Hierarchies of Function Classes Defined by the First-Value Operator
(Extended Abstract)

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Abstract

The first-value operator assigns to any sequence of partial functions of the same type a new such function. Its domain is the union of the domains of the sequence functions, and its value at any point is just the value of the first function in the sequence which is defined at that point.
In this paper, the first-value operator is applied to establish hierarchies of classes of functions under various settings. For effective sequences of computable discrete functions, we obtain a hierarchy connected with Ershov's one within Δ_2^0. The non-effective version over real functions is connected with the degrees of discontinuity and yields a hierarchy related to Hausdorff's difference hierarchy in the Borel class Δ_2^B. Finally, the effective version over approximately computable real functions forms a hierarchy which provides a useful tool in computable analysis.

Keywords: Hierarchies of functions, degree of discontinuity, computable analysis, effective descriptive set theory, Hausdorff hierarchy, Ershov hierarchy

1 Introduction and basic notions

Let $\mathcal{F} = (f_\xi)_{\xi < \alpha}$ be a finite or transfinite sequence of partial functions of some type. This means that $f_\xi : A \rightarrowtail B$, for sets A and B, α is an ordinal number, and ξ runs through the set $\{\xi : \xi < \alpha\} = \{\xi : \xi \in \alpha\} = \alpha$. The

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first-value operator, Φ, assigns to $\mathcal{F}$ the function $\overline{f} : A \rightarrowtail B$ defined by

$$\overline{f}(x) \simeq \begin{cases} f_{\xi_x}(x) & \text{if } \{\xi < \alpha : f_\xi(x) \downarrow\} \neq \emptyset \text{ and } \xi_x = \min\{\xi < \alpha : f_\xi(x) \downarrow\}, \\ \uparrow & \text{if } f_\xi(x) \uparrow \text{ for all } \xi < \alpha. \end{cases}$$

It was introduced by Epstein-Haas-Kramer [3] in order to describe the Ershov hierarchy of classes of discrete sets $M \subseteq \mathbb{N}^k$. In [6,7], we transferred this idea to Hausdorff's hierarchy establishing a substructure of the Borel class Δ_2^{B}, as well as to the Hausdorff-Ershov hierarchy within Δ_2^{ta}, the effective counterpart of Δ_2^{B}. In all these cases, emphasis was put on functions of type $f : A \rightarrowtail \{0,1\}$ characterizing sets as $M_f = \{x : f(x) = 1\}$.

In the present paper, the interest is directed to classes of functions themselves. The (minimal) length of sequences, which is needed to obtain a function by means of the first-value operator from some basic class $\mathfrak{F}$, determines the *level* (or degree) of that function w.r.t. $\mathfrak{F}$. This approach both yields a unifying view to some well-known concepts of classical computability theory and descriptive set theory, as well as it leads to some new features and tools concerning subjects of effective analysis.

We begin with explaining the fundamentals of the non-effective versions of *first-value hierarchies*. Let $\mathfrak{F}$ be a non-empty basic class of partial functions $f : A \rightarrowtail B$. For any ordinal number α, put

$$\nabla_\alpha(\mathfrak{F}) = \{ \Phi(\mathcal{F}) : \mathcal{F} = (f_\xi)_{\xi < \alpha}, \text{ where } f_\xi \in \mathfrak{F} \text{ for all } \xi < \alpha \}.$$

So, $\nabla_0(\mathfrak{F}) = \{\emptyset\}$, where $\emptyset$ is the *empty function*, and $\nabla_1(\mathfrak{F}) = \mathfrak{F}$. Moreover,

$$\nabla_{\alpha+1}(\mathfrak{F}) = \{ f : f(x) \simeq \begin{cases} g(x) & \text{if } g(x) \downarrow, \\ h(x) & \text{otherwise}, \end{cases} \quad \text{for } g \in \nabla_\alpha(\mathfrak{F}) \text{ and } h \in \mathfrak{F} \}.$$

From $\emptyset \in \mathfrak{F}$, it follows $\nabla_\alpha(\mathfrak{F}) \subseteq \nabla_\beta(\mathfrak{F})$ whenever $\alpha \leq \beta$. If $\bigcup_{\eta < \xi} \mathrm{dom}(f_\eta) \supseteq \mathrm{dom}(f_\xi)$, then the function f_ξ does not influence the result of the sequence, $\Phi(\mathcal{F})$. Thus, given a sequence $\mathcal{F} = (f_\xi)_{\xi < \alpha}$, by transfinite recursion one can define a sequence $\mathcal{F}' = (f'_\xi)_{\xi < \alpha'}$ such that $\alpha' \leq \alpha$, $\bigcup_{\eta < \xi} \mathrm{dom}(f'_\eta) \subset \bigcup_{\eta \leq \xi} \mathrm{dom}(f'_\eta)$ and $\Phi(\mathcal{F}) = \Phi(\mathcal{F}')$. So, for countable universes A, we can restrict ourselves to sequences $\mathcal{F}$ whose lengths α belong to the *second number class*, C_{II}, which consists of all ordinal numbers of finite or countably infinite cardinality. Also if A is a separable topological space and all $f \in \mathfrak{F}$ have open domains, sequences of lengths $\alpha \in \mathrm{C}_{\mathrm{II}}$ are sufficient to obtain all possible results of the first-value operator applied to arbitrary sequences built of functions from $\mathfrak{F}$.

The main subject of this paper is to explore properties of linear hierarchies $(\nabla_\alpha^{\cdots}(\mathfrak{F}))_{\alpha \in \mathrm{C}_{\mathrm{II}}}$, under several settings concerning the universes A and the basic class $\mathfrak{F}$, possibly combined with requirements concerning the sequences $\mathcal{F}$ to which the operator Φ has to be applied. They will be indicated by upper

labels at the ∇-sign.

For a first illustration, let $A = B = \mathbb{N}$ and consider the basic classes

$$\mathfrak{F}_{\mathrm{sing}} = \{\, f : \ f : \mathbb{N} \rightarrowtail \mathbb{N} \,,\ \mathrm{card}(f) \leq 1 \,\} \quad \text{and}$$
$$\mathfrak{F}_{\mathrm{fin}} = \{\, f : \ f : \mathbb{N} \rightarrowtail \mathbb{N} \,,\ \mathrm{card}(f) < \omega \,\}.$$

Obviously, $\nabla_n(\mathfrak{F}_{\mathrm{sing}}) = \{ f : \mathrm{card}(f) \leq n \}$ and $\nabla_n(\mathfrak{F}_{\mathrm{fin}}) = \mathfrak{F}_{\mathrm{fin}}$ if $1 \leq n < \omega$, but $\nabla_\alpha(\mathfrak{F}_{\mathrm{sing}}) = \nabla_\alpha(\mathfrak{F}_{\mathrm{fin}}) = \{ f : \ f : \mathbb{N} \rightarrowtail \mathbb{N} \}$ for all $\alpha \geq \omega$.

Quite analogous results hold for functions of type $f : \mathbb{N}^k \rightarrowtail \mathbb{N}$, $k \geq 1$, as well as for many other similar settings. So, the non-effective versions of first-value hierarchies over discrete universes collapse already at level ω with the largest possible class of partial functions, even for simple basic classes $\mathfrak{F}$.

To obtain more interesting results over discrete universes, the operator Φ will be restricted to effective sequences. This is just the subject of the next section. Then we shall consider real-valued functions over Euclidean spaces, $f : \mathbb{R}^k \rightarrowtail \mathbb{R}$. The non-effective setting studied in Section 3 provides the basic background of the effective version which is elaborated in Section 4.

2 The Ershov hierarchy for discrete functions

Now we are going to study the first-value operator on effective sequences of computable functions of type $f : \mathbb{N}^k \rightarrowtail \mathbb{N}$. The related notions and results are essentially known from the recursion theoretic setting around Ershov's hierarchy, cf. [3].

Effectivity for transfinite sequences is defined by means of constructive ordinals. We report some fundamental facts and denotations employed in the sequel. For more details, the reader may consult [16]. Let $(\phi_n : n \in \mathbb{N})$ be the Kleene numbering of the partial recursive functions.

A *naming system* (of ordinals), S, is given by a numbering $\nu_S : \mathbb{N} \rightarrowtail \mathrm{C}_{\mathrm{II}}$ such that

- $\mathrm{ran}(\nu_S)$ is an initial segment of ordinals, $\mathrm{ran}(\nu_S) = \alpha = \{\xi : \xi < \alpha\}$ for some $\alpha \in \mathrm{C}_{\mathrm{II}}$;
- for any $n \in \mathrm{dom}(\nu_S)$, it is decidable (by a partial recursive function k_S) whether $\nu_S(n) = 0$ or whether $\nu_S(n)$ is a successor or a limit number;
- for any number n, where $\nu_S(n)$ is a successor, a number n' is computable (by a partial recursive function p_S) such that $n \in \nu_S^{-1}(\nu_S(n') + 1)$;
- and for any name $n \in \nu_S^{-1}(\lambda)$ of a limit number λ, an index n' is computable (by a partial recursive function q_S) such that $\phi_{n'}$ is a total function and $(\nu_S(\phi_{n'}(m)))_{m \in \mathbb{N}}$ is an increasing sequence of ordinals converging to λ.

More precisely, $S = (\nu_S, k_S, p_S, q_S)$. An ordinal $\alpha \in \mathrm{C}_{\mathrm{II}}$ is called *con-*

structive iff there is a naming system S for which $\alpha \in \mathrm{ran}(\nu_S)$. S is called *recursively related* iff the set $\{(n, n') : n, n' \in \mathrm{dom}(\nu_S), \nu_S(n) \leq \nu_S(n')\}$ is recursive, it is called *univalent* iff ν_S is injective. To any constructive ordinal α, there is a *recursively related, univalent* (briefly, *r.r.u.*) system S assigning a name to that ordinal. So we can restrict ourselves to such special naming systems which are denoted in the form α/S. For $\alpha \leq \omega$ the canonical naming system with the identical mapping as numbering is always used, w.l.o.g.

For a naming system S, S-*computability* of a function $f : \mathbb{N}^k \times \mathrm{C}_{\mathrm{II}} \rightarrowtail \mathbb{N}$, $k \in \mathbb{N}$, means that there is a recursive function $\varphi : \mathbb{N}^k \times \mathbb{N} \rightarrowtail \mathbb{N}$ such that $\varphi(\vec{x}, n) \simeq f(\vec{x}, \nu_S(n))$ for all $\vec{x} \in \mathbb{N}^k$ and $n \in \mathrm{dom}(\nu_S)$. For a naming system α/S, an α-sequence $\mathcal{F} = (f_\xi)_{\xi < \alpha}$ of functions $f_\xi : \mathbb{N}^k \rightarrowtail \mathbb{N}$ is said to be S-*computable* iff there is an S-computable function $\overline{f} : \mathbb{N}^k \times \mathrm{C}_{\mathrm{II}} \rightarrowtail \mathbb{N}$ such that $\overline{f}(\vec{x}, \xi) \simeq f_\xi(\vec{x})$ for all $\vec{x} \in \mathbb{N}^k$ and all ordinals $\xi < \alpha$. Then, if S is recursively related, the domain of the function $\Phi(\mathcal{F})$ is r.e.: $\mathrm{dom}(\Phi(\mathcal{F})) = \bigcup_{\xi < \alpha} \mathrm{dom}(f_\xi) \in \Sigma_1^0$.

A function $f : \mathbb{N}^k \rightarrowtail \mathbb{N}$ is called α/S-*recursive* iff $f = \Phi(\mathcal{F})$ for an S-computable α-sequence $\mathcal{F}$ of k-ary partial recursive functions; f is said to be α-*recursive* iff it is α/S-recursive for some r.r.u. naming system α/S. So we get the following *Ershov classes* of functions, in the *notation-dependent* and the *notation-independent* version, i.e., with and without "$/S$", respectively:

$$\nabla_{\alpha(/S)}^{\mathrm{E}} = \{f : f \text{ is an } \alpha(/S)\text{-recursive function from some } \mathbb{N}^k \text{ into } \mathbb{N}\}.$$

Obviously, $\nabla_{\alpha(/S)}^{\mathrm{E}} \subseteq \nabla_{(\alpha+1)(/S)}^{\mathrm{E}}$ if S is a naming system for $\alpha + 1$ too. Let $\nabla_{<\alpha(/S)}^{\mathrm{E}} = \bigcup_{\beta < \alpha} \nabla_{\beta(/S)}^{\mathrm{E}}$.

Proposition 2.1 *For all r.r.u. naming systems $(\alpha + 1)/S$ and λ/S, with limit numbers λ,*

$$\nabla_{\alpha/S}^{\mathrm{E}} \subset \nabla_{(\alpha+1)/S}^{\mathrm{E}} \quad \text{and} \quad \nabla_{<\lambda/S}^{\mathrm{E}} \subset \nabla_{\lambda/S}^{\mathrm{E}}.$$

This follows immediately from analog results for the classes $\Delta_{\alpha/S}^{\mathrm{E}}$ of the original Ershov hierarchy, cf. [4,3,7]. These classes consist of the so-called α/S-recursive subsets of $\mathbb{N}^k$, which are characterized by the α/S-recursivity of their total characteristic functions in the above defined sense. Thus, due to the classical results, there are even total $\{0, 1\}$-valued discrete functions in $\nabla_{(\alpha+1)/S}^{\mathrm{E}} \setminus \nabla_{\alpha/S}^{\mathrm{E}}$ and $\nabla_{\lambda/S}^{\mathrm{E}} \setminus \nabla_{<\lambda/S}^{\mathrm{E}}$, respectively. □

The following proposition characterizes the functions occurring in the Ershov hierarchy. It also shows that the notation-independent hierarchy collapses at level ω^2. Recall that $f \leq_{\mathrm{T}} 0'$ means that the function f is $0'$-computable. A function $f : \mathbb{N}^k \rightarrowtail \mathbb{N}$ is called *limit-computable* iff there is a total recursive function $\varphi : \mathbb{N}^{k+1} \rightharpoonup \mathbb{N}$ such that $f(\vec{x}) \simeq \lim_{n \to \infty} \varphi(\vec{x}, n)$.

Proposition 2.2 *For any function $f : \mathbb{N}^k \rightarrowtail \mathbb{N}$ are equivalent:*

(i) $f \in \nabla_{\alpha/S}^{\mathrm{E}}$, *for a constructive ordinal α and a r.r.u. system α/S;*

(ii) $f \leq_{\mathrm{T}} 0'$ *and* $\mathrm{dom}(f)$ *is r.e.;*

(iii) *f is limit-computable and* $\mathrm{dom}(f)$ *is r.e.;*

(iv) $f \in \nabla_{\omega^2}^{\mathrm{E}}$.

The equivalence (ii) $\Leftrightarrow$ (iii) is well-known. (iv) $\Rightarrow$ (i) is trivial, and (i) $\Rightarrow$ (iii) is easily shown. Finally, (iii) $\Rightarrow$ (iv) was proved in [3]. $\quad\square$

3 The Hausdorff hierarchy for real functions

Hausdorff's difference hierarchy within the Borel class Δ_2^{B} over (complete and separable) topological spaces was essentially established in [5]. This topological setting was mainly directed to classes of pointsets and is not directly transferable to function classes. Moreover, in topology it is less customary to deal with partial functions, a basic feature in applying the operator Φ. A considerable progress to a related classification of functions was made by Hertling-Weihrauch and Hertling who introduced and studied the levels of discontinuity of functions, see [10,8,9]. Finally, in [7] we have shown how the Hausdorff hierarchy of classes of pointsets can be characterized by means of the first-value operator. This setting is now transferred to a classification of certain real-valued functions over Euclidean spaces.

By a *real function*, we mean any partial function $f : \mathbb{R}^k \rightarrowtail \mathbb{R}$, for an arbitrary dimension $k \in \mathbb{N}_+$. It is said to be *continuous* iff for all open $G \subseteq \mathbb{R}$ the preimages $f^{-1}[G]$ are open pointsets too. In particular, $\mathrm{dom}(f) = f^{-1}[\mathbb{R}]$ has to be open. The basic class of the hierarchy we are going to introduce will be that of continuous functions,

$$\mathfrak{F}_{\mathrm{cont}} = \{ f : \ f \text{ is a continuous real function} \}.$$

For an ordinal $\alpha \in \mathrm{C}_{\mathrm{II}}$, a real function f is said to be α-*continuous* iff there is an α-sequence $\mathcal{F} = (f_\xi)_{\xi < \alpha}$ of continuous functions $f_\xi \in \mathfrak{F}_{\mathrm{cont}}$ such that $f = \Phi(\mathcal{F})$. Notice that then $\mathrm{dom}(f) = \bigcup_{\xi < \alpha} \mathrm{dom}(f_\xi)$ is open too. The 1-continuous functions are just the members of $\mathfrak{F}_{\mathrm{cont}}$.

The classes of the *Hausdorff hierarchy for real functions* are defined as

$$\nabla_\alpha^{\mathrm{H}} = \{ f : f \text{ is an } \alpha\text{-continuous real function} \} \ \text{ and } \ \nabla_{<\alpha}^{\mathrm{H}} = \bigcup_{\beta < \alpha} \nabla_\beta^{\mathrm{H}}.$$

Proposition 3.1 *For all ordinals α and limit numbers λ with $\alpha, \lambda \in \mathrm{C}_{\mathrm{II}}$,*
$$\nabla_\alpha^{\mathrm{H}} \subset \nabla_{(\alpha+1)}^{\mathrm{H}} \quad and \quad \nabla_{<\lambda}^{\mathrm{H}} \subset \nabla_\lambda^{\mathrm{H}}.$$

This follows from related properties of the usual Hausdorff hierarchy for

the classes Δ_α^H, cf. [7]. They consist of the α-clopen pointsets A, which are just those whose total characteristic functions χ_A are α-continuous. Thus, there are total $\{0,1\}$-valued real functions in $\nabla_{(\alpha+1)}^H \setminus \nabla_\alpha^H$ and $\nabla_\lambda^H \setminus \nabla_{<\lambda}^H$, respectively. □

Another characterization of Hausdorff's hierarchy of functions can be obtained by the technique of *depth analysis*. It goes also back to Hausdorff [5], who showed that the resolvable sets are exactly the members of Δ_2^B. In [6,7], we have combined his method with features of Ershov's hierarchy [4] within Δ_2^0. Before that, Hertling-Weihrauch [10] and Hertling [8,9] had applied Hausdorff's method to functions over topological spaces and introduced related levels of discontinuity. Here we essentially follow their setting, even if the notations are modified in order to remain in accordance with [7] and to prepare for a smooth passage to the effective version in the following section.

For a closed set $F \subseteq \mathbb{R}^k$ and an arbitrary $A \subseteq F$, let $A|_F^\circ$ denote the *interior* of A *relatively to* F. Thus, $A|_F^\circ = \bigcup \{ B \cap F : B \cap F \subseteq A, B \text{ is open in } \mathbb{R}^k \}$.

Given a real function $f : \mathbb{R}^k \rightarrowtail \mathbb{R}$, let the pointsets C_f^ξ, $U_f^\xi \subseteq \mathbb{R}^k$, for all ordinals ξ, be defined by transfinite recursion as follows:

$$C_f^0 = \{ \vec{x} : f(\vec{x}) \downarrow, \text{ and } f \text{ is continuous in } \vec{x} \}|_{\mathbb{R}^k}^\circ ,$$

this is the interior of the domain of continuity of f. For $\xi > 0$, put

$$U_f^\xi = \bigcap_{\eta < \xi} \overline{C_f^\eta} \ \text{ and } \ C_f^\xi = \{ \vec{x} \in U_f^\xi : f(\vec{x}) \downarrow, \ f|_{U_f^\xi} \text{ is continuous in } \vec{x} \}|_{U_f^\xi}^\circ .$$

Herein and throughout the paper, the overline denotes the complement of a set. Since $U_f^0 = \bigcap \emptyset = \mathbb{R}^k$, the initial step is included in the general recursion step. By transfinite induction, it follows that any union $\bigcup_{\eta < \xi} C_f^\eta$ is open in $\mathbb{R}^k$ and all *universes* U_f^ξ are closed pointsets. Moreover, $C_f^\eta \cap C_f^\xi = \emptyset$ if $\eta \neq \xi$, and

$$\mathbb{R}^k = U_f^0 \supseteq U_f^1 \supseteq \ldots \supseteq U_f^\xi \supseteq U_f^{\xi+1} \supseteq \ldots \supseteq \overline{\mathrm{dom}(f)} .$$

Thus, there is a least ordinal $\alpha \in \mathrm{C_{II}}$ such that $U_f^\alpha = U_f^{\alpha+1}$ or, equivalently, $C_f^\alpha = \emptyset$. Then we have $U_f^\eta \supset U_f^\xi$ and $C_f^\eta \neq \emptyset$ for all $\eta < \xi \leq \alpha$, but $U_f^\alpha = U_f^\xi$ and $C_f^\xi = \emptyset$ for all $\xi \geq \alpha$.

The function f is called *resolvable* iff $U_f^\alpha = \overline{\mathrm{dom}(f)}$ for this least ordinal α, at which the sequence of universes becomes stationary. In this case, we have

$$\mathrm{dom}(f) = \bigcup_{\xi < \alpha} C_f^\xi,$$

and we define the *local depth* of a point $\vec{x} \in \mathrm{dom}(f)$ w.r.t. function f by

$$\mathrm{depth}_f(\vec{x}) = \xi \ \text{ iff } \ \vec{x} \in C_f^\xi ,$$

whereas the *global depth* of f is given by

$$\text{udepth}(f) = \min\{\xi : U_f^\xi = \overline{\text{dom}(f)}\}.$$

There are close relationships between resolvability and α-continuity.

Lemma 3.2 *Let* $f = \Phi(\mathcal{F})$ *with* $\mathcal{F} = (f_\xi)_{\xi<\alpha}$, $f_\xi \in \mathfrak{F}_{\text{cont}}$, $\alpha \in \text{C}_{\text{II}}$. *Then, for all ordinals* $\xi < \alpha$,

$$\bigcup_{\eta\leq\xi} \text{dom}(f_\eta) \subseteq \bigcup_{\eta\leq\xi} C_f^\eta.$$

This holds for $\xi = 0$, i.e., $\text{dom}(f_0) \subseteq C_f^0$, since C_f^0 is the largest open set on which f is continuous, f_0 coincides with f on $\text{dom}(f_0)$ and f_0 is continuous.

Now let the assertion be fulfilled for all $\xi' < \xi$. Then,

$$U_f^\xi = \bigcap_{\eta<\xi} \overline{C_f^\eta} = \overline{\bigcup_{\eta<\xi} C_f^\eta} \subseteq \overline{\bigcup_{\eta<\xi} \text{dom}(f_\eta)} = \bigcap_{\eta<\xi} \overline{\text{dom}(f_\eta)}.$$

C_f^ξ is the largest relatively open subset of U_f^ξ on which $f|_{U_f^\xi}$ is continuous. Moreover, $f_\xi(\vec{x}) = f(\vec{x})$ for all $\vec{x} \in \text{dom}(f_\xi) \setminus (\bigcap_{\eta<\xi} \text{dom}(f_\eta)) = \text{dom}(f_\xi) \cap \overline{\bigcap_{\eta<\xi} \text{dom}(f_\eta)} \supseteq \text{dom}(f_\xi) \cap U_f^\xi$, and f_ξ is continuous. Thus, $\text{dom}(f_\xi) \cap U_f^\xi \subseteq C_f^\xi$, and $\bigcup_{\eta\leq\xi} \text{dom}(f_\eta) = \text{dom}(f_\xi) \cup \bigcup_{\eta<\xi} \text{dom}(f_\eta) \subseteq \text{dom}(f_\xi) \cup \bigcup_{\eta<\xi} C_f^\eta \subseteq (\text{dom}(f_\xi) \cap U_f^\xi) \cup \bigcup_{\eta<\xi} C_f^\eta \subseteq \bigcup_{\eta\leq\xi} C_f^\eta$. $\qquad\square$

From Lemma 3.2 one obtains the resolvability of any real function, which is α-continuous for some $\alpha \in \text{C}_{\text{II}}$. Indeed, if $f = \Phi(\mathcal{F})$ with a sequence $\mathcal{F} = (f_\xi)_{\xi<\alpha}$, $f_\xi \in \mathfrak{F}_{\text{cont}}$, then $\bigcup_{\eta<\alpha} \text{dom}(f_\eta) = \text{dom}(f)$. Since $\bigcup_{\eta<\xi} C_f^\eta \subseteq \text{dom}(f)$ for all ordinals ξ, it follows $C_f^\alpha = \emptyset$. Thus, f is resolvable.

Moreover, by Lemma 3.2, $\text{udepth}(f)$ is a lower bound of the lengths of sequences of continuous functions, $\mathcal{F}$, with $\Phi(\mathcal{F}) = f$.

Lemma 3.3 *If a real function* f *is resolvable, then there is a sequence* $\mathcal{F} = (f_\xi)_{\xi<\text{udepth}(f)}$ *such that* $f = \Phi(\mathcal{F})$ *and, moreover,*

$(+) \quad f_\xi \in \mathfrak{F}_{\text{cont}}$ *and* $\bigcup_{\eta\leq\xi} \text{dom}(f_\eta) = \bigcup_{\eta\leq\xi} C_f^\eta$ *for all* $\xi < \text{udepth}(f)$.

Put $f_0 = f|_{C_f^0}$, and the requirement $(+)$ is fulfilled for $\xi = 0$.

Suppose f_η has been defined for all $\eta < \xi$ such that $(+)$ holds. Then we have $\bigcup_{\eta<\xi} \text{dom}(f_\eta) = \bigcup_{\eta<\xi} C_f^\eta$. By definition of C_f^ξ, there is an open set $G \subseteq \mathbb{R}^k$ such that $C_f^\xi = U_f^\xi \cap G$, i.e., $G \subseteq \bigcup_{\eta\leq\xi} C_f^\eta \subseteq \text{dom}(f)$. Moreover, $f|_{C_f^\xi}$ is a continuous function on C_f^ξ, considered as a closed subset of the normal space G. Thus, by the Tietze-Urysohn theorem, see [2], $f|_{C_f^\xi}$ can be extended to a continuous function f_ξ over G. So we have $\text{dom}(f_\xi) = G$ and $f_\xi(\vec{x}) = f(\vec{x})$ for all $\vec{x} \in C_f^\xi$. It follows $\bigcup_{\eta\leq\xi} \text{dom}(f_\eta) = \bigcup_{\eta\leq\xi} C_f^\eta$, and $(+)$ is fulfilled. $\qquad\square$

Proposition 3.4 *A real function is resolvable iff it is* α-*continuous for some*

$\alpha \in C_{II}$. *More precisely, for any* $\alpha \in C_{II}$,

$$\nabla_\alpha^H = \{f : f \text{ is a resolvable real function, and } \text{udepth}(f) \leq \alpha\}.$$

This only summarizes some results obtained so far.   □

A sequence of real functions, $\mathcal{F} = (f_\xi)_{\xi < \alpha}$, is called *greedy* iff, for $f = \Phi(\mathcal{F})$ and any $\vec{x} \in \text{dom}(f)$, $\vec{x} \in \text{dom}(f_\xi)$ whenever $\text{depth}_f(\vec{x}) = \xi$.

Notice that the length α of a greedy sequence for function f can obviously be restricted to $\text{udepth}(f)$. Now Lemma 3.3 can be expressed as follows.

Corollary 3.5 *Any resolvable real function can be obtained as* $\Phi(\mathcal{F})$*, with a greedy sequence* $\mathcal{F}$ *of continuous functions.*   □

The following result, shown by Hertling [8] for total functions over metric spaces, stresses that the resolvable functions belong to a rather low level of complexity within the framework of descriptive set theory. Here the notion of Γ-measurable function is employed, cf. [11,15]: Given a class of pointsets, Γ, a real function is called Γ-*measurable* iff $f^{-1}[G] \in \Gamma$ for all open $G \subseteq \mathbb{R}$.

Lemma 3.6 *Any resolvable real function is* Δ_2^B-*measurable.*

This can be proved by means of set resolvability, cf. [8]. Here we sketch a direct proof by means of Proposition 3.2, which can be effectivized in a rather straightforward way, cf. Section 4.

Let $f = \Phi(\mathcal{F})$ with a sequence $\mathcal{F} = (f_\xi)_{\xi<\alpha}$, $f_\xi \in \underline{\mathfrak{F}_{\text{cont}}}$, and let $G \subseteq \mathbb{R}$ be an open set. Then $f^{-1}[G] = \bigcup_{\xi<\alpha}(f_\xi^{-1}[G] \cap \bigcap_{\eta<\xi} \overline{\text{dom}(f_\eta)})$. The open pointsets $f_\xi^{-1}[G]$ are unions of countably many closed sets: $f_\xi^{-1}[G] = \bigcup_{i<\omega} F_{\xi,i}$, with closed $F_{\xi,i}$. The sets $F'_\xi = \bigcap_{\eta<\xi} \overline{\text{dom}(f_\eta)}$ are closed too. Hence

$$f^{-1}[G] = \bigcup_{\xi<\alpha}((\bigcup_{i<\omega} F_{\xi,i}) \cap F'_\xi) = \bigcup_{\xi<\alpha}\bigcup_{i<\omega}(F_{\xi,i} \cap F'_\xi).$$

This means that $f^{-1}[G] \in \Sigma_2^B$, as a countable union of closed sets. A related representation of the complement is obtained analogously.   □

For any set $A \in \Delta_2^B$, $A \subseteq \mathbb{R}^k$, the real function $f_A = A \times \{1\}$ is Δ_2^B-measurable. However, it cannot be resolvable if $\text{dom}(f_A) = A \notin \Sigma_1^B$. It is an open question whether the conversion of Lemma 3.6 holds for functions with open domains or, equivalently, as one can show, for total functions.

4   The Hausdorff-Ershov hierarchy for real functions

The Hausdorff-Ershov hierarchy for functions is obtained by a suitable combination of ingredients both of Ershov's hierarchy for discrete functions and of the Hausdorff hierarchy for real functions. One feature of effectivity, that

we take over from Ershov's setting, consists in the restriction to constructive ordinals as order types of sequences of real functions. Moreover, these functions have to be computable, and this is defined within the framework of computable analysis, cf. [17]. We briefly recall some related basic ideas and notations. For more details, see [6,7].

Real numbers (and tuples) are represented by fast converging Cauchy sequences of (tuples of) rational numbers. Let be fixed a total standard numbering $\nu_{\mathbb{Q}} = (q_n : n \in \mathbb{N})$ of the set of rational numbers $\mathbb{Q} = \{q_n : n \in \mathbb{N}\}$. It is extended to $\mathbb{Q}^k$ via Cantor's k-tuple function such that $\mathbb{Q}^k = \{\vec{q}_n : n \in \mathbb{N}\}$. For points $\vec{x} = (x_1, \ldots, x_k) \in \mathbb{R}^k$, we prefer the *maximum norm* $\|\vec{x}\| = \max_{\varkappa=1}^{k} |x_\varkappa|$ which is topologically equivalent to the Euclidean norm. The natural topology in $\mathbb{R}^k$ is generated by the base of all *rational open balls*, $\mathrm{Ball}_q(\vec{x}) = \{\vec{y} \in \mathbb{R}^k : \|\vec{x} - \vec{y}\| < q\}$, $q \in \mathbb{Q}$, $\vec{x} \in \mathbb{Q}^k$. Let a numbering of them, $(\mathrm{ball}_n : n \in \mathbb{N})$, be fixed by $\mathrm{ball}_{\langle n,m \rangle} = \mathrm{Ball}_{q_n}(\vec{q}_m)$, for any dimension k determined by the context, where $\langle \cdot, \cdot \rangle$ denotes Cantor's pairing function.

Computability of partial real functions $f : \mathbb{R}^k \rightarrowtail \mathbb{R}$ is defined by means of function-oracle Turing machines, due to Ko-Friedman [13] and Kreitz-Weihrauch [14]. The points $\vec{x} \in \mathbb{R}^k$ are represented by (sequences of indices of) *effectively converging Cauchy sequences*, i.e., by sequences of the set $\mathrm{CF}_{\vec{x}} = \{\sigma \in \mathbb{N}^\omega : \|\vec{q}_{\sigma(n)} - \vec{x}\| < 2^{-n} \text{ for all } n \in \mathbb{N}\}$. A *function-oracle Turing machine (OTM)* $\mathcal{M}$ gets a natural number n as *input* and a sequence $\sigma = (i_0, i_1, i_2, \ldots) \in \mathbb{N}^\omega$ as *oracle*, and it has produced an *output* $\mathcal{M}^\sigma(n) \in \mathbb{N}$ when it halts. In the course of its work, it can put oracle *queries* "m?", for $m \in \mathbb{N}$, which are answered by the $(m+1)$st element i_m of the sequence σ.

A real function is called *(approximately) computable* iff there is an OTM $\mathcal{M}$ such that for all $\vec{x} \in \mathbb{R}^k$:

(i) if $f(\vec{x}) \downarrow$, then $\mathcal{M}^\sigma(n)$ exists for all $\sigma \in \mathrm{CF}_{\vec{x}}$ and $n \in \mathbb{N}$, and it holds $(\mathcal{M}^\sigma(n))_{n\in\mathbb{N}} \in \mathrm{CF}_{f(\vec{x})}$;

(ii) if $f(\vec{x}) \uparrow$ and $\sigma \in \mathrm{CF}_{\vec{x}}$, there is an input $n \in \mathbb{N}$ for which $\mathcal{M}^\sigma(n) \uparrow$.

Ko-Friedman used a more restrictive notion. Instead of (ii), they required

(ii') if $f(\vec{x}) \uparrow$, then $\mathcal{M}^\sigma(n)$ is undefined for all $\sigma \in \mathrm{CF}_{\vec{x}}$ and all $n \in \mathbb{N}$.

We call a function f *KF-computable* iff (i) and (ii') for an OTM $\mathcal{M}$.

Π_m^{ta} is the class of complements of members of Σ_m^{ta}, $m \in \mathbb{N}_+$, and a pointset A belongs to Σ_m^{ta} iff, for a total recursive function $\varphi : \mathbb{N}^m \longrightarrow \mathbb{N}$,

$$(*) \quad A = \begin{cases} \bigcup_{n_1 \in \mathbb{N}} \bigcap_{n_2 \in \mathbb{N}} \cdots \bigcup_{n_m \in \mathbb{N}} \ \mathrm{ball}_{\varphi(n_1, n_2, \cdots, n_m)} & \text{if } m \text{ is odd,} \\ \bigcup_{n_1 \in \mathbb{N}} \bigcap_{n_2 \in \mathbb{N}} \cdots \bigcap_{n_m \in \mathbb{N}} \ \overline{\mathrm{ball}_{\varphi(n_1, n_2, \cdots, n_m)}} & \text{if } m \text{ is even.} \end{cases}$$

If φ is allowed to be an arbitrary discrete function, $(*)$ is the typical representation of a Borel set $A \in \Sigma_m^{\mathrm{B}}$. Whereas the domains of approximately computable functions form exactly the class Π_2^{ta} in the *effective Borel hierarchy* $(\Sigma_m^{\mathrm{ta}} : m \in \mathbb{N}_+)$, the domains of KF-computable functions are just the *r.e. open sets*, i.e., the members of Σ_1^{ta}. More precisely, we have

Lemma 4.1 *A pointset A is r.e. open iff $A = \mathrm{dom}(f)$ for a KF-computable real function f. An approximately computable real function f is KF-computable iff $\mathrm{dom}(f)$ is r.e. open.*

The first part is due to Ko-Friedman [13], the second one follows easily.□

Due to Lemma 4.1 and since computable functions are continuous on their domains, the KF-computable real functions form an appropriate effective counterpart of the class of continuous functions, $\mathfrak{F}_{\mathrm{cont}}$. Thus, we take

$$\mathfrak{F}_{\mathrm{KF}} = \{f : f \text{ is a KF-computable real function}\}$$

as the basic class of the effective first-value hierarchy. Effectivity of a transfinite sequence of functions from $\mathfrak{F}_{\mathrm{KF}}$ is defined w.r.t. some r.r.u. naming system α/S. A sequence $\mathcal{F} = (f_\xi)_{\xi<\alpha}$ is said to be *S-KF-computable* iff there is an OTM $\mathcal{M}$ such that, for all points $\vec{x}$ and ordinals $\xi < \alpha$,

(i) if $f_\xi(\vec{x}) \downarrow$, then $\mathcal{M}^\sigma(\langle m, n\rangle) \downarrow$ for all $\sigma \in \mathrm{CF}_{\vec{x}}$ and all $n \in \mathbb{N}$, where $m = \nu_S^{-1}(\xi)$, and it holds $(\mathcal{M}^\sigma(\langle m, n\rangle))_{n \in \mathbb{N}} \in \mathrm{CF}_{f_\xi(\vec{x})}$;

(ii) if $f_\xi(\vec{x}) \uparrow$, then $\mathcal{M}^\sigma(\langle m, n\rangle) \uparrow$ for all $\sigma \in \mathrm{CF}_{\vec{x}}$, all $n \in \mathbb{N}$ and $m = \nu_S^{-1}(\xi)$.

Obviously, it follows $f_\xi \in \mathfrak{F}_{\mathrm{KF}}$ for all $\xi < \alpha$.

A real function f is called α/S-*toprecursive* iff $f = \Phi(\mathcal{F})$ for an S-KF-computable sequence $\mathcal{F} = (f_\xi)_{\xi<\alpha}$. α-*toprecursivity* means α/S-toprecursivity w.r.t. some r.r.u. naming system α/S. The 1-toprecursive functions are just the KF-computable ones. Each α-toprecursive function is α-continuous and has an r.e. open domain.

For constructive ordinals α (and r.r.u. α/S), the notation-dependent and notation-independent, respectively, versions of the *Hausdorff-Ershov classes of functions* are defined as

$$\nabla_{\alpha(/S)}^{\mathrm{HE}} = \{f : f \text{ is an } \alpha(/S)\text{-toprecursive real function}\}.$$

By reasons of cardinality, we have $\nabla_{\alpha(/S)}^{\mathrm{HE}} \subset \nabla_\alpha^{\mathrm{H}}$, for $\alpha \neq 0$, and even $\bigcup\{\nabla_\alpha^{\mathrm{HE}} : \alpha \text{ is a constructive ordinal}\} \subset \bigcup_{\alpha \in \mathrm{C_{II}}} \nabla_\alpha^{\mathrm{H}}$. Properties of the related hierarchy of classes of pointsets yield

Proposition 4.2 *For all constructive ordinals α, constructive limits λ (and corresponding r.r.u. naming systems $(\alpha+1)/S$ and λ/S, respectively),*

$$\nabla_{\alpha(/S)}^{\mathrm{HE}} \subset \nabla_{(\alpha+1)(/S)}^{\mathrm{HE}} \quad and \quad \nabla_{<\lambda(/S)}^{\mathrm{HE}} \subset \nabla_{\lambda(/S)}^{\mathrm{HE}}.$$

Indeed, a pointset $A \subseteq \mathbb{R}^k$ belongs to $\Delta^{\mathrm{HE}}_{\alpha(/S)}$ in the sense of [7] iff its total characteristic function χ_A belongs to $\nabla^{\mathrm{HE}}_{\alpha(/S)}$. Therefore, using results from [7], we get the strict inclusions. $\square$

To localize the functions occurring in classes of the Hausdorff-Ershov hierarchy, Lemma 3.6 can straightforwardly be effectivized within the framework of computable analysis. We sketch this briefly. For details of the definitions and proof techniques, the reader is referred to [1].

Let $\mathcal{N}_m$ be the set of all discrete total functions $\varphi : \mathbb{N}^m \longrightarrow \mathbb{N}$, $m \in \mathbb{N}_+$. They can be represented by sequences $\sigma_\varphi \in \mathbb{N}^\omega$ in a canonical, bijective way. For example, put $\sigma_\varphi(n) = \varphi(\gamma_m(n))$, where γ_m is an effective standard bijection of $\mathbb{N}$ onto $\mathbb{N}^m$. Thus, the sets $A \in \Sigma^{\mathrm{B}}_m$ can also be represented by the sequences $\sigma \in \mathbb{N}^\omega$, namely if $\sigma = \sigma_\varphi$ for some $\varphi \in \mathcal{N}_m$ satisfying the typical equation $(*)$ given above. Now, a real function f is called *effectively Σ^{B}_m-measurable* iff there is an (approximately) computable function $g : \mathbb{N}^\omega \longrightarrow \mathbb{N}^\omega$ such that, whenever a sequence σ represents an open set $A \in \Sigma^{\mathrm{B}}_1$, $A \subseteq \mathbb{R}$, the sequence $g(\sigma)$ represents the preimage $f^{-1}[A]$ as a member of Σ^{B}_m.

Analogously, we call a real function f *effectively Δ^{B}_m-measurable* iff there are (approximately) computable functions $g, \overline{g} : \mathbb{N}^\omega \longrightarrow \mathbb{N}^\omega$ such that, whenever a sequence σ represents an open set $A \subseteq \mathbb{R}$, the sequence $g(\sigma)$ represents the preimage $f^{-1}[A]$ and $\overline{g}(\sigma)$ represents the complement, $\overline{f^{-1}[A]}$, both as members of Σ^{B}_m. By the effective analogue of the proof of Lemma 3.6, with some technical effort using tools prepared in [1], one shows

Lemma 4.3 *If a real function is α/S-toprecursive w.r.t. some r.r.u. naming system α/S, then it is effectively Δ^{B}_2-measurable.* $\square$

For classes of pointsets, relationships between the Hausdorff-Ershov hierarchy and the Ershov hierarchy have been obtained by considering the discrete parts $A \cap \mathbb{N}^k$ of pointsets $A \subseteq \mathbb{R}^k$. For functions, the situation is not so simple, since the restrictions of real functions $f : \mathbb{R}^k \rightarrowtail \mathbb{R}$ to $\mathbb{N}^k$ are not necessarily discrete. Conversely, non-empty discrete functions, $f : \mathbb{N}^k \rightarrowtail \mathbb{N}$, if they are considered as real ones, are not continuous in the sense of Section 3, since the domains are not open. Nevertheless, the Ershov hierarchy of discrete functions can be embedded into the Hausdorff-Ershov hierarchy by means of the operator ϱ assigning to a discrete function $f : \mathbb{N}^k \rightarrowtail \mathbb{N}$ the continuous function $\varrho(f) : \mathbb{R}^k \rightarrowtail \mathbb{R}$ defined as $\varrho(f) = \bigcup \{ \mathrm{Ball}_{\frac{1}{3}}(\vec{x}) \times \{f(\vec{x})\} : \vec{x} \in \mathrm{dom}(f) \}$.

Proposition 4.4 *For any $f : \mathbb{N}^k \rightarrowtail \mathbb{N}$ and every r.r.u. system α/S,*
$$ f \in \nabla^{\mathrm{E}}_{\alpha/S} \quad \textit{iff} \quad \varrho(f) \in \nabla^{\mathrm{HE}}_{\alpha/S}. $$

Indeed, for an S-computable sequence $\mathcal{F} = (f_\xi)_{\xi < \alpha}$ of discrete functions f_ξ, the sequence of real functions $\varrho(\mathcal{F}) = (\varrho(f_\xi))_{\xi < \alpha}$ is S-computable too, in

the related sense. Moreover, $\Phi(\varrho(\mathcal{F})) = \varrho(\Phi(\mathcal{F}))$. Conversely, if $\varrho(f) = \Phi(\mathcal{F})$ for an S-computable sequence $\mathcal{F} = (f_\xi)_{\xi<\alpha}$ of k-ary real functions, then the sequence $\mathcal{F}' = (f'_\xi)_{\xi<\alpha}$, where $f'_\xi = f_\xi|_{\mathbb{N}^k}$, consists of discrete functions and is S-computable in the sense of Section 2. Moreover, $\Phi(\mathcal{F}') = f$.   □

So, the Hausdorff-Ershov hierarchy of function classes is at least as rich as the related Ershov hierarchy. Propositions 2.1 and 4.4 show that each class $\nabla^{\mathrm{HE}}_{\alpha/S}$ contains continuous functions which do not belong to any lower class $\nabla^{\mathrm{HE}}_{\alpha'/S}$ for $\alpha' < \alpha$. In other words, there are real functions $f \in \nabla^{\mathrm{H}}_1 = \mathfrak{F}_{\mathrm{cont}}$, which have the level α/S in the notation-dependent Hausdorff-Ershov hierarchy, i.e., $f \in \nabla^{\mathrm{HE}}_{\alpha/S} \setminus \nabla^{\mathrm{HE}}_{<\alpha/S}$. So the level of toprecursivity of a real function can be arbitrarily higher than its level of continuity.

Those functions which admit computable greedy sequences, representing them by means of the first-value operator, are of special interest. For such a function, its level in the Hausdorff hierarchy coincides with that in the Hausdorff-Ershov hierarchy. Moreover, for each point $\vec{x}$ from the domain, the computation of the function value in $\vec{x}$ can be performed by computing $f_\xi(\vec{x})$, just at the first stage ξ which is possible at all, by topological reasons.

So, a real function f is called *weakly (α/S-)computable* iff there is an S-KF-computable greedy sequence $\mathcal{F} = (f_\xi)_{\xi<\alpha}$ with $f = \Phi(\mathcal{F})$.

A sharper condition requires that, moreover, if $f_\xi(\vec{x}) \downarrow$, then there is a point $\vec{y}$ close to $\vec{x}$ such that $\mathrm{depth}_f(\vec{y}) = \xi$, thus, $f(\vec{y}) = f_\xi(\vec{y}) \downarrow$.

More precisely, a real function f is called *safely (α/S-)computable* iff there is an S-KF-computable double sequence $\mathcal{D} = (f_{n,\xi})_{n<\omega, \xi<\alpha}$ such that the sequences $\mathcal{F}_n = ((f_{n,\xi})_{\xi<\alpha}$ are greedy and $\Phi(\mathcal{F}_n) = f$, for all $n \in \mathbb{N}$, and whenever $f_{n,\xi}(\vec{x}) \downarrow$, then there is a point $\vec{y} \in \mathrm{Ball}_{2^{-n}}(\vec{x})$ with $\mathrm{depth}_f(\vec{y}) = \xi$. S-KF-computability of $\mathcal{D}$ means that there is an OTM $\mathcal{M}$ such that, for all points $\vec{x}$, numbers $n \in \mathbb{N}$ and ordinals $\xi < \alpha$,

(i) if $f_\xi(\vec{x}) \downarrow$, then $\mathcal{M}^\sigma(\langle n, m, l \rangle) \downarrow$ for all $\sigma \in \mathrm{CF}_{\vec{x}}$, $n, l \in \mathbb{N}$ and $m = \nu_S^{-1}(\xi)$, and it holds $(\mathcal{M}^\sigma(\langle n, m, l \rangle))_{l \in \mathbb{N}} \in \mathrm{CF}_{f_\xi(\vec{x})}$;

(ii) if $f_\xi(\vec{x}) \uparrow$, then $\mathcal{M}^\sigma(\langle n, m, l \rangle) \uparrow$ for all $\sigma \in \mathrm{CF}_{\vec{x}}$, all $n, l \in \mathbb{N}$ and $m = \nu_S^{-1}(\xi)$.

By examples constructed in [7], we have

Lemma 4.5 *To any r.r.u. system α/S, there is a safely α/S-computable function $f_{\alpha/S} : \mathbb{R} \longrightarrow \{0,1\}$ with $\mathrm{udepth}(f_{\alpha/S}) = \alpha$.*   □

So, $f_{\alpha/S}$ witnesses that there are safely α/S-computable, hence also weakly α/S-computable, functions that are of level α in the (non-effective) Hausdorff hierarchy. From the discussion in [7], it also follows that there are weakly computable (total) functions which are not safely computable.

By means of the notion of r.e. closed set, the safely computable functions can be characterized among the weakly computable ones. Recall that a pointset $A \subseteq \mathbb{R}^k$ is said to be *r.e. closed* iff $A = \emptyset$ or there is a total recursive function $\varphi : \mathbb{N}^2 \longrightarrow \mathbb{N}$ such that there is a sequence $(\vec{x}_n)_{n \in \mathbb{N}}$ of points $\vec{x}_n \in \mathbb{R}^k$ satisfying $\|\vec{q}_{\varphi(n,m)} - \vec{x}_n\| < 2^{-m}$ for all $n, m \in \mathbb{N}$, and $A = \mathrm{cl}(\{\vec{x}_n : n \in \mathbb{N}\})$, where cl denotes the *closure* of a set.

If a real function f is safely α/S-computable, then the closures of the domains of continuity, i.e., the sets $\mathrm{cl}(C_f^\xi)$, $\xi < \alpha$, are *uniformly r.e. closed*. This means that there is a recursive function $\varphi : \mathbb{N}^3 \rightarrowtail \mathbb{N}$ such that the functions $\varphi_\xi : \mathbb{N}^2 \rightarrowtail \mathbb{N}$, defined by $\varphi_\xi(n,m) \simeq \varphi(\nu_S^{-1}(\xi), n, m)$, are total and witness that the sets $\mathrm{cl}(C_f^\xi)$ are r.e. closed sets in the above sense.

Conversely, given a weakly α/S-computable function f such that the sets $\mathrm{cl}(C_f^\xi)$, $\xi < \alpha$, are uniformly r.e. closed, the safe α/S-computability can easily be shown. So we have

Lemma 4.6 *A weakly α/S-computable function f is safely α/S-computable iff the sets $\mathrm{cl}(C_f^\xi)$, $\xi < \alpha$, are uniformly r.e. closed.* $\square$

Here we conclude. The notions and results presented in this paper provide a unified framework for non-effective and effective hierarchies of classes of real functions which all are Δ_2^{B}-measurable and effectively Δ_2^{B}-measurable, respectively. This establishes a useful tool in classifying certain functions both from the viewpoint of descriptive set theory as well as of computable analysis. Nevertheless, it should be remarked that we know only some basic facts and relationships so far, many questions are still open.

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