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On new critical point theorems without the Palais–Smale condition

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ABSTRACT

In this paper we prove new theorems on critical point theory based on the weak Ekeland's variational principle.

1. Introduction

The weak Ekeland variational principle is an important tool in critical point theory and nonlinear analysis, and in this paper we will use this principle to establish some new results in critical point theory.

2. Preliminaries

We need the following weak Ekeland variational principle which can be found for example in Ref. 1.

Lemma 1 (Weak Ekeland variational principle). *Let (E, d) be a complete metric space and let $\varphi : E \to \mathbb{R}$ be a lower semicontinuous functional, bounded from below. Then for every $\varepsilon > 0$, there exists a point $u^* \in E$ such that*

$$\varphi(u^*) < \varphi(v) + \varepsilon d(u^*, v), \quad \forall v \in E \text{ such that } \quad v \neq u^*.$$

Definition 1. *We say that a functional $\varphi \in C^1(E, \mathbb{R})$ has a sequence of almost critical points if there exists a sequence $(v_n)_n$ in E such that $\varphi'(v_n) \to 0$ in E^* as $n \to \infty$.*

Lemma 2 (Minimization principle). *([2]) Let E be a Banach space and $\varphi : E \to \mathbb{R}$ a functional, bounded from below and Gâteaux differentiable. Then, there exists a minimizing sequence $(v_n)_n$ of almost critical points of φ in the sense that*

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$$\lim_{n\to+\infty}\varphi(v_n)=\inf_{v\in E}\varphi(v) \quad \text{and} \quad \lim_{n\to+\infty}\varphi'(v_n)=0.$$

A slight modification of Theorem 1.26 in Ref. 3 (see also Corollary 4 in this paper) gives the following lemma with a dilatation type condition.

Lemma 3. *Let* $(E,\|.\|_1)$ *be a Banach space and* $(F,\|.\|_2)$ *be a normed space. If A is a closed set in E,* $f:A\to F$ *is continuous, and*

$$\exists k>0:\exists\theta>0,\quad \|f(x)-f(y)\|_2\geq k\|x-y\|_1^{\theta}\quad \forall x,y\in A,$$

then f(A) is closed.

3. Main results

Theorem 1. *Let E be a reflexive Banach space, Ω be a bounded and weakly closed set of E with* $J\in C^1(E,\mathbb{R})$, *and let J' be strongly continuous on Ω. Suppose also that J satisfies* $\|J'(\varphi(u))\|_{E^*}\leq k\|J'(u)\|_{E^*}$ *for all $u\in\Omega$, where $\varphi:\Omega\to\Omega$ is a function such that $\varphi(u)\neq u$ for all $u\in\Omega$, and $0<k<1$ is a constant . Then J has at least one critical point in Ω.*

Proof. *We first show $J'(\Omega)$ is closed. Let* $g\in\overline{J'(\Omega)}$. *There exist a sequence* $g_n\in J'(\Omega)$, *such that* $\lim_{n\to+\infty}g_n=g$, *and so there exist $(u_n)\subset\Omega$ with* $\lim_{n\to+\infty}J'(u_n)=g$. *Since Ω is bounded and E is reflexive, there exist* $(u_{n_k})\subset(u_n)$ *such that* $u_{n_k}\rightharpoonup u\in\Omega$. *Since J' is strongly continuous then*

$$g=\lim_{n\to+\infty}J'(u_{n_k})=J'(u)\in J'(\Omega).$$

We consider the complete metric space $J'(\Omega)$, and define the functional ψ on $J'(\Omega)$ by

$$\psi:J'(\Omega)\to\mathbb{R}$$
$$J'(u)\to\psi(J'(u))=\|J'(u)\|_{E^*}.$$

Then ψ is lower semicontinuous and bounded from below on $J'(\Omega)$. Let $\varepsilon=\frac{1-k}{1+k}\in(0,1)$. From the weak Ekeland variational principle, there exists u^* in Ω such that

$$\|J'(u^*)\|_{E^*}<\|J'(v)\|_{E^*}+\varepsilon\|J'(u^*)-J'(v)\|_{E^*},\quad \forall v\in\Omega \text{ such that } \quad v\neq u^*.$$

We claim that u^* is a critical point of J. If this is not true then $J'(u^*)\neq 0$. Now

$$\|J'(u^*)\|_{E^*}<\frac{1+\varepsilon}{1-\varepsilon}\|J'(v)\|_{E^*},\quad \forall v\in\Omega, \text{such that} \quad v\neq u^*,$$

and in particular for $v=\varphi(u^*)$, we have

$$\|J'(u^*)\|_{E^*}<\frac{1+\varepsilon}{1-\varepsilon}\|J'(\varphi(u^*))\|_{E^*},$$

i.e.,

$$J'(\varphi(u^*))\|_{E^*}>k\|J'(u^*)\|_{E^*}.$$

This contradicts the hypothesis $\|J'(\varphi(u))\|_{E^*}\leq k\|J'(u)\|_{E^*}$. ■

Theorem 2. *Let E be a Banach space and let the functional* $J\in C^1(E,\mathbb{R})$ *with $J'(E)$ a closed set in E^*. Suppose also that J admits a sequence of almost critical points. Then J has at least one critical point in E.*

Proof. *We consider the complete metric space $J'(E)$ of E^* and define the functional φ on $J'(E)$ by*

$$\varphi:J'(E)\to\mathbb{R}$$
$$J'(u)\to\varphi(J'(u))=\|J'(u)\|_{E^*}.$$

Then φ is lower semicontinuous and bounded from below on $J'(E)$. Let $\varepsilon\in(0,1)$. From the weak Ekeland variational principle, there exists u^* in E such that

$$\|J'(u^*)\|_{E^*}\leq\|J'(v)\|_{E^*}+\varepsilon\|J'(u^*)-J'(v)\|_{E^*},\quad \forall v\in E.$$

We deduce that u^* is a critical point of J. If this is not true then $J'(u^*)\neq 0$. Let (v_n) the sequence of almost critical point of J. Then we obtain that

$$\|J'(u^*)\|_{E^*}\leq\|J'(v_n)\|_{E^*}+\varepsilon\|J'(u^*)-J'(v_n)\|_{E^*},\quad \forall n\in\mathbb{N}.$$

Because $J'(v_n)\to 0$ as $n\to\infty$, by passing to the limit, we obtain that

$$\|J'(u^*)\|_{E^*}\leq\varepsilon\|J'(u^*)\|_{E^*},$$

which is a contradiction. ■

As a consequence of the last theorem, we obtain the following corollary.

Remark 1. *The two geometric conditions in the Mountain pass theorem suffice to get a sequence of almost critical points (see Ref. 2).*

Corollary 1. *Let E be a Banach space, and let* $J\in C^1(E,\mathbb{R})$ *satisfy $J(0)=0$. Assume that $J'(E)$ is a closed set in E^* and there exist positive numbers ρ and α such that*

1. *$J(u)\geq\alpha$ if $\|u\|=\rho$,*
2. *there exists $e\in E$ such that $\|e\|>\rho$ and $J(e)<\alpha$.*

Then J admits at least one critical point u. It is characterized by

$$J'(u)=0,\quad J(u)=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]}J(\gamma(t))$$

where

$$\Gamma=\{\gamma\in C([0,1],E)|\gamma(0)=0,\gamma(1)=e\}.$$

Corollary 2. *Let E be a Banach space and let the functional* $J\in C^1(E,\mathbb{R})$ *satisfy*

$$\exists k>0:\exists\theta>0,\|J'(u)-J'(v)\|_{E^*}\geq k\|u-v\|_E^{\theta}\quad \forall u,v\in E. \tag{1}$$

Suppose also that J admits a sequence of almost critical points. Then J has at least one critical point in E.

Proof. *This is a direct consequence of Theorem 2 using Lemma 3.* ■

Corollary 3. *Let E be a Banach space, and let $J \in C^1(E, \mathbb{R})$ with $J'(E)$ a closed set in E^*. Suppose that J is bounded from below. Then J has at least one critical point.*

Proof. *The minimization principle ensures the existence of almost critical points. The conclusion follows from Theorem 2.* ■

Corollary 4. *Let E be a reflexive Banach space, let Ω a bounded and weakly closed subset of E with $J \in C^1(E, \mathbb{R})$, and let J' be strongly continuous on Ω. Suppose also that J admits a sequence of almost critical point in Ω. Then J has at least one critical point in Ω.*

Proof. *We show $J'(\Omega)$ is closed. Indeed let $g \in \overline{J'(\Omega)}$. There exists $(g_n) \subset J'(\Omega)$ such that $\lim_{n\to+\infty} g_n = g$, and there exists $(u_n) \subset \Omega$ with $\lim_{n\to+\infty} J'(u_n) = g$. Since Ω is bounded and E is reflexive, there exists $(u_{n_k}) \subset (u_n)$ such that $u_{n_k} \rightharpoonup u \in \Omega$. Since J' is strongly continuous then*

$$g = \lim_{n\to+\infty} J'(u_{n_k}) = J'(u) \in J'(\Omega)$$

Following the same steps in the proof of Theorem 2, we obtain the result. ■

4. Application

We consider the functional J defined on $E = H^1(0,1)$ by

$$J(u) = \int_0^1 \left(\int_0^{u(t)} f(t,\xi)\, d\xi \right) dt,$$

where $f \in C([0,1]\times\mathbb{R}, \mathbb{R})$ is a continuous function. Suppose that there exist a function $\phi : \mathbb{R} \to \mathbb{R}$ and $k \in]0, 1[$ such that

$$(f_1) \left| \int_0^1 f(t, \phi(u(t)))h(t)\,dt \right| \le k \left| \int_0^1 f(t, u(t))h(t)\,dt \right|,$$

for all $u, h \in H^1(0,1)$.

One may take as examples of f and ϕ,

$$f(t,u) = q(t)(u+k)^2, \quad \phi(s) = ks + k^2 - k, \quad t \in [0,1], k \in]0,1[$$

where q is a positive function defined on [0, 1].

Theorem 3. *Suppose that f satisfies (f_1). Then J has at least one critical point.*

Proof. *Note that J is well defined and $J \in C^1(H^1(0,1), \mathbb{R})$ with*

$$J'(u).h = \int_0^1 f(t, u(t))h(t)\,dt, \quad \text{for all} \quad u, h \in H^1(0,1).$$

We now show J' is strongly continuous on $\Omega = \overline{B}(0,\rho) \subset H^1(0,1)$.

Let (u_n) a sequence with $(u_n) \subset \Omega$ and $u_n \rightharpoonup u$ (Ω is bounded in $H^1(0, 1)$) and note it converges uniformly to u on [0, 1]. Since Ω is weakly closed, $u \in \Omega$. Let C be the constant of the continuous embedding of $H^1(0, 1)$ in $L^2(0, 1)$. We have

$$\begin{aligned}
\|J'(u_n) - J'(u)\|_{E^*} &= \sup_{\|h\|_{H^1}\le 1} |J'(u_n)h - J'(u)h| \\
&= \sup_{\|h\|_{H^1}\le 1} \left| \int_0^1 (f(t,u_n(t)) - f(t,u(t)))h(t)\,dt \right| \\
&\le \sup_{\|h\|_{H^1}\le 1} \left(\int_0^1 (f(t,u_n(t)) - f(t,u(t))dt)^2 \right)^{\frac{1}{2}} \\
&\quad \left(\int_0^1 h^2(t)\,dt \right)^{\frac{1}{2}} \\
&= \sup_{\|h\|_{H^1}\le 1} \left(\int_0^1 (f(t,u_n(t)) - f(t,u(t))dt)^2 \right)^{\frac{1}{2}} \|h\|_{L^2(0,1)} \\
&\le C \sup_{\|h\|_{H^1}\le 1} \left(\int_0^1 (f(t,u_n(t)) - f(t,u(t))dt)^2 \right)^{\frac{1}{2}} \|h\|_{H^1} \\
&\le C \left(\int_0^1 (f(t,u_n(t)) - f(t,u(t))dt)^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

Let K be the constant of the continuous embedding of $H^1(0, 1)$ in C[0, 1], and note that

$$|f(t,u_n(t)) - f(t,u(t))| \le 2 \sup_{(t,y)\in[0,1]\times[-K\rho,K\rho]} |f(t,y)|,$$

$$\lim_{n\to+\infty} f(t,u_n(t)) = f(t,u(t)).$$

From the Lebesgue dominated convergence theorem, we obtain

$$\lim_{n\to+\infty} \int_0^1 (f(t,u_n(t)) - f(t,u(t))dt)^2)^{\frac{1}{2}} = 0,$$

and so

$$\lim_{n\to+\infty} \|J'(u_n) - J'(u)\|_{E^*} = 0.$$

Finally we show J' satisfies $\|J'(\varphi(u))\|_{E^*} \le k\|J'(u)\|_{E^*}$ for all $u \in \Omega$, where φ is the Nemytskii's operator associated with ϕ. Now from (f_1) we have

$$\begin{aligned}
\|J'(\varphi(u))\|_{E^*} &= \sup_{\|h\|_{H^1}\le 1} \left| \int_0^1 f(t,\phi(u(t)))h(t)\,dt \right| \\
&\le k \sup_{\|h\|_{H^1}\le 1} \left| \int_0^1 f(t,u(t))h(t)\,dt \right| \\
&= k\|J'(u)\|_{E^*}.
\end{aligned}$$

From Theorem 1, J has at least one critical point in Ω.

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