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Coalgebraic Logic over Measurable Spaces: Behavioral and Logical Equivalence

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Abstract

We study the relationship between logical and behavioral equivalence for coalgebras on general measurable spaces. Modal logics are interpreted in these coalgebras using predicate liftings. Prominent examples include stochastic relations and labelled Markov transition systems and corresponding Hennessy–Milner type logics. Local versions of logical and behavioral equivalence are introduced and it is shown that these notions coincide for a wide class of functors. We relate these notions to the corresponding global ones common in model checking. Throughout, we work in general measurable spaces. In contrast to previous work, no topological assumptions on the state spaces are needed.

Keywords: coalgebraic logic, measurable space

1 Introduction

Coalgebras for an endofunctor provide a uniform framework for the study of reactive systems. In this article, we will study coalgebras for functors on the category **Meas** of measurables spaces and maps. The subprobability functor S will play a role similar to the powerset functor in that it allows us to treat stochastic aspects instead of non-deterministic ones.

Prominent examples include stochastic relations as coalgebras for the subprobability functor S and so-called Markov transition systems [4], which arise as coalgebras for the functor $X \longmapsto (SX)^A$ for some set A of actions.

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Modal logic, on the other hand, seems to be an appropriate language to talk about properties of coalgebras, and hence of reactive systems.

Modal operators for coalgebras are introduced using predicate liftings for the functor T [10], that is, natural transformations $\mathcal{B}^n \longrightarrow \mathcal{B} \cdot T$. Here, $\mathcal{B}$ is the functor $\mathbf{Meas}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ which sends every measurable space to its set of measurable subsets.

Two states in two coalgebras are called logically equivalent (with respect to a given family of predicate liftings) if they satisfy exactly the same formulas. As usual, the two states are called behaviorally equivalent if we can find two morphisms whose domains are the given coalgebras, whose codomains coincide, and which map the two given states to the same state.

We show that under condition on the set of predicate liftings, the two notions of equivalence of states mentioned above coincide. This generalizes previous results in this area in that we do not require surjectivity of the coalgebra morphisms, which leads us to local versions of the two notions of equivalence. This seems to be more akin to usual coalgebraic modal logic (over **Set**). We show how to deduce the previous, global, results from ours.

The main technical difficulty is in proving that the logic is *expressive*, that is, that logical equivalence implies behavioral equivalence. This is shown using quite a simple factoring technique. To ensure applicability of this technique, we have to rely on a somewhat technical concept which was—in lack of a better name—called *admissibility*.

In contrast to previous work, we do not impose any topological assumptions on the underlying spaces of the coalgebras. We believe that this makes the exposition more accessible.

Related Work

Expressivity of modal logics for coalgebras over suitable measurable spaces have been studied quite extensively in recent years. The works can be divided into two groups. One deals with coalgebras over measurable spaces which satisfy some topological properties [3,4,6,12].

Results for general measurable spaces have been established in [2,8,5], but only for coalgebras for the subprobability functor S or for S^A. We try to reunite this two groups by establishing expressivity results for general endofunctors on **Meas**.

The basic factoring technique used here stems from [6] and was used in [12] for coalgebras for a general endofunctor on the category of analytic spaces. In [13] the existence of final coalgebras based on Standard Borel spaces was used to extend the expressivity results from [12] to general measurable spaces.

The insight that it might be useful to consider a quotient-structure custommade to the logic at hand (and not the canonical one as used in [6,12]) was first used in [5] and is somewhat hinted at in [2].

2 Preliminaries

We collect here some basic definitions and results from measure theory for the reader's convenience and for easier reference. Nearly all results are well-known.

Measurable spaces

Recall that a measurable space X consists of a set $|X|$ and a σ-algebra $\mathcal{B}X$ on X, that is: a family of subsets of $|X|$ which is closed under complementation, countable intersections, and countable unions. For each family $\mathcal{A}$ of subsets of a set M there is a smallest σ-algebra on M containing $\mathcal{A}$, which we denote by $\sigma(\mathcal{A})$. If $\mathcal{B}(X) = \sigma(\mathcal{A})$ then $\mathcal{A}$ is called a *generator* of $\mathcal{B}(X)$. A *measurable function* $X \to Y$ is given by a function $f : |X| \to |Y|$ such that $f^{-1}[B] \in \mathcal{B}X$ for all $B \in \mathcal{B}(Y)$. In case $\mathcal{B}Y = \sigma(\mathcal{A})$, measurability of f is guaranteed by $f^{-1}[A] \in \mathcal{B}X$ for all $A \in \mathcal{A}$. The category of measurable spaces with measurable functions as morphisms is denoted by **Meas**. Observe that we do not notationally distinguish between a **Meas**-morphisms and its underlying function. Often we will just write X in place of $|X|$.

The assignment $X \mapsto \mathcal{B}X$ defines a functor $\mathbf{Meas}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ with its action on morphisms given by the restriction of the inverse-image function, thus $\mathcal{B}f = f^{-1} : \mathcal{B}Y \longrightarrow \mathcal{B}X$ for $f : X \longrightarrow Y$. Observe that $\mathcal{B}$ is naturally isomorphic to the hom-functor $\mathbf{Meas}(-, \mathbf{2})$, with $\mathbf{2}$ the two-point space in which every subset is measurable. In particular, we obtain:

Lemma 2.1 $\mathcal{B} : \mathbf{Meas}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ *preserves limits and* $\mathcal{B}f$ *is injective provided* f *is surjective.* □

Lemma 2.2 *We have* $\sigma(f^{-1}(\mathcal{A})) = f^{-1}(\sigma(\mathcal{A}))$. □

Special morphisms

Given a family $(Y_i)_I$ of measurable spaces and a family $(f_i : A \to |Y_i|)_I$ of functions with common domain, we define the *initial σ-algebra* with respect this data to be $\mathcal{A} = \sigma(\bigcup_I f_i^{-1}[\mathcal{B}(Y_i)])$. It is characterized by the following property: a function $g : |X| \to A$ is measurable with respect to $\mathcal{B}X$ and $\mathcal{A}$ if and only if all $f_i \cdot g : X \to Y_i$ are measurable. In case $I = \{*\}$, we have $\mathcal{A} = f_*^{-1}[\mathcal{B}(Y_*)]$.

Lemma 2.3 *Let* $(A \xrightarrow{f_i} |Y_i|)_I$ *be a family of functions and assume that each* Y_i *has a generator* $\mathcal{G}_i$*. Then the initial* σ*-algebra with respect to the* f_i *is* $\sigma\left(\bigcup_I f_i^{-1}[\mathcal{G}_i]\right)$*.*

Proof. See [7, Satz 5.2]. $\square$

Dually, we define the *final* σ*-algebra* for $(|X_i| \xrightarrow{g_i} B)_I$ by $\{\, E \subseteq B \mid \forall i \in I : g_i^{-1}[E] \in \mathcal{B}(X_i) \,\}$. It is characterized by the property that measurability of $h : B \to |Y|$ is guaranteed by the measurability of all $h \cdot g_i$.

This can be used to show that **Meas** is complete and cocomplete. Limits are constructed as follows: construct the limit $(L, (l_i))$ of the underlying diagram in **Set** and equip L with the initial σ-algebra with respect to the projections l_i, thus, with the σ-algebra generated by $\bigcup l_i^{-1}[\mathcal{B}X_i]$. Colimits are formed dually using final σ-algebras.

Equivalence relations and invariant sets

Let α be an equivalence relation on a set X. Let $\eta_\alpha : X \longrightarrow X/\alpha$ denote the projection. We obtain an adjunction:

$$PX \underset{\eta_\alpha^{-1}[-]}{\overset{\eta_\alpha[-]}{\rightleftarrows}} P(X/\alpha)\,,$$

where PX denotes the powerset of X, ordered by inclusion. Since η_α is surjective, every $B \in P(X/\alpha)$ is a fixpoint of the above adjunction. On the other hand, the fixpoints of $\eta_\alpha^{-1} \cdot \eta_\alpha[-]$ are easily characterized as the α*-invariant* subsets of X. Here, $A \subset X$ is α-invariant provided $x \in A$ and $x\,\alpha\,x'$ imply $x' \in A$ for each x, x' in X. Write $\mathrm{Inv}(\alpha)$ for the α-invariant subsets of X. The adjunction restricts to a pair of mutually inverse functions

$$\mathrm{Inv}(\alpha) \underset{\eta_\alpha^{-1}[-]}{\overset{\eta_\alpha[-]}{\rightleftarrows}} P(X/\alpha). \tag{1}$$

Fix a family $\mathcal{A}$ of subsets of a set X. Define an equivalence relation $\mathrm{Eq}(\mathcal{A})$ by:

$$(x, x') \in \mathrm{Eq}(\mathcal{A}) \iff \forall A \in \mathcal{A} : [x \in A \iff x' \in A].$$

Obviously, every $A \in \mathcal{A}$ is $\mathrm{Eq}(\mathcal{A})$-invariant.

Lemma 2.4 *We have* $\mathrm{Eq}(\mathcal{A}) = \mathrm{Eq}(\sigma\mathcal{A})$*.* $\square$

Separable Measurable spaces

We say that a family $\mathcal{A}$ of subsets of a set X *separates points* if whenever $x \neq x'$ then there exists $A \in \mathcal{A}$ with $x \in A$, $x' \notin A$, or vice versa.

Lemma 2.5 *$\mathcal{A}$ separates points if, and only if, $\sigma(\mathcal{A})$ separates points.* □

Definition 2.6 A measurable space X is called *separable* if $\mathcal{B}X$ separates points. The full subcategory of **Meas** spanned by the separable objects is denoted by **Sep**.

Proposition 2.7 **Sep** *is closed under mono-sources and coproducts in* **Meas**

Proof. Let $(f_i : X \longrightarrow Y_i)_I$ be a mono-source in **Meas** and assume $x \neq x'$ in X. There exists $i \in I$ with $f_i(x) \neq f_i(x')$, hence $B \in \mathcal{B}Y_i$ with (say) $f_i(x) \in B$, $f_i(x') \notin B$. Hence $x \in f_i^{-1}[B]$, $x' \notin f_i^{-1}[B]$, and $f_i^{-1}[B] \in \mathcal{B}X$ since f_i is measurable. Closure under coproducts is obvious. □

Subprobability measures

A *subprobability measure* on a measurable space X is a σ-additive function $\mathcal{B}X \to [0, 1]$. The set of all subprobability measures on X becomes a measurable space SX when equipped with the initial σ-algebra with respect to $(\mathrm{ev}_A)_{A\in\mathcal{B}X}$ with $\mathrm{ev}_A : SX \to [0, 1]$, $\mu \mapsto \mu(A)$. Here, $[0, 1]$ is equipped with the σ-algebra generated by the open subsets. Another generator of this σ-algebra is $\{\, [r, 1] \mid r \in \mathbb{Q} \cap [0, 1] \,\}$.

This subprobability construction gives rise to a functor $S : \mathbf{Meas} \to \mathbf{Meas}$ by setting

$$Sf(\mu)(B) = \mu(f^{-1}[B])$$

for $f : X \to Y$ in **Meas**, $\mu \in SX$, $B \in \mathcal{B}Y$.

Lemma 2.8 *Each SX is separable.*

Proof. Observe that $(\mathrm{ev}_A)_{A\in\mathcal{B}X}$ is a mono-source. Thus the claim follows from Proposition 2.7 □

Lemma 2.9 *Let $\mathcal{A} \subset \mathcal{B}X$ be closed under finite intersections with $\sigma(\mathcal{A}) = \mathcal{B}X$. Then $\mathcal{B}(SX)$ is generated by the set $\{\, \mathrm{ev}_A^{-1}[r, 1] \mid A \in \mathcal{A}, r \in \mathbb{Q} \cap [0, 1] \,\}$.*

Proof. Combine Lemma 3.6 from [14] with Lemma 2.3. □

3 Coalgebras and Models

Let $T : \mathbf{Meas} \longrightarrow \mathbf{Meas}$ be a functor. A *T-coalgebra* $\mathbb{A} = (A, d)$ is given by measurable space A and a **Meas**-morphism $d : A \to TA$, called the *dynamics*. A morphism of coalgebras $(A, d) \to (A', d')$ is given by a **Meas**-morphism

$f : A \to A'$ such that

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "d"'] & A' \arrow[d, "d'"] \\ TA \arrow[r, "Tf"'] & TA' \end{tikzcd}$$

commutes. This leads to the category $\mathbf{Coalg}\, T$ of T-coalgebras and morphisms.

Examples 1 (i) *Coalgebras for the subprobability functor are stochastic relations.*

(ii) *Coalgebras for the functor on* **Meas** *given by* $X \mapsto (SX)^{\mathrm{Act}}$ *for some set* Act *are labelled Markov processes.*

We fix a set Var of *variables*, and define *models* with respect to T and Var as follows:

Definition 3.1 A (T, Var)*-model* consists of a T-coalgebra (A, d) together with a *valuation*; that is, a function $\mathsf{Var} \xrightarrow{V} \mathcal{B}(A)$. A morphism $(A, d, V) \xrightarrow{f} (A', d', V')$ is given by a coalgebra morphism $(A, d) \xrightarrow{f} (A', d')$ which satisfies $\mathcal{B}(f) \cdot V' = V$.

This leads us to a category $\mathbf{Mod}(T, \mathsf{Var})$ of models. Observe that we have an isomorphism $\mathbf{Mod}(T, \emptyset) \cong \mathbf{Coalg}\, T$.

From now on we fix functor T, and the set of variables; $\mathbf{Mod}(T, \mathsf{Var})$ will simply be denoted by $\mathbf{Mod}$.

It is well-known that the obvious forgetful functor $\mathbf{Coalg}\, T \longrightarrow \mathbf{Meas}$ creates colimits; see [1]. Thus, $\mathbf{Coalg}\, T$ is cocomplete since $\mathbf{Meas}$ is so. This generalizes to models:

Proposition 3.2 *The obvious forgetful functor* $\mathbf{Mod} \longrightarrow \mathbf{Coalg}\, T$ *creates colimits. Hence also the forgetful functor* $\mathbf{Mod} \longrightarrow \mathbf{Meas}$ *creates colimits.*

Proof. Let $D : \mathbf{D} \longrightarrow \mathbf{Mod}$ be a (small) diagram. Write $D(i) = (X_i, d_i, V_i)$ and let $((X, d), c_i)$ denote the colimit of the (X_i, d_i) in $\mathbf{Coalg}\, T$. Observe that $(\mathcal{B}X, \mathcal{B}c_i)$ is a limit in $\mathbf{Set}$. Since Dd is a model morphism for each $d : i \to j$ in $\mathbf{D}$ the collection (V_i) of valuations forms a natural cone, hence there is a unique $V : \mathsf{Var} \longrightarrow \mathcal{B}X$ with $\mathcal{B}c_i \cdot V = V_i$ for each i. We claim that (X, d, V) is a colimit of D in $\mathbf{Mod}$. Let $(f_i : D_i \longrightarrow (Y, e, W))$ be a natural cocone, and let $f : X \longrightarrow Y$ be the induced $\mathbf{Coalg}\, T$-morphism. Thus, $f \cdot c_i = f_i$ holds. We have

$$\mathcal{B}c_i \cdot \mathcal{B}f \cdot W = \mathcal{B}f_i \cdot W = V_i = \mathcal{B}c_i \cdot V,$$

where the first equation holds since f_i is a model morphism. Thus, $\mathcal{B}f \cdot W = V$ follows from the fact that $(\mathcal{B}c_i)$ is a limit-source, that is: $(X, d.V) \xrightarrow{f}$

(Y, e, W) is a model-morphism. □

Corollary 3.3 **Mod** *is cocomplete.* □

4 Predicate Liftings

Let T : **Meas** → **Meas** be a functor. An *n-ary predicate lifting* for T is a natural transformation $\lambda : \mathcal{B}^n \to \mathcal{B} \cdot T$. We write $n = \mathsf{ar}(\lambda)$.

Examples 2 (i) *Let r be a rational number and define $\lambda^r_X : \mathcal{B}X \longrightarrow \mathcal{B}SX$ via:*

$$\lambda^r_X(B) = \{\, \mu \mid \mu(B) \geq r \,\} = \mathrm{ev}_B^{-1}[r, 1].$$

(λ^r) is easily seen to be natural. We write $\Lambda_1 = \{\, \lambda^r \mid r \in \mathbb{Q} \cap [0, 1] \,\}$.

(ii) *Let* Act *be a countable set of "actions". For each rational r and each $a \in \mathrm{Act}$, we define a predicate lifting $\lambda^{r,a}$ for S^{Act} via*

$$\lambda^{r,a}_X(B) = \{\, (\mu_i)_{i \in \mathrm{Act}} \mid \mu_a(B) \geq r \,\}.$$

We write $\Lambda_{\mathrm{Act}} = \{\, \lambda^{r,a} \mid a \in \mathrm{Act}, r \in \mathbb{Q} \cap [0, 1] \,\}$.

(iii) *Let T denote the functor given by $X \mapsto S(X \times X)$ with its obvious action on morphisms. We define a family $(\kappa^q)_{q \in \mathbb{Q} \cap [0,1]}$ via*

$$\kappa^q_X(A, B) = \{\, \mu \in S(X \times X) \mid \mu(A \times B) \geq q \,\} = \mathrm{ev}^{-1}_{A \times B}[q, 1]. \quad (2)$$

The Logic Induced by a Family of Predicate Liftings

Fix a set Λ of predicate liftings. We define the logic induced by Λ (and Var) by the following grammar:

$$\phi ::= \top \mid \phi_1 \wedge \phi_2 \mid v \mid \langle \lambda \rangle (\phi_1, \ldots, \phi_{\mathsf{ar}(\lambda)})$$

for $v \in \mathsf{Var}$, $\lambda \in \Lambda$.

Remark 4.1 It is possible to enrich the logic by using further Boolean connectives (modelled by natural transformations $\mathcal{B}^n \longrightarrow \mathcal{B}$) and fixpoint-operators (modelled by natural transformations $\mathcal{B}^\omega \longrightarrow \mathcal{B}$); see [6] for a discussion. We refrain from doing so, the result go through verbatim.

Interpreting the Logic in a Model

Given a model $\mathcal{M} = (A, d, V)$, we may interpret $\mathcal{L}(\Lambda)$ by assigning a set $[\![\phi]\!] \in \mathcal{B}A$ to every formula ϕ as follows:

$$\begin{aligned} [\![\top]\!]_{\mathcal{M}} &= A \\ [\![\phi_1 \wedge \phi_2]\!]_{\mathcal{M}} &= [\![\phi_1]\!]_{\mathcal{M}} \cap [\![\phi_2]\!]_{\mathcal{M}} \\ [\![v]\!]_{\mathcal{M}} &= V(v) \\ [\![\langle\lambda\rangle(\phi_1, \ldots, \phi_{\mathsf{ar}(\lambda)})]\!]_{\mathcal{M}} &= \mathcal{B}d \cdot \lambda_A([\![\phi_1]\!]_{\mathcal{M}}, \ldots, [\![\phi_{\mathsf{ar}(\lambda)}]\!]_{\mathcal{M}}) \end{aligned}$$

For a state x of $\mathcal{M}$ we write $\mathsf{Th}_{\mathcal{M}}(x) = \{\, \phi \in \mathcal{L} \mid x \in [\![\phi]\!]_{\mathcal{M}} \,\}$.

Proposition 4.2 *If $f : \mathcal{M} \longrightarrow \mathcal{N}$ is a model morphism, then we have $\mathcal{B}f([\![\phi]\!]_{\mathcal{N}}) = [\![\phi]\!]_{\mathcal{M}}$ for each ϕ.*

Proof. The proof proceeds by structural induction on formula ϕ and makes use of the naturality of the predicate liftings. For details see [6], where the unimodal case was treated. □

Corollary 4.3 *If $\mathcal{M} \xrightarrow{f} \mathcal{N}$, then $\mathsf{Th}_{\mathcal{M}}(x) = \mathsf{Th}_{\mathcal{N}}(f(x))$.* □

An equivalence relation

Fix a model $\mathcal{M} = (A, d, V)$. Let ℓ denote the equivalence on A which is determined by the extensions of the formulas. Thus,

$$x \ell x' \iff \forall \phi : [x \in [\![\phi]\!] \iff x' \in [\![\phi]\!]],$$

or, more compressed, $\ell = \mathrm{Eq}(\{\, [\![\phi]\!] \mid \phi \in \mathcal{L} \,\})$. Observe that we have $x \ell x' \iff \mathsf{Th}_{\mathcal{M}}(x) = \mathsf{Th}_{\mathcal{M}}(x')$.

We write $\mathcal{E}_{\mathcal{M}} = \sigma(\{\, \eta_\ell[[\![\phi]\!]_{\mathcal{M}}] \mid \phi \text{ formula} \,\})$.

Lemma 4.4 *We have $\{\, \eta_\ell[[\![\phi]\!]] \mid \phi \text{ formula} \,\} = \{\, A \subset X/\ell \mid \eta_\ell^{-1}[A] = [\![\phi]\!] \text{ for some } \phi \,\}$. Both sets are closed under finite intersections.*

Proof. Just use the fact that each $[\![\phi]\!]$ is ℓ-invariant and apply the bijections from (1). The second claim follows since the set of validity-sets is closed under finite intersection and the inverse image function preserves finite intersections. □

Lemma 4.5 *$\eta_\ell : X \longrightarrow (X/\ell, \mathcal{E}_{\mathcal{M}})$ is measurable.*

Proof. Obvious by Lemma 4.4. □

In general, $\mathcal{E}_{\mathcal{M}}$ *will not be the final σ-algebra* with respect to $\mathcal{B}X$ and η_ℓ. In the following, $\eta_{\mathcal{M}}$ will always denote the measurable function whose domain is $(X/\ell, \mathcal{E}_{\mathcal{M}})$.

Lemma 4.6 *Fix a measurable space Y and a generator $\mathcal{A}$ of $\mathcal{B}Y$. A function $f : X/\ell \longrightarrow |Y|$ is measurable if for each $A \in \mathcal{A}$ there exists ϕ such that $\eta^{-1} \cdot f^{-1}[A] = [\![\phi]\!]$.*

Proof. Obvious since it suffices to check measurability on a generator. □

We will introduce two technical conditions on sets of predicate liftings: separability and admissibility. Our notion of separable predicate liftings is a specialization of the one used in coalgebraic logic over **Set**, cf. [11]. It was first introduced in [6] for sets of unary predicate liftings.

Admissibility is another technical condition which we need to impose in order to prove measurability of an induced function. It is introduced below and related to separability.

Separating Predicate Liftings

Let $\lambda : \mathcal{B}^n \longrightarrow \mathcal{B} \cdot T$ be an n-ary predicate lifting. We define an equivalence relation $\equiv^{\lambda}_{\mathcal{M}}$ on TX by setting

$$\equiv^{\lambda}_{\mathcal{M}} = \mathrm{Eq}\left(\{ \lambda_X([\![\phi_1]\!]_{\mathcal{M}}, \ldots, [\![\phi_n]\!]_{\mathcal{M}}) \mid \phi_1, \ldots, \phi_n \in \mathcal{L} \}\right).$$

Definition 4.7 A set Λ of predicate liftings is said to *separate* a model $\mathcal{M}$ if we have

$$\bigcap_{\lambda \in \Lambda} \equiv^{\lambda}_{\mathcal{M}} \subseteq \ker T(\eta_{\mathcal{M}}).$$

Λ is said to be *separating* if it separates every model $\mathcal{M}$.

Examples 3 *The sets Λ_1 and Λ_{Act} are separating. This follows from Lemma 4.9 (below) and Proposition 4.11.*

Admissible predicate liftings

Definition 4.8 We call a set Λ of predicate liftings *admissible* if the set

$$\{ \lambda_{X/\ell}(\eta[\![\phi_1]\!], \ldots, \eta[\![\phi_{\mathsf{ar}(\lambda)}]\!]) \mid \lambda \in \Lambda, \phi_1, \ldots, \phi_{\mathsf{ar}(\lambda)} \in \mathcal{L} \}$$

generates $\mathcal{B}(T(X/\ell, \mathcal{E}_{\mathcal{M}}))$ for every model $\mathcal{M}$.

Lemma 4.9 *For each set* Act*, the family Λ_{Act} is admissible.*

Proof. Write $Q = (X/\ell, \mathcal{E}_{\mathcal{M}})$. First consider the case $\mathrm{Act} = \{*\}$. We know from Lemma 2.9 and Lemma 4.4 that $\mathcal{B}SQ$ is generated by

$$\mathcal{D} = \{ \mathrm{ev}^{-1}_{\eta[\![\phi]\!]}[r, 1] \mid \phi \in \mathcal{L}, r \in \mathbb{Q} \cap [0, 1] \}.$$

Since $\mathrm{ev}^{-1}_{\eta[\![\phi]\!]}[r,1] = \lambda^r_Q(\eta[\![\phi]\!])$ holds, this is just the condition for admissibility.

For a general set Act, observe that we have

$$\begin{aligned}
\mathcal{C} &= \{\, \lambda^{r,a}_Q(\eta[\![\phi]\!]) \mid a \in \mathrm{Act}, r \in \mathbb{Q}\cap[0,1], \phi \in \mathcal{L}\,\} \\
&= \bigcup_{a\in\mathrm{Act}} \{\, \lambda^{r,a}_Q(\eta[\![\phi]\!]) \mid r \in \mathbb{Q}\cap[0,1], \phi \in \mathcal{L}\,\} \\
&= \bigcup_{a\in\mathrm{Act}} \pi_a^{-1}\left(\{\, \lambda^{r}_Q(\eta[\![\phi]\!]) \mid r \in \mathbb{Q}\cap[0,1], \phi \in \mathcal{L}\,\}\right) \\
&= \bigcup_{a\in\mathrm{Act}} \pi_a^{-1}(\mathcal{D})
\end{aligned}$$

holds, where $\pi_a : SQ^{\mathrm{Act}} \longrightarrow SQ$ is the ath projection. Since $\mathcal{D}$ generates $\mathcal{B}SQ$, $\mathcal{C}$ generates $\mathcal{B}(SQ^{\mathrm{Act}})$ by Lemma 2.3. □

Lemma 4.10 *The set $(\kappa^r)_{r\in\mathbb{Q}\cap[0,1]}$ is admissible.*

Proof. For measurable spaces X and Y the set $\{\, A\times B \mid A \in \mathcal{B}X, B \in \mathcal{B}Y\,\}$ is a closed under finite intersections and a generator for $\mathcal{B}(X\times Y)$. Thus, the claim follows from Lemma 2.9. □

Proposition 4.11 *If each TX is separated then every admissible set of predicate liftings is separated.*

Proof. Fix a model $\mathcal{M} = (X, d, V)$, $\lambda \in \Lambda$ of arity n and $\phi_1, \ldots, \phi_n \in \mathcal{L}$. We have, for each $t \in TX$:

$$\begin{aligned}
T\eta(t) \in \lambda_{X/\ell}(\eta[\![\phi_1]\!], \ldots, \eta[\![\phi_n]\!]) &\iff t \in \mathcal{B}T\eta \cdot \lambda_{X/\ell}(\eta[\![\phi_1]\!], \ldots, \eta[\![\phi_n]\!]) \\
&\iff t \in \lambda_X \cdot (\mathcal{B}\eta)^n(\eta[\![\phi_1]\!], \ldots, \eta[\![\phi_n]\!]) \\
&\iff t \in \lambda_X([\![\phi_1]\!], \ldots, [\![\phi_n]\!])
\end{aligned}$$

where the last equality holds since each $[\![\phi_i]\!]$ is invariant.

Take $t, t' \in TX$ with $t \equiv^\lambda_{\mathcal{M}} t'$ for all $\lambda \in \Lambda$. We need to show $T\eta(t) = T\eta(t')$. By separability of $T(X/\ell, \mathcal{E}_{\mathcal{M}})$ and Lemma 2.5, it suffices to show

$$(T\eta(t), T\eta(t')) \in \mathrm{Eq}\left(\{\, \lambda_{X/\ell}(\eta[\![\phi_1]\!], \ldots, \eta[\![\phi_{\mathsf{ar}(\lambda)}]\!]) \mid \lambda \in \Lambda, \phi_1, \ldots, \phi_{\mathsf{ar}(\lambda)} \in \mathcal{L}\,\}\right).$$

From the above calculation it follows that this last condition is equivalent to $(t, t') \in \equiv^\lambda_{\mathcal{M}}$. □

The congruence theorem

Theorem 4.12 *Let Λ be a separating and admissible set of predicate liftings. For every model $\mathcal{M}$ there exists a model structure $\mathcal{M}/\ell$ over $X/\ell_{\mathcal{M}}$ such that $\eta_{\mathcal{M}} : \mathcal{M} \longrightarrow \mathcal{M}/\ell_{\mathcal{M}}$ is a model morphism.*

Proof. We write $\mathcal{M} = (X, d, V)$, ℓ for $\ell_{\mathcal{M}}$, and Q for $(X/\ell_{\mathcal{M}}, \mathcal{E}_{\mathcal{M}})$. We claim that $x\ell x'$ implies $(d(x), d(x')) \in \ker T\eta$. By separatedness of Λ it suffices to show that we have $d(x) \equiv^{\lambda}_{\mathcal{M}} d(x')$ for each $\lambda \in \Lambda$.

Take $\lambda \in \Lambda$, write $n = \mathsf{ar}(\lambda)$ and fix $\phi_1, \ldots, \phi_n \in \mathcal{L}$. We obtain:

$$\begin{aligned} d(x) \in \lambda_X([\![\phi_1]\!], \ldots, [\![\phi_n]\!]) &\iff x \in \mathcal{B}d \cdot \lambda_X([\![\phi_1]\!], \ldots, [\![\phi_n]\!]) \\ &\iff x \in [\![\langle\lambda\rangle(\phi_1, \ldots, \phi_1)]\!] \end{aligned}$$

hence $(d(x), d(x')) \in \equiv^{\lambda}_{\mathcal{M}}$ follows from $x\ell x'$. Hence, there exists a unique function $q : Q \longrightarrow TQ$ for which

$$\begin{array}{ccc} X & \xrightarrow{\eta} & Q \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle q} \\ TX & \xrightarrow[T\eta]{} & TQ \end{array}$$

commutes. We need to show that q is measurable.

Fix any λ of arity n and $\phi_1, \ldots, \phi_n \in \mathcal{L}$ and observe:

$$\begin{aligned} \eta^{-1} \cdot q^{-1} \cdot \lambda_Q(\eta[\![\phi_1]\!], \ldots, \eta[\![\phi_n]\!]) &= \mathcal{B}d \cdot \mathcal{B}T\eta \cdot \lambda_Q(\eta[\![\phi_1]\!], \ldots, \eta[\![\phi_n]\!]) \\ &= \mathcal{B}d \cdot \lambda_X \cdot (\mathcal{B}\eta)^n(\eta[\![\phi_1]\!], \ldots, \eta[\![\phi_n]\!]) \\ &= \mathcal{B}d \cdot \lambda_X(\eta^{-1}[\eta[\![\phi_1]\!]], \ldots \eta^{-1}[\eta[\![\phi_n]\!]]) \\ &\stackrel{\dagger}{=} \mathcal{B}d \cdot \lambda_X([\![\phi_1]\!], \ldots, [\![\phi_n]\!]) \\ &= [\![\langle\lambda\rangle(\phi_1, \ldots, \phi_n)]\!], \end{aligned}$$

where (†) holds by invariance of $[\![\phi_i]\!]$. Measurability of q follows from admissibility of Λ and Lemma 4.6.

We are left to define a valuation W on Q. For $v \in \mathsf{Var}$ we set $W(v) = \eta[\![v]\!]_{\mathcal{M}} \in \mathcal{E}_{\mathcal{M}}$. By ℓ-invariance of $[\![v]\!]_{\mathcal{M}}$ we obtain $\mathcal{B}\eta \cdot W(v) = \eta^{-1}[\eta V(v)] = V(v)$, thus W makes η a model morphism. Uniqueness of W with this property holds by injectivity of $\mathcal{B}\eta$. □

Definition 4.13 The *reduct* $\mathrm{Red}(\mathcal{M})$ of $\mathcal{M}$ is the model (Q, q, W) constructed in Theorem 4.12.

Thus, the underlying set of Q is given by $X/\ell_{\mathcal{M}}$ and we have $\mathcal{B}Q = \mathcal{E}_{\mathcal{M}} = \sigma(\{\, \eta[[\![\phi]\!]_{\mathcal{M}}] \mid \phi \in \mathcal{L}\,\})$.

Over analytic spaces

Recall that a measurable space is called *analytic* if its measurable subsets arise as the Borel sets of a topological space which is the continuous image of a metrizable topological space with a countable base.

In case the underlying measurable space of the model $\mathcal{M}$ is analytic—which is the blanket assumption in [4,6,12]—we can form the reduct $\mathrm{Red}(\mathcal{M})$ without relying on admissibility of Λ.

Lemma 4.14 *Let $\mathcal{M} = (A, d, V)$ be a model with A analytic and assume that $\mathcal{L}$ is countable. Then $\mathcal{E}_{\mathcal{M}}$ is the final σ-algebra with respect to the projection $\eta_{\ell_{\mathcal{M}}}$.*

Proof. See [5, Corollary 3]. □

Observe that $\mathcal{L}$ is countable provided both Λ and Var are countable.

Theorem 4.15 *Let Λ be a countable, separating set of predicate liftings and let Var be countable. For every model $\mathcal{M} = (A, d, V)$ with A analytic there exists a model structure $\mathcal{M}/\ell$ over A/ℓ such that $\eta : \mathcal{M} \longrightarrow \mathcal{M}/\ell$ is a model morphism.*

Proof. We proceed as in the proof of Theorem 4.12 to define the quotient dynamics q. Observe that well-definedness of q just makes use of separatedness of Λ. Measurability of q follows immediately from Lemma 4.14 since we have $q \cdot \eta = T\eta \cdot d$ and the latter function is measurable. □

5 Logical and Behavioral Equivalence

Definition 5.1 We fix models $\mathcal{M}$, $\mathcal{M}'$ and states a in $\mathcal{M}$, a' in $\mathcal{M}'$.

(i) We say that the states a and a' are *logically equivalent* if $\mathsf{Th}_{\mathcal{M}}(a) = \mathsf{Th}_{\mathcal{M}'}(a')$.

(ii) We say that the models $\mathcal{M}$ and $\mathcal{M}'$ are *logically equivalent* if $\{\, \mathsf{Th}_{\mathcal{M}}(a) \mid a \in A \,\} = \{\, \mathsf{Th}_{\mathcal{M}'}(a') \mid a' \in A' \,\}$.

(iii) We say that the states a and a' *behaviorally equivalent* if there exists a model $\mathcal{N}$ and a cospan

$$\mathcal{M} \xrightarrow{f} \mathcal{N} \xleftarrow{g} \mathcal{M}' \tag{3}$$

of model morphisms such that $f(a) = g(a')$ holds;

(iv) We say that the models $\mathcal{M}$ and $\mathcal{M}'$ are *behaviorally equivalent* if there exists a cospan (3) with f and g *surjective* model morphisms.

Observe that the notions of equivalence introduced above naturally divide themselves into two groups: *local* or state-based notions (1,3), and *global* notions (2,4). Proposition 4.2 and Corollary 4.3 entail that:

- (local) behavioral equivalence of states implies their (local) logical equivalence.

- (global) behavioral equivalence of models implies their (global) logical equivalence;

We will now show that under suitable conditions, all four implications can be reversed. So far, work on modal logics for stochastic relations was somewhat concentrated on the global aspects. We think that the local notions (as customary for coalgebras on **Set**) are more in the spirit of classical modal logic, which is intrinsically local.

Theorem 5.2 *Assume that Λ is separating and admissible. Fix models $\mathcal{M}$ and $\mathcal{N}$. If states of x of $\mathcal{M}$ and y of $\mathcal{N}$ are logically equivalent, then they are behaviorally equivalent.*

Proof. We form first the coproduct $\mathcal{M}+\mathcal{N}$ of $\mathcal{M}$ and $\mathcal{N}$ and then the reduct $\mathrm{Red}(\mathcal{M}+\mathcal{N})$ according to Theorem 4.12. Thus, we obtain the following diagram of model morphisms

$$\begin{tikzcd} \mathcal{M} \arrow[r, "i_{\mathcal{M}}"] & \mathcal{M}+\mathcal{N} \arrow[d, "\eta"] & \mathcal{N} \arrow[l, "i_{\mathcal{N}}"'] \\ & \mathrm{Red}(\mathcal{M}+\mathcal{N}) & \end{tikzcd} \tag{4}$$

By Corollary 4.3, we have $\mathsf{Th}_{\mathcal{M}+\mathcal{N}}(i_{\mathcal{M}}(x)) = \mathsf{Th}_{\mathcal{M}}(x) = \mathsf{Th}_{\mathcal{N}}(y) = \mathsf{Th}_{\mathcal{M}+\mathcal{N}}(i_{\mathcal{N}}(b))$. Therefore, $\eta \cdot i_{\mathcal{M}}(x) = \eta \cdot i_{\mathcal{N}}(y)$; that is, $\mathrm{Red}(\mathcal{M}+\mathcal{N})$ witnesses that a and b are behaviorally equivalent. $\square$

The following result on the equivalence of the global properties is in fact an easy consequence of Theorem 5.2:

Theorem 5.3 *Assume that Λ is separating and admissible. Then logical equivalence of models implies behavioral equivalence of models.*

Proof. Let $\mathcal{M}$ and $\mathcal{N}$ be logically equivalent and form the diagram (4). We claim that $\eta \cdot i_{\mathcal{M}}$ and $\eta \cdot i_{\mathcal{N}}$ are surjective. Consider $\eta \cdot i_{\mathcal{M}}$ and take any state $[z]$ in $\mathrm{Red}(\mathcal{M}+\mathcal{N})$. In case z is in the image of $i_{\mathcal{M}}$, we are done. Otherwise, $z = i_{\mathcal{N}}(y)$ for some state of $\mathcal{N}$. We find a state x of $\mathcal{M}$ with $\mathsf{Th}_{\mathcal{M}}(x) = \mathsf{Th}_{\mathcal{N}}(y)$, hence $\mathsf{Th}_{\mathcal{M}+\mathcal{N}}(i_{\mathcal{M}}(x)) = \mathsf{Th}_{\mathcal{M}+\mathcal{N}}(i_{\mathcal{N}}(y))$, that is $\eta \cdot i_{\mathcal{M}}(x) = \eta \cdot i_{\mathcal{N}}(y) = \eta(z) = [z]$. $\square$

6 Conclusion and Further Work

We have established expressivity-results for coalgebraic logic over general measurable spaces. In doing so, we have improved over previously published work in this area in three aspects:

(i) we were able to work without any topological assumptions on the state space;

(ii) we work with a general functor on the category of measurable spaces without relying on what was called left or right coalgebras [6];

(iii) we considered local versions of behavioral and logical equivalence which deal with single states as opposed to the whole models. This seems to be in the spirit of coalgebraic modal logic over the category of sets. In fact, the global properties are simple consequences of the local ones.

One obvious extension of the work presented here would be to include a discussion of bisimilarity as well. This is hindered by the fact that the subprobability functor does not preserve weak pullbacks.

Another possible extension of the results presented here stems from the observation that a measurable space can be seen as a set equipped with a basis for a zero-dimensional topology on it. In particular, the technical notion of admissibility can be rephrased as a continuity condition for these topologies. This hints at a possible unification of our work and the work on Stone coalgebras [9].

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References

[1] J. Adamék. Introduction to coalgebra. *Theory and Applications of Categories*, 14(8):157–199, 2005.

[2] V. Danos, J. Desharnais, F. Laviolette, and P. Panangaden. Bisimulation and cocongruence for probabilistic systems. *Information and Computation*, 204:503–523, 2006.

[3] J. Desharnais, A. Edalat, and P. Panangaden. Bisimulation for labelled Markov processes. *Information and Computation*, 179:163–193, 2002.

[4] E.-E. Doberkat. *Stochastic Relations. Foundations for Markov Transition Systems*. Chapman & Hall/CRC Press, Boca Raton, New York, 2007.

[5] E.-E. Doberkat. Behavioral and logical equivalence of stochastic Kripke models in general measurable spaces. In *Proceedings TAMC 2009*, volume 5532 of *Lect. Notes Computer Science*, pages 192–200, 2009.

[6] E.-E. Doberkat and C. Schubert. Coalgebraic logic for stochastic right coalgebras. *Ann. Pure Appl. Logic*, 2009. doi:10.1016/j.apal.2008/06/018.

[7] J. Elstrodt. *Maß- und Integrationstheorie*. Springer-Verlag, Berlin-Heidelberg-New York, 2 edition, 1999.

[8] B. Jacobs and A. Sokolova. Exemplaric expressivity of modal logics. *Journal of Logic and Computation*, 2009. http://dx.doi.org/10.1093/logcom/exn093.

[9] C. Kupke, A. Kurz, and Y. Venema. Stone coalgebras. *Theoretical Computer Science*, 327:109–134, 2004.

[10] D. Pattinson. Expressive logics for coalgebras via terminal sequence induction. *Notre Dame J. Formal Logic*, 45(1):19 – 33, 2004.

[11] L. Schröder. Expressivity of modal logics: The limits and beyond. *Theoretical Computer Science*, 390:230–247, 2008.

[12] C. Schubert. Coalgebraic logic over analytic spaces. Technical Report 170, Chair for Software Technology, Technische Universität Dortmund, October 2007.

[13] C. Schubert. Final coalgebras for measure-polynomial functors. In *Proceedings TAMC 2009*, volume 5532 of *Lect. Notes Computer Science*, pages 325–334, 2009.

[14] I. D. Viglizzo. *Coalgebras on measurable spaces.* PhD thesis, Indiana University, 2005.