Available online at www.sciencedirect.com

SciVerse ScienceDirect

AASRI Procedia 1 (2012) 299 – 304

AASRI Procedia

www.elsevier.com/locate/procedia

2012 AASRI Conference on Computational Intelligence and Bioinformatics

# The Average Errors for Bernstein-Kantorovich Operators on the r-fold Integrated Wiener Space

Liu Ting, Jiang Yanjie*

*Department of Mathematics and Physics, North China Electric Power University, Baoding 071003, People's Republic of China*

**Abstract**

In this paper, we discuss the average errors of function weighted approximation by the Bernstein-Kantorovich operators. The strongly asymptotic orders for the average errors of the Bernstein-Kantorovich operators sequence are determined on the r-fold integrated Wiener Space.


Selection and/or peer review under responsibility of American Applied Science Research Institute

*Keywords:* *Bernstein-Kantorovich operators; weighted $L_p$-norm; r-fold integrated Wiener space; average error.*

## 1. Introduction

Let $F$ be a real separable Banach space equipped with a probability measure $\mu$ on the Borel sets of $F$. Let $X$ be another normed space such that $F$ is continuously embedded in $X$. By $\|.\|$ we denote the norm in X. Any $T: F \to X$ such that $f \mapsto \| f - T(f) \|$ is a measurable mapping is called an approximation operator. The p-average error of $T$ is defined as

$$e_p(T, \|\cdot\|, F, \mu) = \left( \int_F \|f - T(f)\|^p \, \mu(df) \right)^{\frac{1}{p}}.$$

* Corresponding author. Tel.: +86-0312-7523326.
*E-mail address:* jiangyj@126.com.


doi:10.1016/j.aasri.2012.06.046

Let $F_0 = \{f \in C[0,1] : f(0) = 0\}$. For every $f \in F_0$ set $\|f\|_C = \max_{0\le t\le 1}|f(t)|$. Then $(F_0, \|\cdot\|_C)$ becomes a separable Banach space. Denote by $B(F_0)$ the Borel class of $(F_0, \|\cdot\|_C)$ and by $\omega_0$ the Wiener measure on $B(F_0)$ $(see[1])$.

Let $r \ge 0$ be an integer .For all $g \in F_0$, define $(T_0 g)(t) = g(t)$,and

$$(T_r g)(t) = \int_0^t g(u)\cdot\frac{(t-u)^{r-1}}{(r-1)!}du, r \ge 1.$$

Thus we have

$$(T_r g)(x) \in F_r = \{f \in C^{(r)}[0,1] : f^{(k)}(0) = 0, k = 0,1,\cdots,r\}.$$

It is well known that $T_r$ is a bijective mapping from $F_0$ to $F_r$. The $r$-fold integrated Wiener measure $\omega_r$ on $F_r$ is defined by induced measure $\omega_r = T_r\omega_0$, i.e., for $A \subset F_r$,

$$\omega_r(A) = \omega_0(\{g, T_r g \in A\}).$$

From [1] we know

$$\int_{F_r} f(s) f(t)\omega_r(df) = \int_0^1 \frac{(s-u)_+^r (t-u)_+^r du}{(r!)^2}, \tag{1}$$

where $z_+ = z$ if $z > 0$ and $z_+ = 0$ otherwise.

For $\rho \in L_1[0,1], \rho \ge 0$, the weighted $L_p$-norm of $f \in C[0,1]$ is defined by

$$\|f\| = \|f\|_{p,\rho} = \left(\int_0^1 |f(t)|^p \cdot \rho(t)dt\right)^{\frac{1}{p}}.$$

Let

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k} = C_n^k x^k (1-x)^{n-k}, k = 0,1,\cdots,n.$$

For $f \in C[0,1]$ the well-know Bernstein-Kantorovich polynomials of $f$ is given $(see[2])$ by

$$K_n(f,x) = \sum_{k=0}^n p_{n,k}(x)(n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt. \tag{2}$$

## 2. Main result

Many mathematicians have investigated the approximation behavior of Bernstein-Kantorovich operators on $L_p[0,1], 1 \le p < \infty$. Recently Xu Guiqiao [3] studied the simultaneous approximation average errors for Bernstein operators on the r-fold integrated Wiener space. Motivated by [3], we consider the average errors of function weighted approximation by the Bernstein-Kantorovich operators on the r-fold integrated Wiener space. We obtain the following:

**Theorem 1** Let $1 \le p < \infty$, $r > 1$, $K_n(f,x)$ be given by (2). If $\rho \in L_1[0,1]$, $\rho(x) > 0$ and $\rho(x)$ is continuous on $(-1,1)$, then we have

$$e_p(K_n, \|\cdot\|, F_r, \omega_r) = C_{p,\rho,r}(n+1)^{-1} + o(n^{-1}),$$

where

$$C_{p,\rho,r}=\left(\nu_p\int_0^1\left(\frac{x^{2r-1}(1-2x)^2}{4(2r-1)((r-1)!)^2}+\frac{x^{2r-1}(1-x)^2}{4(2r-3)((r-2)!)^2}+\frac{x^{2r-1}(1-2x)(1-x)}{4((r-1)!)^2}\right)^{\frac{p}{2}}\rho(x)\,dx\right)^{\frac{1}{p}}$$

and

$$\nu_p=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}|x|^p e^{-\frac{x^2}{2}}dx.$$

Here and in the following the notation $a_n=o(b_n)$ for sequences $\{a_n\}$ and $\{b_n\}$ means that $\lim_{n\to\infty}a_n/b_n=0.$

## 3. Proof of Theorem 1

*Proof of Theorem 1*. From [1] we get

$$e_p^p\left(K_n,\|\cdot\|,F_r,\omega_r\right)=\nu_p\int_0^1\left(\int_{F_r}\left|f(x)-K_n(f,x)\right|^2\omega_r(df)\right)^{\frac{p}{2}}\rho(x)\,dx. \tag{3}$$

By (1) and (2), a direct computation shows

$$\begin{aligned}K_n(f,x)-f(x)&=\sum_{k=0}^{n}p_{n,k}(x)(n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}f(t)dt-\sum_{k=0}^{n}p_{n,k}(x)(n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}f(x)dt\\&=(n+1)\sum_{k=0}^{n}p_{n,k}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}\left(f(t)-f(x)\right)dt.\end{aligned}$$

For $r>1$, by Taylor formula we have

$$f(t)-f(x)=(t-x)f'(x)+(t-x)^2\frac{f''(x)}{2}+(t-x)^2\left(\frac{f''(\xi_k)-f''(x)}{2}\right), \tag{4}$$

where $\xi_k$ is between in $t$ and $x$. Hence

$$\begin{aligned}\left|f''(\xi_k)-f''(x)\right|&\le\omega\left(f'',\max\left\{\left|x-\frac{k}{n+1}\right|,\left|x-\frac{k+1}{n+1}\right|\right\}\right)\\&=\omega\left(f'',\frac{\left|x-\frac{k}{n+1}\right|+\left|x-\frac{k+1}{n+1}\right|+\left|\left|x-\frac{k}{n+1}\right|-\left|x-\frac{k+1}{n+1}\right|\right|}{2}\right)\\&\le\frac{3}{2}\omega\left(f'',\left|x-\frac{k}{n+1}\right|\right)+\frac{3}{2}\omega\left(f'',\left|x-\frac{k+1}{n+1}\right|\right)+\frac{3}{2}\omega\left(f'',\frac{1}{n+1}\right),\end{aligned}$$

where $\omega(f,t)$ is the modulus of continuity of $f$ in the uniform norm. Hence, by (4) and a simple computation we obtain

$$\begin{aligned}K_n(f,x)-f(x)&=(n+1)\sum_{k=0}^{n}p_{n,k}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}(t-x)f'(x)dt+(n+1)\sum_{k=0}^{n}p_{n,k}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}(t-x)^2\frac{f''(x)}{2}dt\\&\quad+(n+1)\sum_{k=0}^{n}p_{n,k}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}(t-x)^2\left(\frac{f''(\xi_k)-f''(x)}{2}\right)dt\\&=I_1(x)+I_2(x)+I_3(x).\end{aligned} \tag{5}$$

Note that

$$\sum_{k=0}^{n} p_{n,k}(x)=1,\quad \sum_{k=0}^{n} k p_{n,k}(x)=nx,\quad \sum_{k=0}^{n} k^2 p_{n,k}(x)=n^2x^2+nx(1-x),$$

a simple computation we obtain

$$\begin{aligned} I_1(x)&=(n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}(t-x)f'(x)\,dt \\ &=\frac{f'(x)}{2(n+1)}\sum_{k=0}^{n}(2k+1)p_{n,k}(x)-xf'(x)=\frac{1-2x}{2(n+1)}f'(x), \end{aligned} \tag{6}$$

$$\begin{aligned} I_2(x)&=(n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}(t-x)^2\frac{f''(x)}{2}\,dt \\ &=\left(\frac{x-x^2}{2(n+1)}+\frac{2x^2-2x+\frac{1}{3}}{2(n+1)^2}\right)f''(x), \end{aligned} \tag{7}$$

and

$$\begin{aligned} |I_3(x)|&\le (n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}(t-x)^2\left|\frac{f''(\xi_k)-f''(x)}{2}\right|dt \\ &\le \frac{3(n+1)}{4}\sum_{k=0}^{n}\omega\left(f'',\left|x-\frac{k}{n+1}\right|\right)p_{n,k}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}(t-x)^2\,dt \\ &\quad+\frac{3(n+1)}{4}\sum_{k=0}^{n}\omega\left(f'',\left|x-\frac{k+1}{n+1}\right|\right)p_{n,k}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}(t-x)^2\,dt \\ &\quad+\frac{3(n+1)}{4}\sum_{k=0}^{n}\omega\left(f'',\frac{1}{n+1}\right)p_{n,k}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}(t-x)^2\,dt \\ &\le \frac{C\omega\left(f'',\frac{1}{\sqrt[12]{n^5}}\right)}{n}. \end{aligned} \tag{8}$$

From (5)-(8), we have

$$\begin{aligned} |K_n(f,x)-f(x)|^2&=\left(\frac{(x-x^2)^2}{4(n+1)^2}+\frac{\left(2x^2-2x+\frac{1}{3}\right)^2}{4(n+1)^4}+\frac{(x-x^2)\left(2x^2-2x+\frac{1}{3}\right)}{2(n+1)^3}\right)f''^2(x) \\ &\quad+\frac{(1-2x)^2}{4(n+1)^2}f'^2(x)+\left(\frac{(1-2x)(x-x^2)}{2(n+1)^2}+\frac{(1-2x)\left(2x^2-2x+\frac{1}{3}\right)}{2(n+1)^3}\right)f'(x)f''(x) \end{aligned}$$

$$+I_3^2(x)+\frac{1-2x}{n+1}f'(x)I_3(x)+\left(\frac{x-x^2}{n+1}+\frac{2x^2-2x+\frac{1}{3}}{(n+1)^2}\right)f''(x)I_3(x). \tag{9}$$

Let $f=T_rg$, a direct computation shows

$$\begin{aligned}&\int_{F_r}\left(\frac{(x-x^2)^2}{4(n+1)^2}+\frac{\left(2x^2-2x+\frac{1}{3}\right)^2}{4(n+1)^4}+\frac{(x-x^2)\left(2x^2-2x+\frac{1}{3}\right)}{2(n+1)^3}\right)f''^2(x)\omega_r(df)\\&=\left(\frac{(x-x^2)^2}{4(n+1)^2}+\frac{\left(2x^2-2x+\frac{1}{3}\right)^2}{4(n+1)^4}+\frac{(x-x^2)\left(2x^2-2x+\frac{1}{3}\right)}{2(n+1)^3}\right)\int_{F_{r-2}}f^2(x)\omega_{r-2}(df)\\&=\frac{x^{2r-1}(1-x)^2}{4(2r-3)((r-2)!)^2(n+1)^2}+o\left(\frac{1}{n^2}\right),\end{aligned} \tag{10}$$

$$\begin{aligned}\int_{F_r}\frac{(1-2x)^2}{4(n+1)^2}f'^2(x)\omega_r(df)&=\frac{(1-2x)^2}{4(n+1)^2}\int_{F_0}((T_{r-1}g)(x))^2\omega_0(dg)\\&=\frac{x^{2r-1}(1-2x)^2}{4(2r-1)((r-1)!)^2(n+1)^2},\end{aligned} \tag{11}$$

and

$$\begin{aligned}&\int_{F_r}\left(\frac{(1-2x)(x-x^2)}{2(n+1)^2}+\frac{(1-2x)\left(2x^2-2x+\frac{1}{3}\right)}{2(n+1)^3}\right)f'(x)f''(x)\omega_r(df)\\&=\frac{x^{2r-1}(1-2x)(1-x)}{4((r-1)!)^2(n+1)^2}+o\left(\frac{1}{n^2}\right).\end{aligned} \tag{12}$$

From [4] we know

$$\int_{F_0}\omega\left(g,\frac{1}{n}\right)\omega_0(dg)\le C\left(\frac{\ln n}{n}\right)^{\frac{1}{2}}.$$

By a simple computation we get

$$\begin{aligned}\int_{F_r}I_3^2(x)\omega_r(df)&\le\frac{C}{n^2}\int_{F_r}\omega\left(f'',\frac{1}{\sqrt[12]{n^5}}\right)^2\omega_r(df)\\&\le\frac{C}{n^2}\int_{F_0}\left(2^{r-2}\omega\left((T_rg)^{(r)},\frac{1}{\sqrt[12]{n^5}}\right)\right)^2\omega_0(dg)\\&=\frac{C\cdot 2^{2r-4}}{n^2}\int_{F_0}\omega\left(g,\frac{1}{\sqrt[12]{n^5}}\right)^2\omega_0(dg)\end{aligned}$$

$$\leq \frac{C\cdot 2^{2r-4}}{n^2}\cdot\frac{\ln n^{\frac{5}{12}}}{n^{\frac{5}{12}}}=o\left(\frac{1}{n^2}\right), \tag{13}$$

$$\begin{aligned}\int_{F_r}\frac{1-2x}{n+1}f^{'}(x)I_3(x)\omega_r(df) &\leq \left|\frac{1-2x}{n+1}\right|\int_{F_r}\left|f^{'}(x)I_3(x)\right|\omega_r(df)\\ &\leq \left|\frac{1-2x}{n+1}\right|\left(\int_{F_r}f^{'2}(x)\omega_r(df)\right)^{\frac{1}{2}}\left(\int_{F_r}I_3^2(x)\omega_r(df)\right)^{\frac{1}{2}}\\ &\leq \left|\frac{1-2x}{n+1}\right|\cdot\left(\frac{x^{2r-1}}{(2r-1)\left((r-1)!\right)^2}\right)^{\frac{1}{2}}\cdot o\left(\frac{1}{n}\right)\\ &= o\left(\frac{1}{n^2}\right),\end{aligned} \tag{14}$$

and

$$\begin{aligned}&\int_{F_r}\left(\frac{x-x^2}{n+1}+\frac{2x^2-2x+\frac{1}{3}}{(n+1)^2}\right)f^{"}(x)I_3(x)\omega_r(df)\\ &\leq \left|\frac{x-x^2}{n+1}+\frac{2x^2-2x+\frac{1}{3}}{(n+1)^2}\right|\int_{F_r}\left|f^{"}(x)I_3(x)\right|\omega_r(df)\\ &\leq \left|\frac{x-x^2}{n+1}+\frac{2x^2-2x+\frac{1}{3}}{(n+1)^2}\right|\left(\int_{F_r}f^{"2}(x)\omega_r(df)\right)^{\frac{1}{2}}\left(\int_{F_r}I_3^2(x)\omega_r(df)\right)^{\frac{1}{2}}\\ &\leq \left|\frac{x-x^2}{n+1}+\frac{2x^2-2x+\frac{1}{3}}{(n+1)^2}\right|\cdot\left(\frac{x^{2r-3}}{(2r-3)\left((r-2)!\right)^2}\right)^{\frac{1}{2}}\cdot o\left(\frac{1}{n}\right)=o\left(\frac{1}{n^2}\right).\end{aligned} \tag{15}$$

From (3) and (9)-(15), we obtain the desired estimate of Theorem 1.

## Acknowledgements

This work was supported by a grant from Hebei province higher school science and technology research (Z2010160).

## References

[1] Klaus Ritter, Average-case analysis of numerical problems. New York : Springer-Verlag Berlin Heidelberg; 2000.
[2] Devore R A, Lorentz G G. Constructive Approximation. Berlin: Springer-Verlag;1993.
[3] XU Guiqiao. The Simultaneous Approximation Average Errors for Bernstein Operators on the r-fold Integrated Wiener Space. 2012; to appear.
[4] Sun Yongsheng, Wang Chengyong. Average error bounds of best approximation of continuous functions on the Wiener space. J. Complexity 1995;11: 74-104.