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Backbone Coloring of Graphs with Galaxy Backbones [1]

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Abstract

A (proper) k-coloring of a graph $G = (V, E)$ is a function $c : V(G) \to \{1, \ldots, k\}$ such that $c(u) \neq c(v)$ for every $uv \in E(G)$. Given a graph G and a subgraph H of G, a q-backbone k-coloring of (G, H) is a k-coloring c of G such that $q \leq |c(u) - c(v)|$ for every edge $uv \in E(H)$. The q-backbone chromatic number of (G, H), denoted by $\mathrm{BBC}_q(G, H)$, is the minimum integer k for which there exists a q-backbone k-coloring of (G, H). Similarly, a circular q-backbone k-coloring of (G, H) is a function $c\colon V(G) \to \{1, \ldots, k\}$ such that, for every edge $uv \in E(G)$, we have $|c(u)-c(v)| \geq 1$ and, for every edge $uv \in E(H)$, we have $k - q \geq |c(u) - c(v)| \geq q$. The circular q-backbone chromatic number of (G, H), denoted by $\mathrm{CBC}_q(G, H)$, is the smallest integer k such that there exists such coloring c.
In this work, we first prove that if G is a 3-chromatic graph and F is a galaxy, then $\mathrm{CBC}_q(G, F) \leq 2q + 2$. Then, we prove that $\mathrm{CBC}_3(G, M) \leq 7$ and $\mathrm{CBC}_q(G, M) \leq 2q$, for every $q \geq 4$, whenever M is a matching of a planar graph G. Moreover, we argue that both bounds are tight. Such bounds partially answer open questions in the literature. We also prove that one can compute $\mathrm{BBC}_2(G, M)$ in polynomial time, whenever G is an outerplanar graph with a matching backbone M. Finally, we show a mistake in a proof that $\mathrm{BBC}_2(G, M) \leq \Delta(G) + 1$, for any matching M of an arbitrary graph G [Miškuf *et al.*, 2010] and we present how to fix it.

Keywords: Graph Coloring; Circular Backbone Coloring; Planar Graphs; Brooks' Type Theorem.

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1 Introduction

For basic notions and undefined terminology we refer to [3]. Let $G = (V, E)$ be a simple graph. Given a positive integer k, we denote the set $\{1, \cdots, k\}$ by $[k]$. A *(proper) k-coloring of* G is a function $c : V(G) \to [k]$ such that $c(u) \neq c(v)$ for every edge $uv \in E(G)$. We say G is *k-colorable* if there exists a k-coloring of G. The *chromatic number* of G, denoted by $\chi(G)$, is the smallest k for which G is k-colorable. We say G is *k-chromatic* if $\chi(G) = k$ and that it is k-colorable if it admits a k-coloring.

Let G be a graph and H a subgraph of G. We say that (G, H) is a *pair*, where H is called *backbone* of G. Given two positive integers q and k, a *q-backbone k-coloring of* (G, H) is a k-coloring c of G for which $|c(u) - c(v)| \geq q$ for every $uv \in E(H)$. The *q-backbone chromatic number of* (G, H), denoted by $\mathrm{BBC}_q(G, H)$, is the minimum k for which there exists a q-backbone k-coloring of (G, H). Problems regarding backbone colorings were first introduced by Broersma *et al.* [5], based on coloring problems related to frequency assignment.

Observe that if c is a proper k-coloring of G, then g defined by $g(v) = q \cdot c(v) - (q - 1)$ is a q-backbone $(q \cdot k - q + 1)$-coloring of (G, H) for any spanning subgraph H of G. Hence

$$\mathrm{BBC}_q(G, H) \leq q \cdot \chi(G) - q + 1.$$

We now consider special backbone k-colorings where the color space is circular, i.e. it behaves as $\mathbb{Z}/k$. To be precise, given a graph G, a subgraph H of G, and a positive integer q, a *circular q-backbone k-coloring of* (G, H) is a k-coloring $c \colon V(G) \to \{1, \ldots, k\}$ such that $q \leq |c(u) - c(v)| \leq k - q$, for every $uv \in E(H)$. The *circular q-backbone chromatic number* of (G, H), denoted by $\mathrm{CBC}_q(G, H)$, is the smallest k for which there exists a circular q-backbone k-coloring of (G, H).

Note that any circular q-backbone k-coloring is a q-backbone k-coloring. Conversely, a q-backbone k-coloring yields a circular q-backbone $(k + q - 1)$-coloring. Therefore we get

$$\mathrm{BBC}_q(G, H) \leq \mathrm{CBC}_q(G, H) \leq \mathrm{BBC}_q(G, H) + q - 1, \text{ and also}$$

$$q \cdot \chi(H) \leq \mathrm{CBC}_q(G, H) \leq q \cdot \chi(G).$$

Havet *et al.* [8] conjectured the following problem:

Conjecture 1.1 *If G is a planar graph and F is a galaxy in G, then* $\mathrm{CBC}_q(G, F) \leq 2q + 2$.

Recall that a *star* on n vertices is a tree with $n - 1$ leaves and the remaining vertex is called the *central vertex* of the star (in the case which a star is only an edge, we choose one of its ends as central vertex.). A *galaxy* is a graph whose connected components are stars.

We prove that Conjecture 1.1 holds for 3-chromatic graphs, even for those which are not planar. Recall that for planar graphs, by Grötzsch's theorem, this includes triangle-free planar graphs. It is also known that there are linear-time algorithms to find 3-colorings of triangle-free planar graphs [9]. By combining such algorithms with our result, we deduce that a q-backbone coloring with $2q + 2$ colors can be obtained in linear time. We do not know whether our bound is tight when G is triangle-free. It is trivially tight when G is not triangle-free as it suffices to take a K_4 with $K_{1,3}$ as backbone. Therefore, we pose the question:

Problem 1.2 *If G is a triangle-free 3-chromatic graph and F a galaxy in G, then* $\mathrm{CBC}_q(G, F) \leq 2q + 1$*?*

With respect to matching backbones, Havet *et al.* [8] asked the following question:

Problem 1.3 *If G is a planar graph, M is a matching in G, and q is a positive integer, $q \geq 3$, is it true that* $\mathrm{CBC}_q(G, M) \leq 2q + 1$*?*

In the same article, they prove that it is NP-complete to decide whether $\mathrm{CBC}_2(G, M) \leq k$ for $k \in \{4, 5\}$ when M is a matching in a planar graph G. This is why the problem above does not consider $q = 2$. In fact, they show that if G is a planar graph with girth at least 5 and M is a matching in G, then $\mathrm{CBC}_q(G, M) \leq 2q+1$. This is why they propose to investigate the following relaxation of Problem 1.3.

Problem 1.4 *If G is a planar graph with girth at least 4 and M is a matching in G, is it true that* $\mathrm{CBC}_q(G, M) \leq 2q + 1$*?*

We prove a stronger version of Problem 1.3, namely we prove that the bound $2q+1$ holds for $q = 3$, and that for $q \geq 4$, the bound can be improved to $2q$. Observe that if M is non-empty, then $2q$ is the best possible, because, for every planar graph G, non-empty matching M and positive integer q, we have $\mathrm{CBC}_q(G, M) \geq 2q$.

Therefore, by our result there is always equality if $q \geq 4$, and if $q = 3$ then $\mathrm{CBC}_q(G, M)$ equals either 6 or 7. We also give an example where the upper bound 7 can be attained. So, we pose the following question:

Problem 1.5 *Given a planar graph G and a non-empty matching M, can one decide in polynomial time whether* $\mathrm{CBC}_3(G, M) = 6$*?*

In [4], Broersma *et al.* proved that for $q+1 \leq \chi(G) \leq 2q$, we have $\mathrm{BBC}_q(G, M) \leq 2\chi(G) - 2$, where M is a matching of G. This and the fact that $\mathrm{CBC}_q(G, H) \leq \mathrm{BBC}_q(G, H) + q - 1$, gives us that $\mathrm{CBC}_2(G, M) \leq 5$, whenever G is a 3-chromatic graph and M is a matching of G. But since triangle-free planar graphs are 3-colorable by Grötzsch's Theorem, the case $q = 2$ of Problem 1.4 is known. The other cases follow from our result.

Using the same result and the fact that an outerplanar graph is 3-chromatic, we have that $\mathrm{BBC}_2(G, M) \leq 4$, whenever G is an outerplanar graph and M is a matching of G. In this work, we prove that one can compute $\mathrm{BBC}_2(G, M)$ in polynomial time, whenever G is an outerplanar graph with a matching backbone M.

Apart from Problem 1.5, the only remaining questions concerning q-backbone chromatic number of (G, M) are for $q = 2$. Broersma *et al.* [5] proved that $\mathrm{BBC}_2(G, M) \leq 6$ and ask: 1) can this result be proved without using the Four

Color Theorem? and 2) Can this be improved to 5? Both questions are still open, although in [1] some partial answers are given. They hint for positive answers by proving that if G has no induced cycles of length 4 or 5, then $\mathrm{CBC}_2(G, M) \leq 5$, and that if G is diamond-free, then $\mathrm{CBC}_2(G, M) \leq 6$ (none of their proofs use the Four Color Theorem). Given that the original questions posed by Broersma *et al.* [5] seem to be very hard, we ask the following simpler question that could be a good intermediate step for a definite answer for their questions.

Problem 1.6 *Let G be a planar C_4-free graph, and M be a matching in G. Is it true that* $\mathrm{CBC}_2(G, M) \leq 5$*?*

Finally, with respect to general graphs, Miškuf *et al.* [10] presented a proof for a Brooks' type theorem for BBC, i.e. for any graph G and any matching M in G, we have $\mathrm{BBC}_2(G, M) \leq \Delta(G) + 1$. We found a mistake in their proof and we present here how to fix it.

2 Galaxy backbones

In this section, we want to prove that $\mathrm{CBC}_q(G, F) \leq 2q + 2$ when F is a galaxy of a 3-chromatic graph.

Theorem 2.1 *If G is a 3-chromatic graph and F is a galaxy, then*

$$\mathrm{CBC}_q(G, F) \leq 2q + 2.$$

Proof. Let $c\colon V(G) \to [3]$ be a 3-coloring of G. Define $L_i = \{v \in V(G) \mid c(v) = i \text{ and } d_F(v) = 1\}$ and for each $v \in L_i$, consider $\overline{v}$ the vertex such that $v\overline{v} \in E(F)$. We now define a circular q-backbone coloring $c'\colon V(G) \to [2q + 2]$ as follows:

(i) If $v \in c^{-1}(1)$, then $c'(v) = 1$.

(ii) If $v \in c^{-1}(2)$, then

$$c'(v) = \begin{cases} q + 1, \text{ if } v \in L_2 \text{ and } c(\overline{v}) = 1; \\ 2q + 2, \text{ if } v \in L_2 \text{ and } c(\overline{v}) = 3; \\ q + 3, \text{ otherwise.} \end{cases}$$

(iii) If $v \in c^{-1}(3)$, then

$$c'(v) = \begin{cases} 2, \text{ if } v \in L_3 \text{ and } c(\overline{v}) = 2; \\ q + 2, \text{ otherwise.} \end{cases}$$

First, we prove that c' is a proper coloring. In fact, $C_1, C_2, C_3 \subset V(G)$ be partitions of the 3-partition induced by c and consider $C'_i = \{v \in V(G) : c'(v) = i\}$, for each $i \in \{1, 2, q+1, q+2, q+3, 2q+2\}$. Observe that $C'_{q+1}, C'_{q+3}, C'_{2q+2} \in C_2$, $C'_{q+2}, C'_2 \in C_3$ and $C_1 = C'_1$. Then, C'_i is a independent set, for all $i \in \{1, 2, q + 1, q + 2, q + 3, 2q + 2\}$. This give us that c' must be a proper coloring of G. Now, we prove that it is a circular backbone coloring. For this, we prove that, given a central vertex v, all of its neighbors in the backbone are colored with an appropriate color. First observe that v receives color 1, $q+3$ or $q+2$. In the first case, the colors allowed for its neighbors in F are $q+1$ or $q+2$. In the second case, all the its neighbors in F are colored with colors 1 or 2. Finally, in the last case, all the its neighbors in F are colored with colors 1 or $2q+2$. Hence c' is a circular q-backbone coloring of (G, F).

□

3 Matching backbones

In this section our goal is to prove the upper bounds for planar graphs G with matching backbones M.

The proof of the upper bounds follow by contradiction as we suppose the existence a minimal counter-example. Let us formally define this notion. Given a pair (G, H), a *subpair* (G', H') of (G, H) is a pair such that $H' \subseteq H$ and $G' \subseteq G$. We say that (G', H') is a *proper subpair* of (G, H) if it is a subpair of (G, H) and $H' \subset H$ and $G' \subset G$. A pair (G, H) is called (k, q)-minimal if $\mathrm{CBC}_q(G, H) > k$, but $\mathrm{CBC}_q(G', H') \leq k$ for every proper subpair (G', H') of (G, H). Note that if $\mathrm{CBC}_q(G, H) > k$, then there exists a subpair (G', H') of (G, H) that is (k, q)-minimal.

Let $\mathbb{Z}_k$ be a color space. A subset $S \subseteq \mathbb{Z}_k$, a positive integer q and $i \in \mathbb{Z}_k$, we say that the color i is *q-bad for S* if $S \subseteq \{i - q + 1, \cdots, i + q - 1\}$.

Given a plane graph G, we denote by $\mathcal{F}(G)$ the set of faces of G and by $d(f)$ the degree of a face f in G. Now, given a pair (G, H) and a vertex $u \in V(G)$, the *(k, q)-total degree of u in (G, H)* is given by:

$$d^t_{(G,H),k,q}(u) = d_G(u) + (2q - 2)d_H(u).$$

If G, H, k, q are clear from context, we omit them in the notation.

Lemma 3.1 *Let (G, H) be a (k, q)-minimal pair, with $k \geq 2q$. If $uv \in E(H)$, then*

$$d^t(u) + d^t(v) \geq 2k + 2q - 2.$$

Sketch of the proof.

First, write $d^t(u) = k + \ell$ and $d^t(v) = k + \ell'$. Then, we proved that $d^t(u) \geq k$, for every $u \in V(G)$, so that $\ell, \ell' \geq 0$.

Let f be a circular q-backbone k-coloring of $(G - u - v, H - u - v)$. Note that $a_f(u) = k - (k + \ell - 2q + 1) = 2q - (\ell + 1)$, and analogously $a_f(v) = 2q - (\ell' + 1)$. By contradiction, suppose that $d^t(u) + d^t(v) \leq 2k + 2q - 3$. Then, we get:

$$k + \ell + k + \ell' \leq 2k + 2q - 3 \Leftrightarrow \ell + \ell' \leq 2q - 3.$$

Therefore, $a_f(v) \geq 2q - 1 - (2q - 3 - \ell) = \ell + 2$. Then, we also proved that if $S \subseteq \mathbb{Z}_k$ has cardinality $2q - p$, where $p \geq 0$ and $k \geq 2q$, there are at most p colors in $\mathbb{Z}_k$ that are q-bad for S. So we have that at most $\ell+1$ colors are q-bad for $A_f(u)$. Therefore, there exists a color $c \in A_f(v)$ that is not q-bad for $A_f(u)$, a contradiction. $\square$

The lemma below follows directly from Euler's Formula.

Lemma 3.2 *Let G be a plane graph, M be a matching of G, and q be a positive integer. Then,*

$$\sum_{v \in V(G)} (d^t(v) - 2q - 2) + \sum_{f \in \mathcal{F}(\mathcal{G})} (d(f) - 4) \leq -8.$$

Theorem 3.3 *Let G be a plane graph, M be a matching in G, and q be a positive integer. Then:*

$$\mathrm{CBC}_3(G, M) \leq 7, \text{ and}$$

$$\mathrm{CBC}_q(G, M) \leq 2q, \text{ if } q \geq 4.$$

Sketch of the proof.

First, we prove that it holds when M is a perfect matching and $q \geq 4$. For this, suppose otherwise and let (G, H) be a minimal counter-example. We apply the discharging method. Start by giving charge $d^t(u)$ to every $u \in V(G)$ and $d(f) - 4$ to every $f \in \mathcal{F}(G)$. Let $\alpha = 2q + 2$. By Lemma 3.1, for each $uv \in M$ we have:

$$d^t(u) + d^t(v) \geq 2k + 2q - 2 = 6q - 2 = 2\alpha + 2q - 6.$$

This tells us that the sum of the charges on each edge of the matching is enough to ensure that every vertex can end up with non-negative charge, with surplus of $2q - 6 \geq 2$ on each edge of the matching. Because M is a perfect matching, we get a surplus of at least n which is clearly bigger than the number of triangles, thus contradicting Lemma 3.2. One can verify that when $q = 3$ and $k = 2q + 1$ we can apply a similar argument.

Now, suppose (G, M) is a pair, and M is not a perfect matching. Then, we can add, for each vertex u that is not saturated by M, a pendant vertex u' to G and M in order to obtain a pair (G', M') containing (G, M) and such that M' is a perfect matching of G'. The lemma follows by the previous paragraph and the fact that $(G, M) \subseteq (G', M')$. $\square$

Both upper bounds provided by Theorem 3.3 are tight, under the hypothesis that $M \neq \emptyset$. For $q \geq 4$, if $M \neq \emptyset$, then we have $\mathrm{CBC}_q(G, M) \geq 2q$, for any graph G. In [8], they present an example to show that there exists a planar graph G_3 and a perfect matching M_3 of G_3 such that $\mathrm{BBC}_2(G_3, M_3) = 5$. The same example also satisfies $\mathrm{CBC}_3(G_3, M_3) = 7$.

Proposition 3.4 $\mathrm{CBC}_3(G_3, M_3) = 7$.

Proof. The upper bound is provided by Theorem 3.3. To prove the lower bound, suppose, by contradiction, that there exists a circular 3-backbone 6-coloring c of (G_3, M_3). Observe that, if uv is an edge of M_3, then $\{c(u), c(v)\}$ is either $\{1, 4\}$ or

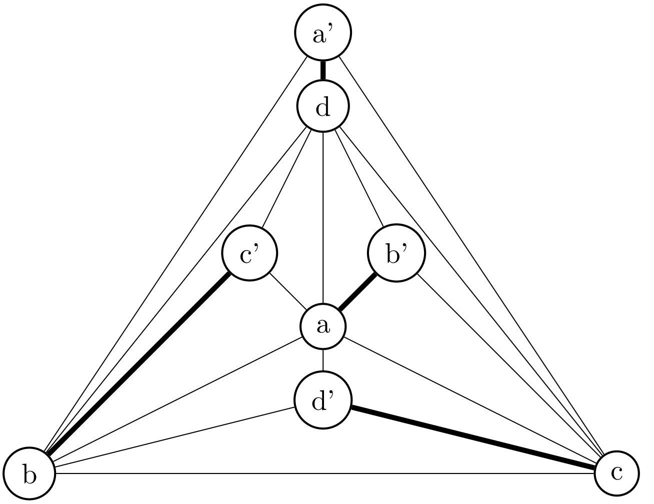


Fig. 1. $\mathrm{CBC}_3(G_3, M_3) = 7$

$\{2,5\}$ or $\{3,6\}$. If the first case (resp. second, third) occurs, we say that uv is an 14-edge (resp. a 25-edge, a 36-edge).

One may assume, without loss of generality, that ab' is a 14-edge. Since d is a neighbor of both a and b', d has a different color, and without loss of generality, we may assume that $a'd$ is a 25-edge. Then, since the only non-neighbor of c is c', we deduce that $c'd$ is a 36-edge. Consequently, since b is adjacent to a', d, c, d', then bc' must be a 14-edge. This is a contradiction, because a is adjacent to b and c' and $c(a) \in \{1,4\}$. □

4 Polynomial-time algorithm for outerplanar graphs

In this section, we give a polynomial algorithm that, given an outerplanar graph G and a matching M of G, decides whether $BBC(G,M) \leq 3$. Since $BBC(G,M) \leq 2$ if and only if G is bipartite and M is empty, and because $BBC(G,M)$ is always at most 4, this implies that one can compute $BBC(G,M)$ in polynomial time.

Theorem 4.1 *Let G be a connected outerplanar graph on n vertices and m edges, and let $M \subseteq E(G)$ be a matching in G. Then, deciding if $BBC(G,M) \leq 3$ can be done in time $O(m+n)$.*

Proof. A *tree decomposition* of a graph G is a pair $\mathcal{D} = (T, \{X_t\}_{t \in V(T)})$ such that: T is a rooted tree; for every vertex $v \in V(G)$, there exists $t \in V(T)$ such that $v \in X_t$; for every edge $uv \in E(G)$, there exists $t \in V(T)$ such that $\{u,v\} \subseteq X_t$; for every $v \in V(G)$, the subset $\{t \in V(T) \mid v \in X_t\}$ induces a subtree of T. A tree decomposition is *nice* if the vertices of T can be classified as one of the following types.

(i) *leaf*: t is a leaf in T;

(ii) *forget*: t has exactly one child t' and there exists $u \in V(G)$ such that $X_t = X_{t'} \setminus \{u\}$;

(iii) *introduce*: t has exactly one child t' and there exists $u \in V(G)$ such that $X_t = X_{t'} \cup \{u\}$; and

(iv) *join*: t has exactly two children, t_1 and t_2, and $X_t = X_{t_1} = X_{t_2}$.

The *width* of a tree decomposition $\mathcal{D}$ is the maximum size of a subset X_t minus one. The *treewidth of* G is the minimum width of a tree decomposition of G and it is denoted by $\text{tw}(G)$. It is known that if G is outerplanar, then $\text{tw}(G) \leq 2$, and that a (nice) tree decomposition $\mathcal{D} = (T, \{X_t\}_{t\in V(T)})$ of width $\text{tw}(G)$ such that $|V(T)| = O(n)$ can be computed in time $\mathcal{O}(n+m)$ [2,7]. Here, we use such a tree decomposition to solve our problem. For this, for each node $t \in V(T)$, denote by T_t the subtree of T rooted at t; by V_t the subset $\bigcup_{t'\in V(T_t)} X_{t'}$; by (G_t, H_t) the pair $(G[V_t], H[V_t])$; and for each coloring $f : X_t \to \{1, 2, 3\}$, define the following:

$$B_t(f) = \begin{cases} 1\ , & \text{if there is a 2-backbone 3-coloring } f' \text{ of } (G_t, H_t) \text{ such that } f \subseteq f'; \\ 0\ , & \text{otherwise.} \end{cases}$$

Now, we present how to compute each $B_t(f)$, given that the values are computed in a post-order traversal of T. Hence, consider a node t and a 3-coloring f of $G[X_t]$. If t is a leaf, then $B_t(f) = 1$ if and only if f is a 2-backbone 3-coloring of $(G[X_t], H[X_t])$, which can be tested in constant time, since $|X_t| \leq 3$. So, suppose otherwise and consider that we know the values of $B_{t'}(f)$ for each child t' of a node t. We analyse all the possible cases according to the type of t:

(i) t is forget: let t' be the child of t and $u \in V(G)$ be such that $X_t = X_{t'} \setminus \{u\}$. Observe that $(G_t, H_t) = (G_{t'}, H_{t'})$. Thus, we get that there exists a 2-backbone 3-coloring f' of (G_t, H_t) that extends f if, and only if, f' is a 2-backbone 3-coloring of $(G_{t'}, H_{t'})$ that contains f. Hence, if we define f_i as $f \cup \{(u, i)\}$ for each $i \in \{1, 2, 3\}$, we get that:

$$B_t(f) = 1 \text{ if, and only if, } B_{t'}(f_i) = 1 \text{ for some } i \in \{1, 2, 3\}.$$

(ii) t is introduce: let t' be the child of t and $u \in V(G)$ be such that $X_t = X_{t'} \cup \{u\}$. Then, there exists a 2-backbone 3-coloring f' of (G_t, H_t) that extends f if, and only if, f is a 2-backbone 3-coloring of $(G[X_t], H[X_t])$ and there exists a 2-backbone 3-coloring f'' of $(G_{t'}, H_{t'})$ that extends $f' = f_{\restriction X_{t'}}$ (f restricted to $X_{t'}$). Hence, we get that:

$$B_t(f) = 1 \text{ if, and only if, } B_{t'}(f') = 1.$$

(iii) t is join: let t_1, t_2 be the children of t. Because X_t separates $G_{t_1} - X_t$ from $G_{t_2} - X_t$, we get that the union of two 2-backbone 3-colorings of G_{t_1} and G_{t_2} that agree on X_t is a 2-backbone 3-coloring of (G_t, H_t). Thus:

$$B_t(f) = 1 \text{ if, and only if, } B_{t_1}(f) = B_{t_2}(f) = 1.$$

Now, observe that each step can be done in constant time, because there are at most 3^3 possible colorings of each X_t. Since there are $\mathcal{O}(n)$ nodes in T, we get that, once we find the tree decomposition of G, one can compute all the values $B_t(f)$ in $\mathcal{O}(n)$ time. Once we arrive at the root r of T, the answer to whether $BBC(G, M) \leq 3$ is “yes” if and only if $B_r(f) = 1$, for some 2-backbone 3-coloring f of X_r. □

We mention that, after the revision and acceptance of this extended abstract, we found an error in our previous proof. Nevertheless, the result is still true and we found a much simpler and general proof. Observe that the above proof can be generalized for any graph G with bounded treewidth, for any fixed k, and for any possible backbone H. This, combined with the fact that $BBC(G, H) \leq 2\chi(G) - 1$, which equals 5 when G is an outerplanar graph, means that the backbone coloring problem on outerplanar graphs is polynomial-time solvable for every possible backbone H.

5 Brooks' Type Theorem

This section is devoted to correcting a proof of a Brook's Type Theorem demonstrated by Miškuf et al. in [10]:

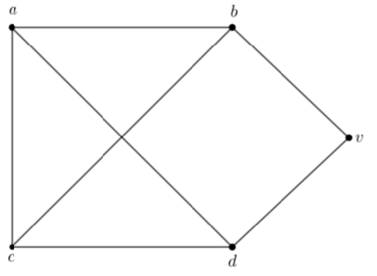
Theorem 5.1 *Let M be a matching in a graph G of maximum degree $\Delta(G)$. Then $BBC_2(G, M) \leq \Delta(G) + 1$.*

For a better understanding of their proof, it is necessary to define a special structure. Let x, y be two non-adjacent neighbors of a vertex v in a connected graph G such $G - x - y$ is connected. Then we say that $(v; x, y)$ is a *fork*. That being said, the following lemma was required:

Lemma 5.2 *Let G be a 2-connected graph with all the vertices of the same degree $d \geq 3$ except a particular vertex v which is of degree $< d$. Then, G has a fork $(w; x, y)$ such that $v \neq x$ and $v \neq y$.*

Assuming that G is neither an odd cycle nor a complete graph, they prove the Theorem 5.1 considering an order $v_1, \ldots, v_n$ of the vertices of G such that each v_i $(i < n)$ has a succeeding neighbor. So, they claim that G is a regular graph and M is a perfect matching. Otherwise, we may choose for v_n a vertex that is of degree $< \Delta$ or that it is not incident with an edge of M. In both cases, the procedure will color also the vertex v_n. Finally, they use the Lemma 5.2 to conclude that G is 2-connected. So, since G is neither an odd cycle nor a complete graph, they use the Theorem 5.3 below to ensure the existence of a fork in G that give us an order to color the vertices of G using at most $\Delta(G) + 1$ colors.

In contrast, we found out that Lemma 5.2 is not true. A simple counterexample is described in the following figure:



Observe that the above graph satisfies the hypotheses of the Lemma 5.2, but the only forks in this graph are $(b; a, v)$, $(b; c, v)$, $(d; a, v)$ and $(d; c, v)$, which contradicts the Lemma 5.2.

On the other hand, the theorem is still true and we find a way to fix the demonstration made in [10]. Before presenting it and to better understand our proof, we recall some definitions and results on connectivity.

Theorem 5.3 *(Bryant [6]) For a 2-connected graph the following three statements are equivalent:*

(i) G is a complete graph or a cycle;

(ii) the removal from G of any two non-adjacent vertices disconnects it;

(iii) the removal from G of any two vertices at distance 2 apart disconnects it.

Notice that the above theorem claims that each 2-connected graph distinct from a cycle and a complete graph contains a fork.

A *block* of a graph G is a maximal connected subgraph of G that has no cut-vertex. If G itself is connected and has no cut-vertex, then G is a block. The *block-cutpoint graph* of a connected graph G is the graph whose vertices are the blocks and the cut-vertices of G, with an cut-vertex v adjacent to an block B if and only if $v \in V(B)$. Observe that a block-cutpoint graph of G is a tree and all its leaves are blocks of G. Each block of G which is a leaf of the block-cutpoint graph of G is called *leaf block*.

Given a leaf block B of G, we say that a vertex $u \in V(B)$ is *internal* if it is not a cut-vertex. Also, if G is 1-connected and the block-cutpoint graph of G is a path, we say that G has a *path structure*.

Now, we can prove the following:

Theorem 5.4 *Let M be a matching in a graph G of maximum degree Δ. Then* $\mathrm{BBC}_2(G, M) \leq \Delta + 1$.

Proof. The proof follows the same steps as Brooks' theorem. We just need to be careful in the case G has cut-vertices, as we may not be able to combine backbone colorings of the blocks of G into a coloring of G.

Without loss of generality, we assume that G is connected. In case G is not regular, then one can create an order $\sigma = v_1, \ldots, v_{n(G)}$ over $V(G)$ such that each v_i

has at least one neighbor v_j with $j > i$ for every $i \in \{1, \ldots, n-1\}$ and $d_G(v_{n(G)}) < \Delta$. By applying the greedy algorithm over this ordering of $V(G)$, the obtained coloring uses at most $\Delta + 1$ colors as each vertex v_j has at most $\Delta - 1$ colored neighbors and at most one edge of the backbone M has v_j as endpoint. Thus, we assume that G is regular.

If G has a fork $(z; x, y)$, then we can produce an order $\sigma = (x, y, v_3, \ldots, v_{n-1}, z)$ of $V(G)$ such that each v_i has at least one neighbor v_j with $j > i$ for every $i \in \{3, \ldots, n-1\}$, and z has two non-adjacent neighbors, namely x and y, such that when we use the greedy algorithm on G using the order σ the vertices x and y have the same color. Then, we can use this order to, given a matching M in G, construct a 2-backbone coloring of (G, M) that uses at most $\Delta + 1$ colors. Consequently, we also assume that G has no fork.

If G is a complete graph or a cycle, then the upper bound holds (see [10] for details). Thus, we consider that G is a k-regular graph with no forks and G is not a complete graph nor a cycle. Consequently, observe that $k \geq 3$. Note also that G cannot be 2-connected, due to Theorem 5.3.

Let B be a leaf-block of G and u be the only cut-vertex of G belonging to $V(B)$.

Case 1: $\kappa(B - u) = 1$. In this case, we claim that G has a fork, contradicting our hypothesis. In fact, note that u has a neighbor in each leaf-block of $B - u$ that is not a cut-point of $B - u$. If $d_B(u) \geq 3$, let x and y be neighbors of u in distinct leaf-blocks of $B - u$, and that are not cut-vertices of $B - u$. Observe that $(u; x, y)$ is a fork of G. Otherwise, $d_B(u) = 2$ and the block-cutpoint tree of $B - u$ is a path. Since G is $k \geq 3$ regular, note that each leaf block in $B - u$ has at least 4 vertices. In case $B - u$ has only two blocks B_1 and B_2, let the unique cutpoint of $B - u$ be z. Since those are the only blocks, note that z must have two neighbors $y \in V(B_1)$ and $x \in V(B_2)$ such that neither y nor x is a neighbor of u. Then, $(z; x, y)$ is a fork of G. Finally, if $B - u$ has at least three blocks, let B_1 be a leaf-block, B_2 be the only block sharing the cut-vertex z with B_1. Once more, one may choose $y \in N_{B_1}(z)$ such that y is not a neighbor of u and any vertex x in B_2 (even if B_2 is a single edge) and then $(z; x, y)$ is a fork of G.

Case 2: $B - u$ is 2-connected. We shall prove that u has exactly two neighbors x and y in B and xy is the only edge that does not belong to $B - u$. In the sequel, we use such structure to build an ordering over $V(G)$ such that the greedy algorithm uses at most $\Delta + 1$ colors in a 2-backbone coloring of G. We claim that $B - u$ has a fork $(z; x, y)$. If not, by Theorem 5.3, $B - u$ should be either a complete graph or a cycle. It is not possible as G is k-regular and u has neighbors in $B - u$ and $G - V(B)$. Moreover, Theorem 5.3 ensures that every two non-adjacent vertices form a fork. Since G has no fork, the only possibility to such fork exist in $B - u$ is that x and y be the only neighbors of u in $B - u$. Thus, the leaf-block B has the structure we claimed: u has exactly two neighbors x and y in B and xy is the only edge that does not belong to $B - u$. Finally, one can construct and ordering over $V(G)$ such that the two first vertices are the neighbors of u in B, then we have all vertices of $V(B) \setminus u, x, y$, in the sequel we place all vertices of $V(G) \setminus V(B)$ in such a way that each vertex has a neighbor that appears latter in the sequence, and the last vertex is u. Observe that each vertex on such sequence either has one neighbor that appears latter in the sequence, or is a neighbor of x and y which will be colored with the same color. Thus, the greedy algorithm uses at most $\Delta + 1$ colors to build a 2-backbone coloring of (G, M).

□

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