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Representation of FS-domains Based on Closure Spaces

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Abstract

In this paper, we propose the notion of FS-closure spaces by incorporating an additional structure into a given closure space, which provides a concrete representation of FS-domains. Furthermore, we prove that the category of FS-closure spaces with approximable mappings as morphisms is equivalent to that of FS-domains with Scott-continuous functions as morphisms.

Keywords: FS-domain, FS-closure space, Closure operator, Categorical equivalence.

1 Introduction

A *closure space* is a pair (X, γ) consisting of a set X and a closure operator γ on X, where the closure operator is an isotone, extensive and idempotent map on the powerset of X. Closure spaces have played an important role in restructuring lattices and various order structures. The technique by adding a special structure into a given closure space may be traced back to the early works of Birkhoff's famous representation theorem for finite distributive lattices [3] and Stone's duality theorem for Boolean algebras [9]. These famous results also encourage researchers to investigate the interrelation between lattices and closure spaces. In [4], Edelman obtained that a lattice is meet-distributive if and only if it is a lattice of closed sets of closure space with the anti-exchange property. Erné [5] developed a uniform approach to representing various complete lattices by closure spaces from the categorical viewpoint. Guo and Li [7] proposed the notion of F-augmented closure spaces by adding a family of finite subsets into the closure space, which essentially establishes the

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representation of algebraic domains. Recently, Wang and Li [8] discuss the relationship between continuous domains and closure spaces. They introduce the notion of F-augmented generalized closure spaces by adding a map into a given closure space, and give a concrete representation of continuous domains.

FS-domains were introduced by A. Jung in [1,2], and proved that the category of FS-domains is a maximal Cartesian closed full subcategory of continuous domains. As is well known, a Cartesian closed category is of great significance that as a formal system with the same expressive power as a typed λ-calculus. Based on the basic fact, in this paper, we focus on the representation of FS-domains.

The paper is organized as follows: In Section 2, we recall some basic notions in domain theory. In Section 3, we introduce the concept of FS-closure spaces. Moreover, we prove that every FS-domain arises as the set of F-regular open sets of some FS-closure space. In Section 4, we obtain the main result that the category of FS-closure spaces with approximable mappings as morphisms is equivalent to that of FS-domains with Scott-continuous functions as morphisms.

2 Preliminaries

For any set X, we write $F \sqsubseteq X$ to mean that F is a finite subset of X. $\mathcal{P}(X)$ and $\mathcal{F}(X)$ are always used to denote the powerset of X and the family of all finite subsets of X, respectively. Let $(L, \leq)$ be a poset. A subset D of L is called *directed*, if it is nonempty and every finite subset of D has an upper bound in D. We use $\sqcup D$ to denote the *least upper bound* of a directed subset D. A poset is called a *dcpo* if every directed subset has a least upper bound. Given $x,\ y \in L$, we say x is *way below* y (in symbol $x \ll y$) if for any directed subset $D \subseteq L$ with $\sqcup D$ exists, $y \leq \sqcup D$ always implies $x \leq d$ for some $d \in D$. For any $x \in L$, we use $\twoheaddownarrow x$ to denote the set $\{y \in L \mid y \ll x\}$. A subset $B \subseteq L$ is called a *basis* of L if for every $x \in L$, $\twoheaddownarrow x \cap B$ is a directed subset and $x = \sqcup(\twoheaddownarrow x \cap B)$. A dcpo is a *continuous domain* if it has a basis.

Definition 2.1 [6] A function $f : L \to L^{'}$ between dcpos is said to be *Scott-continuous* if for any directed subset D of L, $f(\sqcup D) = \sqcup f(D)$.

We denote by $[L \to L^{'}]$ the set of all Scott-continuous functions from L to $L^{'}$.

Definition 2.2 [6] Let L be a dcpo.

(i) An *approximate identity* for a dcpo L is a directed set $\mathcal{D} \subseteq [L \to L]$ satisfying $\sup \mathcal{D} = 1_L$, the identity on L.

(ii) A Scott-continuous function $f : L \to L$ is *finitely separating* if there exists a finite set M_f such that for each $x \in L$, there exists $m \in M_f$ such that $f(x) \leq m \leq x$.

(iii) L is called an *FS-domain* if there is an approximate identity for L consisting of finitely separating functions.

Lemma 2.3 [6] Let L be a dcpo.

(i) If $\mathcal{D} \subseteq [L \to L]$ is an approximate identity for L, then $\mathcal{D}' = \{f^2 = f \circ f : f \in \mathcal{D}\}$ is also an approximate identity.

(ii) If $f \in [L \to L]$ is finitely separating, then $f(x) \ll x$ for all $x \in L$.

Definition 2.4 [8] Let (X, γ) be a closure space. A pair $(X, \tau \circ \gamma)$ is called a *generalized closure space*, if $\tau \circ \gamma$ is the composition map of γ and τ, where τ is a map on $\mathcal{P}(X)$ satisfies the following conditions, for any $A, B \subseteq X$:

(i) $\tau(\gamma(A)) \subseteq \gamma(A)$;

(ii) $\tau(\tau(\gamma(A))) = \tau(\gamma(A))$;

(iii) $\tau(\gamma(A)) \subseteq \tau(\gamma(B))$ whenever $A \subseteq B$.

For simplicity, we write $\langle A \rangle$ for $\tau(\gamma(A))$.

Definition 2.5 [8] Let $(X, \tau \circ \gamma)$ be a generalized closure space and $\mathcal{F}$ a nonempty family of finite subsets of X. The triplet $(X, \tau \circ \gamma, \mathcal{F})$ is called an *F-augmented generalized closure space* if, for any $F \in \mathcal{F}$ and $M \sqsubseteq \langle F \rangle$, there exists $F_1 \in \mathcal{F}$ such that $M \subseteq \langle F_1 \rangle$ and $F_1 \subseteq \langle F \rangle$.

Definition 2.6 [8] Let $(X, \tau \circ \gamma, \mathcal{F})$ be an F-augmented generalized closure space. A nonempty subset U of X is called an *F-regular open set* of $(X, \tau \circ \gamma, \mathcal{F})$ if, for any $M \sqsubseteq U$, there exists some $F \in \mathcal{F}$ such that $M \subseteq \langle F \rangle \subseteq U$.

For convenience, we use $\mathcal{R}(X)$ to denote the family of all F-regular open sets of $(X, \tau \circ \gamma, \mathcal{F})$.

Proposition 2.7 *[8] Let $(X, \tau \circ \gamma, \mathcal{F})$ be an F-augmented generalized closure space.*

(i) For any $F \in \mathcal{F}$, $\langle F \rangle$ is an F-regular open set of $(X, \tau \circ \gamma, \mathcal{F})$.

(ii) U is an F-regular open set of $(X, \tau \circ \gamma, \mathcal{F})$ if and only if $\{\langle F \rangle \mid F \in \mathcal{F}, F \subseteq U\}$ is directed and $U = \bigcup\{\langle F \rangle \mid F \in \mathcal{F}, F \subseteq U\}$.

(iii) If $\{U_j\}_{j \in J}$ is a directed family of F-regular open sets of $(X, \tau \circ \gamma, \mathcal{F})$, then $\bigcup_{j \in J} U_j$ is an F-regular open set of $(X, \tau \circ \gamma, \mathcal{F})$.

Theorem 2.8 *[8] Let $(X, \tau \circ \gamma, \mathcal{F})$ be an F-augmented generalized closure space. Then $(\mathcal{R}(X), \subseteq)$ is a continuous domain.*

Given a continuous domain $(L, \leq)$ with a basis B_L, for any $A \subseteq B_L$, define

$$\gamma(A) = \downarrow A \cap B_L, \tau(A) = \twoheaddownarrow A \cap B_L.$$

Let $\mathcal{F}_L$ be the family of all finite subsets of B_L with a greatest element under the induced order $\leq$. Then for any $F \in \mathcal{F}_L$, we have $\vee F \in F$ and

$$\langle F \rangle = (\twoheaddownarrow \vee F) \cap B_L.$$

Theorem 2.9 *[8] Let $(L, \leq)$ be a continuous domain with a basis B_L. Then $(B_L, \tau \circ \gamma, \mathcal{F}_L)$ is an F-augmented generalized closure space. And $(L, \leq)$ is isomorphic to $(\mathcal{R}(B_L), \subseteq)$.*

Definition 2.10 [8] Let $(X, \tau \circ \gamma, \mathcal{F})$ and $(X', \tau' \circ \gamma', \mathcal{F}')$ be two F-augmented generalized closure spaces. A relation $\Theta \subseteq \mathcal{F} \times X'$ is an *approximable mapping* from $(X, \tau \circ \gamma, \mathcal{F})$ to $(X', \tau' \circ \gamma', \mathcal{F}')$, if the following hold:

(i) $F\Theta F' \Rightarrow F\Theta\langle F'\rangle$,

(ii) $F \sqsubseteq \langle F_1\rangle, F\Theta M' \Rightarrow F_1\Theta M'$,

(iii) $F\Theta M' \Rightarrow (\exists G \in \mathcal{F}, G' \in \mathcal{F}')(G \subseteq \langle F\rangle, M' \subseteq \langle G'\rangle, G\Theta G')$,

for any $F, F_1 \in \mathcal{F}, F' \in \mathcal{F}'$ and $M' \sqsubseteq X'$, where $F\Theta M'$ means that $F\Theta x'$ for any $x' \in M'$.

Given an F-augmented generalized closure space $(X, \tau \circ \gamma, \mathcal{F})$, define a relation $\mathrm{id}_X \subseteq \mathcal{F} \times X$ by

$$\mathrm{id}_X = \{(F, x) \in \mathcal{F} \times X \mid x \in \langle F\rangle\}.$$

It is obvious that id_X is an approximable mapping from $(X, \tau \circ \gamma, \mathcal{F})$ to itself.

Theorem 2.11 *[8] The category* **DOM** *of continuous domains with Scott-continuous functions is equivalent to the category* **FGC** *of F-augmented generalized closure spaces with approximable mappings.*

3 FS-closure spaces

In this section, we give a special type of F-augmented generalized closure space which is called FS-closure space, and use this notion to obtain the representation of FS-domains.

Definition 3.1 An *FS-closure space* is an F-augmented generalized closure space $(X, \tau \circ \gamma, \mathcal{F})$ which satisfies: there exists a directed family $\{\Theta_j\}_{j\in J}$ of approximable mappings for $(X, \tau \circ \gamma, \mathcal{F})$ such that :

(i) $\bigcup_{j\in J} \Theta_j = \mathrm{id}_X$,

(ii) For every Θ_j, we have a finite subset family $\mathcal{M}_j \subseteq \mathcal{F}$ such that for each $F \in \mathcal{F}$, there exists $M \in \mathcal{M}_j$, $F\Theta_j x$ implies $M \subseteq \langle F\rangle$ and $x \in \langle M\rangle$.

Throughout this paper, we use $\mathcal{R}(X)$ to denote the family of all F-regular open sets (Definition 2.6) of FS-closure space $(X, \tau \circ \gamma, \mathcal{F})$.

Theorem 3.2 *Let $(X, \tau \circ \gamma, \mathcal{F})$ be an FS-closure space. Then $(\mathcal{R}(X), \subseteq)$ is an FS-domain.*

Proof. Theorem 2.8 has shown that $(\mathcal{R}(X), \subseteq)$ is a continuous domain. To finish the proof, it is sufficient to show that there is an approximate identity for $\mathcal{R}(X)$ consisting of finitely separating functions. By hypothesis, there exists a directed family $\{\Theta_j\}_{j\in J}$ of approximable mappings for $(X, \tau \circ \gamma, \mathcal{F})$ satisfies the conditions in Definition 3.1. For every Θ_j, define $\phi_{\Theta_j} : \mathcal{R}(X) \to \mathcal{R}(X)$ by $\phi_{\Theta_j}(U) = \{x \in X \mid (\exists F \in \mathcal{F})F \subseteq U \ \&\ F\Theta_j x\}$. From [8, Theorem 4.4], we know that ϕ_{Θ_j} is a Scott-continuous function.

We firstly prove that $\{\phi_{\Theta_j}\}_{j\in J}$ is an approximate identity for $\mathcal{R}(X)$. The directivity of $\{\phi_{\Theta_j}\}_{j\in J}$ just follows immediately from the definition of ϕ_{Θ_j}. Suppose

$U \in \mathcal{R}(X)$, we have

$$\begin{aligned}(sup_{j\in J}\phi_{\Theta_j})(U) &= sup_{j\in J}\phi_{\Theta_j}(U)\\ &= \bigcup_{j\in J}\phi_{\Theta_j}(U)\\ &= \bigcup_{j\in J}\{x \in X \mid (\exists F \in \mathcal{F})\ F \subseteq U\ \&\ F\Theta_j x\}\\ &= \{x \in X \mid (\exists F \in \mathcal{F})\ F \subseteq U\ \&\ (F, x) \in \bigcup_{j\in J}\Theta_j\}\\ &= \{x \in X \mid (\exists F \in \mathcal{F})\ F \subseteq U\ \&\ F\mathrm{id}_X x\}\\ &= \{x \in X \mid (\exists F \in \mathcal{F})\ F \subseteq U\ \&\ x \in \langle F\rangle\}\\ &= U.\end{aligned}$$

This means that $sup_{j\in J}\phi_{\Theta_j} = \mathrm{id}_{\mathcal{R}(X)}$.

We now prove that ϕ_{Θ_j} is finitely separating for every j. By Definition 3.1, for every Θ_j, we have a finite subset family $\mathcal{M}_j \subseteq \mathcal{F}$ such that for each $F \in \mathcal{F}$, there exists $M \in \mathcal{M}_j$, $F\Theta_j x$ implies $M \subseteq \langle F\rangle$ and $x \in \langle M\rangle$. Suppose $U \in \mathcal{R}(X)$, from Proposition 2.7, we obtain that $U = \bigcup\{\langle F\rangle \mid F \in \mathcal{F}, F \subseteq U\}$ and $\{\langle F\rangle \mid F \in \mathcal{F}, F \subseteq U\}$ is directed. Set $\mathcal{D}_M = \{\langle F\rangle \mid F \in \mathcal{F}, F \subseteq U, M \in \mathcal{M}_j, \forall F\Theta_j x \Rightarrow M \subseteq \langle F\rangle\ \&\ x \in \langle M\rangle\}$. It follows that $U = \bigcup\bigcup_{M\in\mathcal{M}_j}\mathcal{D}_M$ and $\bigcup_{M\in\mathcal{M}_j}\mathcal{D}_M$ is directed. Since $\mathcal{M}_j$ is a finite subset family of $\mathcal{F}$, there is a $M_0 \in \mathcal{M}_j$ such that $\mathcal{D}_{M_0}$ is a cofinal subset of $\bigcup_{M\in\mathcal{M}_j}\mathcal{D}_M$. We denote $\mathcal{M}_j^{'} = \{\langle M\rangle \mid M \in \mathcal{M}_j\}$. It is clear that $\mathcal{M}_j^{'}$ is a finite subset family of $\mathcal{R}(X)$. We finish the proof by checking that $\phi_{\Theta_j}(U) \subseteq \langle M\rangle \subseteq U$ for some $M \in \mathcal{M}_j$. In fact, suppose $x \in \phi_{\Theta_j}(U)$, then there exists $F \in \mathcal{F}$ such that $F \subseteq U$ and $F\Theta_j x$. It follows that $\langle F\rangle \subseteq \langle F_0\rangle$ for some $F_0 \in \mathcal{F}$ and $\langle F_0\rangle \in \mathcal{D}_{M_0}$. Therefore, $\phi_{\Theta_j}(U) \subseteq \langle M_0\rangle \subseteq U$. □

Given an FS-domain L, and its approximate identity $\{\phi_j\}_{j\in J}$ for L consisting of finitely separating functions. For every ϕ_j, define a relation $\Theta_{\phi_j} \subseteq \mathcal{F}_L \times L$ by

$$F\Theta_{\phi_j}x \Leftrightarrow x \ll \phi_j^2(\vee F).$$

Lemma 3.3 *Let L be an FS-domain. Then Θ_{ϕ_j} is an approximable mapping on $(L, \tau\circ\gamma, \mathcal{F}_L)$ for every $j \in J$.*

Proof. From Definition 2.10, suppose that $F, F^{'} \in \mathcal{F}_L$ and $F\Theta_{\phi_j}F^{'}$. Then by the definition of Θ_{ϕ_j}, we have $x \ll \phi_j^2(\vee F)$ for every $x \in F^{'}$. Since $F^{'}$ is finite and $\vee F^{'} \in F^{'}$, it follows that $\vee F^{'} \ll \phi_j^2(\vee F)$. Thus $F\Theta_{\phi_j}\langle F^{'}\rangle$.

Let $F \sqsubseteq \langle F^{''}\rangle$ and $F\Theta_{\phi_j}M$, then $x \ll \vee F^{''}$ and $m \ll \phi_j^2(\vee F)$ for every $x \in F, m \in M$. As ϕ_j is order-preserving, $\phi_j^2(\vee F) \le \phi_j^2(\vee F^{''})$. Thus $m \ll \phi_j^2(\vee F^{''})$ for any $m \in M$, which implies $F^{''}\Theta_{\phi_j}M$.

Assume that $F\Theta_{\phi_j}M$, then $m \ll \phi_j^2(\vee F)$ for all $m \in M$. By the interpolation property, there exists $a \in L$ such that $m \ll a \ll \phi_j^2(\vee F)$. Notice that $\phi_j(\vee F) = \phi_j(\vee(\twoheaddownarrow \vee F)) = \vee\phi_j(\twoheaddownarrow \vee F)$. Moreover, $\phi_j^2(\vee F) = \vee\phi_j^2(\twoheaddownarrow \vee F)$, then there exists

$b \in \downarrow\!\!\!\downarrow \vee F$ such that $a \ll \phi_j^2(b)$. Set $G = \{b\}$ and $G' = \{a\}$. It is clear that $G, G' \in \mathcal{F}_L$ such that $G \subseteq \langle F \rangle$, $M \subseteq \langle G' \rangle$ and $G\Theta_{\phi_j}G'$. □

Theorem 3.4 *Let $(L, \leq)$ be an FS-domain. Then $(L, \tau \circ \gamma, \mathcal{F}_L)$ is an FS-closure space. Moreover, $(L, \leq)$ is order isomorphic to $(\mathcal{R}(L), \subseteq)$.*

Proof. By Theorem 2.9, $(L, \tau \circ \gamma, \mathcal{F}_L)$ is an F-augmented generalized closure space and $(L, \leq)$ is order isomorphic to $(\mathcal{R}(L), \subseteq)$. Then it suffices to prove that $\{\Theta_{\phi_j}\}_{j \in J}$ satisfies the conditions in Definition 3.1. From Lemma 3.3, we know that Θ_{ϕ_j} is an approximable mapping on the F-augmented generalized closure space $(L, \tau \circ \gamma, \mathcal{F}_L)$ for ever $j \in J$. We claim that $\{\Theta_{\phi_j}\}_{j \in J}$ is a directed family with respect to inclusion order. For any $\Theta_{\phi_{j_1}}, \Theta_{\phi_{j_2}}$ where $j_1, j_2 \in J$, by definition of Θ_{ϕ_j}, for any $(F, x) \in \Theta_{\phi_{j_i}}$ if and only if $x \ll \phi_{j_i}^2(\vee F)$ for $i = 1, 2$. Because $\{\phi_j\}_{j \in J}$ is directed, there exists $j \in J$ such that $\phi_{j_1}, \phi_{j_2} \leq \phi_j$. Now we prove that $\Theta_{\phi_{j_1}}, \Theta_{\phi_{j_2}} \subseteq \Theta_{\phi_j}$ for this j. Indeed, suppose $(F, x) \in \Theta_{\phi_{j_i}}$, $i = 1, 2$, then $x \ll \phi_{j_i}^2(\vee F) \leq \phi_j^2(\vee F)$, which implies $(F, x) \in \Theta_{\phi_{j_i}}$. Thus $\Theta_{\phi_{j_1}}, \Theta_{\phi_{j_2}} \subseteq \Theta_{\phi_j}$.

We claim that $\bigcup_{j \in J} \Theta_{\phi_j} = \mathrm{id}_L$. Assume that $(F, x) \in \mathrm{id}_L$, then $x \in \langle F \rangle = \downarrow\!\!\!\downarrow \vee F$. Since $\vee F = sup_{j \in J} \phi_j^2(\vee F)$, there exists $j \in J$ such that $x \ll \phi_j^2(\vee F)$. Thus $(F, x) \in \bigcup_{j \in J} \Theta_{\phi_j}$. Conversely, if $(F, x) \in \bigcup_{j \in J} \Theta_{\phi_j}$, then $(F, x) \in \Theta_{\phi_j}$ for some $j \in J$. It follows that $x \ll \phi_j^2(\vee F) \ll \vee F$. Hence $x \in \downarrow\!\!\!\downarrow \vee F = \langle F \rangle$. Therefore, $(F, x) \in \mathrm{id}_L$.

For every $j \in J$, since ϕ_j is finitely separating function of L, there exists a finite subset $M_j \subseteq L$ such that for each $x \in L$, there is $m \in M_j$ such that $\phi_j(x) \leq m \leq x$. We set $\mathcal{M}_j = \{\{\phi_j(m)\} \mid m \in M_j\}$, then $\mathcal{M}_j \subseteq \mathcal{F}_L$ is a finite subset family. For every $F \in \mathcal{F}_L$, there is $m \in M_j$ such that $\phi_j(\vee F) \leq m \leq \vee F$. If $F\Theta_{\phi_j}x$, which implies $x \ll \phi_j^2(\vee F) \leq \phi_j(m) \leq \phi_j(\vee F) \ll \vee F$. Therefore, $\{\phi_j(m)\} \subseteq \langle F \rangle$ and $x \in \langle\{\phi_j(m)\}\rangle$. □

4 The categorical equivalence between FS-closure spaces and FS-domains

In this section, we investigate the connection between approximable mappings and Scott-continuous functions. Moreover, we establish the equivalence between FS-closure spaces and FS-domains from the categorial point of view.

Given an FS-closure space $(X, \tau \circ \gamma, \mathcal{F})$, define a relation $\mathrm{id}_X \subseteq \mathcal{F} \times X$ by

$$(F, x) \in \mathrm{id}_X \Leftrightarrow x \in \langle F \rangle.$$

Given two approximable mappings $\Theta : (X, \tau \circ \gamma, \mathcal{F}) \to (X', \tau' \circ \gamma', \mathcal{F}')$ and $\Theta' : (X', \tau' \circ \gamma', \mathcal{F}') \to (X'', \tau'' \circ \gamma'', \mathcal{F}'')$, define a relation $\Theta' \circ \Theta \subseteq \mathcal{F} \times X''$ by

$$F(\Theta' \circ \Theta)x'' \Leftrightarrow (\exists G \in \mathcal{F}')\ (F\Theta G \ \&\ G\Theta' x'').$$

Proposition 4.1 *FS-closure spaces and approximable mappings form a category that is denoted as* **FSC**.

Proof. Routine checks verify that $\Theta' \circ \Theta$ is an approximable mapping from $(X, \tau \circ \gamma, \mathcal{F})$ to $(X'', \tau'' \circ \gamma'', \mathcal{F}'')$ and id_X is an approximable mapping from $(X, \tau \circ \gamma, \mathcal{F})$ to itself. □

Lemma 4.2 *Let $(L, \leq)$ and $(L', \leq')$ be FS-domains. For any Scott-continuous function $\phi : L \to L'$, define a relation $\Theta_\phi \subseteq \mathcal{F}_L \times L'$ by*

$$(F, x') \in \Theta_\phi \Leftrightarrow x' \ll' \phi^2(\vee F).$$

Then Θ_ϕ is an approximable mapping from $(L, \tau \circ \gamma, \mathcal{F}_L)$ to $(L', \tau' \circ \gamma', \mathcal{F}_{L'})$.

Conversely, suppose $(X, \tau \circ \gamma, \mathcal{F})$ and $(X', \tau' \circ \gamma', \mathcal{F}')$ be FS-closure spaces. For any approximable mapping Θ from $(X, \tau \circ \gamma, \mathcal{F})$ to $(X', \tau' \circ \gamma', \mathcal{F}')$, define a map $\phi_\Theta : \mathcal{R}(X) \to \mathcal{R}(X')$ by

$$\phi_\Theta(U) = \{x' \in X' \mid (\exists F \in \mathcal{F})\ F \subseteq U \ \&\ F\Theta x'\}.$$

Then ϕ_Θ is a Scott-continuous function from $(\mathcal{R}(X), \subseteq)$ to $(\mathcal{R}(X'), \subseteq)$.

Proof. The proof is similar to that of [8, Theorem 4.4, Theorem 4.6]. □

Proposition 4.3 *$\mathfrak{F}$: **FSC** $\to$ **FSdom** is a functor which maps every FS-closure space $(X, \tau \circ \gamma, \mathcal{F})$ to $\mathcal{R}(X)$ and approximable mapping $\Theta : (X, \tau \circ \gamma, \mathcal{F}) \to (X', \tau' \circ \gamma', \mathcal{F}')$ to $\phi_\Theta : \mathcal{R}(X) \to \mathcal{R}(X')$, where ϕ_Θ is defined in Lemma 4.2.*

Proof. Based on Theorem 3.2 and Lemma 4.2, $\mathfrak{F}$ is well-defined. We check that $\mathfrak{F}$ preserves the identity morphism. For any $U \in \mathcal{R}(X)$,

$$\begin{aligned}
\mathfrak{F}(\mathrm{id}_X)(U) &= \phi_{\mathrm{id}_X}(U) \\
&= \{x \in X \mid (\exists F \in \mathcal{F})\ F \subseteq U \ \&\ (F, x) \in \mathrm{id}_X\} \\
&= \{x \in X \mid (\exists F \in \mathcal{F})\ F \subseteq U \ \&\ x \in \langle F \rangle\} \\
&= U \\
&= \mathrm{id}_{\mathcal{R}(X)}.
\end{aligned}$$

Let Θ be an approximable mapping from $(X, \tau \circ \gamma, \mathcal{F})$ to $(X', \tau' \circ \gamma', \mathcal{F}')$ and Θ' an approximable mapping from $(X', \tau' \circ \gamma', \mathcal{F}')$ to $(X'', \tau'' \circ \gamma'', \mathcal{F}'')$. For any $U \in \mathcal{R}(X)$ and $x'' \in X''$, we have

$$\begin{aligned}
x'' \in \mathfrak{F}(\Theta' \circ \Theta)(U) &\Leftrightarrow x'' \in \phi_{\Theta' \circ \Theta}(U) \\
&\Leftrightarrow (\exists F \in \mathcal{F})\ F \subseteq U \ \&\ F(\Theta' \circ \Theta)x'' \\
&\Leftrightarrow (\exists F \in \mathcal{F}, G \in \mathcal{F}')\ F \subseteq U \ \&\ F\Theta G \ \&\ G\Theta' x'' \\
&\Leftrightarrow (\exists G \in \mathcal{F}')\ G \subseteq \phi_\Theta(U) \ \&\ G\Theta' x'' \\
&\Leftrightarrow x'' \in \mathfrak{F}(\Theta')(\mathfrak{F}(\Theta)(U)).
\end{aligned}$$

It implies $\mathfrak{F}(\Theta^{'} \circ \Theta) = \mathfrak{F}(\Theta^{'}) \circ \mathfrak{F}(\Theta)$. □

Now, we obtain the main result of this paper.

Theorem 4.4 **FSC** *and* **FSdom** *are categorically equivalent.*

Proof. According to Theorem 3.4, it is sufficient to show that the functor $\mathfrak{F}$ is full and faithful.

We claim that $\mathfrak{F}$ is full. Let $(X, \tau\circ\gamma, \mathcal{F})$ and $(X^{'}, \tau^{'}\circ\gamma^{'}, \mathcal{F}^{'})$ be FS-closure spaces. For any Scott-continuous map $\phi : \mathcal{R}(X) \to \mathcal{R}(X^{'})$, define a relation $\Theta_\phi \subseteq \mathcal{F} \times X^{'}$ by

$$F\Theta_\phi x^{'} \Leftrightarrow x^{'} \in \phi(\langle F\rangle).$$

It is straightforward to check that Θ_ϕ is an approximable mapping from $(X, \tau\circ\gamma, \mathcal{F})$ to $(X^{'}, \tau^{'}\circ\gamma^{'}, \mathcal{F}^{'})$. Now we only need to prove that $\mathfrak{F}(\Theta_\phi) = \phi$. Suppose $U \in \mathcal{R}(X)$,

$$\begin{aligned}
\mathfrak{F}(\Theta_\phi)(U) &= \phi_{\Theta_\phi}(U)\\
&= \{x^{'} \in X^{'} \mid (\exists F \in \mathcal{F})\ F \subseteq U \ \&\ F\Theta_\phi x^{'}\}\\
&= \{x^{'} \in X^{'} \mid (\exists F \in \mathcal{F})\ F \subseteq U \ \&\ x^{'} \in \phi(\langle F\rangle)\}\\
&= \bigcup\{\phi(\langle F\rangle) \mid F \in \mathcal{F} \ \&\ F \subseteq U\}\\
&= \phi(\bigcup\{\langle F\rangle \mid F \in \mathcal{F} \ \&\ F \subseteq U\})\\
&= \phi(U).
\end{aligned}$$

This implies that $\mathfrak{F}$ is full.

We claim that $\mathfrak{F}$ is faithful. Suppose that $\Theta, \Theta^{'}$ be two approximable mappings from $(X, \tau\circ\gamma, \mathcal{F})$ to $(X^{'}, \tau^{'}\circ\gamma^{'}, \mathcal{F}^{'})$ such that $\phi_\Theta = \phi_{\Theta'}$. For any $F \in \mathcal{F}$ and $x^{'} \in X^{'}$, we have

$$\begin{aligned}
(F, x^{'}) \in \Theta &\Leftrightarrow (\exists G \in \mathcal{F}) G \subseteq \langle F\rangle \ \&\ G\Theta x^{'}\\
&\Leftrightarrow x^{'} \in \phi_\Theta(\langle F\rangle)\\
&\Leftrightarrow x^{'} \in \phi_{\Theta'}(\langle F\rangle)\\
&\Leftrightarrow (F, x^{'}) \in \Theta^{'}.
\end{aligned}$$

Then $\Theta = \Theta^{'}$, and hence $\mathfrak{F}$ is faithful. □

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