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# Finitely Bounded Effective Computability

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**Abstract**

One of the most important property of the computability is the certainty. For example a set of natural numbers is computable if there is a Turing machine which decides certainly if a given natural number belongs to the set or not, and no wrong answers are tolerant. On the other hand, the finitely bounded computability discussed in this paper allows finitely many mistakes which can be eventually corrected during an effective procedure. We will show especially that the class of finitely bounded computable real numbers has very interesting properties, and it is a real closed field containing even properly in the class of d-c.e. real numbers.

*Keywords:* Computable real numbers, D-c.e. real numbers, Bounded computability, Hierarchy.

## 1 Introduction

In computability theory, computable objects are described by effective procedures like Turing machines. These effective procedures do not make mistakes. For example, a set $A$ of natural numbers is computable means that there is a Turing machine which decides, for any given natural number $n$, if $n \in A$ holds or not. And a real number $x$ is computable if there is a Turing machine which, for any given natural number $n$, computes a rational approximation $x_n$ to $x$ within the given error bound $2^{-n}$. In all these cases, the individual results produced by the effective procedure cannot be corrected anymore after they have been achieved. In this sense the computability requires certainty. In practice, however, this ideal property is very difficult to be guaranteed. A more common and reasonable scenario in real world is that mistakes may occur, but the numbers of possible mistakes can be bounded somehow. This observation motivates the investigation of finitely bounded computability.

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Actually, the well-known $k$-c.e. ($k$-computably enumerable) sets of natural numbers for constant $k$ are good examples of finitely bounded computable objects mentioned above. By definition a set $A$ is c.e. if it has a computable enumeration which enumerates all its elements one after another. In other words, there is an effective procedure which puts all its elements into $A$. "Try and correct" is not allowed in this case. On the other hand, if there is an effective procedure which enumerates all elements of a set $A$ in which up to $k$ corrections for each element are allowed, then $A$ is $k$-c.e. This can be more precisely called $k$-bounded computably enumerable. Analogously, if the characteristic function $\chi_A$ of a set $A$ can be computed by a Turing machine in which up to $k$-corrections of the outputs are allowed for any input, then we can achieve naturally the notion of $k$-bounded computable sets.

Similarly, for any computable real number $x$, we have a computable sequence $(x_s)$ of rational numbers which approximates $x$ effectively in the sense that the distance between the $s$-th approximation $x_s$ and $x$ is bounded by $2^{-s}$ for all $s$. Namely, there is an effective procedure which, for any input $s$, outputs a rational approximation to $x$ within the error bound $2^{-s}$. If, for any error bound $2^{-s}$, up to $k$ exceptions to this condition are allowed, then the limit is called $k$-bounded effectively computable or simply $k$-effectively computable ($k$-e.c.), and $x$ is called bounded effectively computable (bec) if it is $k$-e.c. for some constant $k$. We will see that, for different constant $k$ we have a proper hierarchy of $k$-c.e. real numbers. More interestingly, the class of bounded effectively computable reals forms a very interesting class of real numbers. It is closed under arithmetical operations and total computable real functions.

Our paper is organized as follows. In section 2 we discuss the sets of natural numbers of bounded effective computability. The bounded effective computable real numbers are introduced and investigated in section 3. The last section 4 discusses the Turing degrees related to the effective bounded computable real numbers.

## 2 Bounded Computable Sets

The first version of finitely bounded computability comes from the $k$-computable enumerability for the constants $k$. By definition, a set $A$ of natural numbers is *computably enumerable* (c.e., for short) if it has a computable enumeration $(A_s)$ which is a computable sequence of finite sets such that $A_0 = \emptyset$, $A_s \subseteq A_{s+1}$ for all $s$ and $\lim_{s\to\infty} A_s = A$. Intuitively, $A$ is c.e. if there is an effective procedure which enumerates all elements of $A$. The computable enumerability has been generalized by Putnam [5], Gold [3] and Ershov [2] to the $k$-computable enumerability as follows: A set $A$ is *$k$-computably enumerable* ($k$-c.e., for short) if there exists a $k$-computable enumeration of $A$ which is a computable sequence $(A_s)$ of finite sets such that and

(1) $\quad A_s = \emptyset \ \& \ \lim\limits_{s\to\infty} A_s = A,$ and

(2) $\quad (\forall n)\,(|\{s : A_s(n) \neq A_{s+1}(n)\}| \leq k)\,.$

Here and also in the following we identify a set with its characteristic function. Thus we have $A(n) = 1$ iff $n \in A$ and $A(n) = 0$ iff $n \notin A$. Roughly speaking, a set

$A$ is $k$-c.e. if there is an effective procedure which enumerates all elements of $A$ in which up to $k$ corrections for any natural numbers are allowed.

If we are interested more in the computability instead of the enumerability, we can change the definition of $k$-c.e. sets slightly and achieve naturally the following.

**Definition 2.1** For any constant $k \in \mathbb{N}$, a set $A$ is called *$k$-b.c.* (for *$k$-bounded computable*), if there is a computable sequence $(A_s)$ of finite sets which converges *$k$-bounded effectively* ($k$-b.e., for short) to $A$ in the sense that

(3) $(\forall n)\left(|\{s \geq n : A_s(n) \neq A_{s+1}(n)\}| \leq k\right)$.

Notice that, the definition of $k$-b.c. sets differs from that of $k$-c.e. in the following two points. Firstly, the condition $A_0 = \emptyset$ is deleted. The reason is simple. The $k$-computable enumerability interests in how an element be enumerated into the set by an effective procedure and hence we have to begin with an empty set. However, for the $k$-bounded computability, we are interested only in how the value $A(n)$ can be eventually determined, and the initial value $A(n)$ at the beginning does not play a role. Secondly, only the stages $s \geq n$ are counted in (3). This ignores the inessential changes of $A_s(n)$ before the stage $n$ which can be effectively determined in advance. As a result of this slight change, any infinite computable sets are 0-b.c. Otherwise, if a computable sequence $(A_s)$ of finite sets converges to an infinite computable set $A$, then there is at least one $s$ such that $A_s(n) \neq A_{s+1}(n)$ for any $n \in A/A_0$ . Simply letting such kind of changes occur before the stage $s := n$ guarantees that any computable sets are 0-b.c. Furthermore, the stage $n$ can be equivalently replaced by $g(n)$ for any increasing computable function $g$.

Instead of the sequences of finite sets, we can also consider the computable functions of two arguments which corresponds to a computable sequence of (not necessarily finite) sets. This leads to another equivalent definition of $k$-b.c. sets in the following proposition. In this case the condition $s \geq n$ is not necessary anymore. Some other simple properties of $k$-b.c. sets are also included in the proposition.

**Proposition 2.2** *Let $k$ be a constant, and let $A$ be a set of natural numbers.*

(i) *$A$ is $k$-b.c. if and only if there is a computable function $f$ such that, for all $n$,*
$$\lim_{s\to\infty} f(n,s) = A(n) \ \& \ |\{s : f(n,s) \neq f(n,s+1)\}| \leq k;$$
(ii) *If $A$ is $k$-c.e. or co-$k$-c.e., then $A$ is $k$-b.c.*

(iii) *If $A$ is $k$-b.c., the $A$ is $(k+1)$-c.e.*

(iv) *If $A$ is $k$-b.c., then so is the complement $\overline{A}$.*

(v) *$A$ is $k$-b.c. if and only if there is an increasing computable function $g$ and a computable sequence $(A_s)$ of finite sets which converges to $A$ such that*
$$(\forall n)(|\{s \geq g(n) : A_{s+1}(n) \neq A_s(n)\}| \leq k).$$

Analogous to the well known fact that a set $A$ is computable if and only if $A$ as well as its complement $\overline{A}$ are c.e., we have the following result.

**Theorem 2.3** *For any constant $k$, a set $A$ is $k$-b.c. if and only if both $A$ and its complement $\overline{A}$ are $(k+1)$-c.e.*

**Proof.** By Proposition 2.2.(iii), we need only to prove the "if" part. Suppose that $(A_s)$ and $(B_s)$ are $(k+1)$-enumerations of $A$ and $\overline{A}$, respectively. Since $\lim_s A_s(n) = A(n) \neq \overline{A}(n) = \lim_s B_s(n)$ for all $n$, we can define a computable increasing total function $v : \mathbb{N} \to \mathbb{N}$ by

$$\begin{cases} v(0) := \min\{s : A_s(0) \neq B_s(0)\}; \\ v(n+1) := \min\{s > v(n) : A_s(n+1) \neq B_s(n+1)\}. \end{cases}$$

Then we define a computable function $f$ by

$$f(s,n) : \begin{cases} A_{v(n)}(n) & \text{if } A_{v(n)}(n) = 1 \ \& \ s \leq v(n); \\ A_s(n) & \text{if } A_{v(n)}(n) = 1 \ \& \ s > v(n); \\ 1 \dot{-} B_{v(n)}(n) & \text{if } A_{v(n)}(n) = 0 \ \& \ s \leq v(n); \\ 1 \dot{-} B_s(n) & \text{if } A_{v(n)}(n) = 0 \ \& \ s > v(n). \end{cases}$$

Obviously, we have $\lim_s f(s,n) = A(n)$. Suppose that $A_{v(n)}(n) = 1$ for a given $n$. If $s \geq v(n)$ such that $f(s,n) \neq f(s+1,n)$, then $A_s(n) \neq A_{s+1}(n)$. Since $A_{v(n)}(n) = 1 \neq 0 = A_0(n)$ and $(A_s)$ is an $(k+1)$-enumeration, there are at most $k$ such stages $s \geq v(n)$. The same holds if $A_{v(n)}(n) = 0$. This implies that there are at most $k$ different $s \geq v(n)$ such that $f(s,n) \neq f(s+1,n)$. For $s < v(n)$ we have $f(s,n) = f(s+1,n)$ by the definition of $f$. Thus there are at most $k$ different $s$ totally such that $f(s,n) \neq f(s+1,n)$ for all $n$, and hence $A$ is $k$-b.c. by Proposition 2.2.(iii). □

Denote by $k$-**BC** the class of all $k$-b.c. sets and let **BC** : $\bigcup_{k\in\mathbb{N}} k$-**BC** be the class of *bounded computable* sets. The class **BC** is closed under the set-operations of union, intersection, difference, complement and join, where $A$ join $B$ is the set $A \oplus B : \{2n : n \in A\} \cup \{2n+1 : n \in B\}$. From the Ershov's hierarchy theorem and Proposition 2.2, we have also a hierarchy theorem for **BC** that $k$-**BC** $\subsetneq$ $(k+1)$-**BC** for all $k$. In addition, the following theorem shows that the class of $k$-**BC** is not simply the union of the classes of $k$-c.e. and co-$k$-c.e. sets which can be easily proved by a simple diagonalization.

**Theorem 2.4** *For any constant $k$, there is a $k$-b.c. set which is neither $k$-c.e. nor co-$k$-c.e.*

## 3 Bounded Effectively Computable Real Numbers

In this section we discuss the finitely bounded computability of real numbers. A straightforward way to introduce the notion of finitely bounded computable real numbers is to consider the real numbers of $k$-b.c. binary expansion. Namely, we can call $x$ "$k$-bounded computable" if $x = x_A := \sum_{i\in A} 2^{-(i+1)}$ for a $k$-b.c. set $A$. Unfortunately, the "bounded computable reals" defined in this way do not have good mathematical properties. For example, there are c.e. sets $B$ and $C$ such that binary expansion $A$ of $x_A := x_B - x_C$ is not even have an $\omega$-c.e. Turing degree (see [7,9]).

Instead of binary expansion we consider the Cauchy representation of real numbers, and define the bounded computability of real numbers as follows which is the special case of $h$-effective computability of real numbers for functions $h$ discussed in [8].

**Definition 3.1** A real $x$ is *$k$-effectively computable* ($k$-e.c., for short) if there is a computable sequence $(x_s)$ of rational numbers which converges to $x$ *$k$-effectively* in the sense that there are at most $k$ non-overlapping index-pairs $(s, t)$ such that

(4) $s, t \geq n \ \&\ |x_s - x_t| \geq 2^{-n}$.

The class of all $k$-e.c. real numbers is denoted by $k$-**EC**. In addition, we call a real $x$ *bounded effectively computable* (bec, for short) if it is $k$-e.c. for a constant $k$ and the class of all bec real numbers is denoted by **BEC**.

**Lemma 3.2** *Let $i, j$ be any natural numbers.*

(i) *If $x$ and $y$ are $i$-e.c. and $j$-e.c., respectively, then $x + y, x \times y$ and $x/y$ (for $y \neq 0$) are all $(i + j)$-e.c.*

(ii) *If $k > 0$, then the class $k$-**EC** is not closed under addition.*

**Proof.** (i). Let $(x_s)$ and $(y_s)$ be computable sequences of rational numbers which converge $i$-effectively and $j$-effectively to $x$ and $y$, respectively. We consider here only the product $xy$. The situations for other operations are similar. Choose a constant $c$ such that $|x_s|, |y_s| \leq 2^c$ for all $s$. For any natural numbers $s, t$ and $n$, if $|x_s - x_t| \leq 2^{-n}$ and $|y_s - y_t| \leq 2^{-n}$, then we have

(5) $|x_s y_s - x_t y_t| \leq |x_s||y_s - y_t| + |y_t||x_s - x_t| \leq 2^{-(n-(c+1))}$.

Define a computable sequence $(z_s)$ of rational numbers by $z_s := x_{s'} y_{s'}$ for $s' := s + c + 1$. The sequence $(z_s)$ converges obviously to $xy$. We show now that this sequence converges $(i + j)$-effectively.

For any $n$ and $s, t \geq n$, if the inequality $|z_s - z_t| > 2^{-n}$ holds, then, we have either $|x_{s'} - x_{t'}| \geq 2^{-(n+c+1)}$ or $|y_{s'} - y_{t'}| \geq 2^{-(n+c+1)}$ by (5) where $s' := s + c + 1$ and $t' := t + c + 1$. Since $(x_s)$ and $(y_s)$ converge $i$-effectively and $j$-effectively, respectively, the number of non-overlapping index-pairs $(s, t)$ of these properties is bounded by $i + j$. That is, the sequence $(z_s)$ converges $(i + j)$-effectively, and hence $xy$ is $(i + j)$-effectively computable.

(ii). It suffices to show that, for any $k \geq 0$, there are $k$-e.c. real $x$ and 1-e.c. real $y$ such that $x + y$ is not $k$-c.e. We can construct two computable sequences $(x_s)$ and $(y_s)$ of rational numbers which converge $k$-effectively and 1-effectively to $x$ and $y$, respectively such that their sum $x + y$ is different from any $k$-effectively computable real numbers. That is, $x + y$ satisfies the following conditions:

$$R_e: \quad (\varphi_e(s))_s \text{ converges } k\text{-effectively to } z_e \implies x + y \neq z_e.$$

The construction of the sequences $(x_s)$ and $(y_s)$ applies the standard jump technique. To satisfy a single requirement $R_e$, we choose two rational intervals $I_1$ and $I_2$ such that the distance between them is $2^{-n}$ for some natural number $n$. As default, let $x_0$ be the middle point of $I_1$ and $y_0 := 0$. We change $x_s$ to be the middle point of

$I_2$ whenever the sequence $(\varphi_e(t))$ enters the interval $I_1$ after stage $n$ while the $y_s$ remains being unchanged. Redefine $x_t$ to be the middle point of interval $I_1$ if the sequence $(\varphi_e(s))_s$ enters the interval $I_2$ at a later stage $t$. This kind jumps of $x_s$ are allowed at most $k$ times. After $k$ jumps of $x_s$, we can increase or decrease $y_s$ by $2^{-n}$ once to force the sum $x_s + y_s$ leaves the interval $I_1$ or $I_2$ depending on the sequence $(\varphi_e(t))$ enters $I_1$ or $I_2$. In this way, the sequences $(x_s)$ and $(y_s)$ converge $k$-effectively and 1-effectively, respectively, but the limit $x + y$ is different from the possible limit of the sequence $(\varphi_e(t))$ if it converges $k$ effectively.

To satisfy all requirements simultaneously, a finite injury priority construction suffices. □

¿From Lemma 3.2 the following corollary follows immediately.

**Corollary 3.3** *The class* **BEC** *of bounded effectively computable real numbers is closed under arithmetical operations and hence is a field.*

Now we are going to show that the class **BEC** is also closed under the total computable real functions. To this end, we prove a technical lemma at first.

**Lemma 3.4** *If $(x_s)$ is a computable sequence of real numbers which converges $k$-effectively to $x$, then $x$ is $k$-e.c.*

**Proof.** By definition, a sequence $(x_s)$ of real numbers is computable means that there is a computable double sequence $(x_{s,t})$ of rational numbers such that, for all $s, t$, $|x_s - x_{s,t}| \leq 2^{-t}$. Suppose that the computable sequence $(x_s)$ of reals converges $k$-effectively to $x$. That is, for any $n$, there are at most $k$ non-overlapping index-pairs $s, t \geq n$ such that $|x_s - x_t| \geq 2^{-n}$. Let $y_s := x_{s+1,s+2}$ for all $s$. Then $(y_s)$ is a computable sequence of rational numbers which converges to $x$. If $s, t \geq n$ are two indices such that $|y_s - y_t| \geq 2^{-n}$, then $|x_{s+1} - x_{t+1}| \geq 2^{-(n+1)}$ because of the following inequality

$$\begin{aligned} |y_s - y_t| &\leq |x_{s+1,s+2} - x_{s+1}| + |x_{s+1} - x_{t+1}| + |x_{t+1} - x_{t+1,t+2}| \\ &\leq 2^{-(n+1)} + |x_{s+1} - x_{t+1}|. \end{aligned}$$

This implies that the sequence $(y_s)$ converges also $k$-effectively and hence $x$ is $k$-e.c. □

**Theorem 3.5** *The class* **BEC** *of bounded effectively computable real numbers is closed under the computable total real functions.*

**Proof.** Suppose that $x \in [0, 1]$ is a bounded effectively computable real number and $f : [0, 1] \to \mathbb{R}$ be a total computable real function. The function $f$ has a computable modulus function $e : \mathbb{N} \to \mathbb{N}$ (see e.g., [4]) such that

(6) $|x - y| \leq 2^{-e(n)} \Longrightarrow |f(x) - f(y)| \leq 2^{-n}$.

for any $n$ and $x, y \in [0, 1]$. Let $(x_s)$ be a computable sequence of rational numbers which converges $k$-effectively to $x$ for some constant $k$. By the sequential computability of $f$, the sequence $(f(x_s))$ is a computable sequence of real numbers which converges to $f(x)$. From (6) it is not difficult to see that the sequence $(f(x_s))$ converges $k$-effectively too. By Lemma 3.4, $f(x)$ is a $k$-e.c. real and hence **BEC** is closed under total computable real functions. □

We have seen that the class **BEC** has very nice mathematical as well as very interesting computability theoretical properties. Now we will show that this is actually a proper subset of weakly computable real numbers discussed in [1]. In addition, the $k$-e.c. real numbers firstly lead also to an interesting hierarchy of c.e. reals. According to [1], a real $x$ is weakly computable or d-c.e. (difference of c.e.) if there are c.e. reals $y, z$ such that $x = y - z$, where c.e. reals are the limits of increasing computable sequences of rational numbers. It is shown in [1] that $x$ is d-c.e. iff there is a computable sequence $(x_s)$ of rational numbers which converges to $x$ weakly effectively in the sense that the sum $\sum_{n\in\mathbb{N}} |x_n - x_{n+1}|$ is finite and the class **WC** of all d-c.e. reals is actually the arithmetical closure of c.e. reals. The next theorem implies that the classes **BEC** is different from **WC**.

**Theorem 3.6** *There is a c.e. real number $x$ which is not $k$-e.c. for any natural number $k$. Therefore we have* $\mathbf{SC} \nsubseteq \mathbf{BEC}$.

**Proof.** We construct an increasing computable sequence $(x_s)$ of rational numbers converging to a real $x$ which satisfies, for all $e = \langle i, j\rangle$, the requirements

$$R_e : (\varphi_i(s))_s \text{ converges } j\text{-effectively to } y_e \Longrightarrow x \neq y_e.$$

To satisfy a single requirement $R_e$ for $e = \langle i, j\rangle$, we choose a rational interval $I_{e-1}$ which is divided into $2j+1$ equidistant subintervals $J_t$ for $t = 1, 2, \cdots, 2j+1$ of the length $2^{-n_e}$ for some natural number $n_e$. As default define $I_e := J_2$ and let $x_0$ to be the middle point of the interval $I_e$. As long as the sequence $(\varphi_i(s))_s$ does not enter the interval $I_e$, we define $x_s$ equal to $x_0$. Otherwise, if the sequence $(\varphi_i(s))_s$ enters the interval $I_e$ after the stage $n_e$, then we redefine $I_e := J_4$ and define the new $x_s$ as the middle point of this new interval $I_e$. That is, $x_s$ is increased by $2^{-n_e+1}$. If, at a late stage, the sequence $(\varphi_i(s))_s$ enters the new interval $I_e$, then we redefine $I_e := J_6, J_8 \cdots$, and so on until $I_e : J_{2j}$. Each time we define $x_s$ as the middle point of the actual interval $I_e$. This guarantees that the limit of the sequence $(x_s)$ is different from $\lim_{s\to\infty} \varphi_i(s)$ if the sequence $(\varphi_i(s))$ converges $j$-effectively.

By the standard priority construction we can achieve an increasing computable sequence $(x_s)$ whose limit satisfies all requirements $R_e$ and hence it is not $k$-e.c. for any constant $k$. □

**Corollary 3.7** *The class* **BEC** *is a proper subset of the class of weakly computable real numbers, i.e.,* $\mathbf{BEC} \subsetneq \mathbf{WC}$.

**Proof.** It suffices to prove the inclusion $\mathbf{BEC} \subseteq \mathbf{WC}$. Let $x$ be a $k$-e.c. real and let $(x_s)$ be a computable sequence of rational numbers which converges $k$-effectively to $x$. We want to show that the sum $\sum_{s=0}^{\infty} |x_s - x_{s+1}|$ is finite.

By the $k$-effective convergence of $(x_s)$, there are at most $k$ indices $s \in [n-k, n]$ such that $|x_s - x_{s+1}| > 2^{-(n-k)}$, for any natural number $n$. This means that, there is at least one $s \in [n-k, n]$ such that $|x_s - x_{s+1}| \leq 2^{-(n-k)}$. In general, for any $i \leq n-k$, there are at least $i+1$ indices $s \in [n-k-i, n]$ such that $|x_s - x_{s+1}| \leq 2^{-(n-k-i)}$. This implies that, there are $n-k$ different indices $s_0, s_1, \cdots, s_{n-k} \leq n$ such that $|x_{s_i} - x_{s_i+1}| \leq 2^{-i}$ for all $i \leq n-k$. Choose a constant $c$ such that $|x_s - x_{s+1}| \leq c$ for all $s \in \mathbb{N}$. Then we have

$$\sum_{s=0}^{n}|x_s - x_{s+1}| \leq \sum_{i=0}^{n-k}|x_{s_i} - s_{s_i+1}| + ck \leq \sum_{i=0}^{n-k}2^{-i} + ck \leq 2 + ck.$$

Therefore $\sum_{s=0}^{\infty}|x_s - x_{s+1}| \leq 2 + ck$, i.e., $(x_s)$ converges weakly effectively to $x$ and hence $x$ is weakly computable. Thus, $\mathbf{BEC} \subseteq \mathbf{WC}$. □

In the computable analysis, c.e. reals are regarded as the first weaken version of the computable real numbers. Theorem 3.6 shows that not every c.e. real is bounded effectively computable. Of course, this does not mean that the bounded effective computability is weaker than the computable enumerability of real numbers. Actually, as it is shown in [8], there is a bounded effectively computable real which is not c.e. That is, the classes **SC** and **BEC** are incomparable. However, if we consider the c.e. real numbers which fall into the classes $k$-**EC** for different constant $k$, then we achieve a Ershov-style hierarchy of c.e. reals as shown in the next theorem.

**Theorem 3.8** *For any constant $k$, there is a c.e. real which is $(k+1)$-e.c. but not $k$-e.c., and there is a c.e. set which is not $k$-e.c. for any constant $k$.*

**Proof.** We prove only the first assertion. The second part can be proved in an analogous way.

For any constant $k$, we construct a computable sequence $(x_s)$ of rational numbers which converges $(k+1)$-effectively to a c.e. real $x$ and $x$ is not $k$-e.c. The limit $x$ has to satisfy the following requirements

$$R_e: \quad (\varphi_e(s))_s \text{ converges } k\text{-effectively to } y_e \Longrightarrow x \neq y_e.$$

The strategy to satisfy a single requirement $R_e$ is as follows. Choose a rational interval $I$ and divide it into $2k+3$ equidistance subintervals $I_i$ for $i = 0, 1, 2k$. Suppose that the lengths of $I_i$ are $2^{-n_0}$. Choose $I_1$ as default and define $x_s$ as the middle point of $I_1$. If the sequence $(\varphi_e(s))$ enters an interval $I_{2i+1}$ for some $s \geq n_0$, then we move to the interval $I_3$ and define $x_s$ as the middle of this new interval. If it is necessary, we can move to the intervals $I_5, I_7, \cdots$ and so on. However, at most $k+1$ moves are allowed and this suffices to guarantee that the limit $x := \lim_s x_s$ is different from the possible limit $y_e := \lim_s \varphi_e(s)$ if it converges $k$-effectively.

By a standard finite injury priority construction we can construct a computable sequence $(x_s)$ which converges $(k+1)$-effectively to $x$ and $x$ satisfies all requirements. □

# 4 The Turing Degrees

In this section we investigate the Turing degrees which contain bounded effectively computable real numbers.

**Theorem 4.1** *There is a 1-e.c. real which is not of an $\omega$-c.e. Turing degree.*

**Proof.** We will construct a computable sequence $(A_s)$ of finite sets such that the computable sequence $(x_{A_s})$ of rational numbers converges 1-effectively to $x_A$ whose binary expansion $A$ is not Turing equivalent to any $\omega$-c.e. set.

Let $(V_{e,s})$ be an uniformly computable enumeration of all sequences of finite sets. If $B$ is an $\omega$-c.e. set, then there is an index $e$ and a computable total function $\varphi_i$ such that the sequence $(V_{e,s})_s$ is a $\varphi_i$-enumeration of $B$, i.e.,

$$|\{s : V_{e,s}(n) \neq V_{e,s+1}(n)\}| \leq \varphi_i(n)$$

for all $n$. In order to guarantee that the set $A$ is not Turing equivalent to any $\omega$-c.e. set, it suffices to satisfy the following requirements

$$R_{V,\varphi,\Gamma,\Delta}: \quad (V_s) \text{ is a } \varphi\text{-enumeration of } V \Longrightarrow A \neq \Gamma^V \text{ or } V \neq \Delta^A$$

for all computable sequence $(V_s)$ converging to $V$, all computable functions $\varphi$ and all computable functionals $\Gamma$ and $\Delta$. Let $(R_e)$ be a computable enumeration of all requirements $R_{V,\varphi,\Gamma,\Delta}$. We assign $R_i$ a higher priority than $R_j$ if $i < j$. In the following, the use function of the computations $\Gamma^V$ and $\Delta^A$ are denoted by the corresponding small cases $\gamma$ and $\delta$, respectively.

Let's explain the idea how to satisfy a single requirement $R_e := R_{V,\varphi,\Gamma,\Delta}$. Choose a natural number $n_e$ which is not yet in $A$. Our goal is to guarantee that the following condition holds only for at most finite many $s$

(7) $A_s(n_e) = \Gamma^V(n_e)[s]$ & $V \upharpoonright \gamma(n_e)[s] = \Delta^A \upharpoonright \gamma(n_e)[s]$.

To this end, we firstly wait for a stage $s_1 \geq n_e$ such that condition (7) holds for $s := s_1$. If such kind of stage $s_1$ does not exist, then we are done because $n_e$ witnesses that the requirement is satisfied. Otherwise, at stage $s_1 + 1$, we destroy condition (7) by putting $n_e$ into $A$ (i.e., define $A_{s_1+1} := A_{s_1} \cup \{n_e\}$). Notice that we have in increment $x_{A_{s+1}} = x_{A_s} + 2^{-n_e}$ in this case.

Then we wait for a new stage $s_2 > s_1$ such that condition (7) holds for $s := s_2$. Similarly we are done if no such kind $s_2$ exists. Otherwise, we destroy the condition at stage $s_2 + 1$ by deleting $n_e$ from $A_{s_2}$. Because the sequence $(x_s)$ has already a jump of size $2^{-n_e}$ at stage $s_1+1 \geq n_e$, we cannot simply delete $n_e$ from $A$. Otherwise the sequence does not converges 1-effectively. In order to delete $n_e$ with a smaller size of jump, we put all natural numbers $n \in (n_e, s_2]$ in to $A_{s_2+1}$. That is, we define

$$A_{s_2+1} := (A_{s_2} \backslash \{n_e\}) \cup \{n_e + 1, n_2 + 2, \cdots, s_2\}.$$

In this way, we have only a relatively small decrement $x_{A_{s_2+1}} x_{A_{s_2}} - 2^{-s_2}$.

Finally, we wait again for a new stage $s_3 > s_2$ such that condition (7) holds for $s := s_3$. If such $s_3$ exists, then we destroy condition (7) again by put $n_e$ newly into $A$, and at the same time all natural numbers $n \in (n_e, s_2]$ are deleted from $A_{s_2}$. In other words, we have

$$A_{s_3+1} := (A_{s_3} \backslash \{n_e + 1, n_2 + 2, \cdots, s_2\}) \cup \{n_e\}.$$

In this case, we have an increment $x_{A_{s_3+1}} = x_{A_{s_3}} + 2^{-s_2}$. Especially, the set $A_{s_3}$ is recovered to that of stage $s_1$, that is, $A_{s_3} = A_{s_1}$. Accordingly we have $\gamma(n_e)[s_3] = \gamma(n_e)[s_1]$ and $V \upharpoonright \gamma(n_e)[s_3] = V \upharpoonright \gamma(n_e)[s_1]$. This implies that the initial segment $V \upharpoonright \gamma(n_e)$ is changed between the stages $s_1$ and $s_3$. If $\varphi$ is a total computable function and $(V_s)$ is a $\varphi$-enumeration of $V$, then this kind of changes can happen at most $\sum_{i=0}^{\gamma(n_e)} \varphi(i)$ times. Thus we can repeat the above process at most $\sum_{i=0}^{\gamma(n_e)} \varphi(i)$

times to satisfy the requirement $R_e$ while the sequence $(x_{A_s})$ converges still 1-effectively.

To satisfy all requirements simultaneously, we need a finite injury priority construction. Two important points should be mentioned here. Firstly, in order to carry out the strategy for $R_e$, the initial segment $A \upharpoonright \delta\gamma(n_e)$ should be preserved. Namely, the membership $A(n)$ cannot be changed by the strategy of any requirements of a priority lower than $R_e$ for any elements less than $\delta\gamma(n_e)$. This guarantees that at the stage $s_3$, the initial segment $V \upharpoonright \gamma(n_e)$ can be recovered to that of the stage $s_1$. Secondly, for the 1-effective convergence of $(x_{A_s})$, the strategy for $R_i$ should be initialized — i.e., choose a new witness $n_i$ and wait newly for properly $s_1$, $s_2$, $s_3$ and so on, if a requirement $R_e$ of higher priority takes some actions. The further details are omitted here. □

In the next we show a hierarchy theorem of $k$-c.e. Turing degrees which extended the result of Theorem 3.8. To simplify the notation we identify a real number $x$ with its characteristic binary sequence. By definition, two real numbers $x$ and $y$ are Turing equivalent means that there are indices $i$ and $j$ such that $x(n) = \Phi_i^y(n)$ and $y(n) = \Phi_j^x(n)$ for all $n$, where $(\Phi_e)$ is an effective enumeration of computable partial functionals. This implies immediately that there are (maybe different) $i$ and $j$ such that $x \upharpoonright n = \Phi_i^y(n)$ & $y \upharpoonright \Phi_j^x(n)$ for all $n$. In this case, we say that $x$ and $y$ are $(i,j)$-*Turing equivalent* (denote by $x \equiv_T^{(i,j)} y$). In other words, we have

$$x \equiv_T^{(i,j)} y :\iff (\forall n)\left(x \upharpoonright n = \Phi_i^y(n) \;\&\; y \upharpoonright \Phi_j^x(n)\right)$$

For the $(i,j)$-Turing equivalence we have the following important lemma.

**Lemma 4.2 (Rettinger and Zheng [6])** *For any rational interval $I_0$ and any natural numbers $i, j, t$ there are two open rational intervals $I \subseteq I_0$ and $J$ such that*

$$(8)\quad (\forall x, y)\left(x \equiv_T^{(i,j)} y \Longrightarrow (x \in I \Longrightarrow y \in J) \;\;\&\;\; (y \in J \Longrightarrow x \in I_0)\right).$$

We say that an intervals $I$ is $(i,j)$-*reducible* to another interval $J$ (denoted by $I \preceq^{(i,j)} J$) if they satisfy the following condition

$$(\forall x, y)\left(x \in I \;\&\; x \equiv_T^{(i,j)} y \Longrightarrow y \in J\right).$$

By Lemma 4.2, there are $I \subseteq I_0$ and $J$ such that $I \preceq^{(i,j)} J$ for any given interval $I_0$. If all elements of $I_0$ are not $(i,j)$-Turing equivalent to some element, then this holds trivially. Actually, the Lemma 4.2 holds even in an more effective sense. Namely, if there exists $x \in I_0$ which is $(i,j)$-Turing equivalent to some $y$, then the intervals $I$ and $J$ which satisfy condition (8) can be effectively found. This fact will be used in the proof of the following theorem.

**Theorem 4.3** *For any constant $k$, there is a $(k+1)$-e.c. real number which is not Turing equivalent to any $k$-e.c. reals.*

**Proof.** We construct a computable sequence $(x_s)$ of rational numbers which converges $(k+1)$-effectively to a non-$k$-e.c. real $x$. The limit $x$ has to satisfy all the following requirements

$$R_{\langle i,j,k\rangle}: \quad (\varphi_k(s)) \text{ converges } k\text{-effectively to } y_k \Longrightarrow x \not\equiv_T^{(i,j)} y_k.$$

To satisfy a single requirement $R_e$, we are going to find a rational interval $I_e$ such that all $x \in I_e$ satisfy $R_e$. This interval is called a witness interval of $R_e$. We can choose arbitrarily an interval $I$ and three disjoint subintervals $I^1, I^2, I^3 \subseteq I$. If one of these subintervals satisfies the following condition: it does not contain an element which is $(i,j)$-Turing equivalent to some real $y$, then we can simply let this subinterval as witness interval and we are done. Otherwise, by Lemma 4.2, we can effectively find three intervals $J^1, J^2$ and $J^3$ such that $I^t \preceq^{(i,j)} J^t$ for $t = 1, 2, 3$. W.l.o.g. we can assume that the intervals $J^1, J^2$ and $J^3$ are disjoint (Otherwise we can consider the subintervals of $I^t$, if it is necessary). Then, at least two of them are separated by a positive distance. Suppose that $J^1$ and $J^2$ are separated by, say, $2^{-b}$ for a natural number $b$. Accordingly, the maximal distance between the intervals $I^1$ and $I^2$ is bounded by $2^{-a}$ for a natural number $a$. Now we can choose $I^1$ as candidate of witness interval. If $\varphi_k(s)$ enters $J_1$ for $s \geq b$, then we change the witness interval to be $I^2$. If $\varphi_k(s)$ enters $J^2$ later on, then choose $I^1$ as witness interval again, and so on. At most $k$ changes suffice to satisfy the requirement $R_e$.

We need only a standard finite injury priority construction to construct the computable sequence $(x_s)$ and $x_s$ is chosen from the actually smallest witness intervals defined at the stage $s$. The details are omitted here. □

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