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Different Approaches for Implementing Quantum Search by Adiabatic Evolution

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Abstract

In this paper, three different approaches for implementing a quantum search algorithm by adiabatic evolution are shown. As expected, either one of them can provide a quadratic speed up as opposed to the classical search algorithm. This implies that adiabatic evolution based quantum computation gives more feasibilities than the quantum circuit model, although the equivalence between them has already been proven in the corresponding literature.

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1. Introduction

In around 2000, a new quantum computational model based on adiabatic evolution was proposed by Farhi, et al [1], upon which some previously well known quantum algorithms can be rebuilt, such as Grover's quantum search [2], Shor's algorithm [3].

The main idea behind an adiabatic quantum algorithm is as follows: suppose the Hamiltonian $H(t)$ can fully characterize the time evolution of a quantum system. Assume that $|\varphi(t)\rangle$ is the immediate state of the system which is evolving according to the Schrödinger equation:

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$$i \frac{d}{dt} |\varphi(t)\rangle = H(t) |\varphi(t)\rangle \quad (1)$$

Define $|E(0,t)\rangle$ and $|E(1,t)\rangle$ as the ground state and first excited state of the system Hamiltonian $H(t)$ with corresponding eigenvalues $E(0,t)$ and $E(1,t)$ respectively. The adiabatic theorem in quantum physics tells us that if we prepare the system in the ground state of the initial Hamiltonian H_i and let it evolve slowly along certain path to the final Hamiltonian H_f , then

$$|\langle E(0,T) | \varphi(T) \rangle|^2 \geq 1 - \varepsilon^2 \quad (0 < \varepsilon \ll 1) \quad (2)$$

provided that

$$\frac{D_{\max}}{g_{\min}} \leq \varepsilon, \quad (3)$$

in which

$$D_{\max} = \max_{0 \leq t \leq T} \left| \langle E(1,t) | \frac{dH(t)}{dt} | E(0,t) \rangle \right| \quad (4)$$

is a measurement of the evolving rate of the Hamiltonian, and

$$g_{\min} = \min_{0 \leq t \leq T} [E(1,t) - E(0,t)] \quad (5)$$

is the minimum gap between the two lowest eigenvalues. Generally, the required running time T for a typical adiabatic quantum algorithm will be mainly determined by g_{\min} so long as D_{\max} is polynomially bounded. For convenience, we let the time-dependent Hamiltonian $H(t)$ be reparameterized as:

$$H(s) = H(t/T), \quad (0 \leq s = s(t) \leq 1) \quad (6)$$

and the adiabatic evolution path widely used have a simple linear form of

$$H(s) = (1-s)H_i + sH_f. \quad (7)$$

In this paper, we study the adiabatic evolution of a quantum system which has the following form:

$$H_i = I - |\alpha\rangle\langle\alpha| \quad (8)$$

is the initial Hamiltonian, and

$$H_f = I - |\beta\rangle\langle\beta| \quad (9)$$

is the final Hamiltonian. For simplicity, we define the equation for later using:

$$\delta = |\langle\alpha|\beta\rangle| \neq 0 \quad (10)$$

It is not easy to calculate the eigenvalues of $H(s)$ in the computational basis in general. We use the following orthonormal basis to eliminate the difficulty:

$$|1\rangle = |\alpha\rangle, \quad (11)$$

$$|2\rangle = \frac{1}{a}(|\beta\rangle - \langle\alpha|\beta\rangle|\alpha\rangle). \quad (12)$$

Now the Hamiltonian considered here has the equivalent matrix representation:

$$H(s) = \begin{bmatrix} -s\delta^2 + s & -sa\langle\alpha|\beta\rangle \\ -sa\langle\alpha|\beta\rangle^* & -sa^2 + 1 \end{bmatrix}, \quad (13)$$

It is easy to get the two lowest eigenvalues of $H(s)$:

$$E(i,t) = \frac{1}{2} (1 \pm \sqrt{1 - 4(1 - \delta^2)s(1-s)}), \quad i = 0, 1. \quad (14)$$

When a global adiabatic quantum algorithm is applied to complete the task of evolving from the initial state $|\alpha\rangle$ to the final state $|\beta\rangle$, it works as follows:

Algorithm I.

Step 1: Construct the time-dependent Hamiltonian $H(s)$ given by Eq. (7) with H_i and H_f given by Eq. (8) and Eq. (9).

Step 2: Start the quantum system in the state $|\varphi(0)\rangle$ which is given by $|\alpha\rangle$.

Step 3: Evolve according to Eq. (1) for a time T to arrive at $|\varphi(T)\rangle$.

Step 4: Measure the system and get the result state.

Step 5: Repeat several times in order to have a desired probability of success.

It is not difficult to see that the above procedure takes a time complexity of $O(\delta^{-2})$ and no speed up can be thus obtained as compared to the classical search algorithm. However, in the current context below, three different strategies are demonstrated for remedying this fault, and each of them can provide a quadratic speed up over the best classical algorithms, thus recovering the advantage of adiabatic evolution based quantum search.

The rest of the paper is organized as follows: in section II, three different approaches for the problem studied in this paper are shown by three subsections. Conclusions are drawn in the final section to end the whole paper.

2. Different approaches for the speed up of the problem

In this section, we show three different kinds of adiabatic evolution algorithms to arrive at the goal of quadratic speed up for the problem while the global adiabatic evolution fails to do this.

2.1. A local adiabatic evolution approach

Local adiabatic evolution was first proposed in Ref. [4] in order to remove the failure of providing a quadratic speed up over the classical search by a global adiabatic evolution when implementing Grover's quantum search algorithm. Applying the idea to our problem defined before, the algorithm consists of the following steps:

Algorithm II.

Step 1: The initial state of a quantum system is prepared to be the stable ground state of H_i .

Step 2: Instead of using a linear evolution function $s(t)$ in the global adiabatic evolution, we adapt the evolution rate ds/dt to the local adiabatic condition. That is, by dividing the evolution time T into infinitesimal time intervals dt and applying the adiabatic condition locally to each of these intervals, we thus vary the evolution rate continuously in time.

Step 3: Measure the state of the system and get the result.

Now we turn to estimate the time complexity for the algorithm above. In a local adiabatic evolution, we know that the adiabatic condition now changes to:

$$\left| \frac{\langle E(1,t) | \frac{dH(s)}{dt} | E(0,t) \rangle}{[E(1,t) - E(0,t)]^2} \right| \leq \varepsilon, \quad t \in [0, T] \quad (15)$$

Thus, the time complexity of the current algorithm to execute the adiabatic evolution for the problem can be bounded as:

$$T = \frac{1}{\varepsilon} \int_0^1 \frac{1}{g(s)} ds, \quad (16)$$

in which $g(s)$ is energy gap of the two lowest eigenvalues of the Hamiltonian $H(s)$. By some integral calculations we easily get

$$T = O(\delta^{-1}), \quad (17)$$

a quadratic speedup over the global adiabatic evolution can be gained, as expected.

2.2. A partial adiabatic evolution approach

Partial adiabatic evolution was initiated by Tulsi in ref. [5]. We now give the partial adiabatic evolution to execute the task defined before. More concretely, the algorithm is made up of the following steps:

Algorithm III.

- Step 1: The initial state of a quantum system is prepared to be the stable ground state $|\alpha\rangle$ of H_i .
 Step 2: At time $t = 0$, the Hamiltonian of the system is suddenly changed to $H(s^-)$, without disturbing the initial state.
 Step 3: The Hamiltonian evolves from $H(s^-)$ to $H(s^+)$ linearly in time over duration T .
 Step 4: Measure the state of the system and check the result.
 Step 5: Repeat these four steps until the target state can be found.

In the above procedure, $[s^-, s^+]$ is the narrow interval, and they are specified as:

$$s^- = \frac{1}{2} - c\delta, \quad s^+ = \frac{1}{2} + c\delta, \quad (18)$$

in which c is a small positive integer (e.g. three or five). Additionally, it can be proven that the condition constraints for a partial adiabatic evolution can be satisfied by our setting here, which means that the construction here which is a little different from that in Ref. [5] is also a valid partial adiabatic algorithm.

We are now in a position to give the time complexity about the above partial adiabatic algorithm. It is estimated as:

$$T' = \frac{T}{P} = O(\delta^{-1}), \quad (19)$$

in which

$$T = \frac{s^+ - s^-}{\mathcal{G}_{\min}} \quad (20)$$

represents one-round evolution time, and

$$P = \left| \langle \alpha | E(0, s^-) \rangle \right|^2 \times \left| \langle \beta | E(0, s^+) \rangle \right|^2 > \frac{1}{4} \quad (21)$$

is one-round success probability of the algorithm. Therefore, a quadratic speedup over global adiabatic scheme has also been gained.

2.3. A modified global adiabatic evolution approach

Can a global adiabatic evolution algorithm provide a speedup like the two approaches given above? At a first glimpse, we may say no to this question. Using some related knowledge in linear algebra, we can propose the following modified form for the evolution Hamiltonian:

$$\overline{H}(s) = [(1-s)H_i + sH_f] \times y, \quad (22)$$

where y is a coefficient that will be determined. By some simple calculations, we could get the energy gap between the two lowest eigenvalues of the Hamiltonian $H(s)$:

$$g_{\min} = y\delta, \quad (23)$$

which implies a time complexity of

$$\overline{T} = O(y\delta)^{-2} \quad (24)$$

for the current algorithm. And it is easy to check that a quadratic speed up can be obtained if we set the coefficient as $y = \delta^{-1/2}$. But at the same time, it is easy to check that whole energy range of the quantum system Hamiltonian increases by a factor of $\delta^{-1/2}$ as well. Physically, this phenomenon can be interpreted as injecting a large piece of energy can speed up the adiabatic evolution procedure, therefore shortening the evolution procedure.

3. Conclusion

In this paper, we have shown different approaches for the adiabatic evolution of the problem studied in the context. At first glance, each of them can provide a quadratic speed up as compared to the global adiabatic evolution and classical search algorithms. However, this just implies that adiabatic evolution based quantum computation has more feasibilities than the traditional quantum circuit model in which such a speed up is usually not easy to achieve by different methods, although the equivalence between them can be found in related literature, such as [6].

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