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A_α-Spectrum of a Firefly Graph

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Abstract

Let G be a connected graph of order n, $A(G)$ is the adjacency matrix of G and $D(G)$ is the diagonal matrix of the row-sums of $A(G)$. In 2017, Nikiforov [8] defined the convex linear combinations $A_\alpha(G)$ of $A(G)$ and $D(G)$ by

$$A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G), \quad 0 \leq \alpha \leq 1.$$

In this paper, we obtain a partial factorization of the A_α-characteristic polynomial of the firefly graph which explicitly gives some eigenvalues of the graph.

Keywords: Eigenvalues, A_α matrix, firefly Graphs

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1 Introduction

Let G be a simple graph of order n with vertex set $V(G)$ and edge set $E(G)$, such that $|E(G)| = m$. We denote the complete graph with n vertices by K_n. The set of *neighbours* of a vertex v in G is denoted by $N_G(v)$ and $N_G[v] = N_G(v) \cup \{v\}$. The *degree* of a vertex v of G, $d(v)$, is defined by $|N_G(v)|$. Two distinct vertices u and v are called *true twins* in G if $N_G[u] = N_G[v]$ and are called *false twins* if $N_G(u) = N_G(v)$ and u is not adjacent to v, see [7]. The *signless Laplacian* matrix of G is defined by $Q(G) = A(G) + D(G)$, where $D(G)$ is the diagonal matrix of the degrees and $A(G) = [a_{ij}]$ is the *adjacency* matrix of G, where $a_{ij} = 1$ if v_i is adjacent to v_j and $a_{ij} = 0$ otherwise. Recently Nikiforov [8] defined for any real $\alpha \in [0, 1]$, the convex linear combinations $A_\alpha(G)$ of $A(G)$ and $D(G)$ by

$$A_\alpha(G) = \alpha D(G) + (1 - \alpha)A(G).$$

It is easy to see that $A(G) = A_0(G)$, $D(G) = A_1(G)$ and $Q(G) = 2A_{\frac{1}{2}}(G)$. The A_α-*characteristic polynomial* of G is defined by $P_{A_\alpha(G)}(x) = \det(xI - A_\alpha(G))$ and its roots are called the eigenvalues of $A_\alpha(G)$. As usual, we shall index the eigenvalues of $A_\alpha(G)$ in non-increasing order and denote them as $\lambda_1(A_\alpha(G)) \geq \lambda_2(A_\alpha(G)) \geq \cdots \geq \lambda_n(A_\alpha(G))$. The spectrum of $A_\alpha(G)$ is defined as the multiset of eigenvalues with their algebraic multiplicity and denoted by $Spec(A_\alpha(G))$. To simplify we use the A_α and $\lambda_i(A_\alpha)$ notation when there is no risk of ambiguity.

As defined by Aouchiche *et al.* [1], a *firefly graph* $F_{s,r,t}$ is a graph on $n = 2r + s + 2t + 1$ vertices that consists of s pendant edges, r triangles, and t pendant paths of length 2, all of them sharing a common vertex. Let $\mathcal{F}_n$ be the set of all firefly graphs with n vertices. Note that $\mathcal{F}_n$ contains the star $S_n \simeq F_{s,0,0}$, the stretched stars ($\simeq F_{s,0,t}$), the friendship graphs ($\simeq F_{0,r,0}$) and the butterfly graphs ($\simeq F_{s,r,0}$). The relevance of studying this family relates to the fact that many extremal graphs for functions depending on the eigenvalues of graph matrices belong to $\mathcal{F}_n$. For unicyclic graphs, Hong [4] determined the unique graph, $F_{n-3,1,0}$, with maximum largest eigenvalue of $A(G)$. Fan *et al.* [2] determined the unique graph, $F_{n-3,1,0}$, with minimum least eigenvalue of $A(G)$ among all unicyclic graphs of order n when $n \geq 12$. Petrović *et al.* [9] determined the unique graph, $F_{n-3,1,0}$, with minimum least eigenvalue of $A(G)$ among the cacti with n vertices ($n \geq 12$) and k cycles, where $0 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$. Moreover, Li *et al* [6] characterized graphs, $F_{n-\lfloor \frac{n-1}{2} \rfloor -1, \lfloor \frac{n-1}{2} \rfloor, 0}$, with the largest signless Laplacian spectral radius among all the cacti with n vertices.

Here, we address the problem of finding all the eigenvalues of $A_\alpha(F_{s,r,t})$, which fills a literature gap and generalizes the eigenvalues of the adjacency and signless Laplacian matrix of a firefly graph for a convenient α.

The paper is organized such that preliminary results are presented in the next section and the main proofs are in Section 3.

2 Preliminaries results

First, we present the equitable partition theorem of a matrix which can be found at Horn and Johnson [5] and the Propositions 2.2 and 2.3 that show the eigenvalues of $A_\alpha(K_n)$, $A_\alpha(S_n)$ and upper bounds for $\lambda_1(A_\alpha)$, respectively.

Proposition 2.1 **([5])** *Let M be a matrix of order n defined by*

$$M = \begin{bmatrix} M_{1,1} & M_{1,2} & \cdots & M_{1,k} \\ M_{2,1} & M_{2,2} & \cdots & M_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ M_{k,1} & M_{k,2} & \cdots & M_{k,k} \end{bmatrix},$$

where $M_{i,j}$, $1 \leq i, j \leq k$, is a submatrix of order $n_i \times n_j$ such that the sum of each of its rows is equal to $c_{i,j}$. If $\overline{M} = [c_{i,j}]_{k\times k}$, then the eigenvalues of $\overline{M}$ are also eigenvalues of M.

Proposition 2.2 **([8])** *For $\alpha \in [0, 1]$, we have*

(i) $Spec(A_\alpha(K_n)) = \{n - 1, (\alpha n - 1)^{[n-1]}\}$;

(ii) $Spec(A_\alpha(S_n)) = \left\{\frac{1}{2}(\alpha n + \beta), \alpha^{[n-2]}, \frac{1}{2}(\alpha n - \beta)\right\}$, where $\beta = \sqrt{\alpha^2 n^2 + 4(n-1)(1-2\alpha)}$.

Proposition 2.3 **([8])** *If G is a graph of order n and has m edges, then*

$$\lambda_1(A_\alpha(G)) \geq \sqrt{\frac{1}{n}\sum_{u\in V(G)} d^2(u)} \quad \text{and} \quad \lambda_1(A_\alpha(G)) \geq \frac{2m}{n}.$$

Equality holds in the second inequality if and only if G is regular. If $\alpha > 0$, equality holds in the first inequality if and only if G is regular.

Proposition 2.4 states that the existence of twin vertices in G implies the presence of certain eigenvalues, λ, in the spectrum of $A_\alpha(G)$ and also provides a lower bound for the multiplicity, $m(\lambda)$, of such eigenvalues.

Proposition 2.4 *Let G be a graph on $n \geq 2$ vertices, with v_i and v_{j_p}, $1 \leq p \leq r$, twin vertices in G.*

(i) If $v_i \not\sim v_{j_p}$ then $\alpha d(v_i) \in Spec(A_\alpha(G))$ and $m(\alpha d(v_i)) \geq r$.

(ii) If $v_i \sim v_{j_p}$ then $\alpha(d(v_i) + 1) - 1 \in Spec(A_\alpha(G))$ and $m(\alpha(d(v_i) + 1) - 1) \geq r$.

Proof. For a given $p \in \{1, 2, \ldots, r\}$, let v_i and v_{j_p} be twin vertices in G. Consider the vector $\mathbf{x}^{(p)} \in \mathbb{R}^n$ with entries

$$[\mathbf{x}^{(p)}]_k = \begin{cases} 1, & \text{if } k = i; \\ -1, & \text{if } k = j_p; \\ 0, & \text{otherwise.} \end{cases}$$

Since $A_\alpha(G) = A_\alpha$ we have, for each $\ell \in \{1, 2, , \ldots, n\}$,

$$[A_\alpha \mathbf{x}^{(p)}]_\ell = \sum_{k=1}^{n} [A_\alpha]_{\ell k}[\mathbf{x}^{(p)}]_k = [A_\alpha]_{\ell i} - [A_\alpha]_{\ell j_p}. \tag{1}$$

Now, consider the following three cases:

Case 1 $\ell = i$.

In this case,

$$[A_\alpha \mathbf{x}^{(p)}]_i = [A_\alpha]_{ii} - [A_\alpha]_{ij_p} = \alpha d(v_i) - [A_\alpha]_{ij_p},$$

so,

$$[A_\alpha \mathbf{x}^{(p)}]_i = \begin{cases} \alpha(d(v_i) + 1) - 1, & \textit{se } v_i \sim v_{j_p}; \\ \alpha d(v_i), & \textit{se } v_i \nsim v_{j_p}. \end{cases}$$

Case 2 $\ell = j_p$.

In this case,

$$[A_\alpha \mathbf{x}^{(p)}]_{j_p} = [A_\alpha]_{j_p i} - [A_\alpha]_{j_p j_p} = [A_\alpha]_{j_p i} - \alpha d(v_{j_p}),$$

and,

$$[A_\alpha \mathbf{x}^{(p)}]_{j_p} = \begin{cases} -\alpha(d(v_{j_p}) + 1) + 1, & \textit{if } v_i \sim v_{j_p}; \\ -\alpha d(v_{j_p}), & \textit{if } v_i \nsim v_{j_p}. \end{cases}$$

Case 3 $\ell \notin \{i, j_p\}$.

Since v_i and v_{j_p} are twin vertices, we have $[A_\alpha]_{\ell i} = [A_\alpha]_{\ell j_p}$. Then, for equation (1) $[A_\alpha \mathbf{x}^{(p)}]_\ell = 0$.

Therefore, of the three previous cases we have

$$A_\alpha \mathbf{x}^{(p)} = \begin{cases} (\alpha d(v_i) + \alpha - 1)\mathbf{x}^p, & \text{if } v_i \sim v_{j_p}; \\ \alpha d(v_i)\mathbf{x}^{(p)}, & \text{if } v_i \nsim v_{j_p}. \end{cases}$$

It is easy to see that $\{\mathbf{x}^{(p)}\}_{p=1}^{r}$ is linearly independent. Therefore $m(\lambda) \geq r$, for $\lambda \in \{\alpha d(v_i), \alpha d(v_i) + \alpha - 1\}$.

□

3 Main results

In this section, we present the results involving the eigenvalues of graphs in the family $\mathcal{F}_n$. There is a convenient vertex labelling of a graph $F_{s,r,t} \in \mathcal{F}_n$ such that $A_\alpha = A_\alpha(F_{s,r,t})$ can be written as

$$A_\alpha = \begin{bmatrix} \alpha(t+s+2r) & (1-\alpha)\mathbf{J}_{1\times s} & (1-\alpha)\mathbf{J}_{1\times 2r} & (1-\alpha)\mathbf{J}_{1\times t} & \mathbf{0}_{1\times t} \\ (1-\alpha)\mathbf{J}_{s\times 1} & \alpha\mathbf{I}_s & \mathbf{0}_{s\times 2r} & \mathbf{0}_{s\times t} & \mathbf{0}_{s\times t} \\ (1-\alpha)\mathbf{J}_{2r\times 1} & \mathbf{0}_{2r\times s} & B_{2r} & \mathbf{0}_{2r\times t} & \mathbf{0}_{2r\times t} \\ (1-\alpha)\mathbf{J}_{t\times 1} & \mathbf{0}_{t\times s} & \mathbf{0}_{t\times 2r} & 2\alpha\mathbf{I}_t & (1-\alpha)\mathbf{I}_t \\ \mathbf{0}_{t\times 1} & \mathbf{0}_{t\times s} & \mathbf{0}_{t\times 2r} & (1-\alpha)\mathbf{I}_t & \alpha\mathbf{I}_t \end{bmatrix}, \qquad (2)$$

where

$$B_{2r} = \begin{bmatrix} 2\alpha\mathbf{I}_r & (1-\alpha)\mathbf{I}_r \\ (1-\alpha)\mathbf{I}_r & 2\alpha\mathbf{I}_r \end{bmatrix}. \qquad (3)$$

The Figure 1 displays the firefly graph $F_{3,2,2}$ with the adopted labelling.

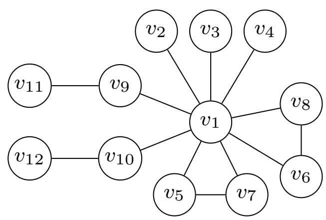

Fig. 1. Firefly graph $F_{3,2,2}$

Remark 3.1 Let $G \simeq F_{s,r,t}$.The graph G has exactly one vertex of degree equal to $2r+s+t$, $2r+t$ vertices of degree equal to 2 and $s+t$ vertices of degree equal to 1. For $\alpha = 1$ the eigenvalues of $A_1(G) = D(G)$ are $d(v)$, $v \in V(G)$, and for $\alpha = 0$ the eigenvalues of $A_0(G) = A(G)$ can be see in [3].

In Proposition 3.2, we were able to prove that the occurrence of some eigenvalues of a firefly graph depends on the existence of certain induced subgraphs.

Proposition 3.2 *Given the nonnegative integers r, s and t, let $G \simeq F_{s,r,t}$ and $\alpha \in (0,1)$. If $t \geq 2$ then $\theta_1 = \frac{3\alpha+\sqrt{5\alpha^2-8\alpha+4}}{2}$ and $\theta_2 = \frac{3\alpha-\sqrt{5\alpha^2-8\alpha+4}}{2}$ are eigenvalues of G, both with multiplicity at least $t-1$. Moreover, if $r \geq 2$ then $\alpha+1$ is an eigenvalue of G with multiplicity at least $r-1$.*

Proof. Given $\lambda \in \{\theta_1, \theta_2\}$ suppose $t \geq 2$ and, for each $i \in \{1, 2, \ldots, t-1\}$, consider the vector $\mathbf{x}^{(i)}$ with $2r+s+2t+1$ entries, where

$$[\mathbf{x}^{(i)}]_j = \begin{cases} \dfrac{\lambda-\alpha}{1-\alpha}, & \text{if } j = s+2r+2; \\ -\dfrac{\lambda-\alpha}{1-\alpha}, & \text{if } j = s+2r+2+i; \\ 1, & \text{if } j = s+2r+t+2; \\ -1, & \text{if } j = s+2r+t+2+i; \\ 0, & \text{otherwise.} \end{cases}$$

In this way, the entries of the vector $A_\alpha(G)\mathbf{x}^{(i)} - \lambda\mathbf{x}^{(i)}$ are given by

$$\left[A_\alpha(G)\mathbf{x}^{(i)} - \lambda\mathbf{x}^{(i)}\right]_j = \begin{cases} \dfrac{\lambda^2-3\alpha\lambda+\alpha^2+2\alpha-1}{\alpha-1}, & \text{if } j = s+2r+2; \\ -\dfrac{\lambda^2-3\alpha\lambda+\alpha^2+2\alpha-1}{\alpha-1}, & \text{if } j = s+2r+2+i; \\ 0, & \text{otherwise.} \end{cases}$$

Since λ is a root of the polynomial $x^2-3\alpha x+\alpha^2+2\alpha-1$, it follows that $\mathbf{x}^{(i)}$ is an associated eigenvector to λ. Since $\{\mathbf{x}^{(i)}\}_{i=1}^{t-1}$ is a linearly independent set, the multiplicity of λ is at least $t-1$.

Now, suppose $r \geq 2$ and denote by e_k the vector with $s+2r+2t+1$ coordinates whose k-th entry is equal to 1 and the others entries are zero. For each j, $s+2 \leq j \leq s+r$, it is easy to show that the vector $z_j = e_j - e_{j+1} + e_{j+r} - e_{j+r+1}$ is an eigenvector of $A_\alpha(G)$ associated with the eigenvalue $\alpha+1$. So, $\alpha+1$ is an eigenvalue of $A_\alpha(G)$ with multiplicity at least $r-1$.

□

Remark 3.3 As described in the Proposition 3.2, we use the notations θ_1 and θ_2 to represent the roots of the polynomial $x^2-3\alpha x+\alpha^2+2\alpha-1$.

For $s \geq 1$, $G \simeq F_{s,0,0} \simeq S_s$ and the eigenvalues of $A_\alpha(S_s)$ can be seen in the Proposition 2.2.

Proposition 3.4 *Let $G \simeq F_{0,r,0}$. If $r \geq 1$ and $\alpha \in (0,1)$ then*

$$P_{A_\alpha(G)}(x) = (x-\alpha-1)^{r-1}(x-3\alpha+1)^r(x^2-(2\alpha r+\alpha+1)x+(6\alpha-2)r).$$

Moreover, if x_1 and x_2 denote the roots of the quadratic factor of $P_{A_\alpha(G)}(x)$ then

$$\begin{cases} x_2 \le 3\alpha - 1 < \alpha + 1 < x_1, \text{ if } 0 < \alpha \le \frac{1}{3}; \\ \\ 3\alpha - 1 \le x_2 < \alpha + 1 < x_1, \text{ if } \frac{1}{3} < \alpha < 1. \end{cases}$$

Proof. For $G \simeq F_{0,r,0}$, we have

$$A_\alpha(G) = \begin{bmatrix} 2\alpha r & (1-\alpha)\mathbf{J}_{1\times 2r} \\ (1-\alpha)\mathbf{J}_{2r\times 1} & B_{2r} \end{bmatrix},$$

where B_{2r} is the matrix given in (3).

Applying the Propositions 2.4 for each triangles of G, we obtain that $3\alpha - 1$ is an eigenvalue of $A_\alpha(G)$ with multiplicity at least r and from Proposition 3.2, for $r \ge 2$, $\alpha + 1$ is an eigenvalue of $A_\alpha(G)$ with multiplicity at least $r - 1$. If $r = 1$, the result follow by Proposition 2.2.

From Proposition 2.1, the spectrum of matrix

$$M = \begin{bmatrix} 2\alpha r & 2r(1-\alpha) \\ 1-\alpha & 1+\alpha \end{bmatrix},$$

whose characteristic polynomial is $g(x) = x^2 - (2\alpha r + \alpha + 1)x + (6\alpha - 2)r$, is contained in the spectrum of $A_\alpha(G)$. Since $3\alpha - 1$ and $\alpha + 1$ are not roots of $g(x)$, we have $P_{A_\alpha(G)}(x) = (x - \alpha - 1)^{r-1}(x - 3\alpha + 1)^r g(x)$.

As G has order $n = 2r+1$ and size $m = 3r$, we have $\overline{d} = \dfrac{6r}{2r+1} = 3 - \dfrac{3}{2r+1} \ge 2$. From Proposition 2.3, $\lambda_1(A_\alpha) \ge 2$. Let x_1 and x_2 be the roots of $g(x)$, with $x_2 < x_1$. Since $3\alpha - 1 < \alpha + 1 < 2$, for $\alpha \in (0, 1)$, we conclude that $x_1 = \lambda_1(A_\alpha)$. Now, we have $g(\alpha + 1) = -2r(\alpha - 1)^2 < 0$ for all $r \ge 1$ and $\alpha \in (0, 1)$. On the other hand, $g(3\alpha - 1) = -2(r - 1)(\alpha - 1)(3\alpha - 1)$, thus $g(3\alpha - 1) \le 0$ if $\alpha \in (0, \frac{1}{3}]$ and $g(3\alpha - 1) > 0$ if $\alpha \in (\frac{1}{3}, 1)$. So it easy to see that

$$\begin{cases} x_2 \le 3\alpha - 1 < \alpha + 1 < x_1, \text{ if } 0 < \alpha \le \frac{1}{3}; \\ \\ 3\alpha - 1 \le x_2 < \alpha + 1 < x_1, \text{ if } \frac{1}{3} < \alpha < 1, \end{cases}$$

and the equalities only hold if $r = 1$ or $\alpha = \frac{1}{3}$.

□

Proposition 3.5 *Let $G \simeq F_{0,0,t}$. If $t \ge 1$ and $\alpha \in (0, 1)$ then*

$$P_{A_\alpha(G)}(x) = (x^2 - 3\alpha x + \alpha^2 + 2\alpha - 1)^{t-1} h(x),$$

where $h(x) = x^3 - \alpha(t+3)x^2 + [\alpha^2 t + (\alpha^2+2\alpha-1)(t+1)]x - 2\alpha(2\alpha-1)t$*. If* x_1*,* x_2 *and* x_3 *are the roots of* $h(x)$ *then* $x_3 < \theta_2 < x_2 < \theta_1 < x_1$*.*

Proof. For $G \simeq F_{0,0,t}$, we have

$$A_\alpha(G) = \begin{bmatrix} \alpha t & (1-\alpha)\mathbf{J}_{1\times t} & \mathbf{0}_{1\times t} \\ (1-\alpha)\mathbf{J}_{t\times 1} & 2\alpha \mathbf{I}_t & (1-\alpha)\mathbf{I}_t \\ \mathbf{0}_{t\times 1} & (1-\alpha)\mathbf{I}_t & \alpha \mathbf{I}_t \end{bmatrix}.$$

From Proposition 3.2, if $t \geq 2$, θ_1 and θ_2 are eigenvalues of $A_\alpha(G)$, both with multiplicity at least $t-1$. From Proposition 2.1, the spectrum of matrix

$$M = \begin{bmatrix} \alpha t & t(1-\alpha) & 0 \\ 1-\alpha & 2\alpha & 1-\alpha \\ 0 & 1-\alpha & \alpha \end{bmatrix},$$

whose characteristic polynomial is $h(x) = x^3 - \alpha(t+3)x^2 + [\alpha^2 t + (\alpha^2+2\alpha-1)(t+1)]x - 2\alpha(2\alpha-1)t$, is contained in the spectrum of $A_\alpha(G)$. As θ_1 and $\theta_2 = 3\alpha - \theta_1$ are not roots of $h(x)$, we have $P_{A_\alpha(G)}(x) = (x^2 - 3\alpha x + \alpha^2 + 2\alpha - 1)^{t-1} h(x)$. Since $h(\theta_1) = -(\alpha-1)^2(\theta_1 - \alpha)t < 0$ and $h(\theta_2) = (\alpha-1)^2(\theta_1 - 2\alpha)t > 0$ for all $t \geq 1$ and $\alpha \in (0,1)$, there is a root of $h(x)$ in (θ_2, θ_1). As $\lim_{x\to\infty} h(x) = \infty$ and $\lim_{x\to-\infty} h(x) = -\infty$, the previous inequalities imply $x_3 < \theta_2 < x_2 < \theta_1 < x_1$. If $t = 1$, $G \simeq P_3$ and the result follows.

□

Proposition 3.6 *Let* $G \simeq F_{s,r,0}$*. If* $s \geq 1$*,* $r \geq 1$ *and* $\alpha \in (0,1)$ *then*

$$P_{A_\alpha(G)}(x) = (x-\alpha)^{s-1}(x - 3\alpha + 1)^r (x - \alpha - 1)^{r-1} h(x),$$

where $h(x) = x^3 - (\alpha s + 2\alpha r + 2\alpha + 1)x^2 + ((\alpha^2 + 3\alpha - 1)(s + 2r) + \alpha^2 + \alpha)x - (2\alpha^2 + \alpha - 1)s + 2\alpha r(1 - 3\alpha)$*. If* x_1*,* x_2 *and* x_3 *are the roots of polynomial* $h(x)$ *then*

$$\begin{cases} x_3 < 3\alpha - 1 < \alpha < x_2 < \alpha + 1 < x_1, & \text{if } 0 < \alpha \leq \frac{1}{3}; \\ \min\{x_3, 3\alpha-1\} < \max\{x_3, 3\alpha - 1\} < \alpha < x_2 < \alpha + 1 < x_1, & \text{if } \frac{1}{3} < \alpha < \frac{1}{2}; \\ x_3 < \alpha < 3\alpha - 1 < x_2 < \alpha + 1 < x_1, & \text{if } \frac{1}{2} \leq \alpha < 1. \end{cases}$$

Proof. For $G \simeq F_{s,r,0}$, we have

$$A_\alpha(G) = \begin{bmatrix} \alpha(s+2r) & (1-\alpha)\mathbf{J}_{1\times s} & (1-\alpha)\mathbf{J}_{1\times 2r} \\ (1-\alpha)\mathbf{J}_{s\times 1} & \alpha\mathbf{I}_s & \mathbf{0}_{s\times 2r} \\ (1-\alpha)\mathbf{J}_{2r\times 1} & \mathbf{0}_{2r\times s} & B_{2r} \end{bmatrix},$$

where B_{2r} is the matrix given in (3).

From the vector obtained in the proof of Proposition 2.4 it is possible to obtain r linearly independent eigenvectors related to the eigenvalue $3\alpha-1$ and $s-1$ linearly independent eigenvectors associated to the eigenvalue α. By Proposition 3.2, $\alpha+1$ is an eigenvalue of $A_\alpha(G)$ with multiplicity at least $r-1$ and, from Proposition 2.1, the eigenvalues of the reduced matrix

$$M = \begin{bmatrix} \alpha(s+2r) & (1-\alpha)s & (1-\alpha)2r \\ 1-\alpha & \alpha & 0 \\ 1-\alpha & 0 & 1+\alpha \end{bmatrix},$$

whose characteristic polynomial is $h(x) = x^3 - (\alpha s + 2\alpha r + 2\alpha + 1)x^2 + ((\alpha^2+3\alpha-1)(s+2r) + \alpha^2 + \alpha)x - (2\alpha^2+\alpha-1)s + 2\alpha r(1-3\alpha)$, is contained in the spectrum of $A_\alpha(G)$. Note that $h(\alpha+1) = -2(\alpha-1)^2 r < 0$ and $h(\alpha) = (\alpha-1)^2 s > 0$ for all $\alpha \in (0,1)$, $s \geq 1$ and $r \geq 1$. So, $h(x)$ has a root, x_2, in $(\alpha, \alpha+1)$. As $\lim\limits_{x\to\infty} h(x) = \infty$ and $h(\alpha+1) < 0$, we concluded that the largest root of $h(x)$, x_1, satisfies $\alpha+1 < x_1$. We have $h(3\alpha-1) = 2(1-\alpha)[(3\alpha^2-3\alpha+1)s + (6\alpha^2-5\alpha+1)(r-1)]$. As $3\alpha^2-3\alpha+1 > 0$ for all $\alpha \in (0,1)$ and $6\alpha^2-5\alpha+1 \geq 0$ for $\alpha \in \left(0, \frac{1}{3}\right] \cup \left[\frac{1}{2}, 1\right)$ we have $h(3\alpha-1) > 0$ for all $\alpha \in \left(0, \frac{1}{3}\right] \cup \left[\frac{1}{2}, 1\right)$. As $\lim\limits_{x\to-\infty} h(x) = -\infty$ we have $x_3 < \min\{3\alpha-1, \alpha\}$. Similarly, when $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right)$, it is shown that $\max\{x_3, 3\alpha-1\} < \alpha$. It is easy to sort the eigenvalues of $A_\alpha(G)$ and the result follows.

□

Proposition 3.7 *Let $G \simeq F_{s,0,t}$. If $s \geq 1$ and $t \geq 1$ then*

$$P_{A_\alpha(G)}(x) = (x-\alpha)^{s-1}(x^2 - 3\alpha x + \alpha^2 + 2\alpha - 1)^{t-1} h(x),$$

where $h(x) = x^4 - \alpha(s+t+4)x^3 + [(3\alpha-1)(\alpha+1)(s+t+1) + \alpha^2]x^2 - \alpha[(\alpha^2 + 2\alpha - 1)(s+2t+1) + (2\alpha-1)(3s+t)]x + (2\alpha-1)[\alpha^2(s+2t) + (2\alpha-1)s]$. Moreover, the polynomial $h(x)$ has four distinct roots, x_1, x_2, x_3 and x_4, such that $x_4 < \theta_2 < x_3 < \alpha < x_2 < \theta_1 < x_1$.

Proof. For $G \simeq F_{s,0,t}$, we have

$$A_\alpha(G) = \begin{bmatrix} \alpha(s+t) & (1-\alpha)\mathbf{J}_{1\times s} & (1-\alpha)\mathbf{J}_{1\times t} & \mathbf{0}_{1\times t} \\ (1-\alpha)\mathbf{J}_{s\times 1} & \alpha\mathbf{I}_s & \mathbf{0}_{s\times t} & \mathbf{0}_{s\times t} \\ (1-\alpha)\mathbf{J}_{t\times 1} & \mathbf{0}_{t\times 1} & 2\alpha\mathbf{I}_t & (1-\alpha)\mathbf{I}_t \\ \mathbf{0}_{t\times 1} & \mathbf{0}_{t\times s} & (1-\alpha)\mathbf{I}_t & \alpha\mathbf{I}_t \end{bmatrix}.$$

From Proposition 2.4, α is eigenvalue of $A_\alpha(G)$ with multiplicity at least $s-1$. If $t=1$, θ_1 and θ_2 are not eigenvalues of $A_\alpha(G)$. From Proposition 3.2, if $t \geq 2$, θ_1 and θ_2 are eigenvalues of $A_\alpha(G)$, both with multiplicity at least $t-1$. From Proposition 2.1, the eigenvalues of the reduced matrix

$$M = \begin{bmatrix} \alpha(s+t) & (1-\alpha)s & (1-\alpha)t & 0 \\ 1-\alpha & \alpha & 0 & 0 \\ 1-\alpha & 0 & 2\alpha & 1-\alpha \\ 0 & 0 & 1-\alpha & \alpha \end{bmatrix},$$

whose characteristic polynomial is $h(x)$, is contained in the spectrum of $A_\alpha(G)$. For θ_1 and $\theta_2 = 3\alpha - \theta_1$, defined in Proposition 3.2, we have

$$h(\alpha) = (\alpha-1)^4 s > 0,\ h(\theta_1) = -\frac{(\alpha-1)^2 t}{2}(3\alpha^2 - 4\alpha + 2 + \alpha\sqrt{5\alpha^2 - 8\alpha + 4}) < 0$$

and $h(\theta_2) = -\dfrac{2(\alpha-1)^6 t}{3\alpha^2 - 4\alpha + 2 + \alpha\sqrt{5\alpha^2 - 8\alpha + 4}} < 0$, for all $\alpha \in (0,1)$, $s \geq 1$ and $t \geq 1$.

Since α, θ_1 and θ_2 are not roots of $h(x)$, we have $P_{A_\alpha(G)}(x) = (x-\alpha)^{s-1}(x^2 - 3\alpha x + \alpha^2 + 2\alpha - 1)^{t-1}h(x)$. The above inequalities imply that the polynomial $h(x)$ has a root in the interval (θ_2, α) and other root in the interval (α, θ_1). Moreover, as $h(\theta_1) < 0$ and $\lim_{x\to\infty} h(x) = \infty$, there is a root of $h(x)$ in the interval (θ_1, ∞). Similarly, we conclude that $h(x)$ has the smallest root in the interval $(-\infty, \theta_2)$.

□

The next propositions, whose proofs are similar to the previous results, complete all cases in the family $\mathcal{F}_n$.

Proposition 3.8 *Let $G \simeq F_{0,r,t}$. If $r \geq 1$, $t \geq 1$ and $\alpha \in (0,1)$ then*

$$P_{A_\alpha(G)}(x) = (x - 3\alpha + 1)^r (x - \alpha - 1)^{r-1} (x^2 - 3\alpha x + \alpha^2 + 2\alpha - 1)^{t-1} h(x),$$

where $h(x) = x^4 - (\alpha(t+2r+4)+1)x^3 + ((3\alpha^2+3\alpha-1)(t+2r+1) + \alpha^2 + 2\alpha)x^2 - ((\alpha^3 + 2\alpha^2 + \alpha - 1)(2t+2r+1) + (2\alpha^2 - 3\alpha + 1)(t+8r) + (14\alpha - 6)r)x + 2\alpha t(2\alpha^2 + \alpha - 1) + 2r(3\alpha^3 + 5\alpha^2 - 5\alpha + 1)$. If x_1, x_2, x_3 and x_4 are the roots of polynomial $h(x)$ then

$$\begin{cases} x_4 < \theta_2 < \min\{x_3, 3\alpha - 1\} < \max\{x_3, 3\alpha - 1\} < \theta_1 < x_2 < \alpha + 1 < x_1, & \text{if } 0 < \alpha < \frac{1}{3}; \\ x_4 < \theta_2 < 3\alpha - 1 < x_3 < \theta_1 < x_2 < \alpha + 1 < x_1, & \text{if } \frac{1}{3} \leq \alpha < 1. \end{cases}$$

Proposition 3.9 *Let $G \simeq F_{s,r,t}$. For $s \geq 1$, $r \geq 1$, $t \geq 1$ and $\alpha \in (0,1)$,*

$$P_{A_\alpha(G)}(x) = (x-\alpha)^{s-1}(x-(3\alpha-1))^r(x-(\alpha+1))^{r-1}(x^2-3\alpha x+\alpha^2+2\alpha-1)^{t-1}g(x),$$

where $g(x) = x^5 + (-\alpha t - 2\alpha r - \alpha s - 5\alpha - 1)x^4 + ((4\alpha^2+3\alpha-1)t + (8\alpha^2+6\alpha-2)r + (4\alpha^2+3\alpha-1)s + 8\alpha^2+6\alpha-1)x^3 + ((-5\alpha^3-11\alpha^2+2\alpha+1)t + (-8\alpha^3-28\alpha^2+10\alpha)r + (-4\alpha^3-13\alpha^2+3\alpha+1)s - 5\alpha^3-8\alpha^2+1)x^2 + ((2\alpha^4+12\alpha^3+\alpha^2-3\alpha)t + (2\alpha^4+28\alpha^3+2\alpha^2-10\alpha+2)r + (\alpha^4+11\alpha^3+7\alpha^2-8\alpha+1)s + \alpha^4+3\alpha^3+\alpha^2-\alpha)x + (-4\alpha^4-2\alpha^3+2\alpha^2)t + (-6\alpha^4-10\alpha^3+10\alpha^2-2\alpha)r + (-2\alpha^4-5\alpha^3+\alpha^2+3\alpha-1)s$. *Moreover, $g(x)$ has five roots x_1, x_2, x_3, x_4, x_5 which are arranged as*

$$x_5 < \theta_2 < x_4 < \min\{3\alpha-1, \alpha\} < \max\{3\alpha-1, \alpha\} < x_3 < \theta_1 < x_2 < \alpha+1 < x_1$$

and the result follows.

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