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$\mathcal{E}^2$-computability of e, π and Other Famous Constants

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Abstract

We show that e, π and other remarkable real numbers are limits of $\mathcal{E}^2$-computable sequences of rational numbers having a polynomial rate of convergence (as usual, $\mathcal{E}^2$ denotes the second Grzegorczyk class). However, only the rational numbers are limits of $\mathcal{E}^2$-computable sequences of rational numbers with an exponential rate of convergence

Keywords: computable real number, second Grzegorczyk class, e, π, Liouville's number, Euler's constant.

1 Introduction

The notion of computable real number is often introduced in the following way: a real number α is called computable if there exists a computable sequence $r_0, r_1, r_2, \dots$ of rational numbers such that $|r_n - \alpha| \le 2^{-n}$ for any natural number n (cf. for example [2,6]). Of course, some acceptable definition of computability for sequences of rational numbers is presupposed, say, the sequence $r_0, r_1, r_2, \dots$ is called computable if there are one-argument recursive functions f, g and h such that

$$r_n = \frac{f(n) - g(n)}{h(n) + 1} \tag{1}$$

for all natural numbers n.[2]

One could introduce subrecursive versions of computability of real numbers by replacing the class of recursive functions with some appropriate subclass of it, for

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[2] Of course such computability of a sequence of rational numbers is stronger than its computability as a sequence of computable real numbers. For instance, let φ be a two-argument recursive function such that the set $\{n \in \mathbb{N} | \exists m(\varphi(n,m) = 0)\}$ is non-recursive, and let r_n be 2^{-k} with $k = \mu m(\varphi(n,m) = 0)$ for any n in the set in question, and r_n be 0 for all other n in $\mathbb{N}$. Then the sequence of the rational numbers $r_0, r_1, r_2, \dots$ is computable as a sequence of computable real numbers, but it is not computable in the above sense.

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instance with some of the classes $\mathcal{E}^m$ introduced in [1]. The replacements with classes $\mathcal{E}^m$, where $m \geq 3$, turn out to be reasonable, since one gets sufficiently large subsets of the set of the computable real numbers. However, the result of a replacement with $\mathcal{E}^2$ is quite different, as the following proposition shows.

Proposition 1.1 *Let α be a real number, and let there exist one-argument functions f, g and h such that h belongs to the class $\mathcal{E}^2$ and for any n in $\mathbb{N}$ the rational number r_n defined by means of (1) satisfies $|r_n - \alpha| \leq 2^{-n}$. Then α is a rational number.*

Proof. Since

$$|r_n - r_{n+1}| \geq \frac{1}{(h(n)+1)(h(n+1)+1)},$$

whenever $r_n \neq r_{n+1}$, and any function from $\mathcal{E}^2$ is dominated by some polynomial, there exists a polynomial $p(n)$ such that

$$p(n)|r_n - r_{n+1}| \geq 1,$$

whenever $r_n \neq r_{n+1}$. Since

$$|r_n - r_{n+1}| \leq |r_n - \alpha| + |r_{n+1} - \alpha| \leq 3 \cdot 2^{-n-1},$$

this polynomial will satisfy the inequality

$$3p(n) \geq 2^{n+1}$$

for all n such that $r_n \neq r_{n+1}$, and therefore only finitely many such n can exist. □

Remark 1.2 A weaker result in this direction can be obtained by using Liouville's approximation theorem. Its application proves the above proposition under the additional assumption that α is an algebraic number (the possibility of such an application of Liouville's theorem is implicitly indicated in footnote 2 of [3]).

To get a reasonable definition of the notion of $\mathcal{E}^2$-computable real number, we note that 2^{-n} can be replaced with $(n+1)^{-1}$ in the definition of computability of a real number, since the definition obtained in this way will be equivalent to the other one. The same holds also for $\mathcal{E}^m$-computability of real numbers in the case of $m \geq 3$. We suggest to adopt such a definition also for $\mathcal{E}^2$-computability, namely: a sequence of rational numbers $r_0, r_1, r_2, \ldots$ is called *$\mathcal{E}^2$-computable* if there exist one-argument functions f, g and h belonging to $\mathcal{E}^2$ such that for any n in $\mathbb{N}$ the equality (1) holds, and a real number α is called *$\mathcal{E}^2$-computable* if there exists an $\mathcal{E}^2$-computable sequence of rational numbers $r_0, r_1, r_2, \ldots$ such that $|r_n - \alpha| \leq (n+1)^{-1}$ for all n in $\mathbb{N}$.

Remark 1.3 It is easy to prove the $\mathcal{E}^2$-computability of any real number α such that $p(n)(r_n - \alpha)$ is bounded for some non-constant polynomial $p(n)$ and some sequence $r_0, r_1, r_2, \ldots$ defined by means of (1) with functions f, g and h belonging to $\mathcal{E}^2$.

As shown in [5], the set of all $\mathcal{E}^2$-computable real numbers is a field containing the real roots of any non-constant polynomial with coefficients from this field. Since this implies the $\mathcal{E}^2$-computability of all real algebraic numbers, it is natural to ask whether there exist $\mathcal{E}^2$-computable transcendental numbers. A positive answer to this question is given in the present paper, in particular the numbers e and π will be shown to be $\mathcal{E}^2$-computable.

2 $\mathcal{E}^2$-computability of the number e

For any natural number k, let

$$s_k = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{k!}. \tag{2}$$

Since

$$s_k < e < s_k + \frac{1}{k!k}$$

for all positive integers k, to assure the inequality $|s_k - e| < (n+1)^{-1}$ for a given n in $\mathbb{N}$, it is sufficient to choose k in such a way that $k!k \geq n+1$. A simple choice would be $k = n+1$, but unfortunately the sequence $r_0, r_1, r_2, \ldots$, where $r_n = s_{n+1}$, is not $\mathcal{E}^2$-computable[3]. Therefore we shall proceed in a more sophisticated though natural way, namely we shall use the numbers $r_n = s_{k_n}$, where k_n is the least k satisfying $k!k \geq n+1$. These rational numbers also form a sequence $r_0, r_1, r_2, \ldots$ such that $|r_n - e| < (n+1)^{-1}$ for all n in $\mathbb{N}$. We shall prove its $\mathcal{E}^2$-computability.

Let us consider the two-argument function f_0 and the one-argument function f_1 in $\mathbb{N}$ that are defined by the equalities

$$f_0(k,n) = \min(k!, n+1), \quad f_1(n) = \max\{k \in \mathbb{N} \,|\, k!k \leq n\}.$$

These functions belong to $\mathcal{E}^2$ thanks to the equalities

$$f_0(0,n) = 1, \quad f_0(k+1,n) = \min(f_0(k,n)(k+1), n+1),$$
$$f_1(n) = \max\{k \in \mathbb{N} \,|\, k \leq n,\ f_0(k,n)k \leq n\}$$

(the second one of them follows from the equality $(k+1)! = k!(k+1)$, and for checking the third one it is appropriate to observe that any of the inequalities $k!k \leq n$ and $f_0(k,n)k \leq n$ implies the equality $k! = f_0(k,n)$, thus these two inequalities are equivalent). Clearly $k_n = f_1(n) + 1$.

Let f_2 be the two-argument function in $\mathbb{N}$ defined by the equality

$$f_2(k,n) = \min(k!s_k, n+1).$$

This function also belongs to $\mathcal{E}^2$, since we have the equalities

$$f_2(0,n) = 1, \quad f_2(k+1,n) = \min(f_2(k,n)(k+1)+1, n+1)$$

[3] This statement follows from Proposition 1.1 by the inequality $|s_{n+1} - e| < 2^{-n}$ and the irrationality of the number e (cf. also the Some comments and acknowledgments, where a direct proof of the statement is given).

(the second one of them follows from the equality $(k+1)!s_{k+1} = k!s_k(k+1)+1$). Now, by the equality $k_n = f_1(n)+1$, we have

$$r_n = \frac{k_n!s_{k_n}}{k_n!} = \frac{f_1(n)!s_{f_1(n)}(f_1(n)+1)+1}{f_1(n)!(f_1(n)+1)}.$$

Since $f_1(n)$ is one of the numbers k with $k!k \leq n$, the inequality $f_1(n)! \leq n+1$ holds (even in the case of $n=0$). Thus $f_1(n)! = f_0(f_1(n), n)$ and

$$f_1(n)!s_{f_1(n)} \leq (n+1)s_{f_1(n)} < (n+1)e < 3(n+1),$$

hence $f_1(n)!s_{f_1(n)} \leq 3n+2$ and therefore $f_1(n)!s_{f_1(n)} = f_2(f_1(n), 3n+1)$. Consequently,

$$r_n = \frac{f(n)}{h(n)+1},$$

where

$$f(n) = f_2(f_1(n), 3n+1)(f_1(n)+1)+1, \quad h(n) = f_0(f_1(n), n)(f_1(n)+1)-1.$$

Since the functions f and h belong to the class $\mathcal{E}^2$, the $\mathcal{E}^2$-computability of the sequence $r_0, r_1, r_2, \ldots$ and of the number e are thus established.

3 $\mathcal{E}^2$-computability of Liouville's number

As well-known, the first examples of transcendental real numbers were constructed by Liouville. The most famous of them is the sum of the infinite series

$$\sum_{m=1}^{\infty} \frac{1}{10^{m!}}.$$

This number is called now Liouville's number or Liouville's constant. It is sometimes denoted by L, and we shall adopt this notation here. Let

$$s_k = \sum_{m=1}^{k} \frac{1}{10^{m!}}$$

for $k = 1, 2, 3, \ldots$, and let $s_0 = 0$. Since

$$s_k < L < s_k + \frac{1}{10^{k!k}}$$

for all k, to assure the inequality $|s_k - L| < (n+1)^{-1}$ for a given n in $\mathbb{N}$, it is sufficient to choose k in such a way that $10^{k!k} \geq n+1$. We shall denote by k_n the least k satisfying the last inequality, and by setting $r_n = s_{k_n}$, $n = 0, 1, 2, \ldots$, we get a sequence $r_0, r_1, r_2, \ldots$ of rational numbers such that $|r_n - L| < (n+1)^{-1}$ for all n in $\mathbb{N}$. We shall prove the $\mathcal{E}^2$-computability of this sequence.

Let us consider the two-argument function f_3 and the one-argument function f_4 in $\mathbb{N}$ that are defined as follows:

$$f_3(m,n) = \min(10^m, n+1), \quad f_4(n) = \max\{k \in \mathbb{N} \mid 10^{k!k} \leq n\} \text{ for } n > 0, \quad f_4(0) = 0.$$

These functions belong to $\mathcal{E}^2$ thanks to the equalities

$$f_3(0,n) = 1, \quad f_3(m+1,n) = \min(10 f_3(m,n), n+1),$$
$$f_4(n) = \max\{k \in \mathbb{N} \mid k \leq n,\ f_3(f_0(k,n)k, n)\operatorname{sg} n \leq n\},$$

where f_0 is the same function as in Section 2 (to check the last of these equalities, it is appropriate to observe that in the case of $n > 0$ any of the inequalities $10^{k!k} \leq n$ and $f_3(f_0(k,n)k,n)\operatorname{sg} n \leq n$ implies the equality $10^{k!k} = f_3(f_0(k,n)k,n)\operatorname{sg} n$, thus these two inequalities are equivalent). Evidently $k_n = (f_4(n)+1)\operatorname{sg} n$.

Let f_5 be the two-argument function in $\mathbb{N}$ defined by the equality

$$f_5(k,n) = \min(10^{k!} s_k, n+1).$$

This function also belongs to $\mathcal{E}^2$, since we have the equalities

$$f_5(0,n) = 0, \quad f_5(k+1,n) = \min(f_5(k,n) f_3(f_0(k,n)k,n) + 1, n+1)$$

(the second one of them follows from the equality $10^{(k+1)!} s_{k+1} = 10^{k!} s_k 10^{k!k} + 1$).

Suppose now that n is a positive integer. Then we have $k_n = f_4(n) + 1$, hence

$$r_n = \frac{10^{k_n!} s_{k_n}}{10^{k_n!}} = \frac{10^{f_4(n)!} s_{f_4(n)} 10^{f_4(n)! f_4(n)} + 1}{10^{f_4(n)! f_4(n)} 10^{f_4(n)!}}.$$

Since $f_4(n)! \leq 10^{f_4(n)! f_4(n)} \leq n$, the equalities

$$f_4(n)! = f_0(f_4(n), n), \quad 10^{f_4(n)! f_4(n)} = f_3(f_0(f_4(n),n) f_4(n), n)$$

hold. If $n \geq 10$ then $f_4(n) \geq 1$, hence we have also $10^{f_4(n)!} \leq 10^{f_4(n)! f_4(n)} \leq n$, therefore

$$10^{f_4(n)!} = f_3(f_0(f_4(n),n), n)$$

in this case. In the same case we have also

$$10^{f_4(n)!} s_{f_4(n)} \leq n s_{f_4(n)} < nL < n,$$

therefore

$$10^{f_4(n)!} s_{f_4(n)} = f_5(f_4(n), n).$$

Thus for any $n \geq 10$ we have

$$r_n = \frac{f(n)}{h(n)+1},$$

where f and h are defined for all natural numbers n by means of the equalities

$$f(n) = f_5(f_4(n), n)f_3(f_0(f_4(n), n)f_4(n), n) + 1,$$
$$h(n) = f_3(f_0(f_4(n), n)f_4(n), n)f_3(f_0(f_4(n), n), n) - 1.$$

Since the functions f and h belong to the class $\mathcal{E}^2$, this is sufficient for a conclusion about the $\mathcal{E}^2$-computability of the sequence $r_0, r_1, r_2, \ldots$ and of the number L.

4 $\mathcal{E}^2$-computability of the number π

The author does not see a way for proving the $\mathcal{E}^2$-computability of the number π by the method used in the previous two sections. However, another method that has a larger field of applicability can be used, namely replacing the terms of the series with appropriate approximations of them.

The equality

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots$$

shows that

$$\pi = \frac{8}{1 \cdot 3} + \frac{8}{5 \cdot 7} + \frac{8}{9 \cdot 11} + \ldots$$

After setting

$$s_k = \sum_{m=0}^{k} \frac{8}{(4m+1)(4m+3)}$$

we have the inequalities

$$s_k < \pi < s_k + \frac{8}{4k+5},$$

hence

$$s_{4n+3} < \pi < s_{4n+3} + \frac{1}{2(n+1)}.$$

For any m and n in $\mathbb{N}$, let $f_6(m, n)$ be the greatest integer not exceeding the number

$$\frac{64(n+1)^2}{(4m+1)(4m+3)}.$$

Then f_6 is a function belonging to $\mathcal{E}^2$, and for all m and n in $\mathbb{N}$ the inequalities

$$\frac{f_6(m,n)}{8(n+1)^2} \leq \frac{8}{(4m+1)(4m+3)} < \frac{f_6(m,n)+1}{8(n+1)^2}$$

hold. Therefore, if we set

$$r_n = \frac{1}{8(n+1)^2} \sum_{m=0}^{4n+3} f_6(m, n),$$

then

$$r_n \leq s_{4n+3} < r_n + \frac{1}{2(n+1)},$$

hence

$$r_n < \pi < r_n + \frac{1}{n+1}.$$

To complete the proof, it is sufficient to show the $\mathcal{E}^2$-computability of the function

$$f_7(n) = \sum_{m=0}^{4n+3} f_6(m,n).$$

This can be done by showing the $\mathcal{E}^2$-computability of the function

$$f_8(k,n) = \sum_{m=0}^{k} f_6(m,n),$$

and its $\mathcal{E}^2$-computability can be seen by observing that

$$f_7(n) = 8(n+1)^2 r_n < 8(n+1)^2\pi < 26(n+1)^2,$$

and $f_6(m,n) = 0$ for any m greater than $4n+3$, hence $f_8(k,n) < 26(n+1)^2$ for all k and n in $\mathbb{N}$.

Remark 4.1 To make the proof as simple as possible, we used a simple representation of π that, unfortunately, is not convenient for its numerical computation. Actually other representations of π could be also used.

5 $\mathcal{E}^2$-computability of Euler's constant

To prove that Euler's constant γ is $\mathcal{E}^2$-computable, we shall use its representation

$$\gamma = \sum_{m=1}^{\infty}\left(\frac{1}{m} - \ln\left(1+\frac{1}{m}\right)\right),$$

as well as the equality

$$\frac{1}{m} - \ln\left(1+\frac{1}{m}\right) = \frac{1}{2m^2} - \frac{1}{3m^3} + \frac{1}{4m^4} - \frac{1}{5m^5} + \frac{1}{6m^6} - \frac{1}{7m^7} + \dots$$

From here, we get the equality

$$\gamma = \sum_{m=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{m^{2j}}\left(\frac{1}{2j} - \frac{1}{(2j+1)m}\right)$$

and we see that

$$0 < \sum_{j=k+1}^{\infty}\frac{1}{m^{2j}}\left(\frac{1}{2j} - \frac{1}{(2j+1)m}\right) < \frac{1}{2(k+1)m^{2(k+1)}}$$

for $m = 1, 2, 3, \ldots,\ \ k = 0, 1, 2, 3, \ldots$ Let

$$s_k = \sum_{m=1}^{k} \sum_{j=1}^{k} \frac{1}{m^{2j}} \left(\frac{1}{2j} - \frac{1}{(2j+1)m} \right)$$

for any positive integer k. We have

$$\gamma = s_k + \sum_{m=1}^{k} \sum_{j=k+1}^{\infty} \frac{1}{m^{2j}} \left(\frac{1}{2j} - \frac{1}{(2j+1)m} \right) + \sum_{m=k+1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{m^{2j}} \left(\frac{1}{2j} - \frac{1}{(2j+1)m} \right),$$

hence

$$s_k < \gamma < s_k + \sum_{m=1}^{k} \frac{1}{2(k+1)m^{2(k+1)}} + \sum_{m=k+1}^{\infty} \frac{1}{2m^2} < s_k + \frac{1}{k+1} + \frac{1}{2k} \leq s_k + \frac{2}{k+1}.$$

Therefore

$$s_{4n+3} < \gamma < s_{4n+3} + \frac{1}{2(n+1)}$$

for any n in $\mathbb{N}$.

For any j, m and n in $\mathbb{N}$, let $f_9(j, m, n)$ be the greatest integer not exceeding the number

$$\frac{2(n+1)(4n+3)^2}{m^{2j}} \left(\frac{1}{2j} - \frac{1}{(2j+1)m} \right)$$

if $j > 0$ and $m > 0$, and let $f_9(j, m, n)$ be 0 otherwise. Then for all positive integers j, m and all n in $\mathbb{N}$ the inequalities

$$\frac{f_9(j, m, n)}{2(n+1)(4n+3)^2} \leq \frac{1}{m^{2j}} \left(\frac{1}{2j} - \frac{1}{(2j+1)m} \right) < \frac{f_9(j, m, n) + 1}{2(n+1)(4n+3)^2}$$

hold. Therefore, if we set

$$r_n = \frac{1}{2(n+1)(4n+3)^2} \sum_{m=1}^{4n+3} \sum_{j=1}^{4n+3} f_9(j, m, n),$$

then

$$r_n \leq s_{4n+3} < r_n + \frac{1}{2(n+1)},$$

hence

$$r_n < \gamma < r_n + \frac{1}{n+1}.$$

To complete the proof, it is sufficient to show the $\mathcal{E}^2$-computability of the function

$$f_{10}(n) = \sum_{m=1}^{4n+3} \sum_{j=1}^{4n+3} f_9(j, m, n).$$

To achieve this, we shall first prove the $\mathcal{E}^2$-computability of the function f_9. We note that $f_9(j,m,n) = 0$, whenever $m^{2j} \geq 2(n+1)(4n+3)^2$. With regard to this, we consider the function

$$f_{11}(j,m,n) = \min(m^{2j}, 2(n+1)(4n+3)^2).$$

This function belongs to the class $\mathcal{E}^2$ thanks to the equalities

$$f_{11}(0,m,n) = 1, \quad f_{11}(j+1,m,n) = \min(f_{11}(j,m,n)m^2, 2(n+1)(4n+3)^2).$$

Now the $\mathcal{E}^2$-computability of f_9 can be seen by observing that for non-zero values of j and m the value $f_9(j,m,n)$ is the greatest integer not exceeding the number

$$\frac{2(n+1)(4n+3)^2}{f_{11}(j,m,n)}\left(\frac{1}{2j} - \frac{1}{(2j+1)m}\right).$$

Once the $\mathcal{E}^2$-computability of f_9 is established, the $\mathcal{E}^2$-computability of f_{10} can be easily derived from the fact that

$$f_{10}(n) = 2(n+1)(4n+3)^2 r_n < 2(n+1)(4n+3)^2\gamma < 2(n+1)(4n+3)^2.$$

For instance, one may use the equality $f_{10}(n) = f_{12}(4n+3, 4n+3, n)$, where f_{12} is defined as follows: we consider the function f_9' such that $f_9'(j,m,n) = f_9(j,m,n)$ if $j \leq 4n+3$, $m \leq 4n+3$, and $f_9'(j,m,n) = 0$ otherwise, then we set

$$f_{12}(k,l,n) = \sum_{m=1}^{k}\sum_{j=1}^{l} f_9'(j,m,n)$$

and, making use of the inequalities

$$\sum_{j=1}^{l} f_9'(j,m,n) < 2(n+1)(4n+3)^2, \quad f_{12}(k,l,n) < 2(n+1)(4n+3)^2,$$

we show that f_{12} belongs to $\mathcal{E}^2$.

6 Some comments and acknowledgments

Although our proofs concern only four concrete real numbers, the methods used in the proofs or similar ones can be applied in many other cases. It seems that $\mathcal{E}^2$-computability of real numbers is present much more often than one could expect.

Several characterizations of the class $\mathcal{E}^2$ are known that are in the terms of computational complexity, for instance the characterization from [4] according to which a function belongs to $\mathcal{E}^2$ iff it can be computed on a linear tape bounded Turing machine in the case of binary encoding of inputs and outputs. As the referee indicated, such characterizations could be useful for comparison with already known

results and for further studies, and, in particular, the characterization from [4] allows relating complexity of real functions as in [2,6] to $\mathcal{E}^2$-computability. The author thanks the referee for his or her remarks.

References

[1] Grzegorczyk, A., "Some Classes of Recursive Functions," Dissertationes Math. (Rozprawy Mat.), **4**, Warsaw, 1953.

[2] Ko, Ker-I, "Complexity Theory of Real Functions," Progress in Theoretical Computer Science, Birkhäuser, Boston, 1991.

[3] Peshev, P., and D. Skordev, *A subrecursive refinement of the fundamental theorem of algebra*, Lecture Notes in Computer Science, **3988** (2006), 435–444.

[4] Ritchie, R. W., *Classes of predictably computable functions*, Transactions of the AMS, **106** (1963), 139–173.

[5] Skordev, D., *Computability of real numbers by using a given class of functions in the set of the natural numbers*, Math. Log. Quart. **48** (2002), Suppl. 1, 91–106.

[6] Weihrauch, K., "Computable Analysis. An Introduction," Springer–Verlag, Berlin/Heidelberg, 2000.

7 Appendix

For any natural number k, let s_k be the approximation of e defined by (2). The integer $k!s_k$ is never divisible by 3. This can be shown by means of an inductive proof of the following statement: the remainder of the division of $k!s_k$ by 3 is 1 if k is divisible by 3, and this remainder is 2 otherwise (the equalities $0!s_0 = 1$ and $(k+1)!s_{k+1} = k!s_k(k+1)+1$ are used in the proof).

Now consider any representation of the numbers s_k in the form

$$s_k = \frac{p_k}{q_k+1}, \quad k = 0, 1, 2, \dots ,$$

where $p_0, p_1, p_2, \dots$ and $q_0, q_1, q_2, \dots$ are natural numbers. Then $(k!s_k)(q_k+1) = k!p_k$ for all k. For any natural number l, if $k \geq 3l$ then $k!$ is divisible by 3^l, hence q_k+1 is also divisible by 3^l, thus $q_k + 1 \geq 3^l$ holds. Therefore if the sequence $q_0, q_1, q_2, \dots$ or some infinite subsequence of it is regarded as a one-argument function in $\mathbb{N}$ then this function cannot be dominated by a polynomial, hence it does not belong to $\mathcal{E}^2$.

Remark 7.1 Although the sequence $s_0, s_1, s_2, \dots$ is not $\mathcal{E}^2$-computable as a sequence of rational numbers, it is $\mathcal{E}^2$-computable as a sequence of $\mathcal{E}^2$-computable real numbers, namely there exist two-argument functions f and h belonging to $\mathcal{E}^2$ such that

$$\left| \frac{f(m,n)}{h(m,n)+1} - s_m \right| \leq \frac{1}{n+1}$$

for all natural numbers m and n. To show the existence of such functions, let us set

$$k_{m,n} = \min\{k \in \mathbb{N} \mid k = m \;\text{ or }\; k!k \geq n+1\}$$

for any m and n in $\mathbb{N}$. Then

$$s_{k_{m,n}} \le s_m < s_{k_{m,n}} + \frac{1}{n+1},$$

and

$$s_{k_{m,n}} = \frac{f(m,n)}{h(m,n)+1}$$

holds with appropriately chosen f and h in $\mathcal{E}^2$. They can be constructed as follows. We consider the two-argument function f_1' such that

$$f_1'(m,n) = \max\{k \in \mathbb{N} \mid k < m,\ k!k \le n\}$$

if $m > 0$, and $f_1'(m,n) = 0$ otherwise. Then $k_{m,n} = (f_1'(m,n)+1)\mathrm{sg}\, m$, and the function f_1' belongs to $\mathcal{E}^2$ since in the case of $m > 0$ we have

$$f_1'(m,n) = \max\{k \in \mathbb{N} \mid k < m,\ f_0(k,n)k \le n\},$$

where f_0 is the same function as in Section 2. Having the function f_1' at our disposal, we set

$$\begin{aligned} f(m,n) &= f_2(f_1'(m,n),n)(f_1'(m,n)+1)+1, \\ h(m,n) &= f_0(f_1'(m,n),n)(f_1'(m,n)+1)-1 \end{aligned}$$

in the case of $m > 0$, where f_2 has the same meaning as in Section 2, and we additionally set $f(0,n) = 1$, $h(0,n) = 0$.