Available online at www.sciencedirect.com

SciVerse ScienceDirect

Electronic Notes in Theoretical Computer Science

Electronic Notes in Theoretical Computer Science 278 (2011) 31–45

www.elsevier.com/locate/entcs

Two-sorted Point-Interval Temporal Logics

Philippe Balbiani[1]

Institut de Recherche
en Informatique de Toulouse
Université Paul Sabatier,
Toulouse, France

Valentin Goranko[2]

Department of Informatics and Mathematical Modeling
Technical University of Denmark,
Lyngby, Denmark

Guido Sciavicco[3]

Department of Computer Engineering
Middle East Technical University, Northern Cyprus Campus, and
Department of Information Engineering and Communications
University of Murcia,
Murcia, Spain

Abstract

There are two natural and well-studied approaches to temporal ontology and reasoning: point-based and interval-based. Usually, interval-based temporal reasoning deals with points as particular, duration-less intervals. Here we develop explicitly two-sorted point-interval temporal logical framework whereby time instants (points) and time periods (intervals) are considered on a par, and the perspective can shift between them within the formal discourse. We focus on fragments involving only modal operators that correspond to the inter-sort relations between points and intervals. We analyze their expressiveness, comparative to interval-based logics, and the complexity of their satisfiability problems. In particular, we identify some previously not studied and potentially interesting interval logics.

Keywords: point and interval temporal logics, decidability, complexity.

1 Introduction

The predominant approach in the studies of temporal reasoning and logics has been based on the assumption of time points (instants) as the primary temporal on-

[1] Email: philippe.balbiani@irit.fr

[2] Email: vfgo@imm.dtu.dk

[3] Email: guido@um.es

doi:10.1016/j.entcs.2011.10.004

tological entities. However, there have also been active studies of interval-based temporal reasoning and logics over the past 2 decades, starting with the seminal work of Halpern and Shoham [11] introducing the multi-modal logic, that we will call $\mathcal{HS}$, comprising modal operators for all possible relations (known as Allen's relations [1]) between two intervals in a linear order, and followed by a series of publications studying expressiveness and decidability/undecidability and complexity of the fragments of $\mathcal{HS}$, e.g., [10,4]. For a detailed philosophical study of both approaches – point-based and interval-based – see [17]. Many studies on interval logics have considered the so-called 'non-strict' interval semantics, allowing point-intervals (with coinciding endpoints) along with proper ones, and thus encompassing the instant-based approach, too; see e.g., [11,10,4]. Yet, little has been done so far on formal treatment of both temporal primitives, points and intervals, *on a par*, in a common, two-sorted framework. The present paper purports to provide a systematic such treatment. Our work is motivated by several observations:

- Natural languages incorporate both ontologies on a par, without assuming the primacy of one over the other, and have the capacity to shift smoothly the perspective from instants to intervals and vice versa within the same discourse.
- There are various temporal scenarios which neither of the two ontologies alone can grasp properly. In particular, sometimes neither the treatment of intervals as sets of their internal points, nor the treatment of points as 'instantaneous' intervals, is really adequate. For example, a sentence like *'Ever since he met her for the first time, he could not stop thinking about her and kept calling her several times every night until she would give him a brush-off, and then after being silent for a while he would phone again...'* cannot be properly represented in either instant-only or interval-only framework. As another example, consider a typical safety requirement of an intelligent systems that controls a rail crossing: *'At the exact moment in which the train passes over the sensor, the rail crossing bar starts to close; the bar will start to open again a while after the train passes over the second sensor'*.
- The technical identification of intervals with sets of their internal points, or of points as instantaneous intervals leads also to conceptual problems, e.g. of confusing events and fluents. *Instantaneous events* are represented by time intervals and should be distinguished from *instantaneous holding of fluents*, which are evaluated at time points. Formally, the point a should be distinguished from the interval $[a, a]$ and the truths in these should not necessarily imply each other.
- Moreover, the area of artificial intelligence is concerned with purely practical problems related to the formal representation and reasoning of intelligent agents on various temporal and spatial aspects such as *position, motion, actions, processes, events, fluents,* etc. Some of these can be adequately represented in either of the rival ontologies, while others become awkward, if not meaningless, in one or the other of them.
- Finally, we note that, while differences in the expressiveness have been found

between the strict and non-strict semantics for some interval logics (see [7], for example), so far no distinction in the decidability of the satisfiability has been found. Therefore, we believe that an attempt to systemize the role of points, intervals, and their interaction, would make good sense not only from a purely ontological point of view, but also from algorithmic and computational perspectives.

There have been several logical studies of the relationship between instants and intervals, including [12,17,15]. One of the conceptual precursors of our present study is [8] where Galton introduces a two-sorted 'aspectual calculus' involving points and events on a par. Further, in [9] Galton argues that Allen's interval-based theory of time and action is inadequate for representing *continuous change* and advocates the necessity of adding time instants in their own capacity to it. Other explicit two-sorted point-interval formal studies of time of which we are aware include the system IP from [18,16], based on the first-order theory of point-interval structures, and the system LNint from [6], where, however, the interval type plays a secondary role since formulae are always evaluated at points, and the time line is assumed to be discrete (the set of integers).

Here we develop explicitly two-sorted modal approach to the point-interval temporal reasoning whereby time instants (points) and time periods (intervals) are considered on a par, and the perspective can shift between them within the formal discourse. We compare that language with those interval logics with non-strict semantics, that we already know to be right on the border between decidable and undecidable. One of the most important examples on the decidable side is that of Propositional Neighborhood Logic (PNL) with non-strict semantics [4], that corresponds to the fragment of $\mathcal{HS}$ with the modal operator for *meets* and *met-by* only, plus the modal constant π for point-interval; In [14] it has been shown that PNL is almost maximal (amongst the fragments of $\mathcal{HS}$) w.r.t. decidability on the class of all linear orders, while the pair of operators corresponding to Allen's relations *ends* and *ended by* constitutes the most interesting exception, as they can be added to PNL without losing the decidability when interpreted over finite models or models based on the set of natural numbers $\mathbb{N}$. Here, we first introduce the logic $\mathcal{PI}$ comprising modal operators for all possible binary relations between points and intervals. It is easy to see that $\mathcal{PI}$ is at l east as expressive as $\mathcal{HS}$, and therefore it is undecidable under the same assumptions where the latter is. Then, we focus on the fragment $\mathcal{PI}_{mix}$ involving only the modal operators that correspond to the inter-sort relations between points and intervals. We analyze the expressiveness and the complexity of the satisfiability problem of $\mathcal{PI}_{mix}$ and some of its fragments. In particular, we identify some previously not studied, in the context of fragments of $\mathcal{HS}$, and potentially interesting interval logics whose decidability/undecidability status can be deduced from already known results for fragments of $\mathcal{HS}$ and at least one of them has an unexpectedly low complexity.

2 Preliminaries: the Logic $\mathcal{HS}$

In the classical interval setting, given a linearly ordered set $\mathbb{D} = \langle D, <\rangle$, an interval (also called *non-strict* interval) is defined as a pair $[a, b]$, where $a, b \in D$ and $a \leq b$. The logic $\mathcal{HS}$, introduced in [11], is defined over a set of propositional letters $\mathcal{AP}$, denoted by $p, q, \ldots$, by using the classical propositional operators $\neg, \vee$ (whereas the remaining ones can be considered as shortcuts), and a modal operator for each of the 12 Allen's relation, that is, each possible binary relation between two intervals on linear orders (excluding equality, the modal operator for which is trivial). The standard notation for such modal operators is as follows: $\langle A\rangle$ (in the non-strict semantics, it is usually denoted by $\Diamond_r$) for the relation *meets*, $\langle B\rangle$ for *begins*, $\langle E\rangle$ for *ends*; moreover and, for each $\langle X\rangle \in \{\langle A\rangle, \langle B\rangle, \langle E\rangle\}$, $\langle\overline{X}\rangle$ denotes its inverse (in the non-strict semantics, $\Diamond_l$ denotes the inverse of $\Diamond_r$), and the modal operators for the remaining six operators can easily be defined in terms of the above ones. In this setting, a $\mathcal{HS}$-*model* M is defined as $M = \langle \mathbb{D}, \mathbb{I}(\mathbb{D})^+, \mathcal{V}\rangle$, where $\mathbb{I}(\mathbb{D})^+$ is the set of all non-strict intervals over $\mathbb{D}$, and $\mathcal{V} : \mathbb{I}(\mathbb{D})^+ \to 2^{\mathcal{AP}}$ is a *labeling* function. The semantics of $\mathcal{HS}$-formulae φ is as follows:

- $M, [a, b] \Vdash p$ iff $p \in \mathcal{V}([a, b])$;
- $M, [a, b] \Vdash \pi$ iff $a = b$;
- $M, [a, b] \Vdash \langle A\rangle\psi$ (resp., $\Diamond_r\psi$) iff there exists $c > b$ (resp., $c \geq b$) such that $M, [b, c] \Vdash \psi$;
- $M, [a, b] \Vdash \langle B\rangle\psi$ iff there exists $a \leq c < b$ such that $M, [a, c] \Vdash \psi$;
- $M, [a, b] \Vdash \langle E\rangle\psi$ iff there exists $a < c \leq b$ such that $M, [c, b] \Vdash \psi$;
- $M, [a, b] \Vdash \langle D\rangle\psi$ iff there exist $a < c < d < b$ such that $M, [c, d] \Vdash \psi$;
- $M, [a, b] \Vdash \langle O\rangle\psi$ iff there exist $a < c \leq b < d$ such that $M, [c, d] \Vdash \psi$;
- $M, [a, b] \Vdash \langle L\rangle\psi$ iff there exist $a < b < c \leq d$ such that $M, [c, d] \Vdash \psi$.

where the semantics of classical operators is as standard. The semantics of the inverse operators can be easily deduced from the above clauses; for example, we have:

- $M, [a, b] \Vdash \langle\overline{A}\rangle\psi$ (resp., $\Diamond_l\psi$) iff there exists $c < b$ (resp., $c \leq b$) such that $M, [c, b] \Vdash \psi$;
- $M, [a, b] \Vdash \langle\overline{B}\rangle\psi$ iff there exists $c > b$ such that $M, [a, c] \Vdash \psi$;
- $M, [a, b] \Vdash \langle\overline{E}\rangle\psi$ iff there exists $c < a$ such that $M, [c, a] \Vdash \psi$.

With every subset $\{\langle X_1\rangle, \ldots, \langle X_k\rangle\}$ of $\mathcal{HS}$ modal operators, we associate the fragment $\mathsf{X_1X_2\ldots X_k}$ of the logic $\mathcal{HS}$ that features all and only those modal operators, possibly extended with the modal constant π, denoted in this case by $\mathsf{X_1X_2\ldots X_k}^\pi$, if π is not definable in that particular fragment. In the recent literature, PNL (resp., PNL$^\pi$) is also used to denote the fragment $\mathrm{A\overline{A}}$ (resp., $\Diamond_r\Diamond_l^\pi$). Recent results concerning decidability, undecidability, and expressive power of fragments of $\mathcal{HS}$ include [7,14,2].

In [11], the satisfiability problem for the logic $\mathcal{HS}$ has been proved to be undecidable over all interesting classes of linearly ordered sets. In the classification of decidability/undecidability for most of the interesting fragments of $\mathcal{HS}$ for both the strict and non-strict semantics (i.e., classes of models that exclude, resp., include point-intervals), the case of PNL/PNL$^\pi$, is probably the most interesting one on the decidable side. Its satisfiability problem is decidable in finite, discrete, and dense case, as well as in the class of all linearly ordered sets, among others; in all these cases, it is NEXPTIME-complete . More recently, the fragment A$\overline{\text{A}}$E$\overline{\text{E}}$ has been shown to be decidable in the finite case and over the set of natural numbers [14], but its complexity is non-elementary. EXPSPACE-completeness holds, among others, for the fragment AB$\overline{\text{B}}$ in the same cases as PNL/PNL$^\pi$ (see [5] for a complete survey of these decidable fragments). Finally, in the view of our classification of the expressive power of some of the languages that we introduce in the following sections, we also note that the fragment A$\overline{\text{A}}$D (and, therefore, A$\overline{\text{A}}$D$^\pi$ and $\Diamond_r\Diamond_r$D$^\pi$) is an important case of undecidable fragment, very close to the decidability/undecidability border [2].

3 Syntax and Semantics of $\mathcal{PI}$ and its Fragments

3.1 Point and Interval Relations

Given a linearly ordered set $\mathbb{D} = \langle D, < \rangle$, we call the elements of D *points* and define an interval as an ordered pair $[a, b]$ of points in D, where $a < b$. Now, as we have mentioned above, there are 13 possible relations, including equality, between any two intervals. From now on, we call these *interval-interval* relations. Besides, there are 3 different relations that may hold between any two points (*before, equal,* and *after*), called hereafter *point-point* relations, and 5 different relations that may hold between a point and an interval and vice-versa, namely *before*, *beginning point*, *during*, *ending point*, and *after*, called hereafter *point-interval* relations. Intuitively, our language will follow the same principle as the logic $\mathcal{HS}$, discussed in the previous section: one modal operator for each one of the 19 relations, excluding the equalities between points and between intervals. To provide a uniform and simple notation, we first distinguish among two main types of modal operators: those evaluated at points, denoted by single square brackets $\langle\rangle$, and those evaluated at intervals, denoted by double square brackets $\langle\langle\rangle\rangle$. Now, consider an interval $[b, c]$: it generates a partition of the set D into five regions (see [13]): the region 0, of those points before b, the region 1 that contains b only, the region 2 (between b and c), 3 (only c), and 4 (the remaining regions). Using that notation, for $k \in \{0, 1, 2, 3, 4\}$, a point modality may belong to one of two categories: 'point to point', of the type $\langle k \rangle$, that refers to any point in the relation k with the current one – in which case the regions $1, 2, 3$ coincide and we use the number 2 to indicate any of them – and 'point to interval', of the type $\langle kk' \rangle$, that refers to any interval such that its beginning point is in the region k, and its ending point in the region k', with respect to the current point. Likewise, an interval modality can be of 'interval to point' type $\langle\langle k \rangle\rangle$ referring to any point in the area k w.r.t. the current interval, or of 'interval to interval' type $\langle\langle kk'\rangle\rangle$, in which case it becomes a syntactic variation of the respective $\mathcal{HS}$ modality in the strict semantics.

3.2 Syntax and Semantics of the Point-Interval Logic $\mathcal{PI}$

The language of the *Point-Interval Logic* $\mathcal{PI}$ comprises the classical connectives $\neg$ and $\vee$ (the rest are considered definable), two sorts of propositional letters, namely the set of *point* propositional letters $\mathcal{AP}_{po}$ and the set of *interval* ones $\mathcal{AP}_{int}$, and unary modalities of each of the types specified above. For technical convenience we will assume that each of $\mathcal{AP}_{po}$ and $\mathcal{AP}_{int}$ is a copy of the set of propositional letters $\mathcal{AP}$ from the language of $\mathcal{HS}$. We will denote typical elements of $\mathcal{AP}$ by $p, q, \ldots$, respective typical elements of $\mathcal{AP}_{po}$ by $p_{po}, q_{po}, \ldots$, and respective typical elements of $\mathcal{AP}_{int}$ by $p_{int}, q_{int}, \ldots$.

The logic $\mathcal{PI}$ has two sorts of formulae: *point formulae* and *interval formulae.* Point formulae are obtained by the following grammar:

$$\varphi_{po} ::= p_{po} \mid \neg\varphi_{po} \mid \varphi_{po} \vee \psi_{po} \mid \langle pp\rangle\varphi_{po} \mid \langle pi\rangle\varphi_{int},$$

where $\langle pp\rangle$ represent any point-to-point modality, and $\langle pi\rangle$ is any point-to-interval modality. Similarly, interval formulae are formed by the following grammar:

$$\varphi_{int} ::= p_{int} \mid \neg\varphi_{int} \mid \varphi_{int} \vee \psi_{int} \mid \langle\langle ii\rangle\rangle\varphi_{int} \mid \langle\langle ip\rangle\rangle\varphi_{po},$$

where the modal operators are interval-to-point or interval-to-interval. The formulae of the type $\langle pp\rangle\varphi_{po}$, $\langle pi\rangle\varphi_{int}$, $\langle\langle ip\rangle\rangle\varphi_{po}$, and $\langle\langle ii\rangle\rangle\varphi_{int}$ are called (respectively, point or interval) *diamond formulae.* The respective *box formulae* are defined, as usual, as their duals, e.g. $[ij]\psi := \neg\langle ij\rangle\neg\psi$ [4]. Lastly, a $\mathcal{PI}$ *formula* is a point-formula or an interval-formula.

A *point-interval structure* is a pair $F = \langle\mathbb{D}, \mathbb{I}(\mathbb{D})\rangle$ where $\mathbb{D} = \langle D, <\rangle$ is a linear order and $\mathbb{I}(\mathbb{D})$ is the set of all strict intervals in $\mathbb{D}$. A $\mathcal{PI}$*-model* is a tuple $M = (\mathbb{D}, \mathbb{I}(\mathbb{D}), \mathcal{V}_{po}, \mathcal{V}_{int})$ where $(\mathbb{D}, \mathbb{I}(\mathbb{D})$ is a point-interval structure and $\mathcal{V}_{po} : \mathbb{D} \to 2^{\mathcal{AP}_{po}}$ and $\mathcal{V}_{int} : \mathbb{I}(\mathbb{D}) \to 2^{\mathcal{AP}_{int}}$ are *valuations* [5] assigning to each point (respectively, interval) the set of point (respectively, interval) propositional letters that are true of it. The truth (satisfaction) relation is defined in a $\mathcal{PI}$-model by a mutual recursion on point and interval formulae as follows (the clauses for the classical connectives are standard):

- $M, a \Vdash p_{po}$ iff $p_{po} \in \mathcal{V}_{po}(a)$;
- $M, a \Vdash \langle 0\rangle\varphi$ iff there exists $b < a$ such that $M, b \Vdash \varphi$;
- $M, a \Vdash \langle 4\rangle\varphi$ iff there exists $b > a$ such that $M, b \Vdash \varphi$;

[4] The typographic similarity between a box $[ij]$ and an interval $[a, b]$ is unfortunate, but should not cause confusion.

[5] Usually, in modal logic valuations are functions assigning sets of possible worlds to propositional letters, and the functions defined here are called 'labeling functions', but in this paper we will use the term 'labeling function' for another purpose.

- $M, a \Vdash \langle 00 \rangle \varphi$ iff there exist b, c such that $b < c < a$, and that $M, [b, c] \Vdash \varphi$;
- $M, a \Vdash \langle 02 \rangle \varphi$ iff there exist b, c such that $b < c = a$, and that $M, [b, c] \Vdash \varphi$;
- $M, a \Vdash \langle 04 \rangle \varphi$ iff there exist b, c such that $b < a < c$, and that $M, [b, c] \Vdash \varphi$;
- $M, a \Vdash \langle 24 \rangle \varphi$ iff there exist b, c such that $b = a < c$, and that $M, [b, c] \Vdash \varphi$;
- $M, a \Vdash \langle 44 \rangle \varphi$ iff there exist b, c such that $a < b < c$, and that $M, [b, c] \Vdash \varphi$,

and respectively:

- $M, [a, b] \Vdash p_{int}$ iff $p_{int} \in \mathcal{V}_{int}([a, b])$;
- $M, [a, b] \Vdash \langle\langle 0 \rangle\rangle \varphi$ iff there exists $c < a$ such that $M, c \Vdash \varphi$;
- $M, [a, b] \Vdash \langle\langle 1 \rangle\rangle \varphi$ iff $M, a \Vdash \varphi$;
- $M, [a, b] \Vdash \langle\langle 2 \rangle\rangle \varphi$ iff there exist c such that $a < c < b$, and that $M, c \Vdash \varphi$;
- $M, [a, b] \Vdash \langle\langle 3 \rangle\rangle \varphi$ iff $M, b \Vdash \varphi$;
- $M, [a, b] \Vdash \langle\langle 4 \rangle\rangle \varphi$ iff there exist c such that $b < c$, and that $M, c \Vdash \varphi$,

An interval- (resp., point-) $\mathcal{PI}$-formula ϕ is *satisfiable* if there exists a $\mathcal{PI}$-model and an interval (resp., a point) in it that satisfies ϕ. Note that the clauses for the 'interval-to-interval' modalities are identical to those for the $\mathcal{HS}$ modalities in the strict semantics.

4 Expressiveness of fragments of $\mathcal{PI}$

In this section we systematically compare the expressiveness of $\mathcal{PI}$ and its fragments to that of $\mathcal{HS}$ and its fragments.

4.1 *Transformations between models of $\mathcal{PI}$ and $\mathcal{HS}$*

In order for us to compare the expressiveness of fragments of $\mathcal{PI}$ and $\mathcal{HS}$ we need to specify transformations of two-sorted $\mathcal{PI}$ models into $\mathcal{HS}$-models, and vice-versa.

First, let $M = (\mathbb{D}, \mathbb{I}(\mathbb{D}), \mathcal{V}_{po}, \mathcal{V}_{int})$ be a point-interval model based on some linearly ordered domain $\mathbb{D} = \langle D, < \rangle$. The *corresponding non-strict $\mathcal{HS}$-model* $\tau(M)$ is obtained by taking the set of all non-strict intervals over $\mathbb{D}$: $\mathbb{I}(\mathbb{D})^+ = \{[a, b] \mid a, b \in \mathcal{D}, a \leq b\}$ and defining $\mathcal{V}$ as follows. For each proper interval $[a, b]$, where $a < b$, we put $\mathcal{V}([a, b]) = \{p \in \mathcal{AP} \mid p_{int} \in \mathcal{V}_{int}([a, b])$. Likewise, for each point interval $[a, a]$ we put $\mathcal{V}([a, a]) = \{p \in \mathcal{AP} \mid p_{po} \in \mathcal{V}_{po}(a)$. Conversely, given any $\mathcal{HS}$-model $M = \langle \mathbb{D}, \mathbb{I}(\mathbb{D})^+, \mathcal{V} \rangle$, the *corresponding $\mathcal{PI}$-model* $\sigma(M)$ is obtained as follows. First, we consider the set of all strict intervals over $\mathbb{D}$, $\mathbb{I}(\mathbb{D}) = \{[a, b] \mid a, b \in \mathcal{D}, a < b\}$. Then, with every propositional letter $p \in \mathcal{AP}$ we associate two distinct new propositional letters: p_{po} and p_{int}. Now, for each $p \in \mathcal{AP}$ and all points $a \in \mathbb{D}$ such that $p \in \mathcal{V}([a, a])$, we put $p_{po} \in \mathcal{V}_{po}(a)$; respectively, for all strict intervals $[a, b] \in I(\mathbb{D})$ such that $p \in \mathcal{V}([a, b])$, we put $p_{int} \in \mathcal{V}_{int}([a, b])$. Finally, we define $\sigma(M) = \langle \mathbb{D}, \mathbb{I}(\mathbb{D}), \mathcal{V}_{po}, \mathcal{V}_{int} \rangle$.

Further, with a slight abuse of notation we will use τ and σ to denote respective syntactic translations of formulae of the one language to the other, such that the translation τ maps each the propositional letters p_{po} and p_{int} to p, while the translation σ splits into two parts: σ^p, producing point formulae by mapping each propositional letter p to p_{po} and σ^i, producing interval formulae by mapping each propositional letter p to p_{int}. Note that τ and σ are so far just notation; further we will define different such translations, to match different fragments of $\mathcal{HS}$ and $\mathcal{PI}$.

4.2 Comparing expressiveness of some fragments of $\mathcal{PI}$ and $\mathcal{HS}$

Now, we can compare the expressive power of standard interval logics and two-sorted ones, by following the general terminology for comparing expressiveness of logics with respect to the transformations τ and σ, as follows.

Let L be any fragment of $\mathcal{PI}$, and L' any fragment of $\mathcal{HS}$; we say that L' is *at least as expressive as* L, denoted by $L \preceq L'$, iff τ can be extended to a truth preserving syntactic translation from L to L', that is, one that maps every L formula to L' formula, such that:

(i) for every $\mathcal{PI}$-model M, point $a \in M$ and a point formula ψ of L:
$M, a \models \psi$ iff $\tau(M), [a, a] \models \tau(\psi)$,

(ii) for every $\mathcal{PI}$-model M, interval $[a, b] \in M$ and an interval formula ψ of L:
$M, [a, b] \models \psi$ iff $\tau(M), [a, b] \models \tau(\psi)$.

Likewise, let L be a fragment of $\mathcal{HS}$ and L' a fragment of $\mathcal{PI}$. We say that L' is *at least as expressive as* L, denoted $L \preceq L'$, iff σ can be extended to a truth preserving syntactic translation from L to L', that is, one that maps every L formula to L' formula, such that:

(i) for every $\mathcal{HS}$-model M, point interval $[a, a] \in M$ and a formula ψ of L:
$M, [a, a] \models \psi$ iff $\sigma(M), a \models \sigma^p(\psi)$,

(ii) for every $\mathcal{HS}$-model M, strict interval $[a, b] \in M$ and a formula ψ of L:
$M, [a, b] \models \psi$ iff $\sigma(M), [a, b] \models \sigma^i(\psi)$.

In each of the cases above we say that L' is *more expressive than* L, denoted $L \prec L'$ if $L \preceq L'$, and not $L' \preceq L$. Respectively, we say that L and L' are *expressively incomparable*[6] if neither $L \preceq L'$ nor $L' \preceq L$, and they are *expressively equivalent*, denoted by $L \equiv L'$, if and only if $L \preceq L'$ and $L' \preceq L$.

Hereafter we only consider in detail the fragment $\mathcal{PI}_{mix}$ of $\mathcal{PI}$ that comprises the *inter-sort* modalities, that is, the 'point-to-interval' and the 'interval-to-point' ones, and some notable fragments of it. We consider $\mathcal{PI}_{mix}$ to be the most interesting of the fragments of $\mathcal{PI}$ because it captures precisely the expressiveness of the interaction between points and intervals. We also denote by $\mathcal{PI}_{mix}\ ^{-\langle\langle k\rangle\rangle}$ the fragment of $\mathcal{PI}_{mix}$ devoid of the interval-to-point modal operator $\langle\langle k\rangle\rangle$, and by $\mathcal{PI}_{mix}\ ^{-\langle kk'\rangle}$ the fragment of $\mathcal{PI}_{mix}$ devoid of the point-to-interval modal operator $\langle kk'\rangle$. Moreover, for any two-sorted language L, we write $L^{>}$ to denote the sub-language

[6] Note that these definitions are given in terms of the specific transformations τ and σ. However, it is easy to see that these definitions are as general as possible.

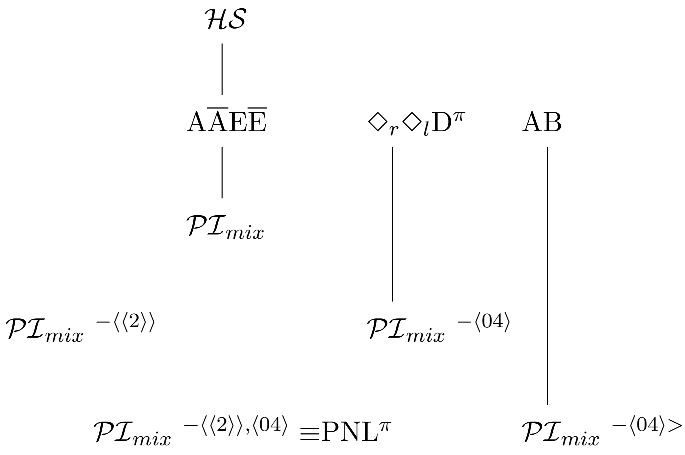

Figure 1. Some interesting fragments of $\mathcal{PI}$ and their expressiveness relation with fragments of $\mathcal{HS}$.

of L that only retains the 'future' operators of L, that is, those that refer to the regions to the right of the current point or interval. In Fig. 1 we have indicated expressiveness relations between each pair of fragments of $\mathcal{HS}$ and $\mathcal{PI}$ connected by a line, where the relation $\preceq$ applies between the lower to the higher fragment. With a mild abuse of notation, we will use $\prec$ and $\preceq$ to compare two fragments of $\mathcal{PI}$ or of $\mathcal{HS}$, in the obvious way (only syntactic translation needed).

Theorem 4.1 *All expressiveness relations $\preceq$ between fragments of $\mathcal{HS}$ and $\mathcal{PI}$, represented in Fig. 1 hold true.*

Proof All relations $\preceq$ on pairs of fragments of $\mathcal{HS}$ and on pairs of fragments of $\mathcal{PI}$ are trivial. As for the remaining cases, it is sufficient to define suitable truth-preserving translations τ and σ, as discussed above. We do that below for each case. Notice that, in the non-strict semantics π is definable in a $\mathcal{HS}$-fragment in presence of $\langle B\rangle$ ($\pi \equiv [B]\bot$) or $\langle E\rangle$ ($\pi \equiv [E]\bot$). Also, observe that $\Diamond_r\varphi \equiv \langle E\rangle(\varphi \wedge \pi) \vee \langle A\rangle\varphi$ and $\Diamond_l\varphi \equiv \langle\overline{A}\rangle(\varphi \vee \langle E\rangle(\pi \wedge \varphi))$ (a similar definition can be devised in presence of $\langle B\rangle$ instead of $\langle E\rangle$).

– $\mathcal{PI}_{mix} \preceq$A$\overline{\mathrm{A}}E\overline{\mathrm{E}}$:
The following translation τ works (the easy details are left to the reader).

- $\tau(\langle 00\rangle\psi) = \pi \wedge \Diamond_l(\neg\pi \wedge \Diamond_l(\neg\pi \wedge \tau(\psi)))$;
- $\tau(\langle 02\rangle\psi) = \pi \wedge \Diamond_l(\neg\pi \wedge \tau(\psi))$;
- $\tau(\langle 04\rangle\psi) = \pi \wedge \Diamond_r(\neg\pi \wedge \langle\overline{E}\rangle\tau(\psi))$;
- $\tau(\langle 24\rangle\psi) = \pi \wedge \Diamond_r(\neg\pi \wedge \tau(\psi))$;
- $\tau(\langle 44\rangle\psi) = \pi \wedge \Diamond_r(\neg\pi \wedge \Diamond_r(\neg\pi \wedge \tau(\psi)))$;
- $\tau(\langle\langle 0\rangle\rangle\psi) = \neg\pi \wedge \Diamond_l(\neg\pi \wedge \Diamond_l(\pi \wedge \tau(\psi)))$;
- $\tau(\langle\langle 1\rangle\rangle\psi) = \neg\pi \wedge \Diamond_l(\pi \wedge \tau(\psi))$;
- $\tau(\langle\langle 2\rangle\rangle\psi) = \neg\pi \wedge \langle E\rangle(\neg\pi \wedge \Diamond_l(\pi \wedge \tau(\psi)))$;

- $\tau(\langle\langle 3\rangle\rangle\psi) = \neg\pi \wedge \Diamond_r(\pi \wedge \tau(\psi))$;
- $\tau(\langle\langle 4\rangle\rangle\psi) = \neg\pi \wedge \Diamond_r(\neg\pi \wedge \Diamond_r(\pi \wedge \tau(\psi)))$,

Note that the relationship $\mathcal{PI}_{mix} \preceq \mathrm{A\overline{A}B\overline{B}}$ holds, too, by symmetry; nevertheless, using $\mathrm{A\overline{A}E\overline{E}}$ gives us the decidability of $\mathcal{PI}_{mix}$ over the natural numbers - the relationship $\mathcal{PI}_{mix} \preceq \mathrm{A\overline{A}B\overline{B}}$ allows us to say that $\mathcal{PI}_{mix}$ is decidable over the (less interesting) set of negative natural numbers [14].

– $\mathcal{PI}_{mix}{}^{-\langle 04\rangle} \preceq \Diamond_r\Diamond_l \mathrm{D}^\pi$:
For this claim it suffices to modify the definition of τ from the previous case as follows, taking into account that $\langle 04\rangle$ is no longer part of the language:

- $\tau(\langle\langle 2\rangle\rangle\psi) = \neg\pi \wedge \langle D\rangle(\pi \wedge \tau(\psi))$;

– $\mathcal{PI}_{mix}{}^{-\langle 04\rangle>} \preceq \mathrm{AB}$:
Again, by modifying τ above:

- $\tau(\langle\langle 1\rangle\rangle\psi) = \neg\pi \wedge \langle B\rangle(\pi \wedge \tau(\psi))$;
- $\tau(\langle\langle 2\rangle\rangle\psi) = \neg\pi \wedge \langle B\rangle(\neg\pi \wedge \langle A\rangle(\pi \wedge \tau(\psi)))$.

– $\mathcal{PI}_{mix}{}^{-\langle\langle 2\rangle\rangle,\langle 04\rangle} \equiv \mathrm{PNL}^\pi$:
To show $\mathcal{PI}_{mix}{}^{-\langle 04\rangle,\langle\langle 2\rangle\rangle} \preceq \mathrm{PNL}^\pi$ we define τ as in the translation to $\mathrm{A\overline{A}E\overline{E}}$ above. To show $\mathrm{PNL}^\pi \preceq \mathcal{PI}_{mix}{}^{-\langle 04\rangle,\langle\langle 2\rangle\rangle}$ we define σ as follows:

- $\sigma^p(\Diamond_r\psi) = \langle 24\rangle\sigma^i(\psi) \vee \sigma^p(\psi)$;
- $\sigma^p(\Diamond_l\psi) = \langle 02\rangle\sigma^i(\psi) \vee \sigma^p(\psi)$;
- $\sigma^i(\Diamond_r\psi) = \langle\langle 3\rangle\rangle(\langle 24\rangle\sigma^i(\psi) \vee \sigma^p(\psi))$;
- $\sigma^i(\Diamond_l\psi) = \langle\langle 1\rangle\rangle(\langle 02\rangle\sigma^i(\psi) \vee \sigma^p(\psi))$
- $\sigma^p(\pi) = \top, \quad \sigma^i(\pi) = \bot$.

$\square$

We note that the expressive embeddings above are not claimed here to be strict. Proving their strictness requires proving respective non-expressibility results (which, in general, depend on the particular class of structures in which the languages are interpreted) for which there is no space here. However, related classification of the expressive power of fragments of $\mathcal{HS}$ has recently appeared in [7], and some of the expressive embedding results above can be proven strict by using the model-theoretic techniques applied there.

5 Decidability and complexity of fragments of $\mathcal{PI}$

Clearly $\mathcal{PI}$ is at least as expressive as $\mathcal{HS}$, and therefore its satisfiability problem is undecidable over most interesting classes of linearly ordered sets. Also, the more expressive fragments of $\mathcal{PI}$ are at least as expressive as some known undecidable fragments of $\mathcal{HS}$, and thus are undecidable themselves. For example, any fragment of $\mathcal{PI}$ including at least the pair of modalities $\langle\langle 34\rangle\rangle$,$\langle\langle 22\rangle\rangle$ (resp., the modality

$\langle\langle 22\rangle\rangle$, or the pair of modalities $\langle\langle 14\rangle\rangle$,$\langle\langle 03\rangle\rangle$) is undecidable in every interesting class of models, as it includes the $\mathcal{HS}$-fragment AD (resp., O, $\overline{\mathrm{BE}}$). On the other hand, a number of fragments of $\mathcal{PI}$ are readily embeddable in already known decidable fragments of $\mathcal{HS}$. Still, several fragments of $\mathcal{PI}$ give rise to essentially new decidability and complexity problems. Because of space constraints, we will consider in detail only one such case.

5.1 Some complexity bounds for $\mathcal{PI}_{mix}$ and its fragments

Here we use the comparative expressiveness results from the previous section to immediately obtain results on decidability and complexity upper bounds of fragments of $\mathcal{PI}$, by using respective known results for fragments of $\mathcal{HS}$.

Proposition 5.1

(i) *The satisfiability problem for $\mathcal{PI}_{mix}$ interpreted in the class of all finite models and in the class of models based on the set of natural numbers $\mathbb{N}$ is decidable, but with a non-elementary time complexity upper bound [14].*

(ii) *The satisfiability problem for $\mathcal{PI}_{mix}{}^{-\langle 04\rangle>}$, interpreted in each of the classes of finite, discrete, dense, all linearly ordered models, as well as over models based on $\mathbb{N}$, is decidable in EXPSPACE [5];*

(iii) *The satisfiability problem for $\mathcal{PI}_{mix}{}^{-\langle\langle 2\rangle\rangle,\langle 04\rangle}$, in each of the classes of all finite, discrete, dense, and all linearly ordered models, as well as over models based on $\mathbb{N}$, is NEXPTIME-complete [4].*

Note that matching lower bounds for most of these cases are not known yet, because $\mathcal{PI}_{mix}$ and its fragments do not have precise expressively matching fragments of $\mathcal{HS}$.

In the rest of this section we will adapt model-theoretic arguments used in [3,4] to obtain a new decidability and complexity result for the satisfiability problem for $\mathcal{PI}_{mix}{}^{-\langle\langle 2\rangle\rangle}$ interpreted over finite models by proving bounded-model property with respect to models of exponential size. For that we will use the more general notion of fulfilling labeling structures.

5.2 Fulfilling labeling structures for $\mathcal{PI}_{mix}{}^{-\langle\langle 2\rangle\rangle}$

To begin with, note that satisfiability of an interval formula φ is equivalent to satisfiability of the point formula $\langle 24\rangle\varphi$; therefore it suffices to consider satisfiability of point formulas. Also, notice that two of the interval-to-point modalities are definable in terms of the others:

$$\langle\langle 4\rangle\rangle\psi \equiv \langle\langle 3\rangle\rangle\langle 24\rangle\langle\langle 3\rangle\rangle\psi, \quad \langle\langle 0\rangle\rangle\psi \equiv \langle\langle 1\rangle\rangle\langle 02\rangle\langle\langle 1\rangle\rangle\psi,$$

and, therefore, we will treat them as shortcuts. Similarly, two of the point-to-interval modalities are definable, too:

$$\langle 44\rangle\psi \equiv \langle 24\rangle\langle\langle 3\rangle\rangle\langle 24\rangle\psi, \quad \langle 00\rangle\psi \equiv \langle 02\rangle\langle\langle 1\rangle\rangle\langle 02\rangle\psi.$$

Definition 5.2 The *closure* of φ is the set $CL(\varphi)$ of all subformulae of φ and their negations, where we identify every $\neg\neg\psi$ with ψ. We denote by $CL_{po}(\varphi)$ the subset of point formulae of $CL(\varphi)$ and by $CL_{int}(\varphi)$ the subset of interval formulae of $CL(\varphi)$.

Definition 5.3 A *point φ-atom* is a set $A \subseteq CL_{po}(\varphi)$ such that for every $\psi \in CL_{po}(\varphi)$, $\psi \in A$ iff $\neg\psi \notin A$ and for every $\psi_1 \vee \psi_2 \in CL_{po}(\varphi)$, $\psi_1 \vee \psi_2 \in A$ iff $\psi_1 \in A$ or $\psi_2 \in A$. An *interval φ-atom* is defined likewise, using $CL_{int}(\varphi)$ instead of $CL_{po}(\varphi)$.

We denote the set of point φ-atoms by A^{φ}_{po} and the set of interval φ-atoms by A^{φ}_{int}.

Definition 5.4 Let φ be a $\mathcal{PI}_{mix}{}^{-\langle\langle 2\rangle\rangle}$-formula. A *labeling structure (LS) for* φ is a tuple $\mathcal{L} = (\mathbb{D}, \mathbb{I}(\mathbb{D}), \mathcal{L}_{po}, \mathcal{L}_{int})$, where $(\mathbb{D}, \mathbb{I}(\mathbb{D}))$ is a point-interval structure and $\mathcal{L}_{po}$ and $\mathcal{L}_{int}$ are *labeling functions* defined respectively as $\mathcal{L}_{po} : D \to A^{\varphi}_{po}$ and $\mathcal{L}_{int} : \mathbb{I}(\mathbb{D}) \to A^{\varphi}_{int}$ and satisfying the following properties:

- For every $[b, c] \in \mathbb{I}(\mathbb{D})$ and formula $[24]\psi$, if $[24]\psi \in \mathcal{L}_{po}(b)$ then $\psi \in \mathcal{L}_{int}([b, c])$;
- For every $a \in \mathbb{D}$, $[b, c] \in \mathbb{I}(\mathbb{D})$ such that $b < a < c$, and formula $[04]\psi$, if $[04]\psi \in \mathcal{L}_{po}(a)$ then $\psi \in \mathcal{L}_{int}([b, c])$;
- For every $[a, b] \in \mathbb{I}(\mathbb{D})$ and formula $[02]\psi$, if $[02]\psi \in \mathcal{L}_{po}(b)$ then $\psi \in \mathcal{L}_{int}([a, b])$,
- For every $[a, b] \in \mathbb{I}(\mathbb{D})$ and formula $[[3]]\psi$, if $[[3]]\psi \in \mathcal{L}_{int}([a, b])$ then $\psi \in \mathcal{L}_{po}(b)$;
- For every $[a, b] \in \mathbb{I}(\mathbb{D})$ and formula $[[1]]\psi$, if $[[1]]\psi \in \mathcal{L}_{int}([a, b])$ then $\psi \in \mathcal{L}_{po}(a)$.

Hereafter, by 'labeling structure' we will mean a labeling structure for some formula φ.

Note that every interval model M induces a LS, with labeling functions:

$$\psi \in \mathcal{L}_{po}(a) \text{ iff } M, a \Vdash \psi \quad \text{and} \quad \psi \in \mathcal{L}_{int}([a, b]) \text{ iff } M, [a, b] \Vdash \psi.$$

Labeling structures can be thought of as *quasi-models*, in which the truth of formulae containing no modal operators is determined by the labeling functions. Furthermore, the labeling functions respect the semantics of the box operators. To obtain 'true models', we must also guarantee that the labeling is in accordance with the semantics of the diamond operators, too. To this end, we introduce the following notion.

Definition 5.5 An LS $\mathcal{L} = (\mathbb{D}, \mathbb{I}(\mathbb{D}), \mathcal{L}_{po}, \mathcal{L}_{int})$ for a formula φ is *fulfilling* (a FLS) iff:

- For every formula $\langle 24\rangle\psi$ in $CL_{po}(\varphi)$ and point $a \in \mathbb{D}$, if $\langle 24\rangle\psi \in \mathcal{L}_{po}(a)$, then there exists an interval $[a, b] \in \mathbb{I}(\mathbb{D})$ such that $\psi \in \mathcal{L}_{int}([a, b])$;
- For every formula $\langle 02\rangle\psi$ in $CL_{po}(\varphi)$ and point $a \in \mathbb{D}$, if $\langle 02\rangle\psi \in \mathcal{L}_{po}(a)$, then there exists an interval $[b, a] \in \mathbb{I}(\mathbb{D})$ such that $\psi \in \mathcal{L}_{int}([b, a])$;
- For every formula $\langle 04\rangle\psi$ and point $a \in \mathbb{D}$, if $\langle 04\rangle\psi \in \mathcal{L}_{po}(a)$, then there exists an interval $[b, c] \in \mathbb{I}(\mathbb{D})$ such that $b < a < c$ and $\psi \in \mathcal{L}_{int}([b, c])$;
- For every formula $\langle\langle 1\rangle\rangle\psi$ in $CL_{int}(\varphi)$ and interval $[a, b] \in \mathbb{I}(\mathbb{D})$, if $\langle\langle 1\rangle\rangle\psi \in \mathcal{L}_{int}([a, b])$, then $\psi \in \mathcal{L}_{po}(a)$;

- For every formula $\langle\langle 3\rangle\rangle\psi$ in $CL_{int}(\varphi)$ and interval $[a,b] \in \mathbb{I}(\mathbb{D})$, if $\langle\langle 3\rangle\rangle\psi \in \mathcal{L}_{int}([a,b])$, then $\psi \in \mathcal{L}_{po}(b)$.

Definition 5.6 A point (resp. interval) formula ψ is *satisfied* by a point (resp. interval) in a given FLS if it belongs to its label. A point (resp. interval) formula ψ is *satisfied* by an LFS if it is satisfied by some point (resp. interval) in it.

Proposition 5.7 *For every $\mathcal{PI}_{mix}{}^{-\langle\langle 2\rangle\rangle}$-formula, satisfiability in a point-interval model is equivalent to satisfiability in some fulfilling labeling structure.*

Definition 5.8 Given an LS $\mathcal{L} = (\mathbb{D}, \mathbb{I}(\mathbb{D}), \mathcal{L}_{po}, \mathcal{L}_{int})$ and a point $a \in D$, the *modal type of a in $\mathcal{L}$* is the set $\mathcal{R}(a)$ consisting of all diamond point formulae in the label $\mathcal{L}_{po}(a)$. We will call the formulae in $\mathcal{R}(a)$ *requests* at the point a.

5.3 Decidability and NEXPTIME-completeness of $\mathcal{PI}_{mix}{}^{-\langle\langle 2\rangle\rangle}$ in the finite

Now we will show that every finite FLS satisfying a given formula of $\mathcal{PI}_{mix}{}^{-\langle\langle 2\rangle\rangle}$ can be cut down to size exponentially bounded above by the size of the formula.

Let us define $m = |CL(\varphi)|$, and observe what follows:

(i) If $\mathcal{R}(\varphi)$ is the set of modal types for points in $\mathcal{L}$, then $|\mathcal{R}(\varphi)| \in O(2^m)$.

(ii) The set of diamond formulas of the type $\langle 02\rangle\xi$ or $\langle 24\rangle\xi$ in a given modal type $\mathcal{R}(a)$ can be satisfied by using at most m distinct endpoints of intervals.

For lack of space we have to omit the proof of the next, main technical result.

Lemma 5.9 *Let $\mathcal{L} = (\mathbb{D}, \mathbb{I}(\mathbb{D}), \mathcal{L}_{po}, \mathcal{L}_{int})$, where $D = (D, <)$, be a finite FLS for a point formula φ, satisfying φ at some point a. Suppose that there exists a point $e \neq a \in D$ with modal type $\mathcal{R} = \mathcal{R}(e)$ such that there are at least $m^2 + m + 1$ points with modal type $\mathcal{R}$ before e, and at least $m^2 + m + 1$ such points after e in $(D, <)$. Then, there exists a FLS $\bar{\mathcal{L}} = (\bar{\mathbb{D}}, \mathbb{I}(\bar{\mathbb{D}}), \bar{\mathcal{L}_{po}}, \bar{\mathcal{L}_{int}})$ satisfying φ such that $\bar{D} = D\setminus\{e\}$.*

Corollary 5.10 *Satisfiability of a $\mathcal{PI}_{mix}{}^{-\langle\langle 2\rangle\rangle}$-formula φ on the class of finite linear orderings is equivalent to its satisfiability on the class of finite linear orderings of size at most $2^{m+1}(m^2 + m + 1)$, where $m = |CL(\varphi)|$.*

Corollary 5.11 *Satisfiability of $\mathcal{PI}_{mix}{}^{-\langle\langle 2\rangle\rangle}$ on the class of all finite linear orderings is decidable in NEXPTIME, and hence it is NEXPTIME-complete.*

5.4 Small ultimately periodic model property of $\mathcal{PI}_{mix}{}^{-\langle\langle 2\rangle\rangle}$ on $\mathbb{N}$

Here, we will extend the previous result to decidability of $\mathcal{PI}_{mix}{}^{-\langle\langle 2\rangle\rangle}$ on $\mathbb{N}$. Again, for lack of space, we can only provide a sketch.

Definition 5.12 An LS $\mathcal{L} = (\mathbb{N}, \mathbb{I}(\mathbb{N}), \mathcal{L}_{po}, \mathcal{L}_{int})$ is *ultimately periodic* (UPLS) with *prefix length* $K \geq 0$ and *period length* P if $\mathcal{L}_{po}(c+P) = \mathcal{L}_{po}(c)$ for every point $c \geq K$ and for every interval $[c,d]$:

- If $c \geq K$, then $\mathcal{L}_{int}([c+P, d+P]) = \mathcal{L}_{int}([c,d])$;
- If $d \geq K$, then $\mathcal{L}_{int}([c, d+P]) = \mathcal{L}_{int}([c,d])$,

Note that every finite LS can be regarded as an UPLS with a period length 0. Also, note that every UPLS is finitely representable: it suffices to define its labeling functions only on all points $c < K+P$ and on all intervals $[c,d]$ such that $d < K+P$ or $d < K + 2P$ and $K < c < K + P$, and then it can be uniquely extended by periodicity.

Claim 5.13 (Small Periodic Model Property) *If φ is any formula of $\mathcal{PI}_{mix}$ $^{-\langle\langle 2\rangle\rangle}$ satisfiable in $\mathbb{N}$, then there exists a (possibly finite) ultimately periodic FLS satisfying φ with lengths of the prefix and of the period at most exponential in $m = |CL(\varphi)|$.*

Proof [**sketch**] Given an interval model based on $\mathbb{N}$ and satisfying φ, we take its corresponding FLS and transform it to an ultimately periodic FLS for φ, by identifying sufficiently long prefix and period in it, and then modifying the rest of the model to make it ultimately periodic. That produces an LS which we make a fulfilling one by applying defect-repairing technique similar to the one in the proof of Lemma 5.9, but now fixing simultaneously all defects involving points on the same periodic orbit, and in a uniform way. Once the ultimately periodic FLS for φ is obtained, we reduce both the prefix and the period down to size, by applying again the defect-repairing technique in a uniform way. Thus, the satisfiability problem for $\mathcal{PI}_{mix}$ $^{-\langle\langle 2\rangle\rangle}$ over $\mathbb{N}$ is NEXPTIME-complete, too. □

6 Concluding remarks

In this paper we have considered a new approach to interval logics where time instants (points) and time intervals are treated as separate sorts, and have introduced the two-sorted point-interval logic $\mathcal{PI}$ that formalizes that approach. We have then focused on its fragment $\mathcal{PI}_{mix}$, involving only the inter-sort modalities and its subfragments, and have analyzed the expressiveness of its fragments with respect to fragments of $\mathcal{HS}$. Using results and adapting techniques for the latter, and we have obtained some decidability and complexity results for the former.

A number of open problems remain, including: determining the decidability status and exact complexity of the satisfiability for the fragments of $\mathcal{PI}$ on the most natural classes of models (based on all, dense, discrete, finite, etc. linear models) and obtaining complete classification of these fragments with respect to their expressiveness, analogous to (and generalizing) the one for the fragments of $\mathcal{HS}$ recently completed in [7]. The current work is intended as a beginning of their systematic exploration.

Acknowledgement

Guido Sciavicco was partially supported by the Spanish MEC project TIN2009-14372-C03-01, and the Spanish fellowship 'Ramon y Cajal' RYC-2011-07821. Valentin Goranko has been supported by a grant 24/2011-CSU 8.2.1 through the bilateral Programme for French-Danish scientific collaboration, and by the HYLO-

CORE project, funded by the Danish Natural Science Research Council. We also thank the reviewers for some corrections and useful remarks.

References

[1] Allen, J. F., *Maintaining knowledge about temporal intervals*, Communications of the ACM **26** (1983), pp. 832–843.

[2] Bresolin, D., D. Della Monica, V. Goranko, A. Montanari and G. Sciavicco, *Decidable and Undecidable Fragments of Halpern and Shoham's Interval Temporal Logic: Towards a Complete Classification*, in: *Proc. of 15th International Conferernce on Logic for Programming, Artificial Intelligence, and Reasoning (LPAR)*, LNCS **5330** (2008), pp. 590–604.

[3] Bresolin, D., D. Della Monica, V. Goranko, A. Montanari and G. Sciavicco, *Metric propositional neighborhood logics on natural numbers*, Software and Systems Modeling (2011), pp. 1–20, 10.1007/s10270-011-0195-y.
URL http://dx.doi.org/10.1007/s10270-011-0195-y

[4] Bresolin, D., V. Goranko, A. Montanari and G. Sciavicco, *Propositional interval neighborhood logics: Expressiveness, decidability, and undecidable extensions*, Annals of Pure and Applied Logic **161** (2009), pp. 289–304.

[5] Bresolin, D., A. Montanari, P. Sala and G. Sciavicco, *What's decidable about Halpern and Shoham's interval logic? the maximal fragment abbl*, in: *Proc. of the 26th Symposium on Logic in Computer Science (LICS 2011)*, 2011, p. 387–396.

[6] de Guzman, I. and C. Rossi, *Lnint: A temporal logic that combines points and intervals and the absolute and relative approaches*, JIGPL **3** (1995), p. 745–764.

[7] Della Monica, D., V. Goranko, A. Montanari and G. Sciavicco, *Expressiveness of the Interval Logics of Allens Relations on the Class of all Linear Orders: Complete Classification*, in: *Proc. of the 22nd International Joint Conference on Artificial Intelligence (IJCAI 2011)*, 2011, p. 845–850.

[8] Galton, A., "The Logic of Aspects," Clarendon Press, 1984.

[9] Galton, A., *A critical examination of Allen's theory of action and time*, Artificial Intelligence **42** (1990), p. 159–198.

[10] Goranko, V., A. Montanari and G. Sciavicco, *A road map of interval temporal logics and duration calculi*, Journal of Applied Non-Classical Logics **14** (2004), pp. 9–54.

[11] Halpern, J. and Y. Shoham, *A propositional modal logic of time intervals*, Journal of the ACM **38** (1991), pp. 935–962.

[12] Hamblin, C., *Instants and intervals*, in: J. Fraser, F. Haber and G. Mueller, editors, *The Study of Time (Volume 1)*, Springer (1972), p. 324–331.

[13] Ligozat, G., *On generalized interval calculi*, in: *Proc. of the 9th National Conference on Artificial Intelligence (AAAI 91)*, 1991, pp. 234–240.

[14] Montanari, A., G. Puppis and P. Sala, *Maximal decidable fragments of Halpern and Shoham's modal logic of intervals*, in: *Proc. of the 37th International Colloquium on Automata, Languages and Programming - Part II (ICALP)*, LNCS **6199**, 2010, pp. 345–356.

[15] Roeper, P., *Intervals and tenses*, Journal of Philosophical Logic **9** (1980), p. 451–469.

[16] Schwalb, E. and L. Vila, *Temporal constraints: A survey*, Constraints: An International Journal **3** (1998), p. 129–149.

[17] van Benthem, J., "The Logic of Time (2nd Edition)," Kluwer Academic Press, 1991.

[18] Vila, L., *IP: An instant-period based theory of time*, in: R. Rodriguez, editor, *Proc. ECAI'94 Workshop on Spatial and Temporal Reasoning*, 1994.