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Refined Bounds on Kolmogorov Complexity for ω-Languages

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Abstract

The paper investigates bounds on various notions of complexity for ω–languages. We understand the complexity of an ω–languages as the complexity of the most complex strings contained in it. There have been shown bounds on simple and prefix complexity using fractal Hausdorff dimension. Here these bounds are refined by using general Hausdorff measure originally introduced by Felix Hausdorff. Furthermore a lower bound for a priori complexity is shown.

Keywords: Kolmogorov complexity, ω–languages, measures

1 Introduction

Algorithmic complexity was introduced to investigate the amount of information of strings. It measures the information of a string as the length of the shortest programme that outputs the string. A comprehensive work on the variants of complexity is the book [4] by Li and Vitániy. The approach on ω–languages we follow is to find complexity bounds for the most complex ω–words contained in the ω–language. Here we find another witness to the obvious assumption that large ω–languages contain complex ω–words. The notion of 'large' used in this paper is taken from geometric measure theory. In [3] Felix Hausdorff introduced the general fractal Hausdorff measure which allows a detailed investigation of infinite sets having Lebesgue measure zero. The concepts of fractal geometry are described at full length in Falconer's book [2]. In the papers [8] and [1] have been proved lower bounds for simple and prefix complexity depending on the Hausdorff dimension. We use the general Hausdorff measures to improve these bounds and to find a lower bound on a priori complexity for ω–languages.

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2 Notation and Preliminary Results

In this section we briefly recall the concept of Kolmogorov complexity of (in)finite words and measures (of ω-languages). For more detailed information the reader is referred to the textbooks [4] and [2]. In the following X is a finite alphabet with cardinality $|X| = r$. By X^* we denote the set (monoid) of words on X, including the *empty word* ε, and X^ω is the set of infinite words (ω–words) over X. For $w \in X^*$ and $\eta \in X^* \cup X^\omega$ let $w \cdot \eta$ be their concatenation. We extend this concatenation in the obvious way to subsets $W \subseteq X^*$ and $B \subseteq X^* \cup X^\omega$. For a language W let $W^* := \bigcup_{n\in\mathbb{N}} W^n$ be the *submonoid* of X^* generated by W, and by $W^\omega := \{w_1 \cdots w_n \cdots \mid w_n \in W \setminus \{\varepsilon\}\}$ we denote the subset of X^ω formed by concatenating words of W. Furthermore $|w|$ is the *length* of the word $w \in X^*$ and $\mathrm{pref}(B)$ is the set of all finite prefixes of strings in $B \subseteq X^* \cup X^\omega$, we abbreviate $w \in \mathrm{pref}(\{\eta\})$ by $w \sqsubseteq \eta$. By $\xi[n]$ we denote the prefix of $\xi \in X^* \cup X^\omega$ of length n. And again for any language W let $W^\delta := \{\xi \mid |\mathrm{pref}(\xi) \cap W| = \infty\}$ the subset of ω–words of X^ω containing infinitely many prefixes of W, called δ–limit of W.

It is useful to consider the set X^ω as a metric space (Cantor space) (X^ω, ρ) of all ω–words over the alphabet X where the metric is ρ is defined as follows

$$\rho(\xi, \eta) := \inf\{r^{-|w|} \mid w \sqsubset \xi \wedge w \sqsubset \eta\}$$

The open (and simultaneously closed) balls in (X^ω, ρ) are the sets of the form $w \cdot X^\omega$, where $w \in X^*$. The diameter of these balls is $d(w \cdot X^\omega) = r^{-|w|}$.

Programme size complexity defines the complexity of a finite string to be the length of a shortest programme which prints the string. Let $\varphi : X^* \to X^*$ be a partial–recursive function. The complexity of a word $w \in X^*$ with respect to φ is defined as

(1) $$\mathrm{K}_\varphi(w) := \{|\pi| \mid \pi \in X^* \wedge \varphi(\pi) = w\}$$

It is well known that there is an optimal partial–recursive function $\mathfrak{U}$, that is, a function satisfying that for every partial–recursive function φ

(2) $$\exists c_\varphi \forall w (w \in X^* \to \mathrm{K}_{\mathfrak{U}}(w) \leq \mathrm{K}_\varphi(w) + c_\varphi)$$

We fix an optimal function $\mathfrak{U}$ and further on we call the complexity with respect to this function KS. The conditional complexity $\mathrm{K}_\varphi(w|n)$ is length of a shortest programme which outputs w under the additional input n (w.r.t. function φ). For every $w \in X^*$ and $n \in \mathbb{N}$ holds $\mathrm{KS}(w|n) \leq \mathrm{KS}(w) + c$ true.

If we solely consider partial–recursive functions φ with prefix–free domains $\mathrm{dom}(\varphi) \subseteq X^*$ we obtain an optimal function in the same way. The complexity function with respect to this (fixed) optimal function is called KP.

The third notion of complexity this paper deals with is a priori complexity. It is obtained in the following way. Consider a semimeasure m on X^*, that is a function which satisfies $m(\varepsilon) \leq 1$ and $m(w) \geq \sum_{x\in X} m(wx)$, for any $w \in X^*$. In [10] Levin proved the existence of a universal semicomputable semimeasure m', that is for all semimeasures m there is a constant c_m such that

(3) $$\forall w \in X^* \qquad m(w) \leq c_m \cdot m'(w)$$

Then the a priori complexity is defined as $\mathrm{KA}(w) = -\log_r m'(w)$. A well known property of KA is

Proposition 2.1 *The function* KA *is a minimal upper semicomputable total function satisfying*

$$\sum_{w \in M} r^{-\mathrm{KA}(w)} \leq 1, \qquad \text{for any prefix–free set } M \subseteq X^*.$$

Accordingly, the complexity of an infinite word ξ is a function mapping the natural number n to the complexity of the n–length prefix of ξ.

Definition 2.2 Let $\xi \in X^\omega$.

(i) The function $\mathrm{KS}(\xi[\cdot]) : \mathbb{N} \to \mathbb{N}$ is called *simple Kolmogorov complexity* of ξ.

(ii) The function $\mathrm{KS}(\xi[\cdot]|\cdot) : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is called *conditional complexity* of ξ.

(iii) The function $\mathrm{KP}(\xi[\cdot]) : \mathbb{N} \to \mathbb{N}$ is called *prefix complexity* of ξ.

(iv) The function $\mathrm{KA}(\xi[\cdot]) : \mathbb{N} \to \mathbb{N}$ is called *a priori complexity* of ξ.

Throughout the paper the notation follows mainly Uspensky and Shen in [9].

3 Generalisation of Hausdorff Measure

As mentioned above, we achieve complexity bounds by a fractal (Hausdorff–) measure. For the purpose of defining the desired measures we need to characterise dimension–functions, that is, functions that "behave well" in the neighbourhood of zero. The behaviour below zero is not important for the definition of our measures. In detail we have the following requirements

Definition 3.1 A function $h : [0, \infty) \to [0, \infty)$ is called *dimension function* if

(i) $h(t) > 0$ for all $t > 0$, $h(0) = 0$,

(ii) h is increasing for $t \geq 0$, and

(iii) h is continuous from the right for all $t \geq 0$.

Now we use the usual construction of an outer measure.

Definition 3.2 For $F \subseteq X^\omega$ and h a dimension function

$$\mathcal{H}^h(F) := \lim_{l \to \infty} \inf \left\{ \sum_{w \in W} h(r^{-|w|}) \mid F \subseteq W \cdot X^\omega \wedge \forall w (w \in W \to |w| \geq l) \right\}$$

is called the Hausdorff h–measure (or h–measure) of F.

Here the condition $\forall w (w \in W \to |w| \geq l)$ means, that the diameters of the covering sets are at most r^{-l}. A well–known family satisfying the conditions of Definition 3.1 is the family of exponential functions $h(t) = t^\alpha$, with $0 \leq \alpha \leq 1$. The measures derived from these functions are the common α–dimensional (Hausdorff–) measures which we call $\mathbb{L}_\alpha$ throughout the paper.

First we examine how the different measures are related to each other. Given two dimension functions g and h a comparison of $\mathcal{H}^h$ and $\mathcal{H}^g$ can be achieved by simply comparing the behaviour of the functions g and h close to zero. The following lemma gives a relation between the behaviour of the dimension functions g and h and the corresponding measures $\mathcal{H}^h$ and $\mathcal{H}^g$.

Lemma 3.3 ([3])
Let g, h dimension functions and $F \subseteq X^\omega$.

(i) *If $\frac{h(t)}{g(t)} \longrightarrow 0$ for $t \longrightarrow 0$, then $\mathcal{H}^g(F) < \infty$ implies $\mathcal{H}^g(F) = \infty$ and $\mathcal{H}^h(F) > 0$ implies $\mathcal{H}^g(F) = \infty$.*

(ii) *If $c_1 \cdot g(t) \leq h(t) \leq c_2 \cdot g(t)$ for constants $c_1, c_2 > 0$ and sufficiently small t, then for every ω–language F it holds $c_1 \cdot \mathcal{H}^g(F) \leq \mathcal{H}^h(F) \leq c_2 \cdot \mathcal{H}^g(F)$.*

¿From this lemma we derive that if $g(t) \leq h(t)$ for sufficiently small t, then $\mathcal{H}^g(F) \leq \mathcal{H}^h(F)$. Especially if $g(t) = c \cdot h(t)$ for some $c > 0$, then $\mathcal{H}^g(F) = c \cdot \mathcal{H}^h(F)$. Additionally, the second part gives us the following equivalence of the measures $\mathcal{H}^g$ and $\mathcal{H}^h$: the measures $\mathcal{H}^g$ and $\mathcal{H}^h$ are simultaneously zero, positive or infinite, respectively.

First we prove under which conditions an ω–language has non–zero h–measure, and subsequently use these conditions to prove our bounds. The proof of this result follows the line of Lemma 3.8 in [8].

Theorem 3.4 *Let $V \subseteq X^*$ and h be a dimension function. Then*

$$\sum_{v \in V} h(r^{-|v|}) < \infty \quad \textit{implies} \quad \mathcal{H}^h(V^\delta) = 0.$$

Proof. Let $V^{(i)} := \{v \mid v \in V \wedge |A(v) \cap V| = i+1\}$. Then $V^{(i)}$ contains exactly those words of V, having $i+1$ prefixes in V. Thus for every $i \in \mathbb{N}$ we have $V^{(i)} \cdot X^\omega \supseteq V^\delta$ and V is disjoint union of all $V^{(i)}$. Further on

$$\mathcal{H}^h(V^\delta) \leq \sum_{v \in V^{(i)}} h(r^{-|v|}), \qquad \text{for all } i \in \mathbb{N}.$$

And since the sum $\sum_{v \in V} h(r^{-|v|})$ converges the right hand side tends to zero for large i. □

4 Refinement of Complexity Bounds

Simple Kolmogorov complexity

In this section we derive our announced refinement of bounds. First we investigate simple Kolmogorov complexity. We want to improve the result from [8]. There it is stated that for an ω–language $F \subseteq X^\omega$ with $\mathbb{L}_\alpha(F) > 0$ and an arbitrary function $f : \mathbb{N} \to \mathbb{N}$ which is growing not too slow, that is $\sum_{i \in \mathbb{N}} r^{-f(i)} < \infty$, there is a $\xi \in F$ satisfying

$$\mathrm{KS}(\xi[n]) \geq_{\text{a.e.}} \alpha \cdot n - f(n) \tag{4}$$

In particular this shows that $\mathrm{KS}(\xi[n]) \geq_{\text{a.e.}} \alpha \cdot n - (1+\varepsilon)\log_r(n)$

The next result states, that the gap between the complexity of the most complex ω–words of a language having non–zero h–measure and function $-\log_r(h(r^{-n}))$ is at most $(1+\varepsilon)\cdot \log n$.

Theorem 4.1 (Refinement of Inequality 4)
Let $F \subseteq X^\omega$ and $f : \mathbb{N} \to \mathbb{N}$ an arbitrary function satisfying $\sum_{i\in\mathbb{N}} r^{-f(i)} < \infty$. Then $\mathcal{H}^h(F) > 0$ implies

$$\exists \xi(\xi \in F \wedge \mathrm{KS}(\xi[n]) \geq_{\text{a.e.}} -\log_r(h(r^{-n})) - f(n)$$

Proof. Define the set of all ω–words with high (conditional) complexity with respect to dimension–function h and function f as

$$E(h,f) := \{\xi \mid \mathrm{KS}(\xi[n] \mid n) \geq_{\text{a.e.}} -\log_r(h(r^{-n})) - f(n)\}.$$

Its complement consists of all ω–words having at least infinitely many prefixes w with complexity less than $-\log_r(h(r^{-|w|})) - f(|w|)$. Thus this complement is the δ-limit of the set $V := \{v \mid \mathrm{KS}(v \mid |v|) < -\log_r(h(r^{-n})) - f(n)\}$. Counting the number of elements of a fixed length n in this set gives us an estimate for $|V \cap X^n| = |\{w \mid |w| = n \wedge \mathrm{KS}(w \mid n) < -\log_r(h(r^{-n})) - f(n)\}| \leq r^{-\log_r(h(r^{-n}))-f(n)}$. In order to utilise Lemma 3.4 we consider the following sum:

$$\begin{aligned} \sum_{v\in V} h(r^{-|v|}) &= \sum_{i\in\mathbb{N}} |V \cap X^n| \cdot h(r^{-i}) \\ &\leq \sum_{i\in\mathbb{N}} r^{-\log_r(h(r^{-i}))-f(i)} \cdot h(r^{-i}) = \sum_{i\in\mathbb{N}} r^{-f(i)} \end{aligned} \tag{5}$$

Due to our assumed properties of f the sum (5) is finite. Thus Lemma 3.4 yields $\mathcal{H}^h(V^\delta) = \mathcal{H}^h(X^\omega \setminus E(h,f)) = 0$. Hence $\mathcal{H}^h(F) = \mathcal{H}^h(F \cap E(h,f))$, for any ω–language $F \subseteq X^\omega$. This gives us $F \cap E(h,f) \neq \emptyset$, whenever $\mathcal{H}^h(F) > 0$, which proves our theorem. □

Consider an ω–language F with $\mathbb{L}_\alpha(F) = \infty$ and $\mathbb{L}_{\alpha'}(F) = 0$, for any $\alpha' > \alpha$. Thus there exists a $\xi \in F$ fulfilling Inequality 4, that is $K(\xi[n]) \geq_{\text{a.e.}} \alpha \cdot n - f(n)$. Now take h as a function converging faster to zero than $r^{-\alpha\cdot n}$ satisfying $\mathcal{H}^h(F) > 0$ and

$$\forall \varepsilon > 0 \qquad r^{-\alpha\cdot n} > h(r^{-n}) > r^{-(\alpha+\varepsilon)\cdot n}$$

for all $n > n_0$ and some $n_0 \in \mathbb{N}$. Then our refined bound states there is a $\xi \in F$ with $K(\xi[n]) \geq_{\text{a.e.}} -\log h(r^{-n}) - f(n) > \alpha \cdot n - f(n)$. Thus the bound has been raised.

On the other hand let $\mathbb{L}_\alpha(F) = 0$ and $\mathbb{L}_{\alpha'}(F) = \infty$, for any $\alpha' < \alpha$. Thus for any $\varepsilon > 0$ there is $\xi \in F$ with $K(\xi[n]) \geq_{\text{a.e.}} (\alpha - \varepsilon)\cdot n - f(n)$. Now let h be again a dimension–function fulfilling

$$\forall \varepsilon > 0 \qquad r^{-(\alpha-\varepsilon)\cdot n} > h(r^{-n}) > r^{-\alpha\cdot n}$$

and $\mathcal{H}^h(F) > 0$. As above we can now raise the bound from $(\alpha - \varepsilon) \cdot n - f(n)$ to $-\log h(r^{-n}) - f(n)$.

Example 4.2 Let $X = \{a,b\}$, $F := \{a,b\} \cdot \prod_{i=0}^{\infty} \left(\{a,b\}^{2^i - 1} \cdot a \right)$ and f with $\sum_{i\in\mathbb{N}} r^{-f(i)} < \infty$ fixed. One can show that $\mathbb{L}_1(F) = 0$ and $\mathbb{L}_{1-\varepsilon}(F) = \infty$, for any $\varepsilon > 0$. Thus there is a $\xi \in F$ such that for any $\varepsilon > 0$

$$K(\xi[n]) \geq_{\text{a.e.}} (1 - \varepsilon) \cdot n - f(n)$$

Now consider the dimension function $h(r^{-|w|}) = r^{-|w|} \cdot |w|$. The h–measure of F is $\mathcal{H}^h(F) = 1$. Hence by Theorem 4 there is a $\xi \in F$ such that

$$K(\xi[n]) \geq_{\text{a.e.}} -\log(r^{-n} \cdot n) - f(n) = n - \log n - f(n).$$

This bound is almost everywhere strictly greater than $(1 - \varepsilon) \cdot n - f(n)$.

If $0 < \mathbb{L}_\alpha(F) < \infty$ Theorem 4.1 yields no improvement. This can be seen by Lemma 3.3.

In the proof of Theorem 4.1 we have shown that the measure of the set of ω–words of low conditional complexity is zero. Thus we can formulate a similar result for conditional complexity.

Corollary 4.3 *Let $F \subseteq X^\omega$, $\mathcal{H}^h(F) > 0$ and $f : \mathbb{N} \to \mathbb{N}$ an arbitrary function satisfying $\sum_{i\in\mathbb{N}} r^{-f(i)} < \infty$. Then there is a $\xi \in F$ satisfying $\mathrm{KS}(\xi[n] \mid n) \geq -\log_r(h(r^{-n})) - f(n)$ for almost every $n \in \mathbb{N}$.*

Prefix complexity

Due to the fact that the domains of prefix functions have to be prefix–codes, the prefix–complexity of a ω–word is higher than its simple or conditional complexity. The known bound in Equation 6 is taken from [1]. Again, let $F \subseteq X^\omega$ an ω–language having $\mathbb{L}_\alpha(F) > 0$ and $c > 0$ an arbitrary constant. Then there is a $\xi \in \mathbb{N}$ satisfying

$$\mathrm{KP}(\xi[0 \ldots n]) \geq_{\text{a.e.}} \alpha \cdot n - c \tag{6}$$

To refine this bound for prefix complexity we take an approach similar to the one for simple complexity. Here again we replace the linear function $\alpha \cdot n$ by $-\log_r(h(r^{-n}))$.

Theorem 4.4 (Refinement of Inequality 6)
Let $F \subseteq X^\omega$, $\mathcal{H}^h(F) > 0$ and $c > 0$ a constant. Then there is a $\xi \in F$ fulfilling $\mathrm{KP}(\xi[0 \ldots n]) \geq_{\text{a.e.}} -\log_r(h(r^{-n})) - c$.

Proof. We prove this by showing that the set of ω–words having infinitely many prefixes of lower complexity is a $\mathcal{H}^h$–null set. We consider this set as the union of δ–limits of the sets $W_c = \{w \mid \mathrm{KP}(w) \leq -\log(h(r^{-|w|})) + c\}$, depending on the constant $c \in \mathbb{N}$. In order to utilise Lemma 3.4 again we estimate the following bound

$$1 > \sum_{w \in X^*} r^{-\mathrm{KP}(w)} > \sum_{w \in W_c} r^{-\mathrm{KP}(w)} \geq \sum_{w \in W_c} r^{-c} \cdot r^{\log h(r^{-|w|})} \geq r^{-c} \cdot \sum_{w \in W_c} h(r^{-|w|})$$

The first part is known as Kraft's inequality. Since c is a constant we have $\sum_{w \in W_c} h(r^{-|w|}) < \infty$, thus Lemma 3.4 yields $\mathcal{H}^h(W_c^\delta) = 0$, for arbitrary $c \in \mathbb{N}$. Now from $\mathcal{H}^h(F) > 0$ it follows that $F \setminus \bigcup_{c \in \mathbb{N}} W_c^\delta \neq \emptyset$, which in turn shows our assertion. $\square$

The same arguments as for simple complexity prove that we achieved a refinement of Inequality 6. A similar result for computable functions h is proved in [6, Theorem 2.6]. Analysing the proof of Theorem 4.4 it yields the following.

Theorem 4.5 *Let $F \subseteq X^\omega$ and $\mathcal{H}^h(F) > 0$. Then it holds*

(i) $\mathcal{H}^h \left(\bigcup_{c \in \mathbb{N}} \{\xi \mid \mathrm{KP}(\xi[0 \ldots n]) \leq_{\text{i.o.}} -\log(h(r^{-|w|})) - c\} \right) = 0$

(ii) *There is a $\xi \in F$, such that* $\lim_{n \to \infty} \mathrm{KP}(\xi[0 \ldots n]) - (-\log(h(r^{-|w|}))) = \infty$

A priori complexity

The last complexity we investigate is a priori complexity. It is known from [4] and [9] that a priori complexity is upper bounded by prefix complexity but incomparable to simple complexity. The result we show states that ω–languages F having positive $\mathcal{H}^h$–measure contain ω–words of complexity at least $-\log_r(h(r^{-n}))$ up to a constant dependent on $\mathcal{H}^h(F)$.

Theorem 4.6 *Let $F \subseteq X^\omega$ and $\mathcal{H}^h(F) > 0$. Then for any constant $c > -\log \mathcal{H}^h(F)$* [2] *there is a $\xi \in F$ such that*

$$\mathrm{KA}(\xi[0 \ldots n]) \geq_{\text{a.e.}} -\log_r(h(r^{-n})) - c.$$

Proof. As in the proof of Theorem 4.1 we define the set of ω–words not fulfilling the asserted inequality as the δ-limit of $W_c = \{w \mid \mathrm{KA}(w) \leq -\log(h(r^{-n})) - c\}$. By V_m we denote those words of W_c having at least length m

$$V_m = \{w \mid |w| \geq m \wedge \mathrm{KA}(w) \leq -\log(h(r^{-|w|})) - c\}$$

Then $V_{m+1} \subseteq V_m$, for any $m \in \mathbb{N}$. Let $\overline{V}_m$ be the set of all words in V_m which have no prefix in V_m. Then $\overline{V}_m$ is a prefix code and $\overline{V}_m \cdot X^\omega$ covers W_c^δ. Using Proposition 2.1 we can estimate the $\mathcal{H}^h$–measure of W_c^δ as follows

$$\begin{aligned}\mathcal{H}^h(W_c^\delta) &= \lim_{n \to \infty} \inf \left\{ \sum_{v \in V} h(r^{-|v|}) \mid V \cdot X^\omega \supseteq W_c \wedge \underline{l}(V) \geq n \right\} \\ &\leq \lim_{m \to \infty} \sum_{v \in \overline{V}_m} h(r^{-|v|}) = \lim_{m \to \infty} \sum_{v \in \overline{V}_m} r^{\log h(r^{-|v|})}\end{aligned}$$

[2] Here it is understood that $-\log \infty = -\infty$

$$\leq \lim_{m\to\infty} \sum_{v\in\overline{V}_m} r^{-\mathrm{KA}(v)-c} \leq r^{-c}$$

Now if $c > -\log_r \mathcal{H}^h(F)$ we have $\mathcal{H}^h(W_c^\delta) \leq r^{-c} < \mathcal{H}^h(F)$, and, consequently, $\mathcal{H}^h(F \setminus W_c^\delta) > 0$. Thus the set $F \setminus W_c^\delta$ is not empty, which in turn proves our assertion.

□

Note that this result is not similar to Theorem 4.4. In contrast to prefix complexity the bound for a priori complexity is not valid for all constants. The constant depends on the ω–language and its $\mathcal{H}^h$–measure. The following example states that the difference between a priori complexity and $-\log(h(r^{-|w|}))$ may not grow unboundedly.

Example 4.7 Let $X = \{0,1,2\}$ and $F = (X \cdot 0)^\omega$. Then we have $\mathbb{L}_{\frac{1}{2}}(F) = 1$. Consequently there is a $\xi \in F$ such that $\mathrm{KA}(\xi[n]) \geq_{\text{a.e.}} \frac{1}{2} \cdot n - c$, for arbitrary $c > 0$. On the other hand one can easily see that

$$\mathrm{KA}(x_1 0 x_2 0 \ldots x_{\frac{n}{2}} 0) \leq \mathrm{KA}(x_1 x_2 \ldots x_{\frac{n}{2}}) + c' \leq \frac{1}{2} \cdot n + c''. \tag{7}$$

Thus a similar result like the one in the second part of Theorem 4.5 for KA is not valid.

Now let $V = \{1,2\}^* \cdot 0$ and $F' = V \cdot F$. Then V is a prefix code and therefore (see [5])

$$\mathbb{L}_{\frac{1}{2}}(F') = \sum_{v\in V} r^{\frac{1}{2}\cdot|v|} \cdot \mathbb{L}_{\frac{1}{2}}(F) = \sum_{v\in V} r^{\frac{1}{2}\cdot|v|} = \infty$$

and $\mathbb{L}_{\frac{1}{2}+\varepsilon}(F') = 0$, for any $\varepsilon > 0$. Since $\mathrm{KA}(v \cdot \xi[n]) \leq \mathrm{KA}(\xi[n]) + c_v$ all ω–words in F' have a linear upper a priori bound $\mathrm{KA}(\xi'[n]) \leq \frac{1}{2} \cdot n + c_{\xi'}$. This shows that even for ω–languages of infinite measure the linear lower bound of Theorem 4.5 is, in general, not improvable.

5 Conclusion

We have seen that the known bounds of simple (conditional) and prefix complexity could be refined using a more general measure than the α–dimensional measure. From the result on a priori complexity one sees that the lower bound is valid for monotone complexity (see [4] or [9]), too. Moreover, it is known (see [9]) that the difference between a priori and monotone complexity is bounded by a slow growing recursive function. This leads to the conjecture that one cannot obtain better lower bounds for monotone complexity than for a priori complexity. Our last example states that in general the lower bound for a priori complexity cannot be improved.

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