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Formal Contexts for Algebraic Domains

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Abstract

In this paper, we investigate the representation of algebraic domains by means of Formal Concept Analysis. For a formal context, we can define a large number of consistent sets. Associated with each consistent set, there is a set of F-approximable concepts which are selected from the well known approximable concepts. By virtue of F-approximable concepts, formal contexts and algebraic domains are able to interpret each other. Moreover, by analyzing the finitely consistent sets, the algebraic bifinite domains, algebraic $L-$domains are exactly located at the corresponding formal contexts, respectively.

Keywords: formal context, F-approximable concept, algebraic domain, categorical equivalence.

1 Introduction

Domain theory was introduced by Dana Scott [5] in the 1970s and has become an important branch of order theory. As the theory of ordered topological structures used in denotational semantics, it has major applications in computer science, where it is used to specify denotational semantics, especially for functional programming languages.

In order to provide a more concrete way for the usual domain-theoretical approach to the semantics of programming languages, the notion of information system was first developed in [6]. By the notion of information elements, information systems provide a concrete representation of Scott domains. During the last few

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decades, many other kinds of information systems have been developed to represent various kinds of domain structures, such as continuous domains [15], algebraic domains [7], L-domains [8], bifinite domains [2] and so on.

Another research field which is closely related to information systems, and thus related to domain theory, is formal concept analysis (for short, FCA). It was pioneered by Wille [4] and was built on applied lattice and order theory [3]. The core idea of classical FCA is to extract the hierarchy structure of concepts inherent in relational data.

Recently, in [10,13], G-Q. Zhang, etc. studied in detail the connections between information systems and FCA. In their work, based on the observation of the mismatch between the formal concepts and information element of the associated information systems, they initially proposed the notion of formal approximable concept which is a generalization of classical formal concepts. It has been verified that approximable concepts exactly capture the class of algebraic lattices which are special domain structures. Subsequently, P. Hitzler and G-Q Zhang [13] further established the categorical equivalence between a specified category of formal contexts and the category of algebraic lattices. These results indicate the significant potential capability of FCA in characterizing domain structures. However, we have not seen any work on representing domain structures other than algebraic lattices by the tool of FCA in the literature.

In this paper, we investigate the representation of algebraic domains by means of FCA. For this purpose, we propose a new notion of F-approximable concept, which can be viewed as a selection of approximable concepts under the constraint of a consistent set employed on a formal context. We further explore the connection between F-approximable concepts and algebraic domains. It is shown that all F-approximable concepts of a formal context form an algebraic domain, and conversely, every algebraic domain is isomorphic to the set of F-approximable concepts of some special formal context.

The remainder of this paper is organized as follows. In Section 2, we briefly recall some necessary preliminaries of domain theory. In Section 3, we develop the notion of F-approximable concept based on G-Q. Zhang's approximable concepts and obtain the representation theory of algebraic domains via F-approximable concepts.

In Section 4, we propose a new type of morphism, named conditional formal context morphisms, and study the associated category of formal contexts (denoted as **Cct**). We eventually obtain the equivalence of **Cct** and the category of algebraic domains.

Section 5 additionally investigates the corresponding formal contexts of some special algebraic domains. In particular, we present sufficient and necessary conditions of formal contexts to represent algebraic bifinite domains, algebraic L-domains. Section 6 gives conclusions.

2 Preliminaries: Domains

For the convenience of the reader, we recall some notions in domain theory. Let T be a poset. Given a subset A of T, we write $\downarrow A$ for the set $\{t \mid t \in T \exists a \in A(t \leq a)\}$ (without confusion, $\downarrow x$ for $\downarrow\{x\}$), and dually, $\uparrow A$ for $\{t \mid t \in T \exists a \in A(a \leq t)\}$. We denote by $ub(A)$ the set of all upper-bounds of A and the minimal elements of $ub(A)$ are called minimal upper-bounds of A. T is said to be a complete lattice if every subset A (include empty set) of T has a least upper-bound in T. T is called bounded complete if each subset which has an upper-bound has a least upper-bound.

A nonempty subset D of T is said to be directed if for two arbitrary elements d_1 and d_1 of D, there exists a $d_3 \in D$ such that $d_1 \leq d_3$ and $d_2 \leq d_3$. A *dcpo* is a poset such that each directed set of it has a least upper bound. If a *dcpo* has a least element, we call it pointed. A *dcpo* T with a least element is called a pointed $L-$domain if for every $x \in T$, $\downarrow x$ is a complete lattice.

Let $x, y \in T$, x is said to approximate y (in symbol $x \ll y$) if and only if for every directed set $D \subseteq T$, $y \leq \bigvee D$ means that there is a $d \in D$ such that $x \leq d$. Let $\Downarrow x$ denote the set $\{t \in T \mid t \ll x\}$. x is compact if $x \ll x$ (We use $\kappa(T)$ to denote the set of all compact elements of T). A subset β of T is said to be a basis of T if for every $x \in T$, $\Downarrow x \cap \beta$ is a directed set and $x = \bigvee(\Downarrow x \cap \beta)$. A *dcpo* is called a continuous domain if it has a basis. In particular, a *dcpo* is called an algebraic domain if all compact elements of it form a basis. A complete lattice is called an algebraic lattice if it is also an algebraic domain. This paper mainly discusses algebraic domains. Bounded complete algebraic domains are called Scott domains.

A monotonic function f between two posets T_1 and T_2 is Scott continuous if f preserves all joins of directed sets, that is, for all directed sets D of T_1, the existence of $\bigvee D$ implies $f(\bigvee D) = \bigvee f(D)$. We use $[T_1 \to T_2]$ to denote the set of Scott continuous functions from T_1 to T_2. The function space $[T_1 \to T_2]$ naturally form a poset under the pointwise order.

Definition 2.1 An approximate identity for a *dcpo* T is a directed set $D \subseteq [T \to T]$ satisfying $\bigvee D = i_T$, the identity on T.

Definition 2.2 An algebraic domain is called a bifinite domain if it has an approximate identity consisting of maps with finite range.

Theorem 2.3 ([2]) *For an algebraic domain T with a least element, the following statements are equivalent:*

(i) *T is an algebraic $L-$domain.*

(ii) *For each upper-bound x of a finite subset A of $\kappa(T)$ there is a unique minimal upper-bound of A below x.*

(iii) *For each upper-bound x of a pair A of compact elements of $\kappa(T)$ there is a unique minimal upper-bound of A below x.*

All pointed algebraic domains and the Scott continuous functions between them form a category $\mathbf{ALG}_\perp$. The category $\mathbf{ALG}_\perp$ has two maximal cartesian closed full subcategory **B** in which all pointed algebraic bifinite domains are objects, and full subcategory **L** in which all pointed algebraic $L-$domains are objects. For more backgrounds about domain theory and category theory, the reader may refer to [9,17,20].

3 Revisiting formal concepts and approximable concepts

A formal context is a triple $(P_o, P_a, \models_P)$ where P_o and P_a are two sets and $\models \subseteq P_o \times P_a$. The elements of P_o and P_a are often called objects and attributes, respectively. FCA studies the relationship between objects and the properties. Galois connection [12] is the basic mathematical technique for extracting concepts from formal contexts.

Definition 3.1 Let P and Q be ordered sets. A pair (f, g) of maps $f : P \to Q$ and $g : Q \to P$ is a Galois connection between P and Q if, for all $p \in P$ and $q \in Q$,

$$\textbf{\textit{(Gal)}}\ f(p) \leq q \Leftrightarrow p \leq g(q).$$

Proposition 3.2 ([3]) *Assume (f, g) is a Galois connection between P and Q. Let $p \in P$ and $q \in Q$. Then*

$(G1)$ *$p \leq g \circ f(p)$ and $f \circ g(q) \leq q$.*

$(G2)$ *Both f and g are monotone.*

$(G3)$ *$f(p) = f \circ g \circ f(p)$ and $g(q) = g \circ f \circ g(q)$.*

Conversely, a pair (f, g) of maps $f : P \to Q$ and $g : Q \to P$ satisfying $(G1)$ and $(G2)$ for all $p \in P$ and $q \in Q$ sets up a Galois connection between P and Q.

Given a formal context $P = (P_o, P_a, \models_P)$, we define two operators α_P and ω_P as follows:

- $\alpha_P : 2^{P_o} \to 2^{P_a}$ with $\alpha_p(A) = \{a \in P_a \mid \forall o \in A, o \models_P a\}$
- $\omega_P : 2^{P_a} \to 2^{P_o}$ with $\omega_P(B) = \{o \in P_o \mid \forall b \in B, o \models_P b\}$.

We write η_P for the composition $\omega_P \circ \alpha_P : 2^{P_o} \to 2^{P_o}$. As is well known, (α_P, ω_P) is a Galois connection between $(2^{P_o}, \subseteq)$ and $(2^{P_a}, \supseteq)$. A concept of the formal context $(P_o, P_a, \models)$ is defined to be a pair (A, B) such that $\alpha_P(A) = B$ and $\omega_P(B) = A$, where A and B are usually called the extent and the intent of the concept (A, B), respectively. The set of all concepts of $(P_o, P_a, \models_P)$ is denoted by $\mathfrak{B}[(P_o, P_a, \models_P)]$. For two arbitrary concepts (A_1, B_1) and (A_2, B_2), we write $(A_1, B_1) \leq (A_2, B_2)$ if $A_1 \subseteq A_2$. Then $\leq$ is an order on $\mathfrak{B}[(P_o, P_a, \models_P)]$. Clearly,

$$(A_1, B_1) \leq (A_2, B_2) \Leftrightarrow A_1 \subseteq A_2 \Leftrightarrow B_1 \supseteq B_2.$$

Generally, letting P and Q be two sets, two arbitrary maps $f : P \to Q$ and $g : Q \to P$ induce a map $H : P \times Q \to P \times Q$ defined by $H(x, y) = (g(y), f(x))$. Writing $fix(H)$ for the set $\{(x, y) \in P \times Q \mid H(x, y) = (x, y)\}$, and $fix_P(H)$ for $\{x \in P \mid \exists y \in Q, (x, y) \in fix(H)\}$, and $fix_Q(H)$ for $\{y \in Q \mid \exists x \in P, (x, y) \in$

$fix(H)\}$, one has:

Proposition 3.3 *If $fix(H) \neq \emptyset$, then the restriction of f to $fix_P(H)$ is a bijection from $fix_P(H)$ to $fix_Q(H)$, and its inverse map is exactly the restriction of g to $fix_Q(H)$.*

In order to be succinct, when speaking of a formal context P in the remainder of this paper, we always explicitly or non-explicitly have defined the corresponding sets P_o, P_a and $\models_P$.

Proposition 3.4 *Given a formal context P, the set $\{A \subseteq P_o \mid \eta_P(A) = A\}$ under the inclusion is order-isomorphic to $\mathfrak{B}[P]$.*

As is well known, $\mathfrak{B}[P]$ under the relation $\leq$ forms a complete lattice. Furthermore, if defining $\gamma : P_o \to \mathfrak{B}[P]$ and $\mu : P_a \to \mathfrak{B}[P]$ by

$$\gamma(o) = (\omega_P \circ \alpha_P(\{o\}), \alpha_P(\{o\})) \text{ and } \mu(a) = (\omega_P(\{a\}), \alpha_P \circ \omega_P(\{a\})),$$

then we can obtain the following central result in FCA:

Theorem 3.5 (Wille [4]) *Let P be a formal context and $L = \mathfrak{B}[P]$ be the associated complete lattice. Then the maps γ and μ are such that $\gamma(P_o)$ is join-dense in L, the set $\mu(P_a)$ is meet-dense in L, and $o \models a$ is equivalent to $\gamma(o) \leq \mu(a)$ for each $o \in P_a$ and $a \in P_a$. For the other direction, let L be a complete lattice and let P_o and P_a be sets and assume that there exist maps $\gamma : P_o \to L$ and $\mu : P_a \to L$ such that $\gamma(P_o)$ is join-dense in L and $\mu(P_a)$ is meet-dense in L. Define $o \models a \Leftrightarrow \gamma(o) \leq \mu(a)$. Then L is order-isomorphic to $\mathfrak{B}(P_o, P_a, \models)$. In particular, any complete lattice L is order-isomorphic to the concept lattice $\mathfrak{B}[(L, L, \leq)]$*

Zhang and Shen [10] introduced the notion of approximable concept to represent algebraic lattices. For a formal context P, a subset A of P_o is called an approximable concept if and only if $\eta_P(X) \subseteq A$ for all finite subsets X of A. Let $\mathcal{A}[P]$ denote the set of all approximable concepts of a formal context P. All approximable concepts have the following properties:

- The directed union of approximable concepts is again an approximable concept.
- Given any set A of objects, the smallest approximable concept containing A is given by $\eta_P(A) = \bigcup\{\eta_P(X) | X \subseteq_{fin} A\}$.
- η_P is an inductive closure operator, i.e., it preserves directed unions. From this is follows immediately that the set of approximable concepts (i.e., the fixpoints of η_P) forms an algebraic lattice in which the compact elements are the closures of finite sets.

Theorem 3.6 (Zhang and Shen [10]) *For any formal context P, $\mathcal{A}[P]$ is an algebraic lattice. Conversely, every algebraic lattice L is order-isomorphic to $\mathcal{A}[(L, \kappa(L), \leq)]$, where the isomorphism is given by $A \mapsto \sup A$ for any $A \in \mathcal{A}[(L, \kappa(L), \leq)]$.*

There are strong relations between approximable concepts and concepts, in fact, the extent of a concept is also an approximable concept, but the inverse may be false. From the definitions, one immediately has,

Lemma 3.7 *Let P be a formal context. Then:*

(i) *For every $X \subseteq 2^{P_o}$, $(\eta_P(X), \alpha_P(X)) \in \mathfrak{B}[P]$.*

(ii) *For every $(A, B) \in \mathfrak{B}[P]$, A is an approximable concept.*

The above facts lead to two functions $\overline{f} : \mathfrak{B}[P] \to \mathcal{A}[P]$ sending an arbitrary $(A, B) \in \mathfrak{B}[P]$ to A, and $\underline{f} : \mathcal{A}[P] \to \mathfrak{B}[P]$ with $\underline{f}(A) = (\eta_P(A), \alpha_P(A))$ for all $A \in \mathcal{A}[P]$.

Theorem 3.8 *Let P be a formal context. Then the map pair $(\underline{f}, \overline{f})$ is a Galois connection between $\mathcal{A}[P]$ and $\mathfrak{B}[P]$.*

Hitzler and Zhang [13] introduced a certain morphism between two formal contexts to investigate the approximable concepts from the point of view of category theory. Let $Fin(A)$ denote the set of all finite subsets of a set A.

Definition 3.9 [13] Given two formal contexts P and Q, a formal context morphism M from P to Q is a relation $M \subseteq Fin(P_o) \times Fin(Q_o)$, such that the following conditions are satisfied for all $X, X' \in Fin(P_o)$ and $Y, Y' \in Fin(Q_o)$:

(cm1) $\emptyset M \emptyset$.

(cm2) XMY and XMY' imply $XM(Y \cup Y')$.

(cm3) $\eta_P(X) \supseteq X'$ and $X'MY'$ and $\eta_Q(Y') \supseteq Y$ imply XMY.

All formal contexts and formal context morphisms between them form a category **Cxt** in which the composition of two formal context morphisms is the usual composition of them as relations.In [13], it is shown that the resulting category **Cxt** is equivalent to that of algebraic lattices and Scott-continuous morphisms.

For a formal context P, a set $\mathscr{F}_P$ which consists of some finite subsets of P_o is called consistent if for each $F \in \mathscr{F}_P$ and each finite set $A \subseteq \eta_P(F)$, there is $F' \in \mathscr{F}_P$ such that $A \subseteq F' \subseteq \eta_P(F)$. A conditional formal context $(P_o, P_a, \models, \mathscr{F}_P)$ is a formal context $(P_o, P_a, \models)$ endowed with a consistent set $\mathscr{F}_P$. For the conditional formal context $(P_o, P_a, \models, \mathscr{F}_P)$, a set $X \subseteq P_o$ is called finitely consistent if for every finite subset Y of X there is an $F \in \mathscr{F}_P$ such that $Y \subseteq F \subseteq X$. We write $\mathcal{W}[(P_o, P_a, \models, \mathscr{F}_P)]$ for the set of all finitely consistent sets of $(P_o, P_a, \models, \mathscr{F}_P)$.

From now on, when P refers to a conditional formal context (shortly, cfc), we mean that all sets P_o, P_a, $\models_P$ and $\mathscr{F}_P$ corresponding to P have been given.

Definition 3.10 Let P be a cfc. A subset X of P_o is call an F-approximable concept if X satisfies the following statements:

(i) X is finitely consistent

(ii) For every $F \in \mathscr{F}_P$, $F \subseteq X$ implies $\eta_P(F) \subseteq X$.

The set of all F-approximable concepts of P is denoted by $\mathcal{F}[P]$.

Example 3.11 Let the formal context P be defined as follows. The set $\{\{A, B\}, \{C, D\}, \{B, C, D\}\}$ is a consistent set. Note that $\{A, B, C, D\}$ is not an

$F-$approximable concept because it is not finitely consistent, but an approximable concept.

	a	b	c	d
A	•	•		
B		•	•	•
C			•	•
D			•	•

Proposition 3.12 *Let P be a cfc. Then:*

(1) *Every F-approximable concept is an approximable concept.*

(2) *For every directed set $\{D_i \in \mathcal{F}[P] \mid i \in I\}$, $\bigvee_{i\in I} D_i = \bigcup_{i\in I} D_i$.*

(3) *$\mathcal{F}[P]$ is an algebraic domain with a basis $\{\eta_P(F) \mid F \in \mathscr{F}_P\}$.*

Corollary 3.13 *The embedding $i : \mathcal{F}[P] \to \mathcal{A}[P]$ which sends X to X for every $X \in \mathcal{F}[P]$ is Scott continuous.*

Now we consider another direction. For an arbitrary algebraic domain L, it not only naturally generates a formal context $P[L] = (\kappa(L), L, \leq)$, but also induces a set $\mathscr{F}_L$ defined by

$$\mathscr{F}_L = \{F \in Fin(\kappa(L)) \mid \exists a \in F \forall b \in F (b \leq a)\}.$$

With respect to the formal context $P[L]$, by the straightforward computation there are:

(i) For all $X \subseteq \kappa(L)$, $\alpha_{P[L]}(X) = \bigcap\{\uparrow a \mid a \in X\}$.

(ii) For all $X \subseteq \kappa(L)$, $\eta_{P[L]}(X) = \bigcap\{\downarrow b \cap \kappa(L) \mid b \in \bigcap\{\uparrow a \mid a \in X\}\}$.

Lemma 3.14 *$(\kappa(L), L, \leq, \mathscr{F}_L)$ is a cfc.*

Theorem 3.15 *For any cfc P, $\mathcal{F}[P]$ is an algebraic domain. Conversely, every algebraic domain L is order-isomorphic to $\mathcal{F}[(\kappa(L), L, \leq, \mathscr{F}_L)]$, where the order-isomorphism is given $A \mapsto \sup A$ for any $A \in \mathcal{F}[(\kappa(L), L, \leq, \mathscr{F}_L)]$.*

Proof. The former assertion is just Proposition 3.12(3) and we only need to show the latter.

Let $A \in \mathcal{F}[(\kappa(L), L, \leq, \mathscr{F}_L)]$. Then there is $\{F_i \mid i \in I\} \subseteq \mathscr{F}_L$ and $\bigcup\{F_i \mid i \in I\} = A$, where $\{F_i \mid i \in I\}$ is directed under inclusion. Given a pair $x, y \in A$, we are able to locate $i_x, i_y, i_z \in I$ such that $x \in F_{i_x}$ and $y \in F_{i_y}$ and $F_{i_x} \cup F_{i_y} \subseteq F_{i_z}$. By the definition of $\mathscr{F}_L$, we also find a $t_{i_z} \in F_{i_z}$ such that $x \leq t_{i_z}$ and $y \leq t_{i_z}$. Therefore, A is a directed subset of L and has a least upper bound $\bigvee A$ in L.

We first claim $A = (\downarrow \bigvee A) \cap \kappa(L)$.

It is sufficient to prove that $(\downarrow \bigvee A) \cap \kappa(L) = \bigcup\{\eta_{P[L]}(F_i) \mid i \in I\}$. In fact, for every $i \in I$, there is an $a_i \in F_i$ with $b \leq a_i$ for all $b \in F_i$, so, $\eta_{P[L]}(F_i) = \downarrow a_i \cap \kappa(L) \subseteq (\downarrow \bigvee A) \cap \kappa(L)$. Let $a \in (\downarrow \bigvee A) \cap \kappa(L)$. Then $a \leq \bigvee A$ implies

$a \le \bigvee\{\bigvee F_i \mid i \in I\} = \bigvee\{a_i \mid i \in I\}$, hence there is $i_0 \in I$ such that $a \le a_{i_0}$ by the compactness of a. That is, $a \in \eta_{P[L]}(F_{i_0})$. We completed our claim.

Since $A = (\downarrow \bigvee A) \cap \kappa(L)$ for every $A \in \mathcal{F}[(\kappa(L), L, \le, \mathscr{F}_L)]$, $A \in \{\downarrow t \cap \kappa(L) \mid t \in L\}$. conversely, for an arbitrary $t \in L$, we show that $\downarrow t \cap \kappa(L)$ is an $F-$approximable concept with respect to $(\kappa(L), L, \le, \mathscr{F}_L)$. For every finite subset F of $\downarrow t \cap \kappa(L)$, there is an $a_F \in \downarrow t \cap \kappa(L)$ such that $b \le a_F$ for all $b \in F$ since $\downarrow t \cap \kappa(L)$ is directed. This implies $F \cup \{a_F\} \in \mathscr{F}_L$ and $F \subseteq F \cup \{a_F\} \subseteq \downarrow t \cap \kappa(L)$. Therefore $\downarrow t \cap \kappa(L)$ is finitely consistent. Note that $\eta_{P[L]}(F \cup \{a_F\}) = \eta_{P[L]}(\{a_F\}) = \downarrow a_F \cap \kappa(L) \subseteq \downarrow t \cap \kappa(L)$. Up to now, $\mathcal{F}[(\kappa(L), L, \le, \mathscr{F}_L)] = \{\downarrow t \cap \kappa(L) \mid t \in L\}$ is proved. Therefore, the assertion of order-isomorphism is easily inferred from the standard domain theory. □

In this section we adopt the function symbols α_P and β_P instead of the symbol ( )' used in the original FCA, because we emphasize the single side concepts for clustering data. But readers can find the counterparts in FCA community after carefully checking out all our terminologies.

4 Morphisms between conditional formal contexts

It is necessary to define a certain morphism between formal contexts in order to investigate the categorical aspects of formal contexts.

Definition 4.1 Given two formal contexts P and Q, a conditional formal context morphism (in short, $cfcm$) R from P and Q is a relation $R \subseteq \mathscr{F}_P \times Q_o$, such that the following conditions hold for all $X, X' \in \mathscr{F}_P$ and $Y \in Fin(Q_o)$ and $a \in Q_o$:

$(c1)$ $XRY \Rightarrow \exists F' \in \mathscr{F}_Q(XRF' \supseteq Y)$,

$(c2)$ $\eta_P(X) \supseteq X' \wedge X'RY \wedge a \in \eta_Q(Y) \Rightarrow XRa$,

where XRY means XRb for every $b \in Y$.

Let R_1 be a $cfcm$ from P to Q and R_2 be a $cfcm$ from Q to S. Define relation $R_2 \circ R_1 \subseteq \mathscr{F}_P \times S_o$ by

$$\forall X \in \mathscr{F}_P \forall s \in S_o(X(R_2 \circ R_1)s) \Leftrightarrow \exists X' \in \mathscr{F}_Q(XR_1X' \wedge X'R_2s).$$

Lemma 4.2 *the relation $R_2 \circ R_1$ is a $cfcm$ from P to S.*

Lemma 4.3 *All $cfcs$ as objects and all $cfcms$ between them form a category named* **Cct**.

Lemma 4.4 *Let M be a formal context morphism from P to Q in the sense of [13]. Then for all $X \in Fin(P_o)$ and $Y \in Fin(Q_o)$, $XMY \Leftrightarrow \forall y \in Y(XM\{y\})$.*

Let **Acct** denote the full subcategory of **Cct** in which all objects are $((P_o, P_a, \models_P), Fin(P_o))$. Define two categorical functions $\mathcal{R}$: **Acct** $\to$ **Cxt** and $\mathcal{M}$: **Cxt** $\to$ **Acct** as follows:

- $\mathcal{R}$ acts on objects: $\mathcal{R}[((P_o, P_a, \models_P), Fin(P_o))] = (P_o, P_a, \models_P)$.
  $\mathcal{R}$ acts on morphisms: Let R be a $cfcm$ from P to Q. For all $X \in Fin(P_o)$ and $Y \in Fin(Q_O)$, $X\mathcal{R}[R]Y \Leftrightarrow \forall y \in Y(XRy)$.

- $\mathcal{M}$ acts on objects: $\mathcal{M}[(P_o, P_a, \models_P)] = ((P_o, P_a, \models_P), Fin(P_o))$.
  $\mathcal{M}$ acts on morphisms: Let M be a formal context morphism from P to Q. For all $X \in Fin(P_o)$ and $a \in Q_O$, $X\mathcal{M}[M]a \Leftrightarrow XM\{a\}$.

Lemma 4.5 *Both $\mathcal{R}$ and $\mathcal{M}$ are functors.*

Given an arbitrary cfc P, for all $A \in \mathcal{W}[P]$, $\{\eta_P(X) \mid X \in \mathscr{F}_P, X \subseteq A\}$ is a directed subset of $\mathcal{W}[P]$, so, $\bigcup\{\eta_P(X) \mid X \in \mathscr{F}_P, X \subseteq A\} \in \mathcal{W}[P]$. This enables us to introduce a function $\overline{\eta}_P : \mathcal{W}[P] \to \mathcal{W}[P]$ such that for all $A \in \mathcal{W}[P]$, $\overline{\eta}_P(A) = \bigcup\{\eta_P(X) \mid X \in \mathscr{F}_P, X \subseteq A\}$. $\overline{\eta}_P$ is an idempotent Scott continuous function, moreover, one should note that $\overline{\eta}_P(X) = \eta_P(X)$ for all $X \in \mathscr{F}_P$.

Proposition 4.6 *Let P and Q be two cfc and let $fix_\eta(P, Q)$ denote the set $\{g \in [\mathcal{W}[P] \to \mathcal{W}[Q]] \mid \overline{\eta}_Q \circ g \circ \overline{\eta}_P = g\}$. Then:*

(1) For every Scott continuous function f from $\mathcal{W}[P]$ to $\mathcal{W}[Q]$. The relation $C(f) \subseteq \mathscr{F}_P \times Q_o$ defined by $XC(f)q \Leftrightarrow q \in \overline{\eta}_Q \circ f \circ \overline{\eta}_P(X)$ is a $cfcm$ from P to Q.

(2) The restriction of C to $fix_\eta(P, Q)$ is bijective from $fix_\eta(P, Q)$ to $hom(P, Q)$ where $hom(P, Q)$ denotes the set of all $cfcms$ from P to Q.

Lemma 4.7 *Let $R : P \to Q$ be a $cfcm$. Then $\mathcal{F}[R] : \mathcal{F}[P] \to \mathcal{F}[Q] : x \mapsto \{b \mid \exists X \in \mathscr{F}_P(X \subseteq x \wedge (XRb))\}$ is a Scott continuous function.*

Lemma 4.8 *$\mathcal{F}$ is a faithful and full functor from* **Cct** *to* **Alg**.

Theorem 4.9 **Cct** $\sim$ **Alg**.

Proof. It is an immediate conclusion of the combination of Theorem 3.15, Lemma 4.8 and Theorem 1(3) of Chapter 4 in [17].

5 Cartesian closed categories

Domain theory stems from programming language semantics. Only Cartesian closed full subcategories of $\mathbf{ALG}_\perp$ are of interest in this case since any advanced algorithmic language allows the formation of function spaces as data type, so, we investigate the pointed bifinite domains, pointed $L-$domains in this section.

Lemma 5.1 *Let P be a cfc. Then $\mathcal{F}[P]$ has the bottom if and only if*

$$(PT) \quad \emptyset \in \mathscr{F}_P \vee \exists X_0 \in \mathscr{F}_P \forall X \in \mathscr{F}_P(X_0 \subseteq \eta_P(X)).$$

Lemma 5.2 *Let P be a cfc. An $x \in \mathcal{W}[P]$ is an F-approximable concept if and only if $\overline{\eta}_P(x) = x$.*

Definition 5.3 Let P be a cfc and let $F \in Fin(P_o)$. $Z \in \mathscr{F}_P$ with $F \subseteq Z$ is called a local least upper-bound of F if for all $V, Y \in \mathscr{F}_P$, $F \subseteq Y$ and $Z \cup Y \subseteq \eta_P(V)$ imply $Z \subseteq \eta_P(Y)$.

We use $\sup(F)$ to denote the set of all local least upper-bounds of F. Clearly, for each $F \in \mathscr{F}_P$, $F \in \sup(F)$ from the above definition. Consider the subsequent condition:

(L): $\forall X \in \mathscr{F}_P \forall F \in Fin(P_o)(F \subseteq \eta_P(X) \Rightarrow \exists Z \in \sup(F)(Z \subseteq \eta_P(X)))$.

Proposition 5.4 *For every pointed $L-$domain, $P[L] = ((\kappa(L), L, \leq), \mathscr{F}_L)$ meets (L).*

Proof. Let $F \in Fin(\kappa(L))$ and $X \in \mathscr{F}_L$ such that $F \subseteq \eta_{P(L)}(X)$. Then $F \subseteq \downarrow \bigvee X \cap \kappa(L)$. From the definition of $\mathscr{F}_L$, there is an $a_X \in X$ such that $a \leq a_X$ for all $a \in X$. Since L is an $L-$domain, there is a unique minimal upper bound z of F in $\downarrow a_X$ from Theorem 2.3. But $z \in \kappa(L)$ from Proposition 1.9 [2], Put $Z = F \cup \{z\}$. Then $Z \in \mathscr{F}_L$, and $Z \subseteq \eta_{P(L)}(X)$.

Now let $Y \in \mathscr{F}_L$ and $Y \supseteq F$. If $Y \cup Z \subseteq \eta_{P(L)}(V)$ for some $V \in \mathscr{F}_L$. Note that $\bigvee Y$ and z are the upper bounds of F in $\downarrow \bigvee V$. From the choice of z , we have $z \leq \bigvee Y$, hence $Z \subseteq \eta_{P(L)}(Y)$. We have proved $Z \in \sup(F)$.

We have done. □

Proposition 5.5 *Let P be a cfc which satisfies (L). Then for each $x \in \mathcal{F}[P]$, $\downarrow x$ is join-complete.*

Proof. Let $x \in \mathcal{F}[P]$ and $x_1, x_2 \subseteq x$. We only need to show that x_1 and x_2 have a least upper bound in $\downarrow x$.

Put $z = \{a \mid \exists F \in Fin(x_1 \cup x_2) \exists X \in \mathscr{F}_P \exists Z \in \sup(F)(X \subseteq x \wedge Z \subseteq \eta_P(X) \wedge a \in \eta_P(Z))\}$.

Claim 1. $z \in \mathcal{W}[P]$

Given an arbitrary $F \in Fin(z)$. Then for every $t \in F$, there are $X_t \subseteq x$ and $F_t \subseteq Fin(x_1 \cup x_2)$ and $Z_t \in \sup(F_t)$ such that $Z_t \subseteq \eta_P(X_t)$ and $t \in \eta_P(Z_t)$. Put $F' = \bigcup\{F_t \mid t \in F\}$ and $X' = \bigcup\{X_t \mid t \in F\}$. We have $X_F \in \mathscr{F}_P$ with $X' \subseteq X_F \subseteq x$, whence $F' \subseteq \eta_P(X_F)$. By (L), there is $Z_{F'} \in \sup(F')$ such that $Z_{F'} \subseteq \eta_P(X_F)$. Since for every $t \in F$, $F_t \subseteq F' \subseteq Z_{F'}$ and $Z_t \cup Z_{F'} \subseteq \eta_P(X_F)$, $Z_t \subseteq \eta_P(Z_{F'})$ by the definition of Z_t. Therefore, $F \subseteq \eta_P(Z_{F'})$, and hence there is $Y \in \mathscr{F}_P$ such that $F \subseteq G \subseteq \eta_P(Z_{F'})$. Note that for each $g \in G$, $g \in \eta_P(Z_{F'})$ and $Z_{F'} \subseteq \eta_P(X_F)$ and $X_F \subseteq x$ and $F' \subseteq x_1 \cup x_2$. $G \subseteq z$ by the definition of z.

Claim 2. $z \in \mathcal{F}[P]$

(Following Claim 1) $F \subseteq \eta_P(Z_{F'})$ implies $\eta_P(F) \subseteq \eta_P^2(Z_{F'}) = \eta_P(Z_{F'})$, that is, $a \in \eta_P(Z_{F'})$ for every $a \in \eta_P(F)$. Thus $\eta_P(F) \subseteq z$.

Claim 3. z is a least upper bound of x_1 and x_2 in $\downarrow x$.

We first show $x_i \subseteq z$ for $i = 1, 2$. Let $a \in x_i$. Then there is $Y \in \mathscr{F}_P$ and $Y \subseteq x_i$ such that $a \in \eta_P(Y)$. $Y \subseteq x_i \subseteq x$ means that there exists $Y' \in \mathscr{F}_P$ and $Y' \subseteq x$ with $Y \subseteq \eta_P(Y')$. Note that $Y \in \sup(Y)$. Thus $a \in z$ since $Y \subseteq x_i \subseteq x_1 \cup x_2$.

Let $y \in \mathcal{F}P$ such that $x_1, x_2 \subseteq y \subseteq x$ and $a \in z$. Then there are $X \subseteq x$ and $F \subseteq Fin(x_1 \cup x_2)$ and $Z \in \sup(F)$ such that $Z \subseteq \eta_P(X)$ and $a \in \eta_P(Z)$. $F \subseteq Fin(x_1 \cup x_2)$ implies that there is V such that $F \subseteq V \in \mathscr{F}_P$ and $V \subseteq y \subseteq x$. Again we have another $X' \in \mathscr{F}_P$ such that $X \cup V \subseteq X' \subseteq x$. Thus $X \cup V \subseteq \eta_P(X')$. $Z \subseteq \eta_P(X)$ infers $Z \subseteq \eta_P(X \cup V) \subseteq \eta_P^2(X') = \eta_P(X')$. Thus $Z \cup V \subseteq \eta_P(X')$. We have $a \in \eta_P(V) \subseteq y$ by the definition of Z. We have obtained $z \subseteq y$. □

Let's consider two subcategories of **Cctb**, **Cctl** of the category **Cct**, where the objects of these subcategories are:

- **Cctb**: The cfcs with (PT) and (B)
- **Cctl**: The cfcs with (PT) and (L).

Theorem 5.6 (1) **Cctb** $\sim$ **B**; (2) **Cctl** $\sim$ **L**.

Proof. We repeatedly employ Theorem 1(3) of Chapter 4 in [17]. Then: (1) It is a immediate conclusion of Theorem 3.15, Lemma 4.8 and Proposition **??**. (2) It is clear by Theorem 3.15, Lemma 4.8, Propositions 5.4 and 5.5.

6 Conclusions

This work relates two independent fields FCA and algebraic domains. A basic idea is the approximation of targets along a road which is called a consistent set, so that we can work out of the hierarchy of complete lattices. The clustering data in format of domains is not only a kind of organization of data but also a procedure of approximation. All results of categorical equivalence in this work are confined in algebraic domains, but we look forward to outcomes with respect to continuous *dcpo*s.

References

[1] A. Edalat, R. Heckmann. A computational model for metric space. Theoretical Computer Science, 1998, 193(1-2): 53-73.

[2] A. Jung. Cartesian closed categories of Domains. PhD thesis, Techinische Hochschule Darmstsdt, 1988.

[3] B. A. Davey, H. A. Priestley. Introduction to Lattices and Order (Second edition), Cambridge University Press, 2002.

[4] B. Ganter, R. Wille. Formal Concept Analysis - Mathematical Foundations, Springer, Berlin, 1999.

[5] D. Scott. Outline of a mathematical theory of computation. In 4th Annual Princeton Conference on Information Sciences and Systemss. 169-176, 1970.

[6] D. Scott. Domains for denotational semantics. In: Automata, Languages and Programming, M. Nielsen *et al.*(Eds.), Lecture notes in computer science, 1982, 140: 109-129.

[7] D. Spreen, L. Xu, X. Mao. Information systems revisited: The general continuous case. Theoretical Computer Science, 2008, 405:176-187.

[8] D. Spreen. Representation L-Domains as information systems. www.uni-siegen.de /fb6 /tcs /team /spreen /public.

[9] G. Gierz, K. H. Hofmann, *et al.* Continuous lattices and Domains. Cambridge university press, 2003.

[10] G-Q. Zhang, G. Shen. Approximable concepts, Chu spaces, and information systems. In de Paiva and Pratt (Guset Edition). Theory and Applications of Categories Special Volume on Chu Spaces: Theory and Applications, Vol. 17, No. 5, pp: 80-102, 2006.

[11] J. Blanck. Domain representations of topological spaces. Theoretical Computer Science, 2002, 247: 229-255.

[12] M. Erné, J. Koslowski, A. Melton, G.Strecker. A primer on Galois connections. Vol. 704. Annals of the New York Academy of Sciences; 1993. pp: 103-125.

[13] P. Hitzler, G-Q. Zhang. A Cartesion closed category of approximable concept structures. In: Proceedings of the International Conference on Conceptual Structures, July 2004, in: Lecture Notes in Computer Science, Vol. 3127 (2004) 170-185.

[14] R. Heckmann. Approximation of metric spaces by partial metric spaces. Applied Categorical Structures, 1999, 7: 71-83.

[15] R. Hoofman. Continuous Information systems. Information and Computation, 1993, 105: 42-71.

[16] R. Bělohlávek. Concept lattices and order in fuzzy logic. Ann. Pure Appl. Logic. 2004, 128: 277-298.

[17] S. M. Lane. Categories for the working mathematician. Springer-Verlag, New York, 1971.

[18] S. Krajči. A generalzied concept lattice. Logic Journal of IGPL, 2005, 13(5):543-550.

[19] S. Pollandt. Fuzzy Begriffe. Springer-Verlag, Berlin/Heidelberg, 1997.

[20] S. Abramsky. Domain theory in a logic form. In Symposium on Logic in Computer Science, Computer Society Press of the IEEE, 1987.

[21] Y. Ershov. Computable functionals of finie type. Algebra and Logic, 11(4): 203-242, 1972.