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Open and Closed Questions in Decision-making

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Abstract

By asking questions, an agent can modify the range of options from which a decision is made.[12] introduced a logic for reasoning about this role of question-asking in the decision-making process. The base logic is a modal logic with an operator D interpreted by: $D\varphi$ iff after any rational choice that the agent can make, φ holds. On top of this, we proposed an analysis of questions as dynamic operators $[?Q]$ and $[!Q]$ which alter the range of options available to the agent in various ways. In the present paper, we provide a complete axiomatisation for this dynamic logic, and extend the analysis to complex questions. A particular feature of our approach is that it does not assume that an agent's preferences are transitive. Here we give a characterisation of the transitivity of the preference order in terms of invariance under changes of the order in which questions are asked. This is applied to a notorious case of transitivity failure: Condorcet's voting paradox.

Keywords: rational choice, preference, closed and open questions, hybrid logic, transitivity of preferences, Condorcet, dynamic logic

1 Questions in decision-making

To decide is to choose between a number of options. The range of options available is to some extent determined by contextual factors beyond the control of the agent who makes the decision. But it is also determined in part by the agent's cognitive

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state. In particular, the agent must know which options are available to her. In this paper, we will focus on exchange decisions, in which an agent has made a particular decision and then considers changing her mind as the result of new information. The new information will come from asking certain questions about the alternatives available. For example, suppose you have been walking to work every day along the south side of the river. Today, however, you wonder whether there is a better way to go, and consider whether crossing the bridge and walking along the north side would be better. In other words, you ask a question about whether taking the north or south side of the river makes any difference. This kind of 'open' question is typical in decision-making. One uses such questions to broaden the space of alternatives and so increase ones chances of arriving at an optimal decision. Of course, the trade-off is that consideration of more alternatives carries costs in time, cognitive processing, and other costs associated with information-gathering. (One might have to buy a map!)

In [12], we introduced a logic for reasoning about the role of question-asking in the decision-making process. The logic was motivated by the analysis of the following example, which we repeat here, in order to explain our application of the logic.

> Alice is considering moving house. She is unhappy with the fact that her house is far from the bus stop. She searches the listings for a house that is better located and sees several that she likes better. She goes to visit the one of them with Betty, her good friend. When Betty sees the house, she says 'what about a garden?' This is not a question that Alice had considered before. Her own house doesn't have one, but she is influenced by Betty to go back to the listings and check out houses with gardens. Eventually, she finds a house and moves. It has a nice big garden. But a few months later, she visits Chandra, a friend of Betty's who lives in a concrete house. Alice finds it quite charming. Her new house is timber-framed, like her old house and every house she has ever lived in. That night, she goes back to the listings. Alice is so impressed with the houses she looks at. Why had she never thought of houses being made out of anything but wood? Next month, she has moved into her new plaster house, which looks very modern and stylish. But after a while she happens to walk past her old house - the first one. Taken by it's quaint charm and worn woodwork, she realises that she prefers it to her new house.

The story illustrates how the process of practical decision-making is guided by the questions one asks. Alice may well have asked very direct questions, such as 'does it have a roof?' and 'can I afford it?' but often it is the more open ended questions such as 'what about a garden?' that helped her to enlarge the options available to her, and it is these questions that we will focus on. Eventually, it is clear that something has clearly gone wrong with Alice's decision-making. In the excitement of the search, something was missing. An easy answer is that Alice's preferences are clearly not transitive. She is therefore irrational, in some sense. But the apparent plausibility of each of her decisions suggests there is more to be said.

An example of a similar kind, not considered in [12], involves majority voting, which notoriously fails in general to preserve the transitivity of preferences. This can produce some odd results. Consider the following example.

> Three friends are going out together for the evening and need to pick a designated driver. First they assume that Andrew will drive, but he objects and suggest Barry instead. They agree that Andrew will drive unless the majority prefer that Barry drives. But they vote for Barry to drive. This time Barry objects because Charles has not even been considered. They decide that Barry should drive unless the majority prefer that Charles drives, in which case Charles is picked as the driver. This happens and Charles is selected, despite the fact that the majority prefer Andrew (the original choice) as the designated driver.

This is a version of Condorcet's Paradox.[4] Assuming that no one prefers himself to be the driver, the preferences of the three friends must have been as follows:

	$A < B$	$B < C$	$A < C$	
Andrew	yes	yes	yes	$A < B < C$
Barry	no	yes	no	$B < C < A$
Charles	yes	no	no	$C < A < B$
Majority	yes	yes	no	

The propositional variables A, B and C stand for 'Andrew drives', 'Barry drives,' and 'Charles drives,' respectively. The preference ordering for the majority is $A < B < C < A$ which is clearly intransitive. We can model this in the manner shown in Figure 1. The friends move from one selection of a driver to the next by asking

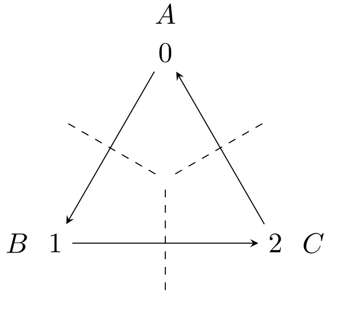

Fig. 1. Condorcet drivers

questions? Initially, Andrew is presumed to be the driver, then he asks 'what about Barry?' At this point the dashed line dividing A and B fades to reveal B as the preferred driver. They ask whether it will be Andrew or Barry, and the vote

[4] This is the problem with majority voting observed by the Marquis de Condorcet in the late 18th century, which shows how three or more agents voting on an issue with more than two options are not guaranteed to produce a transitive preference order.

determines it to be Barry. This decision re-establishes our line between A and B. But then Barry asks 'what about Charles?' The line between B and C now fades, with the result that Charles is selected. We will describe the various operations on the model brought about by the questions asked.

2 A dynamic logic of preferential choice: the atomic case

[12] introduces a logic for reasoning about decision-making when the agent's preferences may fail to be transitive. We suppose that every decision is made in a context in which some factors are allowed to vary and others remain constant. In other words, we follow von Wright [11] and others (especially Girard, in [6]) in taking judgements based on preference to be *ceteris paribus*. Recent work on the logic of *ceteris paribus* operators ([9] and [7]) provided a technical approach that was adapted in [12], with some changes. Specifically, we take an *atomic ceteris paribus choice model* $M = \langle F, V, P\rangle$ to consist of a choice model $\langle F, V\rangle$ together with a set P of propositional variables, which we interpret as the atomic facts that are held constant *ceteris paribus*. This means that the range of options available to the agent consists of those within the equivalence class of the present situation, where 'equivalence' is defined by

$$u \approx v \quad \text{iff} \quad \text{for each } p \in P,\ u \in V(p) \text{ iff } v \in V(p)$$

We require P to be cofinite, so that only a finite number of propositional variables are allowed to vary.

These structures are described by the language

$$i \;\mid\; p \;\mid\; \neg \;\mid\; \wedge \;\mid\; \Box \;\mid\; D \;\mid\; U$$

where $i \in \textsc{Nom}$, a set of *nominals*, $p \in \mathsf{Prop}$, a set of *propositional variables* and $\Box, D$, and U are unary modal operators. The nominals are included in the language so that we can reason about particular agent's preferences concerning particular states/entities. We abbreviate $\Diamond = \neg\Box\neg$, $\langle D\rangle = \neg D\neg$ and $E = \neg U\neg$ to get the corresponding existential modalities. The semantic conditions for our language must be restricted to observe the restriction to $\approx$:

$$\begin{array}{lll}
M, w \models p & \text{iff} & w \in V(p) \\
M, w \models i & \text{iff} & w \in V(i) \\
M, w \models \neg\varphi & \text{iff} & M, w \not\models \varphi \\
M, w \models (\varphi \wedge \psi) & \text{iff} & M, w \models \varphi \text{ and } M, w \models \psi \\
M, w \models \Box\varphi & \text{iff} & M, v \models \varphi \text{ for each } v \approx w \text{ such that } w \leq v \\
M, w \models U\varphi & \text{iff} & M, v \models \varphi \text{ for each } v \approx w \\
M, w \models D\varphi & \text{iff} & M, v \models \varphi \text{ for each } v \approx w \text{ such that } v \text{ is also } \leq \cap \approx\text{-maximal}
\end{array}$$

To say that a state u is '$\leq \cap \approx$-maximal' means that any other state v within the $\approx$-equivalence class of u that is at least as good as u ($u \leq v$) is also no better than u, ($v \leq u$). This is our analysis of the condition under which it is rational to choose u *ceteris paribus.*[5] Note, in particular, that we do not require the $\leq$ relation to be transitive. The 'universal' modality U is also restricted *ceteris paribus*. The following axiomatization is proved complete in [12]:

Theorem 2.1 *The set of formulas valid over the class of atomic* ceteris paribus *preference models is axiomatized by basic hybrid logic, $\mathcal{H}$, together with*

Universal Equivalence	$i \to Ei$	Reflexivity	$i \to \Diamond i$
	$i \to UEi$	Inclusion	$\Diamond i \to Ei$
	$EEi \to Ei$	Preferential Choice	$\langle D \rangle i \quad \leftrightarrow \quad E(i \wedge \Box\Diamond i)$

Call this the logic $\mathcal{PC}$ of Preferential Choice.

To this base, we add some dynamic operators. We model the effect of asking the open question 'what about p?' as the operation of changing the model $M = \langle F, V, P \rangle$ to the model $[?p]M = \langle F, V, P \setminus \{p\} \rangle$ in which p has been subtracted from the set of propositional variables that must be held constant when we search for alternatives. Or, in other words, the range of alternative from which the agent may choose has expanded to include those that differ with respect to the value for p. More generally, for a finite set Q of propositional variables, we define

$$[?Q]M = \langle F, V, P \setminus Q \rangle$$

This operation on models is expressed by the operator $[?Q]$ with

$$M, w \models [?Q]\varphi \quad \text{iff} \quad [?Q]M, w \models \varphi$$

This enables us to give the following analysis of Alice's decision-making (repeated from [12]). Consider the representation of Alice's preferences shown in Figure 2. We model M Alice's decisions about moving from house 0 using the frame shown,

[5] [12] has more discussion of the adequacy of this account.

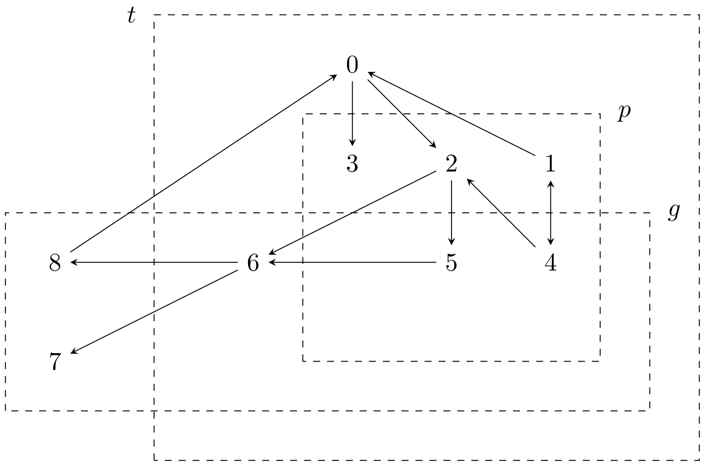

Fig. 2. Alice's story

F, with nominals for each of the houses. t, g, p are propositional variables. Initially, before Alice considers moving, all of these are fixed, so we take $P = \mathsf{Prop}$. When Alice is unhappy about living far from the bus stop, she searches for houses and considers $1, 2$, and 3. These options are available because she has asked the question $[?p]$. She rejects 1 and we have $M, 0 \models [?p]D(2 \vee 3)$ She decides to move to house 2, but her friend Betty asks 'what about a garden?' $[?g]$. Then the listings for houses is enlarged and since $M, 2 \models [?p][?g]D6$ she moves to house 6. We'll return to the example when we can say more about what happens next.

3 A flexible fragment of *Ceteris Paribus* Logic

There is a systematic relationship between the current approach and the *Ceteris Paribus* Logic of [9] and [7], which uses the language L_{cp} defined by

$$p \quad | \quad \neg \quad | \quad \wedge \quad | \quad \langle\Gamma\rangle$$

where Γ is a set of formulas of L_{cp}.[6] Interpreted in a preference model M, the semantic condition for $\langle\Gamma\rangle$ is

$$\begin{array}{lll} M, w \models \langle\Gamma\rangle\varphi & \text{iff} & M, v \models \varphi \text{ for all } v \geq w \text{ and for all } \psi \in \Gamma \\ & & \qquad\qquad\qquad M, w \models \psi \text{ iff } M, v \models \psi \end{array}$$

The axiomatisation of the set of validities of L_{cp} is still an open problem but it is solved in the case of the 'flexible fragment'. Details of this fragment are given in [7]. It is enough for our purposes to note that φ is flexible if every modal operator in φ is of the form $\langle\Gamma\rangle$ where Γ is a cofinite set of propositional variables. Call these the *cofinite* formulas. But now if we consider the atomic *ceteris paribus* preference model $M' = \langle F, V, P\rangle$ then

[6] This circular definition can be straightened out with a transfinite recursive definition. L_{cp} is a proper class but this does not cause a problem in practice. See [7] for details.

$$M', w \models [?Q]\Diamond\varphi \text{ iff } M, w \models \langle P \setminus Q\rangle\varphi$$

and since Q is finite and P is cofinite, $P \setminus Q$ is also cofinite. This connection between the two languages inspires consideration of the following axioms for our dynamic operators:

$$\begin{array}{lll} \textsf{Question Distribution} \vdash [?Q]p & \leftrightarrow & p \\ \vdash [?Q]\neg\varphi & \leftrightarrow & \neg[?Q]\varphi \\ \vdash [?Q](\varphi \wedge \psi) & \leftrightarrow & ([?Q]\varphi \wedge [?Q]\psi) \\ \vdash [?Q_1][?Q_2]\varphi & \leftrightarrow & [?Q_1 \cup Q_2]\varphi \end{array}$$

Applying these axioms from left to right to push question operators to minimal scope, it is easy to see that the $\Diamond$-fragment of our language is logically equivalent to the cofinite fragment of L_{cp}.[7] This is enough to establish the axiomatisability of the $\Diamond$-fragment. To go further in this direction, we would have to extend flexible L_{cp} to a multi-modal setting, which is beyond the scope of this paper. Instead, we will formulate axioms for atomic *ceteris paribus* preference logic directly, using the methods employed for flexible L_{cp}.

The main idea is to reduce all question operators occurring in a formula to multiple occurrences of a single question operator. This can be done using the following valid equivalences:

Question Expansion

$$\vdash [?Q]\Diamond\varphi \leftrightarrow (p \wedge [?p, Q]\Diamond(p \wedge \varphi)) \vee (\neg p \wedge [?p, Q]\Diamond(\neg p \wedge \varphi))$$
$$\vdash [?Q]E\varphi \leftrightarrow (p \wedge [?p, Q]E(p \wedge \varphi)) \vee (\neg p \wedge [?p, Q]E(\neg p \wedge \varphi))$$
$$\vdash [?Q]D\varphi \leftrightarrow (p \wedge [?p, Q]D(p \wedge \varphi)) \vee (\neg p \wedge [?p, Q]D(\neg p \wedge \varphi))$$

Now for any formula φ, let

$$Q_\varphi = \bigcup \{Q \mid [?Q] \text{ occurs in } \varphi\}$$

In other words, Q_φ is the set of all propositional variables questioned in φ. By repeated application of Question Distribution and Question Expansion from left to right, we can compute a logically equivalent formula φ' in which the only question operator that occurs is $[?Q_\varphi]$. But now we can apply Question Distribution from right to left to find a logically formula of the form $[?Q_\varphi]\varphi''$ where φ'' contains no question operators. But

$$\langle F, V, P\rangle, w \models [?Q_\varphi]\varphi'' \quad \text{iff} \quad \langle F, V, P \setminus Q_\varphi\rangle, w \models \varphi''$$

This suggest a final rule of inference for questions:

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[7] The axioms allow us to show that any formula in the $\Diamond$-fragment is equivalent to one in which the only occurrences of questions are immediately in front of modal operators, and these can be translated into L_{cp} operators. In the reverse directly, when Γ is confinite, $\mathsf{Prop} \setminus \Gamma$ is finite and so we have that $[?\mathsf{Prop} \setminus \Gamma]\Diamond$ is equivalent to $\langle\Gamma\rangle$.

Question Generalisation if $\vdash \varphi$ then $\vdash [?Q]\varphi$

which the above shows to be sound. Thus, if φ is consistent, so is $[?Q_\varphi]\varphi''$, and so is φ'', which contains no question operators and so is satisfiable by Theorem 2.1. This establishes

Theorem 3.1 *The set of valid formulas of the language of* ceteris paribus *preference logic with atomic question operators is axiomatised by* $\mathcal{PC}$ *plus* Question Distribution*.* Question Expansion *and* Question Generalisation*. Call this the logic a*$\mathcal{QPC}$*.*

Now this is the logic for arbitrary (atomic) *ceteris paribus* frames, but it is also interesting to see how the transitive frames can be axiomatised. It is, of course, well known that this can be done with $\Diamond\Diamond i \rightarrow \Diamond i$. But, more interestingly, it can also be done with the following principle:

$$\text{Question Permutation } [?p]\Diamond[?q]\Diamond i \leftrightarrow [?q]\Diamond[?p]\Diamond i$$

This says that the order of asking questions in a sequence of choices is irrelevant. Our examples of Alice and the Condorcet drivers both depend on this *not* being valid, and we have seen how they suffer as a result.

Theorem 3.2 Question Permutation *is valid on a* ceteris paribus *choice frame iff it is transitive.*

Proof. Suppose $[?p]\Diamond[?q]\Diamond i \leftrightarrow [?q]\Diamond[?p]\Diamond i$ is valid on F, which is not transitive. Then F must contain three states $1, 2$ and 3 such that $1 \leq 2 \leq 3$ but $1 \not\leq 3$. Then we can get a counterexample, $M = \langle F, \mathsf{Prop}, V\rangle$ by assigning V as shown:

$$\begin{array}{ccccc} p, q & & q & & i \\ 1 & \longrightarrow & 2 & \longrightarrow & 3 \end{array}$$

Then $M, 1 \models [?p]\Diamond[?q]\Diamond i$ but $M, 1 \not\models [?q]\Diamond[?p]\Diamond i$.

Conversely, suppose for contradiction that $[?p]\Diamond[?q]\Diamond i \leftrightarrow [?q]\Diamond[?p]\Diamond i$ is invalid on a transitive frame F. Without loss of generality, assume there is a model $M = \langle F, V, \{p, q\}\rangle$ such that $M, w \models [?p]\Diamond[?q]\Diamond i$ and $M, w \not\models [?q]\Diamond[?p]\Diamond i$. So $\langle F, V, \{q\}\rangle, w \models \Diamond[?q]\Diamond i$ and so there is a $v \approx_{\{q\}} w$ such that $w \leq v$ and $\langle F, V, \{q\}\rangle, v \models [?q]\Diamond i$. So again there is a $u \approx_\emptyset v$ such that $v \leq u$ and $\langle F, V, \emptyset\rangle, u \models i$.

Now by transitivity, $w \leq u$. And as we know $w \approx_{\{q\}} v$, $v \approx_\emptyset u$, and $\approx_{\{q\}} \subseteq \approx_\emptyset$. So $w \approx_\emptyset u$. Then we have $\langle F, V, \emptyset\rangle, w \models \Diamond i$ and $\langle F, V, \{p\}\rangle, w \models [?p]\Diamond i$. From $w \leq w$, and $w \approx w$, $\langle F, V, \{p\}\rangle, w \models \Diamond[?p]\Diamond i$ holds and so $M, w \models [?q]\Diamond[?p]\Diamond i$ holds, contradicting our assumption. □

4 Open, closed and compound questions

We have interpreted the question 'what about p?' as an *open* question, inviting not an answer but a further investigation of situations in which p may or may not be the case. But one can also interpret the question in a *closed* way, as a request for an answer. Closure is achieved by finding out whether p is the case and then sticking to the answer. We can easily add an operation $[!p]$ (more generally $[!Q]$) to our language that has this effect. Let L^c be the extension of L to include closed question operators. Given a model $M = \langle F, V, P \rangle$, let

$$[!Q]M = \langle F, V, P \cup Q \rangle$$

and define [8]

$$M, u \models [!Q]\varphi \quad \text{iff} \quad [!Q]M, u \models \varphi$$

Closed questions allow us to give a model of decisions that are actually taken. If we ask about Q and make a decisive choice, the options opened by $[?Q]$ are typically closed. We represent this by

$$[?Q]D[!Q]$$

The closing of the question marks the end of the decision-making process. This is no clearer than with examples in which the decision involves a clear action, such as moving house. [9] Alice deciding to move to a house with a garden and then actually moving is represented as

$$[?g]D[!g]$$

After this point, she no longer considers houses without a garden. If Alice kept the garden question open, she would not have moved the second time (move to the house 8), which would have not been a stable option. That is

$$M, 0 \models [?p]\langle D \rangle [?g]\langle D \rangle [!g][?t]\langle D \rangle 8 \text{ but not } M, 0 \models [?p]\langle D \rangle [?g]\langle D \rangle [?t]\langle D \rangle 8$$

Finally, we can model the last part of Alice's story, when she discovers that she prefers her old house 0 to the new one 8. That is

$$M, 0 \models [?p]\langle D \rangle [?g]\langle D \rangle [!g][?t][\langle D \rangle [?g][!p]D0$$

As for open questions, closed questions satisfy distribution principles that allow us to find a logical equivalent of any formula in which the only occurrences of $[!Q]$ are immediately before a modal operator: [10]

Closed Question Distribution
$$\begin{aligned}
&\vdash [!Q]p && \leftrightarrow p \\
&\vdash [!Q]\neg\varphi && \leftrightarrow \neg[!Q]\varphi \\
&\vdash [!Q](\varphi \wedge \psi) && \leftrightarrow ([!Q]\varphi \wedge [!Q]\psi) \\
&\vdash [!Q_1][!Q_2]\varphi && \leftrightarrow [!Q_1 \cup Q_2]\varphi \\
&\vdash [!Q_1][?Q_2]\varphi && \leftrightarrow [?Q_2 \setminus Q_1][!Q_1 \setminus Q_2]\varphi
\end{aligned}$$

Form this point, closed questions are even easier to axiomatize that open questions, because they satisfy the following reduction axioms, that allow us to remove all closed questions from the formula:

[8] Note that if P is cofinite and Q is finite then $P \cup Q$ is also cofinite.
[9] The following analysis of the continuation of Alice's story is also repeated from [12].
[10] The last of these is valid because $(P \cup Q_1) \setminus Q_2 = (P \setminus (Q_2 \setminus Q_1)) \cup (Q_1 \setminus Q_2)$.

Closed Question Reduction

$$\vdash [!p, Q]\Diamond\varphi \leftrightarrow (p \wedge [!Q]\Diamond(p \wedge \varphi)) \vee (\neg p \wedge [!Q]\Diamond(\neg p \wedge \varphi))$$
$$\vdash [!p, Q]E\varphi \leftrightarrow (p \wedge [!Q]E(p \wedge \varphi)) \vee (\neg p \wedge [!Q]E(\neg p \wedge \varphi))$$
$$\vdash [!p, Q]D\varphi \leftrightarrow (p \wedge [!Q]D(p \wedge \varphi)) \vee (\neg p \wedge [!Q]D(\neg p \wedge \varphi))$$

The parallel with Ceteris Paribus logic continues, as the closed question fragment of our logic is equivalent to the finite fragment of L_{cp}, which was axiomatised in a similar way in [9].

Theorem 4.1 *$a\mathcal{QPC}$ together with the axioms of* Closed Question Distribution *and* Closed Question Reduction *is a complete axiomatisation of the set of valid formulas of L^c.*

Extending our language to compound questions is a little more difficult. To ask the open question 'what about p?' when P is the set of variables being held constant has a clear interpretation, as we have seen. It means simply that we allow p to vary while keeping everything else in P constant. Compound open questions present more of a problem. If we ask 'what about $(p \vee q)$?' then again, we would like to allow the truth value of $(p \vee q)$ to vary while keeping everything else constant. But in this case, it is difficult to see what 'everything else' means. p can be kept constant while $(p \vee q)$ varies, and so can q, but if both are held constant, $(p \vee q)$ will also be constant. An initial reaction to this problem is to suppose that to allow $(p \vee q)$ to vary is just to allow both p and q to vary. In that case $[?(p \vee q)]$ would be equivalent to $[?p, q]$. More generally, we might say that to allow φ to vary (for some purely Boolean formula φ) is to allow Q to vary, where Q is the set of propositional variables contained in φ.

This is not such a bad solution, but it has some limitations. Firstly, taking the set of propositional variables that occur in φ is too crude. The truth-value of $((p \wedge q) \vee (p \wedge \neg q))$ for example, is not at all effected by the truth-value of q, and so allowing it to vary only requires that p be allowed to vary. Secondly, there is a non-trivial interaction with closed questions. Suppose we asked the closed question $[!(p \wedge q)]$. The result should be that the truth value of $(p \wedge q)$ is kept constant. A parallel argument to the one above would suggest taking $[!(p \wedge q)] = [!p, q]$. But now if we ask the open question $[?(p \vee q)]$ we would like to allow the truth-value of $(p \vee q)$ to vary *while keeping that of $(p \wedge q)$ constant*. But if $[?(p \vee q)] = [?p, q]$ and $[!(p \wedge q)] = [!p, q]$ then $[!(p \wedge q)][?(p \vee q)] = [!p, q][?p, q] = [?p, q]$, and so we lose the requirement that $(p \wedge q)$ is held constant.

So, let us first extend our language to the language LQ:

$$i \mid p \mid \neg \mid \wedge \mid \Box \mid U \mid [?Q]\varphi \mid [!Q]\varphi$$

where Q is a finite subset of L.[11] A *ceteris paribus question model* $M = \langle F, V, \Gamma \rangle$ consists of a preference model $\langle F, V \rangle$ and a set Γ of L-formulas. We define the

[11] Questions that contain statements about questions are beyond the scope of this paper!

ceteris paribus relation in the same way as in L_{cp}, namely

$$u \approx v \quad \text{iff} \quad \text{for each } \varphi \in \Gamma,\ M, u \models \varphi \text{ iff } M, v \models \varphi$$

That is to say, states u and v are equivalent *ceteris paribus* iff they have the same value for all formulas in Γ.

To progress further, we will need say more about how holding some formulas constant effects the value of other formulas. [12] A *state description* of given a finite set of formulas $\{\varphi_1, \dots, \varphi_n\}$ is a formula of the form $\pm\varphi_1 \wedge \dots \wedge \pm\varphi_n$ where $\pm\varphi_i$ is either φ_i or $\neg\varphi_i$. A set of formulas Γ *determines* formula φ iff it contains a non-empty finite subset Γ_0 such that for every state description σ of Γ_0, either $(\sigma \to \varphi)$ or $(\sigma \to \neg\varphi)$ is a theorem of $\mathcal{PC}$. [13] Γ determines a set Q of formulas iff it determines each formula in Q.

Lemma 4.2 *If Γ determines Q then in any model M and states u, v, if $u \approx v$ then for all $\varphi \in Q$, $M, u \models \varphi$ iff $M, v \models \varphi$*

Proof. Suppose $u \approx v$ in M and Γ determines Q. Then there is a finite set $\{\varphi_1, \dots, \varphi_n\} \subseteq \Gamma$ that determines Q. So define

$$\sigma \;=\; (\pm\varphi_1 \wedge \dots \wedge \pm\varphi_n)$$

where $\pm\varphi_i$ is φ_1 if $M, u \models \varphi_i$ and is $\neg\varphi_1$ if $M, u \not\models \varphi_i$. Then $M, u \models \sigma$. Now let $\varphi \in Q$. Then either $(\sigma \to Q)$ or $(\sigma \to \neg Q)$ is a theorem of $\mathcal{PC}$. Without loss of generality, suppose $(\sigma \to Q)$ is the theorem. Then $M, u \models \varphi$. But $u \approx v$ and each $\varphi_i \in \Gamma$, so $M, u \models \varphi_i$ iff $M, v \models \varphi_i$, and so $M, v \models \sigma$ and $M, v \models \varphi$. $\square$

For example, the set $\{p, q\}$ clearly determines $(p \vee q)$ but $\{(p \wedge q), p\}$ does not. To see this, $(p \wedge q)$ is clearly not determined by $\{(p \wedge q)\}$ or by {p} and for $\{(p \wedge q), p\}$, note that $\neg(p \wedge q) \wedge \neg p$ implies neither $(p \vee q)$ nor $\neg(p \vee q)$.

Now, say that a subset $\Gamma' \subseteq \Gamma$ is *Q-releasing* iff Γ' is a maximally non-Q-determining subset of Γ, i.e., Γ' does not determine Q and for any Γ'' that also does not determine Q, if $\Gamma' \subseteq \Gamma'' \subseteq \Gamma$ then $\Gamma'' = \Gamma'$.

If P is the set of propositional variables occurring in some Boolean formula φ, then $\mathsf{Prop} \setminus P$ does not determine φ. But it may not be φ-releasing. For example, if φ is the formula $((p \wedge q) \vee (p \wedge \neg q))$ that we considered earlier, then $\mathsf{Prop} \setminus \{p, q\}$ is not φ-releasing, because the larger set $\mathsf{Prop} \setminus \{p\}$ also fails to determine φ.

Moreover, a set Γ of formulas may have more than one φ-releasing subset. For example, take $\Gamma = \{(p \wedge q), p, q\}$ and $\varphi = (p \vee q)$. Then both $\{(p \wedge q), p\}$ and $\{(p \wedge q), q\}$ are φ-releasing subsets of Γ.

The concept of Q-releasing is exactly what we need to define the semantics of open compound questions. Given such a question $[?Q]$ define the following relation between models $M = \langle F, V, \Gamma\rangle$ and $M = \langle F, V, \Gamma'\rangle$:

$$M[?Q]M' \text{ iff } \Gamma' \text{ is a } Q\text{-releasing subset of } \Gamma \text{ (in } M)$$

[12] The problem is similar to the problem of contraction in belief revision.

[13] One could extend the concept of determination using stronger logics, of course, but $\mathcal{PC}$ is sufficient for present purposes.

Then we use this relation between models to give the semantics for $[?Q]$.

$$M, u \models [?Q]\varphi \quad \text{iff} \quad M', u \models \varphi \text{ for every } M[?Q]M'$$

The new language is a conservative extension of the old in the following sense:

Lemma 4.3 *If $M = \langle F, V, P\rangle$ is an atomic* ceteris paribus *question preference model then it is also a* ceteris paribus *question model and for any formula φ of L,*

$$M, u \models \varphi \text{ (as an atomic model) iff } M, u \models \varphi \text{ (as a general model)}$$

Proof. We need only observe that if Q is a set propositional variables, then P' is a Q-releasing subset of P iff $P' = P \setminus Q$. □

The logical behaviour of the operator $[?Q]$ for non-atomic Q is, however, quite different. In particular, when Q is atomic, $[?Q]$ is self-dual: $[?Q]$ is logically equivalent to $\neg[?Q]\neg$. But does not hold for non-atomic Q. We therefore write $\langle ?Q\rangle$ for the dual $\neg[?Q]\neg$.

In the *ceteris paribus* question preference model $M = \langle F, V, \{(p \wedge q), p, q\}\rangle$, the relation $\approx_M$ is the identity relation because all three states differ on the values given to either p or q. Now we have seen that {(p∧q),p,q} has two $(p \vee q)$-releasing subsets, $\{(p \wedge q), p\}$ and $\{(p \wedge q), q\}$, so

$$M[?(p \vee q)]M_q \text{ and } M[?(p \vee q)]M_p$$

where $M_q = \langle F, V, \{(p \wedge q), p\}\rangle$ and $M_p = \langle F, V, \{(p \wedge q), q\}\rangle$. And the *ceteris paribus* relations in M_p and M_q differ.

These formal properties of our models capture a common insight. When asking an open question, there is more than one way to proceed, depending on which *ceteris paribus* assumptions are dropped. For closed questions, the extension to compound formulas is much easier. We simply define

$$\langle F, V, \Gamma\rangle, u \models [!Q]\varphi \quad \text{iff} \quad \langle F, V, \Gamma \cup Q\rangle, u \models \varphi$$

Now, let's go back to the Condorcet Drivers Problem. At the beginning the friends assume that Andrew will drive, and so the equivalence classes determined by the initial Γ are as shown by the dashed lines of Figure 1. We'll suppose that $\Gamma = \{A, B, C\}$. When Andrew asks 'what about Barry?' he is opening the question $[?(A \vee B)]$. This ensures that the friends will prefer Barry:

$$M, 0 \models [?(A \vee B)]DB$$

They vote decisively for Barry, who is selected. This is a decision so the original question is closed $[!(A \vee B)]$. Later, after a voting about $(B \vee C)$, Charles is selected, because

$$M, 0 \models [?(A \vee B)]D[!(A \vee B)][?(B \vee C)]DC$$

despite the fact that they prefer A to C. We could even go further, adding a new open question $[?(A \vee C)]$ to ask about this, with the result that

$$M, 0 \models [?(A \vee B)]D[!(A \vee B)][?(B \vee C)]D[!(B \vee C)][?(A \vee C)]DA$$

5 Closing Remarks and Related Work

The work begun in [12] considers one way in which questions are in involved in decision-making, namely, the control of the range of options being considered. We analyse this effect by taking questions to be dynamic operators that alter the range of *ceteris paribus* conditions restricting the preference relation, so combining ideas from preference logic and the logic of questions, but in a way that has not been considered before. [14] There is now a growing literature on the logic of questions, some of which also uses dynamic model logic. Johan van Benthem and Ştefan Minică's [2] is the central reference. This was adapted in [3] by one of the present authors to address examples similar to the ones considered here. Nonetheless, the particular approach adopted in [12] and developed here is novel in addressing the interaction between question-asking and *ceteris paribus* conditions. We believe that this is an important conceptual connection that underlies the role of questions in inquiry, more generally. A natural direction to take this would be to look at the role of questions in directing scientific research. Experimental generalisations are always expressed as *ceteris paribus* laws and 'open' questions of the kind considered here are typical. For example, when observing the inverse correlation between the pressure and volume of a gas in a high school physics experiment on Boyle's Law, the student must pay particular attention to the *ceteris paribus* clause 'in conditions of constant temperature'. To ask the question 'What about the temperature?' is to remove the restriction to constant temperature, and calls for further experiments to discover the resulting, more complicated, dependencies.

A second theme to the research reported here is that of reasoning about agents with non-transitive preferences. Intransitive preferences have been considered in the foundations of rational choice theory, most notable by Anand in [1], who argues for an alternative to the standard account in which intransitive preferences are permitted. His view remains controversial as a normative account but there is little doubt that transitivity of preference is simply false as a descriptive account of preference. We showed that the 'local' rationality of decisions based on *ceteris paribus* preference is compatible with a lack of transitivity, and gave two detailed examples of how this can lead to strange results. We also characterised transitivity of preference in terms of invariance under changes in the order in which questions are asked, including a version of Condorcet's notorious voting paradox.

One final connection to be made is between our approach to the dynamics of complex questions and research on belief revision. Removing a complex *ceteris paribus* condition, so allowing it to vary, shares properties with the operation of 'contracting' a belief set with a complex belief. Both are essentially non-deterministic, for similar reasons. Further research is needed to chart out the details of this comparison.

[14] An alternative approach would be to focus on the changing awareness of the agent making decisions as a result of obtaining answers to her questions. Existing work on logics of awareness include [5], [8], [4] and [10]. As far as we are aware, there is no direct technical relationship between these approaches and ours.

References

[1] Paul Anand. *Foundations of Rational Choice Under Risk*. Oxford University Press, 1993.

[2] Johan van Benthem and Ştefan Minică. Toward a dynamic logic of questions. In E. Pacuit X. He, J. Horty, editor, *Logic Rationality and Interaction*, pages 27–41, Chongqing, China, August 2009. Springer.

[3] Meiyun Guo and Zuojun Xiong. A dynamic preference logic with issue-management. *Studies in Logic*, 1, Spring 2011.

[4] Martin Meier Aviad Heifetz and Burkhard Schipper. Interactive unawareness. *Journal of Economic Theory*, 2007.

[5] Oliver Board and Kim-Sau Chung. Object-based unawareness. In Wiebe van der Hoek Giacomo Bonanno and Michael Wooldridge, editors, *Proceedings of The 7th Conference on Logic and the Foundations of Game and Decision Theory (LOFT)*, pages 35–41, 2006.

[6] P. Girard. *Modal Logics for Belief and Preference Change*. PhD thesis, Stanford University, 2008.

[7] Jeremy Seligman and Patrick Girard. Being flexible about ceteris paribus, 2011. Available from: http://auckland.academia.edu/JeremySeligman/Papers/672679.

[8] Giacomo Sillari. Models of awareness. In M. Wooldridge G. Bonanno, W. van der Hoek, editor, *Proceedings of The 7th Conference on Logic and the Foundations of Game and Decision Theory (LOFT)*, number 2006, pages 209–218.

[9] Johan van Benthem, Patrick Girard, and Olivier Roy. Everything else being equal: a modal logic for *ceteris paribus* preferences. *Journal of philosophical logic*, August 2008. Available from: http://www.springerlink.com/content/p756008882505667/.

[10] Hans van Ditmarsch and Tim French. Becoming aware of propositional variables. In M. Banerjee and A. Seth, editors, *ICLA 2011*, number 6521 in LNAI, pages 204–218. Springer-Verlag.

[11] G.H. von Wright. *The Logic of Preference*. Edinburgh, 1963.

[12] Zuojun Xiong and Jeremy Seligman. How questions guide choices: a preliminary logical investigation. In Dianhui Wang and Mark Reynolds, editors, *The 24th Australasian Joint Conference on Artificial Intelligence*, Lecture Notes in Artificial Intelligence. Springer-Verlag, 2011.