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Denotational Semantics of Call-by-name Normalization in Lambda-mu Calculus

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Abstract

We study normalization in the simply typed lambda-mu calculus, an extension of lambda calculus with control flow operators. Using an enriched version of the Yoneda embedding, we obtain a categorical normal form function for simply typed lambda-mu terms, which gives a special kind of a call-by-name denotational semantics particularly useful for deciding equalities in the lambda-mu calculus.

Keywords: Yoneda embedding, categorical semantics, categories of continuations, lambda-mu calculus, normalization, partial equivalence relations

1 Introduction

We study normalization of terms in the simply typed $\lambda\mu$-calculus introduced by Parigot in [13]. This calculus is an extension of the simply typed λ-calculus with operators that influence the sequential control flow during the evaluation of a term. The primary reason for this extension was to provide a constructive notion of a classical natural deduction proof. Moreover, subsequent studies showed that the $\lambda\mu$-calculus can also be realized as a calculus of continuations, and that control operators of certain functional programming languages can be formalized by means of its μ-abstraction mechanism. A very instructive discussion of possible meanings of $\lambda\mu$-terms can be found in [14], §4.2. The suggested background reading in λ-calculus is [2] and [9].

Our approach to normalization is based on a categorical technique called the normalization by the Yoneda embedding, whose motivation is clearly explained in

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Introduction to the paper [4]. This technique employs category theory, and our reference on this subject is Mac Lane's book [10].

Before embarking in the technical exposition we would like to discuss some important topics which are particularly relevant in the context of our paper.

Continuations and classical logic. Continuations in programming languages generalize the notion of a control flow. In functional languages continuations are especially important tools providing rich expressivity. A characteristic example is the operator `call-with-current-continuation`, or `call/cc`, of the language Scheme. A common approach to semantics of languages with continuations (such as those providing labels and jumps, e.g., our simply typed $\lambda\mu$-calculus) involves a translation of a given language into a language that represents continuations as functions. Such translations are known as continuation passing style (CPS) translations. Studies of continuations in programming languages and CPS translations of λ-calculus commenced in early 70's (the relevant references could be found in [16]); e.g., in 1975 Plotkin introduced a call-by-name variant of the CPS translation. Notably, the well-known Gödel's and Kolmogoroff's $\neg\neg$-translations of classical logic into intuitionistic one correspond, on the level of propositions-as-types, respectively to call-by-name CPS translation with values and call-by-value CPS translation, which were studied starting from mid 80's by Felleisen and his co-workers in relationship to the λ-calculus with control operator, cf. [16]. In our paper we employ yet another CPS translation, an elegant call-by-name CPS translation studied in [8,6,14] which was motivated by Lafont's $\neg\neg$-translation of classical logic into the $\neg\wedge$-fragment of intuitionistic logic. Classical logic has several variants of proof-theoretic semantics given by the above mentioned translations of classical into constructive logic; such semantics were studied, e.g., in [5,11,12,16]. Our work can also be seen as a contribution to those studies of proof-theoretic semantics of classical logic.

Minimal-sized definition for response categories and categories of continuations. Our normalization method is based on an enriched case of categories of continuations. These categories are traditionally constructed from so-called categories of responses, as it was done, e.g., by Selinger in [14]. Categories of responses are essentially a categorical version of CPS semantics of λ-calculus. In [15] Selinger remarks on the minimal-sized definition for categories of responses. The enriched version of a category of responses we define in this paper is in fact based on the minimal-sized definition from the latter source. In our case the minimality is crucial since redundancies in the definition of a category of responses, such as presence of coproducts, can lead to a failure of the normalization function. This can happen because the Yoneda embedding which is used extensively in our semantics does not preserve coproducts. Therefore our definition of an enriched response category corresponds to a simple generalization of a cartesian closed category without coproducts, in which only a single fixed object is required to have exponentials.

Denotational semantics of the $\lambda\mu$-calculus. Lambda calculus is usually introduced as a theory of computable functions. The relationship of this theory to actual functions, e.g., functions between sets, is established by means of a suitable denotational semantics. Denotational semantics gives meaning to a language, in our case the simply typed $\lambda\mu$-calculus, by assigning mathematical objects as values to its terms. If M is a $\lambda\mu$-term, we will write $[\![M]\!]$ for the meaning of M under a given interpretation function $[\![-]\!]$, and $\equiv$ will be some congruence relation on interpretations ($\equiv$ can be an equality, but in general it suffices to be just a decidable congruence relation such as α-congruence). Consider some equality predicate $=_t$ on $\lambda\mu$-terms. Given a particular interpretation function, soundness is the property that $M =_t N$ implies $[\![M]\!] \equiv [\![N]\!]$. Then completeness is the property $[\![M]\!] \equiv [\![N]\!]$ implies $M =_t N$.

The decision problem for the $\lambda\mu$-calculus can be formulated as follows: For any possibly open $\lambda\mu$-terms M and N of type A, an object context Γ, and a control context Δ, decide whether $M =_{\Gamma,\Delta} N$, where $=_{\Gamma,\Delta}$ denotes the equality of $\lambda\mu$-terms in the context of Γ and Δ. With each $\lambda\mu$-term M in context Γ, Δ we associate its *abstract normal form* $\mathrm{nf}_{\Gamma,\Delta}(M)$, for which there exists a reverse function $\mathrm{fn}_{\Gamma,\Delta}$ from normal forms to terms, such that the following properties hold:

$$\begin{array}{ll} \text{(NF1)} & \mathrm{fn}_{\Gamma,\Delta}(\mathrm{nf}_{\Gamma,\Delta}(M)) =_{\Gamma,\Delta} M, \\ \text{(NF2)} & M =_{\Gamma,\Delta} N \quad \text{implies} \quad \mathrm{nf}_{\Gamma,\Delta}(M) \equiv \mathrm{nf}_{\Gamma,\Delta}(N). \end{array}$$

Note that nf is allowed not to be injective and hence there is no inverse function nf^{-1} in general. Since the conditions (NF1) and (NF2) imply $M =_{\Gamma,\Delta} N$ if and only if $\mathrm{nf}_{\Gamma,\Delta}(M) \equiv \mathrm{nf}_{\Gamma,\Delta}(N)$ (that is the soundness and completeness property), comparing such abstract normal forms can yield a denotational semantics and a decision procedure for the $\lambda\mu$-calculus, with (NF1) corresponding to the completeness property and (NF2) corresponding to the soundness property. Our (NF1) and (NF2) are similar to those appeared in [1] and applied there to normalization by the Yoneda embedding in simply typed λ-calculus with coproducts. However, our categorical techniques are different from those of [1].

Normalization by the Yoneda embedding and normalization by evaluation. The fact that normalization by the Yoneda embedding is closely related to the algorithm of normalization by evaluation due to Berger and Schwichtenberg [3] was noted in [4]. The correspondence is that the free interpretation, $\llbracket - \rrbracket$ in our notation, corresponds to the "evaluation functional" of [3], and the components ι and ι^{-1} of the natural isomorphism between the interpretation of generators by the Yoneda embedding and its free extension $\llbracket - \rrbracket$ correspond respectively to the functionals "procedure → expression", $p{\to}e$, and "make self evaluating", *mse*. The difference of course is that, unlike in the normalization by evaluation method, in normalization by Yoneda one does not mention any rewriting techniques.

In Sec. 2, following Kelly [7] we give definitions for a special instance of enriched category theory, category theory enriched over the category with objects being sets equipped with partial equivalence relations and morphisms being functions preserving these relations. Specifically, in §2.4 we develop an enriched version of categories of continuations. The idea of such an enrichment appeared in [4]; however, our definitions are different and in fact are just instances of more general definitions given by Kelly. In Sec. 3, we define the simply typed $\lambda\mu$-calculus, the call-by-name CPS translation and the categorical interpretation function. The aim of Sec. 3 is to show how the $\lambda\mu$-calculus can be embedded into the $\lambda^{R\times}$-calculus. Finally, in Sec. 4 we obtain the normal form function for simply typed $\lambda\mu$-terms. In §4.3, we characterize the obtained normal form function and sketch the proofs for soundness and completeness theorems, thus showing that our normal form function induces a special kind of a call-by-name denotational semantics for the $\lambda\mu$-calculus which is particularly useful for deciding equalities in the $\lambda\mu$-calculus.

2 PSet-enriched category theory

2.1 *Per-sets and* **PSet**-*categories*

A **per-set** A is a pair $A = (|A|, \sim_A)$, where $|A|$ is a set and $\sim_A$ is a partial equivalence relation (per) on $|A|$. A **per-function** between the per-sets $A = (|A|, \sim_A)$ and $B = (|B|, \sim_B)$ is a function $f: |A| \to |B|$ such that $a \sim_A a'$ implies $f(a) \sim_B f(a')$, for all $a, a' \in |A|$.

Specifically, we will need the following kinds of per-sets:

- the one-point per-set $\mathbf{1} = (\{*\}, \sim_\mathbf{1})$, where $* \sim_\mathbf{1} *$;
- the cartesian product of two per-sets A and B, that is the per-set $A \times B = (|A| \times |B|, \sim_{A\times B})$, where $\langle a, b\rangle \sim_{A\times B} \langle a', b'\rangle$ if $a \sim_A a'$ and $b \sim_B b'$, for all $a, a' \in |A|$ and $b, b' \in |B|$;
- the exponential of a per-set B by a per-set A, that is the per-set $B^A = (|B|^{|A|}, \sim_{B^A})$, where $f \sim_{B^A} g$ if, for all $a, a' \in |A|$, $a \sim_A a'$ implies $f(a) \sim_B g(a')$, for all $f, g \in |B|^{|A|}$.

A **cartesian category** is a category with finite products and the terminal object. A **cartesian closed category** (**ccc**) $\mathcal{C}$ is a cartesian category in which each functor $- \times A: \mathcal{C} \to \mathcal{C}$ has a right adjoint $(-)^A$.

Per-sets and per-functions form a cartesian closed category, denoted $\mathbf{PSet}_0$, whose objects are (small) per-sets and whose morphisms are per-functions between per-sets. The cartesian closedness of $\mathbf{PSet}_0$ means that there is an adjunction

$$\mathbf{PSet}_0(C \times A, B) \cong \mathbf{PSet}_0(C, B^A), \tag{1}$$

which is a bijection natural in C and B, with unit $d: C \to (C \times A)^A$ and counit $e: B^A \times A \to B$.

A **PSet-enriched category** (or, shorter, a **PSet-category**) $\mathcal{A}$ consists of

- a set $\mathrm{ob}(\mathcal{A})$ of objects,
- a hom-object $\mathcal{A}(A, B) \in \mathrm{ob}(\mathbf{PSet}_0)$, for each pair of objects of $\mathcal{A}$ (the elements of the hom-object are called morphisms from A to B),
- a per-function $\circ = \circ_{A,B,C} : \mathcal{A}(B, C) \times \mathcal{A}(A, B) \to \mathcal{A}(A, C)$, for each triple of objects $A, B, C \in \mathrm{ob}(\mathcal{A})$ (called the composition law of A, B, C),
- and a per-function $\mathrm{id} = \mathrm{id}_A : \mathbf{1} \to \mathcal{A}(A, A)$, for each object $A \in \mathrm{ob}(\mathcal{A})$ (called the identity element of A);

each of the above subject to the associativity and unit axioms expressed by the commutativity of the following two diagrams:

$$\begin{tikzcd}[column sep=small]
(\mathcal{A}(C,D)\times\mathcal{A}(B,C))\times\mathcal{A}(A,B) \arrow[rr, "a"] \arrow[d, "\circ\times 1"'] & & \mathcal{A}(C,D)\times(\mathcal{A}(B,C)\times\mathcal{A}(A,B)) \arrow[d, "1\times\circ"] \\
\mathcal{A}(B,D)\times\mathcal{A}(A,B) \arrow[dr, "\circ"'] & & \mathcal{A}(C,D)\times\mathcal{A}(A,C) \arrow[dl, "\circ"] \\
& \mathcal{A}(A,D) &
\end{tikzcd} \tag{2}$$

and

$$\begin{tikzcd}
\mathcal{A}(B,B)\times\mathcal{A}(A,B) \arrow[r, "\circ"] & \mathcal{A}(A,B) & \mathcal{A}(A,B)\times\mathcal{A}(A,A) \arrow[l, "\circ"'] \\
\mathbf{1}\times\mathcal{A}(A,B) \arrow[u, "\mathrm{id}\times 1"] \arrow[ur, "l"'] & & \mathcal{A}(A,B)\times\mathbf{1} \arrow[u, "1\times\mathrm{id}"'] \arrow[ul, "r"]
\end{tikzcd} \tag{3}$$

For **PSet**-categories $\mathcal{A}$ and $\mathcal{B}$, a **PSet-functor** $F: \mathcal{A}\to\mathcal{B}$ consists of

- a function $F: \mathrm{ob}(\mathcal{A}) \to \mathrm{ob}(\mathcal{B})$,
- and, for each pair $A, B \in \mathrm{ob}(\mathcal{A})$, a map $F_{A,B}: \mathcal{A}(A,B) \to \mathcal{B}(FA,FB)$;

subject to the compatibility with composition and identities expressed by the commutativity of

$$\begin{tikzcd}
\mathcal{A}(B,C)\times\mathcal{A}(A,B) \arrow[r, "\circ"] \arrow[d, "F\times F"'] & \mathcal{A}(A,C) \arrow[d, "F"] \\
\mathcal{B}(FB,FC)\times\mathcal{B}(FA,FB) \arrow[r, "\circ"'] & \mathcal{B}(FA,FC)
\end{tikzcd}
\quad\text{and}\quad
\begin{tikzcd}
& \mathcal{A}(A,A) \arrow[dd, "F"] \\
I \arrow[ur, "\mathrm{id}"] \arrow[dr, "\mathrm{id}"'] & \\
& \mathcal{B}(FA,FA)
\end{tikzcd} \tag{4}$$

The **PSet**-functor $F : \mathcal{A} \to \mathcal{B}$ is said to be **fully faithful** if each $F_{A,B}$ is an isomorphism.

For **PSet**-functors $F, G: \mathcal{A}\to\mathcal{B}$, a **PSet-natural transformation** $\alpha: F \to G: \mathcal{A} \to \mathcal{B}$ is an $\mathrm{ob}(\mathcal{A})$-indexed family of components $\alpha_A: \mathbf{1} \to \mathcal{B}(FA,GA)$ satisfying the **PSet-naturality condition** expressed by the commutativity of

$$\begin{tikzcd}
& \mathbf{1}\times\mathcal{A}(A,B) \arrow[r, "\alpha_B\times F"] & \mathcal{B}(FB,GB)\times\mathcal{B}(FA,FB) \arrow[dr, "\circ"] & \\
\mathcal{A}(A,B) \arrow[ur, "l^{-1}"] \arrow[dr, "r^{-1}"'] & & & \mathcal{B}(FA,GB) \\
& \mathcal{A}(A,B)\times\mathbf{1} \arrow[r, "G\times\alpha_A"'] & \mathcal{B}(GA,GB)\times\mathcal{B}(FA,GA) \arrow[ur, "\circ"'] &
\end{tikzcd} \tag{5}$$

For $\alpha: F \to G: \mathcal{A} \to \mathcal{B}$ and $\beta: G \to H: \mathcal{A} \to \mathcal{B}$, their "vertical" composite

$\beta \circ \alpha$ has the component $(\beta \circ \alpha)_A$ given by

$$\mathbf{1} \cong \mathbf{1} \times \mathbf{1} \xrightarrow{\beta_A \times \alpha_A} \mathcal{B}(GA, HA) \times \mathcal{B}(FA, GA) \xrightarrow{\circ} \mathcal{B}(FA, HA) \tag{6}$$

The composite of α with $Q: \mathcal{B} \to \mathcal{C}$ has for its component $(Q\alpha)_A$ the composite

$$\mathbf{1} \xrightarrow{\alpha_A} \mathcal{B}(FA, GA) \xrightarrow{Q} \mathcal{C}(QFA, QGA) \tag{7}$$

while the composite of α with $R: \mathcal{D} \to \mathcal{A}$ has for its component $(\alpha R)_D$ simply α_{RD}.

Given two **PSet**-categories $\mathcal{A}$ and $\mathcal{B}$, and two **PSet**-functors $F: \mathcal{A} \to \mathcal{B}$ and $U: \mathcal{B} \to \mathcal{A}$, a **PSet-adjunction** between F (the left adjoint) and U (the right adjoint) consists of **PSet**-natural transformations $\eta: 1 \to FU$ (the unit) and $\varepsilon: UF \to 1$ (the counit) satisfying the equations $F\varepsilon \circ \eta F = 1$ and $\varepsilon U \circ U\eta = 1$.

2.2 *The* **PSet**-*category* **PSet**

Now we will give the standard argument in the style of [7] to exhibit the **PSet**-category **PSet**.

Lemma 2.1 *There is a* **PSet**-*category* **PSet**, *whose objects are per-sets and where* $\mathbf{PSet}(A, B) = B^A$.

Proof. Putting $C = \mathbf{1}$ in the adjunction (1), using the isomorphism $l: \mathbf{1} \times A \cong A$, and writing P for the ordinary set-valued functor $\mathbf{PSet}_0(\mathbf{1}, -): \mathbf{PSet}_0 \to \mathbf{Set}$, we get the natural isomorphism

$$\mathbf{PSet}_0(A, B) \cong P(B^A). \tag{8}$$

which gives an equivalence between $\mathbf{PSet}_0$ and the underlying ordinary category of the **PSet**-category, which we will denote **PSet**, whose objects are those of $\mathbf{PSet}_0$, and whose hom-object $\mathbf{PSet}(A, B)$ is B^A. Since B^A is thus exhibited as a lifting through P of the hom-set $\mathbf{PSet}_0(A, B)$, it is the internal hom of A and B of **PSet**. The important point here is that the internal hom of **PSet** makes **PSet** itself into a **PSet**-category. Its composition law $\circ: B^A \times A^C \to B^C$ corresponds under (1) to the composite

$$(B^A \times A^C) \times C \xrightarrow{a} B^A \times (A^C \times C) \xrightarrow{1 \times e} B^A \times A \xrightarrow{e} B \tag{9}$$

and the identity element $\mathrm{id}_A : \mathbf{1} \to A^A$ corresponds under (1) to $l: \mathbf{1} \times A \to A$. Verification of the axioms (2) and (3) is easy since the definition (9) of $\circ$ is equivalent to $e(\circ \times 1) = e(1 \times e)a$. □

2.3 *The* **PSet**-*functor category* $\mathcal{B}^{\mathcal{A}}$ *and the* **PSet**-*enriched Yoneda lemma*

Let $\mathcal{A}$ and $\mathcal{B}$ be **PSet**-categories. The **PSet-functor category** $\mathcal{B}^{\mathcal{A}}$ is defined as follows:

- objects of $\mathcal{B}^{\mathcal{A}}$ are all **PSet**-functors from $\mathcal{A}$ to $\mathcal{B}$;
- for two such functors $F, G : \mathcal{A} \to \mathcal{B}$, their hom-object $\mathcal{B}^{\mathcal{A}}(F, G)$ is a per-set of $\mathrm{ob}(\mathcal{A})$-indexed families of components α_A in $|\mathcal{B}(FA, GA)|$, with the per on families defined by $\alpha \sim_{\mathcal{B}^{\mathcal{A}}(F,G)} \beta$ if α and β satisfy the **PSet**-naturality condition and, for all A, $\alpha_A \sim_{\mathcal{B}(FA,GA)} \beta_A$;
- $\circ_{F,G,H}(\alpha, \beta)$ is defined componentwise by $\circ_{F,G,H}(\alpha, \beta)_A = \circ_{FA,GA,HA}(\alpha_A, \beta_A)$;
- $\mathrm{id}_F : \mathbf{1} \to \mathcal{B}^{\mathcal{A}}(F, F)$ is defined componentwise by $(\mathrm{id}_F)_A = \mathrm{id}_{FA}$.

We define the **PSet**-enriched Yoneda functor $Y : \mathcal{A} \to \mathbf{PSet}^{\mathcal{A}^{\mathrm{op}}}$ by $YA = \mathcal{A}(-, A)$. Below we instantiate Kelly's $\mathcal{V}$-enriched (strong) Yoneda lemma [7] with our data. The parameter $\mathcal{V}$ becomes **PSet**. The Yoneda lemma is given for reference purposes, and hence without proof which can be found in [7] (for the case of the covariant Yoneda functor). In this paper we use the Yoneda lemma by way of its Corollary 2.3.

Lemma 2.2 (PSet-enriched Yoneda) *Let $\mathcal{A}$ be a* **PSet***-category, K an object of $\mathcal{A}$, and $F : \mathcal{A}^{\mathrm{op}} \to$* **PSet***. The transformation $\phi_A : \; FK \to (FA)^{YKA}$,* **PSet***-natural in A, expresses FK as the equalizer*

$$\mathbf{PSet}^{\mathcal{A}^{\mathrm{op}}}(YK, F) \xrightarrow{\;E\;} \prod_{A \in \mathrm{ob}(\mathcal{A})} (FA)^{YKA} \underset{\sigma}{\overset{\rho}{\rightrightarrows}} \prod_{A,B \in \mathrm{ob}(\mathcal{A})} \left((FB)^{YKA}\right)^{\mathcal{A}^{\mathrm{op}}(A,B)}$$

where $\rho_{A,B}$ and $\sigma_{A,B}$ are the transformations of $\left((F-)^{YKA}\right)_{A,B}$ and $\left((FB)^{YK-}\right)_{B,A}$ respectively; so that we have the following isomorphism **PSet***-natural in K and F:*

$$\phi : \; FK \cong \mathbf{PSet}^{\mathcal{A}^{\mathrm{op}}}(YK, F).$$

Corollary 2.3 *For any* **PSet***-category $\mathcal{A}$, the* **PSet***-functor $Y : \mathcal{A} \to \mathbf{PSet}^{\mathcal{A}^{\mathrm{op}}}$ is fully faithful.*

Due to this corollary Y is called the **PSet-enriched Yoneda embedding**. The fact that it is full justifies that a morphism $YA \to YB$ in $\mathbf{PSet}^{\mathcal{A}^{\mathrm{op}}}$ is essentially the same as $A \to B$ in $\mathcal{A}$. Note that $\mathbf{PSet}^{\mathcal{A}^{\mathrm{op}}}$ always exists since, by definition, $\mathcal{A}$ is locally small (its hom-objects are small per-sets).

2.4 **PSet**-*categories of continuations*

A **PSet-category of responses** is a **PSet**-category $\mathcal{C}$ with finite products and exponentials of the form R^A for a fixed **object of responses** R and any object A of $\mathcal{C}$. The latter means that there is a canonical isomorphism

$$\mathcal{C}(B, R^A) \cong \mathcal{C}(B \times A, R) \tag{10}$$

PSet-natural in A.

Given a **PSet**-category of responses $\mathcal{C}$ (and the exponentiable object R of $\mathcal{C}$), we *construct* its **PSet-category of continuations** $\mathcal{C}_R$ as follows:

- objects of $\mathcal{C}_R$ are n-tuples of objects of $\mathcal{C}$, for $n \geq 0$;
- for all $A = \langle A_1, \dots, A_n \rangle$ and $B = \langle B_1, \dots, B_m \rangle$ in $\mathrm{ob}(\mathcal{C}_R)$, their hom-object $\mathcal{C}_R(A, B)$ is $\mathcal{C}(\prod_i R^{A_i}, \prod_j R^{B_j})$.

Therefore $\mathcal{C}_R$ is the full subcategory of $\mathcal{C}$ on objects of the kind $R^{A_1} \times \dots \times R^{A_n}$, which we abbreviate to $\mathbf{R}^A$. It follows that, in $\mathcal{C}_R$, the composition $\circ_{A,B,C} : \mathcal{C}_R(B, C) \times \mathcal{C}_R(A, B) \to \mathcal{C}_R(A, C)$ coincides with the composition $\circ_{\mathbf{R}^A, \mathbf{R}^B, \mathbf{R}^C} : \mathcal{C}(\mathbf{R}^B, \mathbf{R}^C) \times \mathcal{C}(\mathbf{R}^A, \mathbf{R}^B) \to \mathcal{C}(\mathbf{R}^A, \mathbf{R}^C)$ in $\mathcal{C}$, and the identity element $\mathrm{id}_A : \mathbf{1} \to \mathcal{C}_R(A, A)$ coincides with the identity element $\mathrm{id}_{R^A} : \mathbf{1} \to \mathcal{C}(\mathbf{R}^A, \mathbf{R}^A)$ in $\mathcal{C}$. Thus $\mathcal{C}_R$ satisfies the axioms of **PSet**-categories (2) and (3) in the trivial way.

We note that **PSet**-categories of continuations are cartesian closed **PSet**-categories. Indeed, given $\mathcal{C}_R$, one has finite products in $\mathcal{C}_R$ as a consequence of it being a full subcategory of $\mathcal{C}$. Next, if $A = \langle A_1, \dots, A_n \rangle$ and $B = \langle B_1, \dots, B_m \rangle$ then the object $B^A = \prod_j R^{B_j \times \prod_i R^{A_i}}$ is their exponential in $\mathcal{C}$, and thus $\mathcal{C}_R$ is closed under exponentiation, and there is an isomorphism

$$\mathbf{PSet}(\prod_k R^{C_k} \times \prod_i R^{A_i}, \prod_j R^{B_j}) \cong \mathbf{PSet}(\prod_k R^{C_k}, \prod_j R^{B_j \times \prod_i R^{A_i}}), \tag{11}$$

PSet-natural in C_k and B_j, giving rise to a **PSet**-adjunction with unit η and counit ε, respectively:

$$\begin{aligned} \eta &: \prod_k R^{C_k} \to \prod_k R^{C_k \times \prod_i R^{A_i}} \times \prod_i R^{A_i \times \prod_i R^{A_i}}, \\ \varepsilon &: \prod_j R^{B_j \times \prod_i R^{A_i}} \times \prod_i R^{A_i} \to \prod_j R^{B_j}. \end{aligned}$$

Hence we will use standard notation for the **PSet**-categorical analogues of structural ccc-morphisms.

Given two **PSet**-categories of continuations $\mathcal{C}_R$ and $\mathcal{D}_{R'}$, and a **PSet**-functor from the first to the second, an obvious question arises about whether F in its image preserves the structure of the first category, for instance, whether the exponentiable object of responses retains its qualities in the image. We make this precise in the following definition. Given two **PSet**-categories of continuations $\mathcal{C}_R$ and $\mathcal{D}_{R'}$, a **PSet-functor of PSet-categories of continuations**, or, in short, a **PSet-coc functor**, is a **PSet**-functor $F : \mathcal{C}_R \to \mathcal{D}_{R'}$, together with **PSet**-natural isomorphisms, for $n \geq 0$,

$$\begin{aligned} r^R_{\langle A_1, \dots, A_n \rangle} &: \quad R'^{FA_1} \times \dots \times R'^{FA_n} \xrightarrow{\cong} F(R^{A_1} \times \dots \times R^{A_n}) \\ r^\times_{A_1, \dots, A_n} &: \quad R'^{FA_1 \times \dots \times FA_n} \xrightarrow{\cong} R'^{F(A_1 \times \dots \times A_n)} \end{aligned} \tag{12}$$

commuting with the morphism structure in all the evident ways. Note that despite some notational clumsiness arising from the presence of the exponentiable object, the meaning of the isomorphisms is very clear since $r^R_{\langle A_1, \dots, A_n \rangle}$ is an element of the hom-object

$$\mathcal{D}_{R'}(\langle FA_1, \dots, FA_n \rangle, F\langle A_1, \dots, A_n \rangle),$$

and $r^{\times}_{A_1,\dots,A_n}$ is an element of the hom-object

$$\mathcal{D}_{R'}(FA_1 \times \cdots \times FA_n, F(A_1 \times \cdots \times A_n)).$$

In the following we will be especially interested in actions of the Yoneda embedding Y on a **PSet**-category of continuations $\mathcal{C}_R$. In this situation the image of Y on $\mathcal{C}_R$ is a **PSet**-category which will be later shown to be a **PSet**-category of continuations. However, it is more convenient to define Y not just on $\mathcal{C}_R$ but on $\mathcal{C}$ itself, so we have $Y : \mathcal{C} \to \mathbf{PSet}^{\mathcal{C}^{\mathrm{op}}}$, where $\mathbf{PSet}^{\mathcal{C}^{\mathrm{op}}}$ is obtained as a usual **PSet**-functor category and is clearly a **PSet**-category of responses with the object of responses being $YR = \mathcal{C}(-, R)$. As for the corresponding **PSet**-category of continuations, denoted $\mathbf{PSet}^{\mathcal{C}^{\mathrm{op}}}_{YR}$, we obtain it as follows[3]:

- objects of $\mathbf{PSet}^{\mathcal{C}^{\mathrm{op}}}_{YR}$ are n-tuples $\langle YA_1, \dots, YA_n\rangle$, for $n \geq 0$ and $\langle A_1, \dots, A_n\rangle \in \mathrm{ob}(\mathcal{C}_R)$;
- hom-objects of $\mathbf{PSet}^{\mathcal{C}^{\mathrm{op}}}_{YR}$ are $\mathbf{PSet}^{\mathcal{C}^{\mathrm{op}}}_{YR}(\langle YA_1, \dots, YA_n\rangle, \langle YB_1, \dots, YB_m\rangle) = \mathbf{PSet}^{\mathcal{C}^{\mathrm{op}}}((YR)^{YA_1} \times \cdots \times (YR)^{YA_n}, (YR)^{YB_1} \times \cdots \times (YR)^{YB_m})$, for $n, m \geq 0$, $\langle A_1, \dots, A_n\rangle \in \mathrm{ob}(\mathcal{C}_R)$ and $\langle B_1, \dots, B_m\rangle \in \mathrm{ob}(\mathcal{C}_R)$;
- the composition law and the identity element are defined in the obvious way;
- the object of responses is $YR = \mathcal{C}(-, R) \in \mathrm{ob}(\mathbf{PSet}^{\mathcal{C}^{\mathrm{op}}})$.

The statement about the object of responses might seem not straightforward, therefore we will give it some more attention. Note that in $\mathbf{PSet}^{\mathcal{C}^{\mathrm{op}}}$ the object of responses $YR = \mathcal{C}(-, R)$ is isomorphic to $(YR)^{Y\mathbf{1}} = \mathcal{C}(-, R)^{\mathcal{C}(-,\mathbf{1})}$ since $Y\mathbf{1}$ is terminal in $\mathbf{PSet}^{\mathcal{C}^{\mathrm{op}}}$. Also observe that, for any $A \in \mathrm{ob}(\mathcal{C})$, the isomorphism $\mathcal{C}(-, R^A) \cong \mathcal{C}(-, R^A)^{\mathcal{C}(-,\mathbf{1})} \cong \mathcal{C}(-, R)^{\mathcal{C}(-,A)}$ holds. This allows one to consider the full subcategory of $\mathbf{PSet}^{\mathcal{C}^{\mathrm{op}}}$ on objects of the kind $\mathcal{C}(-, R)^{\mathcal{C}(-,A_1)} \times \cdots \times \mathcal{C}(-, R)^{\mathcal{C}(-,A_n)}$, which is precisely our $\mathbf{PSet}^{\mathcal{C}^{\mathrm{op}}}_{YR}$. The result can be stated in the following lemma.

Lemma 2.4 *For a* **PSet**-*category of continuations* $\mathcal{C}_R$, *the* **PSet**-*category* $\mathbf{PSet}^{\mathcal{C}^{\mathrm{op}}}_{YR}$ *is a* **PSet**-*category of continuations with the object of responses being* $\mathcal{C}(-, R) \in \mathrm{ob}(\mathbf{PSet}^{\mathcal{C}^{\mathrm{op}}})$.

The Yoneda embedding $Y : \mathcal{C} \to \mathbf{PSet}^{\mathcal{C}^{\mathrm{op}}}$ restricts on $\mathcal{C}_R$ to a **PSet**-functor $Y_R : \mathcal{C}_R \to \mathbf{PSet}^{\mathcal{C}^{\mathrm{op}}}_{YR}$ which consists of

- a function $Y_R : \mathrm{ob}(\mathcal{C}_R) \to \mathrm{ob}(\mathbf{PSet}^{\mathcal{C}^{\mathrm{op}}}_{YR})$ which sends an n-tuple $\langle A_1, \dots, A_n\rangle$ to the n-tuple $\langle YA_1, \dots, YA_n\rangle$;
- for each pair of tuples $A, B \in \mathrm{ob}(\mathcal{C}_R)$, a map $(Y_R)_{A,B} : \mathcal{C}_R(A, B) \to \mathbf{PSet}^{\mathcal{C}^{\mathrm{op}}}_{YR}(Y_R A, Y_R B)$ which sends each $f \in \mathcal{C}_R(A, B) = \mathcal{C}(R^{A_1} \times \cdots \times R^{A_n}, R^{B_1} \times \cdots \times R^{B_m})$ to $Yf \in \mathbf{PSet}^{\mathcal{C}^{\mathrm{op}}}_{YR}((YR)^{YA_1} \times \cdots \times (YR)^{YA_n}, (YR)^{YB_1} \times \cdots \times (YR)^{YB_m})$, and such that, for $f, g \in \mathcal{C}_R(A, B)$, $Y_R(f) \sim Y_R(g)$ if and only if $f \sim g$.

This Y_R is not actually the Yoneda embedding any more, but it is still an embedding and, moreover, a **PSet**-coc functor since one has the required **PSet**-natural

[3] The notation $\mathbf{PSet}^{\mathcal{C}^{\mathrm{op}}}_{YR}$ is handy but ambiguous in that YR can be understood differently; the only correct understanding is $[\mathcal{C}^{\mathrm{op}}, \mathbf{PSet}]_{YR}$.

isomorphisms in $\mathbf{PSet}_{YR}^{\mathcal{C}^{\mathrm{op}}}$ due to the fact that Y preserves finite products:

$$\begin{aligned} r^R_{\langle A_1,\dots,A_n\rangle} :\ & \mathcal{C}(-,R)^{\mathcal{C}(-,A_1)} \times \cdots \times \mathcal{C}(-,R)^{\mathcal{C}(-,A_n)} \xrightarrow{\cong} \mathcal{C}(-,R^{A_1} \times \cdots \times R^{A_n}) \\ r^\times_{A_1,\dots,A_n} :\ & \mathcal{C}(-,R)^{\mathcal{C}(-,A_1)\times\cdots\times\mathcal{C}(-,A_n)} \xrightarrow{\cong} \mathcal{C}(-,R)^{\mathcal{C}(-,A_1\times\cdots\times A_n)} \end{aligned}$$

Hence one has the following lemma.

Lemma 2.5 *Let $\mathcal{C}$ be a* **PSet***-category of responses with the object of responses R, so that $\mathcal{C}_R$ is a* **PSet***-category of continuations. The restriction $Y_R : \mathcal{C}_R \to \mathbf{PSet}_{YR}^{\mathcal{C}^{\mathrm{op}}}$ of the* **PSet***-categorical Yoneda embedding $Y : \mathcal{C} \to \mathbf{PSet}^{\mathcal{C}^{\mathrm{op}}}$ on the full subcategory $\mathcal{C}_R$ is a* **PSet***-coc functor.*

3 $\lambda\mu$-calculus

3.1 Syntax of the $\lambda\mu$-calculus

Let $\sigma, \sigma_1, \dots$ range over a set Σ_T of **type constants**. Types, ranged over by $A, B, \dots$, are constructed by the grammar:

$$A ::= \sigma \mid B^A \mid \bot$$

Let V_o and V_c be two given countable disjoint sets of **object variables** $x, y, \dots$ and **control variables** $\alpha, \beta, \dots$, respectively. Let Σ_K be a set of typed **object constants** $c^A, c_1^B, \dots$ The pair (Σ_T, Σ_K) is called a **signature of the $\lambda\mu$-calculus**, and denoted by Σ. Terms, ranged over by $M, N, \dots$, are constructed by the grammar:

$$M ::= x \mid c^A \mid MN \mid \lambda x^A.M \mid [\alpha]M \mid \mu\alpha^A.M$$

Terms of the form MN, $\lambda x^A.M$, $\mu\alpha^A.M$ and $[\alpha]M$ are called respectively an **application**, a **λ-abstraction**, a **μ-abstraction** and a **named term**. In the terms $\lambda x^A.M$ and $\mu\alpha^A.M$, the object variable x, respectively the control variable α, is bound.

Now we define typing of the $\lambda\mu$-calculus. An **object context** is a finite, possibly empty sequence $\Gamma = x_1{:}B_1, x_2{:}B_2, \dots, x_n{:}B_n$ of pairs of an object variable and a type, such that $x_i \neq x_j$, for all $i \neq j$. A **control context** $\Delta = \alpha_1{:}A_1, \alpha_2{:}A_2, \dots, \alpha_m{:}A_m$ is defined analogously. A **typing judgement** is an expression of the form $\Gamma \vdash M : A \mid \Delta$. Such a judgement has an interpretation in sequent calculus for classical logic, with $\vdash$ interpreted as entailment symbol (formally corresponding to implication) and $\mid$ interpreted as disjunction. **Valid typing judgements** are derived using the typing rules in Table 1. An **equation** is an expression of the form $\Gamma \vdash M = N : A$, where $\Gamma \vdash M : A \mid \Delta$ and $\Gamma \vdash N : A \mid \Delta$ are valid typing judgements. We do not discuss here what a *valid* equation could be because we use an analogous notion of call-by-name equivalence introduced in §3.2.

$$\frac{}{\Gamma \vdash x : A \mid \Delta} \text{ if } x{:}A \in \Gamma \qquad \frac{}{\Gamma \vdash c^A : A \mid \Delta}$$

$$\frac{\Gamma \vdash M : B^A \mid \Delta \qquad \Gamma \vdash N : A \mid \Delta}{\Gamma \vdash MN : B \mid \Delta} \qquad \frac{\Gamma, x{:}A \vdash M : B \mid \Delta}{\Gamma \vdash \lambda x^A.M : B^A \mid \Delta}$$

$$\frac{\Gamma \vdash M : A \mid \Delta}{\Gamma \vdash [\alpha]M : \bot \mid \Delta} \text{ if } \alpha{:}A \in \Delta \qquad \frac{\Gamma \vdash M : \bot \mid \alpha{:}A, \Delta}{\Gamma \vdash \mu\alpha^A.M : A \mid \Delta}$$

$$\frac{\Gamma \vdash M : A \mid \Delta}{\Gamma' \vdash M : A \mid \Delta'} \text{ if } \Gamma \subseteq \Gamma',\ \Delta \subseteq \Delta'$$

Table 1
Typing rules of the $\lambda\mu$-calculus

3.2 Call-by-came continuation passing style translation

Consider the $\lambda\mu$-calculus over a given signature $\Sigma = (\Sigma_T, \Sigma_K)$. We will review the call-by-name semantics of this calculus by the continuation passing style (CPS) translation obtained in [6] and [14]. The target language of the CPS translation is a $\lambda^{R\times}$-calculus, that is a λ-calculus with products and a distinguished type R of responses. We assume that the target calculus has

- a type constant $\tilde{\sigma}$ for each type constant $\sigma \in \Sigma_T$ of the $\lambda\mu$-calculus;
- a pair of types K_A (the type of continuations of type A) and C_A (the type of computations of type A) for each type A of the $\lambda\mu$-calculus, defined as follows:

$$\begin{aligned} K_\sigma &= \tilde{\sigma} \qquad \text{where } \sigma \text{ is a type constant} \\ C_A &= R^{K_A} \\ K_{B^A} &= C_A \times K_B \\ K_\bot &= 1 \end{aligned}$$

- a constant $\tilde{c}$ for each object constant $c^A \in \Sigma_K$ of the $\lambda\mu$-calculus;
- a distinct variable $\tilde{x}$ and a distinct variable $\tilde{\alpha}$ for each object variable x and each control variable α, respectively.

The call-by-name CPS translation $\underline{M}$ of a typed term M is given in Table 2.

$$\begin{aligned} \underline{x} &= \lambda k^{K_A}.\tilde{x}k \quad \text{where } x : A \\ \underline{c^A} &= \lambda k^{K_A}.\tilde{c}k \\ \underline{MN} &= \lambda k^{K_B}.\underline{M}\langle \underline{N}, k\rangle \quad \text{where } M : B^A,\ N : A \\ \underline{\lambda x^A.M} &= \lambda \langle \tilde{x}, k\rangle^{K_{B^A}}.\underline{M}k \quad \text{where } M : B \\ \underline{[\alpha]M} &= \lambda k^{K_\bot}.\underline{M}\tilde{\alpha} \quad \text{where } M : A \\ \underline{\mu\alpha^A.M} &= \lambda \tilde{\alpha}^{K_A}.\underline{M} * \quad \text{where } M : \bot \end{aligned}$$

Table 2
The call-by-name CPS translation of the $\lambda\mu$-calculus

As can be easily seen, the CPS translation **respects the typing** in the sense that a judgement of the $\lambda\mu$-calculus

$$x_1{:}B_1, \ldots, x_n{:}B_n \vdash M : A \mid \alpha_1{:}A_1, \ldots, \alpha_m{:}A_m \tag{13}$$

is translated to the judgement of the $\lambda^{R\times}$-calculus

$$\tilde{x}_1{:}C_{B_1}, \ldots, \tilde{x}_n{:}C_{B_n}, \tilde{\alpha}_1{:}K_{A_1}, \ldots, \tilde{\alpha}_m{:}K_{A_m} \vdash \underline{M} : C_A \ . \tag{14}$$

Analogously, it respects the typing of equations. Therefore we can consistently use the notation $\underline{\Gamma \vdash M : A \mid \Delta}$ for the translation of a typing judgement, and $\underline{\Gamma \vdash M = N : A \mid \Delta}$ for the translation of an equation.

Let M and N be the terms of the $\lambda\mu$-calculus such that $\Gamma \vdash M : A \mid \Delta$ and $\Gamma \vdash N : A \mid \Delta$. We say that M and N are **call-by-name equivalent**, and then write $M =_{\Gamma,\Delta} N$, if $\underline{\Gamma \vdash M = N : A \mid \Delta}$, that is if the CPS translation of M is $\beta\eta$-equivalent in the context to the CPS translation of N. The **call-by-name $\lambda\mu$-theory determined by a $\lambda^{R\times}$-theory** $\mathcal{T}$ is then defined to be the set of all equations E of the $\lambda\mu$-calculus such that $\underline{E} \in \mathcal{T}$.

3.3 Categorical call-by-name interpretation of the $\lambda\mu$-calculus

The target $\lambda^{R\times}$-calculus of the above call-by-name CPS translation will now be interpreted in a **PSet**-category of responses $\mathcal{C}$. We give the **PSet**-enriched generalization of the construction that was originally developed for the ordinary category case in [6] and [14].

Since the CPS translation is type respecting, a typing $\lambda\mu$-judgement of the form (13) gives rise to a morphism in $\mathcal{C}$: $C_{B_1} \times \cdots \times C_{B_n} \times K_{A_1} \times \cdots \times K_{A_m} \to C_A$, which in turn, using $C_A = R^{K_A}$ and then using the canonical isomorphism (10), amounts to a morphism $R^{K_{B_1}} \times \cdots \times R^{K_{B_n}} \to R^{K_A \times K_{A_1} \times \cdots \times K_{A_m}}$, which now lies within the **PSet**-category of continuations $\mathcal{C}_R$. Note that we do not mention the computation types of the form C_A any more. Henceforth we will simply write A instead of K_A in the context of the CPS translation.

Let $\mathcal{C}_R$ be a **PSet**-category of continuations, and let $\Sigma = (\Sigma_T, \Sigma_K)$ be the signature of the $\lambda\mu$-calculus. Assume now a choice of an object $\tilde{\sigma} \in \mathrm{ob}(\mathcal{C})$ for every type constant $\sigma \in \Sigma_T$. Each type constructor is interpreted by the corresponding object constructor of **PSet**-categories of continuations:

$$\begin{aligned} [\![\sigma]\!] &= \tilde{\sigma}, \quad \text{where } \sigma \text{ is a type constant},\\ [\![B^A]\!] &= R^{[\![A]\!]} \times [\![B]\!],\\ [\![\bot]\!] &= \mathbf{1} \end{aligned} \tag{15}$$

If $\Gamma = x_1{:}B_1, \ldots, x_n{:}B_n$ is an object context, its interpretation in the **PSet**-category of continuations $\mathcal{C}$ is $R^{[\![B_1]\!]} \times \cdots \times R^{[\![B_n]\!]}$ which we abbreviate simply to $\prod_i R^{[\![B_i]\!]}$, and we denote the l-th projection map by $\pi_l : \prod_i R^{[\![B_i]\!]} \to R^{[\![B_l]\!]}$. If $\Delta = \alpha_1{:}A_1, \ldots, \alpha_m{:}A_m$ is a control context, its interpretation in $\mathcal{C}$ is $R^{[\![A_1]\!] \times \cdots \times [\![A_m]\!]}$, abbreviated to $R^{\prod_j [\![A_j]\!]}$, and we denote its l-th weakening map by $w_l = R^{\pi_l} : R^{[\![A_l]\!]} \to R^{\prod_j [\![A_j]\!]}$. Assume also a choice in $\mathcal{C}_R$ of a morphism $\tilde{c} : \mathbf{1} \to R^{[\![A]\!]}$ for each object constant $c^A \in \Sigma_K$. Now we can interpret a typing judgement $\Gamma \vdash M : A \mid \Delta$ as a morphism $[\![\Gamma \vdash M : A \mid \Delta]\!] : \prod_i R^{[\![B_i]\!]} \to R^{[\![A]\!] \times \prod_j [\![A_j]\!]}$ which can be abbreviated to $[\![M]\!]$ when the context allows. The inductive definition for the **PSet**-categorical interpretation is given in Table 3.

$$
\begin{aligned}
[\![\Gamma \vdash x_l : B_l \mid \Delta]\!] &= \prod_i R^{[\![B_i]\!]} \xrightarrow{\pi_l} R^{[\![B_l]\!]} \xrightarrow{w} R^{[\![B_l]\!] \times \prod_j [\![A_j]\!]} \\
[\![\Gamma \vdash c^A : A \mid \Delta]\!] &= \prod_i R^{[\![B_i]\!]} \xrightarrow{0} \mathbf{1} \xrightarrow{\tilde{c}} R^{[\![A]\!]} \xrightarrow{w} R^{[\![A]\!] \times \prod_j [\![A_j]\!]} \\
[\![\Gamma \vdash MN : B \mid \Delta]\!] &= \prod_i R^{[\![B_i]\!]} \xrightarrow{\langle [\![M]\!], [\![N]\!] \rangle} R^{R^{[\![A]\!]} \times [\![B]\!] \times \prod_j [\![A_j]\!]} \times R^{[\![A]\!] \times \prod_j [\![A_j]\!]} \\
&\xrightarrow{\varepsilon^{\prod_j [\![A_j]\!]}} R^{[\![B]\!] \times \prod_j [\![A_j]\!]} \\
[\![\Gamma \vdash \lambda x^A . M : B^A \mid \Delta]\!] &= \prod_i R^{[\![B_i]\!]} \xrightarrow{[\![M]\!]^*} R^{R^{[\![A]\!]} \times [\![B]\!] \times \prod_j [\![A_j]\!]} \\
[\![\Gamma \vdash [\alpha_l] M : \bot \mid \Delta]\!] &= \prod_i R^{[\![B_i]\!]} \xrightarrow{[\![M]\!]} R^{[\![A_l]\!] \times \prod_j [\![A_j]\!]} \xrightarrow{(w_l)^{\prod_j [\![A_j]\!]}} R^{\prod_j [\![A_j]\!] \times \prod_j [\![A_j]\!]} \\
&\xrightarrow{R^{\langle \prod_j \mathrm{id}_{[\![A_j]\!]}, \prod_j \mathrm{id}_{[\![A_j]\!]} \rangle}} R^{\prod_j [\![A_j]\!]} \xrightarrow{\cong} R^{\mathbf{1} \times \prod_j [\![A_j]\!]} \\
[\![\Gamma \vdash \mu\alpha^A . M : A \mid \Delta]\!] &= \prod_i R^{[\![B_i]\!]} \xrightarrow{[\![M]\!]} R^{\mathbf{1} \times [\![A]\!] \times \prod_j [\![A_j]\!]} \xrightarrow{\cong} R^{[\![A]\!] \times \prod_j [\![A_j]\!]}
\end{aligned}
$$

Table 3
The call-by-name interpretation of the $\lambda\mu$-calculus in a **PSet**-category of responses

Until now the description of the **PSet**-categorical interpretation did not differ from the ordinary categorical one [14] in any significant way. But the present interpretation does differ if one considers $\lambda\mu$-theories. In this case we define partial equivalence relations between morphisms to be given by the call-by-name equivalence in a $\lambda\mu$-theory, with the composition law defined componentwise, and with the obvious choice of identity elements (each of these being the unique morphism from a fixed variable of a given type to itself). This will be made precise in the definition of a syntactic **PSet**-category of continuations in 4.1. Meanwhile, analogously to the case of the ordinary categorical interpretation (cf. [14], Lemma 5.4) we relate the CPS and the **PSet**-categorical interpretations of the $\lambda\mu$-calculus by the following lemma, which can be proved by the straightforward induction on the complexity of $\lambda\mu$-terms.

Lemma 3.1 *Given a* **PSet***-category of continuations* $\mathcal{C}_R$*, the* **PSet***-categorical call-by-name interpretation of the* $\lambda\mu$*-calculus in* $\mathcal{C}_R$ *coincides with the interpretation of the call-by-name CPS translation in* $\mathcal{C}_R$*.*

From this lemma we immediately obtain the following soundness and completeness result for $\lambda^{R\times}$-theories (whose corresponding ordinary category case is Proposition 5.5 of [14]).

Corollary 3.2 *The $\lambda^{R\times}$-theories induced on the $\lambda\mu$-calculus by the* **PSet**-*categorical call-by-name interpretation are precisely the $\lambda^{R\times}$-theories induced by the call-by-name CPS translation.*

Therefore $\lambda\mu$-theories intuitively can be seen as embedded into $\lambda^{R\times}$-theories, and interpreted semantically as $\lambda^{R\times}$-theories of a special kind.

4 Categorical semantics of normalization in $\lambda\mu$-calculus

4.1 Syntactic **PSet**-*category of continuations and canonical call-by-name interpretation of the $\lambda\mu$-calculus*

Let x be a fixed object variable. We say that a $\lambda^{R\times}$-judgement is in **standard form** if it has the form

$$x{:}A_1 \times \cdots \times A_n \vdash M : B_1 \times \cdots \times B_m,$$

that is if the object context declares exactly the one variable x. Every $\lambda^{R\times}$-judgement $x_1{:}R^{A_1}, \ldots, x_n{:}R^{A_n} \vdash M : R^{B_1} \times \cdots \times R^{B_m}$ has a standard form

$$x{:}R^{A_1} \times \cdots \times R^{A_n} \vdash (\lambda x_1 \ldots x_n.M)(\pi_1 x) \ldots (\pi_n x) : R^{B_1} \times \cdots \times R^{B_m}. \tag{16}$$

Therefore the call-by-name CPS translation of every $\lambda\mu$-judgement has a yet simpler standard form

$$x{:}R^{A_1} \times \cdots \times R^{A_n} \vdash (\lambda x_1 \ldots x_n.M)(\pi_1 x) \ldots (\pi_n x) : R^{B}. \tag{17}$$

Using the notion of a standard form, we will define structural operations of **PSet**-categories of continuations in a way similar to defining structural operations of cartesian closed categories by typing judgements of simply typed lambda calculus. The following lemma can be easily checked case by case.

Lemma 4.1 *The structural operations of a* **PSet**-*category of responses $\mathcal{C}$ with the object of responses R are defined by the operations on typing judgements shown in Table 4, with pers on hom-objects defined by* id $\sim$ id, $0 \sim 0$, $\pi_l \sim \pi_l$, $\varepsilon \sim \varepsilon$, $[\![\Gamma \vdash M : A \mid \Delta]\!] \sim [\![\Gamma \vdash N : A \mid \Delta]\!]$ *if M and N are call-by-name equivalent, and such that all the operations respect $\sim$.*

The following lemma is just an instantiation of Lemma 4.1 (see also Lemma 5.6 of [14] for a closely related case of the structural operations of control categories).

Lemma 4.2 *The structural operations of a* **PSet**-*category of continuations $\mathcal{C}_R$ are defined by the operations on typing judgements shown in Table 5, with pers on hom-objects defined by* id $\sim$ id, $0 \sim 0$, $\pi_l \sim \pi_l$, $\varepsilon \sim \varepsilon$, $[\![\Gamma \vdash M : A \mid \Delta]\!] \sim [\![\Gamma \vdash N : A \mid \Delta]\!]$ *if M and N are call-by-name equivalent, and such that all the operations respect $\sim$.*

$$
\begin{array}{ll}
\mathrm{id} = x{:}A \vdash x : A & \dfrac{f = x{:}A \vdash M : B \qquad g = x{:}B \vdash N : C}{g \circ f = x{:}A \vdash (\lambda x^B.N)M : C} \\
0 = x{:}A \vdash * : \mathbf{1} & \dfrac{f = x{:}A \vdash M : B \qquad g = x{:}A \vdash N : C}{\langle g, f\rangle = x{:}A \vdash \langle M, N\rangle : B \times C} \\
\pi_l = x{:}\prod_i A_i \vdash \pi_l x : A_l & \dfrac{f = x{:}A \times B \vdash M : R^C}{f^* = x{:}A \vdash \lambda y^B.(\lambda x^{A\times B}.M)\langle x, y\rangle : R^{B\times C}} \\
\varepsilon = x{:}R^{A\times B} \times A \vdash (\pi_1 x)(\pi_2 x) : R^B &
\end{array}
$$

Table 4
Operations of a **PSet**-category of responses on typing judgements

$$
\begin{array}{ll}
\mathrm{id} = x{:}\mathbf{R}^A \vdash x : \mathbf{R}^A & \dfrac{f = x{:}\mathbf{R}^A \vdash M : \mathbf{R}^B \qquad g = x{:}\mathbf{R}^B \vdash N : \mathbf{R}^C}{g \circ f = x{:}\mathbf{R}^A \vdash (\lambda x^{\mathbf{R}^B}.N)M : \mathbf{R}^C} \\
0 = x{:}\mathbf{R}^A \vdash * : \mathbf{1} & \dfrac{f = x{:}\mathbf{R}^A \vdash M : \mathbf{R}^B \qquad g = x{:}\mathbf{R}^A \vdash N : \mathbf{R}^C}{\langle g, f\rangle = x{:}\mathbf{R}^A \vdash \langle M, N\rangle : \mathbf{R}^B \times \mathbf{R}^C} \\
\pi_l = x{:}\prod_i R^{A_i} \vdash \pi_l x : R^{A_l} & \dfrac{f = x{:}\mathbf{R}^A \times \mathbf{R}^B \vdash M : R^C}{f^* = x{:}\mathbf{R}^A \vdash \lambda y^{\mathbf{R}^B}.(\lambda x^{\mathbf{R}^A\times\mathbf{R}^B}.M)\langle x, y\rangle : R^{\mathbf{R}^B\times C}} \\
\varepsilon = x{:}R^{\mathbf{R}^A\times B} \times \mathbf{R}^A \vdash (\pi_1 x)(\pi_2 x) : R^B &
\end{array}
$$

Table 5
Operations of a **PSet**-category of continuations on typing judgements

Given a $\lambda\mu$-signature $\Sigma = (\Sigma_T, \Sigma_K)$, we construct the **syntactic PSet-category of continuations** $\mathcal{C}_R^\Sigma$ as follows. First, we define its underlying **PSet**-category of responses $\mathcal{C}^\Sigma$ to consist of

- objects $\prod_{i=1}^n A_i$, for $n \geq 0$, where either $A_i \in \Sigma_T \cup \{R\}$ or $A_i = R^B$, for B an object;
- hom-objects $\mathcal{C}^\Sigma(A, B)$ containing an element $f \in \mathcal{C}^\Sigma(A, B)$, for each well-typed standard form $\lambda^{R\times}$-judgement $x{:}A \vdash M : B$, with the per of morphisms from A to B being the least per containing id $\sim$ id (if $A = B$), $0 \sim 0$ (if $B = \mathbf{1}$), $\pi_l \sim \pi_l$ (if $B = A_l$), $\varepsilon \sim \varepsilon$ (if $A = R^C \times C$ and $B = R$), $[\![\Gamma \vdash M : A \mid \Delta]\!] \sim [\![\Gamma \vdash N : A \mid \Delta]\!]$ if M and N are call-by-name equivalent, and closed under the structural operations of **PSet**-categories of responses shown in Table 4;
- for each triple of objects A, B, C, the composition law $\circ = \circ_{A,B,C}$ defined componentwise: for $f = (x{:}A \vdash M : B)$ and $g = (x{:}B \vdash N : C)$, their composition is $g \circ f = (x{:}A \vdash (\lambda x^B.N)M : C)$, — and interacting with pers of the corresponding hom-objects as follows: for $f, f' \in \mathcal{C}^\Sigma(A, B)$ and $g, g' \in \mathcal{C}^\Sigma(B, C)$, it holds that $g \circ f \sim_{\mathcal{C}^\Sigma(A,C)} g' \circ f'$ if and only if $f \sim_{\mathcal{C}^\Sigma(A,B)} f'$ and $g \sim_{\mathcal{C}^\Sigma(B,C)} g'$;
- for each object A, the identity element $\mathrm{id} = \mathrm{id}_A = (x{:}A \vdash x : A)$.

Second, we construct the required syntactic **PSet**-category of continuations $\mathcal{C}_R^\Sigma$ as follows:

- objects are n-tuples of objects of $\mathcal{C}^\Sigma$, for $n \geq 0$;

- for $A, B \in \mathrm{ob}(\mathcal{C}^{\Sigma}_{R})$, their hom-object $\mathcal{C}^{\Sigma}_{R}(A,B) = \mathcal{C}^{\Sigma}_{R}(\langle A_1,\dots,A_n\rangle, \langle B_1,\dots,B_m\rangle)$ is the hom-object $\mathcal{C}^{\Sigma}(R^{A_1} \times \cdots \times R^{A_n}, R^{B_1} \times \cdots \times R^{B_m})$ of the underlying **PSet**-category of responses, named by $x{:}\mathbf{R}^A \vdash M : \mathbf{R}^B$;
- composition laws and identity elements are borrowed from $\mathcal{C}^{\Sigma}$ as usual.

There is a **free call-by-name interpretation** $[\![-]\!]^0$ (sometimes also called a canonical call-by-name interpretation) of the $\lambda\mu$-calculus with signature Σ in $\mathcal{C}^{\Sigma}_{R}$, defined by $\tilde{\sigma} = \sigma$ and $\tilde{c} = x{:}\mathbf{1} \vdash c : R^A$, for each $c^A \in \Sigma_K$. It has the property that the interpretation of each typing judgement is call-by-name equivalent to its standard form. The pair $(\mathcal{C}^{\Sigma}_{R}, [\![-]\!]^0)$ is determined up to isomorphism by the following universal property: For each call-by-name interpretation $[\![-]\!]$ in $\mathcal{D}_{R'}$, which agrees with $[\![-]\!]^0$ on generators, there is a unique (up to **PSet**-natural isomorphism) **PSet**-coc functor

$$Q : \mathcal{C}^{\Sigma}_{R} \to \mathcal{D}_{R'} \tag{18}$$

such that $Q[\![A]\!]^0 = [\![A]\!]$ for all $\lambda\mu$-types A, and $Q[\![\Gamma \vdash M : A \mid \Delta]\!]^0 = [\![\Gamma \vdash M : A \mid \Delta]\!]$ for all well-typed judgements $\Gamma \vdash M : A \mid \Delta$.

Assume that $H : \mathcal{C}^{\Sigma}_{R} \to \mathcal{D}_{R'}$ is another **PSet**-coc functor such that $H[\![-]\!]^0 = [\![-]\!]$. We will exhibit a **PSet**-natural isomorphism $\iota : Q \to H$ by induction on the complexity of an object of $\mathcal{C}^{\Sigma}_{R}$ using the structural **PSet**-natural isomorphisms r^R and $r^{\times}$ from (12):

$$\begin{aligned}
&\iota_{\langle\rangle} = r^R_{\langle\rangle}, \quad \iota^{-1}_{\langle\rangle} = 0_{H\langle\rangle}, && \iota_{\langle\sigma\rangle} = \iota^{-1}_{\langle\sigma\rangle} = \mathrm{id}_{\langle\sigma\rangle}, \quad \iota_{\langle\mathbf{1}\rangle} = \iota^{-1}_{\langle\mathbf{1}\rangle} = \mathrm{id}_{\langle\mathbf{1}\rangle}, \\
&\iota_{\langle R^A\rangle} = r^R_{\langle A\rangle}(\iota_{\langle\mathbf{1}\rangle}\varepsilon\langle\pi_1, \iota^{-1}_{\langle A\rangle}\pi_2\rangle)^*, && \iota^{-1}_{\langle R^A\rangle} = (\iota^{-1}_{\langle\mathbf{1}\rangle}(H\varepsilon)r^{\times}_{R^A,A}\langle\pi_1, \iota_{\langle A\rangle}\pi_2\rangle)^*, \\
&\iota_{\langle A_1\times\cdots\times A_n\rangle} = \iota_{\langle A_1\rangle} \times \cdots \times \iota_{\langle A_n\rangle}, && \iota^{-1}_{\langle A_1\times\cdots\times A_n\rangle} = \iota^{-1}_{\langle A_1\rangle} \times \cdots \times \iota^{-1}_{\langle A_n\rangle}, \\
&\iota_{\langle A_1,\dots,A_n\rangle} = \langle \iota_{\langle A_1\rangle}, \dots, \iota_{\langle A_n\rangle}\rangle, && \iota^{-1}_{\langle A_1,\dots,A_n\rangle} = \langle \iota^{-1}_{\langle A_1\rangle}, \dots, \iota^{-1}_{\langle A_n\rangle}\rangle,
\end{aligned} \tag{19}$$

where $n > 1$. The above given components of ι define the required **PSet**-natural isomorphism. The fact is easy to establish by checking the condition (5) routinely.

4.2 The normal form function

Henceforth let Y denote the **PSet**-categorical Yoneda embedding $Y : \mathcal{C}^{\Sigma} \to \mathbf{PSet}^{(\mathcal{C}^{\Sigma})^{\mathrm{op}}}$ defined as in 2.3 and discussed further in 2.4. Recall that Y_R is the restriction of Y on $\mathcal{C}^{\Sigma}_{R}$. We let $\{\!|-|\!\} : \mathcal{C}^{\Sigma}_{R} \to \mathbf{PSet}^{(\mathcal{C}^{\Sigma})^{\mathrm{op}}}_{YR}$ be the **PSet**-coc functor freely extending the interpretation of base types by Y_R. We deliberately chose the bracketed notation for the free extension functor to emphasize the fact that $\{\!|-|\!\}$ is also an interpretation (moreover, a free interpretation) and, besides, to improve readability.

By the universal property of the pair $(\mathcal{C}^{\Sigma}_{R}, [\![-]\!]^0)$, there is a natural isomorphism $\iota : \{\!|-|\!\} \xrightarrow{\cong} Y_R$. Hence, by Corollary 2.3, for each hom-object of $\mathcal{C}^{\Sigma}_{R}$, we can construct the inverse of the interpretation $\llbracket - \rrbracket$ on this hom-object according to the diagram

$$\begin{tikzcd} \mathcal{C}_R^\Sigma(A,B) \arrow[rr, "\llbracket - \rrbracket"] \arrow[dr, shift left, "Y_R"] & & \mathbf{PSet}_{YR}^{(\mathcal{C}^\Sigma)^{\mathrm{op}}}(\llbracket A \rrbracket, \llbracket B \rrbracket) \arrow[dl, "\iota_B \circ \llbracket - \rrbracket \circ \iota_A^{-1}"] \\ & \mathbf{PSet}_{YR}^{(\mathcal{C}^\Sigma)^{\mathrm{op}}}(Y_R A, Y_R B) \arrow[ul, shift left, "-_A(\mathrm{id}_A)"] & \end{tikzcd} \tag{20}$$

Hence, for any $f \in \mathcal{C}_R^\Sigma(A,B)$, we obtain a **PSet**-natural transformation

$$YA \xrightarrow{\iota_A^{-1}} \llbracket A \rrbracket \xrightarrow{\llbracket f \rrbracket} \llbracket B \rrbracket \xrightarrow{\iota_B} YB$$

which, if further evaluated at A, gives

$$\mathcal{C}_R^\Sigma(A,A) \xrightarrow{\iota_{A,A}^{-1}} \llbracket A \rrbracket A \xrightarrow{\llbracket f \rrbracket_A} \llbracket B \rrbracket A \xrightarrow{\iota_{B,A}} \mathcal{C}_R^\Sigma(A,B).$$

Thus we define the normal form function to be

$$\mathrm{nf}(f) = \iota_{B,A}(\llbracket f \rrbracket_A(\iota_{A,A}^{-1}(\mathrm{id}_A))), \tag{21}$$

where $f \in \mathcal{C}_R^\Sigma(A,B)$ is a morphism named by a $\lambda^{R\times}$-judgement $x{:}\mathbf{R}^A \vdash M : \mathbf{R}^B$, and id_A is the morphism named by $x{:}\mathbf{R}^A \vdash x : \mathbf{R}^A$. Note that in (21) we could equally write the judgements naming the morphisms f and id_A. The "functional" notation $\mathrm{nf}(f)$ is especially handy if we note that the typing contexts are implicitly given in f, thus one should not write them explicitly.

In the setting of a syntactic **PSet**-category of continuations $\mathcal{C}_R^\Sigma$, nf is a per-function on $\lambda^{R\times}$-terms (and not just a function on $\beta\eta$-convertibility classes of $\lambda^{R\times}$-terms, as it would be if one considered only the underlying ordinary categories).

4.3 Characterization of categorical normal forms

We will check whether our nf from 4.2 satisfies the characteristic properties (NF1) and (NF2) from Introduction.

Recall from 3.2 that, for $\lambda\mu$-terms M and N, we write $M = N$ if M and N are call-by-name equivalent. Without loss of generality we assume that interpretations of $\lambda\mu$-terms in the syntactic **PSet**-category of continuations $\mathcal{C}_R^\Sigma$ are given in their standard forms.

Theorem 4.3 (Completeness, NF1) *There is a function* fn *from abstract normal forms to terms such that, for a well-typed $\lambda\mu$-judgement $\Gamma \vdash M : C \mid \Delta$, it holds that* $\mathrm{fn}(\mathrm{nf}(\llbracket \Gamma \vdash M : C \mid \Delta \rrbracket^0)) =_{\Gamma,\Delta} M$.

Proof (sketch). Let $\Gamma = x_1{:}A_1, \ldots, x_n{:}A_n$, $\Delta = \alpha_1{:}B_1, \ldots, \alpha_m{:}B_m$ and $f =$

$[\![\Gamma \vdash M : C \mid \Delta]\!]^0$, and let $f \in \mathcal{C}_R^{\Sigma}(A, B)$ be of the form

$$x{:}R^{A_1} \times \cdots \times R^{A_n} \vdash (\lambda x_1 \ldots x_n.\underline{M})(\pi_1 x) \ldots (\pi_n x) : R^{C \times B_1 \times \cdots \times B_m}$$

(by our global assumption interpretations are given in standard form). Since $\mathcal{C}_R^{\Sigma}$ has the canonical structure of **PSet**-categories of continuations, we can use induction on the complexity of $\underline{M}$ to prove that $\mathrm{nf}(f) \sim g$ for some g in standard form:

$$g \;=\; x{:}R^{A_1} \times \cdots \times R^{A_n} \vdash (\lambda x_1 \ldots x_n.\underline{N})(\pi_1 x) \ldots (\pi_n x) : R^{C \times B_1 \times \cdots \times B_m}.$$

Since nf is a per preserving function, it follows that $f \sim g$, i.e., the $\lambda^{R\times}$-term naming f is $\beta\eta$-equivalent to the $\lambda^{R\times}$-term naming $\mathrm{nf}(f)$. Therefore $\underline{M}$ and $\underline{N}$ are $\beta\eta$-equivalent, and hence M and N are call-by-name equivalent. Thus we can put $\mathrm{fn}(\mathrm{nf}([\![\Gamma \vdash M : C \mid \Delta]\!]^0)) := N$. □

In the following soundness theorem we employ the method developed in [4], Section 3.4, for proving the *uniqueness* property of categorical normal forms there. We modify this method to suit the case of **PSet**-categories of continuations.

Theorem 4.4 (Soundness, NF2) *If M and N are of type C, and $M =_{\Gamma,\Delta} N$, it holds that* $\mathrm{nf}([\![\Gamma \vdash M : C \mid \Delta]\!]^0) \equiv \mathrm{nf}([\![\Gamma \vdash N : C \mid \Delta]\!]^0)$.

Proof. Let $f = [\![\Gamma \vdash M : C \mid \Delta]\!]^0$, $g = [\![\Gamma \vdash N : C \mid \Delta]\!]^0$ and $f, g \in \mathcal{C}_R^{\Sigma}(A, B)$. Let $\mathcal{C}_R^{\Sigma,\equiv}$ denote the **PSet**-category of continuations which has the same objects and the same underlying sets of morphisms as $\mathcal{C}_R^{\Sigma}$, but whose pers $\equiv$ on morphisms are given by α-congruence on terms naming the morphisms. Observe that the **PSet**-category of responses $\mathcal{C}^{\Sigma,\equiv}$ is already given with $\mathcal{C}_R^{\Sigma,\equiv}$ by construction. By analogy with (20), consider a **PSet**-coc functor $\llbracket - \rrbracket^{\equiv} : \mathcal{C}_R^{\Sigma} \to \mathbf{PSet}_{YR}^{(\mathcal{C}^{\Sigma,\equiv})^{\mathrm{op}}}$ freely extending the interpretation of objects of $\mathcal{C}_R^{\Sigma}$ by Yoneda $Y_R^{\equiv} : \mathcal{C}_R^{\Sigma} \to \mathbf{PSet}^{(\mathcal{C}^{\Sigma,\equiv})^{\mathrm{op}}}$. Being a **PSet**-coc functor, $\llbracket - \rrbracket^{\equiv}$ is such that $f \sim g$ implies $\llbracket f \rrbracket^{\equiv} \equiv \llbracket g \rrbracket^{\equiv}$. By the universal property of $(\mathcal{C}_R^{\Sigma}, [\![-]\!]^0)$, it holds that $\iota_{B,A} \llbracket f \rrbracket_A^{\equiv} \iota_{A,A}^{-1}(\mathrm{id}_A) \equiv \iota_{B,A} \llbracket g \rrbracket_A^{\equiv} \iota_{A,A}^{-1}(\mathrm{id}_A)$. Clearly, the ordinary categories underlying $\mathbf{PSet}_{YR}^{(\mathcal{C}^{\Sigma})^{\mathrm{op}}}$ and $\mathbf{PSet}_{YR}^{(\mathcal{C}^{\Sigma,\equiv})^{\mathrm{op}}}$ have the same structure of categories of continuations. Therefore $\iota_{B,A} \llbracket f \rrbracket_A^{\equiv} \iota_{A,A}^{-1}(\mathrm{id}_A) = \iota_{B,A} \llbracket f \rrbracket_A \iota_{A,A}^{-1}(\mathrm{id}_A)$ and $\iota_{B,A} \llbracket g \rrbracket_A^{\equiv} \iota_{A,A}^{-1}(\mathrm{id}_A) = \iota_{B,A} \llbracket g \rrbracket_A \iota_{A,A}^{-1}(\mathrm{id}_A)$. Hence $\mathrm{nf}(f) \equiv \mathrm{nf}(g)$. □

5 Conclusions

We have shown that the method of normalization by the Yoneda embedding [4,1] can be successfully applied to the problem of normalization of simply typed $\lambda\mu$-terms. We obtained a solution for this problem by developing an apparatus of categories of continuations enriched over the category of sets with partial equivalence relations and functions preserving these relations. As a result we constructed a sound and complete denotational semantics of call-by-name normalization in simply typed $\lambda\mu$-calculus. An important role in this semantics was played by the call-by-name continuation passing style translation obtained in [6,14].

Future research should be dedicated to algorithmization of the soundness and completeness theorems (for the purpose of extracting functional programs from soundness and completeness proofs) and to extensions of the normalization method onto more expressive calculi such as the simply typed $\lambda\mu$-calculus with products and coproducts.

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