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On Computable Compact Operators on Banach Spaces

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Abstract

We develop some parts of the theory of compact operators from the point of view of computable analysis. While computable compact operators on Hilbert spaces are easy to understand, it turns out that these operators on Banach spaces are harder to handle. Classically, the theory of compact operators on Banach spaces is developed with the help of the non-constructive tool of sequential compactness. We demonstrate that a substantial amount of this theory can be developed computably on Banach spaces with computable Schauder bases that are well-behaved. The conditions imposed on the bases are such that they generalize the Hilbert space case. In particular, we prove that the space of compact operators on Banach spaces with monotone, computably shrinking and computable bases is a computable Banach space itself and operations such as composition with bounded linear operators from left are computable. Moreover, we provide a computable version of the Theorem of Schauder on adjoints in this framework and we discuss a non-uniform result on composition with bounded linear operators from right.

Keywords: Computable functional analysis, Banach spaces, compact operators.

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1 Introduction

In functional analysis an operator $T : X \to Y$ on Banach spaces X, Y is called *compact*, if it is a linear operator such that the closure of the image TB_X of the closed unit ball $B_X := \{x \in X : ||x|| \leq 1\}$ is compact in Y. It is easy to see that any compact operator is necessarily bounded. Compact operators are particularly important, as for infinite-dimensional X the unit ball B_X is not compact and thus compactness of the operator can often compensate this "defect".

Classically, one can prove a number of basic properties of compact operators on Banach spaces, some of which are summarized here:

(1) T, S compact $\Longrightarrow aT, T + S$ compact,

(2) T compact, S linear bounded $\Longrightarrow ST, TS$ compact,

(3) T compact $\iff T'$ compact.

The first property basically says that compact operators are closed under linear operations, the second one says that compact operators are closed under composition with linear bounded operators and the third property, which is called the Theorem of Schauder, ensures that the adjoint $T' : Y' \to X', f \mapsto fT$ of any compact operator is compact again.

We would like to prove computable versions of these results in the framework of computable analysis. For the special case of Hilbert spaces this has partially been done in [4]. However, in the general case of Banach spaces the classical theory is developed using the concept of sequential convergence that is highly non-constructive. Therefore, we need a substitute for this concept in case of computable analysis. It seems that there is no obvious way how to establish all the above mentioned properties for compact operators in the general case of computable Banach spaces. In particular, closure of compact operators under addition cannot easily be proved in general. However, we show that in case of Banach spaces with well-behaved computable Schauder bases, one can establish a reasonable theory of compact operators computably that includes, in particular, computable versions of most of the above mentioned properties.

It was a long standing open question in functional analysis, called *Banach's basis problem*, whether every infinite-dimensional separable Banach space has a Schauder basis and for all natural examples of spaces this seems to be the case. In particular, the space $\mathcal{C}[0, 1]$ has a Schauder basis and it is known that any separable Banach space is a subspace of $\mathcal{C}[0, 1]$. It was only proved in 1973 by Per Enflo that there exists an infinite-dimensional separable Banach space without a basis (in fact, the space is even reflexive and lacks the approximation property, see [10]). Regarding the difficulties to construct examples of such spaces it seems not to be a too strict requirement to restrict the theory of computable compact operators to Banach spaces with bases.

We close this section with a brief survey on the organization of this paper. In the following section we recall some basic definitions regarding computable Banach spaces and compact subsets. In Section 3 we introduce Banach spaces with computable bases and we prove some elementary properties of these spaces. Moreover we discuss the special case of monotone bases and we show that in Banach spaces with computable monotone bases the natural projections are computably compact and the coordinate functionals have computable norms. In Section 4 we discuss properties of the dual space and we define the concept of a computable dual basis. Moreover, we prove that any computable, computably shrinking and monotone basis is necessarily a computable dual basis. In Section 5 we introduce a representation of compact operators on computable Banach spaces that combines the information on the operator as continuous function and on the image of the unit ball as compact set. We prove some results regarding composition with bounded linear operators from left and the Theorem of Schauder. In Section 6 we study the space of compact operators as a Banach space and we provide sufficient conditions such that this space is a computable Banach space and such that the resulting Cauchy representation is equivalent to the representation introduced in Section 5. Finally, in Section 7 we discuss composition with bounded linear operators from right and we provide sufficient conditions for a non-uniform result. In the Conclusions we summarize all our results under assumptions that are sufficient for all results simultaneously.

2 Computable Banach Spaces and Compact Sets

In this section we briefly define some concepts from computable analysis and we refer the reader to [11] for some more detailed introduction. In the following we assume that Banach spaces are defined over the field $\mathbb{F}$, which might either be $\mathbb{R}$ or $\mathbb{C}$.

Definition 2.1 [Computable Banach space] A *computable Banach space* $(X, ||\ ||, e)$ is a separable Banach space $(X, ||\ ||)$ together with a fundamental sequence $e : \mathbb{N} \to X$ (i.e. the linear span of range(e) is dense in X) such that the induced metric space is a computable metric space that makes the linear operations (addition and multiplication with scalars) computable.

The induced computable metric space is the space (X, d, α_e) where d is given by $d(x, y) := ||x - y||$ and $\alpha_e : \mathbb{N} \to X$ is defined by $\alpha_e\langle k, \langle n_0, ..., n_k\rangle\rangle := \sum_{i=0}^{k} \alpha_{\mathbb{F}}(n_i)e_i$. Here $\alpha_{\mathbb{F}}$ is a standard numbering of $\mathbb{Q}_{\mathbb{F}}$ where $\mathbb{Q}_{\mathbb{F}} = \mathbb{Q}$ in case of $\mathbb{F} = \mathbb{R}$ and $\mathbb{Q}_\mathbb{F} = \mathbb{Q}[i]$ in case of $\mathbb{F} = \mathbb{C}$. We assume that there is some $n \in \mathbb{N}$ with $\alpha_\mathbb{F}(n) = 0$.

In general, a space (X, d, α) is called *a computable metric space*, if (X, d) is a metric space with a dense sequence α such that $d \circ (\alpha \times \alpha)$ is a computable double sequence. If not mentioned otherwise, then we assume that all computable Banach spaces X are represented by their Cauchy representation δ_X (of the induced metric space). The *Cauchy representation* $\delta_X :\subseteq \Sigma^\omega \to X$ of a computable metric space X is defined such that a sequence $p \in \Sigma^\omega$ represents a point $x \in X$, if it encodes a sequence $(\alpha(n_i))_{i\in\mathbb{N}}$, which rapidly converges to x, where rapid means that $d(\alpha(n_i), \alpha(n_j)) < 2^{-j}$ for all $i > j$. Here Σ^ω denotes the set of infinite sequences over some finite set Σ (the *alphabet*) and Σ^ω is endowed with the product topology with respect to the discrete topology on Σ.

In general a *representation* of a set X is a surjective map $\delta :\subseteq \Sigma^\omega \to X$. Here the inclusion symbol "$\subseteq$" indicates that the corresponding map might be partial. Given representations $\delta :\subseteq \Sigma^\omega \to X$ and $\delta' :\subseteq \Sigma^\omega \to Y$, a map $f :\subseteq X \to Y$ is called (δ, δ')*–computable*, if there exists a computable map $F :\subseteq \Sigma^\omega \to \Sigma^\omega$ such that $\delta' F(p) = f\delta(p)$ for all $p \in \mathrm{dom}(f\delta)$. Analogously, one can define computability for multi-valued functions $f :\subseteq X \rightrightarrows Y$. In this case the equation above has to be replaced by the condition $\delta' F(p) \in f\delta(p)$. Here a function $F :\subseteq \Sigma^\omega \to \Sigma^\omega$ is called *computable* if there exists a Turing machine which computes F. Similarly, one can define the concept of *continuity* with respect to representations, where the computable function F is replaced by a continuous function.

Cauchy representations of computable metric spaces X are known to be *admissible* and for such representations continuity with respect to representations coincides with ordinary continuity. If X, Y are computable metric spaces, then we assume that the space $\mathcal{C}(X, Y)$ of continuous functions $f : X \to Y$ is represented by $[\delta_X \to \delta_Y]$, which is a canonical function space representation. This representation satisfies two characteristic properties, evaluation and type conversion, which can be performed computably (see [11] for details). If $Y = \mathbb{F}$, then we write for short $\mathcal{C}(X) = \mathcal{C}(X, \mathbb{F})$.

We say that a representation δ is *computably reducible* to another representation δ' of the same set, in symbols $\delta \le \delta'$, if there is a computable function $F :\subseteq \Sigma^\omega \to \Sigma^\omega$ such that $\delta(p) = \delta' F(p)$ for all $p \in \mathrm{dom}(\delta)$. This is equivalent to the fact that the identity $\mathrm{id} : X \to X$ is (δ, δ')–computable. Two representations are said to be *computably equivalent*, if they are mutually computably reducible to each other, in symbols $\delta \equiv \delta'$.

Since we want to handle compact operators, we also have to represent compact subsets. For any metric space X we denote by $\mathcal{K}(X)$ the set of nonempty compact subsets of X. A straightforward way to obtain a representation of $\mathcal{K}(X)$ is to use the Cauchy representation $\delta_{\mathcal{K}(X)}$ which is induced by the computable metric space that is given by the *Hausdorff metric*

$$d_{\mathrm{H}}:\mathcal{K}(X)\times\mathcal{K}(X)\to\mathbb{R},(A,B)\mapsto\max\left\{\sup_{a\in A}\mathrm{dist}_B(a),\sup_{b\in B}\mathrm{dist}_A(b)\right\}$$

If (X,d,α) is a computable metric space, then we can use as a dense subset of $\mathcal{K}(X)$ the set of finite subsets of range(α). We obtain the following result.

Proposition 2.2 *Let (X,d,α) be a computable metric space and let α_F be a standard numbering of the finite subsets of* range(α). *Then $(\mathcal{K}(X),d_{\mathrm{H}},\alpha_F)$ is a computable metric space.*

A proof of this result, several characterizations and other results on representations of compact subsets of metric spaces can be found in Corollary 4.11 and Theorems 4.12 and 4.13 in [5]. In the following we assume that $\mathcal{K}(X)$ is endowed with the Cauchy representation of $(\mathcal{K}(X),d_{\mathrm{H}},\alpha_F)$. A compact subset $K\subseteq X$ of a computable metric space X is called *recursive*, if it is a computable point in $\mathcal{K}(X)$. There is the weaker concept of a co-r.e. compact subset $K\subseteq X$, which is a subset such that one can computably enumerate all finite covers consisting of rational open balls and there are corresponding weaker representations of compact subsets (see [5] for details). It is known that a set $K\subseteq X$ is recursive compact if and only if it is co-r.e. compact and there exists a computable sequence that is dense in K.

In [8] it has been proved that a subset $K\subseteq X$ of a computable metric space (X,d,α) is recursive compact if and only if it is closed, located and computably totally bounded. Here a set $A\subseteq X$ is called *located*, if its distance function $d_A:X\to\mathbb{R},x\mapsto\inf_{y\in A}d(x,y)$ is computable and a set $A\subseteq X$ is called *computably totally bounded*, if there exists a computable modulus of total boundedness $t:\mathbb{N}\to\mathbb{N}$ that does the following: given a precision k it determines a number $t(k)=\langle m,\langle c_0,...,c_m\rangle\rangle$ such that $A\subseteq\bigcup_{i=0}^m B(\alpha(c_i),2^{-k})$. This result even holds uniformly, i.e. given a name for the distance function and a name for the modulus of total boundedness for a compact set K, one can compute a Hausdorff name of K and vice versa. Here by $B(y,r):=\{x\in X:d(x,y)<r\}$ we denote the *open ball* with center y and radius r and by $\overline{B}(y,r):=\{x\in X:d(x,y)\leq r\}$ the *closed balls*. For normed spaces X we use the short notation $B_X:=\overline{B}(0,1)$ for the closed unit ball, as mentioned earlier.

One essential property of compact subsets is that the continuous image of compact subsets is compact again and this result holds effectively and with respect to the stronger as well as the weaker representation of compact subsets (see Theorem 3.3 in [12]).

3 Banach Spaces With Monotone Bases

Let X be a Banach space. A sequence $(e_i)_{i\in\mathbb{N}}$ in X is called a *Schauder basis* of X (or a *basis* for short), if any point x can be uniquely represented as $x = \sum_{i=0}^{\infty} x_i e_i$ with $x_i \in \mathbb{F}$. If X is a computable Banach space, then we will say that $(e_i)_{i\in\mathbb{N}}$ is a *computable basis*, if it is a Schauder basis that is computable in X. It is clear that for any computable Banach space $(X, ||\ ||, e)$ with computable basis f, it follows that f is a fundamental sequence in X and the space $(X, ||\ ||, f)$ is a computable Banach space that is computably equivalent to the space $(X, ||\ ||, e)$. Here two computable Banach spaces with the same underlying set are called *computably equivalent*, if the corresponding Cauchy representations are computably equivalent (i.e. if the identity is computable in both directions). Any Banach space with a basis is necessarily infinite-dimensional and separable. It is easy to see that typical Banach spaces of this type, such as ℓ_p for $1 \leq p < \infty$, c_0, $\mathcal{C}[0,1]$ and others have bases and natural bases are typically computable.

For any Banach space with a basis $(e_i)_{i\in\mathbb{N}}$ there are associated linear operators, called *natural projections*:

$$P_n : X \to X,\ \sum_{i=0}^{\infty} x_i e_i \mapsto \sum_{i=0}^{n} x_i e_i.$$

The number $c := \sup_{n\in\mathbb{N}} ||P_n||$ is always finite and called the *basis constant* of the basis $(e_i)_{i\in\mathbb{N}}$. The basis $(e_i)_{i\in\mathbb{N}}$ is called *monotone*, if its basis constant is $c = 1$. For any Banach space with basis $(e_i)_{i\in\mathbb{N}}$ one can define the *coordinate functionals*

$$e'_n : X \to \mathbb{F},\ \sum_{i=0}^{\infty} x_i e_i \mapsto x_n.$$

It is well-known that $||e'_i|| \cdot ||e_i|| \leq 2c$ for all i.

The following result shows that the natural projections and coordinate functionals are always computable.

Proposition 3.1 *Let X be a computable Banach space with computable basis e. Then the corresponding sequence of natural projections $(P_n)_{n\in\mathbb{N}}$ is a computable sequence in $\mathcal{C}(X,X)$ and the corresponding sequence $(e'_n)_{n\in\mathbb{N}}$ of coordinate functionals is a computable sequence in $\mathcal{C}(X)$.*

Now we consider Banach spaces with monotone bases and we prove that in this case the natural projections are not only computable operators, but computably compact operators in some sense, defined later. Moreover, we will be able to conclude that the norms of the coordinate functionals form a computable sequence. In order to prove these results, we need some further preparation. The first observation is that the unit ball B_X of every computable Banach space X is always located, i.e. its distance function $d_{B_X} : X \to \mathbb{R}$ is computable. The next observation is that compactness can be transferred effectively from subspaces to the whole space. We say that a subspace Y of a computable Banach space X is a *computable subspace*, if it is a computable Banach space with respect to some fundamental sequence that is computable in X. This is equivalent to the condition that Y is a computable Banach space and the inclusion map $\mathrm{in}_Y : Y \hookrightarrow X$ is computable.

Lemma 3.2 *Let X be a computable Banach space with a computable subspace $Y \subseteq X$. Then the inclusion map* $\mathrm{in} : \mathcal{K}(Y) \hookrightarrow \mathcal{K}(X), K \mapsto K$ *is computable.*

In the next step we mention that the unit balls B_n of the n–dimensional subspaces X_n of a computable Banach space X form a computable sequence in $\mathcal{K}(X)$. We leave the proof to the reader.

Lemma 3.3 *Let X be a computable Banach space with a computable basis e. Then the sequence $(B_n)_{n\in\mathbb{N}}$ of sets*

$$B_n := \left\{ x = \sum_{i=0}^{n} x_i e_i \in X : ||x|| \leq 1 \right\}$$

is computable in $\mathcal{K}(X)$.

In the next step we observe that for monotone bases we obtain $P_n B_X = B_n$.

Lemma 3.4 *Let X be a Banach space with a monotone basis e. Then*

$$P_n B_X = B_n \subseteq B_X.$$

Proof. We claim that $P_n B_X = B_n$. By definition of B_n it is clear that "$\supseteq$" holds. For the other inclusion, let $x = \sum_{i=0}^{\infty} x_i e_i \in B_X$. Then $P_n x = \sum_{i=0}^{n} x_i e_i$ and since e is monotone, one obtains $||P_n x|| \leq ||P_n|| \cdot ||x|| \leq 1$, i.e. $P_n x \in B_n$. It is also clear that $B_n \subseteq B_X$ holds. $\square$

Now we can conclude that the natural projections are computably compact in a sense defined later. Here we express the result directly.

Corollary 3.5 *Let X be a computable Banach space with a computable and monotone basis e. Then $(P_n B_X)_{n\in\mathbb{N}}$ is a computable sequence in $\mathcal{K}(X)$.*

¿From this corollary we are able to conclude that the coordinate functionals form a computable sequence in X'. We will prove a more general result in Proposition 6.4.

Proposition 3.6 *Let X be a computable Banach space with a computable and monotone basis e. Then $(||e'_n||)_{n\in\mathbb{N}}$ is a computable sequence.*

4 Dual Spaces

In this section we discuss some properties of the dual spaces X' of computable Banach spaces X. The following representation is a natural representation of X'.

Definition 4.1 [Dual space representation] Let X be a separable Banach space. We define a representation $\delta_{X'}$ of the dual space X' by

$$\delta_{X'}\langle p, q\rangle = f :\iff [\delta_X \to \delta_{\mathbb{F}}](p) = f \text{ and } \delta_{\mathbb{R}}(q) = ||f||.$$

That is, a name of a functional f contains information on both, the functional f as a continuous function and its operator norm $||f||$. If not mentioned otherwise, we will assume that X' is represented by $\delta_{X'}$. As a consequence of Proposition 3.1 and Proposition 3.6 we can conclude that the coordinate functionals form a computable sequence of functionals, provided the corresponding basis is monotone.

Corollary 4.2 *Let X be a computable Banach space with a computable and monotone basis e. Then $(e'_n)_{n\in\mathbb{N}}$ is a computable sequence in X' with respect to $\delta_{X'}$.*

Unfortunately, the dual space X' of a computable Banach space is not necessarily a computable Banach space again. In [2] we have seen, for instance, that the dual space ℓ_∞ of ℓ_1 cannot be considered as a computable Banach space in the ordinary sense (one can either get a computable addition or a computable norm, but not both). However, the dual space X' of any computable Banach space is a *general computable Banach space* in some sense (see Corollary 21.30 in [2]). For our purposes here it is convenient to have the notion of a computable dual space as captured in the next definition.

Definition 4.3 [Computable dual space] Let X be a computable Banach space. We say that X has a *computable dual space* X', if there exists a sequence $e^* : \mathbb{N} \to X'$ such that

(1) $(X', ||\ ||, e^*)$ is a computable Banach space,

(2) $\delta_{X'}$ is computably equivalent to the Cauchy representation of $(X', ||\ ||, e^*)$.

In case of a Banach space with a basis e we have a canonical choice for e^*. Given a Banach space with basis e, the coordinate functionals e' can form a basis of X'. Actually, this is the case if and only if e is shrinking (see Proposition 4.4.7 in [10]). Here a basis e is called *shrinking*, if $\lim_{n\to\infty} ||f||_{(n)} = 0$ for all $f \in X'$, where

$$||f||_{(n)} := \sup\left\{\frac{|fx|}{||x||} : x \in \overline{\mathrm{span}\{e_i : i > n\}}, x \neq 0\right\}.$$

In case that a basis e is shrinking, one obtains a Cauchy representation of X'.

Definition 4.4 [Cauchy representation of the dual space] Let X be a Banach space with a shrinking basis e. Then the coordinate functionals e' form a basis of X' and by $\delta^{\mathrm{C}}_{X'}$ we denote the corresponding Cauchy representation of $(X', ||\ ||, e')$.

For a reflexive Banach space X any basis e is shrinking (see Theorem 4.4.15 in [10]). Now the interesting question is whether and under which conditions the representations $\delta_{X'}$ and the Cauchy representation $\delta^{\mathrm{C}}_{X'}$ of X' are computably equivalent. We introduce a special type of bases for which this is the case.

Definition 4.5 [Computable dual basis] Let X be a computable Banach space. A basis e is called a *computable dual basis*, if

(1) e is a computable and shrinking basis of X,

(2) $\delta_{X'} \equiv \delta^{\mathrm{C}}_{X'}$.

Note that the basis e has to be shrinking in order to guarantee that e' is a basis of X' and that $\delta^{\mathrm{C}}_{X'}$ exists. A computable Banach space X with a computable dual basis, in particular, has a computable dual space X'. It would be desirable to have conditions on e and X which guarantee that conditions (2) follows from (1). In order to prove a result in this direction we use the following notion.

Definition 4.6 [Shrinking modulus] Let X be a Banach space with basis e and let $f \in X'$. Then $m : \mathbb{N} \to \mathbb{N}$ is called a *shrinking modulus* of f, if $||f||_{(m(k))} < 2^{-k}$ for all $k \in \mathbb{N}$.

Using this notion we can define computably shrinking bases.

Definition 4.7 [Computably shrinking] Let X be a computable Banach space with a computable basis e. Then e is called *computably shrinking*, if there exists a $(\delta_{X'}, \delta_{\mathbb{N}^\mathbb{N}})$–computable map $S : X' \rightrightarrows \mathbb{N}^\mathbb{N}$ such that for each fixed $f \in X'$ any $m \in S(f)$ is a shrinking modulus for $f \in X'$.

Now we can formulate the following characterization. We leave the proof to the reader.

Proposition 4.8 *Let X be a computable Banach space with a computable and shrinking basis e. Then the following conditions are equivalent:*

(1) $\delta^{\mathrm{C}}_{X'} \leq \delta_{X'}$,

(2) $(X', ||\ ||, e')$ *is a computable Banach space,*

(3) $||\ || : X' \to \mathbb{R}$ *is* $(\delta^{\mathrm{C}}_{X'}, \delta_{\mathbb{R}})$*–computable.*

Moreover, the following conditions are equivalent to each other:

(4) $\delta_{X'} \leq \delta^{\mathrm{C}}_{X'}$,

(5) e *is computably shrinking.*

Now we can conclude the following characterization of computable dual bases. The first two conditions are equivalent by the previous proposition, the second condition obviously implies the third condition and the third implies the first by the Stability Theorem 21.23 and Corollary 21.30 in [2] and Proposition 3.1.

Corollary 4.9 *Let X be a computable Banach space with a computable and shrinking basis e. Then the following conditions are equivalent:*

(1) e *is a computable dual basis (i.e.* $\delta^{\mathrm{C}}_{X'} \equiv \delta_{X'}$*),*

(2) $(X', ||\ ||, e')$ *is a computable Banach space and* e *is computably shrinking,*

(3) $+ : X' \times X' \to X'$ *is* $(\delta_{X'}, \delta_{X'}, \delta_{X'})$*–computable and* $(||e'_i||)_{i\in\mathbb{N}}$ *is a computable sequence.*

With the next result we show that for monotone and computable bases the norm on the dual space is always computable. We will prove a more general result in Proposition 6.4.

Proposition 4.10 *Let X be a computable Banach space with a computable, shrinking and monotone basis e. Then* $||\ || : X' \to \mathbb{R}$ *is* $(\delta^{\mathrm{C}}_{X'}, \delta_{\mathbb{R}})$*–computable and* $(X', ||\ ||, e')$ *is a computable Banach space.*

This leads to the following characterization of computable dual bases in the monotone case (as a corollary of Corollary 4.9, Proposition 4.10 and Proposition 3.6).

Corollary 4.11 *Let X be a computable Banach space with a computable, shrinking and monotone basis e. Then the following conditions are equivalent:*

(1) e *is a computable dual basis (i.e.* $\delta^{\mathrm{C}}_{X'} \equiv \delta_{X'}$*),*

(2) e is computably shrinking,

(3) $+ : X' \times X' \to X'$ is $(\delta_{X'}, \delta_{X'}, \delta_{X'})$–computable.

It is important to mention that the third condition does not depend on the basis. Therefore, for a fixed computable Banach space X either any computable, shrinking and monotone basis is automatically computably shrinking or no such basis has this property. We obtain the following conclusion.

Corollary 4.12 *Let X be a computable Banach space. Any computable, computably shrinking and monotone basis e of X is a computable dual basis.*

The following result provides an important criterion that is naturally satisfied in many applications and guarantees that our requirements on bases are satisfied.

Theorem 4.13 (Dual basis theorem) *Let X be a computable Banach space with a computable and monotone basis e. If X is reflexive and has a computable dual space, then the basis e is computably shrinking (and, thus, a computable dual basis).*

Proof. Any basis of a reflexive space is automatically shrinking by Theorem 4.4.15 in [10]. By Corollary 4.11 it remains to prove that addition $+ : X' \times X' \to X'$ is computable on the dual space X' with respect to $\delta_{X'}$. But since X has a computable dual space, the dual space X' is a computable Banach space and its Cauchy representation is computably equivalent to $\delta_{X'}$. Since the Cauchy representation makes the addition computable, the desired result follows. □

The next observation is that for computable Hilbert spaces X (i.e. computable Banach spaces that are, additionally, Hilbert spaces) everything is satisfied automatically.

Proposition 4.14 *Let X be an infinite-dimensional computable Hilbert space. Then X has a computable orthonormal basis e and any such basis is a computable, monotone and computably shrinking Schauder basis.*

Proof. Let X be an infinite-dimensional computable Hilbert space. In Lemma 3.1 of [7] it has been proved that any computable Hilbert space X has a computable orthonormal basis e. Then any point x can be uniquely represented by its Fourier series as $x = \sum_{i=0}^{\infty} \langle x, e_i \rangle e_i$. In particular, e is a computable Schauder basis. Moreover, any Hilbert space is reflexive and any computable Hilbert space has a computable dual space (see Theorem 4.7 in [7], which is a consequence of the computable Fréchet-Riesz Theorem). By Theorem 4.13 the desired result follows. □

There is an example of a computable Hilbert space with a computable Schauder basis that is not a computable dual basis. This example in particular shows that we cannot simply omit monotonicity in Propositions 3.6 and 4.10, Corollary 4.11 and Theorem 4.13. Besides Hilbert spaces there are many natural examples of Banach spaces that admit a computable dual basis. We mention the example of the ℓ_p spaces.

Example 4.15 The sequence spaces ℓ_p with computable $p > 1$ and equipped with the norm $||(x_n)_{n\in\mathbb{N}}||_p := \sqrt[p]{\sum_{i=0}^{\infty} ||x_i||^p}$ and the basis e, defined by $e_i(j) := \delta_{ij}$, form a computable Banach space and the basis e is computable and monotone. It follows from Theorem 4.13 that this basis is also computably shrinking, as the ℓ_p–spaces are reflexive and they admit computable dual spaces by the computable version of the Theorem of Landau (see Theorem 21.33.2 in [2]).

Spaces like ℓ_1 or $\mathcal{C}[0,1]$ that do not have a separable dual space cannot have a computable dual basis. However, the two mentioned spaces still admit computable bases.

5 Computably Compact Operators

In this section we introduce a representation of compact operators on separable Banach spaces and we provide computable versions of some elementary properties of compact operators. In particular, we prove that compact operators are closed with respect to composition in some sense and we provide a computable version of the Theorem of Schauder.

Definition 5.1 [Image representation] Let X, Y be separable Banach spaces. We define a representation $\delta_{\mathcal{B}_\infty(X,Y)}$ of the set $\mathcal{B}_\infty(X,Y)$ of compact operators $T : X \to Y$ by

$$\delta_{\mathcal{B}_\infty(X,Y)}\langle p, q\rangle = T :\iff [\delta_X \to \delta_Y](p) = T \text{ and } \delta_{\mathcal{K}(Y)}(q) = \overline{TB_X}.$$

That is, a name $\langle p, q\rangle$ of a compact operator T contains two types of information, a name p of T as a continuous map and a name q of the closure of the image of the unit ball as a compact set. If not mentioned otherwise, we will assume that $\mathcal{B}_\infty(X,Y)$ is represented by $\delta_{\mathcal{B}_\infty(X,Y)}$. It follows from Proposition 3.1 and Corollary 3.5 that the projections are computably compact operators, provided the corresponding basis is monotone.

Corollary 5.2 *Let X be a computable Banach space with a computable and monotone basis e. Then $(P_n)_{n\in\mathbb{N}}$ is a computable sequence in $\mathcal{B}_\infty(X,X)$ with respect to $\delta_{\mathcal{B}_\infty(X,X)}$.*

Another operation that obviously becomes computable with respect to this representation is composition with bounded linear operators from the left.

Proposition 5.3 (Composition) *Let X, Y, Z be computable Banach spaces. Then composition*

$$\circ :\subseteq \mathcal{C}(Y,Z) \times \mathcal{B}_\infty(X,Y) \to \mathcal{B}_\infty(X,Z), (S,T) \mapsto ST,$$

restricted to linear operators S, is $([\delta_Y \to \delta_Z], \delta_{\mathcal{B}_\infty(X,Y)}, \delta_{\mathcal{B}_\infty(X,Z)})$–computable.

Proof. First of all, it is clear that composition of continuous operators is $([\delta_Y \to \delta_Z], [\delta_X \to \delta_Y], [\delta_X \to \delta_Z])$–computable (see Exercise 3.3.11 in [11]). Moreover, the continuous image of compact sets can be computed (see Theorem 3.3 in [12]). That is $\overline{(ST)B_X} = S(\overline{TB_X}) \in \mathcal{K}(Z)$ can be computed, given a $\overline{TB_X} \in \mathcal{K}(Y)$. □

The next observation is that the computably compact operators $T \in \mathcal{B}_\infty(X,\mathbb{F})$ are exactly the computable functionals $T \in X'$ and this holds uniformly.

Proposition 5.4 (Functionals as compact operators) *Let X be a computable Banach space. Then $X' = \mathcal{B}_\infty(X,\mathbb{F})$ and the representations $\delta_{X'}$ and $\delta_{\mathcal{B}_\infty(X,\mathbb{F})}$ are computably equivalent.*

Proof. For any functional $f : X \to \mathbb{F}$ we obtain $\overline{fB_X} = \overline{B}(0, ||f||)$. Since the map $R : \mathbb{R} \to \mathcal{K}(\mathbb{F}), r \mapsto \overline{B}(0,r)$ is computable and has a computable right inverse, the desired result follows. □

¿From this result one can easily conclude that there are classically compact operators (in fact, functionals) that are computable but not computably compact: in Example 4.6 of [7] a computable functional $f : \ell_2 \to \mathbb{R}$ without computable norm $||f||$ is constructed. For arbitrary compact operators we get a result that, in one direction, is similar to the previous one, namely the operator bound of any compact operator can be computed. We will see below that one cannot get any similar result in the other direction.

Proposition 5.5 (Operator norm of compact operators) *Let X, Y be computable Banach spaces. Then the map*

$$||\ || : \mathcal{B}_\infty(X,Y) \to \mathbb{F}, T \mapsto ||T||$$

is computable.

Proof. For any linear bounded operator $T : X \to Y$ we obtain

$$||T|| = \sup_{||x|| \leq 1} ||Tx|| = \sup ||\overline{TB_X}||.$$

Given a name of a compact operator $T \in \mathcal{B}_\infty(X,Y)$ we can compute a name for $\overline{TB_X} \in \mathcal{K}(Y)$. Since the continuous image of a compact set is computable (see Theorem 3.3 in [12]), we can also compute $||\overline{TB_X}|| \in \mathcal{K}(\mathbb{R})$. Finally, we note that the suprema over compact subsets of $\mathbb{R}$ is computable (see Lemma 5.2.6 in [11]) and thus the desired result follows. □

The previous results lead to the question whether any classically compact operator that is computable and has a computable norm is already computably compact (as it is the case for functionals). The following example shows that this is not the case, not even for Hilbert spaces.

Example 5.6 Let $a = (a_k)_{k\in\mathbb{N}}$ be a computable sequence of reals such that $||a||_2 < 1$, but $||a||_2$ is non-computable. We use ℓ_2 over $\mathbb{F} = \mathbb{R}$ and we define a linear bounded operator $T : \ell_2 \to \ell_2$ using the matrix representation

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & a_0 & a_1 & a_2 & a_3 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then T is compact since its range is two-dimensional, T is obviously computable as the sequence a is computable, the norm $||T|| = 1$ is computable, but T is not computably compact and the image TB_{ℓ_2} is not even located since $d_{TB_{\ell_2}}(e_1) = 1 - ||a||_2$ is not computable.

As a next result we formulate a computable version of the Theorem of Schauder. The classical Theorem of Schauder states that for any compact operator $T : X \to Y$ the *adjoint*

$$T' : Y' \to X', f \mapsto fT$$

is compact again. In [7] we have already presented a computable version of this theorem for computable Hilbert spaces. In case of Banach spaces we have to face the problem that there are computable Banach spaces (such as ℓ_1) whose dual spaces (such as ℓ_∞) are not computable Banach spaces. In order to avoid additional technicalities, we formulate two separate results here that will give us together a computable version of the Theorem of Schauder for spaces whose dual spaces are well-behaved.

Proposition 5.7 *Let X, Y be computable Banach spaces. Then the operation*

$$C :\subseteq \mathcal{B}_\infty(X,Y) \times \mathcal{C}(Y) \to X', (T,f) \mapsto fT,$$

restricted to linear functionals f, is $(\delta_{\mathcal{B}_\infty(X,Y)}, [\delta_Y \to \delta_{\mathbb{F}}], \delta_{X'})$–computable.

Proof. Given $T \in \mathcal{B}_\infty(X,Y)$ and $f \in \mathcal{C}(Y)$, we can compute $fT \in \mathcal{B}_\infty(X,\mathbb{F})$ by Proposition 5.3 and thus $fT \in X'$ by Proposition 5.4. $\square$

Note that this result implies by type conversion that the adjoint map considered as map $A : \mathcal{B}_\infty(X,Y) \to \mathcal{C}(Y',X'), T \mapsto T'$ is computable. However, in order to get a name of $T' : Y' \to X'$ as a continuous map, it is sufficient that the input information f is given without the norm $||f||$ (i.e. it is sufficient to supply $f \in \mathcal{C}(Y)$, not necessarily $f \in Y'$). In order to get the adjoint T' as a compact operator, we have to compute, additionally, the image $\overline{T'B_{Y'}}$ as a compact set in X'. According to the previous result it is sufficient to show that $B_{Y'}$ is computably compact in $\mathcal{C}(Y)$. This is, more or less, the statement of the computable version of the Theorem of Banach-Alaoglu. We use a version of this result provided in [3]. If we now combine the previous results, we get the following version of the the Theorem of Schauder.

Theorem 5.8 (Computable Theorem of Schauder) *Let X, Y be computable Banach spaces with computable dual spaces X', Y'. Then the map*

$$A : \mathcal{B}_\infty(X,Y) \to \mathcal{B}_\infty(Y',X'), T \mapsto T'$$

that maps any compact operator $T : X \to Y$ to its adjoint T', is computable.

Proof. By Proposition 5.7 we can conclude by type conversion that the adjoint map $A_0 :\subseteq \mathcal{B}_\infty(X,Y) \to \mathcal{C}(\mathcal{C}(Y),X'), T \mapsto T'$ is computable. By the computable Theorem of Banach-Alaoglu there exists a computable metric space Z, a computable map $f : Z \to \mathcal{C}(Y)$ and a computably compact set $K \subseteq Z$ such that $f(Z) = B_{Y'}$. It follows from the fact that we can compute continuous images of compact sets (by Theorem 3.3 in [12]) that we can also compute $\overline{T'B_{Y'}} = T'B_{Y'} = T'f(Z) \in \mathcal{K}(X')$. Altogether, this means that we can compute $T' \in \mathcal{B}_\infty(Y',X')$. $\square$

6 The Space of Compact Operators

In this section we want to study the space $\mathcal{B}_\infty(X,Y)$ of compact operators somewhat further. Endowed with the operator norm this space is a Banach space itself. However, it is not necessarily separable and even in cases where it is separable, it is not clear that it is a computable Banach space. This is problematic as one would like to perform operations like addition on compact operators. We will provide conditions under which the space turns out to be a computable Banach space and the corresponding Cauchy representation is computably equivalent to the representation discussed in the previous section.

The approximation property plays a particular role in this context. A Banach space X is said to have the *approximation property*, if for every compact set $K \subseteq X$ and every $\varepsilon > 0$ there exists an operator $T : X \to Y$ of finite rank such that $||Tx - x|| < \varepsilon$ for every $x \in K$.

Let us assume that X, Y are separable Banach spaces. If the dual space X' or the space Y has the approximation property, then the set $\mathcal{F}(X, Y)$ of finite rank operators is dense in $\mathcal{B}_\infty(X, Y)$ (see the Theorems of Grothendieck: Theorem 1.e.4(v) and Theorem 1.e.5 in [9]). In particular, for a Banach space X with a shrinking basis e the dual space X' has the approximation property, since for such a space e' is a basis for X' and any space with a basis has the approximation property by Theorem 4.1.33 in [10]. In this situation there is a natural Cauchy representation of the set of compact operators.

Definition 6.1 [Cauchy representation] Let X be a Banach space with a shrinking basis e and let $(Y, || \, ||, f)$ be a separable Banach space. We define a numbering α_{ef} of some finite rank operators $T : X \to Y$ by

$$\alpha_{ef}\langle k, \langle n_0, ..., n_k\rangle, \langle l_0, ..., l_k\rangle\rangle(x) := \sum_{i=0}^{k} \alpha_{e'}(n_i)(x)\alpha_f(l_i).$$

Then range(α_{ef}) is dense in the set $\mathcal{B}_\infty(X, Y)$ of compact operators. We denote by $\delta^{\mathrm{C}}_{\mathcal{B}_\infty(X,Y)}$ the Cauchy representation of $\mathcal{B}_\infty(X, Y)$ induced by α_{ef}.

It follows from the computable version of the Fréchet-Riesz Theorem proved in [7] that for Hilbert spaces the Cauchy representation provided here is equivalent to the one introduced in [4].

The most interesting question for us is under which conditions the space $(\mathcal{B}_\infty(X, Y), || \, ||, \alpha_{ef})$ is a computable Banach space and when the corresponding Cauchy representation $\delta^{\mathrm{C}}_{\mathcal{B}_\infty(X,Y)}$ is equivalent to $\delta_{\mathcal{B}_\infty(X,Y)}$. We start to discuss the question when the Cauchy representation can be computably translated into $\delta_{\mathcal{B}_\infty(X,Y)}$. First of all, we extract the information on compact operators as bounded operators.

Proposition 6.2 *Let X be a computable Banach space with a shrinking and computable basis e and let Y be a computable Banach space. Then the injection*

$$\mathrm{inj} : \mathcal{B}_\infty(X, Y) \hookrightarrow \mathcal{C}(X, Y)$$

is $(\delta^{\mathrm{C}}_{\mathcal{B}_\infty(X,Y)}, [\delta_X \to \delta_Y])$–computable.

In a second step we want to consider the information on the image of the unit ball. Therefore it is helpful to use the following lemma.

Lemma 6.3 *Let X, Y be Banach spaces and let $S, T : X \to Y$ be compact operators. Then*

$$d_{\mathrm{H}}(\overline{SB_X}, \overline{TB_X}) \leq ||S - T||.$$

Now we are prepared to prove the following result. Here we have to assume that the basis of the space X is even monotone.

Proposition 6.4 *Let X be a computable Banach space with a monotone, shrinking and computable basis e and let $(Y, || \, ||, f)$ be a computable Banach space. Then the image map*

$$\mathrm{im} : \mathcal{B}_\infty(X, Y) \to \mathcal{K}(Y), T \mapsto \overline{TB_X}$$

is $(\delta^{\mathrm{C}}_{\mathcal{B}_\infty(X,Y)}, \delta_{\mathcal{K}(Y)})$–computable.

Proof. By the previous Lemma 6.3 it suffices to consider the case that $T = \alpha_{ef}(n) = \sum_{i=0}^{k} \alpha_{e'}(n_i)\alpha_f(l_i)$ for some $n = \langle k, \langle n_0, ..., n_k\rangle, \langle l_0, ..., l_k\rangle\rangle$. We can assume that each n_i has the form $n_i = \langle h_i, \langle m_{i0}, ..., m_{ih_i}\rangle\rangle$ and thus

$$Tx = \sum_{i=0}^{k} \alpha_{e'}(n_i)(x)\alpha_f(l_i) = \sum_{i=0}^{k} \alpha_f(l_i) \sum_{j=0}^{h_i} \alpha_{\mathbb{F}}(m_{ij})e'_j(x).$$

Now we let $h := \max\{h_0, ..., h_k\}$ and we claim that $TP_hB_X = \overline{TB_X}$. It is clear that "$\subseteq$" holds, as $P_hB_X \subseteq B_X$ by Lemma 3.4. For "$\supseteq$" it suffices to prove that $TB_X \subseteq TP_hB_X$, as P_hB_X is compact and hence TP_hB_X is closed. Let $y \in TB_X$. Then there exists some $x = \sum_{i=0}^{\infty} x_ie_i \in B_X$ with $x_i \in \mathbb{F}$ and $Tx = y$. With $x' := P_hx = \sum_{i=0}^{h} x_ie_i$ we obtain $Tx = Tx'$. Since $x' \in P_hB_X$, we can conclude that $y \in TP_hB_X$.

We prove that given n we can compute $P_hB_X \in \mathcal{K}(X)$. By Corollary 3.5 it holds that $(P_nB_X)_{n\in\mathbb{N}}$ is a computable sequence in $\mathcal{K}(X)$. Since the continuous image of a compact set can be computed (by Theorem 3.3 in [12]) it follows that $\overline{TB_X} \in \mathcal{K}(Y)$ can be computed. □

Whenever X is a computable Banach space with a monotone, shrinking and computable basis e, then we obtain the following proposition.

Proposition 6.5 *Let X be a computable Banach space with a monotone, shrinking and computable basis e and let Y be a computable Banach space. Then $(\mathcal{B}_\infty(X, Y), || \, ||, \alpha_{ef})$ is a computable Banach space and $\delta^{\mathrm{C}}_{\mathcal{B}_\infty(X,Y)} \leq \delta_{\mathcal{B}_\infty(X,Y)}$.*

Proof. The computable reducibility of the representations follows directly from Proposition 6.2 and Proposition 6.4. It is clear that addition and multiplication with scalars is computable in the dense subset with respect to α_{ef}.

Moreover, the norm $||\ || : \mathcal{B}_\infty(X,Y) \to \mathbb{R}$ is computable with respect to $\delta_{\mathcal{B}_\infty(X,Y)}$ and $\delta^{\mathrm{C}}_{\mathcal{B}_\infty(X,Y)}$. Altogether, this implies that $(\mathcal{B}_\infty(X,Y), ||\ ||, \alpha_{ef})$ is a computable Banach space. □

Now we continue to discuss the translation in the other direction. First of all, we prove that any computable Banach space with computable basis has the effective approximation property. We use the following notion for that purpose.

Definition 6.6 [Basis modulus] Let X be a Banach space with a basis and corresponding natural projections P_n. Let $K \subseteq X$ be a subset. Then $m : \mathbb{N} \to \mathbb{N}$ is called a *basis modulus* for K, if $||P_{m(k)}x - x|| < 2^{-k}$ for any $x \in K$.

Now we can prove the following result.

Proposition 6.7 (Effective approximation property) *Let X be a computable Banach space with a computable basis e. Then there is a computable multi-valued map $\mathrm{AP} : \mathcal{K}(X) \rightrightarrows \mathbb{N}^{\mathbb{N}}$ such that for each fixed $K \in \mathcal{K}(X)$ any $m \in \mathrm{AP}(K)$ is a basis modulus of K.*

This result, applied to the target space Y, now allows to formulate a sufficient condition for the translation of the representations of the space of compact operators in the other direction.

Proposition 6.8 *Let X be a computable Banach space with a computable and computably shrinking basis e and let Y be a computable Banach space with a computable basis f. Then $\delta_{\mathcal{B}_\infty(X,Y)} \leq \delta^{\mathrm{C}}_{\mathcal{B}_\infty(X,Y)}$.*

Proof. Given $T \in \mathcal{B}_\infty(X,Y)$ with respect to $\delta_{\mathcal{B}_\infty(X,Y)}$ and $k \in \mathbb{N}$, we have to compute some $n \in \mathbb{N}$ with $||T - \alpha_{ef}(n)|| \leq 2^{-k}$.

First of all, we can use the $\delta_{\mathcal{B}_\infty(X,Y)}$–name of T to compute a basis modulus $m : \mathbb{N} \to \mathbb{N}$ for $\overline{TB_X} \in \mathcal{K}(Y)$ by Proposition 6.7. Then

$$||P_{m(k+1)}T - T|| = \sup\{||P_{m(k+1)}Tx - Tx|| : x \in B_X\} \leq 2^{-k-1}$$

for all $k \in \mathbb{N}$. By Proposition 3.1 we can compute $P_{m(k+1)} \in \mathcal{C}(X,X)$, given k, and by Proposition 5.3 we can compute $T_k := P_{m(k+1)}T \in \mathcal{B}_\infty(X,Y)$ with respect to $\delta_{\mathcal{B}_\infty(X,Y)}$. We obtain $T_kx = \sum_{i=0}^\infty f_i'(T_kx)f_i = \sum_{i=0}^{m(k+1)}(f_i'T_k)xf_i$. By Proposition 3.1 the coordinate functionals $f_i' \in \mathcal{C}(Y)$ are computable as continuous functions and by Proposition 5.7 it follows that $f_i'T_k \in X'$ can be computed for any $i = 0, ..., m(k+1)$ with respect to $\delta_{X'}$.

Since e is computably shrinking, we can translate $\delta_{X'}$ into the Cauchy representation of $(X', ||\ ||, e')$ by Proposition 4.8 and thus we can compute

$n_0, ..., n_{m(k+1)} \in \mathbb{N}$ such that

$$||f_i' T_k - \alpha_{e'}(n_i)|| < \frac{2^{-k-1}}{||f_i|| \cdot (m(k+1)+1)}$$

and some $n \in \mathbb{N}$ such that $\alpha_{ef}(n)(x) = \sum_{i=0}^{m(k+1)} \alpha_{e'}(n_i)(x) f_i$. Then we obtain for all $x \in B_X$

$$\begin{aligned} ||Tx - \alpha_{ef}(n)(x)|| &\leq ||Tx - T_k x|| + \left|\left| T_k x - \sum_{i=0}^{m(k+1)} \alpha_{e'}(n_i)(x) f_i \right|\right| \\ &\leq 2^{-k-1} + \sum_{i=0}^{m(k+1)} ||f_i' T_k - \alpha_{e'}(n_i)|| \cdot ||x|| \cdot ||f_i|| < 2^{-k} \end{aligned}$$

and thus $||T - \alpha_{ef}(n)|| \leq 2^{-k}$. □

Now we are prepared to prove our main result. It provides sufficient conditions under which our two representations for the space of compact operators are equivalent and the space is a computable Banach space itself. It is a direct consequence of Proposition 6.5 and Proposition 6.8.

Theorem 6.9 *Let X be a computable Banach space with a monotone, computably shrinking and computable basis e and let Y be a computable Banach space with a computable basis f. Then $(\mathcal{B}_\infty(X, Y), || \; ||, \alpha_{ef})$ is a computable Banach space and $\delta^{\mathrm{C}}_{\mathcal{B}_\infty(X,Y)} \equiv \delta_{\mathcal{B}_\infty(X,Y)}$.*

If we apply this result to the case $Y = \mathbb{F}$, then it follows that the dual space representations $\delta_{X'}$ and $\delta^{\mathrm{C}}_{X'}$ are computably equivalent. In this sense, a well-behaved dual space is necessary for a well-behaved space of compact operators. By Proposition 4.14 we can conclude that our main theorem holds, in particular, in the situation of Hilbert spaces.

Corollary 6.10 *Let X and Y be infinite-dimensional computable Hilbert spaces with computable orthonormal bases e and f, respectively. Then $\delta^{\mathrm{C}}_{\mathcal{B}_\infty(X,Y)} \equiv \delta_{\mathcal{B}_\infty(X,Y)}$ and $(\mathcal{B}_\infty(X, Y), || \; ||, \alpha_{ef})$ is a computable Banach space.*

7 Biduals and Composition from Right

In this section we finally address the question whether the composition of compact operators with bounded operators from right is a computable operation. For this purpose we need the following definition.

Definition 7.1 [Computable reflexivity] A computable Banach space X is said to be *computably embeddable into its bidual*, if the partial inverse ι^{-1} of the canonical map

$$\iota : X \to X'', x \mapsto (f \mapsto fx)$$

is $(\delta_{X''}, \delta_X)$–computable. Moreover, X is said to be *computably reflexive*, if X is reflexive and X is computably embeddable into its bidual.

Here we assume that

$$\delta_{X''}\langle p, q\rangle = x :\iff [\delta_{X'} \to \delta_{\mathbb{F}}](p) = x \text{ and } \delta_{\mathbb{R}} = ||x||.$$

By definition a Banach space X is called *reflexive*, if ι is surjective. It is easy to see that the map ι itself is always $(\delta_X, \delta_{X''})$–computable. This is because evaluation on X' is computable with respect to $\delta_{X'}$ and ι is an isometry.

Now we can formulate the following result on composition from right. Since the adjoint T' of a computable operator T is not necessarily computable (see Example 6.5 of [7]), we need an extra assumption on the adjoint in the following result. We leave the proof to the reader.

Theorem 7.2 *Let X, Y, Z be computable Banach spaces with computable dual spaces and let $\mathcal{T} \subseteq \mathcal{C}(X, Y)$ be a set of linear bounded operators such that the adjoint operation $A :\subseteq \mathcal{C}(X, Y) \to \mathcal{C}(Y', X'), T \mapsto T'$ is computable restricted to $\mathcal{T}$. If X' and Z' have computable dual spaces and Z is computably embeddable into its bidual then the composition*

$$\circ :\subseteq \mathcal{B}_\infty(Y, Z) \times \mathcal{C}(X, Y) \to \mathcal{B}_\infty(X, Z), (S, T) \mapsto ST,$$

restricted to $\mathcal{T}$ in the second component, is computable.

8 Conclusions

In this paper we have shown that the theory of compact operators can be developed computably for Banach spaces with bases that satisfy some reasonable additional conditions. In the following corollary we summarize our results under assumptions that are sufficient to conclude all desirable properties of compact operators simultaneously. If a Banach space has a computable, computably shrinking and monotone basis, then the space has a computable dual space by Corollary 4.12. Therefore, the conditions as given below are sufficient for all our results.

Corollary 8.1 *Let X, Y, and Z be computable Banach spaces with computable, computably shrinking and monotone bases. Then the following hold:*

(1) $+ : \mathcal{B}_\infty(X, Y) \times \mathcal{B}_\infty(X, Y) \to \mathcal{B}_\infty(X, Y), (T, S) \mapsto T + S$ is computable,

(2) $\cdot : \mathbb{F} \times \mathcal{B}_\infty(X, Y) \to \mathcal{B}_\infty(X, Y), (a, T) \mapsto aT$ is computable,

(3) $|| \, || : \mathcal{B}_\infty(X,Y) \to \mathbb{R}, T \mapsto ||T||$ *is computable,*

(4) $A : \mathcal{B}_\infty(X,Y) \to \mathcal{B}_\infty(Y',X'), T \mapsto T'$ *is computable.*

(5) $\circ :\subseteq \mathcal{C}(Y,Z) \times \mathcal{B}_\infty(X,Y) \to \mathcal{B}_\infty(X,Z), (S,T) \mapsto ST$ *is computable.*

If, additionally, X and Z are computably reflexive, then the following holds:

(6) *if $T : X \to Y$ and $T' : Y' \to X'$ are computable, then the operation $\widehat{T} : \mathcal{B}_\infty(Y,Z) \to \mathcal{B}_\infty(X,Z), S \mapsto ST$ is computable.*

All the conditions mentioned in this corollary are automatically satisfied for infinite-dimensional computable Hilbert spaces X, Y, Z (with computable orthonormal bases that always exist). In case of a finite-dimensional Banach space X, it is easy to see that the unit ball B_X is computably compact and thus the representation $\delta_{\mathcal{B}_\infty(X,Y)}$ is computably equivalent to the representation $[\delta_X \to \delta_Y]$ in this case. All the results above can easily be proved directly in this special case.

There are a number of questions that have not been treated in this paper. For instance, it would be interesting to know whether computable compact operators could be applied in the same way as their classical counterparts, for instance, for the solution of integral or differential equations. Finally, it would be desirable to come to a better understanding of the computability conditions imposed on dual spaces and computable reflexivity. To settle the logical relations between all required notions and to construct corresponding counterexamples was beyond the scope of the present work. A particularly interesting example would be a computable Banach space X with a separable dual space X' such that addition $+ : X' \times X' \to X'$ is not computable with respect to $\delta_{X'}$. It would be desirable that this space has a computable monotone and shrinking basis at the same time. It is not yet clear whether such an example exists at all.

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