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Admissible Representations of Probability Measures

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Abstract

In a recent paper, probabilistic processes are used to generate Borel probability measures on topological spaces X that are equipped with a representation in the sense of Type-2 Theory of Effectivity. This gives rise to a natural representation of the set $\mathcal{M}(X)$ of Borel probability measures on X. We compare this representation to a canonically constructed representation which encodes a Borel probability measure as a lower semicontinuous function from the open sets to the unit interval. This canonical representation turns out to be admissible with respect to the weak topology on $\mathcal{M}(X)$. Moreover, we prove that for countably based topological spaces X the representation via probabilistic processes is equivalent to the canonical representation and thus admissible with respect to the weak topology on $\mathcal{M}(X)$.

Keywords: Measure Theory, Probabilistic Processes, Type 2 Theory of Effectivity, Admissible Representations

1 Introduction

Measures are the traditional tool in mathematics for assigning weights to the subsets of a topological space X (cf. [1,2,3,4]). They have to fulfil certain well-behavedness conditions like *modularity* (requiring $\mu(A) + \mu(B) = \mu(A \cup B) + \mu(A \cap B)$). A well-known topology on the set of measures is the *weak topology*, which is closely related to integration with respect to measures. The existence of a reasonable topology is indispensable for a space to be able to be handled by the Type 2 Theory of Effectivity (TTE). In this paper we study Borel

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measures over qcb-spaces (which are topological spaces that are quotients of countably-based spaces) and discuss how they can be dealt with in TTE.

The general idea of the Type-2 Theory of Effectivity ([12]) for computing on non-discrete spaces, like the real numbers, is to endow any given space with a suitable representation. A representation equips the elements of the represented space with names which are infinite words over some alphabet Σ. The actual computation is performed on these names. The property of *admissibility* is introduced to guarantee that representations induce a reasonable notion of computability on the represented space (cf. Subsection 2.1).

We investigate two representations for the space of Borel probability measures over a represented topological space X. Both are derived from the original representation of X. The first representation, which we call the *canonical representation*, is obtained by using established construction schemes for representations: a Borel probabilitistic measure on X is uniquely determined by its values on the opens of X and this restriction is a lower semicontinuous function. So Borel probabilistic measures can be represented via a canonical function space representation for lower semicontinuous functions (cf. Subsection 3.1). The second representation is based on the notions of *probabilistic process* and *probabilistic name* introduced in [9]. The idea is to probabilistically generate an element x of X by probabilistically producing, via a sequence of independent choices, an infinite word representing x. This sequence can be encoded as elements of Σ^ω, leading to our second representation (cf. Subsection 3.2).

We prove in Section 3 that the canonical representation is admissible w.r.t. the weak topology on the set of Borel probability measures. Moreover, we characterise the canonical representation (up to computable equivalence) as $\leq_{\mathrm{cp}}$-complete in the class of representations admitting computability of integration w.r.t. Borel measures. This generalises a corresponding result by K. Weihrauch ([11]) for the unit interval. In Section 4 we prove our main result stating that for countably based T_0-spaces the canonical representation and the representation via probabilistic names are equivalent. This implies that the representation via probabilistic names is admissible w.r.t. the weak topology too. Beforehand, we give in Section 2 some background from Type-2 Theory and from Measure Theory.

2 Preliminaries

Throughout the paper, we assume Σ to be a finite alphabet containing the symbols $\mathtt{0}, \mathtt{1}$. We denote the set of finite words over Σ by Σ^*, the set of words of length n by Σ^n and the set $\{p \,|\, p : \mathbb{N} \to \Sigma\}$ of infinite words by Σ^ω. We write $\sqsubseteq$ for the reflexive prefix relation on $\Sigma^* \cup \Sigma^\omega$. For a word $w \in \Sigma^*$ and a subset $W \subseteq \Sigma^*$ we denote by $w\Sigma^\omega$ the set $\{p \in \Sigma^\omega \mid w \sqsubseteq p\}$, by $W\Sigma^\omega$ the set $\bigcup_{w \in W} w\Sigma^\omega$ and by $\lg(w)$ the length of w.

We denote the topology of a topological space X by $\mathcal{O}(X)$ and its underlying set by the symbol X as well. The set of continuous functions from X to another topological space Y is denoted by $\mathcal{C}(X,Y)$. On Σ^ω we consider the Cantor topology $\mathcal{O}(\Sigma^\omega) := \{W\Sigma^\omega \mid W \subseteq \Sigma^*\}$.

We define a transitive relation $\lessdot$ on the real numbers by $a \lessdot b :\Longleftrightarrow a < b \vee a = b = 0$. The set of dyadic rational numbers is denoted by $\mathbb{D}$.

2.1 Background from Type-2 Theory

We recall some definitions and facts from Type-2 Theory of Effectivity (TTE). More details can be found in [12,8].

The basic idea of TTE is to represent infinite objects like real numbers, functions or sets by infinite words over some alphabet Σ. The corresponding partial surjective function $\delta :\subseteq \Sigma^\omega \to X$ is called a *representation* of set X.

Given two representations $\delta :\subseteq \Sigma^\omega \to X$ and $\gamma :\subseteq \Sigma^\omega \to Y$, a total function $f : X \to Y$ is called (δ,γ)-*computable* or *relative computable w.r.t.* δ and γ, if there is a Type-2-computable function $g :\subseteq \Sigma^\omega \to \Sigma^\omega$ *realising* g, which means $\gamma(g(p)) = f(\delta(p))$ for all $p \in \mathrm{dom}(\delta)$, where $\mathrm{dom}(\delta)$ denotes the domain of δ. If there are ambient representations of X and Y, then we simply say that f is *computable* rather than f is (δ,γ)-*computable*. The function f is called (δ,γ)-*continuous*, if there is a continuous function g realising f w.r.t. δ and γ. Relative computability and relative continuity for multivariate functions are defined similarly.

Given a representation δ' of a superset X' of X, δ is called *computably reducible to* δ' (in symbols $\delta \leq_{\mathrm{cp}} \delta'$), if the embedding of X into X' is (δ,δ')-computable. We say that δ and δ' are *computable equivalent*, in symbols $\delta \equiv_{\mathrm{cp}} \delta'$, if $\delta \leq_{\mathrm{cp}} \delta' \leq_{\mathrm{cp}} \delta$. Topological reducibility and topological equivalence are defined analogously and denoted by, respectively, $\leq_{\mathrm{t}}$ and $\equiv_{\mathrm{t}}$. Note that computably (topologically) equivalent representations induce the same class of relatively computable (continuous) functions.

The property of admissibility is defined to reconcile relative continuity with mathematical continuity. We call $\gamma :\subseteq \Sigma^\omega \to Y$ *admissible* w.r.t. a topology τ_Y on Y, if γ is continuous w.r.t. this topology and every continuous representation $\phi :\subseteq \Sigma^\omega \to Y$ satisfies $\phi \leq_{\mathrm{t}} \gamma$. If γ is admissible w.r.t. τ_Y, then a total function is (δ,γ)-continuous if and only if it is continuous w.r.t. to the *quotient topology* $\tau_\delta := \big\{U \subseteq X \,\big|\, \exists V \in \mathcal{O}(\Sigma^\omega).\, V \cap \mathrm{dom}(\delta) = \delta^{-1}[U]\big\}$ on X and τ_Y (cf. [7]). From [12] we obtain canonical constructions of a representation $[\delta, \gamma]$ of $X \times Y$ and a representation $[\delta \to \gamma]$ of the set of (δ, γ)-continuous total functions. Both constructions preserve admissibility.

The category of sequential topological spaces having an admissible representation is known to be equal to the category QCB_0 of T_0-quotients of countably based spaces (*qcb_0-spaces*). Importantly, QCB_0 is cartesian closed.

We equip the unit interval $\mathbb{I} = [0, 1]$ with two representations $\varrho_{\mathbb{I}_=}$ and $\varrho_{\mathbb{I}_<}$. They are the restriction to $\mathbb{I}$ of the respective representations of $\mathbb{R}$ from [12, Definition 4.1.3]. The first one is admissible w.r.t. the Euclidean topology $\mathcal{O}(\mathbb{I}_=)$ and the second one is admissible w.r.t. the lower topology $\mathcal{O}(\mathbb{I}_<) := \big\{(x, 1], [0, 1], \emptyset \,\big|\, x \in [0, 1)\big\}$ on $\mathbb{I}$. As the ambient representation of Σ^*, we will use $\varrho_{\Sigma^*} :\subseteq \Sigma^\omega \to \Sigma^*$ defined by $\varrho_{\Sigma^*}(\mathtt{0}a_1\mathtt{0} \ldots \mathtt{0}a_k\mathtt{11}\ldots) := a_1 \ldots a_k$, which is admissible w.r.t. the discrete topology on Σ^*.

2.2 Background from Measure Theory

Let X be a set. A *lattice* over X is a collection of subsets of X which contains the empty set and is closed under finite intersections and finite unions. An algebra over X is a lattice over X containing X and closed under complement, whereas a *σ-algebra* over X is an algebra that is closed under countable unions and countable intersections.

A *(probabilistic) valuation* ν on a lattice is a function from the lattice into the unit interval $\mathbb{I} = [0, 1]$ which is *strict* (i.e. $\nu(\emptyset) = 0$), monotone, *modular* (i.e. $\nu(U) + \nu(V) = \nu(U \cup V) + \nu(U \cap V)$) and *probabilistic* (i.e. $\nu(X) = 1$). In this paper we are only interested in probabilistic valuations and measures. Therefore we will omit the adjective "probabilistic" in the following. Usually, one allows the weight of X to be any number in $[0, \infty]$. A *measure* on a σ-algebra $\mathcal{A}$ is a valuation μ on $\mathcal{A}$ that is *σ-additive*, i.e. $\mu(\biguplus_{i \in \mathbb{N}} A_i) = \sum_{i \in \mathbb{N}} \mu(A_i)$ for every pairwise disjoint sequence $(A_i)_i$ in $\mathcal{A}$.

Given a topological space X, we are mainly interested in Borel measures. Borel measures are measures defined on the smallest σ-algebra $\mathcal{B}(X)$ containing $\mathcal{O}(X)$. The elements of $\mathcal{B}(X)$ are called the *Borel sets* of X. We denote the set of Borel probabilistic measures on X by $\mathcal{M}(X)$. By σ-additivity, any Borel measure on X restricts to a ω-continuous valuation on $\mathcal{O}(X)$. A valuation $\nu : \mathcal{O}(X) \to \mathbb{I}$ is called *ω-continuous*, if $\nu(\bigcup_{i \in \mathbb{N}} U_i) = \sup_{n \in \mathbb{N}} \nu(\bigcup_{i=0}^{n} U_i)$ holds for every sequence $(U_i)_{i \in \mathbb{N}}$ of opens. This continuity notion is equivalent to topological continuity w.r.t. the ω-Scott topology on the lattice of opens (cf. [10]) and the lower topology $\mathcal{O}(\mathbb{I}_<) = \big\{(x, 1], [0, 1], \emptyset \,\big|\, x \in [0, 1)\big\}$ on the unit interval.

Every Borel measure $\mu \in \mathcal{M}(X)$ induces an *outer measure* $\mu^* : 2^X \to \mathbb{I}$ defined by $\mu^*(M) := \inf \big\{\mu(B) \,\big|\, B \in \mathcal{B}(X),\, B \supseteq M\big\}$. Subsets $M \subseteq X$ satisfying $\mu^*(M) + \mu^*(X \setminus M) = 1$ are called *μ-measurable*. Measurability of M w.r.t.

μ is equivalent to the existence of Borels sets C, D satisfying $C \subseteq M \subseteq D$ and $\mu(C) = \mu(D)$. The outer measure μ^* restricts to a measure on the σ-algebra of μ-measurable sets. A subset $M \subseteq X$ is called *universally measurable*, if M is ν-measurable for all Borel measures $\nu : \mathcal{B}(X) \to \mathbb{I}$. Obviously, all Borel sets are universally measurable.

3 Representions for Borel Measures

Let X be a sequential topological space with admissible representation δ. We equip the set $\mathcal{M}(X)$ of Borel measures on X with two representation, the first one is obtained by standard constructions of representations, the second one is based on probabilistic processes.

3.1 The canonical representation

Any Borel measure $\mu : \mathcal{B}(X) \to \mathbb{I}$ restricts to a valuation $\mu|_{\mathcal{O}(X)}$ on the lattice of opens of X, which is continuous with respect to the ω-Scott topology [2] $\mathcal{O}^2(X)$ on $\mathcal{O}(X)$ and the lower topology on $\mathbb{I}$, because by σ-additivity $\mu(\bigcup_{i\in\mathbb{N}} O_i) = \sup_{i\in\mathbb{N}} \mu(O_i)$ holds for any increasing sequence $(O_i)_i$ of open sets. This opens up the possibility to represent Borel measures via a canonical function space representation of $\mathcal{C}(\mathcal{O}(X), \mathbb{I}_<)$.

A canonical representation of $\mathcal{O}(X)$ which is admissible w.r.t. the ω-Scott topology is constructed by

$$\delta^{\oplus}(q) = U \;:\Longleftrightarrow\; [\delta \to \varrho_{\mathbb{I}_<}](p) = cf_U\,,$$

where $cf_U : X \to \mathbb{I}_<$ denotes the characteristic function of U (mapping the elements of U to 1 and the elements of $X \setminus U$ to 0). Since any Borel measure is uniquely determined by its restriction to $\mathcal{O}(X)$ and this restriction is lower semicontinuous, we can define a representation $\delta^{\mathcal{M}_\mathrm{C}}$ of $\mathcal{M}(X)$ by

$$\delta^{\mathcal{M}_\mathrm{C}}(p) = \mu \;:\Longleftrightarrow\; [\delta^{\oplus} \to \varrho_{\mathbb{I}_<}](p) = \mu|_{\mathcal{O}(X)}\,.$$

We call $\delta^{\mathcal{M}_\mathrm{C}}$ the *canonical* representation of $\mathcal{M}(X)$. From [12, Ex. 3.3.7] we conclude:

Proposition 3.1 *The operator $\gamma \mapsto \gamma^{\mathcal{M}_\mathrm{C}}$ preserves computable as well as topological equivalence of representations.*

The canoncial representation $\delta^{\mathcal{M}_\mathrm{C}}$ is admissible, because it is obtained by constructions preserving admissibility (cf. [8]). We now show that $\delta^{\mathcal{M}_\mathrm{C}}$ is ad-

[2] The ω-Scott topology $\mathcal{O}^2(X)$ on $\mathcal{O}(X)$ is defined as the family of all subsets $H \subseteq \mathcal{O}(X)$ such that $U \in H$ and $U \subseteq V \in \mathcal{O}(X)$ imply $V \in H$ and $\bigcup_{n\in\mathbb{N}} U_n \in H$ implies $\exists n. U_n \in H$ for any increasing sequence $(U_n)_n$ of opens.

missible w.r.t. the weak topology on $\mathcal{M}(X)$. The *weak topology* τ_{w} on $\mathcal{M}(X)$ is defined as the weakest topology such that, for all lower semicontinuous functions $f\colon X \to \mathbb{I}_<$, the function $\mu \mapsto \int f\,d\mu$ is lower semicontinuous. Integration of a lower semicontinuous function f w.r.t. to a Borel measure μ is defined by the Riemann integral

$$\int f d\mu := \int_0^1 \mu(\{x \in X \mid f(x) > t\})\,dt \tag{1}$$
$$= \sup\Big\{\sum_{i=1}^{k}(a_i - a_{i-1}) \cdot \mu(f^{-1}(a_i, \infty]) \,\Big|\, 0 = a_0 < a_1 < \ldots < a_k \leq 1\Big\}.$$

Note that the mapping $t \mapsto \mu(\{x \in X \mid f(x) > t\})$ is Riemann-integrable by being decreasing and right-continuous (cf. [1]). This *horizontal integral* is monotone and satisfies $\int f\,d\mu + \int g\,d\mu = 2\int (f+g)/2\,d\mu$ (cf. [6]).

The following characterisation of the weak topology, which we formulate for bases of hereditarily Lindelöf spaces, belongs to the folklore of measure theory.

Lemma 3.2 *Let X be a hereditarily Lindelöf space and $\mathcal{B}$ be a base of X closed under finite union. Then the family $\mathcal{D}$ of sets $\big\{\mu \in \mathcal{M}(X) \,\big|\, \mu(B) > q\big\}$, where $B \in \mathcal{B}$ and $q \in \mathbb{Q}$, is a subbase of the weak topology on X.*

Proof. Since we have $\int cf_U\,d\mu = \mu(U)$ for all $\mu \in \mathcal{M}(X)$ and $U \in \mathcal{O}(X)$, every element of the family $\mathcal{D}$ is contained in the weak topology.

Let $f \in \mathcal{C}(X, \mathbb{I}_<)$ and $\mu \in \mathcal{M}(X)$. For any rational number $z < \int f\,d\mu$, there are $k \geq 1$ and rational numbers $0 = a_0 < a_1 < \ldots < a_k \leq 1$, $b_1, \ldots, b_k$ with $\mu(f^{-1}(a_i, 1]) > b_i$ and $\sum_{i=1}^{k}(a_i - a_{i-1}) \cdot b_i \geq z$. Since X is hereditarily Lindelöf, the open set $f^{-1}(a_i, 1]$ is a countable union of elements in $\mathcal{B}$, hence by continuity of μ there is some $B_i \in \mathcal{B}$ with $B_i \subseteq f^{-1}(a_i, 1]$ and $\mu(B_i) > b_i$. It follows $\int f\,d\nu > z$ for all Borel measures $\nu \in \bigcap_{i=1}^{k}\big\{\eta \in \mathcal{M}(X) \,\big|\, \eta(B_i) > b_i\big\}$.$\Box$

With the help of this lemma, we show that $\delta^{\mathcal{M}_{\mathrm{C}}}$ is admissible w.r.t. to the weak topology on $\mathcal{M}(X)$.

Theorem 3.3 *Let δ be an admissible representation of a sequential space X. Then $\delta^{\mathcal{M}_{\mathrm{C}}}$ is admissible w.r.t. the weak topology on $\mathcal{M}(X)$.*

For the proof, we need the following proposition about admissibility of the canonical function space representation $[\delta \to \gamma]$. It follows from Lemma 4.2.2 and Proposition 4.2.5 in [8].

Proposition 3.4 *Let δ and γ be admissible representations of sequential spaces X and Y, respectively. Then $[\delta \to \gamma]$ is admissible w.r.t. to the simple topology on the set $\mathcal{C}(X, Y)$ of continuous functions between X and Y which* *has as its subbase the family of sets* $\{f \in \mathcal{C}(X,Y) \,|\, \forall n \leq \infty.\, f(x_n) \in V\}$, *where* $(x_n)_{n\leq\infty}$ *is a convergent sequence of* X *and* $V \in \mathcal{O}(Y)$.

Proof of Theorem 3.3: At first we show that $\delta^{\oplus}$ is indeed admissible w.r.t. the ω-Scott topology. From Proposition 3.4 it follows that $\delta^{\oplus}$ is admissible w.r.t. the topology τ_1 on $\mathcal{O}(X)$ which has as its subbase the family of all sets of the form

$$\{U \in \mathcal{O}(X) \,|\, \{x_\infty, x_n \,|\, n \in \mathbb{N}\} \subseteq U\}\,,$$

where $(x_n)_n$ is a sequence converging in X to some $x_\infty \in X$.

Let H be an ω-Scott open set and let $(U_n)_n$ be a sequence of opens converging with respect to τ_1 to some element $U_\infty \in H$. It is not difficult to verify that for any $m \in \mathbb{N}$ the set $U'_m := \bigcap_{j\geq m} U_j \cap U_\infty$ is sequentially open (and thus open in X, because X is sequential) and that $U_\infty = \bigcup_{m\in\mathbb{N}} U'_m$. Since H is ω-Scott open, there is some $m_0 \in \mathbb{N}$ with $U'_{m_0} \in H$ implying $U_n \in H$ for every $n \geq m_0$, because $U'_{m_0} \subseteq U_n$. Hence $(U_n)_n$ converges to U_∞ w.r.t. the ω-Scott topology. Since conversely the ω-Scott topology contains τ_1, it induces the same convergence relation on $\mathcal{O}(X)$ as τ_1. Thus $\delta^{\oplus}$ is admissible w.r.t. the ω-Scott topology as well. Note that the ω-Scott topology is known to be sequential, thus it is equal to the final topology of $\delta^{\oplus}$ by [7, Theorem 7].

By Proposition 3.4, $\delta^{\mathcal{M}_C}$ is admissible w.r.t. the topology τ_2 on $\mathcal{M}(X)$ that has as a subbase the family of all sets of the form

$$B_{(U_n),z} := \{\nu \in \mathcal{M}(X) \,|\, \forall n \leq \infty.\, \nu(U_n) > z\}\,,$$

where $z \in \mathbb{R}$ and $(U_n)_n$ is a sequence of opens that converges w.r.t. the ω-Scott topology to some $U_\infty \in \mathcal{O}(X)$. Clearly, τ_2 contains the weak topology as a subset. To show that the converse holds as well, let $\mu \in B_{(U_n),z}$. As mentioned above, the sets $U'_m := \bigcap_{j\geq m} U_j \cap U_\infty$ are open and satisfy $U_\infty = \bigcup_{m\in\mathbb{N}} U'_m$. Since $\mu|_{\mathcal{O}(m)}$ is continuous, there is some $m_0 \in \mathbb{N}$ with $\mu(U'_{m_0}) > z$. Obviously

$$\mu \in \{\nu \in \mathcal{M}(X) \,|\, \nu(U'_{m_0}) > z\} \cap \bigcap_{i<m_0} \{\nu \in \mathcal{M}(X) \,|\, \nu(U_i) > z\} \subseteq B_{(U_n),z}\,.$$

Thus $B_{(U_n),z}$ belongs to the weak topology, implying that the weak topology is equal to τ_2. Therefore $\delta^{\mathcal{M}_C}$ is admissible w.r.t. the weak topology. □

We do not know whether the weak topology on $\mathcal{M}(X)$ is sequential, which would imply that the weak topology is equal to the final topology of the canonical representation. However, for countably based spaces X the weak topology on $\mathcal{M}(X)$ is countably based by virtue of Lemma 3.2 and thus sequential. We obtain by [7, Theorem 7]:

Corollary 3.5 *For every admissible representation* δ *of a countably based space* X *the final topology of* $\delta^{\mathcal{M}_C}$ *is equal to the weak topology on* $\mathcal{M}(X)$.

The canonical representation $\delta^{\mathcal{M}_C}$ turns out to be $\leq_{\mathrm{cp}}$-complete in the set of all representations γ of $\mathcal{M}(X)$ that admit computability of horizontal integration. This generalises the corresponding result by K. Weihrauch for the unit interval (cf. [11, Theorem 2.6]).

Proposition 3.6 *Let γ be a representation of a subset $\mathcal{M}$ of $\mathcal{M}(X)$.*

- (i) *The integral operator $\int : \mathcal{C}(X, \mathbb{I}_<) \times \mathcal{M} \to \mathbb{I}_<$ is $([\delta \to \varrho_{\mathbb{I}_<}], \gamma, \varrho_{\mathbb{I}_<})$-computable if and only if $\gamma \leq_{\mathrm{cp}} \delta^{\mathcal{M}_C}$.*
- (ii) *The integral operator $\int : \mathcal{C}(X, \mathbb{I}_<) \times \mathcal{M} \to \mathbb{I}_<$ is $([\delta \to \varrho_{\mathbb{I}_<}], \gamma, \varrho_{\mathbb{I}_<})$-continuous if and only if $\gamma \leq_{\mathrm{t}} \delta^{\mathcal{M}_C}$.*

Proof. We only show (i), the proof of (ii) is similar.

"$\Longleftarrow$": It suffices to consider the case $\gamma = \delta^{\mathcal{M}_C}$. Let $f := [\delta \to \varrho_{\mathbb{I}_<}](p)$ and $\mu := \delta^{\mathcal{M}_C}(q)$. For any rational number $z < \int f\, d\mu$, there are $k \geq 1$ and rational numbers $0 = a_0 < a_1 < \ldots < a_k \leq 1$, $b_1, \ldots, b_k$ with $\mu(f^{-1}(a_i, 1]) > b_i$ and $\sum_{i=1}^{k}(a_i - a_{i-1}) \cdot b_i \geq z$. From the name p and any rational number a we can compute a $\delta^{\oplus}$-name r of the open set $f^{-1}(a, 1])$ and then from the names q and r some $\varrho_{\mathbb{I}_<}$-name s of $\mu(f^{-1}(a, 1])$. This name s effectively encodes a list of rationals below $\mu(f^{-1}(a, 1])$. Therefore we can effectively produce a list of rationals below $\int f\, d\mu$ and hence $\varrho_{\mathbb{I}_<}$-name of $\int f\, d\mu$. This means that $\int$ is $([\delta \to \varrho_{\mathbb{I}_<}], \delta^{\mathcal{M}_C}, \varrho_{\mathbb{I}_<})$-computable.

"$\Longrightarrow$": Let $\mu := \gamma(p)$ and $U := \delta^{\oplus}(q)$. Since $\mu(U) = \int cf_U\, d\mu$ and $[\delta \to \varrho_{\mathbb{I}_<}](q) = cf_U$, any computable realiser $g :\subseteq \Sigma^\omega \times \Sigma^\omega \to \Sigma^\omega$ of $\int$ also realises the function $(\mu, O) \mapsto \mu(O)$ w.r.t. γ, $\delta^{\oplus}$ and $\varrho_{\mathbb{I}_<}$. Hence we can compute a $[\delta^{\oplus} \to \varrho_{\mathbb{I}_<}]$-name of the valuation $\mu|_{\mathcal{O}(X)}$, which by definition is a $\delta^{\mathcal{M}_C}$-name of μ. □

3.2 The representation via probabilistic processes

We now define a representation $\delta^{\mathcal{M}_N}$ of $\mathcal{M}(X)$ based on probabilistic processes on infinite words. Probabilistic processes provide a natural way of generating Borel measures on represented spaces. The idea is to randomly generate an element p of Σ^ω by performing a sequence of independent choices. If the probability of p being in the domain of the considered representation is 1, then this equates to randomly choose an element of the represented space X, see [9] for details.

Formally, a *probabilistic process* is a function $\pi : \Sigma^* \to \mathbb{I}$ satisfying

$$\pi(\epsilon) = 1 \text{ and } \pi(w) = \sum_{a \in \Sigma} \pi(wa) \text{ for all } w \in \Sigma^* . \tag{2}$$

A probabilistic process π induces a Borel measure $\hat{\pi}$ on Σ^ω which is defined on the open sets of Σ^ω by

$$\hat{\pi}(W\Sigma^\omega) := \sum_{w\in W} \pi(w) \quad \text{for all prefix-free sets } W \subseteq \Sigma^*. \tag{3}$$

By Lemma 4 in [9], Equation (3) indeed determines a unique Borel measure on Σ^ω.

Given a Borel measure $\mu : \mathcal{B}(X) \to \mathbb{I}$ on X, we say that a probabilistic process π is a *δ-probabilistic name for μ*, if

$$\mu(B) = \hat{\pi}^*(\delta^{-1}[B]) \quad \text{holds for every Borel set } B \in \mathcal{B}(X)\,.$$

Remember that $\hat{\pi}^*$ denotes the outer measure generated by $\hat{\pi}$. If additionally $\Sigma^\omega \setminus \mathrm{dom}(\delta)$ is a $\hat{\pi}$-null set, i.e. $\hat{\pi}^*(\Sigma^\omega \setminus \mathrm{dom}(\delta)) = 0$, then π is called a *strong probabilistic name for μ.*

By Lemma 6 and Proposition 7 in [9], π is a δ-probabilistic name for μ if and only if for every $U \in \mathcal{O}(X)$ and $V \in \mathcal{O}(\Sigma^\omega)$

$$V \cap \mathrm{dom}(\delta) = \delta^{-1}[U] \ \text{ implies } \ \hat{\pi}(V) = \mu(U)\,.$$

In particular this implies $\hat{\pi}^*(\mathrm{dom}(\delta)) = 1$. Conversely, if $\hat{\pi}^*(\mathrm{dom}(\delta))$ is equal to 1, then $B \mapsto \hat{\pi}^*(\delta^{-1}[B])$ yields a Borel measure on X. We denote by $\mathcal{M}_{\mathrm{N}}(X)$ (by $\mathcal{M}_{\mathrm{S}}(X)$) the set of Borel measures that have an ordinary (respectively strong) δ-probabilistic name. Note that both sets $\mathcal{M}_{\mathrm{N}}(X)$ and $\mathcal{M}_{\mathrm{S}}(X)$ are independent of the choice of the admissible representation δ by [9, Corollary 12]. It is not known whether every Borel measure on any admissibly representable space has a probabilistic name. However, if X is countably based or, more generally, if X has a countable pseudobase consisting of universally measurable sets, then $\mathcal{M}_{\mathrm{N}}(X)$ is equal to $\mathcal{M}(X)$ by [9, Proposition 13]. By the following lemma, $\mathcal{M}_{\mathrm{S}}(X)$ is not necessarily equal to $\mathcal{M}_{\mathrm{N}}(X)$.

Lemma 3.7 *A subspace Y of Σ^ω satisfies $\mathcal{M}_{\mathrm{S}}(Y) = \mathcal{M}_{\mathrm{N}}(Y)$ if and only if Y is a universally measurable subset of Σ^ω.*

Proof. We define $\gamma :\subseteq \Sigma^\omega \to Y$ defined by $\mathrm{dom}(\gamma) = Y$ and $\gamma(p) = p$. Obviously, γ is an admissible representation of Y.

Let Y be universally measurable. Any γ-probabilistic name π for a Borel measure μ on Y is simulaneously a strong γ-probabilistic name for Y, because $\hat{\pi}^*(\Sigma^\omega \setminus \mathrm{dom}(\gamma)) = \hat{\pi}^*(\Sigma^\omega \setminus Y) = 1 - \hat{\pi}^*(Y) = 0$, as Y is $\hat{\pi}$-measurable. Thus $\mathcal{M}_{\mathrm{S}}(Y) = \mathcal{M}_{\mathrm{N}}(Y)$.

Now let μ be a Borel measure on Σ^ω such that Y is not μ-measurable. Choose some Borel set $G \subseteq \mathcal{B}(\Sigma^\omega)$ with $Y \subseteq G$ and $\mu^*(Y) = \mu(G)$. Then $\mu(G) > 0$, because otherwise Y would be measurable by being a null-set. Obviously, $\nu : \mathcal{B}(\Sigma^\omega) \to \mathbb{I}$ defined by $\nu(B) := \mu(B \cap G)/\mu(G)$ is a Borel measure on Σ^ω with $\nu^*(Y) = \nu(\Sigma^\omega) = 1$.

Assume $\nu^*(\Sigma^\omega \setminus Y) = 0$. Then there is some $N \in \mathcal{B}(\Sigma^\omega)$ with $\Sigma^\omega \setminus Y \subseteq N$

and $\nu(N) = 0$, hence $\mu(N \cap G) = 0$. Since $\Sigma^\omega \setminus N \subseteq Y \subseteq G$ and $\mu(\Sigma^\omega \setminus N) = 1 - \mu(N) = \mu(N \cup G) - \mu(N) = \mu(G) - \mu(N \cap G) = \mu(G)$, this implies that Y is μ-measurable, a contradiction.

Hence $\nu^*(\Sigma^\omega \setminus Y) + \nu^*(Y) > 1$ meaning that Y is not ν-measurable. Clearly, the function $\pi : \Sigma^* \to \mathbb{I}$ with $\pi(w) := \nu(w\Sigma^\omega)$ is a probabilistic process satisfying $\hat{\pi} = \nu$. By Lemma 6 and Theorem 8 in [9], π is a γ-probabilistic name of the Borel measure $\eta : \mathcal{B}(Y) \to \mathbb{I}$ defined by $\eta(B) := \hat{\pi}^*(\gamma^{-1}[B])$. Since γ is injective, π is the only γ-probabilistic name of η. Hence η does not have any strong γ-probabilistic name. We conclude $\mathcal{M}_{\mathrm{S}}(Y) \neq \mathcal{M}_{\mathrm{N}}(Y)$. □

Using the ambient representations ϱ_{Σ^*} of Σ^* and $\varrho_{\mathbb{I}_=}$ of $\mathbb{I}_=$, we define representations $\delta^{\mathcal{M}_{\mathrm{N}}}$ of $\mathcal{M}_{\mathrm{N}}(X)$ and $\delta^{\mathcal{M}_{\mathrm{S}}}$ of $\mathcal{M}_{\mathrm{S}}(X)$ by

$$\delta^{\mathcal{M}_{\mathrm{N}}}(p) = \mu \;:\Longleftrightarrow\; [\varrho_{\Sigma^*} \to \varrho_{\mathbb{I}_=}](p) \text{ is a } \delta\text{-probabilistic name for } \mu,$$
$$\delta^{\mathcal{M}_{\mathrm{S}}}(p) = \mu \;:\Longleftrightarrow\; [\varrho_{\Sigma^*} \to \varrho_{\mathbb{I}_=}](p) \text{ is a strong } \delta\text{-probabilistic name for } \mu\,.$$

Trivially, $\delta^{\mathcal{M}_{\mathrm{S}}} \leq_{\mathrm{cp}} \delta^{\mathcal{M}_{\mathrm{N}}}$. If $\mathrm{dom}(\delta)$ is universally measurable, then both representations are equal. Proposition 10 in [9] implies that both operators preserve computable equivalence of representations. With similar proofs, one can show preservation of topological equivalence.

Proposition 3.8 *The operators $\gamma \mapsto \gamma^{\mathcal{M}_{\mathrm{N}}}$ and $\gamma \mapsto \gamma^{\mathcal{M}_{\mathrm{S}}}$ preserve computable as well as topological equivalence of representations.*

4 Comparison of the Representations of Borel Measures

In this section we will investigate the relationship between the canonical representation of $\mathcal{M}(X)$ and the representation via probabilistic names. In particular, we will show that they are topologically equivalent for countably based T_0-spaces.

From Theorem 16 in [9] we know that the integral operator $\int : \mathcal{C}(X, \mathbb{I}_<) \times \mathcal{M}_{\mathrm{N}}(X) \to \mathbb{I}_<$ is $([\delta \to \varrho_{\mathbb{I}_<}], \delta^{\mathcal{M}_{\mathrm{N}}}, \varrho_{\mathbb{I}_<})$-computable. By Proposition 3.6 we obtain:

Proposition 4.1 *Let δ be an admissible representation of a sequential space X. Then $\delta^{\mathcal{M}_{\mathrm{N}}}$ is computably reducible to $\delta^{\mathcal{M}_{\mathrm{C}}}$, i.e. $\delta^{\mathcal{M}_{\mathrm{N}}} \leq_{\mathrm{cp}} \delta^{\mathcal{M}_{\mathrm{C}}}$.*

As consequence we obtain that, in the case $\mathcal{M}_{\mathrm{N}}(X) = \mathcal{M}(X)$, the final topology of $\delta^{\mathcal{M}_{\mathrm{N}}}$ is finer than (or equal to) the weak topology on $\mathcal{M}(X)$.

4.1 The Case of Countably Based Spaces

We now work towards showing that for countably based spaces the canonical representation and the representation via ordinary probabilistic names are topologically equivalent. The idea is to prove this equivalence at first for Scott's graph model $\mathbb{P}$. The underlying set of $\mathbb{P}$ is the family of all subsets of $\mathbb{N}$, topologised by the Scott topology $\mathcal{O}(\mathbb{P})$ on the dcpo $(\mathbb{P}, \subseteq)$. The family of sets $\uparrow E := \{y \in \mathbb{P} \,|\, E \subseteq y\}$, where E is a finite subset of $\mathbb{N}$, forms a base of the Scott topology on $\mathbb{P}$. We equip $\mathbb{P}$ with a representation $\varrho_{\mathbb{P}} : \Sigma^\omega \to \mathbb{P}$ defined by

$$\varrho_{\mathbb{P}}(p) := \big\{m \in \mathbb{N} \,\big|\, \exists i \in \mathbb{N}.\, p\langle i, m\rangle = 1\big\}\,,$$

where $\langle \cdot, \cdot \rangle : \mathbb{N}^2 \to \mathbb{N}$ denotes the computable bijection defined by $\langle a, b\rangle := b + \sum_{i=1}^{a+b} i$. It is admissible: continuity of $\varrho_{\mathbb{P}}$ is obvious; for a given continuous function $\phi :\subseteq \Sigma^\omega \to \mathbb{P}$, the continuous function $g : \Sigma^\omega \to \Sigma^\omega$ defined by

$$g(p)(\langle a, b\rangle) := \begin{cases} 1 \text{ if } \phi(q) \in \uparrow\{b\} \text{ for all } q \in dom(\phi) \text{ with } \forall i \leq a.\, q(i) = p(i) \\ 0 \text{ otherwise} \end{cases}$$

translates ϕ into $\varrho_{\mathbb{P}}$.

The space $\mathbb{P}$ is known to be universal for the class of countably based T_0-spaces in the sense that each of them is isomorphic to some subspace of $\mathbb{P}$. Given a numbering $\beta : \mathbb{N} \to B$ of a countable subbase B of X, an embedding $\iota_\beta : X \to \mathbb{P}$ can be defined by $\iota_\beta(x) := \{i \in \mathbb{N} \,|\, x \in \beta(i)\}$. This embedding leads to a representation $\delta_\beta := \iota_\beta^{-1} \circ \varrho_{\mathbb{P}}$ of X which can easily be verified to be admissible w.r.t. the topology of X. A representation of X constructed in this way is called a *standard representation* of X.

For the the proof of $\varrho_{\mathbb{P}}^{\mathcal{M}_{\mathrm{C}}} \leq_{\mathrm{cp}} \varrho_{\mathbb{P}}^{\mathcal{M}_{\mathrm{N}}}$ we need the following Splitting Lemma (cf. [3]), which we formulate for rational numbers rather than real numbers.

Lemma 4.2 (Splitting Lemma) *Let $(r_a)_{a\in A}$ and $(s_b)_{b\in B}$ be two finite sequences of non-negative rational numbers and $Z \subseteq A \times B$ be a relation such that, for all $I \subseteq A$,*

$$\sum_{i\in I} r_i \leq \sum\{s_j \,|\, \exists i \in I.\, (i, j) \in Z\}\,. \tag{4}$$

Then one can compute non-negative rationals $(t_{i,j})_{(i,j)\in Z}$ satisfying

$$\sum_{(a,j)\in Z} t_{a,j} = r_a \quad \text{and} \quad \sum_{(i,b)\in Z} t_{i,b} \leq s_b \tag{5}$$

for all $a \in A$ and $b \in B$. The algorithm produces dyadic rationals, if the input consists of dyadic rationals.

Proof. [Sketch] This Splitting Lemma follows from the computable Max-Flow Min-Cut Theorem, see [5] for details. Assuming that A, B and $\{\bot,\top\}$ are disjoint, one considers the directed graph which has $A\cup B\cup\{\bot,\top\}$ as its set of vertices and $E := (\{\bot\}\times A)\cup Z\cup(B\times\{\top\})$ as its set of edges. The capacity assigned to an edge $(\bot,a)$ is r_a, to an edge $(a,b)\in Z$ is ∞, and to an edge $(b,\top)$ is s_b. Condition (4) ensures that the minimal cut of this graph (with $\bot$ being the source and $\top$ being the sink) is $\sum_{a\in A} r_a$. By the computable Max-Flow Min-Cut Theorem one can compute a feasible flow $t : E \to \mathbb{Q}\cap[0,\infty)$ through the graph with maximal value $\sum_{a\in A} r_a$. The numbers $(t(i,j))_{(i,j)\in Z}$ are easily verified to satisfy Condition (5). □

Proposition 4.3 *The representations* $\varrho_{\mathbb{P}}^{\mathcal{M}_{\mathrm{N}}}$, $\varrho_{\mathbb{P}}^{\mathcal{M}_{\mathrm{S}}}$ *and* $\varrho_{\mathbb{P}}^{\mathcal{M}_{\mathrm{C}}}$ *are computably equivalent.*

Proof. From Proposition 4.1 we already know $\varrho_{\mathbb{P}}^{\mathcal{M}_{\mathrm{N}}} \leq_{\mathrm{cp}} \varrho_{\mathbb{P}}^{\mathcal{M}_{\mathrm{C}}}$. Since the complement of the domain of $\varrho_{\mathbb{P}}$ has measure 0 by being empty, $\varrho_{\mathbb{P}}^{\mathcal{M}_{\mathrm{S}}}$ and $\varrho_{\mathbb{P}}^{\mathcal{M}_{\mathrm{N}}}$ are equal. The difficult part of the proof is to show $\varrho_{\mathbb{P}}^{\mathcal{M}_{\mathrm{C}}} \leq_{\mathrm{cp}} \varrho_{\mathbb{P}}^{\mathcal{M}_{\mathrm{N}}}$.

Idea of the proof of $\varrho_{\mathbb{P}}^{\mathcal{M}_{\mathrm{C}}} \leq_{\mathrm{cp}} \varrho_{\mathbb{P}}^{\mathcal{M}_{\mathrm{N}}}$

Given a name p of a Borel measure $\mu = \varrho_{\mathbb{P}}^{\mathcal{M}_{\mathrm{C}}}(p)$ under the canonical representation, we construct at first for every $n\in\mathbb{N}$ a valuation ν_n on the smallest algebra $\mathcal{A}_n$ on $\mathbb{P}$ containing $\big\{\uparrow\{0\},\dots,\uparrow\{n\}\big\}$ such that [3]

$$\nu_n(B) \leq \nu_{n+1}(B) \lessdot \mu(B) \quad \text{and} \quad \sup_{i\geq n} \nu_n(B) = \mu(B) \tag{6}$$

holds for every open set $B \in \mathcal{B}_n := \mathcal{A}_n \cap \mathcal{O}(\mathbb{P})$. From the sequence $(\nu_n)_n$ we then compute a probabilistic process $\pi_p : \Sigma^* \to \mathbb{D}\cap\mathbb{I}$ satisfying

$$\nu_k(B) = \sum \big\{\pi_p(u) \,\big|\, u \in \Sigma^{\ell(k)} \text{ and } \uparrow(u^+) \subseteq B\big\} \tag{7}$$

for all $B\in\mathcal{B}_k$, where $\ell(k) := \sum_{i=1}^{k+1} i$ and

$$u^+ := \varrho_{\mathbb{P}}(u0^\omega) = \big\{m\in\mathbb{N} \,\big|\, \exists i\in\mathbb{N}.(\langle i,m\rangle < \lg(u),\, u\langle i,m\rangle = 1)\big\}\,.$$

This probabilistic process π_p turns out to be a $\varrho_{\mathbb{P}}$-probabilistic name for μ.

Note that ℓ is chosen in such a way that for all $F\subseteq\{0,\dots,k+1\}$ there is exactly one word $w\in\Sigma^{k+2}$ such that for all $u\in\Sigma^{\ell(k)}$ we have $uw\in\Sigma^{\ell(k+1)}$ and $(uw)^+ = u^+\cup F$. This enables us to construct π_p in such a way that Property (7) is satisfied. The algebra $\mathcal{A}_n$ is obtained by closing the family $\mathcal{C}_n := \big\{C_F^n \,\big|\, F\subseteq\{0,\dots,n\}\big\}$ of prime crescents

$$C_F^n := \big(\textstyle\bigcap_{i\in F}\uparrow\{i\}\big) \setminus \big(\bigcup_{i\in\{0,\dots,n\}\setminus F}\uparrow\{i\}\big)$$

[3] The relation $\lessdot$ is defined by $a \lessdot b \;:\Longleftrightarrow\; a<b \,\vee\, a=b=0$.

under finite union.

Construction of the valuations ν_n

In order to construct the valuations ν_n, we at first compute from the input name p of the Borel measure μ for every open $B \in \bigcup_{n\in\mathbb{N}} \mathcal{B}_n$ a sequence $(d_{B,n})_{n\in\mathbb{N}}$ of dyadic rationals approximating $\mu(B)$ from below, i.e. $(d_{B,n})_{n\in\mathbb{N}}$ satisfies

$$0 \leq d_{B,n} \leq d_{B,n+1} \lessdot \mu(B) \quad \text{and} \quad \sup_{i\in\mathbb{N}} d_{B,i} = \mu(B)\,. \tag{8}$$

We sketch how this can be achieved: Given some $r \in \mathrm{dom}(\varrho_{\mathbb{P}})$, we can semi-decide $\varrho_{\mathbb{P}}(r) \in B$ by searching a prefix w of r satisfying $\uparrow(w^+) \subseteq B$. This implies that we can produce a $[\varrho_{\mathbb{P}} \to \varrho_{\mathbb{I}_<}]$-name of B. The $\varrho_{\mathbb{P}}^{\mathcal{M}_{\mathrm{C}}}$-name p allows us the computation of a $\varrho_{\mathbb{I}_<}$-name of $\mu(B)$. The latter codes a sequence $(r_{B,n})_n$ of rationals converging to $\mu(B)$ from below. From $(r_{B,n})_n$ we can extract in a computable way a sequence $(d_{B,n})_n$ with the required properties. Note that $(d_{B,n})_{B,n}$ depends on the name p, not only on the valuation μ encoded by p.

From $(d_{B,n})_{B,n}$, we construct the sequence $(\nu_n)_n$ recursively as follows.

"$n = 0$": We set $\nu_0(\uparrow\{0\}) := d_{\uparrow\{i\},0}$ and $\nu_0(\mathbb{P} \setminus \uparrow\{i\}) := 1 - \nu_0(\uparrow\{i\})$.

"$n \to n+1$": We construct ν_{n+1} from ν_n by searching a finite function $q : \mathcal{C}_{n+1} \to \mathbb{D}$ and a number $m \geq n+1$ satisfying

(a) $d_{B,n+1} \leq \sum\{q(C) \,|\, C \in \mathcal{C}_{n+1},\, C \subseteq B\} \leq d_{B,m}$ for all $B \in \mathcal{B}_{n+1}$,

(b) $\nu_n(B) \leq \sum\{q(C) \,|\, C \in \mathcal{C}_{n+1},\, C \subseteq B\}$ for all $B \in \mathcal{B}_n$ and

(c) $\sum_{C\in\mathcal{C}_{n+1}} q(C) = 1$

and by setting $\nu_{n+1}(M) := \sum\{q(C) \,|\, C \in \mathcal{C}_{n+1},\, C \subseteq M\}$ for $M \in \mathcal{A}_{n+1}$. Note that Conditions (a) and (b) imply Property (6), whereas Condition (c) ensures that ν_{n+1} is probabilistic.

To show that such a pair (q, m) exists, choose a number $t \in (0, 1)$ satisfying $d_{B,n+1} \leq t \cdot \mu(B)$ for all $B \in \mathcal{B}_{n+1}$ and $\nu_n(B) \leq t \cdot \mu(B)$ for all $B \in \mathcal{B}_n$. For every prime crescent $C \in \mathcal{C}_{n+1} \setminus \{C_\emptyset^{n+1}\}$ we choose a dyadic rational number $q(C)$ with $t \cdot \mu(C) \leq q(C) \lessdot \mu(C)$ and set $q(C_\emptyset^{n+1}) := 1 - \sum_{C\neq C_\emptyset^{n+1}} q(C) \geq 0$. One easily verifies that q satisfies (b), (c) and $d_{B,n+1} \leq \sum_{C\subseteq B} q(C) \lessdot \mu(B)$ for all $B \in \mathcal{B}_{n+1}$. Since $\mathcal{B}_{n+1}$ is finite and $\sup_{i\in\mathbb{N}} d_{B,i} = \mu(B)$, there is some m satisfying (a).

Construction of the probabilistic process π_p

We construct π_p recursively and define in step k the values $\pi_p(w)$ for words w with $\ell(k-1) < \lg(w) \leq \ell(k)$.

"$k = 0$": Obviously, $\Sigma^{\ell(0)} = \{\mathtt{1}, \mathtt{0}\}$, $\mathcal{C}_0 = \big\{\uparrow\{0\}, \mathbb{P}\setminus\uparrow\{0\}\big\}$ and $\mathcal{A}_0 = \{\emptyset, \mathbb{P}\} \cup \mathcal{C}_0$. We set $\pi_p(\mathtt{1}) := \nu_0(\uparrow\{0\})$, $\pi_p(\mathtt{0}) := 1 - \pi_p(\mathtt{1})$ and $\pi_p(\epsilon) := 1$.

"$k \to k+1$": In order to apply the Splitting Lemma, we define a relation $Z \subseteq \Sigma^{\ell(k)} \times \mathcal{C}_{k+1}$ by $(u, C) \in Z :\Longleftrightarrow \uparrow(u^+) \supseteq C$ and set $r_u := \pi_p(u)$ and $s_C := \nu_{k+1}(C)$. Let $I \subseteq \Sigma^{\ell(k)}$. Since $B := \bigcup\big\{\uparrow(u^+)\,\big|\, u \in I\big\} \in \mathcal{B}_n$, we obtain

$$\begin{aligned}
\textstyle\sum_{u\in I} r_u &\leq \textstyle\sum \big\{\pi_p(u)\,\big|\, u \in \Sigma^{\ell(k)}, \uparrow(u^+) \subseteq B\big\} = \nu_k(B) \\
&\leq \nu_{k+1}(B) = \textstyle\sum \big\{\nu_{k+1}(C)\,\big|\, C \in \mathcal{C}_{k+1},\, C \subseteq B\big\} \\
&= \textstyle\sum \big\{\nu_{k+1}(C)\,\big|\, C \in \mathcal{C}_{k+1},\, \exists u \in I.\, C \subseteq \uparrow(u^+)\big\} \\
&= \textstyle\sum \big\{s_C\,\big|\, \exists u \in I.\, (u, C) \in Z\big\}.
\end{aligned}$$

By the Splitting Lemma 4.2, we can compute dyadic rationals $(t_{u,C})_{(u,C)\in Z}$ such that

$$\sum_{(u,D)\in Z} t_{u,D} = \pi_p(u) \quad \text{and} \quad \sum_{(v,C)\in Z} t_{v,C} \leq \nu_{k+1}(C) \tag{9}$$

for all $u \in \Sigma^{\ell(k)}$ and $C \in \mathcal{C}_{k+1}$. Since

$$\begin{aligned}
1 &= \textstyle\sum \big\{\nu_{k+1}(C) \,|\, C \in \mathcal{C}_{k+1}\big\} \geq \textstyle\sum \big\{ \sum_{(u,C)\in Z} t_{u,C}\,\big|\, C \in \mathcal{C}_{k+1}\big\} \\
&= \textstyle\sum \big\{ \sum_{(u,C)\in Z} t_{u,C}\,\big|\, u \in \Sigma^{\ell(k)}\big\} = \textstyle\sum\{\pi_p(u) \,|\, u \in \Sigma^{\ell(k)}\} = 1\,,
\end{aligned}$$

we have even

$$\sum_{(v,C)\in Z} t_{v,C} = \nu_{k+1}(C) \qquad \text{for all } C \in \mathcal{C}_{k+1}. \tag{10}$$

Using $w^\oplus := \{i < \lg(w) \,|\, w(i) = \mathtt{1}\}$, we define π_p on $\Sigma^{\ell(k+1)}$ unambiguously by

$$\pi_p(uw) := \begin{cases} t_{u,C^{k+1}_{w^\oplus}} \text{ if } u^+ \subseteq w^\oplus \\[2ex] \quad 0 \quad\; \text{otherwise} \end{cases}$$

for $u \in \Sigma^{\ell(k)}$ and $w \in \Sigma^{k+2}$. Moreover, for words v with $\ell(k) < \lg(v) < \ell(k+1)$ we set

$$\pi_p(v) := \textstyle\sum \big\{\pi_p(vw)\,\big|\, vw \in \Sigma^{\ell(k+1)}\big\}.$$

For $u \in \Sigma^{\ell(k)}$ and $w \in \Sigma^{k+2}$ we have $u^+ \subseteq w^\oplus \iff (u, C^{k+1}_{w^\oplus}) \in Z$, $uw \in \Sigma^{\ell(k+1)}$ and $(uw)^+ = u^+ \cup w^\oplus$. Equation (9) yields

$$\begin{aligned}\pi_p(u) &= \sum \left\{t_{u,C} \,\middle|\, (u, C) \in Z\right\} = \sum \left\{t_{u,C^{k+1}_{w^\oplus}} \,\middle|\, w \in \Sigma^{k+2},\, u^+ \subseteq w^\oplus\right\} \\ &= \sum \left\{\pi_p(uw) \,\middle|\, w \in \Sigma^{k+2},\, u^+ \subseteq w^\oplus\right\} = \sum \left\{\pi_p(uw) \,\middle|\, w \in \Sigma^{k+2}\right\} \\ &= \sum \left\{\pi_p(uw) \,\middle|\, uw \in \Sigma^{\ell(k+1)}\right\}.\end{aligned}$$

This implies $\pi_p(v) = \sum_{a \in \Sigma} \pi_p(va)$ for all words v with $\lg(v) < \ell(k+1)$, hence π_p is a probabilistic process.

For every prime crescent $C \in \mathcal{C}_{k+1}$ there is exactly one word $x_C \in \Sigma^{k+2}$ with $C = C^{k+1}_{x_C^\oplus}$. We obtain by Equation (10)

$$\begin{aligned}\nu_{k+1}(C) &= \sum \left\{t_{u,C} \,\middle|\, (u, C) \in Z\right\} = \sum \left\{\pi_p(ux_C) \,\middle|\, u \in \Sigma^{\ell(k)},\, u^+ \subseteq x_C^\oplus\right\} \\ &= \sum \left\{\pi_p(uw) \,\middle|\, u \in \Sigma^{\ell(k)},\, w \in \Sigma^{k+2},\, (uw)^+ = x_C^\oplus\right\} \\ &= \sum \left\{\pi_p(v) \,\middle|\, v \in \Sigma^{\ell(k+1)},\, v^+ = x_C^\oplus\right\} = \sum \left\{\pi_p(v) \,\middle|\, v \in \Sigma^{\ell(k+1)},\, C^{k+1}_{v^+} = C\right\}.\end{aligned}$$

Hence

$$\begin{aligned}&\nu_{k+1}(B) = \sum \left\{\nu_{k+1}(C) \,\middle|\, C \in \mathcal{C}_{k+1},\, C \subseteq B\right\} \\ &= \sum \left\{\pi_p(v) \,\middle|\, v \in \Sigma^{\ell(k+1)},\, C^{k+1}_{v^+} \subseteq B\right\} = \sum \left\{\pi_p(v) \,\middle|\, v \in \Sigma^{\ell(k+1)},\, \uparrow(v^+) \subseteq B\right\}\end{aligned}$$

for all $B \in \mathcal{B}_{k+1}$, guaranteeing Equation (7).

Proof of π_p being a probabilistic name for μ

Let $U \in \mathcal{O}(X)$ and $V := \varrho_{\mathbb{P}}^{-1}[U]$. For $n \in \mathbb{N}$ let $U_n := \bigcup\left\{\uparrow F \,\middle|\, F \subseteq \{0, \ldots, n\},\, \uparrow F \subseteq U\right\}$, $V_n := \bigcup\left\{w\Sigma^\omega \,\middle|\, w \in \Sigma^{\ell(n)},\, \uparrow(w^+) \subseteq U\right\}$. and $V = \bigcup_{n \in \mathbb{N}} V_n$. Clearly $U = \bigcup_{n \in \mathbb{N}} U_n$, $U_n \in \mathcal{B}_n$, $V = \bigcup_{n \in \mathbb{N}} V_n$, and $\hat{\pi}_p(V) = \sup_{n \in \mathbb{N}} \hat{\pi}_p(V_n)$. By (7) we have

$$\begin{aligned}\nu_n(U_n) &= \sum \left\{\pi_p(w) \,\middle|\, w \in \Sigma^{\ell(n)},\, \uparrow(w^+) \subseteq U_n\right\} \\ &= \sum \left\{\pi_p(w) \,\middle|\, w \in \Sigma^{\ell(n)},\, \uparrow(w^+) \subseteq U\right\} = \hat{\pi}_p(V_n).\end{aligned}$$

Hence $\hat{\pi}_p(V_n) \leq \mu(U_n) \leq \mu(U)$ implying $\hat{\pi}_p(V) \leq \mu(U)$. Now let $\varepsilon > 0$. Since $\mu|_{\mathcal{O}(X)}$ is a continuous valuation, there is some $m \in \mathbb{N}$ such that $\mu(U_m) \geq \mu(U) - \varepsilon/2$. By (6) there is some $n \geq m$ with $\nu_n(U_m) > \mu(U_m) - \varepsilon/2$. Hence

$$\hat{\pi}_p(V) \geq \hat{\pi}_p(V_n) = \nu_n(U_n) \geq \nu_n(U_m) \geq \mu(U) - \varepsilon.$$

We conclude $\hat{\pi}_p(V) = \mu(U)$. Therefore π_p is $\varrho_{\mathbb{P}}$-probabilistic name for μ.

Computability of $p \mapsto \pi_p$

The function $(p, w) \mapsto \pi_p(w)$ is computable, because $\pi_p(w)$ only depends on finitely many values of the sequences $(d_{B,n})_{B,n}$ and $(\nu_n)_n$, which are computable in p, B, n. Hence there is a computable function $g :\subseteq \Sigma^\omega \to \Sigma^\omega$ with $\pi_p = [\varrho_{\Sigma^*} \to \varrho_{\mathbb{I}_<}](g(p))$. Since π_p is a probablistic name of $\mu = \varrho_{\mathbb{P}}^{\mathcal{M}_{\mathrm{C}}}(p)$, g translates $\varrho_{\mathbb{P}}^{\mathcal{M}_{\mathrm{C}}}$ into $\varrho_{\mathbb{P}}^{\mathcal{M}_{\mathrm{N}}}$. This concludes the proof of $\varrho_{\mathbb{P}}^{\mathcal{M}_{\mathrm{C}}} \le_{\mathrm{cp}} \varrho_{\mathbb{P}}^{\mathcal{M}_{\mathrm{N}}}$. □

For transferring the equivalence result from $\mathbb{P}$ to arbitrary countably based spaces, we need the property that a probabilistic names π for a measure μ satisfies $\mu^*(M) = \hat{\pi}^*(\varrho_{\mathbb{P}}^{-1}[M])$ for all subsets M of $\mathbb{P}$, not only for Borel sets.

Lemma 4.4 *Let $\pi : \Sigma^* \to \mathbb{I}$ be a $\varrho_{\mathbb{P}}$-probabilistic name for a Borel measure μ on $\mathbb{P}$. Then $\mu^*(M) = \hat{\pi}^*(\varrho_{\mathbb{P}}^{-1}[M])$ for all $M \subseteq \mathbb{P}$.*

Proof. Let $M \subseteq \mathbb{P}$. There are Borel sets $A \in \mathcal{B}(\mathbb{P})$ and $B \in \mathcal{B}(\Sigma^\omega)$ with $M \subseteq A$, $\mu(A) = \mu^*(M)$, $\varrho_{\mathbb{P}}^{-1}[M] \subseteq B$ and $\hat{\pi}(B) = \hat{\pi}^*(\varrho_{\mathbb{P}}^{-1}[M])$. The graph of $\varrho_{\mathbb{P}}$ is a Borel set of $\Sigma^\omega \times \mathbb{P}$, because

$$\mathrm{Graph}(\varrho_{\mathbb{P}}) = \bigcap_{i \in \mathbb{N}} \left(\varrho_{\mathbb{P}}^{-1}[\uparrow\{i\}] \times \uparrow\{i\} \,\cup\, \varrho_{\mathbb{P}}^{-1}[\mathbb{P} \setminus \uparrow\{i\}] \times (\mathbb{P} \setminus \uparrow\{i\}) \right).$$

Hence the set

$$\varrho_{\mathbb{P}}[\Sigma^\omega \setminus B] = \left\{ z \in \mathbb{P} \,\middle|\, \exists p \in \Sigma^\omega.\, (p, z) \in \mathrm{Graph}(\varrho_{\mathbb{P}}) \cap (\Sigma^\omega \setminus B) \times \mathbb{P} \right\}$$

is the projection of a Borel set of $\Sigma^\omega \times \mathbb{P}$ onto $\mathbb{P}$. Since Σ^ω is a separable complete metric space, $\varrho_{\mathbb{P}}[\Sigma^\omega \setminus B]$ is a universally measurable subset of $\mathbb{P}$ by [1, Proposition 8.4.4]. Thus there are Borel sets $C, D \in \mathbb{P}$ with $C \subseteq \varrho_{\mathbb{P}}[\Sigma^\omega \setminus B] \subseteq D$ and $\mu(C) = \mu(D)$. Since $M \subseteq \mathbb{P} \setminus \varrho_{\mathbb{P}}[\Sigma^\omega \setminus B]$ and $\varrho_{\mathbb{P}}^{-1}[\mathbb{P} \setminus D] \subseteq B$, it follows

$$\begin{aligned} \mu^*(M) \le \mu^*(\mathbb{P} \setminus \varrho_{\mathbb{P}}[\Sigma^\omega \setminus B]) \le \mu(\mathbb{P} \setminus C) = \mu(\mathbb{P} \setminus D) = \hat{\pi}^*(\varrho_{\mathbb{P}}^{-1}[\mathbb{P} \setminus D]) \\ \le \hat{\pi}(B) = \hat{\pi}^*(\varrho_{\mathbb{P}}^{-1}[M]) \le \hat{\pi}^*(\varrho_{\mathbb{P}}^{-1}[A]) = \mu(A) = \mu^*(M), \end{aligned}$$

hence $\mu^*(M) = \hat{\pi}^*(\varrho_{\mathbb{P}}^{-1}[M])$. □

Now we are ready to prove:

Theorem 4.5 *For a standard representation δ_β of a countably based T_0-space X, the representations $\delta_\beta^{\mathcal{M}_{\mathrm{N}}}$ and $\delta_\beta^{\mathcal{M}_{\mathrm{C}}}$ are computably equivalent.*

Proof. From Proposition 4.1 we already know $\delta_\beta^{\mathcal{M}_{\mathrm{N}}} \le_{\mathrm{cp}} \delta_\beta^{\mathcal{M}_{\mathrm{C}}}$.

Let $\mu : \mathcal{B}(X) \to \mathbb{I}$ be a Borel measure on X. Then $\hat{\mu} : \mathcal{B}(\mathbb{P}) \to \mathbb{I}$ defined by $\tilde{\mu}(D) := \mu(\iota_\beta^{-1}[D])$ is a Borel measure on $\mathbb{P}$. We show that any $\varrho_{\mathbb{P}}$-probabilistic name π for $\tilde{\mu}$ is simultaneously a δ_β-probabilistic name for μ. Let $B \in \mathcal{B}(\mathbb{P})$.

There is some Borel set $B^\# \in \mathcal{B}(\mathbb{P})$ with $B = \iota_\beta^{-1}[B^\#]$. We calculate

$$\begin{aligned}\tilde{\mu}^*(\iota_\beta[B]) &= \inf\left\{\tilde{\mu}(D) \,\middle|\, D \in \mathcal{B}(\mathbb{P}),\, D \supseteq \iota_\beta[B]\right\}\\ &= \inf\left\{\mu(\iota_\beta^{-1}[D]) \,\middle|\, D \in \mathcal{B}(\mathbb{P}),\, D \supseteq \iota_\beta[B]\right\}\\ &\geq \mu(B) = \tilde{\mu}(B^\#) \geq \tilde{\mu}^*(\iota_\beta[B])\,.\end{aligned}$$

By Lemma 4.4 we obtain $\hat{\pi}^*(\delta_\beta^{-1}[B]) = \hat{\pi}^*(\varrho_{\mathbb{P}}^{-1}[\iota_\beta[B]]) = \tilde{\mu}^*(\iota_\beta[B]) = \mu(B)$. Therefore π is a δ_β-probabilistic name for μ.

We conclude that any $\varrho_{\mathbb{P}}^{\mathcal{M}_{\rm N}}$-name $q \in \Sigma^\omega$ of $\tilde{\mu}$ is also a $\delta_\beta^{\mathcal{M}_{\rm N}}$-name of μ. On the other hand, it is easy to see that any $\delta_\beta^{\mathcal{M}_{\rm C}}$-name $p \in \Sigma^\omega$ of μ is simultaneously a $\varrho_{\mathbb{P}}^{\mathcal{M}_{\rm C}}$-name of $\tilde{\mu}$. Therefore the translator from $\varrho_{\mathbb{P}}^{\mathcal{M}_{\rm C}}$ into $\varrho_{\mathbb{P}}^{\mathcal{M}_{\rm N}}$ provided by Proposition 4.3 also translates $\delta_\beta^{\mathcal{M}_{\rm C}}$ into $\delta_\beta^{\mathcal{M}_{\rm N}}$. □

Note that by Lemma 3.7 the representation $\delta_\beta^{\mathcal{M}_{\rm S}}$ need not be equivalent to $\delta_\beta^{\mathcal{M}_{\rm N}}$. We obtain by Propositions 3.1 and 3.8:

Theorem 4.6 *For any admissible representation δ of a countably based T_0-space X, the canonical representation $\delta^{\mathcal{M}_{\rm C}}$ of the Borel measures on X and the representation $\delta^{\mathcal{M}_{\rm N}}$ via probabilistic names are topologically equivalent.*

The function translating $\delta^{\mathcal{M}_{\rm C}}$ into $\delta^{\mathcal{M}_{\rm N}}$ produces probabilistic names with dyadic rational weights. Thus it would be sufficient to define probabilistic processes as a function π from Σ^* to $\mathbb{I} \cap \mathbb{D}$ satisfying Equation (2). One can show that under this definition $\mathcal{M}_{\rm N}(X) = \mathcal{M}(X)$ remains true for all admissibly representable spaces X which have a pseudobase consisting of universally measurable sets.

Theorems 4.6, 3.3 and Corollary 3.5 yield:

Theorem 4.7 *For any admissible representation δ of a countably based T_0-space X, the representation $\delta^{\mathcal{M}_{\rm N}}$ via probabilistic name is admissible w.r.t. the weak topology on $\mathcal{M}(X)$. The final topology of $\delta^{\mathcal{M}_{\rm N}}$ is equal to the weak topology.*

The signed-digit representation $\varrho_{\rm sd}$ of the reals as well as the representation $\varrho_{\mathbb{I}_=}$ of the unit interval are computably equivalent to a respective standard representation (cf. [12]). By Theorem 4.5 along with Propositions 3.1 and 3.8 this implies $\varrho_{\rm sd}^{\mathcal{M}_{\rm C}} \equiv_{\rm cp} \varrho_{\rm sd}^{\mathcal{M}_{\rm N}}$ and $\varrho_{\mathbb{I}_=}^{\mathcal{M}_{\rm C}} \equiv_{\rm cp} \varrho_{\mathbb{I}_=}^{\mathcal{M}_{\rm N}}$. Thus the representation $\varrho_{\mathbb{I}_=}^{\mathcal{M}_{\rm N}}$ via probabilistic names of the Borel measures on the unit interval is computably equivalent to the representation considered in [11], because both are $\leq_{\rm cp}$-complete in the set of representations of $\mathcal{M}(\mathbb{I}_=)$ admitting computability of integration.

5 Discussion

We have shown that probabilistic names induce a representation for Borel measures which, in the case of countably based T_0-spaces, is admissible w.r.t. the familiar weak topology. This question had remained unsolved in the preceding paper [9]. Therefore the representation via probabilistic names is very natural, justifying the concept of probabilistic names as generators of Borel measures. An open problem is to characterise the class of non countably based qcb_0-spaces for which the representation via probabilistic names is admissible w.r.t. the weak topology.

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