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The Relationships Between KM-fuzzy Quasi-metric Spaces and the Associated Posets of Formal Balls

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Abstract

The purpose of this paper is to investigate the connections between KM-fuzzy quasi-metric spaces and the associated order structures of formal balls. We introduce the notions of Yoneda T-completeness and Yoneda S-completeness on KM-fuzzy quasi-metric spaces. In both cases, the associated posets of formal balls are directed complete. In particular, for a subclass of KM-fuzzy quasi-metric spaces, Yoneda T-completeness and Yoneda S-completeness are characterized respectively by directed completeness of the associated posets of formal balls.

Keywords: KM-fuzzy quasi-metric space, Yoneda T-complete, Yoneda S-complete, Formal ball.

1 Introduction

Recently, several authors paid attention to the connections between the theory of fuzzy metric spaces and domain theory [18,21,22]. This topic wishes to provide order-theoretical approaches for the study of fuzzy metric spaces. The KM-fuzzy metric spaces due to Kramosil and Michálek [13] and the GV-fuzzy metric spaces due to George and Veeramani [6] are two types of frequently used fuzzy metric spaces. These concepts are originally inspired by the notion of probabilistic metric spaces [26]. Recently, Mardones-Pérez and de Prada Vicente considered in [18] a variant of KM-fuzzy metric spaces in which the required conditions of KM-fuzzy metrics are weaker than that of GV-fuzzy metrics. As it was pointed in [5,16], many

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topological types of results related to GV-fuzzy metric spaces are preserved in this context, for example, a Hausdorff topology can also be obtained. From now on, a KM-fuzzy metric space is always in the sense of [18].

The notion of formal balls plays an important role in the interplay between the theory of fuzzy metric spaces and domain theory. In order to generalize some important results of [3] to the fuzzy metric setting, Ricarte and Romaguera [22] presented the notion of standard completeness on KM-fuzzy metric spaces. It has been shown that a KM-fuzzy metric space $(X, M, \wedge)$ is standard complete if and only if the associated poset of formal balls is directed complete. Subsequently, through different methods, Mardones-Pérez and de Prada Vicente [18] further investigated the relationships between completeness of KM-fuzzy metric spaces $(X, M, \wedge)$ and directed completeness of the associated posets of formal balls. The computational model [12] for each KM-fuzzy metric space $(X, M, \wedge)$ was also established. Moreover, the idea in [3] has been extended to quasi-metric spaces [1,25], uniform spaces [24] and fuzzy partial metric spaces [27].

In [10], Gregori and Romaguera generalized the fuzzy metric spaces [6,13] to the fuzzy quasi-metric setting. Several authors have contributed to the development of fuzzy quasi-metric spaces [2,5,8,9,11,23]. In this paper, we will focus on the connections between completeness of KM-fuzzy quasi-metric spaces and directed completeness of the associated posets of formal balls. The priori work due to [21] showed that if the associated poset of formal balls in a KM-fuzzy quasi-metric space $(X, M, \wedge)$ is directed complete, then $(X, M, \wedge)$ is standard complete. However, different from the conclusions of [22] for fuzzy metric case, the converse of this result does not hold in general.

In this paper, we introduce the notions of Yoneda T-completenenss and Yoneda S-completenenss which imply the T-completeness in the sense of [18] and standard completeness (briefly, S-completenenss) in the sense of [22], respectively, on KM-fuzzy quasi-metric spaces. Furthermore, inspired by the methods used in [18,21], we prove that for each Yoneda $T(S)$-complete KM-fuzzy quasi-metric space $(X, M, \wedge)$ the associated poset of formal balls is directed complete. Hence the results give an answer to the problem involved in [21]. Conversely, if the associated poset of formal balls is directed complete, then $(X, M, \wedge)$ is $T(S)$-complete. Finally, the characterization of Yoneda $T(S)$-completenenss on KM-fuzzy metric spaces by directed completenenss of the associated posets of formal balls are obtained.

2 Preliminaries

Firstly, let us recall some basic notions about quasi-metric spaces which can be found in [4,14]. A quasi-pseudo-metric space (X, d) is a set X together with a non-negative real-valued function $d : X^2 \longrightarrow \mathbb{R}^+$ (called quasi-pseudo-metric) such that, for every $x, y, z \in X$, (i) $d(x, x) = 0$ and (ii) $d(x, y) \leq d(x, z) + d(z, y)$. If d satisfies the additional condition: (iii) $d(x, y) = d(y, x) = 0$ implies that $x = y$, then d is called a quasi-metric on X. If d is a quasi-metric on X, then the function d^{-1} defined by $d^{-1}(x, y) = d(y, x)$, is also a quasi-metric on X. A quasi-metric is a metric provided that $d(x,y)=d(y,x)$. Each quasi-pseudo-metric d on X induces a topology τ_d on X which has the family of open balls $\{B_d(x,r) : x \in X, r > 0\}$ as a base, where $B_d(x,r) = \{y \in X : d(x,y) < r\}$ for all $x \in X$ and $r > 0$. A net $\{x_i\}_{i\in D}$ of a quasi-pseudo-metric space (X,d) is called left K-Cauchy, if for each $\varepsilon > 0$ there exists $k_\varepsilon \in D$ such that $d(x_i,x_j) < \varepsilon$ whenever $k_\varepsilon \le i \le j$. A quasi-pseudo-metric space (X,d) is said to be left K-complete if every left K-Cauchy net has a limit with respect to the topology τ_d.

According to [26], a *continuous t-norm* is a continuous function $* : [0,1]\times[0,1] \to [0,1]$ that satisfies the following conditions for every $a,b,c,d \in [0,1]$: (1) $a*b = b*a$; (2) $a*(b*c) = (a*b)*c$; (3) $a*1 = a$; (4) $a*b \le c*d$ whenever $a \le c$ and $b \le d$. It is well-known that $a*b \le a \wedge b = \min\{a,b\}$ for the continuous minimum t-norm $\wedge$.

In [18], Mardones-Pérez and de Prada Vicente considered a variant of KM-fuzzy (pseudo-)metric spaces [13] as follows.

Definition 2.1 **([18])** A *KM-fuzzy metric space* is a triple $(X,M,*)$, where X is a non-empty set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times [0,+\infty)$ satisfying for all $x,y,z \in X$ and $t,s > 0$ that the following conditions:

(Fm-1) $M(x,y,0) = 0$;

(Fm-2) $M(x,x,t) = 1$ for all $t > 0$;

(Fm-3) $M(x,y,t) = M(y,x,t) = 1$ for all $t > 0$ implies $x = y$;

(Fm-4) $M(x,y,t) = M(y,x,t)$ for all $t > 0$;

(Fm-5) $M(x,z,t+s) \ge M(x,y,t) * M(y,z,s)$;

(Fm-6) $M(x,y,-) : [0,+\infty) \to [0,1]$ is left continuous, and $\lim_{t\to+\infty} M(x,y,t) = 1$.

For a KM-fuzzy metric space $(X,M,*)$, the fuzzy set M is called a KM-fuzzy metric on X. The conditions (Fm-2) and (Fm-4) imply that the function $M(x,y,-) : [0,+\infty) \to [0,1]$ is non-decreasing for all $x,y \in X$. The following notions that will be useful in the sequel can be found in [5,10,16].

If the condition (Fm-3) is dropped, we obtain a *KM-fuzzy pseudo-metric space.*

If the condition (Fm-4) is dropped, we obtain a *KM-fuzzy quasi-metric space.*

Similarly, if both conditions (Fm-3) and (Fm-4) are dropped, it will be called a *KM-fuzzy quasi-pseudo-metric space.*

For each KM-fuzzy quasi-pseudo-metric space $(X,M,*)$, the family $\mathfrak{B} = \{U(t,\varepsilon) : t > 0, \varepsilon \in (0,1]\}$, where $U(t,\varepsilon) = \{(x,y) \in X^2 : M(x,y,t) > 1-\varepsilon\}$ generates a quasi-uniformity $\mathfrak{U}_M$ on X [5,8,10]. Furthermore, the family $\mathcal{B} = \{U(\frac{1}{n},\frac{1}{n}) : n \in \mathbb{N}\}$ is a countable base for $\mathfrak{U}_M$. The associated topology of $\mathfrak{U}_M$ is denoted by $\mathfrak{T}_M$. For each KM-fuzzy quasi-metric space, there is a natural dual topology $\mathfrak{T}_{M^{-1}}$ on X, where the KM-fuzzy quasi-metric M^{-1} is given by $M^{-1}(x,y,t) = M(y,x,t)$ for all $x,y \in X$ and $t \in [0,+\infty)$.

Proposition 2.2 **([16, Corollary 4.3])** *Let $(X,M,*)$ be a KM-fuzzy quasi-metric space. Each of the following families is a neighbourhood base at $x \in X$ for the topology $\mathfrak{T}_M$.*

(1) $\mathcal{N}_x = \{U(t,\varepsilon)[x] : t > 0, \varepsilon \in (0,1]\}$, *where* $U(t,\varepsilon)[x] = \{y \in X : M(x,y,t) > 1-\varepsilon\}$.

(2) $\mathcal{N}_x = \{U(t)[x] : t > 0\}$, *where* $U(t)[x] = \{y \in X : M(x,y,t) > 1-t\}$.

Proposition 2.3 *For a KM-fuzzy quasi-metric space* $(X, M, *)$*, a net* $\{x_i\}_{i\in D}$ *converges to* x *with respect to the topology* $\mathfrak{T}_M$ *if and only if* $\lim\limits_{i\in D} M(x, x_i, t) = 1$ *for all* $t > 0$ *with respect to the usual topology on* $[0,1]$.

Definition 2.4 Let $(X, M, *)$ be a KM-fuzzy quasi-metric space. An element $x \in X$ is said to be a *Yoneda limit* of the net $\{x_i\}_{i\in D}$ if $M(x,y,t) = \sup\limits_{j\in D}\inf\limits_{i\geq j} M(x_i, y, t)$ for all $y \in X$ and $t > 0$.

Every KM-fuzzy quasi-metric M on X generates a T_0 topology $\mathfrak{T}_M$, and thus the limit of a net in the space is not always unique if it exists.

Proposition 2.5 *In a KM-fuzzy quasi-metric space* $(X, M, *)$*, the Yoneda limit of a net is unique if it exists. Moreover, if* x *is a Yoneda limit of the net* $\{x_i\}_{i\in D}$ *in* $(X, M, *)$*, then* $\{x_i\}_{i\in D}$ *converges to* x *with respect to the topology* $\mathfrak{T}_{M^{-1}}$.

Proof. We omit the proof of uniqueness of the Yoneda limit of a net $\{x_i\}_{i\in D}$, which can be deduced from Definition 2.4. For each $t > 0$, since x is a Yoneda limit of $\{x_i\}_{i\in D}$, we have $1 = M(x,x,t) = \sup\limits_{j\in D}\inf\limits_{i\geq j} M(x_i, x, t)$ for all $t > 0$. For each $\varepsilon \in (0,1)$ and $t > 0$, there exists $i_0 \in D$ such that $\inf\limits_{i\geq i_0} M(x_i, x, t) > 1-\varepsilon$. Then $M(x_i, x, t) > 1-\varepsilon$ for all $i \geq i_0$, which follows from Proposition 2.2 that $\{x_i\}_{i\in D}$ converges to x with respect to the topology $\mathfrak{T}_{M^{-1}}$. □

Definition 2.6 (compare [9, Definition 5.1]) Let $(X, M, *)$ be a KM-fuzzy quasi-metric space. A net $\{x_i\}_{i\in D}$ in X is said to be *left K-Cauchy* if for each $\varepsilon \in (0,1)$ and $t > 0$ there exists $k \in D$ such that $M(x_i, x_j, t) > 1-\varepsilon$ for any $k \leq i \leq j$. A KM-fuzzy quasi-metric space $(X, M, *)$ will be called *Yoneda complete* if any left K-Cauchy net has a Yoneda limit. A KM-fuzzy quasi-metric space $(X, M, *)$ is said to be *left K-complete* if any left K-Cauchy net has a limit with respect to the topology $\mathfrak{T}_M$.

The notion of Yoneda completeness on quasi-metric spaces was presented in [14], and its important role in domain theory was further revealed in [1,7,25]. Recently, the notion of Yoneda completeness on ordered fuzzy sets was introduced and investigated in [15]. The next proposition shows that the notion of "Yoneda limit" may be a proper generalization of that of "limit".

Proposition 2.7 *If* $(X, M, *)$ *is a KM-fuzzy metric space, then for each net* $\{x_i\}_{i\in D}$*, its limit is exactly the Yoneda limit.*

Proof. Let $(X, M, *)$ be a KM-fuzzy metric space. Suppose that x is a Yoneda limit of $\{x_i\}_{i\in D}$. The element x is clearly a limit of $\{x_i\}_{i\in D}$ with respect to the topology $\mathfrak{T}_M$ from Proposition 2.5.

Conversely, let x be a limit of $\{x_i\}_{i\in D}$. For each $\varepsilon \in (0,1)$ and $t > 0$, there exists $i_0 \in D$ such that $M(x, x_i, \varepsilon t) > 1-\varepsilon$ for all $i \geq i_0$. Then we have

$$\begin{aligned} M(x_i, y, t) \geq & M(x_i, x, \varepsilon t) * M(x, y, (1-\varepsilon)t) = M(x, x_i, \varepsilon t) * M(x, y, (1-\varepsilon)t) \\ \geq & (1-\varepsilon) * M(x, y, (1-\varepsilon)t) \end{aligned}$$

for all $i \geq i_0$, which follows that

$$\sup_{j\in D} \inf_{i\geq j} M(x_i, y, t) \geq \inf_{i\geq i_0} M(x_i, y, t) \geq (1-\varepsilon) * M(x, y, (1-\varepsilon)t).$$

By the condition (Fm-6) and continuity of t-norm $*$, let $\varepsilon \to 0$, then we get

$$\sup_{j\in D} \inf_{i\geq j} M(x_i, y, t) \geq 1 * M(x, y, t) = M(x, y, t).$$

for all $y \in X$ and $t > 0$. Assume that there exist $y_0 \in X$ and $t_0 > 0$ satisfying

$$\sup_{j\in D} \inf_{i\geq j} M(x_i, y_0, t_0) > M(x, y_0, t_0), \text{ obviously, } M(x, y_0, t_0) < 1.$$

Thus, there exists $j_0 \in D$ such that $M(x_i, y_0, t_0) > M(x, y_0, t_0)$ for all $i \geq j_0$.

For each $s > 0$, since x is a limit of $\{x_i\}_{i\in D}$, there exists $k_0 \geq j_0 \in D$ such that $M(x, x_{k_0}, s) > M(x, y_0, t_0)$.

Now, for the index k_0, we have $M(x_{k_0}, y_0, t_0) > M(x, y_0, t_0)$. By (Fm-6), there is a $t_1 < t_0$ such that $M(x_{k_0}, y_0, t_1) > M(x, y_0, t_0)$.

Then, we claim that

$$M(x, y_0, s + t_1) \geq M(x, x_{k_0}, s) \wedge M(x_{k_0}, y_0, t_1) > M(x, y_0, t_0).$$

However, the free choice of $s > 0$ allows us to take $s = t_0 - t_1 > 0$. Then we obtain $M(x, y_0, t_0) > M(x, y_0, t_0)$, which is impossible.

Therefore, $M(x, y, t) = \sup_{j\in D} \inf_{i\geq j} M(x_i, y, t)$ for all $y \in X$ and $t > 0$. □

The following corollary is direct by Proposition 2.7.

Corollary 2.8 *If $(X, M, *)$ is a KM-fuzzy metric space, then it is Yoneda complete if and only if it is left K-complete.*

The problem of associating each fuzzy pseudo-metric in the sense of George and Veeramani [6] with a family of ordinary pseudo-metrics was investigated in [20,28]. Later on, it has been proved in [16] that each KM-fuzzy pseudo-metric under the continuous t-norm $\wedge$ can also be identified with certain family of ordinary pseudo-metrics. Let us recall the main notions and results involved in [16,18].

A family of real-valued maps $\{d_i : i \in I \subseteq \mathbb{R}\}$ will be called lower semicontinuous (shortly, LSC) if for any $i \in I$, $d_i = \bigwedge_{j>i} d_j$.

A KM-fuzzy (quasi-)pseudo-metric space $(X, M, *)$ will be called (quasi-)pseudo-metrically generated if there exists a LCS family of (quasi-)pseudo-metrics $\{d_a : a \in$

$[0,1)\}$ on X, such that for any $x, y \in X$ and $t \in [0,+\infty)$,

$$M(x, y, t) = \bigvee\{a \in [0,1) : d_a(x, y) < t\} \tag{1}$$

Given a KM-fuzzy pseudo-metric $(X, M, *)$, for any $a \in [0,1)$, the map $m_a : X^2 \to [0,+\infty)$ is defined as follows:

$$m_a(x, y) = \bigvee\{t \in [0,+\infty) : M(x, y, t) \leq a\}. \tag{2}$$

Proposition 2.9 ([16, Lemma 3.5]) *Let $(X, M, *)$ be a KM-fuzzy pseudo-metric space. For any $a \in [0,1)$, $x, y \in X$ and $t \in [0,+\infty)$, the subsequent equivalence holds:*

$$M(x, y, t) \leq a \Longleftrightarrow t \leq m_a(x, y).$$

Theorem 2.10 ([16, Theorem 3.13]) *Any KM-fuzzy pseudo-metric space $(X, M, *)$ is pseudo-metrically generated if and only if M is a KM-fuzzy pseudo-metric on X under the t-norm $\wedge$.*

In [17, Remark 1.4], under the usual assumption $\bigvee \emptyset = 0$, the authors pointed out that any KM-fuzzy pseudo-metric space $(X, M, \wedge)$ can also be pseudo-metrically generated by the family of pseudo-metrics $C_M = \{m_a : a \in (0,1)\}$. The family C_M and each associated pseudo-metric space (X, m_a) will be called the characteristic family and characteristic space of the KM-fuzzy pseudo-metric space $(X, M, \wedge)$, respectively.

As pointed out in [16], the proofs of Proposition 2.9 and Theorem 2.10 are independent from the conditions (Fm-3) and (Fm-4), this fact permits us to reword them to the fuzzy quasi-metric setting and obtain the subsequent results.

Theorem 2.11 *Let $(X, M, \wedge)$ be a KM-fuzzy quasi-metric space and each map of the family $\{m_a : a \in (0,1)\}$ be defined as (2) above. Then we have*

(1) $M(x, y, t) > a$ if and only if $m_a(x, y) < t$.

(2) $(X, M, \wedge)$ is quasi-pseudo-metrically generated by the characteristic family of quasi-pseudo-metrics $\{m_a : a \in (0,1)\}$, i.e., $M(x, y, t) = \bigvee\{a \in (0,1) : m_a(x, y) < t\}$.

Remark 2.12 For more details of Theorem 2.11, we recommend the reader to [19]. Moreover, it is interesting to note that there exist fuzzy quasi-metric spaces such that any map of the characteristic family is not an ordinary quasi-metric. An extraordinary example was given in [18].

Due to Theorem 2.11, we will consider the fuzzy quasi-metric space under the minimum t-norm $\wedge$ in this paper.

From [16, Proposition 4.5], we know that $\mathfrak{T}_M = \bigvee\limits_{a \in (0,1)} \tau_{m_a}$. The following conclusions can be deduced by Theorem 2.11.

Corollary 2.13 *Let $(X, M, \wedge)$ be a KM-fuzzy quasi-metric space.*

(1) A net $\{x_i\}_{i\in D}$ is left K-Cauchy in $(X, M, \wedge)$ if and only if it is left K-Cauchy in the characteristic space (X, m_a) for all $a \in (0, 1)$.

(2) $(X, M, \wedge)$ is left K-complete if and only if (X, m_a) is left K-complete for all $a \in (0, 1)$.

3 The connections between KM-fuzzy quasi-metric spaces and formal balls

Firstly, let us recall several basic notions about domain theory and formal balls [3,7]. Let $(P, \leq)$ be a poset. The notation $\bigvee A$ denotes the supremum of any subset $A \subseteq P$. A non-empty subset $D \subseteq P$ is *directed* if for any $x, y \in D$, there exists a $z \in D$ such that $x, y \leq z$. If any directed subset $D \subseteq P$ has a supremum in P, then the poset P is called a *directed complete poset* (briefly, *dcpo*).

A *formal ball* on a non-empty set X is a pair (x, r) with $x \in X$ and $r \in [0, +\infty)$. The set of all formal balls will be denoted by BX. In [3], Edalat and Heckmann proved that each complete metric space (X, d) can be characterized as: the related poset of formal balls $(BX, \sqsubseteq_d)$ is directed complete, where the partial order $\sqsubseteq_d$ on BX is defined by $(x, r) \sqsubseteq_d (y, s) \iff d(x, y) \leq r - s$.

3.1 Yoneda T-completeness and formal balls

Definition 3.1 Let $(X, M, \wedge)$ be a KM-fuzzy quasi-metric space and $C_M = \{m_a : a \in (0, 1)\}$ be its characteristic family. Define on BX the following relation:

$$(x, r) \sqsubseteq^1_M (y, s) \iff (x, r) \sqsubseteq_{m_a} (y, s) \iff m_a(x, y) \leq r - s, \text{ for any } a \in (0, 1).$$

Then we obtain the following results, which can be similarly proved as [18, Proposition 2.3].

Proposition 3.2 *Let $(X, M, \wedge)$ be a KM-fuzzy quasi-metric space. Then*

(1) $(BX, \sqsubseteq^1_M)$ is a poset.

(2) If $(x, r) \sqsubseteq^1_M (y, s)$, then $s \leq r$.

Any directed subset of $(BX, \sqsubseteq^1_M)$ in a fuzzy quasi-metric space $(X, M, \wedge)$ will be denoted by $\Gamma = \{(x_i, r_i)\}_{i\in D}$, where D is a directed set and for each $i, j \in D$, $(x_i, r_i) \sqsubseteq^1_M (x_j, r_j)$ if and only if $i \leq j$. As a weak version of Yoneda completeness, we now introduce the notion of Yoneda T-completeness and investigate its role in connecting KM-fuzzy quasi-metric spaces with the associated posets of formal balls.

Definition 3.3 Let $(X, M, \wedge)$ be a KM-fuzzy quasi-metric space. A net $\{x_i\}_{i\in D}$ is said to be *LT-Cauchy* if for each $\varepsilon \in (0, 1)$, there exists $k_\varepsilon \in D$ such that $m_a(x_i, x_j) < \varepsilon$ whenever $k_\varepsilon \leq i \leq j$ and $a \in (0, 1)$. A KM-fuzzy quasi-metric space $(X, M, \wedge)$ will be called *Yoneda T-complete* if any LT-Cauchy net has a Yoneda limit. The KM-fuzzy quasi-metric space $(X, M, \wedge)$ is *T-complete* if every left LT-Cauchy sequence has a limit with respect to the topology $\mathfrak{T}_{M^{-1}}$.

Proposition 3.4 *For any KM-fuzzy quasi-metric spaces $(X, M, \wedge)$, we have*

(1) Every LT-Cauchy net $\{x_i\}_{i\in D}$ is left K-Cauchy.

(2) If $(X, M, \wedge)$ is Yoneda T-complete, then it is T-complete.

Proof.

(1) Let $\varepsilon \in (0,1)$ and $t > 0$. Since $\{x_i\}_{i\in D}$ is an LT-Cauchy net, there exists $k_{t\varepsilon} \in D$ such that $m_a(x_i, x_j) < \min\{\varepsilon, t\}$ for any $a \in (0,1)$ whenever $k_{t\varepsilon} \le i \le j$. Then, by Theorem 2.11, we deduce $M(x_i, x_j, t) = \bigvee\{a \in (0,1) : m_a(x_i, x_j) < t\} \geq \bigvee\{a \in (0,1) : m_a(x_i, x_j) < \min\{\varepsilon, t\}\} = 1 > 1 - \varepsilon$ for any $k_{t\varepsilon} \le i \le j$. Therefore, the net $\{x_i\}_{i\in D}$ is left K-Cauchy.

(2) The result of (1) allows us to obtain by Proposition 2.5 that every left LT-Cauchy sequence is convergent with respect to the topology $\mathfrak{T}_{M^{-1}}$. □

Proposition 3.5 *Let $(X, M, \wedge)$ be a KM-fuzzy quasi-metric space and $\Gamma = \{(x_i, r_i)\}_{i\in D}$ be a directed subset of $(BX, \sqsubseteq_M^1)$. Then the net $\{x_i\}_{i\in D}$ is LT-Cauchy.*

Proof. Let $r = \inf\limits_{i\in D} r_i$. For each $\varepsilon \in (0,1)$, there exists $k_\varepsilon \in D$ such that $r_{k_\varepsilon} < r+\varepsilon$. Then for all $k_\varepsilon \le i \le j \in D$ and $a \in (0,1)$, we have $m_a(x_i, x_j) \le r_i - r_j \le r_i - r < r_{k_\varepsilon} - r < \varepsilon$. Consequently, $\{x_i\}_{i\in D}$ is a LT-Cauchy net. □

Lemma 3.6 *Suppose that $(X, M, \wedge)$ is a Yoneda T-complete KM-fuzzy quasi-metric space and $\Gamma = \{(x_i, r_i)\}_{i\in D}$ is a directed subset of $(BX, \sqsubseteq_M^1)$. If x is the Yoneda limit of $\{x_i\}_{i\in D}$ and $r = \inf\limits_{i\in D} r_i$, then $\bigvee \Gamma = (x, r)$.*

Proof. For each $\varepsilon \in (0,1)$, since x is the Yoneda limit of $\{x_i\}_{i\in D}$, we have $\sup\limits_{j\in D}\inf\limits_{i\ge j} M(x_i, x, \varepsilon) = M(x, x, \varepsilon) = 1$. Then for each $a \in (0,1)$ there exists $j_{a,\varepsilon}$ satisfying $M(x_j, x, \varepsilon) > 1 - (1-a) = a$, or equivalently, $m_a(x_j, x) < \varepsilon$ for all $j \ge j_{a,\varepsilon}$ by Theorem 2.11. For each $i \in D$, we choose a $k \in D$ such that $k \ge i, j_{a,\varepsilon}$. Then $m_a(x_i, x) \le m_a(x_i, x_k) + m_a(x_k, x) < r_i - r_k + \varepsilon \le r_i - r + \varepsilon$. Thus we have $m_a(x_i, x) \le r_i - r$ for all $a \in (0,1)$, i.e., $(x_i, r_i) \sqsubseteq_M^1 (x, r)$, which implies that (x, r) is an upper bound of Γ.

Let (y, s) be any upper bound of Γ. It remains to show $(x, r) \sqsubseteq_M^1 (y, s)$. For any $a \in (0,1)$, we can choose an element $b \in (0,1)$ with $b > a$. For each $\varepsilon \in (0,1)$, there exists $k_\varepsilon \in D$ such that $r_i \le r_{k_\varepsilon} < r + \varepsilon$ for all $i \ge k_\varepsilon$. Thus $m_b(x_i, y) \le r_i - s < r - s + \varepsilon$ for all $i \ge k_\varepsilon$. By Theorem 2.11, it is equivalent to $M(x_i, y, r - s + \varepsilon) > b$ for all $i \ge k_\varepsilon$. So, we deduce $M(x, y, r - s + \varepsilon) = \sup\limits_{j\in D}\inf\limits_{i\ge j} M(x_i, y, r - s + \varepsilon) \geq \inf\limits_{i\ge k_\varepsilon} M(x_i, y, r - s + \varepsilon) \ge b > a$, which implies that $m_a(x, y) < r - s + \varepsilon$. Clearly, we have $m_a(x, y) \le r - s$ for any $a \in (0,1)$, i.e., $(x, r) \sqsubseteq_M^1 (y, s)$. □

Theorem 3.7 *If $(X, M, \wedge)$ is a Yoneda T-complete KM-fuzzy quasi-metric space, then $(BX, \sqsubseteq_M^1)$ is a dcpo.*

Proof. Combining Proposition 3.5 and Lemma 3.6. □

Proposition 3.8 *Assume that $(BX, \sqsubseteq_M^1)$ is a dcpo and $\Gamma = \{(x_i, r_i)\}_{i\in D}$ is a directed subset such that $\bigvee \Gamma = (x, r)$. Then we have $\bigvee\limits_{i\in D} (x_i, r_i + \alpha) = (x, r + \alpha)$ for all $\alpha \ge 0$.*

Proof. The subset $\{(x_i, r_i+\alpha)\}_{i\in D}$ is clearly directed and $(x_i, r_i+\alpha) \sqsubseteq^1_M (x, r+\alpha)$ for each $i \in D$. Considering that $(BX, \sqsubseteq^1_M)$ is a dcpo, suppose $\bigvee_{i\in D}(x_i, r_i+\alpha) = (y, s)$. Then $(y, s) \sqsubseteq^1_M (x, r+\alpha)$, and by Proposition 3.2, $s \geq r+\alpha$, i.e., $s-\alpha \geq r \geq 0$.

For each $i \in D$, since $(x_i, r_i+\alpha) \sqsubseteq^1_M (y, s)$, we obtain $m_a(x_i, y) \leq r_i + \alpha - s = r_i - (s-\alpha)$ for any $a \in (0, 1)$. It follows that $(y, s-\alpha)$ is an upper bound of Γ. Thus we conclude $(x, r) \sqsubseteq^1_M (y, s-\alpha)$, which implies $m_a(x, y) \leq r-(s-\alpha) = (r+\alpha)-s$ for any $a \in (0, 1)$, i.e., $(x, r+\alpha) \sqsubseteq^1_M (y, s)$. Therefore, $(x, r+\alpha) = (y, s)$, which completes the proof. □

Theorem 3.9 *Let $(X, M, \wedge)$ be a KM-fuzzy quasi-metric space. If $(BX, \sqsubseteq^1_M)$ is a dcpo and $\Gamma = \{(x_i, r_i)\}_{i\in D}$ be a directed set such that $\bigvee \Gamma = (x, r)$, then the net $\{x_i\}_{i\in D}$ converges to x with respect to the topology $\mathfrak{T}_{M^{-1}}$ and $r = \inf_{i\in D} r_i$.*

Proof. Let $\beta = \inf_{i\in D} r_i$. We first show that $r = \beta$. Obviously, $\{(x_i, r_i-\beta)\}_{i\in D}$ is directed. Suppose that $\bigvee_{i\in D}(x_i, r_i-\beta) = (y, s)$, because $(BX, \sqsubseteq^1_M)$ is a dcpo. Then $0 \leq s \leq \inf_{i\in D}(r_i-\beta) = \inf_{i\in D} r_i - \beta = 0$. Hence $s = 0$. According to Proposition 3.8, we get $(x, r) = \bigvee_{i\in D}(x_i, r_i) = \bigvee_{i\in D}(x_i, (r_i-\beta)+\beta) = (y, \beta)$, which follows that $r = \beta$.

Since $r = \inf_{i\in D} r_i$, for each $\varepsilon \in (0, 1)$, there exists $i_\varepsilon \in D$ such that $r_i \leq r_{i_\varepsilon} < r+\varepsilon$ for all $i \geq i_\varepsilon$. By (x, r) being an upper bound of Γ, for each $a \in (0, 1)$, we obtain $m_a(x_i, x) \leq r_i - r < \varepsilon$, i.e., $M(x_i, x, \varepsilon) > a$ for all $i \geq i_\varepsilon$. Let $a = 1-\varepsilon$. Then we have $M(x_i, x, \varepsilon) > 1-\varepsilon$ for all $i \geq i_\varepsilon$. By Proposition 2.2, we conclude that $\{x_i\}_{i\in D}$ converges to x with respect to the topology $\mathfrak{T}_{M^{-1}}$. □

Corollary 3.10 *If $(BX, \sqsubseteq^1_M)$ is a dcpo, then the KM-fuzzy quasi-metric space $(X, M, \wedge)$ is T-complete.*

Proof. Let $\{x_i\}_{i\in\mathbb{N}}$ be a LT-Cauchy sequence. For each $k \in \mathbb{N}$, there exists $i_k > i_{k-1} \in \mathbb{N}$ such that $m_a(x_i, x_j) < 2^{-(k+1)}$ for all $i_k \leq i \leq j$ and $a \in (0, 1)$. We only need to prove that the subsequence $\{x_{i_k}\}_{k\in\mathbb{N}}$ converges to $x \in X$, which implies that $\{x_i\}_{i\in\mathbb{N}}$ converges to x with respect to the topology $\mathfrak{T}_{M^{-1}}$.

Since $m_a(x_{i_k}, x_{i_{k+1}}) < 2^{-(k+1)} = 2^{-k} - 2^{-(k+1)}$ for all $a \in (0, 1)$, the subset $\{(x_{i_k}, 2^{-k})\}_{k\in\mathbb{N}}$ of $(BX, \sqsubseteq^1_M)$ is increasing. As the poset $(BX, \sqsubseteq^1_M)$ is a dcpo, by Theorem 3.9, we conclude that $\{x_{i_k}\}_{k\in\mathbb{N}}$ converges to an element $x \in X$ with respect to the topology $\mathfrak{T}_{M^{-1}}$. Therefore, $(X, M, \wedge)$ is T-complete. □

The following example due to [21, Example 6.4] verifies that the converse of Corollary 3.10 is not true in general.

Example 3.11 Let $\mathcal{A}$ be the family of all non-empty countable subsets of $\mathbb{R}$. Define a fuzzy set M on $\mathcal{A}^2 \times [0, +\infty)$ by

$$M(A, B, t) = \begin{cases} 1 & \text{if } A \subseteq B, \text{ and } t > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then the triple $(\mathcal{A}, M, \wedge)$ is a KM-fuzzy quasi-metric space. Furthermore, each LT-Cauchy sequence $\{A_i\}_{i\in\mathbb{N}}$ converges to $\bigcup_{i\in\mathbb{N}} A_i$ with respect to the topology $\mathfrak{T}_{M^{-1}}$. Thus, $(\mathcal{A}, M, \wedge)$ is T-complete. Suppose

$\mathcal{T} = \{A : A$ is a non-empty finite subset of $\mathbb{R}$ consisting of irrational numbers$\}$. We observe that the subset $D = \{(A, 0) : A \in \mathcal{T}\}$ of $(B\mathcal{A}, \sqsubseteq^1_M)$ is directed but has no upper bound. Thus $(B\mathcal{A}, \sqsubseteq^1_M)$ is not a dcpo.

However, it is interesting to point out that the subset $\mathcal{T}$ is directed (with inclusion order) and can be regarded itself as a LT-Cauchy net in $(\mathcal{A}, M, \wedge)$. Since $\mathcal{T}$ is not convergent with respect to the topology $\mathfrak{T}_{M^{-1}}$, by Proposition 2.5, the KM-fuzzy quasi-metric spaces $(\mathcal{A}, M, \wedge)$ is not Yoneda T-complete.

We note that Corollary 3.10 fails to show that whether the KM-fuzzy quasi-metric space $(X, M, \wedge)$ is Yoneda T-complete. Nevertheless, if condition (Fm-4) is required, we have the next characterizations.

Theorem 3.12 *Let $(X, M, \wedge)$ be a KM-fuzzy metric space. The following conditions are equivalent:*

(1) $(X, M, \wedge)$ is Yoneda T-complete;

(2) $(X, M, \wedge)$ is T-complete;

(3) $(BX, \sqsubseteq^1_M)$ is a dcpo.

Proof.

(1) $\Rightarrow$ (3): By Theorem 3.7.

(3) $\Rightarrow$ (2): Straightforward from Corollary 3.10.

(2) $\Rightarrow$ (1): Let $\{x_i\}_{i\in D}$ be a LT-Cauchy net. Similar to that construction in Corollary 3.10, there is a LT-Cauchy sequence $\{x_{i_k}\}_{k\in\mathbb{N}}$ of $\{x_i\}_{i\in D}$, which satisfies that for each $k \in \mathbb{N}$, there exists $i_k \geq i_{k-1} \in D$ such that $m_a(x_i, x_j) < 2^{-(k+1)}$ for all $i_k \leq i \leq j \in D$ and $a \in (0, 1)$. Immediately, $\{x_{i_k}\}_{k\in\mathbb{N}}$ is a LT-Cauchy sequence.

Fix $\varepsilon \in (0, 1)$. For each $t > 0$, there exists $k_1 \in \mathbb{N}$ satisfying $2^{-(k_1+1)} < t/2$. Then $m_a(x_i, x_j) < 2^{-(k_1+1)} < t/2$ for all $i_{k_1} \leq i \leq j \in D$ and $a \in (0, 1)$. In particular, let $a = 1 - \varepsilon$. Thus, by Theorem 2.11, we have $M(x_i, x_j, t/2) > 1 - \varepsilon$ for all $i_{k1} \leq i \leq j$. Furthermore, since $(X, M, \wedge)$ is T-complete, $\{x_{i_k}\}_{k\in\mathbb{N}}$ converges to an element $x \in X$ with respect to the topology $\mathfrak{T}_{M^{-1}}$. Thus there exists $i_{k_2} \geq i_{k_1} \in D$ such that $M(x_{i_{k_2}}, x, t/2) > 1 - \varepsilon$ by Proposition 2.3.

Then $M(x, x_j, t) \geq M(x, x_{i_{k_2}}, t/2) \wedge M(x_{i_{k_2}}, x_j, t/2) = M(x_{i_{k_2}}, x, t/2) \wedge M(x_{i_{k_2}}, x_j, t/2) > (1 - \varepsilon) \wedge (1 - \varepsilon) = 1 - \varepsilon$ for all $j \geq i_{k_2}$. Therefore, x is a limit of $\{x_i\}_{i\in D}$. It is immediately obtained by Proposition 2.7 that $\{x_i\}_{i\in D}$ has x as a Yoneda limit. $\square$

3.2 Yoneda S-completeness and formal balls

In this subsection, we will introduce the notion of Yoneda S-completeness and investigate its role in connecting the KM-fuzzy quasi-metric spaces and the associated posets of formal balls. Since the converse of Theorem 3.18(2) in the sequel is not true in general (see [21, Proposition 6.3]), we aim to give an answer to this question.

It is fair to say that the proofs in this subsection exist no internal differences to the first approach and hence they are omitted.

Definition 3.13 ([18]) Let $(X, M, \wedge)$ be a KM-fuzzy quasi-metric space and $\{m_a : a \in (0,1)\}$ be its characteristic family. Define on BX the following relation:

$$(x,r) \sqsubseteq_M^2 (y,s) \Longleftrightarrow (x, \frac{a}{1-a}r) \sqsubseteq_{m_a} (y, \frac{a}{1-a}s) \Longleftrightarrow m_a(x,y) \leq \frac{a}{1-a}(r-s)$$

for any $a \in (0,1)$.

Remark 3.14 In [21], the author defined a partial order on the set of formal balls BX as: $(x,r) \sqsubseteq_M (y,s) \Longleftrightarrow M(x,y,t) \geq \dfrac{t}{t+r-s}$ for all $t > 0$. As showed in [18, Remark 2.20], we have $\sqsubseteq_M^2 = \sqsubseteq_M$.

Similar to Proposition 3.2, we have the following statements.

Proposition 3.15 *Let $(X, M, \wedge)$ be a KM-fuzzy quasi-metric space. Then*

(1) $(BX, \sqsubseteq_M^2)$ is a poset.

(2) If $(x,r) \sqsubseteq_M^2 (y,s)$, then $s \leq r$.

Definition 3.16 Let $(X, M, \wedge)$ be a KM-fuzzy quasi-metric space. A net $\{x_i\}_{i\in D}$ is said to be *LS-Cauchy* if for each $\varepsilon \in (0,1)$, there exists $k_\varepsilon \in D$ such that $m_a(x_i, x_j) < \dfrac{a}{1-a}\varepsilon$ for any $a \in (0,1)$ whenever $k_\varepsilon \leq i \leq j$. A KM-fuzzy quasi-metric space $(X, M, \wedge)$ will be called *Yoneda S-complete* if any LS-Cauchy net has a Yoneda limit. The KM-fuzzy quasi-metric space $(X, M, \wedge)$ is called *standard complete* (briefly, *S-complete*) if any left LS-Cauchy sequence has a limit with respect to the topology $\mathfrak{T}_{M^{-1}}$.

The notion of S-completeness on KM-fuzzy quasi-metric spaces was firstly given in [21, Definition 6.2]. In particular, for a KM-fuzzy metric space, Mardones-Pérez and de Prada Vicente [18] showed that the notion of S-completeness is equivalent to that of R-completeness defined therein.

Proposition 3.17 *For any KM-fuzzy quasi-metric spaces $(X, M, \wedge)$, we have*

(1) Every LS-Cauchy net $\{x_i\}_{i\in D}$ is left K-Cauchy.

(2) If $(X, M, \wedge)$ is Yoneda S-complete, then it is S-complete.

Following similar arguments in Proposition 3.7, Corollary 3.10 and Theorem 3.12, we have the results in the sequel.

Theorem 3.18 *Let $(X, M, \wedge)$ be a KM-fuzzy quasi-metric space. Then*

(1) If $(X, M, \wedge)$ is a Yoneda S-complete, then $(BX, \sqsubseteq_M^2)$ is a dcpo.

(2) If $(BX, \sqsubseteq_M^2)$ is a dcpo, then $(X, M, \wedge)$ is S-complete.

The result of Theorem 3.18(2) was firstly obtained in [21]. It has been told in [21] that the converse of (2) does not hold in general. Hence we give it a sufficient condition for the associated poset $(BX, \sqsubseteq_M^2)$ of formal balls to be directed complete.

Theorem 3.19 *Let $(X, M, \wedge)$ be a KM-fuzzy metric space. Then the following statements are equivalent:*

(1) $(X, M, \wedge)$ *is Yoneda S-complete;*

(2) $(X, M, \wedge)$ *is S-complete;*

(3) $(BX, \sqsubseteq^2_M)$ *is a dcpo.*

4 Conclusions

In this paper, we concentrate on the relationships between fuzzy quasi-metric spaces and the related order structures of formal balls. For this purpose, we introduce the notions of Yoneda T-completeness and Yoneda S-completeness on KM-fuzzy quasi-metric spaces. The relationships between completeness on fuzzy quasi-metric spaces and the associated posets of formal balls are discussed. Moreover, Yoneda T-completeness and Yoneda S-completeness on a subclass of KM-fuzzy quasi-metric spaces are characterized, respectively, with the aid of order-theoretical properties of the associated posets of formal balls.

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