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Z-abstract Basis [1]

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Abstract

In this paper, the notion of a Z-abstract basis is defined and its properties are discussed. It is proved that for a union-complete subset system, the basis of a Z-continuous poset, with the restricted $\ll_Z$-relation, is a Z-abstract basis and the rounded Z-ideal completion of the Z-abstract basis is isomorphic to the Z-continuous poset. Conversely, the rounded Z-ideal completion of a Z-abstract basis is a Z-continuous poset.

Keywords: Z-subset system; Z-continuous poset; Z-basis; Z-abstract basis

1 Introduction

The use of Z-subset systems to study the common properties of classes of posets such as completely distributive lattices and continuous posets first appeared in [1]. Responding to the suggestion in [1], H. Bandelt and M. Erné in [2] studied Z-continuous posets, and this research was carried on by P. Venugopalan in [3]. Z-subset system is a generalization of directed set system. Since the concept was defined, many theorems in domain theory have been given the corresponding conclusions with respect to Z-subset system. As is known to the mathematicians in domain theory, abstract basis is an important concept because of its relation to continuous posets. Can the corresponding concept of abstract basis related to Z-subset system been defined? How is its

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relation to Z-continuous posets? In this paper, we introduce the concept of Z-abstract basis which is a generalization of the notion of an abstract basis for continuous domains. The basic properties of Z-abstract basis and its relation to Z-continuous posets are discussed.

Definition 1.1 [4] A subset system on **Poset** is a functor $Z : \mathbf{Poset} \longrightarrow \mathbf{Set}$ satisfying the following conditions:

(i) For any poset P, $Z(P) \subseteq 2^P$;

(ii) If P and Q are two posets and $f : P \longrightarrow Q$ a monotone mapping, then for any $A \in Z(P)$, $Z(f)(A) = f(A) \in Z(Q)$;

(iii) $Z(P)$ contains a nonempty, nonsingleton set for some poset P.

Remark 1.2 [4] (i) For any poset P and $p \in P$, we have $\{p\} \in Z(P)$.

(ii) If P is a poset and $Q \subseteq P$, then $Z(Q) \subseteq Z(P)$.

(iii) For each P and each $\{x, y\} \subseteq P$ such that $x \leq y$, we have $\{x, y\} \in Z(P)$.

Definition 1.3 [4] A poset P is said to be Z-complete if for each $S \in Z(P)$, $\vee S$ exists.

Definition 1.4 [4] A subset I of a poset P is said to be a Z-ideal of P if it is a lower set generated by some $S \in Z(P)$.

We shall denote by $ZI(P)$ the set of all Z-ideals of P, ordered by inclusion.

A Z-ideal of P is said to be a principal ideal of P if it is generated by some singleton subset of P.

Definition 1.5 [4] A subset system Z is union-complete iff for each poset P, $ZI(P)$ ordered by inclusion, is Z-complete and for each $\mathcal{S} \in Z(ZI(P))$, $\vee\mathcal{S} = \cup\mathcal{S}$.

Definition 1.6 [4] Let P be a poset, x, y two elements of P. We say that x is way below y with respect to Z, in symbol, $x \ll_Z y$, iff for any $S \in Z(P)$ such that $\vee S$ exists, the relation $y \leq \vee S$ implies the existence of an element $s \in S$ with $x \leq s$. For any element x of a poset P we write $J(x) = \{u \in P : u \ll_Z x\}$.

Definition 1.7 [4] A poset P is Z-continuous if for every $p \in P$ there is a set $S_p \in Z(P)$ such that $S_p \subseteq J(p)$ and $p = \vee S_p$.

Definition 1.8 [5] Let P be a poset and Z a subset system. A subset $B \subseteq P$ is called a Z-basis of P, if for each $x \in P$, there exists $S \in Z(B)$ such that $S \subseteq J(x)$ and $x = \vee S$.

Proposition 1.9 [5] *Let P be a Z-continuous poset, $B \subseteq P$. The following conditions are equivalent:*

(i) B is a Z-basis of P;

(ii) $\forall\, x \in P$, $J(x) \cap B \in ZI(B)$ and $x = \vee(J(x) \cap B)$.

If Z is a union-complete subset system, then (i) and (ii) are also equivalent to the following conditions:

(iii) $\forall\, x \in P$, there exists $S \in Z(B)$ such that $\downarrow_B S = J(x) \cap B$ and for every $y \ll_Z x$, there exists $b \in B$ such that $y \leq b \ll_Z x$;

(iv) $\forall\, x \in P$, there exists $S \in Z(B)$ such that $\downarrow_B S = J(x) \cap B$ and for every $y \ll_Z x$, there exists $b \in B$ such that $y \ll_Z b \ll_Z x$.

Definition 1.10 [6] An abstract basis is given by a set B together with a transitive relation $\prec$ satisfying the following interpolation property:

$$\text{(INT)}\ \forall \mid F \mid < \infty,\ F \prec z \Rightarrow \exists\, y \prec z \text{ such that } F \prec y$$

where $F \prec y$ means $\forall\, t \in F,\ t \prec y$. (INT) is also called full transitivity.

For an abstract basis $(B, \prec)$, taking $F = \emptyset$, the above interpolation property implies that for all $y \in B$, there is a $z \in B$ such that $z \prec y$. If we take F to be a singleton, then the interpolation properties implies that for all $x \prec y$, there is some $z \in B$ such that $x \prec z \prec y$, a narrow meaning of interpolation property.

2 $Z-$abstract Basis and Main Results

In order to define Z-abstract basis, we introduce the following category **TS** and define a Z-subset system on it which is a generalization of that on **Poset**.

The category whose objects are sets equipped with a transitive relation and whose morphisms are maps preserving the transitive relations will be denoted by **TS**.

Remark 2.1 **TS** is a subcategory of **Poset**.

In fact, every poset is a set with a partial order which is transitive, whence $\mathcal{O}b(\mathbf{Poset}) \subseteq \mathcal{O}b(\mathbf{TS})$. Also, each monotone map preserves the transitive relation and so $\mathcal{M}or(\mathbf{Poset}) \subseteq \mathcal{M}or(\mathbf{TS})$.

Definition 2.2 A subset system on **TS** is a functor $Z : \mathbf{TS} \longrightarrow \mathbf{Set}$ satisfying the following conditions:

(i) For any **TS**-object $(S, \prec)$, $Z(S) \subseteq 2^S$;

(ii) If S and T are two **TS**-objects and $f : S \longrightarrow T$ a transitive relation-preserving mapping, then for any $A \in Z(S)$, $Z(f)(A) = f(A) \in Z(T)$;

(iii) $Z(S)$ contains a nonempty, nonsingleton set for some **TS**-object $(S, \prec)$.

Definition 2.3 A subset I of a **TS**-object $(B, \prec)$ is said to be a Z-ideal of

B if it is a $\prec$-lower set generated by some $S \in Z(B)$, that is, $I = \downarrow S$ which is defined as $\{x \in B : \text{there exists } s \in S \text{ such that } x \prec s\}$.

We shall denote by $ZI(B)$ the set of all Z-ideals of B, ordered by inclusion.

A Z-ideal of B is said to be a principal ideal of B if it is generated by some singleton subset of B, that is, there exists $a \in B$ such that $I = \downarrow a$ which is defined as $\{x \in B : x \prec a\}$.

Definition 2.4 A subset system Z is union-complete iff for each **TS**-object $(B, \prec)$, $ZI(B)$ ordered by inclusion, is Z-complete and for each $\mathcal{S} \in Z(ZI(B))$, $\vee\mathcal{S} = \cup\mathcal{S}$.

Definition 2.5 We define an Z-abstract basis to be a nonempty set B together with a binary relation $\prec$ which is transitive and satisfies the following properties:

(i) $\forall\, y \in B$, there is a $z \in B$ such that $z \prec y$;

(ii) $\forall\, x, y \in B$ with $x \prec y$, there is some $z \in B$ such that $x \prec z \prec y$;

(iii) $\forall\, x \in B$, the set $\downarrow x = \{y \in B : y \prec x\} \in ZI(B)$.

Definition 2.6 A Z-ideal I of a Z-abstract basis B is called a rounded Z-ideal if for any $y \in I$, there exists $x \in I$ such that $y \prec x$.

The set of all rounded Z-ideals of B ordered by set inclusion is called the rounded Z-ideal completion of B, denoted by $RZI(B)$.

Theorem 2.7 *Let B be a Z-basis of a Z-complete Z-continuous poset P and Z a union-complete subset system such that all the Z-sets are nonempty . Then*

(i) If we restrict the way-below relation $\ll_Z$ on P to the Z-basis B, then it satisfies the axioms of a Z-abstract basis.

(ii) For every $x \in P$, $I_x = J(x) \cap B$ is a rounded Z-ideal of B.

(iii) $\varphi : x \mapsto I_x : P \longrightarrow RZI(B)$ is an order isomorphism, the inverse map being $\phi : I \mapsto \vee I : RZI(B) \longrightarrow P$.

Proof. (i) (1) It is clear that $\ll_Z$-relation is transitive on B.

(2) $\forall\, y \in B$, $J(y) \cap B \in ZI(B)$. So, there exists $z \in B$ such that $z \ll_B y$.

(3) If $x, y \in B$ with $x \ll_Z y$, then, from Proposition 1.9, there is some $z \in B$ such that $x \ll_Z z \ll_Z y$;

(4) By Proposition 1.9, we know that for all $x \in B$, the set $\downarrow_B x = J(x) \cap B \in ZI(B)$.

Thus, $(B, \ll_Z)$ is a Z-abstract basis.

(ii) (1) For all $x \in B$, the set $\downarrow_B x = J(x) \cap B \in ZI(B)$;

(2) For any $y \in J(x) \cap B$, by Proposition 1.9, there exists $z \in J(x) \cap B$ such that $y \ll_Z z$.

Thus, $I_x = J(x) \cap B$ is a rounded Z-ideal of B.

(iii) On one hand, for $x \in P$, $\phi(\varphi(x)) = \vee I_x = \vee(J(x) \cap B) = x$; on the other hand, for $I \in RZI(B)$, $\varphi(\phi(I)) = J(\vee I) \cap B = I$. In fact, if $y \in I \subseteq B$, then there exists $z \in I$ such that $y \ll_Z z \leq \vee I$, and hence $y \in J(\vee I) \cap B$. If $y \in J(\vee I) \cap B$, then $y \ll_Z \vee I \leq \vee I$, and hence $y \in I$. $\square$

Corollary 2.8 [6] *Let B be a base of a domain L.*

(i) If we restrict the way-below relation $\ll$ on L to the base B, then it satisfies the axioms of an abstract basis.

(ii) For every $x \in L$, $I_x = \Downarrow x \cap B$ is a rounded ideal of B.

(iii) $x \mapsto I_x : L \longrightarrow RId\ B$ is an order isomorphism, the inverse map being $I \mapsto \vee I : RId\ B \longrightarrow L$.

Theorem 2.9 *Let $(B, \prec)$ be a Z-abstract basis and Z a union-complete subset system.*

(i) The rounded ideal completion $(RZI(B), \subseteq)$ is a Z-complete poset.

(ii) For $b \in B$, $\downarrow b = \{y \in B : y \prec b\}$ is a rounded Z-ideal of B.

(iii) For $I, J \in RZI(B)$, $I \ll_Z J$ in $RZI(B)$ iff there are $x, y \in J$ with $x \prec y$ such that $I \subseteq \downarrow x \subseteq \downarrow y \subseteq J$. In particular, if $a \prec b$, then $\downarrow a \ll_Z \downarrow b$.

(iv) The rounded Z-ideal completion $RZI(B)$ is a Z-continuous poset with the Z-ideals $\downarrow b$, $b \in B$, as a Z-basis.

Proof. (i) Let $\{I_k : k \in K\} \in Z(RZI(B)) \subseteq Z(ZI(B))$.

(1) Since Z is a union-complete subset system, $\bigcup_{k \in K} I_k \in ZI(B)$.

(2) $\forall\, x \in \bigcup_{k \in K} I_k$, there exists $k \in K$ such that $x \in I_k$. Since I_k is a rounded Z-ideal of B, there exists $y \in I_k \subseteq \bigcup_{k \in K} I_k$ such that $x \prec y$.

Thus, $(RZI(B), \subseteq)$ is a Z-complete poset.

(ii) Firstly, from the definition of Z-abstract basis, we know that for all $b \in B$, the set $\downarrow b = \{y \in B : y \prec b\} \in ZI(B)$.

Secondly, $\forall\, y \in \downarrow b$, there is some $z \in \downarrow b$ such that $y \prec z$,

Thus, $\downarrow b$ is a rounded Z-ideal of B.

(iii) (1) Since J is a $\prec$-lower set, $\forall\, x \in J$, $\downarrow x \subseteq J$ and so $\bigcup_{x \in J} \downarrow x \subseteq J$; Conversely, $\forall\, x \in J$, there exists $y \in J$ such that $x \prec y$. Then $x \in \downarrow y$ and so $J \subseteq \bigcup_{x \in J} \downarrow x$. Thus, $J = \bigcup_{x \in J} \downarrow x$.

(2) Since $J \in RZI(B) \subseteq ZI(B)$, there exists $S \in Z(B)$ such that $J = \downarrow S$. The map $i : (x \mapsto \downarrow x) : B \longrightarrow RZI(B)$ preserves the $\prec$-relation since $\prec$ is a transitive relation. Hence, for $S \in Z(B)$, the image of S, $i(S) = \{\downarrow x : x \in S\} \in Z(RZI(B))$. Also, $\bigcup_{x \in J} \downarrow x = \bigcup_{x \in S} \downarrow x$. Thus, $J = \bigcup_{x \in J} \downarrow x = \bigcup_{x \in S} \downarrow x$.

If $I \ll_Z J$ in $RZI(B)$, then there exists $x \in S \subseteq J$ such that $I \subseteq \downarrow x$. Since J is a rounded Z-ideal, for x, there exists $y \in J$ such that $x \prec y$. Thus $I \subseteq \downarrow x \subseteq \downarrow y \subseteq J$.

Conversely, if there are $x, y \in J$ with $x \prec y$ such that $I \subseteq \downarrow x \subseteq \downarrow y \subseteq J$, then $I \ll_Z J$ in $RZI(B)$. In fact, let $\{I_k : k \in K\} \in Z(RZI(B)) \subseteq Z(ZI(B))$ and $J \subseteq \bigcup_{k \in K} I_k$. $y \in J \subseteq \bigcup_{k \in K} I_k$ implies that there is some $k \in K$ such that $y \in I_k$, and hence $I \subseteq \downarrow x \subseteq \downarrow y \subseteq I_k$. Thus, $I \ll_Z J$ in $RZI(B)$.

(iv) From the proof of (ii), for each $I \in RZI(B)$, $A = \bigcup_{x \in S} \downarrow x$, where $S \in Z(B)$, $A = \downarrow S$ and $\{\downarrow x : x \in S\} \in Z(RZI(B))$. Also, $\forall\ x \in S$, $\downarrow x \ll_Z A$. Hence, the rounded Z-ideal completion $RZI(B)$ is a Z-continuous poset with the Z-ideals I_b, $b \in B$, as a Z-basis. □

Corollary 2.10 [6] *Let B be any abstract basis.*

(i) The set RId B of the rounded ideals ordered by inclusion is a dcpo (with directed suprema given by union).

(ii) For $b \in B$, $\downarrow b = \{a \in B : a \prec b\}$ is a rounded ideal of B.

(iii) $I \ll J$ holds in RId B iff there exists elements $a \prec b$ in B such that $I \subseteq \downarrow a \subseteq \downarrow b \subseteq J$. In particular, $\downarrow a \ll \downarrow b$ iff $a \prec b$.

(iv) The rounded ideals form a domain with the ideals $\downarrow b$, $b \in B$, as a basis.

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