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Computable Analysis of the Abstract Cauchy Problem in a Banach Space and Its Applications (I)

Klaus Weihrauch[1]

Department of Mathematics and Computer Science
University of Hagen
Hagen, Germany

Ning Zhong[2]

Department of Mathematical Sciences
University of Cincinnati
Cincinnati, USA

Abstract

We study computability of the abstract linear Cauchy problem

$$du(t)/dt = Au(t), \quad u(0) = x \in X, \tag{1}$$

where A is a linear operator, possibly unbounded, on a Banach space X. We give necessary and sufficient conditions for A such that the solution operator $K : x \mapsto u$ of the problem (1) is computable. For studying computability we use the representation approach to Computable Analysis developed by Weihrauch and others. This approach is consistent with the model used by Pour-El/Richards.

Keywords: computable analysis, partial differential equations, Cauchy problem

[1] Email: `klaus.weihrauch@fernuni-hagen.de`
[2] Email: `Ning.Zhong@uc.edu`

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1 Introduction

The study of numerical algorithms for solutions of differential equations of physics is central in numerical analysis. The numerical technology for solving differential equations include finite-difference methods and finite-elements methods. The first method approximates a differential operator by a difference quotient, while the second approximates the solution itself by functions that can be broken up into simple pieces. The aim is to find numerically stable algorithms that rapidly converge to the correct solution. But can such an algorithm be invented for every differential equation? In a more formal way, the question becomes whether or not it is always possible to compute physical processes modeled by differential equations. This question is studied in computable analysis, which is the study of computability and complexity of continuous problems based on Turing machines. In the context of computable analysis, the solution of a differential equation is computable if there is a Turing machine (oracle or Type-2) that computes approximations converging to the solution from approximations to given parameters. Therefore, if a solution is computable, then the existence of convergent algorithms for numerical solution is guaranteed. Moreover, proofs of computability usually give rise to Turing algorithms, although very intricate at times, these Turing algorithms may possibly be translated into numerical algorithms.

There have been many studies on computability of solutions of partial differential equations (PDEs) (see, for example, [1,2,4,5,8,7,12,14,13] and references therein). The majority of results obtained so far deal with individual equations, for example, linear heat, wave or Schrödinger equation, and the KdV equation. This may have to be the case for nonlinear equations, for different nonlinear equations generally have little in common and they may have to be dealt with on a case-by-case basis. But how about linear PDEs? Is there any Turing algorithm that solves a class of linear PDEs? In this paper, we study the problem via semigroup theory. We show how to compute on Turing machines a semigroup, uniformly continuous or strongly continuous, from its infinitesimal generator and vise versa. Then, by making use of these constructions, we present two Turing algorithms for computing the solution operator of the initial-value problem of any linear parabolic equation, homogeneous or inhomogeneous.

Semigroup theory is the study of abstract Cauchy problems (also called initial-value problems) of first-order ordinary differential equations with values in Banach spaces, driven by linear, but possibly unbounded operators. An abstract Cauchy problem can be written in the form of

$$\begin{cases} \frac{d}{dt}u(t) = Au(t), \qquad t > 0, \\ \\ u(0) = x, \end{cases} \tag{2}$$

where X is a Banach space and $A: X \to X$ a linear operator. The initial-value $x \in X$ is given. So if $A = \sum_{j=1}^{3} \partial^2/\partial x_j^2$ is the Laplace operator and $X = L^2(\mathbb{R}^3)$, then the problem (2) is the initial-value problem of the heat equation

$$\frac{du}{dt} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2}, \; x \in \mathbb{R}^3 \;\&\; t > 0, \; u(0) = f \in X.$$

Here A is a linear but unbounded operator. Or if $A = i\sum_{j=1}^{3} \partial^2/\partial x_j^2$, $i = \sqrt{-1}$, then the problem (2) corresponds to the initial-value problem of the Schrödinger equation

$$\frac{du}{dt} = i\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2}\right), \; x \in \mathbb{R}^3 \;\&\; t > 0, \; u(0) = f \in X.$$

Again A is unbounded on $L^2(\mathbb{R}^3)$. By Pour-El and Richards' First Main Theorem [9], A is uncomputable in the sense that there exist computable functions $f \in X$, (weakly) twice differentiable, such that $Af = A(f) \in X$ are not computable.

The main question studied classically regarding the abstract Cauchy problem (2) is the following one: Under what conditions on A, does (2) admit a unique solution? This problem has been well studied. It is known that if A is the infinitesimal generator of a uniformly (resp. strongly) continuous semigroup $W(t)$, $t \geq 0$, of bounded linear operators, $W(t): X \to X$, then (2) admits a unique (resp. mild) solution $u(t)$ for every $x \in X$ and this solution can be written as $u(t) = W(t)x$. In such cases, if one can compute the semigroup $W(t)$ from A, then one can compute the solution of (2). What information on A are needed to compute $W(t)$ and vice versa is a main concern of the present paper.

The paper is organized as follows. In Section 2, we briefly review some basic definitions and facts related to C_0 semigroups of bounded linear operators in Banach spaces. In Section 3, we introduce the model for computation and related definitions. In Section 4, we show how to compute a uniformly continuous semigroup from its bounded linear infinitesimal generator and vice versa.

In Section 5, several Turing algorithms are constructed to compute strongly continuous semigroups and analytic semigroups from their infinitesimal generators. In Section 6, two Turing algorithms are presented for computing the solution operators of linear parabolic equation, homogeneous or inhomogeneous.

2 Infinitesimal Generators

In this section, we summarize some definitions and facts regarding semigroups. The reader is referred to [6] for more details. Let X be a Banach space. A one parameter family $W(t)$, $0 \le t < \infty$, of bounded linear operators from X to X is called a semigroup of bounded linear operators on X if

(i) $W(0) = I$ (I is the identity operator on X),

(ii) $W(t+s) = W(t)W(s)$ for every $t,\ s \ge 0$ (the semigroup property).

The linear operator A defined by

$$\mathrm{dom}(A) = \{x \in X : \lim_{t \to 0^+} \frac{W(t)x - x}{t} \text{ exists}\}$$

and

$$Ax = \lim_{t \to 0^+} \frac{W(t)x - x}{t} = \frac{d^+}{dt} W(t)x \bigg|_{t=0} \tag{3}$$

for $x \in \mathrm{dom}(A)$ is called the infinitesimal generator of the semigroup $W(t)$. A semigroup $W(t)$, $t \ge 0$, of bounded linear operators on X is called *uniformly continuous* if

$$\lim_{t \to 0^+} \|W(t) - I\| = 0. \tag{4}$$

The following theorems reveal the relationship between a uniformly continuous semigroup $W(t)$ and its infinitesimal generator A.

Theorem 2.1 *A linear operator A is the infinitesimal generator of a uniformly continuous semigroup $W(t)$, $t \ge 0$, if and only if A is a bounded linear operator.*

Theorem 2.2 *Let $W(t)$, $t \ge 0$, be a uniformly continuous semigroup of bounded linear operators. Then*

(i) *there exists a constant $\omega \ge 0$ such that*

$$\|W(t)\| \le e^{\omega t} \qquad \text{for any } t \ge 0,$$

(ii) *there exists a unique bounded linear operator A such that $W(t) = e^{At}$,*

(iii) *the operator A in part (b) is the infinitesimal generator of* $\{W(t)\}_{t\geq 0}$,

(iv) *the map* $t \to W(t)$ *is differentiable in norm and*

$$\frac{d}{dt}W(t) = AW(t) = W(t)A.$$

A semigroup $W(t)$, $t \geq 0$, of bounded linear operators on X is called a *strongly continuous* semigroup or a C_0 semigroup, if

$$\lim_{t\to 0^+} W(t)x = x \quad \text{for every } x \in X. \tag{5}$$

The resolvent set $\rho(A)$ of a linear operator A is the set of all complex numbers β for which $\beta I - A$ is invertible, i.e., $R(\beta, A) := (\beta I - A)^{-1} : X \to X$ is a bounded linear operator.

The following two theorems show the relation between the semigroup and its infinitesimal generator, which is one of the fundamental problems in the theory of semigroups of operators.

Theorem 2.3 *For $\theta \geq 0$ and $M \geq 1$, a linear operator A is the infinitesimal generator of a C_0 semigroup $W(t)$ satisfying $\|W(t)\| \leq Me^{\theta t}$, if and only if*

(i) *A is a closed map and dom(A) is dense in X,*

(ii) *the resolvent set $\rho(A)$ of A contains the interval (θ, ∞) and*

$$\|R(\lambda, A)^n\| \leq \frac{M}{(\lambda - \theta)^n} \quad \text{for } \lambda > \theta,\ n = 1, 2 \cdots$$

For convenience we call a triple (A, θ, M) a piece of type-IG information for A (IG refers to "infinitesimal generator"), if the conditions 2.3.i and 2.3.ii hold for (A, θ, M). Theorem 2.3 shows that a piece of type-IG information for A is necessary and sufficient for A to be the infinitesimal generator of a C_0 semigroup.

Theorem 2.4 *Let $W(t)$, $t \geq 0$, be a C_0 semigroup of bounded linear operators with the infinitesimal generator A. Then*

(i) *there exist constants $\theta > 0$ and $M \geq 1$ such that*

$$\|W(t)\| \leq Me^{\theta t} \qquad \text{for } 0 \leq t < \infty.$$

(ii) *for every $x \in X$, $t \to W(t)x$ is a continuous function from $[0, \infty)$ into X.*

(iii) *for every $x \in$ dom(A), $W(t)x \in$ dom(A) and*

$$\frac{d}{dt}W(t)x = AW(t)x = W(t)Ax.$$

(iv) *for every $x \in dom(A)$,*

$$W(t)x - W(s)x = \int_s^t W(\tau)Axd\tau = \int_s^t AW(\tau)xd\tau.$$

A C_0 semigroup $W(t)$ can be constructed from its infinitesimal generator A as follows:

$$W(t)x = \lim_{\lambda\to\infty} e^{tA_\lambda}x, \quad x \in X, \tag{6}$$

where A_λ, for every $\lambda > \theta$, is the *Yosida approximation* defined by $A_\lambda = \lambda AR(\lambda, A)$. Notice that

$$A_\lambda = \lambda AR(\lambda, A) = \lambda^2 R(\lambda, A) - \lambda I \quad \text{and} \tag{7}$$

$$A_\lambda x = \lambda R(\lambda, A)Ax \quad \text{for} \quad x \in \mathrm{dom}(A)\,. \tag{8}$$

3 Computability on Banach Spaces

In this article, we use the representation approach to study computability in analysis and differential equations (TTE) [10]. In TTE "concrete" computability on the set Σ^* of finite words and the set Σ^ω of infinite sequences of symbols from a finite alphabet Σ is defined explicitly by means of Turing machines. Computable functions are continuous w.r.t the discrete topology on Σ^* and Cantor topology on Σ^ω. As usual we use canonical tupling functions $\langle\,\rangle$ mapping finite or infinite tuples of elements from Σ^* or Σ^ω injectively to Σ^* or Σ^ω. These functions as well as the projections of their inverses are computable [10].

Computability on abstract sets M is introduced by *notations* $\nu :\subseteq \Sigma^* \to M$ or *representations* $\delta :\subseteq \Sigma^\omega \to M$ where finite or infinite sequences of symbols, respectively, are used as *names* of the "abstract" points $x \in M$. For given naming systems $\gamma_i :\subseteq Y_i \to M_i$, $Y_i \in \{\Sigma^*, \Sigma^\omega\}$ for $i = 0, 1, \ldots, n$, a multi-function $f :\subseteq M_1 \times \ldots \times M_n \rightrightarrows M_0$ is $(\gamma_1, \ldots, \gamma_n, \gamma_0)$-computable (-continuous, respectively), if there is a computable (continuous, respectivey) function $h :\subseteq Y_1 \times \ldots \times Y_n \to Y_0$ (called a *realization* of f) such that $\gamma_0 \circ h(y_1, \ldots y_n) \in f(x_1, \ldots, x_n)$ whenever $(\gamma_1(y_1), \ldots, \gamma_n(y_n)) = (x_1, \ldots, x_n) \in \mathrm{dom}(f)$. Reducibility and equivalence are defined by $\gamma_1 \le \gamma_2$, iff $M_1 \subseteq M_2$ and the identity mapping is (γ_1, γ_2)-computable, and $\gamma_1 \equiv \gamma_2$ iff $\gamma_1 \le \gamma_2$ and $\gamma_2 \le \gamma_1$. Naming systems induce the same computability, iff they are equivalent.

For many familiar spaces studied in analysis there are canonical naming systems. Let $\nu_{\mathbb{N}} :\subseteq \Sigma^* \to \mathbb{N}$ be some standard notation of the natural numbers and let $\rho :\subseteq \Sigma^\omega \to \mathbb{R}$ be the Cauchy representation of the real numbers [10]. The canonical product of γ_1 and γ_2 is defined by

$[\gamma_1, \gamma_2]\langle y_1, y_2\rangle := (\gamma_1(y_1), \gamma_2(y_2))$. A function is $(\gamma_1, \gamma_2, \gamma_0)$-computable , iff it is $([\gamma_1, \gamma_2], \gamma_0)$-computable. There is also a canonical representation $[\gamma_1 \to \gamma_2]$ of the set of (γ_1, γ_2)-continuous functions $f : M_1 \to M_2$. The representation $[\gamma_1 \to \gamma_2]$ admits evaluation and type conversion as follows:

Lemma 3.1 (type conversion [10]) *Let $\delta_i :\subseteq \Sigma^\omega \to M_i$ $(i = 1, 2, 3)$ be representations. For any function $f : M_1 \times M_2 \to M_3$ define $T(f) : M_1 \to M_3^{M_2}$ by $T(f)(x)(y) := f(x, y)$. Then*

$$T \circ [[\delta_1, \delta_2] \to \delta_3] \equiv [\delta_1 \to [\delta_2 \to \delta_3]]\,. \tag{9}$$

In particular, f is $([\delta_1, \delta_2], \delta_3)$-computable, iff $T(f)$ is $(\delta_1, [\delta_2 \to \delta_3])$-computable. For $\delta_1 = [\delta_2 \to \delta_3]$ we obtain that the evaluation function $\mathrm{ev} : (f, x) \mapsto f(x)$ is $([[\delta_2 \to \delta_3], \delta_2], \delta_3)$-computable, hence $([\delta_2 \to \delta_3], \delta_2, \delta_3)$-computable.

For $\gamma :\subseteq Y \to X$ the representation $[\gamma]^\omega$ of the set X^ω of sequences on X is defined by $\gamma\langle y_0, y_1, y_2, \ldots\rangle := (\gamma(y_0), \gamma(y_1), \gamma(y_2), \ldots)$. We have

$$[\gamma]^\omega \equiv [\nu_{\mathbb{N}} \to \gamma]\,. \tag{10}$$

Definition 3.2 [Computable metric space [10]] A tuple (X, d, D, α) is called a computable metric space if

(i) $d : X \times X \to \mathbb{R}$ is a metric on X and D is a dense set in X,

(ii) $\alpha :\subseteq \Sigma^* \to D$ is a notation with recursive domain,

(iii) the restriction of d to $D \times D$ is (α, α, ρ)-computable.

The Cauchy representation $\delta_X :\subseteq \Sigma^\omega \to X$ of a computable metric space (X, d, D, α) is defined as follows: $\delta_X(p) = x \iff p = \langle w_0, w_1, w_2 \ldots\rangle$, $(w_i \in \mathrm{dom}(\alpha)$ such that

$$d(x, \alpha(w_i)) \le 2^{-i} \quad (\text{for all } i \in \mathbb{N})\,.$$

Thus, a Cauchy name $p \in \Sigma^\omega$ of an $x \in X$ encodes a sequence in the dense set D that converges to x rapidly.

For a continuous function between computable metric spaces a modulus of continuity at a point can be computed. The following lemma generalizes Lemma 6.2.8 in [10], see also [11].

Lemma 3.3 (modulus of continuity) *For computable metric spaces X and Y with Cauchy representations δ_X and δ_Y, respectively, the function* $\mathrm{MOD} : (f, x, m) \rightrightarrows k$ *mapping every continuous function $f : X \to Y$, every $x \in X$ and every $m \in \mathbb{N}$ to some $k \in \mathbb{N}$ such that $d(f(x), f(y)) \le 2^{-m}$ if $d(x, y) \le 2^{-k}$ is $([\delta_X \to \delta_Y], \delta_X, \nu_{\mathbb{N}}, \nu_{\mathbb{N}})$-computable.*

The main theorems in this article will be proved for *computable Banach spaces* which are special computable metric spaces.

Definition 3.4 [Computable Banach space [9]] A tuple $\mathcal{B} = (X, \| \, \|, D, \alpha)$ is called a computable Banach space if

(i) $(X, \| \, \|)$ is a Banach space,

(ii) (X, d, D, α) such that $d(x, y) = \|x - y\|$ is a computable metric space,

(iii) with respect to the representation ρ of $\mathbb{R}$ and the Cauchy representation δ_X of X, addition $(x, y) \to x + y$ and scalar multiplication $(a, x) \to ax$ are computable operations.

In many applications there is a canonical *effective generating set* [9] $e : \mathbb{N} \to X$ such that the rational linear span D of its range is dense in X. In such case, usually a "good" notation α of D can be constructed from the numbering e canonically.

In the following $\mathcal{B} = (X, \| \, \|, D, \alpha)$ will be a fixed computable Banach space with Cauchy representation δ_X. The limit of rapidly converging sequences on $\mathcal{B}$ is computable (Exercise 8.1.8 in [10]):

Lemma 3.5 *The function* $\mathrm{Lim} :\subseteq X^\omega \to X$ *such that* $\mathrm{Lim}((x_i)_{i \in \mathbb{N}}) = x \iff (\forall i) \, \|x - x_i\| \le 2^{-i}$ *is* $([\delta_X]^\omega, \delta_X)$*-computable (and equivalently* $([\nu_{\mathbb{N}} \to \delta_X], \delta_X)$*-computable).*

Since the representation $\delta_X :\subseteq \Sigma^\omega \to X$ is admissible, a function $f : X \to X$ is continuous, iff it is (δ_X, δ_X)-continuous. Therefore, the standard representation $[\delta_X \to \delta_X] :\subseteq \Sigma^\omega \to C(X, X)$ is a representation of the set of all continuous functions from X to X. In particular, it represents the linear continuous functions.

We define a representation δ_G of the set LCG of linear operators on $\mathcal{B}$ with non-empty closed graph as follows:

$$\delta_G(p) = A :\iff \begin{cases} \text{there are } p_i, q_i \in \mathrm{dom}(\delta_X) \text{ such that } \; p = \langle p_0, q_0, p_1, q_1, \ldots \rangle \\ \text{and } \{(\delta_X(p_i), \delta_X(q_i)) \mid i \in \mathbb{N}\} \text{ is dense in graph}(A) \end{cases}$$

Thus a δ_G-name of $A \in \mathrm{LCG}$ is a list of a dense set in its graph. Similarly, we also define a representation δ_{GG} of the set LCG $\times$ LCG as follows:

$$\delta_{GG}(p) = (A,B) :\Longleftrightarrow \begin{cases} \text{there are } p_i, q_i, r_i \in \mathrm{dom}(\delta_X) \text{ such that} \\ p = \langle p_0, q_0, r_0, p_1, q_1, r_1, \ldots \rangle, \\ \{(\delta_X(p_i), \delta_X(q_i)) \mid i \in \mathbb{N}\} \text{ is dense in graph}(A), \\ \text{and}\{(\delta_X(p_i), \delta_X(r_i)) \mid i \in \mathbb{N}\} \text{ is dense in graph}(B) \end{cases}$$

From a $[\delta_X \to \delta_X]$-name of a continuous linear operator A, an upper bound of its operator norm $\|A\| = \sup_{\|x\|=1} \|f(x)\|$ and a dense subset of its graph can be computed. On the other hand, for a continuous linear operator $A : X \to X$, its values on a dense set and an upper bound of its norm are sufficient to compute Ax for all $x \in X$, i.e., to find a $[\delta_X \to \delta_X]$-name.

Lemma 3.6 *For continuous linear functions $A : X \to X$,*

(i) *The multi-function $A \rightrightarrows b$ such that $\|A\| \leq b$ is $([\delta_X \to \delta_X], \rho)$-computable.*

(ii) *the function $A \mapsto A$ is $([\delta_X \to \delta_X], \delta_G)$-computable*

(iii) *the function $(A, b) \mapsto A$ for $\|A\| \leq b$ is $(\delta_G, \rho, [\delta_X \to \delta_X])$-computable,*

Proof. i. Apply Lemma 3.3 to $(A, 0, 0)$, from A we can compute some $k \in \mathbb{N}$ such that $\|x\| \leq 2^{-k}$ implies $\|Ax\| = \|Ax - A0\| \leq 2^{-0} = 1$, hence $\|A\| \leq 2^k$.

ii. The set $\{(x, Ax) \mid x \in \mathrm{range}(\alpha)\}$ is dense in graph(A). Since dom(α) is recursive and $\alpha \leq \delta_X$, we can easily compute a δ_G-name from a $[\delta_X \to \delta_X]$-name of A.

iii. Let $\delta_G(p) = A$ and $p = \langle p_0, q_0, p_1, q_1, \ldots \rangle$. Since the set $\{(\delta_X(p_i), \delta_X(q_i)) \mid i \in \mathbb{N}\}$ is dense in graph(A), for any x (given by a δ_X-name) we can find some i such that $\|x - \delta_X(p_i)\| < 2^{-i}/b$. Then

$$\|Ax - \delta_X(q_i)\| = \|Ax - A\delta_X(p_i)\| \leq \|A\|\|x - \delta_X(p_i)\| \leq 2^{-i}.$$

Therefore, the multi-function $(A, b, x) \rightrightarrows (y_i)_{i\in\mathbb{N}}$ such that $Ax = \mathrm{Lim}(y_i)_{i\in\mathbb{N}}$ is $(\delta_G, \rho, \delta_X, [\delta_X]^\omega)$-computable, where $y_i = \delta_X(q_i)$. By Lemma 3.5, the function $(A, b, x) \mapsto Ax$ is $(\delta_G, \rho, \delta_X, \delta_X)$-computable. Apply the type conversion lemma 3.1. □

In the following we will use the fixed naming systems $\nu_\mathbb{N}$ for the natural numbers, ρ for the real numbers, ρ_+, the restriction of ρ to the set of non-negative real numbers, for real "time" $t \geq 0$, δ_X for the Banach space $\mathcal{B}$, $[\delta_X \to \delta_X]$ for bounded linear (continuous linear) operators on X, δ_G for (unbounded) linear operators on X with non-empty closed graph and $\delta_S :=$

$[\rho_+ \to [\delta_X \to \delta_X]]$ for the C_0 semigroups of bounded linear operators $W(t)$, $t \geq 0$. Occasionally, we will say "computable" instead of $(\delta, \dots)$-computable if no misunderstanding is possible. Recall that an operator is called computable if there is a Turing machine that translates input names to output names. For example, when we say that the operator $A \mapsto W$ mapping every linear bounded operator A to its generated uniformly continuous semigroup W is $([\delta_X \to \delta_X], \delta_S)$-computable, it is meant that there is a Turing machine that translates every $[\delta_X \to \delta_X]$-name of A to a δ_S-name of W.

4 Uniformly Continuous Semigroups

In this section, we show how to compute a uniformly continuous semigroup $W(t)$ from its infinitesimal generator A and vice versa.

Since W is uniformly continuous, there is some k such that $\|W(t) - I\| \leq 1/2$ for $0 \leq t \leq 2^{-k}$. We will use such a number k for computing A from W. Since we do not yet know whether such a number k can be computed from W in general we add it as a portion of information to W.

Theorem 4.1 (i) *The operator $S : A \rightrightarrows (W, k)$, mapping every continuous linear operator A to its generated uniformly continuous semigroup W and some k such that $\|W(t) - I\| \leq 1/2$ for $0 \leq t \leq 2^{-k}$, is $([\delta_X \to \delta_X], [\delta_S, \nu_{\mathbb{N}}])$-computable.*

(ii) *The operator $\overline{S} : (W, k) \mapsto A$, mapping every uniformly continuous semigroup W and every $k \in \mathbb{N}$ such that $\|W(t) - I\| \leq 1/2$ for $0 \leq t \leq 2^{-k}$ to its infinitesimal generator A, is $([\delta_S, \nu_{\mathbb{N}}], [\delta_X \to \delta_X])$-computable.*

Proof. i. By Lemma 3.6.ii from A we can compute some $b \in \mathbb{N}$ such that $\|A\| \leq b$. Then, from A, b, a real number $t \geq 0$, a point $x \in X$ and a number $k \in \mathbb{N}$ we will compute some $y_k \in X$ such that $\|e^{tA}x - y_k\| \leq 2^{-k}$. For this purpose, find a number $j \in \mathbb{N}$ such that

$$2 \cdot t \cdot b \leq j \quad \text{and} \quad \frac{(t \cdot b)^j}{j!}\|x\| \leq 2^{-k-1} \quad \text{and compute} \quad y_k := \sum_{i<j} \frac{(tA)^i}{i!} x\,.$$

For each k we have

$$\begin{aligned}
\|e^{tA}x - y_k\| &= \|\textstyle\sum_{i \geq j} \frac{(tA)^i}{i!} x\| \leq \sum_{i \geq j} \frac{(t\|A\|)^i}{i!}\|x\| \\
&\leq \textstyle\sum_{i \geq j} \frac{(t \cdot b)^i}{i!}\|x\| \leq \frac{(t \cdot b)^j}{j!}\|x\| \sum_{l=0}^{\infty} (\frac{t \cdot b}{j})^l \\
&\leq \quad 2^{-k-1} \cdot 2 \quad = 2^{-k}\,.
\end{aligned}$$

Since the evaluation $ev : (A, x) \mapsto Ax$ is computable w.r.t. the representation $[\delta_X \to \delta_X]$, the multi-function ("multi" since j cannot be computed uniquely) $(A, t, x, k) \rightrightarrows y_k$ is $([\delta_X \to \delta_X], \rho_+, \delta_X, \nu_{\mathbb{N}}, \delta_X)$-computable. Using Lemma 3.1 we conclude that the multi-function $(A, t, x) \rightrightarrows (y_k)_{k\in\mathbb{N}}$ is $([\delta_X \to \delta_X], \rho_+, \delta_X, [\nu_{\mathbb{N}} \to \delta_X])$-computable. Since $\|e^{tA}x - y_k\| \le 2^{-k}$ by Lemma 3.5 the function $(A, t, x) \mapsto e^{tA}x$ is $([\delta_X \to \delta_X], \rho_+, \delta_X, \delta_X)$-computable. By Lemma 2.2.ii and 2.2.iii, $W(t)x = e^{tA}x$. Thus $(A, t, x) \rightrightarrows W(t)x$ is $([\delta_X \to \delta_X], \rho_+, \delta_X, \delta_X)$-computable. Applying type conversion twice we obtain that $A \mapsto W$ is $([\delta_X \to \delta_X], \delta_S)$-computable.

It remains to show that some appropriate k can be computed from A. From A we can compute some upper bound b of $\|A\|$ by Lemma 3.6 and from b some $k \in \mathbb{N}$ such that $2^{-k} \le 1/(4b)$. Then for $0 \le t \le 2^{-k}$,

$$\|W(t) - I\| = \|\sum_{i\ge 1} \frac{(tA)^i}{i!}\| \le \sum_{i\ge 1} \|(tA)^i\| \le \sum_{i\ge 1} \frac{1}{4^i} < \frac{1}{2}\,.$$

ii. On the other hand, assume $\|W(t) - I\| \le 1/2$ for $t \le 2^{-k}$. First we will show $\|A\| \le 2^k$. We obtain

$$\begin{aligned} \|I - 2^k \int_0^{2^{-k}} W(s)\,ds\| &= 2^k \| \int_0^{2^{-k}} (I - W(s))\,ds\| \\ &\le 2^k \int_0^{2^{-k}} \|I - W(s)\|\,ds \\ &\le 2^k \cdot \frac{1}{2} \cdot 2^{-k} \qquad = \frac{1}{2}\,. \end{aligned}$$

Let $B := 2^k \int_0^{2^{-k}} W(s)\,ds$. Since B is continuous and $\|I - B\| \le 1/2$, $B^{-1} = (I - (I - B))^{-1} = \sum_{i\in\mathbb{N}} (I - B)^i$ exists and $\|B^{-1}\| \le 1 + 1/2 + 1/4 + \ldots \le 2$. From Theorem 1.1.2 [6] we know $A = (W(2^{-k}) - I)(\int_0^{2^{-k}} W(s)\,ds)^{-1}$ and therefore

$$\|A\| \le 1/2 \cdot \|(2^{-k}B)^{-1}\| = 1/2 \cdot \|2^k B^{-1}\| \le 1/2 \cdot 2^k \cdot 2 = 2^k.$$

We will compute A from W using (3). Since $W(t)(x) = e^{tA}x$, for $\|x\| \le 2^l$ and $t \le 2^{-2k-l-n-1}$,

$$\begin{aligned}\|\frac{W(t)x-x}{t}-Ax\| &= \|\frac{1}{t}\sum_{i\geq 2}\frac{(tA)^i}{i!}x\| \\ &= \|tA^2\sum_{i\geq 0}\frac{(tA)^i}{(i+2)!}x\| \\ &\leq 2^{-2k-l-n-1}\cdot 2^{2k}\sum_{i\geq 0}\frac{(1/2)^i}{(i+2)!}2^l \\ &< 2^{-n}\,.\end{aligned}$$

First, from W, k, x and n compute some l such that $\|x\| \leq 2^l$ and then $y_n := (W(2^{-2k-l-n-1})x - x)\cdot 2^{2k+l+n+1}$. Then $\|y_n - Ax\| \leq 2^{-n}$. Since evaluation is computable w.r.t. representations $[.\to .]$, the multi-function $(W,k,x) \rightrightarrows (y_n)_n$ is $(\delta_S, \nu_{\mathbb{N}}, \delta_X, [\nu \to \delta_X])$-computable. By Lemma 3.1 the function $(W,k,x)\mapsto Ax$ is $(\delta_S,\nu_{\mathbb{N}},\delta_X,\delta_X)$-computable. Again by type conversion, the operator $(W,k)\mapsto A$ for $\|W(t)-I\| \leq 1/2$ if $t \leq 2^{-k}$ is $(\delta_S,\nu_{\mathbb{N}},[\delta_X\to\delta_X])$-computable. □

As a special case of Theorem 4.1, the solution operator of the Cauchy problem $du(t)/dt = Au(t)$, $t>0$, and $u(0)=x$ is computable from A and x for bounded linear operator A.

Corollary 4.2 *Assume that $A : X \to X$ is a bounded linear operator. Then the solution operator $S : (A,x)\mapsto u$ of the Cauchy problem $du(t)/dt = Au(t)$, $t>0$, and $u(0)=x$, is $([\delta_X\to\delta_X],\delta_X,[\rho\to\delta_X])$-computable.*

Proof. Since $u(t) = W(t)x$, where $W(t)$, $t\geq 0$, is the uniformly continuous semigroup generated by A, by Theorem i, W can be computed from A. Consequently, u can be computed from A and x. □

5 C_0 Semigroups

Throughout this section we assume that $\mathcal{B} = (X, \|\ \|, D, \alpha)$ is a computable Banach space and $A :\subseteq X \to X$ an unbounded linear operator. For any piece of type-IG information (A,θ,M) for A, if in addition a δ_G-name of A (a code for a countable dense subset in the graph of A), a ρ-name of θ (a code for a rational sequence rapidly convergent to θ) and a ρ-name of M are available, (A,θ,M) will be referred as a piece of type-IG data for A. One of the main results in this section is a computable version of Theorem 2.3, which shows that a piece of type-IG data is sufficient to compute the C_0 semigroup generated by A. To prove this result we will need several lemmas.

Lemma 5.1 (i) *The function $F_1 : (A,\theta,M,\lambda)\mapsto B$, where (A,θ,M) is a piece of Type-IG information, $\lambda>\theta$ and $B = R(\lambda,A)$, is*

$(\delta_G, \rho, \rho, \rho, [\delta_X \to \delta_X])$-computable.

(ii) *The function $(R(\lambda, A), \lambda) \mapsto A$ for continuous $R(\lambda, A)$ and $\lambda > 0$ is $([\delta_X \to \delta_X], \rho, \delta_G)$-computable.*

Proof. For any $x, y \in X$, $(x, y) \in \mathrm{graph}(A) \iff (\lambda x - y, x) \in \mathrm{graph}(\lambda I - A)^{-1}$. On $X \times X$ we consider the maximum distance $d((x, y), (x', y')) = \max(\|x - x'\|, \|y - y'\|)$.

i. Suppose, $\{(x_i, y_i) \mid i \in \mathbb{N}\}$ is dense in graph(A). Consider $\varepsilon > 0$ and $(w, x) \in \mathrm{graph}(\lambda I - A)^{-1}$. Then $(x, \lambda x - w) \in \mathrm{graph}(A)$ and for some i, $d((x, \lambda x - w), (x_i, y_i)) < \varepsilon/(\lambda + 1)$, hence $\|x - x_i\| < \varepsilon/(\lambda + 1)$ and $\|\lambda x - w - y_i\| < \varepsilon/(\lambda + 1)$. We obtain

$$\begin{aligned} d((w, x), (\lambda x_i - y_i, x_i)) &= \max(\|\lambda x_i - w - y_i\|, \|x - x_i\|) \\ &\leq \max(\lambda\|x_i - x\| + \|\lambda x - w - y_i\|, \|x - x_i\|) \\ &< \varepsilon. \end{aligned}$$

Therefore, $\{(\lambda x_i - y_i, x_i) \mid i \in \mathbb{N}\}$ is dense in graph$(\lambda I - A)^{-1}$. From this we can conclude that the function $(A, \lambda) \mapsto R(\lambda, A)$ is $(\delta_G, \rho, \delta_G)$-computable. Since $\|R(\lambda, A)\| \leq M/(\lambda - \theta)$, by Lemma 3.6 we can compute a $[\delta_X \to \delta_X]$-name of $R(\lambda, A)$ from (A, θ, M, λ).

ii. Since $\{\alpha(i) \mid i \in \mathbb{N}\}$ is dense in X and $R(\lambda, A)$ is continuous, $\{(\alpha(i), R(\lambda, A) \circ \alpha(i)) \mid i \in \mathbb{N}\}$ is dense in graph$(R(\lambda, A))$. As above from this set we can compute a countable set dense in graph(A). □

Lemma 5.2 *Define a multi-function F_3 such that $N \in F_3(A, \theta, M, x, T, k)$, iff (A, θ, M) is a piece of type-IG information, $x \in X$, $T > 0$, $k \in \mathbb{N}$, and $\|W(t)x - W(s)x\| \leq 2^{-k}$ if $s, t \leq T$ and $|t - s| \leq 2^{-N}$, where $W(t)$ is the C_0 semigroup generated by A. Then F_3 is $(\delta_G, \rho, \rho, \delta_X, \nu_{\mathbb{N}}, \nu_{\mathbb{N}}, \nu_{\mathbb{N}})$-computable.*

Proof. By Thm. 2.4.iv for $s, t \leq T$ and $x \in \mathrm{dom}(A)$,

$$\|W(t)x - W(s)x\| = \|\int_s^t W(\tau)Axd\tau\| \leq Me^{T\theta}\|Ax\|\,|t - s|\,.$$

For $x \in X$ and $y \in \mathrm{dom}(A)$,

$$\begin{aligned} \|W(t)(x) - W(s)(x)\| &\leq \|(W(t) - W(s))(y)\| + \|(W(t) - W(s))(x - y)\| \\ &\leq Me^{T\theta}\|Ay\|\,|s - t| + 2Me^{\theta T}\|x - y\|. \end{aligned}$$

For given A, M, θ, x, T, k, where A is given by some δ_G-name $p = \langle p_0, q_0, p_1, q_1, \ldots\rangle$, first select some i such that $2Me^{\theta T}\|x - \delta_X(p_i)\| \leq 2^{-k-1}$ and then some N such that $Me^{T\theta}\|\delta_X(q_i)\| \cdot 2^{-N} \leq 2^{-k-1}$. Such a number i

exists since $\mathrm{dom}(A)$ is dense in X. Notice that $A\delta_X(p_i) = \delta_X(q_i)$. Then $\|W(t)(x) - W(s)(x)\| \le 2^{-k}$ if $|t-s| \le 2^{-N}$. □

Lemma 5.3 *Define a multifunction F_4 such that $l \in F_4(A,\theta,M,x,k)$, iff (A,θ,M) is a piece of type-IG information, $x \in X$, $k \in \mathbb{N}$, and $\|\lambda R(\lambda,A)x - x\| \le 2^{-k}$ for $\lambda \ge l$. Then F_4 is $(\delta_G,\rho,\rho,\delta_X,\nu_{\mathbb{N}},\nu_{\mathbb{N}})$-computable.*

Proof. Since $\lambda R(\lambda,A) - I)a = R(\lambda,A)Aa$ for $a \in \mathrm{dom}(A)$, for any $x \in X$, $a \in \mathrm{dom}(A)$ and $\lambda > 2\theta$,

$$\begin{aligned}
\|\lambda R(\lambda,A)x - x\| &\le \|(\lambda R(\lambda,A) - I)a\| + \|(\lambda R(\lambda,A) - I)(x-a)\| \\
&= \|R(\lambda,A)Aa\| + \|(\lambda R(\lambda,A) - I)(x-a)\| \\
&\le \frac{M}{\lambda-\theta}\|Aa\| + \|\lambda R(\lambda,A) - I\|\|x-a\| \\
&\le \frac{M}{\lambda-\theta}\|Aa\| + (\|\lambda R(\lambda,A)\| + 1)\|x-a\| \\
&\le \frac{M}{\lambda-\theta}\|Aa\| + (\frac{\lambda M}{\lambda-\theta} + 1)\|x-a\| \\
&\le \frac{M}{\lambda-\theta}\|Aa\| + (2M+1)\|x-a\| \,.
\end{aligned}$$

For given (A,θ,M,λ,x,k) where A is given by some δ_G-name $p = \langle p_0,q_0,p_1,q_1,\ldots\rangle$, first select some i such that $(2M+1)\|x - \delta_X(p_i)\| \le 2^{-k-1}$ and then choose some l such that $l > 2\theta$ and $(M/(l-\theta))\|\delta_X(q_i)\| \le 2^{-k-1}$ (notice that $A\delta_X(p_i) = \delta_X(q_i)$). Then $\|\lambda R(\lambda,A)x - x\| \le 2^{-k}$ for $\lambda \ge l$. □

We can now prove our computable version of Theorem 2.3.

Theorem 5.4 (i) *The operator $\overline{Q} : (W,\theta,M) \mapsto A$ mapping every C_0 semigroup $W(t)$ and constants $\theta \ge 0$ and $M \ge 1$ such that $\|W(t)\| \le Me^{\theta t}$ to its infinitesimal generator A is $(\delta_S,\rho,\rho,\delta_G)$-computable.*

(ii) *The operator $Q :\subseteq (A,\theta,M) \mapsto W$ mapping every piece of type-IG information (A,θ,M) to the C_0 semigroup $W(t)$ generated by A is $(\delta_G,\rho,\rho,\delta_S)$-computable.*

Proof. i. As mentioned in the proof of Theorem 2.3 [6],

$$R(\lambda,A)x = \int_0^\infty e^{-\lambda t}W(t)x\,dt$$

for every $\lambda > \theta$. As a straightforward generalization of Thm. 6.4.1 in [10] the operator $I : (f,a,b) \mapsto \int_a^b f(x)\,dx$ for continuous $f : [0;\infty) \to X$ and $a,b \ge 0$ is $([\rho_+ \to \delta_X],\rho,\rho,\delta_X)$-computable.

Since $(W, t, x) \mapsto W(t)x$ is $(\delta_S, \rho_+, \delta_X, \delta_X)$-computable, the function $(W, \lambda, x) \mapsto (t \mapsto e^{-\lambda t}W(t)x)$ is $(\delta_S, \rho, \delta_X, [\rho_+ \to \delta_X])$-computable and therefore, $(W, \lambda, x, a, b) \mapsto \int_a^b e^{-\lambda t}W(t)x\,dt$ is $(\delta_S, \rho, \delta_X, \rho, \rho_+, \delta_X)$-computable.

Since $\|W(t)\| \le Me^{\theta t}$, for any $0 < a < b$,

$$\| \int_a^b e^{-\lambda t}W(t)x\,dt\| \le \int_a^b Me^{-(\lambda-\theta)t}\|x\|\,dt = \Big[-\frac{M\|x\|}{\lambda-\theta}e^{-(\lambda-\theta)t}\Big]_a^b .$$

For $k \in \mathbb{N}$ and W, θ, M, λ, x find a number $T \in \mathbb{N}$ such that $\frac{M\|x\|}{\lambda-\theta}e^{-(\lambda-\theta)T} < 2^{-k}$ and determine $y_k := \int_0^T e^{-\lambda t}W(t)x\,dt$. The multi-function $(W, \theta, M, \lambda, x, k) \rightrightarrows (y_k)_{k\in\mathbb{N}}$ is computable and

$$\|y_k - R(\lambda, A)x\| = \| \int_T^\infty e^{-\lambda t}W(t)x\,dt\| \le \frac{M\|x\|}{\lambda-\theta}e^{-(\lambda-\theta)T} < 2^{-k} .$$

By Lemma 3.5, $(W, \theta, M, \lambda, x) \mapsto R(\lambda, A)x$ is computable and by Lemma 3.1, the function $(W, \theta, M, \lambda) \mapsto R(\lambda, A)$ is $(\delta_S, \rho, \rho, [\delta_X \to \delta_X])$-computable. As a next step of computation for θ find some $\lambda \in \mathbb{N}$ such that $\theta < \lambda$. Therfore, $\overline{Q}$ is computable.

ii. We consider the three cases ($M = 1$ and $\theta = 0$), ($M \ge 1$ and $\theta = 0$) and ($M \ge 1$ and $\theta \ge 0$.

Case 1: ($M = 1$ and $\theta = 0$)
We will apply (6), $W(t)x = \lim_{\lambda\to 0^+} e^{\lambda A_\lambda}x$, to compute W from A. By Theorems 2.3 and 2.4,

$$\|W(t)\| \le 1 \quad \text{and} \quad \|R(\lambda, A)\| \le \frac{1}{\lambda} \quad \text{for} \quad \lambda > 0 . \tag{11}$$

For the Yosida approximation A_λ by (7) we obtain

$$\|e^{tA_\lambda}\| = e^{-t\lambda}\|e^{t\lambda^2 R(\lambda,A)}\| \le e^{-t\lambda}e^{t\lambda^2\|R(\lambda,A)\|} \le 1. \tag{12}$$

For any $\lambda > 0$ and $\mu > 0$, it is clear from the definitions that e^{tA_λ}, e^{tA_μ}, A_λ and A_μ commute with each other and consequently, for any $a \in \mathrm{dom}(A)$,

$$\begin{aligned}
\|e^{tA_\lambda}a - e^{tA_\mu}a\| &= \left\| \int_0^1 \frac{d}{ds}(e^{tsA_\lambda}e^{t(1-s)A_\mu}a)ds \right\| \\
&\le \int_0^1 t\|e^{tsA_\lambda}e^{t(1-s)A_\mu}\,(A_\lambda a - A_\mu a)\,\|ds \\
&\le \int_0^1 t\|e^{tsA_\lambda}\|\|e^{t(1-s)A_\mu}\|\|A_\lambda a - A_\mu a\|ds \qquad (13)\\
&\le t\|A_\lambda a - A_\mu a\|. \qquad (14)
\end{aligned}$$

Now for every $x \in X$ and any $a \in \mathrm{dom}(A)$ by (8), (12) and (13),

$$\begin{aligned}
&\|e^{tA_\lambda}x - e^{tA_\mu}x\| \\
&= \|e^{tA_\lambda}(x-a) + e^{tA_\mu}(a-x) + (e^{tA_\lambda} - e^{tA_\mu})a\| \\
&\le \|e^{tA_\lambda}(x-a)\| + \|e^{tA_\mu}(a-x)\| + \|(e^{tA_\lambda} - e^{tA_\mu})a\| \\
&\le \|e^{tA_\lambda}\|\|(x-a)\| + \|e^{tA_\mu}\|\|(a-x)\| + t\|A_\lambda a - A_\mu a\| \\
&\le 2\|x-a\| + t\|A_\lambda a - A_\mu a\| \\
&\le 2\|x-a\| + t\|\lambda R(\lambda,A)Aa - \mu R(\mu,A)Aa\| \\
&\le 2\|x-a\| + t\|\lambda R(\lambda,A)Aa - Aa\| + t\|\mu R(\mu,A)Aa - Aa\|\,.
\end{aligned} \tag{15}$$

Next, from (A,t,x,k) we compute some $y_k \in X$ such that $\|y_k - W(t)x\| \le 2^{-k}$. Let A be given by some δ_G-name $\langle p_0, q_0, p_1, q_1, \ldots\rangle$. Since $\mathrm{dom}(A)$ is dense in X we can find some i such that

$$\|x - \delta_X(p_i)\| \le 2^{-k-2}\,. \tag{16}$$

Notice that $A\delta_X(p_i) = \delta_X(q_i)$. By Lemma 5.3 we can compute some $l \in \mathbb{N}$ such that

$$t\|\lambda R(\lambda,A)\delta_X(q_i) - \delta_X(q_i\| \le 2^{-k-2} \tag{17}$$

for $\lambda \ge l$. From (15) with $a := \delta_X(p_i)$ we obtain $\|e^{tA_\lambda}x - e^{tA_\mu}x\| \le 2^{-k}$ for $\lambda, \mu \ge l$ and therefore,

$$\|e^{tA_l}x - W(t)x\| \le 2^{-k}\,.$$

Next we compute $y_k := e^{tA_l}x$ from (A,t,x,k). By Lemma 5.1 the function $(A,\lambda) \mapsto R(\lambda,A)$ is $(\delta_G, \rho, [\delta_X \to \delta_X])$-computable (by assumption, $\theta = 0$ and $M = 1$), therefore, $(A,\lambda) \mapsto \lambda^2 R(\lambda,A) - \lambda I = A_\lambda$ is $(\delta_G, \rho, [\delta_X \to \delta_X])$-computable. In Theorem 4.1 we have proved that $(B,t,x) \mapsto e^{tB}x$ is $([\delta_X \to \delta_X], \rho_+, \delta_X, \delta_X)$-computable. We conclude that the function $(A,\lambda,t,x) \mapsto e^{tA_\lambda}x$ is $(\delta_G, \rho, \rho_+, \delta_X, \delta_X)$-computable. Therefore, from (A,t,x) we can compute a sequence $(y_k)_{k\in\mathbb{N}}$ such that $\|y_k - W(t)x\| \le 2^{-k}$. By Lemma 3.5 we can compute its limit $W(t)x$. Therefore $(A,t,x) \mapsto W(t)x$ is $(\delta_G, \rho_+, \delta_X, \delta_X)$-computable. By type conversion (Lemma 3.1) $(A,t) \mapsto W(t)$ is $(\delta_G, \rho_+, [\delta_X \to \delta_X])$-computable, and again by type conversion, $A \mapsto W$ is $(\delta_G, [\rho_+ \to [\delta_X \to \delta_X]])$-computable, that is, (δ_G, δ_S)-computable.

Case 2: ($M \ge 1$ and $\theta = 0$)

In this case, we will introduce a new norm on the space X, denoted as $\|\cdot\|_{new}$. This norm is equivalent to the original norm $\|\cdot\|$ but it rescales M back to 1. Define

$$\|x\|_{new} = \sup_{t\ge 0} \|W(t)x\|.$$

Since $\|W(t)\| \le Me^{\theta t} = M$,

$$\|x\| \le \|x\|_{new} \le M\|x\| \tag{18}$$

Thus, $\|\cdot\|_{new}$ indeed is a norm on X and it is equivalent to the original norm $\|\cdot\|$ on X. Furthermore,

$$\|W(t)x\|_{new} = \sup_{s\geq 0}\|W(s)W(t)x\| = \sup_{s\geq 0}\|W(s+t)x\| \leq \sup_{s\geq 0}\|W(s)x\| = \|x\|_{new},$$

which implies that $W(t)$ is a C_0 semigroup of contractions on X endowed with the norm $\|\cdot\|_{new}$. Consequently, Case 2 can be reduced to Case 1 in the new norm $\|\cdot\|_{new}$. In particular, (15) is now valid for the new norm $\|\cdot\|_{new}$ instead of $\|\cdot\|$. From this and (18) we obtain

$$\begin{aligned} & \|e^{tA_\lambda}x - e^{tA_\mu}x\| \\ \leq\, & \|e^{tA_\lambda}x - e^{tA_\mu}x\|_{new} \\ \leq\, & 2\|x-a\|_{new} + t\|\lambda R(\lambda,A)Aa - Aa\|_{new} + t\|\mu R(\mu,A)Aa - Aa\|_{new} \\ \leq\, & 2M\|x-a\| + tM\|\lambda R(\lambda,A)Aa - Aa\| + tM\|\mu R(\mu,A)Aa - Aa\|\,. \end{aligned} \tag{19}$$

We obtain an method to compute W from A if we replace (16) and (17) in the above algorithm for Case 1 by

$$M\|x - \delta_X(p_i)\| \leq 2^{-k-2} \;\text{ and }\; tM\|\lambda R(\lambda,A)\delta_X(q_i) - \delta_X(q_i\| \leq 2^{-k-2}\,.$$

Case 3: ($M \geq 1$ and $\theta \geq 0$)
We introduce a function U by

$$U(t) = e^{-\theta t}W(t)\,.$$

It can be shown easily that U is a C_0 semigroup. Since

$$\|U(t)\| = \|e^{-\theta t}W(t)\| = e^{-\theta t}\|W(t)\| \leq e^{-\theta t}Me^{\theta t} = M$$

U satisfies the assumption of Case 2. We determine its infinitesimal generator B of U by (3):

$$Bx = \lim_{t\mapsto 0^+}\frac{e^{-\theta t}W(t)x - x}{t} = \lim_{t\mapsto 0^+}\left(\frac{W(t)x - x}{t} + \frac{(e^{-\theta t}-1)W(t)x}{t}\right) = Ax - \theta x\,,$$

that is, $B = A - \theta I$. For determining W from A we compute B from A, U from B and finally W from U.

- The function $(A,\theta)\mapsto A-\theta I$ is (δ_G,ρ,δ_G)-computable:
  Suppose, $p = \langle p_0, q_0, p_1, q_1, \ldots\rangle$, $A = \delta_G(p)$ and $\rho(q) = \theta$. Then the set $\{(\delta_X(p_i), \delta_X(q_i)) \mid i \in \mathbb{N}\}$ is dense in graph(A) and hence the set $\{(\delta_X(p_i), \delta_X(q_i) - \theta\delta_X(p_i) \mid i \in \mathbb{N}\}$ is dense in graph($A-\theta I$). From (p,q,i) we can compute q_i' such that $\delta_X(q_i') = \delta_X(q_i) - \theta\delta_X(p_i)$. Hence, from (p,q) we can compute some q' such that $\delta_G(q') = $ graph($A-\theta I$). Therefore, $(A,\theta)\mapsto A-\theta I$ is (δ_G,ρ,δ_G)-computable.

- The function $(B, M) \mapsto U$ is $(\delta_G, \rho, \delta_S)$-computable by Case 2.
- The function $(U, \theta) \mapsto W$ is $(\delta_S, \rho, \delta_S)$-computable:
  By Lemma 3.1 the function $(U, t, x) \mapsto U(t)x$ is $(\delta_S, \rho_+, \delta_X, \delta_X)$-computable. Since the function $(\theta, t, y) \mapsto e^{-\theta t}y$ is $(\rho, \rho_+, \delta_X, \delta_X)$-computable, the function $(U, \theta, t, x) \mapsto e^{-\theta t}U(t)x = W(t)x$ is $(\delta_S, \rho, \rho_+, \delta_X, \delta_X)$-computable. Therefore by Lemma, 3.1 $(U, \theta) \mapsto W$ is $(\delta_S, \rho, \delta_S)$-computable.

Combining the three computations we obtain that $(A, \theta, M) \mapsto W$ is $(\delta_G, \rho, \rho, \delta_S)$-computable. □

We notice that for $M \leq 1$, if $||R(\lambda, A)|| \leq 1/(\lambda - \theta)$ for any $\lambda > \theta$, then $||R(\lambda, A)^n|| \leq 1/(\lambda - \theta)^n$ for all $n \in \mathbb{N}$. In such cases, we call (A, θ) a piece of type-IGC information for A (IGC refers to "infinitesimal generator of a contraction semigroup"). Theorem 5.4 leads to the following algorithms computing contraction semigroups from their generators and vice versa.

Theorem 5.5 *1. The map $(A, \theta) \mapsto W$ is $(\delta_G, \rho, \delta_S)$-computable, provided (A, θ) is a piece of type-IGC information for A, where $W(t)$ is a C_0 semigroup generated by A satisfying $||W(t)|| \leq e^{\theta t}$.*

2. The map $W \mapsto (A, \theta)$, where (A, θ) is a piece of type-IGC information for A, is $(\delta_S, \delta_G, \rho)$-computable, provided $||W(t)|| \leq e^{\theta t}$ for all $t \geq 0$.

In Theorem 5.4.ii, a Turing algorithm is constructed that computes, on any given piece of type-IG data for A, a C_0 semigroup $W(t)$ generated by A satisfying $||W(t)|| \leq Me^{\theta t}$. This algorithm can be used to solve the following abstract Cauchy problem.

Corollary 5.6 *There is a Turing algorithm that computes the solution u of the initial-value problem $du(t)/dt = Au(t)$ for $t > 0$ and $u(0) = x$ from any given piece of type-IG data for A plus any δ_X-name for the initial value $x \in X$.*

Proof. If A is the infinitesimal generator of a C_0 semigroup $W(t)$, then the solution $u(t) = W(t)x$. □

We note that, compared to the Cauchy problem where A is a bounded linear operator, more information is required in order to compute the solution when A is unbounded.

One of the fundamental problems in the theory of semigroup of operators is the relation between the semigroup $W(t)$ and its infinitesimal generator A. However, from the point of view of applications to partial differential equations it is more interesting to obtain $W(t)$ from A. The reason goes as follows: If A is the infinitesimal generator of a C_0 semigroup $W(t)$, then for $x \in X$, $W(t)x$

is the (mild) solution of the initial-value problem

$$\frac{du}{dt} - Au = 0, \quad u(0) = x$$

(see, for example, Section 4.1 of [Paz83]). Since our motivation is to invent Turing algorithms for computing solutions of PDEs, in the remaining of the paper, we will be mainly studying how to compute the semigroup from its infinitesimal generator for various abstract Cauchy problems of the form:

$$\begin{cases} \frac{d}{dt}u(t) = Au(t) + Bu(t), & t > 0, \\ \\ u(0) = x, \end{cases} \tag{20}$$

where B, called a perturbation of A, is a linear operator from X to X. Thus, if $A + B$ is the infinitesimal generator of a C_0 semigroup $W(t)$ of bounded linear operators, then (20) admits a unique solution and this solution is $u(t) = W(t)x$. In such cases, if one has a piece of type-IG data for $A+B$, then one can compute the solution $u(t)$ by making use of the Turing algorithm constructed in Theorem 5.4.ii. In many applications, however, it might be much easier to get a piece of type-IG data for A than the same type of data for $A + B$. In the following, we will be interested in the possibility of using various types of data for A to compute the C_0 semigroup generated by $A + B$. When B is a bounded linear operator, any piece of type-IG data for A together with a $[\delta_X \to \delta_X]$-name of B is sufficient to compute the semigroup generated by $A + B$ as the following theorem demonstrates.

Theorem 5.7 *Assume that $B : X \to X$ is a bounded linear map. Then the map $(A, \theta, M, B) \mapsto W$, where $W(t)$, $t \geq 0$, is the C_0 semigroup generated by $A + B$, is $(\delta_G, \rho, \rho, [\delta_X \to \delta_X], \delta_S)$ -computable, provided that (A, θ, M) is a piece of type-IG information for A.*

Proof. To compute $W(t)$, by Theorem 5.4.ii, it suffices to have a piece of type-IG data for $A + B$. It is known classically that if (A, θ, M) is a piece of type-IG information, then $(A + B, \theta + Mb, M)$ will be a piece of type-IG information for $A + B$ for any bounded linear operator B, where b is an arbitrary upper bound of $||B||$ (Theorem 3.1.1 [6]). Thus, it suffices to show that (i) an upper bound of $||B||$ can be computed from any given $[\delta_X \to \delta_X]$-name of B and (ii) a δ_G-name of $A + B$ can be computed from any given δ_G-name of A together with a $[\delta_X \to \delta_X]$-name of B. It follows from Lemma 3.6 that (i) is true. The proof for (ii): for any dense set $\{(a_i, A(a_i))\}_{i\in\mathbb{N}}$ in the graph of A, it can be easily verified that the set $\{(a_i, A(a_i) + B(a_i))\}_{i\in\mathbb{N}}$ is dense in the graph of $A + B$. Since $\mathrm{dom}(B) = X \supset \mathrm{dom}(A)$ and $(B, x) \mapsto B(x)$ is $([\delta_X \to \delta_X], \delta_X, \delta_X)$-computable, (ii) holds. □

Corollary 5.8 *Assume that $A : X \to X$ is a densely defined closed (unbounded) linear operator and $B : X \to X$ is a bounded linear operator. Then the solution operator $S : (A, \theta, M, B, x) \mapsto u$ of the initial-value problem (20) is $(\delta_G, \rho, \rho, [\delta_X \to \delta_X], \delta_X, [\rho \to \delta_X])$-computable, provided (A, θ, M) is a piece of type-IG information for A.*

When $B : X \to X$ is an unbounded linear operator, in order to compute the solution of the problem (20), B will be assumed to be an A-bounded closed linear operator, that is, $\mathrm{dom}(B) \supseteq \mathrm{dom}(A)$ and there are two constants $a \geq 0$ and $b \geq 0$ such that

$$\|Bx\| \leq a\|Ax\| + b\|x\| \qquad \text{for } x \in \mathrm{dom}(A). \tag{21}$$

We call the pair (a, b) an A-bound of B. As for the operator A, some alternation in data type is also needed. A triple (A, M, δ) is called a piece of type-IGA information for A (IGA refers to "infinitesimal generator of an analytic semigroup") if (1) $A : X \to X$ is a closed, densely defined operator, (2) $0 < \delta < \pi/2$, $\rho(A) \supset \Sigma_\delta = \{\lambda \in \mathbb{C} : |\arg\lambda| < \frac{\pi}{2} + \delta\} \cup \{0\}$, and (3) $||R(\lambda, A)|| \leq M/|\lambda|$ for all $\lambda \in \Sigma_\delta$ and $\lambda \neq 0$. Recall that $\rho(A)$ denotes the resolvent set of A. In addition, if a δ_G-name of A, a ρ-name of M and a ρ-name of δ are available, (A, M, δ) will be referred as a piece of type-IGA data for A. Classically, it is known that if (A, M, δ) is a piece of type-IGA information for A, then A is the infinitesimal generator of an analytic semigroup. In the following lemma, we show that much more follows from a piece of type-IGA information.

Lemma 5.9 *The map $(A, M, \delta) \mapsto (W, C)$ is $(\delta_G, \rho, \rho, \delta_S, \rho)$-computable, provided that (A, M, δ) is a piece of of type-IGA information for A, where $W(t)$ is the semigroup generated by A satisfying $||W(t)|| \leq C$.*

Proof. Let $\nu = \frac{\pi}{2} + \frac{\delta}{2}$. Clearly ν is computable from δ. Then the semigroup $W(t)$ generated by A can be written in the form of

$$W(t)x = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} R(\lambda, A) x d\lambda$$

(Theorem 1.7.7 [Paz83]), where Γ is a smooth curve in Σ_δ running from $\infty e^{-i\nu}$ to $\infty e^{i\nu}$. In the following, we show how to compute $W(t)x$ for any $x \in X$ and $t > 0$. Fix $t > 0$.

(a) Since $R(\lambda, A)$ is analytic in Σ_δ, without changing the value of the integral, we may shift Γ to Γ_t, where $\Gamma_t = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ and $\Gamma_1 = \{re^{-i\nu} : t^{-1} \leq r < \infty\}$, $\Gamma_2 = \{t^{-1}e^{i\phi} : -\nu \leq \phi \leq \nu\}$ and $\Gamma_3 = \{re^{i\nu} : t^{-1} \leq r < \infty\}$. Let c_t be the constant $c_t = t\sin(\nu - \frac{\pi}{2})$. Let $\Gamma^k = \Gamma_1^k \cup \Gamma_2 \cup \Gamma_3^k$, where $\Gamma_1^k = \{re^{-i\nu} : t^{-1} \leq r \leq k\}$ and $\Gamma_3^k = \{re^{i\nu} : t^{-1} \leq r \leq k\}$, $k \geq t^{-1}$. By Lemma 5.1.i the map $(A, M, \delta, \lambda) \mapsto R(\lambda, A)$, $\lambda \in \Sigma_\delta$, is $(\delta_G, \rho, \rho, \rho^2, [\delta_X \to \delta_X])$-computable, and consequently the map $(A, M, \delta, t, x, \lambda) \mapsto e^{\lambda t}R(\lambda, A)x$ is $(\delta_G, \rho, \rho, \rho_+, \delta_X, \rho^2, \delta_X)$-computable and, therefore, $(A, M, \delta, t, x, k) \mapsto I_k(t)x = \frac{1}{2\pi i}\int_{\Gamma^k} e^{\lambda t}R(\lambda, A)xd\lambda$ is $(\delta_G, \rho, \rho, \rho_+, \delta_X, \nu_{\mathbb{N}}, \delta_X)$-computable. Furthermore,

$$\begin{aligned}
&||W(t) - I_k(t)||_{L(X)} \\
&= \sup_{||x||=1}\left|\left|\frac{1}{2\pi i}\int_{\Gamma_1\setminus\Gamma_1^k} e^{\lambda t}R(\lambda, A)xd\lambda + \int_{\Gamma_3\setminus\Gamma_3^k} e^{\lambda t}R(\lambda, A)xd\lambda\right|\right|_X \\
&\leq \frac{1}{2\pi}\int_k^\infty |e^{tre^{-i\nu}}| \cdot \big|\big|R(re^{-i\nu}, A)\big|\big|\, d|re^{-i\nu}| \\
&\qquad + \frac{1}{2\pi}\int_k^\infty |e^{tre^{i\nu}}| \cdot \big|\big|R(re^{i\nu}, A)\big|\big|\, d|re^{i\nu}| \\
&\leq \frac{1}{\pi}\int_k^\infty e^{tr\cos\nu}\frac{M}{r}dr \\
&= \frac{M}{\pi}\int_k^\infty e^{-tr\sin(\nu-\pi/2)}\frac{1}{r}dr \qquad (\frac{\pi}{2} < \nu < \frac{\pi}{2} + \delta < \pi) \\
&= \frac{M}{\pi}\int_{kc_t}^\infty \frac{e^{-s}}{s}ds.
\end{aligned}$$

Since the integral $\int_k^\infty \frac{e^{-s}}{s}ds$ converges effectively to 0 as $k \to \infty$, $I_k(t)$ effectively converges to $W(t)$ in k for $t > 0$ in the uniform operator norm. This proves that $(A, M, \delta, t, x) \mapsto W(t)x$ is $(\delta_G, \rho, \rho, \rho, \delta_X, \delta_X)$-computable. By type conversion twice, $(A, M, \delta) \mapsto W$ is $(\delta_G, \rho, \rho, \delta_S)$-computable.

(b) Use the path Γ_t and a similar estimate as of (a), one can verifies that $||W(t)|| \leq 2C_1 + C_2$, where C_1 is any rational number satisfying $C_1 \geq \frac{M}{2\pi}\int_{kc_t}^\infty \frac{e^{-s}}{s}ds$ and C_2 any rational number satisfying $C_2 \geq \frac{M}{2\pi}\int_{-\nu}^\nu e^{\cos\phi}d\phi$. Since the two integrals are computable from M and ν, so are the constants $C = 2C_1 + C_2$. □

Theorem 5.10 *Assume that B is an A-bounded closed linear operator. Then the map $((A, B), M, \delta, a, b) \mapsto (W, C, \omega)$, where $W(t)$ is the C_0 semigroup generated by $A + B$ satisfying $||W(t)|| \leq Ce^{\omega t}$, is $(\delta_{GG}, \rho, \rho, \rho, \rho, \delta_S, \rho, \rho)$-computable, provided that (A, M, δ) is a piece of type-IGA information for A and (a, b) is an A-bound of B satisfying $0 \leq a \leq \frac{1}{3}(1 + M)^{-1}$.*

Proof. By Lemma 5.9 it suffices to compute a piece of type-IG data for $A+B$ from the given information.

(a) Since B is A-bounded, the following estimate holds for any $x \in X$:

$$\begin{aligned}||BR(\lambda, A)x|| &\leq a||AR(\lambda, A)x|| + b||R(\lambda, A)x|| \\ &\leq a||x|| + (a|\lambda| + b)(M/|\lambda|)||x|| \\ &= (a(M+1) + bM/|\lambda|)||x|| \leq \frac{2}{3}||x||\end{aligned}$$

for any $\lambda \in \Sigma_\delta$ and $|\lambda| > 3Mb$ (recall that $0 \leq a \leq \frac{1}{3}(M+1)^{-1}$). Thus $||BR(\lambda, A)|| \leq 2/3$, and consequently $I - BR(\lambda, A)$ is invertible with $||(I - BR(\lambda, A))^{-1}|| \leq \sum_{k=0}^{\infty} ||BR(\lambda, A)^k|| = 3$. A simple calculation shows that

$$R(\lambda, A+B) = R(\lambda, A)(I - BR(\lambda, A))^{-1}, \ \lambda \in \Sigma_\delta \text{ and } |\lambda| > 3Mb$$

The above representation of $R(\lambda, A+B)$ immediately leads to the estimate

$$||R(\lambda, A+B)|| \leq \frac{3M}{|\lambda|}, \ \lambda \in \Sigma_\delta \text{ and } |\lambda| > 3Mb$$

Let $\beta = 6Mb$. Then for any $\theta \in \Sigma_\delta$, $|\theta + \beta| > 3Mb$. Since for any $\theta \in \Sigma_\delta$, $R(\theta, A+B-\beta I) = (\theta I - (A+B-\beta I))^{-1} = ((\theta+\beta)I - (A+B))^{-1} = R(\theta + \beta, A+B)$, it follows that $\rho(A+B-\beta I) \supset \Sigma_\delta$. Consider Σ_α, where $\alpha = \delta/2$. For any $\theta \in \Sigma_\alpha$, write $\theta = \theta_1 + i\theta_2$. If $\theta_1 \geq 0$, then $|\theta + \beta| \geq |\theta|$. If $\theta_1 < 0$, let α_θ be the acute angle between $\mathbf{0\boldsymbol{\theta}}$ (the vector from the origin of the complex plane $\mathbb{C}$ to θ) and the positive or negative i-axis. Then $|\theta + \beta| \geq |\theta| \cos \alpha_\theta$. Since $0 < \alpha_\theta \leq \alpha < \pi/2$, one obtains $|\theta + \beta| \geq |\theta| \cos \alpha_\theta \geq |\theta| \cos \alpha$. Thus for any $\theta \in \Sigma_\alpha$, $||R(\theta, A+B-\beta I)|| = ||R(\theta+\beta, A+B)|| \leq 3M/|\theta+\beta| \leq \frac{3M/\cos\alpha}{|\theta|}$, which in turn implies that $(A+B-\beta I, 3M/\cos(\delta/2), \delta/2)$ is a piece of type-IGA information for $A+B-\beta I$.

(b) Obviously $3M/\cos(\delta/2)$ is computable from M and δ. Thus, to compute a piece of type-IGA data for $A+B-\beta I$, it suffices to show that a δ_G-name of $A+B-\beta I$ can be computed from M, b and a δ_{GG}-name of (A, B). Let $\{(a_i, Aa_i)\}_{i\in\mathbb{N}}$ be a dense set of the graph of A and $\{(a_i, B(a_i)\}_{i \in mathbbN}$ a dense set of the graph of B. Then for any $n \in \mathbb{N}$ and any $x \in \mathrm{dom}(A)$, there is an a_i such that $d((x, Ax), (a_i, Aa_i)) = \max(||x - a_i||, ||A(x - a_i)||) < 2^{-(n+4)}/c$, where $c = \frac{1}{3}(1+M)^{-1} + b$. Moreover,

$$\begin{aligned}d(Bx, Ba_i) &= ||B(x - a_i)|| \\ &\leq a||A(x-a_i)|| + b||x - a_i|| < 2^{-(n+2)},\end{aligned}$$

which implies that $d(Ax+Bx, Aa_i+Ba_i) \leq d(Ax+Bx, Aa_i+Bx) + d(Aa_i + Bx, Aa_i + Ba_i) = ||Ax - Aa_i|| + ||Bx - Ba_i|| \leq 2^{-(n+1)}$, and consequently, $d((x, Ax+Bx), (a_i, Aa_i+Ba_i)) \leq 2^{-n}$. Thus $\{(a_i, Aa_i+Ba_i)\}_{i\in\mathbb{N}}$ is a dense set of the graph of $A+B$ and therefore $\{(a_i, Aa_i+Ba_i-\beta a_i)\}_{i\in\mathbb{N}}$ is a dense set of the graph of $A+B-\beta I$. Since $\beta = 6Mb$ is computable from M and b, and both addition $(x,y)\mapsto x+y$ and scalar multiplication $(c,x)\mapsto cx$ are computable operators on the computable Banach space X, a δ_G-name of $A+B-\beta I$ can be computed. Let $W_\beta(t)$ be the semigroup generated by $A+B-\beta I$. Then by Lemma 5.9, a δ_S-name of W_β and a ρ-name of C satisfying $||W_\beta(t)|| \le C$ can then be computed.

(c) Let $W(t) = e^{\beta t}W_\beta(t)$. Then by making use of a similar argument as that used in the proof of case 3 of Theorem 5.4.ii, it can be verified that $A+B$ is the infinitesimal generator of $W(t)$ satisfying $||W(t)|| \le Ce^{\beta t}$, and $W(t)$ is computable from $W_\beta(t)$. $\square$

Corollary 5.11 *Assume that $A : X \to X$ is a densely defined closed (unbounded) linear operator and $B : X \to X$ is A-bounded. Then the solution operator $S : ((A,B), M, \delta, a, b, x) \to u$ of the initial-value problem (20) is $(\delta_{GG}, \rho, \rho, \rho, \rho, \delta_X, [\rho \to \delta_X])$-computable, provided (A, M, δ) is a piece of type-IGA information for A and (a,b) is an A-bound of B satisfying $0 \le a \le \frac{1}{3}(1+M)^{-1}$.*

6 Applications

In this section, results obtained in the previous section are applied to compute solutions of homogeneous and inhomogeneous linear parabolic equations.

First we recall the definition of parabolic equations. Consider the following differential operator of order $2m$:

$$\mathcal{A}(x,D) = \sum_{|\alpha|\le 2m} a_\alpha(x) D^\alpha,$$

where the coefficients $a_\alpha(x)$ are sufficiently smooth complex-valued functions of x in $\mathbb{R}^n$, $\alpha = (\alpha_1, \ldots, \alpha_n)$ with non-negative integer components is a multi-index of order $|\alpha| = \alpha_1 + \ldots + \alpha_n$, and $D^\alpha f = D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n} f$. The principle part $\mathcal{A}'(D)$ of $\mathcal{A}(x,D)$ is the operator

$$\mathcal{A}'(D) = \sum_{|\alpha|=2m} a_\alpha D^\alpha,$$

where the coefficients a_α with $|\alpha| = 2m$ are constants.

Definition 6.1 The operator $\mathcal{A}(x, D)$ is called strongly elliptic if there exists a constant $c > 0$ such that

$$(-1)^m \mathcal{A}'(\xi) \geq c|\xi|^{2m}$$

for all $\xi \in \mathbb{R}^n$.

In the following, we assume that a_α are real numbers if $|\alpha| = 2m$, and $a_\alpha \in C_0^\infty(\mathbb{R}^n)$ if $|\alpha| \leq 2m-1$, where $C_0^\infty(\mathbb{R}^n)$ is the set of all infinitely differentiable functions on $\mathbb{R}^n$ with compact supports. We call a tuple a $\delta_{\mathcal{A}}$-name of $\mathcal{A}$ if it lists a ρ-name for each a_α, $|\alpha| = 2m$, and a $\delta_{\mathcal{D}}$-name for each a_α, $|\alpha| \leq 2m-1$ (the definition of $\delta_{\mathcal{D}}$ can be found in [15]). Speaking roughly, a $\mathcal{A}$-name of $\mathcal{A}$ is a collection of data on the coefficients of the operator $\mathcal{A}$. We note that the data-type used are standard for numbers and C_0^∞ functions.

The following initial-value problem is called an initial-value problem of a parabolic equation if $\mathcal{A}$ is a strongly elliptic operator:

$$\begin{cases} \partial_t u + \mathcal{A}(x, D)u = 0, & x \in \mathbb{R}^n,\ t > 0, \\ \\ u(x, 0) = \phi(x) \in L^2(\mathbb{R}^n). \end{cases} \tag{22}$$

Theorem 6.2 *Assume that $\mathcal{A}(x, D)$ is a strongly elliptic operator. Then, the solution operator $K : (\mathcal{A}, \phi) \mapsto u$ of the problem (22) is $(\delta_{\mathcal{A}}, \delta_{L^2(\mathbb{R}^n)}, [\rho \to \delta_{L^2(\mathbb{R}^n)}])$-computable.*

Proof. First we define two operators $A, B : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ as follows: $\mathrm{dom}(A) = H^{2m}(\mathbb{R}^n)$ and for every $u \in \mathrm{dom}(A)$, $Au = -\mathcal{A}'(D)u$; $\mathrm{dom}(B) = H^{2m-1}(\mathbb{R}^n)$ and for every $u \in \mathrm{dom}(B)$,

$$Bu = -(\mathcal{A}(x, D) - \mathcal{A}'(D)u) = -\sum_{|\alpha|<2m} a_\alpha(x) D^\alpha u$$

(see, for example, [3] for definitions of $L^p(\mathbb{R}^n)$ and $H^m(\mathbb{R}^n)$). The initial value problem (22) can now be written in the form of the following abstract Cauchy problem in $L^2(\mathbb{R}^n)$:

$$\begin{cases} \frac{d}{dt}u = Au + Bu, & t > 0, \\ \\ u(0) = \phi. \end{cases} \tag{23}$$

We prove in the following that (a) $(A, \sqrt{2}, \pi/4)$ is a piece of type-IGA information for A and, moreover, a piece of type-IGA data can be computed from any given $\delta_{\mathcal{A}}$-name of $\mathcal{A}$; (b) B is A-bounded and, furthermore, an A-bound (a, b) with $0 < a < \frac{1}{3}(1+\sqrt{2})^{-1}$ can be computed from any given $\delta_{\mathcal{A}}$-name of $\mathcal{A}$; (c) A δ_{GG}-name of $A+B$ can be computed from any $\delta_{\mathcal{A}}$-name of $\mathcal{A}$. Then, by Theorem 5.10, the semigroup $W(t)$ generated by $A+B$ can be computed. Since $u(t) = W(t)\phi$, the solution u is computable from $\mathcal{A}$ and ϕ.

(a) It is known classically that the operator $A : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a densely defined closed linear operator. It remains to show that (a1) $\rho(A) \supset \Sigma_{\pi/4} = \{\lambda : |\arg\lambda| < \frac{\pi}{2} + \frac{\pi}{4}\} \cup \{0\}$ and (a2) $||R(\lambda, A)|| \leq \sqrt{2}/|\lambda|$ for any $\lambda \in \Sigma_{\pi/4}$ and $\lambda \neq 0$.

The proof for (a1): Since $\mathcal{A}$ is strongly elliptic, $0 \in \rho(A)$. For any $\lambda \in \Sigma_{\pi/4}$ with $\lambda \neq 0$, write $\lambda = \lambda_1 + \lambda_2 i$. If $(\lambda I - A)u = 0$ for some $u \in H^{2m}(\mathbb{R}^n)$, then, taking the Fourier transform of both sides yields $(\lambda + (-1)^m \mathcal{A}'(\xi))\hat{u} = 0$. Since either $\lambda_1 + (-1)^m \mathcal{A}'(\xi) \neq 0$ for all $\xi \in \mathbb{R}^n$ or $\lambda_2 \neq 0$, $\hat{u} \equiv 0$. Thus, $u \equiv 0$. This proves that for any $\lambda \in \Sigma_{\pi/4}$, $\lambda I - A$ is one-to-one. Next we show that $\lambda I - A : H^{2m}(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is also surjective for any $\lambda \in \Sigma_{\pi/4}$. For any $f \in L^2(\mathbb{R}^n)$, let $u = \mathcal{F}^{-1}\left(\frac{\hat{f}}{\lambda + (-1)^m \mathcal{A}'(\xi)}\right)$, where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform. Notice that u is well defined due to the fact that for any $\lambda \in \Sigma_{\pi/4}$, $\lambda + (-1)^m \mathcal{A}'(\xi) \neq 0$ for all $\xi \in \mathbb{R}^n$. It can be easily verified that $(\lambda I - A)u = f$. By applying the fact that $u \in H^{2m}(\mathbb{R}^n)$ if and only if $(1 + |\xi|^2)^m \hat{u} \in L^2(\mathbb{R}^n)$, one readily sees that $u = \mathcal{F}^{-1}(\hat{f}/(\lambda + (-1)^m \mathcal{A}'(\xi))) \in H^{2m}(\mathbb{R}^n)$. Thus, $\lambda I - A$ is surjective. (a1) is proved.

The proof for (a2): For any $\lambda \in \Sigma_{\pi/4}$ with $\lambda \neq 0$ and any $f \in L^2(\mathbb{R}^n)$, assume that $R(\lambda, A)f = g$. Then $f = (\lambda I - A)g$. Performing the Fourier transform yields $\hat{g} = \hat{f}/(\lambda + (-1)^m \mathcal{A}'(\xi))$. Moreover, since $(-1)^m \mathcal{A}'(\xi) \geq c|\xi|^2 m$ for all $\xi \in \mathbb{R}^n$, $c > 0$, the estimate $|\lambda + (-1)^m \mathcal{A}'(\xi)| \geq |\lambda| \cos \pi/4$ holds for all $\xi \in \mathbb{R}^n$. Thus, $||\hat{g}||_{L^2(\mathbb{R}^n)} = ||\hat{f}/(\lambda + (-1)^m \mathcal{A}'(\xi))||_{L^2(\mathbb{R}^n)} \leq \frac{\sqrt{2}}{|\lambda|}||\hat{f}||_{L^2(\mathbb{R}^n)}$. Combining with the fact that $||\hat{f}||_{L^2(\mathbb{R}^n)} = ||f||_{L^2(\mathbb{R}^n)}$ for all $f \in L^2(\mathbb{R}^n)$, we have the following estimates: $||g||_{L^2(\mathbb{R}^n)} = ||R(\lambda, A)f||_{L^2(\mathbb{R}^n)} \leq \frac{\sqrt{2}}{|\lambda|}||f||_{L^2(\mathbb{R}^n)}$ for any $f \in L^2(\mathbb{R}^n)$, which yields $||R(\lambda, A)|| \leq \sqrt{2}/|\lambda|$ for any $\lambda \in \Sigma_{\pi/4}$. (a2) is proved.

To get a piece of type-IGA data of A, it suffices to show that a δ_G-name of A can be computed from any given $\delta_{\mathcal{A}}$-name of $\mathcal{A}$. Since, for any α with $|\alpha| = 2m$, the differentiation $\partial^\alpha : H^{2m}(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, $f \mapsto \partial^\alpha f$, is $(\delta_{H^{2m}(\mathbb{R}^n)}, \delta_{L^2(\mathbb{R}^n)})$-computable, where $\partial^\alpha f$ is the weak derivative of f of order α, and the operator $A : H^{2m}(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a finite sum of scaler multiples of ∂^α, it follows that $(\mathcal{A}, f) \mapsto Af$ is $(\delta_{\mathcal{A}}, \delta_{H^{2m}(\mathbb{R}^n)}, \delta_{L^2(\mathbb{R}^n)})$-computable. Consequently, a δ_G-name of A that encodes the dense set $\{(P_j, AP_j) : j \in \mathbb{N}\}$ can be computed from any $\delta_{\mathcal{A}}$-name of $\mathcal{A}$, where $\{P_j\}_{j\in\mathbb{N}}$ is the set of "rationally smoothly truncated" rational polynomials. The set $\{P_j\}_{j\in\mathbb{N}}$ is dense in

$H^k(\mathbb{R}^n)$ for any $k \in \mathbb{N}$.

(b) By interpolation inequalities for intermediate derivatives (see, for example, Theorem 4.13 [Ada75]), for any $\epsilon > 0$, there exists a constant $C = C_\epsilon$, depending only on ϵ and computable from ϵ and m, such that for any $u \in H^{2m}(\mathbb{R}^n)$,

$$||Bu||_{L^2(\mathbb{R}^n)} \leq \epsilon ||Au||_{L^2(\mathbb{R}^n)} + C_\epsilon ||u||_{L^2(\mathbb{R}^n)}.$$

It follows that B is A-bounded if one selects an $\epsilon < \frac{1}{3}(1+\sqrt{2})^{-1}$.

(c) Since the multiplication, $(\varphi, f) \mapsto \varphi f$, of any C_0^∞ function φ and any L^2 function f is $(\delta_{\mathcal{D}}, \delta_{L^2(\mathbb{R}^n)}, \delta_{L^2(\mathbb{R}^n)})$-computable ([15]), and the differentiations $\partial^\alpha : H^{2m-1}(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ are $(\delta_{H^{2m-1}(\mathbb{R}^n)}, \delta_{L^2(\mathbb{R}^n)})$- computable for any α with $|\alpha| \leq 2m-1$, it follows that $(\mathcal{A}, f) \mapsto Bf$ is $(\delta_{\mathcal{A}}, \delta_{H^{2m-1}(\mathbb{R}^n)}, \delta_{L^2(\mathbb{R}^n)})$-computable. Consequently, a δ_G-name of B that encodes the dense set $\{(P_j, BP_j) : j \in \mathbb{N}\}$ can be computed from any $\delta_{\mathcal{A}}$-name of $\mathcal{A}$. Thus a δ_{GG}-name of $A+B$ can be computed. □

Corollary 6.3 *Let $T > 0$ be a computable real number. Then the solution operator $K_H : L^2(\mathbb{R}^3) \to C([0,T], L^2(\mathbb{R}^3))$ of the initial-value problem of the heat equation*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2}, \quad u(x_1, x_2, x_3, 0) = f(x_1, x_2, x_3)$$

is $(\delta_{L^2(\mathbb{R}^3)}, [\rho \to \delta_{L^2(\mathbb{R}^3)}])$-computable.

Next we consider the inhomogeneous parabolic initial-value problem

$$\begin{cases} \frac{du(t)}{dt} + \mathcal{A}u(t) = f(t), & t > 0 \\ u(0) = x, & x \in X \end{cases} \tag{24}$$

where $f : [0,T] \to X$ is continuous and $\mathcal{A}$ is a strongly elliptic operator. The next theorem shows that the solution of (24) is computable from the parameters defining the problem (i.e. $\mathcal{A}$, f and ϕ) in certain spaces.

Theorem 6.4 *Assume that $X = L^2(\mathbb{R}^n)$, $\mathcal{A}(x, D)$ is a strongly elliptic operator, and T is a computable real number. Then the solution operator $K : (\mathcal{A}, f, x) \mapsto u$ of the problem (24) is $(\delta_{\mathcal{A}}, [\rho \to \delta_{L^2(\mathbb{R}^n)}], \delta_{L^2(\mathbb{R}^n)}, [\rho \to \delta_{L^2(\mathbb{R}^n)}])$-computable.*

Proof. It follows from Theorem 6.2 that a C_0 semigroup $W(t)$ can be computed from $\mathcal{A}$ such that $W(t)x$ is the solution of the homogeneous problem associated to (24) (i.e. setting $f = 0$). Since the solution $u(t)$ of (24) can be written as follows

$$u(t) = W(t)x + \int_0^t W(t-s)f(s)ds$$

(Definition 4.2.3 [6]), it suffices to show that the integral can be computed from W, f and t. A similar argument as of the proof for Theorem 5.4.i yields the computability of the integral. □

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