Available online at www.sciencedirect.com

ScienceDirect

Electronic Notes in Theoretical Computer Science 333 (2017) 89–101

Electronic Notes in Theoretical Computer Science

www.elsevier.com/locate/entcs

Characterizations of Various Continuities of Posets Via Approximated Elements

Xuxin Mao[2]

College of Science
Nanjing University of Aeronautics and Astronautics
Nanjing 210016, P. R. China

Luoshan Xu[1,3]

Mathematics Department
Yangzhou University
Yangzhou 225002, P. R. China

Abstract

In this paper, various continuities of posets which may not be dcpos are considered. The concepts of approximated elements and hyper-approximated elements on posets are introduced. New characterizations of continuous posets and hypercontinuous posets are given. Meanwhile, as a generalization of approximated elements, the concept of quasi-approximated elements on dcpos is introduced and some characterizations of quasicontinuous domains are also obtained. It is proved that under some reasonable conditions, the set $\mathrm{B}(L)$ (resp., $\mathrm{QB}(L)$) of approximated elements (resp., quasi-approximated elements) in the induced order of a dcpo L is a continuous domain (resp., a quasicontinuous domain).

Keywords: continuous poset; hypercontinuous poset; approximated element; interpolation property; quasicontinuous domain

1 Introduction

The notion of continuous lattices as a model for the semantics of programming languages was introduced by Scott in [10]. Later, a more general notion of continuous directed complete partially ordered sets (i.e., continuous dcpos or domains) was introduced and extensively studied (see [1], [5], [6]). Since some naturally arising

[1] Supported by the Natural Science Foundation of China (11671008, 11101212), the Fund of University Speciality Construction (PPZY2015B109) of Jiangsu Province and the University Science Research Project of Jiangsu Province (15KJD110006)

[2] Email: xuxinmao@yahoo.com

[3] Email: luoshanxu@hotmail.com

https://doi.org/10.1016/j.entcs.2017.08.008

posets are important but fail to be directed complete, there are more and more occasions to study posets which fail to possess all directed suprema (see [7]-[9], [11]-[14]). Lawson in [6] gave a remarkable characterization that a dcpo is continuous iff its Scott topology is completely distributive. By the technique of embedded bases and sobrification via the Scott topology, Xu in [11] successfully embedded continuous posets into continuous domains and proved that a poset is continuous iff its Scott topology is completely distributive. Quasicontinuous domains were introduced by Gierz, Lawson and Stralka (see [3]) as a common generalization of both generalized continuous lattices (see [2]) and continuous domains. It was proved that quasicontinuous domains have many properties similar to that of continuous domains and quasicontinuous domains equipped with the Scott topologies are precisely the spectra of distributive hypercontinuous lattices. In terms of intrinsic topologies on posets, Mao and Xu in [8] introduced the concept of hypercontinuous posets and quasicontinuous posets and proved that a poset is quasicontinuous iff its Scott topology is a hypercontinuous lattice.

In [15], Zhao introduced the concept of weakly approximated elements on complete lattices and gave several characterizations of continuous lattices and completely distributive lattices. According to Zhao, an element x of a complete lattice L is said to be weakly approximated if it holds that $x = \bigvee \Downarrow x$. Zhao derived a novel characterization [15, Theorem 3] of continuous lattice that a complete lattice L is continuous iff L satisfies (i) the interpolation property (INT) for the way-below relation $\ll$ on L and (ii) for any $x, y \in L$, $x \neq y$ implies $\Downarrow x \neq \Downarrow y$. He also constructed two counterexamples (see [15, p.163]) to show that none of the conditions (i) and (ii) may be omitted.

In this paper, we introduce the concepts of approximated elements and hyper-approximated elements on posets and discuss some properties and relations of approximated elements and hyper-approximated elements. With these new concepts, we give several characterizations of continuous posets and hypercontinuous posets, generalizing relevant results in [15]. Meanwhile, as a generalization of approximated elements, the concept of quasi-approximated elements on dcpos is also introduced and some new characterizations of quasicontinuous domains are obtained. We will prove that under some reasonable conditions, the set $\mathrm{B}(L)$ (resp., $\mathrm{QB}(L)$) of approximated elements (resp., quasi-approximated elements) in the induced order of a dcpo L is a continuous domain (resp., a quasicontinuous domain).

2 Preliminaries

We quickly recall some basic notions and results (see [1], [3], [8] and [11]). Let $(L, \leq)$ be a poset. A *principal ideal* (resp., *principal filter*) is a set of the form $\downarrow x = \{y \in L \mid y \leqslant x\}$ (resp., $\uparrow x = \{y \in L \mid x \leqslant y\}$). For $A \subseteq L$, we write $\downarrow A = \{y \in L \mid \exists\, x \in A,\, y \leqslant x\}$, $\uparrow A = \{y \in L \mid \exists\, x \in A,\, x \leq y\}$. A subset A is a(n) *lower set* (resp., *upper set*) if $A = \downarrow A$ (resp., $A = \uparrow A$). We say that z is a(n) *lower bound* (resp., *upper bound*) of A if $A \subseteq \uparrow z$ (resp., $A \subseteq \downarrow z$). The set of lower bounds of A is denoted by $\mathrm{lb}(A)$. The supremum of A is denoted by $\bigvee A$ or $\sup A$. The infimum of A is denoted by $\bigwedge A$ or $\inf A$. A nonempty subset D of L is *directed* if every finite subset of D has an upper bound in D. A poset L is a *directed complete partially ordered set* (dcpo, for short) if every directed subset of L has a supremum. A *complete lattice* is a poset in which every subset has a supremum.

In a poset L, we say that x *approximates* y, written $x \ll y$ if whenever D is a directed set that has a supremum $\sup D \geqslant y$, then $x \leqslant d$ for some $d \in D$. For $x \in L$, we write $\twoheaddownarrow x = \{z \in L \mid z \ll x\}$ and $\twoheaduparrow x = \{z \in L \mid x \ll z\}$. The poset L is said to be *continuous* if every element is the directed supremum of elements that approximate it, i.e., for all $x \in L$, the set $\twoheaddownarrow x$ is directed and $x = \bigvee \twoheaddownarrow x$. A continuous poset which is also a dcpo is called a *continuous domain* or *domain*. A continuous poset which is also a complete lattice is called a *continuous lattice*.

A subset A of a poset L is *Scott closed* if $\downarrow A = A$ and for any directed set $D \subseteq A$, $\sup D \in A$ whenever $\sup D$ exists. The complements of the Scott closed sets form a topology, called the *Scott topology* and denoted by $\sigma(L)$. It is well-known that for a continuous poset the Scott topology has a base of all sets of the form $\twoheaduparrow x$. The topology generated by the complements of all principal filters $\uparrow x$ (resp., principal ideals $\downarrow x$) is called the *lower topology* (resp., *upper topology*) and denoted by $\omega(L)$ (resp., $\nu(L)$).

Proposition 2.1 (see [1,13]) *Let L be a continuous poset. Then the following are true:*
(1) The relation $\ll$ has the interpolation property: $x \ll z \Longrightarrow \exists y \in L$ such that $x \ll y \ll z$;
(2) The Scott open filters form a topological basis of $\sigma(L)$.

Let L be a poset. We define a binary relation $\prec_{\nu(L)}$ on L such that $x \prec_{\nu(L)} y \Longleftrightarrow y \in int_{\nu(L)}\uparrow x$, where the interior is taken in the upper topology $\nu(L)$.

Proposition 2.2 (see [8, Proposition 3.1]) *Let L be a poset. Then for all u, x, y, $z \in L$:*
(1) *$x \prec_{\nu(L)} y$ implies $x \ll y$;*
(2) *$u \leqslant x \prec_{\nu(L)} y \leqslant z$ implies $u \prec_{\nu(L)} z$;*
(3) *$x \prec_{\nu(L)} z$ and $y \prec_{\nu(L)} z$ imply $x \vee y \prec_{\nu(L)} z$ whenever the join $x \vee y$ exists in L;*
(4) *$\perp \prec_{\nu(L)} x$ whenever L has a smallest element $\perp$.*

Definition 2.3 (see [8]) *A poset L is called a hypercontinuous poset if for all $x \in L$, the set $\Downarrow_{\prec_{\nu(L)}} x = \{y \in L \mid y \prec_{\nu(L)} x\}$ is directed and $x = \sup \Downarrow_{\prec_{\nu(L)}} x$.*

Lemma 2.4 (see [8, Proposition 3.5]) *Let L be a poset. Then the following statements are equivalent:*
(1) *L is a hypercontinuous poset;*
(2) *L is a continuous poset and for all x, $y \in L$, $x \ll y$ implies $x \prec_{\nu(L)} y$;*
(3) *L is a continuous poset and the Scott topology $\sigma(L)$ is the upper topology $\nu(L)$.*

Remark 2.5 *Applying Proposition 2.2(1) and Lemma 2.4, one can easily see that the relation $\prec_{\nu(L)}$ on a hypercontinuous poset L is precisely the approximation relation $\ll$ of L. By Proposition 2.1(1), the relation $\prec_{\nu(L)}$ on a hypercontinuous poset L satisfies the interpolation property: $x \prec_{\nu(L)} z \Longrightarrow \exists y \in L$ such that*

$x \prec_{\nu(L)} y \prec_{\nu(L)} z$.

For a set X, we use $\mathcal{P}(X)$ to denote the powerset of X and $\mathcal{P}_{fin}(X)$ to denote the set of all nonempty finite subsets of X. For a dcpo L, we define a preorder $\leq$ (sometimes called *Smyth preorder*) on $\mathcal{P}(L)\setminus\{\emptyset\}$ by $G \leq H$ iff $\uparrow H \subseteq \uparrow G$ for all $G, H \subseteq L$. That is, $G \leq H$ iff for each $y \in H$ there is an element $x \in G$ with $x \leq y$. We say that a nonempty family $\mathcal{F}$ of subsets of L is *directed* if it is directed in the Smyth preorder. More precisely, $\mathcal{F}$ is directed if for all F_1, $F_2 \in \mathcal{F}$, there exists $F_3 \in \mathcal{F}$ such that F_1, $F_2 \leq F_3$, i.e., $F_3 \subseteq \uparrow F_1 \cap \uparrow F_2$. We say that G is *way below* H or G *approximates* H and write $G \ll H$ if for every directed set $D \subseteq L$, $\sup D \in \uparrow H$ implies $d \in \uparrow G$ for some $d \in D$. We write $G \ll x$ for $G \ll \{x\}$ and $y \ll H$ for $\{y\} \ll H$. For all $G \subseteq L$, we write $\Downarrow G = \{y \in L \mid y \ll G\}$ and $\Uparrow G = \{y \in L \mid G \ll y\}$.

Proposition 2.6 *(see [1,3]) Let L be a dcpo. Then*
(1) $\forall\, G, H \subseteq L$, $G \ll H \Longrightarrow G \leq H$;
(2) $\forall\, G, H \subseteq L$, $G \ll H \Longleftrightarrow \forall\, h \in H, G \ll h$;
(3) $\forall\, E, F, G, H \subseteq L$, $E \leq F \ll G \leq H \Longrightarrow E \ll H$;
(4) $\forall\, x, y \in L$, $\{x\} \ll \{y\} \Longleftrightarrow x \ll y$.

Lemma 2.7 (see [1, Corollary III-3.4]) *Let $\mathcal{F}$ be a directed family of nonempty finite sets in a dcpo. If $G \ll H$ and $\bigcap_{F\in\mathcal{F}} \uparrow F \subseteq \uparrow H$, then $F \subseteq \uparrow G$ for some $F \in \mathcal{F}$.*

As a generalization of continuous domains, the following definition gives the concept of quasicontinuous domains.

Definition 2.8 (see [1,3]) *A dcpo L is called a quasicontinuous domain if it satisfies the following two conditions:*
(1) *For each $x \in L$, $\mathrm{fin}(x) = \{F \subseteq L \mid F$ is finite and $F \ll x\}$ is a directed family;*
(2) *For all x, $y \in L$, if $x \not\leq y$, then there exists $F \in \mathrm{fin}(x)$ such that $y \notin \uparrow F$.*

One can also derive the interpolation property for quasicontinuous domains.

Proposition 2.9 (see [1, Proposition III-3.5]) *Let L be a quasicontinuous domain, $H \subseteq L$ and $x \in L$. If $H \ll x$, then there exists a finite set F such that $H \ll F \ll x$.*

3 Continuous posets and approximated elements

In this section, we introduce the concept of approximated elements and give several characterizations of continuous posets.

Definition 3.1 *Let L be a poset and $x \in L$. If there is a directed set $D_x \subseteq \Downarrow x$ such that $\bigvee D_x = x$, then x is called an approximated element. The set of all approximated elements of L is denoted by $\mathrm{B}(L)$.*

Proposition 3.2 *Let L be a poset and $x \in L$. Then x is an approximated element iff the set $\Downarrow x$ is directed and $\bigvee \Downarrow x = x$.*

Proof. $\Longleftarrow$: Straightforward.

$\Longrightarrow$: If x is an approximated element, then there is a directed set $D_x \subseteq \Downarrow x$ such that $\bigvee D_x = x$. It is easy to prove that $\bigvee \Downarrow x = x$. We next show that the set $\Downarrow x$ is directed. For all $a, b \in \Downarrow x$, since $a \ll x = \bigvee D_x$ and D_x is directed, there is $d_a \in D_x$ such that $a \leqslant d_a$. Similarly, there is $d_b \in D_x$ such that $b \leqslant d_b$. By the directedness of D_x, there is some $d \in D_x \subseteq \Downarrow x$ such that $d_a, d_b \leqslant d$, showing that the set $\Downarrow x$ is directed. □

Remark 3.3 *In view of the fact that in a complete lattice L, for any $x \in L$ one has that $\Downarrow x$ is directed. So, by Proposition 3.2, an element x of L is a weakly approximated element (in the sense of [15]) iff it is an approximated element in the sense of Definition 3.1, revealing that the notions of approximated elements and weakly approximated elements coincide in a complete lattice. However, for a poset L, we will see in Proposition 3.6 that the requirement for the directedness of $D_x \subseteq \Downarrow x$ is indeed important. This is the main reason for us to give the concept of approximated elements.*

Proposition 3.4 *Let L be a poset. Then L is a continuous poset iff every element of L is an approximated element.*

Proof. It follows from the definition of continuous poset and Proposition 3.2. □

Recall that a basis of a poset L is a set $B \subseteq L$ such that for every $x \in L$, the subset $B \cap \Downarrow x$ is directed and $x = \bigvee(B \cap \Downarrow x)$. It is well-known that a poset is continuous iff it has a basis. By Proposition 3.4, we have immediately the following

Corollary 3.5 *A poset L has a basis if and only if every element of L is an approximated element.*

One will be interested in giving a direct proof of the above corollary. Clearly, the existence of a basis for L implies that every element of L is approximated. Conversely, for all $x \in L$, choose a directed set $D_x \subseteq \Downarrow x$ such that $\bigvee D_x = x$. Let $B = \bigcup_{x \in L} D_x$. Then for all $x \in L$, clearly $\bigvee(B \cap \Downarrow x) = x$. To show $B \cap \Downarrow x$ is directed, let $s, t \in B \cap \Downarrow x$. Then there exists an element $d \in D_x \subseteq B \cap \Downarrow x$ such that $s, t \leqslant d$, revealing that $B \cap \Downarrow x$ is directed. Thus B is a basis of L.

Proposition 3.6 *Let L be a dcpo. If $\mathrm{B}(L)$ is nonempty, then $\mathrm{B}(L)$ is closed under the directed sups of L and thus a dcpo. Moreover, for all $x, y \in \mathrm{B}(L)$, $x \ll y$ implies $x \ll_{\mathrm{B}(L)} y$, where $\ll_{\mathrm{B}(L)}$ means the approximation relation on $\mathrm{B}(L)$.*

Proof. To show that $\mathrm{B}(L)$ is closed under the directed sups, let $S \subseteq \mathrm{B}(L)$ be a directed set. For each $s \in S$, by Definition 3.1, there is a directed set $D_s \subseteq \Downarrow s$ such that $\bigvee D_s = s$. Since $s \leqslant \bigvee S$, we have $D_s \subseteq \Downarrow s \subseteq \Downarrow(\bigvee S)$. Let $D_{\bigvee S} = \bigcup_{s \in S} D_s$. Clearly, $D_{\bigvee S} \subseteq \Downarrow(\bigvee S)$. For all $x_a, x_b \in D_{\bigvee S}$, there exist $a, b \in S$ such that $x_a \in D_a \subseteq \Downarrow a$ and $x_b \in D_b \subseteq \Downarrow b$. By the directedness of S, there is $c \in S$ such that $a, b \leqslant c$. So, $x_a \ll c$ and $x_b \ll c$. Since D_c is directed and $c = \bigvee D_c$, there exist $d, e \in D_c$ such that $x_a \leqslant d$ and $x_b \leqslant e$. By the directedness of D_c, there is $w \in D_c \subseteq D_{\bigvee S}$ such that $d, e \leqslant w$. Thus there is $w \in D_{\bigvee S}$ such that $x_a, x_b \leqslant w$, showing the directedness of $D_{\bigvee S}$. Clearly, $\bigvee S$ is an upper bound of the set $D_{\bigvee S} = \bigcup_{s \in S} D_s$. Let t be any upper bound of $D_{\bigvee S}$. Then for each $s \in S$, $s = \bigvee D_s \leqslant t$. So, $\bigvee S \leqslant t$, showing that $\bigvee S$ is the least upper bound of the set $D_{\bigvee S}$. By Definition 3.1, $\bigvee S \in \mathrm{B}(L)$, showing that $\mathrm{B}(L)$ is closed under the directed sups. With the above result, the second part of the proposition is clear. □

Proposition 3.7 *Let L be a poset and the relation $\ll$ on L has the interpolation property. If for all $x \in L$, the set $\Downarrow x$ is directed and $\bigvee \Downarrow x$ exists, then there exists an approximated element $B(x)$ satisfying:* (i) $\Downarrow B(x) = \Downarrow x$; (ii) $\forall y \in L$, $\Downarrow y = \Downarrow x$ *implies* $y \geqslant B(x)$.

Proof. Let L be a poset. If for all $x \in L$, the set $\Downarrow x$ is directed and $\bigvee \Downarrow x$ exists, then let $B(x) = \bigvee \Downarrow x$. We next show that $B(x)$ is an approximated element satisfying condition (i) and (ii). Since $B(x) \leqslant x$, we have $\Downarrow B(x) \subseteq \Downarrow x$. Let $a \in \Downarrow x$. By the interpolation property of the relation $\ll$, there is $b \in L$ such that $a \ll b \ll x$. Thus we have $a \ll b \leqslant B(x)$ and $a \in \Downarrow B(x)$. This shows that $\Downarrow x \subseteq \Downarrow B(x)$ and thus $\Downarrow B(x) = \Downarrow x$. So, the element $B(x)$ satisfies condition (i). By condition (i), it is easy to prove that $B(x)$ is an approximated element. For all $y \in L$, if $\Downarrow y = \Downarrow x$, then $y \geqslant \bigvee \Downarrow x = B(x)$. Thus, $B(x)$ also satisfies condition (ii). □

Theorem 3.8 *Let L be a dcpo and the relation $\ll$ on L has the interpolation property. If for all $x \in L$, the set $\Downarrow x$ is directed, then $\mathrm{B}(L)$ in the induced order is a continuous domain.*

Proof. For each $u \in L$, by Proposition 3.7, there exists an approximated element $B(u)$ such that $\Downarrow B(u) = \Downarrow u$ and $B(u) \leqslant u$. So, $\mathrm{B}(L)$ is nonempty. By Proposition 3.6, $\mathrm{B}(L)$ in the induced order is a dcpo. For each $x \in \mathrm{B}(L)$, let $D_x = \Downarrow x \cap \mathrm{B}(L)$. It follows from Proposition 3.6 that $D_x \subseteq \Downarrow_{\mathrm{B}(L)} x$, where the set $\Downarrow_{\mathrm{B}(L)} x = \{y \in \mathrm{B}(L) \mid y \ll_{B(L)} x\}$. By the directedness of $\Downarrow x$, there is $v \in \Downarrow x$ such that $v \ll x$. By Proposition 3.7 again, there exists an approximated element $B(v)$ such that $B(v) \leqslant v \ll x$. This shows that $B(v) \in \Downarrow x \cap B(L) = D_x$ and thus D_x is nonempty. For all $a, b \in D_x$, by the directedness of $\Downarrow x$, there is $c \in \Downarrow x$ such that $a, b \leqslant c \ll x$. Since the relation $\ll$ on L has the interpolation property, there is $e(x)$ such that $a, b \leqslant c \ll e(x) \ll x$. By Proposition 3.7, there exists an approximated element $B(e(x))$ such that $\Downarrow B(e(x)) = \Downarrow e(x)$ and $B(e(x)) \leqslant e(x)$. So, $B(e(x)) \in \Downarrow x \cap \mathrm{B}(L) = D_x$ and $a, b \leqslant c \ll B(e(x))$. This shows the directedness of the set D_x.

We next show that $\bigvee_{\mathrm{B}(L)} D_x = x$, where the subscript $\mathrm{B}(L)$ means to take relevant operations in poset $\mathrm{B}(L)$. Clearly, x is an upper bound of D_x. Let t be any upper bound of D_x in $B(L)$. For all $z \in \Downarrow x$, since the relation $\ll$ on L has the interpolation property, there is $h(x)$ such that $z \ll h(x) \ll x$. By Proposition 3.7, there exists an approximated element $B(h(x))$ such that $\Downarrow B(h(x)) = \Downarrow h(x)$ and $B(h(x)) \leqslant h(x)$. Thus $z \ll B(h(x)) \leqslant h(x) \ll x$ and $B(h(x)) \in \Downarrow x \cap \mathrm{B}(L) = D_x$. This shows that $z \leqslant t$ and hence $\bigvee \Downarrow x = x \leqslant t$. So, x is the least upper bound of D_x in $\mathrm{B}(L)$, i.e., $x = \bigvee_{\mathrm{B}(L)} D_x$. By Definition 3.1, x is an approximated element of $B(L)$. It follows from the arbitrariness of x and Proposition 3.4 that $B(L)$ in the induced order is a continuous domain. □

Remark 3.9 *Note that in a complete lattice L, for each $x \in L$, the set $\Downarrow x$ is automatically directed. So, Theorem 3.8 is a generalization of [15, Theorem 2] for continuous lattices.*

The following example shows that the requirement for the set $\Downarrow x$ is directed can not be omitted from Theorem 3.8.

Example 3.10 *In the dcpo L obtained by pasting the two top elements in the two paralleled closed intervals $[0,1]$, $\Downarrow x$ is empty for each $x \in L$ and thus not directed. The set $\mathrm{B}(L)$ of all approximated elements is empty and is not a continuous domain.*

Zhao constructed [15, Example 2] a complete lattice $L = \{a\} \cup [0,1]$ with $a \notin [0,1]$. He defined the partial order on L such that $1/2 < a < 1$ and that the order of elements in $[0,1]$ is in the ordinary order of $[0,1]$. For this L, one sees that $\mathrm{B}(L) = [0,1]$ is continuous but the relation $\ll$ does not have the interpolation property (INT). Considering Zhao's example, we pose the following problem.

Problem Does Theorem 3.8 still hold if the condition "the relation $\ll$ on L has the interpolation property" is removed?

Theorem 3.11 *Let L be a poset. If for all $x \in L$, the set $\Downarrow x$ is directed and $\bigvee \Downarrow x$ exists, then L is continuous iff L satisfies the following two conditions: (1) the relation $\ll$ on L has the interpolation property; (2) $\forall\, x,\, y \in L,\ x \neq y \Longrightarrow \Downarrow x \neq \Downarrow y$.*

Proof. $\Longleftarrow$: Let $x \in L$. By Proposition 3.7, there exists an approximated element $B(x) \leqslant x$ such that $\Downarrow B(x) = \Downarrow x$. Applying condition (2), we have $x = B(x)$. Thus $L = \mathrm{B}(L)$. By Proposition 3.4, L is a continuous poset.

$\Longrightarrow$: By the definition of continuous posets and Proposition 2.1(1). □

Remark 3.12 *Theorem 3.11 deals with general posets. So, Theorem 3.11 generalizes the result [15, Theorem 3] mentioned in the introduction section.*

Definition 3.13 *Let L be a poset and $\Phi \subseteq \mathcal{P}(L)$ be a family of subsets of L. We say that Φ strongly separates points of L if for all $x,\, y \in L$ with $x \nleqslant y$, there exist $A \in \Phi$ and $u \in \mathrm{lb}(A)$ such that $x \in A$ and $u \nleqslant y$.*

Lemma 3.14 *Let L be a poset and $x,\, y \in L$ with $x \nleqslant y$. If there is a Scott open set U and $z \in \mathrm{lb}(U)$ such that $x \in U$ and $z \nleqslant y$, then $z \ll x$.*

Proof. Let $D \subseteq L$ be directed with existing $\sup D \geqslant x \in U$. By the Scott openness of U, we have $\sup D \in U$ and hence there is $d \in U \cap D$ such that $z \leqslant d$, showing that $z \ll x$. □

Theorem 3.15 *Let L be a poset. If for all $x \in L$, the set $\Downarrow x$ is directed, then L is continuous iff the family of all Scott open filters strongly separates points of L.*

Proof. $\Longleftarrow$: For each $x \in L$, we only need to show that $x = \bigvee \Downarrow x$. Clearly, x is an upper bound of the set $\Downarrow x$. Let t be any upper bound of $\Downarrow x$. Suppose that $x \nleqslant t$. Then there is a Scott open filter U and $z \in \mathrm{lb}(U)$ such that $x \in U$ and $z \nleqslant t$. By Lemma 3.14, $z \ll x$ but $z \nleqslant t$, a contradiction to the assumption that t is an upper bound of $\Downarrow x$. Therefore, $x \leqslant t$. This shows that x is the least upper bound of $\Downarrow x$, as desired.

$\Longrightarrow$: Let L be a continuous poset. For all $x,\, y \in L$ with $x \nleqslant y$, there is $z \ll x$ such that $z \nleqslant y$. It follows from $x \in \Uparrow z \in \sigma(L)$ and Proposition 2.1 (2) that there is a Scott open filter U such that $x \in U \subseteq \Uparrow z \subseteq \uparrow z$. So, $z \in \mathrm{lb}(U)$ and $z \not\leqslant y$. By Definition 3.13, the family of all Scott open filters strongly separates points of L.□

Corollary 3.16 (see [15, Theorem 1]) *A complete lattice L is continuous iff for all $x, y \in L$ with $x \not\leqslant y$, there exists a Scott open filter U such that $x \in U$ and $\bigwedge U \not\leqslant y$.*

Proof. Note that in a complete lattice L, the set $\twoheaddownarrow x$ is automatically directed for each $x \in L$. Then the corollary follows immediately from Theorem 3.15. □

By this corollary, one sees that for complete lattices, the concept of strongly separation in Definition 3.13 coincides with Zhao's [15, Definition 1]).

4 Hypercontinuity of posets and hyper-approximated elements

In this section, we introduce the concept of hyper-approximated elements and give several characterizations of hypercontinuous posets.

Definition 4.1 *Let L be a poset and $x \in L$. If there is a directed set $D_x \subseteq \Downarrow_{\prec_{\nu(L)}} x$ such that $\bigvee D_x = x$, then x is called a hyper-approximated element. The set of all hyper-approximated elements of L is denoted by* $\mathrm{HB}(L)$.

Proposition 4.2 *Let L be a poset and $x \in L$. If x is a hyper-approximated element, then x is an approximated element.*

Proof. Follows from Proposition 2.2(1), Definition 3.1 and Definition 4.1. □

Proposition 4.3 *Let L be a poset and $x \in L$. Then x is a hyper-approximated element iff the set $\Downarrow_{\prec_{\nu(L)}} x$ is directed and $\bigvee \Downarrow_{\prec_{\nu(L)}} x = x$.*

Proof. $\Longleftarrow$: Straightforward.

$\Longrightarrow$: If x is a hyper-approximated element, then there exists a directed set $D_x \subseteq \Downarrow_{\prec_{\nu(L)}} x$ such that $\bigvee D_x = x$. It is easy to prove that $\bigvee \Downarrow_{\prec_{\nu(L)}} x = x$. We next show that the set $\Downarrow_{\prec_{\nu(L)}} x$ is directed. For all $a, b \in \Downarrow_{\prec_{\nu(L)}} x$, by Proposition 2.2(1), we have $a \ll x$ and $b \ll x$. Since $a \ll x = \bigvee D_x$ and D_x is directed, there is $d_a \in D_x$ such that $a \leqslant d_a$. Similarly, there is $d_b \in D_x$ such that $b \leqslant d_b$. By the directedness of D_x, there is some $d \in D_x \subseteq \Downarrow_{\prec_{\nu(L)}} x$ such that $d_a, d_b \leqslant d$. This shows that there is $d \in \Downarrow_{\prec_{\nu(L)}} x$ such that $a, b \leqslant d$. Thus the set $\Downarrow_{\prec_{\nu(L)}} x$ is directed. □

Proposition 4.4 *Let L be a poset. Then L is a hypercontinuous poset iff every element of L is a hyper-approximated element.*

Proof. Follows from Definition 2.3 and Proposition 4.3. □

Proposition 4.5 *Let L be a poset and the relation $\prec_{\nu(L)}$ on L has the interpolation property. If for all $x \in L$, the set $\Downarrow_{\prec_{\nu(L)}} x$ is directed and $\bigvee \Downarrow_{\prec_{\nu(L)}} x$ exists, then there exists a hyper-approximated element $\mathrm{HB}(x)$ satisfying:* (i) $\Downarrow_{\prec_{\nu(L)}} \mathrm{HB}(x) = \Downarrow_{\prec_{\nu(L)}} x$; (ii) $\forall\, y \in L$, $\Downarrow_{\prec_{\nu(L)}} y = \Downarrow_{\prec_{\nu(L)}} x$ *implies* $y \geqslant \mathrm{HB}(x)$.

Proof. For all $x \in L$, let $\mathrm{HB}(x) = \bigvee \Downarrow_{\prec_{\nu(L)}} x$. We next show that $\mathrm{HB}(x)$ is a hyper-approximated element satisfying condition (i) and (ii). Since $\mathrm{HB}(x) \leqslant x$, we have $\Downarrow_{\prec_{\nu(L)}} \mathrm{HB}(x) \subseteq \Downarrow_{\prec_{\nu(L)}} x$. Let $a \in \Downarrow_{\prec_{\nu(L)}} x$. By the interpolation property of the relation $\prec_{\nu(L)}$, there is $b \in L$ such that $a \prec_{\nu(L)} b \prec_{\nu(L)} x$. Therefore, $a \prec_{\nu(L)} b \leqslant \mathrm{HB}(x)$ and $a \in \Downarrow_{\prec_{\nu(L)}} \mathrm{HB}(x)$. This shows that $\Downarrow_{\prec_{\nu(L)}} x \subseteq \Downarrow_{\prec_{\nu(L)}} \mathrm{HB}(x)$ and thus $\Downarrow_{\prec_{\nu(L)}} \mathrm{HB}(x) = \Downarrow_{\prec_{\nu(L)}} x$. So, the element $\mathrm{HB}(x)$ satisfies condition (i). By condition (i), it is easy to prove that $\mathrm{HB}(x)$ is a hyper-approximated element. For all $y \in L$, if $\Downarrow_{\prec_{\nu(L)}} y = \Downarrow_{\prec_{\nu(L)}} x$, then $y \geqslant \bigvee \Downarrow_{\prec_{\nu(L)}} x = \mathrm{HB}(x)$. Thus, $\mathrm{HB}(x)$ satisfies condition (ii), as desired. □

Theorem 4.6 *Let L be a poset. If for all $x \in L$, the set $\Downarrow_{\prec_{\nu(L)}} x$ is directed and $\bigvee \Downarrow_{\prec_{\nu(L)}} x$ exists, then L is a hypercontinuous poset iff L satisfies the following two conditions:* (1) *the relation $\prec_{\nu(L)}$ on L has the interpolation property;* (2) $\forall\, x, y \in L$, $x \neq y \Longrightarrow \Downarrow_{\prec_{\nu(L)}} x \neq \Downarrow_{\prec_{\nu(L)}} y$.

Proof. $\Longleftarrow$: Let $x \in L$. By Proposition 4.5, there is a hyper-approximated element $\mathrm{HB}(x) \leqslant x$ such that $\Downarrow_{\prec_{\nu(L)}} \mathrm{HB}(x) = \Downarrow_{\prec_{\nu(L)}} x$. Applying condition (2), we have $x = \mathrm{HB}(x)$. Thus $L = \mathrm{HB}(L)$. By Proposition 4.4, L is a hypercontinuous poset.

$\Longrightarrow$: By Definition 2.3 and Remark 2.5. □

Theorem 4.7 *Let L be a poset and the relation $\prec_{\nu(L)}$ on L has the interpolation property. If for all $x \in L$, the set $\Downarrow_{\prec_{\nu(L)}} x$ is directed, then L is a hypercontinuous poset iff the family of all $\nu(L)$-open filters strongly separates points of L.*

Proof. $\Longleftarrow$: For each $x \in L$, we only need to show that $x = \bigvee \Downarrow_{\prec_{\nu(L)}} x$. Clearly, x is an upper bound of the set $\Downarrow_{\prec_{\nu(L)}} x$. Let t be any upper bound of $\Downarrow_{\prec_{\nu(L)}} x$. Suppose that $x \nleqslant t$. Then there is a $\nu(L)$-open filter U and $z \in \mathrm{lb}(U)$ such that $x \in U \subseteq \Uparrow z$ and $z \nleqslant t$. Since U is $\nu(L)$-open, we have $x \in int_{\nu(L)} \uparrow z$. So, $z \prec_{\nu(L)} x$ but $z \nleqslant t$, a contradiction to the assumption that t is an upper bound of $\Downarrow_{\prec_{\nu(L)}} x$. Therefore, $x \leqslant t$. This shows that x is the least upper bound of $\Downarrow_{\prec_{\nu(L)}} x$, as desired.

$\Longrightarrow$: By Lemma 2.4 and Theorem 3.15. □

5 Quasicontinuous domains and quasi-approximated elements

In this section, we introduce the concept of quasi-approximated elements and give several characterizations of quasicontinuous domains.

Definition 5.1 *Let L be a dcpo and $x \in L$. If there is a directed family $\mathcal{D}_x \subseteq \mathrm{fin}(x)$ such that $\Uparrow x = \bigcap\{\uparrow F \mid F \in \mathcal{D}_x\}$, then x is called a quasi-approximated element. The set of all quasi-approximated elements of L is denoted by* $\mathrm{QB}(L)$.

Proposition 5.2 *Let L be a dcpo and $x \in L$. If x is an approximated element, then x is a quasi-approximated element.*

Proof. Let L be a dcpo and $x \in L$. If x is an approximated element, then there is a directed set $S_x \subseteq \Downarrow x$ such that $\bigvee S_x = x$. Let $\mathcal{D}_x = \{\{d\} \mid d \in S_x\}$. By Proposition

2.6(4), $\mathcal{D}_x \subseteq \mathrm{fin}(x)$. It is straightforward to show that $\mathcal{D}_x$ is a directed family and $\Uparrow x = \bigcap\{\Uparrow d \mid \{d\} \in \mathcal{D}_x\}$. So, x is a quasi-approximated element. □

Proposition 5.3 *Let L be a dcpo and $x \in L$. Then x is a quasi-approximated element iff the family* $\mathrm{fin}(x)$ *is directed and* $\Uparrow x = \bigcap\{\uparrow F \mid F \in \mathrm{fin}(x)\}$.

Proof. $\Longleftarrow$: Straightforward.

$\Longrightarrow$: If x is a quasi-approximated element, then there exists a directed family $\mathcal{D}_x \subseteq \mathrm{fin}(x)$ such that $\Uparrow x = \bigcap\{\uparrow H \mid H \in \mathcal{D}_x\}$. It is easy to verify that $\Uparrow x = \bigcap\{\uparrow F \mid F \in \mathrm{fin}(x)\}$. We next show that the family $\mathrm{fin}(x)$ is directed. For all $F_1, F_2 \in \mathrm{fin}(x)$, since $F_1 \ll x$ and $\Uparrow x = \bigcap\{\uparrow H \mid H \in \mathcal{D}_x\}$. By Lemma 2.7, there is $H_1 \in \mathcal{D}_x$ such that $H_1 \subseteq \uparrow F_1$. Similarly, there is $H_2 \in \mathcal{D}_x$ such that $H_2 \subseteq \uparrow F_2$. By the directedness of $\mathcal{D}_x$, there is some $H \in \mathcal{D}_x$ such that $H_1, H_2 \leqslant H$. Thus $\uparrow H \subseteq \uparrow H_1 \cap \uparrow H_2 \subseteq \uparrow F_1 \cap \uparrow F_2$. This shows that there is $H \in \mathcal{D}_x \subseteq \mathrm{fin}(x)$ such that $F_1, F_2 \leqslant H$. Therefore, the family $\mathrm{fin}(x)$ is directed, as desired. □

Proposition 5.4 *Let L be a dcpo. Then L is a quasicontinuous domain iff every element of L is a quasi-approximated element.*

Proof. By Definition 2.8 and Proposition 5.3. □

Proposition 5.5 *Let L be a dcpo. If* $\mathrm{QB}(L)$ *is nonempty, then* $\mathrm{QB}(L)$ *is closed under the directed sups of L and thus a dcpo. Moreover, for all $x \in \mathrm{QB}(L)$ and $F \in \mathcal{P}_{fin}(\mathrm{QB}(L))$, $F \ll x$ implies $F \ll_{\mathrm{QB}(L)} x$, where $\ll_{\mathrm{QB}(L)}$ means the approximation relation on* $\mathrm{QB}(L)$.

Proof. We show that $\mathrm{QB}(L)$ is closed under the directed sups of L. Let $S \subseteq \mathrm{QB}(L)$ be a directed set. For each $s \in S$, by Definition 5.1, there is a directed family $\mathcal{D}_s \subseteq \mathrm{fin}(s)$ such that $\Uparrow s = \bigcap\{\uparrow F \mid F \in \mathcal{D}_s\}$. Since $s \leqslant \bigvee S$, we have $\mathcal{D}_s \subseteq \mathrm{fin}(s) \subseteq \mathrm{fin}(\bigvee S)$. Let $\mathcal{D}_{\bigvee S} = \bigcup_{s\in S} \mathcal{D}_s$. Clearly, $\mathcal{D}_{\bigvee S} \subseteq \mathrm{fin}(\bigvee S)$. For all F_a, $F_b \in \mathcal{D}_{\bigvee S}$, there exist $a, b \in S$ such that $F_a \in \mathcal{D}_a \subseteq \mathrm{fin}(a)$ and $F_b \in \mathcal{D}_b \subseteq \mathrm{fin}(b)$. By the directedness of S, there is $c \in S$ such that $a, b \leqslant c$. So, $F_a \ll c$ and $F_b \ll c$. Since $\mathcal{D}_c$ is a directed family and $\Uparrow c = \bigcap\{\uparrow F \mid F \in \mathcal{D}_c\}$, by Lemma 2.7, there exist $F_1, F_2 \in \mathcal{D}_c$ such that $F_1 \subseteq \uparrow F_a$ and $F_2 \subseteq \uparrow F_b$. This shows that $F_a \leqslant F_1$ and $F_b \leqslant F_2$. By the directedness of $\mathcal{D}_c$, there is $F_3 \in \mathcal{D}_c \subseteq \mathcal{D}_{\bigvee S}$ such that $F_1, F_2 \leqslant F_3$. Thus there is $F_3 \in \mathcal{D}_{\bigvee S}$ such that $F_a, F_b \leqslant F_3$. This shows the directedness of $\mathcal{D}_{\bigvee S}$. Clearly, $\Uparrow(\bigvee S) \subseteq \bigcap\{\uparrow F \mid F \in \mathcal{D}_{\bigvee S}\}$. Let $t \in \bigcap\{\uparrow F \mid F \in \mathcal{D}_{\bigvee S}\}$. Then for all $s \in S$, we have $t \in \bigcap\{\uparrow F \mid F \in \mathcal{D}_s\} = \Uparrow s$. Thus, $\bigvee S \leqslant t$ and $t \in \Uparrow(\bigvee S)$. This shows that $\bigcap\{\uparrow F \mid F \in \mathcal{D}_{\bigvee S}\} \subseteq \Uparrow(\bigvee S)$. So, $\Uparrow(\bigvee S) = \bigcap\{\uparrow F \mid F \in \mathcal{D}_{\bigvee S}\}$. By Definition 5.1, $\bigvee S \in \mathrm{QB}(L)$, showing that $\mathrm{QB}(L)$ is closed under the directed sups.

Moreover, let $x \in \mathrm{QB}(L)$, $F \in \mathcal{P}_{fin}(\mathrm{QB}(L))$. Suppose that $F \ll x$. Let $S \subseteq \mathrm{QB}(L)$ be a directed set such that $\bigvee_{\mathrm{QB}(L)} S \geqslant x$, where $\bigvee_{\mathrm{QB}(L)} S$ means the supremum of S in $\mathrm{QB}(L)$. Since $\mathrm{QB}(L)$ is closed under the directed sups in L, we have $\bigvee S = \bigvee_{\mathrm{QB}(L)} S \geqslant x$. It follows from $F \ll x$ that $S \cap \uparrow F \neq \emptyset$. This implies that $S \cap \uparrow_{\mathrm{QB}(L)} F \neq \emptyset$, where the subscript $\mathrm{QB}(L)$ means to take relevant operations in poset $\mathrm{QB}(L)$. Thus, $F \ll_{\mathrm{QB}(L)} x$. □

Proposition 5.6 *Let L be a dcpo and the relation $\ll$ on L has the interpolation property as stated in Proposition 2.9. If for all $x \in L$, the family* $\mathrm{fin}(x)$ *is directed and $\bigcap\{\Uparrow F \mid F \in \mathrm{fin}(x)\} = \Uparrow t$ for some t, then there exists a quasi-approximated element* $\mathrm{QB}(x)$ *satisfying:* (i) $\mathrm{fin}(\mathrm{QB}(x)) = \mathrm{fin}(x)$; (ii) $\forall y \in L$, $\mathrm{fin}(y) = \mathrm{fin}(x)$ *implies* $y \geqslant \mathrm{QB}(x)$.

Proof. Let L be a dcpo. If for all $x \in L$, $\mathrm{fin}(x)$ is directed and $\bigcap\{\Uparrow F \mid F \in \mathrm{fin}(x)\} = \Uparrow t$ for some t, then let $\mathrm{QB}(x) = t$. We next show that $\mathrm{QB}(x)$ is a quasi-approximated element satisfying condition (i) and (ii). It follows from $\mathrm{QB}(x) \leqslant x$ and Proposition 2.6 that $\mathrm{fin}(\mathrm{QB}(x)) \subseteq \mathrm{fin}(x)$. Let $F \in \mathrm{fin}(x)$. By the interpolation property of the relation $\ll$ as stated in Proposition 2.9, there is a finite set E such that $F \ll E \ll x$. Thus we have $F \ll E \leqslant \mathrm{QB}(x)$ and $F \in \mathrm{fin}(\mathrm{QB}(x))$. This shows that $\mathrm{fin}(x) \subseteq \mathrm{fin}(\mathrm{QB}(x))$ and thus $\mathrm{fin}(x) = \mathrm{fin}(\mathrm{QB}(x))$. So, the element $\mathrm{QB}(x)$ satisfies condition (i). By condition (i), it is easy to prove that $\mathrm{QB}(x)$ is a quasi-approximated element. For all $y \in L$, if $\mathrm{fin}(y) = \mathrm{fin}(x)$, then $y \in \bigcap\{\Uparrow F \mid F \in \mathrm{fin}(x)\} = \Uparrow t = \uparrow \mathrm{QB}(x)$. Thus $y \geqslant \mathrm{QB}(x)$ and $\mathrm{QB}(x)$ satisfies condition (ii), as desired. □

Lemma 5.7 *Let L be a dcpo and the relation $\ll$ on L has the interpolation property as stated in Proposition 2.9. If for all $x \in L$, the family* $\mathrm{fin}(x)$ *is directed and $\bigcap\{\Uparrow F \mid F \in \mathrm{fin}(x)\} = \Uparrow t$ for some t, then for each $F \in \mathrm{fin}(x)$ there exists $\mathrm{QB}(F) \in \mathcal{P}_{fin}(\mathrm{QB}(L))$ such that $F \ll \mathrm{QB}(F) \ll x$.*

Proof. Let $x \in L$. For each $F \in \mathrm{fin}(x)$, by the interpolation property as stated in Proposition 2.9 of the relation $\ll$ on L, there is nonempty finite set E_F such that $F \ll E_F \ll x$. It follows from Proposition 2.6(2) that for all $e \in E_F$, we have $F \ll e$. By Proposition 5.6, there exists a quasi-approximated element $\mathrm{QB}(e)$ such that $\mathrm{fin}(\mathrm{QB}(e)) = \mathrm{fin}(e)$ and $\mathrm{QB}(e) \leqslant e$. Let $\mathrm{QB}(F) = \{\mathrm{QB}(e) \mid e \in E_F\} \in \mathcal{P}_{fin}(\mathrm{QB}(L))$. It is straightforward to prove that $F \ll \mathrm{QB}(F) \ll x$. □

Theorem 5.8 *Let L be a dcpo and the relation $\ll$ on L has the interpolation property as stated in Proposition 2.9. If for all $x \in L$, the family* $\mathrm{fin}(x)$ *is directed and $\bigcap\{\Uparrow F \mid F \in fin(x)\} = \Uparrow t$ for some t, then* $\mathrm{QB}(L)$ *in the induced order is a quasicontinuous domain.*

Proof. For each $u \in L$, by Proposition 5.6, there exists a quasi-approximated element $\mathrm{QB}(u)$ such that $\mathrm{fin}(\mathrm{QB}(u)) = \mathrm{fin}(u)$ and $\mathrm{QB}(u) \leqslant u$. Thus $\mathrm{QB}(L)$ is nonempty. By Proposition 5.5, $\mathrm{QB}(L)$ in the induced order is a dcpo. For all $x \in \mathrm{QB}(L)$ and $F \in \mathrm{fin}(x)$, by Lemma 5.7, there exists $\mathrm{QB}(F) \in \mathcal{P}_{fin}(\mathrm{QB}(L))$ such that $F \ll \mathrm{QB}(F) \ll x$. Let $\mathcal{D}_x = \{\mathrm{QB}(F) \mid F \in \mathrm{fin}(x)\}$. It follows from Proposition 5.5 that $\mathcal{D}_x \subseteq \mathrm{fin}_{\mathrm{QB}(L)}(x)$, where the set $\mathrm{fin}_{\mathrm{QB}(L)}(x) = \{H \in \mathcal{P}_{fin}(\mathrm{QB}(L)) \mid H \ll_{\mathrm{QB}(L)} x\}$. Since the family $\mathrm{fin}(x)$ is nonempty, the family $\mathcal{D}_x$ is nonempty. For all $\mathrm{QB}(F_1)$, $\mathrm{QB}(F_2) \in \mathcal{D}_x$, we have $\mathrm{QB}(F_1) \ll x$ and $\mathrm{QB}(F_2) \ll x$. By the directedness of $\mathrm{fin}(x)$ and the interpolation property as stated in Proposition 2.9 of the relation $\ll$ on L, there is $F_3 \in \mathrm{fin}(x)$ such that $\mathrm{QB}(F_1), \mathrm{QB}(F_2) \ll F_3 \ll x$. By Lemma 5.7, there exists $\mathrm{QB}(F_3) \in \mathcal{P}_{fin}(\mathrm{QB}(L))$ such that $F_3 \ll \mathrm{QB}(F_3) \ll x$. This shows that there exists $\mathrm{QB}(F_3) \in \mathcal{D}_x$ such that $\mathrm{QB}(F_1), \mathrm{QB}(F_2) \leqslant \mathrm{QB}(F_3)$. Thus the family $\mathcal{D}_x$ is directed.

We next show that $\uparrow_{\mathrm{QB}(L)} x = \bigcap\{\uparrow_{\mathrm{QB}(L)} \mathrm{QB}(F) \mid \mathrm{QB}(F) \in \mathcal{D}_x\}$, where the subscript $\mathrm{QB}(L)$ means to take relevant operations in poset $\mathrm{QB}(L)$. Since $\mathcal{D}_x \subseteq \mathrm{fin}_{\mathrm{QB}(L)}(x)$, we have $\uparrow_{\mathrm{QB}(L)} x \subseteq \bigcap\{\uparrow_{\mathrm{QB}(L)} \mathrm{QB}(F) \mid \mathrm{QB}(F) \in \mathcal{D}_x\}$. Let $z \in \bigcap\{\uparrow_{\mathrm{QB}(L)} \mathrm{QB}(F) \mid \mathrm{QB}(F) \in \mathcal{D}_x\}$. By the construction of $\mathcal{D}_x$, for all $F \in \mathrm{fin}(x)$, we have $F \ll \mathrm{QB}(F) \leqslant z$ and thus $z \in \Uparrow F$. This shows that $z \in \bigcap\{\Uparrow F \mid F \in \mathrm{fin}(x)\}$. Since $x \in \mathrm{QB}(L)$ is a quasi-approximated element, we have $z \in \bigcap\{\Uparrow F \mid F \in \mathrm{fin}(x)\} = \Uparrow x$. Thus $z \in \Uparrow x \cap \mathrm{QB}(L) = \uparrow_{\mathrm{QB}(L)} x$. This shows that

$$\bigcap\{\uparrow_{\mathrm{QB}(L)} \mathrm{QB}(F) \mid \mathrm{QB}(F) \in \mathcal{D}_x\} \subseteq \uparrow_{\mathrm{QB}(L)} x.$$

Therefore, $\uparrow_{\mathrm{QB}(L)} x = \bigcap\{\uparrow_{\mathrm{QB}(L)} \mathrm{QB}(F) \mid \mathrm{QB}(F) \in \mathcal{D}_x\}$. By Definition 5.1, x is a quasi-approximated element of $\mathrm{QB}(L)$. It follows from the arbitrariness of x and Proposition 5.4 that $\mathrm{QB}(L)$ in the induced order is a quasicontinuous domain. □

Theorem 5.9 *Let L be a dcpo. If for all $x \in L$, the family $\mathrm{fin}(x)$ is directed and $\bigcap\{\Uparrow F \mid F \in \mathrm{fin}(x)\} = \Uparrow t$ for some t, then L is a quasicontinuous domain iff L satisfies the following two conditions:* (1) *the relation $\ll$ on L has the interpolation property as stated in Proposition 2.9;* (2) $\forall\, x, y \in L$, $x \neq y \Longrightarrow \mathrm{fin}(x) \neq \mathrm{fin}(y)$.

Proof. $\Longleftarrow$: Let $x \in L$. By Proposition 5.6, there exists a quasi-approximated element $\mathrm{QB}(x) \leqslant x$ such that $\mathrm{fin}(\mathrm{QB}(x)) = \mathrm{fin}(x)$. Applying condition (2), we have $x = \mathrm{QB}(x)$. Thus $L = \mathrm{QB}(L)$. By Proposition 5.4, L is a quasicontinuous domain.

$\Longrightarrow$: By Definition 2.8 and Proposition 2.9. □

6 Conclusion remarks and future development

In this paper, in order to characterize (1) continuous posets, (2) hypercontinuous posets, and (3) quasicontinuous domains, we introduce notions of (hyper) approximated elements for posets and quasi-approximated elements for dcpos by modifying Zhao's concept of weakly approximated elements in [15]. We extend Zhao's results in [15] for complete lattices in each of the above settings of posets in a systematic manner.

It should be stated that quasi-approximated elements are now only defined for dcpos. How to define quasi-approximated elements in general posets remains open. A recent result established in [4] that a dcpo L is quasicontinuous iff the poset $\mathcal{Q}_f L$ of nonempty finitely generated upper sets ordered by reverse inclusion $\supseteq$ is continuous. It is not clear, though, whether this result can be generalized to posets with the concept of quasicontinuous posets in [8]. However, using the above result, it is often convenient to pass from quasicontinuous domains to continuous posets via the poset $\mathcal{Q}_f L$. By this trick one may view the single element x as its principal filter $\Uparrow x$ in the poset $\mathcal{Q}_f L$. Then it may be true that an element $x \in L$ is quasi-approximated iff $\Uparrow x$ is approximated in $\mathcal{Q}_f L$. In this way, on one hand, for a general poset L, an element $x \in L$ is quasi-approximated can be defined by that $\Uparrow x$ is approximated in the poset $\mathcal{Q}_f L$; on the other hand, the results reported in Section 5 can be viewed as direct corollaries of those reported in Section 3 and generalized to the setting of posets. For the details, we leave them as future work.

Acknowledgement

The authors would like to thank the reviewers. Their valuable comments and suggestions make the presentation of this paper much improved.

References

[1] G. Gierz, K. Hofmann, K. Keimel, J. D. Lawson, M. Mislove and D. S. Scott, Continuous Lattices and Domains. Cambridge University Press, Cambridge, 2003.

[2] G. Gierz and J. D. Lawson. Generalized continuous and hypercontinuous lattices. Rocky Mountain J. Math., 1981, 11: 271-296.

[3] G. Gierz, J. D. Lawson, and A. Stralka. Quasicontinuous posets. Houston Journal of Mathematics, 1983, 9(2): 191-208.

[4] R. Heckmann and K. Keimel. Quasicontinuous Domains and the Smyth Powerdomain. Electronic Notes in Theoretical Computer Science, 2013, 298: 215-232.

[5] R. E. Hoffmann. Continuous posets and adjoint sequences. Semigroup Forum, 1979, 18: 173-188.

[6] J. D. Lawson. The duality of continuous posets. Houston J. of Mathematics, 1979, 5: 357-394.

[7] J. D. Lawson and Luoshan Xu, Posets Having Continuous Intervals. Theoretical Computer Science, 2004, 316: 89-103.

[8] Xuxin Mao, Luoshan Xu. Quasicontinuity of posets via Scott topology and sobrification. Order, 2006, 23(4): 359-369.

[9] M. Mislove, Local DCPOs, local CPOs and local completions, Electronic Notes in Theoretical Computer Science 1999, 20: 1-14.

[10] D. S. Scott. Continuous Lattices. Lecture Notes in Mathematics 274, Springer-Verlag, Berlin, 1972, 97-136.

[11] Luoshan Xu. Continuity of posets via Scott topology and sobrification. Topology and its Applications, 2006, 153: 1886-1894.

[12] Luoshan Xu, Xuxin Mao. Strongly continuous posets and the local Scott topology. J. Math. Anal. Appl, 2008, 345: 816-824.

[13] Han Zhang. A Note on Continuous Partially Ordered Sets. Semigroup Forum, 1993, 47: 101-104.

[14] Bin Zhao, Yihui Zhou. The category of supercontinuous poset. J. Math. Anal. Appl, 2006, 320: 632-641.

[15] Dongsheng Zhao. Several Characterizations of Continuous Lattices and Completely Distributive Lattices. Chinese Quarterly Journal of Mathematics, 1990, 5(1-2): 162-165. (in Chinese)