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Contraction Maps on Ifqm-spaces with Application to Recurrence Equations of Quicksort

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Abstract

Recently, C. Alaca, D. Turkoglu and C. Yildiz [Chaos, Solitons and Fractals, 2006], have proved intuitionistic fuzzy versions of the celebrated Banach fixed point theorem and Edelstein fixed point theorem respectively, by means of a notion of intuitionistic fuzzy metric space which is based on the concept of fuzzy metric space due to I. Kramosil and J. Michalek [Kybernetika, 1975]. In this paper we generalize the notions of intuitionistic fuzzy metric space by Alaca, Turkoglu and Yildiz to the quasi-metric setting and we present an intuitionistic fuzzy quasi-metric version of the Banach contraction principle. We apply this approach to deduce the existence of solution for the recurrence equations associated to the analysis of Quicksort algorithm in the framework of intuitionistic fuzzy quasi-metric spaces (ifqm-spaces, in short).

Keywords: fuzzy quasi-metric space, intuitionistic fuzzy quasi-metric space, fixed point.

1 Introduction

In [17] J.H. Park introduced and studied a notion of intuitionistic fuzzy metric space that generalizes the concept of fuzzy metric space due to A. George and P. Veeramani [8]. These spaces were initially motivated from a physic point of view in the context of the two-slit experiment as the foundation of E-infinity of high energy physics, recently studied by M.S. El Naschie in [2], [3], [4], [5], [6], etc. On the other hand, and almost simultaneously, C. Alaca, D. Turkoglu and C. Yildiz [1] have proved intuitionistic fuzzy versions of the celebrated Banach fixed point theorem and the Edelstein fixed point theorem by using a notion of intuitionistic

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fuzzy metric space which is based on the concept of fuzzy metric space introduced by I. Kramosil and J. Michalek [13].

In this paper we generalize the notions of intuitionistic fuzzy metric space by Alaca, Turkoglu and Yildiz to the quasi-metric setting and we present an intuitionistic fuzzy quasi-metric version of the Banach contraction principle which is applied to deduce the existence of solution for the recurrence equation which is typically associated to the complexity analysis of Quicksort.

Our basic reference for quasi-metric spaces is [7].

Following the modern terminology, by a quasi-metric on a nonempty set X we mean a nonnegative real valued function d on $X \times X$ such that for all $x, y, z \in X$:

(i) $x = y$ if and only if $d(x, y) = d(y, x) = 0$;

(ii) $d(x, z) \leq d(x, y) + d(y, z)$.

A quasi-metric space is a pair (X, d) such that X is a nonempty set and d is a quasi-metric on X.

Each quasi-metric d on X generates a T_0 topology τ_d on X which has as a base the family of open balls $\{B_d(x, r) : x \in X,\ r > 0\}$, where $B_d(x, r) = \{y \in X : d(x, y) < r\}$ for all $x \in X$ and $r > 0$.

A topological space (X, τ) is said to be quasi-metrizable if there is a quasi-metric d on X such that $\tau = \tau_d$.

Given a quasi-metric d on X, then the function d^{-1} defined on $X \times X$ by $d^{-1}(x, y) = d(y, x)$, is also a quasi-metric on X, called the conjugate of d, and the function d^s defined on $X \times X$ by $d^s(x, y) = \max\{d(x, y), d^{-1}(x, y)\}$ is a metric on X.

A quasi-metric space (X, d) is said to be bicomplete if (X, d^s) is a complete metric space. In this case, we say that d is a bicomplete quasi-metric on X.

According to [21], by a continuous t-norm we mean a binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ which satisfies the following conditions: (i) $*$ is associative and commutative; (ii) $*$ is continuous; (iii) $a * 1 = a$ for every $a \in [0, 1]$; (iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0, 1]$.

By a continuous t-conorm we mean a binary operation $\diamond : [0, 1] \times [0, 1] \to [0, 1]$ which satisfies the following conditions: (i) $\diamond$ is associative and commutative; (ii) $\diamond$ is continuous; (iii) $a \diamond 0 = a$ for every $a \in [0, 1]$; (iv) $a \diamond b \leq c \diamond d$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0, 1]$.

It is well known, and easy to see, that if $*$ is a continuous t-norm and $\diamond$ is a continuous t-conorm, then for all $a, b, c, d \in [0, 1]$:

$$a * b \leq a \wedge b \leq a \vee b \leq a \diamond b.$$

Definition 1.1 [9]. A KM-fuzzy quasi-metric on a nonempty set X is a pair $(M, *)$ such that $*$ is a continuous t-norm and M is a fuzzy set in $X \times X \times [0, \infty)$ such that for all $x, y, z \in X$:

(i) $M(x, y, 0) = 0$;

(ii) $x = y$ if and only if $M(x, y, t) = M(y, x, t) = 1$ for all $t > 0$;

(iii) $M(x,z,t+s) \geq M(x,y,t) * M(y,z,s)$ for all $t,s \geq 0$;
(iv) $M(x,y,\_) : [0,\infty) \to [0,1]$ is left continuous.

Note that a KM-fuzzy quasi-metric $(M,*)$ satisfying for all $x,y \in X$ and $t > 0$ the symmetry axiom $M(x,y,t) = M(y,x,t)$, is a fuzzy metric in the sense of Kramosil and Michalek [13]. In the following, KM-fuzzy quasi-metrics will be simply called fuzzy quasi-metrics.

A triple $(X,M,*)$ where X is a nonempty set and $(M,*)$ is a fuzzy (quasi-)metric on X, is said to be a fuzzy (quasi-)metric space. It was shown in Proposition 1 of [9] that if $(M,*)$ is a fuzzy quasi-metric on X, then for each $x,y \in X$, $M(x,y,\_)$ is nondecreasing, i.e. $M(x,y,t) \leq M(x,y,s)$ whenever $t \leq s$.

If $(M,*)$ is a fuzzy quasi-metric on X, then $(M^{-1},*)$ is also a fuzzy quasi-metric on X, where M^{-1} is the fuzzy set in $X \times X \times [0,\infty)$ defined by $M^{-1}(x,y,t) = M(y,x,t)$. Moreover, if we denote by M^i the fuzzy set in $X \times X \times [0,\infty)$ given by $M^i(x,y,t) = \min\{M(x,y,t), M^{-1}(x,y,t)\}$, then $(M^i,*)$ is a fuzzy metric on X [9].

Similarly to the fuzzy metric case (compare [9]), each fuzzy quasi-metric $(M,*)$ on X generates a T_0 topology τ_M on X which has as a base the family of open balls $\{B_M(x,r,t) : x \in X,\ 0 < r < 1,\ t > 0\}$, where $B_M(x,r,t) = \{y \in X : M(x,y,t) > 1-r\}$.

Example 1.2 Let (X,d) be a quasi-metric space. For $a,b \in [0,1]$ let $a \cdot b$ be the usual multiplication, and let M_d be the function defined on $X \times X \times [0,\infty)$ by $M_d(x,y,0) = 0$ and

$$M_d(x,y,t) = \frac{t}{t + d(x,y,t)} \qquad \text{whenever} \quad t > 0.$$

It is easily seen [9] that $(M_d,\cdot)$ is a fuzzy quasi-metric on X which will be called the fuzzy quasi-metric induced by d. Moreover $\tau_d = \tau_{M_d}$ and $\tau_{d^{-1}} = \tau_{(M_d)^{-1}}$, and hence $\tau_{d^s} = \tau_{(M_d)^i}$ on X. If d is a metric, then $(M_d,\cdot)$ is obviously a fuzzy metric on X (compare [8]).

Definition 1.3 (compare [8]). A sequence $(x_n)_n$ in a fuzzy (quasi-)metric space $(X,M,*)$ is called a Cauchy sequence if for each $\varepsilon \in (0,1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n,x_m,t) > 1-\varepsilon$ for all $n,m \geqslant n_0$. (Cauchy sequences in fuzzy quasi-metric spaces are called bi-Cauchy sequences in [10]).

Definition 1.4 (compare [8]). A fuzzy metric space $(X,M,*)$ is called complete if every Cauchy sequence is convergent with respect to τ_M.

Definition 1.5 [9]. A fuzzy quasi-metric space $(X,M,*)$ is called bicomplete if the fuzzy metric space $(X,M^i,*)$ is complete.

Remark 1.6 It follows from the preceding definitions that a fuzzy quasi-metric space $(X,M,*)$ is bicomplete if and only if every Cauchy sequence converges with respect to τ_{M^i}.

2 Intuitionistic Fuzzy Quasi-metric Spaces (ifqm-spaces)

Definition 2.1 An intuitionistic fuzzy quasi-metric space (in the following, an ifqm-space) is a 5-tuple $(X, M, N, *, \diamond)$ such that X is a (nonempty) set, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm and M, N are fuzzy sets in $X \times X \times [0, \infty)$ satisfying the following conditions, for all $x, y, z \in X$:

(a) $M(x, y, t) + N(x, y, t) \leq 1$ for all $t \geq 0$;
(b) $M(x, y, 0) = 0$;
(c) $x = y$ if and only if $M(x, y, t) = M(y, x, t) = 1$ for all $t > 0$;
(d) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ for all $t, s \geq 0$;
(e) $M(x, y, \_) : [0, \infty) \to [0, 1]$ is left continuous;
(f) $N(x, y, 0) = 1$;
(g) $x = y$ if and only if $N(x, y, t) = N(y, x, t) = 0$ for all $t > 0$;
(h) $N(x, y, t) \diamond N(y, z, s) \geq N(x, z, t + s)$ for all $t, s \geq 0$;
(i) $N(x, y, \_) : [0, \infty) \to [0, 1]$ is left continuous.

In this case we say that $(M, N, *, \diamond)$ is an intuitionistic fuzzy quasi-metric (in the following, an ifqm) on X.

If in addition, M and N satisfy that $M(x, y, t) = M(y, x, t)$ and $N(x, y, t) = N(y, x, t)$ for all $x, y \in X$ and $t > 0$, then $(M, N, *, \diamond)$ is called an intuitionistic fuzzy metric on X and $(X, M, N, *, \diamond)$ is called an intuitionistic fuzzy metric space.

Note that the authors of [1] require conditions $\lim_{t\to\infty} M(x, y, t) = 1$ and $\lim_{t\to\infty} N(x, y, t) = 0$ in their notion of intuitionistic fuzzy metric space; however these conditions are not necessary in our context.

Remark 2.2 It is clear that if $(X, M, N, *, \diamond)$ is an ifqm-space, then $(X, M, *)$ is a fuzzy quasi-metric space in the sense of Definition 1.

If $(M, N, *, \diamond)$ is an ifqm on X, then $(M^{-1}, N^{-1}, *, \diamond)$ is also an ifqm on X, where M^{-1} is the fuzzy set in $X \times X \times [0, \infty)$ defined by $M^{-1}(x, y, t) = M(y, x, t)$ and N^{-1} is the fuzzy set in $X \times X \times [0, \infty)$ defined by $N^{-1}(x, y, t) = N(y, x, t)$. Moreover, if we define M^i as above and denote by N^s the fuzzy set in $X \times X \times [0, \infty)$ given by $N^s(x, y, t) = \max\{N(x, y, t), N^{-1}(x, y, t)\}$ then $(M^i, N^s, *, \diamond)$ is an intuitionistic fuzzy metric on X.

In order to construct a suitable topology on an ifqm-space $(X, M, N, *, \diamond)$ it seems natural to consider "balls" $B(x, r, t)$ defined, similarly to [17] and [1], by:

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r, N(x, y, t) < r\}$$

for all $x \in X$, $r \in (0, 1)$ and $t > 0$.

Then, one can prove, as in [17], that the family of sets of the form $\{B(x, r, t) : x \in X, r \in (0, 1), t > 0\}$ is a base for a topology $\tau_{(M,N)}$ on X.

The following result is analogous to Proposition 1 of [11].

Proposition 2.3 *Let $(X, M, N, *, \diamond)$ be an ifqm-space. Then, for each $x \in X$, $r \in (0,1)$, $t > 0$, we have $B(x, r, t) = B_M(x, r, t)$.*

Proof. It is clear that $B(x, r, t) \subseteq B_M(x, r, t)$.

Now suppose that $y \in B_M(x, r, t)$. Then $M(x, y, t) > 1 - r$, so, by condition (a) of Definition 5, we have:

$$N(x, y, t) \leq 1 - M(x, y, t) < 1 - (1 - r) = r.$$

Consequently $y \in B(x, r, t)$. This concludes the proof. $\square$

From Proposition 1 we immediately deduce the following results.

Corollary 2.4 *Let $(X, M, N, *, \diamond)$ be an ifqm-space. Then $\tau_{(M,N)} = \tau_M$, $\tau_{(M^{-1},N^{-1})} = \tau_{M^{-1}}$ and $\tau_{(M^i,N^s)} = \tau_{M^i}$ on X.*

Corollary 2.5 *Let $(x_n)_n$ be a sequence in an ifqm-space $(X, M, N, *, \diamond)$ and let $x \in X$. Then, the following statements are equivalent.*

(1) *The sequence $(x_n)_n$ converges to x with respect to $\tau_{(M^i,N^s)}$.*
(2) *The sequence $(x_n)_n$ converges to x with respect to τ_{M^i}.*
(3) $\lim_{n\to\infty} M^i(x, x_n, t) = 1$ *for all* $t > 0$.
(4) $\lim_{n\to\infty} M^i(x, x_n, t) = 1$ *and* $\lim_{n\to\infty} N^s(x, x_n, t) = 0$ *for all* $t > 0$.

Corollary 2.6 *Let $(X, M, N, *, \diamond)$ be an ifqm-space. Then $(X, \tau_{(M,N)})$ is a quasi-metrizable topological space and $(X, \tau_{(M^i,N^s)})$ is a metrizable topological space.*

Definition 2.7 A sequence $(x_n)_n$ in an ifqm-space $(X, M, N, *, \diamond)$ is called a Cauchy sequence if for each $\varepsilon \in (0,1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$, and $N(x_n, x_m, t) < \varepsilon$, for all $n, m \geqslant n_0$.

Proposition 2.8 *A sequence in an ifqm-space $(X, M, N, *, \diamond)$ is a Cauchy sequence if and only if it is a Cauchy sequence in the fuzzy quasi-metric space $(X, M, *)$.*

Proof. Clearly, every Cauchy sequence in $(X, M, N, *, \diamond)$ is a Cauchy sequence in $(X, M, *)$.

Conversely, suppose that $(x_n)_n$ is a Cauchy sequence in $(X, M, *)$. Fix $\varepsilon \in (0,1)$ and $t > 0$; then, there is $n_0 \in \mathbb{N}$ such that $M^i(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \geq n_0$. Hence

$$N^s(x_n, x_m, t) \leq 1 - M^i(x_n, x_m, t) < \varepsilon$$

for all $n, m \geq n_0$. So $\lim_{n\to\infty} M^i(x_n, x_m, t) = 1$ and $\lim_{n\to\infty} N^s(x_n, x_m, t) = 0$. Therefore $(x_n)_n$ is a Cauchy sequence in $(X, M, N, *, \diamond)$. $\square$

Remark 2.9 In [17] Park introduces the notion of a complete intuitionistic fuzzy metric space. It is proved in [11] that an intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ is complete if and only if $(X, M, *)$ is complete.

Definition 2.10 An ifqm-space $(X, M, N, *, \diamond)$ is called bicomplete if $(X, M^i, N^s, *, \diamond)$ is a complete intuitionistic fuzzy metric space.

By Remark 2.9 and Definition 1.5 we deduce the following result.

Proposition 2.11 *An ifqm-space $(X, M, N, *, \diamond)$ is bicomplete if and only if the fuzzy quasi-metric space $(X, M, *)$ is bicomplete.*

Definition 2.12 [7]**.** A quasi-metric d on a set X is called non-Archimedean if $d(x, z) \leqslant \max\{d(x, y), d(y, z)\}$ for all $x, y, z \in X$.

The notion of a non-Archimedean fuzzy metric space was introduced by Sapena [20]. We give natural generalizations of this concept to the (intuitionistic) quasi-metric setting.

Definition 2.13 [18] A fuzzy quasi-metric space $(X, M, *)$ such that $M(x, z, t) \geq \min\{M(x, y, t), M(y, z, t)\}$ for all $x, y, z \in X$, $t > 0$, is called a non-Archimedean fuzzy quasi-metric space, and $(M, *)$ is called a non-Archimedean fuzzy quasi-metric on X.

Definition 2.14 An ifqm-space $(X, M, N, *, \diamond)$ such that $(M, *)$ is an non-Archimedean fuzzy quasi-metric on X and $N(x, z, t) \leqslant \max\{N(x, y, t), N(y, z, t)\}$ for all $x, y, z \in X$, $t > 0$, is called a non-Archimedean ifqm-space, and $(M, N, *, \diamond)$ is called a non-Archimedean ifqm on X.

3 Contraction Maps and Fixed Point Theorems

In the last years some authors have studied several kinds of contraction maps and fixed point theorems in (inuitionistic) fuzzy metric and fuzzy quasi-metric spaces (see, for instance, [1], [16], [12], [18], [19], etc.). Next we present some new results in this setting which will be useful later on.

Theorem 3.1 *Let $(X, M, *)$ be a bicomplete non-Archimedean fuzzy quasi-metric space and let $T : X \to X$ be a self-mapping such that*

$$M(Tx, Ty, t) \geqslant 1 - k + kM(x, y, t)$$

for all $x, y \in X$ and $t > 0$ (with $k \in (0, 1)$). Then T has a unique fixed point.

Proof. It immediately follows that

$$M^i(Tx, Ty, t) \geqslant 1 - k + kM^i(x, y, t)$$

for all $x, y \in X$ and $t > 0$.

Fix $x \in X$. Then for each $n \in \mathbb{N}$ and $t > 0$ we have:

$$\begin{aligned} M^i(T^n x, T^{n+1}x, t) &\geqslant 1 - k + kM^i(T^{n-1}x, T^n x, t) \\ &\geqslant 1 - k + k(1 - k + kM^i(T^{n-2}x, T^{n-1}x, t)) \\ &\geqslant (1 - k^2) + k^2(1 - k + kM^i(T^{n-3}x, T^{n-2}x, t)) \\ &\geqslant (1 - k^n) + k^n M^i(x, Tx, t). \end{aligned}$$

So $M^i(T^n x, T^{n+1}x, t) \geqslant 1 - k^n$ for all $n \in \mathbb{N}$.

Consequently for each $n, m \in \mathbb{N}$ (we assume without loss of generality that $m = n + j$ for some $j \in \mathbb{N}$), we deduce:

$$\begin{aligned} M^i(T^n x, T^m x, t) &= M^i(T^n x, T^{n+j}x, t) \\ &\geqslant \min\{M^i(T^n x, T^{n+1}x, t), ..., M^i(T^{n+j-1}x, T^{n+j}x, t)\} \\ &\geqslant \min\{(1 - k^n), ..., (1 - k^{n+j-1})\} = 1 - k^n. \end{aligned}$$

Given $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $k^{n_0} < \varepsilon$. So for $n, m \geqslant n_0$ it follows that:

$$M^i(T^n x, T^m x, t) \geqslant 1 - k^n \geqslant 1 - k^{n_0} > 1 - \varepsilon.$$

Therefore the sequence $(T^n x)_n$ is a Cauchy sequence in $(X, M, *)$. Hence, there is $y \in X$ such that $T^n x \to y$ with respect to τ_{M^i}.

Since $M^i(Ty, T^{n+1}x, t) \geqslant 1 - k + kM^i(y, T^n x, t)$ and $\lim_n M^i(y, T^n x, t) = 1$ it follows that $\lim_n M^i(Ty, T^{n+1}x, t) = 1$. Therefore $T^n x \to Ty$ with respect to τ_{M^i}; so $y = Ty$.

Finally, suppose that $z \in X$ satisfies $Tz = z$. Then $M^i(y, z, t) = M^i(Ty, Tz, t) \geqslant 1 - k + kM^i(y, z, t)$, so $(1 - k)M^i(y, z, t) \geqslant 1 - k$, and thus $M^i(y, z, t) = 1$ for all $t > 0$. We conclude that $y = z$. Hence y is the unique fixed point of T. □

Theorem 3.2 *Let $(X, M, N, *, \diamond)$ be a bicomplete non-Archimedean ifqm-space and let $T : X \to X$ be a self-mapping such that:*

$$M(Tx, Ty, t) \geqslant 1 - k + kM(x, y, t)$$

for all $x, y \in X$ and $t > 0$, (with $k \in (0, 1)$). Then T has a unique fixed point.

Proof. Apply Proposition 2.11 and Theorem 3.1. □

Example 3.3 Let (X, d) be a quasi-metric space. It is immediate to show that d is a non-Archimedean quasi-metric if and only if $(M_d, 1-M_d, \cdot, \diamond)$ is a non-Archimedean ifqm, where $a \diamond b = 1 - [(1 - a) \cdot (1 - b)]$ for all $a, b \in [0, 1]$.

The following result, whose easy proof is omitted, permits us to construct in an easy way a non-Archimedean ifqm from a bounded non-Archimedean quasi-metric d, which is different from the ifqm as defined in Example 3.3 (compare [18], Proposition 1).

Proposition 3.4 *Let d be a non-Archimedean quasi-metric on a set X such that $d(x, y) \leq 1$ for all $x, y \in X$. Let*

$$M_{d1}(x, y, 0) = 0 \quad \textit{for all } x, y \in X,$$

$$M_{d1}(x,y,t) = 1 - d(x,y) \quad \textit{for all } x,y \in X \textit{ and } t > 0,$$

$$N_{d1}(x,y,0) = 1 \quad \textit{for all } x,y \in X,$$

$$N_{d1}(x,y,t) = d(x,y) \quad \textit{for all } x,y \in X \textit{ and } t > 0,$$

Then the following statements hold.

(1) $(M_{d1}, N_{d1}, *, \diamond)$ *is a non-Archimedean ifqm on* X*, where by* $*$ *we denote the continuous t-norm and by* $\diamond$ *we denote any t-conorm associated to* $*$*, and given by* $a \diamond b = 1 - [(1-a) * (1-b)]$*, for all* $a, b \in [0,1]$

(2) For each $x, y \in X$*,* $t \in (0,1)$ *and* $\varepsilon \in (0,1)$ *:*

$$M(x,y,t) > 1-\varepsilon \textit{ and } N(x,y,t) < \varepsilon \Leftrightarrow d(x,y) < \varepsilon \Leftrightarrow M(x,y,t) > 1-\varepsilon$$

(3) $\tau_{(M_{d1},N_{d1})} = \tau_d = \tau_{M_{d1}}$*, and* $\tau_{((M_{d1})^{-1},(N_{d1})^{-1})} = \tau_{d^{-1}} = \tau_{(M_{d1})^{-1}}$

(4) A sequence in X *is Cauchy in* $(X, M_{d1}, N_{d1}, *, \diamond)$ *if and only if it is Cauchy in* (X, d)*.*

(5) $(X, M_{d1}, N_{d1}, *, \diamond)$ *is bicomplete if and only if* (X, d) *is bicomplete.*

4 Application to Recurrence Equations of Quicksort

Let Σ be a nonempty alphabet. Let Σ^∞ be the set of all finite and infinite sequences ("words") over Σ, where we adopt the convention that the empty sequence ϕ is an element of Σ^∞. Denote by $\sqsubseteq$ the prefix order on Σ^∞, i.e. $x \sqsubseteq y \Leftrightarrow x$ is a prefix of y.

Now, for each $x \in \Sigma^\infty$ denote by $\ell(x)$ the length of x. Then $\ell(x) \in [1,\infty]$ whenever $x \neq \phi$ and $\ell(\phi) = 0$. For each $x, y \in \Sigma^\infty$ let $x \sqcap y$ be the common prefix of x and y.

Thus, the function $d_\sqsubseteq$ defined on $\Sigma^\infty \times \Sigma^\infty$ by

$$d_\sqsubseteq(x,y) = 0 \quad \text{if } x \sqsubseteq y,$$

$$d_\sqsubseteq(x,y) = 2^{-\ell(x \sqcap y)} \quad \text{otherwise},$$

is a quasi-metric on Σ^∞. (We adopt the convention that $2^{-\infty} = 0$).

Actually $d_\sqsubseteq$ is a non-Archimedean quasi-metric on Σ^∞ (see, for instance, [15] Example 8 (b)).

We also observe that the non-Archimedean metric $(d_\sqsubseteq)^s$ is the Baire metric on Σ^∞, i.e.

$$(d_\sqsubseteq)^s(x,y) = 2^{-\ell(x \sqcap y)}$$

for all $x, y \in \Sigma^\infty$ such that $x \neq y$.

It is well known that $(d_\sqsubseteq)^s$ is complete. Therefore $d_\sqsubseteq$ is bicomplete.

The quasi-metric $d_\sqsubseteq$, which was introduced by M.B. Smyth [22], will be called the Baire quasi-metric. Observe that condition $d_\sqsubseteq(x,y) = 0$ can be used to distinguish between the case that x is a prefix of y and the remaining cases.

Next we construct an example of a bicomplete non-Archimedean ifqm on Σ^∞ that is related to the Baire quasi-metric defined above and for which Theorem 3.2 applies (compare [18] Proposition 4).

Remark 4.1 *Let $d_\sqsubseteq$ be the Baire quasi-metric on Σ^∞. Then $d_\sqsubseteq$ is a non-Archimedean quasi-metric on Σ^∞ and $d_\sqsubseteq(x,y) \leq 1$ for all $x,y \in \Sigma^\infty$. Let*

$$M_{d_\sqsubseteq 1}(x,y,0) = 0 \quad \textit{for all } x,y \in \Sigma^\infty,$$

$$Md_{\sqsubseteq 1}(x,y,t) = 1 - d_\sqsubseteq(x,y) \quad \textit{for all } x,y \in \Sigma^\infty \textit{ and } t > 0,$$

$$Nd_{\sqsubseteq 1}(x,y,0) = 1 \quad \textit{for all } x,y \in \Sigma^\infty,$$

$$Nd_{\sqsubseteq 1}(x,y,t) = d_\sqsubseteq(x,y) \quad \textit{for all } x,y \in \Sigma^\infty \textit{ and } t > 0.$$

It follows from Proposition 3.4 that $(\Sigma^\infty, M_{d_\sqsubseteq 1}, N_{d_\sqsubseteq 1}, *, \diamond)$ is a bicomplete non-Archimedean ifqm-space, where $*$ is any continuous t-norm and $\diamond$ is its associated continuous t-conorm.

Example 4.2 Next we apply Remark 4.1 and Theorem 3.2 to the complexity analysis of Quicksort algorithm. The average case analysis of Quicksort is discussed in [14], where the following recurrence equation is obtained:

$$T(1) = 0, \quad \text{and}$$

$$T(n) = \frac{2(n-1)}{n} + \frac{n+1}{n}T(n-1), \quad n \geq 2.$$

Consider as an alphabet Σ the set of nonnegative real numbers, i.e. $\Sigma = [0,\infty)$. We associate to T the functional $\Phi : \Sigma^\infty \to \Sigma^\infty$ given by $(\Phi(x))_1 = T(1)$ and

$$(\Phi(x))_n = \frac{2(n-1)}{n} + \frac{n+1}{n}x_{n-1}$$

for all $n \geq 2$ (if $x \in \Sigma^\infty$ has length $n < \infty$, we write $x := x_1x_2...x_n$, and if x is an infinite word we write $x := x_1x_2...$).

Next we show that Φ is a contraction (in the sense of Theorem 3.2) on the bicomplete ifqm-space $(\Sigma^\infty, M_{d_\sqsubseteq 1}, N_{d_\sqsubseteq 1}, *, \diamond)$, with contraction constant $k = 1/2$. (Although the technique of the proof of this fact is similar to the one used in [18], p. 2201-2202, we give it here for the sake of completeness.)

To this end, we first note that, by construction, we have $\ell(\Phi(x)) = \ell(x) + 1$ for all $x \in \Sigma^\infty$ (in particular, $\ell(\Phi(x)) = \infty$ whenever $\ell(x) = \infty$).

Furthermore, it is clear that

$$x \sqsubseteq y \Longleftrightarrow \Phi(x) \sqsubseteq \Phi(y),$$

and consequently

$$\Phi(x \sqcap y) \sqsubseteq \Phi(x) \sqcap \Phi(y)$$

for all $x,y \in \Sigma^\infty$. Hence

$$\ell(\Phi(x \sqcap y)) \leq \ell(\Phi(x) \sqcap \Phi(y))$$

for all $x, y \in \Sigma^\infty$. We have:

$$\begin{aligned} M_{d_\sqsubseteq 1}(\Phi(x), \Phi(y), t) &= 1 - 2^{-\ell(\Phi x \sqcap \Phi y)} \\ &\geq 1 - 2^{-\ell(\Phi(x \sqcap y))} = 1 - 2^{-(\ell(x \sqcap y)+1)} \\ &= 1 - \frac{1}{2} 2^{-\ell(x \sqcap y)} = 1 - \frac{1}{2} + \frac{1}{2} M_{d_\sqsubseteq 1}(x, y, t) \end{aligned}$$

for all $t > 0$.

Therefore Φ is a contraction on $(\Sigma^\infty, M_{d_\sqsubseteq 1}, N_{d_\sqsubseteq 1}, *, \diamond)$ with contraction constant $1/2$. So, by Theorem 3.2, Φ has a unique fixed point $z = z_1 z_2 ...$, which is obviously the unique solution to the recurrence equation T, i.e. $z_1 = 0$ and

$$z_n = \frac{2(n-1)}{n} + \frac{n+1}{n} z_{n-1}$$

for all $n \geq 2$.

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