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Electronic Notes in Theoretical Computer Science 308 (2014) 289–307

Electronic Notes in Theoretical Computer Science

www.elsevier.com/locate/entcs

Towards a Quantum Domain Theory: Order-enrichment and Fixpoints in W*-algebras

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Abstract

We discuss how the theory of operator algebras, also called operator theory, can be applied in quantum computer science. From a computer scientist point of view, we explain some fundamental results of operator theory and their relevance in the context of domain theory. In particular, we consider the category $\mathbf{W}^*$ of W*-algebras together with normal sub-unital maps, provide an order-enrichment for this category and exhibit a class of its endofunctors with a canonical fixpoint.

Keywords: W*-algebras, operator theory, domain theory, fixpoint theorem, quantum computation

Introduction

Our aim here is to use the theory of operator algebras to study the differences and similarities between probabilistic and quantum computations, by unveiling their domain-theoretic and topological structure. To our knowledge, the deep connection between the theory of operator algebras and domain theory was not fully exploited before. This might be due to the fact that the theory of operator algebras, mostly unknown to computer scientists, was developed way before the theory of domains.

Our main contribution is a connection between two different communities: the community of theoretical computer scientists, who use domain theory to study program language semantics (and logic), and the community of mathematicians and theoretical physicists, who use a special class of algebras called W*-algebras to study quantum mechanics.

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http://dx.doi.org/10.1016/j.entcs.2014.10.016

Our main purpose was to pave the way to a study of quantum programming language semantics, where types are denoted by W*-algebras, terms are denoted by normal completely positive maps, recursive terms and weakest preconditions are denoted by fixed points (via $\mathbf{Dcpo}_{\perp!}$-enrichment), and recursive types are denoted by fixed points of endofunctors (via algebraical compactness).

We list here the main points of the paper, that will be detailed and explained later on:

- Positive maps can be ordered by a generalized version of the Löwner order [8], considered in the finite-dimensional case by Selinger in [13]: for two positive maps f and g, $f \sqsubseteq g$ iff $(g - f)$ is positive [Definition 2.1]. We also consider completely positive maps in Section 4.
- Hom-sets of normal (positive) sub-unital maps between W*-algebras are directed-complete with this so-called Löwner order [Theorem 2.2].
- The category of $\mathbf{W}^*$ of W*-algebras together with normal sub-unital maps is order-enriched [Theorem 2.9].
- The notion of von Neumann functor is introduced to denote the locally continuous endofunctors on $\mathbf{W}^*$ which preserve multiplicative maps [Definition 3.1].
- For such functors, we provide a canonical way to construct a fixpoint, by showing that the category $\mathbf{W}^*$ is algebraic compact (for the class of von Neumann functors) [Theorem 3.6]. Our proof are in the lines of Smyth-Plotkin [14].

1 A short introduction to operator theory

In this section, we will introduce two structures, known as C*-algebras and W*-algebras. We refer the interested reader to [15] for more details.

1.1 C-algebras*

A Banach space is a normed vector space where every Cauchy sequence converges. A Banach algebra is a linear associative algebra A over the complex numbers $\mathbb{C}$ with a norm $\|\cdot\|$ such that its norm $\|\cdot\|$ is submultiplicative (i.e. $\forall x, y \in A, \|xy\| \leq \|x\| \, \|y\|$) and turns A into a Banach space. A Banach algebra A is unital if it has a unit, i.e. if it has an element 1 such that $a1 = 1a = a$ holds for every $a \in A$ and $\|1\| = 1$.

A *-algebra is a linear associative algebra A over $\mathbb{C}$ with an operation $(-)^* : A \to A$ such that for all $x, y \in A$, the following equations holds: $(x^*)^* = x$, $(x + y)^* = (x^* + y^*)$, $(xy)^* = y^* x^*$ and $(\lambda x)^* = \overline{\lambda} x^* \quad (\lambda \in \mathbb{C})$.

A C*-algebra is a Banach *-algebra A such that $\|x^* x\| = \|x\|^2$ for all $x \in A$. This identity is sometimes called the C*-identity, and implies that every element x of a C*-algebra is such that $\|x\| = \|x^*\|$.

Consider now a unital C*-algebra A. An element $x \in A$ is self-adjoint if $x = x^*$. An element $x \in A$ is positive if it can be written in the form $x = y^* y$, where $y \in A$.

We write $A_{\mathrm{sa}} \hookrightarrow A$ (resp. $A^+ \hookrightarrow A$) for the subset of self-adjoint (resp. positive) elements of A.

For every C*-algebra, the subset of positive elements is a convex cone and thus induces a partial order structure on self-adjoint elements, see [15, Definition 6.12]. That is to say, one can define a partial order on self-adjoint elements of a C*-algebra A as follows: $x \leq y$ if and only if $y - x \in A^+$.

From now on, we will consider the following kind of maps of C*-algebras. Let $f : A \to B$ be a linear map of C*-algebras:

P The map f is *positive* if it preserves positive elements and therefore restricts to a function $A^+ \to B^+$. A positive map $A \to \mathbb{C}$ will be called a state on A. It should be noted that positive maps of C*-algebras preserve the order on self-adjoint elements.

M The map f is *multiplicative* if $\forall x, y \in A, f(xy) = f(x)f(y)$;

I The map f is *involutive* if $\forall x \in A, f(x^*) = f(x)^*$;

U The map f is *unital* if it preserves the unit;

sU The map f is *sub-unital* if the inequality $0 \leq f(1) \leq 1$ holds;

cP For every C*-algebra A, one can easily define pointwise a C*-algebra $\mathcal{M}_n(A)$ from the set of n-by-n matrices whose entries are elements of A. The map f is *completely positive* if for every $n \in \mathbb{N}$, $\mathcal{M}_n(f) : \mathcal{M}_n(A) \to \mathcal{M}_n(B)$ defined for every matrix $[x_{i,j}]_{i,j \leq n} \in \mathcal{M}_n(A)$ by $\mathcal{M}_n(f)([x_{i,j}]_{i,j \leq n}) = [f(x_{i,j})]_{i,j \leq n}$ is positive.

For every Hilbert space H, the Banach space $\mathcal{B}(H)$ of bounded linear maps on H is a C*-algebra. The space $\mathcal{C}_0(X)$ of complex-valued continuous functions, that vanish at infinity, on a locally compact Hausdorff space X is a common example of commutative C*-algebra.

Self-adjoint and positive elements of $\mathcal{B}(H)$ can be defined alternatively through the inner product of H, as in the following standard theorem (see [3, II.2.12,VIII.3.8]):

Theorem 1.1 *Let H be a Hilbert space and $T \in \mathcal{B}(H)$. Then:*

(i) *T is self-adjoint if and only if $\forall x \in H, \langle Tx|x\rangle \in \mathbb{R}$;*

(ii) *T is positive if and only if T is self-adjoint and $\forall x \in H, \langle Tx|x\rangle \geq 0$.*

1.2 W-algebras*

We will denote by $\mathcal{B}(H)$ (resp. $\mathrm{Ef}(H)$) the collection of all bounded operators (resp. positive bounded operators below the unit) on a Hilbert space H. There are several standard topologies that one can define on a collection $\mathcal{B}(H)$ (see [15] for an overview).

Definition 1.2 The operator norm $\|T\|$ is defined for every bounded operator T in $\mathcal{B}(H)$ by: $\|T\| = \sup\{\|T(x)\| \mid x \in H, \|x\| \leq 1\}$. The norm topology is the topology induced by the operator norm on $\mathcal{B}(H)$. A sequence of bounded operators (T_n) converges to a bounded operator T in this topology if and only if $\|T_n - T\| \underset{n\to\infty}{\longrightarrow} 0$.

The strong operator topology on $\mathcal{B}(H)$ is the topology of pointwise convergence in the norm of H: a net of bounded operators $(T_\lambda)_{\lambda\in\Lambda}$ converges to a bounded operator T in this topology if and only if $\|(T_\lambda - T)x\| \longrightarrow 0$ for each $x \in H$. In that case, T is said to be strongly continuous.

The weak operator topology on $\mathcal{B}(H)$ is the topology of pointwise weak convergence in the norm of H: a net of bounded operators $(T_\lambda)_{\lambda\in\Lambda}$ converges to a bounded operator T in this topology if and only if $\langle (T_\lambda - T)x|y\rangle \longrightarrow 0$ for $x, y \in H$. In that case, T is said to be weakly continuous.

It is known that, for an arbitrary Hilbert space H, the weak operator topology on $\mathcal{B}(H)$ is weaker than the strong operator topology on $\mathcal{B}(H)$, which is weaker than the norm topology on $\mathcal{B}(H)$. However, when H is finite-dimensional, the weak topology, the strong topology and the norm topology coincide. Moreover, for the strong and the weak operator topologies, the use of nets instead of sequences should not be considered trivial: it is known that, for an arbitrary Hilbert space H, the norm topology is first-countable whereas the other topologies are not necessarily first-countable, see [15, Chapter II.2].

The commutant of $A \subset \mathcal{B}(H)$ is the set A' of all bounded operators that commutes with those of A: $A' = \{T \in \mathcal{B}(H) \mid \forall S \in A, TS = ST\}$. The bicommutant of A is the commutant of A' and will be denoted by A''.

The following theorem is a fundamental result in operator theory as it remarkably relates a topological property (being closed in two operator topologies) to an algebraic property (being its own bicommutant).

Theorem 1.3 (von Neumann bicommutant theorem) *Let A be a unital *-subalgebra of $\mathcal{B}(H)$ for some Hilbert space H. The following conditions are equivalent:*

(i) $A = A''$.

(ii) *A is closed in the weak topology of $\mathcal{B}(H)$.*

(iii) *A is closed in the strong topology of $\mathcal{B}(H)$.*

A W*-algebra (or von Neumann algebra) is a C*-algebra which satifies one (hence all) of the conditions of the von Neumann bicommutant theorem. The collections of bounded operators on Hilbert spaces are the most trivial examples of W*-algebras. The function space $L^\infty(X)$ for some standard measure space X and the space $\ell^\infty(\mathbb{N})$ of bounded sequences are common examples of commutative W*-algebras.

For every C*-algebra A, we denote by A' the dual space of A, i.e. the set of all linear maps $\phi : A \to \mathbb{C}$. It is known that a C*-algebra A is a W*-algebra if and only if there is a Banach space A_*, called pre-dual of A, such that $(A_*)' = A$, see [11, Definition 1.1.2].

A positive map $\phi : A \to B$ between two C*-algebras is normal if every increasing net $(x_\lambda)_{\lambda\in\Lambda}$ in A^+ with least upper bound $\bigvee x_\lambda \in A^+$ is such that the net $(\phi(x_\lambda))_{\lambda\in\Lambda}$ is an increasing net in B^+ with least upper bound $\bigvee \phi(x_\lambda) = \phi(\bigvee x_\lambda)$.

1.3 Direct sums and tensors of W*-algebras

The direct sum of a family of C*-algebras $\{A_i\}_{i\in I}$ is defined as the C*-algebra

$$\bigoplus_{i\in I} A_i = \left\{ (a_i)_i \in \prod_{i\in I} A_i \mid \sup_{i\in I} \|a_i\| < \infty \right\}$$

where the operations are defined component-wise and with a norm defined by $\|(a_i)_{i\in I}\|_\infty = \sup \|a_i\|$.

The direct sum of a family of W*-algebras $\{A_i\}_{i\in I}$ is the W*-algebra $\bigoplus_{i\in I} A_i$ defined as the dual of the C*-algebras $\bigoplus_{i\in I} A_{i*}$, such that A_i is the dual of A_{i*}, seen as a C*-algebra.

The spatial tensor product $A\overline{\otimes}B$ of two W*-algebras A with universal normal representations $\pi_A : A \to \mathcal{B}(H)$ and $\pi_B : B \to \mathcal{B}(K)$ can be defined as the subalgebra of $\mathcal{B}(H\otimes K)$ generated by the operators $m\otimes n \in \mathcal{B}(H\otimes K)$ where $(m,n) \in A\times B$.

Proposition 1.4 *For a W*-algebra A, one has the following properties:*

- $A \oplus 0 = A = 0 \oplus A$;
- $A\overline{\otimes}0 = 0 = 0\overline{\otimes}A$;
- $A\overline{\otimes}\mathbb{C} = A = \mathbb{C}\overline{\otimes}A$;
- $A\overline{\otimes}(\bigoplus_{i\in I} B_i) = \bigoplus_{i\in I}(A\overline{\otimes}B_i)$ *for every family of W*-algebras* $\{A_i\}_{i\in I}$.

2 Rediscovering the domain-theoretic structure of W*-algebras

The W*-algebras together with the normal sub-unital maps (or NsU-maps), i.e. positive and Scott-continuous maps, give rise to a category $\mathbf{W}^*$, which is a subcategory of the category $\mathbf{C}^*$ of C*-algebras together with positive sub-unital maps.

In this section, after recalling some standard notion of domain theory, we will show that positive sub-unital maps can be ordered in such a way that the category $\mathbf{W}^*$ will be $\mathbf{Dcpo}_{\perp!}$-enriched. The decision of considering normal sub-unital maps instead of normal completely positive maps will be discussed in Section 4.1.

2.1 A short introduction to domain theory

A non-empty subset Δ of a poset P is called directed if every pair of elements of Δ has an upper bound in Δ. We denote it by $\Delta \subseteq_{dir} P$. A poset P is a directed-complete partial order (dcpo) if each directed subset has a least upper bound. A function $\phi : P \to Q$ between two posets P and Q is strict if $\phi(\perp_P) = \perp_Q$, is monotonic if it preserves the order and Scott-continuous if it preserves directed joins. We denote by $\mathbf{Dcpo}_\perp$ (resp. $\mathbf{Dcpo}_{\perp!}$) the category with dcpos with bottoms as objects and Scott-continuous maps (resp. strict Scott-continuous maps) as morphisms.

2.2 A Löwner order on positive maps

Since positive elements are self-adjoint, one can define the following order on positive maps of C*-algebras.

Definition 2.1 [Löwner partial order] For positive maps $f, g : A \to B$ between C*-algebras A and B, we define pointwise the following partial order $\sqsubseteq$, which turns out to be an infinite-dimensional generalization of the Löwner partial order [8] for positive maps: $f \sqsubseteq g$ if and only if $\forall x \in A^+, f(x) \leq g(x)$ if and only if $\forall x \in A^+, (g-f)(x) \in B^+$ (i.e. $g - f$ is positive).

One might ask if, for arbitrary C*-algebras A and B, the poset $(\mathbf{C}^*(A, B), \sqsubseteq)$ is directed-complete. The answer turns out to be no, as shown by our following counter-example:

Let us consider the C*-algebra $C([0,1]) := \{f : [0,1] \to \mathbb{C} \mid f \text{ continuous}\}$.

The hom-set $\mathbf{C}^*(\mathbb{C}, C([0,1]))$ is isomorphic to $C([0,1])$ if one considers the functions $F : \mathbf{C}^*(\mathbb{C}, C([0,1])) \to C([0,1])$ and $G : C([0,1]) \to \mathbf{C}^*(\mathbb{C}, C([0,1]))$ respectively defined by $F(f) = f(1)$ and $G(g) = \lambda\alpha \in \mathbb{C}.\alpha \cdot g$.

We define an increasing chain $(f_n)_{n\geq 0}$ of $C([0,1])$ define for every $n \in \mathbb{N}$ by

$$f_n(x) = \begin{cases} 0 & \text{if } 0 \leq x < \frac{1}{2} \\ (x - \frac{1}{2})2^{n+1} & \text{if } \frac{1}{2} \leq x \leq \frac{1}{2} + 2^{-(n+1)} \\ 1 & \text{if } \frac{1}{2} + 2^{-(n+1)} < x \leq 1 \end{cases}$$

Suppose that there is a least upper bound ϕ in $C([0,1])$ for this chain. Then, $\phi(x) = 0$ if $x < \frac{1}{2}$. Moreover, $\lim_{n\to\infty}\left(\frac{1}{2} + 2^{-(n+1)}\right) = \frac{1}{2}$ implies that $\phi(x) = 1$ if $x > \frac{1}{2}$. It follows that $\phi(x) \in \{0,1\}$ if $x \neq \frac{1}{2}$.

By the Intermediate Value Theorem, the continuity of the function ϕ on the interval $[0,1]$ implies that there is a $c \in [0,1]$ such that $\phi(c) = \frac{1}{2}$. From $\phi(c) \notin \{0,1\}$, we obtain that $c = \frac{1}{2}$. That is to say $\phi(\frac{1}{2}) = \frac{1}{2}$, which is absurd since $f_n(\frac{1}{2}) = 0$ for every $n \in \mathbb{N}$.

It follows that there is no least upper bound for this chain in $C([0,1])$ and therefore $C([0,1])$ is not chain-complete. However:

Theorem 2.2 *For W*-algebras A and B, the poset $(\mathbf{W}^*(A, B), \sqsubseteq)$ is directed-complete.*

The proof of this theorem will be postponed until after the following lemmas.

Lemma 2.3 ([10], Corollary 1) *Let $f \in \mathbf{C}^*(A, B)$ and $x \in A^+$. Then, $f(x) \leq \|x\| \cdot 1$. Therefore, $\|f(x)\| \leq \|x\|$.*

The following result is known in physics as Vigier's theorem [16]. A weaker version of this theorem can be found in [13]. It is important in this context because it establishes the link between limits in topology and joins in order theory.

Lemma 2.4 *Let H be a Hilbert space. Let $(T_\lambda)_{\lambda\in\Lambda}$ be an increasing net of* $\mathrm{Ef}(H)$. *Then the least upper bound $\bigvee T_\lambda$ exists in* $\mathrm{Ef}(H)$ *and is the limit of the net $(T_\lambda)_{\lambda\in\Lambda}$ in the strong topology.*

Proof. For any operator $U \in \mathcal{B}(H)$, the inner product $\langle Ux|x\rangle$ is real if and only if U is self-adjoint (by Theorem 1.1). Thus, for each $x \in H$, the net $(\langle T_\lambda x|x\rangle)_{\lambda\in\Lambda}$ of real numbers is increasing, bounded by $\|x\|^2$ and thus convergent to a limit $\lim_\lambda \langle T_\lambda x|x\rangle$ since $\mathbb{R}$ is bounded-complete.

By polarization on norms, $\langle T_\lambda x|y\rangle = \frac{1}{2}(\langle T_\lambda(x+y)|(x+y)\rangle - \langle T_\lambda x|x\rangle - \langle T_\lambda y|y\rangle)$ for any $\lambda \in \Lambda$. Then, for all $x, y \in H$, the limit $\lim_\lambda \langle T_\lambda x|y\rangle$ exists and thus we can define pointwise an operator $T \in \mathrm{Ef}(H)$ by $\langle Tx|y\rangle = \lim_\lambda \langle T_\lambda x|y\rangle$ for $x, y \in H$.

Indeed, T is the limit of the net $(T_\lambda)_{\lambda\in\Lambda}$ in the weak topology, and therefore in the strong topology since a bounded net of positive operators converges strongly whenever it converges weakly (see [1, I.3.2.8]).

Moreover, T is an upper bound for the net $(T_\lambda)_{\lambda\in\Lambda}$ since $T_\lambda \leq T$ for every $\lambda \in \Lambda$. By Theorem 1.1, if there is a self-adjoint operator $S \in B(H)$ such that $T_\lambda \leq S$ for every $\lambda \in \Lambda$, then $\langle T_\lambda x|x\rangle \leq \langle Sx|x\rangle$ for every $\lambda \in \Lambda$. Thus, $\langle Tx|x\rangle = \lim_\lambda \langle T_\lambda x|x\rangle \leq \langle Sx|x\rangle$. Then, $\langle (S-T)x|x\rangle \geq 0$ for every $x \in H$. By Theorem 1.1, $S - T$ positive and thus $T \leq S$. It follows that T is the least upper bound of $(T_\lambda)_{\lambda\in\Lambda}$. □

Corollary 2.5 *For every W*-algebra A, the poset $[0,1]_A$ is directed-complete.*

Proof. Let A be a W*-algebra. By definition, A is a strongly closed subalgebra of $\mathcal{B}(H)$, for some Hilbert space H. Then, let $(T_\lambda)_{\lambda\in\Lambda}$ be an increasing net in $[0,1]_A \subseteq \mathrm{Ef}(H)$. By Lemma 2.4, $(T_\lambda)_{\lambda\in\Lambda}$ converges strongly to $\bigvee T_\lambda \in \mathrm{Ef}(H)$. It follows that $\bigvee T_\lambda \in [0,1]_A$ because $[0,1]_A$ is strongly closed. Thus, $[0,1]_A$ is directed-complete. □

This corollary constitutes a crucial step in the proof of Theorem 2.2, as it unveils a link between the topological properties and the order-theoretic properties of W*-algebras.

Lemma 2.6 *Any positive map $f : A \to B$ between C*-algebras is completely determined and defined by its action on $[0,1]_A$.*

Proof. A positive map of C*-algebras $f : A \to B$ restrict by definition to a map $f : A^+ \to B^+$. Since f preserves the order $\leq$ on positive elements, it restricts to $[0,1]_A \to [0,1]_B$:

Let $x \in A^+ \setminus \{0\}$. From $x \leq \|x\| 1$, we can see that $\frac{1}{\|x\|}x \in [0,1]_A$ and thus $f(\frac{1}{\|x\|}x) \in [0,1]_B$. Moreover, $f(x) = \|x\| f(\frac{1}{\|x\|}x)$. This statement can be extended to every element in A since each $y \in A$ is a linear combination of four positive elements (see [1, II.3.1.2]), determining $f(y) \in B$. □

We can now show that the poset $(\mathbf{W}^*(A,B), \sqsubseteq)$ is directed-complete for every pair A and B of W*-algebras.

Proof. [Proof of Theorem 2.2] Let A and B be two W*-algebras. By Corollary 2.5, $[0,1]_A$ and $[0,1]_B$ are directed-complete.

We now consider an increasing net $(f_\lambda)_{\lambda\in\Lambda}$ of NsU-maps from A to B, increasing in the Löwner order. Then, for every $x \in A^+$, there is an increasing net $(f_\lambda(x))_{\lambda\in\Lambda}$ bounded by $\|x\| \cdot 1$ (by Lemma 2.3).

Moreover, for every non-zero element $x \in A^+$, from the fact that $[0,1]_B$ is directed-complete, we obtain that the increasing net $(f_\lambda(\frac{x}{\|x\|}))_{\lambda\in\Lambda}$ has a join $\bigvee f_\lambda(\frac{x}{\|x\|})$ in $[0,1]_B$ and thus we can define pointwise the following upper bound $f : [0,1]_A \to [0,1]_B$ for the increasing net $(f'_\lambda)_{\lambda\in\Lambda}$ of NsU-maps from $[0,1]_A$ to $[0,1]_B$ such that, for every $\lambda \in \Lambda$, $f'_\lambda(x) = f_\lambda(\frac{x}{\|x\|})(x \neq 0)$: $f(\frac{x}{\|x\|}) = \bigvee f_\lambda(\frac{x}{\|x\|}) \quad (x \in A^+ \setminus \{0\})$

This upper bound f is a positive sub-unital map by construction and can be extended to an upper bound $f : A \to B$ for the increasing net $(f_\lambda)_{\lambda\in\Lambda}$: for every nonzero $x \in A^+$, the increasing sequence $(f_\lambda(x))_{\lambda\in\Lambda} = (\|x\| \, f_\lambda(\frac{x}{\|x\|}))_{\lambda\in\Lambda}$ has a join $\bigvee f_\lambda(x) = \|x\| \bigvee f_\lambda(\frac{x}{\|x\|})$ in B^+ and thus one can define pointwise an upper bound $f : A \to B$ for $(f_\lambda)_{\lambda\in\Lambda}$ by $f(x) = \bigvee f_\lambda(x)$ for every $x \in A^+$.

We now need to prove that the map f is normal, by exchange of joins.
Let $(x_\gamma)_{\gamma\in\Gamma}$ be an increasing bounded net in A^+ with join $\bigvee_\gamma x_\gamma$. For every $\gamma' \in \Gamma$, we observe that $x_{\gamma'} \leq \bigvee_\gamma x_\gamma$ and thus, $f(x_\gamma) \leq f(\bigvee_\gamma x_\gamma)$ (recall that f preserves the order). As seen earlier, since $[0,1]_B$ is directed-complete, the increasing net $(f(x_\gamma))_{\gamma\in\Gamma}$, which is equal by definition to the increasing net $(\bigvee_\lambda(f_\lambda(x_\gamma)))_{\gamma\in\Gamma}$, has a join in B^+ defined by $\bigvee_\gamma f(x_\gamma) = \bigvee_{\gamma\in\Gamma, x_\gamma\neq 0} \|x_\gamma\| \, f(\frac{1}{\|x_\gamma\|} x_\gamma)$ if there is a $\gamma'' \in \Gamma$ such that $x_{\gamma''} \neq 0$ and by $\bigvee_\gamma f(x_\gamma) = 0$ otherwise. It follows that $\bigvee_\gamma f(x_\gamma) \leq f(\bigvee_\gamma x_\gamma)$.

We have to prove now that $f(\bigvee_\gamma x_\gamma) \leq \bigvee_\gamma f(x_\gamma)$. Since each map f_λ $(\lambda \in \Lambda)$ is normal, we obtain that $f(\bigvee_\gamma x_\gamma) = \bigvee_\lambda(f_\lambda(\bigvee_\gamma x_\gamma)) = \bigvee_\lambda(\bigvee_\gamma(f_\lambda(x_\gamma))$. Moreover, for $\gamma' \in \Gamma$ and $\lambda' \in \Lambda$, $f_{\lambda'}(x_{\gamma'}) \leq \bigvee_\lambda f_\lambda(x_{\gamma'}) \leq \bigvee_\gamma(\bigvee_\lambda f_\lambda(x_\gamma))$. Then, $\bigvee_\gamma f_{\lambda'}(x_\gamma) \leq \bigvee_\gamma(\bigvee_\lambda f_\lambda(x_\gamma))$ and thus $\bigvee_\lambda(\bigvee_\gamma f_\lambda(x_\gamma)) \leq \bigvee_\gamma(\bigvee_\lambda f_\lambda(x_\gamma))$. It follows that $f(\bigvee_\gamma x_\gamma) = \bigvee_\lambda(\bigvee_\gamma f_\lambda(x_\gamma)) \leq \bigvee_\gamma(\bigvee_\lambda f_\lambda(x_\gamma)) = \bigvee_\gamma f(x_\gamma)$.

Let $g \in \mathbf{W}^*(A,B)$ be an upper bound for the increasing net $(f_\lambda)_{\lambda\in\Lambda}$. For $\lambda' \in \Lambda$ and $x \in A^+$, $f_{\lambda'}(x) \leq g(x)$. Then, $\forall x \in A^+, f(x) = \bigvee f_\lambda(x) \leq g(x)$, i.e. $f \sqsubseteq g$. It follows that f is the join of $(f_\lambda)_{\lambda\in\Lambda}$. □

This theorem generalizes the fact that the effects of a W*-algebra A (i.e. the positive unital maps from $\mathbb{C}^2$ to A) form a directed-complete poset [15, III.3.13-16]. Moreover, it turns out that Theorem 2.2 can be slightly generalized to the following theorem, with a similar proof.

Theorem 2.7 *Let A and B be two C*-algebras.*
If $[0,1]_B$ is directed-complete, then the poset $(\mathbf{C}^*(A,B), \sqsubseteq)$ is directed-complete.

2.3 $\mathbf{Dcpo}_{\perp!}$*-enrichment for W*-algebras*

In this section, we will provide a $\mathbf{Dcpo}_{\perp!}$-enrichment for the category $\mathbf{W}^*$ and discuss the domain-theoretic properties of C*-algebras.

Definition 2.8 Let $\mathbf{C}$ be a category for which every hom-set is equipped with the structure of a poset. $\mathbf{C}$ is said to be $\mathbf{Dcpo}_{\perp!}$-enriched if its hom-sets are dcpos with bottom and if the composition of homomorphisms is strict and Scott-continuous, i.e. the pre-composition $(-) \circ f : \mathbf{C}(B,C) \to \mathbf{C}(A,C)$ and the post-composition $h \circ (-) : \mathbf{C}(A,B) \to \mathbf{C}(A,C)$ are strict and Scott-continuous for homomorphisms $f : A \to B$ and $h : B \to C$.

Theorem 2.9 *The category $\mathbf{W}^*$ is a $\mathbf{Dcpo}_{\perp!}$-enriched category.*

Proof.

For every pair (A,B) of W*-algebras, $\mathbf{W}^*(A,B)$ together with the Löwner order is a dcpo with zero map as bottom, and therefore $\mathbf{W}^*(A,B) \in \mathbf{Dcpo}_{\perp!}$.

In particular, for every W*-algebra A, $\mathbf{W}^*(A,A) \in \mathbf{Dcpo}_{\perp!}$. We consider now for every W*-algebra A a map $I_A : 1_A = \{\perp_A\} \to \mathbf{W}^*(A,A)$ such that $I_A(\perp) \in \mathbf{W}^*(A,A)$ is the identity map on A. The map I_A is clearly strict Scott-continuous for every W*-algebra A.

Then, what need to be proved is that, given three W*-algebras A, B, C, the composition $\circ_{A,B,C} : \mathbf{W}^*(B,C) \times \mathbf{W}^*(A,B) \to \mathbf{W}^*(A,C)$ is Scott-continuous (the strictness of the composition can be easily verified).

We now consider a NsU-map $f : A \to B$ and the increasing net $(g_\lambda)_{\lambda\in\Lambda}$ in $\mathbf{W}^*(B,C)$, with join $\bigvee_\lambda g_\lambda \in \mathbf{W}^*(B,C)$. One can define an upper bound pointwise by $u(x) = ((\bigvee_\lambda g_\lambda) \circ f)(x)$ for the increasing net $(g_\lambda \circ f)_{\lambda\in\Lambda}$ in $\mathbf{W}^*(A,C)$. It is easy to check that u is a join for the increasing net $(g_\lambda \circ f)_{\lambda\in\Lambda}$: for every upper bound $v \in \mathbf{W}^*(A,C)$ of the increasing net $(g_\lambda \circ f)_{\lambda\in\Lambda}$, we have that $\forall \lambda \in \Lambda, g_\lambda \circ f \sqsubseteq v$, i.e. $\forall \lambda \in \Lambda, \forall x \in A^+, g_\lambda(f(x)) \leq v(x)$ and thus $\forall x \in A^+, u(x) = ((\bigvee_\lambda g_\lambda) \circ f)(x) = (\bigvee_\lambda g_\lambda)(f(x)) \leq v(x)$, which implies that $u \sqsubseteq v$. It follows that the pre-composition is Scott-continuous and, similarly, the post-composition is Scott-continuous. □

In operator theory, a C*-algebra is monotone-complete (or monotone-closed) if it is directed-complete for bounded increasing nets of positive elements. The notion of monotone-completeness goes back at least to Dixmier [4] and Kadison [5] but, to our knowledge, it is the first time that the notion of monotone-completeness is explicitly related to the notion of directed-completeness. The interested reader will find in Appendix A a more detailed correspondence between operator theory and order theory.

It is natural to ask if all monotone-complete C*-algebras are W*-algebras. Dixmier proved that every W*-algebra is a monotone-complete C*-algebra and that the converse is not true [4]. For an example of a subclass of monotone-complete C*-algebras which are not W*-algebras, we refer the reader to a recent work by Saitô and Wright [12].

3 A fixpoint theorem for endofunctors on W*-algebras

In this section, we will show that it is possible to exhibit a fixpoint for a specific class of endofunctors on W*-algebras, that we will define later.

For this purpose, we first observe that the one-element W*-algebra $\mathbf{0} = \{0\}$ is the zero object, i.e. initial and terminal object, of the category $\mathbf{W}^*$: a NsU-map

$f : A \to \mathbf{0}$ must be defined by $f(x) = 0$; a NsU-map $g : \mathbf{0} \to A$ is linear and thus $g(0) = 0_A$ must holds.

We will now consider the following class of functors and then show that they admit a canonical fixpoint.

Definition 3.1 A von Neumann functor is a locally continuous endofunctor on $\mathbf{W}^*$ which preserves multiplicative maps.

Example 3.2 The identity functor and the constant functors on $\mathbf{W}^*$ are locally continuous and so does any (co)product of locally continuous functors. It is also clear that all those functors preserve multiplicative maps.

Secondly, our proofs and structures will use the notion of embedding-projection pairs, that we will define as follows.

Definition 3.3 An embedding-projection pair is a pair of arrows $\langle e, p\rangle \in \mathbf{C}(X, Y) \times \mathbf{C}(Y, X)$ in a $\mathbf{Dcpo}_\perp$-enriched category $\mathbf{C}$ such that $p \circ e = \mathrm{id}_X$ and $e \circ p \leq \mathrm{id}_Y$.

For two pairs $\langle e_1, p_1\rangle$, $\langle e_2, p_2\rangle$, it can be shown that $e_1 \leq e_2$ iff $p_2 \leq p_1$, which means that one component of the pair can uniquely determine the other one. We denote by e^P the projection corresponding to a given embedding e and p^E the embedding corresponding to a given projection p. It should be noted that $(e \circ f)^P = f^P \circ e^P$, $(p \circ q)^E = q^E \circ p^E$ and $\mathrm{id}^P = \mathrm{id}^E = \mathrm{id}$.

The category $\mathbf{C}^E$ of embeddings of a $\mathbf{Dcpo}_{\perp!}$-enriched category $\mathbf{C}$ is the subcategory of $\mathbf{C}$ that has objects of $\mathbf{C}$ has objects and embeddings as arrows. It should be noted that this category is itself a $\mathbf{Dcpo}_{\perp!}$-enriched category. Dually, one can define the category $\mathbf{C}^P = (\mathbf{C}^E)^{op}$ of projections of a $\mathbf{Dcpo}_{\perp!}$-enriched category $\mathbf{C}$.

An endofunctor F on a $\mathbf{Dcpo}_{\perp!}$-enriched category $\mathbf{C}$ is locally continuous (resp. locally monotone) if $F_{X,Y} : \mathbf{C}(X, Y) \to \mathbf{C}(FX, FY)$ is Scott-continuous (resp. monotone).

We can now consider the following setting. Let $F : \mathbf{W}^* \to \mathbf{W}^*$ be a von Neumann functor. Consider the ω-chain $\Delta = (D_n, \alpha_n)_n$ for which $D_0 = \mathbf{0}$, the embedding $\alpha_0 : D_0 \to FD_0$ is the unique NsU-map from D_0 to FD_0, and the equalities $\alpha_{n+1} = F\alpha_n$ and $D_{n+1} = FD_n$ hold for every $n \geq 0$.

Since the endofunctor F is locally monotone, if for some $n \in \mathbb{N}$ there is an embedding-projection pair $\left\langle \alpha_n^E, \alpha_n^P \right\rangle$, the pair $\left\langle \alpha_{n+1}^E, \alpha_{n+1}^P \right\rangle = \left\langle F\alpha_n^E, F\alpha_n^P \right\rangle$ is also an embedding-projection pair. It follows that the ω-chain Δ is well-defined.

Definition 3.4 Consider the collection $D = \{(x_n)_n \in \bigoplus_n D_n \mid \forall n \geq 0, \alpha_n^p(x_{n+1}) = x_n\}$. It forms a poset together with the order $\leq_D$ defined by $(x_n)_n \leq_D (y_n)_n \equiv \forall n \geq 0, x_n \leq_{D_n} y_n$. Moreover, the collection D can be seen as a *-algebra:

From the fact that the projection $\alpha_0^P : D_1 \to D_0 = \mathbf{0}$ is trivially a multiplicative map (which maps everything to the unique element of $\mathbf{0}$) and that the functor F preserves multiplicative maps, we can conclude that for every $n \geq 0$, the projection $\alpha_{n+1}^P = F\alpha_n^P = \cdots = F^{n+1}\alpha_0^P$ is a NMIsU-map. In fact, it can be shown that the embeddings $\alpha_n : D_n \to D_{n+1}$ are NMIsU-maps as well, by the same reasoning. Moreover, it should also be noted that embeddings and projections are strict, i.e.

preserves 0.

Considering these facts, one can verify that the collection D forms a *-algebra with operations defined component-wise on the family of W*-algebras $\{D_n\}_{n\geq 0}$.

- The unit is defined by $(1_n)_n = (1_{D_n})_n$. From the fact that the embeddings α_n^E are NsU-maps, i.e. $\alpha_n^E(1) \leq 1$ for every $n \in \mathbb{N}$, we deduce that $1 = (\alpha_n^P \circ \alpha_n^E)(1) \leq \alpha_n^P(1)$ for every $n \in \mathbb{N}$ (recall that the projection α_n^P is an order-preserving map). Hence, every projection α_n^P is a unital map and thus $\alpha_n^P(1_{n+1}) = \alpha_n^P(1_n)$ holds for every $n \in \mathbb{N}$.
- The addition is defined by $(x_n)_n +_D (y_n)_n = (x_n +_{D_n} y_n)_n$ for all $(x_n)_n, (y_n)_n \in D$. This operation is well-defined: since the projections α_n^P are linear maps, one can observe that $\alpha_n^P(0_{n+1}) = 0_n$ and $\alpha_n^P(x_{n+1} + y_{n+1}) = \alpha_n^P(x_{n+1}) + \alpha_n^P(y_{n+1}) = x_n + y_n$ for every $n \in \mathbb{N}$. Moreover, by the triangle inequality, $\sup_n \|x_n + y_n\| \leq \sup_n \|x_n\| + \sup_n \|y_n\| < \infty$.
- The scalar multiplication is defined by $\lambda(x_n)_n = (\lambda x_n)_n$ for every $\lambda \in \mathbb{C}$ and and every $(x_n)_n \in D$. It is easy to verify that this operation is well-defined: $\alpha_n^P(\lambda x_{n+1}) = \lambda\alpha_n^P(x_{n+1}) = \lambda x_n$ for every $n \in \mathbb{N}$ (by linearity of α_n^P) and $\|(\lambda x_n)_n\|_\infty = \lambda \|(x_n)_n\|_\infty < \infty$.
- The multiplication is defined by $(x_n)_n \cdot_D (y_n)_n = (x_n \cdot_{D_n} y_n)_n$ for all $(x_n)_n, (y_n)_n \in D$. This operation is well-defined since the projections α_n^P are multiplicatives: $\alpha_n^P(x_{n+1} \cdot y_{n+1}) = \alpha_n^P(x_{n+1}) \cdot \alpha_n^P(y_{n+1}) = x_n \cdot y_n$.
  Moreover, the Banach spaces D_n are submultiplicatives and thus the following inequality holds: $\sup_n \|x_n \cdot y_n\| \leq (\sup_n \|x_n\|)(\sup_n \|y_n\|) < \infty$.
- The involution is defined by $((x_n)_n)^* = (x_n^*)_n$ for every $(x_n)_n \in D$.
  This operation is well-defined since the projections α_n^P are involutives: $\alpha_n^P(x_{n+1}^*) = \alpha_n^P(x_{n+1})^* = x_n^*$. Moreover, as a direct consequence of the C*-identity of the C*-algebras D_n, the following equality holds: $\sup_n \|x_n^*\| = \sup_n \|x_n\| < \infty$.

Proposition 3.5 *The *-algebra D forms a C*-algebra.*

Proof. Since every D_n is a C*-algebra, the C*-identity holds for D as well: $\|(x_n)_n^*(x_n)_n\|_\infty = \|(x_n^* x_n)_n\|_\infty = \sup_n \|x_n^* x_n\| = \sup_n \|x_n\|^2 = (\|(x_n)_n\|_\infty)^2$

Consider a Cauchy sequence $((x_{m,n})_n)_m \in D$. It follows that for every $n' \in \mathbb{N}$, the following proposition holds: $\forall \varepsilon > 0, \exists M \in \mathbb{N}, \forall m, m' \geq M, \|x_{m,n'} - x_{m',n'}\| \leq \sup_n \|x_{m,n} - x_{m',n}\| < \varepsilon$.

We are required to prove that the Cauchy sequence $((x_{m,n})_n)_m \in D$ converges, by constructing its limit that we will denote by $(l_n)_n$. We can first deduce that, for every $n \in \mathbb{N}$, the sequence $(x_{m,n})_m$ in D_n is a Cauchy sequence, which converges to a limit $l_n \in D_n$ since D_n is a W*-algebra (and therefore a Banach space). Then, we obtain a sequence $(l_n)_n \in \prod_{n\in\mathbb{N}} D_n$. Since PsU-maps (and therefore NMIsU-maps) are contractive, we can conclude that $(l_n)_n \in D$ by the following arguments.

Let $n \in \mathbb{N}$ and $\varepsilon > 0$. From the fact that the inequality $\|x_{m,n+1} - l_{n+1}\| < \varepsilon$ holds, we deduce that $\|\alpha_n^P(x_{m,n+1}) - \alpha_n^P(l_{n+1})\| = \|\alpha_n^P(x_{m,n+1} - l_{n+1})\| \leq \|x_{m,n+1} - l_{n+1}\| < \varepsilon$ and thus, $l_n = \lim_{m\to\infty} x_{m,n} = \lim_{m\to\infty} \alpha_n^P(x_{m,n+1}) =$

$\alpha_n^P(l_{n+1})$. Moreover, $\|(l_n)_n\|_\infty = \sup_n \|l_n\| < \infty$ since l_n is the limit of a Cauchy sequence (recall that every Cauchy sequence is bounded). □

This leads us to the following new result.

Theorem 3.6 *The category $\mathbf{W}^*$ is algebraically compact for the class of von Neumann functors, i.e. every von Neumann functor F admits a canonical fixpoint and there is an isomorphism between the initial F-algebra and the inverse of the final F-coalgebra.*

The proof of this theorem involves the following notions.

Definition 3.7 An ω-chain in a category $\mathbf{C}$ is a sequence of the form $\Delta = D_0 \xrightarrow{\alpha_0} D_1 \xrightarrow{\alpha_1} \cdots$

Given an object D in a category $\mathbf{C}$, a cocone $\mu : \Delta \to D$ for the ω-chain Δ is a sequence of arrows $\mu_n : D_n \to D$ such that the equality $\mu_n = \mu_{n+1} \circ \alpha_n$ holds for every $n \geq 0$.

A colimit (or colimiting cocone) of the ω-chain Δ is an initial cocone from Δ to D, i.e. it has the following universal property: for every cocone $\mu' : \Delta \to D'$, there exists a unique map $f : D \to D'$ such that the equality $f \circ \mu_n = \mu'_n$ holds for every $n \geq 0$.

Dually, we will consider ω^{op}-chains $\Delta^{\mathrm{op}} = D_0 \xleftarrow{\beta_0} D_1 \leftarrow \cdots$ in a category, cones $\gamma : \Delta^{\mathrm{op}} \leftarrow D$ and limits (or limiting cones) for an ω^{op}-chain Δ^{op}.

Proof. [Proof of Theorem 3.6] Together with its previously defined order, the C*-algebra D is monotone-complete, since all the W*-algebras D_n are monotone-complete. Moreover, a separating set of normal states can be defined for D, on the separating set of normal states of the W*-algebras D_n. We can then conclude by Theorem A.4 that D forms a W*-algebra and we are now required to prove that it can be turn into a colimit for the diagram Δ.

We now define a cocone $\Delta \to D$ which arrows are embeddings $\mu_n : D_n \to D(n \geq 0)$ defined by:

- $\mu_n(x) = ((\alpha_0^P \circ \cdots \circ \alpha_{n-1}^P)(x), (\alpha_1^P \circ \cdots \circ \alpha_{n-1}^P)(x), \ldots, \alpha_{n-1}^P(x), x, \alpha_n^E(x), (\alpha_{n+1}^E \circ \alpha_n^E)(x), \ldots)$ for every $x \in D_n$. It is easy to check that for every $m \in \mathbb{N}$ if we define a sequence $(y_n)_n = \mu_m(x)$, then the equation $\alpha_n^P(y_{n+1}) = y_n$ holds for every $n \in \mathbb{N}$. Moreover, as a positive map, the embedding μ_n is contractive and thus $\|\mu_n(x)\|_\infty \leq \sup_n \|x\| = \|x\|_\infty < \infty$.
- $\mu_n^P((x_n)_n) = x_n$ for every $(x_n)_n \in D$.

Indeed, it is easy to check that those projections μ_n^P are NMIU-maps by construction and that the corresponding embeddings μ_n^E are NMIsU-maps by construction.

Then, one can see that $\mu_n^P(\mu_n^E(x)) = x$ for every $x \in D_n$ and that $\mu_n^E(\mu_n^P((x_n)_n)) \leq (x_n)_n$ for every $(x_n)_n \in D$ since:

(i) For every $m \in \mathbb{N}$ such that $0 \leq m < n$, $(\alpha_m^P \circ \cdots \circ \alpha_{n-1}^P)^E(x_m) = (\alpha_{n-1}^E \circ \cdots \circ \alpha_m^E)(x_m) = x_n$, which implies that $x_m = (\alpha_m^P \circ \cdots \circ \alpha_{n-1}^P)((\alpha_m^P \circ \cdots \circ \alpha_{n-1}^P)^E(x_m)) = (\alpha_m^P \circ \cdots \circ \alpha_{n-1}^P)(x_n)$ and thus $((\mu_n^E \circ \mu_n^P)((x_n)_n))_m = x_m$ for every $m \leq n$;

(ii) From the fact that $\alpha_n^E \circ \alpha_n^P \leq \mathrm{id}_{D_{n+1}}$ for every $n \in \mathbb{N}$, we obtain that $\alpha_n^E(x_n) = \alpha_n^E(\alpha_n^P(x_{n+1})) \leq x_{n+1}$ for every $n \in \mathbb{N}$ and thus by induction, $\alpha_m^E \circ \cdots \circ \alpha_{n+1}^E \circ \alpha_n^E(x_n) \leq \alpha_m^E \circ \cdots \circ \alpha_{n+1}^E(x_{n+1}) \leq \cdots \leq x_m$ for every $m \geq n$. Thus, $((\mu_n^E \circ \mu_n^P)((x_n)_n))_m \leq x_m$ for every $m \geq n$.

Moreover, for every $n \geq 0$, we observe that $\mu_n = \mu_{n+1} \circ \alpha_n$ since $\alpha_n^P(\mu_{n+1}^P((x_n)_n)) = \alpha_n^P(x_{n+1}) = x_n = \mu_n^P((x_n)_n)$ and thus $\mu_n^P = \alpha_n^P \circ \mu_{n+1}^P = (\mu_{n+1} \circ \alpha_n)^P$.

As stated in [9], the fact that F is locally continuous implies that $\bigvee_n(\mu_n \circ \mu_n^P) = \mathrm{id}_D$, and thus $\mu : \Delta \to D$ is a colimiting cocone for Δ by [14, Theorem 2,Proposition A]. Dually, one can show that $\mu^P : D \to \Delta^P$ is a limiting cone for Δ^P, the cone of projections $D_0 \xleftarrow{\alpha_0^P} D_1 \xleftarrow{\alpha_1^P} \cdots$.

Since F is locally continuous, it is therefore locally monotone. It follows that :

- For every $n \in \mathbb{N}$, $\langle F\mu_n, F\mu_n^P \rangle$ is an embedding-projection pair;
- The chain $\{F\mu_n \circ F\mu_n^P\}_n$ is increasing with join $\bigvee_n(F\mu_n \circ F\mu_n^P) = \mathrm{id}_{FD}$.

From [14, Theorem 2] again, we conclude that $F\mu : F\Delta \to FD$ is a colimiting cocone (and dually $F\mu^P : FD \to F\Delta^P$ is a limiting cone). Then, we observe that $F\Delta$ is obtained by removing the first arrow from Δ (recall that $F\alpha_n = \alpha_{n+1}$). Finally, the fact that two colimiting cocone with the same vertices are isomorphic implies that D and FD share the same limit and the same colimit and that there is an isomorphism $\phi : D \to FD$, i.e. the functor F admits a fixpoint. □

We will now consider as example the construction of the natural numbers.

Example 3.8 The functor defined by $FX = X \oplus \mathbb{C}$ gives the chain of embeddings $\mathbf{0} \to \mathbb{C} \to \mathbb{C}^2 \to \mathbb{C}^3 \to \cdots$, where $\mathbb{C}^n$ is the direct sum of n copies of $\mathbb{C}$. The relation $\alpha_{n+1}^E = \alpha_n^E \oplus \mathrm{id}_{\mathbb{C}}$ holds for every $n \in \mathbb{N}$ and thus by induction, $\alpha_n^E = \alpha_0^E \oplus \mathrm{id}_{\mathbb{C}^n}$. Hence, for every $n \in \mathbb{N}$, $\alpha_n^E : \mathbb{C}^n = \mathbf{0} \oplus \mathbb{C}^n \to \mathbb{C}^{n+1}$ is defined by $\alpha_n^E(c_1, \ldots, c_n) = (0, c_1, \ldots, c_n)$.

Similarly, for every $n \in \mathbb{N}$, $\alpha_n^P = \alpha_0^P \oplus \mathrm{id}_{\mathbb{C}^n} : \mathbb{C}^{n+1} \to \mathbb{C}^n$ is defined by $\alpha_n^P(c_1, c_2, \ldots, c_n) = (c_2, \ldots, c_n)$ and thus the property $\alpha_n^P(x_{n+1}) = x_n$ holds for every $(x_n)_n \in \bigoplus_{i \geq 1} \mathbb{C}^i = \bigoplus_{i \geq 0} \mathbb{C}$.

It follows that $D = \bigoplus_{i \geq 0} \mathbb{C} = \ell^\infty(\mathbb{N})$ for this functor. More generally, if one consider a functor $FX = X \oplus A$ where A is a W*-algebra, then $D = \bigoplus_{i \geq 0} A$.

4 Streams of qubits

We will now consider the functor $FX = (X \overline{\otimes} A) \oplus \mathbb{C}$ to represent the construction of a list of unbounded length whose elements are in a W*-algebra A. It should be noted that in this setting, the functor $FX = (X \overline{\otimes} M_2) \oplus \mathbb{C}$ represent the construction of a list of unbounded length whose elements are qubits.

The functor defined by $FX = (X \overline{\otimes} A) \oplus \mathbb{C}$ gives the chain of embeddings $\mathbf{0} \to \mathbb{C} \to A \oplus \mathbb{C} \to 2 \cdot A \oplus A \oplus \mathbb{C} \to \cdots$ where $n \cdot A$ denotes the spatial tensor of n

copies of A. Assume that this functor has a canonical fixpoint D (this point will be discussed in the next subsection).

From the relation $\alpha^E_{n+1} = (\alpha^E_n \otimes \mathrm{id}_A) \oplus \mathrm{id}_{\mathbb{C}} (n \in \mathbb{N})$, we obtain by induction that $\alpha^E_n = (\alpha^E_0 \otimes \mathrm{id}_{n \cdot A}) \oplus \mathrm{id}_{(n-1)\cdot A} \oplus \cdots \oplus \mathrm{id}_A \oplus \mathrm{id}_{\mathbb{C}}$ for every $n \in \mathbb{N}$. It follows that an embedding $\alpha^E_n : n \cdot A \oplus \cdots \oplus A \oplus \mathbb{C} \to (n+1) \cdot A \oplus \cdots \oplus A \oplus \mathbb{C}$ $(n \in \mathbb{N})$ is defined by $\alpha^E_n(\langle a^n_1, \ldots, a^n_n \rangle, \ldots, \langle a^1_1 \rangle, x) = (\langle 0, a^{n-1}_1, \ldots, a^{n-1}_n \rangle, \langle a^{n-1}_1, \ldots, a^{n-1}_n \rangle, \ldots, \langle a^1_1 \rangle, x)$.

It is clear that the corresponding projection is $\pi_{2,(n+1)\cdot A \oplus \cdots \oplus A \oplus \mathbb{C}}$ and thus $D = \bigoplus_{i \geq 0} i \cdot A$ (where, by convention, we denote $\mathbb{C}$ by $0 \cdot A$).

Remark 4.1 It is well known that $\overline{\bigotimes}_{i\geq 1} A$ is the colimit of the (trivial) diagram $A \xrightarrow{-\overline{\otimes} A} A \overline{\otimes} A \to \cdots$ in $\mathbf{W^*}_{\mathrm{NMIU}}$, the category of W*-algebras together with NMIU-maps. However in our framework, the functor $F = -\overline{\otimes} A$ is associated to the diagram $\mathbf{0} \to \mathbf{0}\overline{\otimes} A = \mathbf{0} \to \mathbf{0}\overline{\otimes} A = \mathbf{0} \to \cdots$.

4.1 Remarks about complete positivity

Unfortunately, the functor $FX = (X \overline{\otimes} A) \oplus \mathbb{C}$ does not preserve NsU-maps. However, one might consider restricting to NcPsU-maps to consider such functor. There is a $\mathbf{Dcpo}_{\perp !}$-enrichment for the category $\mathbf{W^*}_{\mathrm{cP}}$ of W*-algebras together with NcPsU-maps, investigated independently by Cho [2], who proposed the following variation of the Löwner order :

$f \sqsubseteq_{cP} g$ if and only if $g - f$ is completely positive, i.e. $\forall n. \forall x. \mathcal{M}_n(f)(x) \leq \mathcal{M}_n(g)(x)$.

In fact, the following proposition shows that our domain-theoretic structure do not change if one restricts to completely-positive maps.

Proposition 4.2 *Let f and g be two NsU-maps from a W*-algebra A to a W*-algebra B. If f and g are completely positive maps, then the relation $f \sqsubseteq_{cP} g$ holds if and only if the relation $f \sqsubseteq g$ (Definition 2.1) holds.*

Proof. If $g - f$ is completely positive with f and g completely positive (i.e. $f \sqsubseteq_{cP} g$), it is therefore positive and thus it is clear that $f \sqsubseteq_{cP} g$ implies $f \sqsubseteq g$.

Conversely, we will now show that $f \sqsubseteq g$ implies $f \sqsubseteq_{cP} g$ when f and g are completely positive maps. It is equivalent to show that if f and $f + g$ are completely positive maps, then g is positive implies that g is completely positive.

By the Hahn-Banach theorem, if we consider $P = \mathbf{W}^*(A, B)$ as a normed vector space (defined pointwise) and $cP = \mathbf{W^*}_{\mathrm{cP}}(A, B)$ as a linear subspace of P and if we consider an element $z \notin P \setminus \mathrm{span}(cP)$, then there is a (continuous) linear map $\varphi : P \to \mathbb{R}$ with $\varphi(x) = 0$ for every $x \in cP$ and $\varphi(z) = 1$.

We will now apply this fact. If the map g is just positive and not completely positive, we obtain that $\varphi(g) = 1$ and therefore $\varphi(f + g) = \varphi(f) + \varphi(g) = 0 + 1 = 1$ by linearity. But this is absurd since, by assumption, the map $f + g$ is completely positive, and thus $\varphi(f + g) = 0$. It follows that g is completely positive. □

Then, it is easy to see that every directed join of completely positive maps is completely positive map as well (using the fact that $\mathcal{M}(\bigvee_i f_i) = \bigvee_i \mathcal{M}(f_i)$ for every direct set of completely positive maps $\{f_i\}_i$).

Moreover, as shown in [7], the direct sums and the spatial tensor products of W*-algebras can be turned into endofunctors $-\oplus- : \mathbf{W^*}_{\mathrm{cP}} \times \mathbf{W^*}_{\mathrm{cP}} \to \mathbf{W^*}_{\mathrm{cP}}$ and $-\overline{\otimes}- : \mathbf{W^*}_{\mathrm{cP}} \times \mathbf{W^*}_{\mathrm{cP}} \to \mathbf{W^*}_{\mathrm{cP}}$, which are von Neumann functors.

Concluding remarks

The theorem 2.2 provides a $\mathbf{Dcpo}_{\perp!}$-enrichment for the category $\mathbf{W}^*$ of W*-algebras with NsU-maps, while the theorem 3.6 gives a canonical fixpoint for every multiplicative map-preservering locally continuous endofunctor on $\mathbf{W}^*$. We believe that these two theorems are encouraging enough to consider further investigations of the semantics of quantum computation, using W*-algebras.

Acknowledgment

The author would like to thank Bart Jacobs, Sam Staton, Robert Furber, Bas Westerbaan, and Sander Uijlen for helpful discussions. This research has been financially supported by the European Research Council (ERC) under the QCLS grant (Quantum Computation, Logic & Security).

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A Correspondence between operator theory and order theory

In this section, we will provide the following correspondence table between operator theory and order theory, where A and B are C*-algebras.

Operator Theory	Order theory	Reference
A monotone-closed	$[0,1]_A$ directed-complete	A.2
$f : A \to B$ NsU-map	$f : [0,1]_A \to [0,1]_B$ Scott-continuous PsU-map	A.3
A W*-algebra	$[0,1]_A$ dcpo with a separating set of normal states	A.4

In the standard litterature [1,15], monotone-closed C*-algebras and normal maps are defined as follows.

Definition A.1 A C*-algebra A is monotone-closed (or monotone-complete) if every bounded increasing net of positive elements of A has a join in A^+.

A positive map $\phi : A \to B$ between C*-algebras is normal (or a N-map) if every increasing net $(x_\lambda)_{\lambda\in\Lambda}$ in A^+ with a join $\bigvee x_\lambda \in A^+$ is such that the net $(\phi(x_\lambda))_{\lambda\in\Lambda}$ is an increasing net in B^+ with join $\bigvee \phi(x_\lambda) = \phi(\bigvee x_\lambda)$.

In the standard definition of the notion of monotone-closedness, the increasing nets are not required to be bounded by the unit, like in the definitions we used in this thesis. We will now show that we can assume that the upper bound is the unit, without loss of generality.

Proposition A.2 *A C*-algebra A is monotone-closed if and only if the poset $([0,1]_A, \leq)$ is directed-complete.*

Proof. Let A be a C*-algebra.

If A is monotone-closed, then, by definition every increasing net of positive elements bounded by 1 has a join in $[0,1]_A$ and therefore, the poset $([0,1]_A, \leq)$ is directed-complete.

Conversely, suppose that $[0,1]_A$ is directed-complete. We now consider an increasing net of positive elements $(a_\lambda)_{\lambda\in\Lambda}$ in A^+, bounded by a nonzero positive element $b \in A^+$. Then, it restricts to an increasing net $(\frac{a_\lambda}{\|b\|})_{\lambda\in\Lambda}$ in $[0,1]_A$ since $b \leq \|b\| \cdot 1$. By assumption, the increasing net $(\frac{a_\lambda}{\|b\|})_{\lambda\in\Lambda}$ has a join $\bigvee_\lambda \frac{a_\lambda}{\|b\|} \in [0,1]_A$ and thus $\|b\| \bigvee_\lambda \frac{a_\lambda}{\|b\|}$ is an upper bound for $(a_\lambda)_{\lambda\in\Lambda}$.

Let $c \in A^+$ be an upper bound for the increasing net $(a_\lambda)_{\lambda\in\Lambda}$ such that $c \leq b$. For every $\lambda' \in \Lambda$, $a_{\lambda'} \leq c \leq b \leq \|b\| \cdot 1$ and thus $\frac{c}{\|b\|}$ is an upper bound for the increasing net $(\frac{a_\lambda}{\|b\|})_{\lambda\in\Lambda}$. It follows that $\bigvee_\lambda \frac{a_\lambda}{\|b\|} \leq \frac{c}{\|b\|}$ and therefore, $\|b\| \bigvee_\lambda \frac{a_\lambda}{\|b\|} \leq c$. Thus, $\|b\| \bigvee_\lambda \frac{a_\lambda}{\|b\|}$ is the join of the increasing net $(a_\lambda)_{\lambda\in\Lambda}$ bounded by b and we can conclude that A is monotone-closed. □

In this thesis, we have chosen to use the standard definition of normal maps. However, one can say that a PsU-map is normal if its restriction $f : [0,1]_A \to [0,1]_B$ is Scott-continuous.

Proposition A.3 *A PsU-map $f : A \to B$ between C*-algebras is normal if and only if its restriction $f : [0,1]_A \to [0,1]_B$ is Scott-continuous.*

Proof. Let $f : A \to B$ be a positive map between two C*-algebras A and B.

If f is normal, then by definition every increasing net $(x_\lambda)_{\lambda\in\Lambda}$ in $[0,1]_A \subseteq A^+$ with join $\bigvee x_\lambda \in [0,1]_A$ is such that the net $(f(x_\lambda))_{\lambda\in\Lambda}$ is an increasing net in $[0,1]_B \subseteq \mathcal{B}^+$ with join $\bigvee f(x_\lambda) = f(\bigvee x_\lambda) \in [0,1]_B$. That is to say, the restriction $f : [0,1]_A \to [0,1]_B$ is Scott-continuous.

Conversely, suppose that the restriction $f : [0,1]_A \to [0,1]_B$ is Scott-continuous. Let $(x_\lambda)_{\lambda\in\Lambda}$ be an increasing net in A^+ with a nonzero join $y \in A^+$. Since $y \leq \|y\| \cdot 1$, it restricts to an increasing net $(\frac{x_\lambda}{\|y\|})_{\lambda\in\Lambda}$ in $[0,1]_A$ with a join $\frac{y}{\|y\|}$. From the Scott-continuity of $f : [0,1]_A \to [0,1]_B$, we deduce that the net $(f(\frac{x_\lambda}{\|y\|}))_{\lambda\in\Lambda}$ is an increasing net in $[0,1]_B$ with join $\bigvee f(\frac{x_\lambda}{\|y\|}) = f(\frac{y}{\|y\|}) \in [0,1]_B$. It follows that the net $(f(x_\lambda))_{\lambda\in\Lambda}$, which is equal to $(\|y\|\, f(\frac{x_\lambda}{\|y\|}))_{\lambda\in\Lambda}$ by linearity, is an increasing net in B^+ with an upper bound $\|y\| \bigvee f(\frac{x_\lambda}{\|y\|}) = f(\|y\|\, \frac{y}{\|y\|}) = f(y) \in B^+$.

Suppose that $z \in B^+$ is an upper bound for the increasing net $(f(x_\lambda))_{\lambda\in\Lambda}$. From the fact that $f(x_{\lambda'}) \leq z$ and therefore $f(\frac{x_{\lambda'}}{\|y\|}) = \frac{f(x_{\lambda'})}{\|y\|} \leq \frac{z}{\|y\|}$ for every $\lambda' \in \Lambda$, we obtain that $f(\frac{y}{\|y\|}) \leq \frac{z}{\|y\|}$ and thus $f(y) \leq z$. It follows that $f(y)$ is the join of the increasing net $(f(x_\lambda))_{\lambda\in\Lambda}$. Hence, we can conclude that the map f is normal. □

It is known that a C*-algebra A is a W*-algebra if and only if it is monotone-complete and admits sufficiently many normal states, i.e. the set of normal states of A separates the points of A, see [15, Theorem 3.16]. By combining this fact and Proposition A.2, one can provide an order-theoretic characterization of W*-algebras, as in the following theorem.

Theorem A.4 *Let A be a C*-algebra.*

Then A is a W-algebra if and only if its set of effects $[0,1]_A$ is directed-complete with a separating set of normal states (i.e. $\forall x \in A, \exists f \in \mathbf{W}^*(A,[0,1]_{\mathbb{C}}), f(x) \neq 0$).*

The proof will be postponed until after the following theorem, which can be found in [15], and the following lemma.

Theorem A.5 *Every C*-algebra A admits a faithful (i.e. injective) representation, i.e. an injective *-homomorphism $\pi : A \to \mathcal{B}(H)$ for some Hilbert space H. A C*-algebra A is a W*-algebra if and only if there is a faithful representation $\pi : A \to \mathcal{B}(H)$, for some Hilbert space H, such that $\pi(A)$ is a strongly-closed subalgebra of $\mathcal{B}(H)$.*

Lemma A.6 *For each W*-algebra A, there is an isomorphism $A \simeq \mathrm{span}(\mathcal{NS}(A))'$, where $\mathcal{NS}(A) = \mathbf{W}^*(A, [0,1])$ is the collection of normal states of A.*

Proof. We now consider the map $\zeta_X : X \to X''$ defined by $\zeta_X(x)(\phi) = \phi(x)$ for $x \in X$ and $\phi \in X'$. Let A be a W*-algebra. We observe that $\zeta_{A_*} : A_* \to A'$ is a "canonical embedding" of A_* into A' and it can be proved that A_* is a linear subspace of A' generated by the normal states of A, i.e. $\zeta_{A_*}(A_*) = \mathrm{span}(\mathcal{NS}(A))$, see the proof of [11, Theorem 1.13.2]. Then, we can now consider the induced surjection $\zeta_{A_*} : A_* \to \mathrm{span}(\mathcal{NS}(A))$, which turns out to be injective (and thus bijective): for every pair $(x, y) \in A_* \times A_*$ such that $x \neq y$, there is a $f \in \mathcal{NS}(A)$ such that $\zeta_{A_*}(x)(f) = f(x) \neq f(y) = \zeta_{A_*}(y)(f)$, which implies that $\zeta_{A_*}(x) \neq \zeta_{A_*}(y)$.

Then for every W*-algebra, from $A_* \simeq \mathrm{span}(\mathcal{NS}(A))$ for every W*-algebra A, we obtain that $A = (A_*)' \simeq \mathrm{span}(\mathcal{NS}(A))'$.

□

Proof. [Proof of Theorem A.4] Let A be a C*-algebra.

Suppose that A is a W*-algebra. Then, by Corollary 2.5, $[0,1]_A$ is a dcpo and thus A is monotone-complete by Proposition A.2. Moreover, we know by Lemma A.6 that there is an isomorphism $\zeta_A : A \to \mathrm{span}(\mathcal{NS}(A))'$ defined by $\zeta_A(a)(\varphi) = \varphi(a)$ for $a \in A$ and $\varphi \in A'$. Therefore, ζ_A is injective and thus for every pair (x, y) of distinct elements of A, $\zeta_A(x) \neq \zeta_A(y)$, which means that there is a $\varphi \in \mathcal{NS}(A)$ such that $\varphi(x) = \zeta_A(x)(\varphi) \neq \zeta_A(y)(\varphi) = \varphi(y)$. It follows that the set $\mathcal{NS}(A)$ is a separating set for A.

Conversely, suppose that A is monotone-closed and admits its normal states as a separating set.

There is a representation $\pi : A \to B(H)$, for some Hilbert space H, induced by the normal states on A, by the Gelfand-Naimark-Segal (GNS) construction [15, Theorem I.9.14, Definition I.9.15]:

- Every normal state ω on A induces a representation $\pi_\omega : A \to \mathcal{B}(H_\omega)$ such that there is a vector ξ_ω such that $\omega(x) = \langle \pi_\omega(x)\xi_\omega | \xi_\omega \rangle$ for every $x \in A$
- We define a Hilbert space H, which is the direct sum of the Hilbert spaces H_ω, where ω is a normal state on A.
- The representation $\pi : A \to \mathcal{B}(H)$ is defined pointwise for every $x \in A$: $\pi(x)$ is the bounded operator on H defined as the direct sum of the bounded operators $\pi_\omega(x)$ on H_ω, where ω is a normal state on A.

By assumption, the set of normal states of A is a separating set for A and thus, for every pair of distincts elements x, y in A, there is a state ρ on A such that $\langle \pi_\rho(x)\xi_\rho | \xi_\rho \rangle = \rho(x) \neq \rho(y) = \langle \pi_\rho(y)\xi_\rho | \xi_\rho \rangle$ and thus $\pi_\rho(x) \neq \pi_\rho(y)$ for some state ρ on A. It follows that $\pi(x) \neq \pi(y)$ and hence, the representation π is faithful.

Let ρ be a normal state on A. Since A is monotone-closed, every directed set $(\rho(x_\lambda))_{\lambda \in \Lambda}$ in $\mathcal{B}(H)$ has a join $\bigvee_\lambda \rho(x_\lambda) = \rho(\bigvee_\lambda x_\lambda)$. According to the definition we gave earlier of π_ρ, this imply that $\pi_\rho(x_\lambda)$ converges weakly to $\bigvee_\lambda \pi_\rho(x_\lambda)$. Since a bounded net of positive operators converges strongly whenever it converges weakly (see [1, I.3.2.8]), it turns out that $\bigvee_\lambda \pi_\rho(x_\lambda)$ is the strong limit of $(\pi_\rho(x_\lambda))_{\lambda \in \Lambda}$ in

$\mathcal{B}(H_\rho)$. Hence, the strong limit of $(\pi(x_\lambda))_{\lambda\in\Lambda}$ in $\mathcal{B}(H)$ exists in $\mathcal{B}(H)$ and is defined as the direct sum of the strong limit of the nets $(\pi_\omega(x_\lambda))_{\lambda\in\Lambda}$ where ω is a normal state on A. Thus, $\pi(A)$ is strongly closed in $\mathcal{B}(H)$ and thus A is a W*-algebra (by Theorem A.5). □

It is important to note that, in one of the very first articles about W*-algebras [5], Kadison defined W*-algebras as monotone-closed C*-algebras which separates the points. However, to our knowledge, this definition never became standard.