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Stochastic Bounds for the Max Flow in a Network with Discrete Random Capacities

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Abstract

We show how to obtain stochastic bounds for the strong stochastic ordering and the concave ordering of the maximal flow in a network where the capacities are non negative discrete random variables. While the deterministic problem is polynomial, the stochastic version with discrete random variables is NP-hard. The monotonicity of the Min-Cut problem for these stochastic orderings allows us to simplify the input distributions and obtain bounds on the results. Thus we obtain a tradeoff between the complexity of the computations and the precision of the bounds. We illustrate the approach with some examples.

Keywords: Maximal flow, Random capacity, Stochastic ordering, Increasing convex ordering

1 Introduction

Due to the large number of sensors available in smart cities, smart buildings and in transportation systems, we have a huge volume of data for our models. Quite

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often, these information change with time, due to noise, contention, incidents. They could not be seen as deterministic anymore and we have to deal with the apparent randomness of our measures. For instance the variation of the delay in a transportation system is more much related to congestion than to noise in the measurement process.

Here we propose a method to deal with this randomness for a classical problem in operation research: the computation of the maximal flow. This problem is also important for the analysis of a network reliability. Indeed, it is well known that the two terminals reliability problem is a special case of the maximal flow problem with 0/1 capacity. Let $G = (V, E)$ a capacited directed graph (or network) where the capacity of directed edge e is an integer (it could be 0 and in this case the edge is broken). The capacity of edge e is denoted as $w(e)$. Let N_v be the number of edges. Here we assume that the capacities are random discrete variables.

Computing the maximal flow of such a capacited network is polynomial when the capacities are deterministic. Unfortunately, it is not true anymore when the edges are associated with random variables (see [3] for a survey on the complexity for various delays and flow problems for networks or graphs with random discrete costs or durations). The two terminals reliability problem is proved to be NP-Hard and the stochastic maximal flow is NP-Hard as well. As it is difficult to solve the problem, several approaches have been proposed to obtain approximations or bounds. We briefly reviewed some of these approaches and put more emphasis of the methods associated with stochastic ordering.

Note that it is still possible to solve the problem for small instances when the discrete variables take values in very small sets. It is sufficient to use the Total Probability Theorem after conditioning on the states of all the random variables. Let X_i be the discrete multivariate random vector associated with the distribution of the capacity. We assume that the support of X_i is finite. Let S_i be the size of the support of X_i (i.e. the number of atoms in the distribution) and $Pr(X_i = d_i)$ the probability that edge i has capacity d_i. We denote by Ω the Cartesian product of the support of the input distributions. Under the independence assumptions, the probability of $(d_1, ..., d_k)$ is given by:

$$Pr(d_1, ..., d_{Nv}) = \prod_{i=1}^{N_v} Pr(X_i = d_i). \tag{1}$$

And the size of Ω is $\prod_{i=1}^{N_v} S_i$.

Assume that $MFlow(d_1, .., d_{Nv})$ is the maximal flow when the capacities are deterministic and equal to d_i for edge i. The total probability law gives a formal description of the distribution of the Maximal Flow:

$$Pr(MaxFlow = k) = \sum_{(d_1,...,d_{Nv}) \in \Omega} Pr(d_1, ..., d_{Nv}) 1_{MFlow(d_1,..,d_{Nv})=k},$$

where 1_X is the indicator function for condition X. As already mentioned, computing $MFlow(d_1, .., d_{Nv})$ is a polynomial problem, and computing $Pr(d_1, ..., d_{Nv})$ is easily done with Eq. 1. Therefore the hard part of the problem is the size of Ω. Note that, to avoid the confusion, $MFlow$ will denote the deterministic problem while $MaxFlow$ will be the random variable associated with the maximal flow when the capacities are random.

Thus we advocate computing bounds with input distributions which have a smaller number of atoms. The number of atoms we want to keep is a parameter of the algorithms we have designed. Thus the number of atoms we keep in the bounds gives us a tradeoff between the complexity and the accuracy [2]. We use two key properties. First the strong stochastic ordering and the increasing concave ordering are used to design upper or lower bounding distributions of the capacity of the edges which have a smaller number of atoms. Second, we prove the monotonicity of the Max-Flow problem for the strong stochastic order and the increasing convex order. Due to the monotonicity, computing the max-flow for input bounds is easier because they have less atoms and provides a stochastic bound on the results.

Following Fulkerson's approach of stochastic PERT networks [11], it is often suggested to use the expectations of the random variables as the inputs of a deterministic problem (assuming that the expectations are integers or fractions). Such an approach provides un upper bound for the expectation of the maximal flow (see section 2). Monte Carlo simulation was proposed and improved by Fishman to estimate the cumulative distribution of the maximal flow [7]. Sarangan et al. [15] had proposed an algorithm to compute the distribution of the minimum capacity (and thus MaxFlow). First they replace each random variable by its expectation. They obtain a minimal cut-set for the deterministic problem and they compute the distribution of the capacity of the chosen cut-set taking into account the input distributions. They approximate the value of the stochastic maximal flow by this distribution. Recently, Hastings had proposed in her PHD [13] a new method based on a symbolic description of paths or sets of edges (for instance cuts) and an automatic derivation of stochastic bounds (for large models) or exact results (for small or simple models). These bounds are based on associated random variables [6] and strong stochastic bounds. In an intuitive formulation, associated random variables are positively correlated and this is a natural property exhibited by paths or cut set (because they share edges). Associated random variables were previously used for the analysis of the completion time of a task graph or a PERT network [17]. The methodology proposed by Hastings can be used for many optimization problems on graphs associated with min, max and "+" operators (for instance, Shortest Path, Completion Time and Maximal Flow).

The technical part of the paper is as follows. In section 2, we present a brief introduction to strong stochastic bounds and increasing convex stochastic bounds. We show how to build discrete distributions which are lower or upper bound in the sense of these ordering. We also present basic algorithms to change the size of the distributions while building a bound. These results have already been published in [2,4] and they are given here for the sake of readability. The methods allow to build many stochastic bounds for the max-flow with a low complexity. Therefore in Section 3, we show how to combine them to obtain a more accurate bound. Section 4 is devoted to the numerical results.

2 Strong Stochastic Bounds, Convex/Concave Stochastic Bounds

In the following, S_X will denote the support of the distribution or the random variable. $|S|$ will be the size of set S. δ_x will denote a Dirac distribution with an atom in x.

In [2,4] we have proposed to reduce the number of atoms while keeping some quantitative and qualitative information on the results. This is obtained through the use of stochastic orderings. We begin with the definition of the orders we will use in this paper (see [14] and [16] for more information).

Definition 2.1 [strong stochastic ordering] Let X and Y be two random variables, $X \preceq_{st} Y$ if for all increasing function Φ, $\mathbb{E}[\phi(X)] \leq \mathbb{E}[\phi(Y)]$ if the expectations exist.

The stochastic comparison of random variables also implies that a strict inequality between their expectations as seen below.

Proposition 2.2 *Let X and Y be two random variables, such that $X \preceq_{st} Y$. If $\mathbb{E}[X] = \mathbb{E}[Y]$ then $X =_{st} Y$.*

We also use some orders associated with variability of the random variables to obtain tighter bounds. Let us first consider convex order which is defined as follows.

Definition 2.3 [stochastic convex ordering] Let X and Y be two random variables, $X \preceq_{cx} Y$ if for all convex function ϕ, $\mathbb{E}[\phi(X)] \leq \mathbb{E}[\phi(Y)]$ if the expectations exist [12].

Here we will use the concave ordering which is easily derived from the convex ordering.

Definition 2.4 [stochastic concave ordering] Let X and Y be two random variables, $X \preceq_{cv} Y$ if $Y \preceq_{cx} X$

Definition 2.5 [increasing convex ordering] Let X and Y be two random variables, $X \preceq_{icx} Y$ if for all increasing convex function ϕ, $\mathbb{E}[\phi(X)] \leq \mathbb{E}[\phi(Y)]$ if the expectations exist.

Definition 2.6 [increasing concave ordering] Let X and Y be two random variables, $X \preceq_{icv} Y$ if for all increasing concave function ϕ, $\mathbb{E}[\phi(X)] \leq \mathbb{E}[\phi(Y)]$ if the expectations exist.

Corollary 2.7 *Thus, we have:*

- *$X \preceq_{cx} X$, $X \preceq_{cv} X$ and $X \preceq_{st} X$.*
- *If $X \preceq_{st} Y$, then for all k, $\mathbb{E}[X^k] \leq \mathbb{E}[Y^k]$.*
- *If $X \preceq_{cx} Y$, then for all $k > 1$, $\mathbb{E}[X^k] \leq \mathbb{E}[Y^k]$. Taking into account that $\mathbb{E}[X] = \mathbb{E}[Y]$, we get $Var[X] \leq Var[Y]$.*
- *$X \preceq_{cx} Y$, if and only if $\mathbb{E}[X] = \mathbb{E}[Y]$ and $X \preceq_{icx} Y$.*

- *If* $X \preceq_{st} Y$ *then* $X \preceq_{icv} Y$
- *If* $X \preceq_{st} Y$ *then* $X \preceq_{icx} Y$
- *If* $X \preceq_{cv} Y$ *then* $X \preceq_{icv} Y$

These orders differ considerably when we consider the expectations of the random variables. Thus, using the convex ordering instead of the strong stochastic ordering we keep constant the expectation and we hope that we only introduce a small bias when we deal with bounds instead of the measurements.

The maximal flow of a network is monotone related to various stochastic orderings. Let us define first the generic $\Psi-$monotony.

Definition 2.8 [$\Psi-$Monotony] A function f is $\Psi-$monotone if for all X and Y random variables such that $X \preceq_{\psi} Y$, then $f(X) \preceq_{\psi} f(Y)$.

Due to the definitions of the orderings we considered by set of functions, the following property holds:

Proposition 2.9 *If f is increasing, then it is $st-monotone$. Similarly, f is increasing and concave then it is monotone for the increasing concave ordering.*

In this paper, we will prove that the problems we consider are monotone for the strong ordering or monotone for the increasing concave ordering.

Proposition 2.10 *The MaxFlow problem is monotone for the strong stochastic ordering.*

Proof: As the maximal flow is equal to the minimal cut (the so called Max-Flow=Min Cut Theorem), we define the problem as follows.

$$MaxFlow = Min_C \sum_{e \in C} w(e)$$

where C is a cut of network G and e is an arbitrary edge of C. Therefore as it is defined with the "min" and "+" operators which are increasing, the problem is monotone for the strong stochastic ordering .

Proposition 2.11 *Similarly, the MaxFlow problem is monotone for the increasing concave ordering.*

Proof: because they are all defined with the "min" and "+" operators which are increasing and concave.

Combining the $\Psi-$monotone property and the approach based on conditioning suggest the following method: algorithmically reduce the number of atoms in the distributions to get bounds on the inputs and obtain bounds of the outputs due to the monotone property. We have studied this approach for the strong stochastic ordering [2] and the convex/concave ordering [4]. Such an approach was shown to be valuable for network performance modeling [1], operation research [2], reliability modeling [10]. This approach will be detailed in Section 2.

Our approach relies on the following theorem:

Theorem 2.12 *Let us consider a network with arbitrary random discrete capacity*

$\mathbf{D_i}$ *for directed edge* i*. Let us consider some other distributions* $\mathbf{L}_i$*. If for all directed edge* i*,* $\mathbf{L}_i \preceq_{cv} \mathbf{D_i}$*, then*

$$MAXFLOW(\mathbf{L}_1, \mathbf{L}_2, .., \mathbf{L}_{Nv}) \preceq_{icv} MAXFLOW(\mathbf{D}_1, \mathbf{D}_2, .., \mathbf{D}_{Nv}).$$

Proof: First $\mathbf{L}_i \preceq_{cv} \mathbf{D_i}$ implies that $\mathbf{L}_i \preceq_{icv} \mathbf{D_i}$. Then, the inequality is a consequence of Prop. 2.11 on the monotonicity of the MaxFlow problem for the increasing concave ordering.

In the literature, a well-known approach consists in taking the expectation for all the random variables and solve the deterministic problem (assuming that the expectations are integers). Using this approach adds a systematic bias which must be known.

Theorem 2.13 *If we replace each random variables by its expectation and compute the maximum flow, we obtain an upper bound of the expectation of the exact distribution.*

$$\mathbb{E}[MAXFLOW(\mathbf{D}_1, \mathbf{D}_2, .., \mathbf{D}_{Nv})] \leq MAXFLOW(\mathbb{E}[\mathbf{D}_1], \mathbb{E}[\mathbf{D}_2], .., \mathbb{E}[\mathbf{D}_{Nv}]).$$

Proof: First we know from Property 2.15 that $X \preceq_{icv} \mathbb{E}(X)$ for all random variable X. As MaxFlow problem is defined with increasing and concave function, we have:

$$MaxFlow(X) \preceq_{icv} MaxFlow(\mathbb{E}(X))$$

Taking the expectations of both random variables , we get (because of the definition of the increasing concave ordering)

$$\mathbb{E}(MaxFlow(X)) \leq \mathbb{E}(MaxFlow(\mathbb{E}(X)))$$

As $MaxFlow(\mathbb{E}(X))$ is deterministic, we have:

$$\mathbb{E}(MaxFlow(\mathbb{E}(X))) = MaxFlow(\mathbb{E}(X)).$$

And finally,

$$\mathbb{E}(MaxFlow(X)) \leq MaxFlow(\mathbb{E}(X)).$$

Note that the expectation of the distribution is an upper bound for the concave order. Thus one may expect that using Theorem 2.12 instead one may obtain more accurate concave bounds. The upper bound based on the expectation of all the distributions is the worst bound based on concave ordering of the inputs. Finally we add some well-known properties which will be useful to prove our algorithms. Their proofs and more results on these stochastic orderings can be found in the literature [14,16].

Proposition 2.14 (Stop Loss) *Let* X *and* Y *be two random variables,* $X \preceq_{cx} Y$ *if and only if* $\mathbb{E}[X] = \mathbb{E}[Y]$ *and, for all* d *we have,* $\mathbb{E}[(X-d)^+] \leq \mathbb{E}[(Y-d)^+]$*.*

Proposition 2.15 (Expectation) *Let* X *be a random variable with finite expectation, then* $E[X] \preceq_{cx} X$ *and* $X \preceq_{cv} E[X]$*.*

This property is very important as it provides the basic action to obtain upper bound for the concave ordering and lower bound for the convex ordering.

Proposition 2.16 (Mixing) *Let X, Y and Θ three random variables such that $[X|\Theta = a] \preceq_{cx} [Y|\Theta = a]$ for all a in the support of Θ, then $X \preceq_{cx} Y$.*

2.1 Basic algorithms for strong stochastic ordering

The main question is to build stochastic upper and lower bounding distributions for the input distributions. Let us assume that the size of an arbitrary initial distribution is N and that we want to obtain a bound of size K. We first propose some elementary actions which build upper or lower bounds for the strong stochastic ordering with one atom less. Applying these actions $N - K$ times we will get distributions with size K.

Lemma 2.17 *[Upper bounding distribution] We consider an arbitrary discrete distribution (say* **D1***). We consider two atoms a and b of* **D1** *(without loss of generality we assume that $a < b$) defined by their positive probabilities p_a and p_b. We consider discrete distribution* **D2** *defined as follows:*

- *q_i is the probability of atom i in* **D2**
- *The atoms of* **D2** *are the atoms of* **D1** *except a*
- *for all atoms i of* **D2** *except b, $q_i = p_i$,*
- *$q_b = p_a + p_b$.*

Then, $\mathbf{D1} \preceq_{\mathbf{st}} \mathbf{D2}$.

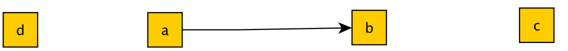

Fig. 1. Fusion of two atoms for an upper bound for the strong order

Lemma 2.18 *[Lower bounding distribution] We consider again an arbitrary discrete distribution (say* **D1***) and two atoms arbitrary a and b of* **D1** *(without loss of generality we assume that $a < b$) with by their positive probabilities p_a and p_b. We consider discrete distribution* **D3** *defined as follows:*

- *q_i is the probability of atom i in* **D3**
- *The atoms of* **D3** *are the atoms of* **D1** *except b*
- *for all atoms i of* **D2** *except a, $q_i = p_i$,*
- *$q_a = p_a + p_b$.*

Then, $\mathbf{D3} \preceq_{\mathbf{st}} \mathbf{D1}$.

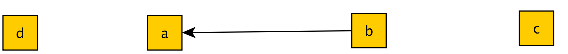

Fig. 2. Fusion of two atoms for a lower bound for the strong order

Example 2.19 Let **D1** be a discrete distribution defined on $\mathcal{H}1 = \{1, 2, 4, 5, 8, 9\}$ with following probabilities $[0.2, 0.1, 0.1, 0.2, 0.3, 0.1]$. Then applying two times

Lemma 2.17 on atoms 2, 4 for the first iteration and atoms 5 and 9 for the second, we obtain a distribution **D2** with 4 atoms: $\mathcal{H}2 = \{1, 4, 8, 9\}$ with probability $[0.2, 0.2, 0.3, 0.3]$. One can easily obtain $\mathbb{E}[\mathbf{D1}] = \mathbf{5.1}$ and $\mathbb{E}[\mathbf{D2}] = \mathbf{6.1}$. Both distributions are depicted in Fig. 3.

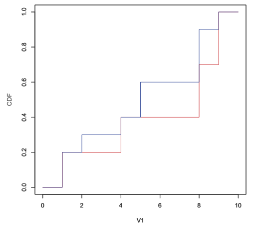

Fig. 3. Upper bound for the strong order. The CDF of **D1** is depicted in blue. Its upper bound in in red. Remark that an upper bound in strong ordering implies that the CDF of the upper bound is always below the initial distribution.

So we can obtain upper and lower bounds for the input distributions of our problem (i.e. the capacities) with a very low complexity. It remains to decide which atoms to combine to obtain an accurate bound and also which input distributions have to be replaced by a bound. The first problem has partially been solved in [2] for the strong stochastic order and in [4] for the convex and concave order. We do not address the last problem in this paper but we show in Section 3 how to improve bounds by combining them.

In [2] we have also found an algorithm which computes the optimal upper bounds in the following sense. For an arbitrary distribution D with size N and any positive increasing reward function r, we proved in [2] an algorithm to find the distributions $D1$ and $D2$ with size $K < N$ such that

- $\mathbf{D1} \preceq_{\mathbf{st}} \mathbf{D} \preceq_{\mathbf{st}} \mathbf{D2}$
- **D1** and **D2** are optimal bounds according to the expectation of function r.

The optimality of **D1** means that if we found a distribution **D3** such that $\mathbf{D3} \preceq_{\mathbf{st}} \mathbf{D}$ and $\sum_i \mathbf{r(i)D1(i)} \leq \sum_{\mathbf{i}} \mathbf{r(i)D3(i)} \leq \sum_{\mathbf{i}} \mathbf{r(i)D(i)}$, then $\mathbf{D3} = \mathbf{D1}$ or $\mathbf{D3} = \mathbf{D}$. The optimality of **D2** is defined in a similar manner. Note that, as function $\mathbf{r}$ is increasing, $\mathbf{D1} \preceq_{\mathbf{st}} \mathbf{D}$ implies that $\sum_i \mathbf{r(i)D1(i)} \leq \sum_{\mathbf{i}} \mathbf{r(i)D(i)}$.

2.2 Basic algorithms for concave and convex orderings

We now prove some lemmas on the basic operations of fusion of atoms to obtain lower bounds (Lemma 2.20) and upper bounds (Lemma 2.21) for the convex ordering. Note that we only present the simplest actions, one can find in [4] more complex algorithms to design bounds for this stochastic order.

Lemma 2.20 *We consider an arbitrary discrete distribution (say* **D1***) with two atoms a and b (without loss of generality we assume that $a < b$) defined by the following positive probabilities p_a and p_b. We build distribution* **D2** *as follows:*

- *q_i is the probability of atom i in* **D2**

- *The atoms of* $\mathbf{D2}$ *are the atoms of* $\mathbf{D1}$ *except* a *and* b *which are omitted and* M *which may be added.*
- *for all atoms* i *of* $\mathbf{D2}$ *except* a, b *and* M, $q_i = p_i$,
- $q_M = p_a + p_b + p_M$ (p_M *is* 0 *when* M *is not an atom of* $\mathbf{D1}$).

Then, $\mathbf{D2} \preceq_{\mathbf{cx}} \mathbf{D1}$. *We also have* $\mathbf{D1} \preceq_{\mathbf{cv}} \mathbf{D2}$.

Proof: it is a simple application of property 2.15 and 2.16 Note that M may already be an atom of $\mathbf{D1}$. In that particular case, the number of atoms in $\mathbf{D2}$ is reduced by 2. Otherwise, it is only reduced by 1.

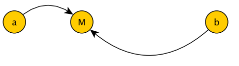

Fig. 4. Fusion of two atoms for a lower bound for the convex order and an upper bound for the concave order

It is worthy to remark that a lower bound for the concave ordering of a distribution with more than 2 atoms has at least two atoms. Therefore when we design Icv bounds for the maximal flow, we must use strong stochastic bounds of the input distributions (due to item 5 of Corollary 2.7) if the complexity of using concave bounds of these distributions is too high.

Lemma 2.21 *We consider an arbitrary discrete distribution (say* $\mathbf{D3}$*) with at least three atoms* a, b, c *(without loss of generality we assume that* $a < b < c$*) defined by the positive probabilities* p_a, p_b *and* p_c. *Let us build* $\mathbf{D4}$ *a new distribution with probabilities denoted as* q_i *such that*

- *The atoms of* $\mathbf{D4}$ *are the atoms of* $\mathbf{D1}$ *except* b *which is omitted,*
- *for all atoms* i *of* $\mathbf{D2}$ *except* a *and* c, $q_i = p_i$,
- q_a *and* q_c *are defined by*

$$q_a + q_c = p_a + p_b + p_c$$

and

$$aq_a + cq_c = ap_a + bp_b + cp_c.$$

Then, $\mathbf{D4}$ *is an upper bound for the convex stochastic ordering of* $\mathbf{D3}$: $\mathbf{D3} \preceq_{\mathbf{cx}} \mathbf{D4}$. *We also have* $\mathbf{D4} \preceq_{\mathbf{cv}} \mathbf{D3}$

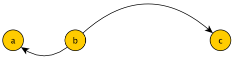

Fig. 5. Upper bounding distribution.

See [4] for a proof. Remark that this operation is the discrete analog of the fusion operation studied by Eltan and Hill [5].

Example 2.22 Consider again $\mathbf{D1}$ a discrete distribution defined on $\mathcal{H}1 = \{1, 2, 4, 5, 8, 9\}$ with following probabilities $[0.2, 0.1, 0.1, 0.2, 0.3, 0.1]$. We apply Lemma 2.21 on atoms 1, 2, 5 for the first iteration. We obtain a distribution

$bfD2$ with 5 atoms: $\mathcal{H}2 = \{1, 4, 5, 8, 9\}$ with probability $[0.275, 0.1, 0.225, 0.3, 0.1]$. One can easily check that $\mathbb{E}[\mathbf{D1}] = \mathbf{5.1} = \mathbb{E}[\mathbf{D2}]$. Now we apply again Lemma 2.21 to split atom 8 on atoms 5 and 9. We get support $\mathcal{H}3 = \{1, 4, 5, 9\}$ and probabilities $[0.275, 0.1, 0.3, 0.325]$ All the distributions are depicted in Fig. 6.

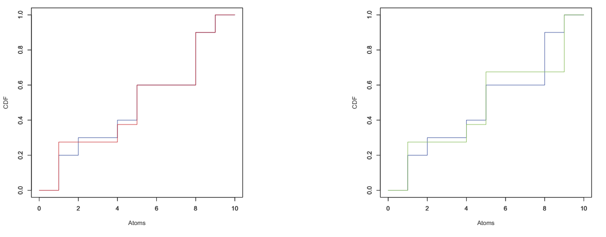

Fig. 6. Upper bound for the concave order. The CDF of **D1** is depicted in blue in both figures. The first upper bound in in red in the left figure. The second upper bound for the concave order in the green in the right figure. Remark that the CDF of the upper bounds in the concave order cross the CDF of the initial distribution.

We also have an optimal bound for the increasing concave upper bound which has been proved in [4] which is based on the basic actions we have presented. More precisely, for a arbitrary distribution **D** with N positive atoms and convex function $r(x) = x^2$, we prove an algorithm to find **D2** such that

- $\mathbf{D} \preceq_{\mathbf{cx}} \mathbf{D2}$
- **D2** has size $K < N$.
- **D2** is an optimal bound according to the expectation of function $r(x) = x^2$ (i.e. the second moment).

Furthermore, we propose an algorithm to find a lower bound **D1** with size K but we do not prove the optimality of our method. This method can be easily modified to deal with any arbitrary increasing convex function r.

When we deal with capacity in a maximal flow problem, one must take into account that the capacities must be an integer for the Ford and Fulkerson Algorithm to converge. Thus, when we add one new atom by the fusion operation detailed in Lemma 2.20, this atom is not an integer in general. Therefore we proceed as follows:

(i) Let **D** be the initial distribution.

(ii) First, we compute the bounding distributions using the fusion operation and we do not care about the expectation being integer or not

(iii) We proceed until the distribution has fixed size K . Let **D1** be this distribution. By construction we have: $\mathbf{D} \preceq_{\mathbf{icv}} \mathbf{D1}$.

(iv) Finally, we modify all the atoms d of **D1** to obtain a new distribution (say **D2**).
  - If atom d of **D1** is an integer, we keep it in **D2**
  - If atom d is not an integer, it is mapped to the next integer in **D2**. Several atoms may be merged in that operation.

Clearly, $\mathbf{D1} \preceq_{\mathbf{st}} \mathbf{D2}$.

(v) Thus, $\mathbf{D1} \preceq_{\mathbf{icv}} \mathbf{D2}$ (see Corollary 2.7) and $\mathbf{D} \preceq_{\mathbf{icv}} \mathbf{D2}$ by transitivity.

Example 2.23 Let $\mathbf{D1}$ be a discrete distribution defined on $\mathcal{H}1 = \{1, 2.5, 2.6, 3, 3.4, 5\}$ with following probabilities $[0.2, 0.1, 0.1, 0.2, 0.3, 0.1]$. Then atoms 1 and 5 are kept unchanged. Atoms 2.5 and 2.6 are merged with atom 3 and become atom 3 in the bound while atom 3.4 becomes atom 4 of $\mathbf{D2}$. The support of $\mathbf{D2}$ is $\{1, 3, 4, 5\}$ and the distribution is $[0.2, 0.4, 0.3, 0.1]$.

This example shows one of the property of the concave ordering.

Proposition 2.24 *The extreme atoms of the initial distributions (say* $\mathbf{D}$*) are kept in the upper concave bounds (i.e. they have a positive probability) computed by the splitting operation described in Lemma 2.21. Indeed at each step, we consider three ordered atoms and the atom in the middle is removed. Note that this property is also true when we compute the upper bounding distribution (for the increase concave ordering) of the maximal flow by the Total Probability method.*

3 Combining Several Distributions

In the previous section we have found how to compute bounds for the input distributions. Let N_i be the size of the random variable describing capacity of edge i. We know how to bound these distributions with lower or upper bounds with size K_i. The question is to chose K_i for all directed edge i. Clearly, changing the K_i gives a new bounding distribution for the MaxFlow and a natural question is to combine all these bounds into a more accurate one. Of course, the way to combine these distributions depend on the ordering and so on the bounding algorithms.

3.1 Strong order

Thus the general problem is the following. Assume that we have obtained two upper bounds Y and Z of X for the strong stochastic ordering. How to compute a new distribution W such that: $X \preceq_{st} W$, $W \preceq_{st} Y$ and $W \preceq_{st} Z$? Clearly W will be more accurate than both Y and Z.

Remember that $X \preceq_{st} Y$ implies that for all a in the support of the distributions $Pr(Y \leq a) \leq Pr(X \leq a)$. By assumptions we also have: $Pr(Z \leq a) \leq Pr(X \leq a)$. Therefore, for all a

$$max(Pr(Z \leq a), Pr(Y \leq a)) \leq Pr(X \leq a).$$

Let us define W by $Pr(W \leq a) = max(Pr(Z \leq a), Pr(Y \leq a))$. Clearly W satisfies all the properties required. As we deal with discrete distributions, we only have to compute $Pr(W \leq a)$ for the support of $S_Z \cup S_Y$. And we have: $S_W \subset S_Z \cup S_Y$. Algorithm 1 performs such a computation. It assumes that the distributions are represented as sorted lists.

In the code of this algorithm, CDF is a scalar to store the current value of the CDF up to the current atom Procedure Insert(W,a,f) performs an insertion in the structure W (the output distribution) of an atom a with sum of probability f. This can be done with a constant complexity using sorted lists.

Algorithm 1 Algorithm to combine two upper bounds for the strong stochastic ordering.

Input: input distributions **Z**, and **Y** given by their atoms S_Y and S_Z and their probability vectors $Pr_Z[]$ and $Pr_Y[]$
Output: Output distribution **W**

$CDF_Z = 0$. $CDF_Y = 0$. $CDF_W = 0$.
for all atoms a in $S_Y \cup S_Z$ **do**
  if $(Pr_Z(a) > 0)$ **then**
    CDF_Z += $Pr_Z[a]$
  end if
  if $(Pr_Y(a) > 0)$ **then**
    CDF_Y += $Pr_Y[a]$
  end if
  f = max(CDF_Z,CDF_Y)
  if $(f > CDF_W)$ **then**
    $CDF_W = f$
    Insert(W,a,f)
  end if
end for

Proposition 3.1 *Consider two upper bounding distributions the size of which are $N1$ and $N2$. Algorithm 1 computes a combination of two upper bounding distributions which is more accurate and it requires $0(N1+N2)$ steps if the distributions are stored as sorted lists. The size of the resulting distribution is smaller than $N1+N2$ but this value is tight.*

Note that it is possible to combine with the same approach two lower bounds for the strong stochastic ordering. Assume that we have obtained two lower bounds Y and Z of X for the strong stochastic ordering. We define W by its CDF $Pr(W \leq a) = min(Pr(Z \leq a), Pr(Y \leq a))$. Then we have: $Pr(X \leq a) \leq Pr(Y \leq a)$ and $Pr(X \leq a) \leq Pr(Z \leq a)$ (by assumptions) and $Pr(X \leq a) \leq Pr(W \leq a) = min(Pr(Z \leq a), Pr(Y \leq a))$. Again W is more accurate than both Y and Z.

Example 3.2 Let **D1** be a discrete distribution defined on $\mathcal{H}1 = \{1, 2, 4, 5, 8, 9\}$ with probabilities $[0.1, 0.2, 0.1, 0.3, 0.2, 0.1]$ and **D2** be another discrete distribution defined on $\mathcal{H}2 = \{1, 3, 4, 6, 7\}$ with probabilities $[0.2, 0.1, 0.1, 0.4, 0.2]$. Assume that both distributions are upper bounds for the strong stochastic order of a distribution **D**. Then one can combine distributions **D1** and **D2** to obtain a more accurate bound. Using Algorithm 1, we obtain a bounding distribution defined on $\mathcal{H}3 = \{1, 2, 4, 5, 6, 7\}$ with probability vector $[0.2, 0.1, 0.1, 0.3, 0.1, 0.2]$. All three distributions are depicted in Fig. 7. Clearly the CDF of the new bound is the maximum of the two CDFs.

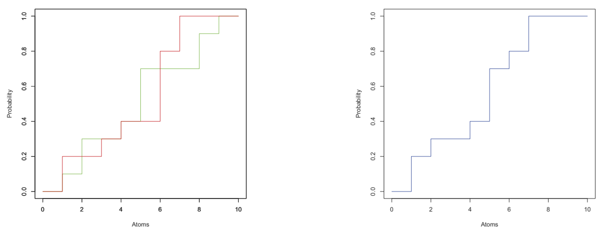

Fig. 7. Combining upper bounding distributions. On the left, two upper bounds. On the right, the new upper bound obtained by combining these two bounds.

3.2 Concave order

We show how we can combine two upper bounds to prove a more accurate upper bounding distribution. The method is based on the stop loss property. An equivalent formulation of this property for increasing concave ordering follows: $X \preceq_{icv} Y$ if for all d, $\mathbb{E}[min(X,d)] \geq \mathbb{E}[min(Y,d)]$. This property will be used to develop algorithms when the random variables are discrete.

Definition 3.3 We define the stop loss function of random variable (or distribution) Y as follows:

$$SL_Y(y) = \mathbb{E}[min(Y,y)].$$

Proposition 3.4 *The Stop Loss function of Y is an affine function by intervals. The boundaries of the intervals are the atoms of distribution Y. Furthermore at an atom d_i, the difference of the slopes of two consecutive affine functions gives the probability of the atom.*

Proof: Algebraic manipulation. Consider two consecutive atoms $d1$ and $d2$ of Y. We assume that $d1 < y \leq d2$. By construction:

$$SL_Y(y) = \mathbb{E}[min(Y,y)] = \sum_{i \in S_Y} min(i,y)Pr(i) = y * \sum_{i \geq d2} Pr(i) + \sum_{i \leq d1} i * Pr(i)$$

When $y \leq d1$ where $d1$ is the smallest atom of Y we have $SL_Y(y) = y$ while $SL_Y(y) = \mathbb{E}[Y]$ when y is larger than all the atoms of the distribution. Thus the result is proved for all the intervals. Clearly, the function is also continuous at the boundaries. Now let us compute the two curves crossing at $d2$. The first curve has equation $y * \sum_{i \geq d2} Pr(i) + \sum_{i \leq d1} i * Pr(i)$ while the second will be $y * \sum_{i \geq d3} Pr(i) + \sum_{i \leq d2} i * Pr(i)$, assuming that $d3$ is the next atom after $d2$. Therefore the difference of the slopes is $Pr(d2)$. It is also interesting to note that the slopes are decreasing when we progress along the intervals.

Example 3.5 Consider again **D** a discrete distribution defined on $\mathcal{H}1 = \{1,2,4\}$ with following probabilities $[0.3, 0.3, 0.4]$. The Stop Loss function of **D1** is defined as follows

- $y \leq 1$, $SL_Y(y) = \mathbb{E}[min(Y,y)] = \mathbb{E}[(y)]$ because $min(Y,y) = y$ due to the assumption. Thus, $SL_Y(y) = y$.
- $1 < y \leq 2$, $SL_Y(y) = \mathbb{E}[min(Y,y)] = Pr(Y = 1) * 1 + Pr(Y = 2) * y + Pr(Y =$

$4) * y = 0.7 * y + 0.3$.

- $2 < y \leq 4$, $SL_Y(y) = \mathbb{E}[min(Y, y)] = Pr(Y = 1) * 1 + Pr(Y = 2) * 2 + Pr(Y = 4) * y = 0.4 * y + 0.9$.
- $y > 4$, $min(Y, y) = Y$. Therefore $SL_Y(y) = \mathbb{E}[Y] = 2.5$. The function is represented Fig. 8

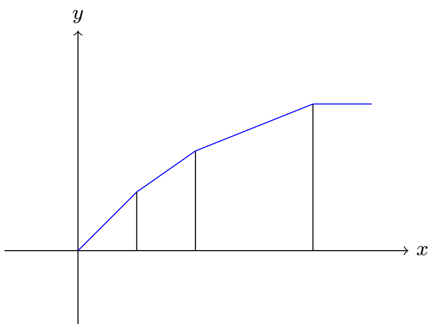

Fig. 8. The stop loss function.

Proposition 3.6 *Let Y and Z be two upper bounds of X for the increasing concave stochastic ordering. We define distribution W by its step function as follows:*

$$SL_W(y) = min(SL_Z(y), SL_Y(y)).$$

Then W is an upper bound of X for the increasing concave ordering.

Proof: First, we have by assumption: for all y, $SL_X(y) \leq SL_Z(y)$ and $SL_X(y) \leq SL_Y(y)$. Therefore $SL_X(y) \leq min(SL_Z(y), SL_Y(y)) = SL_W(y)$. We will show in Prop. 3.7 that W is a proper distribution of probability. First we have to detail the construction of SL_W.

We give an algorithm to compute $SL_W(y)$. Let us consider an arbitrary interval and its boundaries $d1$ and $d2$ which are consecutive atoms in $S_Y \cup S_Z$. Without loss of generality we assume that $d1 < d2$. Between d_1 and d_2 the stop loss function of Z (i.e. $SL_Z(x)$) is an affine function $a_Z(d1)x + b_Z(d1)$. Similarly $SL_Y(x)$ is an affine function $a_Y(d1)x + b_Y(d1)$. Therefore for the interval $(d1, d2]$ we have to compute the minimum of two affine functions. Two cases may occur:

- The two functions cross in the interval. Thus the minimum is not an affine function on the interval. Instead it is an affine function between $d1$ and c and

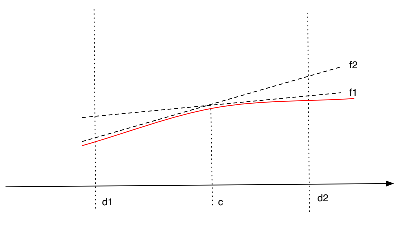

Fig. 9. Affine functions $f1$ and $f2$ cross inside interval $(d1, d2]$ at c

another affine function between c and $d2$ where c is the intersection of the two affine functions $f1$ and $f2$. Thus we have to compute the intersection and the description of the minimum. Clearly, we have:

$$c = \frac{b_Y(d1) - b_Z(d1)}{a_Z(d1) - a_Y(d1)}$$

and the minimum is easily determined by the smallest function at $d1$.

- Affine functions do not cross inside the interval. Therefore one of them is dominating the other on the interval. And the minimum is the affine function which is the dominated one.

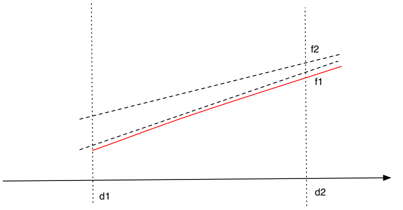

Fig. 10. Affine functions $f1$ and $f2$ do not cross inside interval $(d1, d2]$.

Now we have to show how to extract a distribution of probability from $SL_W(y)$.

The algorithm builds the Stop Loss function by an iteration on the successive intervals. In the following algorithm, the method "Add(SL,d1,d2,f)" appends function f inside interval $(d1, d2]$ in the description of Stop Loss function SL.

Proposition 3.7 *One can derive a probability distribution from* $SL_W(y) = min(SL_Y(y), SL_Z(y))$. *The support is included in the union of the support of Z and Y and the set of intersection nodes. The probability of an atom is obtained by the difference between the slopes of the function before and after the atom. Atoms with a null probability are removed from the support.*

It is important to remark that we can not use this method to improve increasing concave lower bounds. Of course, one can build $\max(SL_Z(y),\ SL_Y(y))$ but it is easy to remark that the slopes of these functions may be increasing in some cases. And this property implies that we cannot extract a proper distribution from $\max(SL_Z(y),\ SL_Y(y))$.

Proposition 3.8 *Consider two upper bounding distributions for the increasing concave ordering, the size of which are $N1$ and $N2$. Algorithm 2 computes a combination of two upper bounding distributions for the increasing concave ordering, which is more accurate and it requires* $0(N1 + N2)$ *steps if the distributions are stored as sorted lists. The size of the resulting distribution may be as large as $N1 + N2$. Remark that two arrays of size $N1$ or $N2$ are sufficient to store one Stop Loss Function.*

Example 3.9 Consider the following two distributions **D1** with support $\mathcal{H}1 = \{1, 2, 4\}$ and probabilities $[0.3, 0.3, 0.4]$ and **D2** with support $\mathcal{H}1 = \{1, 3, 4\}$ and probabilities $[0.4, 0.1, 0.5]$ (see Fig. 11). The Stop Loss functions intersect between 2 and 3, more precisely at 2.5. Therefore the bound obtained by combining **D1**

Algorithm 2 Algorithm to combine two upper bounds for the increasing concave stochastic ordering.

Input: input distributions **Z**, and **Y**
Output: Output distribution **W**

Compute the Stop Loss functions for Z: $f_Z(x)$ for all intervals.
Compute the Stop Loss functions for Y: $f_Y(x)$ for all intervals.
Init $SL_W = \emptyset$
for all intervals $(d1, d2]$ based on consecutive atoms $d1$, $d2$ in $S_Y \cup S_Z$ **do**
  Let $f1$ be the affine function associated with Z and $f2$ be the affine function associated with Y.
  if $f1$ and $f2$ cross in c inside $(d1, d2]$ **then**
    if $f1(d1) < f2(d1)$ **then**
      $Add(SL_W, d1, c, f1)$
      $Add(SL_W, c, d2, f2)$
    else
      $Add(SL_W, d1, c, f2)$
      $Add(SL_W, c, d2, f1)$
    end if
  else
    if $f1(d1) < f2(d1)$ **then**
      $Add(SL_W, d1, d2, f1)$
    else
      $Add(SL_W, d1, d2, f2)$
    end if
  end if
  Obtain W from its Stop Loss function SL_W using the difference of the slopes.
end for

and **D2** has a support included in $\{1, 2, 2.5, 3, 4\}$. We give in Table 1 the Stop Loss functions for both distributions.

Interval	$SL_Y(y)$	$SL_Z(y)$
$[0,1)$	y	y
$[1,2)$	$0.7*y+0.3$	$0.6*y+0.4$
$[2,3)$	$0.4*y+0.9$	$0.6*y+0.4$
$[3,4)$	$0.4*y+0.9$	$0.5*y+0.7$
$[4,+\infty)$	2.5	2.7

Interval	$SL_W(y)$
$[0,1)$	y
$[0,2)$	$0.6*y+0.4$
$[2,2.5)$	$0.6*y+0.4$
$[2.5,3)$	$0.4*y+0.9$
$[3,4)$	$0.4*y+0.9$
$[4,+\infty)$	2.5

Table 1
Stop Loss functions for Y, Z and W.

We now build the Stop Loss function SL_W. From these function, it can be seen that the probability of atom 3 and atom 2 is zero and we remove them from the support. Finally, the support of the combined distribution is $\{1, 2.5, 4\}$, and the probability vector is $[0.4, 0.2, 0.4]$.

Remark that the intersection of the two stop loss functions may not be an integer.

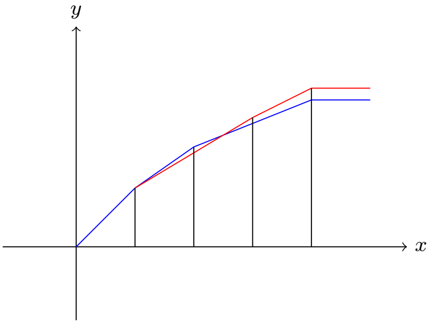

Fig. 11. The stop loss functions cross between 2 and 3.

Even if it is not a valid capacity, it improves the accuracy of the expectation of the maximal flow.

The algorithms we design to bound the input distributions can be used in many applications where measurements are available. They will be added in the next version of our performance evaluation tool (XBorne [9,8]).

4 Numerical Results

We study a small graph (depicted Fig. 12) to illustrate the approach and the algorithms. As the size of the graph and the distributions is not very large, we can compare the bounds with the exact results.

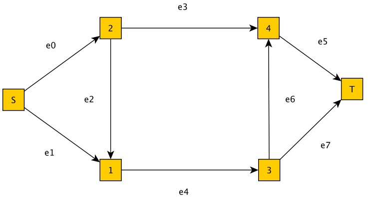

Fig. 12. The example graph.

The input distributions for the edge capacity are gathered in Table 2. The graph has 8 edges. Each input distribution has 8 atoms. Therefore the total probability approach leads to the evaluation of $8^8 = 2^{24}$ Maximal Flow problems with deterministic capacity.

In the following we present the results for several bounding strategies. They are grouped by the number of deterministic flows (i.e. $Mflows$) we have to solve. As expected this provides a tradeoff between the complexity of the computation (i.e. the number of deterministic maximal flows problems) and the accuracy of the bound. To compare bounds, we compute the expectation of the maximal flow for each strategy. The exact expectation of the maximal flow is 6.38. Indeed, as the complexity is not that large, we also solve the initial problem: we compute with the Total Probability approach the distribution of the maximal flow. The complete distribution is depicted in Fig. 13. We have drawn in separate pictures the tail and the head of the distribution to be clearer.

Edge	Support of the distribution	Probability	Expectation
e0	{2, 3, 7, 8, 10, 11, 15, 16}	[0.1, 0.1, 0.2, 0.1, 0.1, 0.2, 0.1, 0.1]	9
e1	{1, 2, 3, 4, 8, 9, 10, 13}	[0.1, 0.2, 0.1, 0.1, 0.2, 0.1, 0.1, 0.1]	6
e2	{1, 2, 3, 4, 8, 9, 10, 13}	[0.1, 0.2, 0.1, 0.1, 0.2, 0.1, 0.1, 0.1]	6
e3	{1, 2, 3, 4, 7, 8, 9, 12}	[0.2, 0.1, 0.2, 0.1, 0.1, 0.1, 0.1, 0.1]	5
e4	{1, 2, 3, 4, 7, 8, 9, 12}	[0.2, 0.1, 0.2, 0.1, 0.1, 0.1, 0.1, 0.1]	5
e5	{1, 2, 3, 4, 8, 9, 10, 13}	[0.1, 0.2, 0.1, 0.1, 0.2, 0.1, 0.1, 0.1]	6
e6	{1, 2, 3, 4, 7, 8, 9, 12}	[0.2, 0.1, 0.2, 0.1, 0.1, 0.1, 0.1, 0.1]	5
e7	{1, 2, 3, 4, 6, 9, 11, 16}	[0.1, 0.2, 0.1, 0.1, 0.1, 0.1, 0.1, 0.2]	7

Table 2
The capacity distributions for the edges of the simple graph.

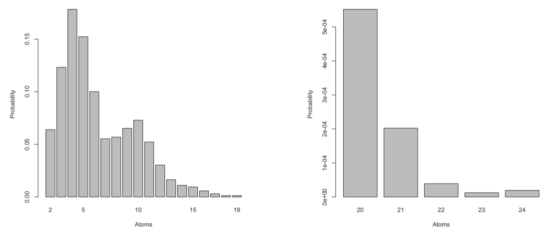

Fig. 13. Exact distribution of the maximal flow. Head on the left, tail on the right.

.

4.1 *Bounds with 1 Mflow*

Clearly, as the problem is monotone for the strong ordering, replacing each distribution by a Dirac on an atom equal to the maximum of the distribution will provide a deterministic problem the solution of which will be an upper bound of the distribution of the flow. Similarly, replacing all the input distributions by a Dirac on the smaller atom gives a lower bound.

Furthermore note that the approach based on the deterministic problem with capacity equal to the expectation of the random variables leads to a maximal flow which is an upper bound for the concave ordering. as shown by Theorem 2.13. All these bounds only require to solve 1 deterministic maximal flow problem. The expectations are reported in the next table. The computation time is smaller than $1ms$.

Exact	Upper St 1 atom	Lower St 1 atom	Upper Icv 1 atom
6.38	24	2	10

Table 3
Comparisons for the expectation for the bounding distributions (strong ordering and increasing convex ordering).

The lower bound and the upper bound based on two atoms require 14 ms on the same ordinary laptop.

4.2 Bounds with 8 $Mflows$

We first present a very simple strategy for the strong stochastic bounds (both upper and lower bounds). All random variables (except one) are replaced by their extreme atom associated with a probability equal to 1. So we obtain 8 upper bounding distributions and 8 lower bounding ones which are reported in the following figures (we have omitted the upper bounds based on edges $e2$ and $e6$ because they consist in a δ_{24} distribution. Note that these bounds are very easily obtained as the number of cases is now the number of atoms in the input distribution for the capacity which has not been changed (8 atoms in this example). The lower bounding distributions are not represented: they are all equal to δ_2.

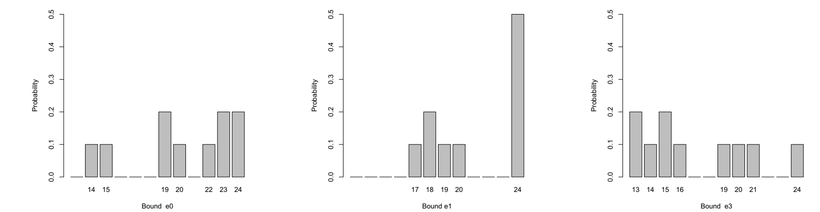

Fig. 14. Upper bound for strong ordering. Left figure (only e0 is unchanged), center (only e1 is unchanged), figure on the right (only e3 is unchanged)

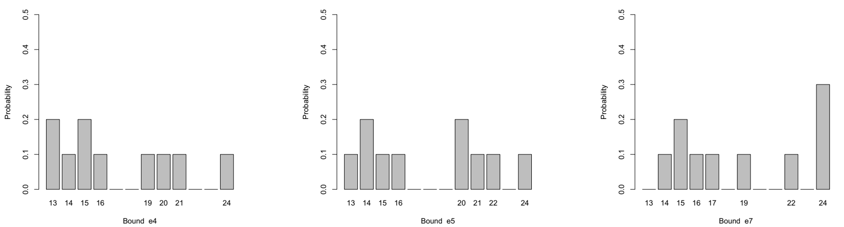

Fig. 15. Upper bound for strong ordering. Left figure (only e4 is unchanged), center (only e5 is unchanged), figure on the right (only e7 is unchanged)

The strategy for the upper bound in the increasing concave ordering is similar. All the random variables (except one) are replaced by a Dirac distribution located at their expectation. We also report the 5 bounding distributions, the remaining ones are all equal to δ_{10}. The bounds using increasing concave ordering are clearly much more accurate than the bounds based on strong stochastic ordering.

To obtain lower bounds for increasing concave ordering, one must consider lower bounds of the inputs for this ordering. But the simplest lower bound for the concave ordering have two atoms. These bounds contain the smallest and the largest atom of the initial distribution and have the same expectation. For instance for the lower bound for the capacity of edge $e0$, we use a distribution with two atoms in 2 and 9 and a probability vector equal to $[0.5, 0.5]$. To keep the same strategy (all distribution except one aggregated into one single atom), one must use strong stochastic bounds and we get the same result. To obtain a distinct lower bound

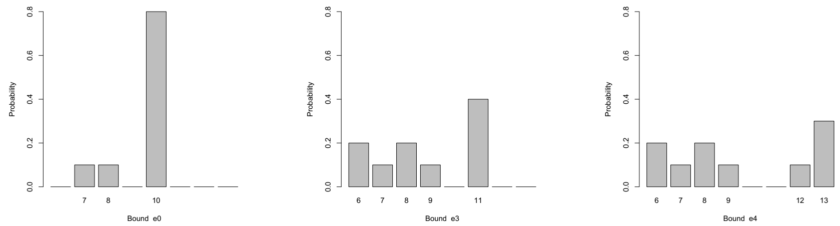

Fig. 16. Upper bound for increasing concave ordering ordering. Left figure (only e0 is unchanged), center (only e3 is unchanged), figure on the right (only e4 is unchanged)

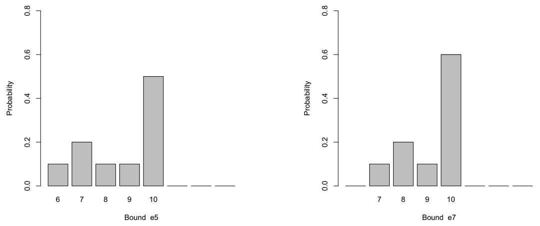

Fig. 17. Upper bound for increasing concave ordering ordering. Left (only e5 is unchanged), right (only e7 is unchanged)

for the increasing concave bound of the maximal flow, we need to compute 2^8 deterministic maximal flow problems (i.e. a concave bounds with two atoms per distribution and 8 edges). Therefore these bounds are reported in the next section with all the bounds using 256 computations of max flow.

For each method we have reported the more accurate result and the less accurate result.

Edge	e0	e1	e2	e3	e4	e5	e6	e7
St Upper bound	20.3	21.2	24	17	17	17	24	19
St Lower bound	2	2	2	2	2	2	2	2
Icv Upper bound	9.5	10	10	8.8	9.5	8.7	10	9.2

Table 4
Expectations for St and Icv Bounds based on 8 deterministic flows.

It is clear that icv upper bounds are always better (for the expectations) than the st upper bounds.

4.3 Bounds with 2^8 $Mflows$

To obtain better bounds we investigate now two directions. First we use another constant aggregation scheme of the input distributions with more atoms. We bound each input distribution by a distribution with two atoms, according to the strong ordering or the convex ordering. Second, we use some properties of the graph to design an aggregation pattern which does not use the same number of atoms for each input distributions.

4.3.1 Constant Aggregation Scheme

As mentioned previously we build stochastic bounds with two atoms for each input distribution. We present two sets of bounds for each input distribution. Note that the results of these new schemes and the former ones cannot be compared as the inputs (i.e. the input bounds) are not comparable in general for the two schemes.

St Upper Bound:

Each input distribution is divided into two groups of consecutive atoms. For the strong upper bound, we aggregate all the atoms in a group into the largest atom. The probability of this remaining atom is the sum of the probabilities of the elements of the group. For the first scheme, the two groups have the same size (i.e. 4 atoms here). For the second scheme, the first group contains the two smallest atoms. The table contains the atoms and the probabilities for both schemes.

Edge	1*st* scheme				2*nd* scheme			
	Atom		probability		Atom		probability	
e0	8	16	0.5	0.5	3	16	0.2	0.8
e1	4	13	0.5	0.5	2	13	0.3	0.7
e2	4	13	0.5	0.5	2	13	0.3	0.7
e3	4	12	0.6	0.4	2	12	0.3	0.7
e4	4	12	0.5	0.5	2	12	0.3	0.7
e5	4	13	0.5	0.5	2	13	0.3	0.7
e6	4	12	0.6	0.4	2	12	0.3	0.7
e7	4	16	0.5	0.5	2	16	0.3	0.7

Table 5
St Upper Bounds for the Input Distributions with 2 atoms.

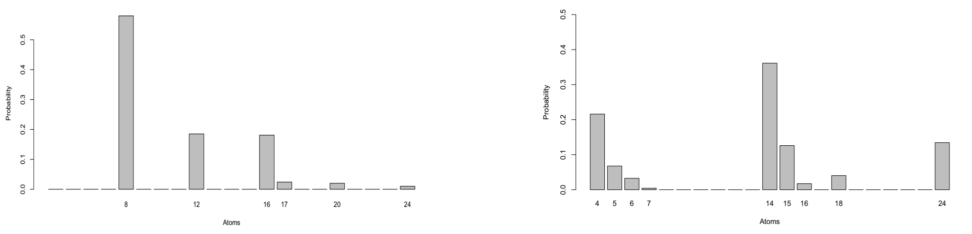

Fig. 18. St Upper Bound (first scheme on the left, second scheme on the right).

St Lower Bound

We use similar schemes to bound the input distributions. We divide into two groups the atoms and we keep the smallest atom to concentrate the probability of the elements of a group. The distributions are given in Table 6.

Icv Upper Bound

This is done with an iterative application of Lemma 2.20 for the upper bound. We chose the atoms to be merged such that the resulting atom is an integer. Note that the results of this new scheme (depicted in Fig. 20) and the former ones (in

Edge	1st scheme				2nd scheme			
	Atom		probability		Atom		probability	
e0	2	10	0.5	0.5	2	15	0.8	0.2
e1	1	8	0.5	0.5	1	10	0.8	0.2
e2	1	8	0.5	0.5	1	10	0.8	0.2
e3	1	7	0.6	0.4	1	9	0.8	0.2
e4	1	7	0.5	0.5	1	9	0.8	0.2
e5	1	8	0.5	0.5	1	10	0.8	0.2
e6	1	7	0.6	0.4	1	9	0.8	0.2
e7	1	6	0.5	0.5	1	11	0.7	0.3

Table 6
St lower Bounds for the Input Distributions with 2 atoms

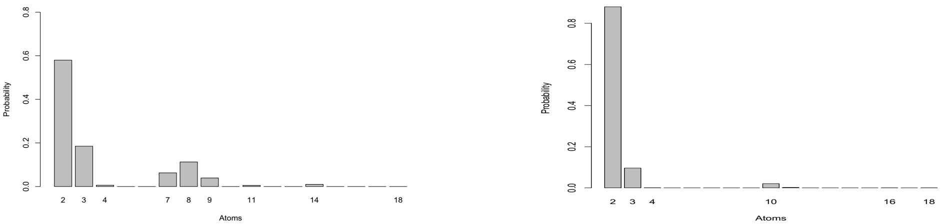

Fig. 19. St Lower Bound (first scheme on the left, second scheme on the right)

Fig. 16 and Fig. 17) cannot be compared as the inputs (i.e. the input bounds) are not comparable for the two schemes. The upper bounding distributions of the inputs are given in Table 7.

Edges	First Atom	Probability	Last Atom	Probability
e0	5	0.2	10	0.8
e1	2	0.2	7	0.8
e2	2	0.2	7	0.8
e3	1	0.2	6	0.8
e4	1	0.2	6	0.8
e5	2	0.2	7	0.8
e6	1	0.2	6	0.8
e7	6	0.8	11	0.2

Table 7
Icv upper bounds of the input distributions with two atoms.

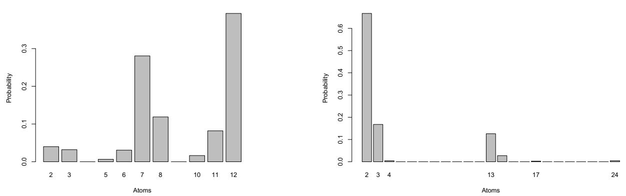

Fig. 20. Icv Upper Bound (left), Icv Lower Bound (right).

Icv Lower bound:

Icv lower bounds are easily obtained by considering concave lower bounds for the input distributions. Such bounds on the input distributions with two atoms are unique. Indeed the two extreme atoms must be in the concave lower bound. For instance, the concave lower bound for the distribution of the capacity of edge $e0$ contains two atoms, 2 and 16 both with probability 0.5.

Edge	First Atom	Probability	Second Atom	Probability
e0	2	0.5	16	0.5
e1	1	0.583	13	0.417
e2	1	0.583	13	0.417
e3	1	0.64	12	0.36
e4	1	0.64	12	0.36
e5	1	0.583	13	0.417
e6	1	0.64	12	0.36
e7	1	0.6	16	0.4

Table 8
Icv Lower bounds for the Input Distributions with 2 atoms.

As mentioned previously, the smallest and the largest atoms of the exact distribution 2 and 24 are also in the lower increasing concave bound (see the right part of Fig. 20. for the lower increasing concave bound and Fig. 13 for the exact result). To present a synthetic comparison of these results, we now give the expectations of the distributions we have computed in the following table.

Order	St 1st Scheme	St 2nd Scheme	Icv
Sup	10.81	12.61	9.09
Inf	3.63	2.29	4.04

Table 9
Expectations for the bounds based on 256 deterministic max flows

Clearly the bounds based on the increasing concave ordering are better than the bounds obtained by the the strong ordering and they are also more accurate than the approach based on the expectation of all the random variables modeling the inputs.

4.3.2 Aggregation based on the Minimal Cut

Following [15] and [13], an edge is important for the accuracy of the bound when it is in the cut-set. We now try a simple heuristic to decide which input distributions must be aggregated and which one must keep all its atoms. Note that all the previous bounds are based on a global aggregation pattern which is not related to the graph properties. Here we experiment with the following heuristic: keep more atoms for the input distributions associated with the edges of a minimal cut of the deterministic problem associated with the expected capacities (here they are $e3$ and $e4$). Of course, we keep the same global number of max flow computations (i.e. 256 in this section) to compare with the other analysis. We both present st-bounds and icv-bounds for three schemes.

We keep distributions associated to $e3$ or $e4$ unchanged. Some distributions are bounded by distributions with with two atoms and the last ones are bounded by Dirac distributions such as the number of deterministic problems is always 2^8. For the Dirac distributions, the atoms depend on the ordering and the bounds (extreme atoms for the st bounds or expectations for the Icv upper bound, Icv lower bounds are replaced by st lower bounds as usual when we deal with Dirac distributions). For the bounding distributions with two atoms, we use the first scheme for st-bounds (see Table 5 and 6) as it provides better results. For Icv Upper bounds, the input distributions with 4 atoms are taken from Table 10 and the distributions with two atoms in Table 7. In the following figures, the bounds will be named according to number of atoms for the edges.

Edges	Upper Bounds								Lower Bounds							
	Atom				Probability				Atom				Probability			
e0	5	7	10	13	0.2	0.2	0.4	0.2	2	7	10	16	0.18	0.290	0.320	0.21
e1	2	7	8	11	0.4	0.2	0.2	0.2	1	3	8	13	0.20	0.280	0.360	0.16
e2	2	7	8	11	0.4	0.2	0.2	0.2	1	3	8	12	0.40	0.280	0.360	0.16
e3	1	3	8	10	0.2	0.4	0.2	0.2	1	3	7	12	0.25	0.325	0.265	0.16
e4	1	3	8	10	0.2	0.4	0.2	0.2	1	3	7	12	0.25	0.325	0.265	0.16
e5	2	7	8	11	0.4	0.2	0.2	0.2	1	3	8	13	0.20	0.280	0.360	0.16
e6	1	3	8	10	0.2	0.4	0.2	0.2	1	3	7	12	0.25	0.325	0.265	0.16
e7	2	5	10	16	0.4	0.2	0.2	0.2	1	3	6	16	0.20	0.270	0.250	0.28

Table 10
Lower and Upper Concave Bounds of the input distributions with 4 atoms.

First Scheme:

We keep two edges with initial distributions, one edge with a distribution with 4 atoms and the remaining ones bounded by a Dirac distribution.

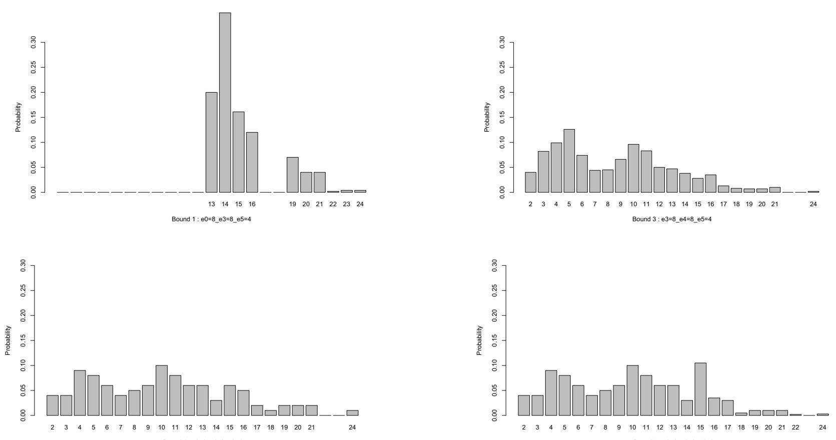

Fig. 21. St upper bounds first scheme.

We do not represent some lower bounding distributions for the strong stochastic ordering because they have one or two atoms.

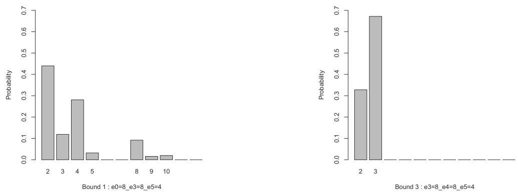

Fig. 22. St Lower bounds first scheme.

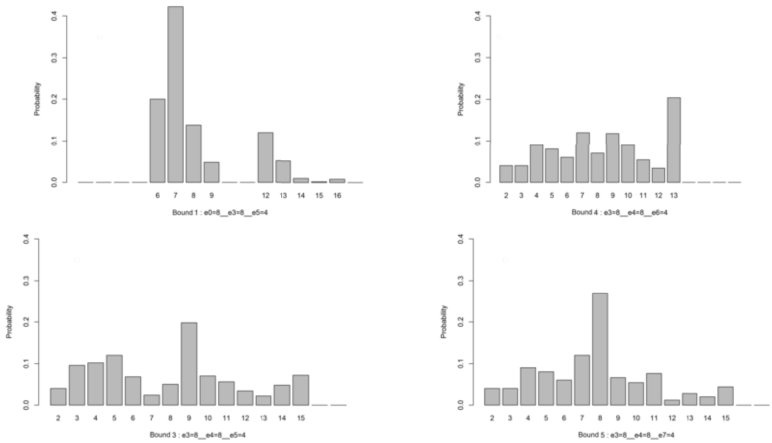

Fig. 23. Icv upper bounds first scheme.

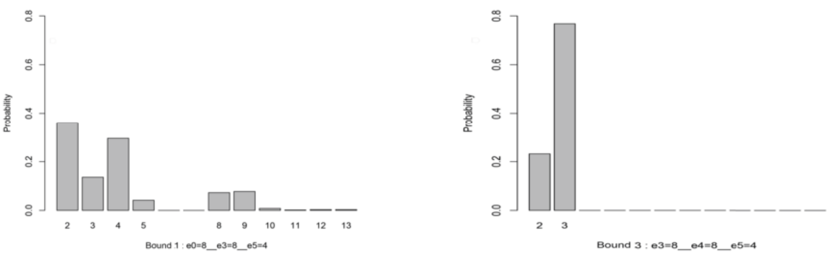

Fig. 24. Icv Lower bounds first scheme. Only two distributions are depicted.

Configuration	e0=8,e3=8,e5=4	e3=8,e4=8,e5=4	e3=8,e4=8,e6=4	e3=8,e4=8,e7=4
St Upper Bound	15.163	8.55	10	9.791
St Lower Bound	3.601	2.672	2	2
Icv Upper Bound	8.10	7.897	8.336	7.738
Icv Lower Bound	3.9784	2.768	2	2.64

Table 11
Expectation of the Max flow for the bounds, Scheme 1.

Clearly, upper Icv bounds are better than strong stochastic bounds. For lower bounds, as Icv bounds of the input are often based on strong stochastic bounds we obtain quite similar results for the expectation of the distribution of the max flow.

Second Scheme

We now keep only one edge with its initial distribution. Two edges are associated with bounding distributions on 4 atoms while one edge is modeled by a distribution on 2 atoms and all the remaining have a Dirac distribution.

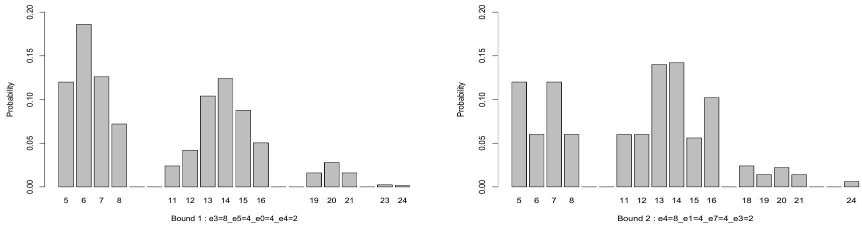

Fig. 25. St Upper bound, Scheme 2.

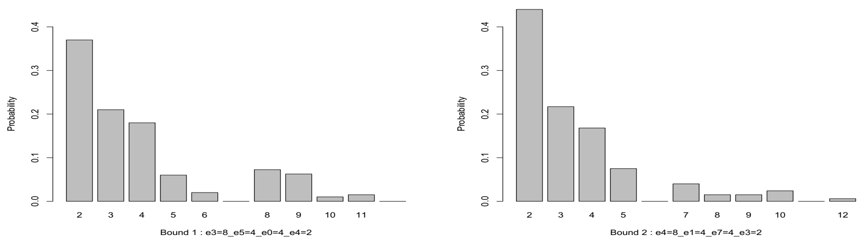

Fig. 26. St Lower bound, Scheme 2.

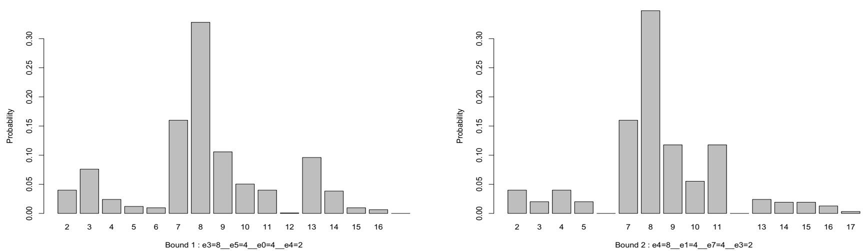

Fig. 27. Icv Upper bound, Scheme 2.

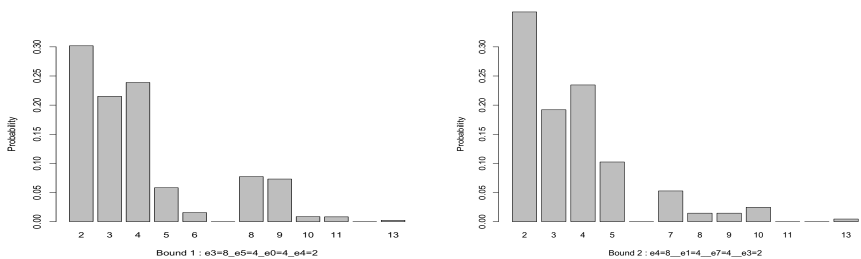

Fig. 28. Icv Lower bound, Scheme 2.

Ordering	Upper Bounds Configuration		Lower Bounds Configuration	
	e3=8,e5=4,e0=4,e4=2	e4=8,e1=4,e7=4,e3=2	e3=8,e5=4,e0=4,e4=2	e4=8,e1=4,e7=4,e3=2
St Bounds	10.44	11.516	3.9175	3.425
Icv Bounds	8.256	8.3036	4.086	3.668

Table 12
Expectation of the Max flow for the bounds, Scheme 2.

Third Scheme

Finally, we keep both edges of the minimal cut (i.e. $e3$ and $e4$) with their initial input distributions. Two other edges have a bounding distribution with two atoms. The remaining edges, as usual, are associated with Dirac distributions. The figures are labelled with the name of the edges associated with distributions

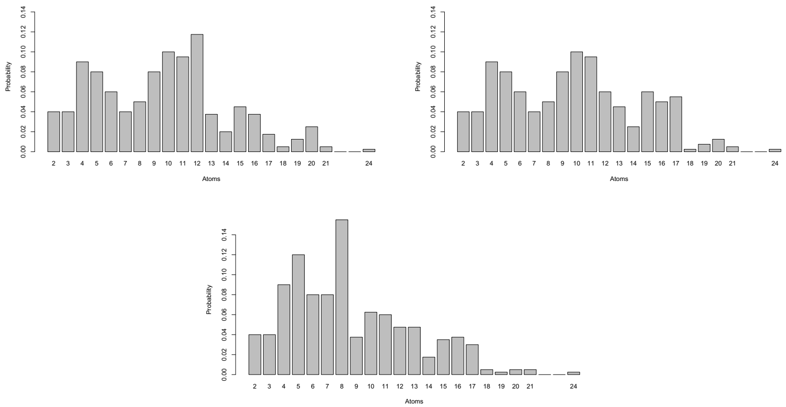

Fig. 29. St Upper bounds: upper left (e0, e1), upper right (e0, e7), bottom (e5, e7), Third Scheme.

Lower bounding distributions (both st and icv) are not represented because they only have one or two atoms.

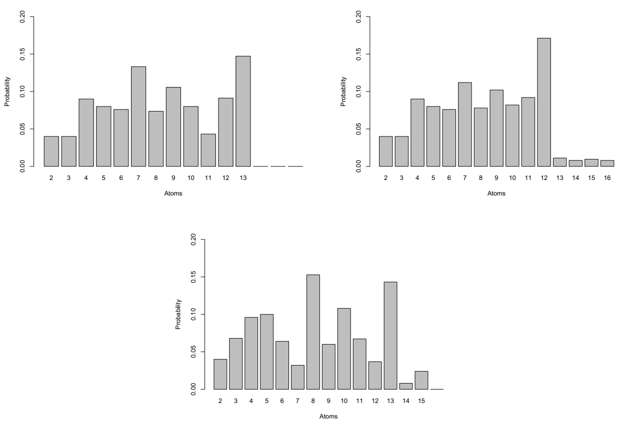

Fig. 30. Icv Upper bounds: upper left (e0, e1), upper right (e0, e7), bottom (e5, e7), Third Scheme.

Configuration with 2 atoms)	Upper Bound			Lower Bound	
	e0 and e1	e0 and e7	e5 and e7	e0 and e7	e5 and e7
St Bounds	9.51	9.66	8.56	2.4	2.68
Icv Bounds	8.17	8.158	8.1328	2.32	2.586

Table 13
Expectation of the Max flow for the bounds, Scheme 3.

5 Concluding Remarks

Our method allows us to obtain bounds very easily and with a tradeoff between accuracy and the computing efforts. Clearly, one can derive some conclusions from these first analysis:

- St upper bounds are less accurate than Icv upper bounds with the same number of atoms in the input bounds.
- Increasing the size of the input distributions increase the accuracy of the results.
- Heuristic based on the minimal cut-set are not really better than the constant aggregation scheme for the upper bounds.
- It is difficult to improve the lower bounds (both St and Icv). Concave bounds require an input distribution with at least two atoms leading to a problem with 2^8 deterministic flows to compute. Increasing the size of the input distributions does not help significantly. The initial St lower bound is 2 and the strategy with 8 deterministic flows leads to distributions which all have an expectation equal to 2.

We now want to develop new heuristics to chose the input variables which must be bounded and the number of remaining atoms in the input bounds. This also suggests to develop iterative techniques to improve the accuracy of the bounds by considering a sequence of bounds of the inputs for some important edges.

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