An Axiomatic Approach to Computing the Connectivity of Synchronous and Asynchronous Systems

(Preliminary Version)

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Abstract

We present a unified, axiomatic approach to proving lower bounds for the k-set agreement problem in both synchronous and asynchronous message-passing models. The proof involves constructing the set of reachable states, proving that these states are highly connected, and then appealing to a well-known topological result that high connectivity implies that set agreement is impossible. We construct the set of reachable states in an iterative fashion using a round operator that we define, and our proof of connectivity is an inductive proof based on this iterative construction and simple properties of the round operator.

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1 Introduction

The consensus problem [18] has received a great deal of attention. In this problem, $n + 1$ processors begin with input values, and all must agree on one of these

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values as their output value. Fischer, Lynch, and Paterson [7] surprised the world by showing that solving consensus is impossible in an asynchronous system if one processor is allowed to fail. This leads one to wonder if there is any way to weaken consensus to obtain a problem that can be solved in the presence of $k-1$ failures but not in the presence of k failures. Chaudhuri [5] defined the k-set agreement problem and conjectured that this was one such problem, and a trio of papers [4,13,19] proved that she was right. The k-set agreement problem is a generalization of consensus, where we relax the requirement that processors agree on a single value: the set of output values chosen by the processors may contain as many as k distinct values, and not just 1.

Set agreement (and in particular consensus) has been studied in both synchronous and asynchronous models of computation, but mostly independently. Indeed, prior proofs for these models appeared to have little in common, as reflected by the organization of a main textbook in the area [14], where the first part is devoted to synchronous systems and the second part of the book to asynchronous systems. Recent work has been uncovering more and more features and structure in common to both models [8,12,15,16]. However, these results are in the form of transformations between models, or proofs that have a similar structure in both models. Only [15] describes an abstract model that encompasses both models, with clearly identified properties that are needed to carry out consensus impossibility results. To go from consensus to set agreement a big step in complexity is encountered, since one must deal with higher dimensional topology instead of just graphs, as discovered by the trio of papers [4,13,19] mentioned above. The contribution of this paper is to present a new axiomatic approach were set consensus impossibility proofs can be derived in a uniform manner for both synchronous and asynchronous models.

All known proofs for the set agreement lower bound depend — either explicitly or implicitly — on a deep connection between computation and topology. These proofs essentially consider the simplicial complex representing all possible reachable states of a set agreement protocol, and then argue about the connectivity of this complex. These lower bounds for set agreement follow from the observation that set agreement cannot be solved if the complex of reachable states is sufficiently highly-connected. This connection between connectivity and set agreement has been established both in a generic way [11] and in ways specialized to particular models of computation [1,4,6,10,11,12,19]. Once the connection has been established, however, the problem reduces to reasoning about the connectivity of a protocol's reachable complex.

The primary contribution of this work is a new, substantially simpler proof of how the connectivity of the synchronous and asynchronous complexes evolve over time. Our proof depends on two key insights:

(i) The notion of a *round operator* that maps a global state to the set of global states reachable from this state by one round of computation, an operator satisfying a few simple algebraic properties.

(ii) The notion of an *absorbing poset* organizing the set of global states into a

partial order, from which the connectivity proof follows easily using the round
operator's algebraic properties.

We believe this new approach has several novel and elegant features. First, we
are able to isolate a small set of elementary combinatorial properties of the round
operator that suffice to establish the connection with classical topology in a model-
independent way. Second, these properties require only local reasoning about how
the computation evolves from one round to the next. Finally, most connectivity
arguments can be difficult to follow because they mix semantic, combinatorial, and
topological arguments. Those arguments are cleanly separated here. The round
operator definition relies on semantics: it is a combinatorial restatement of the
properties of the synchronous model. Once the round operator is defined, however,
we no longer appeal to properties of the original model. We reason in a purely com-
binatorial way about intersections of global states, and how they can be placed in a
partial order. Once these combinatorial arguments are in place, we appeal directly
to well-known theorems of topology to establish connectivity. These topology theo-
rems are treated as "black boxes," in the sense that we apply them directly without
any need to make additional topological arguments. Furthermore, our absorbing
posets are very similar to shellable complexes [3], yet one more link between the
work of topologists and distributed computing.

Most of our proofs appear in the appendix.

2 Preliminaries

2.1 Models

We consider two (standard) message-passing models, the *synchronous* and *asyn-
chronous models*. In both models, we restrict our attention to computations with
a round structure: the initial state of each processor is its input value, and com-
putation proceeds in a sequence of rounds. In each round, each processor sends
messages to other processors, receives messages sent to it by the other processors
in that round, performs some internal computation, and changes state. We assume
that processors are following a full-information protocol, which means that each
processor sends its entire local state to every processor in every round. This is a
standard assumption to make when proving lower bounds. A processor can fail by
crashing in the middle of a round, in which case it sends its state only to a subset
of the processors in that round. Once a processor crashes, it never sends another
message after that.

In the synchronous model [2,14], all processors execute round r at the same
time, and processor P fails to receive a message from processor Q, then Q must
have crashed, either in that round or in the previous round.

In the asynchronous model, there is no bound on processor step time nor on
message delivery time, so a crashed processor cannot be distinguished from a slow
processor. Our results, however, depend only on the unbounded message delivery
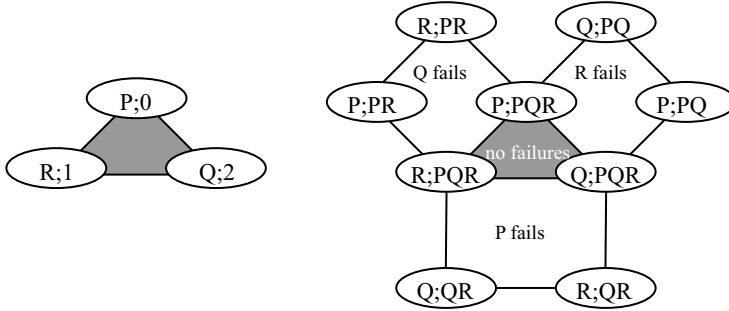time. Since our goal is to prove impossibility results, we are free to restrict our

Fig. 1. A global state S and the set $\mathcal{S}_1(S)$ of global states after one round from S.

attention to executions in which processors take steps at a regular pace, and only message delivery times are delayed. In the behaviors we consider, messages from one processor to another are delivered in FIFO order, but when one message from P to Q is delivered, all outstanding messages from P to Q are delivered at the same time.

It is convenient to recast the asynchronous model in the following *omissions-failure* form. In each round, at most f processors exhibit faulty behavior, although this set may change from round to round. In each round, the nonfaulty processors broadcast their states to all processors (including the faulty processors). Each faulty processor broadcasts its state to some subset of the processors, and may omit to send to the others. Processors never crash. It can be shown that k-set agreement lower bounds in this omissions failure model carry over to the standard asynchronous crash-failure model; see [9] for a similar argument.

2.2 Combinatorial Topology

We represent the local state of a processor with a vertex labeled with that processor's id and its local state. We represent a global state as a set of labeled vertexes, labeled with distinct processors, representing the local state of each processor in that global state. In topology, a *simplex* is a set of vertexes, and a *complex* is a set of simplexes that is closed under containment. The *dimension* of a simplex is equal to its number of vertexes minus one. Applications of topology to distributed computing often assume that these vertexes are points in space and that the simplex is the convex hull of these points in order to be able to use standard topology results. As you read this paper, you might find it helpful to think of simplexes in this way, but in the purely combinatorial work done in this paper, a simplex is just a set of vertexes.

As an example, consider the simplex and complex illustrated in Figure 1. On the left side, we see a simplex representing an initial global state in which processors P, Q, and R start with input values 0, 2, and 1. Each vertex is labeled with

a processor's id and its local state (which is just its input value in this case). On the right we see a complex representing the set of states that arise after one round of computation from this initial state in the synchronous model if one processor is allowed to crash. The labeling of the vertexes is represented schematically by a processor id such as P and a string of processor ids such as PQ. The string PQ is intended to represent the fact that P heard from processors P and Q during the round but not from R, since R failed that round. (We are omitting input values on the right for notational simplicity.) The simplexes that represent states after one round are the 2-dimensional triangle in the center and the 1-dimensional edges that radiate from the triangle (including the edges of the triangle itself). The central triangle represents the state after a round in which no processor fails. Each edge represents a state after one processor failed. For example, the edge with vertexes labeled $P; PQR$ and $Q; PQ$ represent the global state after a round in which R fails by sending a message to P and not sending to Q: P heard from all three processors, but Q did not hear from R.

What we do in this paper is define round operators like the round operator $\mathcal{S}_1$ that maps the simplex S on the left of Figure 1 to the complex $\mathcal{S}_1(S)$ on the right, and then reason about the connectivity of $\mathcal{S}_1(S)$. Informally, connectivity in dimension 0 is just ordinary graph connectivity, and connectivity in higher dimensions means that there are no "holes" of that dimension in the complex. When we reason about connectivity, we often talk about the connectivity of a simplex S when we really mean the connectivity of the induced complex consisting of S and all of its faces. For example, both of the complexes in Figure 1 are 0-connected since they are connected in the graph-theoretic sense. In fact, the complex on the left is also 1-connected, but the complex on the right is not since there are "holes" formed by the three cycles of 1-dimensional edges.

Given a simplex S, a *labeling* of S from a set V is a new simplex constructed by replacing each vertex s of S with a pair (s, v), where $v \in V$.

Given a simplex S and a set V, we define the *pseudosphere* $\mathcal{P}(S, V)$ to be this set of labelings of S with elements of V. (We call $\mathcal{P}(S, V)$ a pseudosphere because is has some of the topological properties of a sphere.) The face S is called the *base simplex* of the pseudosphere, and given a simplex T of a pseudosphere $\mathcal{P}(S, V)$, we define $base(T)$ to be the base simplex S of the pseudosphere.

The input complex for k-set agreement is $\mathcal{P}(S, V)$, the pseudosphere in which each vertex is labeled with an input from a set V, where $|V| > k$. The set of all reachable states of a protocol P with initial states $\mathcal{P}(S, V)$ is the protocol complex $\mathcal{C} = \mathcal{C}(\mathcal{P}(S, V))$. The fundamental connection between k-set agreement and connectivity is expressed in the following theorem (e.g. [11]):

Theorem 2.1 *Let P be a protocol, and let $\mathcal{C}$ be its protocol complex. If $\mathcal{C}$ is $(k-1)$-connected, then P cannot solve k-set agreement.*

Thus, our main task will be to prove that $\mathcal{C}$ is $(k-1)$-connected. Proving that a union of complexes is connected is made easier by the following theorem [4]. Notice

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4 Actually this theorem is a well-known corollary of the Mayer-Vietoris sequence, which is described in

that if A and B are complexes then both $A \cup B$ and $A \cap B$ are complexes.

Theorem 2.2 (Mayer-Vietoris) *Let A, B be two complexes. Then $A \cup B$ is c-connected if A and B are c-connected and $A \cap B$ is $(c-1)$-connected.*

Think about the special one-dimensional case of this statement: a graph that is the union of subgraphs A, B is 0-connected (connected in the graph theoretic sense) if A and B are 0-connected and $A \cap B$ is -1-connected (nonempty).

To prove that a complex $\mathcal{C}$ is c-connected, we split $\mathcal{C}$ into subcomplexes with less and less simplexes, and apply repeatedly the Mayer-Vietoris theorem. At the bottom of this recursion, we get complexes with just one simplex, and use the following fact.

Theorem 2.3 *A simplex of dimension at least ℓ is $(\ell-1)$-connected.*

In this paper all we need to assume from topology is the previous two theorems. Both are very basic algebraic topology facts that appear in standard textbooks such as [17,20].

3   Absorbing Posets and Round Operators

The *codimension* of two simplexes S_0 and S_1 is a measure of how much they have in common defined by

$$codim(S_0, S_1) = \max_i \{\dim(S_i) - \dim(\cap_j S_j)\}$$

where $\dim(\emptyset) = -1$ is the dimension of the empty simplex. Two useful properties of this definition are that if $S \subseteq T$ then

$$codim(S, T) = \dim(T) - \dim(S),$$

and if $S \subseteq X \subseteq T$ then

$$codim(S, T) = codim(S, X) + codim(X, T).$$

Let $\mathcal{S}$ be a nonempty set of simplexes, and $\preceq$ a partial order on $\mathcal{S}$.

Definition 3.1 We say that $(\mathcal{S}, \preceq)$ is an *absorbing poset* if for every two simplexes S and T in $\mathcal{S}$ with $T \npreceq S$ there is a T_S in $\mathcal{S}$, $T_S \preceq T$ such that

$$S \cap T \subseteq T_S \cap T \tag{1}$$
$$codim(T_S, T) = 1 \tag{2}$$
$$codim(S \cap T, T_S) \leq codim(S, T). \tag{3}$$

The first two properties say that when considering pairwise intersections of simplexes — as we will frequently do in our Mayer-Vietoris arguments — pairs of high

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most algebraic topology textbooks; see for example [20] Chapter 4, Section 6.

codimension are "absorbed" by pairs of low codimension, and we can restrict our attention to pairs of simplexes of codimension one. The third property just says that T_S satisfies the same property that S and T do, namely, $codim(S \cap T, X) \leq codim(S, T)$ for $X = S, T$. An absorbing poset is almost equivalent to a *shellable complex* [3]. In a shellable complex, Equations 2 and 3 apply only to principal faces ("facets") of the complex, while our construction allows one simplex in $\mathcal{S}$ to be a proper face of another. It follows that every absorbing poset induces a shellable complex, but not vice-versa.

Lemma 3.2 *If $\mathcal{A}$ is a set of simplexes such that every pair of simplexes has codimension 1, then $(\mathcal{A}, <)$ is an absorbing poset, where $<$ is any total order on $\mathcal{A}$.*

Proof. For any simplexes S and T in $\mathcal{A}$ such that $S < T$, pick $T_S = S$. Substituting S for T_S, it is easy to check that the three conditions of Definition 3.1 are satisfied:

$$S \cap T \subseteq S \cap T$$
$$codim(S, T) = 1$$
$$codim(S \cap T, S) \leq codim(S, T).$$

$\square$

3.1 Axioms

A *simplicial operator* $\mathcal{Q}$ is a family of maps. Each map $\mathcal{Q}_\ell$ carries a simplex of dimension $m \geq \ell$ to a nonempty set of simplexes, where each simplex has dimension at most m. The subscript ℓ is the operator's *degree*. For $\ell < 0$, it is convenient to define $\mathcal{Q}_\ell(S)$ to be the empty set. Note that $\mathcal{Q}_\ell(\emptyset) = \emptyset$ for all ℓ, and $\mathcal{Q}_0(S) \neq \emptyset$ for any nonempty simplex S.

Simplicial operators extend naturally to sets of simplexes. If $\mathcal{A}$ is a set of simplexes,

$$\mathcal{Q}_\ell(\mathcal{A}) = \bigcup_{A \in \mathcal{A}} \mathcal{Q}_\ell(A). \tag{4}$$

The semantic interpretation of simplicial operators varies from model to model. In the synchronous message-passing model, ℓ is the number of processors that can crash in each round. In the asynchronous model, ℓ is the number of processors that remain partially silent in each round.

We use $\mathcal{Q}_k \mathcal{Q}_\ell(S)$ to denote the composition of $\mathcal{Q}_k$ and $\mathcal{Q}_\ell$ applied to S, $\mathcal{Q}_\ell^r(S)$ to denote the r-fold composition of $\mathcal{Q}_\ell$ applied to S, and $\|\mathcal{Q}_\ell^r(S)\|$ to denote the simplicial complex induced by the set $\mathcal{Q}_\ell^r(S)$ (i.e., closed under containment).

We assume that simplicial operators satisfy the following axioms. The first axiom says that the states reachable after the failure of ℓ processors are reachable after the failure of even more processors.

Axiom 3.3

$$\mathcal{Q}_\ell(S) \subseteq \mathcal{Q}_m(S)$$

when $\ell \leq m$.

The second axiom describes multi-round executions. We introduce a model-specific, nonnegative, integer-valued linear function ϕ. Informally, $\phi(f)$ is the number of failures needed in a round to hide the existence of f faulty processors. We will see that in the synchronous model, faulty processors crash, so $\phi(f) = 0$. In the asynchronous model, faulty processors fail to send messages, so $\phi(f) = f$.

Axiom 3.4 *Let $k \geq \ell$, and let $S \subseteq T$, where $c = codim(S, T)$. For all $r > 0$,*

$$for \ all \ S' \in \mathcal{Q}_{k-\phi(c)}^r \mathcal{Q}_{\ell-c}(S), \ there \ exists \ T' \in \mathcal{Q}_k^r \mathcal{Q}_\ell(T)$$
$$such \ that \ S' \subseteq T' \ and \ codim(S', T') \leq c.$$

Executions from initial state T in which the processes not in S are silenced at every round look the same to processes in S as executions with fewer failures in which the silenced processes do not participate.

Axiom 3.5 *Let $k \geq \ell$. For all $r > 0$, if $c = codim(S_0, S_1)$,*

$$\|\mathcal{Q}_k^r \mathcal{Q}_\ell(S_0)\| \cap \|\mathcal{Q}_k^r \mathcal{Q}_\ell(S_1)\| \subseteq \|\mathcal{Q}_{k-\phi(c)}^r \mathcal{Q}_{\ell-c}(S_0 \cap S_1)\|.$$

The right-hand-side of this equation is the set of states for processors that cannot tell whether the initial state was S_0 or S_1. The processors that can tell the difference must be silenced in the first round, requiring an extra c failures, and must be kept silent for the remaining rounds, requiring $\phi(c)$ extra failures in each subsequent round.

The last axiom says there is a partial order such that

Axiom 3.6 *For every simplex S, $\mathcal{Q}_\ell(S)$ is an absorbing poset.*

4 Theorems and Lemmas

The following is an immediate condequence of Axiom 3.4.

Lemma 4.1 *Let $i \geq j$. For all $r > 0$, if $S \subseteq T$,*

$$\|\mathcal{Q}_i^r \mathcal{Q}_j(S)\| \subseteq \|\mathcal{Q}_{i+\phi(c)}^r \mathcal{Q}_{j+c}(T)\|$$

where $c = codim(S, T)$.

Lemma 4.2 *Let $k \geq \ell$. For all $r > 0$, if $c = codim(S_0, S_1)$,*

$$\|\mathcal{Q}_k^r \mathcal{Q}_\ell(S_0)\| \cap \|\mathcal{Q}_k^r \mathcal{Q}_\ell(S_1)\| = \|\mathcal{Q}_{k-\phi(c)}^r \mathcal{Q}_{\ell-c}(S_0 \cap S_1)\|.$$

Proof. The "$\subseteq$" direction is immediate from Axiom 3.5. Because $S_0 \cap S_1 \subseteq S_1$, and $codim(S_0 \cap S_1, S_1) \leq c$, Axiom 3.4 implies that

$$\|\mathcal{Q}_{k-\phi(c)}^r \mathcal{Q}_{\ell-c}(S_0 \cap S_1)\| \subseteq \|\mathcal{Q}_k^r \mathcal{Q}_\ell(S_0)\|,$$

and symmetrically for $\|\mathcal{Q}_k^r \mathcal{Q}_\ell(S_1)\|$. □

Lemma 4.3 *If $(\mathcal{S}, \preceq)$ is an absorbing poset, and S, T, and T_S are defined as in Definition 3.1, then*

$$codim(S \cap T, T_S \cap T) < codim(S, T).$$

Proof. Because $S \cap T \subseteq T_S \cap T$, $codim(S \cap T, T_S \cap T)$ is just the number of vertexes in $T_S \cap T$ but not in $S \cap T$.

There are two cases to consider. First, suppose there is a vertex in T but not in T_S. It follows that

$$codim(S \cap T, T_S \cap T) < codim(S \cap T, T) \le codim(S, T).$$

Second, suppose instead that $T \subset T_S$. Because T and T_S are distinct, there is vertex in T_S but not in T. It follows that

$$codim(S \cap T, T) < codim(S \cap T, T_S).$$

By Equation 3,
$$codim(S \cap T, T_S) \le codim(S, T).$$

Combining these inequalities yields the bound. □

The next lemma states that every state reachable with a certain number of failures is also reachable with more failures.

Lemma 4.4
$$\mathcal{Q}_j^r \mathcal{Q}_k(S) \subseteq \mathcal{Q}_\ell^r \mathcal{Q}_m(S).$$

when $j \le \ell$ and $k \le m$.

Proof. We argue by induction on $r \ge 0$. When $r = 0$, the claim follows from Axiom 3.3.

Suppose $r > 0$. Since $\mathcal{Q}_j^{r-1} \mathcal{Q}_k(S) \subseteq \mathcal{Q}_\ell^{r-1} \mathcal{Q}_m(S)$ by the induction hypothesis, we have

$$\mathcal{Q}_j^r \mathcal{Q}_k(S) = \mathcal{Q}_j \mathcal{Q}_j^{r-1} \mathcal{Q}_k(S) \subseteq \mathcal{Q}_j \mathcal{Q}_\ell^{r-1} \mathcal{Q}_m(S) \subseteq \mathcal{Q}_\ell \mathcal{Q}_\ell^{r-1} \mathcal{Q}_m(S) = \mathcal{Q}_\ell^r \mathcal{Q}_m(S).$$

□

Lemma 4.5 *Let $(\mathcal{S}, \preceq)$ be an absorbing poset, and let $T \in \mathcal{S}$ be a maximal simplex with respect to $\preceq$. We claim that the following sets are both absorbing posets: $(\mathcal{L}, \preceq)$, where $\mathcal{L} = \{L | L \in \mathcal{S} - \{T\}\}$, and $(\mathcal{M}, \preceq)$, where $\mathcal{M} = \{T\}$.*

Lemma 4.6 *Let $(\mathcal{A}, \preceq)$ be an absorbing poset containing more than one simplex, and let $A \in \mathcal{A}$ be a maximal simplex with regards to $\preceq$. For each $B \ne A$ in $\mathcal{A}$, there exists a $A_B \in \mathcal{A}$ satisfying the three conditions of Definition 3.1. We claim that the set*

$$\mathcal{B} = \{A_B \cap A | B \in \mathcal{A} - \{A\}\}$$

is an absorbing poset for any total order $<$ on the elements of $\mathcal{A} - \{A\}$

Lemma 4.7 *If every simplex in $\mathcal{Q}_k^r \mathcal{Q}_\ell(\mathcal{A})$ has dimension at least d, then so does every simplex in $\mathcal{A}$.*

Lemma 4.8 *Let $\mathcal{Q}_k^r \mathcal{Q}_\ell$ be a composition of simplicial operators where $k \geq \ell$. If $(\mathcal{S}, \preceq)$ is an absorbing poset then for every two simplexes S and T in $\mathcal{S}$ with $T \not\preceq S$ there is a T_S in $\mathcal{S}$ with $T_S \preceq T$, such that*

$$\|\mathcal{Q}_k^r \mathcal{Q}_\ell(S)\| \cap \|\mathcal{Q}_k^r \mathcal{Q}_\ell(T)\| \subseteq \|\mathcal{Q}_k^r \mathcal{Q}_\ell(T_S)\| \cap \|\mathcal{Q}_k^r \mathcal{Q}_\ell(T)\|$$
$$codim(T_S, T) = 1$$
$$codim(S \cap T, T_S) \leq codim(S, T).$$

Lemma 4.9 *If $(\mathcal{A}, \preceq)$ is an absorbing poset where ℓ is the minimum dimension of any simplex in $\mathcal{A}$, then $\|\mathcal{A}\|$ is $(\ell - 1)$-connected.*

Theorem 4.10 *Let $\mathcal{Q}_k^r \mathcal{Q}_\ell$ be a composition of simplicial operators where $k \geq \ell$, and $(\mathcal{A}, \preceq)$ an absorbing poset. If every simplex in $\mathcal{Q}_k^r \mathcal{Q}_\ell(\mathcal{A})$ has dimension at least ℓ, then $\|\mathcal{Q}_k^r \mathcal{Q}_\ell(\mathcal{A})\|$ is $(\ell - 1)$-connected.*

5 The Synchronous Model

We assume a standard synchronous message-passing model with crash failures [2,14]. The system has $n + 1$ processors, and at most f of them can crash in any given execution. Each processor begins in an initial state consisting of its input value, and computation proceeds in a sequence of rounds. In each round, each processor sends messages to other processors, receives messages sent to it by the other processors in that round, performs some internal computation, and changes state. We assume that processors are following a full-information protocol, which means that each processor sends its entire local state to every processor in every round. This is a standard assumption to make when proving lower bounds. A processor can fail by crashing in the middle of a round, in which case it sends its state only to a subset of the processors in that round. Once a processor crashes, it never sends another message after that

A simplex X is *between* two simplexes T and R if $T \subseteq X \subseteq R$. We use $[T : R]$ to denote the set of simplexes between T and R.

Definition 5.1 Given simplexes S, T, and R, the *pseudosphere* $\mathcal{P}(S, [T : R])$ is the set of all possible labelings of S with simplexes between T and R.

We call this set a pseudosphere because the induced complex has some of the topological properties of a sphere. The simplex S is called the *base simplex* of the pseudosphere, and given a simplex X of a pseudosphere $\mathcal{P}(S, [T : R])$, we define $base(X)$ to be S.

Given a simplex S and a set D of processors, let $F = S/D$ be the face of S obtained from S by deleting the vertexes labeled with processors in D. The set of states reachable from S by one round of synchronous computation in which the processors in D fail can be represented by the pseudosphere $\mathcal{P}(F, [F : S])$, the set of all possible labelings of F with simplexes between F and S.

Next, we define the failure operator. Given a simplex S and an integer $\ell \geq 0$, the *ℓ-failure operator* $\mathcal{F}_\ell(S)$ maps S to the set of all faces F of S with $codim(F, S) \leq \ell$, which is the set of all faces obtained by deleting at most ℓ vertexes from S. This models the sets of at most ℓ processors that can fail in one round of computation from S.

Definition 5.2 For every integer $\ell \geq 0$, the *synchronous round operator* $\mathcal{S}_\ell(S)$ is defined by

$$\mathcal{S}_\ell(S) = \bigcup_{F \in \mathcal{F}_\ell(S)} \mathcal{P}(F, [F : S]).$$

We now check that the synchronous round operator satisfies our axioms.

Lemma 5.3 $\mathcal{S}_\ell$ *satisfies Axiom 3.3:*

$$\mathcal{S}_\ell(S) \subseteq \mathcal{S}_m(S)$$

when $\ell \leq m$.

Proof. Since $\ell \leq m$ implies $\mathcal{F}_\ell(S) \subseteq \mathcal{F}_m(S)$, it follows that

$$\mathcal{S}_\ell(S) = \bigcup_{F \in \mathcal{F}_\ell(S)} \mathcal{P}(F, [F : S]) \subseteq \bigcup_{F \in \mathcal{F}_m(S)} \mathcal{P}(F, [F : S]) = \mathcal{S}_m(S).$$

$\square$

In this model, the integer-valued linear function ϕ is simply $\phi(f) = 0$.

Lemma 5.4 $\mathcal{S}_\ell$ *satisfies Axiom 3.4: Let $k \geq \ell$, and let $S \subseteq T$, where $c = codim(S, T)$. For all $r > 0$,*

$$\text{for all } S' \in \mathcal{S}^r_{k-\phi(c)} \mathcal{S}_{\ell-c}(S), \text{ there exists } T' \in \mathcal{S}^r_k \mathcal{S}_\ell(T)$$
$$\text{such that } S' \subseteq T' \text{ and } codim(S', T') \leq c.$$

Lemma 5.5 $\mathcal{S}_\ell$ *satisfies Axiom 3.5: Let $k \geq \ell$. For all $r > 0$, if $c = codim(S_0, S_1)$,*

$$\|\mathcal{S}^r_k \mathcal{S}_\ell(S_0)\| \cap \|\mathcal{S}^r_k \mathcal{S}_\ell(S_1)\| \subseteq \|\mathcal{S}^r_k \mathcal{S}_{\ell-c}(S_0 \cap S_1)\|.$$

To show that $\mathcal{S}_\ell$ satisfies Axiom 3.6, we impose a partial order on simplexes of $\mathcal{S}_\ell(S)$. Recall that

$$\mathcal{S}_\ell(S) = \bigcup_{F \in \mathcal{F}_\ell(S)} \mathcal{P}(F, [F : S]).$$

This expression suggests a lexicographic order. We will combine a total order on simplexes F in $\mathcal{F}_\ell(S)$ with a partial order on simplexes of each $\mathcal{P}(F, [F : S])$.

We assume a total order $\leq_{\mathrm{id}}$ on processor ids, which induces a total order on the vertexes of a simplex. We begin by imposing a lexicographic total order on the faces F of S. First we order the faces by decreasing dimension, so that large faces occur before small faces. Then we order faces of the same dimension with a rather arbitrary rule based on our total order on processor ids: we order F before G if the

smallest processor id labeling vertexes in F and not G comes before the smallest processor id labeling G and not F. Formally:

Definition 5.6 Define the total order $<_f$ on the faces of a simplex S by $F <_f G$ if

(i) $\dim(F) > \dim(G)$ or

(ii) $\dim(F) = \dim(G)$ and $p_F <_{\text{id}} p_G$ where

$$p_F = \min\{ids(F) - ids(G)\} \quad \text{and} \quad p_G = \min\{ids(G) - ids(F)\}.$$

Define $F \leq_f G$ if $F <_f G$ or $F = G$.

Next we order the simplexes in a pseudosphere $\mathcal{P}(F, [F : S])$ using the following face ordering: we order A before B if, for each vertex v of the base simplex F, the face of S labeling v in A comes before the face of S labeling v in B. Formally:

Definition 5.7 Define the partial order $\preceq_p$ on the simplexes of a pseudosphere $\mathcal{P}(F, [F : S])$ by $A \preceq_p B$ if and only if $A_v \leq_f B_v$ for each vertex v in F, where A_v and B_v are the simplexes labeling the vertex v in A and B.

Now we order $\mathcal{S}_\ell(S)$ lexicographically using the face and pseudosphere orders: we order the simplexes in a pseudosphere $\mathcal{P}(F, [F : S])$ before the simplexes in a pseudosphere $\mathcal{P}(G, [G : S])$ if F is ordered before G in the face ordering, and we order the simplexes within a single pseudosphere using the pseudosphere ordering. Formally:

Definition 5.8 Define the partial order $\preceq_r$ on the simplexes in $\mathcal{S}_\ell(S)$ by $A \preceq_r B$ if and only if

(i) *different pseudospheres:* $base(A) <_f base(B)$ or

(ii) *same pseudosphere:* $base(A) = base(B)$ and $A \preceq_p B$

Theorem 5.9 $\mathcal{S}_\ell$ *satisfies Axiom 3.6: For every simplex S, $(\mathcal{S}_\ell(S), \preceq_r)$ is an absorbing poset.*

Theorem 5.10 *Assume $n + 1 \geq f + k + 1$. No synchronous protocol for k-set agreement halts in fewer than $\lfloor f/k \rfloor + 1$ rounds in the presence of f crash failures.*

Proof. Suppose there is a protocol that halts in fewer than $\lfloor f/k \rfloor + 1$ rounds, and assume without loss of generality that it halts in exactly $r = \lfloor f/k \rfloor$ rounds in every execution. Consider the subset of executions in which at most k processors halt in every round. For the input complex $\mathcal{P}(S, V)$, the set of final states of such executions is $\mathcal{S}_k^r(\mathcal{P}(S, V))$. Every simplex in this complex has dimension at least k. By Theorem 4.10, this complex is $(k - 1)$-connected, and by Theorem 2.1, the protocol cannot solve k-set agreement. □

6 Asynchronous Model

Informally, the asynchronous round operator $\mathcal{A}_\ell(S)$ is defined as follows. There are at most ℓ faulty processors in each round, although the set of faulty processors

can change from round to round. Faulty processors never crash, but they can omit sending messages. In each round, all nonfaulty processors send their states to all the processors (including faulty ones), while the faulty processors send messages to an arbitrary subset of processors (perhaps none).

Definition 6.1 For every integer $\ell \geq 0$, the *asynchronous round operator* $\mathcal{A}_\ell(S)$ is defined by

$$\mathcal{A}_\ell(S) = \bigcup_{F \in \mathcal{F}_\ell(S)} \mathcal{P}(S, [F : S]).$$

The difference between the synchronous and asynchronous round operators is that every pseudosphere has the same base simplex S in the definition of the asynchronous operator. Notice that S and every simplex in $\mathcal{A}_\ell(S)$ have the same dimension. With the asynchronous round operator, every processor in S (not just those labeling a face F of S) is labeled with states that include all nonfaulty processors and some subset of faulty processors.

We now check that the asynchronous round operator satisfies our axioms.

Lemma 6.2 $\mathcal{A}_\ell$ satisfies Axiom 3.3:

$$\mathcal{A}_\ell(S) \subseteq \mathcal{A}_m(S)$$

when $\ell \leq m$.

Proof. Since $\ell \leq m$ implies $\mathcal{F}_\ell(S) \subseteq \mathcal{F}_m(S)$, it follows that

$$\mathcal{A}_\ell(S) = \bigcup_{F \in \mathcal{F}_\ell(S)} \mathcal{P}(S, [F : S]) \subseteq \bigcup_{F \in \mathcal{F}_m(S)} \mathcal{P}(S, [F : S]) = \mathcal{A}_m(S).$$

$\square$

In this model, the integer-valued linear function ϕ is simply $\phi(f) = f$.

Lemma 6.3 $\mathcal{A}_\ell$ satisfies Axiom 3.4: Let $k \geq \ell$, and let $S \subseteq T$, where $c = codim(S, T)$. For all $r > 0$,

for all $S' \in \mathcal{A}_{k-\phi(c)}^r \mathcal{A}_{\ell-c}(S)$, there exists $T' \in \mathcal{A}_k^r \mathcal{A}_\ell(T)$
such that $S' \subseteq T'$ and $codim(S', T') \leq c$.

Lemma 6.4 $\mathcal{A}_\ell$ satisfies Axiom 3.5: Let $k \geq \ell$. For all $r > 0$, if $c = codim(S_0, S_1)$,

$$\|\mathcal{A}_k^r \mathcal{A}_\ell(S_0)\| \cap \|\mathcal{A}_k^r \mathcal{A}_\ell(S_1)\| \subseteq \|\mathcal{A}_{k-c}^r \mathcal{A}_{\ell-c}(S_0 \cap S_1)\|.$$

To show that $\mathcal{A}_\ell$ satisfies Axiom 3.6, we impose a partial order on simplexes of $\mathcal{A}_\ell(S)$. Recall that

$$\mathcal{A}_\ell(S) = \bigcup_{F \in \mathcal{F}_\ell(S)} \mathcal{P}(S, [F : S]).$$

If $F' \subseteq F$, then $\mathcal{P}(S, [F' : S]) \subseteq \mathcal{P}(S, [F : S])$, so we can restrict our attention to faces of codimension ℓ. Unlike in the synchronous model, where simplexes have

varying dimensions, all simplexes in this set are labelings of S, and all have dimension n.

We use the same total order $\leq_{id}$ on processor ids, the same total order $<_f$ on the faces of a simplex S. Next we order the simplexes in $\mathcal{A}_\ell(S)$ using this face ordering: we order A before B if, for each vertex v of the base simplex S the face of F labeling v in A comes before the face of F labeling v in B. Formally:

Definition 6.5 Define the partial order $\preceq_p$ on the simplexes of $\mathcal{A}_\ell(S)$ by $A \preceq_p B$ if and only if $A_v \leq_f B_v$ for each vertex v in S, where A_v and B_v are the simplexes labeling the vertex v in A and B.

Theorem 6.6 $\mathcal{A}_\ell$ *satisfies Axiom 3.6: For every simplex S, $(\mathcal{A}_\ell(S), \preceq_p)$ is an absorbing poset.*

Theorem 6.7 *No asynchronous protocol for k-set agreement exists in the presence of k crash failures.*

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Proofs from Section 4

Lemma 4.5 Let $(\mathcal{S}, \preceq)$ be an absorbing poset, and let $T \in \mathcal{S}$ be a maximal simplex with respect to $\preceq$. We claim that the following sets are both absorbing posets: $(\mathcal{L}, \preceq)$, where $\mathcal{L} = \{L | L \in \mathcal{S} - \{T\}\}$, and $(\mathcal{M}, \preceq)$, where $\mathcal{M} = \{T\}$.

Proof. Because $(\mathcal{S}, \preceq)$ is an absorbing poset, for every S in $\mathcal{L}$, there is $T_S \in \mathcal{S}$ such that $T_S \preceq T$ and satisfying the three absorbing poset conditions. Because T was chosen to be a maximal element of $\mathcal{S}$ and since $T_S \preceq T$, it follows that $T_S \in \mathcal{S} - \{T\} = \mathcal{L}$.

The singleton set $\{T\}$ is trivially an absorbing poset. □

Lemma 4.6 Let $(\mathcal{A}, \preceq)$ be an absorbing poset containing more than one simplex, and let $A \in \mathcal{A}$ be a maximal simplex with regards to $\preceq$. For each $B \neq A$ in $\mathcal{A}$, there exists a $A_B \in \mathcal{A}$ satisfying the three conditions of Definition 3.1. We claim that the set

$$\mathcal{B} = \{A_B \cap A | B \in \mathcal{A} - \{A\}\}$$

is an absorbing poset for any total order $<$ on the elements of $\mathcal{A} - \{A\}$

Proof. For any two distinct simplexes B_0 and B_1 in $\mathcal{B}$, $codim(B_0, B_1) = 1$, and the claim follows from Lemma 3.2. □

Lemma 4.7 If every simplex in $\mathcal{Q}_k^r \mathcal{Q}_\ell(\mathcal{A})$ has dimension at least d, then so does every simplex in $\mathcal{A}$.

Proof. By induction on r. When $r = 0$, for every A in $\mathcal{A}$ and B in $\mathcal{Q}_\ell(\mathcal{A})$, $\dim(B) \leq \dim(A)$ by the definition of simplicial operators.

When $r > 1$, $\mathcal{Q}_k^r \mathcal{Q}_\ell(\mathcal{A}) = \mathcal{Q}_k(\mathcal{Q}_k^{r-1} \mathcal{Q}_\ell(\mathcal{A}))$. By the definition of a simplicial operator $\mathcal{Q}_k$, every simplex in $\mathcal{Q}_k^{r-1} \mathcal{Q}_\ell(\mathcal{A})$ has dimension at least d since every simplex in $\mathcal{Q}_k(\mathcal{Q}_k^{r-1} \mathcal{Q}_\ell(\mathcal{A}))$ does. By the induction hypothesis, every simplex in $\mathcal{A}$ has dimension at least d since every simplex in $\mathcal{Q}_k^{r-1} \mathcal{Q}_\ell(\mathcal{A})$ does. □

Lemma 4.8 Let $\mathcal{Q}_k^r \mathcal{Q}_\ell$ be a composition of simplicial operators where $k \geq \ell$. If $(\mathcal{S}, \preceq)$ is an absorbing poset then for every two simplexes S and T in $\mathcal{S}$ with $T \not\preceq S$ there is a T_S in $\mathcal{S}$ with $T_S \preceq T$, such that

$$\|\mathcal{Q}_k^r \mathcal{Q}_\ell(S)\| \cap \|\mathcal{Q}_k^r \mathcal{Q}_\ell(T)\| \subseteq \|\mathcal{Q}_k^r \mathcal{Q}_\ell(T_S)\| \cap \|\mathcal{Q}_k^r \mathcal{Q}_\ell(T)\|$$
$$codim(T_S, T) = 1$$
$$codim(S \cap T, T_S) \leq codim(S, T).$$

Proof. Because $(\mathcal{S}, \preceq)$ is an absorbing poset, for every two simplexes S and T in $\mathcal{S}$ with $T \not\preceq S$ there is a T_S in $\mathcal{S}$ with $T_S \preceq T$ satisfying Equations 1, 2, and 3. By Lemma 4.2,

$$\|\mathcal{Q}_k^r \mathcal{Q}_\ell(S)\| \cap \|\mathcal{Q}_k^r \mathcal{Q}_\ell(T)\| = \|\mathcal{Q}_{k-\phi(c)}^r \mathcal{Q}_{\ell-c}(S \cap T)\|.$$

where $c = codim(S, T)$.

Let $d = codim(S \cap T, T_S \cap T)$. By Lemma 4.1, and linearity of ϕ,

$$\|\mathcal{Q}_{k-\phi(c)}^r \mathcal{Q}_{\ell-c}(S \cap T)\| \subseteq \|\mathcal{Q}_{k-\phi(c-d)}^r \mathcal{Q}_{\ell-c+d}(T_S \cap T)\|.$$

By Lemma 4.3, $d < c$, so $\ell - c + d \leq \ell - 1$ and $k - \phi(c-d) \leq k - \phi(1)$ since ϕ is nonnegative. By Lemma 4.4,

$$\|\mathcal{Q}_{k-\phi(c-d)}^r \mathcal{Q}_{\ell-c+d}(T_S \cap T)\| \subseteq \|\mathcal{Q}_{k-\phi(1)}^r \mathcal{Q}_{\ell-1}(T_S \cap T)\|.$$

By Lemma 4.2,

$$\|\mathcal{Q}_{k-\phi(1)}^r \mathcal{Q}_{\ell-1}(T_S \cap T)\| = \|\mathcal{Q}_k^r \mathcal{Q}_\ell(T_S)\| \cap \|\mathcal{Q}_k^r \mathcal{Q}_\ell(T)\|$$

Combining these inclusions yields the desired result.   □

Lemma 4.9 If $(\mathcal{A}, \preceq)$ is an absorbing poset where ℓ is the minimum dimension of any simplex in $\mathcal{A}$, then $\|\mathcal{A}\|$ is $(\ell - 1)$-connected.

Proof. Suppose that $\mathcal{A}$ contains m simplexes. We argue by induction on m. When $m = 1$, a single simplex of dimension at least ℓ is $(\ell - 1)$-connected (Theorem 2.3).

Suppose $m > 1$ and the induction hypothesis holds for $m - 1$. Let T be a maximal element of $\mathcal{A}$ with respect to $\preceq$, and let

$$\mathcal{L} = \{S | S \in \mathcal{A} - \{T\}\} \qquad \text{and} \qquad \mathcal{M} = \{T\}.$$

By Lemma 4.5, both $\mathcal{L}$ and $\mathcal{M}$ are absorbing posets. By construction, each has fewer than m simplexes, all of which have dimension at least ℓ. It follows from the induction hypothesis that complexes $\|\mathcal{L}\|$ and $\|\mathcal{M}\|$ are each $(\ell - 1)$-connected

We now claim that $\|\mathcal{L}\| \cap \|\mathcal{M}\|$ is $(\ell - 2)$-connected. Since $(\mathcal{A}, \preceq)$ is an absorbing poset, for every L in $\mathcal{L}$, there exists a T_L in $\mathcal{L}$ such that

$$L \cap T \subseteq T_L \cap T$$
$$codim(T_L, T) = 1.$$

Let $\mathcal{N} = \{T_L \cap T | L \in \mathcal{L}\}$. Note that $\|\mathcal{N}\| = \|\mathcal{L}\| \cap \|\mathcal{M}\|$. Lemma 4.6 implies that $\mathcal{N}$ is an absorbing poset containing fewer than m simplexes. By hypothesis, every simplex in $\mathcal{Q}_k^r \mathcal{Q}_\ell(\mathcal{A})$ has dimension at least ℓ. Because $\mathcal{L} \subseteq \mathcal{A}$, so does every simplex in $\mathcal{Q}_k^r \mathcal{Q}_\ell(\mathcal{L})$. By construction, for every simplex N in $\mathcal{N}$, there is a simplex $T_S \in \mathcal{L}$ such that $codim(N, T_S) \leq 1$. Axiom 3.4 implies that for every simplex N' in $\mathcal{Q}_{k-\phi(1)}^r \mathcal{Q}_\ell(\mathcal{N})$, there exists a simplex L' in $\mathcal{Q}_k^r \mathcal{Q}_\ell(\mathcal{L})$ such that $codim(N', L') \leq 1$. In particular, $\dim(N') \geq \ell - 1$, so every simplex in $\mathcal{Q}_{k-\phi(1)}^r \mathcal{Q}_\ell(\mathcal{N})$, has dimension at least $\ell - 1$. The induction hypothesis for m implies that $\mathcal{Q}_{k-\phi(1)}^r \mathcal{Q}_\ell(\mathcal{N})$ is $(\ell - 1)$-connected.

We have shown that $\|\mathcal{L}\|$ and $\|\mathcal{M}\|$ are each $(\ell - 1)$-connected, and $\|\mathcal{L}\| \cap \|\mathcal{M}\|$ is $(\ell - 2)$ connected, so $\|\mathcal{L}\| \cup \|\mathcal{M}\|$ is $(\ell - 1)$-connected by Mayer Vietoris.   □

Theorem 4.10 Let $\mathcal{Q}_k^r \mathcal{Q}_\ell$ be a composition of simplicial operators where $k \geq \ell$, and $(\mathcal{A}, \preceq)$ an absorbing poset. If every simplex in $\mathcal{Q}_k^r \mathcal{Q}_\ell(\mathcal{A})$ has dimension at least ℓ, then $\|\mathcal{Q}_k^r \mathcal{Q}_\ell(\mathcal{A})\|$ is $(\ell - 1)$-connected.

Proof. We prove a stronger induction hypothesis: If $k \geq \ell \geq \hat{\ell}$ and every simplex in $\mathcal{Q}_k^r \mathcal{Q}_\ell(\mathcal{A})$ has dimension at least $\hat{\ell}$, then $\|\mathcal{Q}_k^r \mathcal{Q}_\ell(\mathcal{A})\|$ is $(\hat{\ell} - 1)$-connected. Since $\mathcal{Q}_k^r \mathcal{Q}_\ell$ is an $r + 1$ fold composition of simplicial operators, we can view the case of $r + 1 = 0$ or $r = -1$ as the special case of no operators at all: $\mathcal{Q}_k^r \mathcal{Q}_\ell(\mathcal{A}) = \mathcal{A}$. We argue by induction on $r \geq -1$. When $r = -1$, the claim follows from Lemma 4.9.

Suppose $r \geq 0$ and the induction hypothesis holds for $r - 1$. We argue by induction on m, the number of simplexes in $\mathcal{A}$. When $m = 1$, let A be the unique simplex in $\mathcal{A}$. If $\mathcal{B} = \mathcal{Q}_\ell(A)$, then

$$\mathcal{Q}_k^r \mathcal{Q}_\ell(\mathcal{A}) = \mathcal{Q}_k^r(\mathcal{B}) = \mathcal{Q}_k^{r-1} \mathcal{Q}_k(\mathcal{B}).$$

The set $\mathcal{B}$ is an absorbing poset by Axiom 3.6. Since $k \geq k \geq \hat{\ell}$ and every simplex in $\mathcal{Q}_k^{r-1} \mathcal{Q}_k(\mathcal{B}) = \mathcal{Q}_k^r(\mathcal{B}) = \mathcal{Q}_k^r \mathcal{Q}_\ell(\mathcal{A})$ has dimension at least $\hat{\ell}$, the induction hypothesis for r implies that $\mathcal{Q}_k^r \mathcal{Q}_\ell(\mathcal{A}) = \mathcal{Q}_k^r(\mathcal{B}) = \mathcal{Q}_k^{r-1} \mathcal{Q}_k(\mathcal{B})$ is $(\hat{\ell} - 1)$-connected.

Suppose $m > 1$ and the induction hypothesis holds for $m' < m$. Let T be a maximal element of $\mathcal{A}$ with respect to $\preceq$, and let

$$\mathcal{L} = \{S | S \in \mathcal{A} - \{T\}\} \qquad \text{and} \qquad \mathcal{M} = \{T\}.$$

By Lemma 4.5, both $\mathcal{L}$ and $\mathcal{M}$ are absorbing posets, each containing fewer than m simplexes, all of dimension at least $\hat{\ell}$. It follows from the induction hypothesis for m that complexes $\|\mathcal{Q}_k^r \mathcal{Q}_\ell(\mathcal{L})\|$ and $\|\mathcal{Q}_k^r \mathcal{Q}_\ell(\mathcal{M})\|$ are each $(\hat{\ell} - 1)$-connected.

We now show that

$$\|\mathcal{Q}_k^r \mathcal{Q}_\ell(\mathcal{L})\| \cap \|\mathcal{Q}_k^r \mathcal{Q}_\ell(\mathcal{M})\|$$

is $(\hat{\ell} - 2)$-connected. Notice that

$$\|\mathcal{Q}_k^r \mathcal{Q}_\ell(\mathcal{L})\| \cap \|\mathcal{Q}_k^r \mathcal{Q}_\ell(\mathcal{M})\| = \left(\bigcup_{S \in \mathcal{L}} \|\mathcal{Q}_k^r \mathcal{Q}_\ell(S)\| \right) \cap \|\mathcal{Q}_k^r \mathcal{Q}_\ell(T)\|$$
$$= \bigcup_{S \in \mathcal{L}} \left(\|\mathcal{Q}_k^r \mathcal{Q}_\ell(S)\| \cap \|\mathcal{Q}_k^r \mathcal{Q}_\ell(T)\| \right).$$

Because $(\mathcal{A}, \preceq)$ is an absorbing poset, Lemma 4.8 implies that for every S in $\mathcal{L}$ there is a T_S in $\mathcal{L}$ such that

$$\|\mathcal{Q}_k^r \mathcal{Q}_\ell(S)\| \cap \|\mathcal{Q}_k^r \mathcal{Q}_\ell(T)\| \subseteq \|\mathcal{Q}_k^r \mathcal{Q}_\ell(T_S)\| \cap \|\mathcal{Q}_k^r \mathcal{Q}_\ell(T)\|$$
$$codim(T_S, T) \leq 1.$$

It follows that

$$\|\mathcal{Q}_k^r \mathcal{Q}_\ell(\mathcal{L})\| \cap \|\mathcal{Q}_k^r \mathcal{Q}_\ell(\mathcal{M})\| = \bigcup_{S \in \mathcal{L}} \left(\|\mathcal{Q}_k^r \mathcal{Q}_\ell(S)\| \cap \|\mathcal{Q}_k^r \mathcal{Q}_\ell(T)\| \right)$$
$$\subseteq \bigcup_{S \in \mathcal{L}} \left(\|\mathcal{Q}_k^r \mathcal{Q}_\ell(T_S)\| \cap \|\mathcal{Q}_k^r \mathcal{Q}_\ell(T)\| \right)$$

and it is easy to see that

$$\bigcup_{S \in \mathcal{L}} \left(\|\mathcal{Q}_k^r \mathcal{Q}_\ell(T_S)\| \cap \|\mathcal{Q}_k^r \mathcal{Q}_\ell(T)\| \right) \subseteq \bigcup_{S \in \mathcal{L}} \left(\|\mathcal{Q}_k^r \mathcal{Q}_\ell(S)\| \cap \|\mathcal{Q}_k^r \mathcal{Q}_\ell(T)\| \right)$$
$$= \|\mathcal{Q}_k^r \mathcal{Q}_\ell(\mathcal{L})\| \cap \|\mathcal{Q}_k^r \mathcal{Q}_\ell(\mathcal{M})\|$$

since every T_S is in $\mathcal{L}$, so

$$\|\mathcal{Q}_k^r \mathcal{Q}_\ell(\mathcal{L})\| \cap \|\mathcal{Q}_k^r \mathcal{Q}_\ell(\mathcal{M})\| = \bigcup_{S \in \mathcal{L}} \left(\|\mathcal{Q}_k^r \mathcal{Q}_\ell(T_S)\| \cap \|\mathcal{Q}_k^r \mathcal{Q}_\ell(T)\| \right)$$

with $codim(T_S, T) = 1$ for each T_S.

By Lemma 4.2,

$$\bigcup_{S \in \mathcal{L}} \left(\|\mathcal{Q}_k^r \mathcal{Q}_\ell(T_S)\| \cap \|\mathcal{Q}_k^r \mathcal{Q}_\ell(T)\| \right) = \bigcup_{S \in \mathcal{L}} \|\mathcal{Q}_{k-\phi(1)}^r \mathcal{Q}_{\ell-1}(T_S \cap T)\|.$$

Let $\mathcal{N} = \{T_S \cap T | S \in \mathcal{L}\}$.

$$\bigcup_{S \in \mathcal{L}} \|\mathcal{Q}_{k-\phi(1)}^r \mathcal{Q}_{\ell-1}(T_S \cap T)\| = \|\mathcal{Q}_{k-\phi(1)}^r \mathcal{Q}_{\ell-1}(\mathcal{N})\|.$$

Lemma 4.6 states that $\mathcal{N}$ is an absorbing poset. Moreover, $\mathcal{N}$ is an absorbing poset containing fewer than m simplexes, where each simplex has dimension at least $\hat{\ell} - 1$. It follows that we can apply the induction hypothesis for m to deduce that $\|\mathcal{Q}_{k-\phi(1)}^r \mathcal{Q}_{\ell-1}(\mathcal{N})\|$ is $(\hat{\ell} - 2)$-connected.

We have shown that $\|\mathcal{Q}_k^r \mathcal{Q}_\ell(\mathcal{L})\|$ and $\|\mathcal{Q}_k^r \mathcal{Q}_\ell(\mathcal{M})\|$ are each $(\hat{\ell} - 1)$-connected, and that $\|\mathcal{Q}_k^r \mathcal{Q}_\ell(\mathcal{L})\| \cap \|\mathcal{Q}_k^r \mathcal{Q}_\ell(\mathcal{M})\|$ is $(\hat{\ell} - 2)$-connected, so their union

$$\|\mathcal{Q}_k^r \mathcal{Q}_\ell(\mathcal{L})\| \cup \|\mathcal{Q}_k^r \mathcal{Q}_\ell(\mathcal{M})\| = \|\mathcal{Q}_k^r \mathcal{Q}_\ell(\mathcal{A})\|$$

is $(\hat{\ell} - 1)$-connected by Mayer-Vietoris.   □

Proofs from Section 5

Lemma 5.4 $\mathcal{S}_\ell$ satisfies Axiom 3.4: Let $k \geq \ell$, and let $S \subseteq T$, where $c = codim(S, T)$. For all $r > 0$,

$$\text{for all } S' \in \mathcal{S}_{k-\phi(c)}^r \mathcal{S}_{\ell-c}(S), \text{ there exists } T' \in \mathcal{S}_k^r \mathcal{S}_\ell(T)$$
$$\text{such that } S' \subseteq T' \text{ and } codim(S', T') \leq c.$$

Proof. We show the stronger property that

$$\mathcal{S}_k^r \mathcal{S}_{\ell-c}(S) \subseteq \mathcal{S}_k^r \mathcal{S}_\ell(T).$$

We argue by induction on r. Let $r = 0$. If $A \in \mathcal{S}_{\ell-c}(S)$, then A is a labeling of F with simplexes in $[F : S]$ for some face F of S satisfying $codim(F, S) \leq \ell - c$.

$$codim(F, T) = codim(F, S) + codim(S, T)$$
$$\leq (\ell - c) + c = \ell.$$

It follows that $A \in \mathcal{S}_\ell(T)$.

Let $r > 0$. If $A \in \mathcal{S}_k^r \mathcal{S}_{\ell-c}(S)$, then there is a $B \in \mathcal{S}_k^{r-1} \mathcal{S}_{\ell-c}(S)$ such that A is a labeling of F with simplexes in $[F : B]$ for some face F of B satisfying $codim(F, B) \leq k$. By the induction hypothesis, $B \in \mathcal{S}_k^{r-1} \mathcal{S}_\ell(T)$. Moreover, $codim(F, B) \leq k$. It follows that $A \in \mathcal{S}_k^r \mathcal{S}_\ell(T)$.   □

Lemma 5.5 $\mathcal{S}_\ell$ satisfies Axiom 3.5: Let $k \geq \ell$. For all $r > 0$, if $c = codim(S_0, S_1)$,

$$\|\mathcal{S}_k^r \mathcal{S}_\ell(S_0)\| \cap \|\mathcal{S}_k^r \mathcal{S}_\ell(S_1)\| \subseteq \|\mathcal{S}_k^r \mathcal{S}_{\ell-c}(S_0 \cap S_1)\|.$$

Proof. We argue by induction on r that if $A_0 \cap A_1$ is nonempty for A_i in $\mathcal{S}_k^r \mathcal{S}_\ell(S_i)$, then there exists an A in $\mathcal{S}_k^r \mathcal{S}_{\ell-c}(S_0 \cap S_1)$ such that $A_0 \cap A_1 \subseteq A$ and $\dim(A) = \max\{\dim(A_0), \dim(A_1)\}$.

Suppose $r = 0$. Each A_i is a labeling of a face F_i of S_i by simplexes in $[F_i : S_i]$ where $codim(F_i, S_i) \leq \ell$. Their intersection $A_0 \cap A_1$ is a labeling of a face of $F_0 \cap F_1$ by simplexes in $[F_0 \cup F_1 : S_0 \cap S_1]$. (Because $A_0 \cap A_1$ is nonempty, $F_0 \cup F_1 \subseteq S_0 \cap S_1$.) Denote the simplex S_0 or S_1 of maximal dimension by S_s, and denote the simplex A_0 or A_1 of maximal dimension by A_a, which implies that F_a is the base simplex F_0 or F_1 of maximal dimension. We can extend this labeling $A_0 \cap A_1$ of this face of $F_0 \cap F_1$ to a labeling A of the maximal base simplex F_a by labeling each unlabeled

vertex with $F_0 \cup F_1$. We know that $F_0 \cup F_1$ is a face of $S_0 \cap S_1$, so F_a is a face of $S_0 \cap S_1$ which is a face of S_s, and we have

$$codim(F_a, S_s) = codim(F_a, S_0 \cap S_1) + codim(S_0 \cap S_1, S_s) = codim(F_a, S_0 \cap S_1) + c$$

so

$$\begin{aligned} codim(F_a, S_0 \cap S_1) &= codim(F_a, S_s) - c \\ &= (\dim(S_s) - \dim(F_a)) - c \leq (\dim(S_s) - \dim(F_s)) - c \\ &= codim(F_s, S_s) - c \leq \ell - c. \end{aligned}$$

It follows that $A \in \mathcal{S}_{\ell-c}(S_0 \cap S_1)$, and $\dim(A) = \max \{\dim(A_0), \dim(A_1)\}$ by construction.

Suppose $r > 0$. There exist $B_i \in \mathcal{S}_k^{r-1}\mathcal{S}_\ell(S_i)$ such that each A_i is a labeling of a face F_i of B_i by simplexes in $[F_i : B_i]$ where $codim(F_i, B_i) \leq k$. $A_0 \cap A_1$ is a labeling of a face of $F_0 \cap F_1$ by simplexes in $[F_0 \cup F_1 : B_0 \cap B_1]$. (Because A is nonempty, $F_0 \cup F_1 \subseteq B_0 \cap B_1$.) Denote the simplex B_0 or B_1 of maximal dimension by B_b, and denote the simplex A_0 or A_1 of maximal dimension by A_a, which implies that F_a is the base simplex F_0 or F_1 of maximal dimension. By the induction hypothesis, there exists $B \in \mathcal{S}_k^{r-1}\mathcal{S}_{\ell-c}(S_0 \cap S_1)$ such that $B_0 \cap B_1 \subseteq B$ and $\dim(B) = \dim(B_b)$. We can extend this labeling $A_0 \cap A_1$ of this face of $F_0 \cap F_1$ to a labeling A of the maximal base simplex F_a by labeling each unlabeled vertex with $F_0 \cup F_1$. We know that $F_0 \cup F_1$ is a face of $B_0 \cap B_1 \subseteq B$, so F_a is a face of a face of B and B_b, and we have

$$\begin{aligned} codim(F_a, B) &= \dim(B) - \dim(F_a) \\ &= \dim(B_b) - \dim(F_a) \leq \dim(B_b) - \dim(F_b) \\ &= codim(F_b, B_b) \leq k. \end{aligned}$$

So $A \in \mathcal{S}_k(B) \in \mathcal{S}_k^r\mathcal{S}_{\ell-c}(S_0 \cap S_1)$, and $\dim(A) = \max \{\dim(A_0), \dim(A_1)\}$ by construction. $\qquad \square$

Theorem 5.9 $\mathcal{S}_\ell$ *satisfies Axiom 3.6: For every simplex S, $(\mathcal{S}_\ell(S), \preceq_r)$ is an absorbing poset.*

Proof. Let A and B be two simplexes in $\mathcal{S}_\ell(S)$ satisfying $B \not\preceq_r A$. We consider two cases depending on whether A and B are in the same pseudosphere or not.

Case 1: Suppose A and B are in the same pseudosphere $\mathcal{P}(F, [F : S])$ for some face F of S. The simplexes A and B are labelings of F with simplexes between F and S, so let A_v and B_v denote the label of vertex v in A and B for every vertex $v \in F$. There must be some vertex v with $A_v <_f B_v$ since $B \not\preceq_r A$. Notice that the face B_v must not be the entire simplex S, since S is ordered before every proper face of S, including A_v. Let B_A be B with the label of v changed from B_v to S, and notice that B_A and B are in the same pseudosphere. We have $B_A \prec_r B$ since the label of v in B_A is ordered before the label of v in B, and the labels of all other vertices are equal. We have $A \cap B \subseteq B_A$ since v is not in $A \cap B$ due to the conflicting

labels for v, while all other vertexes of B and hence of $A \cap B$ are in B_A. We have $codim(B_A, B) = 1$ since B_A and B differ only in the label of v. Finally, since $A \cap B$ is a face of both B_A and B, and since B_A and B have the same dimension, we have

$$
\begin{aligned}
codim(A \cap B, B_A) &= \dim(B_A) - \dim(A \cap B) \\
&= \dim(B) - \dim(A \cap B) \\
&\leq \max \{\dim(A) - \dim(A \cap B), \dim(B) - \dim(A \cap B)\} \\
&= codim(A, B)
\end{aligned}
$$

Case 2: Suppose A and B are in different pseudospheres $\mathcal{P}(F_A, [F_A : S])$ and $\mathcal{P}(F_B, [F_B : S])$ for distinct faces F_A and F_B of S. First, we can assume without loss of generality that every vertex of $B - A$ is labeled with S, since if there is any vertex of $B - A$ that is not labeled with S, then we can take B_A to be B with this label replaced by S, and it is easy to check that $B_A \prec_r B$, $A \cap B \subseteq B_A$, $codim(B, B_A) = 1$, and $codim(A \cap B, B_A) \leq codim(A, B)$. Second, the two base simplexes of A and B must be ordered $F_A <_f F_B$ since $B \not\prec_r A$. Let us consider the two cases induced by the ordering $F_A <_f F_B$.

Case 2a: Suppose $\dim(F_A) > \dim(F_B)$. Since $\dim(F_A) > \dim(F_B)$, the set $F_A - F_B$ must be nonempty, so choose any vertex $v \in F_A - F_B$. Since $A \in \mathcal{P}(F_A, [F_A : S])$, the simplex A must be a labeling of F_A with simplexes between F_A and S. Since v is a vertex of F_A, this means that v appears in all simplexes labeling A, and hence in all simplexes labeling $A \cap B$. Since we have assumed that S is the label of every vertex in $B - A$, and since S certainly contains the vertex v, the vertex v appears in all labels of $B - A$. It follows that v appears in every simplex labeling B. Let B_A be the simplex consisting of B together with the vertex v labeled with S, and notice that B_A is a simplex in $\mathcal{S}_\ell(S)$. We have $B_A \prec_r B$ since $\dim(B_A) = \dim(B) + 1 > \dim(B)$. We have $A \cap B \subseteq B_A$ since $A \cap B \subseteq B \subseteq B_A$. We have $codim(B, B_A) = 1$ since B and B_A differ only in v. Finally, since $A \cap B$ is a face of both B and B_A, and since $\dim(B_A) = \dim(B) + 1 \leq \dim(A)$, we have

$$
\begin{aligned}
codim(A \cap B, B_A) &= \dim(B_A) - \dim(A \cap B) \\
&\leq \dim(A) - \dim(A \cap B) \\
&\leq \max \{\dim(A) - \dim(A \cap B), \dim(B) - \dim(A \cap B)\} \\
&= codim(A, B).
\end{aligned}
$$

Case 2b: Suppose $\dim(F_A) = \dim(F_B)$, in which case we have $p_A \prec_p p_B$ where $p_A = \min \{ids(F_A) - ids(F_B)\}$ and $p_B = \min \{ids(F_B) - ids(F_A)\}$. Let v_A and v_B be the vertexes for processors p_A and p_B in the faces F_A and F_B of S. Let F_C be the face of S obtained from F_B by replacing v_B with v_A, and let C be the labeling of F_C obtained by labeling v_A with S and every other vertex with its label in B. Since A is a labeling of F_A with simplexes between F_A and S, and since v_A is a vertex of F_A, the vertex v_A appears in every simplex labeling A and hence $A \cap B$; and since we are assuming that every vertex of $B - A$ is labeled with S which certainly contains v_A, it follows that every vertex of $B - A$ is labeled with a simplex containing v_A; and

hence it follows that every label in B contains F_C. It follows that $C \in \mathcal{S}_\ell(S)$ since C is a labeling of a face F_C of S with simplexes between F_C and S. We have $C \prec_r B$ since

$$\min \{ids(F_C) - ids(F_B)\} = p_A \prec_p p_B = \min \{ids(F_B) - ids(F_C)\}.$$

We have $A \cap B \subseteq C$ and $codim(B, C) = 1$, and we have

$$
\begin{aligned}
codim(A \cap B, C) &= \dim(C) - \dim(A \cap B) \\
&= \dim(B) - \dim(A \cap B) \\
&\leq \max \{\dim(A) - \dim(A \cap B), \dim(B) - \dim(A \cap B)\} \\
&= codim(A, B).
\end{aligned}
$$

Taking $B_A = C$, we are done. $\qquad\square$

Proofs from Section 6

Lemma 6.3 $\mathcal{A}_\ell$ satisfies Axiom 3.4: Let $k \geq \ell$, and let $S \subseteq T$, where $c = codim(S, T)$. For all $r > 0$,

$$\text{for all } S' \in \mathcal{A}^r_{k-\phi(c)} \mathcal{A}_{\ell-c}(S), \text{ there exists } T' \in \mathcal{A}^r_k \mathcal{A}_\ell(T)$$
$$\text{such that } S' \subseteq T' \text{ and } codim(S', T') \leq c.$$

Proof. We must show that every simplex $A \in \mathcal{A}^r_{k-c} \mathcal{A}_{\ell-c}(S)$ is a face of simplex $B \in \mathcal{A}^r_k \mathcal{A}_\ell(T)$ of codimension at most c.

We argue by induction on r. Let $r = 0$. If $A \in \mathcal{A}_{\ell-c}(S)$, then A is a labeling of S with simplexes in $[F : S]$ for some face F of S satisfying $codim(F, S) \leq \ell - c$. Extend A to a labeling B of T by labeling each remaining vertex with F. Since F is a face of S which is in turn a face of T, we have

$$
\begin{aligned}
codim(F, T) &= codim(F, S) + codim(S, T) \\
&\leq (\ell - c) + c = \ell.
\end{aligned}
$$

It follows that $A \subseteq B \in \mathcal{A}_\ell(T)$ with codimension at most c.

Let $r > 0$. By the induction hypothesis, every simplex $A \in \mathcal{A}^{r-1}_{k-c} \mathcal{A}_{\ell-c}(S)$ is a face of simplex $B \in \mathcal{A}^{r-1}_k \mathcal{A}_\ell(T)$ where $codim(A, B) \leq c$. Every simplex A' in $\mathcal{A}_{k-c}(B)$ is a labeling of A with simplexes in $[F : A]$, where $codim(F, A) \leq k - c$. Extend A' to a labeling B' of B by labeling each remaining vertex with F. We know that

$$codim(F, B) = codim(F, A) + codim(A, B),$$

and that $codim(F, A) \leq k - c$. Because A is a face of B we have

$$
\begin{aligned}
codim(A, B) &= \dim(B) - \dim(A) \\
&= \dim(T) - \dim(S) \leq c.
\end{aligned}
$$

The second equality follows from the fact that the asynchronous round operator does not decrease dimension. It follows that $codim(F, B) \leq k$, so $A' \subset B' \in \mathcal{A}_k^r \mathcal{A}_\ell(S_i)$, and $codim(A', B') \leq c$. $\qquad \square$

Lemma 6.4 $\mathcal{A}_\ell$ satisfies Axiom 3.5: Let $k \geq \ell$. For all $r > 0$, if $c = codim(S_0, S_1)$,

$$\|\mathcal{A}_k^r \mathcal{A}_\ell(S_0)\| \cap \|\mathcal{A}_k^r \mathcal{A}_\ell(S_1)\| \subseteq \|\mathcal{A}_{k-c}^r \mathcal{A}_{\ell-c}(S_0 \cap S_1)\|.$$

Proof. We argue by induction on r that if $A_0 \cap A_1$ is nonempty for simplexes A_i in $\mathcal{A}_k^r \mathcal{A}_\ell(S_i)$, then there exist simplexes A_i' in $\mathcal{A}_k^r \mathcal{A}_\ell(S_i)$ with $codim(A_0', A_1') = c$, $A_0 \cap A_1 \subseteq A_0' \cap A_1'$, $\dim(A_i) = \dim(A_i')$ for each i, and $A_0' \cap A_1' \in \mathcal{A}_{k-c}^r \mathcal{A}_{\ell-c}(S_0 \cap S_1)$.

Let $r = 0$. Each A_i is a labeling of S_i by simplexes in $[F_i, S_i]$, where $codim(F_i, S_i) \leq \ell$. The intersection $A_0 \cap A_1$ is a labeling of some $S \subseteq S_0 \cap S_1$ by simplexes in $[F_0 \cup F_1, S_0 \cap S_1]$. We can extend this labeling of $S \subseteq S_i$ to a labeling A_i' of S_i by labeling the vertexes in $(S_0 \cap S_1) - S$ with $F_0 \cup F_1$ and the remaining vertexes with S_i. Notice that these labelings agree on $S_0 \cap S_1$ and differ on the remaining vertexes, so $codim(A_0', A_1') = codim(S_0, S_1) = c$. Notice also that $\dim(A_i) = \dim(S_i) = \dim(A_i')$ and that $A_0 \cap A_1 \subseteq A_0' \cap A_1'$ by construction. Finally, the simplex $A_0' \cap A_1'$ is a labeling of $S_0 \cap S_1$ by simplexes of $[F_0 \cup F_1 : S_0 \cap S_1]$. Because

$$codim(F_0 \cup F_1, S_i) = codim(F_0 \cup F_1, S_0 \cap S_1) + codim(S_0 \cap S_1, S_i),$$

$codim(F_0 \cup F_1, S_0 \cap S_1) \leq \ell - c$. It follows that $A_0' \cap A_1' \in \mathcal{A}_{\ell-c}(S_0 \cap S_1)$.

Suppose $r > 0$. A_i is a labeling of a simplex $B_i \in \mathcal{A}_k^{r-1} \mathcal{A}_\ell(S_i)$ by simplexes in $[F_i : B_i]$ where $codim(F_i, B_i) \leq k$. By the induction hypothesis, there exist simplexes $B_i' \in \mathcal{A}_k^{r-1} \mathcal{A}_\ell(S_0)$ such that, $codim(B_0', B_1') = c$, $B_0 \cap B_1 \subseteq B_0' \cap B_1'$, $\dim(B_i) = \dim(B_i')$ for each i, and $B_0' \cap B_1' \in \mathcal{A}_{k-c}^{r-1} \mathcal{A}_{\ell-c}(S_0 \cap S_1)$.

The intersection $A_0 \cap A_1$ is a labeling of some $B \subseteq B_0 \cap B_1 \subseteq B_0' \cap B_1'$ by simplexes in $[F_0 \cup F_1, B_0 \cap B_1]$. We can extend this labeling of $B \subseteq B_i'$ to a labeling A_i' of B_i' by labeling the vertexes in $(B_0' \cap B_1') - B$ with $F_0 \cup F_1$ and the remaining vertexes with B_i'. Notice that these labelings agree on $B_0' \cap B_1'$ and differ on the remaining vertexes, so $codim(A_0', A_1') = codim(B_0', B_1') = c$. Notice also that $\dim(A_i) = \dim(B_i) = \dim(B_i') = \dim(A_i')$ and that $A_0 \cap A_1 \subseteq A_0' \cap A_1'$ by construction. Finally, the simplex $A_0' \cap A_1'$ is a labeling of $B_0' \cap B_1'$ by simplexes of $[F_0 \cup F_1 : B_0 \cap B_1]$ and therefore of $[F_0 \cup F_1 : B_0' \cap B_1']$. Because B_i and B_i' have the same dimension by hypothesis, $codim(F_i, B_i') = codim(F_i, B_i) \leq k$. Thus $codim(F_0 \cup F_1, B_i') \leq k$, and

$$codim(F_0 \cup F_1, B_i') = codim(F_0 \cup F_1, B_0' \cap B_1') + codim(B_0' \cap B_1', B_i'),$$

implying that $codim(F_0 \cup F_1, B_0' \cap B_1') \leq k - c$. It follows that $A_0' \cap A_1' \in \mathcal{A}_{k-c}^r \mathcal{A}_{\ell-c}(S_0 \cap S_1)$. $\qquad \square$

Theorem 6.6 $\mathcal{A}_\ell$ satisfies Axiom 3.6: For every simplex S, $(\mathcal{A}_\ell(S), \preceq_p)$ is an absorbing poset.

Proof. Let A and B be two simplexes in $\mathcal{A}_\ell(S)$ satisfying $B \not\preceq_p A$. Let A_v and B_v denote the label of vertex v in A and B for every vertex $v \in S$. There must be some vertex v with $A_v <_f B_v$ since $B \not\preceq_p A$. Notice that the face B_v must not be the entire simplex S, since S is ordered before every proper face of S, including A_v. Let B_A be B with the label of v changed from B_v to S, and notice that B_A and B are in the same pseudosphere. We have $B_A \preceq_p B$ since the label of v in B_A is ordered before the label of v in B, and the labels of all other vertices are equal. Equation 1 is satisfied: $A \cap B \subseteq B_A \cap B$ since v is not in $A \cap B$ due to the conflicting labels for v, while all other vertexes of B and hence of $A \cap B$ are in B_A. Equation 2 is satisfied: $codim(B_A, B) = 1$ since B_A and B differ only in the label of v. Finally, Equation 3 is satisfied: since $A \cap B$ is a face of both B_A and B, and since B_A and B have the same dimension,

$$\begin{aligned}
codim(A \cap B, B_A) &= \dim(B_A) - \dim(A \cap B) \\
&= \dim(B) - \dim(A \cap B) \\
&\leq \max\left\{\dim(A) - \dim(A \cap B), \dim(B) - \dim(A \cap B)\right\} \\
&= codim(A, B)
\end{aligned}$$

$\square$

Theorem 6.7 No asynchronous protocol for k-set agreement exists in the presence of k crash failures.

Proof. Suppose there is a protocol that halts in r rounds. Consider the subset of executions in which at most k processors are silent in each round. For the input complex $\mathcal{P}(S, V)$, the set of final states of such executions is $\mathcal{A}_k^r(\mathcal{P}(S, V))$. Every simplex in this complex has dimension $n > k$. By Theorem 4.10, this complex is $(k-1)$-connected, and by Theorem 2.1, the protocol cannot solve k-set agreement.$\square$