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Simulations as Homotopies

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Abstract

We exhibit a model structure on **2-Cat**, obtained by transfer from **sSet** across the adjunction $C_2 \circ Sd^2 \dashv Ex^2 \circ N_2$. A certain class of homotopies in this model structure turns out to be in 1-to-1 correspondence with strong simulations among labeled transitions systems, formalising the geometric intuition of simulations as deformations. The correspondence still holds in the cubical setting, characterising simulations of higher-dimensional transition systems (HDTS).

Keywords: Algebraic topology, model categories, 2-categories, cubical and simplicial sets, labeled transition systems, simulation, higher-dimensional transition systems

1 Introduction

Thorough understanding of computational agents with coordinated activities overlapping in time, also known as *concurrent processes*, is crucial in today's world of number-crunching supercomputers and critical systems. A popular approach to concurrent processes is to consider them in terms of process calculi, which are rewriting systems equipped with algebraic rules subject to

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inductive/coinductive reasoning (c.f. [19]). This approach, essentially an attempt to generalize the λ-calculus, did clarify an impressive number of issues, though from a very specific viewpoint, tightly bound to a formal syntax. It is therefore desirable to develop a more general approach.

This work investigates the potential of algebro-topological techniques in classical concurrency theory. Specifically, we employ categorical homotopy theory à la Quillen based on the notion of *model category* (c.f. [22] [15][14]). In order to fix the ideas, we first focus on *labeled transition systems* (cf. [20]). The latter have been extensively studied from a categorical angle (cf. for instance [16]), so which category and which model structure (cf. [22]) for *homotopies of labeled transition systems*? The present account is based on our recent discovery of a model structure on the category **2-Cat** (c.f. [17]). An notion of homotopy with respect to this model structure agrees on relevant instances with a specific yet less widespread characterisation of simulation (cf. [12]).

Once the 1-dimensional case laid out, we treat the general case i.e. higher-dimensional transition systems a.k.a. HDTS's. Intuitively, the latter are groups of computational agents exhibiting varying degrees of coordination. Let $cSet$ be the category of cubical sets. The category of HDTS's is a certain subcategory of the slice category $cSet/L$ for a suitable $L \in cSet$. Consider for instance the HDTS

$$\begin{tikzcd}
 & a \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] \arrow[dd, Rightarrow, "\alpha\beta"] & \\
a' \arrow[dr, "\beta"'] & & a'' \arrow[dl, "\alpha"] \\
 & b &
\end{tikzcd}$$

consisting of 2 agents with uncoordinated parallel activity $\Rightarrow$ (the 2-cube) leading from *state* a to state b, *labeled* by $\alpha\beta$ and with suitably labeled *interleavings* (the 1-cubes), the whole being coherent by virtue of the face relations. The labels, taken from L, indicate the nature of the activities. Observe that if the agents were coordinated the 2-cube $\Rightarrow$ would be missing, i.e. the standard cubical homology of this automaton would be non-trivial in dimension 1. A simulation of HDTS's can be seen as a lifting in the category of cubical 2-categories, for which we have established an appropriate model category structure. The latter is the cubical version of the above-mentioned 1-dimensional case.

The paper is organized as follows. Section 2 provides some background material, in particular on barycentric subdivision as well as on 2-categories and their (2-categorical) nerves. Section 3 introduces labeled transition systems and their simulations. The link between the latter and oplax transformations as noticed by Hermida is explained. Section 4 goes into the heart of the matter. A novel model structure on **2-Cat** is described. A simulation is then characterized as a right homotopy. The reader unfamiliar with the lore of model categories may wish at this point to consult say the first two chapters of [15] in order to get acquainted with the jargon. Section 5 generalizes the setup to cubical sets and cubical 2-categories. Section 6 draws some conclusive remarks.

2 2-Categories

We introduce some facts and terminology related to the topic of 2-categories, in a rather lengthy manner drawn from [25]. It is not a standard way to present 2-categories, yet it allows a better understanding, in particular of the process of 2-categorification to be introduced in section 4.

2.1 2-graphs

Definition 2.1 *Let $\mathbb{A}$ be a category. A preglobular object A in $\mathbb{A}$ is a $\mathbb{N}$-indexed sequence*

$$\cdots \quad A_i \underset{cod_{i-1}}{\overset{dom_{i-1}}{\rightrightarrows}} A_{i-1} \quad \cdots$$

of objects and morphisms subject to the identities

$$\begin{aligned} dom_i \circ dom_{i+1} &= dom_i \circ cod_{i+1} \\ cod_i \circ dom_{i+1} &= cod_i \circ cod_{i+1} \end{aligned}$$

A is n-truncated if $i < n$. An n-graph is a n-truncated preglobular set.

Remark 2.2 *Since an n-graph is just a presheaf, **n-Grph** is a topos for each $n \in \mathbb{N}$. In particular, **n-Grph** is complete and cocomplete.*

Definition 2.3

(i) A graph is a 1-graph with $dom \overset{def}{=} dom_0$ and $cod \overset{def}{=} cod_0$. Let H be a graph and $a, b \in H_0$, then

$$H\,(a,b) \overset{def}{=} \{u \in H_1 \mid dom\,(u) = a \wedge cod\,(u) = b\}$$

(ii) Let G be a 2-graph. As in the case of graphs, the elements of G_0 are called vertices *or* objects *and those of G_1* arrows*,* edges *or* 1-morphisms*.*

The elements of G_2 *are called* 2-cells *or* 2-morphisms. *G's underlying graph* $|G|$ *is given by its 1-truncation* $G_1 \underset{cod}{\overset{dom}{\rightrightarrows}} G_0$.

(iii) Given $x, y \in G_0$, $G(x, y)$*is the graph with*

$$G(x,y)_0 \overset{def}{=} \{f \in G_1 \mid dom_0(f) = x \wedge cod_0(f) = y\}$$
$$G(x,y)_1 \overset{def}{=} \{\alpha \in G_2 \mid dom_1(\alpha), cod_1(\alpha) \in G(x,y)_0\}$$

and with $dom_{x,y}, cod_{x,y} : G(x,y)_1 \to G(x,y)_1$ *given by*

$$dom_{x,y}(\alpha) \overset{def}{=} dom_1(\alpha)$$
$$cod_{x,y}(\alpha) \overset{def}{=} cod_1(\alpha)$$

Properties and concepts defined with respect to $G(x, y)$ (and of its more structured counterparts to be introduced below) are called *local*. For instance, a morphism of graphs $h : G \to H$ is *locally injective if* $h_1 |_{G(x,y)}$ is an injective function for each $x, y \in G_0$.

2.2 Derivation schemes and sesquicategories.

Definition 2.4 *A derivation scheme is a 2-graph D such that the underlying graph* $|D|$ *is a category. The composition in* $|D|$ *is denoted* $\circ$ *and written infix in the evaluation order. Morphisms of derivation schemes are morphisms of 2-graphs that are functors on the underlying categories.*

Proposition 2.5 *Derivation schemes and their morphisms form the category* ***Der***. *There is an adjunction*

$$\begin{tikzcd} \textbf{\textit{Der}} \arrow[r, bend right, "U_{der}"'] \arrow[r, phantom, "\perp"] & \textbf{\textit{2-Grph}} \arrow[l, bend right, "F_{der}"'] \end{tikzcd}$$

Let G be a 2-graph. The free derivation scheme $F_{der}G$ is given by

$$|F_{der}(G)| = \mathcal{F}_{\mathbf{Cat}}(|G|)$$

Let $x, y \in G_0$. A situation involving an $\alpha \in G(x,y)_1$ such that $dom(\alpha) = f$ and $cod(\alpha) = g$ is customarily drawn as

$$\begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & y \end{tikzcd}$$

Definition 2.6 *A sesquicategory* $\mathbb{S}$ *is a derivation scheme such that* $\mathbb{S}(x, y)$ *is a category for all* $x, y \in \mathbb{S}_0$. *The composition in* $\mathbb{S}(x, y)$ *is denoted* $\bullet$ *and* *is written infix in the evaluation order. For each* $x', x, y \in \mathbb{S}_0$ *there is the operation*

$$W_{left} : \mathbb{S}\left(x', x\right)_0 \times \mathbb{S}\left(x, y\right)_1 \to \mathbb{S}\left(x', y\right)_1$$

and for each $x, y, y' \in \mathbb{S}_0$ *there is the operation*

$$W_{right} : \mathbb{S}\left(x, y\right)_1 \times \mathbb{S}\left(y, y'\right)_0 \to \mathbb{S}\left(x, y'\right)_1$$

Both operations are called whiskering *and are denoted* $\circ$ *by abuse of notation.* W_{left} *is subject to the identities*

(i) *given*

$$\begin{tikzcd} x \arrow[r, "id"] & x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & y \end{tikzcd}$$

the equation

$$\alpha \circ id_x = \alpha$$

holds;

(ii) *given*

$$\begin{tikzcd} x' \arrow[r, "f"] & x \arrow[r, bend left, "u"] \arrow[r, bend right, "u"'] \arrow[r, phantom, "\Downarrow id"] & y \end{tikzcd}$$

the equation

$$id_u \circ f = id_{u \circ f}$$

holds;

(iii) *given*

$$\begin{tikzcd} x'' \arrow[r, "f'"] & x' \arrow[r, "f"] & x \arrow[r, bend left, "u"] \arrow[r, bend right, "u"'] \arrow[r, phantom, "\Downarrow\alpha"] & y \end{tikzcd}$$

the equation

$$\alpha \circ (f \circ f') = (\alpha \circ f) \circ f'$$

holds;

(iv) *given*

$$\begin{tikzcd} x' \arrow[r, "f"] & x \arrow[r, bend left=60, "u"] \arrow[r, "v" description] \arrow[r, bend right=60, "w"'] \arrow[r, phantom, shift left=2, "\Downarrow\alpha"] \arrow[r, phantom, shift right=2, "\Downarrow\beta"] & y \end{tikzcd}$$

the equation

$$(\beta \bullet \alpha) \circ f = (\beta \circ f) \bullet (\alpha \circ f)$$

holds;

(v) the rules governing W_{right} are defined symmetrically;

(vi) given

$$\begin{tikzcd} x \arrow[r, "f"] & x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & y \arrow[r, "g"] & z \end{tikzcd}$$

the equation $g \circ (\alpha \circ f) = (g \circ \alpha) \circ f$ holds.

Morphisms of sesquicategories, called sesquifunctors, *are morphisms of the underlying derivation schemes which are locally functors and which preserve whiskering.*

The equations of a sesquicategory guarantee in particular that there is no harm to write the 2-cells as strings like

$$g_m \circ \cdots \circ g_1 \alpha f_n \cdots f_1$$

Proposition 2.7 *Sesquicategories and sesquifunctors organize in the category* ***Sesqu****. There is an adjunction*

$$\begin{tikzcd} \textbf{\textit{Sesqu}} \arrow[r, bend right, "U_{sesqu}"'] \arrow[r, phantom, "\perp"] & \textbf{\textit{Der}} \arrow[l, bend right, "F_{sesqu}"'] \end{tikzcd}$$

A free sesquicategory $\mathcal{F}\mathbb{D}$ over a derivation scheme $\mathbb{D}$ is given by formally adding all the whiskering composites and all the vertical composites in a consistent way. That is, the 2-cells $\mathcal{F}\mathbb{D}_2$ along with their whiskering and vertical composition can be presented by the set of generators $\mathbb{D}_2$, the rules

$$\frac{\alpha \in \mathbb{D}\,(x, y) \quad f : x' \to x}{\alpha \circ f \in \mathcal{F}\mathbb{D}\,(x', y)} \qquad \frac{\alpha \in \mathbb{D}\,(x, y) \quad g : y \to y'}{g \circ \alpha \in \mathcal{F}\mathbb{D}\,(x, y')}$$

$$\frac{\iota, \kappa \in \mathcal{F}\mathbb{D}\,(x, y) \quad cod_{xy}\,(\iota) = dom_{xy}\,(\kappa)}{\kappa \bullet \iota \in \mathcal{F}\mathbb{D}\,(x, y)}$$

and the equations of definition 2.6.

Recall next that a congruence $\sim$ on a category $\mathbb{A}$ is a family

$$\{\sim_{X,Y} \subseteq \mathbb{A}\,(X, Y) \times \mathbb{A}\,(X, Y)\}_{X,Y \in \mathbb{A}_0}$$

of equivalence relations such that

$$f \sim g \Rightarrow v \circ f \sim v \circ g$$

and

$$f \sim g \Rightarrow f \circ u \sim g \circ u$$

Definition 2.8 *Let $\mathbb{S}$ be a sesquicategory. A* sesquicongruence *on $\mathbb{S}$ is a local congruence which is preserved by whiskering.*

Observe that any congruence on $\mathbb{S}$'s underlying category is a sesquicongruence.

Proposition 2.9 *An arbitrary intersection of sesquicongruences is again a sesquicongruence. Any family of local relations $\sim_{x,y}$ on a sesquicategory $\mathbb{S}$ generate a sequicongruence $\sim$. The quotient $\mathbb{S}/\sim$ is again a sesquicategory.*

2.3 2-categories.

Definition 2.10 *Let $\mathbb{S}$ be a sesquicategory and $x, y, z \in \mathbb{S}_0$. The latter satisfy the* interchange law *if any diagram of the form*

$$\begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & y \arrow[r, bend left, "f'"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow\alpha'"] & z \end{tikzcd}$$

verifies the equation

$$(g' \circ \alpha) \bullet (\alpha' \circ f) = (f' \circ \alpha) \bullet (\alpha' \circ g) \qquad (*)$$

*A 2-category is a sesquicategory in which the interchange law holds for every triple of objects. A 2-functor is a sesquifunctor among 2-categories. 2-categories and 2-functors are bundled in the category **2-Cat**.*

A 2-category $\mathcal{A}$ admits in particular a "horizontal" composition of 2-cells where $\alpha' \circ \alpha$ is given by either side of $(*)$, giving rise to a family of functors

$$\_ \circ \_ : \mathcal{A}(y, z) \times \mathcal{A}(x, y) \to \mathcal{A}(x, z)$$

indexed by triples $x, y, z \in \mathcal{A}_0$. This is the way 2-categories are usually introduced in the literature (c.f. [3]). Any category is a 2-category with identities as only 2-cells.

Proposition 2.11 *There is an adjunction*

$$\begin{tikzcd} \textbf{\textit{2-Cat}} \arrow[r, bend right, "U_{\textbf{\textit{2-Cat}}}"'] \arrow[r, phantom, "\perp"] & \textbf{\textit{Sesqu}} \arrow[l, bend right, "F_{\textbf{\textit{2-Cat}}}"'] \end{tikzcd}$$

It is easy to see that constructing the free 2-category on a sesquicategory amounts to quotienting the latter by the sesquicongruence generated by the equations enforcing the interchange law for all triples of objects. We thus have the series of adjunctions

$$\begin{tikzcd} \mathbf{2\text{-}Cat} \arrow[r, bend right, "U_{\mathbf{2\text{-}Cat}}"'] \arrow[r, phantom, "\perp"] & \mathbf{Sesqu} \arrow[l, bend right, "F_{\mathbf{2\text{-}Cat}}"'] \arrow[r, bend right, "U_{sesqu}"'] \arrow[r, phantom, "\perp"] & \mathbf{Der} \arrow[l, bend right, "F_{sesqu}"'] \arrow[r, bend right, "U_{der}"'] \arrow[r, phantom, "\perp"] & \mathbf{2\text{-}Grph} \arrow[l, bend right, "F_{der}"'] \end{tikzcd}$$

Definition 2.12 *Let G be a 2-graph and $\mathcal{F} \overset{def}{=} F_{\textbf{\textit{2-Cat}}} \circ F_{\textbf{\textit{Sesqu}}} \circ F_{\textbf{\textit{Der}}}$. The* free 2-category $\mathcal{F}G$ on G *is given by this functor.*

*2.4 Limits and colimits in **2-Cat**.*

Proposition 2.13 ***2-Cat** is complete and cocomplete.*

Products, equalizers and coproducts are easy. The existence of coequalizers can be seen at hand of a result in enriched category theory. In the 1970's, John Gray's student Harvey Wolff showed that, given a symmetric monoidal category $\mathcal{V}$, the category of small $\mathcal{V}$-categories $\mathcal{V}$-Cat is monadic over the category of small $\mathcal{V}$-graphs. A corollary thereof is that $\mathcal{V}$-Cat is cocomplete provided $\mathcal{V}$ is (c.f. [28]). This result applies to the present case since 2-categories are **Cat**-categories and the well-known fact that **Cat** is cocomplete can be shown using the same argument (viz. a category is a **Set**-category).

Following Gray (c.f. [11, I.1.3,p.2]), the argument for **Cat** can be sketched as follows. Let

$$(\_)_0 : \mathbf{Cat} \to \mathbf{Set}$$

be the "underlying set functor". The latter has a right adjoint sending a set to the corresponding trivial connected groupoid, i.e. a category with all homsets containing precisely one morphism. Let

$$\begin{tikzcd} \mathbb{A} \arrow[r, shift left, "F"] \arrow[r, shift right, "G"'] & \mathbb{B} \end{tikzcd}$$

be a diagram in **Cat**. If this diagram admits a coequalizer, then the underlying set of the latter has to be (in bijection with) $\mathbb{K}_0$ in the coequalizer diagram

$$\begin{tikzcd} \mathbb{A}_0 \arrow[r, shift left, "F_0"] \arrow[r, shift right, "G_0"'] & \mathbb{B}_0 \arrow[r, "K_0"] & \mathbb{K}_0 \end{tikzcd}$$

in **Set**, this since $(\_)_0$ is a left adjoint. Next, given $M \neq N \in \mathbb{K}_0$, let $E_{M,N}$ be the coproduct

$$\coprod \mathbb{B}\left(B_1, B_1'\right) \times \cdots \times \mathbb{B}\left(B_n, B_n'\right)$$

over all finite sequences $B_1, B_1', \ldots B_n, B_n'$ such that $K_0\left(B_1\right) = M$, $K_0\left(B_i'\right) = K_0\left(B_{i+1}\right)$ for all $1 \leq i \leq n$ and $K_0\left(B_n'\right) = N$. Let

$$c_{M,N} : E_{M,N} \to E_{M,N}$$

be the morphism given by universal property from the homset-wise compositions

$$c_{B,B',B''} : \mathbb{B}(B, B') \times \mathbb{B}(B', B'') \to \mathbb{B}(B, B'')$$

for all triples (B, B', B''). On the other hand, let

$$mix_{M,N} : D_{M,N} \to E_{M,N}$$

be the morphism given by universal property from all possible insertions of the morphisms

$$F_{A,A'} : \mathbb{A}(A, A') \to \mathbb{B}(FA, FA')$$

and

$$G_{A,A'} : \mathbb{A}(A, A') \to \mathbb{B}(GA, GA')$$

Let $K_{M,N}$ be the coequalizer object of the pair $(c \circ mix, mix)$. The assignment $\mathbb{K}(M, N) \overset{def}{=} K_{M,N}$ yields a category $\mathbb{K}$ with $\mathbb{K}_0$ as set of objects. This category is the coequalizer object of the diagram above. The purpose of the construction is to take into account possible "new" morphisms arising from identifications of objects. The case $M = N$ is treated similarly, by adding the terminal to the coproduct $E_{M,N}$ and choices of units to mix.

Proposition 2.14 (Gray) *The functor $\mathcal{U}_1$: **2-Cat** → **Cat** which forgets the 2-cells has a right adjoint.*

The right adjoint turns a homset in a trivial connected groupoid.

By proposition 2.14 an argument formally identical to the above considerations shows that **2-Cat** has all coequalizers. All one needs to do is to replace sets with categories. When analyzing both variants, it is striking that only morphisms, limits and colimits in **Set** respectively in **Cat** are involved. It is precisely the reason why the argument works in a uniform way for **Cat** and **2-Cat**. Wolff showed that it is the case for all monoidal closed $\mathcal{V}$'s.

As a useful alternative, proposition 2.14 paves the way to a **2-Cat**-version of Gabriel's and Zisman's construction of colimits in **Cat** (c.f. [6, "Dictionary", p.4]). The advantage here is that the calculation of a colimit can be carried out *directly,* without having to express it in terms of coequalizers and coproducts as above. This provides a finer control over the construction.

3 Transition Systems and their Simulations

3.1 Transition systems

Definition 3.1 *Let Σ be a set. A transition system*

$$\mathbf{S} = (S, i, \to \subseteq S \times \Sigma \times S)$$

over the alphabet Σ *consists of a set of* states S*, of an* initial state $i \in S$ *and of a* transition relation $\to \subseteq S \times \Sigma \times S$.

A morphism of transition systems over Σ is a label-preserving function among the sets of states.

Proposition 3.2 *Transition systems over Σ and their morphisms form a category TS_Σ.*

Although definition 3.1 is the usual one, transition systems can be presented in many equivalent ways. In what follows we will use two other variants.

3.2 Simulations and Open Maps.

Proposition 3.3 *Let the coslice $\boldsymbol{pGrph} \stackrel{def}{=} \mathbf{1} \setminus \boldsymbol{Grph}$ be the category of pointed graphs and*

$$(\_)_1 : \boldsymbol{pGrph} \to \boldsymbol{Set}$$

the functor sending a graph to its set of edges. This functor has a right adjoint

$$(\_)_\bullet : \boldsymbol{Set} \to \boldsymbol{pGrph}$$

sending a set Σ on the one-vertex graph $\Sigma_\bullet$ such that $(\Sigma_\bullet)_1 = \Sigma$.

Definition 3.4 *Let Σ be a set. The* locally injective slice $\boldsymbol{pGrph} /_i\, \Sigma_\bullet \subseteq \boldsymbol{pGrph} / \Sigma_\bullet$ *is the full subcategory of the slice $\boldsymbol{pGrph} / \Sigma_\bullet$ with the locally injective graph morphisms as objects.*

Proposition 3.5 *There is an isomorphism of categories*

$$TS_\Sigma \cong \boldsymbol{pGrph} /_i\, \Sigma_\bullet$$

Consider a transition system $\mathbf{S} \in TS_\Sigma$. Let $G_\mathbf{S}$ be a graph with S as its set of vertices and with

$$\left\{ x \xrightarrow{(x,\alpha,y)} y \;\middle|\; (x, \alpha, y) \in \to \right\}$$

as its set of edges. Observe that there is the graph morphism $s : G_\mathbf{S} \to \Sigma_\bullet$ given on edges by the assignment $(x, \alpha, y) \mapsto \alpha$. It is then immediate that the assignment $\mathbf{S} \mapsto s$ extends to a functor $TS_\Sigma \to \mathbf{pGrph} /_i\, \Sigma_\bullet$ and easy to see that this functor is an isomorphism of categories.

In what follows, we will be using the same notation for either point of view provided by proposition 3.5. As a matter of notation, we write $x\sigma y$ as a shorthand indicating that the pair $(x, y) \in X \times Y$ are related by $\sigma \subseteq X \times Y$.

Definition 3.6 *Let* $\mathbf{S}$ *and* $\mathbf{S}'$ *be transition systems. A* presimulation $\mathbf{S} \rightsquigarrow \mathbf{S}'$ *is a relation* $\sigma : S \nrightarrow S'$ *such that*

$$\forall \alpha \in \Sigma.\, x\sigma x' \;\wedge\; \exists y \in S.x \to^{\alpha} y \Rightarrow \exists y' \in S'.x' \to'^{\alpha} y' \;\wedge\; y\sigma y'$$

It is a simulation *if*

$$i\sigma i'$$

also holds.

Definition 3.7 *Let* $\mathbb{S}$ *and* $\mathbb{A} \subseteq \mathbb{B}$ *be categories.* $\mathbb{A}$ *is a* $\mathbb{S}$*-skeletal subcategory of* $\mathbb{B}$ *if one (hence all) of its skeletons is isomorphic to* $\mathbb{S}$*.*

Let Δ be the simplicial category, i.e. the category of finite ordinals and monotone maps. In definition 3.8 to follow, finite ordinals are seen as graphs pointed at 0.

Definition 3.8 *Let* $\mathbb{P} \subsetneqq$ ***pGrph*** *be a* Δ*-skeletal subcategory. A morphism of pointed graphs* $h : G \to H$ *is* open *if* h_0 *is surjective and* $h \in RLP\,(\mathbb{P}_1)$*.*

Proposition 3.9 *The following are equivalent*

(i) *The morphism of pointed graphs* $h : G \to H$ *is open;*

(ii) h_0 is surjective and for each $H \ni f : x \to y$ and $a \in G$ with $h\,(a) = x$ there is $G \ni u : a \to b$ such that $h\,(u) = f$.

Proposition 3.10 *Let* $\mathbf{S}$ *and* $\mathbf{S}'$ *be transition systems. There is a simulation* $\mathbf{S} \nrightarrow \mathbf{S}'$ *if and only if there is a commuting square*

$$\begin{tikzcd} & R \arrow[dl, "r_1"'] \arrow[dr, "r_2"] & \\ G \arrow[dr, "\mathbf{S}"'] & & H \arrow[dl, "\mathbf{S}'"] \\ & \Sigma_\bullet & \end{tikzcd}$$

in ***pGrph*** *with* r_1 *open.*

Observe that the square in proposition 3.10 is a span in $\mathbf{pGrph} \,/\!\!_i\, \Sigma_\bullet$. Proposition 3.10 is a variant of the celebrated characterization of bisimulations due to Joyal, Winskel & Nielsen (c.f. [16]).

3.3 A Relational Structure

Observe that, putting the issue of the initial state aside, the usual presentation of a labeled transition system $\mathbf{S} = (\to \subseteq S \times \Sigma \times S)$ amounts to an indexed set of relations$(\to^{\alpha} \subseteq S \times S)_{\alpha\in\Sigma}$. Given the 2-category of sets and relations $\mathbf{Rel}$ (in fact a category with homsets ordered by inclusion of relations), it is not that hard to see that $(\to^{\alpha} \subseteq S \times S)_{\alpha \in \Sigma}$ gives rise to a morphism of monoids $\overline{\mathbf{S}} : \Sigma^* \to \mathbf{Rel}\,(S, S)$. This morphism can also be seen as a (2-)functor $\overline{\mathbf{S}} : \Sigma^* \to \mathbf{Rel}$. On the other hand, the presimulation condition is equivalent to

$$\forall \alpha \in \Sigma.\ \to^{\alpha} \circ \sigma^{op} \subseteq \sigma^{op} \circ \to'^{\alpha} \qquad (*)$$

Claudio Hermida used this observations in order to characterize presimulations (c.f. [12]).

Definition 3.11 *Let $\mathcal{A}$ be a 2-category and $f, g \in \mathcal{A}_1$.*

(i) A lax square (u_0, u_1, α) from f to g *is given by the diagram*

$$\begin{tikzcd}
x \arrow[r, "u_0"] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha \Nearrow"] & y \arrow[d, "g"] \\
x' \arrow[r, "u_1"'] & y'
\end{tikzcd}$$

(ii) A cylinder (θ_0, θ_1) from $(u_0, u_1, \alpha)$to (v_0, v_1, β) is given by the diagram

$$\begin{tikzcd}
x \arrow[r, bend left, "v_0"] \arrow[r, bend right, "u_0"'] \arrow[r, phantom, "\theta_0 \Nearrow"] \arrow[d, "f"'] \arrow[dr, phantom, "\beta \Nearrow"] & y \arrow[d, "g"] \\
x' \arrow[r, bend left, "v_1"] \arrow[r, bend right, "u_1"'] \arrow[r, phantom, "\theta_1 \Nearrow"] \arrow[ur, phantom, "\alpha \Nearrow"] & y'
\end{tikzcd}$$

where $(g \circ \theta_0) \bullet \alpha = \beta \bullet (\theta_1 \circ f)$

Proposition 3.12 *Let $\mathcal{A}$ be a 2-category. There is a 2-category $\mathcal{C}yl\,(\mathcal{A})$ given by the data*

(i) Objects: $\mathcal{A}_1$

(ii) Arrows: lax squares

(iii) 2-cells: cylinders

equipped with a 2-functor $\langle dom, cod \rangle : Cyl\,(\mathcal{A}) \to \mathcal{A} \times \mathcal{A}$.

Following Jean Bénabou, we call $Cyl\,(\mathcal{A})$ the *2-category of cylinders over $\mathcal{A}$* (cf. [2]). The name stems from the "geometry" of 2-cells. Notice that $Cyl\,(\mathcal{A})$ is a “lax” generalization of the familiar category of arrows.

Definition 3.13 *Let $F, G : \mathcal{A} \to \mathcal{B}$ be 2-functors. An oplax transformation $\alpha : F \Rightarrow G$ is given by the data*

(i) for each $x \in \mathcal{A}$ a morphism $\mathcal{B} \ni \alpha_x : F\,(x) \to G\,(x)$;

(ii) for each morphism $\mathcal{A} \ni f : x \to y$ *a 2-cell*

$$\begin{tikzcd} F(x) \arrow[r, "\alpha_x"] \arrow[d, "F(f)"'] \arrow[dr, phantom, "\swarrow\alpha_f"] & G(x) \arrow[d, "G(f)"] \\ F(y) \arrow[r, "\alpha_y"'] & G(y) \end{tikzcd}$$

subject to the coherence conditions

(i) $\alpha_{f'} \bullet (G(\theta) \circ \alpha_x) = (\alpha_y \circ F(\theta)) \bullet \alpha_f$ *for each 2-cell* $\theta : f \Rightarrow f' : x \to y$*;*

(ii) $(\alpha_g \circ F(f)) \bullet (G(g) \circ \alpha_f) = \alpha_{g \circ f}$ *for each* $f : x \to y$ *and* $g : y \to z$*.*

The way Bénabou originally introduced lax and oplax transforms in the more general setting of bicategories and lax functors was in terms of a *classifier*, viz. a *bicategory of cylinders* (c.f. [2]). In the 2-categorical setting of interest here we state it as a characterization:

Proposition 3.14 *The following are equivalent*

(i) There is an oplax transformation $\alpha : F \Rightarrow G$*;*

(ii) There is a 2-functor $\overline{\sigma} : \mathcal{A} \to Cyl(\mathcal{B})$ *such that*

$$\begin{tikzcd} & Cyl(\mathcal{B}) \arrow[d, "<dom{,}cod>"] \\ \mathcal{A} \arrow[ur, "\bar{\sigma}"] \arrow[r, "<G{,}F>"'] & \mathcal{B} \times \mathcal{B} \end{tikzcd}$$

commutes.

Now a concise way to express the presimulation condition $(*)$ is

Theorem 3.15 (Hermida)*Let* $\mathbf{S}$ *and* $\mathbf{T}$ *be transition systems. The following are equivalent*

(i) there is a simulation $\mathbf{S} \nrightarrow \mathbf{T}$*;*

(ii) there is an oplax transformation $\overline{\mathbf{T}} \Rightarrow \overline{\mathbf{S}}$*.*

4 A homotopical Characterization of Simulation

Lemma 4.1 *Let* $F : \mathbb{C} \to \mathbb{A}$ *be a functor and* $A \in \mathbb{A}$*. The assignment*

$$A \mapsto \mathbb{A}(F(\_), A)$$

determines a functor $F_* : \mathbb{A} \to \boldsymbol{Set}^{\mathbb{C}^{op}}$*. If* $\mathbb{A}$ *is cocomplete then* F_* *has a left adjoint* $F_!$ *and* F *factors through* $F_!$ *by the Yoneda embedding* $y : \mathbb{C} \to \boldsymbol{Set}^{\mathbb{C}^{op}}$*.*

Indeed, if it exists, $F_! = Lan_y F$ is the pointwise left Kan extension of F along y.

$$\begin{tikzcd} \mathbf{Set}^{\mathbb{C}^{op}} \arrow[dr, bend left, "F_!"] \arrow[dr, phantom, "\vdash"] & \\ \mathbb{C} \arrow[u, "y"] \arrow[r, "F"'] & \mathbb{A} \arrow[ul, bend left, "F_*"] \end{tikzcd}$$

The condition of $\mathbb{A}$ being cocomplete is thus sufficient but not necessary, yet it is verified in most of the cases of interest, including the ones encountered in this paper.

4.1 2-nerve and 2-categorification.

Definition 4.2 *Let $[n] \in \Delta$ and δn be the derivation scheme given by the data*

(i) $|\delta n| \stackrel{def}{=} \mathcal{F}([n])$;

(ii) $\delta n_2 \stackrel{def}{=} \{\ll i, j, k \gg \mid 0 \leq i < j < k \leq n\}$ *where*

$$dom_1 (\ll i, j, k \gg) = \langle j, k\rangle \circ \langle i, j\rangle$$

and

$$cod_1 (\ll i, j, k \gg) = \langle i, k\rangle$$

The 2-category $\lceil n \rceil$ is $(\mathcal{F}_{\boldsymbol{2\text{-}Cat}} \circ \mathcal{F}_{sesqu})(\delta n)$ quotiented by the relations

$$\ll i, k, l \gg \bullet (\langle k, l\rangle \circ \ll i, j, k \gg) = \ll i, j, l \gg \bullet (\ll j, k, l \gg \circ \langle i, j\rangle)$$

Remark 4.3 *The 2-category $\lceil n \rceil$ can be described as follows:*

(i) objects: $\{0, \dots n\}$;

(ii) morphisms: totally ordered sequences $\langle k < \cdots < l\rangle : k \to l$ such that $0 \leq k$ and $l \leq n$;

(iii) 2-cells: superset relation on the above sequences;

along with the obvious compositions.

Proposition 4.4 *The construction $\lceil \_ \rceil : \Delta \to$ **2-Cat** is functorial and determines an adjunction*

$$C_2 \dashv N_2$$

The functoriality is immediate while $C_2 \stackrel{def}{=} \lceil \_ \rceil_!$ and $N_2 \stackrel{def}{=} \lceil \_ \rceil_*$ (c.f. lemma 4.1). Following Ross Street, we call the $\lceil n \rceil$'s *2-orientals* (c.f. [24]). N_2 is called *2-nerve* and C_2 *2-categorification*.

Remark 4.5 *Given a simplicial set K, $C_2(K)$ is the free 2-category on the 2-graph determined by $(K_i)_{0 \leq i \leq 2}$ (and the relevant faces), quotiented by the relations given by K_3 and those given by the relevant degeneracies.*

As an example, let $\mathbb{A}$ be a category and $N_1 : \mathbf{Cat} \to \mathbf{sSet}$ the usual categorical nerve. $C_2 N_1 (\mathbb{A})$can be characterized as follows: the objects are those of $\mathbb{A}$, the arrows are generated by those of $\mathbb{A}$ (they are formal composites), while the 2-cells are generated by the collection

$$\alpha_{f,g} : g \circ f \Rightarrow g \circ_{\mathbb{A}} f$$

subject to the relations

$$\alpha_{g \circ f, h} \bullet (h \circ \alpha_{f,g}) = \alpha_{f, h \circ g} \bullet (\alpha_{g,h} \circ f)$$

Definition 4.6 *Let $\mathcal{A}$ and $\mathcal{B}$ be 2-categories and $F : \mathcal{A} \to \mathcal{B}$ a morphism of 2-graphs. F is a* normal lax functor *provided*

(i) it is locally a functor;

(ii) it preserves horizontal identities;

(iii) for any $f \in \mathcal{A}(x, y)$ and $g \in \mathcal{A}(y, z)$ there is a 2-cell

$$\gamma_{f,g} : F(g) \circ F(f) \Rightarrow F(g \circ f)$$

such that, given $h \in \mathcal{A}(z, a)$, the equation

$$\gamma_{g \circ f} \bullet (F(h) \circ \gamma_{f,g}) = \gamma_{f, h \circ g} \bullet (\gamma_{g,h} \circ F(f))$$

holds.

The equations in condition (iii) are referred to as *coherence conditions.* It is well-known that lax functors compose and that this composition is associative. A 2-functor is evidently a special case of a lax one.

Remark 4.7 *Let $\boldsymbol{NLax}([n], \mathcal{A})$be the set of normal lax functors from $[n]$ to $\mathcal{A}$. Then*

$$N_2(\mathcal{A})_n = \boldsymbol{NLax}([n], \mathcal{A})$$

and N_2 acts on 2-functors by postcomposition.

4.2 The standard model structure on ***sSet.***

Definition 4.8 *Let $\mathbb{M}$ be a category. $\mathcal{L}, \mathcal{R} \subseteq \mathbb{M}_1$ form a* weak factorization system $(\mathcal{L}, \mathcal{R})$ *if*

(i) any morphism $f \in \mathbb{M}_1$ factors as $f = r \circ l$ with $r \in \mathcal{R}$ and $l \in \mathcal{L}$;

(ii) $\mathcal{R} = RLP(\mathcal{L})$ and $\mathcal{L} = LLP(\mathcal{R})$.

Definition 4.9 $\mathbb{M}$ *is a* model category *if it is complete, cocomplete and has three distinguished classes of morphisms $\mathcal{C}, \mathcal{W}, \mathcal{F} \subseteq \mathbb{M}_1$ such that* *(i)* $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$*and* $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ *are weak factorization systems;*

(ii) $\mathcal{C}$, $\mathcal{F}$ *and* $\mathcal{W}$ *are closed under retracts in* $\mathbb{M}^{\rightarrow}$*;*

(iii) *if two of the morphisms in a commuting triangle are in* $\mathcal{W}$ *so is the third one.*

It is firmly established terminology to call morphisms in $\mathcal{F}$ *fibrations* with $\twoheadrightarrow$ as notation, those in $\mathcal{C}$ *cofibrations* with $\rightarrowtail$ as notation and those in $\mathcal{W}$ *weak equivalences* with $\xrightarrow{\sim}$ as notation. It is also customary to call morphisms in $\mathcal{F} \cap \mathcal{W}$ *acyclic fibrations* and those in $\mathcal{C} \cap \mathcal{W}$ *acyclic cofibrations*.

Given $g : \Delta \to \mathbf{Top}$ the functor assigning the affine n-tetrahedron to $[n]$, the *geometric realization* $|K|$ of a simplicial set K is given by $|K| \overset{def}{=} g_!(K)$ (c.f. lemma 4.1). It is well known that **sSet** is a model category with weak equivalences precisely those simplicial maps which induce isomorphisms of homotopy groups under the geometric realization, and with cofibrations being all the monos.

Definition 4.10 *Let* $\mathbb{M}$ *be a cocomplete category and* $I \subseteq \mathbb{M}_1$*.*

(i) *Let* λ *be an ordinal. A* (λ, I)-suite in $\mathbb{M}$ *is a cocontinuous functor* $\lambda \to \mathbb{M}$ *such that all its values on morphisms are in* I*.*

(ii) $A \in \mathbb{M}$ *is* small with respect to I *if there is a cardinal* κ *such that the covariant hom-functor* $\mathbb{M}(A, \_)$ *preserves colimits of all* (λ, I)*-suites for all regular cardinals* $\lambda \geq \kappa$ *.*

(iii) I permits the small object argument *if the domains of morphisms in* I *are small with respect to* I*.*

Definition 4.11 *A model category* $\mathbb{M}$ *is* cofibrantly generated *if there are sets of morphisms* $I, J \subseteq \mathbb{M}_1$ *permitting the small object argument and such that*

$$\mathcal{F} \cap \mathcal{W} = RLP(I)$$

and

$$\mathcal{F} = RLP(J)$$

I is called the set of the generating cofibrations while J is called the set of generating acyclic cofibrations, this since

Proposition 4.12 *Morphisms in* I *are cofibrations while those in* J *are acyclic cofibrations.*

It is well-known that **sSet** is cofibrantly generated. Given the *standard* n*-simplex* $\Delta[n] \overset{def}{=} \Delta(\_, [n])$, it is easy to see that it has precisely one non-degenerate simplex $\langle n \rangle \overset{def}{=} (0, \ldots, n)$ in dimension n and precisely $n+1$ non-degenerate simplices $\langle k \rangle \overset{def}{=} (0, \ldots, n) \backslash (k)$ in dimension $n-1$. The n*-boundary*

$\partial\Delta[n]$ is $\Delta[n]$ with $\langle n\rangle$ removed while the *k-th horn* $\Lambda^k[n]$ is $\partial[n]$ with $\langle k\rangle$ removed. The set of **sSet**'s generating cofibrations is the set of inclusions

$$\{\partial[n] \hookrightarrow \Delta[n]\}_{n\in\mathbb{N}}$$

while the set of **sSet**'s generating acyclic cofibrations is the set of inclusions

$$\left\{\Lambda^k[n] \hookrightarrow \Delta[n]\right\}_{\substack{0\le k\le n \\ n\in\mathbb{N}\setminus\{0\}}}$$

*4.3 Local presentability of **2-Cat**.*

Definition 4.13 *Let $\mathbb{C}$ be a category and $\mathcal{M} \subseteq \mathbb{C}_1$ be the collection of all monos. An epi e is* strong *if $e \in LLP(\mathcal{M})$. Suppose further that $\mathbb{C}$ has coproducts. A family $(G_i)_{i\in I}$ of objects indexed by the set I is a* strong family of generators *provided*

$$[f]_{i\in I, f\in\mathbb{C}(G_i,C)} : \coprod_{i\in I, f\in\mathbb{C}(G_i,C)} dom(f) \to C$$

is a strong epi for each $C \in \mathbb{C}$. Such a family is called a generator *if it is indexed by the singleton set.*

Lemma 4.14 *Let $\mathbf{2_2}$ be the 2-category*

$$\begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & y \end{tikzcd}$$

*$\mathbf{2_2}$ is a strong generator in **2-Cat**.*

Definition 4.15 *A cocomplete category $\mathbb{C}$ is* locally α-presentable *for a regular cardinal α if it has strong family of generators $(G_i)_{i\in I}$ such that the covariant hom-functor $\mathbb{C}(G_i, \_)$ preserves α-filtered colimits for all $i \in I$. The smallest α for which it is the case is called the* rank of presentability *of $\mathbb{C}$ and denoted $\pi(\mathbb{C})$. $\mathbb{C}$ is called* locally presentable *if it is locally α-presentable for some regular cardinal α.*

Proposition 4.16 ***2-Cat** is locally presentable with $\pi(\textbf{2-Cat}) = \aleph_0$.*

2-Cat is cocomplete by proposition 2.13 and has a strong generator by lemma 4.14. It is easy to see that **2-Cat** $(\mathbf{2_2}, \_)$ preserves filtered colimits.

Proposition 4.17 *Let $\mathbb{C}$ be a locally presentable category. There is a set of objects $\mathcal{G} \subseteq \mathbb{C}_0$ such that*

(i) *the covariant hom-functor* $\mathbb{C}(A, \_)$ *preserves filtered colimits for every* $A \in \mathcal{G}$*;*

(ii) *every object in* $\mathbb{C}$ *is a filtered colimit of those in* $\mathcal{G}$*.*

The set $\mathcal{G}$ is usually different from the strong family of generators required by the definition. It is however obtained from the latter by a transfinite construction completing (the full subcategory determined by) $\mathcal{G}$ with respect to α-limits for a regular cardinal $\alpha > \pi(\mathbb{C})$.

Locally presentable categories were introduced and extensively studied by Peter Gabriel and Friedrich Ulmer in their beautiful 1970's treatise (c.f. [5]). They became very popular among homotopy theorists in the 1990's since in a locally presentable category every object is small with respect to any set of morphisms.

4.4 A Thomason model structure on *2-Cat*.

Definition 4.18 *Let* $\mathbb{M}$ *be a model category and*

$$\begin{tikzcd} \mathbb{C} \arrow[r, bend right, "R"'] \arrow[r, phantom, "\perp"] & \mathbb{M} \arrow[l, bend right, "L"'] \end{tikzcd}$$

be an adjunction. R creates *a model structure on* $\mathbb{C}$ *if there is a model structure on* $\mathbb{C}$ *such that* $\mathcal{F}_{\mathbb{C}} = R^{-1}(\mathcal{F}_{\mathbb{M}})$ *and* $\mathcal{W}_{\mathbb{C}} = R^{-1}(\mathcal{W}_{\mathbb{M}})$*.*

Proposition 4.19 *Let* $\mathbb{M}$ *be a cofibrantly generated model category with* J *the set of generating acyclic cofibrations and* $L \dashv R$ *be as in definition 4.18 with in addition* $\mathbb{C}$ *a locally presentable category. Suppose further that*

(i) R *preserves filtered colimits;*

(ii) *for any* $f \in J$ *and for any pushout* g *of* $L(f)$*,* $R(g)$*is a weak equivalence.*

Then R *creates a cofibrantly generated model structure on* $\mathbb{C}$*.*

A slightly stronger version of proposition 4.19 appears in Tibor Beke's [1].

Definition 4.20 *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be 2-categories s.t.* $\mathcal{A} \subseteq \mathcal{B}$*.* $\mathcal{A}$ *is a* 2-sieve *if for any* $a \in \mathcal{A}_0$

(i) $cod(f) = a \Rightarrow f \in \mathcal{A}_1$ *for all* $f \in \mathcal{B}_1$*;*

(ii) $cod \circ dom(\alpha) = (cod \circ cod)(\alpha) = a \Rightarrow \alpha \in \mathcal{A}_2$ *for all* $\alpha \in \mathcal{B}_2$*.*

2-cosieves are defined dually.

Definition 4.21 *Let* $\widetilde{\textbf{2-Cat}}$ *be the category of 2-categories and normal lax functors. Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be* 2*-categories. An inclusion* $i : \mathcal{A} \hookrightarrow \mathcal{B}$ *is a* weak immersion *if* *(i)* $\mathcal{A}$ *is a 2-sieve;*

(ii) *there is a 2-cosieve* $\mathcal{W}$ *such that* $\mathcal{A} \subseteq \mathcal{W} \subseteq \mathcal{B}$*;*

(iii) $i : \mathcal{A} \hookrightarrow \mathcal{W}$ *admits a retraction* r*;*

(iv) *there is a normal lax functor* $\varepsilon : [1] \times \mathcal{W} \to \mathcal{W}$ *such that*

$$\begin{tikzcd} \mathcal{W} \arrow[r, "i_0"] \arrow[dr, "id_{\mathcal{W}}"'] & {[1] \times \mathcal{W}} \arrow[d, "\varepsilon"] & \mathcal{W} \arrow[l, "i_1"'] \arrow[dl, "i \circ r"] \\ & \mathcal{W} & \end{tikzcd}$$

commutes in $\widetilde{\textbf{2-Cat}}$ *and further that* $\varepsilon|_{[1]\times\mathcal{A}}$ *is strict and* $\varepsilon(0 \leq 1, id_a) = id_a$ *for all* $a \in \mathcal{A}$*.*

Definition 4.22 *Let* $\mathbb{M}$ *be a model category. A weak pushout square in* $\mathbb{M}$ *is a commuting square such that the comparison map from the inscribed pushout is a weak equivalence:*

$$\begin{tikzcd} A \arrow[rr] \arrow[dd] & & B \arrow[dl] \arrow[dd] \\ & B +_A C \arrow[dr, "\sim"] & \\ C \arrow[ur] \arrow[rr] & & D \end{tikzcd}$$

Lemma 4.23 *The image under* N_2 *of a pushout square of a weak immersion along an arbitrary 2-functor is a weak pushout square.*

Theorem 4.24 $Ex^2 \circ N_2$ *creates a model structure on 2-Cat.*

Proof. It is well-known that **2-Cat** is finitely presentable and that Ex preserves filtered colimits. It is easy to see that N_2 preserves filtered colimits, so it remains to establish condition (ii) of proposition 4.19. Let $i_{k,n} : \Lambda^k[n] \hookrightarrow \Delta[n]$ be a horn inclusion. It can be shown that $C_2(Sd^2(i_{k,n}))$ is a weak immersion and that $N_2C_2(Sd^2(i_{k,n}))$ is a weak equivalence in **sSet**, so the assertion follows from lemma 4.23 by 2-of-3. □

Clearly, lemma 4.23 is the "workhorse" here. We call the model structure of theorem 4.24 the *2-Thomason model structure* (c.f. [17]) since it is conceptually similar to a model structure on **Cat** due to R.W.Thomason (cf. [26]).

4.5 Simulations as homotopies.

Definition 4.25 *Let* $\mathbb{M}$ *be a model category.*

(i) P *is a* path object *on* B *if there is a commuting diagram*

$$\begin{tikzcd} & P \arrow[dr, "p"] & \\ B \arrow[ur, "\sim"] \arrow[rr, "\Delta"'] & & B \times B \end{tikzcd}$$

(ii) Given $f, g : A \to B$, *there is a* right homotopy $f \simeq g$ *if there is a path object over* B *such that* $\langle f, g\rangle$ *factors through* p*:*

$$\begin{tikzcd} & P \arrow[d, "p"] \\ A \arrow[ur] \arrow[r, "<f,g>"'] & B \times B \end{tikzcd}$$

Recall from proposition 3.12 that $Cyl\,(\mathcal{A})$ is Bénabou's "2-category of cylinders" which classifies oplax transformations.

Proposition 4.26 *$Cyl\,(\mathcal{A})$ is a path object on $\mathcal{A}$ in the 2-Thomason model structure.*

Definition 4.27 *Let* $\mathbf{S}$ *and* $\mathbf{T}$ *be transition systems. Let* ***pRel*** *be the 2-category of pointed sets and pointed relations. The 2-category* $\Psi_{\mathbf{S},\mathbf{T}}$ *is given as follows.*

- *(i) objects: the pointed sets* (S, ι_S) *and* (T, ι_T)
- *(ii) morphisms: generated by* $\left(\overline{\mathbf{S}}\,(\alpha)\right)_{\alpha\in\Sigma} \cup \left(\overline{\mathbf{T}}\,(\alpha)\right)_{\alpha\in\Sigma} \cup \boldsymbol{pRel}(T, S)_0$ *(c.f. section 3.3)*
- *(iii) 2-cells: generated by* $\boldsymbol{pRel}(T, S)_1$

The endomorphisms of $\Psi_{\mathbf{S},\mathbf{T}}$ are the transition relations, thus not pointed.

Lemma 4.28 *The following are equivalent*

- *(i) There is a simulation* $\mathbf{S} \nrightarrow \mathbf{T}$*;*
- *(ii) there is a* $\widehat{\sigma} : \Sigma^* \to Cyl\,(\Psi_{\mathbf{S},\mathbf{T}})$ *such that*

$$\begin{tikzcd} & Cyl(\Psi^{\mathbf{S},\mathbf{T}}) \arrow[d, "<dom,cod>"] \\ \Sigma^* \arrow[ur, "\hat{\sigma}"] \arrow[r, "<\hat{\mathbf{T}},\hat{\mathbf{S}},>"'] & \Psi^{\mathbf{S},\mathbf{T}} \times \Psi^{\mathbf{S},\mathbf{T}} \end{tikzcd}$$

commutes.

We are now in position to characterize precisely simulations as homotopies.

Theorem 4.29 *The following are equivalent*

- *(i) there is a simulation* $\mathbf{S} \nrightarrow \mathbf{T}$*;*
- *(ii) there is a right homotopy* $\widehat{\mathbf{T}} \simeq \widehat{\mathbf{S}}$ *in the 2-Thomason model structure.*

5 Higher-dimensional Transition Systems

In this section we address the issue of true concurrency in transition systems. Traditionally, as for instance in the seminal work of Robin Milner (c.f. [20]), concurrency is modeled by non-deterministic choice. By definition, this *interleaved semantic* does not distinguish between actions overlapping in time and actions chosen non-deterministically for other reasons. In real life however, non-deterministic choice without concurrency is in fact at least as common as concurrency itself. It can be explicitly programmed, say in stochastic algorithms, or be implicit, as in the case of sequential programs reacting to user input or to data measured by some sensors. On the other hand, representing concurrency purely in terms of non-deterministic choices is only realistic under the assumption that the processes to be represented share no more than *one* processor.

Nonetheless, concurrent systems are ubiquitous in today's world. One can for instance consider the Internet as a concurrent system, yet smaller-scale devices ranging from million-dollar number-crunchers to desktop shared-memory servers equipped with a couple to half a dozen processors are quite common as well. A *truly concurrent semantics* of concurrent systems distinguishes between actions overlapping in time and non-deterministically chosen actions. The approaches achieving this goal in a more or less satisfactory way are manifold, including those based on *petri nets*, *event structures* (c.f. [27],[21],[23]) or even *chemical abstract* machines (c.f. [7]). In this section, we focus on an approach using coherent families of *independence relations* on actions performed by a transition system. It turns out that the former can be organised as cubical sets (c.f. [8],[9]).

A possible way to realize this program is to label transitions with sequences over the alphabet Σ (c.f. [10]). The computational reading behind the setup is as follows: an n-cube A is seen as a transition with label $(\omega_1, \ldots, \omega_n)$ and represents unconstrained parallel activity [1] of n processes p_i leading from some global state [2] to another, each p_i performing the action ω_i. In line with this computational interpretation, we call n-cubes *n-transitions* when appropriate. It may be useful to think of such a situation as a concurrent system where each p_i executes on a distinct processor, accessing local memory only. The faces of A represent *interleaved activity* coherent with the *maximal degree* of concurrency given by the n-transition. That is, faces of A model all possible schedules of its actions when fewer than n processors are available. Degeneracies may be thought of as the opposite situation: there are more processors available

[1] There is no *interaction* like semaphore acquisition or message-passing involved.

[2] The values of all registers and memory locations are generally considered as a *global state*.

than processes running, so some processors idle. A further typical situation involving interleaved execution is *mutual exclusion* [3].

We first introduce higher-dimensional transition systems or *HDTS*'s in a traditionally syntactic way, then develop the topic as in sections 3 and 4.

5.1 Higher-dimensional transition systems.

Definition 5.1 *Given a set* Σ *and* $\omega = (\omega_1, \ldots, \omega_n) \in \Sigma^*$, *let* $|\omega| \stackrel{def}{=} n$. *Suppose from now on* Σ *totally ordered and let*

$$\omega \models (\Sigma, \leq) \stackrel{def}{\Longleftrightarrow} \omega_1 \leq \cdots \leq \omega_n$$

Let further $\star \notin \Sigma$, $\omega = (\omega_1, \ldots, \omega_n) \in (\Sigma \cup \{\star\})^*$, $\underline{\omega} \subseteq \omega$ *be the word obtained from* ω *by removing all the occurrences of* $\star$ *and*

$$!\Sigma_n \stackrel{def}{=} \{\omega \in (\Sigma \cup \{\star\})^* \mid |\omega| = n \wedge \underline{\omega} \models (\Sigma, \leq)\}$$

Finally, let

$$\delta_i (\omega_1, \ldots, \omega_n) \stackrel{def}{=} (\omega_1, \ldots, \omega_{i-1}, \omega_{i+1}, \ldots, \omega_n)$$

and

$$\varepsilon_i (\omega_1, \ldots, \omega_n) \stackrel{def}{=} (\omega_1, \ldots, \omega_{i-1}, \star, \omega_i, \ldots, \omega_n)$$

Definition 5.2 *A higher-dimensional transition system or* HDTS

$$\mathbf{S} = \left(S, i, T = (T_n)_{n\in\mathbb{N}}\right)$$

over the alphabet Σ *consists of a set of* states S, *of an* initial state $i \in S$ *and of a family of* transition relations $T = (T_n)_{n\in\mathbb{N}}$ *where* $T_n \subseteq S \times !\Sigma_n \times S$, *such that given* $n \in \mathbb{N}$, *the projection* $\ell_n : T_n \to !\Sigma_n$, *and* $1 \leq i \leq n$, *there are functions*

$$\partial_i^-, \partial_i^+ : T_n \to T_{n-1}$$

verifying $\ell_{n-1}\left(\partial_i^- (t)\right) = \ell_{n-1}\left(\partial_i^+ (t)\right) = \delta_i (\ell_n (t))$ *and functions*

$$\epsilon_i : T_n \to T_{n+1}$$

verifying $\ell_{n+1} (\epsilon_i (t)) = \varepsilon_i (\ell_n (t))$ *subject to the coherence conditions*

(i) $\partial_i^p \partial_j^q = \partial_{j-1}^q \partial_i^p$ *where* $i < j$ *and* $p, q \in \{-, +\}$;

(ii) $\epsilon_i \epsilon_j = \epsilon_{j+1} \epsilon_i$ *where* $i \leq j$;

[3] There is interaction like semaphore acquisition or message-passing and it is enforced by the operating system.

$$(iii)\ \partial_i^p \epsilon_j = \begin{cases} \epsilon_{j-1}\partial_i^p & \text{where } i \leq j \text{ and } p \in \{-,+\} \\ \epsilon_j \partial_{i-1}^p & \text{where } i > j \text{ and } p \in \{-,+\} \\ id & \text{where } i = j \end{cases}$$

The coherence conditions of definition 5.2 are called *cubical identities.* A transition system is obviously a low-dimensional particular case of a HDTS.

Proposition 5.3 *A morphism of HDTS's over Σ is a function among the sets of states preserving the labels and the initial state. HDTS's and their morphisms form a category $HDTS_\Sigma$.*

Definition 5.4 *Let* $\mathbf{T}$ *and* $\mathbf{T}'$ *be HDTS's. A* simulation $\mathbf{T} \nrightarrow \mathbf{T}'$ *is a relation* $\sigma : T \nrightarrow T'$ *such that*

$$\forall n \in \mathbb{N}.\, x\,(\sigma)\,x' \ \wedge\ x \xrightarrow{(\omega_1,\ldots,\omega_n)} x' \ \Rightarrow\ \exists y' \in T'.\, y\,(\sigma)\,y' \ \wedge\ y \xrightarrow{(\omega_1,\ldots,\omega_n)} y'$$

Observe that the cubical identities ensure that all the interleavings of $x \xrightarrow{(\omega_1,\ldots,\omega_n)} x'$ are simulated too.

5.2 Simulations and open maps.

Definition 5.5 *A cubical set K is a sequence*

$$\cdots \quad K_n \underset{\partial_i^-}{\overset{\partial_i^+}{\underset{\xleftarrow{\epsilon_i}}{\rightrightarrows}}} K_{n-1} \quad \cdots$$

of sets and functions where $0 \leq i \leq n-1$, subject to the identities

(i) $\partial_i^p \partial_j^q = \partial_{j-1}^q \partial_i^p$ *where $i < j$ and $p, q \in \{-,+\}$*

(ii) $\epsilon_i \epsilon_j = \epsilon_{j+1}\epsilon_i$ *where $i \leq j$*

$$(iii)\ \partial_i^p \epsilon_j = \begin{cases} \epsilon_{j-1}\partial_i^p & \text{where } i \leq j \text{ and } p \in \{-,+\} \\ \epsilon_j \partial_{i-1}^p & \text{where } i > j \text{ and } p \in \{-,+\} \\ id & \text{where } i = j \end{cases}$$

Given $A \in K_n$, let $dom_n(A) \overset{def}{=} \underbrace{\partial_0^- \ldots \partial_0^-}_{n\times}$ and $cod_n(A) \overset{def}{=} \partial_0^+ \ldots \partial_{n-1}^+$. Given $x, y \in K_0$, let $K_n(x,y) \overset{def}{=} \{A \in K_n \mid dom_n(A) = x \wedge cod_n(A) = y\}$

Similarly to a simplicial set, which is a presheaf over the simplicial category, a cubical set is obviously a presheaf over a category of “ideal cubes” viz. the cubical category $\square$. Cubical sets are thus objects of the topos **cSet** while the pointed ones are objects of **pcSet**. The ∂'s are called (positive respectively negative) *faces* while the ϵ's are called *degeneracies*. Again in analogy to simplicial sets, elements of K_n are called n-cubes.

Proposition 5.6 *The sequence $(!\Sigma_n)_{n\in\mathbb{N}}$ of definition 5.1 is a cubical set with faces $\partial_i^- = \delta_i$ respectively $\partial_i^+ = \delta_i$ and with degeneracies $\epsilon_i = \varepsilon_i$.*

Definition 5.7 *Let $\boldsymbol{pcSet}/_i\,!\Sigma \subseteq \boldsymbol{pcSet}/\,!\Sigma$ be the full subcategory of the slice $\boldsymbol{pcSet}/\,!\Sigma$ such that, given $k \in \boldsymbol{pcSet}/_i\,!\Sigma$, $k_n\,|_{K(x,y)}$ is injective for each $n \in \mathbb{N}$ and each $x, y \in K_0$.*

Proposition 5.8 *There is an isomorphism of categories*

$$HDTS_\Sigma \;\cong\; \boldsymbol{pcSet}/_i\,!\Sigma$$

The automata with parallelism or HDA (higher-dimensional automata) introduced by Eric Goubault live in the slice $\mathbf{cSet}/_i\,!\Sigma$ (c.f. [10] for the most recent account). A HDTS over Σ is thus a pointed HDA $k : K \to !\Sigma$, subject to a local injectivity condition.

Definition 5.9 *Let $(S, <)$ be a strict total order. Given $x, y \in S$ let*

$$x \prec y \overset{def}{\iff} x < y \wedge \;\nexists z \in S.\, x < z < y$$

Definition 5.10 *Let $\mathbb{P} \subseteq \boldsymbol{pcSet}$ be the full subcategory with objects $P \in \mathbb{P}$ such that*

(i) there is a subset $S \subseteq P_0$ which is a finite strict total order;

(ii) the point is given by the smallest element of S;

(iii) given $x, y \in S$
   - (a) $x \not\prec y \Rightarrow \forall n \in \mathbb{N}.\, P_n(x, y) = \varnothing$;
   - (b) *given $x \prec y$ there is $n_{x,y} \in \mathbb{N}$ such that there is precisely one $T_{x,y} \in P_{n_{x,y}}(x, y)$. This $T_{x,y}$ is not a face and $P_n(x, y) = \varnothing$ for $n < n_{x,y}$;*
   - (c) *given $z \in S$ and $x \prec y \prec z$, $T_{x,y}$ and $T_{y,z}$ have y as the only common face;*

(iv) any other non-degenerate n-cube in P is a face of precisely one $T_{x,y}$.

Objects of $\mathbb{P}$ are called paths.

Definition 5.9 states that paths are series of n-transitions glued at the end-points of their main diagonals as for instance in

$$
\begin{tikzcd}
 & & & & & z_1 \arrow[rr] & & x_4 \\
 & y_2 \arrow[rr] & & x_2 \arrow[r] \arrow[dd] & x_3 \arrow[ur] \arrow[rr] & & z_2 \arrow[ur] & \\
y_1 \arrow[ur] \arrow[rr] & & y_3 \arrow[ur] & & & & & \\
 & y_4 \arrow[uu] \arrow[rr] & & y_6 & & & & \\
x_1 \arrow[uu] \arrow[ur] \arrow[rr] & & y_5 \arrow[uu] \arrow[ur] & & & & &
\end{tikzcd}
$$

where $S = \{x_1 \prec x_2 \prec x_3 \prec x_4\}$. It is easily seen from this example that S induces a partition on $K_0 \setminus S$.

Definition 5.11 $(\boldsymbol{pcSet})_1 \ni h : M \to K$ *is open provided* $h \in RLP\,(\mathbb{P})$ *and* h_0 *is surjective.*

Proposition 5.12 *The following are equivalent.*

(i) h *is open;*

(ii) h_0 *is surjective and* $\forall n \in \mathbb{N},\ A \in K_n.\ \exists C \in M_n.\ h_n\,(C) = A.$

Proposition 5.13 *The following are equivalent.*

(i) there is a simulation $\mathbf{K} \nrightarrow \mathbf{L}$*;*

(ii) there is a span

$$
\begin{tikzcd}
 & R \arrow[dl, "r_1"'] \arrow[dr, "r_2"] & \\
K \arrow[dr, "k"'] & & L \arrow[dl, "l"] \\
 & !\Sigma &
\end{tikzcd}
$$

in $\boldsymbol{pcSet}/_{\!i}\,!\Sigma$ *with* r_1 *open.*

5.3 Simulations as homotopies.

Proposition 5.14 *Let* $\boldsymbol{c2\text{-}Cat} \stackrel{def}{=} \boldsymbol{2\text{-}Cat}^{\square^{op}}$ *be the category of cubical 2-categories. There is a model structure on* $\boldsymbol{c2\text{-}Cat}$ *with* $\mathcal{W}_{\boldsymbol{c2\text{-}Cat}}$ *and* $\mathcal{F}_{\boldsymbol{c2\text{-}Cat}}$ *taken levelwise.*

It is not true that every model structure carries over pointwise to functor categories, yet it is the case for cofibrantly generated structures when the indexing category is small (c.f. Hirschhorn [14, th. 11.6.1]). It follows that the 2-Thomason model structure carries over to **c2-Cat** by virtue of cofibrant generation and the smallness of $\square$. We call the resulting model structure *cubical 2-Thomason model structure.*

Proposition 5.15 *Let $\mathcal{A}$ be a cubical 2-category. There is a cubical 2-category $Cyl\,(\mathcal{A})$ such that $Cyl\,(\mathcal{A})_n = Cyl\,(\mathcal{A}_n)$. This cubical 2-category is a path object in the cubical 2-Thomason model structure.*

An observation similar to the one in section 3 can be made at this point. Let $\mathbf{K}$ be a HDTS over Σ given by $k : K \to !\Sigma$. There is a cubical monoid $!\Sigma^*$ where

$$(!\Sigma^*)_n = !\Sigma_n^*$$

with obvious faces and degeneracies and for each $n \in \mathbb{N}$ a morphism of monoids

$$\overline{k}_n : !\Sigma_n^* \to \mathbf{Rel}\,(K_0, K_0)$$

determined by its action on generators

$$(\omega_1, \ldots, \omega_n) \mapsto \left\{(dom_n\,(A)\,, cod_n\,(A)) \mid A \in k_n^{-1}\,(\omega_1, \ldots, \omega_n)\right\}$$

Lemma 5.16 *Let $\mathbf{K}$ and $\mathbf{L}$ be HDTS's. There is a cubical 2-category $\Psi^{\mathbf{K},\mathbf{L}}$ with $\Psi_n^{\mathbf{K},\mathbf{L}}$ given by the following data*

(i) objects: $K_0 \cup L_0$;

(ii) morphisms: generated from

$$\left\{\overline{k}_n\,(\omega) \mid \omega \in !\Sigma_n^*\right\} \cup \left\{\overline{l}_n\,(\omega) \mid \omega \in !\Sigma_n^*\right\} \cup \boldsymbol{pRel}\,(L_0, K_0)_0$$

(iii) 2-cells: generated from $\boldsymbol{pRel}\,(L_0, K_0)_1$ by whiskering.

Faces and degeneracies are constant on $\mathbf{pRel}\,(L_0, K_0)$ while

$$\begin{aligned}
\partial^p\left(\overline{k}_n\,(\omega)\right) &\stackrel{def}{=} \overline{k}_{n-1}\,(\partial^p\,(\omega)) \\
\partial^p\left(\overline{l}_n\,(\omega)\right) &\stackrel{def}{=} \overline{l}_{n-1}\,(\partial^p\,(\omega)) \\
\epsilon_i\left(\overline{k}_{n-1}\,(\omega)\right) &\stackrel{def}{=} \overline{k}_n\,(\epsilon_i\,(\omega)) \\
\epsilon_i\left(\overline{l}_{n-1}\,(\omega)\right) &\stackrel{def}{=} \overline{l}_n\,(\epsilon_i\,(\omega))
\end{aligned}$$

where the ∂'s on the right hand side are face maps of $!\Sigma^*$, as in

$$\begin{tikzcd}
L_0 \arrow[d, "\overline{l}_n(\omega)"'] & & L_0 \arrow[d, "\overline{l}_{n-1}(\partial^+(\omega))"] \\
L_0 \arrow[d, bend right, "r'"'] \arrow[d, bend left, "r"] \arrow[d, phantom, "\subseteq"] \arrow[rr, mapsto, "\partial^+"] & & L_0 \arrow[d, bend right, "r'"'] \arrow[d, bend left, "r"] \arrow[d, phantom, "\subseteq"] \\
K_0 & & K_0
\end{tikzcd}$$

Theorem 5.17 *The following are equivalent.*

(i) *There is a simulation* $\mathbf{K} \nrightarrow \mathbf{L}$*;*

(ii) *The diagram of cubical 2-categories*

$$\begin{tikzcd} & Cyl(\Psi^{\mathbf{K},\mathbf{L}}) \arrow[d, "<dom,cod>"] \\ !\Sigma^* \arrow[ur, "\bar{\sigma}"] \arrow[r, "<\bar{l},\bar{k}>"'] & \Psi^{\mathbf{K},\mathbf{L}} \times \Psi^{\mathbf{K},\mathbf{L}} \end{tikzcd}$$

commutes.

(iii) *There is a right homotopy* $\bar{l} \simeq \bar{k}$ *with respect to the cubical 2-Thomason model structure.*

6 Conclusion

Given transition systems $\mathbf{S}$ and $\mathbf{T}$, it is at any rate sensible to ask if there is a simulation. A specific instance is of particular interest: as Hermida recently put forward (cf. [13]), it is the case that given $\mathbf{S}$ and a relational modal formula ϕ, the truth of $\phi \models \mathbf{S}$ amounts to a simulation $\Phi \nrightarrow \mathbf{S}$ where Φ is a transition system built from ϕ. Hence, by theorem 4.29, it amounts to a homotopy $\overline{\Phi} \rightsquigarrow \overline{\mathbf{S}}$, so looking for an obstruction can be assimilated to *model-checking*.

It is easy to see that $\langle dom, cod\rangle : \mathcal{C}yl\,(\mathcal{A}) \rightarrow \mathcal{A} \times \mathcal{A}$ is not a fibration in the 2-Thomason model structure, so its fibrant replacement with a *very good* cylinder object is required in order to formulate the relevant lifting problem. What remains to do is to develop an appropriate obstruction theory. There has been some work in this direction by Dwyer *et.al.* (cf. [4]) but their notion of obstruction may be too coarse to be used in this context.

A good notion of obstruction is subject of an ongoing investigation. The future will show if any of this turns out to be of relevance for program verification, in particular the setup should then accommodate fixpoints *i.e.* the modal logic should be able to handle the expressiveness of a modal μ-calculus. Nevertheless, we believe that this line of research might very well lead to new insights and techniques.

Last but not least, the Thomason model structure on 2-Cat is of independent mathematical interest we plan to elaborate upon, in particular to outline the (quite significant) differences with the model structure on 2-Cat recently introduced by Steve Lack (cf. [18]).

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