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Separation of variables in one case of motion of a gyrostat acted upon by gravity and magnetic fields

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ABSTRACT

A version of the integrable problem of motion of a dynamically symmetric gyrostat about a fixed point similar to the Kowalevski top, while acted upon by a combination of uniform gravity and magnetic fields is considered. This version is reduced, in general, to hyperelliptic quadratures. The special case when the gyrostatic momentum is absent is solved in terms of elliptic functions of time.

1. Introduction

The problem of motion of a heavy rigid body about a fixed point has a long history beginning with Euler's work [1]. For most of this history, the main concern of authors was to isolate cases, when the general solution of the equations of motion can be expressed explicitly in terms of functions of time, or, at least, can be reduced to quadratures. This recipe has succeeded in two cases: Euler's case of a body moving by inertia and Lagrange's case of a symmetrical top [2]. Separation of variables in Euler's case was found by Euler himself, but the solution was expressed by Jacobi in terms of his newly invented elliptic functions. Lagrange reduced the case of axisymmetric top to separation of variables involving elliptic integrals. Explicit expression of the solution in terms of time was initiated by Jacobi and can be found with some variations in [3–5].

The following historical turn in rigid body dynamics came in the opposite direction, from the study of the nature of solutions of the equations of motion. Kowalevski [6] isolated the possible cases which share with Euler's and Lagrange's cases

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the property of having their solutions as meromorphic functions of time. It turned out that only one more case satisfies this criterion. That case became known as Kowalevski's. Kowalevski obtained the complementary integral for that case as a quartic polynomial in velocities. She also integrated the equations of motion in terms of hyperelliptic functions of time. Her solution was simplified by Kötter [7] and reconsidered by a series of authors. For a detailed history, see e.g. [8].

When a uniformly rotating rotor has its axis of symmetry fixed in the rigid body, the resulting system is known as a gyrostat. For this system, the generalization of Lagrange's case and its solution was straightforward. Euler's case was generalized by Zhukovsky [9]. The corresponding solution was shown by Volterra to be expressible in terms of sigma functions of Weierstrass [10]. In [11], Wittenburg pointed out another solution in terms of elliptic functions of time. Detailed presentation of the history of general and particular solutions for a heavy gyrostat can be found in [12].

A century after the discovery of Kowalevski's case, its generalization to the problem of gyrostat has been found in a different context. The problem of motion of a gyrostat similar to the Kowalevski top and acted upon by two skew uniform fields (gravity and magnetic) was considered in [13]. A general first integral quartic in velocities and generalizing the Kowalevski integral was found for this generic problem. As, in the general case, the two force fields problem does not admit a symmetry group, the additional cyclic integral does not exist. It turned out that such integral still exists in two special cases [13]. The first case generalizes Kowalevski's case of one field to the gyrostat motion in one field. The second case does not contain the classical case of Kowalevski, since the intensities of the two fields are proportional and can vanish only simultaneously. In the last case the cyclic variable is a complementary angle to the sum (or difference) of the two angles of precession and proper rotation.

In the present paper we accomplish separation of variables for a version of the last case, corresponding to a special value of the cyclic constant proportional to the gyrostatic moment and singled out by the condition that the reduced system becomes time-reversible. We give two algebraic separations of variables. In the first one, the cyclic constant is supposed nonzero and the variables of separation are to be determined as functions of time by solving hyperelliptic Abel–Jacobi equations. This result is based on the analogy established in [14] (see also [15] for further generalizations) of a special class of problems of the gyrostat motion in two fields with the problems of the gyrostat motion in axially symmetric field with zero momentum constant. Thus, the first separation given below corresponds to the algebraic separation [16,17] found by the method proposed in [18,19] for the Goryachev case. This separation is not applicable if the gyrostatic moment in the initial problem is zero. Nevertheless, as it is shown in [14], the equivalent problem of the rigid body motion in an axisymmetric field is the integrable case of Chaplygin [20]. Therefore we give the second separation which transfers the elliptic separation found by Chaplygin to the two-fields problem.

The two types of separation of variables accomplished here make it easy to apply the algorithm of finding the admissible regions for the integral constants and to establish the rough phase topology of the system [21,22]. Moreover, the recent results for separated systems [23] give a method to calculate the exact topological invariants of singular points and all regular iso-energy levels. This will give the complete topological analysis of the problem which will be different (as far as special types of motions are concerned) from the corresponding Goryachev and Chaplygin cases [17,24] since the analogy of these problems with the motion of a gyrostat in two fields does not give a global diffeomorphism of the corresponding phase spaces.

It will also be interesting to analyze all special cases when the trajectories in reduced systems become periodic. For the Goryachev case the corresponding quadratures are found in [17]. In our problem such quadratures can lead to explicit calculation of the orientation matrix and therefore provide the analytical basis for the geometric interpretation of periodic and two-frequency motions of the considered gyrostat in two fields.

2. Equations and integrals

The equation of the motion of a gyrostat acted upon by two homogeneous fields in the general case can be written in the Euler – Poisson form

$$\begin{aligned} \dot{\mathbf{M}} &= \mathbf{M} \times \boldsymbol{\omega} + \mathbf{c}_1 \times \boldsymbol{\alpha} + \mathbf{c}_2 \times \boldsymbol{\beta}, \\ \dot{\boldsymbol{\alpha}} &= \boldsymbol{\alpha} \times \boldsymbol{\omega}, \quad \dot{\boldsymbol{\beta}} = \boldsymbol{\beta} \times \boldsymbol{\omega}. \end{aligned} \tag{1}$$

Here $\boldsymbol{\omega}$ is the angular velocity, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are the characteristic vectors of the force fields (say, the vectors of the gravity force and of the magnetic field strength), $\mathbf{c}_1, \mathbf{c}_2$ are the vectors pointing from the fixed point O to the centers of the fields application. All objects are referred to some moving axes. The kinetic momentum vector $\mathbf{M}$ is connected with the angular velocity by the relation

$$\mathbf{M} = \boldsymbol{\omega}\mathbf{I} + \boldsymbol{\lambda},$$

where $\mathbf{I}$ and $\boldsymbol{\lambda}$ are the inertia tensor at O and the gyrostatic momentum vector. Both $\mathbf{I}$ and $\boldsymbol{\lambda}$ are constant in the moving frame. We consider the components of all vectors as rows, thus obtaining the unusual order of the objects in the above expression for $\mathbf{M}$.

It is known [8] that without changing the plane $O\mathbf{c}_1\mathbf{c}_2$ in the body, one can make the pair of the vectors $\mathbf{c}_1, \mathbf{c}_2$ to be orthonormal. Let us choose the moving frame $O\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ of the principal axes of the inertia tensor. Suppose that the gyrostat is dynamically symmetric $\mathbf{e}_1\mathbf{I}\cdot\mathbf{e}_1 = \mathbf{e}_2\mathbf{I}\cdot\mathbf{e}_2$, $\boldsymbol{\lambda} = \{0, 0, \lambda\}$ and the centers of the fields application lie in the equatorial plane $\mathbf{c}_1\cdot\mathbf{e}_3 = 0$, $\mathbf{c}_2\cdot\mathbf{e}_3 = 0$. In this case (see [25,26]) by some linear change of variables one can make the immovable in space vectors $\boldsymbol{\alpha}, \boldsymbol{\beta}$ to be mutually orthogonal. Then, after the pair $\mathbf{c}_1, \mathbf{c}_2$ is made orthonormal, the modules of the vectors $\boldsymbol{\alpha}, \boldsymbol{\beta}$ contain all scalar information on the interaction of the gyrostat with the fields (e.g. for the gravity field, the module of the corresponding vector is equal to the product of the gyrostat weight and the distance from the mass center to the fixed point). Therefore, we call $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ the intensities of the force fields. In the case of dynamic symmetry any orthonormal pair in the equatorial plane becomes principal for the inertia tensor, so we take $\mathbf{e}_1 = \mathbf{c}_1$, $\mathbf{e}_2 = \mathbf{c}_2$.

In what follows, we assume that the inertia tensor satisfies the Kowalevski conditions

$$\mathbf{I} = \mathrm{diag}\{I_1, I_1, I_3\}, \quad I_1 = I_2 = 2I_3 \tag{2}$$

and the intensities of the orthogonalized fields are equal

$$\boldsymbol{\alpha}^2 = \boldsymbol{\beta}^2 = a^2, \quad \boldsymbol{\alpha}\cdot\boldsymbol{\beta} = 0 \quad (a > 0). \tag{3}$$

We also choose the dimension units in such a way that $I_3 = 1$.

Note that introducing dimensionless variables we can also obtain $a = 1$. Still, we keep the inessential parameter a for a possibility to pass to the limit case $a = 0$, and also for better control on the dimensions of expressions in the obtained formulas.

As shown in [13], under conditions (2), (3) in addition to the energy integral

$$H = \omega_1^2 + \omega_2^2 + \frac{1}{2}\omega_3^2 - \alpha_1 - \beta_2$$

Equation (1) have the first integrals

$$\begin{aligned} K &= \left(\omega_1^2 - \omega_2^2 + \alpha_1 - \beta_2\right)^2 + (2\omega_1\omega_2 + \alpha_2 + \beta_1)^2 \\ &\quad + 2\lambda\left[(\omega_3 - \lambda)\left(\omega_1^2 + \omega_2^2\right) + 2(\alpha_3\omega_1 + \beta_3\omega_2)\right], \\ G &= 2\omega_1\gamma_1 + 2\omega_2\gamma_2 + (\omega_3 + \lambda)(\gamma_3 - a). \end{aligned} \tag{4}$$

Here the vector $\boldsymbol{\gamma} = a^{-1}\,\boldsymbol{\alpha}\times\boldsymbol{\beta}$ augments the pair $\boldsymbol{\alpha}, \boldsymbol{\beta}$ to form a fixed in space orthogonal basis normalized by the value a. In particular, the matrix of direction cosines is

$$\mathrm{Q} = \frac{1}{a}\begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \\ \boldsymbol{\gamma} \end{pmatrix}.$$

Let us consider the action on $SO(3)$ of the subgroup $\{g_\tau\}$ of matrices

$$g_\tau = \begin{pmatrix} \cos\tau & \sin\tau & 0 \\ -\sin\tau & \cos\tau & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathrm{Q} \mapsto \mathrm{Q}(\tau) = g_\tau \mathrm{Q}\, g_\tau^{-1}.$$

by inner automorphisms

This action is not free; the subgroup $\{g_\tau\}$ itself is a stationary subgroup for each of its elements. Suppose that passing to the quotient space we identify the whole subgroup $\{g_\tau\}$ to one point. Then we obtain the fiber bundle of $SO(3)$ over the two-dimensional sphere, but this bundle is not locally trivial [27,28]. Therefore, such symmetry is called a singular symmetry. The singular points on the quotient sphere obviously correspond to the cases

$$\boldsymbol{\gamma} = \pm a\mathbf{e}_3. \tag{5}$$

The plus sign corresponds to the points of the subgroup $\{g_\tau\}$. For the minus sign the covering orbit of the action is twice shorter than all close ones.

The integral G is a cyclic integral generated by the action of $\{g_\tau\}$. Indeed, the instant angular velocity of the rotation $\mathrm{Q}(\tau)$ at $\tau = 0$ equals

$$\mathrm{Q}^{\mathrm{T}}(0)\frac{d}{d\tau}\Big|_{\tau=0}\mathrm{Q}(\tau) = \frac{1}{a}(\boldsymbol{\gamma} - a\mathbf{e}_3) = \frac{1}{a}(\gamma_1, \gamma_2, \gamma_3 - a),$$

therefore, G is, up to the constant multiplier a, the corresponding momentum integral.

3. Reduction by cyclic coordinate

We introduce the Euler angles θ, φ, ψ $(0 \leqslant \theta \leqslant \pi,\ 0 \leqslant \varphi \leqslant 2\pi,\ 0 \leqslant \psi \leqslant 2\pi)$, supposing θ is the angle between $\boldsymbol{\gamma}$ and $\mathbf{e}_3$:

$$\begin{gathered} \alpha_1 = a(\cos\varphi\cos\psi - \sin\varphi\sin\psi\cos\theta), \\ \alpha_2 = -a(\sin\varphi\cos\psi + \cos\varphi\sin\psi\cos\theta), \\ \alpha_3 = a\sin\psi\sin\theta, \\ \beta_1 = a(\cos\varphi\sin\psi + \sin\varphi\cos\psi\cos\theta), \\ \beta_2 = -a(\sin\varphi\sin\psi - \cos\varphi\cos\psi\cos\theta), \\ \beta_3 = -a\cos\psi\sin\theta, \\ \gamma_1 = a\sin\varphi\sin\theta, \ \ \gamma_2 = a\cos\varphi\sin\theta, \ \ \gamma_3 = a\cos\theta, \\ \omega_1 = \dot\psi\sin\varphi\sin\theta + \dot\theta\cos\varphi, \ \ \omega_2 = \dot\psi\cos\varphi\sin\theta - \dot\theta\sin\varphi, \\ \omega_3 = \dot\varphi + \dot\psi\cos\theta. \end{gathered}$$

The Lagrange function for the system (1) is

$$\begin{aligned} L &= \frac{1}{2}\dot\varphi^2 + \dot\theta^2 + \frac{1}{4}(3 - \cos 2\theta)\dot\psi^2 + \dot\varphi\dot\psi\cos\theta + \lambda(\dot\varphi + \dot\psi\cos\theta) \\ &\quad + a\cos(\varphi + \psi)(1 + \cos\theta). \end{aligned}$$

Remark 1. The points (5) correspond to the values $\theta = 0$ and $\theta = \pi$, i.e. to the case

$$\sin\theta = 0. \tag{6}$$

We shall consider only those solutions of the system (1) which do not cross the set of the phase space satisfying (6).

We will now introduce a change of variables

$$\varphi = \Phi - \Psi, \quad \psi = \Psi, \quad \theta = 2\Theta.$$

Then $\Theta \in [0, \pi/2]$, and the angles Φ, Ψ can be taken to vary on the segment $[0, 2\pi]$, since the change $(\varphi, \psi) \mapsto (\Phi, \Psi)$ is given by the integer matrix with determinant 1. Obviously, Ψ is a cyclic coordinate corresponding to the integral G. In the new variables, this integral is

$$G = a\frac{\partial L}{\partial\dot\Psi} = 2a\left[(3 + \cos 2\Theta)\dot\Psi - \dot\Phi - \lambda\right]\sin^2\Theta.$$

It is natural to denote its constant by $2ag$. From the equation of the cyclic integral we find

$$\dot\Psi = \frac{(\dot\Phi + \lambda)\sin^2\Theta + g}{D\sin^2\Theta}.$$

Here we denote

$$D = 3 + \cos 2\Theta = 2\left(2 - \sin^2\Theta\right).$$

Eliminating the cyclic coordinate we construct the Routhian

$$\begin{aligned} R &= \dot\Theta^2 + \frac{\cos^2\Theta}{2D}\dot\Phi^2 + \frac{2\lambda\cos^2\Theta - g}{2D}\dot\Phi + \frac{a}{2}\cos\Phi\cos^2\Theta \\ &\quad - \frac{\left(g + \lambda\sin^2\Theta\right)^2}{4D\sin^2\Theta}. \end{aligned}$$

Note that the solutions crossing the set (6) indeed need some special consideration since the values $\sin\Theta = 0$ and $\cos\Theta = 0$ cause obvious singularities either in the potential or in the kinetic energy of the reduced system.

The term linear in $\dot\Phi$ does not affect the Lagrange equations if

$$0\equiv\frac{\partial}{\partial\Theta}\left(\frac{2\lambda\cos^2\Theta-g}{2D}\right)=-\frac{2\lambda+g}{2D^2}\sin2\Theta,$$

i.e. for the value

$$g=-2\lambda. \tag{7}$$

In the sequel we shall consider only this case. Under condition (7), the reduced system is a natural mechanical system with

$$R=\dot\Theta^2+\frac{\cos^2\Theta}{2D}\dot\Phi^2+\frac{a}{2}\cos\Phi\cos^2\Theta-\frac{\lambda^2}{4\sin^2\Theta}. \tag{8}$$

For Θ and Φ, we have the pair of differential equations

$$\ddot\Phi=\frac{4\tan\Theta}{D}\dot\Phi\dot\Theta-\frac{aD\sin\Phi}{2},$$
$$\ddot\Theta=-\frac{\sin2\Theta}{2D^2}\dot\Phi^2+\frac{1}{4}\left(\lambda^2\frac{\cos\Theta}{\sin^3\Theta}-a\cos\Phi\sin2\Theta\right).$$

The cyclic variable is obtained by integrating the equation for $\dot\Psi$

$$\dot\Psi=\frac{\dot\Phi}{D}-\frac{\lambda}{2\sin^2\Theta}, \tag{9}$$

4. Separation of variables for non-zero gyrostatic momentum

It is known that in the corresponding problem of the dynamics of a rigid body in one axially symmetric force field [29] there exists a separation of variables. One of the possible ways to obtain a separation was pointed out in [30]. Complete real algebraic separation for the Goryachev case was given in [16,17]. Here we give a real separation for the gyrostat in the double field with singular symmetry under condition (7).

It is convenient to pass from H, K to the "shifted" first integrals

$$\tilde H=\frac{1}{4}\left(H+\frac{\lambda^2}{2}\right),\quad \tilde K=\frac{1}{4}(K+2\lambda^2H)$$

We denote by h and k the corresponding integral constants. In explicit form we have

$$\tilde H=\dot\Theta^2+\frac{\cos^2\Theta}{2D}\dot\Phi^2-\frac{a}{2}\cos\Phi\cos^2\Theta+\frac{\lambda^2}{4\sin^2\Theta},$$
$$\begin{aligned}\tilde K={}&4\dot\Theta^4+\frac{\sin^4 2\Theta}{4D^4}\dot\Phi^4+\frac{2\sin^2 2\Theta}{D^2}\dot\Phi^2\dot\Theta^2+\frac{8a\sin\Phi\cos\Theta\sin^3\Theta}{D}\dot\Phi\dot\Theta\\&+2\left[\frac{\lambda^2}{\sin^2\Theta}+2a\cos\Phi\sin^2\Theta\right]\dot\Theta^2+\frac{2\cos^2\Theta}{D^2}[\lambda^2-2a\cos\Phi\sin^4\Theta]\\&\times\dot\Phi^2+\frac{\lambda^4}{4\sin^4\Theta}+a^2\sin^4\Theta.\end{aligned}$$

Now we formulate the main result. Define the variables z_1, z_2 as the roots of the quadratic equation

$$z^2\sin^2\Theta-\lambda^2 z+2\lambda^2\left(\dot\Theta^2+\frac{\sin^2 2\Theta}{4D^2}\dot\Phi^2\right)-k\sin^2\Theta+\frac{\lambda^4}{2\sin^2\Theta}=0$$

and introduce the following two-valued ramified functions of z_1 and z_2

$$\begin{aligned}&p_1=\sqrt{k+a\lambda^2-z_1^2},\quad q_1=\sqrt{z_1^2-k+a\lambda^2},\\&r_1=\sqrt{z_1(z_1-\lambda^2)+2\lambda^2h-k},\\&p_2=\sqrt{k+a\lambda^2-z_2^2},\quad q_2=\sqrt{k-a\lambda^2-z_2^2},\\&r_2=\sqrt{z_2(\lambda^2-z_2)-2\lambda^2h+k}.\end{aligned} \tag{10}$$

Theorem 1. On common levels of the integrals

$$\tilde H=h,\quad \tilde K=k$$

the angles Θ, Φ are expressed via the variables z_1, z_2 by the formulas

$$\begin{aligned}&\sin\Theta=\frac{\lambda}{\sqrt{z_1+z_2}},\quad \cos\Theta=\frac{\sqrt{z_1+z_2-\lambda^2}}{\sqrt{z_1+z_2}},\\&\sin\frac{\Phi}{2}=\frac{p_2q_1r_1+p_1q_2r_2}{\lambda\sqrt{2a}(z_1-z_2)\sqrt{z_1+z_2}\sqrt{z_1+z_2-\lambda^2}},\\&\cos\frac{\Phi}{2}=\frac{p_1q_2r_1-p_2q_1r_2}{\lambda\sqrt{2a}(z_1-z_2)\sqrt{z_1+z_2}\sqrt{z_1+z_2-\lambda^2}},\end{aligned} \tag{11}$$

and the dependency of z_1, z_2 on time is described by the following differential equations

$$\lambda(z_2-z_1)\dot z_1=\sqrt{Z(z_1)},\quad \lambda(z_2-z_1)\dot z_2=\sqrt{Z(z_2)}, \tag{12}$$

where $Z(z)$ is a polynomial of degree six defined as

$$Z(z)=\left[a^2\lambda^4-(k-z^2)^2\right]\left[z(z-\lambda^2)+2\lambda^2h-k\right].$$

Remark 2. The differential equations for the auxiliary variables can obviously be written in the form of Abel–Jacobi equations

$$\frac{dz_1}{\sqrt{Z(z_1)}}-\frac{dz_2}{\sqrt{Z(z_2)}}=0,\quad \frac{z_1dz_1}{\sqrt{Z(z_1)}}-\frac{z_2dz_2}{\sqrt{Z(z_2)}}=-\frac{1}{\lambda}dt.$$

Solutions can be expressed in hyperelliptic functions of time.

Remark 3. In the above notation we consider the radicals $\sqrt{z_1+z_2-\lambda^2}$ and $\sqrt{z_1+z_2}$ to be always positive. Indeed, it follows from the definition of z_1, z_2 that $z_1+z_2\geqslant\lambda^2>0$. But the case $z_1+z_2=\lambda^2$ gives $\cos\Theta=0$. Such solutions are excluded according to Remark 1. Thus, these radicals (and obviously the constant $\sqrt{2a}>0$) do not generate multiple values in (11). On the contrary, if we fix some rectangle of oscillation of the separation variables z_1, z_2 in such a way that the expressions (11) are all real, then some of the radicals (10) periodically change their signs. In formulas (10), (11), where for consistency we must choose $\sqrt{Z(z_1)}=p_1q_1r_1$ and $\sqrt{Z(z_2)}=p_2q_2r_2$, we can make any formal substitution of the type $p_i\to -p_i$ or $q_i\to -q_i$, thus obtaining a completely equivalent form of the algebraic solution.

The expression for $\sin\Theta, \cos\Theta$ follow straightforwardly from the definition of the auxiliary variables, according to which

$$z_1+z_2=\frac{\lambda^2}{\sin^2\Theta}. \tag{13}$$

Taking into account that

$$z_1z_2=\frac{2\lambda^2}{\sin^2\Theta}\left(\dot\Theta^2+\frac{\sin^2 2\Theta}{4D^2}\dot\Phi^2\right)-k-\frac{\lambda^4}{2\sin^4\Theta}$$

we find

$$\frac{1}{\sin^2\Theta}\left(\dot{\Theta}^2+\frac{\sin^2 2\Theta}{4D^2}\dot{\Phi}^2\right)=\frac{2k-(z_1^2+z_2^2)}{4\lambda^2}. \tag{14}$$

To simplify the calculation let us introduce the variable V by putting

$$\dot{\Theta}=\frac{V}{2\sqrt{z_1+z_2}}. \tag{15}$$

Then eliminating Θ from (13), (14), we obtain

$$\dot{\Phi}=\frac{\left[\lambda^2-2(z_1^2+z_2^2)\right]\sqrt{2k-(z_1^2+z_2^2)-V^2}}{\lambda\sqrt{z_1+z_2}\sqrt{z_1+z_2-\lambda^2}}. \tag{16}$$

Substitution of the found expressions for $\sin\Theta, \cos\Theta, \dot{\Theta}, \dot{\Phi}$ into the equations of the first integrals gives two equations to determine V and $\cos\Phi$ as the functions of z_1, z_2:

$$\begin{aligned}&4a^2\lambda^4V^4+4a\lambda^2\{\lambda^2(z_1^2+z_2^2-2k)+[(k-z_1^2)(k-z_2^2)+\lambda^4]\cos\Phi\}V^2\\&\quad+\left[k^2+a^2\lambda^4+z_1^2z_2^2+a\lambda^2(z_1^2+z_2^2)\cos\Phi-k(2a\lambda^2\cos\Phi+z_1^2+z_2^2)\right]^2\\&=0,\ (z_1+z_2-\lambda^2)(V^2+a\lambda^2\cos\Phi)+k[\lambda^2-2(z_1+z_2)]\\&\quad+2h\lambda^2(z_1+z_2)+\frac{z_1^4-z_2^4}{z_1-z_2}-\lambda^2\frac{z_1^3-z_2^3}{z_1-z_2}=0.\end{aligned}$$

Whence,

$$\begin{aligned}V^2&=\left[(z_1+z_2-\lambda^2)(z_1-z_2)^2(z_1+z_2)\right]^{-1}\{2p_1q_1r_1p_2q_2r_2+2k^3\\&\quad-k^2[\lambda^2(4h-z_1-z_2)+3(z_1^2+z_2^2)]+k[3(z_1^4+z_2^4)\\&\quad+4h\lambda^2(z_1^2+z_2^2)-2\lambda^2(z_1^3+z_2^3)-2a^2\lambda^4]+a^2\lambda^6(4h-z_1-z_2)\\&\quad+a^2\lambda^4(z_1^2+z_2^2)+\lambda^2[z_1^5+z_2^5-2h(z_1^4+z_2^4)]-(z_1^6+z_2^6)\},\cos\Phi\\&=\left[(z_1+z_2-\lambda^2)(z_1-z_2)^2(z_1+z_2)\right]^{-1}-\{2p_1q_1r_1p_2q_2r_2+[a^2\lambda^4\\&\quad-(k-z_1^2)(k-z_2^2)]2k-\lambda^2(4h-z_1-z_2)-(z_1^2+z_2^2)\}.\end{aligned} \tag{17}$$

Let us calculate $\sqrt{\frac{1}{2}(1+\cos\Phi)}$, $\sqrt{\frac{1}{2}(1-\cos\Phi)}$ and $\sqrt{V^2}$. It is possible to avoid double radicals in the resulting formulas by obvious identity

$$\sqrt{A+2\sqrt{B}}=\frac{1}{\sqrt{2}}\left(\sqrt{A+C}+\sqrt{A-C}\right),\quad C=\sqrt{A^2-4B},$$

if C is a whole expression (see Chaplygin's techniques in [20]). Applying this to (17), we obtain the sine and cosine of the half-angle in the form (11), while for V we get

$$V=\frac{p_1q_1r_1+p_2q_2r_2}{(z_1-z_2)\sqrt{z_1+z_2}\sqrt{z_1+z_2-\lambda^2}}. \tag{18}$$

Note that at this point we can choose arbitrary formal signs at these values since only $\cos^2\Theta, V^2$, and $\cos\Phi$ are uniquely defined as the functions of z_i, p_i, q_i, r_i. But the choice of signs now made determine all choices of signs in the consequent expressions (see Remark 3 above).

Substituting (18) for V in (15), (16) we find $\dot{\Theta}, \dot{\Phi}$ in terms of z_1, z_2,

$$\begin{aligned}\dot{\Theta}&=\frac{p_1q_1r_1+p_2q_2r_2}{2(z_1^2-z_2^2)\sqrt{z_1+z_2-\lambda^2}},\\ \dot{\Phi}&=\frac{[2(z_1+z_2)-\lambda^2](p_1q_1r_2-p_2q_2r_1)}{\lambda(z_1^2-z_2^2)(z_1+z_2-\lambda^2)}.\end{aligned} \tag{19}$$

On the other hand, considering the operator $D_t=\dot{z}_1\partial_{z_1}+\dot{z}_2\partial_{z_2}$ of the complete time derivative of a function of z_1, z_2, we calculate from Equations (11)

$$\dot{\Theta}=\frac{D_t\sin\Theta}{\cos\Theta},\quad \dot{\Phi}=2\frac{D_t\sin\frac{\Phi}{2}}{\cos\frac{\Phi}{2}}. \tag{20}$$

Eliminating $\dot{\Theta}, \dot{\Phi}$ in (19) and (20) gives the system of equations linear in $\dot{z}_1, \dot{z}_2$,

$$\begin{aligned}&C_1\dot{z}_1+C_2\dot{z}_2+\frac{[2(z_1+z_2)-\lambda^2](p_1q_1r_2-p_2q_2r_1)}{\lambda}=0,\\&\dot{z}_1+\dot{z}_2+\frac{p_1q_1r_1+p_2q_2r_2}{\lambda(z_1-z_2)}=0.\end{aligned}$$

Here

$$\begin{aligned}C_1&=\frac{r_2}{r_1}(2z_1-\lambda^2)(z_1+z_2)-2z_1(z_1+z_2-\lambda^2)\frac{p_2q_2}{p_1q_1},\\C_2&=\frac{r_1}{r_2}(2z_2-\lambda^2)(z_1+z_2)-2z_2(z_1+z_2-\lambda^2)\frac{p_1q_1}{p_2q_2}.\end{aligned}$$

Solving this system we come to Equation (12).

Thus, in the reduced system under the condition (7) on the cyclic constant and the gyrostatic momentum we obtain the separation of variables and express the trigonometric functions of non-cyclic angles as rational expressions in the basic radicals (10) with coefficients defined by one-valued functions of the separation variables. For the cyclic coordinate (the angle Ψ) the generalized velocity can be presented in the same form. Indeed, from (9) in virtue of the formulas obtained for $\dot{\Theta}, \dot{\Phi}$ we get

$$\dot{\Psi}=-\frac{1}{2\lambda}\left[(z_1+z_2)+\frac{p_2q_2r_1-p_1q_1r_2}{(z_1+z_2-\lambda^2)(z_1-z_2)}\right].$$

After integrating differential Equation (12), we find the right-hand side as a function of time.

To conclude this section let us make one interesting observation. Having in mind the topological analysis of the problem, one has to construct the so-called bifurcation set of the first integrals. Such set is the part of the discriminant set Δ of the polynomial $Z(z)$ in the right-hand side of separated Equation (12) corresponding to real motions. It is easy to show that the set Δ after the normalization $a=1$ and the shift $k'=k-\lambda^2$ turns into the set bearing the bifurcation diagram constructed in [31] for another integrable case of Chaplygin also found in [20] and defined by the Hamiltonian

$$H=\frac{1}{2}\left(\omega_1^2+\omega_2^2+\frac{1}{2}\omega_3^2\right)+\frac{1}{2}(\alpha_1^2-\alpha_2^2)+c\alpha_1 \tag{21}$$

with $\alpha^2=1$ and $2c^2=\lambda^2$. It was shown in [32] that the bifurcation diagram of the axisymmetric problem with the Hamiltonian (21) can be represented as the discriminant set of some polynomial. Still for this Chaplygin's case the complete algebraic separation of variables is not known. It seems likely that such a separation can be obtained based on the obvious analogy between the two problems.

5. The case of zero gyrostatic momentum

The above separation, obviously, takes place only for the case $\lambda\neq0$. In particular, there is no analog for this solution for the two-fields Kowalevski top in the S^1-symmetric case. Nevertheless, when $\lambda=0$ (and then also $g=0$ in virtue of the condition (7)) the Routhian (8) and the first integrals do not have singular terms on the (Θ,Φ)-sphere. It is shown in [14] (see also [27]), that such problem has an analogy with Chaplygin's solution of the problem of motion of a body in a liquid under the Kowalevski condition $I_1 = I_2 = 2I_3$ [20]. We now transfer the results of Chaplygin's classical work [20] to the case under consideration of two fields.

Denote

$$\Omega = \omega_1^2 + \omega_2^2, \quad \varkappa = \sqrt{\tilde{k}}.$$

The latter notation is possible since for $\lambda = 0$ the integral K in (4) is the sum of two squares and, consequently, the integral $\tilde{K}$ is non-negative. Let us introduce the variables τ_1, τ_2 similar to the separated variables of Chaplygin putting

$$\tau_1 = \frac{1}{2a\sin^2\Theta}(\Omega + 2\varkappa), \quad \tau_2 = \frac{1}{2a\sin^2\Theta}(\Omega - 2\varkappa). \tag{22}$$

Let

$$\begin{aligned} &P_1 = \sqrt{\tau_1 + 1}, \ Q_1 = \sqrt{\tau_1 - 1}, \ R_1 = \sqrt{a\tau_1 - \varkappa - 2h}, \\ &P_2 = \sqrt{1 + \tau_2}, \ Q_2 = \sqrt{1 - \tau_2}, \ R_2 = \sqrt{2h - \varkappa - a\tau_2}. \end{aligned} \tag{23}$$

Theorem 2. On common levels of the integrals

$$\tilde{H} = h, \quad \tilde{K} = \varkappa^2$$

the angles Θ, Φ are expressed in terms of the variables τ_1, τ_2 by the formulas

$$\begin{aligned} &\sin\Theta = \frac{\sqrt{2\varkappa}}{\sqrt{a(\tau_1 - \tau_2)}}, \quad \cos\Theta = \frac{\sqrt{a(\tau_1 - \tau_2) - 2\varkappa}}{\sqrt{a(\tau_1 - \tau_2)}}, \\ &\sin\frac{\Phi}{2} = \frac{P_2Q_1R_1 + P_1Q_2R_2}{\sqrt{2(\tau_1 - \tau_2)}\sqrt{a(\tau_1 - \tau_2) - 2\varkappa}}, \\ &\cos\frac{\Phi}{2} = \frac{P_1Q_2R_1 - P_2Q_1R_2}{\sqrt{2(\tau_1 - \tau_2)}\sqrt{a(\tau_1 - \tau_2) - 2\varkappa}}, \end{aligned} \tag{24}$$

The corresponding generalized velocities have the form

$$\begin{aligned} \dot{\Theta} &= \frac{\sqrt{\varkappa}}{\sqrt{2(\tau_1 - \tau_2)}\sqrt{a(\tau_1 - \tau_2) - 2\varkappa}}(P_1Q_1R_1 + P_2Q_2R_2), \\ \dot{\Phi} &= \frac{2[a(\tau_1 - \tau_2) - \varkappa]}{(\tau_1 - \tau_2)[a(\tau_1 - \tau_2) - 2\varkappa]}(P_1Q_1R_2 - P_2Q_2R_1), \end{aligned} \tag{25}$$

and the dependency of τ_1, τ_2 on time is described by the following differential equations

$$\dot{\tau}_1 = -\sqrt{U_1(\tau_1)}, \quad \dot{\tau}_2 = \sqrt{U_2(\tau_2)}, \tag{26}$$

where

$$U_1(\tau_1) = (\tau_1^2 - 1)(a\tau_1 - \varkappa - 2h), \quad U_2(\tau_2) = (1 - \tau_2^2)(2h - \varkappa - a\tau_2).$$

The expression for Ψ is obtained by integrating the relation

$$\dot{\Psi} = \frac{\dot{\Phi}}{D} = \frac{a}{2[a(\tau_1 - \tau_2) - 2\varkappa]}(P_1Q_1R_2 - P_2Q_2R_1),$$

Remark 4. The general solution of Equation (26) is obtained by inverting the following elliptic integrals

$$-\int^{\tau_1} \frac{d\tau_1}{\sqrt{(\tau_1^2 - 1)(a\tau_1 - \varkappa - 2h)}} = t = \int^{\tau_2} \frac{d\tau_2}{\sqrt{(1 - \tau_2^2)(2h - \varkappa - a\tau_2)}}. \tag{27}$$

The explicit expressions in the Jacobi functions depend on the chosen intervals of oscillation of the variables τ_1, τ_2. The possible intervals are defined by the condition that the values (24) and (25) should be real.

In (26), we must choose $\sqrt{U_1(\tau_1)} = P_1Q_1R_1$ and $\sqrt{U_2(\tau_2)} = P_2Q_2R_2$. As in the previous case (see Remark 3), in formulas (24), (25), and (26), we can make any formal substitution of the type $P_i \to -P_i$ or $Q_i \to -Q_i$, thus obtaining a completely equivalent form of the algebraic solution. Our choice was made to have the combinations of the basic radicals in (24), (25) similar to those in (10), (11). This choice gave the minus sign in the first Equation (26) and at the first integral in (27).

Proof. From (22),

$$\Omega = 2\varkappa\frac{\tau_1 + \tau_2}{\tau_1 - \tau_2}, \quad \sin^2\Theta = \frac{2\varkappa}{a(\tau_1 - \tau_2)}. \tag{28}$$

The second equation defines the expressions for $\sin\Theta, \cos\Theta$. Calculate the value Ω in the angle variables,

$$\Omega = 4\dot{\Theta}^2 + \frac{\sin^2 2\Theta}{(3 + \cos 2\Theta)^2}\dot{\Phi}^2. \tag{29}$$

As before, let us introduce an auxiliary variable V,

$$\dot{\Theta} = \frac{\sqrt{\varkappa}V}{\sqrt{2a(\tau_1 - \tau_2)}}. \tag{30}$$

Then from (28), (29) we get

$$\dot{\Phi} = -\frac{2[a(\tau_1 - \tau_2) - \varkappa]\sqrt{a(\tau_1 + \tau_2) - V^2}}{\sqrt{a(\tau_1 - \tau_2) - 2\varkappa}\sqrt{a(\tau_1 - \tau_2)}}. \tag{31}$$

The equations of the first integrals give

$$\begin{aligned} &[a(\tau_1 - \tau_2) - 2\varkappa](V^2 + a\cos\Phi) - a[a(\tau_1 - \tau_2) - \varkappa](\tau_1 + \tau_2) \\ &\quad + 2ah(\tau_1 - \tau_2) = 0, \\ &4V^4 + 4a[(1 + \tau_1\tau_2)\cos\Phi - (\tau_1 + \tau_2)]V^2 \\ &\quad + a^2[(1 + \tau_1\tau_2) - (\tau_1 + \tau_2)\cos\Phi]^2 = 0. \end{aligned} \tag{32}$$

Hence we write

$$\begin{aligned} \cos\Phi &= (\tau_1 - \tau_2)^{-1}[a(\tau_1 - \tau_2) - 2\varkappa]^{-1}\{[a(\tau_1 + \tau_2) - 4h](1 - \tau_1\tau_2) \\ &\quad -2P_1Q_1R_1P_2Q_2R_2\}, \\ V^2 &= (\tau_1 - \tau_2)^{-1}[a(\tau_1 - \tau_2) - 2\varkappa]^{-1}a\{a[\tau_1(\tau_1^2 - 1) - \tau_2(1 - \tau_2^2)] \\ &\quad -2h(\tau_1^2 + \tau_2^2 - 2) - \varkappa(\tau_1^2 - \tau_2^2) + 2P_1Q_1R_1P_2Q_2R_2\}. \end{aligned} \tag{33}$$

Having chosen at this moment one of the possible solutions of system (32), we thus have determined the formal choice of the signs of the two-valued radicals (23) in the following expressions of the phase variables. The radicals themselves can have any sign on a trajectory (moreover, some of them change the sign periodically) but the formal expressions must be consistent. The first equation in (33) leads to the formulas for the sine and cosine of the half-angle as in (24). From the second one we find

$$\begin{aligned} V &= \frac{\sqrt{a}(P_1Q_1R_1 + P_2Q_2R_2)}{\sqrt{\tau_1 - \tau_2}\sqrt{a(\tau_1 - \tau_2) - 2\varkappa}}, \sqrt{a(\tau_1 + \tau_2) - V^2} \\ &= \frac{\sqrt{a}(P_2Q_2R_1 - P_1Q_1R_2)}{\sqrt{\tau_1 - \tau_2}\sqrt{a(\tau_1 - \tau_2) - 2\varkappa}}. \end{aligned}$$

Whence, according to (30) and (31), we obtain the expressions (25) for the generalized velocities.

Substituting the values (24) and (25) in (20), we come to separated Equation (26). Let us emphasize that this separation is elliptic with polynomial of degree 3 in the radicals, therefore it is complete in the sense that both Equation (26) are integrated independently of each other. Of course, the solutions can be easily written out in Jacobi functions.

Finally, we find $\dot{\Psi}$ from (9) by putting $\lambda = 0$.

This completes the expressions for $\Theta, \dot{\Theta}, \Phi, \dot{\Phi}, \dot{\Psi}$ as elliptic functions of time.

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