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Modified Laguerre Wavelets Method for delay differential equations of fractional-order

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ABSTRACT

In this article, Laguerre Wavelets Method (LWM) is proposed and combined with steps Method to solve linear and nonlinear delay differential equations of fractional-order. Computational work is fully supportive of compatibility of proposed algorithm and hence the same may be extended to other physical problems also. A very high level of accuracy explicitly reflects the reliability of this scheme for such problems.

1. Introduction

Fractional differential equations are applied to model wide range of physical problems including nonlinear oscillation of earth quakes [1], fluid-dynamic traffic [2], frequency dependent damping behavior of many viscoelastic materials, signal processing [5] and control theory [6]. Moreover, in several areas of applied mathematics [1,7–11] fractional differential equations are often used. These are also used in the study of epidemics, age-structured population growth [12], automation, traffic flow and in many engineering problems. The basic motivation of this paper is to develop a Laguerre Wavelets Method (LWM) and combine it with the steps Method [13] to solve linear and nonlinear delay differential equations [4] of fractional-order. It is observed that proposed method is fully compatible with the complexity of such problems and is very user-friendly. The error estimates explicitly reveal the very high accuracy level of the suggested technique.

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2. Laguerre Wavelets

Wavelets [2,3,5] constitute a family of functions constructed from dilation and translation of a single function $\psi(x)$ called the mother wavelet. When the dilation parameter a and the translation parameter b vary continuously we have the following family of continuous wavelets as [10]

$$\psi_{a,b}(x) = |a|^{-\frac{1}{2}}\psi\left(\frac{x-b}{a}\right),\ a,\ b\in R,\ a\neq 0.$$

If we restrict the parameters a and b to discrete values as $a = a_0^{-k},\ b = nb_0a_0^{-k},\ a_0 > 1,\ b_0 > 0,$ we have the following family of discrete wavelets

$$\psi_{k,n}(x) = |a|^{\frac{k}{2}}\,\psi(a_0^k x - nb_0),\ k,\ n\in\mathbb{Z},$$

where $\psi_{k,n}$ form a wavelet basis for $L^2(R)$. In particular, when $a_0 = 2$ and $b_0 = 1$, then $\psi_{k,n}(x)$ form an orthonormal basis.

The Laguerre wavelets $\psi_{n,m}(x) = \psi(k, n, m, x)$ involve four arguments $n = 1, 2, \cdots, 2^{k-1}$, k is assumed any positive integer, m is the degree of the Laguerre polynomials and it is the normalized time. They are defined on the interval $[0, 1)$ as

$$\psi_{n,m}(x) = \begin{cases} 2^{\frac{k}{2}}\tilde{L}_m(2^k x - 2n + 1), & \frac{n-1}{2^{k-1}} \le x < \frac{n}{2^{k-1}}, \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

where

$$\tilde{L}_m(x) = \frac{1}{m!}L_m(x), \tag{2}$$

$m = 0, 1, 2, \cdots, M-1$. In eq. (2) the coefficients are used for orthonormality. Here $L_m(x)$ are the Laguerre polynomials of degree m with respect to the weight function $w(x) = 1$ on the interval$[0, \infty]$, and satisfy the following recursive formula

$$L_0(x) = 1, \quad L_1(x) = 1 - x,$$
$$L_{m+2}(x) = \frac{((2m+3-x)L_{m+1}(x) - (m+1)L_m(x))}{m+2},\ m = 0, 1, 2, 3, \cdots.$$

Modified Laguerre Wavelet Method (MLWM): In the present paper, we consider the Delay Differential Equation of the form

$$y^{\alpha}(x) = f(y) + g(x)y\left(\frac{x}{a} - c\right),\ 0 < x < b,\ 1 < \alpha \le 2, \tag{3}$$

$$y(x) = p(x),\ -b \le x \le 0.$$

where $g(x)$ is a source term function, $f(y)$ is a given continuous linear or nonlinear function.

According to the proposed method, first use the method of step to convert the delay differential equation (3) to inhomogeneous ordinary differential equation by using initial function, $p(x)$, Equation (3) implies

$$y^{\alpha}(x) = f(y) + g(x)\,p\left(\frac{x}{a} - c\right),\ 0 < x < b,\ 1 < \alpha \le 2, \tag{4}$$

which is a fractional differential equation and

The solution of the Equation (4) can be expanded as a Laguerre wavelets series as follows:$y(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}c_{n,m}\,\psi_{n,m}(x)$,

where $\psi_{n,m}(x)$ is given by the Equation (1). We approximate $y(x)$ by the truncated series

$$y_{k,M}(x) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{nm}\psi_{n,m}(x). \tag{4a}$$

Then a total number of $2^{k-1}M$ conditions should exist for determination of $2^{k-1}M$ coefficients $c_{10}, c_{11}, \ldots, c_{1M-1}, c_{20}, c_{21}, \ldots, c_{2M-1}, \ldots, c_{2^{k-1}0}, c_{2^{k-1}1}, \cdots, c_{2^{k-1}M-1}$.

Since two conditions are furnished by the initial conditions, namely

$$y_{k,M}(0) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{nm}\psi_{n,m}(0) = p(0),$$
$$\frac{d}{dx}y_{k,M}(0) = \frac{d}{dx}\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{nm}\psi_{n,m}(0) = p'(0). \tag{5}$$

We see that there should be $2^{k-1}M - 2$ extra conditions to recover the unknown coefficients c_{nm}. These conditions can be obtained by substituting Equation (4) in Equation (3);

$$\frac{d^{\alpha}}{dx^{\alpha}}\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-3}c_{nm}\psi_{n,m}(x) = f\left(\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-3}c_{nm}\psi_{n,m}(x)\right) + g(x)\,p\left(\frac{x}{a} - c\right). \tag{6}$$

We, now assume Equation (6) is exact at $2^{k-1}M - 3$ points x_i as follows:

Table 1 – Numerical results of Example 1.

t	Exact solution	Solution by proposed method	Error in proposed method M = 5	Error in proposed method M = 10	Error in proposed method M = 20
0.0	1.00000000000	1.00000000200	2.00000E-09	2.70000E-08	1.00000E-09
0.1	0.90031699980	0.90033016590	1.31661E-05	2.04000E-08	9.00000E-10
0.2	0.80241064730	0.80244245410	3.18068E-05	1.75000E-08	1.10000E-09
0.3	0.70773067800	0.70774110310	1.04251E-05	1.78000E-08	9.00000E-10
0.4	0.61740564790	0.61737305130	3.25966E-05	2.83000E-08	1.00000E-09
0.5	0.53228073020	0.53222793850	5.27917E-05	4.12000E-08	8.00000E-10
0.6	0.45295378910	0.45293810650	1.56826E-05	4.87000E-08	5.00000E-10
0.7	0.37980938990	0.37987859860	6.92087E-05	5.91000E-08	5.00000E-10
0.8	0.31305050400	0.31316715980	1.16656E-04	7.68000E-08	3.00000E-10
0.9	0.25272775330	0.25266423690	6.35164E-05	9.57000E-08	1.00000E-10
1.0	0.19876611040	0.19797297850	7.93132E-04	1.30400E-07	7.00000E-10

Table 2 – Numerical results of Example 2.

t	Exact solution	Solution by proposed method	Error in proposed method
			M = 5
0.0	0.00000000000	−0.00000000141	1.41421E-09
0.1	0.01000000000	0.01000004758	4.75800E-08
0.2	0.04000000000	0.04000009693	9.69300E-08
0.3	0.09000000000	0.09000014701	1.47010E-07
0.4	0.16000000000	0.16000019820	1.98200E-07
0.5	0.25000000000	0.25000025090	2.50900E-07
0.6	0.36000000000	0.36000030550	3.05500E-07
0.7	0.49000000000	0.49000036240	3.62400E-07
0.8	0.64000000000	0.64000042200	4.22000E-07
0.9	0.81000000000	0.81000048480	4.84800E-07
1.0	1.000000000000	1.00000055100	5.51000E-07

$$\frac{d^\alpha}{dx^\alpha}\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-3} c_{nm}\psi_{n,m}(x_i) = f\left(\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-3} c_{nm}\psi_{n,m}(x_i)\right) + g(x_i)\, p\left(\frac{x_i}{a} - c\right). \tag{7}$$

The best choice of the x_i points are the zeros of the shifted Laguerre polynomials of degree $2^{k-1}M - 2$ in the interval $[0,1]$ that is $x_i = \frac{s_i+1}{2}$, where $s_i = \cos\left(\frac{(2i-1)\pi}{2^{k-1}M-1}\right)$, $i = 1, \dots, 2^{k-1}M - 2$.

Combine Equations (5) and (7) to obtain $2^{k-1}M$ linear equations from which we can compute values for the unknown coefficients, c_{nm}. Same procedure is repeated for delay differential equations of first and second order also.

3. Solution procedure

Example 1. Consider the Fractional Delay Differential Equation of the form

$$u^\alpha(t) = -u(t) - e^{-\frac{t}{2}}\sin\left(\frac{t}{2}\right)u\left(\frac{t}{2}\right) - 2e^{-\frac{3t}{4}}\cos\left(\frac{t}{4}\right)\sin\left(\frac{t}{4}\right)u\left(\frac{t}{4}\right),$$
$$0 \le t \le 1,\ \ 0 < \alpha \le 1,$$

subject to the initial condition $u(0) = 1$.

The exact solution of the above system is $u(t) = e^{-t}\cos(t)$.

Table 1 shows the comparison of the absolute error between exact solution and approximate solution for $M = 5,\ 10,\ 20$ and $K = 1$ by Modified Laguerre Wavelet Method (MLWM).

Example 2. Consider the Fractional Delay Differential Equation of the form

$$u^\alpha(t) = \frac{3}{4}u(t) + u\left(\frac{t}{2}\right) - t^2 + 2,\ \ 0 \le t \le 1,\ \ 1 < \alpha \le 2,$$

subject to the initial conditions $u(0) = 0,\ u'(0) = 0$.

The exact solution of the above system is $u(t) = t^2$.

Table 2 shows the comparison of the absolute error between exact solution and approximate solution for $M = 5$, and $k = 1$ by Modified Laguerre Wavelet Method (MLWM).

Table 3 – Numerical results of Example 3.

t	Exact solution	Solution by proposed method	Error in proposed method	Error in proposed method	Error in proposed method
			M = 5	M = 10	M = 20
0.0	1.00000000000	0.99999999950	5.00000E-10	6.30000E-09	0.00000E+00
0.1	0.90483741800	0.90483743970	2.17000E-08	1.49000E-08	1.00000E-10
0.2	0.81873075310	0.81872927080	1.48230E-06	2.35000E-08	1.00000E-10
0.3	0.74081822070	0.74081168180	6.53890E-06	3.17000E-08	1.00000E-10
0.4	0.67032004600	0.67031009600	9.95000E-06	4.00000E-08	0.00000E+00
0.5	0.60653065970	0.60653917080	8.51110E-06	4.82000E-08	1.00000E-10
0.6	0.54881163610	0.54890279800	9.11619E-05	4.82000E-08	1.00000E-10
0.7	0.49658530380	0.49689410400	3.08800E-04	6.50000E-08	0.00000E+00
0.8	0.44932896410	0.45009544900	7.66485E-04	7.39000E-08	1.00000E-10
0.9	0.40656965970	0.40817842780	1.60877E-03	8.27000E-08	3.00000E-10
1.0	0.36787944120	0.37090386940	3.02443E-03	8.62000E-08	2.00000E-10

Table 4 – Numerical results of Example 4.

t	Exact solution	Solution by proposed method	Error in proposed method	Error in proposed method	Error in proposed method
			M = 5	M = 10	M = 20
0.0	1.00000000000	0.99999999950	5.00000E-10	5.20000E-08	2.10000E-08
0.1	0.99500416530	0.99500423090	6.56000E-08	6.33000E-08	2.11000E-08
0.2	0.98006657780	0.98006756710	9.89300E-07	7.78000E-08	2.09000E-08
0.3	0.95533648910	0.95533840950	1.92040E-06	9.32000E-08	2.09000E-08
0.4	0.92106099400	0.92106174490	7.50900E-07	1.10700E-07	2.08000E-08
0.5	0.87758256190	0.87757914590	3.41600E-06	1.27600E-07	2.06000E-08
0.6	0.82533561490	0.82532877020	6.84470E-06	1.44500E-07	2.04000E-08
0.7	0.76484218730	0.76484536080	3.17350E-06	1.62800E-07	2.03000E-08
0.8	0.69670670930	0.69676024620	5.35369E-05	1.79200E-07	2.00000E-08
0.9	0.62160996830	0.62180134050	1.91372E-04	1.95600E-07	1.99000E-08
1.0	0.54030230590	0.54079314330	4.90837E-04	2.21800E-07	1.97000E-08

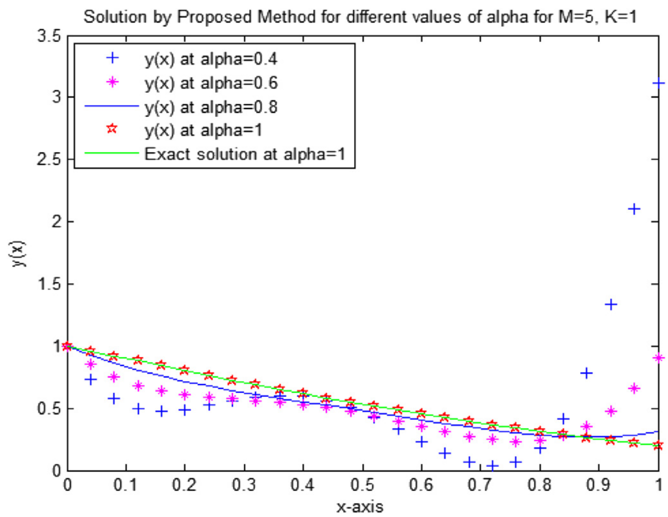

Fig. 1 – Modified Laguerre Wavelets Method solution for fractional DDE given in Example 1 and its comparison with exact solution.

Example 3. Consider the Fractional Delay Differential Equation of the form

$$u^{\alpha}(t) = -u(t) - u(t-0.3) + e^{-t+0.3}, \quad 0 < t \le 1, \ 2 < \alpha \le 3,$$

subject to the initial condition $u(0) = 1$, $u'(0) = -1$, $u''(0) = 1$.

The exact solution of the above system is $y(x) = e^{-t}$.

Table 3 shows the comparison of the absolute error between exact solution and approximate solution for $M = 5, 10, 20$ and $k = 1$ by Modified Laguerre Wavelet Method (MLWM).

Example 4. Consider the Nonlinear Fractional Delay Differential Equation of the form

$$u^{\alpha}(t) = 1 - 2u^2\left(\frac{t}{2}\right), 0 \le x \le 1, \ 1 < \alpha \le 2,$$

subject to the initial condition $u(0) = 1$, $u'(0) = 0$.

The exact solution of the above system is $u(t) = \cos(t)$.

Table 4 shows the comparison of the absolute error between exact solution and approximate solution for $M = 5, 10, 20$ and $k = 1$ by Modified Laguerre Wavelet Method (MLWM).

4. Conclusion

Linear and Nonlinear Delay Differential Equations of fractional-order are successfully tackled by Modified Laguerre Wavelets Method (MLWM). The solutions of the fractional delay differential equation converge to the solution of integer delay differential equation, as shown in Figs. 1–4. According to the Tables, we get more accurate results while increasing M. Computational work and numerical results explicitly reflect that the proposed method (MLWM) is very user-friendly but extremely accurate.

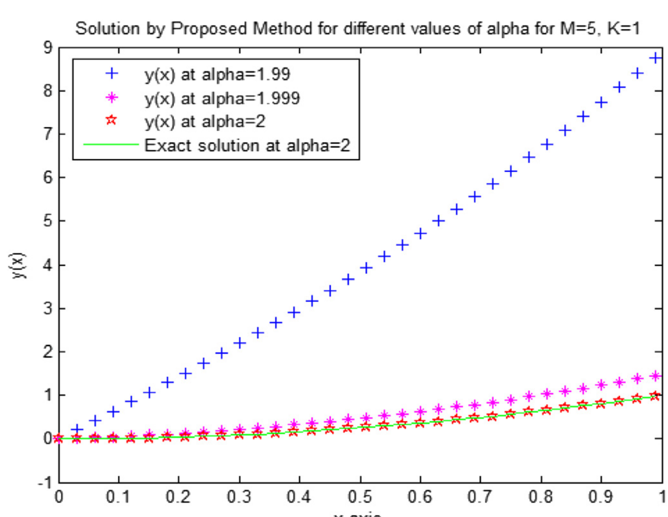

Fig. 2 – Modified Laguerre Wavelets Method solution for fractional DDE given in Example 2 and its comparison with exact solution.

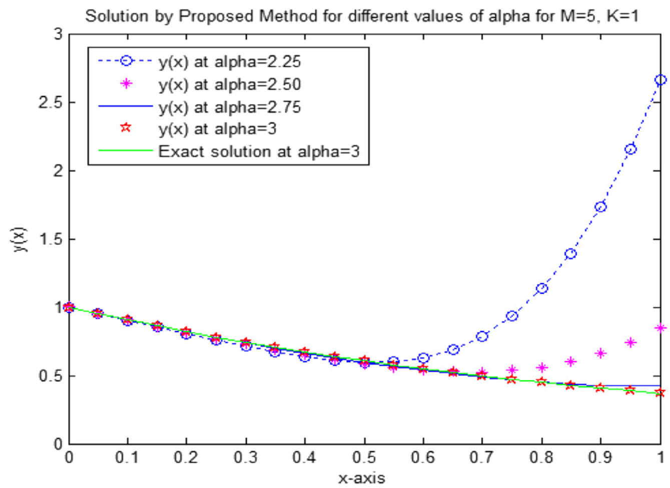

Fig. 3 – Modified Laguerre Wavelets Method solution for fractional DDE given in Example 3 and its comparison with exact solution.

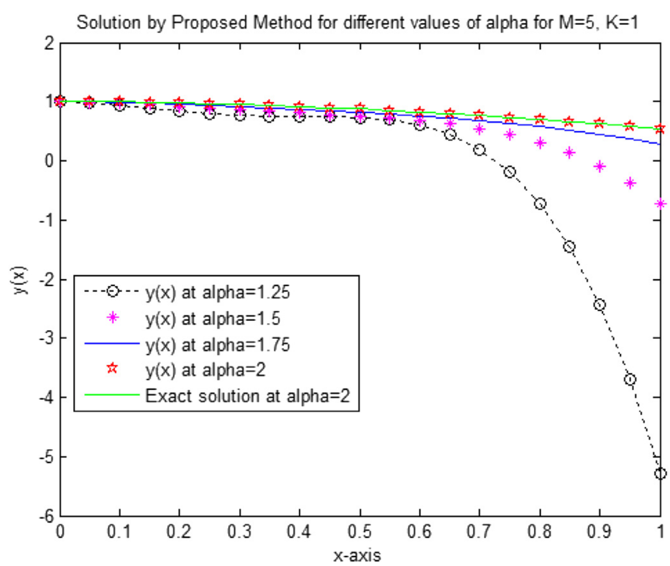

Fig. 4 – Modified Laguerre Wavelets Method solution for fractional DDE given in Example 4 and its comparison with exact solution.

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