n)} F = \mathrm{Loc}_A\left(F\right) \setminus \mathrm{Loc}_A^{(0)}\left(F\right) = \left(\mathrm{Loc}_A\left(F\right)\right)'.
\end{aligned}$$

For (21) we note that for all $n \in \mathbb{N}_+$ we always have the equality

$$\left(\mathrm{Loc}_A\left(F^{\circ}\right)\right)^{(n)} \overset{(1)}{=} \mathrm{Loc}_A\left(F^{\circ(n)}\right) = \mathrm{Loc}_A\left(F^{(n)}\right) \overset{(1)}{=} \left(\mathrm{Loc}_A F\right)^{(n)} = \left(\mathrm{Loc}_A F\right)^{\circ(n)}.$$

Therefore, the relationship of $\mathrm{Loc}_A\left(F^{\circ}\right)$ and $\left(\mathrm{Loc}_A F\right)^{\circ}$ w.r.t. "$\subseteq$", "$\supseteq$" and "$=$" is completely determined by that of $\mathrm{Loc}_A^{(0)}\left(F^{\circ}\right)$ and $\left(\mathrm{Loc}_A F\right)^{\circ(0)}$. As nullary operations have one-element domains, for all $\tilde{F} \subseteq \mathrm{O}_A$ we have $\mathrm{Loc}_A^{(0)}\tilde{F} = \tilde{F}^{(0)}$. We can apply this to see the equalities $\mathrm{Loc}_A^{(0)}\left(F^{\circ}\right) = F^{\circ(0)} = F^{(0)} \cup \left\{ f^{\circ} \;\middle|\; f \in C_1\left(F\right)\right\}$ and $\left(\mathrm{Loc}_A F\right)^{\circ(0)} = \mathrm{Loc}_A^{(0)} F \cup \left\{ f^{\circ} \;\middle|\; f \in C_1\left(\mathrm{Loc}_A F\right)\right\} = F^{(0)} \cup \left\{ f^{\circ} \;\middle|\; f \in C_1\left(\mathrm{Loc}_A F\right)\right\}$. Since $C_1\left(F\right) \subseteq C_1\left(\mathrm{Loc}_A F\right)$, we have $\mathrm{Loc}_A^{(0)}\left(F^{\circ}\right) \subseteq \left(\mathrm{Loc}_A F\right)^{\circ(0)}$ and hence (21). Together with what was said before, now also the equivalence involving (22) is clear.

The last two conditions in the lemma are sufficient for (22) because constants $c_a^{(1)} \in C_1\left(\mathrm{Loc}_A F\right)$ can be interpolated on singletons $\{b\}$ by operations in $F^{(1)}$. $\square$

3.1.5 Behaviour of $^{\circ}$, $'$ towards clone closures

Having dealt with the local closure operators, it is now time to study the behaviour of $^{\circ}$ and $'$ w.r.t. the clone closures in Lemmas 3.13 and 3.14 below.

Lemma 3.13 *For any set $Q \subseteq \mathrm{R}_A$ the following equalities hold:*

$$\left[Q \setminus \{\emptyset\}\right]_{\mathrm{R}_A} = \left[\left[Q\right]_{\mathrm{R}_A} \setminus \{\emptyset\}\right]_{\mathrm{R}_A}, \tag{23}$$

$$\left[Q'\right]_{\mathrm{R}_A} = \left[Q\right]'_{\mathrm{R}_A}, \tag{24}$$

$$\left[Q^{\circ}\right]_{\mathrm{R}_A} = \left[Q\right]^{\circ}_{\mathrm{R}_A}. \tag{25}$$

Proof The inclusion $\left[Q \setminus \{\emptyset\}\right]_{\mathrm{R}_A} \subseteq \left[\left[Q\right]_{\mathrm{R}_A} \setminus \{\emptyset\}\right]_{\mathrm{R}_A}$ in (23) is clear. For the converse we define $S := \left[Q\right]_{\mathrm{R}_A} \cap \left(\left[Q \setminus \{\emptyset\}\right]^{\circ}_{\mathrm{R}_A}\right)$. Due to Lemma 3.3 the set $\left[Q \setminus \{\emptyset\}\right]^{\circ}_{\mathrm{R}_A}$ is a relational clone, thus, $S \in \mathcal{L}_A^*$ and $Q \subseteq S$, whence $\left[Q\right]_{\mathrm{R}_A} \subseteq \left[S\right]_{\mathrm{R}_A} = S \subseteq \left[Q\right]_{\mathrm{R}_A}$. This proves $\left[Q\right]_{\mathrm{R}_A} = S$, which can be reformulated as $\left[Q\right]_{\mathrm{R}_A} \setminus \{\emptyset\} \subseteq \left[Q \setminus \{\emptyset\}\right]_{\mathrm{R}_A}$. It follows $\left[\left[Q\right]_{\mathrm{R}_A} \setminus \{\emptyset\}\right]_{\mathrm{R}_A} \subseteq \left[\left[Q \setminus \{\emptyset\}\right]_{\mathrm{R}_A}\right]_{\mathrm{R}_A} = \left[Q \setminus \{\emptyset\}\right]_{\mathrm{R}_A}$, finishing the proof of (23).

To show (24) we distinguish two cases according to the definition of $'$. Note that

$$\mathrm{Pol}_A^{(0)}(Q\setminus\{\emptyset\}) \overset{2.6}{=} \mathrm{Pol}_A^{(0)}[Q\setminus\{\emptyset\}]_{\mathrm{R}_A} \overset{(23)}{=} \mathrm{Pol}_A^{(0)}\Big[[Q]_{\mathrm{R}_A}\setminus\{\emptyset\}\Big]_{\mathrm{R}_A} \overset{2.6}{=} \mathrm{Pol}_A^{(0)}([Q]_{\mathrm{R}_A}\setminus\{\emptyset\}).$$

Now, if $\mathrm{Pol}_A^{(0)}(Q\setminus\{\emptyset\})=\emptyset$, then $Q'=Q$ and $[Q]'_{\mathrm{R}_A}=[Q]_{\mathrm{R}_A}$, wherefore we have $[Q']_{\mathrm{R}_A}=[Q]_{\mathrm{R}_A}=[Q]'_{\mathrm{R}_A}$ as claimed. Otherwise, it is $\mathrm{Pol}_A^{(0)}(Q\setminus\{\emptyset\})\neq\emptyset$. Then $Q'=Q\setminus\{\emptyset\}$ and $[Q]'_{\mathrm{R}_A}=[Q]_{\mathrm{R}_A}\setminus\{\emptyset\}$, so $[Q\setminus\{\emptyset\}]_{\mathrm{R}_A}=[Q]_{\mathrm{R}_A}\setminus\{\emptyset\}$ needs to be verified. From $\mathrm{Pol}_A^{(0)}(Q\setminus\{\emptyset\})\neq\emptyset$ we get $\emptyset\notin[Q\setminus\{\emptyset\}]_{\mathrm{R}_A}$, using Lemma 3.2(c), i.e. $[Q\setminus\{\emptyset\}]_{\mathrm{R}_A}\subseteq[Q]_{\mathrm{R}_A}\setminus\{\emptyset\}\subseteq\Big[[Q]_{\mathrm{R}_A}\setminus\{\emptyset\}\Big]_{\mathrm{R}_A}\overset{(23)}{=}[Q\setminus\{\emptyset\}]_{\mathrm{R}_A}$.

From $Q^{\circ}\subseteq[Q]^{\circ}_{\mathrm{R}_A}$ we obtain $\left[Q^{\circ}\right]_{\mathrm{R}_A}\subseteq\Big[[Q]^{\circ}_{\mathrm{R}_A}\Big]_{\mathrm{R}_A}=[Q]^{\circ}_{\mathrm{R}_A}$, as $[Q]^{\circ}_{\mathrm{R}_A}\in\mathcal{L}_A^*$ (see Lemma 3.3). Hence, we have shown the first inclusion of (25). Conversely, we have $[Q]^{\circ}_{\mathrm{R}_A}=[Q]_{\mathrm{R}_A}\cup\{\emptyset\}\subseteq\left[Q^{\circ}\right]_{\mathrm{R}_A}$ because $[Q]_{\mathrm{R}_A}\subseteq\left[Q^{\circ}\right]_{\mathrm{R}_A}$ and $\emptyset\in Q^{\circ}\subseteq\left[Q^{\circ}\right]_{\mathrm{R}_A}$. □

Lemma 3.14 *For a set $F\subseteq\mathrm{O}_A$ of finitary operations the following inclusions*

$$\left\langle F'\right\rangle_{\mathrm{O}_A}\subseteq\langle F\rangle'_{\mathrm{O}_A}, \qquad (26) \qquad\qquad \left\langle F^{\circ}\right\rangle_{\mathrm{O}_A}\subseteq\langle F\rangle^{\circ}_{\mathrm{O}_A} \qquad (27)$$

are true. In formula (26) the equality $\langle F'\rangle_{\mathrm{O}_A}=\langle F\rangle'_{\mathrm{O}_A}$ holds if and only if

$$\forall a\in A\colon c_a^{(0)}\in F \implies c_a^{(1)}\in\left\langle F'\right\rangle_{\mathrm{O}_A}. \tag{28}$$

This is clearly the case if $c_a^{(0)}\in F$ entails $c_a^{(1)}\in F$ for $a\in A$, e.g. for clones $F\in\mathcal{L}_A$.

We have the equality $\left\langle F^{\circ}\right\rangle_{\mathrm{O}_A}=\langle F\rangle^{\circ}_{\mathrm{O}_A}$ (in formula (27)) if and only if

$$C_1\left(\langle F\rangle_{\mathrm{O}_A}\right)=\emptyset \text{ or } C_1(F)\neq\emptyset \text{ or } F^{(0)}\neq\emptyset. \tag{29}$$

Proof Since $F'\subseteq\mathrm{O}'_A\in\mathcal{L}_A$, we have $\langle F'\rangle_{\mathrm{O}_A}\subseteq\langle F\rangle_{\mathrm{O}_A}\cap\mathrm{O}'_A=\langle F\rangle'_{\mathrm{O}_A}$, i.e. (26).

From $F\subseteq\langle F\rangle_{\mathrm{O}_A}$ we get $\left\langle F^{\circ}\right\rangle_{\mathrm{O}_A}\subseteq\left\langle\langle F\rangle^{\circ}_{\mathrm{O}_A}\right\rangle_{\mathrm{O}_A}=\langle F\rangle^{\circ}_{\mathrm{O}_A}$ since $\langle F\rangle^{\circ}_{\mathrm{O}_A}\in\mathcal{L}_A$ (see statement (9) of Lemma 3.3). This proves inclusion (27).

Now assume $\langle F'\rangle_{\mathrm{O}_A}=\langle F\rangle'_{\mathrm{O}_A}$ and consider some $a\in A$ where $c_a^{(0)}\in F$. Then we have $c_a^{(0)}\in F\subseteq\langle F\rangle_{\mathrm{O}_A}$, and so $c_a^{(1)}\in\langle F\rangle^{(1)}_{\mathrm{O}_A}={\langle F\rangle'_{\mathrm{O}_A}}^{(1)}\subseteq\langle F\rangle'_{\mathrm{O}_A}=\langle F'\rangle_{\mathrm{O}_A}$ since $\langle F\rangle_{\mathrm{O}_A}$ is a clone and $'$ does not modify its higher arity part. Hence, (28) is necessary.

To prove that this condition is also sufficient for $\langle F'\rangle_{\mathrm{O}_A}=\langle F\rangle'_{\mathrm{O}_A}$, it is useful to note that $\langle F\rangle'_{\mathrm{O}_A}=\langle F\rangle_{\mathrm{O}_A}\cap\mathrm{O}'_A=\langle F\rangle_{\mathrm{O}_A}\cap\bigcup_{n\in\mathbb{N}_+}\mathrm{O}_A^{(n)}=\bigcup_{n\in\mathbb{N}_+}\langle F\rangle^{(n)}_{\mathrm{O}_A}$. Because of (26), to achieve our goal, it suffices to show $\langle F\rangle^{(n)}_{\mathrm{O}_A}\subseteq\langle F'\rangle_{\mathrm{O}_A}$ for every $n\in\mathbb{N}_+$. This can be done by induction on the structure of the n-ary terms (with operation symbols from F), which describe the members of $\langle F\rangle^{(n)}_{\mathrm{O}_A}$.

To characterise $\left\langle F^{\circ}\right\rangle_{\mathrm{O}_A}=\langle F\rangle^{\circ}_{\mathrm{O}_A}$ as in (29), we first show that it is equivalent to $\left\langle F^{\circ}\right\rangle^{\circ}_{\mathrm{O}_A}=\left\langle F^{\circ}\right\rangle_{\mathrm{O}_A}$. Clearly, $\left\langle F^{\circ}\right\rangle^{\circ}_{\mathrm{O}_A}={\langle F\rangle^{\circ}_{\mathrm{O}_A}}^{\circ}=\langle F\rangle^{\circ}_{\mathrm{O}_A}=\left\langle F^{\circ}\right\rangle_{\mathrm{O}_A}$ follows from

$\langle F^{\circ}\rangle_{\mathrm{O}_A} = \langle F\rangle^{\circ}_{\mathrm{O}_A}$. Conversely, assuming $\langle F^{\circ}\rangle^{\circ}_{\mathrm{O}_A} = \langle F^{\circ}\rangle_{\mathrm{O}_A}$, taking into account (27) and $\langle F\rangle_{\mathrm{O}_A} \subseteq \langle F^{\circ}\rangle_{\mathrm{O}_A}$ yields $\langle F\rangle^{\circ}_{\mathrm{O}_A} \subseteq \langle F^{\circ}\rangle^{\circ}_{\mathrm{O}_A} = \langle F^{\circ}\rangle_{\mathrm{O}_A} \subseteq \langle F\rangle^{\circ}_{\mathrm{O}_A}$.

The condition $\langle F^{\circ}\rangle^{\circ}_{\mathrm{O}_A} = \langle F^{\circ}\rangle_{\mathrm{O}_A}$ simply says that $\langle F^{\circ}\rangle_{\mathrm{O}_A}$ is a clone that is closed w.r.t. $^{\circ}$. Such clones have been characterised in Lemma 3.3 in (9), yielding that $\langle F^{\circ}\rangle_{\mathrm{O}_A}$ is closed if and only if $\langle F^{\circ}\rangle^{(0)}_{\mathrm{O}_A} \neq \emptyset$ or $C_1\left(\langle F^{\circ}\rangle_{\mathrm{O}_A}\right) = \emptyset$. We will transform this condition into the one stated in the lemma. If $F^{\circ(0)} \neq \emptyset$, then also $\langle F^{\circ}\rangle^{(0)}_{\mathrm{O}_A} \neq \emptyset$. Conversely, if $F^{\circ(0)} = \emptyset$, then $F^{\circ} \subseteq \mathrm{O}'_A$, i.e. $\langle F^{\circ}\rangle^{(0)}_{\mathrm{O}_A} \subseteq \langle \mathrm{O}'_A\rangle^{(0)}_{\mathrm{O}_A} = \mathrm{O}'_A{}^{(0)} = \emptyset$. Hence, $\langle F^{\circ}\rangle^{(0)}_{\mathrm{O}_A} \neq \emptyset$ if and only if $F^{\circ(0)} \neq \emptyset$. By definition $F^{\circ(0)} = F^{(0)} \cup \{ f^{\circ} \mid f \in C_1(F)\}$, thus $\langle F^{\circ}\rangle^{(0)}_{\mathrm{O}_A}$ is non-empty if and only if $F^{(0)} \neq \emptyset$ or $C_1(F) \neq \emptyset$. This yields the second part of the disjunction in (29). For the first part we show $\langle F\rangle^{(1)}_{\mathrm{O}_A} = \langle F^{\circ}\rangle^{(1)}_{\mathrm{O}_A}$, then implying $C_1\left(\langle F\rangle_{\mathrm{O}_A}\right) = C_1\left(\langle F^{\circ}\rangle_{\mathrm{O}_A}\right)$. The inclusion $\langle F\rangle^{(1)}_{\mathrm{O}_A} \subseteq \langle F^{\circ}\rangle^{(1)}_{\mathrm{O}_A}$ is indeed obvious, the other one can easily be verified by an induction on the unary term operations corresponding to operation symbols in F°. □

3.2 Characterisation of GALOIS *closed clones*

It is the purpose of this subsection to demonstrate that central results about the GALOIS connection Pol – Inv known from the standard theory of clones without nullary operations (see e.g. [17]) continue to hold in the more general setting without any problems. More precisely, we shall prove that the GALOIS closed (relational) clones are exactly the local closures of (relational) clones, even if nullary relations and operations are permitted.

To some extend this fact seems to have become part of folklore in the universal algebraic community within the last decades or so. Yet from time to time one faces some reluctance among authors to rely on these results also for the case of nullary operations, partly driven by the wish to be compatible with traditional clone theory, and partly to avoid running into (sometimes unexpected) minor technical modifications. We therefore consider it useful to have a reliable source for the main theorem regarding the GALOIS theory for clones involving nullary operations, showing that at least in this respect *no* modifications are necessary. In the best case this will lead to an even broader acceptance of clones with nullary operations among universal algebraists.

To establish the mentioned result, one has different possibilities. One can express the operators Pol, Inv, Loc, LOC and the two clone closures $\langle\ \rangle_{\mathrm{O}_A}$ and $[\ ]_{\mathrm{R}_A}$ using their traditional counterparts, arising from the theory without nullary operations, together with the operators $'$ and $^{\circ}$ from Subsection 3.1. This approach has been chosen in [1, Section 3.2], and some of the arising formulæ turn out to be complicated case distinctions. Once this is done, one may exploit the results of the classical theorems (without nullary operations) to get corresponding results for the general case (cf. Section 3.3 of [1]).

To avoid these technical complications, here, we actually follow the lines of the seminal publication [17], in particular results 4.1–4.3, quite closely and give a direct proof for the characterisations of the closure operators $\mathrm{Pol}_A \mathrm{Inv}_A$ and $\mathrm{Inv}_A \mathrm{Pol}_A$.

First we are going to deal with clones of operations, i.e. with the operator $\mathrm{Pol}_A \mathrm{Inv}_A$. The proof of the main result is based on the following lemmas about subalgebras of direct powers of an algebra.

Lemma 3.15 *Let $\mathbf{A} = \langle A; F\rangle$ be an algebra, Y a set and U a subuniverse, i.e. a carrier set of a subalgebra, of $\mathbf{A}^Y$. Then for any set X and any mapping $\beta\colon X \longrightarrow Y$ the set[5] $\{u \circ \beta \mid u \in U\}$ is a subuniverse of $\mathbf{A}^X$.*

In particular, if $Y = A^N$ for some set N and $(b_i)_{i\in N} \in \left(A^K\right)^N$ for some set K, then one may use the tupling $\beta = \left((b_i)_{i\in N}\right) : K \longrightarrow A^N = Y$ to get that the set $\left\{u \circ (b_i)_{i\in N} \;\middle|\; u \in U\right\}$ is a subuniverse of $\mathbf{A}^K$. Specifically, if $N = n, K = k \in \mathbb{N}$ and $b_0, \ldots, b_{n-1} \in A^k$, then $\{u \circ (b_0, \ldots, b_{n-1}) \mid u \in U\}$ is a subuniverse of $\mathbf{A}^k$.

Proof Note that for $n \in \mathbb{N}$ any $f \in F^{(n)}$ acts in $\mathbf{A}^X$ by $f \circ (x_0, \ldots, x_{n-1})$ for all $(x_0, \ldots, x_{n-1}) \in \left(A^X\right)^n$. Thus, the first claim follows from Lemma 2.5(iii): if $(u_0, \ldots, u_{n-1}) \in U^n$, then $f \circ (u_0 \circ \beta, \ldots, u_{n-1} \circ \beta) = (f \circ (u_0, \ldots, u_{n-1})) \circ \beta$ is in $\{u \circ \beta \mid u \in U\}$ since $f \circ (u_0, \ldots, u_{n-1}) \in U$ as U was a subuniverse of $\mathbf{A}^Y$.

The second part of the lemma evidently follows by specialisation. $\square$

Lemma 3.16 *Let $F \subseteq \mathrm{O}_A$ be a clone on a set A and put $\mathbf{A} := \langle A; F\rangle$. Then for any $n \in \mathbb{N}$ the subset $F^{(n)} \subseteq F$ is a subuniverse of the algebra $\mathbf{A}^{A^n}$. Moreover, for any subset $X \subseteq A^n$ the set $\varrho_{X,n} := \left\{ f|_X \;\middle|\; f \in F^{(n)}\right\}$ is a subuniverse of $\mathbf{A}^X$. If, furthermore, X is finite and $\beta\colon k \longrightarrow X$ is any bijection between X and its cardinality $k := |X|$, then $\varrho_{X,n,\beta} := \left\{ f|_X \circ \beta \;\middle|\; f \in F^{(n)}\right\}$ belongs to $\mathrm{Inv}_A^{(k)} F$.*

Proof That $F^{(n)}$ is a subuniverse of $\mathbf{A}^{A^n}$ is an obvious consequence of closedness of F under composition. If $X \subseteq A^n$ is a subset, then using in Lemma 3.15 the mapping $\tilde{\beta}\colon X \longrightarrow A^n$ given by restriction of the identity map yields that $\varrho_{X,n}$ is a subuniverse of $\mathbf{A}^X$. Similarly, if $\beta\colon k \longrightarrow X$ is any bijection between X and its finite cardinality k, then considering $\tilde{\beta} \circ \beta$ in Lemma 3.15 shows that $\varrho_{X,n,\beta}$ is a subuniverse of $\mathbf{A}^k$, i.e. a member of $\mathrm{Inv}_A^{(k)} F$ (cf. Definition 2.3). $\square$

The previous lemma enables us now to characterise the fixed points of the Galois closure $\mathrm{Pol}_A \mathrm{Inv}_A$ as local closures of clones of operations.

Theorem 3.17 (cf. [17, Theorem 4.1]) *For any set $F \subseteq \mathrm{O}_A$ we have the equality $\mathrm{Pol}_A \mathrm{Inv}_A F = \mathrm{Loc}_A \langle F\rangle_{\mathrm{O}_A}$.*

Proof Since we have $\mathrm{Pol}_A \mathrm{Inv}_A F = \mathrm{Pol}_A \mathrm{Inv}_A \langle F\rangle_{\mathrm{O}_A}$ by Lemma 2.6, and $\langle F\rangle_{\mathrm{O}_A}$ is a clone, it is surely sufficient to prove $\mathrm{Pol}_A \mathrm{Inv}_A F = \mathrm{Loc}_A F$ for clones $F \in \mathcal{L}_A$. The inclusion $\mathrm{Pol}_A \mathrm{Inv}_A F \supseteq \mathrm{Loc}_A F$ already follows from $F \subseteq \mathrm{Pol}_A \mathrm{Inv}_A F$ and formula (2) of Lemma 2.8, thus we only need to deal with the converse containment. For this let $n \in \mathbb{N}$ and $g \in \mathrm{Pol}_A^{(n)} \mathrm{Inv}_A F$. In order to prove $g \in \mathrm{Loc}_A F$,

[5] For the notation $u \circ \beta$ recall that tuples in A^Y are modelled as maps from Y to A.

we consider any finite $X \subseteq A^n$ and fix a bijection $\beta\colon k \longrightarrow X$ where $k := |X|$. Certainly, we have $\left\{ e_i^{(n)}|_X \circ \beta \;\middle|\; 1 \leq i \leq n \right\} \subseteq \varrho_{X,n,\beta}$. Moreover, it follows $g \rhd \varrho_{X,n,\beta}$ since $\varrho_{X,n,\beta} \in \mathrm{Inv}_A^{(k)} F$ by Lemma 3.16. Therefore, the relation $\varrho_{X,n,\beta}$ contains $g \circ \left(e_1^{(n)}|_X \circ \beta, \ldots, e_n^{(n)}|_X \circ \beta\right) = g \circ \left(e_1^{(n)}, \ldots, e_n^{(n)}\right)|_X \circ \beta = g|_X \circ \beta$. Hence, by the structure of $\varrho_{X,n,\beta}$ there exists an operation $f \in F^{(n)}$ such that $g|_X \circ \beta = f|_X \circ \beta$, and hence $g|_X = f|_X$ due to bijectivity of β. As this argument works for all finite subsets $X \subseteq A^n$, we can conclude that g belongs to $\mathrm{Loc}_A F$. □

Subsequently, we wish to establish a similar characterisation for relational clones, too. Again, the proof is based on a lemma, this time involving the notion of directedness.

Let X be a set and $\leq \subseteq X \times X$ be a binary transitive relation on X. A sequence $(x_i)_{i\in I} \in X^I$ of elements from X is *weakly directed* (w.r.t. $\leq$) if for any $i, j \in I$ there exists an index $k \in I$ such that $x_i, x_j \leq x_k$. The I-indexed system $(x_i)_{i\in I}$ is said to be *directed* if it is weakly directed and the set I is non-empty. This is easily seen to be equivalent to the condition that for any finite subset $I_0 \subseteq I$ there exists some $k \in I$ such that $x_i \leq x_k$ for all $i \in I_0$.

A system $(X_i)_{i\in I}$ of subsets $X_i \subseteq Y$ of some set Y is said to be *directed* if it is directed in the sense above w.r.t. the relation $\subseteq$ on $\mathfrak{P}(Y)$.

Lemma 3.18 (cf. [17, Proposition 1.13]) *For any set $Q \subseteq \mathrm{R}_A$, the local closure $\mathrm{LOC}_A Q$ is closed w.r.t. unions of directed systems of relations of equal arity, i.e. for $m \in \mathbb{N}$, every set I and $(\varrho_i)_{i\in I} \in \left(\mathrm{LOC}_A^{(m)} Q\right)^I$, the union $\bigcup_{i\in I} \varrho_i$ still belongs to $\mathrm{LOC}_A Q$.*

Proof Let $m \in \mathbb{N}$ be an arity and $(\varrho_i)_{i\in I} \in \left(\mathrm{LOC}_A^{(m)} Q\right)^I$ a directed system, put $\varrho := \bigcup_{i\in I} \varrho_i$. If $B \subseteq \varrho$ is a finite subset, then there exists a subset $I_0 \subseteq I$ such that $B \subseteq \bigcup_{i\in I_0} \varrho_i$. As $(\varrho_i)_{i\in I}$ is directed, there exists some $k \in I$ such that $\varrho_i \subseteq \varrho_k$ holds for all $i \in I_0$, thus $B \subseteq \bigcup_{i\in I_0} \varrho_i \subseteq \varrho_k$. Since ϱ_k belongs to $\mathrm{LOC}_A Q$ there exists a relation $\sigma \in Q^{(m)}$ such that $B \subseteq \sigma \subseteq \varrho_k \subseteq \bigcup_{i\in I} \varrho_i = \varrho$. Hence, we have $\varrho \in \mathrm{LOC}_A Q$. □

Another observation that we are going to need is the following.

Lemma 3.19 *Let $\mathbf{A} = \langle A; G\rangle$ be an algebra and $F \in \mathcal{L}_A$ be a clone of operations such that $\langle G\rangle_{\mathrm{O}_A} \subseteq F \subseteq \mathrm{Pol}_A \mathrm{Inv}_A G$. Then for any finite number $n \in \mathbb{N}$ of tuples $b_0, \ldots, b_{n-1} \in A^k$, $k \in \mathbb{N}$, we have*

$$\langle\{b_0, \ldots, b_{n-1}\}\rangle_{\mathbf{A}^k} = \left\{ f \circ (b_0, \ldots, b_{n-1}) \;\middle|\; f \in F^{(n)} \right\}.$$

Proof We have $\mathrm{Inv}_A G = \mathrm{Inv}_A \mathrm{Pol}_A \mathrm{Inv}_A G \subseteq \mathrm{Inv}_A F \subseteq \mathrm{Inv}_A \langle G\rangle_{\mathrm{O}_A} = \mathrm{Inv}_A G$ by the assumption on F and Lemma 2.6. Moreover, from Lemma 3.16 we get that $F^{(n)}$ is a subuniverse of $\langle A; F\rangle^{A^n}$. Combining this with the second part of Lemma 3.15 for $N = n$, $K = k$ and $U = F^{(n)}$, we can infer that $\sigma := \left\{ f \circ (b_0, \ldots, b_{n-1}) \;\middle|\; f \in F^{(n)} \right\}$

is a subuniverse of $\langle A; F\rangle^k$, i.e. that it belongs to $\mathrm{Inv}_A^{(k)} F = \mathrm{Inv}_A^{(k)} G$. Thus it is indeed a subuniverse of $\mathbf{A}^k$, and it contains $b_0, \ldots, b_{n-1}$ as the projections belong to $F^{(n)}$. Therefore, $\langle\{b_0, \ldots, b_{n-1}\}\rangle_{\mathbf{A}^k} \subseteq \sigma$.

Conversely, let $\varrho \subseteq A^k$ be any subuniverse of $\mathbf{A}^k$, i.e. a relation $\varrho \in \mathrm{Inv}_A^{(k)} G$, fulfilling $\{b_0, \ldots, b_{n-1}\} \subseteq \varrho$. Since $\varrho \in \mathrm{Inv}_A G = \mathrm{Inv}_A F$, every $f \in F^{(n)}$ preserves ϱ, hence $f \circ (b_0, \ldots, b_{n-1})$ must belong to ϱ. As this holds true for any such relation ϱ, we have demonstrated $\sigma \subseteq \langle b_0, \ldots, b_{n-1}\rangle_{\mathbf{A}^k}$. □

We have now collected enough facts to prove the characterisation of the other Galois closure, $\mathrm{Inv}_A \mathrm{Pol}_A$.

Theorem 3.20 (cf. [17, Theorem 4.2]) *For any set $Q \subseteq \mathrm{R}_A$ we have the equality $\mathrm{Inv}_A \mathrm{Pol}_A Q = \mathrm{LOC}_A [Q]_{\mathrm{R}_A}$.*

Proof First, Lemma 2.6 yields the inclusion $[Q]_{\mathrm{R}_A} \subseteq \mathrm{Inv}_A \mathrm{Pol}_A Q$, which implies $\mathrm{LOC}_A [Q]_{\mathrm{R}_A} \subseteq \mathrm{Inv}_A \mathrm{Pol}_A Q$ by formula (2) of Lemma 2.8.

For the remaining inclusion let $k \in \mathbb{N}$ be an arity and $\sigma \in \mathrm{Inv}_A^{(k)} \mathrm{Pol}_A Q$, i.e. a subuniverse of $\mathbf{A}^k$ where $\mathbf{A} = \langle A; \mathrm{Pol}_A Q\rangle$. Since taking the least subuniverse generated by a set is an algebraic closure operator, $\sigma = \langle\sigma\rangle_{\mathbf{A}^k} = \bigcup\{\langle B\rangle_{\mathbf{A}^k} \mid B \subseteq \sigma \text{ finite}\}$. Since any union of finitely many finite subsets of σ is again a finite subset of σ, the union on the right-hand side is clearly directed. Hence, in view of Lemma 3.18, in order to prove $\sigma \in \mathrm{LOC}_A [Q]_{\mathrm{R}_A}$, it suffices to demonstrate $\langle B\rangle_{\mathbf{A}^k} \in \mathrm{LOC}_A [Q]_{\mathrm{R}_A}$ for any finite $B \subseteq \sigma$. Indeed, we will even show $\langle\{b_0, \ldots, b_{n-1}\}\rangle_{\mathbf{A}^k} \in [Q]_{\mathrm{R}_A}$ for any $n \in \mathbb{N}$ and $\{b_0, \ldots, b_{n-1}\} \subseteq \sigma$. Let this data be fixed now. For any $m \in \mathbb{N}$ and $\varrho \in Q^{(m)}$ we define the index set $I_{m,\varrho} := \{(m, \varrho, \mathbf{r}) \mid \mathbf{r} = (r_0, \ldots, r_{n-1}) \in \varrho^n\}$ and we let $I := \bigcup_{m\in\mathbb{N}} \bigcup_{\varrho\in Q^{(m)}} I_{m,\varrho}$. Given $(m, \varrho, \mathbf{r}) \in I$ we define mappings

$$\begin{aligned} \alpha_{(m,\varrho,\mathbf{r})}\colon\ m &\longrightarrow A^n & \beta\colon\ k &\longrightarrow A^n \\ i &\longmapsto (r_0(i), \ldots, r_{n-1}(i)), & j &\longmapsto \beta(j) := (b_0(j), \ldots, b_{n-1}(j)). \end{aligned}$$

Certainly, we have $\sigma_{b_0,\ldots,b_{n-1}} := \bigwedge^{\beta}_{\left(\alpha_{(m,\varrho,\mathbf{r})}\right)_{(m,\varrho,\mathbf{r})\in I}} (\varrho)_{(m,\varrho,\mathbf{r})\in I} \in [Q]_{\mathrm{R}_A}$ since all the arguments ϱ belong to Q by the choice of the indexing. Now we simply prove that

$$\begin{aligned}
&\sigma_{b_0,\dots,b_{n-1}}\\
&= \left\{ (f(\beta(0)),\dots,f(\beta(k-1))) \;\middle|\; \begin{array}{l} f \in A^{A^n} \wedge \forall\,(m,\varrho,\mathbf{r}) \in I\colon \\ \quad \big(f(\alpha_{(m,\varrho,\mathbf{r})}(0)),\dots,f(\alpha_{(m,\varrho,\mathbf{r})}(m-1))\big) \in \varrho \end{array} \right\}\\
&= \left\{ f \circ (b_0,\dots,b_{n-1}) \;\middle|\; \begin{array}{l} f \in A^{A^n} \wedge \forall m \in \mathbb{N}\,\forall\,\varrho \in Q^{(m)}\ \big(\forall\,(r_0,\dots,r_{n-1}) \in \varrho^n\colon \\ \hfill f \circ (r_0,\dots,r_{n-1}) \in \varrho\big) \end{array} \right\}\\
&= \left\{ f \circ (b_0,\dots,b_{n-1}) \;\middle|\; f \in A^{A^n} \wedge \forall m \in \mathbb{N}\,\forall\,\varrho \in Q^{(m)}\colon f \rhd \varrho \right\}\\
&= \left\{ f \circ (b_0,\dots,b_{n-1}) \;\middle|\; f \in A^{A^n} \wedge f \in \bigcap_{m\in\mathbb{N}} \operatorname{Pol}_A Q^{(m)} \right\}\\
&= \left\{ f \circ (b_0,\dots,b_{n-1}) \;\middle|\; f \in \operatorname{Pol}^{(n)}_A Q \right\}\\
&= \langle\{b_0,\dots,b_{n-1}\}\rangle_{\mathbf{A}^k}\,,
\end{aligned}$$

where we have applied Lemma 3.19 to get the last equality. Thus, it is indeed $\langle\{b_0,\dots,b_{n-1}\}\rangle_{\mathbf{A}^k} = \sigma_{b_0,\dots,b_{n-1}} \in [Q]_{\mathrm{R}_A}$ as desired. □

Corollary 3.21 *If A is a finite set, then for any $F \subseteq \mathrm{O}_A$ and $Q \subseteq \mathrm{R}_A$ we have $\operatorname{Pol}_A \operatorname{Inv}_A F = \langle F\rangle_{\mathrm{O}_A}$ and $\operatorname{Inv}_A \operatorname{Pol}_A Q = [Q]_{\mathrm{R}_A}$.*

Proof The claim is an immediate consequence of Theorems 3.17 and 3.20, and the remark on page 11 about local closure operators on finite sets. □

We finish with the discussion of a few other results that are related to the Galois connection Pol – Inv or the lattice of all clones. Of course, the focus is again on nullary operations.

First, we mention that the report [17, Theorems 4.1, 4.2] also characterises the fixed points of the Galois connections induced by the operators $\operatorname{Pol} - \operatorname{Inv}^{(s)}$ and $\operatorname{Pol}^{(s)} - \operatorname{Inv}$, where $s \in \mathbb{N}_+$ is a positive integer and the definition of the mentioned operators omits nullary operations and relations. There the resulting closures are so-called *s-local closures* of the least generated (relational) clone, where the s-local closure operators are defined analogously to Definition 2.7, but incorporate an upper bound on the cardinality of the local set B to be interpolated. The characterisation from [17] continues to hold for functions. For sets of relations a small modification is necessary (see parts (b) and (c) of the subsequent proposition).

Since the s-local closure operators are less important than the standard local closures, we have not defined them explicitly in this article. Therefore, we will only mention how the characterisations of [17] look like after including nullary operations. The proofs employ similar arguments as used for Theorems 3.17 and 3.20. Alternatively, one can find proofs of the following result in [1, Lemmas 3.21, 3.22], which translate the results from [17] via the operators $^\circ$ and $'$.

Proposition 3.22 *For $s \in \mathbb{N}_+$, $F \subseteq \mathrm{O}_A$ and $Q \subseteq \mathrm{R}_A$ the following facts are true:*

(a) $\mathrm{Pol}_A \mathrm{Inv}_A^{(s)} F = \text{s-Loc}_A \langle F \rangle_{\mathrm{O}_A}$;

(b) $\mathrm{Inv}_A \mathrm{Pol}_A^{(s)} Q = \left(\text{s-LOC}_A [Q]_{\mathrm{R}_A}\right)^{\circ}$;

(c) $\mathrm{Inv}_A \tilde{F} = \text{s-LOC}_A [Q]_{\mathrm{R}_A}$ *for any set* $\tilde{F} \subseteq \mathrm{O}_A$ *of operations satisfying the condition* $\mathrm{Pol}_A^{(0)} Q \cup \mathrm{Pol}_A^{(s)} Q \subseteq \tilde{F} \subseteq \bigcup_{t=0}^{s} \mathrm{Pol}_A^{(t)} Q =: \mathrm{Pol}_A^{(\leq s)} Q$.

Another important theorem in clone theory that we wish to address is the description of *minimal* clones, i.e. atoms in the clone lattice, on finite base sets A. These are described by types of generating functions of minimal arities, so-called *minimal functions.* The truth of this theorem (see [21]) persists in the general case with nullary operations, since the set of minimal clones remains unchanged. This is due to the fact that any clone F satisfying $F^{(0)} \neq \emptyset$ has a lower cover $F' > \mathrm{J}_A$. Therefore, the "new" clones do not contribute to the set of atoms of the clone lattice.

Similarly, the characterisation of all *maximal* clones (coatoms in the clone lattice, atoms in the lattice of relational clones) on finite carrier sets (see [20,19]) almost stays the same. They are usually described using single generating relations, and these relations can be reused in the general case, too. However, there is one additional minimal relational clone present, namely $[\{\emptyset\}]_{\mathrm{R}_A}$. It corresponds to the maximal clone O'_A, which used to be the top element of the traditional clone lattice. In the general setting it constitutes an additional maximal clone, because it is covered above by O_A.

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