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A Digital Version of the Kakutani Fixed Point Theorem for Convex-valued Multifunctions

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Abstract

In this paper, the concepts of power graphs and power complexes are introduced. The multifunctions for graphs are defined and will be classified. The concept of simplicial mappings for complexes then is extended to multifunctions. A notion of *weak convexity* is defined in the intersection graphs of $(3^n - 1)$-adjacent n-dimensional real digital pictures based on the usual Euclidean convex closure operator. It is shown that any $(3^n - 1)$-adjacent n-dimensional digital picture has the simplicial weak convex almost fixed point property, which may be considered as a digital version of the Kakutani fixed point theorem for convex-valued multifunctions.

Key words: Almost fixed point property; Digital convexity; Digital pictures; Intersection graphs; Kakutani Theorem; Power structures

1 Introduction

In looking for a "digital" analogue of Brouwer's fixed point theorem, it is natural to think of graphs which may be considered to approximate the unit cube, and of graph morphisms (edge-preserving maps from vertices to vertices) thereon. Graphs of the form I_k^n (the strong product of n copies of I_k [2]), for example, could be suitable, where I_k is the linear graph with k vertices and $k - 1$ edges. We see at once that an exact fixed point result is not possible (just consider the order-reversing map on I_4). What we may hope for, instead, is an "almost" fixed point property: any endomorphism maps some vertex to an *adjacent* or *equal* vertex.

Results of this kind have, indeed, been propounded a number of times. In the image-processing literature we have, in particular, Rosenfeld [13], where

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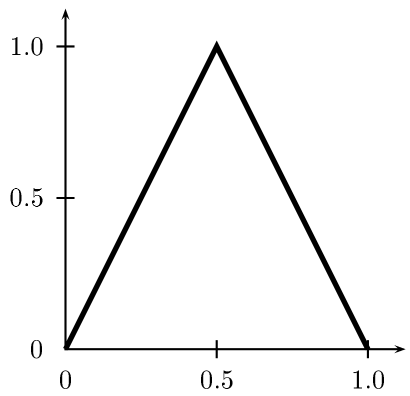


Fig. 1. The diagram of f

the afpp (almost fixed point property) is shown to hold for I_k^n. A more general result had already been proved, by rather elaborate algebraic-topological means, in the unpublished thesis of Poston [11]: any contractible graph has the afpp. Even earlier than this, Pultr had already proved the afpp for contractible graphs, with various associated results, in his little-known paper [12].

These authors' results were conceived, more-or-less explicitly, as analogues of Brouwer's theorem. The term "analogue" here is clearly quite informal. In our approach, in contrast, we seek digital (approximating) results which are "versions" of classical results in a rather strong sense: it should be possible to recover the classical result from the digital result by a limiting process. Clearly, for this to make sense, we have to be able to regard graphs, such as I_k^n, as approximating spaces such as the unit cube I^n in a precise sense. A framework in which this can be done (the approximation of various spaces by inverse limits of graphs) was developed in [15,16]; an extended treatment may be found in [22]. An important aspect of this development is that, in approximating an ordinary continuous function on a (classical) space we in general require multifunctions on the approximating graphs. The point may perhaps be made clear via a simple example: consider a continuous function f from unit interval I to I defined by

$$f(x) = \begin{cases} 2x, & 0 \le x \le 0.5 \\ 2 - 2x, & 0.5 \le x \le 1 \end{cases}$$

where the diagram of f is shown in Fig. 1.

It is easy to see that during the approximation of f, the discrete representations $\dot{f}$ and $\ddot{f}$ of f may become multifunctions, i.e. $\dot{f} : I_4 \to I_4$, $00 \mapsto \{00, 01, 10\}$, $01 \mapsto \{01, 10, 11\}$, $10 \mapsto \{01, 10, 11\}$ & $11 \mapsto \{00, 01, 10\}$; and $\ddot{f} : I_8 \to I_8$, $000 \mapsto \{000, 001, 010\}$, $001 \mapsto \{001, 010, 011, 100\}$, $010 \mapsto \{011, 100, 101, 110\}$, $011 \mapsto \{101, 110, 111\}$, $100 \mapsto \{101, 110, 111\}$, $101 \mapsto \{011, 100, 101, 110\}$, $110 \mapsto \{001, 010, 011, 100\}$ & $111 \mapsto \{000, 001, 010\}$, the Fig. 2 shows the diagrams of $\dot{f}$ and $\ddot{f}$. Hence the discrete representations of a function may become many-valued rather than single-valued.

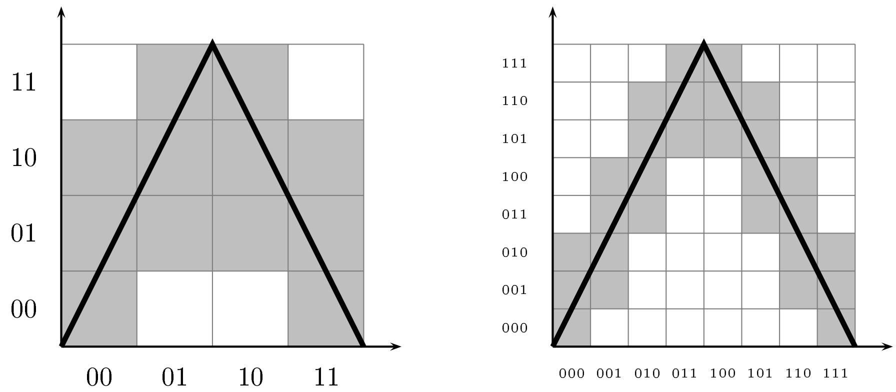


Fig. 2. The diagrams of $\dot{f}$ and $\ddot{f}$

By these considerations, we are led to consider (almost) fixed points of many-valued functions in digital spaces. In topology, the best-known generalization of Brouwer's theorem to multifunctions is the theorem of Kakutani. In this paper, therefore, we shall propose a digital version of Kakutani's theorem. This will be reached in Section 5, after some preparatory material on power graphs and complexes (Section 2 and 3) and convexity in graphs (Section 4).

Note 1 *It will be sufficient for our purpose here to consider the fixed point theorems of Brouwer and Kakutani as asserting that the unit cube $I^n \subseteq \mathbb{R}^n$ has the fixed point property for continuous functions and upper-semi continuous convex-valued multifunctions, respectively. See Section 5 for further details.*

2 Power graphs

By a *graph* G we mean a set $V(G)$ of *vertices* (or *points*), with a reflexive and symmetric relation $E(G)$ the *edges*, also known as a *tolerance graph* in [11,18]. For graphs G_1 and G_2, a *graph homomorphism* $f : G_1 \to G_2$ maps points of G_1 to points of G_2, preserving the tolerance relation.

Let G be a graph. We define two *power* graphs having the non-empty finite subsets of $V(G)$, $P^+_{fin}(V(G))$ as vertices, as follows:

Definition 2.1 The *weak* power graph $P_w(G)$ has the edge set $E_w(G)$, where

$$(A, B) \in E_w(G) \Leftrightarrow \exists x \in A, y \in B, (x, y) \in E(G).$$

The *strong* power graph $P_s(G)$ has the edge set $E_s(G)$, where

$$\begin{aligned}(A, B) \in E_s(G) \Leftrightarrow \forall x \in A, \exists y \in B, (x, y) \in E(G) \ \ \& \\ \forall y \in B, \exists x \in A, (x, y) \in E(G).\end{aligned}$$

These constructions may be simply characterized as follows:

Theorem 2.2 *Let E^+ be a relation on $P^+_{fin}(V(G))$. Consider the properties:*

(1) $(x, y) \in E(G) \Rightarrow (\{x\}, \{y\}) \in E^+$,

$(2)_w$ $(A, B) \in E^+ \Rightarrow \forall C \supseteq B, (A, C) \in E^+$,

$(2)_s$ $(A, B) \in E^+, (C, D) \in E^+ \Rightarrow (A \cup C, B \cup D) \in E^+$.

Then E_w is the least symmetric relation E^+ satisfying (1) *and* $(2)_w$*; while E_s is the least* (*symmetric*) *relation E^+ satisfying* (1) *and* $(2)_s$. □

The preceding definition and proposition will clearly work just as well for directed graphs (that is, for arbitrary binary relations), with just a slight modification of condition $(2)_w$. More significantly, the sort of development familiar from power domain theory can readily be given here as well. In particular, the constructs P_s and P_w are (functorial and) monadic; and from Theorem 2.2 we can derive the further characterization that $(P_s(G), \cup)$ constitutes the free semilattice over G (in the category of graphs and graph homomorphisms **Grph**). We do not give the details here, as none of this development will be needed in the present paper.

Note that whereas the relation E_s corresponds to the so-called "Egli-Milner ordering", E_w has no counterpart in domain theory (since raising an ordering by E_w results in a relation which is not even transitive). Yet it is E_w that will be the focus of our study in this paper. The reason for this may be seen by looking at the approximation of real functions by graph multifunctions, illustrated in Section 1. For example, $\ddot{f}$ is a multifunction defined on the graph I_8 where $V(I_8)$ is the set $\{000, \ldots, 111\}$, and $E(I_8)$ is adjacency in the sequence of binary integers. Edges of I_8 are mapped to edges of $P_w(I_8)$, but not to edges of $P_s(I_8)$. In general, it is "weak" multifunctions that are appropriate for approximating real functions.

Let G and H be graphs. A multifunction f of G into H is a function that assigns to a vertex x of G a non-empty subset $f(x)$ in $V(H)$. Corresponding to the weak and strong relations, we have the following notions of multifunctions:

Definition 2.3 Let G and H be graphs. The multifunction f from G to H, $f : G \to H$, is *weak* (resp. *strong*) if $\hat{f} : G \to P_w(H)$ (resp. $\hat{f} : G \to P_s(H)$) is a graph homomorphism, where $\hat{f}(x) = f(x) \in P^+_{fin}(V(H))$.

Notice that, in common with most of the literature, we write "the multifunction $f : G \to H$" for what is actually a function from $V(G)$ to the power set of $V(H)$. The expression $\hat{f}$ is simply another way of writing f which makes this explicit.

The strong relation (and strong multifunctions) have been considered a number of times in the literature. Of some indirect relevance to our work here are [14,21], where fixed points of strong multifunctions (called "isotone relations" by these authors) on a poset are studied. Grätzer and Whitney [6] generalize the "strong" power construction to n-ary relations. A survey of these developments may be found in [3].

The weak relation and the corresponding multifunctions were studied in [17] and [22], as means for approximating continuous functions. They do not, to our knowledge, appear elsewhere in the literature.

The following propositions show some properties of weak and strong multifunctions.

Proposition 2.4 *Let $f : G \to H, g : H \to L$ be weak (resp. strong) multifunctions. Then the composition $g \circ f : G \to L, x \mapsto g(f(x)) = \{z \in V(L) \mid z \in g(y)$ for some $y \in f(x)\}$, is a weak (resp. strong) multifunction.*

Proof. Suppose $x, y \in V(G)$ such that $(x, y) \in E(G)$:

(1) f and g are weak: Since f is weak, there exist $x' \in f(x)$ and $y' \in f(y)$ such that $(x', y') \in E(H)$. Since g is weak, there exist $x'' \in g(x') \subseteq (g \circ f)(x)$ and $y'' \in g(y') \subseteq (g \circ f)(y)$ such that $(x'', y'') \in E(L)$.

(2) f and g are strong: For any $x'' \in (g \circ f)(x)$, there exists an $x' \in f(x)$ such that $x'' \in g(x')$. Since f is strong, there exists a $y' \in f(y)$ such that $(x', y') \in E(H)$. Hence there exists a $y'' \in g(y') \subseteq (g \circ f)(y)$ such that $(x'', y'') \in E(L)$ since g is strong. Similarly, for any $y'' \in (g \circ f)(y)$, there must exist an $x'' \in (g \circ f)(x)$ such that $(x'', y'') \in E(L)$. □

Proposition 2.5 *If $f : G \to H, g : H \to L$ and $h : L \to T$ are weak (resp. strong) multifunctions, then $(h \circ g) \circ f = h \circ (g \circ f)$.* □

Proposition 2.6 *Every strong multifunction is weak.* □

3 Power complexes

An *abstract simplicial complex* K (briefly a complex) is defined to consist of a collection $V(K)$ of points together with a prescribed collection $\Lambda(K) \subseteq P^+_{fin}(V(K))$ of finite non-empty subsets of $V(K)$ (the *simplexes* of K), satisfying: (1) $x \in V(K) \Rightarrow \{x\} \in \Lambda(K)$; and (2) $\forall \alpha \in \Lambda(K), \beta \subseteq \alpha(K), \beta \neq \emptyset \Rightarrow \beta \in \Lambda(K)$. The concept of simplicial complexes is discussed in most topology texts. A suitable one for our purpose is [5]. A *simplicial mapping* $\phi : K_1 \to K_2$ from the complex K_1 into the complex K_2 is a function which maps points of K_1 to the points of K_2 preserving the simplex structures.

Let K be a complex. As a direct generalization of the preceding Section, we have:

Definition 3.1 The *weak power* complex $P_w(K)$ is $(P^+_{fin}(V(K)), \Lambda_w)$, where

$$\rho = \langle \rho_i \rangle \in \Lambda_w \Leftrightarrow \exists \sigma \in \Lambda(K) \text{ s.t. } \rho_i \cap \sigma \neq \emptyset, \text{for all } i.$$

The *strong power* complex $P_s(K)$ also has the non-empty finite subsets of $V(K)$ as vertices, and the collection Λ_s of simplexes, where

$$\rho = \langle \rho_i \rangle_{i \in I} \in \Lambda_s \Leftrightarrow \forall k \in I, \forall v \in \rho_k, \exists \langle v_i \rangle_{i \in I} \text{ s.t. } v_k = v \;\&\; \langle v_i \rangle_{i \in I} \in \Lambda(K).$$

The power complex constructions may be simply characterized as follows:

Theorem 3.2 *Let Λ^+ be a collection of non-empty finite subsets of $P^+_{fin}(V(K))$. Consider the properties:*

(1) $\langle v_i \rangle \in \Lambda(K) \Rightarrow \langle \{v_i\} \rangle \in \Lambda^+$,

$(2)_w$ $\forall A \in \sigma^+ \in \Lambda^+ \ \&\ A \subseteq B \Rightarrow \rho \in \Lambda^+$, *where* ρ *is obtained by replacing* A *in* σ^+ *by* B,

$(2)_s$ $\langle A_i \rangle_{i=1,\ldots,k}, \langle B_i \rangle_{i=1,\ldots,k} \in \Lambda^+ \Rightarrow \langle A_i \cup B_i \rangle_{i=1,\ldots,k} \in \Lambda^+$.

Then Λ_w *is the least collection* Λ^+ *satisfying* (1) *and* $(2)_w$*; and* Λ_s *is the least collection* Λ^+ *satisfying* (1) *and* $(2)_s$. □

As in the case of graphs, the constructs P_w and P_s may be considered as (monadic) functors in the category **Compx** of complexes and simplicial mappings; again, we do not develop this aspect here.

An important example of the complexes is the *clique complex* $\mathcal{C}(G)$ of a graph G (the vertices of the complex are those of G, and the simplexes are the cliques) [9]. It is important to realize, however, that even when complexes occur, in the first instance, as clique complexes (as in the case below-Theorem 5.3), the application of P_w (or P_s) takes us to a complex which is no longer a clique complex. As is easily checked, a simplex of $P_w(\mathcal{C}(G))$ is not the same as a clique of $P_w(G)$ (the cliques, in general, are a proper subset of the simplexes in the power structure).

It is well-known that the concept of complexes provides an important foundation of algebraic topology so that homology theory can be developed. However, though various power constructions have been extensively studied in the past three decades [3,19], none of this work has any direct bearing on power complexes as defined here.

The following definition generalizes the multifunctions for graphs:

Definition 3.3 Let K_1 and K_2 be complexes, and $s : K_1 \to K_2$ a multifunction from K_1 to K_2. Then s is *weak simplicial* if $\hat{s} : K_1 \to P_w(K_2)$ is simplicial, and s is *strong simplicial* if $\hat{s} : K_1 \to P_s(K_2)$ is simplicial, where $\hat{s}(x) = s(x) \in P^+_{fin}(V(K_2))$.

Intuitively, the simplicial multifunctions are required to preserve the simplicial structures, that is, the image of any simplex of the source complex is a "power simplex" of the target complex. This is indicated in the following proposition which can be easily proved by Definition 3.1 and 3.3.

Proposition 3.4 *Let* K_1 *and* K_2 *be complexes, and* $\langle x_i \rangle$ *any simplex of* K_1. *If* $s : K_1 \to K_2$ *is a simplicial multifunction from* K_1 *to* K_2*, then we have:*

(1) *s is weak* $\Leftrightarrow \forall s(x_i), \exists y_i \in s(x_i)$ *s.t.* $\langle y_i \rangle \in \Lambda(K_2)$.

(2) *s is strong* $\Leftrightarrow \forall x \in \bigcup_i s(x_i), \exists y_i \in s(x_i)$ *s.t.* $x \in \langle y_i \rangle \in \Lambda(K_2)$. □

The following propositions clearly are generalizations of Theorem 2.4, 2.5 and 2.6.

Proposition 3.5 *Let* $s_1 : K_1 \to K_2, s_2 : K_2 \to K_3$ *be weak* (*resp. strong*) *simplicial multifunctions. Then the composition* $s_2 \circ s_1 : K_1 \to K_3, x \mapsto$

$s_2(s_1(x)) = \{z \in V(K_3) \mid z \in s_2(y)$ *for some* $y \in s_1(x)\}$ *is a weak (resp. strong) simplicial multifunction.* $\square$

Proposition 3.6 *If* $s_1 : K_1 \to K_2, s_2 : K_2 \to K_3$ *and* $s_3 : K_3 \to K_4$ *are weak (resp. strong) simplicial multifunctions, then* $(s_3 \circ s_2) \circ s_1 = s_3 \circ (s_2 \circ s_1)$. $\square$

Proposition 3.7 *Every strong simplicial multifunction is weak simplicial.* $\square$

4 Weak convexity

It is well-known that the Euclidean space $\mathbb{R}^n$ can be digitized using a special regular tessellation. The most common tessellation is the (hyper-) cubic tessellation in which (hyper-) cubes are called cells (or grid points). Obviously, two cells are adjacent (or neighbours to each other) if and only if they overlap on some common points in $\mathbb{R}^n$. Given T a (hyper-) cubic tessellation of $\mathbb{R}^n$, then a subset D of T is called an n-dimensional *real digital picture* with *mesh* ε if $\bigcup D$ is a rectangular portion of $\mathbb{R}^n$ and each cell has ε as its side length. Suppose that for each i, m_i is the number of cells along the ith edge (so that the length of edge i is $m_i\varepsilon$). Then we shall denote an n-dimensional real digital picture with these parameters by $\prod_{1\le i\le n}[0,\varepsilon,m_i]$ for $n \in \mathbb{N}$ and $\varepsilon \in \mathbb{R}$, and $[0,\varepsilon,m]^n$ if $m_i = m$ for all i. Especially, in the case that $\varepsilon = 1$, we also denote $\prod_{1\le i\le n}[0,\varepsilon,m_i]$ by $\prod_{1\le i\le n}[0,m_i]$ (or $[0,m]^n$ if $m_i = m$ for all i).

For a characterization of real digital pictures with fixed meshes, it is easy to see that two real digital pictures $\mathcal{G}_1 = \prod_{1\le i\le n_1}[0,\varepsilon_1,x_i]$ and $\mathcal{G}_2 = \prod_{1\le j\le n_2}[0,\varepsilon_2,y_j]$ are isomorphic (with respect to their neighbourhood structures) if and only if (1) $n_1 = n_2$, and (2) for any i, there exists an unique j s.t. $x_i = y_j$. Hence if measurement (mesh of cells) does not play an important role in the context, in the remaining sections except Theorem 4.2 we are able to restrict our attention to real digital pictures with unit mesh $\prod_{1\le i\le n}[0,m_i]$.

Let S be a set and $D = \{c_1,\ldots,c_k\}$ a non-empty collection of distinct non-empty subsets of S whose union is S, we call G the *intersection graph* of D if $V(G) = D$ with c_i and c_j adjacent whenever $i \neq j$ and $c_i \cap c_j \neq \emptyset$ [7]. By $\bigotimes_{1\le i\le n} I_{m_i}$ (I_m^n) we denote the intersection graph induced from $\prod_{1\le i\le n}[0,m_i]$ ($[0,m]^n$); we may call $\bigotimes_{1\le i\le n} I_{m_i}$ an n-dimensional *digital picture* as $\bigotimes_{1\le i\le n} I_{m_i}$ reflects the adjacent (neighbourhood) relationships of cells in $\prod_{1\le i\le n}[0,m_i]$.

The aspect of real digital pictures with unit mesh has been considered a number of times, though in different terminologies, in the literature. In [8], any subset of $\prod_{1\le i\le n}[0,m_i]$ (where n is possibly ∞) is called a molecular space, and it was claimed that every graph can be represented as an intersection graph of molecular spaces. Similar aspects can also be found easily in the study of digital geometry: for example, Kim called any subset of $\prod_{1\le i\le n}[0,m_i]$ a rectangular cellular mosaic [10], and in Chaudhuri and Rosenfeld [4], $\prod_{1\le i\le n}[0,m_i]$ is called the region representation of $\bigotimes_{1\le i\le n} I_{m_i}$ and $\bigotimes_{1\le i\le n} I_{m_i}$ is called the lattice point representation of $\prod_{1\le i\le n}[0,m_i]$. In this paper we use the term

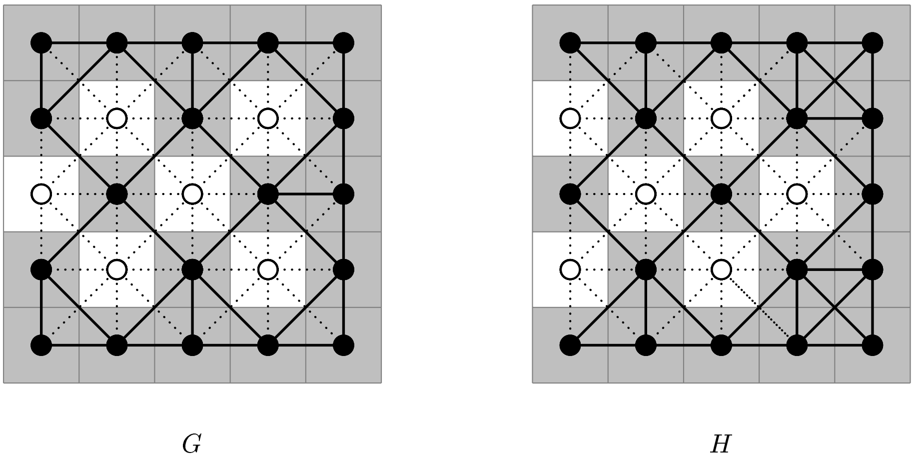


Fig. 3. G and H are weak convex sets of I_5^2 ($[0,5]^2$)

"real digital picture" because it appears that $\bigotimes_{1\leq i\leq n} I_{m_i}$ is the structure most commonly called a "digital picture" in the literature of digital geometry.

Since there is an one-to-one correspondence between cells of $\prod_{1\leq i\leq n}[0,m_i]$ and vertices of $\bigotimes_{1\leq i\leq n} I_{m_i}$, we use the inverse notation "–1" to express their corresponding relations. For example, c^{-1} is the induced subset of $\prod_{1\leq i\leq n}[0,m_i]$ for a given subset c of $\bigotimes_{1\leq i\leq n} I_{m_i}$.

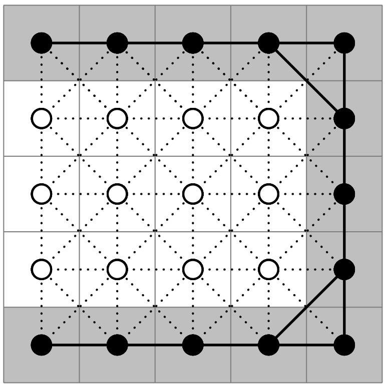
Definition 4.1 Given any subset c of $\prod_{1\leq i\leq n}[0,m_i]$. Then c is said to be a *weak convex* subset of $\prod_{1\leq i\leq n}[0,m_i]$ if for any $x \in \mathrm{conv}(\bigcup c)$, there exists cells $\alpha, \beta \in \prod_{1\leq i\leq n}[0,m_i]$ and $x \in \alpha$ & $\beta \in c$ such that α, β are neighbours, where conv is the usual Euclidean convex closure operator. Furthermore, a given subset c of $\bigotimes_{1\leq i\leq n} I_{m_i}$ is *weak convex* if and only if c^{-1} is weak convex in $\prod_{1\leq i\leq n}[0,m_i]$.

Note that this weak convexity is so "weak" that it does not satisfy the intersection property, even if we assume that the intersection of weak convex subsets is connected. As for an example, the subsets G and H shown in Fig. 3 are weak convex subsets of I_5^2 ($[0,5]^2$). However, it is easy to check that $G \cap H$ shown in Fig. 4 is not weak convex.

We should emphasize that weak convexity is not being proposed as being a "reasonable" notion of convexity in itself. Rather, its purpose is to be as inclusive as possible (with respect to notions of digital or graph convexity), while supporting the almost fixed point property for multifunctions (Section 5). It also serves to characterize the closed bounded Euclidean convex sets, as the following simple result (Theorem 4.2) shows.

Let A be an n-dimensional closed bounded subset of $\mathbb{R}^n$. Without loss of generality, we may assume that A is contained in the space I^n. Define, for all $k \in \mathbb{N}$,

$$A(k) = \{c \in [0, 1/k, k]^n \mid c \cap A \neq \emptyset\}$$



$G \cap H$

Fig. 4. $G \cap H$ is not a weak convex set of I_5^2 $([0,5]^2)$

giving a sequence of subsets of $[0, 1/k, k]^n$. We have the following theorem:

Theorem 4.2 *A is Euclidean convex if and only if for any $k \in \mathbb{N}$, $A(k)$ is a weak convex set in $[0, 1/k, k]^n$.*

Proof. The "only if" is trivial. To show the "if", suppose A is not convex, then since A is closed, there exists an open ball $B \in \mathrm{conv}(A) \setminus A$ with radius δ. Let $l = \lceil 1/\delta \rceil$, then it is clear that $A(k)$ is not a weak convex set in $[0, 1/k, k]^n$ for any $k \geq 3l$. □

5 Almost fixed point property

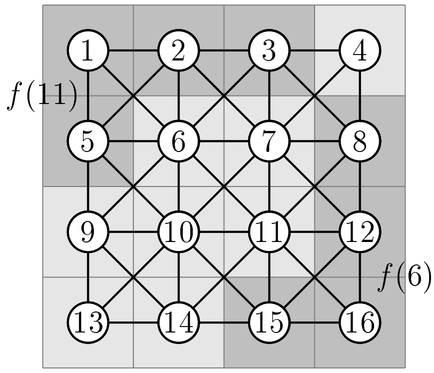
Let K be a graph or complex and $f : K \to K$ a self-mapping multifunction. Then we say that $x \in V(K)$ is an *almost fixed point* of f if $\langle \{x\}, f(x) \rangle \in \Lambda(P_w(K))$ (informally, if x is adjacent to the set $f(x)$). Let ϕ be a property of sets of vertices (points) of K. The structure K is said to possess the *ϕ-almost fixed point property* (ϕ-afpp) if for any multifunction $f : K \to K$ which sends each point of $V(K)$ into a subset of K satisfying ϕ (f is said to be a *ϕ-multifunction* in this case), f has an almost fixed point.

Proposition 5.1 *Let K be a complex. Then K has ϕ-afpp for simplicial strong multifunctions if K has ϕ-afpp for simplicial weak multifunctions.* □

Proposition 5.2 *Let K be a complex and let ϕ, ψ be properties of $P^+_{fin}(V(K))$ satisfying $\phi \Rightarrow \psi$. Then K has ϕ-afpp if K has ψ-afpp.* □

It is easy to see that the graph $\bigotimes_{1 \leq i \leq n} I_{m_i}$ need not have the weak wc-afpp, where wc stands for weak convexity. As for an example, the weak wc-multifunction f defined in Fig. 5 shows that I_4^2 has no weak wc-afpp.

To achieve a satisfactory wc-afpp, we need to consider, not exactly the graph $\bigotimes_{1 \leq i \leq n} I_{m_i}$, but its clique complex:



x	$f(x)$	x	$f(x)$
1	$\{16\}$	13	$\{4\}$
2	$\{15\}$	14	$\{3\}$
3	$\{14\}$	15	$\{2\}$
4	$\{13\}$	16	$\{1\}$
5	$\{12\}$	6	$\{8, 12, 15, 16\}$
8	$\{9\}$	7	$\{9, 13, 14, 15\}$
9	$\{8\}$	10	$\{3, 4, 8, 12\}$
12	$\{5\}$	11	$\{1, 2, 3, 5\}$

Fig. 5. f has no almost fixed point in I_4^2

Theorem 5.3 *$\mathcal{C}(\bigotimes_{1\le i\le n} I_{m_i})$ has the simplicial wc-afpp.*

Proof. Without loss of generality, we can consider $\mathcal{C}(I_m^n)$ only. For any self-mapping simplicial wc-multifunction $f : \mathcal{C}(I_m^n) \to \mathcal{C}(I_m^n)$, we claim that f has an almost fixed point. First note that any infinite path $\mathcal{C}(I_\infty)$ does not have the simplicial wc-afpp, therefore we assume $n_i < \infty$ for all i. The proof given below is divided into three steps and several lemmas:

Step 1. Making a well-defined partition of $\bigcup[0, m]^n$

Let A be a subset of $\mathbb{R}^n$ and $\varepsilon \in \mathbb{R}, \varepsilon > 0$, we denote $\mathcal{B}_\varepsilon(A) = \{x \in \mathbb{R}^n \mid \exists a \in A \text{ s.t. } d_i(x, a) < \varepsilon\}$. We also denote by $\mathcal{S}^j([0, m]^n)$ the collection of all j-dimensional faces of cells of $[0, m]^n$.

We construct a certain covering Ω of $\bigcup[0, m]^n \subseteq \mathbb{R}^n$ that has order $n + 1$: Let $\varepsilon \in \mathbb{R}, 0 < \varepsilon < 0.5$, and define $\Omega = \Omega_0 \uplus \Omega_1 \uplus \cdots \uplus \Omega_n$ to be the collection of n-dimensional rectangular subsets of $\bigcup[0, m]^n$ satisfying

$$\Omega_j = \{(\mathcal{B}_\varepsilon(x) \setminus \bigcup_{-1\le k\le j-1} \Omega_k) \cap (\bigcup[0, m]^n) \mid x \in \mathcal{S}^j([0, m]^n)\}.$$

where $\Omega_{-1} = \emptyset$ and $0 \le j \le n$. Notice that this is an inductive definition w.r.t. j. Informally, Ω_j consists of n-dimensional "approximations" of the j-dimensional faces of cells of $[0, m]^n$, see Fig. 6.

It is clear that $\bigcup[0, m]^n = \bigcup_{0\le k\le n} \Omega_k$, and for any $x \in \Omega_i, y \in \Omega_j, 0 \le i, j \le n, i \ne j$, we have $x \cap y = \emptyset$. Hence Ω is a well-defined partition of $\bigcup[0, m]^n$. The Fig. 6 shows the three different partition elements Ω_0, Ω_1 and Ω_2 of Ω whenever $n = 2$.

Step 2. New (continuous) mappings derived from old (discrete) mappings

Having the simplicial wc-multifunction $f : \mathcal{C}(I_m^n) \to \mathcal{C}(I_m^n)$, we define $F : \bigcup[0, m]^n \to \bigcup[0, m]^n$ as follows. Let $x \in \omega \in \Omega_i$. Then

$$x \mapsto \bigcap\{\mathrm{conv}(\bigcup(f(c^{-1}))^{-1}) \mid c \in [0, m]^n \text{ and } c \cap \omega \ne \emptyset\}$$

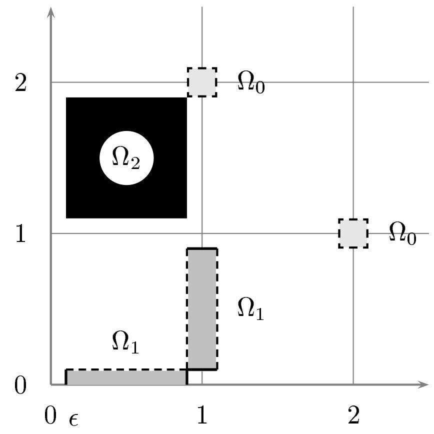


Fig. 6. The three partition elements Ω_0, Ω_1 and Ω_2 whenever $n = 2$

For example, with x as any point in the Ω_0-element which contains $(1, 2)$ in Fig. 6, we would take the convex closures of the images of the four cells (squares) adjacent to $(1, 2)$, and map x to the intersection of these.

Lemma 5.4 *F is well-defined.*

Proof. Clearly, $F(x)$ is unique for any $x \in \bigcup[0, m]^n$ since Ω is a well-defined partition of $\bigcup[0, m]^n$. Furthermore, in order to show that $F(x) \neq \emptyset$ for any $x \in \bigcup[0, m]^n$, it is sufficient to show that $F(y) \neq \emptyset$ for any $y \in \Omega_0$. However, since f is simplicial, therefore $F(y)$ must not be empty. □

Step 3. New (discrete) almost fixed point theorem derived from old (continuous) fixed point theorem

Recall that a multifunction $F : X \to Y$ from the topological space X to the topological space Y is said to be *upper semi-continuous* (usc) if and only if for any open set O of Y, $F^+(O) = \{x \in X \mid F(x) \subseteq O\}$ is an open set of X. For convenience we shall require also, in the remainder of this paper, that $F(x)$ is non-empty and compact for each $x \in X$ [1].

In 1941, Kakutani generalized Brouwer's fixed point theorem for closed n-balls (or compact convex non-empty subsets of $\mathbb{R}^n$) to multifunctions by showing that any closed n-ball (or compact convex non-empty subset of $\mathbb{R}^n$) has the *fixed point property* for usc closed (compact) convex multifunctions.

Returning to the multifunction F, we note the following features:

(1) F is constant over any Ω-element ω since, for any $x \in \omega$, the value $F(x)$ depends only on those (closed) cells C_k which meet ω.

(2) If $\omega' \in \Omega_i$ and $\omega \in \Omega_j$ are Ω-elements satisfying $i \leq j$ and ω', ω are neighbours, then $F(\omega') \subseteq F(\omega)$, since every (closed) cell C_k which meets ω also meets ω'.

(3) For any $x \in \bigcup[0, m]^n$, we can find an open neighbourhood V of x such

that $F(V) = F(x)$. Namely, we take for V the union of the Ω-element $\omega \in \Omega_j$ which contains x with all $\omega' \in \Omega_i$ satisfying $i \leq j$ and ω', ω are neighbours.

Lemma 5.5 *F is compact convex usc.*

Proof. It is clear that $F(x)$ is compact convex for any $x \in \bigcup[0,m]^n$. Moreover, by the feature (3) above, $F^+(A)$ is open for arbitrary $A \subseteq \bigcup[0,m]^n$ (not only for open A). $\square$

Lemma 5.6 *F has a fixed point $x^* \in \bigcup[0,m]^n$, i.e. $x^* \in F(x^*)$.*

Proof. By Lemma 5.4, 5.5 and the Kakutani fixed point theorem. $\square$

Lemma 5.7 *If F has a fixed point $x^* \in \bigcup[0,m]^n$ such that $x^* \in c$ for some cell $c \in [0,m]^n$, then c^{-1} is either a fixed point or an almost fixed point of f.*

Proof. Denote by $\mathcal{D}_c$ the subset $\bigcup(f(c^{-1}))^{-1} \subseteq \bigcup[0,m]^n$ induced by cells in the image of c. (Expressed differently, $\mathcal{D}_c$ is the subset of $\mathbb{R}^n$ covered by $f(c)$.) Since Ω is a partition of $\bigcup[0,m]^n$, therefore there exists an unique $\omega \in \Omega_i, 0 \leq i \leq n$ satisfying $x^* \in \omega$. Consider the following cases:

(1) $i = m$: If $x^* \in \mathcal{D}_c$, then c^{-1} is a fixed point of f. Otherwise (that is, $x^* \in \mathrm{conv}(\mathcal{D}_c) \setminus \mathcal{D}_c$), c^{-1} is an almost fixed point of f since $f(c^{-1})$ is a weak convex set in I^n_m.

(2) $i \neq m$: There exists 2^{n-i} mutually adjacent cells $D = \{c, c_2, \ldots, c_{2^{n-i}}\}$ satisfying $x^* \in \omega \subseteq \bigcup_{c_j \in D} c_j$ and $F(x^*) = \bigcap_{c_j \in D} \mathrm{conv}(\mathcal{D}_{c_j})$. Therefore we have $x^* \in \mathrm{conv}(\mathcal{D}_{c_j})$ for any $c_j \in D$, and hence by the similar argument of (1), it is easy to check that c^{-1} is either a fixed point or an almost fixed point of f. $\square$

Therefore by Lemma 5.6 and 5.7, f has an almost fixed point, and this implies that $\mathcal{C}(I^n_m)$ has the simplicial wc-afpp. So we complete the proof of Theorem 5.3. $\square$

6 Conclusion and future work

The result proved in the preceding section has a certain resemblance with Kakutani's theorem. We still need to consider whether it is a digital "version" of that theorem, according to the criterion suggested in the Introduction. Can the Kakutani theorem be derived from Theorem 5.3 by a limiting argument?

Suppose a non-empty convex-valued multifunction $h : I^n \to I^n$ is given. We seek to approximate h by a sequence of digital functions $h_0, h_1, \ldots, h_i, \ldots$, where for each i, h_i is a multifunction on $I^n(2^i)$, the unit cube subdivided into cells of length 2^{-i}.

For simplicity, in the following we view the cells directly as the vertices. That is, in applying Theorem 5.3, we work with $I^n(2^i)$ as the complex in question, instead of its isomorphic copy $\mathcal{C}(I^n_{2^i})$. For Theorem 5.3 to be applicable, we need the values assumed by h_i to be weak convex, and so we define:

$$h_i(e) = \{v \in I^n(2^i) \mid v \cap \mathrm{conv}(h(e)) \neq \emptyset\}. \tag{1}$$

(The notation $h(e)$ means, of course, $\bigcup_{x \in e} h(x)$.) The multifunction $h_i : I^n(2^i) \to I^n(2^i)$ is simplicial, because a collection of cells of $I^n(2^i)$ is a simplex if and only if the cells have (at least) a point in common, and h takes non-empty values. Moreover, if $e_0 \supseteq e_1 \supseteq \cdots \supseteq e_i \supseteq \cdots$ $(e_i \in I^n(2^i))$ is a descending sequence of cells with intersection $\{x\}$ $(x \in I^n)$, then

$$h(x) \subseteq \bigcap_i (h_i(e_i)).$$

In case we have equality here (for every such descending sequence of cells), we shall say that (h_i) is an approximating sequence for h.

Lemma 6.1 *If the multifunction h is usc, then the sequence (h_i) defined by (1) is an approximating sequence for h.*

Proof. Suppose, for a contradiction, that we have a descending sequence $e_0 \supseteq e_1 \supseteq \cdots \supseteq e_i \supseteq \cdots$ with intersection $\{x\}$, and a point y such that for all i, $y \in h_i(e_i)$, but $y \notin h(x)$. Since $h(x)$ is closed, y is at some positive distance, say ε, from $h(x)$. Let V be a convex $\varepsilon/2$-neighbourhood of $h(x)$. By upper semi-continuity we have a neighbourhood U of x such that, for every $u \in U, h(u) \subseteq V$. Since V is convex, this implies that, for every $e_i \subseteq U$, $\mathrm{conv}(h(e_i)) \subseteq V$. Then for $e_i \subseteq U$ of mesh $< \varepsilon/2$, we shall have $y \notin (h_i(e_i))$.□

Now, given a function h satisfying the conditions of Kakutani's Theorem, we construct the approximating sequence (h_i) of digital functions. By Theorem 5.3, each h_i has one or more almost fixed points (cells). By a standard (though non-constructive) compactness argument we recover, from these collections of almost fixed cells, a fixed point of h. Then we have:

Proposition 6.2 *Theorem 5.3 implies Kakutani's Theorem.* □

Of course, the preceding considerations do not give us a new proof of Kakutani's Theorem, since we used that Theorem in deriving Theorem 5.3. The point is just to show that the criterion for a "digital version" of a classical theorem is satisfied.

A combinatorial proof of Theorem 5.3 is desirable. This is a topic of continuing investigations. The second author has combinatorial proofs of certain restricted versions of the theorem (that is, with restricted notions of convexity). See [20].

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