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Integral of Two-dimensional Fine-computable Functions

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Abstract

We discuss effective integrability and effectivization of Fubini's Theorem for a Fine-computable function $F(x,y)$ on the upper-right open unit square. The core objective is Fine-computability of $f(x) = \int_{[0,1)} F(x,y)dy$ as a function on $[0,1)$.

Keywords: Two-dimensional Fine-computable function, effective integrability, Fubini's Theorem, integral operator

1 Introduction

Notions of Fine-continuity and of Fine-computabilities on $[0,1)$ are defined with respect to the Fine topology (cf. Section 2, [2,3,6]). We have defined effective integrability for Fine-computable functions on $[0,1)$ ([6,8]). In this article, we investigate the notions of Fine-computabilities and effective integrability of functions on the upper-right open unit square $[0,1)^2$.

In classical analysis, the integral operator with a kernel $F(x,y)$, which maps a function $g(x)$ on X to $(Tg)(x) = \int_X g(y)F(x,y)dy$, is a central subject. Measurability and integrability of Tg are fundamental properties to be proved and Fubini's Theorem is a fundamental tool to deal with investigations of such an operator.

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Theorem 1.1 (Fubini's Theorem) *Let $F(x,y) \geqslant 0$ be a measurable and integrable function on the upper-right open unit square $[0,1)^2$. Then the following hold.*

(i) *For almost all x, $F(x,\cdot)$ is measurable and integrable.*

(ii) $\int_{[0,1)} F(x,y)dy$ *and* $\int_{[0,1)} F(x,y)dx$ *are measurable.*

(iii) $\iint_{[0,1)^2} F(x,y)dxdy = \int_{[0,1)}\left(\int_{[0,1)} F(x,y)dy\right)dx = \int_{[0,1)}\left(\int_{[0,1)} F(x,y)dx\right)dy.$

In this article, we discuss an effectivization of Fubini's Theorem for uniformly Fine-computable functions on $[0,1)^2$ (Definition 3.3) and make an introductory consideration on Fine-computable functions (Definition 4.1). We also make some observations on the transformation T. In effectivization, *Fine-computability* and *effective integrability* correspond to classical *measurability* and *integrability* respectively.

From the standpoint of computable analysis, it is expected that $f(x) = \int_{[0,1)} F(x,y)dy$ is defined everywhere on $[0,1)$ and $f(x)$ is Fine-computable for a Fine-computable function $F(x,y)$ on $[0,1)^2$. To secure the former, we assume that $F(x,y)$ is integrable with respect to y for all $x \in [0,1)$.

Since Fine-computable functions are continuous at all dyadically irrational points with respect to the Euclidean topology, they are measurable, and Fubini's Theorem holds classically for integrable Fine-computable functions. Therefore, effectivization of Fubini's Theorem boils down to the proof of Fine-computability of $f(x)$. Hence the proof of this property is the main objective of this paper.

Roughly speaking, continuity of Tg is deduced from that of $F(x,y)$. Hence, by modifying the proof of Fine-computability of $f(x)$, we can easily prove Fine-computability of Tg under some suitable conditions on integrability.

Our main assertions are that Fine-computability of $f(x)$ holds for a "uniformly Fine-computable" function $F(x,y)$ and for a "bounded Fine-computable" function $F(x,y)$, and that we need some additional conditions on general Fine-computable functions.

We make introductory speculations with some examples concerning Fine-computability of $f(x)$.

Example 1.2 (Suggested by Yagishita) Let us define $F(x,y) = \frac{1}{1-y}e^{-(\frac{x}{1-y})^2}$. Then $F(x,y)$ is positive and continuous on $\mathbb{R}\times[0,1)$. It is easy to prove that the restriction of $F(x,y)$ to $[0,1)\times[0,1)$ is Fine-computable.

It holds that $\int_0^1 F(x,y)dx = \int_0^1 \frac{1}{1-y}e^{-(\frac{x}{1-y})^2}dx = \int_0^{\frac{1}{1-y}} e^{-x^2}dx < \sqrt{\pi}$.

Hence $\int_{-1}^1 dy \int_0^1 F(x,y)dx < \infty$.

On the other hand, $F(0,y) = \frac{1}{1-y}$ is not integrable, that is, $f(0) = \int_{[0,1)} F(0,y)dy$ is not defined.

Example 1.2 shows that Fine-computability and integrability of $F(x,y)$ do not assure that $f(x)$ is a total function.

Example 1.3 Let $\alpha(k)$ be a recursive injection whose range is not recursive. Then

$$\varphi(y) = 2^k 2^{-\alpha(k)} \ \text{ if } \ 1-2^{-(k-1)} \leqslant y < 1-2^{-k}, k=1,2,\ldots$$

is Fine-computable and integrable but not effectively integrable (Brattka, [1]).

Define $F(x, y) = \varphi(y)(1 - x)^{\varphi(y)-1}$ and $f(x) = \int_{[0,1)} F(x, y)dy$.

Then, $F(x, y)$ is Fine-computable and not bounded. It holds that $\int_{[0,1)} F(x, y)dx = 1$, $\iint_{[0,1)^2} F(x, y)dxdy = 1$ and $f(x)$ is total.

On the other hand, $f(0) = \int_{[0,1)} F(0, y)dy = \sum_{k=1}^{\infty} 2^{-\alpha(k)}$ is not a computable number, and hence sequential computability for $f(x)$ does not hold.

Example 1.3 shows that Fine-computability of $F(x, y)$ and computability of $\iint_{[0,1)^2} F(x, y)dxdy$ do not imply Fine-computability of $f(x)$ even if it is total.

In Section 2, we review Fine-computability and effective integrability for a function on $[0, 1)$.

In Section 3, we define the two-dimensional Fine-space and notions of Fine-computability and prove that $f(x)$ is uniformly Fine-computable if $F(x, y)$ is uniformly Fine-computable (Theorem 3.6).

In Section 4, we prove that $f(x)$ is Fine-computable for a bounded Fine-computable $F(x, y)$ (Theorem 4.6). We give also a sufficient condition for Fine-computability of $f(x)$, where $F(x, y)$ is Fine-computable but not necessarily bounded. This is an effectivization of a well known classical result.

Consult [2] as to Fine-continuous functions.

2 Preliminaries

We summarize Fine-computability properties on $[0, 1)$ and effective integrability of such functions. (See [6,7,8].) We assume basic knowledge of computability on the Euclidean space (cf. [9]). We use the notations $\mathbb{N} = \{0, 1, \dots\}$ and $\mathbb{N}^+ = \{1, 2, \dots\}$.

A left-closed right-open interval with dyadic end points is called a *dyadic interval.* We call $I(n, k) = [k2^{-n}, (k + 1)2^{-n})$ a *fundamental dyadic interval* (of level n) and $J(x, n)$, the unique fundamental dyadic interval $I(n, k)$ which contains x, the *fundamental dyadic neighborhood* of x (of level n).

Lemma 2.1 ([6]) (1) *The following three properties are equivalent for any* $x, y \in [0, 1)$ *and any nonnegative integer* n.

(i) $y \in J(x, n)$. (ii) $x \in J(y, n)$. (iii) $J(x, n) = J(y, n)$.

(2) *If* $\{x_m\}$ *is Fine-computable, then we can decide effectively whether* $x_m \in I(n, k)$ *or not for all* m.

$\{J(x, n)\}$ satisfies the axioms of the effective uniformity (cf. [10]). We call the topology generated by $\{I(n, k)\}$ the *Fine topology* and put prefix *Fine-* to such notions. We put no prefix to notions which are defined by means of Euclidean topology.

A double sequence of dyadic rationals $\{r_{n,m}\}$ is said to be *recursive* if there exist recursive functions $\alpha(n, m), \beta(n, m)$ such that $r_{n,m} = \beta(n, m)2^{-\alpha(n,m)}$.

Definition 2.2 (1) (Effective Fine-convergence of reals) A double sequence $\{x_{n,m}\}$ is said to *Fine-converge effectively* to $\{x_n\}$ if there exists a recursive function $\alpha(n, k)$ which satisfies that $m \geqslant \alpha(n, k)$ implies $x_{n,m} \in J(x_n, k)$.

(2) (Fine-computable sequence of reals) A sequence of real numbers $\{x_m\}$ in $[0, 1)$ is said to be *Fine-computable* if there exists a recursive double sequence of dyadic rationals $\{r_{n,m}\}$ which Fine-converges effective ly to $\{x_m\}$.

If $x_{n,m} = x_m$ and $x_n = x$, we obtain the definition of effective Fine-convergence of $\{x_m\}$ to x.

Remark 2.3 (1) The original definition of a Fine-computable sequence of real numbers is that $\{r_{n,m}\}$ be a recursive sequence of rational numbers (cf. [11]). The present definition is equivalent to the original one.

(2) The set of computable numbers and that of Fine-computable numbers coincide.

(3) A Fine-computable sequence is (Euclidean) computable.

(4) $\{e_i\}$ will denote an effective enumeration of all nonnegative dyadic rationals in $[0, 1)$. It is an effective separating set of the Fine-space $[0, 1)$ (cf. [5]).

Definition 2.4 (Uniformly Fine-computable sequence of functions, [3,6]) A sequence of functions $\{f_n\}$ is said to be *uniformly Fine-computable* if (i) and (ii) below hold.

(i) (Sequential Fine-computability) The double sequence $\{f_n(x_m)\}$ is computable for any Fine-computable sequence $\{x_m\}$.

(ii) (Effectively uniform Fine-continuity) There exists a recursive function $\alpha(n, k)$ such that, for all n, k and all $x, y \in [0, 1)$, $y \in J(x, \alpha(n, k))$ implies $|f_n(x) - f_n(y)| < 2^{-k}$.

Definition 2.5 (Effectively uniform convergence of functions, [3,6]). A double sequence of functions $\{g_{m,n}\}$ is said to *converge effectively uniformly* to a sequence of functions $\{f_m\}$ if there exists a recursive function $\alpha(m, k)$ such that, for all m, n and k, $n \geqslant \alpha(m, k)$ implies $|g_{m,n}(x) - f_m(x)| < 2^{-k}$ for all $x \in [0, 1)$.

Definition 2.6 (Fine-computable sequence of functions, [6]) A sequence of functions $\{f_n\}$ is said to be *Fine-computable* if it satisfies the following.

(i) $\{f_n\}$ is sequentially Fine-computable.

(ii) (Effective Fine-Continuity) There exists a recursive function $\alpha(n, k, i)$ such that

(ii-a) $x \in J(e_i, \alpha(n, k, i))$ implies $|f_n(x) - f_n(e_i)| < 2^{-k}$,

(ii-b) $\bigcup_{i=1}^{\infty} J(e_i, \alpha(n, k, i)) = [0, 1)$ for each n, k.

Definition 2.7 (Effective Fine-convergence of functions, [6]) We say that a double sequence of functions $\{g_{m,n}\}$ *Fine-converges effectively* to a sequence of functions $\{f_m\}$ if there exist recursive functions $\alpha(m, k, i)$ and $\beta(m, k, i)$, which satisfy

(a) $x \in J(e_i, \alpha(m, k, i))$ and $n \geqslant \beta(m, k, i)$ imply $|g_{m,n}(x) - f_m(x)| < 2^{-k}$,

(b) $\bigcup_{i=1}^{\infty} J(e_i, \alpha(m, k, i)) = [0, 1)$ for each m and k.

Definition 2.8 (Computable sequence of dyadic step functions, [3,6]) A sequence of functions $\{\varphi_n\}$ is called a *computable sequence of dyadic step functions* if there exist a recursive function $\alpha(n)$ and a computable sequence of reals $\{c_{n,j}\}$ $(0 \leqslant j <$

$2^{\alpha(n)}$, $n = 1, 2, \ldots$) such that

$$\varphi_n(x) = \sum_{j=0}^{2^{\alpha(n)}-1} c_{n,j}\chi_{I(\alpha(n),j)}(x),$$

where χ_A denotes the indicator (characteristic) function of A.

Proposition 2.9 ([6]) *Let f be a Fine-computable function. The computable sequence of dyadic step functions $\{\varphi_n\}$, which is defined by*

$$\varphi_n(x) = \sum_{j=0}^{2^n-1} f(j2^{-n})\chi_{I(n,j)}(x), \tag{1}$$

Fine-converges effectively to f.

Moreover, if f is uniformly Fine-computable, then $\{\varphi_n\}$ converges effectively uniformly to f.

We will briefly review effective integrability of functions on $[0, 1)$. See [6,7,8] for details.

Definition 2.10 (Effective integrability of a sequence of functions, [7,8]) A sequence of Fine-computable functions $\{f_n\}$ is called *effectively integrable* if each f_n is integrable and $\{\int_{[0,1)} f_n^+(x)dx\}$ and $\{\int_{[0,1)} f_n^-(x)dx\}$ are computable sequences of real numbers.

A Fine-computable function is said to be *effectively integrable* if the sequence $f, f, \ldots$ is effectively integrable.

Integral on a finite union of fundamental dyadic intervals E is defined to be $\int_{[0,1)} f(x)\chi_E(x)dx$.

It is easy to prove that a computable sequence of dyadic step functions is effectively integrable.

Theorem 2.11 (Effective bounded convergence theorem, [7,8]) *Let $\{g_n\}$ be a uniformly bounded Fine-computable sequence of functions which is effectively integrable and Fine-converges effectively to f. Then, f is Fine-computable and $\{\int_{[0,1)} g_n(x)dx\}$ converges effectively to $\int_{[0,1)} f(x)dx$. As a consequence, f is effectively integrable.*

Theorem 2.12 ([7,8]) *A bounded Fine-computable function is effectively integrable.*

Theorem 2.13 ([7,8]) *Let $\{f_n\}$ be Fine-computable and effectively bounded, that is, there exists a computable sequence of reals $\{M_n\}$ such that $|f_n(x)| \leqslant M_n$ for all x. Then $\{f_n\}$ is effectively integrable.*

Theorem 2.14 (Effective dominated convergence theorem, [7,8]) *Let $\{g_{m,n}\}$ be an effectively integrable Fine-computable sequence which Fine-converges effectively to $\{f_m\}$. Suppose that there exists an effectively integrable Fine-computable function h such that $|g_{m,n}(x)| \leqslant h(x)$. Then, $\{\int_{[0,1)} g_{m,n}(x)dx\}$ converges effectively to $\int_{[0,1)} f_m(x)dx$.*

Proposition 2.15 ([7,8]) *Let f be an effectively integrable Fine-computable function and let I_n be a computable sequence of dyadic intervals such that $\bigcup_{n=1}^{\infty} I_n =$*

$[0,1)$. *Put* $E_n = \bigcup_{i=1}^{n} I_i$. *Then,* $\int_{E_n} f(x)dx$ *converges effectively to* $\int_{[0,1)} f(x)dx$, *or equivalently,* $\int_{E_n^C} f(x)dx$ *converges effectively to zero.*

3 Uniformly Fine-computable functions on $[0,1)^2$

The main objective of this section is to prove uniform Fine-computability of $f(x) = \int_{[0,1)} F(x,y)dy$ for a uniformly Fine-computable function $F(x,y)$ on $[0,1)^2$.

On the upper-right open unit square $[0,1)^2$, we denote $[k2^{-n},(k+1)2^{-n}) \times [\ell 2^{-m},(\ell+1)2^{-m})$ with $I_2(n,m;k,\ell)$ and call it a *fundamental dyadic rectangle*. We also denote $J(x,n)\times J(y,m)$ by $J_2(x,y;n,m)$ and call it a *fundamental dyadic neighborhood* of (x,y). We call the topology generated by the set $\{J_2(e_i,e_j;n,m)\}_{i,j,n,m}$ the *Fine-topology* on $[0,1)^2$ and the space $[0,1)^2$ with this topology the *two-dimensional Fine-space*. Notions of computability on $[0,1)^2$ are defined with respect to the Fine-topology.

Note that $\{J_2(x,y;n,n)\}$ satisfies the axioms of the effective uniformity (cf. [10]), and the topology generated by the set $\{J_2(e_i,e_j;n,n)\}$ is equivalent.

Definition 3.1 (1) A double sequence $\{(x_{p,q},y_{p,q})\}$ from $[0,1)^2$ is said to *Fine-converges effectively* to $\{(x_p,y_p)\}$ if there exists a recursive function $\alpha(p,n,m)$ such that $q \geqslant \alpha(p,n,m)$ implies $(x_{p,q},y_{p,q}) \in J_2(x_p,y_p;n,m)$.

(2) A sequence $\{(x_p,y_p)\}$ is said to be *Fine-computable* if there exist recursive sequences of dyadic rationals $\{s_{p,q}\}$ and $\{t_{p,q}\}$ such that $\{s_{p,q}\}$ and $\{t_{p,q}\}$ Fine-converge effectively to $\{x_p\}$ and $\{y_p\}$ respectively.

Lemma 3.2 (cf. Lemma 2.1) (1) *The following three properties are equivalent for any* $(x,y),(z,w) \in [0,1)^2$ *and any positive integers* n,m.

(i) $(z,w) \in J_2(x,y;n,m)$. (ii) $(x,y) \in J_2(z,w;n,m)$.

(iii) $J(x,y;n,m) = J(z,w;n,m)$.

(2) *If* $\{(x_p,y_p)\}$ *is Fine-computable, then we can decide effectively whether* $(x_p,y_p) \in I_2(n,m;k,\ell)$ *or not.*

In the following, we use the notation $F(x,\cdot)$ to designate the function $F(x,y)$ regarded as a function of y (for each fixed x).

Definition 3.3 (Uniform Fine-computability) A function $F(x,y)$ on $[0,1)^2$ is said to be *uniformly Fine-computable* if it satisfies the following two conditions.

(i) (Sequential computability) $\{F(x_n,y_m)\}$ is a computable double sequence of reals for every Fine-computable sequence $\{(x_n,y_m)\}$.

(ii) (Effective uniform Fine-continuity) There exist recursive functions $\alpha_1(k)$ and $\alpha_2(k)$ such that $(x,y) \in J_2(z,w;\alpha_1(k),\alpha_2(k))$ implies $|F(x,y)-F(z,w)| < 2^{-k}$.

Proposition 3.4 *Let* $F(x,y)$ *be uniformly Fine-computable as a function of* (x,y). *Then the following hold.*

(1) *If* $\{x_n\}$ *is a Fine-computable sequence, then* $\{f_n(y)\} = \{F(x_n,y)\}$ *is a uniformly Fine-computable sequence of functions on* $[0,1)$ *(Definition 2.4).*

(2) *If a Fine-computable sequence* $\{x_{m,n}\}$ *Fine-converges effectively to* $\{x_m\}$*, then* $\{F(x_{m,n},\cdot)\}$ *converges effectively uniformly to* $\{F(x_m,\cdot)\}$ *(Definition 2.5).*

Proof Let $\alpha_1(k)$ and $\alpha_2(k)$ be as in Definition 3.3.

(1) Let $\{y_m\}$ be a Fine-computable sequence of reals. Then $\{f_n(y_m)\} = \{F(x_n, y_m)\}$ is a computable sequence of reals due to the sequential computability of $F(x, y)$. Then, $|f_n(y) - f_n(z)| = |F(x_n, y) - F(x_n, z)| < 2^{-k}$ if $y \in J(z, \alpha_2(k))$, and hence follows effective uniform Fine-continuity.

(2) From the effective Fine-convergence of $\{x_{m,n}\}$ to $\{x_m\}$, there exists a recursive function $\beta(m, \ell)$ such that $n \geq \beta(m, \ell)$ implies $x_{m,n} \in J(x_m, \ell)$.

If we take $\delta(m, k) = \beta(m, \alpha_1(k))$, then $|F(x_{m,n}, y) - F(x_m, y)| < 2^{-k}$ for $n \geq \delta(m, k)$ and all $y \in [0, 1)$. □

It is pointed out in [4] that a uniformly Fine-computable function $g(y)$ on $[0, 1)$ is bounded and has a computable supremum. The latter property holds for a uniformly Fine-computable sequence of functions. These properties are easily deduced from Theorem 2 in [3]. We denote the supremum of $|g|$ by $||g||$.

Similarly, we can prove that a uniformly Fine-computable function $F(x, y)$ takes a computable supremum.

Regarding uniform Fine-computability of $F(x, y)$, we obtain the following theorem.

Theorem 3.5 *For a function* $F(x, y)$*, the following* (i) *and* (ii) *are equivalent.*

(i) $F(x, y)$ *is uniformly Fine-computable.*

(ii) (ii-a) $\{F(x_n,\cdot)\}$ *is a uniformly Fine-computable sequence of functions on* $[0, 1)$ *for any Fine-computable sequence* $\{x_n\}$.

(ii-b) *There exists a recursive function* $\alpha(k)$ *such that,* $y \in J(x, \alpha(k))$ *implies* $||F(x,\cdot) - F(y,\cdot)|| < 2^{-k}$ *for all* k.

Proof (i)⇒(ii): (ii-a) follows immediately from Proposition 3.4 (1).

To prove (ii-b), let us take $\alpha_1(k)$ and $\alpha_2(k)$ in Definition 3.3. If $x \in J(y, \alpha_1(k+1))$, then $(x, z) \in J_2(y, z; \alpha_1(k + 1), \alpha_2(k + 1))$ for all $z \in [0, 1)$. So, $|F(x, z) - F(y, z)| < 2^{-(k+1)}$ and $||F(x,\cdot) - F(y,\cdot)|| < 2^{-k}$.

(ii)⇒(i): Let $\alpha(k)$ be the recursive function in (ii-b). Then, $z \in J(x, \alpha(k))$ implies $||F(x,\cdot) - F(z,\cdot)|| < 2^{-k}$. Put $r_{k,j} = j2^{-\alpha(k)}$ for $j = 0, 1, \ldots, 2^{\alpha(k)} - 1$. By (ii-a), the sequence $\{F(r_{k,j},\cdot)\}$ is a uniform Fine-computable sequence of functions on $[0, 1)$. So, there exists a recursive function $\beta(k, j)$ such that $y \in J(w, \beta(k, j))$ implies $|F(r_{k,j}, y) - F(r_{k,j}, w)| < 2^{-k}$.

Define $\gamma(k) = \max\{\alpha(k + 2), \beta(k + 2, 0), \beta(k + 2, 1), \ldots, \beta(k + 2, 2^{\alpha(k+2)} - 1)\}$ and suppose that $(x, y) \in J_2(z, w; \gamma(k), \gamma(k))$. Since $z \in J(x, \alpha(k + 2))$, there exists a j, such that $[j2^{-\alpha(k+2)}, (j + 1)2^{-\alpha(k+2)})$ contains both x and z. Therefore, we obtain

$$\begin{aligned}
&|F(x, y) - F(z, w)| \\
&\leqslant |F(x, y) - F(r_{k+2,j}, y)| + |F(r_{k+2,j}, y) - F(r_{k+2,j}, w)| + |F(r_{k+2,j}, w) - F(z, w)| \\
&< 3 \cdot 2^{-(k+2)} < 2^{-k}.
\end{aligned}$$

This shows effective uniform Fine-continuity of $F(x,y)$.

Let $\{x_n\}$ and $\{y_m\}$ be Fine-computable sequences. Then $\{F(x_n,\cdot)\}$ is a uniformly Fine-computable sequence of functions. This implies that $\{F(x_n,y_m)\}$ is a computable sequence of reals. □

It is easy to check that a uniformly Fine-computable function on $[0,1)^2$ is Lebesgue integrable and that its integral is a computable number, similarly to the case of uniformly Fine-computable functions on $[0,1)$ ([7]).

Theorem 3.6 (Effective Fubini's Theorem for uniformly Fine-computable functions)

Let $F(x,y)$ be a uniformly Fine-computable function. Then the following hold.

(1) *If $\{x_n\}$ is Fine-computable, then $\{F(x_n,\cdot)\}$ and $\{F(\cdot,x_n)\}$ are uniformly Fine-computable sequences of functions on* $[0,1)$.

(2) $\int_{[0,1)} F(x,y)dy$ *and* $\int_{[0,1)} F(x,y)dx$ *are uniformly Fine-computable functions.*

(3) $\iint_{[0,1)^2} F(x,y)dxdy$ *is a computable number and*

$\iint_{[0,1)^2} F(x,y)dxdy = \int_{[0,1)} dx \int_{[0,1)} F(x,y)dy = \int_{[0,1)} dy \int_{[0,1)} F(x,y)dx.$

Proof. (1) is Proposition 3.4 (1). The equation in (3) is a consequence of classical Fubini's Theorem.

(2) To prove sequential computability, let $\{x_n\}$ be a Fine-computable sequence. Then $\{F(x_n,\cdot)\}$ is a uniformly bounded uniformly Fine-computable sequence of functions. Hence, $\{\int_{[0,1)} F(x_n,y)dy\}$ is a computable sequence of reals by Theorem 2.13.

Effective uniform Fine-continuity follows from the inequality

$|\int_{[0,1)} F(x,y)dy - \int_{[0,1)} F(z,y)dy| \leqslant ||F(x,\cdot) - F(z,\cdot)||$

and Theorem 3.5 (ii-b). □

We can easily extend (2) above as follows.

Theorem 3.7 *Let $F(x,y)$ be a uniformly Fine-computable function on $[0,1)^2$ and let g be an effectively integrable Fine-computable function on $[0,1)$. Then $(Tg)(x) = \int_{[0,1)} g(y)F(x,y)dy$ is uniformly Fine-computable.*

Especially, the operator T maps any uniformly Fine-computable function to a uniformly Fine-computable function.

Proof First, we note that $M = \sup_{(x,y)\in[0,1)^2} |F(x,y)|$ is computable if $F(x,y)$ is uniformly Fine-computable on $[0,1)^2$.

Let $\{x_m\}$ be Fine-computable. Then $\{g(y)F(x_m,y)\}$ is a Fine-computable sequence of functions of y by Theorem 3.6 (1). If we take the approximating computable sequence of dyadic step functions $\{\varphi_{m,n}(y)\}$ which Fine-converges effectively to $\{g(y)F(x_m,y)\}$, obtained by Proposition 2.9, then, it is obviously an effectively integrable Fine-computable sequence and satisfies $|\varphi_{m,n}(y)| \leqslant M|g(y)|$. Hence, $\{\int_{[0,1)} \varphi_{m,n}(y)dy\}$ converges effectively to $\{\int_{[0,1)} g(y)F(x_m,y)dy\}$ by Theorem 2.14. Therefore, $\{\int_{[0,1)} g(y)F(x_m,y)dy\}$ is a computable sequence.

Effective uniform continuity follows from the following inequality;

$|\int_{[0,1)} g(y)F(x,y)dy - \int_{[0,1)} g(y)F(z,y)dy| \leqslant ||F(x,\cdot) - F(z,\cdot)|| \int_{[0,1)} |g(z)|dz.$ □

4 Fine-computable functions on $[0,1)^2$

In the following, we treat Fine-computability of $f(x) = \int_{[0,1)} F(x,y)dy$ for a Fine-computable function $F(x,y)$. First we define Fine-computability of functions on $[0,1)^2$, which is weaker than uniform Fine-computability (Definition 3.3), as follows.

Definition 4.1 (Fine-computable functions on $[0,1)^2$) Let $F(x,y)$ be a function on $[0,1)^2$. F is said to be *Fine-computable* if it satisfies the following (i) and (ii).

(i) F is sequentially computable.

(ii) (Effective Fine-continuity) There exist recursive functions $\alpha_1(k,i,j)$ and $\alpha_2(k,i,j)$ which satisfy

(ii-a) $(x,y) \in J_2(e_i, e_j; \alpha_1(k,i,j), \alpha_2(k,i,j))$ implies $|F(x,y) - F(e_i,e_j)| < 2^{-k}$,

(ii-b) $\bigcup_{i,j=1}^{\infty} J_2(e_i, e_j; \alpha_1(k,i,j), \alpha_2(k,i,j)) = [0,1)^2$ for each k.

We state the Proposition 3.1 in [6] for the case $\{r_i\} = \{e_i\}$.

Proposition 4.2 *A function g on $[0,1)$ is effectively Fine-continuous if and only if there exist a recursive sequence of dyadic rationals $\{r_{k,q}\}$ and a recursive function $\delta(k,q)$ which satisfy the following.*

(a) *$x \in J(r_{k,q}, \delta(k,q))$ implies $|g(x) - g(r_{k,q})| < 2^{-k}$.*

(b) *$\bigcup_{q=1}^{\infty} J(r_{k,q}, \delta(k,q)) = [0,1)$ for each k.*

(c) *The intervals in $\{J(r_{k,q}, \delta(k,q))\}$ are mutually disjoint with respect to q for each k.*

In the proof of Proposition 3.1 in [6], the crucial properties are those of Lemma 2.1, whose two-dimensional version is Lemma 3.2, and the fact that the complement of a finite (disjoint) union of fundamental dyadic intervals can be represented as a finite disjoint union of fundamental dyadic intervals. A similar fact also holds for fundamental dyadic rectangles. So, we can prove the following proposition.

Proposition 4.3 *Effective Fine-continuity of a function F on $[0,1)^2$ is equivalent to the following: There exist a recursive sequence of pairs of dyadic rationals $\{(s_{k,p}, t_{k,p})\}$ and recursive functions $\beta_1(k,p)$, $\beta_2(k,p)$ which satisfy the following three conditions.*

(a) *$(x,y) \in J_2(s_{k,p}, t_{k,p}; \beta_1(k,p), \beta_2(k,p))$ implies $|F(x,y) - F(s_{k,p}, t_{k,p})| < 2^{-k}$.*

(b) *$\bigcup_{p=1}^{\infty} J_2(s_{k,p}, t_{k,p}; \beta_1(k,p), \beta_2(k,p)) = [0,1)^2$ for each k.*

(c) *The fundamental dyadic neighborhoods in $\{J_2(s_{k,p}, t_{k,p}; \beta_1(k,p), \beta_2(k,p))\}$ are mutually disjoint with respect to p for each k.*

Remark 4.4 The conditions (b) and (c) in Proposition 4.3 signify that the upper-right open square $[0,1)^2$ is partitioned into (infinitely many) disjoint rectangles $\{J_2(s_{k,p}, t_{k,p}; \beta_1(k,p), \beta_2(k,p))\}$ for each k. Hence, the following hold:

(a) For any k and any $(x,y) \in [0,1)^2$, there is the unique number $p(k,x,y)$ such that (x,y) is contained in $J_2(s_{k,p(k,x,y)}, t_{k,p(k,x,y)}; \beta_1(k,p(k,x,y)), \beta_2(k,p(k,x,y)))$.

Moreover, $(z,w) \in J_2(s_{k,p(k,x,y)}, t_{k,p(k,x,y)}; \beta_1(k,p(k,x,y)), \beta_2(k,p(k,x,y)))$ implies $p(k,x,y) = p(k,z,w)$.

(b) If $\{(x_n,y_n)\}$ is Fine-computable, then $i(k,n) = p(k,x_n,y_n)$ is a recursive function.

Proposition 4.5 *Let $F(x,y)$ be Fine-computable. Then the following hold.*

(1) If $\{x_m\}$ is a Fine-computable sequence of reals, then $\{F(x_m,\cdot)\}$ is a Fine-computable sequence of functions.

(2) If $\{x_{m,n}\}$ is a Fine-computable sequence of reals and Fine-converges effectively to $\{x_m\}$, then $\{F(x_{m,n},\cdot)\}$ Fine-converges effectively to $\{F(x_m,\cdot)\}$.

Proof. Let us take $\{(s_{k,p},t_{k,p})\}$ and $\beta_1(k,p)$, $\beta_2(k,p)$ in Proposition 4.3.

Proof of (1): We prove (i) and (ii) in Definition 2.6 for $\{F(x_m,\cdot)\}$.

(i): Sequential computability of $\{F(x_m,\cdot)\}$ is an easy consequence of sequential computability of $F(x,y)$.

(ii-a): For each m, k and j, we can find effectively and uniquely such $p = p(m,k,j)$ that (x_m,e_j) is contained in $J_2(s_{k+1,p},t_{k+1,p};\beta_1(k+1,p),\beta_2(k+1,p))$ by Remark 4.4. Define $\alpha(m,k,j) = \beta_2(k+1,p(m,k+1,j))$ and suppose that $y \in J(e_j,\alpha(m,k,j))$.

Then (x_m,y) is also contained in $J_2(s_{k+1,p},t_{k+1,p};\beta_1(k+1,p),\beta_2(k+1,p))$. So, we obtain

$$\begin{aligned}&|F(x_m,y) - F(x_m,e_j)|\\ &\leqslant |F(x_m,y) - F(s_{k+1,p},t_{k+1,p})| + |F(s_{k+1,p},t_{k+1,p}) - F(x_m,e_j)| < 2^{-k}.\end{aligned}$$

(ii-b): Let us take $p = p(k,x_m,y)$ for arbitrary $y \in [0,1)$, as in Remark 4.4. Then, $J_2(s_{k+1,p},t_{k+1,p};\beta_1(k+1,p),\beta_2(k+1,p))$ contains (x_m,e_j) for some dyadic rational e_j. By Remark 4.4 (a), we obtain $p(k,x_m,y) = p(k,x_m,e_j)$ and $J_2(s_{k+1,p},t_{k+1,p};\beta_1(k+1,p),\beta_2(k+1,p)) = J_2(x_m,e_j;\beta_1(k+1,p),\beta_2(k+1,p))$. Hence, $\bigcup_{j=1}^{\infty} J(e_j,\alpha(m,k,j)) = [0,1)$ holds.

Proof of (2): We note first that $\{x_m\}$ is a Fine-computable sequence. Let $\gamma(m,\ell)$ be an effective modulus of effective Fine-convergence. That is, it satisfies that $n \geqslant \gamma(m,\ell)$ implies $x_{m,n} \in J(x_m,\ell)$.

For any m and e_j, we can find effectively and uniquely such $p = p(m,j)$ that $J_2(s_{k+1,p},t_{k+1,p};\beta_1(k+1,p),\beta_2(k+1,p))$ contains (x_m,e_j). We note that $J(s_{k+1,p},\beta_1(k+1,p)) = J(x_m,\beta_1(k+1,p))$ by Lemma 2.1.

If $n \geqslant \gamma(m,\beta_1(k+1,p))$ and $y \in J(t_{k+1,p},\beta_2(k+1,p)) = J(e_j,\beta_2(k+1,p))$, then

$$\begin{aligned}&|F(x_{m,n},y) - F(x_m,y)|\\ &\leqslant |F(x_{m,n},y) - F(s_{k+1,p},t_{k+1,p})| + |F(s_{k+1,p},t_{k+1,p}) - F(x_m,y)| < 2\cdot 2^{-(k+1)} = 2^{-k}.\end{aligned}$$

By Proposition 4.3 (b), $\bigcup_{J(s_{k+1,p},\beta_1(k+1,p))\ni x} J(t_{k+1,p},\beta_2(k+1,p)) = [0,1)$ for all k, and hence, $\bigcup_j J(e_j,\beta_2(k+1,p)) = [0,1)$. This proves the effective Fine-convergence of $\{F(x_{m,n},\cdot)\}$ to $\{F(x_m,\cdot)\}$ with respect to $\alpha(k,j) = \beta_2(k+1,p(m,j))$ and $\delta(k,i) = \gamma(m,\beta_1(k+1,p(m,j)))$ (cf. Definition 2.7). □

In the rest of this section, we investigate Fine-computability of the function

$f(x) = \int_{[0,1)} F(x,y)dy$ for a bounded Fine-computable function $F(x,y)$.

Theorem 4.6 *If $F(x,y)$ is bounded and Fine-computable on $[0,1)^2$, then $f(x) = \int_{[0,1)} F(x,y)dy$ is Fine-computable on $[0,1)$.*

Outline of the proof of Effective Fine-continuity: Let us take $\{(s_{k,p}, t_{k,p})\}$ and $\beta_1(k,p)$, $\beta_2(k,p)$ in Proposition 4.3. Then, we construct a function $N(k,x)$, on $\mathbb{N}^+ \times [0,1)$, functions $h(k,x,\ell)$, $\alpha_1(k,x,\ell)$, $\alpha_2(k,x,\ell)$ on $\mathbb{N}^+ \times [0,1) \times \{1,2,\ldots,N(k,x)\}$ and sequences $u_{k,x,\ell}$, $v_{k,x,\ell}$ for each k, x and $1 \leqslant \ell \leqslant N(k,x)$, which satisfy the following:

(a) Dyadic intervals $\{J(v_{k,x,\ell}, \alpha_2(k,x,\ell))\}_{1\leqslant\ell\leqslant N(k,x)}$ are mutually disjoint.

(b) $\sum_{\ell=1}^{N(k,x)} 2^{-\alpha_2(k,x,\ell)} > 1 - 2^{-k}$.

(c) $y \in J(v_{k,x,\ell}, \alpha_2(k,x,\ell))$ and $z \in J(u_{k,x,\ell}, \alpha_1(k,x,\ell))$ imply $|F(x,y) - F(z,y)| < 2^{-k}$ due to Proposition 4.3 (a) for $1 \leqslant \ell \leqslant N(k,x)$.

(d) $u_{k,x,\ell} \leqslant x < u_{k,x,\ell} + 2^{-\alpha_1(k,x,\ell)}$ for $1 \leqslant \ell \leqslant N(k,x)$.

Define $\xi(k,x) = \max_{1\leqslant\ell\leqslant N(k,x)} u_{k,x,\ell}$ and $\eta(k,x) = \min_{1\leqslant\ell\leqslant N(k,x)} u_{k,x,\ell}+2^{-\alpha_1(k,x,\ell)}$. Then, $[\xi(k,x), \eta(k,x))$ is a dyadic interval and contains x. So, we can define $\gamma(k,x)$ as $\min\{\ell \mid J(x,\ell) \subset [\xi(k,x), \eta(k,x))\}$.

Properties of $\gamma(k,x)$ and $N(k,x)$:

(i) If $z \in J(x, \gamma(k,x))$, then $N(k,z) = N(k,x)$. Moreover, $u_{k,x,\ell} = u_{k,z,\ell}$, $v_{k,x,\ell} = v_{k,z,\ell}$ and $\alpha_i(k,x,\ell) = \alpha_i(k,z,\ell)$ for $1 \leqslant \ell \leqslant N(k,x)$.

(ii) If $y \in \bigcup_{\ell=1}^{N(k,x)} J(v_{k,x,\ell}, \alpha_2(k,x,\ell))$ and $z \in J(x, \gamma(k,x))$, then $|F(x,y) - F(z,y)| < 2^{-k}$.

(iii) $|\bigcup_{n=1}^{N(k,x)} J(v_{k,x,\ell}, \alpha_2(k,x,\ell))| = \sum_{n=1}^{N(k,x)} 2^{-\alpha_2(k,x,\ell)} > 1 - 2^{-k}$.

Moreover, $N(k,e_i)$, $\alpha_i(k,e_i,\ell)$ $(i = 1,2)$ and $\gamma(k,e_i)$ can be regarded as recursive functions. While, $u_{k,e_i,\ell}$ and $v_{k,e_i,\ell}$ can be regarded as computable sequences of dyadic rationals.

From boundedness of $F(x,y)$, there exists an integer K such that $|F(x,y)| < 2^K$ for all (x,y). Now, if we define $\delta(k,i) = \gamma(k+K+2, e_i)$, then δ is a recursive function. Suppose that $x \in J(e_i, \delta(k,i)) = J(e_i, \gamma(k+K+2, e_i))$, and put $E_{k,i} = \bigcup_{\ell=1}^{N(k,e_i)} J(v_{k,e_i,\ell}, \alpha_2(k,e_i,\ell))$. Then, $E_{k,i} = \bigcup_{\ell=1}^{N(k,x)} J(v_{k,x,\ell}, \alpha_2(k,x,\ell))$, and we obtain

$$\begin{aligned}|f(x) - f(e_i)| &\leqslant \int_{E_{k,i}} |F(x,y) - F(e_i,y)|dy + \int_{(E_{k,i})^C} |F(x,y)|dy + \int_{(E_{k,i})^C} |F(e_i,y)|dy \\ &< 2^{-(k+K+2)} + 2 \cdot 2^K 2^{-(k+K+2)} < 2^{-k}.\end{aligned}$$

For all $x \in [0,1)$, $J(x, \delta(k,x))$ contains a dyadic rational, say, e_i. By property (i), $J(x, \delta(k,i)) = J(e_i, \delta(k,i))$. So $x \in J(e_i, \delta(k,i))$ and we obtain $\bigcup_{i=1}^{\infty} J(e_i, \delta(k,i)) = [0,1)$. This proves effective Fine-continuity of $f(x)$. □

Example 1.3 in Introduction shows that the conclusion of Theorem 4.6 does not hold for a Fine-computable function in general. Therefore, for general Fine-computable functions, we need an additional condition on integrability.

We give a sufficient condition that assures the Fine-computability of $f(x)$ for a

Fine-computable function $F(x, y)$.

Theorem 4.7 *If $F(x, y)$ is Fine-computable and there exists an effectively integrable Fine-computable function $g(y)$ which satisfies $|F(x, y)| \leqslant g(y)$ for all x, then $f(x) = \int_{[0,1)} F(x, y)dy$ is Fine-computable.*

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