

# Quasicontinuous Domains and the Smyth Powerdomain

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## Abstract

In Domain Theory quasicontinuous domains pop up from time to time generalizing slightly the powerful notion of a continuous domain. It is the aim of this paper to show that quasicontinuous domains occur in a natural way in relation to the powerdomains of finitely generated and compact saturated subsets. Properties of quasicontinuous domains seem to be best understood from that point of view. This is in contrast to the previous approaches where the properties of a quasicontinuous domain were compared primarily with the properties of the lattice of Scott-open subsets. We present a characterization of those domains that occur as domains of nonempty compact saturated subsets of a quasicontinuous domain.

A set theoretical lemma due to M. E. Rudin has played a crucial role in the development of quasicontinuous domains. We present a topological variant of Rudin's Lemma where irreducible sets replace directed sets. The notion of irreducibility here is that of a nonempty set that cannot be covered by two closed sets except if already one of the sets is covering it. Since directed sets are the irreducible sets for the Alexandroff topology on a partially ordered set, this is a natural generalization. It allows a remarkable characterization of sober spaces.

For this we denote by  $\Omega X$  the space of nonempty compact saturated subsets (with the upper Vietoris topology) of a topological space  $X$ . The following properties are equivalent: (1)  $X$  is sober, (2)  $\Omega X$  is sober, (3)  $X$  is *strongly* well-filtered in the following sense: Whenever  $\mathcal{A}$  is an irreducible subset of  $\Omega X$  and  $U$  an open subset of  $X$  such that  $\bigcap \mathcal{A} \subseteq U$ , then  $K \subseteq U$  for some  $K \in \mathcal{A}$ . This result fills a gap in the existing literature.

**Keywords:** Quasicontinuous Domains, M. E. Rudin's Lemma, Powerdomains of compact saturated subsets, Smyth powerdomain.

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# 1 Introduction

In this paper we deal with the powerspace of compact saturated sets, quasicontinuous domains and variants of Rudin’s Lemma. We intend to show that these three ingredients are inseparably tied together.

Quasicontinuous domains introduced by Gierz, Lawson and Stralka [5] capture many of the essential features of continuous domains. Recently they have attracted increased attention through the remarkable work of J. Goubault-Larrecq [6] and through a paper by Li and Xu [11].

An important result concerning continuous domains is their characterization by properties of their Scott topology. A dcpo is continuous if and only if its lattice of Scott open subsets is completely distributive. Gierz, Lawson and Stralka [5] have characterized quasicontinuous domains by the property that their lattice of Scott-open subsets is hypercontinuous. One of the characterizations of hypercontinuous lattices is that they are images of completely distributive lattices under maps preserving arbitrary meets and directed joins.

A characterization of the lattice of open subsets is equivalent to a characterization of the opposite lattice of closed subsets. The lattice of Scott-closed subsets of a dcpo is often called the Hoare or lower powerdomain of a dcpo. Thus, one can say that Gierz, Lawson and Stralka have characterized quasicontinuous domains through their lower powerdomains.

In this paper we intend to show that quasicontinuous domains should be tied up with the Smyth or upper powerdomain [15,16] rather than the lower powerdomain. We show that among dcpos the quasicontinuous domains can be characterized by the property that the poset of finitely generated upper sets ordered by reverse inclusion is a continuous poset. We claim that this opens useful insights and simpler proofs for known properties (see 4.6). We finish with a characterization of those domains that occur as upper powerdomains of quasicontinuous domains (see Theorem 4.16).

From the beginning, the development of the notion of a quasicontinuous domain was dependent on a set theoretical lemma. In fact, M. E. Rudin provided the appropriate lemma as an answer to a question asked by Gierz, Lawson and Stralka, when they prepared the paper [5], where the notion of a quasicontinuous domain was introduced. In the same spirit, variants of Rudin’s Lemma are the second ingredient of this paper (see Section 3). Rudin’s original lemma is captured in Lemma 3.4 and Corollary 3.5. We also need it in our approach to quasicontinuous domains in Lemma 4.1.

A new topological variant of Rudin’s Lemma is presented in Lemma 3.1; directed sets in Rudin’s original Lemma are viewed as special cases of irreducible sets in topological spaces. This lemma allows a characterization of sober spaces (see Theorem 3.13). We use this theorem for a simplified proof of the sobriety of quasicontinuous posets (see Corollary 4.12).

Theorem 3.13 solves an open problem. A topological space had been called well-filtered<sup>4</sup> [4, I-1.24.1] if, whenever  $\bigcap \mathcal{F} \subseteq U$  for a filter basis  $\mathcal{F}$  of compact saturated

<sup>4</sup> Well-filtered spaces have also been called  $U_K$ -admitting in [10].

sets and an open subset  $U$ , then  $K \subseteq U$  for some  $K \in \mathcal{F}$ . It is known that every sober space is well-filtered. Conversely every locally compact well-filtered space is sober (Theorem [4, II-1.21]). But sobriety is not characterized by well-filteredness in general. There are even examples of dcpos that are well-filtered for their Scott topology but not sober; a first such example is due to Kou Hui [10]<sup>5</sup>. Theorem 3.13 tells us that sobriety is characterized by the property of being strongly well-filtered. By this we mean that, whenever  $\mathcal{A}$  is an irreducible set in the hyperspace of compact saturated subsets (with the upper Vietoris topology) such that  $\bigcap \mathcal{A}$  is contained in an open set  $U$ , then  $K \subseteq U$  for some  $K \in \mathcal{A}$ .

## 2 Preliminaries

### 2.1 Order theoretical notions

For a partially ordered set (= poset)  $P$ , more generally for a preordered set, we fix the following terminology:

$D \subseteq P$  is *directed* if  $D$  is nonempty and if for any  $d_1, d_2$  in  $D$  there is a  $d$  in  $D$  above  $d_1$  and  $d_2$ .

In a poset  $P$ , a directed subset  $D$  may or may not have a least upper bound. We adopt the following convention: if we write  $\bigvee^\uparrow D$  then we mean that  $D$  is a directed subset of  $P$  which has a least upper bound in  $P$  which we denote by  $\bigvee^\uparrow D$ .

$P$  is *directed complete* (a *dcpo*) if every directed subset  $D$  of  $P$  has a least upper bound  $\bigvee^\uparrow D$ .

For  $a \in P$  let  $\uparrow a$  denote the set of all  $x \in P$  with  $a \leq x$  and, for a subset  $A$ , let  $\uparrow A = \bigcup_{a \in A} \uparrow a$ . A subset  $A$  of  $P$  is an *upper set* if  $A = \uparrow A$ . We denote by  $\mathcal{U}X$  the collection of all upper sets in  $X$ . The order dual concepts are  $\downarrow a$ ,  $\downarrow A$  and *lower set*.

For any set  $X$ , we denote by  $\mathfrak{P}X$  the set of all subsets and by  $\mathfrak{P}_f X$  the collection of all nonempty finite subsets; the letters  $F, G, H$  will always denote nonempty finite subsets.

If  $X$  is a partially ordered set, more generally a preordered set, we introduce a preorder  $\sqsubseteq$  on the powerset  $\mathfrak{P}X$ , sometimes called the *Smyth preorder*, by

$$A \sqsubseteq B \iff \uparrow B \subseteq \uparrow A,$$

that is,  $A \sqsubseteq B$  iff for every element  $b \in B$  there is an element  $a \in A$  with  $a \leq b$ . On the collection  $\mathcal{U}X$  of upper sets,  $\sqsubseteq$  is a partial order, namely reverse inclusion. We denote by

$$\eta_X: X \rightarrow \mathfrak{P}X \quad \text{the map} \quad \eta_X(x) = \uparrow x$$

which is an order embedding.

Every topological space  $X$  carries a natural (pre-)order, the *specialization (pre-)order*  $x \leq y$  iff  $x \in \text{cl}\{y\}$ , the closure of the singleton  $\{y\}$ . The previous order theoretical concepts can be applied to the specialization (pre-)order. And when we apply order theoretical notions to topological spaces, they always refer to the

<sup>5</sup> Zhao Dongsheng and Xi Xiaoyong have exhibited simpler examples recently (Oral communication).

specialization (pre-)order. A subset of a topological space that is an upper set for its specialization (pre-)order is also called a *saturated* set.

Conversely, every poset  $X$  can be topologized in various ways. The upper sets form the *Alexandroff topology*  $\mathfrak{U}X$ . A coarser topology is the *Scott topology*  $\sigma X$ : A subset  $U \subseteq X$  is *Scott-open* if  $U$  is an upper set and if  $\bigvee^\uparrow D \in U \Rightarrow D \cap U \neq \emptyset$ , that is, if for every directed set  $D$  with  $\bigvee^\uparrow D \in U$ , there is a  $d \in D$  with  $d \in U$ , provided that  $D$  has a least upper bound in  $X$ . The Scott-open sets form indeed a topology.

## 2.2 Compact and supercompact sets

A subset  $K$  of a topological space  $X$  is *compact* if for all directed families  $(U_i)_{i \in I}$  of opens,  $K \subseteq \bigcup_{i \in I} U_i$  implies  $K \subseteq U_k$  for some  $k$  in  $I$ . It is *supercompact* if for arbitrary families  $(U_i)_{i \in I}$  of opens,  $K \subseteq \bigcup_{i \in I} U_i$  implies  $K \subseteq U_k$  for some  $k$  in  $I$ .

Using that  $K \subseteq U$  if and only if  $K$  does not meet  $C = X \setminus U$ , compactness can also be characterized using closed instead of open sets:

**Fact 2.1** *A set  $K$  is compact iff for all filtered families  $(C_i)_{i \in I}$  of closed sets,  $K$  meets  $\bigcap_{i \in I} C_i$  whenever  $K$  meets all  $C_i$ . A set  $K$  is supercompact iff for arbitrary families  $(C_i)_{i \in I}$  of closed sets,  $K$  meets  $\bigcap_{i \in I} C_i$  whenever  $K$  meets all  $C_i$ .*

Note that a subset  $K$  is compact if and only if its *saturation*, the upper set  $\uparrow K$  generated by  $K$  w.r.t. the specialization (pre-)order, is compact.

**Fact 2.2** *The supercompact saturated sets of a topological space  $X$  are exactly the sets  $\uparrow x$  with  $x$  in  $X$ .*

**Proof.** The sets  $\uparrow x$  are clearly supercompact and saturated. For the opposite direction, let  $S$  be a supercompact upper set. The set  $S$  meets all sets of the family  $(\downarrow a)_{a \in S}$  of closed sets. By supercompactness, it meets  $\bigcap_{a \in S} \downarrow a$ . Let  $x$  be a member of  $S \cap \bigcap_{a \in S} \downarrow a$ . Since  $S$  is an upper set,  $\uparrow x \subseteq S$  holds. On the other hand,  $x$  is in  $\downarrow a$  for all  $a$  in  $S$ , whence  $S \subseteq \uparrow x$ .  $\square$

## 2.3 The Upper Powerspace

On the powerset  $\mathfrak{P}X$  of all subsets of a topological space  $X$  we consider the *upper Vietoris topology*, the topology generated by the sets

$$\square U = \{K \in \mathfrak{P}X \mid K \subseteq U\},$$

where  $U$  ranges over the open subsets of  $X$ . Since

$$\square(U \cap V) = \square U \cap \square V,$$

the sets  $\square U$  form indeed a basis for the upper Vietoris topology. Equivalently, the sets  $\diamond C = \{K \in \mathfrak{P}X \mid K \cap C \neq \emptyset\}$  are closed for all closed sets  $C$  of  $X$  and they form a basis for the closed sets of the upper Vietoris topology. The canonical map  $\eta_X = (x \mapsto \uparrow x): X \rightarrow \mathfrak{P}X$  is a topological embedding. The specialization preorder

for the upper Vietoris topology on  $\mathfrak{P}X$  agrees with the Smyth preorder  $A \sqsubseteq B$ , i.e.,  $\uparrow B \subseteq \uparrow A$ . We consider several subspaces of  $\mathfrak{P}X$ :

- $\mathfrak{P}_f X$ , the space of all nonempty finite subsets of  $X$ ,
- $\mathfrak{K}X$ , the space of all nonempty compact subsets,
- $\mathfrak{Q}_f X$ , the space of all nonempty finitely generated saturated sets  $\uparrow F$ ,  $F \in \mathfrak{P}_f X$ ,
- $\mathfrak{Q}X$ , the space of all nonempty compact saturated subsets of  $X$ .

These spaces are always endowed with the upper Vietoris topology. The specialization preorder is  $\sqsubseteq$  as above; it is a partial order only on  $\mathfrak{Q}X$  and  $\mathfrak{Q}_f X$ .

We also have a semilattice operation on  $\mathfrak{P}X$ , namely  $A \sqcap B = A \cup B$ , and  $\mathfrak{P}_f X$ ,  $\mathfrak{K}X$ ,  $\mathfrak{Q}_f X$ , and  $\mathfrak{Q}X$  are subsemilattices thereof. The basic open neighborhoods  $\square U$  are filters, that is,  $A \sqcap B \in \square U$  if and only if  $A \in \square U$  and  $B \in \square U$ . This implies that the semilattice operation  $\sqcap$  is continuous with respect to the upper Vietoris topology.

## 2.4 Irreducible Sets

Let  $X$  be a topological space. For a subset  $A$  of  $X$ , the following are equivalent:

- (1) For any finite family  $(C_i)_{i \in F}$  of closed sets: if  $A \subseteq \bigcup_{i \in F} C_i$ , then  $A \subseteq C_i$  for some  $i \in F$ .
- (2) For any finite family  $(U_i)_{i \in F}$  of open sets: if  $A$  meets all  $U_i$ , then  $A$  meets  $\bigcap_{i \in F} U_i$ .

For the proof just observe that  $A$  meets  $U$  if and only if  $A \not\subseteq X \setminus U$ .

A subset  $A$  of  $X$  is said to be *irreducible* if it satisfies the equivalent conditions (1) and (2) above. Let us collect some known facts about irreducible sets in a topological space  $X$ .

**Fact 2.3** *A closed set  $A$  in a topological space is irreducible if and only if, for any finite family  $(C_i)_{i \in F}$  of closed sets,  $A = \bigcup_{i \in F} C_i$  implies  $A = C_i$  for some  $i \in F$ .*

Since an open set meets the closure of  $A$  iff it meets  $A$ , we have:

**Fact 2.4** *A set is irreducible iff its closure is irreducible.*

**Fact 2.5** *Let  $f : X \rightarrow Y$  be a continuous map of topological spaces  $X$  and  $Y$ . If  $A$  is irreducible in  $X$ , then its image  $f(A)$  is irreducible in  $Y$ .*

**Proof.** If  $f(A) \subseteq \bigcup_{i \in F} C_i$ , then  $A \subseteq f^{-1}(\bigcup_{i \in F} C_i) = \bigcup_{i \in F} f^{-1}C_i$ , whence  $A \subseteq f^{-1}C_i$  for some  $i$  in  $F$ , and so  $f(A) \subseteq C_i$ .  $\square$

**Fact 2.6** (i) *Every subset of a topological space which is directed with respect to the specialization (pre-)order is irreducible.*

(ii) *The irreducible sets of a poset  $P$  endowed with the Alexandroff topology are exactly the directed subsets.*

**Proof.** (i) Let  $A$  be a directed set. If  $A$  meets open sets  $U_1, \dots, U_n$ , then there are points  $x_i$  in  $A \cap U_i$ . Since  $A$  is directed, there is an upper bound  $x$  of  $x_1, \dots, x_n$  in  $A$ . Since open sets are upper sets,  $x$  is in  $A \cap U_1 \cap \dots \cap U_n$ . Thus  $A$  is irreducible.

(ii) Directed sets are irreducible by (i). For the opposite direction, let  $A$  be an irreducible set and  $x_1, \dots, x_n$  be elements of  $A$ . Then  $A$  meets the upper (hence Alexandroff open) sets  $\uparrow x_1, \dots, \uparrow x_n$ . Since  $A$  is irreducible,  $A \cap \uparrow x_1 \cap \dots \cap \uparrow x_n \neq \emptyset$  follows. Any member of this intersection is a common upper bound of  $x_1, \dots, x_n$  in  $A$ .  $\square$

### 3 Rudin's Lemma and its topological variants

In her original, not easily accessible paper [13], M. E. Rudin formulated the following theorem: *If  $\mathcal{F}$  is a collection of finite subsets of  $P$  which is  $\sqsubseteq$ -directed and converges to 1, then there is a subset of  $\bigcup \mathcal{F}$  which is directed and converges to 1.* Here  $P$  is a poset with a maximal element 1; a  $\sqsubseteq$ -directed family  $\mathcal{F}$  is said to converge to 1 if  $\bigcap_{F \in \mathcal{F}} \uparrow F = \{1\}$ , and a directed set  $D$  is said to converge to 1 if  $\bigcap_{d \in D} \uparrow d = \{1\}$ . M. E. Rudin used transfinite induction for the proof. For the use in domain theory a modified version as in Corollary 3.5 below has become prominent.

#### 3.1 A topological variant of Rudin's Lemma

The original Rudin Lemma deals with directed sets. Fact 2.6 suggests to replace directed sets by irreducible sets in a topological setting.

**Lemma 3.1** (Topological Rudin Lemma) *Let  $X$  be a topological space and  $\mathcal{A}$  an irreducible subset of  $\mathfrak{K}X$  ( $\mathfrak{Q}(X)$ ,  $\mathfrak{Q}_f X$ , respectively). Any closed set  $C \subseteq X$  that meets all members of  $\mathcal{A}$  contains an irreducible closed subset  $A$  that still meets all members of  $\mathcal{A}$ .*

**Proof.** Let  $\mathcal{C}$  be the set of all closed subsets of  $C$  that meet all members of  $\mathcal{A}$ . Then  $\mathcal{C}$  is not empty as it contains  $C$ , and is closed under filtered intersections by 2.1 since all members of  $\mathcal{A}$  are compact. By the order-dual of Zorn's Lemma,  $\mathcal{C}$  contains a minimal element  $A$ . As a member of  $\mathcal{C}$ ,  $A$  is closed and meets all members of  $\mathcal{A}$ . We show that  $A$  is irreducible using 2.3.

So let  $A = \bigcup_{i \in F} C_i$  where  $(C_i)_{i \in F}$  is a finite family of closed sets. Every  $K$  in  $\mathcal{A}$  meets  $A$ , and therefore some  $C_i$ . Hence  $\mathcal{A} \subseteq \bigcup_{i \in F} \diamond C_i$ . Since  $\mathcal{A}$  is irreducible in  $\mathfrak{K}X$  and the sets  $\diamond C_i$  are closed in  $\mathfrak{K}X$  (Section 2.3), we conclude that  $\mathcal{A} \subseteq \diamond C_k$  for some  $k$  in  $F$ . Thus  $C_k$  meets all members of  $\mathcal{A}$ , whence  $C_k$  is in  $\mathcal{C}$  and is a subset of  $A$ . By minimality of  $A$  in  $\mathcal{C}$ ,  $A = C_k$  follows.  $\square$

In the previous Lemma 3.1, one may choose  $C = X$  so that for every irreducible subset  $\mathcal{A}$  of  $\mathfrak{K}X$ ,  $\mathfrak{Q}X$  and  $\mathfrak{Q}_f X$ , respectively, there is an irreducible closed subset of  $X$  that meets all members of  $\mathcal{A}$ .

By 2.6, directed sets are irreducible. Therefore, 3.1 implies the following corollary:

**Corollary 3.2** *Let  $X$  be a topological space and  $\mathcal{A}$  a  $\sqsubseteq$ -directed family of nonempty compact subsets of  $X$ . Any closed set  $C$  that meets all members of  $\mathcal{A}$  contains an irreducible closed subset  $A$  that still meets all members of  $\mathcal{A}$ .*

**Remark 3.3** M. Ern  [3, Proposition 3] had already obtained the following equivalent version of Corollary 3.2:

*For every filtered collection  $\mathcal{A}$  of nonempty compact saturated subsets of a space  $X$ , there is an irreducible (closed) subset  $A$  meeting all members of  $\mathcal{A}$ .*

In his paper, Ern  emphasizes the fact that this result can be proved without using the full strength of Zorn’s lemma (as we did in the proof of 3.1), but only the ultrafilter principle. He also avoids the upper powerspace, but rather embeds the space  $X$  into its sobrification  $X^s$ . The saturations  $\uparrow_{X^s} K$  in  $X^s$  of the  $K \in \mathcal{A}$  form a filtered collection of compact saturated sets which has a nonempty intersection. Picking an element  $a$  in this intersection, the set  $A = X \cap \text{cl}_{X^s}\{a\}$  is a closed irreducible subset of  $X$  meeting all members of  $\mathcal{A}$ . One can also prove this corollary directly by a slight modification of the proof of 3.1. The price for avoiding the upper powerspace is that 3.2 is less general than 3.1 (but still more general than the original Rudin Lemma).

### 3.2 Rudin’s Lemma

We now apply Corollary 3.2 to a space arising from a preorder  $P$  with the Alexandroff topology. In such a space, closed = lower, irreducible = directed, and compact = finitary, where those sets  $K$  are called *finitary* whose up-sets are finitely generated, that is,  $\uparrow K = \uparrow F$  for some finite set  $F$ . We obtain:

**Lemma 3.4** (Order Rudin Lemma) *Let  $P$  be a preorder and  $\mathcal{F}$  a  $\sqsubseteq$ -directed family of finitary upper sets of  $P$ . Any lower set  $L$  that meets all members of  $\mathcal{F}$  has a directed lower subset  $A$  that still meets all members of  $\mathcal{F}$ .*

From this version, it is easy to derive A. Jung’s version of Rudin’s Lemma [9, Theorem 4.11]:

**Corollary 3.5** *If  $(F_i)_{i \in I}$  is a  $\sqsubseteq$ -directed family of nonempty finite sets in a poset  $P$ , then there is a directed subset  $A$  of  $\bigcup_{i \in I} F_i$  that meets all  $F_i$ .*

**Proof.** Let  $Q$  be the poset  $\bigcup_{i \in I} F_i$  with the order inherited from  $P$ . Since all  $F_i$  are non-empty,  $Q$  itself is a lower set that meets all  $F_i$ . By 3.4, it has a directed lower subset  $A$  that still meets all  $F_i$ .  $\square$

In Corollary 3.5 it is essential to restrict to collections  $\mathcal{F}$  of finite subsets. Indeed, if we take an infinite set  $M$  with the discrete order and consider the filter  $\mathcal{F}$  of cofinite subsets, then  $\mathcal{F}$  is directed for reverse inclusion, but of course there is no directed subset  $D$  satisfying  $D \cap F \neq \emptyset$  for all  $F \in \mathcal{F}$ ; indeed, the only directed sets are singleton.

### 3.3 Another variant of Rudin’s Lemma

One may ask the following question: Let  $(F_i)_{i \in I}$  be a  $\sqsubseteq$ -directed family of nonempty finite sets of a poset  $X$ . Is there a directed subset  $D$  of  $\bigcup_i F_i$  which intersects each  $F_i$  in exactly one point? A positive answer would be a strengthening of Jung’s

version 3.5 of Rudin's Lemma, which asserts that there is a directed subset  $D$  of  $\bigcup_i F_i$  which intersects each  $F_i$  in at least one point.

The answer to the question above is negative in general. It is not difficult to come up with a finite counterexample. For treelike directed families, there is a positive answer to our question. For this we use a variant of Rado's Selection Lemma due to R. J. Cowen [2, Theorem 3]:

Let  $\mathcal{F}$  be a set of partial functions defined on subsets of a set  $I$  with the following properties:

- (i)  $\mathcal{F}$  is of finite character, that is,  $f$  belongs to  $\mathcal{F}$  if and only if the restriction of  $f$  to any finite subset of its domain belongs to  $\mathcal{F}$ .
- (ii)  $\{f(i) \mid f \in \mathcal{F}\}$  is finite for each  $i \in I$ .
- (iii) For each finite  $J \subseteq I$ , there exists an  $f \in \mathcal{F}$  whose domain contains  $J$ .

Then  $\mathcal{F}$  contains a function defined on all of  $I$ .

**Lemma 3.6** *Let  $I$  be a directed poset which is treelike in the sense that the upper set of each  $i \in I$  is linearly ordered. Let  $(F_i)_{i \in I}$  be a collection of nonempty finite subsets of a poset  $P$  such that  $F_i \subseteq F_j$  whenever  $i \leq j$ . Then one may choose  $x_i \in F_i$  for every  $i$  such that  $x_i \leq x_j$  whenever  $i \leq j$ .*

**Proof.** We consider the collection  $\mathcal{F}$  of order preserving maps  $f$  defined on subsets  $J$  of  $I$  such that  $f(i) \in F_i$  for all  $i \in J$ . The hypotheses (i), (ii), (iii) of the Cowen Lemma are satisfied: Clearly, this collection  $\mathcal{F}$  is of finite character. For every finite subset  $J$  of  $I$ , we can find an order preserving map  $x$  from  $J$  to  $\bigcup_i F_i$  such that  $x_j \in F_j$  for all  $j \in J$ . For this, we may suppose that  $J$  has a greatest element  $j_0$ . We begin by choosing any  $x_{j_0} \in F_{j_0}$ . We now look at the immediate predecessors  $j_1, \dots, j_k$  of  $j_0$  in  $J$  and we choose  $x_{j_\iota} \in F_{j_\iota}$  such that  $x_{j_\iota} \leq x_{j_0}$  which is possible, since  $\uparrow F_{j_0} \subseteq \uparrow F_{j_\iota}$  for  $\iota = 1, \dots, k$ . For each of the  $j_\iota$  we repeat the same procedure. After finitely many steps we have exhausted the finite set  $J$ . We have used that the directed set  $I$  is a tree: descending paths in the finite subset  $J$  never meet.

We now can apply Cowen's Selection Lemma cited above and we obtain the desired conclusion.  $\square$

**Remark 3.7** Notice that a directed set which is a tree has cofinal chains; just take  $\uparrow x$  for any member  $x$  of the tree. Using König's Lemma, the preceding Lemma 3.6 has been proved by Goubault-Larrecq [6, Lemma 4.12] for the case where  $I$  is the set of natural numbers with its usual order.

### 3.4 The Dcpo Case

The Order Rudin Lemma 3.4 has interesting consequences in a dcpo (see [5]).

**Fact 3.8** *Let  $D$  be a dcpo and  $\mathcal{F}$  a filtered family of nonempty finitely generated upper sets of  $D$ . Any Scott-closed set  $C$  that meets all members of  $\mathcal{F}$  also meets  $\bigcap \mathcal{F}$ .*



**Proof.** Let  $C$  be a Scott-closed, hence lower set that meets all members of  $\mathcal{F}$ . By 3.4, it has a directed subset  $A$  that still meets all members of  $\mathcal{F}$ . The least upper bound  $x$  of  $A$  exists in the dcpo  $D$  and is in  $C$  since  $C$  is Scott-closed. Since  $A$  meets all members of  $\mathcal{F}$  and since these members are upper sets, the upper bound  $x$  of  $A$  is in all of them, i.e.,  $x$  is in  $C \cap \bigcap \mathcal{F}$ .  $\square$

By contraposition and complementing  $C$ , one obtains the following:

**Corollary 3.9** *Let  $D$  be a dcpo and  $\mathcal{F}$  a  $\sqsubseteq$ -directed family of nonempty finite sets of  $D$ . If  $\bigcap_{F \in \mathcal{F}} \uparrow F$  is a subset of a Scott-open set  $U$ , then already some member of  $\mathcal{F}$  is a subset of  $U$ .*

Note that these two statements are based on considering two different topologies on the underlying set: 3.4 is the instance of the Topological Rudin Lemma for the Alexandroff topology, whereas the derivation of 3.8 and 3.9 from 3.4 is based on the Scott topology.

**Corollary 3.10** *Let  $D$  be a dcpo and  $\mathcal{F}$  a filtered family of nonempty finitary upper sets of  $D$ . Then  $\bigcap \mathcal{F}$  is a nonempty compact saturated set.*

**Proof.** Applying 3.8 in the case  $C = X$ , we see that  $\bigcap \mathcal{F}$  is nonempty. In order to show the compactness of  $\bigcap \mathcal{F}$ , suppose that  $(U_i)_i$  is a family of open sets covering  $\bigcap \mathcal{F}$ . By the previous corollary, some  $K \in \mathcal{F}$  is contained in the open set  $\bigcap_i U_i$ . By the compactness of  $K$ , finitely many of the  $U_i$  already cover  $K$ , hence they also cover  $\bigcap \mathcal{F}$ .  $\square$

### 3.5 The Sober Case

The Topological Rudin Lemma itself has analogous consequences in a sober space. Recall that a topological space is *sober*, if every irreducible closed subset  $A$  is the closure of a uniquely determined point  $a$ . Unlike the dcpo case, all arguments are based on a single topology. Thus, the following is not a generalization of 3.8, but a logically unrelated statement.

**Proposition 3.11** *Let  $X$  be a sober space and  $\mathcal{A}$  an irreducible subset of  $\mathfrak{K}X$  ( $\Omega X$ ,  $\Omega_f X$ , respectively). Then any closed subset  $C$  of  $X$  that meets all members of  $\mathcal{A}$  also meets  $\bigcap_{K \in \mathcal{A}} \uparrow K$ , and if  $\bigcap_{K \in \mathcal{A}} \uparrow K$  is a subset of an open set  $U$ , then already some member of  $\mathcal{A}$  is a subset of  $U$ .*

**Proof.** Let  $C$  be a closed set that meets all members of  $\mathcal{A}$ . By 3.1, it has an irreducible closed subset  $A$  that still meets all members of  $\mathcal{A}$ . Since  $X$  is sober,  $A$  is the closure of a unique point  $x$ ,  $A = \text{cl}\{x\} = \downarrow x$ . Then  $x \in A \subseteq C$ , and since  $A$  meets all members of  $\mathcal{A}$ , the greatest element  $x$  of  $A$  belongs to  $\uparrow K$  for all  $K \in \mathcal{A}$ . The statement about the open set follows by contraposition and complementing the closed set.  $\square$

The following lemma is useful in the proof of the subsequent soberness criterion:

**Fact 3.12** *Let  $\mathcal{A}$  be a set of compact (supercompact) subsets of a topological space  $X$  and  $K$  an arbitrary subset of  $X$  with the property that  $K$  is a subset of an open*

set  $U$  iff some member of  $\mathcal{A}$  is a subset of  $U$ . Then  $K$  is compact (supercompact).

**Proof.** Let  $(U_i)_{i \in I}$  be a directed (arbitrary) family of open sets such that  $K \subseteq \bigcup_{i \in I} U_i$ . By hypothesis, there is some  $Q$  in  $\mathcal{A}$  such that  $Q \subseteq \bigcup_{i \in I} U_i$ . Since  $Q$  is compact (supercompact),  $Q \subseteq U_k$  holds for some  $k$  in  $I$ . By the hypothesis again,  $K \subseteq U_k$  follows.  $\square$

We now can prove the main result in this section:

**Theorem 3.13** *For a topological space  $X$ , the following are equivalent:*

- (i)  $X$  is sober.
- (ii)  $X$  is strongly well-filtered, that is, whenever  $\mathcal{A} \subseteq \Omega X$  is an irreducible collection of nonempty compact saturated sets and  $U$  an open subset of  $X$  such that  $\bigcap \mathcal{A} \subseteq U$ , then  $K \subseteq U$  for some  $K \in \mathcal{A}$ .
- (iii)  $\Omega X$  is sober.

**Proof.** The implication (i)  $\Rightarrow$  (ii) holds by 3.11. For (ii)  $\Rightarrow$  (iii), let  $\mathcal{A}$  be an irreducible closed set in  $\Omega X$ . By 3.12,  $K = \bigcap \mathcal{A}$  is compact, i.e., an element of  $\Omega X$ . The property  $K \in \square U$ , i.e.,  $K \subseteq U$ , is equivalent to  $\mathcal{A} \cap \square U \neq \emptyset$  by (ii). This equivalence proves  $\text{cl}_{\Omega X} \{K\} = \mathcal{A}$ .

Finally assume  $\Omega X$  is sober and let  $C$  be an irreducible closed set of  $X$ . Then  $\mathcal{A} = \text{cl}\{\uparrow x \mid x \in C\}$  is an irreducible closed set of  $\Omega X$  by 2.5 ( $((x \mapsto \uparrow x) : X \rightarrow \Omega X$  is continuous) and 2.4. Since  $\Omega X$  is sober, there is a compact saturated set  $K$  such that  $\mathcal{A} = \text{cl}\{K\}$ . Hence  $K \in \square U$  iff  $\{\uparrow x \mid x \in C\}$  meets  $\square U$ . Therefore,  $\{\uparrow x \mid x \in C\}$  and  $K$  satisfy the hypothesis of 3.12, whence  $K$  is supercompact. By 2.2,  $K = \uparrow a$  holds for some  $a$  in  $X$ . For all open sets  $U$ ,  $C$  meets  $U$  iff  $\uparrow x \subseteq U$  for some  $x$  in  $C$ , iff  $K = \uparrow a \subseteq U$ , iff  $a$  in  $U$ . This equivalence implies  $C = \text{cl}\{a\}$ .  $\square$

**Remark 3.14** (1) In Statement (ii) one may replace the collection  $\Omega X$  of all nonempty compact saturated sets by the collection  $\mathfrak{K}X$  of all nonempty compact sets.

(2) Since filtered collections are irreducible, statement (ii) of 3.13 implies the corresponding statement for filtered sets  $\mathcal{F}$  of compact saturated sets: *Whenever  $\mathcal{F}$  is a filtered collection of nonempty compact saturated sets and  $U$  an open set such that  $\bigcap \mathcal{F} \subseteq U$ , then  $Q \subseteq U$  for some  $Q \in \mathcal{F}$ .* In [4, Definition I-1.24.1] a space has been called *well-filtered*, if this latter property holds. This “filtered” version of 3.11 can be derived from 3.2, the filtered version of the Topological Rudin Lemma. In his PhD thesis [8, Problem 6, p. 120], the first author asked the question whether the “filtered” version of 3.11 is equivalent to soberness. The answer is “no” which we already discussed at the end of the introduction. Thus 3.13 shows that the general “irreducible” version of 3.11 is strictly more powerful than the “filtered” version.

(3) The implication (i)  $\Rightarrow$  (iii) in the previous theorem has already been proved by A. Schalk [14, Lemma 7.20].

## 4 Quasicontinuous domains

We present an approach to quasicontinuous dcpos by focussing on the poset  $\mathfrak{Q}_f X$  of nonempty finitely generated sets and on the poset  $\mathfrak{Q}X$  of nonempty compact saturated sets rather than the Scott-open ones. We present simpler proofs of known results and a characterization of those dcpos that are Smyth powerdomains of quasicontinuous domains.

### 4.1 The way-below relation on finite subsets

Throughout let  $X$  be a dcpo. As before,  $\mathfrak{Q}_f X$  denotes the collection of all nonempty finitely generated upper sets ordered by  $\sqsubseteq$ , that is, by reverse inclusion. By  $F, G, H, \dots$  we always denote nonempty finite subsets.

Let us recall the definition of the *way-below* relation on an arbitrary poset  $P$ . For  $x, y \in P$  one writes

$$x \ll y \iff (y \leq \bigvee^\uparrow D \Rightarrow \exists d \in D. x \leq d)$$

that is,  $x \ll y$  if, for every directed subset  $D$  of  $P$  such that  $y \leq \bigvee^\uparrow D$ , there is an element  $d \in D$  with  $x \leq d$ , provided that  $D$  has a least upper bound in  $P$ .

Let us apply this definition to the poset  $\mathfrak{Q}_f X$  of nonempty finitely generated upper sets ordered by reverse inclusion:  $\uparrow G \ll \uparrow H$  iff for every  $\sqsubseteq$ -directed family  $(\uparrow F_i)_i$  such that  $\bigcap_i \uparrow F_i$  is a finitely generated upper set contained in  $\uparrow H$ , there is an  $i$  such that  $F_i \subseteq \uparrow G$ .

We will write  $G \ll H$  if  $\uparrow G \ll \uparrow H$ . The following lemma shows that the way-below relation on the poset  $\mathfrak{Q}_f P$  agrees with the way-below relation defined for finite subsets of a dcpo in [5] and in [4, Definition III-3,1]:

**Lemma 4.1** *For nonempty finite subsets of a dcpo  $X$  one has  $G \ll H$  if and only if, whenever  $\bigvee^\uparrow D \in \uparrow H$  for some directed  $D \subseteq X$ , then  $d \in \uparrow G$  for some  $d \in D$ .*

**Proof.** Suppose first that  $G \ll H$  according to our definition. Consider a directed set  $D$  such that  $\bigvee^\uparrow D \in \uparrow H$ . Then the principal filters  $\uparrow d$ ,  $d \in D$ , form a filtered family of nonempty finitely generated upper sets with  $\bigcap_{d \in D} \uparrow d = \uparrow(\bigvee^\uparrow D) \subseteq \uparrow H$ . Thus, if  $\uparrow G \ll \uparrow H$ , there is a  $d \in D$  such that  $d \in \uparrow G$ .

Conversely, suppose that  $\bigvee^\uparrow D \in \uparrow H \Rightarrow \exists d \in D. d \in \uparrow G$ . In order to show that  $\uparrow G \ll \uparrow H$ , consider any filtered family of nonempty finitely generated upper sets  $(\uparrow F_i)_i$  whose intersection is a finitely generated upper set contained in  $\uparrow H$ . Suppose that none of the  $F_i$  is contained in  $\uparrow G$ . Then the  $F'_i = F_i \setminus \uparrow G$  are nonempty and they still form a  $\sqsubseteq$ -directed family. By Jung's version 3.5 of Rudin's Lemma, there is a directed set  $D \subseteq \bigcup_i F'_i$  such that  $D \cap F'_i \neq \emptyset$  for all  $i$ . Then  $\bigvee^\uparrow D \in \uparrow F'_i \subseteq \uparrow F_i$  for all  $i$ , whence  $\bigvee^\uparrow D \in \bigcap_i \uparrow F_i \subseteq \uparrow H$ . By our hypothesis, this implies  $d \in \uparrow G$  for some  $d \in D$ , which contradicts the fact that  $d$  belongs to some  $F'_i$  which is disjoint from  $\uparrow G$  by its definition. Thus, some  $F_i$  is contained in  $\uparrow G$ .  $\square$

We abbreviate  $G \ll \{y\}$  by  $G \ll y$ . As a special case of the previous lemma we obtain:

**Corollary 4.2**  $G \ll y$  iff  $(y \leq \bigvee^\uparrow D \Rightarrow \exists d \in D. d \in \uparrow G)$ .

In particular,  $\{x\} \ll \{y\}$  in  $\mathfrak{Q}_f X$  iff  $x \ll y$  in  $X$ . Thus the canonical map  $x \mapsto \uparrow x: X \rightarrow \mathfrak{Q}_f X$  is an embedding for the order, for directed suprema and for  $\ll$ .

Using the alternative description of the way-below relation of Lemma 4.1 we see:

**Corollary 4.3** *The way-below relation on  $\mathfrak{Q}_f X$  is preserved by union, that is, for nonempty finite subsets one has:  $F \ll G$  and  $F' \ll G' \Rightarrow F \cup F' \ll G \cup G'$ , or, equivalently:*

$$\uparrow F \ll \uparrow G \text{ and } \uparrow F' \ll \uparrow G' \Rightarrow \uparrow F \sqcap \uparrow F' \ll \uparrow G \sqcap \uparrow G'$$

In particular,  $F \ll G$  if and only if  $F \ll x$  for all  $x \in G$ .

## 4.2 Quasi-continuous dcpos

Recall that a poset  $P$  is called continuous if, for all  $x \in P$ , the set of all  $y \ll x$  is directed and  $x = \bigvee^\uparrow \{y \mid y \ll x\}$ . We now define:

**Definition 4.4** A dcpo  $X$  is called *quasicontinuous* if the poset  $\mathfrak{Q}_f X$  of nonempty finitely generated upper sets ordered by reverse inclusion  $\sqsubseteq$  is continuous.

In the following proposition we show that our definition of quasicontinuity is equivalent to the one given in [5] and [4, Definition III-3.2]:

**Proposition 4.5** *A dcpo  $X$  is quasicontinuous according to our definition if and only if it satisfies condition*

( $*$ ): *for every  $x \in X$  the family of nonempty finite sets  $F \ll x$  is  $\sqsubseteq$ -directed and  $\bigcap_{F \ll x} \uparrow F = \uparrow x$ , that is, whenever  $y \not\ll x$  there is a finite  $F \ll x$  such that  $y \notin \uparrow F$ .*

**Proof.** <sup>6</sup> Suppose first that  $X$  is quasicontinuous according to our definition, that is,  $(\mathfrak{Q}_f X, \sqsubseteq)$  is a continuous poset. Then in particular the  $F \ll x$  form a  $\sqsubseteq$ -directed subset of  $\mathfrak{Q}_f(P)$  and  $\uparrow x = \bigcap \{\uparrow F \mid F \ll x\}$ .

Suppose conversely that condition ( $*$ ) is satisfied. As we have remarked in 4.3, we have  $F \ll G$  iff  $F \ll x$  for all  $x \in G$ . By hypothesis, the set of  $F \ll x$  is a  $\sqsubseteq$ -ideal. In a semilattice, an intersection of finitely many ideals is an ideal. Thus, the set of  $F \ll G$  is  $\sqsubseteq$ -directed. In order to show that  $\bigcap_{F \ll G} \uparrow F = \uparrow G$ , consider any  $z \notin \uparrow G$ . By our hypothesis ( $*$ ), for every  $x \in G$  there is an  $F_x \ll x$  such that  $z \notin \uparrow F_x$ . For the finite set  $F = \bigcup_{x \in G} F_x$  one has  $F \ll G$  by 4.3 and clearly  $z \notin \uparrow F$ . We conclude that  $z \notin \bigcap_{F \ll G} \uparrow F$ .  $\square$

We deduce some properties of quasicontinuous dcpos:

**Properties 4.6** *Let  $X$  be a quasicontinuous dcpo.*

<sup>6</sup> We are indebted to Achim Jung for pointing out a gap in the proof of this proposition in a previous version of this paper.

- (i) *The way-below relation  $F \ll G$  on  $\mathfrak{Q}_f X$  has the interpolation property. In particular, if  $F \ll x$ , then there is a  $G$  such that  $F \ll G \ll x$ . (Compare [4, Proposition III-3.5].)*

Indeed, by definition  $\mathfrak{Q}_f X$  is a continuous poset, and the way-below relation on every continuous poset has the interpolation property.

- (ii) *For every nonempty finite subset  $F$ , the set*

$$\uparrow F = \{x \in X \mid F \ll x\}$$

*is Scott-open. (Compare [4, Proposition III-3.6].)*

**Proof.** Let  $F$  be a nonempty finite set in  $X$ . In order to show that the set  $\uparrow F$  is Scott-open, consider any element  $x_0$  such that  $F \ll x_0$  and suppose that  $x_0 \leq \bigvee_i x_i$  for some directed family  $(x_i)_i$  in  $X$ . By the interpolation property, there is an  $F'$  such that  $F \ll F' \ll x_0$ . Then  $y \leq x_i$  for some  $y \in F'$  and some  $i$ . Since  $F \ll y$ , we conclude that  $F \ll x_i$ , that is,  $x_i \in \uparrow F$ .  $\square$

- (iii) *A subset  $U$  of  $X$  is Scott-open if and only if, for every  $x \in U$ , there is a nonempty finite set  $F \ll x$  such that  $\uparrow F \subseteq U$ . Thus, the sets of the form  $\uparrow F$  for nonempty finite subsets  $F$  form a basis for the Scott topology on  $X$ . (Compare [4, Proposition III-3.6].)*

**Proof.** Let  $U$  be a Scott-open subset of  $X$  and  $x \in U$ . We know that  $\uparrow x = \bigcap_{F \ll x} \uparrow F$ . Since the collection of  $F \ll x$  is  $\sqsubseteq$ -directed, Corollary 3.9 tells us that there is an  $F \ll x$  such that  $\uparrow F \subseteq U$ . Suppose conversely that for every  $x \in U$  there is a finite set  $F \ll x$  such that  $\uparrow F \subseteq U$ . Then  $U$  is the union of the sets  $\uparrow F$ , where  $F$  ranges over the nonempty finite subsets  $F$  of  $U$ . From (ii) we conclude that  $U$  is Scott-open.  $\square$

- (iv) *For every nonempty compact saturated subset  $Q$  of  $X$  and every Scott-open neighborhood  $U$  of  $Q$ , there is a nonempty finite subset  $F \subseteq U$  such that  $Q \subseteq \uparrow F$ .*

**Proof.** Let  $Q$  be nonempty, compact and saturated. Let  $U$  be a Scott-open set containing  $Q$ . By property (iii),  $U$  is the union of the sets  $\uparrow F$ , where  $F$  ranges over the nonempty finite subsets of  $U$ . As the compact set  $Q$  is covered by this collection of basic opens, there are finitely many  $F_i \subseteq U$  such that  $Q$  is covered by the  $\uparrow F_i, i = 1, \dots, n$ . Thus  $F = \bigcup_i F_i$  is a finite subset of  $U$  with the property that  $Q \subseteq \bigcup_i \uparrow F_i = \uparrow F$ .  $\square$

- (v) *A quasicontinuous dcpo  $X$  is locally compact for its Scott topology. (Compare [4, Proposition III-3.7(a)].)*

Indeed, by (iv) every  $x \in X$  has a neighborhood basis of finitely generated upper sets and those are compact.

Every continuous poset has a round ideal completion. A directed lower set is an *ideal*, and an ideal  $I$  in a continuous poset  $P$  is *round* if for every  $a \in I$  there is an

element  $b \in I$  with  $a \ll b$ . For every  $b \in P$ , the set

$$\downarrow b = \{a \in P \mid a \ll b\}$$

is a round ideal. The collection  $\mathfrak{I}P$  of all round ideals ordered by inclusion is called the *round ideal completion* of  $P$ . The map  $b \mapsto \downarrow b: P \rightarrow \mathfrak{I}P$  is an order embedding. The following is well known:

**Lemma 4.7** *The round ideal completion  $\mathfrak{I}P$  of a continuous poset  $P$  is a continuous dcpo. For two round ideals  $I$  and  $J$  one has  $I \ll J$  if and only if there is an element  $b \in J$  such that  $I \subseteq \downarrow b$ .*

For a quasicontinuous dcpo  $X$  the round ideal completion of the continuous poset  $\Omega_f X$  has a concrete description:

**Lemma 4.8** *Let  $X$  be a quasicontinuous dcpo. If we assign to every round ideal  $\mathcal{I}$  of  $\Omega_f X$  the set  $\kappa(\mathcal{I}) = \bigcap \mathcal{I}$ , we obtain an isomorphism of the round ideal completion of  $\Omega_f X$  onto the collection  $\Omega X$  of all nonempty compact saturated subsets of  $X$ .*

**Proof.** For any ideal  $\mathcal{I}$  of  $\Omega_f X$ , the intersection  $\bigcap \mathcal{I}$  is a nonempty compact saturated set by Corollary 3.10. Thus,  $\kappa$  maps round ideals to nonempty compact saturated sets. Clearly,  $\kappa$  is order preserving.

Let conversely  $Q$  be a nonempty compact saturated set. The collection  $\mathcal{I}_Q$  of all  $\uparrow F \in \Omega_f X$  such that  $Q \subseteq \uparrow F$  is a round ideal such that  $\kappa(\mathcal{I}_Q) = Q$  by Property 4.6(iv). Thus  $\kappa$  is surjective. If  $Q$  and  $Q'$  are nonempty compact saturated sets such that  $Q \not\subseteq Q'$ , then there is an open set  $U$  containing  $Q$  but not  $Q'$ . It follows that there is a nonempty finite subset  $F \subseteq U$  such that  $Q \subseteq \uparrow F$ . Thus  $\uparrow F \in \mathcal{I}_Q \setminus \mathcal{I}_{Q'}$ , whence  $\mathcal{I}_Q \not\subseteq \mathcal{I}_{Q'}$ . It follows that  $\kappa$  is an order isomorphism.  $\square$

By Lemma 4.7 and Lemma 4.8 we conclude:

**Proposition 4.9** *For a quasicontinuous dcpo  $X$ , the collection  $\Omega X$  of all nonempty compact saturated subsets ordered by reverse inclusion  $\sqsupseteq$  is a continuous directed complete dcpo. The way-below relation on  $\Omega X$  is given by:  $Q \ll Q'$  iff there is a nonempty finite subset  $F \subseteq Q$  such that  $Q' \subseteq \uparrow F$  iff  $Q$  is a neighborhood of  $Q'$ . The nonempty finitely generated upper sets form a basis.*

**Remark 4.10** Clearly,  $\Omega X$  is also a semilattice for the operation  $Q \sqcap Q' = Q \cup Q'$ , and this semilattice operation preserves the way-below relation:

$$Q \ll K, Q' \ll K' \implies Q \sqcap Q' \ll K \sqcap K'.$$

Indeed, if  $Q$  is a neighborhood of  $K$  and  $Q'$  a neighborhood of  $K'$ , then  $Q \cup Q'$  is a neighborhood of  $K \cup K'$ .

**Lemma 4.11** (Compare [14, Lemma 7.26][6, Corollary 3.6].) *For a quasicontinuous dcpo  $X$ , the upper Vietoris topology agrees with the Scott topology on  $\Omega X$ .*

**Proof.** The basic open sets for the upper Vietoris topology,  $\square U$  for Scott-open  $U \subseteq X$ , are also Scott-open in  $\Omega X$ . Indeed if  $(\uparrow F_i)_i$  is a  $\sqsubseteq$ -directed family such that  $\bigcap_i \uparrow F_i \subseteq U$ , then  $\uparrow F_i \subseteq U$  for some  $i$  by Corollary 3.9.

Conversely, a basic open set of the Scott topology on  $\Omega X$  is of the form  $\{Q \in \Omega X \mid \uparrow F \ll Q\}$  and this set can be rewritten as  $\square(\uparrow F)$ , and  $\uparrow F$  is Scott-open by Property 4.6(ii).  $\square$

**Corollary 4.12** (Compare [4, Proposition III-3.7]) *A quasicontinuous dcpo  $X$  is sober.*

**Proof.** Indeed,  $\Omega X$  is a continuous dcpo, hence sober for its Scott topology. Since the Scott topology agrees with the upper Vietoris topology by Lemma 4.11,  $X$  is sober by Theorem 3.13.  $\square$

For later use let us record the following properties:

**Proposition 4.13** *The canonical embedding  $\eta_X = (x \mapsto \uparrow x): X \rightarrow \Omega X$  is an embedding for the respective Scott, lower and Lawson topologies.*

**Proof.** The map  $\eta_X = (x \mapsto \uparrow x): X \rightarrow \Omega X$  is an embedding of  $X$  (with the Scott topology) into  $\Omega X$  with the upper Vietoris topology which agrees with the Scott topology by Lemma 4.11.

The map  $\eta_X$  is also an embedding for the respective lower topologies: Since every compact saturated set is the intersection of a filtered family of finitely generated upper sets, a subbasis for the closed sets of the lower topology on  $\Omega X$  is given by the sets of the form  $\{Q \in \Omega X \mid Q \subseteq \uparrow F\}$ , where  $F$  ranges over the finite subsets of  $X$ . The inverse image of such a set under  $\eta_X$  is the set  $\{x \in X \mid \uparrow x \subseteq \uparrow F\} = \uparrow F$ , and these sets form a basis for the closed sets for the lower topology on  $X$ .  $\square$

Since the Lawson topology on the continuous dcpo  $\Omega X$  is regular and Hausdorff, these properties are inherited by the Lawson topology on  $X$ . (Compare [4, Proposition III-3.7(b)].)

### 4.3 Abstract characterization of the domains $\Omega X$ for quasicontinuous $X$

We intend to show that the properties collected in Proposition 4.9 and the subsequent remark characterize those dcpos that are isomorphic to the powerdomain of all compact saturated subsets of quasicontinuous dcpos.

For this we have to identify  $X$  in  $\Omega X$ . In  $\Omega X$  we can find the elements  $x$  of  $X$  through the sets of the form  $\uparrow x$ . Can we distinguish these particular compact saturated sets from the others in the domain  $\Omega X$  by an intrinsic property?

Recall that an element  $p$  of a meet-semilattice is called *prime* if  $x \wedge y \leq p$  implies  $x \leq p$  or  $y \leq p$ . If there is a top element, we consider it to be prime as in [4]. The property of being prime extends from finite meets to meets of compact sets:

**Lemma 4.14** *If  $p$  is a prime element in a quasi-continuous meet-semilattice  $S$  and  $Q$  a Scott-compact subset of  $S$  with a greatest lower bound  $\bigwedge Q$  in  $S$  then  $\bigwedge Q \leq p$  implies that  $q \leq p$  for some  $q \in Q$ .*



**Proof.** Assume  $q \not\leq p$  for all  $q \in Q$ . Then for all  $q$  in  $Q$ , there is a finite  $F_q \ll q$  such that  $p \notin \uparrow F_q$ . The sets  $\{x \mid F_q \ll x\}$ ,  $q \in Q$ , form an open cover of  $Q$ . By compactness, there is a finite  $G \subseteq Q$  such that  $Q \subseteq \bigcup_{q \in G} \{x \mid F_q \ll x\}$ . Let  $F$  be the finite set  $\bigcup_{q \in G} F_q$ . Then  $Q \subseteq \uparrow F$ , and so  $p \geq \bigwedge Q \geq \bigwedge F$ . Since  $p$  is prime, there is some  $a$  in  $F$  such that  $p \geq a$ , whence there is some  $q$  in  $G$  such that  $p \in \uparrow F_q$  – a contradiction.  $\square$

We use this lemma for the following:

**Lemma 4.15** *Let  $X$  be a quasicontinuous dcpo. The prime elements of the  $\sqcap$ -semilattice  $\Omega X$  are the principal filters  $\uparrow x$ ,  $x \in X$ .*

**Proof.** All the  $\uparrow x$ ,  $x \in X$ , are prime in  $\Omega X$ . Indeed  $\uparrow x \subseteq Q_1 \cup Q_2$  implies  $x \in Q_1$  or  $x \in Q_2$ , whence  $\uparrow x \subseteq Q_1$  or  $\uparrow x \subseteq Q_2$ . It remains to show that every prime element in  $\Omega X$  is of the form  $\uparrow x$  for some  $x \in X$ .

Consider  $K \in \Omega X$ . The set  $\mathcal{K} = \{\uparrow x \mid x \in K\}$  is a compact subset of  $\Omega X$ . Its union is  $K$ , so  $\mathcal{K}$  has an infimum  $K = \bigcap \mathcal{K}$  in  $\Omega X$ . We now use Lemma 4.14: If  $K$  is prime in  $\Omega X$ , then there is an element  $\uparrow x \in \mathcal{K}$  such that  $\uparrow x \sqsubseteq K$ , which implies that  $K = \uparrow x$  for some  $x \in K$ .  $\square$

We now can formulate our representation theorem:

**Theorem 4.16** *Suppose that*

- (1)  *$L$  is a continuous directed complete  $\wedge$ -semilattice,*
- (2) *the finite meets of prime elements form a basis of  $L$ ,*
- (3) *the way-below relation  $\ll$  on  $L$  is preserved by the semilattice operation  $\wedge$ , that is, if  $a \ll b$  and  $a' \ll b'$  then  $a \wedge a' \ll b \wedge b'$ .<sup>7</sup>*

*Then the prime elements of  $L$  form a quasicontinuous dcpo  $X$  for the induced order and  $L$  is isomorphic to the continuous  $\sqcap$ -semilattice of all compact saturated subsets of  $X$ .*

For the proof of the theorem we use a relaxed notions of primeness. An ideal  $I$  of a  $\wedge$ -semilattice is called prime if  $a \wedge b \in I$  implies  $a \in I$  or  $b \in I$ . An element  $p$  is called *pseudoprime* if there is a prime ideal  $I$  such that  $p = \bigvee^\uparrow I$ . Clearly prime elements are pseudoprime. By [4, Proposition I-3.28] we have:

**Lemma 4.17** *Let  $L$  be a continuous directed complete  $\wedge$ -semilattice. Suppose that  $\wedge$  preserves the way-below relation in  $L$ . Then the pseudoprime elements agree with the prime elements.*

**Proof of Theorem 4.16.** Suppose that  $L$  satisfies the hypotheses of the theorem. Let  $X$  be the set of prime elements of  $L$ . Under our hypotheses the notions prime and pseudoprime agree by Lemma 4.17. We conclude that the join of a directed set  $D$  of prime elements is prime; indeed,  $\downarrow D$  is a prime ideal, whence  $\bigvee^\uparrow D$  is pseudoprime and consequently prime. Thus  $X$  is a sub-dcpo of  $L$ .

<sup>7</sup> In [4], this property is called the *multiplicativity* of the way-below relation.



We denote by  $L_f$  the set of all elements of  $L$  which have a representation as a meet  $f = \bigwedge F$  of a nonempty finite set  $F$  of prime elements.

Now look at a  $p \in X$  and an element  $f \in L_f$  such that  $f \ll p$  in  $L$ . If  $F$  is a finite subset of  $X$  such that  $f = \bigwedge F$ , we show that  $F \ll \{p\}$  in  $X$ . Suppose indeed that  $\bigwedge F \ll p$  in  $L$ . If  $D$  is a directed set in  $X$  such that  $p \leq \bigvee^\uparrow D$ , then there is a  $d \in D$  such that  $\bigwedge F \leq d$  which implies that  $x \leq d$  for some  $x \in F$ , since  $d$  is prime. Thus  $F \ll \{p\}$  in  $X$ .

For  $f, f' \in L_f$  there are finite sets  $F, F'$  in  $X$  such that  $f = \bigwedge F$  and  $f' = \bigwedge F'$ . Then  $f \leq f'$  iff  $F \sqsubseteq F'$ . Indeed, if  $f \leq f'$  then  $\bigwedge F \leq \bigwedge F' \leq p$  for every  $p \in F'$ ; since  $p$  is prime, there is a  $q \in F$  such that  $q \leq p$ , whence  $F \sqsubseteq F'$ . The converse is straightforward.

In order to show that  $X$  is quasicontinuous, consider any  $p \in X$ . The set of all  $f \in L_f$  such that  $f \ll p$  is directed, since the  $f \in L_f$  form a base of  $L$  by hypothesis (2). Thus the set of all nonempty finite subsets  $F$  of  $X$  such that  $\bigwedge F \ll p$  is  $\sqsubseteq$ -directed by the previous paragraph. Now let  $q$  be a prime element with  $p \not\leq q$ . There is an  $f = \bigwedge F \in L_f$  such that  $f \ll p$  but  $f \not\leq q$ . Thus  $F \ll \{p\}$  in  $X$  but  $q \notin \uparrow_X F$ . This shows that  $X$  is a quasicontinuous dcpo by Proposition 4.5.

We now have to show that  $L$  is isomorphic to the domain  $\mathfrak{Q}X$  of Scott-compact saturated subsets of  $X$ . For every  $a \in L$  consider the saturated subset  $\uparrow a \cap X$  of  $X$ . Suppose first  $a \in L_f$ . Then  $a = p_1 \wedge \dots \wedge p_n$  for prime elements  $p_1, \dots, p_n \in X$ . For any  $p \in X$ , one has  $p \geq a$  iff  $p \geq p_i$  for some  $i$ . Thus,  $\uparrow a \cap X$  is the upper set in  $X$  generated by the finite set  $\{p_1, \dots, p_n\}$ , hence a compact saturated subset of  $X$ . An arbitrary  $a \in L$  is the sup of the directed family of elements  $f_j$  in  $L_f$  with  $f_j \ll a$ . Then  $\uparrow a \cap X$  is the intersection of the filtered family  $\uparrow F_j \cap X$  of finitely generated upper sets in  $X$ , hence compact and saturated by 3.10. Thus  $a \mapsto \uparrow a \cap X$  is a map from  $L$  into  $\mathfrak{Q}X$ , which clearly is order preserving.

Conversely, let  $K$  be a Scott-compact saturated subset of  $X$ . Then  $K$  is the intersection of the filtered family  $\uparrow F_j$  of finitely generated upper sets in  $X$  such that  $F_j \ll K$ . We assign to  $K$  the element  $\bigvee_j \bigwedge F_j$  of  $L$  and we have a map from  $\mathfrak{Q}X$  to  $L$  which also is clearly order preserving.

It is straightforward to check that these two maps are inverse to each other, and the proof is complete.  $\square$

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