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# A Connection Between Concurrency and Language Theory

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**Abstract**

We show that three fixed point structures equipped with (sequential) composition, a sum operation, and a fixed point operation share the same valid equations. These are the theories of (context-free) languages, (regular) tree languages, and simulation equivalence classes of (regular) synchronization trees (or processes). The results reveal a close relationship between classical language theory and process algebra.

*Keywords:* Fixed point operations, iteration theories, context-free languages, regular tree languages, synchronization trees, simulation equivalence

## 1 Introduction

Iteration theories [8] capture the equational properties of fixed point operations including the least fixed point operation over continuous or monotone functions over cpo's or complete lattices, in rational algebraic theories [33,40], in theories of monotone functions over partially ordered sets with enough least (pre-)fixed points [19], or the initial fixed point operation over continuous functors over categories with directed colimits [7] or more generally, in theories of functors with enough initial algebras [31], or the unique fixed point operation in Elgot's (pointed) iterative theories [17], or the fixed point operation in theories of trees and synchronization trees, and many other structures.

It was argued in [8,10] that all natural cartesian fixed point models lead to iteration theories. Moreover, it was proved in [37] that essentially every nontrivial subclass of iteration theories obeying a natural condition satisfies exactly the equations of iteration theories.

But several models have an additional structure, such as a nondeterministic choice operation, or more generally, an additive structure, which interacts with the cartesian operations and the fixed point operation in a nontrivial way. The relationship between the iteration theory structure and the additional operations has been the subject of several papers, including [1,9,13,22,12,20,23,24,29,30] and

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the recent [26]. In many cases, it was possible to capture this relationship by a finite number of equational (or sometimes quasi-equational) axioms. As a byproduct of these results, it was possible to give complete (though infinite) sets of equational axioms and finite sets of quasi-equational or more general first-order axioms for various bisimulation and trace based process behaviors, rational power series and regular languages, regular tree languages, and many other models.

The equational theory of simulation equivalence classes of (regular) synchronization trees over a set of action symbols, equipped with the cartesian operations, the least fixed point operation *and sum*, has a finite equational axiomatization relatively to iteration theories [22]. Incidentally, the very same equations hold for continuous or monotone functions over complete lattices equipped with the least fixed point operation and the pointwise binary supremum operation as sum, or in all '($\omega$-)continuous idempotent grove theories'. In this paper, our main new contribution is that two more well-known classes of structures relevant to computer science are of this sort, the theories of (regular) tree languages and the theories of (context-free) languages (Theorem 3.2). In our argument, we will make use of a concrete characterization of the free $\omega$-continuous idempotent grove theories, which is a result of independent interest (cf. Theorem 4.3). The facts proved in the paper reveal a close relationship between models of concurrency, automata and language theory, and models of denotational semantics.

The results of this paper can be formulated in several different formalism including '$\mu$-terms', 'letrec expressions', or cartesian categories. We have chosen the simple language of Lawvere theories, i.e., cartesian categories generated by a single object. The extension of the results to many-sorted theories is straightforward.

## 2 Theories

In any category, we write the composition $f \cdot g$ of morphisms $f : a \to b$ and $g : b \to c$ in diagrammatic order, and we let $\mathbf{1}_a$ denote the identity morphism $a \to a$. For an integer $n \geq 0$, we let $[n]$ denote the set $\{1, \ldots, n\}$. When $n = 0$, this set is empty.

A *(Lawvere) theory* [34,8] is a small category $T$ whose objects are the nonnegative integers such that each object $n$ is the $n$-fold coproduct of object 1 with itself. The hom-set of morphisms $n \to p$ of a theory $T$ is denoted $T(n, p)$. We assume that every theory comes with distinguished coproduct injections $i_n : 1 \to n$, $i \in [n]$, $n \geq 0$. Thus, for any sequence of morphisms $f_1, \ldots, f_n : 1 \to p$, there is a unique morphism $f : n \to p$ with $i_n \cdot f = f_i$, for all $i \in [n]$. We denote this unique morphism $f$ by $\langle f_1, \ldots, f_n \rangle$ and call it the *tupling* of the $f_i$. When $n = 0$, we also write $0_p$. Since 0 is initial object, $0_p$ is the unique morphism $0 \to p$. It is clear that $\mathbf{1}_n = \langle 1_n, \ldots, n_n \rangle$ for all $n \geq 0$. We require that $\mathbf{1}_1 = 1_1$, so that $\langle f \rangle = f$ for all $f : 1 \to p$. Since the object $n + m$ is the coproduct of objects $n$ and $m$ with respect to the coproduct injections

$$\kappa_{n,n+m} = \langle 1_{n+m}, \ldots, n_{n+m} \rangle : n \to n + m$$
$$\lambda_{m,n+m} = \langle (n+1)_{n+m}, \ldots, (n+m)_{n+m} \rangle : m \to n + m,$$

each theory is equipped with a *pairing* operation mapping a pair of morphisms $(f, g)$

with $f : n \to p$ and $g : m \to p$ to $\langle f, g\rangle : n + m \to p$:

$$\begin{tikzcd}
n \arrow[r, "\kappa"] \arrow[dr, "f"'] & n+m \arrow[d, "\langle f,g\rangle"] & m \arrow[l, "\lambda"'] \arrow[dl, "g"] \\
 & p &
\end{tikzcd}$$

The pairing operation is associative and satisfies $\langle f, 0_p\rangle = f = \langle 0_p, f\rangle$ for all $f : n \to p$.

Also, we can define for $f : n \to p$ and $g : m \to q$ the morphism $f \oplus g : n + m \to p + q$ as $\langle f \cdot \kappa_{p,p+q},\ g \cdot \lambda_{q,p+q}\rangle$. Then $f \oplus g$ is the unique morphism $n + m \to p + q$ with

$$\kappa_{n,n+m} \cdot (f \oplus g) = f \cdot \kappa_{p,p+q}$$
$$\lambda_{m,n+m} \cdot (f \oplus g) = g \cdot \lambda_{q,p+q}.$$

$$\begin{tikzcd}
n \arrow[r, "\kappa"] \arrow[d, "f"'] & n+m \arrow[d, "f\oplus g"] & m \arrow[l, "\lambda"'] \arrow[d, "g"] \\
p \arrow[r, "\kappa"'] & p+q & q \arrow[l, "\lambda"]
\end{tikzcd}$$

The $\oplus$ operation is associative, and $0_0 \oplus f = f = f \oplus 0_0$ for all $f : n \to p$. Also,

$$(f \oplus g) \cdot \langle h, k\rangle = \langle f \cdot h,\ g \cdot k\rangle$$

for all $f : n \to p$, $g : m \to q$, $h : p \to r$ and $k : q \to r$.

Each theory $T$ may be seen as a many-sorted algebra, whose set of sorts is the set $\mathbb{N} \times \mathbb{N}$ of all ordered pairs of nonnegative integers, satisfying certain equational axioms, see e.g. [8]. Morphisms of theories are functors preserving objects and distinguished morphisms. It follows that any theory morphism preserves the tupling, (and pairing) operations. The kernels of theory morphisms are called *theory congruences*. The *quotient* $T/\equiv$ of a theory $T$ with respect to a theory congruence is defined as usual. A *subtheory* of a theory $T$ is a theory $T'$ whose set of morphisms is included in the morphisms of $T$ such that the natural embedding of $T'$ into $T$ is a theory morphism $T' \to T$. See [8] for more details.

We end this section by providing some examples.

Let $X = \{x_1, x_2, \ldots\}$ denote a fixed countably infinite set of variables, and let $A$ be a set disjoint from $X$. For each $p \geq 0$, let $X_p = \{x_1, \ldots, x_p\}$. The theory $\mathbf{W}_A$ has as morphisms $1 \to p$ all words in $(A \cup X_p)^*$. A morphism $n \to p$ is an $n$-tuple of morphisms $1 \to p$. For morphisms $u = (u_1, \ldots, u_n) : n \to p$ and $v = (v_1, \ldots, v_p) : p \to q$, we define $u \cdot v = (u_1 \cdot v, \ldots, u_n \cdot v)$, where for each $i \in [n]$, $u_i \cdot v$ is the word obtained from $u_i$ by substituting a copy of $v_j$ for each occurrence of the variable $x_j$ in $u_i$, for all $j \in [p]$. Equipped with this composition operation and the morphisms $\mathbf{1}_n = (x_1, \ldots, x_n) : n \to n$ as identity morphisms, $\mathbf{W}_A$ is a category. In fact, $\mathbf{W}_A$ is a theory with distinguished morphisms $i_n = x_i : 1 \to n$, $i \in [n]$, $n \geq 0$.

Suppose now that $\Sigma = \biguplus_{k\geq 0}\Sigma_k$ is a *ranked set* which is disjoint from $X$. We may view $\Sigma$ as a pure set and form the theory $\mathbf{W}_\Sigma$. Consider the subtheory $\mathbf{Tree}_\Sigma$ of $\mathbf{W}_\Sigma$ consisting of the $\Sigma$-trees (or $\Sigma$-terms). A morphism $1 \to p$ in $\mathbf{Tree}_\Sigma$ is a well-formed word in $(\Sigma \cup X_p)^*$ which is either a variable in $X_p$ or a word of the form $\sigma t_1 \dots t_k$ for a letter $\sigma \in \Sigma_k$ and trees $t_1, \dots, t_k : 1 \to p$. A morphism $n \to p$ is an $n$-tuple of morphisms $n \to p$. It is well-known that the theory $\mathbf{Tree}_\Sigma$ is the free theory, freely generated by $\Sigma$. Indeed, each letter $\sigma \in \Sigma_n$ may be identified with a tree in $\mathbf{Tree}_\Sigma(1, n)$ so that given any theory $T$ and rank preserving function $\varphi : \Sigma \to T$, there is a unique theory morphism $\varphi^\sharp : \mathbf{Tree}_\Sigma \to T$ extending $\varphi$.

**Remark 2.1** *If in the previous example $\Sigma$ is empty, then we obtain the initial theory $\Theta$. A morphism $n \to p$ of this initial theory is a tupling of distinguished morphisms (i.e., variables) and may be identified with a function $[n] \to [p]$, so that composition corresponds to composition of functions. A* base morphism *of a theory $T$ is a morphism that arises as the image of a morphism in the initial theory with respect to the unique theory morphism $\Theta \to T$. For example, the base morphisms $n \to p$ in a theory $\mathbf{W}_A$ are the morphisms of the form $(x_{1\rho}, \dots, x_{n\rho})$, where $\rho$ is a function $[n] \to [p]$. In any nontrivial theory, we may faithfully represent base morphisms $n \to p$ as functions $[n] \to [p]$.*

## 3 Statement of the main result

By taking sets of morphisms of a theory $T$, we may *sometimes* define a new theory $P(T)$. (For a more general construction, the reader is referred to [11].) The morphisms $1 \to p$ in $P(T)$ are all sets $L \subseteq T(1, p)$. A morphism $n \to p$ is an $n$-tuple $(L_1, \dots, L_n)$ of morphisms $1 \to p$, including the tuple $0_{n,p} = (\emptyset, \dots, \emptyset)$. To define composition, suppose that $L : 1 \to p$ and $K = (K_1, \dots, K_p) : p \to q$. Then we define $L \cdot K : 1 \to q$ to be the set of all morphisms $1 \to q$ in $T$ of the form

$$f \cdot \langle g_1, \dots, g_m\rangle$$

such that $f : 1 \to m$ in $T$ and there is a base morphism $\rho : m \to p$ with $f \cdot \rho \in L$ and $g_i \in K_{i\rho}$ for all $i \in [m]$. When $L = (L_1, \dots, L_n) : n \to p$, we define $L \cdot K$ as the morphism $(L_1 \cdot K, \dots, L_n \cdot K) : n \to q$. For each $n \geq 0$, the identity morphism $\mathbf{1}_n$ is the morphism $(\{1_n\}, \dots, \{n_n\}) : n \to n$, and the $i$th distinguished morphism $1 \to n$ is $\{i_n\}$. When $P(T)$ is a theory, we call it a *power-set theory.*

Suppose that $P(T)$ is a power-set theory. We may also equip $P(T)$ with a sum operation, denoted $+$ and defined by component-wise set union. We define

$$L + L' = (L_1 \cup L'_1, \dots, L_n \cup L'_n) : n \to p,$$

for all $L = (L_1, \dots, L_n) : n \to p$ and $L' = (L'_1, \dots, L'_n) : n \to p$ in $P(T)$. It is clear that, equipped with the operation $+$ and the constant $0_{n,p}$, each hom-set $P(T)(n, p)$ is a commutative, idempotent monoid. Moreover,

$$i_n \cdot (L + L') = i_n \cdot L + i_n \cdot L' \tag{1}$$
$$i_n \cdot 0_{n,p} = 0_{1,p} \tag{2}$$
$$(L + L') \cdot K = L \cdot K + L' \cdot K \tag{3}$$
$$0_{n,p} \cdot K = 0_{n,q}, \tag{4}$$

for all $L, L' : n \to p$ and $K : p \to q$. (Here, we adapt the convention that composition has higher precedence than sum.) Thus, $P(T)$ is an *idempotent grove theory*, cf. [8] or Section 4.

The above power-set construction is applicable to the theories $\mathbf{W}_A$ and $\mathbf{Tree}_\Sigma$, yielding the idempotent grove theories $\mathbf{Lang}_A$ of languages over $A$ and $\mathbf{TreeLang}_\Sigma$ of tree languages over $\Sigma$. In $\mathbf{Lang}_A$, composition is the usual operation of 'language substitution'. In $\mathbf{TreeLang}_\Sigma$, it corresponds to the 'OI-substitution' of [16].

Any power-set theory $P(T)$ is naturally equipped with a partial order $\subseteq$ defined by component-wise set inclusion. It is clear that each hom-set $P(T)(n, p)$ is a complete lattice with least element $0_{n,p}$, moreover, the theory operations are monotone, in fact continuous. (Composition preserves all suprema in its first argument, and tupling preserves all suprema in each of its arguments.) Thus, we can define a *dagger operation* $^\dagger : T(P)(n, n + p) \to T(P)(n, p)$ $(n, p \geq 0)$, $L \mapsto L^\dagger$, by taking the least solution of the fixed point equation $X = L \cdot \langle X, \mathbf{1}_p \rangle$. In particular, the theories $\mathbf{Lang}_A$ and $\mathbf{TreeLang}_\Sigma$ are also equipped with a dagger operation. The least subtheory of $\mathbf{Lang}_A$ containing the finite languages which is closed under dagger is the theory $\mathbf{CFL}_A$ of context-free languages, and the least subtheory of $\mathbf{TreeLang}_A$ containing the finite tree languages which is closed under dagger is the theory $\mathbf{Reg}_\Sigma$ of regular tree languages [20,24,32]. Both $\mathbf{CFL}_A$ and $\mathbf{Reg}_\Sigma$ are idempotent grove theories.

We define yet another class of theories equipped with both an additive structure and a dagger operation, the theories of simulation equivalence classes of synchronization trees. A *hyper-tree* consists of a countable set $V$ of vertices and a countable set $E$ of edges, each edge $e$ having a source $v$ in $V$ and an ordered sequence of target vertices $(v_1, \ldots, v_n) \in V^n$, for some $n \geq 0$. There is a distinguished vertex, the *root* $v_0$, such that each vertex $v$ is the target vertex of a unique path from $v_0$ to $v$. An isomorphism between hyper-trees is determined by a bijection between the vertices and a bijection between the edges that jointly preserve the root and the source and target of the edges.

Synchronization trees over a set $A$ of *action symbols* were defined in [39]. A (slight) generalization of synchronization trees for ranked sets is given in [22]. Suppose that $\Sigma$ is a ranked set. A *synchronization tree* $t = (V_t, E_t, \lambda_t) : 1 \to p$ over $\Sigma$ is a hyper-tree with vertex set $V_t$, hyper-edges $E_t$, equipped with a labeling function $\lambda_t : E_t \to \Sigma \cup \{\mathsf{ex}_1, \ldots, \mathsf{ex}_p\}$, where the $\mathsf{ex}_i$ are referred to as the *exit symbols*. Each hyper-edge $e : v \to (v_1, \ldots, v_n)$ with source $v$ and target $(v_1, \ldots, v_n)$ is labeled in $\Sigma_n$, when $n \geq 1$, or by an exit symbol or a symbol in $\Sigma_0$, when $n = 0$. When $t$ is a synchronization tree and $v$ is a vertex of $t$, then the vertices 'accessible' from $v$ (including $v$) span the *subtree* $t|_v$. The edges of $t|_v$ are those edges of $t$ having a source accessible from $v$. An isomorphism between synchronization trees is an isomorphism of the underlying hyper-trees which preserves the labeling. We usually identify isomorphic synchronization trees. A synchronization tree $n \to p$ over $\Sigma$ is an $n$-tuple $(t_1, \dots, t_n)$ of synchronization trees $1 \to p$ over $\Sigma$. A synchronization tree $t : 1 \to p$ is *finite* if its set of edges is finite (and thus its vertex set is also finite), *finitely branching* if each vertex is the source of a finite number of edges, and *regular*, if it has a finite number of subtrees (up to isomorphism) and only a finite number of letters from $\Sigma$ appear as edge labels. A synchronization tree $t : n \to p$ is finite (finitely branching, regular, resp.) if its components are all finite (finitely branching, regular, resp.).

We may identify each letter $\sigma \in \Sigma_n$ with the finite synchronization tree $1 \to n$ having and edge $v_0 \to (v_1, \dots, v_n)$ labeled $\sigma$, where $v_0$ is the root, and an edge originating in $v_i$ labeled $\mathsf{ex}_i$ for each $i \in [n]$. In the same way, we may view each exit symbol $\mathsf{ex}_i$ as a tree $1 \to n$ for each $i \in [n]$, $n \geq 0$.

Synchronization trees over $\Sigma$ form a theory $\mathbf{ST}_\Sigma$. When $t : 1 \to p$ and $t' = (t'_1, \dots, t'_p) : p \to q$, then $t \cdot t' : 1 \to q$ is constructed from $t$ by replacing each edge of $t$ labeled $\mathsf{ex}_i$ for some $i \in [p]$ by a copy of $t'_i$. When $t = (t_1, \dots, t_n) : n \to p$, then $t \cdot t' = (t_1 \cdot t', \dots, t_n \cdot t') : n \to q$. For each $i \in [n]$, the distinguished morphism $i_n$ is the tree having a single edge labeled $\mathsf{ex}_i$. For synchronization trees $t, t' : 1 \to p$, we also define $t + t' : 1 \to p$ as the tree obtained from (disjoint copies of) $t$ and $t'$ by merging the roots. When $t = (t_1, \dots, t_n) : n \to p$ and $t' = (t'_1, \dots, t'_n) : n \to p$, then $t+t' = (t_1+t'_1, \dots, t_n+t'_n) : n \to p$. We define $0_{1,p}$ as the tree $1 \to p$ having no edge, and $0_{n,p} = (0_{1,p}, \dots, 0_{1,p}) : n \to p$, for all $n, p \geq 0$. Clearly, each hom-set of $\mathbf{ST}_\Sigma$ is a commutative monoid and (1)–(4) hold, so that $\mathbf{ST}_\Sigma$ is a *grove theory* [8]. We also define the grove theories $\mathbf{FST}_\Sigma$ of finite and $\mathbf{RST}_\Sigma$ of regular synchronization trees over $\Sigma$.

Suppose that $t$ and $t'$ are synchronization trees $1 \to p$ over $\Sigma$. A *simulation* [35,36] $t \to t'$ is a relation $R \subseteq V_t \times V_{t'}$, relating the roots such that whenever $e : v \to (v_1, \dots, v_n)$ is an edge of $t$ and $vRv'$, then there is an equally labeled edge $e' : v' \to (v'_1, \dots, v'_n)$ of $t'$ such that $v_i R v'_i$ for all $i$. Note that the domain of a simulation $R : t \to t'$ is $V_t$. For later use we also define a *morphism* $t \to t'$ to be a simulation $\tau$ which is a function $V_t \to V_{t'}$. Thus, a morphism is a *functional simulation*. Note that when $R$ is a simulation $t \to t'$, then $R$ contains a function $\tau$ which is a morphism. Indeed, for a vertex $v \in V_f$ at distance $n$ from the root, we define $v\tau$ as follows. When $n = 0$ so that $v$ is the root of $f$, then let $v\tau$ be the root of $g$. Suppose now that $n > 0$ and let $v$ be one of the target vertices of the edge $e : u \to (u_1, \dots, u_m)$ of $f$, say $v = u_i$. Then $u\tau$ is already defined so that $uR(u\tau)$ holds, and there is an (equally labeled) edge $e' : u\tau \to (u'_1, \dots, u'_m)$ with $u_j R u'_j$ for all $j$. We define $v\tau = u'_i$.

It is well-known that simulations compose, so that if $t, t', t'' : 1 \to p$ and $R$ is a simulation $t \to t'$ and $R'$ is a simulation $t' \to t''$, then the relational composition of $R$ and $R'$ is a simulation $t \to t''$. When $t = (t_1, \dots, t_n)$ and $t' = (t'_1, \dots, t'_n)$ are synchronization trees $n \to p$, a simulation $t \to t'$ is an $n$-tuple $(R_1, \dots, R_n)$, where each $R_i$ is a simulation $t_i \to t'_i$. We say that $t$ and $t'$ are *simulation equivalent*, denoted $t \equiv_s t'$, if there are simulations $t \to t'$ and $t' \to t$. The relation $\equiv_s$ is a *grove theory congruence* of $\mathbf{ST}_\Sigma$, i.e., a theory congruence which preserves the sum operation, giving rise to the grove theory $\mathbf{SST}_\Sigma = \mathbf{ST}_\Sigma / \equiv_s$. We will denote the simulation equivalence class of a tree $t$ by $[t]_s$, or sometimes just $[t]$. Moreover, when $t = (t_1, \ldots, t_n) : n \to p$, we identify $[t]_s$ with $([t_1]_s, \ldots, [t_n]_s)$.

We define the relation $t \sqsubseteq_s t'$ for synchronization trees $t, t' : n \to p$ iff there is a simulation $t \to t'$. Also, we define $[t]_s \sqsubseteq_s [t']_s$ iff $t \sqsubseteq_s t'$, since the definition is independent of the choice of the representatives of the equivalence classes. Since simulations compose, the relation $\sqsubseteq_s$ is a pre-order on synchronization trees and a partial order on simulation equivalence classes. Each hom-set of $\mathbf{SST}_\Sigma$ has all countable suprema. Indeed, when $t_i$, $i \in I$, is a countable family of trees $1 \to p$, then $\sup_{i \in I} [t_i]_s = [t]_s$ for the tree $t = \sum_{i \in I} t_i : 1 \to p$ obtained by taking the disjoint union of the $t_i$ and identifying the roots. When $I$ is empty, the sum is the tree $0_{1,p}$. More generally, when $t_i : n \to p$, for all $i \in I$, then $\sup_{i \in I} t_i$ is the tree $t : n \to p$ such that for each $j \in [n]$, $j_n \cdot t = \sum_{i \in I} j_n \cdot t_i$.

The theory operations are $\omega$-continuous, so that we can define a dagger operation. For each $f = [t]_s : n \to n + p$ in $\mathbf{SST}_\Sigma$, $f^\dagger : n \to p$ is the least solution of the fixed-point equation $x = f \cdot \langle x, \mathbf{1}_p \rangle$. The least subtheory of $\mathbf{SST}_\Sigma$ containing the finite synchronization trees which is closed under dagger is the theory $\mathbf{SRST}_\Sigma$ of simulation equivalence classes containing at least one regular tree. Further, we denote by $\mathbf{SFST}_\Sigma$ the subtheory determined by those simulation equivalence classes containing at least one finite synchronization tree. Both $\mathbf{SRST}_\Sigma$ and $\mathbf{SFST}_\Sigma$ are closed under the sum operation, and both of them are grove theories. Note that we may identify $\mathbf{SRST}_\Sigma$ with $\mathbf{RST}_\Sigma / {\equiv_s}$ and $\mathbf{SFST}_\Sigma$ with $\mathbf{FST}_\Sigma / {\equiv_s}$.

**Remark 3.1** *It is known, cf. [8,13], that the dagger operation may also be defined on $\mathbf{ST}_\Sigma$, by taking 'initial solutions' of fixed point equations $x = f \cdot \langle x, \mathbf{1}_p \rangle$ for $f : n \to n+p$. The subtheory $\mathbf{RST}_\Sigma$ is closed under this dagger operation. Moreover, it turns out that simulation equivalence becomes a congruence as does the bisimilarity relation (see below).*

A *term* is a well-formed expression composed of morphism variables and constants for the distinguished morphisms using the theory operations, sum, and dagger. Each term has a source $n$ and a target $p$, for some nonnegative integers $n, p$.

We are now ready to state our main result. We may view each set $A$ as a ranked set where each letter has rank 1.

**Theorem 3.2** *The following conditions are equivalent for terms $t, t' : n \to p$.*

*(i) The identity $t = t'$ holds in all power-set theories $P(T)$, where $T$ is a theory.*

*(ii) The identity $t = t'$ holds in all theories $\mathbf{Lang}_A$ (or $\mathbf{CFL}_A$), where $A$ is a set.*

*(iii) The identity $t = t'$ holds in all theories $\mathbf{TreeLang}_\Sigma$ (or $\mathbf{Reg}_\Sigma$), where $\Sigma$ is a ranked set.*

*(iv) The identity $t = t'$ holds in all theories $\mathbf{SST}_\Sigma$ (or $\mathbf{SRST}_\Sigma$), where $\Sigma$ is a ranked set.*

*(v) The identity $t = t'$ holds in all theories $\mathbf{SST}_A$ (or $\mathbf{SRST}_A$), where $A$ is a set.*

(In (ii) and (v), by a straightforward coding argument, we could as well require that $A$ is a two-element set, or even a singleton set in (v).) The proof of Theorem 3.2 will be completed in Section 5.

Since simulation equivalence is known to be decidable (in polynomial time for finite process graphs, cf. [2,38]), it follows that it is decidable for terms $t, t' : n \to p$ whether $t = t'$ holds in all theories $\mathbf{CFL}_A$. This fact is in contrast with the well-known undecidability of the equivalence problem for context-free grammars. Intuitively, our positive result is due to the fact that we are interested in the equivalence of terms under *all* possible interpretations of the morphism variables as context-free languages. By restricting the interpretations to those mapping a fixed morphism variable $1 \to 2$ to the language $\{x_1 x_2\}$ (or by adding to our operations a constant for this language), we would run into undecidability, in fact the equational theory would not be recursively enumerable.

**Remark 3.3** *Languages and tree languages satisfy*

$$L \cdot \langle L_1 + L_1', \ldots, L_n + L_n' \rangle = \sum_{K_i \in \{L_i, L_i'\}} L \cdot \langle K_1, \ldots, K_n \rangle$$
$$L \cdot \langle L_1, \ldots, 0_{1,p}, \ldots, L_n \rangle = 0_{1,p}$$

*for all $L : 1 \to n$, and $L_i, L_i' : 1 \to p$ whenever each of the variables $x_1, \ldots, x_n$ occurs* exactly once *in each word/tree of $L$. However, these equations do not hold universally.*

# 4 Free $\omega$-continuous idempotent grove theories

Recall from [8] that a grove theory is a theory $T$ with a commutative additive monoid structure $(T(n,p), +, 0_{n,p})$ on each hom-set such that (1)–(4) hold. An idempotent grove theory is a grove theory with an idempotent sum operation. A morphism of (idempotent) grove theories is a theory morphism preserving $+$ and the constants $0_{n,p}$. When $T$ is an idempotent grove theory, we may define a partial order $\leq$ on each hom-set $T(n,p)$ by $f \leq g$ iff $f + g = g$. It is clear that $0_{n,p}$ is the least element of $T(n,p)$ with respect to this partial order, and the tupling and sum operations preserve the order. Composition necessarily preserves the order in the first argument, but not necessarily in the second. When it does, we call $T$ an *ordered idempotent grove theory*. Moreover, when $f, g : n \to p$, then $f \leq g$ iff $i_n \cdot f \leq i_n \cdot g$ for all $i \in [n]$. Thus, the partial order on morphisms $n \to p$ is determined by the order on the morphisms $1 \to p$. Morphisms of idempotent grove theories necessarily preserve the order.

We say that an idempotent grove theory is *$\omega$-continuous* if the supremum $\sup_k f_k$ of each $\omega$-chain $(f_k : n \to p)_k$ exists and composition preserves the supremum of $\omega$-chains in both arguments. It follows that every $\omega$-continuous idempotent grove theory is ordered, and the supremum of every countable family of morphisms $f_i : n \to p$, $i \in I$ exists. Moreover, composition preserves the supremum of all countable families in its first argument. A morphism of $\omega$-continuous idempotent grove theories preserves the supremum of $\omega$-chains.

Examples of $\omega$-continuous idempotent grove theories include all power-set theories $P(T)$ and thus the theories $\mathbf{Lang}_A$, $\mathbf{TreeLang}_\Sigma$, and the theories $\mathbf{SST}_\Sigma$ defined above. In $\mathbf{Lang}_A$ and $\mathbf{TreeLang}_\Sigma$, the relation $\leq$ is the component-wise set inclusion relation $\subseteq$, whereas it is the relation $\sqsubseteq_s$ in $\mathbf{SST}_A$. Each of these theories is equipped with a dagger operation. More generally, we may define a dagger operation in any $\omega$-continuous idempotent grove theory: for a morphism $f : n \to n + p$, $f^\dagger : n \to p$ is the least solution of the equation $x = f \cdot \langle x, \mathbf{1}_p\rangle$ in the variable $x : n \to p$. We have $f^\dagger = \sup_k f^{(k)}$, where $f^{(0)} = 0_{n,p}$ and $f^{(k+1)} = f \cdot \langle f^{(k)}, \mathbf{1}_p\rangle$, for all $k \geq 0$. It is clear that every morphism of $\omega$-continuous idempotent grove theories preserves dagger.

An *ideal* in $\mathbf{FST}_\Sigma(n, p)$ is a nonempty set $Q \subseteq \mathbf{FST}_\Sigma$ which is downward closed with respect to the relation $\sqsubseteq_s$. An $\omega$*-ideal* is an ideal $Q$ which is generated by some $\omega$-chain $(t_k)_k$ of trees $t_k : n \to p$ in $\mathbf{FST}_\Sigma$ with $t_k \sqsubseteq_s t_{k+1}$ for all $k \geq 0$. Note that we may identify any $(\omega)$-ideal $Q \subseteq \mathbf{FST}_\Sigma(n, p)$ with an $n$-tuple of $(\omega)$-ideals $(Q_1, \ldots, Q_n)$, where $Q_i \subseteq \mathbf{FST}_\Sigma(1, p)$ is the set of all $i$th components of the members of $Q$, for each $i \in [n]$. We may recover $Q$ from $(Q_1, \ldots, Q_n)$ as the set $\{t : n \to p : i_n \cdot t \in Q_i \text{ for all } i \in [n]\}$.

We may turn $\omega$-ideals into an idempotent grove theory $\omega\mathbf{SFST}_\Sigma$. The set of morphisms $n \to p$ in $\omega\mathbf{SFST}_\Sigma$ is the collection of all $\omega$-ideals $Q \subseteq \mathbf{FST}_\Sigma(n, p)$. When $Q : n \to p$ and $Q' : p \to q$, then we define $Q \cdot Q' : n \to q$ to be the ideal generated by the set of all trees $f \cdot g$ with $f : n \to p$ in $Q$ and $g : p \to q$ in $Q'$. When $Q$ and $Q'$ are generated by the $\omega$-chains $(f_k)_k$ and $(g_k)_k$, respectively, then $Q \cdot Q'$ is the $\omega$-ideal generated by the $\omega$-chain $(f_k \cdot g_k)_k$. For each $i \in [n]$, $n \geq 0$, the distinguished morphism $1 \to n$ is the ideal generated by the tree $\mathsf{ex}_i$. The sum $Q + Q' : n \to p$ of $Q : n \to p$ and $Q' : n \to p$ is defined as the ideal generated by $\{f + g : f \in Q,\ g \in Q'\}$. It is easy to see that this is again an $\omega$-ideal. The morphism $0_{n,p} : n \to p$ in $\omega\mathbf{SFST}_\Sigma$ is the ideal containing only the tree $0_{n,p}$.

There is a canonical embedding of $\mathbf{SFST}_\Sigma$ into $\omega\mathbf{SFST}_\Sigma$ which maps the simulation equivalence class of a finite tree $t : n \to p$ to the principal $\omega$-ideal $\{t' : n \to p : t' \sqsubseteq_s t\}$. It is easy to see that this defines an (ordered) idempotent grove theory morphism $\mathbf{SFST}_\Sigma \to \omega\mathbf{SFST}_\Sigma$.

An $\omega$-ideal in $\mathbf{SFST}_\Sigma(n, p)$ is defined in the same way as in $\mathbf{FST}_\Sigma(n, p)$ using the partial order $\sqsubseteq_s$. We may identify any $\omega$-ideal $Q \subseteq \mathbf{SFST}_\Sigma(n, p)$ with an $\omega$-ideal $Q' \subseteq \mathbf{FST}_\Sigma(n, p)$ which is the union of all simulation equivalence classes of the trees in $Q$. Using this identification, $\omega\mathbf{SFST}_\Sigma$ is just the completion of $\mathbf{SFST}_\Sigma$ by $\omega$-ideals as defined in [5] [2] It follows from the main result of [5] that $\omega\mathbf{SFST}_\Sigma$ is an $\omega$-continuous idempotent grove theory, and that we have:

**Proposition 4.1** *The theory $\omega\mathbf{SFST}_\Sigma$ is the free $\omega$-continuous idempotent grove theory, freely generated by $\mathbf{SFST}_\Sigma$. Given any $\omega$-continuous idempotent grove theory $T$ and an (ordered) idempotent grove theory morphism $\varphi : \mathbf{SFST}_\Sigma \to T$, there is a unique $\omega$-continuous idempotent grove theory morphism $\varphi^\sharp : \omega\mathbf{SFST}_\Sigma \to T$ extending $\varphi$.*

[2] Actually [5] uses a different representation of $\omega$-ideals.

**Proposition 4.2** *The theory* $\mathbf{SFST}_\Sigma$ *is the free ordered idempotent grove theory, freely generated by* $\Sigma$.

**Proof.** It is known that $\mathbf{FST}_\Sigma$ is the free grove theory, freely generated by $\Sigma$, cf. [8]. Let $\approx$ denote the least grove theory congruence such that $\mathbf{FST}_\Sigma/\approx$ is an ordered idempotent grove theory, and define $f \preceq g$ iff $f + g \approx g$, for all $f, g : n \to p$. Thus, $f \approx g$ iff both $f \preceq g$ and $g \preceq f$ hold. We show that the relations $\equiv_s$ and $\approx$ are equal. The inclusion of $\approx$ in $\equiv_s$ is clear, since $\mathbf{SFST}_\Sigma$ is an ordered idempotent grove theory. To complete the proof, we show that for all $f, g : 1 \to p$ in $\mathbf{FST}_\Sigma$, if $f \sqsubseteq_s g$, then $f \preceq g$. We argue by induction on the height [3] of $f$. When the height of $f$ is 0, $f = 0_{1,p}$ and our claim is clear. Suppose now that the height of $f$ is positive. If the root of $f$ is the source of a single edge, then $f = \sigma \cdot \langle f'_1, \ldots, f'_k\rangle$ or $f = j_p$ for some $\sigma \in \Sigma_k$, $f'_1, \ldots, f'_k : 1 \to p$ and $j \in [p]$. In the first case, since $f \sqsubseteq_s g$, we may write $g$ as $g_0 + \sigma \cdot \langle g'_1, \ldots, g'_k\rangle$ for some $g_0, g'_1, \ldots, g'_k : 1 \to p$ with $f'_i \sqsubseteq_s g'_i$ for all $i \in [k]$. By the induction hypothesis, we have $f'_i \preceq g'_i$ for all $i \in [k]$ and thus $f \preceq g$, since $\mathbf{FST}_\Sigma/\approx$ is an ordered idempotent grove theory. In the second case, when $f = j_p$, $g$ can be written as $g_0 + j_p$, for some $g_0 : 1 \to p$. Thus, $f \preceq g$ again. Suppose finally that the root of $f$ has 2 or more outgoing edges $e$. Then we can write $f$ as a finite sum of summands $f|_e$, where the height of each $f|_e$ is less than or equal to the height of $f$ and has a single edge whose source is the root. By the previous case and the induction hypothesis we have $f|_e \preceq g$ for each $e$. Since sum is idempotent, we conclude that $f \preceq g$. □

By Proposition 4.2 and Proposition 4.1, we immediately have:

**Theorem 4.3** *For each ranked alphabet* $\Sigma$, *the theory* $\omega\mathbf{SFST}_\Sigma$ *is the free* $\omega$*-continuous idempotent grove theory, freely generated by* $\Sigma$.

Our next task is to relate $\omega$-ideals of finite synchronization trees to possibly infinite synchronization trees.

For each tree $t : n \to p$ in $\mathbf{ST}_\Sigma$, let $K(t)$ denote the set of all *finite* trees $t' : n \to p$ with $t' \sqsubseteq_s t$.

**Proposition 4.4** *A set of finite trees* $Q \subseteq \mathbf{FST}_\Sigma(n, p)$ *is an* $\omega$*-ideal iff* $Q = K(t)$ *for some (possibly infinite) tree* $t : n \to p$ *in* $\mathbf{ST}_\Sigma$.

**Proof.** It suffices to prove the claim for $n = 1$. Suppose first that $Q = K(t)$ for some $t : 1 \to p$ in $\mathbf{ST}_\Sigma$. Then $Q$ is the $\omega$-ideal generated by the $\omega$-chain $(t_k)_k$, where $t_k : 1 \to p$ is the prefix of $t$ of height at most $k$ which is determined by those edges whose source is at distance at most $k - 1$ from the root.

Suppose now that $Q$ is the $\omega$-ideal generated by the $\omega$-chain $(t_k)_k$ of finite trees $1 \to p$. Let $t = \sum_{k \geq 0} t_k$. Then for any finite tree $s : 1 \to p$, $s \sqsubseteq_s t$ iff $s \sqsubseteq_s \sum_{i=0}^k t_i$ for some $k \geq 0$ iff $s \sqsubseteq_s t_k$ for some $k \geq 0$. □

**Proposition 4.5** *Suppose that* $t, t' : n \to p$ *in* $\mathbf{ST}_\Sigma$. *If* $t \sqsubseteq_s t'$ *then* $K(t) \subseteq K(t')$. *Moreover, if* $t$ *and* $t'$ *are finitely branching, or simulation equivalent to some finitely branching trees, and if* $K(t) \subseteq K(t')$, *then* $t \sqsubseteq_s t$.

[3] The height is the length of the longest path.

**Proof.** The first statement is obvious. In order to prove the second, we may restrict ourselves to finitely branching trees $1 \to p$. So suppose that $t, t' : 1 \to p$ are finitely branching with $K(t) \subseteq K(t')$. For each $k \geq 0$, let $t_k$ and $t'_k$ denote the (finite) prefixes of $t$ and $t'$ of height at most $k$, so that $t$ is the union of the $t_k$ and, similarly, $t'$ is the union of the $t'_k$, for $k \geq 0$. Since $K(t) \subseteq K(t')$, we have $t_k \sqsubseteq_s t'_k$ for each $k \geq 0$, so that there is a finite nonempty set of morphisms $t_k \to t'_k$ for each $k \geq 0$. Also, the restriction of a morphism $t_{k+1} \to t'_{k+1}$ onto $t_k$ is a morphism $t_k \to t'_k$. By König's lemma, we may select a sequence of morphisms $(\tau_k : t_k \to t'_k)_k$ such that $\tau_k$ is the restriction of $\tau_{k+1}$ for each $k \geq 0$. Since $t$ is the union of the $t_k$ and $t'$ is the union of the $t'_k$, the sequence $(\tau_k)_k$ determines a morphism $\tau : t \to t'$. □

**Corollary 4.6** *If $t, t' : n \to p$ in $\mathbf{ST}_\Sigma$ are simulation equivalent to finitely branching trees, then $t \sqsubseteq_s t'$ iff $K(t) \subseteq K(t')$, and $t \equiv_s t'$ iff $K(t) = K(t')$.*

**Example 4.7** *Let $t$ be the infinitely branching tree $t = \sum_{n\geq 0} \sigma^n \cdot 0_{1,0} : 1 \to 0$, and let $t' = \sigma^\omega : 1 \to 0$, a tree consisting of a single infinite branch with edges labeled $\sigma \in \Sigma_1$. Then $K(t) = K(t')$ but $t \equiv_s t'$ does not hold.*

Since every regular synchronization tree is simulation equivalent to a finitely branching regular tree, we have:

**Corollary 4.8** *Suppose that $t, t' : n \to p$ in $\mathbf{RST}_\Sigma$. Then $t \sqsubseteq_s t'$ iff $K(t) \subseteq K(t')$ and $t \equiv_s t'$ iff $K(t) = K(t')$.*

From Theorem 4.3 and Corollary 4.8, we obtain:

**Corollary 4.9** *Suppose that $\Sigma$ is a ranked set, $T$ is an $\omega$-continuous idempotent grove theory and $\varphi : \Sigma \to T$ is a rank preserving function. Then there is a unique way to extend $\varphi$ to an idempotent grove theory morphism $\varphi^\sharp : \mathbf{SRST}_\Sigma \to T$ preserving dagger.*

**Proof.** Suppose that $T$ is an $\omega$-continuous idempotent grove theory and $\varphi$ is a rank preserving function $\Sigma \to T$. We may extend $\varphi$ to a morphism $\psi : \omega\mathbf{SFST}_\Sigma \to T$ of $\omega$-continuous idempotent grove theories. We know that $\mathbf{SRST}_\Sigma$ embeds in $\omega\mathbf{SFST}_\Sigma$ by the function which maps a regular tree $t : n \to p$ to $K(t)$. It is a routine matter to verify that the embedding preserves the theory operations, the additive structure, and dagger. Thus, we may identify $\mathbf{SRST}_\Sigma$ with a subtheory of $\omega\mathbf{SFST}_\Sigma$. The restriction of $\psi$ to $\mathbf{SRST}_\Sigma$ is the required extension $\varphi^\sharp : \mathbf{SRST}_\Sigma \to T$. □

**Remark 4.10** *Corollary 4.9 is also derivable from a stronger result in [22], where it is shown (using the language of $\mu$-terms) that simulation equivalence classes of regular synchronization trees form the free theories in a class of iteration theories with an additive structure satisfying certain axioms. Our aim here was to derive this result from Theorem 4.3.*

## 5 Proof of the main result

In this section our aim is to prove Theorem 3.2.

Recall that we may view each set $A$ as a ranked set of letters of rank 1. We start by showing that for each ranked set $\Sigma$ there is some set $A$ such that $\mathbf{SST}_\Sigma$ embeds in $\mathbf{SST}_A$ and $\mathbf{SRST}_\Sigma$ embeds in $\mathbf{SRST}_A$.

**Proposition 5.1** *For every ranked set $\Sigma$ there exists a set $A$ and an injective (idempotent) grove theory morphism $\mathbf{SRST}_\Sigma \to \mathbf{SRST}_A$ preserving dagger.*

**Proof.** When $\Sigma$ is a ranked set, define $A = \overline{\Sigma} \cup \{\#\}$, where $\overline{\Sigma} = \{\overline{\sigma} : \sigma \in \Sigma\}$. Our aim is to show that there is an injective dagger preserving grove theory morphism $\mathbf{SRST}_\Sigma \to \mathbf{SRST}_A$.

Consider the function $\varphi$ which maps the simulation equivalence class of the tree corresponding to a letter $\sigma \in \Sigma_n$, $n \geq 0$, to the simulation equivalence class of the synchronization tree

$$s_\sigma = \overline{\sigma} \cdot (\# \cdot 1_n + \#^2 \cdot 2_n + \ldots + \#^n \cdot n_n) : 1 \to n$$

in $\mathbf{ST}_A$. (Recall that $\overline{\sigma}$ has rank 1. The tree $s_\sigma$ has a single edge originating in the root, which is labeled $\overline{\sigma}$. The target of this edge is the source of $n$ branches, a branch of the form $\#^i \cdot i_n$ for each $i \in [n]$.) When $n = 0$, the simulation equivalence class of the tree $\sigma$ is mapped to the equivalence class $[\overline{\sigma} \cdot 0_{1,0}]_s$. By Corollary 4.9, this function can be extended in a unique way to an idempotent grove theory morphism $\varphi : \mathbf{SRST}_\Sigma \to \mathbf{SRST}_A$ preserving dagger.

It is not hard to see that $\varphi$ takes the following concrete form. Suppose that $t : 1 \to p$ in $\mathbf{RST}_\Sigma$. Then $[t]_s\varphi$ is the equivalence class of the (regular) tree $t' : 1 \to p$ in $\mathbf{RST}_A$ obtained from $t$ by replacing each edge labeled $\sigma \in \Sigma$ by a copy of the tree $s_\sigma$. Formally, the set of vertices of $t'$ consists of the vertices of $t$ together with a vertex $[v, (v_1, v_2, \ldots, v_n)]$ and vertices $(v_i, j)$ with $1 < j < i \leq n$, for each hyper-edge $v \to (v_1, \ldots, v_n)$ of $t$. The edges of $t'$ are the exit edges of $t$ labeled as in $t$ together with the following ones, where we suppose that $e : v \to (v_1, \ldots, v_n)$ is a hyper-edge of $t$ labeled $\sigma$.

(i) An edge $v \to [v, (v_1, \ldots, v_n)]$ labeled $\overline{\sigma}$.

(ii) An edge $[v, (v_1, \ldots, v_n)] \to (v_i, 1)$ for each $1 < i \leq n$ labeled $\#$.

(iii) An edge $(v_i, j) \to (v_i, j + 1)$ and an edge $(v_i, i - 1) \to v_i$ labeled $\#$, for all $1 < i \leq n$ and $1 < j < i - 1$.

(iv) An edge $[v, (v_1, \ldots, v_n)] \to v_1$ labeled $\#$.

When $t = (t_1, \ldots, t_n) : n \to p$ and each $[t_i]_s$ is mapped to $[t'_i]_s$, then $[t]_s\varphi = ([t'_1]_s, \ldots, [t'_n]_s)$. Since each $t : 1 \to p$ can be recovered from $t' : 1 \to p$, $[t]_s$ is uniquely determined by $[t']_s$, i.e., $\varphi$ is injective. □

**Remark 5.2** *The above proof can be extended to all synchronization trees to obtain an injective $\omega$-continuous idempotent grove theory morphism $\mathbf{SST}_\Sigma \to \mathbf{SST}_A$.*

**Proposition 5.3** *For each set $A$ there exist a set $B$ and an injective dagger preserving (idempotent) grove theory morphism $\mathbf{SRST}_A \to \mathbf{CFL}_B$.*

**Proof.** Let $B = A \cup \{\#, \$\}$ and consider an idempotent grove theory morphism $\varphi : \mathbf{SRST}_A \to \mathbf{CFL}_B$ preserving dagger defined by the assignment $a \mapsto a(\#x_1\$)^* = \{a, a\#x_1\$, a(\#x_1\$)^2, \ldots\} : 1 \to 1$, so that the image of each letter $a \in A$ is a regular language. We claim that for any regular trees $t, s : 1 \to p$ in $\mathbf{RST}_A$,

$$[t] \sqsubseteq_s [s] \Leftrightarrow [t]\varphi \subseteq [s]\varphi.$$

The implication from left-to-right is immediate from Corollary 4.9. Suppose now that $t \not\sqsubseteq_s s$. We want to prove that $[t]\varphi \not\subseteq [s]\varphi$. We consider only the case $p = 0$ since the argument is similar for $p > 0$. Without loss of generality we may suppose that $t$ and $s$ are finitely branching since every regular tree is similar to a finitely branching regular tree. The $n$-round *simulation game* on the pair $(t, s)$ is played by two players, player I and II. In each round, player I selects an edge originating in the vertex of $t$ entered in the previous round, or in the root in the first round, and player II must respond by selecting an equally labeled edge originating in the vertex of $s$ entered in the previous round, or in the root of $s$ in first round. Player I wins the play if player II cannot respond. Otherwise player II wins. Since $t \not\sqsubseteq_s s$ and our trees are finitely branching, it follows that there is some $n \geq 1$ such that player I wins the $n$-round simulation game on $(t, s)$. We show by induction on $n$ that $[t]\varphi \not\subseteq [s]\varphi$. When $n = 1$, player I can choose an edge originating in the root of $t$ whose label is not matched by the label of any edge originating in the root of $s$. Since the label of this edge is a word in $[t]\varphi$ but not the first letter of any word in $[s]\varphi$, we have $[t]\varphi \not\subseteq [s]\varphi$.

Suppose now that $n > 1$ and that we have established the claim for $n - 1$. Now player I can select an edge originating in the root of $t$, labeled $a \in A$, with target the root of a subtree $t'$ such that for each $a$-labeled edge from the root of $s$ to the root of some subtree $s'$, player I wins the $(n-1)$-round game on $(t', s')$. By the induction hypothesis, this means that $[t']\varphi \not\subseteq [s']\varphi$ for all such subtrees $s'$.

Let $s_1, \ldots, s_k$ $(k > 0)$ be up to isomorphism all those subtrees of $s$ whose roots are the targets of $a$-labeled edges originating in the root of $s$. We have $[t']\varphi \not\subseteq [s_i]\varphi$ for all $i$. Now $[t]\varphi$ contains $a(\#[t']\varphi\$)^k$ as a subset, and all the words in $[s]\varphi$ starting with $a$ are in the set $\{a\} \cup \bigcup_{i\in[k]} \bigcup_{m\geq 1} a(\#[s_i]\varphi\$)^m$. For each $i \in [k]$, let $u_i$ be a word in $[t']\varphi$ which is not in $[s_i]\varphi$. Then the word $a\#u_1\$ \ldots \#u_k\$$ is in $[t]\varphi$ but does not belong to $[s]\varphi$, since it does not belong to any $a(\#[s_i]\varphi)^k$. Thus, $[t]\varphi \not\subseteq [s]\varphi$. □

**Proposition 5.4** *For each set $A$ there exist a ranked set $\Sigma$ and an injective dagger preserving (idempotent) grove theory morphism* $\mathbf{SRST}_A \to \mathbf{Reg}_\Sigma$.

**Proof.** Let $\Sigma_0 = A \cup \{\#, \$\}$, $\Sigma_2 = \{\sigma\}$, and let $\Sigma_n$ be empty if $n = 1$ or $n > 2$. For each $a \in A$, consider a regular tree language $L_a : 1 \to 1$ in $\mathbf{Reg}_\Sigma$ whose 'frontier' is the context-free (in fact, regular) language described in the previous proof. (Such a regular tree language exists, since the context-free languages are exactly the frontier languages of regular tree languages, see e.g. [32].) For example, let $L_a = \{t_0 = a, t_1 = \sigma a \sigma \# \sigma x_1 \$, t_2 = \sigma a \sigma \# \sigma x_1 \sigma \$ \sigma \# \sigma x_1 \$, \ldots\}$. Then let $\psi : \mathbf{SRST}_A \to \mathbf{Reg}_\Sigma$ be the unique dagger preserving morphism of idempotent grove theories determined by the assignment $a \mapsto L_a$ for all $a \in A$, which exists by Corollary 4.9. The morphism $\varphi$ constructed in the proof of Proposition 5.3 factors through $\psi$ by the 'frontier map'. Since $\varphi$ is injective, so is $\psi$. □

We are now ready to prove Theorem 3.2. First note that since for each set $A$, $\mathbf{CFL}_A$ embeds in $\mathbf{Lang}_A$, every identity that holds in all theories $\mathbf{Lang}_A$ holds in the theories $\mathbf{CFL}_A$. Similar facts are true in (iii), (iv) and (v) for the theories $\mathbf{TreeLang}_\Sigma$ and $\mathbf{Reg}_\Sigma$, $\mathbf{SST}_\Sigma$ and $\mathbf{SRST}_\Sigma$, and $\mathbf{SST}_A$ and $\mathbf{SRST}_A$. Clearly, every identity that holds in the theories $\mathbf{SST}_\Sigma$ ($\mathbf{SRST}_\Sigma$, resp.) also holds in the theories $\mathbf{SST}_A$ ($\mathbf{SRST}_A$, resp.). Since each theory $\mathbf{TreeLang}_\Sigma$ embeds in a theory $\mathbf{Lang}_A$, and similarly, each theory $\mathbf{Reg}_\Sigma$ embeds in some theory $\mathbf{CFL}_A$, each condition of (ii) implies the corresponding condition of (iii). By Corollary 4.9, if an identity holds in all theories $\mathbf{SRST}_\Sigma$ then it holds in all $\omega$-continuous idempotent grove theories and thus in all theories appearing in Theorem 3.2. Also, if an identity holds in all power-set theories, then it holds in the theories $\mathbf{Lang}_A$. Thus, to complete the proof it suffices to show that if an identity holds in all theories $\mathbf{Reg}_\Sigma$, then it holds in the theories $\mathbf{SRST}_A$, and that this turn implies that the identity holds in all theories $\mathbf{SRST}_\Sigma$. But these facts follow from Proposition 5.4 and Proposition 5.3.

**Remark 5.5** *The above proof also establishes the fact that an identity holds in all $\omega$-continuous idempotent grove theories iff it holds in all theories $\mathbf{SRST}_\Sigma$ (or the theories mentioned in Theorem 3.2).*

**Remark 5.6** *(Based on [22].) When $A$ is a poset with all countable suprema, the $\omega$-continuous functions over $A$ form an $\omega$-continuous idempotent grove theory $\omega\mathbf{Cont}_A$. A morphism $n \to p$ in this theory is an $\omega$-continuous function $A^p \to A^n$ (note the reversal of the arrow), and composition is function composition (in the reverse order). For each $i \in [n]$, $n \geq 0$, the distinguished morphism $i_n : 1 \to n$ is the $i$th projection function $A^n \to A$. The constant $0_{n,p}$ is the constant function mapping all elements of $A^p$ to the least element of $A^n$, and $f + g$ is the pointwise supremum of $f$ and $g$, for each $f, g : n \to p$ (i.e., $\omega$-continuous functions $f, g : A^p \to A^n$). Note that the order $\leq$ becomes the pointwise partial order. Since $\omega\mathbf{Cont}_A$ is a continuous idempotent grove theory, it comes with the least fixed point operation as dagger operation.*

*Every $\omega$-continuous idempotent grove theory $T$ may be embedded in a theory $\omega\mathbf{Cont}_A$. Given $T$, let $A = \prod_{p\geq 0} T(1,p)$, equipped with the pointwise partial order. The embedding maps a morphism $f : 1 \to n$ to the $\omega$-continuous function $A^p \to A$ defined by*

$$f((g_{1,p})_p, \ldots, (g_{n,p})_p) = (f \cdot \langle g_{1,p}, \ldots, g_{n,p}\rangle)_p.$$

*By this embedding, we conclude that an identity holds in all continuous idempotent grove theories iff it holds in all theories of the sort $\omega\mathbf{Cont}_A$, where $A$ is an $\omega$-continuous poset (or in fact a complete lattice).*

## 6 Axiomatization

By Theorem 3.2 and Remark 5.6, the very same equational calculus may be used in formal calculations involving simulation equivalence classes of (regular) synchronization trees, (context-free) languages, (regular) tree languages, (regular) synchronization trees, power-set theories, $\omega$-continuous idempotent grove theories, or $\omega$-

continuous functions over $\omega$-complete posets, equipped with the theory operations, sum, and dagger. Axiomatic treatments were given in [22] using the formalism of $\mu$-terms. We now transform these results into into the categorical language of this paper.

*Iteration theories* were introduced in 1980 by Bloom, Elgot and Wright [6], and independently by Ésik [18], as a generalization of Elgot's iterative theories [17] and the rational and continuous theories [33,40] of the ADJ group. See [8] for original references. Iteration theories are algebraic theories equipped with a dagger operation satisfying certain equational axioms such as the *fixed point identity*

$$f \cdot \langle f^\dagger, \mathbf{1}_p\rangle = f^\dagger \tag{5}$$

or the *parameter identity*

$$(f \cdot (\mathbf{1}_n \oplus g))^\dagger = f^\dagger \cdot g, \tag{6}$$

where $f : n \to n+p$ and $g : p \to q$. The equational axioms of iteration theories may conveniently be divided into two groups, the 'Conway identities' and the 'commutative identities' [18], which are simplified to the 'group identities' in [21]. For detailed accounts, we refer to [8,10,21]. A morphism of iteration theories is a theory morphism preserving dagger. Iteration theory congruences are defined in the expected way.

A *grove iteration theory* [8,13] is an iteration theory which is a grove theory. Here, we also require that

$$\mathbf{1}_1{}^\dagger = 0_{1,0}. \tag{7}$$

It is known that in a grove iteration theory,

$$(\mathbf{1}_n \oplus 0_p)^\dagger = 0_{n,p}$$

holds for all $n, p \geq 0$. Morphisms of grove iteration theories are both iteration theory morphisms and grove theory morphisms. A grove iteration theory congruence is just an iteration theory congruence.

It is possible to define a *star operation* $^* : T(n, n+p) \to T(n, n+p)$ $(n, p \geq 0)$ in any grove iteration theory. When $f : n \to n+p$, we define

$$f^* = (f \cdot (\mathbf{1}_n \oplus 0_n \oplus \mathbf{1}_p) + (0_n \oplus \mathbf{1}_n \oplus 0_p))^\dagger : n \to n+p.$$

Thus, in particular, $\mathbf{1}_1{}^* = (1_2 + 2_2)^\dagger : 1 \to 1$. It can be seen, cf. [8], that in grove iteration theories $T$, $f^*$ is a solution of the equation

$$x = f \cdot \langle x, 0_n \oplus \mathbf{1}_p\rangle + (\mathbf{1}_n \oplus 0_p),$$

for all $f : n \to n+p$ in $T$. When $p = 0$, this equation becomes $x = f \cdot x + \mathbf{1}_n$. In fact, properties of the dagger operation may be translated into corresponding properties of the star operation and vice versa, see [8,25,28].

A grove iteration theory is *$\omega$-idempotent* if

$$\mathbf{1}_1{}^* = \mathbf{1}_1$$

holds. Any $\omega$-idempotent grove iteration theory is an idempotent grove theory and thus an idempotent grove iteration theory. Indeed, if $T$ is $\omega$-idempotent, then

$$\mathbf{1}_1 + \mathbf{1}_1 = \mathbf{1}_1 \cdot \mathbf{1}_1{}^* + \mathbf{1}_1 = \mathbf{1}_1{}^* = \mathbf{1}_1,$$

so that $f + f = (\mathbf{1}_1 + \mathbf{1}_1) \cdot f = \mathbf{1}_1 \cdot f = f$, for all $f : 1 \to p$. Thus, we can define a partial order as above by $f \leq g$ iff $f + g = g$, for all $f, g : n \to p$. Call an idempotent grove iteration theory *ordered* if the dagger operation is monotone:

$$f^\dagger \leq (f + g)^\dagger, \tag{8}$$

or equivalently, if

$$f \leq g : n \to n + p \Rightarrow f^\dagger \leq g^\dagger : n \to p,$$

for all $f, g : n \to n + p$. It follows that composition is also monotone, since in iteration theories,

$$f \cdot g = (\mathbf{1}_n \oplus 0_p) \cdot \langle 0_n \oplus f \oplus 0_q, 0_{n+p} \oplus g\rangle^\dagger,$$

for all $f : n \to p$ and $g : p \to q$. (It can be seen that an idempotent grove theory is ordered iff composition and the scalar dagger operation $f \mapsto f^\dagger$, $f : 1 \to 1 + p$ are monotone.) Morphisms of (ordered) $\omega$-idempotent grove iteration theories are just grove iteration theory morphisms.

The following results were proved in [22] using the formalism of $\mu$-terms.

**Theorem 6.1** *For each ranked set $\Sigma$, $\mathbf{SRST}_\Sigma$ is the free ordered $\omega$-idempotent grove iteration theory, freely generated by $\Sigma$.*

**Corollary 6.2** *An identity holds in all ordered $\omega$-idempotent grove iteration theories iff it holds in all $\omega$-continuous idempotent grove theories, or in the theories of Theorem 3.2.*

Note that Theorem 6.1 shows that the theories of simulation equivalence classes of regular synchronization theories have a finite axiomatization relatively to iteration theories. (Without the additive structure, they satisfy exactly the iteration theory identities.)

**Remark 6.3** *By removing (8) from the axioms, we obtain a complete axiomatization of 'bisimilarity' of (regular) synchronization trees, cf. [8,13], and by adding to the the axioms the identities*

$$f \cdot (g + h) = f \cdot g + f \cdot h$$
$$f \cdot 0_{p,q} = 0_{n,p},$$

*for all $f : n \to p$ and $g, h : p \to q$, the resulting system is known to be complete for (matrix) theories over $\omega$-continuous idempotent semirings, or regular languages, or theories of binary relations, and many other structures. See [8], where original references may be found. (We note that (8) now becomes redundant.)*

In the presence of some Conway theory identities, the commutative identities (and the group identities) are implied by simpler quasi-equational or other first-order axioms of more general type. Since many of these simpler axioms hold in $\omega$-continuous idempotent grove theories, we may derive several corollaries to Theorem 6.1 and Corollary 6.2.

A *idempotent Park grove theory* is an ordered idempotent grove theory equipped with a dagger operation satisfying the fixed point identity (5), the parameter identity (6) and the *fixed point induction rule*:

$$f \cdot \langle g, \mathbf{1}_p \rangle \leq g \Rightarrow f^\dagger \leq g,$$

for all $f : n \to n + p$ and $g : n \to p$. (The equation (7) now becomes redundant, and the fixed point identity may be simplified to the inequality

$$f \cdot \langle f^\dagger, \mathbf{1}_p \rangle \leq f^\dagger, \quad f : n \to n + p.)$$

For example, all $\omega$-continuous idempotent grove theories are idempotent Park grove theories. It is known, cf. [19], that every idempotent Park grove theory is an ordered $\omega$-idempotent iteration theory. A morphism of idempotent Park grove theories is an idempotent grove theory morphism which preserves dagger.

Using Theorem 6.1 we have:

**Corollary 6.4** *For each ranked set $\Sigma$, $\mathbf{SRST}_\Sigma$ is the free idempotent Park grove theory, freely generated by $\Sigma$.*

**Corollary 6.5** *The following are equivalent for an identity $t = t'$ between terms $t, t' : n \to p$ involving the theory operations, dagger, and sum.*

- *$t = t'$ holds in all $\omega$-continuous idempotent grove theories.*
- *$t = t'$ holds in all ordered $\omega$-idempotent grove iteration theories.*
- *$t = t'$ holds in all idempotent Park grove theories.*

It is well-known that it suffices to require the fixed point induction rule just in the case when $n = 1$, see [4,14] or [8].

By the above results, simulation equivalence classes of regular synchronization trees have a finite implicational axiomatization. Other known implicational or first-order axiomatizations involve the weak functorial implication of [8], or a version [27] of the Scott induction rule. In [25], it is shown that by adding one-sided residuation to the collection of operations, a finite purely equational system may be derived.

## 7 Proof of Theorem 6.1.

In order to make the paper self contained, in this section we provide a proof of Theorem 6.1, originally obtained in [22] using the formalism of $\mu$-expressions. We make use of a corresponding result from [8,13] that concerns bisimilarity.

Recall [35,36] that a simulation $R : f \to g$ for trees $f, g \in \mathbf{ST}_\Sigma$ is a *bisimulation* if the relational inverse of $R$ is a simulation $g \to f$. We say that trees $f, g : 1 \to p$ are *bisimilar* if there is a bisimulation $f \to g$. When $f = (f_1, \ldots, f_n) : n \to p$ and $g = (g_1, \ldots, g_n) : n \to p$, then we say that $f$ is bisimilar to $g$ is each $f_i$ is bisimilar to $g_i$. As mentioned above, bisimilarity, denoted $\equiv_b$, is a grove iteration theory congruence [8], moreover, $\mathbf{BST}_\Sigma = \mathbf{ST}_\Sigma / \equiv_b$ is an $\omega$-idempotent grove iteration theory as is $\mathbf{BRST}_\Sigma = \mathbf{RST}_\Sigma / \equiv_b$. The bisimilarity class of a tree $f : n \to p$ in $\mathbf{ST}_\Sigma$ (or in $\mathbf{RST}_\Sigma$) will be denoted $[f]_b$. Note that we may embed $\Sigma$ into $\mathbf{BRST}_\Sigma$ or $\mathbf{BST}_\Sigma$ by the function mapping a letter $\sigma \in \Sigma_n$ into the bisimilarity class of $\sigma$ seen as a tree $1 \to n$.

The following result was proved in [13,8] (at least in the case when each letter in $\Sigma$ has rank 1).

**Theorem 7.1** *For each ranked alphabet $\Sigma$, $\mathbf{BRST}_\Sigma$ is the free $\omega$-idempotent grove iteration theory, freely generated by $\Sigma$. In more detail, given any $\omega$-idempotent iteration theory $T$ and rank preserving function $\Sigma \to T$, there is a unique morphism of $\omega$-idempotent grove iteration theories $\varphi^\sharp : \mathbf{BRST}_\Sigma \to T$ extending $\varphi$.*

We will use Theorem 7.1 in our characterization of $\mathbf{SRST}_\Sigma$ as the free *ordered* $\omega$-idempotent grove iteration theory, freely generated by $\Sigma$. But first we need some preparations. Recall that a morphism $\tau : f \to g$ for $f, g : 1 \to p$ in $\mathbf{ST}_\Sigma$ is a function $V_f \to V_g$ preserving the root such that whenever $e : u \to (u_1, \ldots, u_n)$ is an edge of $f$ then there is an equally labeled edge $e' : u\tau \to (u_1\tau, \ldots, u_n\tau)$ of $g$. Call the morphism $\tau$ *locally surjective* if for each $u \in V_f$ and $v \in V_g$ with $u\tau = v$, if $e' : v \to (v_1, \ldots, v_n)$ is an edge of $g$, then there exists an equally labeled edge $e : u \to (u_1, \ldots, u_n)$ of $f$ with $u_i\tau = v_i$ for all $i \in [n]$. Clearly, every locally surjective morphism is a surjective function. Moreover, we have:

**Lemma 7.2** *Suppose that $\Sigma$ is a ranked alphabet and $f, g : 1 \to p$ in $\mathbf{ST}_\Sigma$. Then a morphism $\tau : f \to g$ is a bisimulation iff $\tau$ is locally surjective.*

**Lemma 7.3** *Suppose that $\Sigma$ is a ranked alphabet and $f, g : 1 \to p$ in $\mathbf{RST}_\Sigma$ with $f \sqsubseteq_s g$. Then there is a regular tree $f' : 1 \to p$ in $\mathbf{RST}_\Sigma$ such that $f' \equiv_b g$ and $f$ embeds in $f'$ by an injective morphism.*

**Proof.** When $\tau$ is a morphism $f \to g$ and $u$ is a vertex of $f$, let $f|_u\tau$ be the tree determined by those vertices of $g|_{u\tau}$ which are in $V_{f|_u}\tau$, together with all edges $v\tau \to (v_1\tau, \ldots, v_n\tau)$ of $g|_{u\tau}$ such that $v$ is in $V_{f|_u}$. When $n = 0$, we also require that there is an equally labeled edge of $f$ originating in $v$. Also, let $g_u$ be the tree determined by those edges of $g_{u\tau}$ which are not edges of $f|_u\tau$. Thus, $g|_{u\tau} = f|_u\tau + g_u$.

First note that since $f \sqsubseteq_s g$, there is a morphism $\tau : f \to g$ such that if $f|_u$ is isomorphic to $f|_v$ and $g|_{u\tau}$ is isomorphic to $g|_{v\tau}$, for some vertices $u, v$ of $f$, then there exist isomorphisms $\pi : f|_u \to f|_v$ and $\pi' : g|_{u\tau} \to g|_{v\tau}$ with $x\pi\tau = x\tau\pi'$ for all vertices $x$ of $f|_u$, so that also $f|_u\tau$ is isomorphic to $f|_v\tau$ and $g_u$ is isomorphic to $g_v$.

Let $f'$ be obtained from $f$ by adding $g_u$ as a summand to $f|_u$, for each vertex $u$ of $f$. Then $f$ clearly embeds in $f'$ and $f' \equiv_b g$, since there is a locally surjective morphism $f' \to g$. Moreover, $f'$ is regular, since if $f|_u$ is isomorphic to $f|_v$ and $g|_{u\tau}$ is isomorphic to $g|_{v\tau}$, then $f'|_u$ is isomorphic to $f'|_v$. □

**Example 7.4** *As an illustration, let $\sigma$ be a letter of rank 1 and let $\sigma'$ and $\sigma''$ be letters or rank 0. Let $f : 1 \to 0$ be the full binary tree whose edges are all labeled $\sigma$. Let $g : 1 \to 0$ have a single maximal infinite path whose edges are all labeled $\sigma$ and such that each vertex along this path is the source of one more edge. This edge is labeled $\sigma'$ or $\sigma''$ according to whether the vertex is at even or odd distance from the root. Then there is a single morphism $f \to g$. The tree $f'$ can be obtained from $f$ by adding an outedge labeled $\sigma'$ to each vertex at even distance from the root, and an edge labeled $\sigma''$ to each vertex at odd distance from the root.*

**Lemma 7.5** *Suppose that $\approx$ is a grove iteration theory congruence of $\mathbf{BRST}_\Sigma$ such that the quotient $\mathbf{BRST}_\Sigma/\approx$ is an ordered grove iteration theory. For all $f, g : n \to p$ in $\mathbf{RST}_\Sigma$, define $f \preceq g$ when $[f]_b + [g]_b \approx [g]_b$. Moreover, define $f \approx g$*

*iff* $[f]_b \approx [g]_b$, *so that* $f \approx g$ *iff* $f \preceq g$ *and* $g \preceq f$.

*Let* $f, g : 1 \to p$ *in* $\mathbf{RST}_\Sigma$. *If* $f : 1 \to p$ *embeds in* $g : 1 \to p$ *by an injective morphism, then* $f \preceq g$.

**Proof.** Without loss of generality, we may choose an embedding $\tau : f \to g$ such that whenever $f|_u$ is isomorphic to $f|_v$ and $g|_{u\tau}$ is isomorphic to $g|_{v\tau}$, for some vertices $u, v$ of $f$, then there exist isomorphisms $\pi : f|_u \to f|_v$ and $\pi' : g|_{u\tau} \to g|_{v\tau}$ such that $x\pi\tau = x\tau\pi'$ for all vertices $x$ of $f|_u$. In particular, $f|_u\tau$ is isomorphic to $f|_v\tau$ and $g_u$ is isomorphic to $g_v$. (For definitions, see the preceding proof.) Let $f_1 = f, \ldots, f_n$ and $g_1 = g, \ldots, g_m$ be, up to isomorphism, all the subtrees of $f$ and $g$, respectively. Let us label a vertex $u$ of $f$ by the pair $(i, j)$ if $f|_u = f_i$ and $g|_{u\tau} = g_j$. Let $K$ denote the set of all pairs that label some vertex $u$, and consider a bijection $\psi$ between the sets $K$ and $[k]$, where $k$ denotes the number of elements of $K$. We may assume that $(1, 1)\psi = 1$. Let $\psi'$ denote the inverse of $\psi$. Now using the labeling of $f$, let us construct a tree $a : k \to k + p$, $a = \langle a_1, \ldots, a_k\rangle$. Consider a vertex $u$ of $f$ labeled $(i, j)$, say, together with all edges originating in $u$. For each $\sigma \in \Sigma_r$ that occurs as the label of an edge, and for each sequence $(i_1, j_1), \ldots, (i_r, j_r)$ of elements of $K$, let $\ell = \ell(\sigma, (i_1, j_1), \ldots, (i_r, j_r))$ denote the number of outgoing edges of $u$ labeled $\sigma$ with target vertices labeled $(i_1, j_1), \ldots, (i_r, j_r)$ if there is at least one such edge. Consider the tree $(\ell \cdot \sigma \cdot \rho) \oplus 0_p$, where $\rho$ is the base morphism $r \to k$ corresponding to the function $[r] \to [k]$ with $s \mapsto (i_s, j_s)\psi$, for all $s \in [r]$. Here, $\ell \cdot \sigma \cdot \rho$ is the $\ell$-fold sum of the tree $\sigma \cdot \rho$ with itself, which is a countably infinite sum when $\ell = \infty$. Moreover, for each $j \in [p]$, let $\ell_j$ denote the number of edges originating in $u$ labeled $\mathsf{ex}_j$. Then the component $a_{(i,j)\psi}$ of $a$ corresponding to $(i, j)\psi$ is the (finite) sum of all such trees $(\ell \cdot \sigma \cdot \rho) \oplus 0_p$ and $0_k \oplus (\ell_j \cdot j_p)$.

Having constructed $a : k \to k + p$, we construct $b : k \to k + p$. To this end, we label a vertex $v$ of $g$ by $(i, j) \in K$ if $g|_v = g_j$ and there is a vertex $u$ of $f$ with $u\tau = v$ and $f|_u = f_i$. (Note that there is at most one vertex $u$ with $u\tau = v$.) Now the component of $b$ corresponding to $(i, j)\psi$ is the finite sum of trees $(\ell \cdot \sigma \cdot \rho) \oplus 0_p$ and $0_k \oplus (\ell_j \cdot j_p)$ as above, where $\ell$ and the $\ell_j$ are determined using the labeling of $g$, together with the regular tree $0_k \oplus g'_{(i,j)\psi}$ where $g'_{(i,j)\psi}$ is obtained from $g_u$ by removing those outgoing edges of the root labeled in $\Sigma_0$ or by an exit symbol (since these have already been taken care of).

Clearly $a \preceq b$, since $b \approx a + c$ for some regular $c$. For each $s \in [k]$, let $i_s$ be the first component of $s\psi'$, and $j_s$ the second. Then the fixed point equation $x = a \cdot \langle x, \mathbf{1}_p\rangle$ has $(f_{i_1}, \ldots, f_{i_k})$ as its unique solution in $\mathbf{ST}_\Sigma$. Similarly, the equation $y = b \cdot \langle y, \mathbf{1}_p\rangle$ has $(g_{j_1}, \ldots, g_{j_k})$ as its unique solution in $\mathbf{ST}_\Sigma$. Now it is known, cf. [13,8], that $\mathbf{ST}_\Sigma$ and $\mathbf{RST}_\Sigma$ are also grove iteration theories such that bisimilarity is a congruence of $\mathbf{ST}_\Sigma$ (and of $\mathbf{RST}_\Sigma$). Thus, $f = 1_k \cdot a^\dagger$ and $g = 1_k \cdot b^\dagger$. Since $a \preceq b$ and dagger preserves $\preceq$, we conclude that $f \preceq g$. □

**Example 7.6** *In order to illustrate the above argument, let* $\sigma, \sigma'$ *be letters of rank* 1. *Let* $f : 1 \to 0$ *be the infinite tree with a single maximal path and edges labeled* $\sigma$, *and let* $g : 1 \to 0$ *be the full binary tree with each vertex having an outgoing edge labeled* $\sigma$ *and an edge labeled* $\sigma'$. *Then there is an (essentially unique) injective* *morphism* $f \to g$. *Now* $k = 1$, $a : 1 \to 1$ *is the tree* $\sigma$, *and* $b : 1 \to 1$ *is the tree* $\sigma + (0_1 \oplus (\sigma' \cdot g))$. *Clearly,* $a \preceq b$, *and thus* $f = a^\dagger \preceq b^\dagger = g$.

**Example 7.7** *Suppose that* $\sigma$ *is a letter of rank* 1 *and* $\sigma', \sigma''$ *have rank* 0. *Let* $f$ *be the tree* $g$ *of Example 7.4, and define now* $g$ *as the tree obtained from the full binary tree with edges labeled* $\sigma$ *by adding to each vertex an edge labeled* $\sigma'$ *and an edge labeled* $\sigma''$. *Then there is an essentially unique embedding of* $f$ *into* $g$. *We have* $k = 2$, $a = \langle a_1, a_2\rangle : 2 \to 2$ *and* $b = \langle b_1, b_2\rangle : 2 \to 2$ *where*

$$\begin{aligned}
a_1 &= \sigma \cdot 2_2 + (0_2 \oplus \sigma') \\
a_2 &= \sigma \cdot 1_2 + (0_2 \oplus \sigma'') \\
b_1 &= \sigma \cdot 2_2 + (0_2 \oplus \sigma') + (0_2 \oplus \sigma'') + (0_2 \oplus (\sigma \cdot g)) \\
b_2 &= \sigma \cdot 1_2 + (0_2 \oplus \sigma') + (0_2 \oplus \sigma'') + (0_2 \oplus (\sigma \cdot g)),
\end{aligned}$$

*so that* $f = 1_2 \cdot a^\dagger$, $g = 1_2 \cdot b^\dagger$. *Also,* $a \preceq b$ *and thus* $f \preceq g$.

We now complete the proof of Theorem 6.1.

Let $\approx$ denote the least grove iteration theory congruence of $\mathbf{BRST}_\Sigma$ such that the quotient $\mathbf{BRST}_\Sigma/\approx$ is an *ordered* grove iteration theory. For all $f, g : 1 \to p$ in $\mathbf{RST}_\Sigma$, define $f \preceq g$ when $[f]_b + [g]_b \approx [g]_s$. Moreover, define $f \approx g$ iff $[f]_b \approx [g]_b$, so that $f \approx g$ iff $f \preceq g$ and $g \preceq f$.

Our goal is to show that the relations $\sqsubseteq_s$ and $\preceq$ on $\mathbf{RST}_\Sigma$ are equal. It is clear that $\preceq$ is included in $\sqsubseteq_s$. To complete the proof, it suffices to show that $f \preceq g$ whenever $f, g : 1 \to p$ in $\mathbf{RST}_\Sigma$ with $f \sqsubseteq_s g$. But if $f \sqsubseteq_s g$, then by Lemma 7.3 there exists $f' : 1 \to p$ in $\mathbf{RST}_\Sigma$ with $f' \equiv_b g$ and such that there is an injective morphism $f \to f'$. But then $f' \approx g$ by Theorem 7.1, and $f \preceq f'$ by Lemma 7.5. It follows that $f \preceq g$. The proof is complete.

## 8 Summary

We have shown that several apparently different fixed point models, relevant to automata and formal language theory, concurrency and semantics, share the same equational theory, and presented sound and complete equational and quasi-equational axiomatizations. The fixed point models included the theories of (context-free) languages, (regular) tree languages and the simulation equivalence classes of (finite) transition systems. We expect that our results generalize to weighted languages, weighted tree automata [15] and simulation equivalence classes of weighted transition systems.

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