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Uniformly Computable Aspects of Inner Functions

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Abstract

The theory of inner functions plays an important role in the study of bounded analytic functions. Inner functions are also very useful in applied mathematics. Two foundational results in this theory are Frostman's Theorem and the Factorization Theorem. We give a uniformly computable version of Frostman's Theorem. We then claim that the Factorization Theorem is not uniformly computably true. We then claim that for an inner function u, the Blaschke sum of u provides the exact amount of information necessary to compute the factorization of u. Along the way, we discuss some uniform computability results for Blaschke products. These results play a key role in the analysis of factorization. We also give some computability results concerning zeros and singularities of analytic functions. We use Type-Two Effectivity as our foundation.

Keywords: Computable analysis, complex analysis, bounded analytic functions.

1 Introduction

Let $H^\infty(\mathbb{D})$ be the set of all bounded analytic functions from $\mathbb{D}$ into $\mathbb{C}$. For $f \in H^\infty(\mathbb{D})$, let

$$\| f \|_\infty = \sup\{|f(z)| \ : \ z \in \mathbb{D}\}.$$

A function $u \in H^\infty(\mathbb{D})$ is *inner* if

$$\lim_{z \to z_0} |f(z)| = 1$$

for almost every $z_0 \in \partial\mathbb{D}$. It follows from the Factorization Theorem (Theorem 5.2) that if u is an inner function, then $\| u \|_\infty \leq 1$.

A sequence $\{a_n\}_{n=0}^\infty$ of points in $\mathbb{D} - \{0\}$ is a *Blaschke sequence* if

$$(1) \quad \sum_{n=0}^{\infty}(1 - |a_n|) < \infty$$

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The series in (1) is called the *Blaschke sum* of $\{a_n\}_{n=0}^{\infty}$. The inequality in (1) is called the *Blaschke condition*.

For every $a \in \mathbb{D}$, let

$$b_a(z) = \frac{|a|}{a}\frac{a-z}{1-\overline{a}z}.$$

If $A = \{a_n\}_{n=0}^{\infty}$, where $a_n \in \mathbb{D} - \{0\}$ for all n, then let

$$B_A = \prod_{n=0}^{\infty} b_{a_n}.$$

A *Blaschke product* is a product of the form $z^k B_A(z)$. We let $B_{A,k}(z) = z^k B_A(z)$. If A is a Blaschke sequence, then the terms of A are precisely the zeros of B_A, and the order of a zero is the number of times it is repeated in A. On the other hand, if A is not a Blaschke sequence, then B_A is identically 0.

Let $\mathbb{D}^\omega$ be the set of all sequences $\{a_n\}_{n=0}^{\infty}$ such that $a_n \in \mathbb{D}$ for all n.

Two foundational results in the theory of inner functions are that inner functions can be estimated by Blaschke products (Frostman's Theorem) and that every inner function can be written as a product of a Blaschke product and a singular function (the Factorization Theorem). Our chief results are that inner functions can be effectively estimated by Blaschke products, but that factorization of inner functions is not a computable operation. We use Type-Two Effectivity as developed in [12] as our foundation.

Once a function f has been shown to be non-computable, it is natural to ask which additional parameters are necessary for its computation. For example, in [13], it is shown that spectral decomposition of complex matrices is not computable, but that spectral decomposition is computable once the cardinality of the spectrum is given. We will also show that the Blaschke sum of an inner function, together with the order of the zero at zero (if there is any), provide the precise amount of information necessary to compute the factorization of the function. To make this more precise, we make some informal definitions. Suppose that f is a function that is not computable with respect to some selection of notations. We say that the parameter $g(x)$ is *sufficient* for the computability of f if

$$(x, g(x)) \mapsto f(x)$$

is computable. If in addition,

$$(x, f(x)) \mapsto g(x)$$

is computable, then we say that the parameter $g(x)$ is *necessary* for the computability of f.

2 Frostman's Theorem

For $a, z \in \overline{\mathbb{D}}$ with $|a| < 1$, let $M_a(z) = \frac{z-a}{1-\overline{a}z}$. If u is an inner function, then $M_a \circ u$ is called a *Frostman shift* of u.

For each closed $K \subseteq \mathbb{D}$ and each positive measure σ on K, let $U_\sigma : \mathbb{D} \to \mathbb{D}$ be defined by the equation

$$U_\sigma(z) = \int_K \log \frac{1}{|z-\zeta|} d\sigma(\zeta).$$

Definition 2.1 Let $F \subseteq \mathbb{D}$ be closed. We say that F has *zero capacity* if for every positive measure on F, σ, with $\sigma \neq 0$, U_σ is not bounded on any neighborhood of F. Otherwise, we say that F has *positive capacity.* If U is an arbitrary subset of $\mathbb{D}$, then we say that U has positive capacity just in case it has a closed subset with positive capacity; otherwise, we say that it has zero capacity.

Every capacity zero set has measure zero. However, there are measure zero sets with positive capacity. For example, the Cantor middle-third set. See [11] or [5] for a thorough treatment of the topic of capacity.

We say that λf is a *unit multiple of f* if $\lambda \in \partial\mathbb{D}$.[2] A proof of the following can be found in [4]. It was originally proven in Otto Frostman's dissertation [3].

Theorem 2.2 (Frostman's Theorem) *Let u be a non-constant inner function. Then, $M_a \circ u$ is a unit multiple of a Blaschke product for all $a \in \mathbb{D}$ except in a set of capacity zero.*

The set of values of a for which $M_a \circ u$ is not a unit multiple of a Blaschke product is called the *exception set of u*. McLaughlin and Piranian have shown that a subset of $\mathbb{D}$ is an exception set for an inner function if and only if it is an F_σ set with capacity zero [8].

One easily verifies that

(2) $\lim_{a\to 0} |z - M_a(z)| = 0.$

We then obtain the following corollary.

Corollary 2.3 *Let u be an inner function. Then, for each $\epsilon > 0$ there is a unit multiple of a Blaschke product, λb, such that*

$$\| u - \lambda b \|_\infty < \epsilon.$$

For the sake of proving a uniformly computable version of Corollary 2.3, we will need a more explicit result than (2) which we derive now. Fix $\alpha, z \in \overline{\mathbb{D}}$ with $|\alpha| < 1$. By a simple calculation, we obtain

$$z - M_\alpha(z) = \frac{-\overline{\alpha} z^2 + \alpha}{1 - \overline{\alpha} z}.$$

We now note that $|1 - \overline{\alpha} z| \geq 1 - |\alpha||z|$. It then follows that

$$|z - M_\alpha(z)| \leq |\alpha| \frac{|z|^2 + 1}{1 - |\alpha||z|}.$$

Fix $\epsilon > 0$. It then follows from a simple calculation that

$$|\alpha| \frac{|z|^2 + 1}{1 - |\alpha||z|} < \epsilon \;\Leftrightarrow\; |\alpha| < \frac{\epsilon}{|z|^2 + 1 + \epsilon|z|}.$$

[2] In the literature on bounded analytic functions, it is customary to identify a Blaschke product with its unit multiples. However, as this potentially makes a difference in the information content, we can not afford this luxury.

The right side of the latter inequality is minimized on $\overline{\mathbb{D}}$ when $|z| = 1$. It now follows that

(3) $|\alpha| < \dfrac{\epsilon}{2+\epsilon} \ \Rightarrow\ |z - M_\alpha(z)| < \epsilon.$

If $u \in H^\infty(\mathbb{D})$, then for each $r \in (0,1)$ let

$$m_u(r) = \frac{1}{2\pi}\int_0^{2\pi} \log|u(re^{i\theta})|d\theta.$$

The following follows directly from Theorem II.2.4 of [4].

Theorem 2.4 *Let $u \in H^\infty(D)$, and suppose $\| u \|_\infty \leq 1$. Then, the following are equivalent.*

(i) $$\lim_{r\to 1} m_u(r) = 0.$$

(ii) *u is a unit multiple of a Blaschke product.*

The following is a direct consequence of Corollary I.6.6 of [4].

Lemma 2.5 *If u is an inner function, then m_u is increasing.*

We note that if u is an inner function, then

(4) $$\sup_{0<r<1} \frac{1}{2\pi}\int_0^{2\pi} \log|u(re^{i\theta})|d\theta \leq 0.$$

Denote the open disk with center α and radius r by $B_r(\alpha)$.

We now state a uniformly computable version of Frostman's Theorem. The proof will be given in a future paper.

Theorem 2.6 (Uniformly computable Frostman Theorem) *There is a multivalued function $\Psi :\subseteq H^\infty(\mathbb{D}) \times \mathbb{N} \rightrightarrows \mathbb{D}$ that is $([\rho^2 \to \rho^2], \nu_{\mathbb{N}}, \rho^2)$-computable and such that for every non-constant inner function u and every $n \in \mathbb{N}$, $M_\alpha \circ u$ is a unit multiple of a Blaschke product and $\| u - M_\alpha \circ u \|_\infty < 2^{-n}$ for every α in the image of Ψ on u.*

3 Some observations about zeros

If $u \in H^\infty(D)$, and if $\{a_n\}_{n=0}^\infty$ is a sequence of points in $\mathbb{D}$, then let us call $\{a_n\}_{n=0}^\infty$ a *primary zero sequence* of u if its terms are precisely the zeros of u and the number of times each zero of u is repeated is its multiplicity. If $\{a_n\}_{n=0}^\infty$ contains precisely the zeros of u that are not zero, and if the number of times each zero of u is repeated is its multiplicity, then let us call $\{a_n\}_{n=0}^\infty$ a *truncated zero sequence* of u.

The following is a direct consequence of Theorem II.2.1 of [4].

Theorem 3.1 *If u is an inner function with infinitely many zeros, and if u is not identically zero, then its truncated zero sequences are Blaschke sequences.*

Theorem 3.2 (Effective Removable Singularity Theorem) *There is a* $([\rho^2 \to \rho^2]^2, \rho^2, [\rho^2 \to \rho^2])$*-computable map* $\mathcal{E}$ *such that if* f *is an analytic function with domain* $\mathbb{D} - \{z_0\}$ *for some* $z_0 \in \mathbb{D}$*, and if* f *has a removable singularity at* z_0*, then* $\mathcal{E}(f, z_0)$ *is the analytic function obtained from* f *by removing the singularity at* z_0*.*

Proof. Let $\hat{f}$ be the function that results from f by removing the singularity at z_0. We describe how to build a $[\rho^2 \to \rho^2]$-name for $\hat{f}$ from a $[\rho^2 \to \rho^2]$-name of f, p_0, and a ρ^2-name of z_0, p_1. Given a ρ^2-name of $z \in \mathbb{D}$, begin reading down p and p_0 simultaneously until w_1 and w_2 are found such that $z, z_0 \in I_2(w_2)$ or $z_0 \in I_2(w_1)$ and $z \in I_2(w_2)$ and $I_2(w_1) \cap I_2(w_2) = \emptyset$. In either case, $z_0 \notin \partial I_2(w_2)$ and $z \in I_2(w_2)$. Hence, by the Cauchy Integral Formula:

$$\begin{aligned}\hat{f}(z) &= \frac{1}{2\pi i} \int_{\partial I_2(w_2)} \frac{\hat{f}(\alpha)}{\alpha - z} d\alpha \\ &= \frac{1}{2\pi i} \int_{\partial I_2(w_2)} \frac{f(\alpha)}{\alpha - z} d\alpha\end{aligned}$$

Since integration is a computable operator, it follows that we can compute a $[\rho^2 \to \rho^2]$-name of $\hat{f}$. □

Lemma 3.3 *There is a* $(\theta_<, \psi_<, [\rho^2]^\omega)$*-computable multivalued function* Ψ *such that if* $U \subseteq \mathbb{C}$ *is open, and if* $X \subseteq U$ *is infinite, discrete, and closed relative to* U*, then* $(U, \overline{X}) \in dom(\Psi)$ *and every sequence in the image of* Ψ *on* $(U, \overline{X})$ *is a one-to-one enumeration of* X*.*

Proof. This is basically shown in the proof of Lemma 3.3 of [7]. □

Let δ_1 be as in Exercise 6.1.11 of [12].

Theorem 3.4 *There is a* $(\delta_1, \psi_<)$*-computable function* F *such that if* $f :\subseteq \mathbb{C} \to \mathbb{C}$ *is analytic (and hence has open domain), then* $F(f) = f^{-1}\{0\}$*.*

Proof. This is essentially shown in the proof of Theorem 3.4 of [7]. □

Corollary 3.5 *In Theorem 3.4,* F *is* (δ_1, ψ)*-computable. Hence, if* $f :\subseteq \mathbb{C} \to \mathbb{C}$ *is* δ_1*-computable and analytic, then its zero set is* ψ*-computable.*

Proof. By reading down a δ_1-name of f, we can determine which basic open sets are mapped into open sets which do not contain zero. Thus, F is $(\delta_1, \psi_>)$-computable. □

The situation is quite different for continuous functions. See, for example, Theorem 6.3.2 of [12].

Theorem 3.6 *There is a* $([\rho^2 \to \rho^2], [\rho^2]^\omega)$*-computable multivalued function* Ψ *such that for every analytic function with domain* $\mathbb{D}$ *and with infinitely many zeros (but not identically zero),* $u \in dom(\Psi)$ *and every sequence in the image of* Ψ *on* u *is a primary zero sequence of* u*.*

Proof. It follows from Lemma 3.3 and Theorem 3.4 that there is a $([\rho^2 \to \rho^2], [\rho^2]^\omega)$-computable function Φ such that for every analytic f with domain $\mathbb{D}$ and with infinitely many zeros, the terms of $\Phi(f)$ are precisely the zeros of f, and each zero appears exactly once in $\Phi(f)$.

For each $\{b_n\}_{n=0}^\infty \in \mathbb{D}^\omega$, let $\Gamma(\{b_n\}_{n=0}^\infty) = b_0$. Thus, Γ is $([\rho^2]^\omega, \rho^2)$-computable.

Now, let

$$\Psi(f) = (\Gamma(\Phi(\hat{f}_0)), \Gamma(\Phi(\hat{f}_1)), \Gamma(\Phi(\hat{f}_2)), \ldots)$$

where:

$$\begin{aligned} \hat{f}_0 &= f \\ \hat{f}_{n+1} &= \mathcal{E}\left(\frac{\hat{f}_n}{z - \Gamma(\Phi(\hat{f}_n))}, \Gamma(\Phi(\hat{f}_n))\right). \end{aligned}$$

(Here, $\mathcal{E}$ is as in Theorem 3.2.) It follows that if f is analytic with domain $\mathbb{D}$, and if f has infinitely many zeros (but is not identically zero), then each $\hat{f}_n$ is defined and is analytic with domain $\mathbb{D}$. It then follows that $\Psi(f)$ has the required properties.$\square$

4 Uniform computability results for Blaschke products

The results of this section can be summarized as follows. Given a Blaschke sequence $A = \{a_n\}_{n=0}^\infty$, the following problems are computationally equivalent.

- Computing B_A.
- Computing the Blaschke sum of A.
- Computing $B_A(0)$.

That is, given A and any one of the above, we can compute the other two. Furthermore, from B_A we can compute A. It follows that the data $(A, \sum_{n=0}^\infty(1 - |a_n|))$ are both necessary and sufficient to compute B_A.

Theorem 4.1 *The map*

$$\left(\{a_n\}_{n=0}^\infty, \sum_{n=0}^\infty (1 - |a_n|)\right) \mapsto \prod_{n=0}^\infty b_{a_n}$$

is $([\rho^2]^\omega, \rho, [\rho^2 \to \rho^2])$-computable. That is, there is a function $\Psi \subseteq: \mathbb{D}^\omega \times \mathbb{R} \to H^\infty(\mathbb{D})$ that is $([\rho^2]^\omega, \rho, [\rho^2 \to \rho^2])$-computable and such that if $\{a_n\}_{n=0}^\infty$ is a Blaschke sequence, then

$$\Psi\left(\{a_n\}_{n=0}^\infty, \sum_{n=0}^\infty (1 - |a_n|)\right) = \prod_{n=0}^\infty b_{a_n}.$$

Proof. This is essentially shown in the proof of Theorem 4.6 of [7]. $\square$

We note that $b_{a_n}(0) = |a_n|$. Hence, $B_A(0)$ is the product of the moduli of the terms of A.

Theorem 4.2 *The map*

$$\left(\prod_{n=0}^{\infty} b_{a_n}\right) \mapsto (\{a_n\}_{n=0}^{\infty}, \prod_{n=0}^{\infty} b_{a_n}(0))$$

is $([\rho^2 \to \rho^2], [\rho^2]^\omega, \rho)$-computable. That is, there is a function $\Psi \subseteq: H^\infty(\mathbb{D}) \to \mathbb{D}^\omega \times \mathbb{R}$ that is $([\rho^2 \to \rho^2], [\rho^2]^\omega, \rho)$-computable and such that if $\{a_n\}_{n=0}^{\infty}$ is a Blaschke sequence, then

$$\Psi\left(\prod_{n=0}^{\infty} b_{a_n}\right) = (\{a_n\}_{n=0}^{\infty}, \prod_{n=0}^{\infty} b_{a_n}(0)).$$

Proof. This follows from Theorem 3.6. □

We note that Theorem 4.2 yields another way to prove Theorem 3.6 of [7].

Theorem 4.3 *The map*

$$\left(\{a_n\}_{n=0}^{\infty}, \prod_{n=0}^{\infty} b_{a_n}(0)\right) \mapsto \sum_{n=0}^{\infty}(1 - |a_n|)$$

is $([\rho^2]^\omega, \rho, \rho)$-computable. That is, there is a computable function $\Psi \subseteq: \mathbb{D}^\omega \times \mathbb{R} \to \mathbb{R}$ that is $([\rho^2]^\omega, \rho, \rho)$-computable and such that if $\{a_n\}_{n=0}^{\infty}$ is a Blaschke sequence, then

$$\Psi\left(\{a_n\}_{n=0}^{\infty}, \prod_{n=0}^{\infty} b_{a_n}(0)\right) = \sum_{n=0}^{\infty}(1 - |a_n|).$$

A proof will be given in a future paper.

Corollary 4.4 *Suppose $A = \{a_n\}_{n=0}^{\infty}$ is a $[\rho^2]^\omega$-computable Blaschke sequence. Then, B_A is $[\rho^2 \to \rho^2]$-computable if and only if $\sum_{n=0}^{\infty}(1 - |a_n|)$ is a ρ-computable real number.*

Corollary 4.5 *Suppose $A = \{a_n\}_{n=0}^{\infty}$ is a $[\rho^2]^\omega$-computable Blaschke sequence. Then, B_A is $[\rho^2 \to \rho^2]$-computable if and only if $B_A(0)$ is ρ-computable.*

Corollary 4.6 *Suppose A is a $[\rho^2]^\omega$-computable Blaschke sequence. If B_A maps ρ^2-computable complex numbers to ρ^2-computable complex numbers, then it is itself computable.*

We note that the situation is quite different for power series. Namely, there is a power series with a computable sequence of coefficients that maps computable complex numbers to computable complex numbers yet is not computable (see, for example, [2]). (This answers a question posed by the referee of [7].)

5 Factorization

Definition 5.1 Let σ be a positive finite Borel measure on $\partial\mathbb{D}$ that is singular with respect to Lebesgue measure. For all $z \in \mathbb{D}$, let

$$Q(z) = \exp\left\{-\int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\sigma(\theta)\right\}.$$

The function $Q : \mathbb{D} \to \mathbb{C}$ is called a *singular function*.

Every singular function is an inner function. A singular function s has the following properties.

(i) $s(z) \neq 0$ for all $z \in \mathbb{D}$.

(ii) $|s(z)| \leq 1$ for all $z \in \mathbb{D}$.

(iii) $s(0)$ is a positive real number.

The basic facts about singular functions may be found in [4] and [9]. Let σ be the measure on $\partial\mathbb{D}$ defined by the equation

$$\sigma(E) = \begin{cases} 1 \text{ if } 1 \in E \\ 0 \text{ if } 1 \notin E \end{cases}$$

The resulting singular function is

$$z \mapsto \exp\left(-\frac{1+z}{1-z}\right).$$

We will use this function in the proof of Theorem 5.4.

The following is due to Smirnov [10].

Theorem 5.2 (Factorization Theorem) *If u is an inner function, then there exist λ, b, s such that $u = \lambda bs$, $\lambda \in \partial\mathbb{D}$, b is a (possibly finite) Blaschke product, and s is a singular function.*

See [4] or [9] for a proof. It follows that λ, b, s are unique. For, suppose u is an inner function and u has two such factorizations, λbs and $\lambda_0 b_0 s_0$. Since these Blaschke products are completely determined by the zeros of u, it follows that $b = b_0$. Hence, $\lambda s = \lambda_0 s_0$ except possibly at the zeros of u. However, since u is analytic, its zeros are isolated, and so by continuity we can conclude these functions are identical. Thus, $\lambda s(0) = \lambda_0 s_0(0)$. Hence, $|\lambda||s(0)| = |\lambda_0||s_0(0)|$. But, $|\lambda| = |\lambda_0| = 1$ and $s(0), s_0(0) > 0$. It then follows that $s(0) = s_0(0)$. Hence, $\lambda = \lambda_0$. Thus, $s = s_0$.

We call b the *Blaschke factor* of u, and denote it by b_u. We call s the *singular factor* of u, and denote it by s_u. We call λ the *constant factor* of u, and denote it by λ_u. If u is a unit multiple of a Blaschke product, then $s_u \equiv 1$. If u is a unit multiple of a singular function, then we define b_u to be 1.

Throughout the rest of this section, the variable u ranges over inner functions that are not identically zero but have infinitely many zeros. Let us call the map $u \mapsto (\lambda_u, b_u, s_u)$ the *factorization map*. From now on, let us denote the Blaschke sum of a truncated zero sequence of u by Σ_u. (It clearly follows that all truncated zero sequences of such an inner function have the same Blaschke sum.) If $u(0) = 0$, then let k_u denote the order of u's zero at 0; otherwise, let $k_u = 0$. Let us call the the operator $u \mapsto \Sigma_u$ the *Blaschke sum map*.

Theorem 5.3 *The map $(u, \Sigma_u) \mapsto k_u$ is $([\rho^2 \to \rho^2], \rho, \nu_{\mathbb{N}})$-computable.*

The proof will be given in a future paper.

Theorem 5.4 *The factorization map is not* $([\rho^2 \to \rho^2], \rho^2, [\rho^2 \to \rho^2]^2)$*-continuous and hence not* $([\rho^2 \to \rho^2], \rho^2, [\rho^2 \to \rho^2]^2)$*-computable.*

The proof will be given in a future paper.

The following theorem shows that the parameter Σ_u provides the exact amount of information necessary to make the factorization map computable.

Theorem 5.5 (i) *The map*

$$(u, \Sigma_u) \mapsto (\lambda_u, b_u, s_u)$$

is $([\rho^2 \to \rho^2], \rho, \rho^2, [\rho^2 \to \rho^2]^2)$*-computable. That is, there is a map*

$$\Psi :\subseteq H^\infty(\mathbb{D}) \times \times \mathbb{R} \to \mathbb{D} \times H^\infty(\mathbb{D}) \times H^\infty(\mathbb{D})$$

such that Ψ *is* $([\rho^2 \to \rho^2], \rho, \rho^2, [\rho^2 \to \rho^2]^2)$*-computable, and if* u *is any inner function with infinitely many zeros (but not identically zero), then* $(u, \Sigma_u) \in dom(\Psi)$ *and* $\Psi(u, \Sigma_u) = (\lambda_u, b_u, s_u)$*.*

(ii) *The map*

$$(\lambda_u, b_u, s_u) \mapsto \Sigma_u$$

is $(\rho^2, [\rho^2 \to \rho^2]^2, \rho)$*-computable. That is, there is a map*

$$\Psi :\subseteq \mathbb{D} \times H^\infty(\mathbb{D}) \times H^\infty(\mathbb{D}) \to \mathbb{R}$$

such that Ψ *is* $(\rho^2, [\rho^2 \to \rho^2]^2, \rho)$*-computable, and if* u *is any inner function with infinitely many zeros (but not identically zero), then* $(\lambda_u, b_u, s_u) \in dom(\Psi)$ *and* $\Psi(\lambda_u, b_u, s_u) = \Sigma_u$*.*

The proof will be given in a future paper.

Corollary 5.6 *The Blaschke sum map is not* $([\rho^2 \to \rho^2], \nu_{\mathbb{N}}, \rho)$*-continuous.*

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