Available online at www.sciencedirect.com
ScienceDirect

Electronic Notes in Theoretical Computer Science 167 (2007) 3–15

Electronic Notes in Theoretical Computer Science

www.elsevier.com/locate/entcs

Computability and the Implicit Function Theorem

Timothy H. McNicholl[1]

Department of Mathematics
Lamar University
Beaumont, Texas USA

Abstract

We prove computable versions of the Implicit Function Theorem in the single and multivariable cases. We use Type Two Effectivity as our foundation.

Keywords: Computable Analysis, Computability.

1 Introduction

The Implicit Function Theorem guarantees, under certain conditions, the existence of unique local continuous functional solutions to equations of the form

(1) $\phi(x, y) = 0$

given an initial condition of the form

(2) $y(a) = b$.

Under a surprisingly weak assumption, namely the differentiability of ϕ, the differentiability of the solution is also guaranteed. A very simple application of this, encountered by most single-variable Calculus students when they learn to use implicit differentiation to calculate tangent lines to curves (although they are not aware of the statement of the Implicit Function Theorem as it requires notation from multivariable calculus), is given by the equation

[1] Email: mcnichollth@my.lamar.edu

doi:10.1016/j.entcs.2006.08.004

(3) $x^2 + y^2 = 1$

and the initial condition

(4) $y(\frac{1}{\sqrt{2}}) = \frac{1}{\sqrt{2}}$.

Once granted the assumption that there *is* a differentiable functional solution to (3) that satisfies (4) on an open interval containing $\frac{1}{\sqrt{2}}$, one can verify through direct calculation that the derivative of the solution satisfies

$$y' = -\frac{x}{y}.$$

Then, using the initial condition, we can determine that $y'(\frac{1}{\sqrt{2}}) = -1$.

The Implicit Function Theorem also has important applications to differential equations, numerical analysis, and geometric analysis. A thorough discussion of the Implicit Function Theorem, its many variations, and its applications may be found in [2].

Here we state and prove a computable version of the Implicit Function Theorem in its single variable case, which is what we have just broadly described, and in its multivariable case. We use Type Two Effectivity theory as developed in Weihrauch [5] as our foundation. The reasoning for the multivariable case builds on that for the single-variable case. Hence, even though the multivariable case implies the single-variable case, we present both arguments. Our goal is to show that, in general terms, if ϕ, a, and b are computable, then the unique continuous functional solution to (1) that satisfies (2) is computable. In addition, we show that if ϕ has a computable derivative, then this solution has a computable derivative. We also prove uniform versions of these results.

Unless otherwise mentioned, all computability notation is as in Soare [4]. Unless otherwise mentioned, all computable analysis notation is as in Weihrauch [5].

We define a few notations and helpful conventions. First, we write $f :\subseteq A \to B$ if $dom(f) \subseteq A$ and $ran(f) \subseteq B$. If f is a function and X is a set, then

$$f[X] =_{df} \{y \mid \exists x \in dom(f) \cap X \ f(x) = y\}.$$

We note that X is not required to be a subset of the domain of f. Unless otherwise mentioned, the following conventions are followed for sake of brevity.

(i) A *computable real number* is a ρ-computable real number.

(ii) A point $a \in \mathbb{R}^n$ is *computable* if and only if it is ρ^n-computable.

(iii) A function $\phi :\subseteq \mathbb{R}^n \to \mathbb{R}^m$ is *computable* if it is (ρ^n, ρ^m)-computable.

(iv) A finite interval is *computable* if its endpoints are computable. An interval of the form $(-\infty, a)$, $(-\infty, a]$, (a, ∞), or $[a, \infty)$ is *computable* if a is computable.

(v) $C(U)$ is the set of all continuous functions from U into U.

(vi) A function $F :\subseteq C(U) \to \mathbb{R}$ is *computable* if it is (δ_{co}^U, ρ)-computable. That is, if there is a computable function $H :\subseteq \Sigma^\omega \to \Sigma^\omega$ such that $F \circ \delta_{co}^U = \rho \circ H$.

(vii) A function $F :\subseteq C(U) \times \mathbb{R} \to \mathbb{R}$ is *computable* if it is $(\delta_{co}^U, \rho, \rho)$-computable. That is, if there is a computable function $H :\subseteq \Sigma^\omega \to \Sigma^\omega$ such that $F \circ [\delta_{co}^U, \rho] = \rho \circ H$.

(viii) A function $F :\subseteq \mathbb{R} \to C(U)$ is *computable* if it is (ρ, δ_{co}^U)-computable.

2 Single-variable case

Theorem 2.1 (Single-Variable Implicit Function Theorem) *Suppose $E \subseteq \mathbb{R}^2$ is open. Suppose $\phi : E \to \mathbb{R}$ and $\frac{\partial \phi}{\partial y}$ are continuous. Suppose $\phi(a, b) = 0$ and $\frac{\partial \phi}{\partial y}(a, b) \neq 0$. Then, there exist open intervals U, V with $a \in U$ and $b \in V$ such that there exists a unique $f : U \to V$ such that $\phi(x, f(x)) = 0$ for all $x \in U$ and $f(a) = b$. Furthermore, f is continuous.*

We prove the following computable version of this theorem.

Theorem 2.2 (Computable Single-Variable Implicit Function Theorem) *Suppose $E \subseteq \mathbb{R}^2$ is open. Suppose $\phi : E \to \mathbb{R}$ is computable and $\frac{\partial \phi}{\partial y}$ is continuous. Suppose $a, b \in \mathbb{R}$ are computable, $\phi(a, b) = 0$, and $\frac{\partial \phi}{\partial y}(a, b)$ is a non-zero computable number. Then, there exist computable open intervals $U, V \subseteq \mathbb{R}$ with $a \in U$ and $b \in V$ such that there exists a unique $f : U \to V$ such that $\phi(x, f(x)) = 0$ for all $x \in U$ and $f(a) = b$. Furthermore, f is computable.*

We give two proofs of Theorem 2.2. The first is a simple, non-uniform proof.

First proof of Theorem 2.2 Let U, V, and f be as given by Theorem 2.1. Let V' be an open interval such that $V' \subseteq V$, $b \in V'$, and the endpoints of V' are rational. By the continuity of f, there is an open interval U' such that $U' \subseteq U$, $a \in U'$, the endpoints of U' are rational, and $f[U'] \subseteq V'$. Let g be the restriction of f to U'. It follows that for each $x \in U'$, $g(x)$ is the unique number in V' such that $\phi(x, g(x)) = 0$. It now follows from Corollary 6.3.5 of [5] that g is computable. □

Our second proof of Theorem 2.2 is uniform. It uses a computable version of the Contraction Mapping Theorem which we state and prove below. We will need the following definition, which is essentially that given in [1]. Here, and throughout this paper, d denotes the Euclidean distance function.

Definition 2.3 Suppose $f :\subseteq \mathbb{R}^n \to \mathbb{R}^m$ and $k \in \mathbb{R}$. We say that k is a *contraction constant for f* if $0 < k < 1$ and $d(f(x), f(y)) \leq kd(x, y)$ for all $x, y \in dom(f)$.

The following is well-known.

Theorem 2.4 (Contraction Mapping Theorem) *If U is a closed interval and $f \in C(U)$ has a contraction constant, then f has a unique fixed point.*

Theorem 2.5 (Uniformly Computable Contraction Mapping Theorem) *Suppose $U \subseteq \mathbb{R}$ is a closed interval. There is a computable $\Psi :\subseteq C(U) \times (0,1) \to U$ such that for all $(f, k) \in C(U) \times (0,1)$, if k is a contraction constant for f, then $(f, k) \in dom(\Psi)$ and $\Psi(f, k)$ is a fixed point for f.*

Proof. Fix a rational number $p_0 \in U$. For all $f \in C(U)$ and $k \in (0,1)$, let

$$\Psi(f, k) = \lim_{m \to \infty} f^m(p_0).$$

If k is a contraction constant for f, then $\Psi(f, k)$ is defined and is a fixed point for f. It remains to show that Ψ is computable.

For all $f \in C(U)$, let

$$S(f) = (p_0, f(p_0), f^2(p_0), \ldots).$$

It follows from Theorem 3.1.7.2 of [5] that S is $(\delta_{co}^U, [\rho]^\omega)$ computable. For all $f \in C(U)$, $k \in (0,1)$, and $n \in N$, let $e(f, k, n)$ be the least number m such that

$$\frac{|p_0 - f(p_0)|k^m}{1 - k} \leq \frac{1}{2^n}.$$

It follows that e is $(\delta_{co}^U, \rho, \nu_{\mathbb{N}}, \nu_N)$-computable.

Suppose $f \in C(U)$, and suppose $k \in (0,1)$ is a contraction constant for f. For all m, let $p_m = f^m(p_0)$. It follows that for all $m \in \mathbb{N}$

$$|p_m - p_{m+1}| \leq k^m |p_0 - p_1|.$$

It now follows that when $n > m$,

$$|p_m - p_n| \leq |p_m - p_{m+1}| + |p_{m+1} - p_{m+2}| + \ldots + |p_{n-1} - p_n|$$

$$\leq k^m |p_0 - p_1| + k^{m+1} |p_0 - p_1| + \ldots + k^{n-1} |p_0 - p_1|$$

$$\leq |p_0 - p_1| \frac{k^m}{1-k}.$$

It now follows from Theorem 4.3.7 of [5] that Ψ is computable. $\square$

Second proof of Theorem 2.2 We follow the proof in [3]. Define

$$F(x, y) = y - \frac{\phi(x, y)}{\frac{\partial \phi}{\partial y}(a, b)}.$$

Thus, $F : E \to \mathbb{R}^2$ is computable, $\frac{\partial F}{\partial y}(a, b) = 0$, and $\phi(x, y) = 0$ if and only if $F(x, y) = y$. Also, $\frac{\partial F}{\partial y}$ is continuous.

There is a rational $r > 0$ such that $\left|\frac{\partial F}{\partial y}\right| < \frac{1}{2}$ on the open disk in $\mathbb{R}^2$ centered at (a, b) and with radius r. Fix a rational number k in $(0, r)$. Fix a rational number h such that $0 < h < \sqrt{r^2 - k^2}$ and $|F(x, b) - b| < k/2$ if $|x - a| < h$. Define U to be $(a - h, a + h)$ and V to be $(b - k, b + k)$.

Fix $x \in U$. We first note that if $|y - b| \leq k$, then $d((x, y), (a, b)) < r$. The key claim is that if $y, y' \in \overline{V}$, then $|F(x, y) - F(x, y')| \leq \frac{1}{2}|y - y'|$. For, suppose y, y' are distinct elements of $\overline{V}$. By the Mean Value Theorem, there is a number y'' between y and y' such that

$$F(x, y) - F(x, y') = \frac{\partial F}{\partial y}(x, y'')(y - y').$$

The claim then follows from the previously imposed bound on $\frac{\partial F}{\partial y}$. We note that we do not need a computable version of the Mean Value Theorem to establish this claim.

We now note that $F(x, \cdot) : \overline{V} \to \overline{V}$ is a contraction map with contraction constant $\frac{1}{2}$. By Lemma 6.1.7 of [5], the representations δ_{co}^U and $[\rho \to \rho]_U$ are computably equivalent. Since F is computable, it follows from Theorem 2.3.13 of [5] that the map $x \mapsto F(x, \cdot)$ is computable. Let Ψ be as in Theorem 2.5. Define $f(x) = \Psi(F(x, \cdot), \frac{1}{2})$. It follows from Theorem 2.5 that f is computable. Hence, f is continuous. The uniqueness of f follows from the uniqueness clause of Theorem 2.4. $\square$

Theorem 2.6 *In Theorem 2.2, if ϕ is differentiable, and if ϕ' is computable, then U, V can be chosen so that f' is computable.*

Proof. In the proof of Theorem 2.2, choose r so that $\left|\frac{\partial \phi}{\partial y}\right| > 0$ on the open disk in $\mathbb{R}^2$ with center (a, b) and radius r. Let B be this disk.

Let x_0, x be distinct elements of U. Then, $(x_0, f(x_0)), (x, f(x)) \in U \times V \subseteq B$. By the multivariable version of the Mean Value Theorem, there is a point

z on the line segment between $(x_0, f(x_0))$ and $(x, f(x))$ such that

$$\phi(x_0, f(x_0)) - \phi(x, f(x)) = \phi'(z) \cdot (x_0 - x, f(x_0) - f(x)).$$

(Again, we are not using, nor do we need, a computable version of this theorem.) Since $\phi(x_0, f(x_0)) = \phi(x, f(x)) = 0$, it follows that

$$\frac{-\frac{\partial\phi}{\partial x}(z)}{\frac{\partial\phi}{\partial y}(z)} = \frac{f(x_0) - f(x)}{x_0 - x}.$$

As x approaches x_0, z approaches $(x_0, f(x_0))$. Since ϕ' is computable, $\frac{\partial\phi}{\partial x}$ and $\frac{\partial\phi}{\partial y}$ are computable. It then follows that $\frac{\partial\phi}{\partial x}$ and $\frac{\partial\phi}{\partial y}$ are continuous. It then follows that

$$f'(x_0) = \frac{-\frac{\partial\phi}{\partial x}(x_0, f(x_0))}{\frac{\partial\phi}{\partial y}(x_0, f(x_0))}.$$

Since $\frac{\partial\phi}{\partial x}$, $\frac{\partial\phi}{\partial y}$, and f are computable, it follows that f' is computable. □

We now state uniform versions of these results. We will need the following definitions.

Definition 2.7 Let $n, m \geq 1$.

(i) $C_{n,m}$ is the set of all functions $\phi :\subseteq \mathbb{R}^n \to \mathbb{R}^m$ such that $dom(\phi)$ is open and ϕ is continuous.

(ii) $C^1_{n,m}$ is the set of all functions $\phi :\subseteq \mathbb{R}^n \to \mathbb{R}^m$ such that $dom(\phi)$ is open and ϕ' is continuous.

The following is a straightforward modification of the naming system for $C^1(\mathbb{R}^n)$ in [6].

Definition 2.8 Let $n, m \geq 1$. Let $a, b, r_1, r_2 \in \mathbb{Q}$, and suppose $r_1, r_2 > 0$.

(i) R_{a,r_1,b,r_2} is the set of all functions $\phi \in C_{n,m}$ such that $\phi[\overline{B(a, r_1)}] \subseteq B(b, r_2)$.

(ii) Fix i, j such that $1 \leq i, j \leq n$. We define $R^{i,j}_{a,r_1,b,r_2}$ to be the set of all functions $\phi \in C^1_{n,m}$ such that $\frac{\partial\phi_j}{\partial x_i}[\overline{B(a, r_1)}] \subseteq B(b, r_2)$.

(iii) We define $\sigma_{n,m}$ to be $\{R_{a,r_1,b,r_2} \mid a, b, r_1, r_2 \in \mathbb{Q} \ \wedge \ r_1, r_2 > 0\}$.

(iv) We define $\sigma^1_{n,m}$ to be

$$\{R_{a,r_1,b,r_2} \ \cap \ C^1_{n,m} \mid a, b, r_1, r_2 \in \mathbb{Q} \ \wedge \ r_1, r_2 > 0\}$$

$$\cup \ \{R^{i,j}_{a,r_1,b,r_2} \mid a, b, r_1, r_2 \in \mathbb{Q} \ \wedge \ i, j \in \{1, \ldots, n\} \ \wedge \ r_1, r_2 > 0\}.$$

(v) Let $\nu_{n,m}$ be a standard notation of $\sigma_{n,m}$.

(vi) Let $\nu^1_{n,m}$ be a standard notation of $\sigma^1_{n,m}$.

(vii) $\delta_{n,m}$ is the representation of $C_{n,m}$ given by $\nu_{n,m}$.

(viii) $\delta^1_{n,m}$ is the representation of $C^1_{n,m}$ given by $\nu^1_{n,m}$.

Thus, the $\delta^1_{n,m}$ name of a function $\phi \in C^1_{n,m}$ provides the information to compute ϕ as well as ϕ'.

Theorem 2.9 (Non-Differentiable Uniformly Computable Single-Variable Implicit Function Theorem) *There is a $(\delta_{2,1}, \delta_{2,1}, \rho^2, \rho^2, \delta_{1,1})$-computable function $\Psi :\subseteq C_{2,1} \times C_{2,1} \times \mathbb{R}^2 \to \mathbb{R}^2 \times C_{1,1}$ such that if $\phi \in C_{2,1}$, $\phi(a,b) = 0$, $\frac{\partial \phi}{\partial y}$ is continuous, and $\frac{\partial \phi}{\partial y}(a,b) \neq 0$, then $(\phi, \frac{\partial \phi}{\partial y}, a, b) \in dom(\Psi)$. Furthermore, if $(r_1, r_2, f) = \Psi(\phi, \frac{\partial \phi}{\partial y}, a, b)$, then f is the unique function such that $f : (a-r_1, r+r_1) \to (b-r_2, b+r_2)$, $\phi(x, f(x)) = 0$ for all $x \in (a-r_1, a+r_1)$, and $f(a) = b$.*

Proof sketch Most of the work has been done in the proof of Theorem 2.2. In the proof of Theorem 2.2, use the information provided by the $\delta_{2,1}$ name of $\frac{\partial \phi}{\partial y}$ to find r, h, k. In Theorem 2.5, the function Ψ can be obtained uniformly from the interval U. The proof of Theorem 2.6 shows that we can compute a $\delta_{1,1}$ name of f from a $\delta_{2,1}$ name of ϕ. □

Theorem 2.10 (Differentiable Uniformly Computable Single-Variable Implicit Function Theorem) *There is a $(\delta^1_{2,1}, \rho^2, \rho^2, \delta^1_{1,1})$-computable function $\Psi :\subseteq C^1_{2,1} \times \mathbb{R}^2 \to \mathbb{R}^2 \times C^1_{1,1}$ such that if $\phi \in C^1_{2,1}$, $\phi(a,b) = 0$, and $\frac{\partial \phi}{\partial y}(a,b) \neq 0$, then $(\phi, a, b) \in dom(\Psi)$. Furthermore, if $(r_1, r_2, f) = \Psi(\phi, a, b)$, then f is the unique function such that $f : (a-r_1, r+r_1) \to (b-r_2, b+r_2)$, $\phi(x, f(x)) = 0$ for all $x \in (a-r_1, a+r_1)$, and $f(a) = b$.*

Proof sketch Most of the work has already been done. The only addition is that the proof of Theorem 2.6 shows we can compute a $\delta^1_{1,1}$ name of f from a $\delta^1_{1,1}$ name of ϕ once we have first computed a $\delta_{1,1}$ name of f. □

3 The multivariable case

If A is a square matrix, then $det(A)$ denotes the determinant of A.

Theorem 3.1 (Multivariable Implicit Function Theorem) *Let $a \in \mathbb{R}^m$, and let $b \in \mathbb{R}^n$. Let $E \subseteq \mathbb{R}^{m+n}$ be an open set that contains (a,b). Let $\phi : E \to \mathbb{R}^n$ be continuous. Suppose the following hold.*

- $\phi(a,b) = \vec{0}$.

- *For all* $i, j \in \{1, \ldots, n\}$, $\frac{\partial \phi_i}{\partial x_{m+j}}$ *is continuous on* E.
- *det* $\left(\frac{\partial \phi_i}{\partial x_{m+j}}(a,b)\right)_{i,j=1,\ldots,n} \neq 0$.

Then, there exist open $U \subseteq \mathbb{R}^m$ *and* $V \subseteq \mathbb{R}^n$ *such that* $a \in U$, $b \in V$, *and there is a unique continuous function* $f : U \to V$ *such that* $\phi(x, f(x)) = \vec{0}$ *for all* $x \in U$.

Definition 3.2 An open or closed ball $B \subseteq \mathbb{R}^n$ is *computable* if its center and radius are computable.

When B is an open or closed ball in $\mathbb{R}^n$, let $C(B)$ denote the set of all continuous functions from B into B. A function $\Psi :\subseteq C(B) \times (0,1) \to B$ is *computable* if it is $(\delta_{co}^B, \rho, \rho^n)$-computable.

Theorem 3.3 (Uniformly Computable Multivariable Contraction Mapping Theorem) *Let* B *be a closed ball in* $\mathbb{R}^n$. *There is a computable* $\Psi :\subseteq C(B) \times (0,1) \to B$ *such that for all* $(f,k) \in C(B) \times (0,1)$, *if* k *is a contraction constant for* f, *then* $\Psi(f,k)$ *is a fixed point for* f.

The proof is basically identical to the proof of Theorem 2.5.

Theorem 3.4 (Computable Multivariable Implicit Function Theorem) *Let* $a \in \mathbb{R}^m$, *and let* $b \in \mathbb{R}^n$ *be computable. Let* $E \subseteq \mathbb{R}^{m+n}$ *be an open set that contains* (a,b). *Let* $\phi : E \to \mathbb{R}^n$ *be computable. Suppose the following hold.*

- $\phi(a,b) = \vec{0}$.
- *For all* $i, j \in \{1, \ldots, n\}$, $\frac{\partial \phi_i}{\partial x_{m+j}}$ *is continuous on* E.
- *det* $\left(\frac{\partial \phi_i}{\partial x_{m+j}}(a,b)\right)_{i,j=1,\ldots,n} \neq 0$, *and* $\frac{\partial \phi_i}{\partial x_{m+j}}(a,b)$ *is computable for all* $i, j \in \{1, \ldots, n\}$.

Then, there exist computable open $U \subseteq \mathbb{R}^m$ *and* $V \subseteq \mathbb{R}^n$ *such that* $a \in U$, $b \in V$, *and there is a unique function* $f : U \to V$ *such that* $\phi(x, f(x)) = \vec{0}$ *for all* $x \in U$ *and* $f(a) = b$. *Furthermore,* f *is computable.*

As with Theorem 2.2, we can give a simple non-uniform proof of Theorem 3.4. We show later that the proof below is uniform.

Proof. Let y_j denote x_{m+j}. Let

$$D = \left(\frac{\partial \phi_i}{\partial y_j}(a,b)\right)_{i,j=1,\ldots,n}.$$

Let $(c_{i,j})_{i,j=1,\dots,n} = C = D^{-1}$. For each $i \in \{i, \dots, n\}$, let

$$F_i(x, y) = y_i - \sum_{k=1}^{n} c_{i,k}\phi_k(x, y).$$

Let $F = (F_1, \dots, F_n)$. It follows that F is continuously differentiable on E. Since the entries in D are computable, it follows from Proposition 6 of [7] that C is computable. Hence, F is computable. Also, $\phi(x, y) = \vec{0}$ if and only if $F(x, y) = y$. By direct calculation, we have:

$$\frac{\partial F_i}{\partial y_i}(a, b) = 1 - \Sigma_{k=1}^{n} c_{i,k} d_{k,i}$$
$$\frac{\partial F_i}{\partial y_j}(a, b) = -\Sigma_{k=1}^{n} c_{i,k} d_{k,j} \text{ if } i \neq j$$

Since $CD = I_n$, it follows that $\frac{\partial F_i}{\partial y_j}(a, b) = 0$ for all $i, j \in \{1, \dots, n\}$.

For each $r > 0$, let B_r be the open ball with center (a, b) and radius r. Choose a rational number $r > 0$ so that $B_r \subseteq E$ and the following hold.

- For all $i, j \in \{1, \dots, n\}$, $\left|\frac{\partial F_i}{\partial y_j}\right| < \frac{1}{2n^2}$ on B_r.
- $det \left(\frac{\partial \phi_i}{\partial y_j}\right)_{i,j=1,\dots,n}$ is non-zero on B_r.

Now, choose a rational number k such that $0 < k < r$. Finally, choose a rational number h such that $0 < h < \sqrt{r^2 - k^2}$ and $d(F(x, b), b) < k/2$ whenever $x \in R^n$ and $d(x, a) < h$. Let U be the open ball in $\mathbb{R}^m$ with center a and radius h. Let V be the open ball in $\mathbb{R}^n$ with center b and radius k. Clearly, U, V are computable. If $(x, y) \in U \times \overline{V}$, then

$$d((x, y), (a, b))^2 = d(x, a)^2 + d(y, b)^2$$

$$\leq h^2 + r^2 - h^2 = r^2.$$

Hence, $U \times \overline{V} \subseteq B_r$.

We now claim that $d(F(x, y), F(x, y')) \leq \frac{1}{2} d(y, y')$ whenever $(x, y), (x, y') \in U \times \overline{V}$. For, let $(x, y), (x, y) \in U \times \overline{V}$. Without loss of generality, suppose $y \neq y'$. Fix $i \in \{1, \dots, n\}$. By the multivariable version of the Mean Value Theorem, there is a point y_0 on the line segment between y and y' such that

$$F_i(x, y) - F_i(x, y') = \frac{\partial F_i}{\partial y_1}(x, y_0)(y_1 - y'_1) + \ldots + \frac{\partial F_i}{\partial y_n}(x, y_0)(y_n - y'_n).$$

Hence,

$$|F_i(x,y) - F_i(x,y')| \leq \left|\frac{\partial F_i}{\partial y_1}(x,y_0)\right| |y_1 - y_1'| + \ldots + \left|\frac{\partial F_i}{\partial y_n}(x,y_0)\right| |y_n - y_n'|$$

$$\leq \frac{1}{2n^2}(|y_1 - y_1'| + \ldots + |y_n - y_n'|)$$

$$\leq \frac{n}{2n^2} d(y,y') = \frac{1}{2n} d(y,y').$$

It now follows that $d(F(x,y), F(x,y')) \leq \frac{1}{2} d(y,y')$. In addition, by taking $y' = b$, it also follows that

$$d(F(x,y), b) \leq d(F(x,y), F(x,b)) + d(F(x,b), b)$$

$$< \frac{1}{2} d(y,b) + \frac{k}{2}$$

$$\leq k.$$

Hence, for each $x \in U$, $F(x,\cdot) : \overline{V} \to \overline{V}$, and $F(x,\cdot)$ has a contraction constant of $\frac{1}{2}$. Let Ψ be as in Theorem 3.3. Let $f(x) = \Psi(F(x,\cdot), \frac{1}{2})$. It follows that f is computable and $\phi(x, f(x)) = 0$ for all $x \in U$. $\square$

We now discuss differentiability.

Theorem 3.5 *If in Theorem 3.4 we assume ϕ is differentiable on E and that ϕ' is computable, then we may conclude that f is differentiable and f' is computable.*

Proof. Let r, U, V, *etc.* be as in the proof of Theorem 3.4.

There are two parts to this proof. The first is to show that f is differentiable. The second is to show that f' is computable. The first part is not the main concern here and in any case is well-established. A thorough proof may be found in [3]. So, we give the second part only.

To show that f' is computable, it suffices to show that $\frac{\partial f_i}{\partial x_j}$ is computable for all $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$. Fix $j \in \{1, \ldots, m\}$. For each $x \in U$ let

$$g(x) = (x, f(x)).$$

Now, fix $i \in \{1, \ldots, n\}$. We note that $\phi_i \circ g = 0$. If we apply the chain rule to $\phi_i \circ g$ for the purpose of calculating its partial derivative with respect to x_j, we obtain

$$\frac{\partial(\phi_i \circ g)}{\partial x_j} = (\phi_i' \circ g) \cdot \frac{\partial g}{\partial x_j}.$$

(We note that we are not applying, nor do we need, a computable version of the multivariable chain rule.) We have,

$$\frac{\partial g}{\partial x_j} = \left(0, \ldots, 1, \ldots, 0, \frac{\partial f_1}{\partial x_j}, \ldots, \frac{\partial f_n}{\partial x_j}\right)$$

where the 1 appears in the j-th position. We also have

$$\phi_i' = \left(\frac{\partial \phi_i}{\partial x_1}, \ldots, \frac{\partial \phi_i}{\partial x_m}, \frac{\partial \phi_i}{\partial y_1}, \ldots, \frac{\partial \phi_i}{\partial y_n}\right).$$

By combining these results, we obtain

$$0 = \frac{\partial \phi_i \circ g}{\partial x_j} = \frac{\partial \phi_i}{\partial x_j} \circ g + \left(\frac{\partial \phi_i}{\partial y_1} \circ g\right) \frac{\partial f_1}{\partial x_j} + \ldots \left(\frac{\partial \phi_i}{\partial y_n} \circ g\right) \frac{\partial f_1}{\partial x_j}.$$

We now allow i to vary, but keep j fixed. For each $x \in U$, let

$$A_j(x) = \left(\frac{\partial \phi_i}{\partial y_k}(g(x))\right)_{i,k=1,\ldots,n},$$

and let

$$B_j(x) = \begin{pmatrix} -\frac{\partial \phi_1}{\partial x_j}(g(x)) \\ . \\ . \\ . \\ -\frac{\partial \phi_n}{\partial x_j}(g(x)) \end{pmatrix}.$$

It now follows that for each $x \in U$, $\frac{\partial f}{\partial x_j}(x) = \left(\frac{\partial f_1}{\partial x_j}(x), \ldots, \frac{\partial f_n}{\partial x_j}(x)\right)$ is a solution to the system of linear equations in the n variables $\omega_1, \ldots, \omega_n$ given by

$$A_j(x) \cdot \begin{pmatrix} \omega_1 \\ . \\ . \\ . \\ \omega_n \end{pmatrix} = B_j(x).$$

However, by the choice of r, $det(A_j(x)) \neq 0$ for all $x \in U$. Hence, $\frac{\partial f}{\partial x_j}(x)$ is the unique solution to this system. It now follows from Proposition 6 of [7] that $\frac{\partial f}{\partial x_j}$ is computable for each $j \in \{1, \dots, m\}$. Hence, f' is computable. □

We now discuss uniformity.

Theorem 3.6 (Non-Differentiable Uniformly Computable Multivariable Implicit Function Theorem) *There is a* $(\delta_{m+n,n}, (\delta_{m+n,n})^{n^2}, \rho^m, \rho^n, \rho^2, \delta_{m,n})$*-computable* $\Psi :\subseteq C_{m+n,n} \times (C_{m+n,n})^{n^2} \times \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R} \times \mathbb{R} \times C_{m,n}$ *such that if* $\phi \in C_{m+n,n}$, $a \in \mathbb{R}^m$, $b \in \mathbb{R}^n$, $\phi(a,b) = 0$, $\frac{\partial \phi_i}{\partial x_{m+j}}$ *is continuous for all* $i, j \in \{1, \dots, n\}$, *and* $det\left(\frac{\partial \phi_i}{\partial x_{m+j}}(a,b)\right)_{i,j=1,\dots,n} \neq 0$, *then* $(\phi, \left(\frac{\partial \phi_i}{\partial x_{m+j}}\right)_{i,j=1,\dots,n}, a, b) \in dom(\Psi)$. *Furthermore, if* $(r_1, r_2, f) = \Psi(\phi, \left(\frac{\partial \phi_i}{\partial x_{m+j}}\right)_{i,j=1,\dots,n}, a, b)$, *then* f *is the unique function such that* $f : B(a, r_1) \rightarrow B(b, r_2)$, $\phi(x, f(x)) = \vec{0}$ *for all* $x \in B(a, r_1)$, *and* $f(a) = b$.

Proof sketch In the proof of Theorem 3.4, r, h, k can be computed from the $\delta_{m+n,n}$ names of $\frac{\partial \phi_i}{\partial x_{m+j}}$ for $i, j \in \{1 \dots, n\}$. In Theorem 3.3, the function Ψ can be obtained uniformly from the ball B. The proof of Theorem 3.4 shows that we can compute a $\delta_{m,n}$ name of f from a $\delta_{m+n,n}$ name of ϕ and the $\delta_{m+n,n}$ names of $\frac{\partial \phi_i}{\partial x_{m+j}}$ for $i, j \in \{1 \dots, n\}$. □

Theorem 3.7 (Differentiable Uniformly Computable Multivariable Implicit Function Theorem) *There is a* $(\delta^1_{m+n,n}, \rho^m, \rho^n, \rho^2, \delta^1_{m,n})$*-computable* $\Psi :\subseteq C^1_{m+n,n} \times \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R} \times \mathbb{R} \times C^1_{m,n}$ *such that if* $\phi \in C^1_{m+n,n}$, $a \in \mathbb{R}^m$, $b \in \mathbb{R}^n$, $\phi(a,b) = 0$, *and* $det\left(\frac{\partial \phi_i}{\partial x_{m+j}}(a,b)\right)_{i,j=1,\dots,n} \neq 0$, *then* $(\phi, a, b) \in dom(\Psi)$. *Furthermore, if* $(r_1, r_2, f) = \Psi(\phi, a, b)$, *then* f *is the unique function such that* $f : B(a, r_1) \rightarrow B(b, r_2)$, $\phi(x, f(x)) = \vec{0}$ *for all* $x \in B(a, r_1)$, *and* $f(a) = b$.

Proof sketch Most of the work has already been done. The proof of Theorem 3.5 shows that we can compute a $\delta^1_{m,n}$ name of f from a $\delta^1_{m+n,n}$ name of ϕ once we have computed a $\delta_{m,n}$ name of f. □

Acknowledgement

Many thanks to Iraj Kalantari, Ted Mahavier, Alec Matheson, Susan McNicholl, and the referees.

References

[1] Hale, J., “Ordinary Differential Equations”, 2nd ed., Wiley, 1980.

[2] Krantz, S., and H. Parks, “The Implicit Function Theorem”, 1st ed., Birkhäuser, Boston, 2002.

[3] Rosenlicht, M., “Introduction to Analysis”, 1st ed., Dover, 1986.

[4] Soare, R. I., “Recursively Enumerable Sets and Degrees”, 1st ed., Springer-Verlag, Berlin, Heidelberg, 1987.

[5] Weihrauch, K., “Computable Analysis. An Introduction”, 1st ed., Springer-Verlag, Berlin, 2000.

[6] Weihrauch, K., and N. Zhong, *The Wave Propogator is Turing Computable*, Lecture Notes in Computer Science **1644** (1999) “Automata, languages, and programming (Prague, 1999)”, 697 - 706, Springer, Berlin 1999.

[7] Ziegler, M., and V. Brattka, *Computability in linear algebra*, Theoretical Computer Science **326** (2004), 187 - 211.