Computing Solutions of Symmetric Hyperbolic Systems of PDE's

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Abstract

We study the computability properties of symmetric hyperbolic systems of PDE's $A\frac{\partial \mathbf{u}}{\partial t} + \sum\limits_{i=1}^{m} B_i \frac{\partial \mathbf{u}}{\partial x_i} = 0$, $A = A^* > 0$, $B_i = B_i^*$, with the initial condition $\mathbf{u}|_{t=0} = \varphi(x_1, \ldots, x_m)$. Such systems first considered by K.O. Friedrichs can be used to describe a wide variety of physical processes. Using the difference equations approach, we prove computability of the operator that sends (for any fixed computable matrices $A, B_1, \ldots, B_m$ satisfying some natural conditions) any initial function $\varphi \in C^{k+1}(Q, \mathbb{R}^n)$, $k \geq 1$, to the unique solution $\mathbf{u} \in C^k(H, \mathbb{R}^n)$, where $Q = [0,1]^m$ and H is the nonempty domain of correctness of the system.

Keywords: Hyperbolic system, PDE, computability, metric space, norm, matrix pencil, difference scheme, stability, finite-dimensional approximation.

1 Introduction

We study the computability properties of symmetric hyperbolic systems

$$\begin{cases} A\frac{\partial \mathbf{u}}{\partial t} + \sum\limits_{i=1}^{m} B_i \frac{\partial \mathbf{u}}{\partial x_i} = 0, \\ \mathbf{u}|_{t=0} = \varphi(x_1, \ldots, x_m), \end{cases} \quad (1)$$

where $A = A^* > 0$ and $B_i = B_i^*$ are constant symmetric $n \times n$-matrices, $t \geq 0$, $x = (x_1, \ldots, x_m) \in Q = [0, 1]^m$, $\varphi : Q \to \mathbb{R}^n$ and $\mathbf{u} : Q \times [0, +\infty) \rightharpoonup \mathbb{R}^n$ is a partial function. In particular, we prove computability of the operator that sends (for any fixed computable $A, B_1, \ldots, B_m$ satisfying some natural conditions) any initial function $\varphi \in C^{k+1}(Q, \mathbb{R}^n)$, $k \geq 1$ to the unique solution $\mathbf{u} \in C^k(H, \mathbb{R}^n)$ of (1), where $H \neq \emptyset$ is the largest set where the unique solution of the initial value problem (1) exists. Note that $\mathbf{u}$ and φ may be considered as n-tuples of functions; sometimes it will be easier not to use the boldface font to denote such vector-functions.

Such systems first considered by K.O. Friedrichs [2] can be used to describe a wide variety of physical processes like those considered in the theories of elasticity, acoustics, electromagnetism etc. Some of these processes can also be described via the wave equation, but in many cases it is more convenient to solve the equivalent first-order systems (1) which actually form a more general class of PDE's. Some motivation for doing so was given in [2] were the well-posedness of (1) (i.e., the existence, uniqueness and continuous dependence of the solution on initial data) was established. The notion of a hyperbolic system (applicable also to broader classes of systems) is due to I.G. Petrovskii [11].

The Friedrichs method (described in more modern terminology in [4]) to prove the existence theorem is based on finite difference approximations, in contrast with the Cauchy-Kovalevskaya method based on approximations by analytic functions and a careful study of infinite series (see e.g. [13]). This feature of the Friedrichs method is interesting from the computational point of view because it yields (as we show here) algorithms for solving PDE's in the exact sense of computable analysis [15] which are based on methods really used in numerical analysis. Thus, in this way one is filling the gap between the exact approach of computable analysis and heuristic algorithms (the correctness of which is not always clear) widely used in numerical analysis. For other methods of proving computability of PDE's solutions see e.g. [12,16,17].

Our proof relies on the well-known classical theorem of the theory of difference schemes (see e.g. [7]) stating that the approximation and stability properties of a difference scheme imply its convergence to the solution of the correspondent differential equation in the grid norm uniformly on steps (see Section 4 for some additional details). We use this theorem to prove a convergence result (Theorem 4.1) in suitable functional norms.

The well-posedness of the initial value problem (1) in the maximal domain H was established for a general case by I.G. Petrovskii. A proof for systems close to ours based on the difference equation method is presented in [4] where sufficient details are given for the one-dimensional case $m = 1$. We work here with a slightly modified difference scheme described in detail in [5] and widely used in numerical methods (cf. e.g. [8,14] and references therein). Among reasons to use the modified scheme here are the possibilities of generalizing our proofs to some other processes (including shock waves) and of constructing analogous higher-order schemes in a hope to make an insight into complexity of the resulting algorithms.

The set H above (which is, under some restrictions on $A, B_1, \ldots, B_m$, a convex

polyhedron being an intersection of $2m + 1$ semispaces of $\mathbb{R}^{m+1}$, details are given below) is the nonempty domain of existence and uniqueness of the solution **u** of (1). It is computable from $A, B_1, \ldots, B_m$ using the eigenvalues of the so called regular matrix pencils $\lambda A - B_i$ [3] for $i = 1, \ldots, m$. The computability of spectrum of a symmetric matrix follows from [19]. Using this result we show that a canonical form for the one-dimensional case $m = 1$ of (1) known as "form in Riemannian invariants" is computable from A, B_1 and cardinalities of the spectra of the matrix A and of the pencil $\lambda A - B_1$. This leads to a stable scheme and to an algorithm for computing the solution of (1).

Although we need here only *eigenvalues* of the mentioned matrix pencils, the procedure of finding them involves the search of *eigenvectors* of the matrix A and another symmetric matrix as well, and these eigenvectors will be also essentially used in our further proofs. We prove the computability of eigenvectors of regular matrix pencils in Section 3, generalizing the result in [19] that the eigenvectors of a symmetric matrix are computable if and only if the cardinality of spectrum of the matrix is known in advance. Moreover, we observe the existence of efficient algorithms for computing the eigenvalues of a regular matrix pencil $\lambda A - B$ provided that we know the cardinality of spectrum of A. Interestingly, in some practically important cases the cardinality may be found from natural physical reasons, e.g. invariance of considered systems under rotations, as in [8,14].

The rest of the paper is organized as follows. In Section 2 we recall some notation, notions and facts. In Section 3 we prove the computability of the canonical form and of the set H. In Section 4 we describe the difference scheme for (1) and some of its properties. In Section 5 we prove the computability of the solution operator for (1). We conclude in Section 6 with a short discussion of possible future work.

2 Notation, Notions and Known Facts

2.1 Matrix Pencils and Description of H

We need some facts about eigenvalues of the matrix pencils $\lambda A - B_i$, $i = 1, \ldots, m$. For the theory of matrix pencils see e.g. [3]. Recall that a *regular matrix pencil* is a matrix $\lambda A - B$ where λ is a real parameter, A and B are symmetric real $n \times n$-matrices, and $A > 0$. The determinant $det(\lambda A - B)$ is called the *characteristic polynomial* of $\lambda A - B$. The roots of the characteristic polynomial are called *eigenvalues* of the pencil. For any eigenvalue λ of $\lambda A - B$ there is a non-zero vector $\mathbf{z} = (z_1, \ldots, z_n)^T$ (called *eigenvector of $\lambda A - B$* related to this eigenvalue) such that $(\lambda A - B)\mathbf{z} = 0$. The following fact is well-known (see e.g. [3], p. 281):

Theorem 2.1 *For any regular matrix pencil $\lambda A - B$ there are eigenvalues $\lambda_1, \ldots, \lambda_n$ and related to them A-orthogonal eigenvectors $\mathbf{z}_1, \ldots, \mathbf{z}_n$ (this means that $\langle A\mathbf{z}_j, \mathbf{z}_k \rangle = \delta_{j,k}$ for all $j, k = 1, \ldots, n$ where $\delta_{j,k}$ is the Kronecker symbol, i.e., $\delta_{j,k} = 0$ for $j \neq k$ and $\delta_{j,k} = 1$ for $j = k$).*

Now let us recall the structure of the domain of correctness $H \subseteq \mathbb{R}^{m+1}$, i.e.,

the maximal set where, for any $k \geq 1$ and $\varphi \in C^{k+1}(Q, \mathbb{R}^n)$ there exists a unique solution $\mathbf{u} \in C^k(H, \mathbb{R}^n)$ of the initial value problem (1).

The set H is known to be (see e.g. [4]) a nonempty intersection of the semispaces

$$t \geq 0, \ x_i - \lambda_{max}^{(i)} t \geq 0, \ x_i - 1 - \lambda_{min}^{(i)} t \leq 0, \ (i = 1, \ldots, m)$$

of $\mathbb{R}^{m+1}$ where, for each $i = 1, \ldots, m$, $\lambda_{max}^{(i)}$ is the maximal and $\lambda_{min}^{(i)}$ is the minimal eigenvalue of the matrix pencil $\lambda A - B_i$. We are especially interested in the case when H is a compact subset of $Q \times [0, +\infty)$ (obviously, a sufficient condition for this to be true is $\lambda_{max}^{(i)} > 0$ and $\lambda_{min}^{(i)} < 0$ for all $i = 1, \ldots, m$; this is often the case for natural physical systems).

2.2   Spaces under Consideration

Let $\mathbb{R}^m$ be the Euclidean space with the usual norm. As it is known, this space is separable, i.e., it has a countable dense subset (e.g., the set of vectors with rational coordinates). We will consider some subspaces of $\mathbb{R}^m$ (as a metric space) with the induced metric, in particular the m-dimensional unitary cube $Q = [0, 1]^m$. Such subspaces are also separable. In particular, a countable dense subset of Q is formed by the binary-rational vectors $(x_1, \ldots, x_m)$ where $x_i = \frac{y_i}{2^k}$ for some $k \geq 0$ and $y_i \in \{0, 1, \ldots, 2^k\}$.

We will work with several functional spaces most of which are subsets of the set $C(\mathbb{R}^m, \mathbb{R}^n) \simeq C(\mathbb{R}^m, \mathbb{R})^n$ of integrable continuous functions $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ equipped with the L_2-norm. In particular, we deal with the space $C(Q, \mathbb{R}^n) \simeq C(Q, \mathbb{R})^n$ (resp. $C^k(Q, \mathbb{R}^n)$) of continuous (resp. continuously k-time differentiable) functions $\varphi : Q \to \mathbb{R}^n$ equipped with the L_2-norm

$$||\varphi||_{L_2} = \left(\int_Q |\varphi(x)|^2 dx \right)^{\frac{1}{2}}, \ |\varphi(x)|^2 = \langle \varphi, \varphi \rangle = \sum_{i=1}^n \varphi_i^2(x).$$

We will also use the *sup*-norm $||\varphi||_s = sup_{x \in Q} |\varphi(x)|$ on $C(Q, \mathbb{R}^n)$ and the sL_2-norm

$$||u||_{sL_2} = sup_{0 \leq t_0 \leq T} \sqrt{\int_Q |u(x, t_0)|^2 dx}$$

on $C(Q \times [0, T], \mathbb{R}^n)$ where $T > 0$. Whenever we want to emphasize the norm we use notation like $C_{L_2}(Q, \mathbb{R}^n)$, $C_s(Q, \mathbb{R}^n)$ or $C_{sL_2}(Q \times [0, T], \mathbb{R}^n)$. In case when the domain of correctness H is compact and $H \subseteq Q \times [0, T]$, we consider also the space $C_{sL_2}(H, \mathbb{R}^n)$ defined in the same way as the last of the mentioned spaces. The space $C_s(Q, \mathbb{R}^n)$ is known to be complete while the spaces $C_{L_2}(Q, \mathbb{R}^n)$ and $C_{sL_2}(Q \times [0, T], \mathbb{R}^n)$ are not complete.

Note that the space $C(Q, \mathbb{R}^n)$ is separable w.r.t. any of the norms. A countable dense set in $C(Q, \mathbb{R}^n)$ frequently used in dealing with difference equations is formed as follows. Take a rectangular grid on Q with rational coordinates (in fact, the uniform grids G_k with step $\frac{1}{2^k}$ on each coordinate suffice). Relate to any function f_k :

$G_k \to \mathbb{Q}$ on the finite set G_k of grid nodes the continuous extension $\tilde{f}_k : Q \to \mathbb{R}$ of f obtained by piecewise-linear interpolation on each coordinate. Such interpolations known also as multilinear interpolations are the simplest class of splines (see e.g. [18]). Note that the restriction of $\tilde{f}_k$ to any grid cell is a polynomial of degree m. The extensions $\tilde{f}_k$ induce a countable dense set in $C(Q, \mathbb{R}^n)$ with any of the three norms.

For all $\tau > 0$ and integer $k \geq 0, L \geq 1$, let G_k^τ be the grid in $Q \times [0, T]$, $T = L\tau$, with step $h = \frac{1}{2^k}$ on the space coordinates x_i and step τ on the time coordinate t. Just as above, such grids induce a countable dense set in $C(Q \times [0, T], \mathbb{R}^n)$ with any of the three norms.

In the study of difference equations the interaction between the infinite-dimensional space $C(Q, \mathbb{R}^n)$ (with a given norm) and the corresponding finite-dimensional spaces $\mathbb{R}^G$ of grid functions $f : G \to \mathbb{R}$ (with the discrete analog of the given norm) plays a crucial role. The discrete analogs of the norms in $C_s(Q, \mathbb{R}^n)$, $C_{L_2}(Q, \mathbb{R}^n)$ and $C_{sL_2}(Q \times [0, T], \mathbb{R}^n)$ are defined in the natural way. E.g., in the last case the norm of a grid function $f : G_k^\tau \to \mathbb{R}$ is defined by

$$||f||_{sL_2} = max_{0 \leq l\tau \leq T} \left(h^m \sqrt{\sum_{x \in G_k} f^2(x, l\tau)} \right).$$

We also need the Cantor space Σ^ω of infinite words over a finite alphabet Σ containing at least two symbols. This space plays a crucial technical role in the exact definition of computability over metric (and even more general) spaces. It is well-known that Σ^ω is a complete separable metric space.

2.3   Computability Notions

We use the TTE-approach to computability over metric spaces developed in the K. Weihrauch's school (see e.g. [15,16,1] for more details).

Recall that a *computable metric space* is a triple (M, d, ν) where (M, d) is a metric space and $\nu : \omega \to M$ is a numbering of a dense subset $rng(\nu)$ of M such that the set $\{(i, j, q, r) \mid i, j \in \omega, q, r \in \mathbb{Q}, q < d(\nu i, \nu j) < r\}$ is computably enumerable.

The *Cauchy representation* of a computable metric space (M, d, ν) is the partial surjection $\delta_M : \Sigma^\omega \rightharpoonup M$ defined exactly on the elements $p \in \Sigma^\omega$ which code (in a natural way) sequences $\{p_i\}$ of natural numbers such that $d(\nu p_i, \nu p_k) \leq 2^{-k}$ for $i \geq k$ and $\{\nu(p_i)\}_{i \in \omega}$ is convergent and sending any such code p to $\delta_M(p) = lim_i\nu(p_i)$. Following [15], by *Cauchy sequences* we mean in this paper sequences $\{\nu(p_i)\}_{i \in \omega}$ satisfying the condition above. An element $x \in M$ of a computable metric space is called *computable* if $x = \delta_M(p)$ for a computable Cauchy sequence coded by $p \in \Sigma^\omega$

Metric spaces $\mathbb{R}^m$, Σ^ω, $C_s(Q, \mathbb{R}^n)$, $C_{L_2}(Q, \mathbb{R}^n)$ and $C_{sL_2}(Q \times [0, T], \mathbb{R}^n)$ discussed in the previous subsection (w.r.t. the metrics induced by the norms and natural numberings of the dense subsets specified above) are computable. The same applies to spaces $C_s(H, \mathbb{R}^n)$, $C_{L_2}(H, \mathbb{R}^n)$ and $C_{sL_2}(H, \mathbb{R}^n)$ provided that matrices $A, B_1, \ldots, B_m$ are computable (as elements of $\mathbb{R}^{n \times n}$) and $\lambda_{max}^{(i)} > 0$, $\lambda_{min}^{(i)} < 0$ for

all $i = 1, \ldots, m$.

A partial function $f : M \rightharpoonup M_1$ on the elements of computable metric spaces (M, d, ν) and (M_1, d_1, ν_1) is *computable* if there is a computable partial function $\hat{f} : \Sigma^\omega \rightharpoonup \Sigma^\omega$ which represents f w.r.t. the Cauchy representations of M and M_1, i.e., $\delta_{M_1}(\hat{f}(p)) = f(\delta_M(p))$ for each $p \in dom(\delta_M)$. Informally, f is computable if there is an algorithm (realized as a Turing machine sending infinite input words over Σ to infinite output words over Σ) which sends any convergent Cauchy sequence $\{a_i\}$ of elements of $rng(\nu)$ with $lim_i a_i \in domf$ to a convergent Cauchy sequence $\{b_i\}$ of elements of $rng(\nu_1)$ such that $lim_i b_i = f(lim_i a_i)$.

Let G be the grid in Q with step $h = \frac{1}{2^k}$ on each coordinate. From well-known facts of computable analysis [15] it follows that $\varphi \mapsto \varphi|_G$ is a computable operator from $C_s(Q, \mathbb{R}^n)$ to $\mathbb{R}^G$. From well-known properties of the piecewise-linear interpolations (see e.g. [4,18]) it follows that $f \mapsto \tilde{f}$ is a computable operator from $(\mathbb{R}^G)_s$ to $C_{L_2}(Q, \mathbb{R}^n)$ (see also the estimate (11) below).

3   Computing the Canonical Form and Domain

First we show that for any one-dimensional (i.e., for $m = 1$) system (1) of the form

$$A\frac{\partial \mathbf{u}}{\partial t} + B\frac{\partial \mathbf{u}}{\partial x} = 0, \ \mathbf{u} = (u_1, u_2, \ldots, u_n)^T \tag{2}$$

we can compute an equivalent system in the canonical form

$$\frac{\partial \mathbf{v}}{\partial t} + M\frac{\partial \mathbf{v}}{\partial x} = 0, \ \ M = diag\{\mu_1, \mu_2, \ldots, \mu_n\}. \tag{3}$$

Theorem 3.1 *Given as inputs symmetric real matrices A, B with $A > 0$ and the cardinalities of spectra of A and $\lambda A - B$, we can compute a system in the canonical form (3) equivalent to (2).*

Proof. We have to compute the matrix M and the non-degenerate linear transformation T of variables $\mathbf{u} = T\mathbf{v}$. Since matrices A and B are symmetric and $A > 0$, there is a non-degenerate matrix T such that $T^*AT = I_n$ (I_n is the identity n-dimensional matrix) and $T^*BT = diag\{\mu_1, \mu_2, \ldots \mu_n\}$. The matrix T may be constructed in the following way:

a) Applying the orthogonal matrix L, $L^*L = I_n$, formed by the coordinates of the normed eigenvectors of A, transform A to the diagonal form

$$L^*AL = \Lambda = diag\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$$

where $\lambda_j > 0$ are the eigenvalues of A. The unknown functions are then transformed to $\mathbf{u} = L\mathbf{v_1}$ while the matrix B is transformed to $B_1 = L^*BL$.

b) Applying the transform $\mathbf{v_1} = D\mathbf{v_2}$, where $D = \Lambda^{-\frac{1}{2}}$, make the coefficient of $\frac{\partial}{\partial t}$ the identity matrix $D^*\Lambda D = I_n$, $D^*B_1 D = B_2$.

c) Applying the transform $\mathbf{v_2} = K\mathbf{v}$, where K is the orthogonal matrix formed by the coordinates of normed eigenvectors of B_2, make the coefficient of $\dfrac{\partial}{\partial x}$ a diagonal matrix; note that the identity coefficient of $\dfrac{\partial}{\partial t}$ remains unchanged.

Finally, we set $T = LDK$ and $\mathbf{u} = T\mathbf{v}$. In this way, the linear substitution $\mathbf{v} = T^{-1}\mathbf{u}$ transforms the system (2) to the canonical form (3). From the main result of [19] it follows that, given A and the cardinality of spectrum of A, one can compute the eigenvectors of A. Thus, matrices L, Λ, K, B_1, B_2 are computable from A, B and the cardinalities of spectra of A and $\lambda A - B$. □

From well-known facts of computable analysis we immediately obtain

Corollary 3.2 *If A, B are computable symmetric real matrices with $A > 0$ then there exist a computable diagonal matrix M and a computable linear transformation $T, \mathbf{u} = T\mathbf{v}$, such that (3) is equivalent to (2).*

Next we observe that the domain H for the problem (1) is computable from $A, B_1, \ldots, B_m$ (i.e., the vector $(\lambda_{max}^{(1)}, \ldots, \lambda_{max}^{(m)}, \lambda_{min}^{(1)}, \ldots, \lambda_{min}^{(m)})$ from Subsection 2.1 is computable from $A, B_1, \ldots, B_m$).

Since, for each $i = 1, \ldots, m$, $\lambda_{max}^{(i)}$ is the maximal and $\lambda_{min}^{(i)}$ is the minimal eigenvalue of the matrix pencil $\lambda A - B_i$, and maximum and minimum of a vector of reals are computable [15], it suffices to show that the vector $(\lambda_1, \ldots, \lambda_n)$ of eigenvalues of $\lambda A - B$ is computable from A, B. But $(\lambda_1, \ldots, \lambda_n)$ is the vector of roots of the characteristic polynomial, hence it is computable [15].

Again we immediately obtain

Corollary 3.3 *If $A, B_1, \ldots, B_m$ are computable symmetric real matrices with $A > 0$ then $\lambda_{max}^{(1)}, \ldots, \lambda_{max}^{(m)}, \lambda_{min}^{(1)}, \ldots, \lambda_{min}^{(m)}$ are computable reals.*

In case when, along with matrices $A, B_1, \ldots, B_m$, the cardinality of spectrum of A is given we may use the algorithm from the proof of Theorem 3.1 to compute $\lambda_{max}^{(1)}, \ldots, \lambda_{max}^{(m)}, \lambda_{min}^{(1)}, \ldots, \lambda_{min}^{(m)}$ in a more efficient way than just finding the roots of the characteristic polynomials and computing maximum and minimum. Indeed, again it suffices to show that the vector $(\lambda_1, \ldots, \lambda_n)$ of eigenvalues of $\lambda A - B$ is computable from A, B and the cardinality of spectrum of A.

Let L be the orthogonal matrix formed by coordinates of the normed eigenvectors of A. By [19], L is computable. Then $L^*L = I_n$, and the matrix $L^*AL = \Lambda = diag\{\lambda_1, \ldots, \lambda_n\}$ is diagonal, where $\lambda_j > 0$ are the eigenvalues of A. By the proof of Theorem 2.1 in [3], the eigenvalues of $\lambda A - B$ coincide with those of the symmetric matrix $B_2 = D^*B_1D$ where $B_1 = L^*BL$ and $D = \Lambda^{-\frac{1}{2}}$. Therefore, they are computable by well-known efficient algorithms of linear algebra (see e.g. [6]).

4 Finite-Dimensional Approximation

In this section we shortly describe the difference scheme for the initial value problem (1) and formulate some of its properties most of which are known [5,7]. Additional

details will be given in the journal version of the paper.

4.1   Description of the Difference Scheme

The difference scheme may be chosen in various ways. Our scheme taken from [5] is a little more complicated than the scheme used in [4] but it has some useful feature mentioned in the introduction. We describe it in few stages.

1. First we describe some discretization details. Simplifying notation, we stick to the 2-dimensional case $x_1 = x$, $x_2 = y$, $B_1 = B$, $B_2 = C$, i.e. $m = 2$ and $n \geq 2$. For $m \geq 3$ the difference scheme is obtained in the same way as for $m = 2$ but the transfer from $m = 1$ to $m = 2$ is nontrivial.

Consider the uniform rectangular grid G on $Q = [0,1]^2$ defined by the family of lines $\{x = x_j\}, \{y = y_k\}$ where $1 \leq j, k \leq 2^N$ for some natural number N (because of many indices we slightly modified here the grid notation from Subsection 2.2). Let $h = x_j - x_{j-1} = y_k - y_{k-1} = 1/2^N$ be the step of the grid. Relate to any function $g \in \{u_1, \ldots, u_n\}$ and any fixed time point $t = l\tau$, $l \in \mathbb{N}$, the vector of dimension 2^{2N} with the components

$$g_{j-\frac{1}{2},k-\frac{1}{2}} = g\left(\frac{j - \frac{1}{2}}{2^N}, \frac{k - \frac{1}{2}}{2^N}, t\right)$$

equal to the values of g in the centers of grid cells.

Note that, strictly speaking, we work with modifications of the grids G_k in Subsection 2.2 when the centers of grid cells are taken as nodes of the modified grids.

2. Consider the following two auxiliary one-dimensional systems with parameters obtained by fixing any of the variables x, y:

$$A\frac{\partial \mathbf{u}}{\partial t} + B\frac{\partial \mathbf{u}}{\partial x} = 0, \; A\frac{\partial \mathbf{u}}{\partial t} + C\frac{\partial \mathbf{u}}{\partial y} = 0$$

where $\mathbf{u} = (u_1, u_2, \ldots, u_n)^T$. Transform the systems into the canonical forms

$$\frac{\partial \mathbf{v}_x}{\partial t} + M_x \frac{\partial \mathbf{v}_x}{\partial x} = 0, \;\; \frac{\partial \mathbf{v}_y}{\partial t} + M_y \frac{\partial \mathbf{v}_y}{\partial y} = 0. \tag{4}$$

as described in Section 3.

3. Any of the systems (4) in the canonical form consists of n independent equations of the form

$$\frac{\partial w}{\partial t} + \mu \frac{\partial w}{\partial x} = 0 \tag{5}$$

where $w = w(x,t)$ is a scalar function and $\mu \in \mathbb{R}$. Consider for the equation (5) the following difference scheme. The function $w(t_0, x)$ already computed on the time level $t = t_0$ (initially $t = 0$) is substituted by the piecewise-constant function with the values $w_{j-\frac{1}{2}}$ within the corresponding grid cell $x_{j-1} < x \leq x_j$. The "large

values" defined at the boundaries of the grid cells are computed as follows:

$$\mathcal{W}_j = \begin{cases} w_{j-\frac{1}{2}}, & \mu > 0, \\ w_{j+\frac{1}{2}}, & \mu < 0. \end{cases} \tag{6}$$

Values on the next time level $t = t_0 + \tau$ (τ is a time step depending on h as specified below) are then computed as

$$w^{j-\frac{1}{2}} = w_{j-\frac{1}{2}} - \mu \frac{\tau}{h}(\mathcal{W}_j - \mathcal{W}_{j-1}).$$

The grid solutions computed in this way approximate solutions of the differential equation (5) with the first order of accuracy on h (in the usual sense of numerical analysis, see e.g. [7]).

Recall that a difference scheme is *stable* if the corresponding difference operators L_h^τ (that send the grid function $\{w_{j-\frac{1}{2}}\}_{j=1}^{2^N}$ to the grid function $\{w^{j-\frac{1}{2}}\}_{j=1}^{2^N}$) are bounded uniformly on h. The investigation of stability of difference schemes usually relies on Fourier analysis (here we only mention without proofs some known related facts [7] applied to our scheme). The necessary and sufficient condition for our scheme to be stable is $|\mu|\frac{\tau}{h} \le 1$.

4. Taking the scheme above for each equation of the systems (4), we obtain for them schemes of the following form:

$$\frac{\mathbf{v_x}^{j-\frac{1}{2}} - \mathbf{v_x}_{j-\frac{1}{2}}}{\tau} + M\frac{(\mathcal{V}_x)_j - (\mathcal{V}_x)_{j-1}}{h} = 0$$

where lower indices mean step $l\tau$, upper indices mean step $(l+1)\tau$ and $\mathcal{V}_j$ is the vector consisting of the "large values" given by one-dimensional formulas (6).

As shown in [5], making the inverse transformation we can write the following scheme that approximates the system (1) with the first order of accuracy:

$$A\frac{\mathbf{u}^{j-\frac{1}{2},k-\frac{1}{2}} - \mathbf{u}_{j-\frac{1}{2},k-\frac{1}{2}}}{\tau} + B\frac{\mathcal{U}_{j+1,k-\frac{1}{2}} - \mathcal{U}_{j,k-\frac{1}{2}}}{h} + C\frac{\mathcal{U}_{j-\frac{1}{2},k+1} - \mathcal{U}_{j-\frac{1}{2},k}}{h} = 0 \tag{7}$$

where

$$\mathcal{U}_{j-\frac{1}{2},k} = T_x \mathcal{V}_{j-\frac{1}{2},k}, \; \mathcal{U}_{j,k-\frac{1}{2}} = T_y \mathcal{V}_{j,k-\frac{1}{2}}.$$

The stability condition looks now as

$$\tau \left(\frac{1}{\tau_x} + \frac{1}{\tau_y} \right) \le 1, \tag{8}$$

where $\tau_x = max_i\{\mu_i(\mu A - B)\}h$ and $\tau_y = max_i\{\mu_i(\mu A - C)\}h$ are the maximal time steps guaranteeing the stability of the corresponding one-dimensional schemes.

It is easy to check that $\left\{\mathbf{u}_{j-\frac{1}{2},k-\frac{1}{2}}\right\} \mapsto \left\{\mathbf{u}^{j-\frac{1}{2},k-\frac{1}{2}}\right\}$ is a linear operator on the corresponding spaces of grid functions.

4.2   Some Estimates

Here we assume that the domain H is compact, hence $H \subseteq Q \times [0, T]$ for some $T > 0$. Given (8), with the aid of Fourier transform, it can be derived that

$$||\{\mathbf{u}^{j-\frac{1}{2},k-\frac{1}{2}}\}||_A \leq ||\{\mathbf{u}_{j-\frac{1}{2},k-\frac{1}{2}}\}||_A, \tag{9}$$

where

$$||\{\mathbf{u}_{j-\frac{1}{2},k-\frac{1}{2}}\}||_A = \sqrt{h^2 \sum_{j,k} \left\langle A\mathbf{u}_{j-\frac{1}{2},k-\frac{1}{2}}, \mathbf{u}_{k-\frac{1}{2},j-\frac{1}{2}} \right\rangle}.$$

Using an equivalent norm, fixing the initial values at the right hand side of (9) and taking the maximal value of the left hand side w.r.t. t such that $\left(\frac{j-\frac{1}{2}}{2^N}, \frac{k-\frac{1}{2}}{2^N}, t\right) \in H$ for some $j, k \in \{1, 2, \ldots, 2^N\}$ we obtain

$$max_{0 \leq l\tau \leq T} \left(h^2 \sum_{j,k} \mathbf{u}_{j-\frac{1}{2},k-\frac{1}{2}}^2\right) \leq c \cdot \left(h^2 \sum_{j,k} \varphi_{j-\frac{1}{2},k-\frac{1}{2}}^2\right) \tag{10}$$

for some constant c depending only on A, B, C (where $\mathbf{u}^2 = \langle \mathbf{u}, \mathbf{u} \rangle$).

Finally, we derive an estimate for the piecewise-linear interpolation $\widetilde{\varphi|_G}$ of the initial function on the grid G and the piecewise-linear interpolation $\tilde{u}$ of of the grid function u on the grid on $Q \times [0, T]$ obtained from G and the time step τ. It is known [5] and easy to see by a direct computation that

$$max_{0 \leq l\tau \leq T} \int_{H \cap \{t=l\tau\}} |\tilde{u}(x, y, t)|^2 dx dy \leq max_{0 \leq l\tau \leq T} \left(h^2 \sum_{j,k} \mathbf{u}_{j-\frac{1}{2},k-\frac{1}{2}}^2\right). \tag{11}$$

Obviously,

$$h^2 \sum_{j,k} \varphi_{j-\frac{1}{2},k-\frac{1}{2}}^2 \leq h^2 \frac{1}{h^2} max_{j,k} \varphi_{j-\frac{1}{2},k-\frac{1}{2}}^2 \leq sup_{(x,y) \in Q} \widetilde{\varphi|_G}^2 (x, y).$$

The last three estimates imply that for a constant c we have

$$||\tilde{u}|_H||_{sL_2} = max_{0 \leq l\tau \leq T} \sqrt{\int_{H \cap \{t=l\tau\}} |\tilde{u}(x, y, t)|^2 dx dy} \leq$$

$$\leq c \cdot sup_{(x,y) \in Q} \widetilde{\varphi|_G}(x, y) = c \cdot ||\widetilde{\varphi|_G}||_s. \tag{12}$$

4.3   Convergence of the Difference Scheme

Let $\varphi \in C^{k+1}(Q, \mathbb{R}^n)$ for some $k \geq 1$ and let $A, B_1, \ldots, B_m$ be symmetric real matrices such that $A > 0$ and $\lambda_{max}^{(i)} > 0$, $\lambda_{min}^{(i)} < 0$ for all $i = 1, \ldots, m$. By Subsection 2.1, H is compact and there is a unique solution $\mathbf{u} \in C^k(H, \mathbb{R}^n)$ of (1). Let $T > 0$ be the smallest number with $H \subseteq Q \times [0, T]$. For any $k \geq 0$, let $\varphi|_{G_k}$

be the restriction of φ to the grid G_k in Q with step $\frac{1}{2^k}$ (recall that φ is actually an n-tuple of functions, hence the restrictions are componentwise). Note that G_k here denote the modified grid in Subsection 4.1 rather than the grid in Subsection 2.2. For any $k \geq 0$, let τ_k be the time step such that the scheme (7) is stable (τ_k is any number satisfying (8) for $h = \frac{1}{2^k}$ and dividing T). For any $k \geq 0$, let u_k be the solution of the difference equation (7) on the grid G_k^τ in $Q \times [0, T]$ obtained from G_k and the time step τ_k. Recall from Subsection 2.2 that $\tilde{f}$ denotes the piecewise-linear interpolation of a grid function f.

Theorem 4.1 *In notation and assumptions of the previous paragraph there is a constant c depending only on $A, B_1, \ldots, B_m$ such that for all $k \geq 0$ we have $||u_k - u|_{G_k^\tau}||_{sL_2} \leq c \cdot \frac{1}{2^k}$ and $||\tilde{u}_k - u||_{sL_2} \leq c \cdot \frac{1}{2^k}$, where u is the solution of (1).*

Proof. The first estimate is a particular case of a well-known classical theorem of the theory of difference schemes (see e.g. [7]) stating that the approximation and stability properties of a difference scheme imply its convergence to the solution of the correspondent differential equation in the grid norm uniformly on steps.

For the second estimate, we have $||\tilde{u}_k - u||_{sL_2} \leq ||\tilde{u}_k - \widetilde{u|_{G_k^\tau}}||_{sL_2} + ||\widetilde{u|_{G_k^\tau}} - u||_{sL_2}$. By a well-known property of the piecewise-linear interpolations, $||\widetilde{u|_{G_k^\tau}} - u||_s \leq c \cdot \frac{1}{2^k}$ for some constant c, hence also $||\widetilde{u|_{G_k^\tau}} - u||_{sL_2} \leq c \cdot \frac{1}{2^k}$ for some constant c. Since the operator of piecewise-linear interpolation is linear, from (11) we get $||\tilde{u}_k - \widetilde{u|_{G_k^\tau}}||_{sL_2} \leq ||u_k - u|_{G_k^\tau}||_{sL_2}$. Taking into account the first estimate, we obtain $||\tilde{u}_k - \widetilde{u|_{G_k^\tau}}||_{sL_2} \leq c \cdot \frac{1}{2^k}$ for some constant c. This implies second estimate. $\qquad\square$

5 Computing the Solution Operator

We are ready to prove the main result of this paper:

Theorem 5.1 *Let $k \geq 1$, let $A, B_1, \ldots, B_m$ be computable symmetric real matrices with $A > 0$, and let $\lambda_{max}^{(i)} > 0$, $\lambda_{min}^{(i)} < 0$ for all $i = 1, \ldots, m$. Then the operator $\varphi \mapsto \mathbf{u}$ sending any $\varphi \in C^{k+1}(Q, \mathbb{R}^n)$ to the unique solution $\mathbf{u} \in C^k(H, \mathbb{R}^n)$ of (1) is a computable partial function from $C_s(Q, \mathbb{R}^n)$ to $C_{sL_2}(H, \mathbb{R}^n)$.*

Proof. (Sketch) We use notation of Theorem 4.1. First note that, by Corollary 3.3 and Subsection 4.1, T is a computable real. Simplifying technical details, assume T to be a rational number (in the general case, take a computable Cauchy sequence of rationals converging to T). Choose a computable sequence $\{\tau_k\}$ of rational numbers for which the estimates of Theorem 4.1 hold.

According to Subsections 2.3 and 2.2, we have to find an algorithm that, given a sequence $\{f_k\}$ of grid functions $f_k : G_{i_k} \to \mathbb{Q}^n$ such that $\{\tilde{f}_k\}$ is a Cauchy sequence converging in $C_s(Q, \mathbb{R}^n)$ to φ, computes a sequence $\{v_k\}$ of grid functions $v_k : G_{i_k}^\tau \to \mathbb{Q}^n$ such that $\{\tilde{v}_k|_H\}$ is a Cauchy sequence converging in $C_{sL_2}(H, \mathbb{R}^n)$ to u. W.l.o.g. we may assume that the sequence $\{i_k\}$ is increasing (otherwise, choose a suitable subsequence of $\{f_k\}$).

Let v_k be constructed from f_k by the algorithm of the difference equation in

Subsection 4.1. It suffices to show that for some constant c (depending only on $A, B_1, \ldots, B_m$) we have $||\tilde{v}_k|_H - u||_{sL_2} \leq c \cdot \frac{1}{2^k}$ for all k. We have

$$||\tilde{v}_k|_H - u||_{sL_2} \leq ||\tilde{v}_k|_H - \tilde{u}_k|_H||_{sL_2} + ||\tilde{u}_k|_H - u||_{sL_2}. \tag{13}$$

By Theorem 4.1, $||\tilde{u}_k|_H - u||_{sL_2} \leq c \cdot \frac{1}{2^{i_k}}$ for a constant c, hence it remains to get the same estimate for the first summand in (13). Since $||\tilde{f}_k - \varphi||_s \leq c \cdot \frac{1}{2^{i_k}}$ for a constant c, $||f_k - \varphi|_{G_k}||_s \leq c \cdot \frac{1}{2^{i_k}}$ for the same c. Since the scheme operator is linear by the remark at the end of Subsection 4.1, from (10) we obtain $||u_k|_H - v_k|_H||_{sL_2} \leq c \cdot ||f_k - \varphi|_{G_k}||_s$ for a constant c. Since the operator of piecewise-linear interpolation is linear, from (12) we get the estimate for the first summand and hence the statement of the theorem. $\qquad \square$

As usual, we immediately obtain

Corollary 5.2 *Let $k \geq 1$, let $A, B_1, \ldots, B_m$ be computable symmetric real matrices with $A > 0$, let $\lambda_{max}^{(i)} > 0$ and $\lambda_{min}^{(i)} < 0$ for all $i = 1, \ldots, m$, and let $\varphi \in C^{k+1}(Q, \mathbb{R}^n)$ be a computable element of $C_s(Q, \mathbb{R}^n)$. Then the unique solution $\mathbf{u} \in C^k(H, \mathbb{R}^n)$ of (1) is a computable element of $C_{sL_2}(H, \mathbb{R}^n)$.*

6   Concluding Remarks

There are many open questions and directions of possible future research related to this paper. We guess that some generalizations and strengthenings of our results hold (e.g., for more complicated sets in place of Q, for the case when matrices $A, B_1, \ldots, B_m$ are also varied, for Sobolev spaces (which are very popular in the study of equations similar to ours [10]) in place of the spaces discussed above, for other types of differential and difference equations, and so on. It would be also interesting to investigate schemes with higher order of accuracy and algoritms based on the popular finite element method.

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