Relating Two Approaches to Coinductive Solution of Recursive Equations

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Abstract

This paper shows that the approach of [2,12] for obtaining coinductive solutions of equations on infinite terms is a special case of a more general recent approach of [4] using distributive laws.

Keywords: Coinduction, coalgebra, recursive equation.

1   Introduction

The finality principle in the theory of coalgebras is usually called coinduction [8]. It involves the existence and uniqueness of suitable coalgebra homomorphisms to final coalgebras. It was realised early on (see [1,5]) that such coinductively obtained homomorphisms can be understood as solutions to recursive (or corecursive, if you like) equations. The equation itself is incorporated in the commuting square expressing that we have a homomorphism from a certain "source" coalgebra to the final coalgebra. Since this diagram arises from the the source coalgebra, this source can also be identified with the recursive equation.

A systematic investigation of the solution of such equations first appeared in [12], followed by [2]. Their coalgebraic approach simplifies results on recursive equations with infinite terms from [6,7]. More recently, a general and abstract approach is proposed in [4], building on distributive laws. The contribution of this paper is that it shows how the approach of [2] for infinite

terms fits in the general approach of [4] with distributive laws. This involves the identification of suitable distributive laws of the monads of terms over the underlying interface functor.

This paper is organised as follows. Section 2 briefly reviews the approach of [4] based on distributive laws. Section 3 introduces two distributive laws for canonical monads associated with a functor F. The approach of [2] for solutions of equations with infinite terms is then explained in Section 4. Finally, Section 5 shows that this approach is an instance of the distribution-based approach.

2 Distributive laws and solutions of equations

Distributive laws found their first serious application in the area of coalgebras in the work of Turi and Plotkin [15] (see also [14]), providing a joint treatment of operational and denotational semantics. In that setting a distributive laws provides a suitable form of compatibility between syntax and dynamics. It leads to results like: bisimilarity is a congruence, where, of course, bisimilarity is a coalgebraic notion of equivalence, and congruence and algebraic one. The claim of [15] that distributive laws correspond to suitable rule formats for operators is further substantiated in [4]. The idea of using a distributive law in extended forms of coinduction (and hence equation solving) comes from [9], and is further developed in [4]. In this section we present its essentials.

Distributive laws are natural transformations $FG \Rightarrow GF$ between two endofunctors $F, G \colon \mathbb{C} \to \mathbb{C}$ on a category $\mathbb{C}$. These F and G may have additional structure (of a point or copoint, or a monad or comonad, see [10]), that must then be preserved by the distributive law. We shall concentrate on the case of distribution of a monad over a functor, because it seems to be most common and natural—see the example in the next section. We shall recall what this means.

Definition 2.1 *Let (T, η, μ) be a monad on a category $\mathbb{C}$, and $F \colon \mathbb{C} \to \mathbb{C}$ be an arbitrary functor. A **distributive law** of T over F is a natural transformation*

$$TF \overset{\lambda}{\Longrightarrow} FT$$

making for each $X \in \mathbb{C}$ the following two diagrams commute.

$$
\begin{array}{ccc}
& FX & \\
{\scriptstyle \eta_{FX}} \downarrow & \overset{F(\eta_X)}{\searrow} & \\
TFX & \underset{\lambda_X}{\longrightarrow} & FTX
\end{array}
\qquad
\begin{array}{ccccc}
T^2FX & \overset{T(\lambda_X)}{\longrightarrow} & TFTX & \overset{\lambda_{TX}}{\longrightarrow} & FT^2X \\
{\scriptstyle \mu_{FX}} \downarrow & & & & \downarrow {\scriptstyle F(\mu_X)} \\
TFX & & \underset{\lambda_X}{\longrightarrow} & & FTX
\end{array}
$$

The underlying idea is that the monad T describes the terms in some syntax, and that the functor F is the interface for transitions on a state space. Intuitively, the presence of the distributive law tells us that the terms and behaviours interact appropriately. The associated notion of model is a so-called λ-bialgebra.

Definition 2.2 *Let* $\lambda\colon TF \Rightarrow FT$ *be a distributive law, like above. A λ-**bialgebra** consists of an object $X \in \mathbb{C}$ with a pair of maps:*

$$TX \xrightarrow{\ a\ } X \xrightarrow{\ b\ } FX$$

where:

- *a is an Eilenberg-Moore algebra, meaning that it satisfies two equations, namely: $a \circ \eta_X = id$ and $a \circ \mu_X = a \circ T(a)$.*

- *a and b are compatible via λ, which means that the following diagram commutes.*

$$
\begin{array}{ccccc}
TX & \xrightarrow{\ a\ } & X & \xrightarrow{\ b\ } & FX \\
{\scriptstyle T(b)}\downarrow & & & & \uparrow{\scriptstyle F(a)} \\
TFX & & \xrightarrow{\ \ \ \lambda_X\ \ \ } & & FTX
\end{array}
$$

*A **map of λ-bialgebras**, from $(TX \xrightarrow{a} X \xrightarrow{b} FX)$ to $(TY \xrightarrow{c} Y \xrightarrow{d} FY)$ is a map $f\colon X \to Y$ in $\mathbb{C}$ that is both a map of algebras and of coalgebras: $f \circ a = c \circ T(f)$ and $d \circ f = F(f) \circ b$.*

The following result is standard.

Lemma 2.3 *Assume a distributive law $\lambda\colon TF \Rightarrow FT$, and let $\zeta\colon Z \xrightarrow{\cong} FZ$ be a final coalgebra. It carries an Eilenberg-Moore algebra obtained by finality in:*

$$
\begin{array}{ccc}
FTZ & \dashrightarrow^{\ F(\alpha)\ } & FZ \\
{\scriptstyle \lambda_Z}\uparrow & & \uparrow{\scriptstyle \cong}\,\zeta \\
TFZ & & \\
{\scriptstyle T(\zeta)}\uparrow{\scriptstyle \cong} & & \\
TZ & \dashrightarrow_{\ \ \alpha\ \ } & Z
\end{array}
$$

The resulting pair $(TZ \xrightarrow{\alpha} Z \xrightarrow{\zeta} FZ)$ is then a final λ-bialgebra.

Proof By uniqueness one obtains that α is an Eilenberg-Moore algebra. By construction, α and ζ are compatible via λ. Assume an arbitrary λ-bialgebra $(TX \xrightarrow{a} X \xrightarrow{b} FX)$. It induces a unique coalgebra map $f\colon X \to Z$ with

$\zeta \circ f = F(f) \circ b$. One then obtains $f \circ a = \alpha \circ T(f)$ by showing that both maps are homomorphisms from the coalgebra $\lambda_X \circ T(b) \colon TX \to FTX$ to the final coalgebra ζ. $\qquad\square$

The following notion of equation and solution comes from [4].

Definition 2.4 *Assume a distributive law* $\lambda \colon TF \Rightarrow FT$. *A **guarded recursive equation** is an FT-coalgebra $e \colon X \to FTX$. A **solution** to such an equation in a λ-bialgebra $(TY \xrightarrow{a} Y \xrightarrow{b} FY)$ is a map $f \colon X \to Y$ making the following diagram commute.*

(1)

$$
\begin{array}{ccc}
FTX & \xrightarrow{\;FT(f)\;} & FTY \\
{\scriptstyle e}\big\uparrow & & \big\downarrow{\scriptstyle F(a)} \\
 & & FY \\
 & & \big\uparrow{\scriptstyle b} \\
X & \xrightarrow[\;f\;]{} & Y
\end{array}
$$

In ordinary coinduction one obtains solutions for equations $X \to FX$. The power of the above notion of equation $X \to FTX$ lies in the fact that it allows actions on terms. For convenience we shall often call these equations $X \to FTX$ λ**-equations**—even though their formulation does not involve a distributive law λ. But their intended use is in a context with distributive laws.

This notion of solution may seem a bit strange at first, but becomes more natural in light of the following result. It is implicit in [4].

Proposition 2.5 *There exists a bijective correspondence between λ-equations $e \colon X \to FTX$ and λ-bialgebras $(T^2X \xrightarrow{\mu_X} TX \xrightarrow{d} FTX)$ with free algebra μ_X.*

Moreover, let $(TY \xrightarrow{a} Y \xrightarrow{b} FY)$ be a λ-bialgebra. Then there is a bijective correspondence between solutions $f \colon X \to Y$ as in (1) and bialgebra maps $g \colon TX \to Y$—for the associated λ-equations and λ-bialgebras.

Proof Given a λ-equation $e \colon X \to FTX$ we define

$$
\overline{e} = \Big(TX \xrightarrow{T(e)} TFTX \xrightarrow{\lambda_{TX}} FT^2X \xrightarrow{F(\mu_X)} FTX \Big)
$$

This yields, together with the free algebra $\mu_X \colon T^2 X \to TX$ a λ-bialgebra:

$$
\begin{aligned}
F(\mu_X) \circ \lambda_{TX} \circ T(\overline{e}) &= F(\mu_X) \circ \lambda_{TX} \circ T(F(\mu_X) \circ \lambda_{TX} \circ T(e)) \\
&= F(\mu_X) \circ FT(\mu_X) \circ \lambda_{T^2 X} \circ T(\lambda_{TX}) \circ T^2(e) \\
&= F(\mu_X) \circ F(\mu_{TX}) \circ \lambda_{T^2 X} \circ T(\lambda_{TX}) \circ T^2(e) \\
&= F(\mu_X) \circ \lambda_{TX} \circ \mu_{FTX} \circ T^2(e) \\
&= F(\mu_X) \circ \lambda_{TX} \circ T(e) \circ \mu_X \\
&= \overline{e} \circ \mu_X.
\end{aligned}
$$

Conversely, given a λ-bialgebra $(T^2 X \xrightarrow{\mu_X} TX \xrightarrow{d} FTX)$, we define a λ-equation:

$$
\overline{d} = \left(X \xrightarrow{\eta_X} TX \xrightarrow{d} FTX \right)
$$

These operations $e \mapsto \overline{e}$ and $d \mapsto \overline{d}$ are each others inverses:

$$
\begin{aligned}
\overline{\overline{e}} &= \overline{e} \circ \eta_X \\
&= F(\mu_X) \circ \lambda_{TX} \circ T(e) \circ \eta_X \\
&= F(\mu_X) \circ \lambda_{TX} \circ \eta_{FTX} \circ e \\
&= F(\mu_X) \circ F(\eta_{TX}) \circ e \\
&= e.
\end{aligned}
$$

$$
\begin{aligned}
\overline{\overline{d}} &= F(\mu_X) \circ \lambda_{TX} \circ T(\overline{d}) \\
&= F(\mu_X) \circ \lambda_{TX} \circ T(d \circ \eta_X) \\
&= d \circ \mu_X \circ T(\eta_X) \\
&= d.
\end{aligned}
$$

Assume now we have a solution $f \colon X \to Y$ for $e \colon X \to FTX$ like in (1). We take $\overline{f} = a \circ T(f) \colon TX \to Y$. It forms a map of λ-bialgebras, from $(\mu_X, \overline{e})$

to (a, b):

$$
\begin{aligned}
a \circ T(\overline{f}) &= a \circ T(a \circ T(f)) \\
&= a \circ \mu_Y \circ T^2(f) \\
&= a \circ T(f) \circ \mu_X \\
&= \overline{f} \circ \mu_X.
\end{aligned}
$$

$$
\begin{aligned}
F(\overline{f}) \circ \overline{e} &= F(a \circ T(f)) \circ F(\mu_X) \circ \lambda_{TX} \circ T(e) \\
&= F(a) \circ F(\mu_X) \circ FT^2(f) \circ \lambda_{TX} \circ T(e) \\
&= F(a) \circ FT(a) \circ FT^2(f) \circ \lambda_{TX} \circ T(e) \\
&= F(a) \circ \lambda_Y \circ TF(a) \circ TFT(f) \circ T(e) \\
&= F(a) \circ \lambda_Y \circ T(b) \circ T(f) \\
&= b \circ a \circ T(f) \\
&= b \circ \overline{f}.
\end{aligned}
$$

Conversely, assume a λ-bialgebra map $g: TX \to Y$ from (μ_X, d) to (a, b). It yields a map $\overline{g} = g \circ \eta_X: X \to Y$ which is a solution of $\overline{d}$, since:

$$
\begin{aligned}
F(a) \circ FT(\overline{g}) \circ \overline{d} &= F(a) \circ FT(g \circ \eta_X) \circ d \circ \eta_X \\
&= F(g) \circ F(\mu_X) \circ FT(\eta_X) \circ d \circ \eta_X \\
&= F(g) \circ d \circ \eta_X \\
&= b \circ g \circ \eta_X \\
&= b \circ \overline{g}.
\end{aligned}
$$

Finally, it is obvious that $f \mapsto \overline{f}$ and $g \mapsto \overline{g}$ are each others inverses. $\quad\square$

Now we can formulate the main result of this distribution-based approach to solving equations.

Theorem 2.6 *Let $F: \mathbb{C} \to \mathbb{C}$ be a functor with a final coalgebra $Z \xrightarrow{\cong} FZ$. For each monad T with distributive law $\lambda: TF \Rightarrow FT$ there are unique solutions to λ-equations in the final λ-bialgebra $(TZ \to Z \to FZ)$ from Lemma 2.3.*

Proof For a λ-equation $e: X \to FTX$, a solution in $(TZ \to Z \to FZ)$ is by the previous proposition the same thing as a map of λ-bialgebras from the associated $(T^2X \to TX \to FTX)$ to $(TZ \to Z \to FZ)$. Since the latter is final, there is precisely one such solution. $\quad\square$

In Example 3.3 in the next section we present an illustration.

3   Free monads and their distributive laws

In this section we consider an endofunctor $F: \mathbb{C} \to \mathbb{C}$ with two canonical associated monads F^* and F^∞, together with distributive laws λ^* and λ^∞ over F. The first result is not used directly, but provides the setting the second one—which forms the basis for Lemma 5.1 later on.

3.1   The free monad on a functor

Let $F: \mathbb{C} \to \mathbb{C}$ be an arbitrary endofunctor on a category $\mathbb{C}$ with (binary) coproducts $+$. The only assumption we make at this stage is that for each object $X \in \mathbb{C}$ the functor $X + F(-): \mathbb{C} \to \mathbb{C}$ has an initial algebra. We shall use the following notation. The carrier of this initial algebra will be written as $F^*(X)$ with structure map given as:

$$X + F(F^*(X)) \xrightarrow[\cong]{\quad \alpha \quad} F^*(X)$$

Further, we shall write

$$\eta_X = \alpha \circ \kappa_1 \qquad \tau_X = \alpha \circ \kappa_2,$$

so that $\alpha_X = [\eta_X, \tau_X]$.

The mapping $X \mapsto F^*(X)$ is functorial: for $f: X \to Y$ we get:

$$
\begin{array}{ccc}
X + F(F^*(X)) & \xrightarrow{\;\mathrm{id} + F(F^*(f))\;} & X + F(F^*(Y)) \\
{\scriptstyle \alpha_X}\Big\downarrow{\scriptstyle \cong} & & \Big\downarrow{\scriptstyle [\eta_Y \,\circ\, f,\, \tau_Y]} \\
F^*(X) & \xrightarrow[\;F^*(f)\;]{} & F^*(Y)
\end{array}
$$

This means that

$$F^*(f) \circ \eta_X = \eta_Y \circ f \qquad F^*(f) \circ \tau_X = \tau_Y \circ F(F^*(f)),$$

i.e. that $\eta: \mathrm{id} \Rightarrow F^*$ and $\tau: FF^* \Rightarrow F^*$ are natural transformations.

Next we establish that F^* is a monad. The multiplication μ is obtained in:

$$
\begin{array}{ccc}
F^*(X) + F(F^*(F^*(X))) & \xrightarrow{\;\mathrm{id} + F(\mu_X)\;} & F^*(X) + F(F^*(X)) \\
{\scriptstyle \alpha_{F^*(X)}}\Big\downarrow{\scriptstyle \cong} & & \Big\downarrow{\scriptstyle [\mathrm{id},\, \tau_X]} \\
F^*(F^*(X)) & \xrightarrow[\;\mu_X\;]{} & F^*(X)
\end{array}
$$

This yields one of the monad equations, namely $\mu_X \circ \eta_{F^*(X)} = \text{id}$. The related equation $\mu_X \circ F^*(\eta_X) = \text{id}$ follows from uniqueness of algebra maps $\alpha_X \to \alpha_X$:

$$
\begin{aligned}
\mu_X \circ F^*(\eta_X) \circ \alpha_X &= \mu_X \circ [\eta_{F^*(X)} \circ \eta_X, \tau_{F^*(X)}] \circ (\text{id} + F(F^*(\eta_X))) \\
&= [\eta_X, \tau_X \circ F(\mu_X)] \circ (\text{id} + F(F^*(\eta_X))) \\
&= \alpha_X \circ (\text{id} + F(\mu_X \circ F^*(\eta_X))).
\end{aligned}
$$

Similarly, the other requirements making F^* a monad are obtained.

The following standard result sums up the situation.

Proposition 3.1 *Let $F: \mathbb{C} \to \mathbb{C}$ with induced monad (F^*, η, μ) be as described above.*

(i) *The mapping $X \mapsto (F(F^*(X) \xrightarrow{\tau_X} F^*(X))$ forms a left adjoint to the forgetful functor $U: \mathbf{Alg}(F) \to \mathbb{C}$.*

The monad induced by this adjunction is (F^*, η, μ).

(ii) *The mapping $\sigma_X = \tau_X \circ F(\eta_X): F(X) \to F^*(X)$ yields a natural transformation $F \Rightarrow F^*$ that makes F^* the free monad on F.*   □

The next observation shows that the monad F^* of (finite) F-terms fits with the behaviour of F. It follows from a general observation (made for instance in [4]) that distributive laws $F^*G \Rightarrow GF^*$ correspond to ordinary natural transformations $FG \Rightarrow GF$. Hence by taking $G = F$ and the identity $FF \Rightarrow FF$ one gets $F^*F \Rightarrow FF^*$. But here we shall present the explicit construction.

Proposition 3.2 *Let $F: \mathbb{C} \to \mathbb{C}$ have free monad F^*. Then there is a distributive law $\lambda^*: F^*F \Rightarrow FF^*$.*

Proof We define $\lambda_X^*: F^*(FX) \to F(F^*X)$ as follows.

$$
F^*(FX) \xrightarrow[\cong]{\alpha_{FX}^{-1}} FX + F(F^*(FX)) \xrightarrow{[F(\eta_X), F(\mu_X \circ F^*(\sigma_X))]} F(F^*X)
$$

where $\sigma_X = \tau_X \circ F(\eta_X): F(X) \to F^*(X)$ as introduced in Proposition 3.1 (ii).
□

Example 3.3 *Let $Z = \mathbb{R}^{\mathbb{N}}$ be the set of streams of real numbers. It is of course the final coalgebra of the functor $F = \mathbb{R} \times (-)$, via the head and tail operations $\langle \mathsf{hd}, \mathsf{tl} \rangle: Z \xrightarrow{\cong} \mathbb{R} \times Z$. It is shown in [13] that on such streams one can coinductively define binary operators $\oplus$ for sum and $\otimes$ for shuffle product*

satisfying the recursive equations:

$$x \oplus y = \big(hd(x) + hd(y)\big) \cdot \big(tl(x) \oplus tl(y)\big)$$
$$x \otimes y = \big(hd(x) \times hd(y)\big) \cdot \big((tl(x) \otimes y) \oplus (x \otimes tl(y))\big),$$

where $\cdot$ *is prefix.*

It is easy to see that one defines $\oplus$ *by ordinary coinduction, in:*

$$
\begin{array}{ccc}
\mathbb{R} \times (Z \times Z) & \xrightarrow{\ id \times \oplus\ } & \mathbb{R} \times Z \\
{\scriptstyle c_\oplus}\big\uparrow & & \cong\big\uparrow{\scriptstyle \langle hd, tl\rangle} \\
Z \times Z & \dashrightarrow[\oplus] & Z
\end{array}
$$

where the coalgebra $c_\oplus$ *is defined by:*

$$c_\oplus(x, y) = \langle hd(x) + hd(y), \langle tl(x), tl(y)\rangle \rangle.$$

Once we have $\oplus\colon Z \times Z \to Z$ *we show how to obtain* $x \otimes y$ *as a solution of a* λ*-equation. We start from the signature functor* $\Sigma(X) = X \times X$. *There is an obvious distributive law* $\Sigma F \Rightarrow F\Sigma$ *given by* $(\langle r, x\rangle, \langle s, y\rangle) \longmapsto \langle r + s, (x, y)\rangle$. *By a result of [4] it lifts to a distributive law* $\lambda\colon \Sigma^* F \Rightarrow F\Sigma^*$ *involving the associated free monad* Σ^*. *The algebra* $\oplus\colon \Sigma(Z) \to Z$ *yields an Eilenberg-Moore algebra* $[\![-]\!]\colon \Sigma^*(Z) \to Z$, *which is by the same result of [4] a* λ*-bialgebra. Now we obtain* $\otimes$ *as solution in:*

$$
\begin{array}{ccc}
\mathbb{R} \times \Sigma^*(Z \times Z) & \xrightarrow{\ id \times \Sigma^*(\otimes)\ } & \mathbb{R} \times \Sigma^*(Z) \\
{\scriptstyle d_\otimes}\big\uparrow & & \big\downarrow{\scriptstyle id \times [\![-]\!]} \\
 & & \mathbb{R} \times Z \\
 & & \cong\big\uparrow{\scriptstyle \langle hd, tl\rangle} \\
Z \times Z & \dashrightarrow[\otimes] & Z
\end{array}
$$

in which the λ*-equation* $d_\otimes$ *is defined by:*

$$d_\otimes(x, y) = \langle hd(x) \times hd(y), (tl(x), y) \underline{\oplus} (x, tl(y))\rangle,$$

where $\underline{\oplus}$ *is a symbol for sum in the language of terms on pairs from* $Z \times Z$. *Here we exploit the expressive power of the* λ*-approach, because we can now write terms as second component.*

Clearly,

$$hd(x \otimes y) = hd(x) \times hd(y).$$

And, as required:

$$
\begin{aligned}
\mathit{tl}(x \otimes y) &= (\llbracket - \rrbracket \circ \Sigma^*(\otimes) \circ \pi_2 \circ d_\otimes)(x, y) \\
&= (\llbracket - \rrbracket \circ \Sigma^*(\otimes))(\mathit{tl}(x), y) \underline{\oplus}(x, \mathit{tl}(y)) \\
&= \llbracket (\mathit{tl}(x) \otimes y) \underline{\oplus}(x \otimes \mathit{tl}(y)) \rrbracket \\
&= (\mathit{tl}(x) \otimes y) \oplus (x \otimes \mathit{tl}(y)).
\end{aligned}
$$

This concludes the example.

3.2 The free iterative monad on a functor

Let, like in the previous section, $F \colon \mathbb{C} \to \mathbb{C}$ be an arbitrary endofunctor on a category $\mathbb{C}$ with (binary) coproducts $+$. The assumption we now make is that for each object $X \in \mathbb{C}$ the functor $X + F(-) \colon \mathbb{C} \to \mathbb{C}$ has an final coalgebra—instead of an initial algebra. We shall use the following notation. The carrier of this final calgebra will be written as $F^\infty(X)$ with structure map given as:

$$
F^\infty(X) \xrightarrow[\ \cong\ ]{\ \zeta\ } X + F(F^\infty(X))
$$

The sets $F^*(X)$ in the previous section are understood as the set of finite terms of type F with free variables from X. Here we understand $F^\infty(X)$ as the set of both finite and infinite terms (or trees) with free variables in X.

Like before, we shall write:

$$
\eta_X = \zeta^{-1} \circ \kappa_1 \qquad \tau_X = \zeta^{-1} \circ \kappa_2.
$$

Functoriality of F^∞ is obtained as follows. For $f \colon X \to Y$ in $\mathbb{C}$ we get:

$$
\begin{array}{ccc}
Y + F(F^\infty(X)) & \xdashrightarrow{\ \mathrm{id} + F(F^\infty(f))\ } & X + F(F^\infty(Y)) \\[2pt]
{\scriptstyle f + \mathrm{id} \circ \zeta_X} \Big\uparrow & & \cong \Big\uparrow {\scriptstyle \zeta_Y} \\[2pt]
F^\infty(X) & \xdashrightarrow[\ F^\infty(f)\ ]{} & F^\infty(Y)
\end{array}
$$
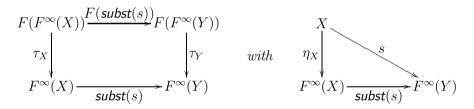
This means that

$$
F^\infty(f) \circ \eta_X = \eta_Y \circ f \qquad F^\infty(f) \circ \tau_X = \tau_Y \circ F(F^\infty(f)),
$$

i.e. that $\eta \colon \mathrm{id} \Rightarrow F^\infty$ and $\tau \colon FF^\infty \Rightarrow F^\infty$ are natural transformations.

It is shown in [11,3] that F^∞ is a monad[1]. The multiplication operation μ is rather complicated, and can best be introduced via substitution $t[s/x]$. What we mean is replacing all occurrences (if any) of the variable x in the term t by the term s, but now for possibly infinite terms. In most general form, this substitution $t[\overrightarrow{s}/\overrightarrow{x}]$ replaces all occurrences of all variables $x \in X$ simultaneously. In this way, substitution may be described as an operation which tells how an X-indexed collection $(s_x)_{x \in X}$ of terms $s_x \in F^\infty(Y)$ acts on a term $t \in F^\infty(X)$. More precisely, substitution becomes an operation $\mathsf{subst}(s): F^\infty(X) \to F^\infty(Y)$, for a function $s: X \to F^\infty(Y)$. As usual, such a substitution operation should respect the term structure—*i.e.* be a homomorphism—and be trivial on variables. Standardly, substitution is defined by induction on the structure of (finite) terms. But since we are dealing here with possibly infinite terms, we have to use coinduction. This makes the substitution more challenging. In general, it is done as follows.

Lemma 3.4 *Let X, Y be arbitrary sets. Each function $s: X \to F^\infty(Y)$ gives rise to a **coalgebraic substitution** operator $\mathsf{subst}(s): F^\infty(X) \to F^\infty(Y)$, namely the unique homomorphism of F-algebras:*

$$
\begin{array}{ccc}
F(F^\infty(X)) \xrightarrow{F(\mathsf{subst}(s))} F(F^\infty(Y)) & & X \\
\quad\downarrow{\scriptstyle \tau_X} \qquad\qquad\qquad \downarrow{\scriptstyle \tau_Y} & \quad with \quad & {\scriptstyle \eta_X}\downarrow \ \searrow{\scriptstyle s} \\
F^\infty(X) \xrightarrow[\mathsf{subst}(s)]{} F^\infty(Y) & & F^\infty(X) \xrightarrow[\mathsf{subst}(s)]{} F^\infty(Y)
\end{array}
$$

Proof We begin by defining a coalgebra structure on the coproduct $F^\infty(Y) + F^\infty(X)$ of terms, namely as the vertical composite on the left below.

$$
\begin{array}{ccc}
Y + F(F^\infty(Y) + F^\infty(X)) & \xrightarrow{\ \mathrm{id}_Y + F(f)\ } & Y + F(F^\infty(Y)) \\
{\scriptstyle [(\mathrm{id}_Y + F(\kappa_1)) \circ \zeta_Y, \kappa_2 \circ F(\kappa_2)]}\Big\uparrow & & \\
F^\infty(Y) + F(F^\infty(X)) & & \\
{\scriptstyle [\kappa_1, s + \mathrm{id}]}\Big\uparrow & & \Big\downarrow{\scriptstyle \cong}\ {\scriptstyle \zeta_Y} \\
F^\infty(Y) + (X + F(F^\infty(X))) & & \\
{\scriptstyle \mathrm{id}_Y + \zeta_X}\Big\uparrow & & \\
F^\infty(Y) + F^\infty(X) & \xrightarrow[\qquad f \qquad]{} & F^\infty(Y)
\end{array}
$$

---

[1] Similar results have been obtained earlier by [12], but for the functor $X \mapsto F(X + -)$.

One first proves that $f \circ \kappa_1$ is the identity, using uniqueness of coalgebra maps $\zeta_Y \to \zeta_Y$. Then, $f \circ \kappa_2$ is the required map $\mathsf{subst}(s)$.   □

In the remainder of this paper we shall make frequent use of this substitution operator $\mathsf{subst}(-)$. Computations with substitution are made much easier with the following elementary results. Proofs are obtained via the uniqueness property of substitution.

Lemma 3.5 *For* $s: X \to F^\infty(Y)$ *we have:*

(i) $\mathsf{subst}(\eta_X) = id_{F(X)}$.

(ii) $\mathsf{subst}(s) \circ F^\infty(f) = \mathsf{subst}(s \circ f)$, *for* $f: Z \to X$.

(iii) $\mathsf{subst}(r) \circ \mathsf{subst}(s) = \mathsf{subst}(\mathsf{subst}(r) \circ s)$, *for* $r: Y \to F^\infty(Z)$.

(iv) $F^\infty(f) = \mathsf{subst}(\eta_Z \circ f)$, *for* $f: Y \to Z$, *and hence* $\mathsf{subst}(F^\infty(f) \circ s) = F^\infty(f) \circ \mathsf{subst}(s)$.

(v) $\mathsf{subst}(s) = [s, \tau_Y \circ F(\mathsf{subst}(s))] \circ \zeta_X$.   □

Proposition 3.6 *The map* $\mu_X = \mathsf{subst}(id_{F^\infty(X)}): F^\infty(F^\infty(X)) \to F^\infty(X)$ *makes the triple* (F^∞, η, μ) *a monad.*

This monad F^∞ *is called the **iterative** monad on* F, *via the natural transformation* $\sigma = \tau \circ F\eta: F \Rightarrow F^\infty$.

In [2] it shown that F^∞ is in fact a *free* iterative monad, in a suitable sense. This freeness is not relevant here.

Proof We check the monad equations, using Lemma 3.5.

$$\mu_X \circ \eta_{F^\infty X} = \mathsf{subst}(id_{F^\infty(X)}) \circ \eta_{F^\infty X}$$
$$= id_{F^\infty(X)}.$$
$$\mu_X \circ F^\infty(\eta_X) = \mathsf{subst}(id_{F^\infty(X)}) \circ F^\infty(\eta_X)$$
$$= \mathsf{subst}(id_{F^\infty(X)} \circ \eta_X)$$
$$= id_{F^\infty(X)}.$$
$$\mu_X \circ F^\infty(\mu_X) = \mathsf{subst}(id_{F^\infty(X)}) \circ F^\infty(\mu_X)$$
$$= \mathsf{subst}(\mu_X)$$
$$= \mathsf{subst}(\mathsf{subst}(id_{F^\infty(X)}) \circ id_{F^\infty(F^\infty(X))})$$
$$= \mathsf{subst}(id_{F^\infty(X)}) \circ \mathsf{subst}(id_{F^\infty(F^\infty(X))})$$
$$= \mu_X \circ \mu_{F^\infty(X)}.$$

□

The following is less standard.

Proposition 3.7 *Consider* $F: \mathbb{C} \to \mathbb{C}$ *with its iterative monad* F^∞.

(i) *There is a distributive law* $\lambda^\infty: F^\infty F \Rightarrow F F^\infty$.

(ii) *The induced mediating map of monads* $F^* \Rightarrow F^\infty$ *commutes with the distributive laws, in the sense that the following diagram commutes.*

$$
\begin{array}{ccc}
F^* F & \longrightarrow & F^\infty F \\
\lambda^* \downarrow & & \downarrow \lambda^\infty \\
F F^* & \longrightarrow & F F^\infty
\end{array}
$$

Proof Like for λ^* we define $\lambda_X^\infty: F^\infty(FX) \to F(F^\infty X)$ as follows:

$$
F^\infty(FX) \xrightarrow[\cong]{\zeta_{FX}} FX + F(F^\infty(FX)) \xrightarrow{[F(\eta_X), F(\mu_X \circ F^\infty(\sigma_X))]} F(F^\infty X)
$$

where $\sigma_X = \tau_X \circ F(\eta_X): F(X) \to F^\infty(X)$ as introduced in Proposition 3.6. It satisfies, like in the proof of Proposition 3.2,

$$
(2) \qquad
\begin{aligned}
\mu_X \circ \sigma_{F^\infty X} &= \mathsf{subst}(\mathrm{id}_{F^\infty X}) \circ \tau_{F^\infty X} \circ F(\eta_{F^\infty X}) \\
&= \tau_X \circ F(\mathsf{subst}(\mathrm{id}_{F^\infty X})) \circ F(\eta_{F^\infty X}) \\
&= \tau_X \circ F(\mathrm{id}_{F^\infty X}) \\
&= \tau_X.
\end{aligned}
$$

Then:

$$
\begin{aligned}
\lambda_X^\infty \circ \eta_{FX} &= [F(\eta_X), F(\mu_X \circ F^\infty(\sigma_X))] \circ \zeta \circ \eta_{FX} \\
&= [F(\eta_X), F(\mu_X \circ F^\infty(\sigma_X))] \circ \kappa_1 \\
&= F(\eta_X).
\end{aligned}
$$

We shall use the following two auxiliary results:

$$
(3) \qquad
\begin{aligned}
\mu_X \circ \sigma_{F^\infty X} \circ \lambda_X^\infty &= \mu_X \circ F^\infty(\sigma_X) \\
F(\tau_X) \circ F(\lambda_X^\infty) &= \lambda_X^\infty \circ \tau_{FX}.
\end{aligned}
$$

We first prove the first equation, and use it immediately to prove the second

one.

$$\mu_X \circ \sigma_{F^\infty X} \circ \lambda_X^\infty$$
$$= [\mu_X \circ \sigma_{F^\infty X} \circ F(\eta_X), \mu_X \circ \sigma_{F^\infty X} \circ F(\mu_X \circ F^\infty(\sigma_X))] \circ \zeta_{FX}$$
$$= [\mu_X \circ F^\infty(\eta_X) \circ \sigma_X, \mu_X \circ F^\infty(\mu_X \circ F^\infty(\sigma_X)) \circ \sigma_{F^\infty FX}] \circ \zeta_{FX}$$
$$= [\mu_X \circ \eta_{F^\infty X} \circ \sigma_X, \mu_X \circ \mu_{F^\infty X} \circ F^\infty F^\infty(\sigma_X) \circ \sigma_{F^\infty FX}] \circ \zeta_{FX}$$
$$= [\mu_X \circ F^\infty(\sigma_X) \circ \eta_{FX}, \mu_X \circ F^\infty(\sigma_X) \circ \mu_{FX} \circ \sigma_{F^\infty FX}] \circ \zeta_{FX}$$
$$= \mu_X \circ F^\infty(\sigma_X) \circ [\eta_{FX}, \tau_{FX}] \circ \zeta_{FX}$$
$$= \mu_X \circ F^\infty(\sigma_X).$$
$$F(\tau_X) \circ F(\lambda_X^\infty)$$
$$\overset{(2)}{=} F(\mu_X \circ \sigma_{F^\infty X} \circ \lambda_X^\infty)$$
$$= F(\mu_X \circ F^\infty(\sigma_X))$$
$$= [F(\eta_X), F(\mu_X \circ F^\infty(\sigma_X))] \circ \kappa_2$$
$$= \lambda_X^\infty \circ \tau_{FX}.$$

Now we are ready to prove that λ^∞ commutes with multiplications.

$$\lambda_X^\infty \circ \mu_{FX}$$
$$= \lambda_X^\infty \circ [\mathrm{id}, \tau_{FX} \circ F(\mu_{FX})] \circ \zeta_{F^\infty FX} \qquad \text{by Lemma 3.5 (v)}$$
$$= [\lambda_X^\infty, \lambda_X^\infty \circ \tau_{FX} \circ F(\mu_{FX})] \circ \zeta_{F^\infty FX}$$
$$\overset{(3)}{=} [\lambda_X^\infty, F(\tau_X \circ \lambda_X^\infty \circ \mu_{FX})] \circ \zeta_{F^\infty FX}$$
$$\overset{(2)}{=} [\lambda_X^\infty, F(\mu_X \circ \sigma_{F^\infty X} \circ \lambda_X^\infty \circ \mu_{FX})] \circ \zeta_{F^\infty FX}$$
$$\overset{(3)}{=} [\lambda_X^\infty, F(\mu_X \circ F^\infty(\sigma_X) \circ \mu_{FX})] \circ \zeta_{F^\infty FX}$$
$$= [\lambda_X^\infty, F(\mu_X \circ \mu_{F^\infty X} \circ F^\infty F^\infty(\sigma_X))] \circ \zeta_{F^\infty FX}$$
$$= [\lambda_X^\infty, F(\mu_X \circ F^\infty(\mu_X \circ F^\infty(\sigma_X)))] \circ \zeta_{F^\infty FX}$$
$$\overset{(3)}{=} [\lambda_X^\infty, F(\mu_X \circ F^\infty(\mu_X \circ \sigma_{F^\infty X} \circ \lambda_X^\infty))] \circ \zeta_{F^\infty FX}$$
$$= [\mathrm{id}, F(\mu_X \circ \mu_{F^\infty X} \circ F^\infty(\sigma_{F^\infty X}))] \circ (\lambda_X^\infty + F(F^\infty \lambda_X^\infty)) \circ \zeta_{F^\infty FX}$$
$$= F(\mu_X) \circ [F(\eta_{F^\infty X}), F(\mu_{F^\infty X} \circ F^\infty(\sigma_{F^\infty X}))] \circ \zeta_{FF^\infty X} \circ F^\infty(\lambda_X^\infty)$$
$$= F(\mu_X) \circ \lambda_{F^\infty X}^\infty \circ F^\infty(\lambda_X^\infty).$$

In order to prove the second point of the proposition we have to disambiguate the notation. Let's write the monad F^* as (F^*, η^*, μ^*) with associated τ^* and σ^*, and F^∞ as $(F^\infty, \eta^\infty, \mu^\infty)$ with τ^∞ and σ^∞. The induced mediating

map $\overline{\sigma^\infty}\colon F^* \Rightarrow F^\infty$ is then given by:

$$
\begin{array}{ccc}
X + F(F^*X) & \xrightarrow{\;\mathrm{id} + F(\overline{\sigma^\infty}_X)\;} & X + F(F^\infty X) \\
{\scriptstyle \alpha_X}\big\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\big\uparrow{\scriptstyle \zeta_X} \\
F^*X & \xdashrightarrow[\;\overline{\sigma^\infty}_X\;]{} & F^\infty X
\end{array}
$$

We already know (from Proposition 3.1) that $\overline{\sigma^\infty}$ is a homomorphism of monads satisfying $\overline{\sigma^\infty} \circ \sigma^* = \sigma^\infty$. Hence $\overline{\sigma^\infty}$ commutes with the distributive laws:

$$
\begin{aligned}
\lambda_X^\infty \circ \overline{\sigma^\infty}_{FX} &= [F(\eta_X^\infty), F(\mu_X^\infty \circ F^\infty(\sigma_X^\infty))] \circ \zeta_{FX} \circ \overline{\sigma^\infty}_{FX} \\
&= [F(\eta_X^\infty), F(\mu_X^\infty \circ F^\infty(\sigma_X^\infty))] \circ \mathrm{id} + F(\overline{\sigma^\infty}_{FX}) \circ \alpha_{FX}^{-1} \\
&= [F(\eta_X^\infty), F(\mu_X^\infty \circ F^\infty(\sigma_X^\infty) \circ \overline{\sigma^\infty}_{FX})] \circ \alpha_{FX}^{-1} \\
&= [F(\eta_X^\infty), F(\mu_X^\infty \circ \overline{\sigma^\infty}_{F^\infty X} \circ F^*(\sigma_X^\infty))] \circ \alpha_{FX}^{-1} \\
&= [F(\eta_X^\infty), F(\mu_X^\infty \circ \overline{\sigma^\infty}_{F^\infty X} \circ F^*(\overline{\sigma^\infty}_X \circ \sigma_X^*))] \circ \alpha_{FX}^{-1} \\
&= [F(\overline{\sigma^\infty}_X \circ \eta_X^*), F(\overline{\sigma^\infty}_X \circ \mu_X^* \circ F^*(\sigma_X^*))] \circ \alpha_{FX}^{-1} \\
&= F(\overline{\sigma^\infty}_X) \circ [F(\eta_X^*), F(\mu_X^* \circ F^*(\sigma_X^*))] \circ \alpha_{FX}^{-1} \\
&= F(\overline{\sigma^\infty}_X) \circ \lambda_X^*.
\end{aligned}
$$

$\square$

4   Iteration and solutions of equations

The material in this section comes (again) from [2]. In Definition 2.4 we have seen an abstract notion of λ-equation and solution. A bit more concretely, for a functor F, a set of recursive equations—often simply called a recursive equation—consists first of all of a set X of recursive variables. For each variable $x \in X$ we have a corresponding term t in an equation $x = t$. We shall allow this term to be infinite. The term t may involve both variables from an already given set Y, and from our new set of recursive variables X. Hence $t \in F^\infty(Y + X)$. Summarising, a **recursive equation** is a map $e\colon X \to F^\infty(Y + X)$. We shall often call such an e a ∞-**equation**, in contrast to a λ-equation $X \to FTX$—as in Definition 2.4.

Definition 4.1 *Let* $F\colon \mathbb{C} \to \mathbb{C}$ *be a functor, with for* $X \in \mathbb{C}$ *a final coalgebra* $F^\infty(X) \xrightarrow{\cong} X + F(F^\infty(X))$.

   *A **solution** for an* ∞*-equation* $e\colon X \to F^\infty(Y + X)$ *is a map* $\mathsf{sol}(e)\colon X \to F^\infty(Y)$ *that produces an appropriate term* $\mathsf{sol}(e)(x)$ *for each recursive variable*

$x \in X$. *This means that substituting the cotuple* $[\eta_Y, \mathsf{sol}(e)]\colon Y + X \to F^\infty(Y)$ *in* e *yields the solution* $\mathsf{sol}(e)$, *i.e.*

$$
\begin{array}{ccc}
\mathsf{sol}(e) & & X \xrightarrow{\;\;e\;\;} F^\infty(Y+X) \\
= \mathsf{subst}([\eta_Y, \mathsf{sol}(e)]) \circ e & \quad in \quad & \Big\downarrow{\scriptstyle \mathsf{subst}([\eta_Y, \mathsf{sol}(e)])} \\
& & \mathsf{sol}(e) \searrow \quad \\
& & F^\infty(Y)
\end{array}
$$

This shows that the solution is a fixed point of $\mathsf{subst}([\eta_Y, -]) \circ e$.

Like for λ-equations, we are interested in unique solutions for ∞-equations. Do they always exist? Not in trivial equations, like $x = x$, where any term is a solution. Such equations are standardly excluded by requiring that the terms of the recursive equation are 'guarded', *i.e.* that its terms are not variables from X. This notion can also be formulated in a general categorical setting: an ∞-equation $e\colon X \to F^\infty(Y+X)$ is called **guarded** if it factors (in a necessarily unique way) as:

$$
\text{(4)} \qquad
\begin{array}{ccc}
& & Y + F(F^\infty(Y+X)) \\
& {\scriptstyle g} \nearrow & \Big\downarrow{\scriptstyle \kappa_1 + \mathrm{id}} \\
& & (Y+X) + F(F^\infty(Y+X)) \\
& & \cong \Big\downarrow{\scriptstyle \zeta^{-1}_{Y+X}} \\
X & \xrightarrow{\quad e \quad} & F^\infty(Y+X)
\end{array}
$$

This says that if we decompose the terms of e using the final coalgebra map, then we do not get variables from X.

Theorem 4.2 ([2]) *Each guarded ∞-equation has a unique solution.*

Proof Assume that a guarded ∞-equation $e\colon X \to F^\infty(Y+X)$ factors as $\zeta^{-1}_{Y+X} \circ (\kappa_1 + \mathrm{id}) \circ g$, for a map $g\colon X \to Y + F(F^\infty(Y+X))$ like in (4). In order to find a solution one first defines, like in the proof of Lemma 3.4, an auxiliary map $h\colon F^\infty(Y+X) + F^\infty(Y) \to F^\infty(Y)$ by coinduction, via an

appropriate structure map on the left-hand-side below.

$$
\begin{array}{c}
Y + F(F^\infty(Y+X) + F^\infty(Y)) \xdashrightarrow{\mathrm{id}_Y + F(h)} Y + F(F^\infty(Y)) \\[2pt]
\big[\mathrm{id} + F(\kappa_1), (\mathrm{id} + F(\kappa_2)) \circ \zeta_Y\big] \uparrow \\[2pt]
(Y + F(F^\infty(Y+X))) + F^\infty(Y) \\[2pt]
[[\kappa_1, g], \kappa_2] + \mathrm{id} \uparrow \qquad\qquad\qquad\qquad \cong \Big\downarrow \zeta_Y \\[2pt]
((Y+X) + F(F^\infty(Y+X))) + F^\infty(Y) \\[2pt]
\zeta_{Y+X} + \mathrm{id} \uparrow \\[2pt]
F^\infty(Y+X) + F^\infty(Y) \xdashrightarrow{\quad h \quad} F^\infty(Y)
\end{array}
$$

The proof then proceeds by showing that $h \circ \kappa_2$ is the identity, and that $h \circ \kappa_1$ is of the form $\mathsf{subst}(k)$ for $k \colon Y + X \to F^\infty(Y)$. The unique solution is then obtained as $\mathsf{sol}(e) = k \circ \kappa_2$. □

5 ∞-equations and solutions as λ-equations and solutions

In this section we put previous results together. We start by fixing an object $Y \in \mathbb{C}$, and defining the associated functors $G^Y, T^Y \colon \mathbb{C} \to \mathbb{C}$ given by

$$
G^Y(X) = Y + F(X) \qquad T^Y(X) = F^\infty(Y+X)
$$

Why do we choose these functors? Well, a guard $X \to Y + F(F^\infty(Y+X))$ like in (4) is now simply a $G^Y T^Y$-coalgebra. We like to understand it as a λ-equation, in order to fit the ∞-equations in the framework of λ-equations. The first requirement is thus to establish the appropriate monad and distribution structure.

It is not hard to see that T^Y is again a monad with unit and multiplication:

$$
\begin{aligned}
\eta_X^Y &= \eta_{Y+X}^\infty \circ \kappa_2 &&: X \longrightarrow Y + X \longrightarrow F^\infty(Y+X) \\
\mu_X^Y &= \mathsf{subst}([\eta_{Y+X}^\infty \circ \kappa_1, \mathrm{id}]) : F^\infty(Y + F^\infty(Y+X)) \longrightarrow F^\infty(Y+X).
\end{aligned}
$$

For convenience we shall drop the superscript Y whenever confusion is unlikely.

Next we note that T^Y is isomorphic to $(G^Y)^\infty$, since each $(G^Y)^\infty(X)$ forms by construction the final coalgebra for the mapping

$$
X \longmapsto X + G^Y(-) \;=\; X + (Y + F(-)) \;\cong\; (Y+X) + F(-).
$$

so that $(G^Y)^\infty(X) \cong F^\infty(Y + X) = T^Y(X)$. Proposition 3.7 then yields the required distributive law. The next lemma describes it concretely.

Lemma 5.1 *In the above situation Proposition 3.7 yields a distributive law*

$$T^Y G^Y \xrightarrow{\;\;\lambda^Y\;\;} G^Y T^Y$$

for each $Y \in \mathbb{C}$. Ommiting the superscript Y, its components are maps of the form:

$$F^\infty(Y + (Y + F(X))) \xrightarrow{\;\lambda_X\;} Y + F(F^\infty(Y + X))$$

Morever, via the two obvious natural transformations $\kappa_2 \colon F \Rightarrow G^Y$ and $F^\infty(\kappa_2) \colon F^\infty \Rightarrow T^Y$ we get a commuting diagram of distributive laws:

$$
\begin{array}{ccc}
F^\infty F & \longrightarrow & T^Y G^Y \\
{\scriptstyle \lambda^\infty} \downarrow & & \downarrow {\scriptstyle \lambda} \\
F F^\infty & \longrightarrow & G^Y T^Y
\end{array}
$$

Proof The distributive law can be described as composite:

$$T^Y G^Y \cong (G^Y)^\infty G^Y \xrightarrow{\text{Proposition 3.7}} G^Y (G^Y)^\infty \cong G^Y T^Y$$

We shall construct this λ_X explicitly. By first applying the final coalgebra map we get:

$$F^\infty(Y + (Y + FX)) \xrightarrow[\cong]{\;\zeta\;} (Y + (Y + FX)) + F F^\infty(Y + (Y + FX))$$

The component on the left of the main $+$ on the right-hand-side readily gives a map to the required target, namely:

$$Y + (Y + FX) \xrightarrow{\;[\kappa_1, \mathrm{id} + F(\eta_X)]\;} Y + F(F^\infty(Y + X))$$

For the component on the right we have to do more work. We are done if we can find a map $F^\infty(Y + (Y + FX)) \to F^\infty(Y + X)$. Such a map can be obtained via substitution from:

$$Y + (Y + FX) \xrightarrow{\;[\eta^\infty_{Y+X} \circ \kappa_1, [\eta^\infty_{Y+X} \circ \kappa_1, \sigma^\infty_{Y+X} \circ F(\kappa_2)]]\;} F^\infty(Y + X)$$

Putting everything together we have the following complicated expression.

$$
\begin{aligned}
\lambda_X = \big[\,[\kappa_1, \mathrm{id} + F(\eta_X)], \\
\kappa_2 \circ F(\mathsf{subst}([\eta^\infty_{Y+X} \circ \kappa_1, [\eta^\infty_{Y+X} \circ \kappa_1, \sigma^\infty_{Y+X} \circ F(\kappa_2)]]))\,\big] \; \circ \zeta_{Y+(Y+FX)}.
\end{aligned}
$$

It is not hard to check that the distributive laws are preserved, as claimed at the end of the lemma.   □

Lemma 5.2 *For each $Y \in \mathbb{C}$, the object $F^\infty(Y)$ carries a final λ^Y-bialgebra structure:*

$$
\begin{array}{ccccc}
T^Y(F^\infty(Y)) & \xrightarrow{\;\xi_Y\;} & F^\infty(Y) & \xrightarrow[\;\cong\;]{\;\zeta_Y\;} & G^Y(F^\infty(Y)) \\
\| & & & & \| \\
F^\infty(Y + F^\infty(Y)) & & & & Y + F(F^\infty(Y))
\end{array}
$$

where $\xi^Y = \mathsf{subst}([\eta^\infty_Y, id])$.

Proof By Lemma 2.3 there is on $F^\infty(Y)$ an Eilenberg-Moore algebra structure $\xi^Y \colon T^Y(F^\infty(Y)) \to F^\infty(Y)$ forming a final λ^Y-bialgebra. We establish that it is of the form $\xi^Y = \mathsf{subst}([\eta^\infty_Y, id])$ by checking that it satisfies the

defining equation in Lemma 2.3. We shall drop superscripts as usual.

$$
\begin{aligned}
&G(\xi_Y) \circ \lambda_{F\infty Y} \circ T(\zeta_Y) \\
&= G(\xi_Y) \circ [\_,\_] \circ \zeta_{Y+(Y+FF\infty Y)} \circ F^\infty(\mathrm{id} + \zeta_Y) \\
&= G(\xi_Y) \circ [\_,\_] \circ ((\mathrm{id} + \zeta_Y) + FF^\infty(\mathrm{id} + \zeta_Y)) \circ \zeta_{Y+F\infty Y} \\
&= (\mathrm{id} + F(\xi_Y)) \circ [\,[\kappa_1, \mathrm{id} + F(\eta_{F\infty Y})] \circ (\mathrm{id} + \zeta_Y), \\
&\qquad\qquad \kappa_2 \circ F(\mathsf{subst}(\_)) \circ FF^\infty(\mathrm{id} + \zeta_Y)\,] \circ \zeta_{Y+F\infty Y} \\
&= [\,[\kappa_1, (\mathrm{id} + F(\xi_Y \circ \eta_{F\infty Y})) \circ \zeta_Y], \\
&\qquad\qquad \kappa_2 \circ F(\xi_Y \circ \mathsf{subst}(\_) \circ F^\infty(\mathrm{id} + \zeta_Y))\,] \circ \zeta_{Y+F\infty Y} \\
&= [\,[\kappa_1, (\mathrm{id} + F(\xi_Y \circ \eta^\infty_{Y+F\infty Y} \circ \kappa_2)) \circ \zeta_Y], \\
&\qquad\qquad \kappa_2 \circ F(\mathsf{subst}(\xi_Y \circ \_ \circ (\mathrm{id} + \zeta_Y)))\,] \circ \zeta_{Y+F\infty Y} \\
&\overset{(*)}{=} [\,[\kappa_1, (\mathrm{id} + F(\mathrm{id}) \circ \zeta_Y], \\
&\qquad\qquad \kappa_2 \circ F(\mathsf{subst}([\eta^\infty_Y, [\eta^\infty_Y, \tau^\infty_Y]] \circ (\mathrm{id} + \zeta_Y)))\,] \circ \zeta_{Y+F\infty Y} \\
&= [\,[\kappa_1, \zeta_Y], \\
&\qquad\qquad \kappa_2 \circ F(\mathsf{subst}([\eta^\infty_Y, \mathrm{id}]))\,] \circ \zeta_{Y+F\infty Y} \\
&= [\,\zeta_Y \circ [\eta^\infty_Y, \mathrm{id}], \\
&\qquad\qquad \zeta_Y \circ \tau^\infty_Y \circ F(\xi_Y)\,] \circ \zeta_{Y+F\infty Y} \\
&= \zeta_Y \circ [\,[\eta^\infty_Y, \mathrm{id}], \tau^\infty_Y \circ F(\xi_Y)\,] \circ \zeta_{Y+F\infty Y} \\
&= \zeta_Y \circ \xi_Y, \qquad \text{by Lemma 3.5 (v).}
\end{aligned}
$$

The marked step $(*)$ in this calculation is explained as follows.

$$
\begin{aligned}
\xi_Y \circ \sigma^\infty_{Y+F\infty Y} \circ F(\kappa_2) &= \mathsf{subst}([\eta^\infty_Y, \mathrm{id}]) \circ \tau^\infty_{Y+F\infty Y} \circ F(\eta^\infty_{Y+F\infty Y}) \circ F(\kappa_2) \\
&= \tau^\infty_Y \circ F(\mathsf{subst}([\eta^\infty_Y, \mathrm{id}])) \circ F(\eta^\infty_{Y+F\infty Y}) \circ F(\kappa_2) \\
&= \tau^\infty_Y \circ F([\eta^\infty_Y, \mathrm{id}]) \circ F(\kappa_2) \\
&= \tau^\infty_Y.
\end{aligned}
$$

$\square$

We are finally in a position to see that ∞-equations and solutions are a special case of λ-equations and solutions. This is our main result.

Theorem 5.3 *Let $F\colon \mathbb{C} \to \mathbb{C}$ be a functor with final coalgebra $F^\infty(X) \overset{\cong}{\longrightarrow} X + F(F^\infty(X))$. Then:*

(i) *A guard $g\colon X \to Y + F(F^\infty(Y + X))$ for an ∞-equation $e\colon X \to F^\infty(Y + X)$ is a λ^Y-equation, for the distributive law λ^Y from Lemma 5.1.*

(ii) *A solution $\mathsf{sol}(e)\colon X \to F^\infty(Y)$ of a guarded ∞-equation e is the same thing as a solution of its guard g—as a λ^Y-equation—in the final λ^Y-bialgebra of Lemma 5.2.*

Proof The first point is obvious, so we concentrate on the second one. We assume that we can write the guarded ∞-equation $e\colon X \to F^\infty(Y + X)$ as $e = \zeta_{Y+X}^{-1} \circ (\kappa_1 + \mathrm{id}) \circ g$, like in (4), where $g\colon X \to Y + F(F^\infty(Y + X))$ is the guard (or λ-equation). We observe for a map $f\colon X \to F^\infty(Y)$,

$$f \text{ is a solution of the } \lambda\text{-equation } g \text{ (see Definition 2.4)}$$
$$\Longleftrightarrow \zeta_Y \circ f = G(\xi_Y) \circ GT(f) \circ g$$
$$\Longleftrightarrow f = \zeta_Y^{-1} \circ G(\xi_Y) \circ GT(f) \circ g$$
$$= [\eta_Y^\infty, \tau_Y^\infty] \circ (\mathrm{id} + F(\xi_Y)) \circ (\mathrm{id} + FF^\infty(\mathrm{id} + f)) \circ g$$
$$= [\eta_Y^\infty, \tau_Y^\infty \circ F(\xi_Y) \circ FF^\infty(\mathrm{id} + f)] \circ g$$
$$= [\eta_Y^\infty, \tau_Y^\infty \circ F(\mathsf{subst}([\eta_Y^\infty, \mathrm{id}]) \circ F^\infty(\mathrm{id} + f))] \circ g$$
$$= [\eta_Y^\infty, \tau_Y^\infty \circ F(\mathsf{subst}([\eta_Y^\infty, \mathrm{id}] \circ (\mathrm{id} + f)))] \circ g$$
$$= [\eta_Y^\infty, \mathsf{subst}([\eta_Y^\infty, f]) \circ \tau_{Y+X}^\infty] \circ g$$
$$= \mathsf{subst}([\eta_Y^\infty, f]) \circ [\eta_{Y+X}^\infty, \tau_{Y+X}^\infty] \circ g$$
$$= \mathsf{subst}([\eta_Y^\infty, f]) \circ \zeta_{Y+X}^{-1} \circ (\kappa_1 + \mathrm{id}) \circ g$$
$$= \mathsf{subst}([\eta_Y^\infty, f]) \circ e$$
$$\Longleftrightarrow f \text{ is a solution of the } \infty\text{-equation } e \text{ (see Definition 4.1).}$$

$\square$

6   Conclusion

We have unified the area of coinductive solutions of equations by showing that one notion developed in [2] (following [12]) is an instance of a more general notion from [4] based on distributive laws.

Acknowledgments

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