Available online at www.sciencedirect.com
ScienceDirect
Electronic Notes in Theoretical Computer Science
Electronic Notes in Theoretical Computer Science 270 (2) (2011) 163–189
www.elsevier.com/locate/entcs

Categorical Properties of The Complex Numbers

Jamie Vicary [1,2]

Computing Laboratory, University of Oxford, United Kingdom

Abstract

Given the success of categorical approaches to quantum theory, it is interesting to consider why the complex numbers are special from a categorical perspective. We describe natural categorical conditions under which the scalars of a monoidal †-category gain many of the features of the complex numbers. Central to our approach are †-*limits*, certain types of limits which are compatible with the †-functor; we explore their properties and prove an existence theorem for them. Our main theorem is that in a nontrivial monoidal †-category with finite †-limits and simple tensor unit, and in which the self-adjoint scalars satisfy a completeness condition, the scalars are valued in the complex numbers, and scalar involution is exactly complex conjugation.

Keywords: Quantum theory, category theory, complex numbers

1 Introduction

The purpose of this paper is to describe a set of properties of a theory of physics, which together imply that the theory makes use of the complex numbers. These properties are phrased in terms of the way that physical processes interact with each other, and as a result are intuitive and physical. Our approach is also robust: we are not concerned with many details of the theory, such as the nature of dynamics, or the way that measurement is described.

To apply our method to a particular theory of physics, we first need to obtain from the theory a family of *systems*, equipped with a family of *processes* which go from one system to another. We will often denote processes as $f : A \longrightarrow B$, which indicates a process f going from system A to system B. It is sometimes useful to imagine that systems are *sets of states*, and processes are *functions* taking states of one system into states of another, but we will not rely on any such interpretation.

[1] Financial support from the EPSRC and the ONR is gratefully acknowledged.

[2] Email: jamie.vicary@comlab.ox.ac.uk

doi:10.1016/j.entcs.2011.01.030

For any two 'head-to-tail' processes $f : A \longrightarrow B$ and $g : B \longrightarrow C$ we require that there exists a composite process $f; g : A \longrightarrow C$, interpreted as the process f followed by the process g. We require that this composition is associative, and for any system A, we require the existence of a 'trivial' process $\mathrm{id}_A : A \longrightarrow A$ which is the identity for composition. These are exactly the axioms of a *category*, and we will make essential use of the tools of category theory to prove our results.

We call this category the *category of processes* associated to a particular physical theory. Of course, very few realistic theories of physics will naturally be presented in terms of a category of processes, but for many theories there will be natural candidates. If any of these have the properties we will describe, then that will indicate that the underlying theory somehow makes use of on the complex numbers. For the case of quantum mechanics, we might take systems to be separable Hilbert spaces and processes to be bounded linear maps; this theory certainly makes use of the complex numbers, and its category of processes satisfies the properties we will describe.

The first property that we require is that each process has an *adjoint*, which can be considered as a formal 'reversal'. We use the term 'adjoint' since this is a generalisation of a familiar operation from quantum theory, taking the adjoint of a bounded linear map between Hilbert spaces. For any $f : A \longrightarrow B$ its adjoint is a process $f^\dagger : B \longrightarrow A$; we require that $(f^\dagger)^\dagger = f$ for any process f, and also that $(f; g)^\dagger = g^\dagger; f^\dagger$ for any composable processes f and g. These properties define a *functor* from our category to itself, and we call this the *†-functor*. A second property that we require is *superposition*: for any two parallel processes $f, g : A \longrightarrow B$ there must exist a third process $f + g : A \longrightarrow B$, where $+$ is an associative, unital, commutative operation with the property that $(f + g); h = f; h + g; h$ for any $h : B \longrightarrow C$ and any system C. Finally, we require a notion of *compound system*: for any two systems A and B there must exist a compound system $A \otimes B$, where $\otimes$ is an associative, unital operation. Its unit is a system I, for which $I \otimes A = A = A \otimes I$ for all systems A. This compounding operation must also be defined on processes, so for all $f : A \longrightarrow B$ and $g : C \longrightarrow D$ there exists a process $f \otimes g : A \otimes C \longrightarrow B \otimes D$; this must interact well with composition, satisfying $(f \otimes g); (h \otimes j) = (f; h) \otimes (g; j)$ for all appropriate processes f, g, h and j.

We now consider 'diagrams' of our processes. A diagram is a finite set of systems and processes, closed under composition, such that for every process its initial and final systems are included, and for every system its identity process included. Here is a drawing of a simple diagram, where for clarity we leave out the identity processes:

$$\begin{tikzcd}
 & D & & \\
A \arrow[ur, bend left, "h"] \arrow[dr, bend left, "g"] & & C \arrow[ul, bend left, "k"] \arrow[ul, bend right, "l"'] & E \arrow[loop above, "m"] \\
 & B \arrow[uu, "j"'] \arrow[ul, bend left, "f"] & &
\end{tikzcd} \tag{1}$$

Suppose that these processes compose in the following way:

$$g;j = h \qquad f;h = j \qquad g;f = \mathrm{id}_A \qquad f;g = \mathrm{id}_B \qquad m;m = m \qquad (2)$$

Then our processes are closed under composition, and the diagram is well-defined.

For any diagram we can consider its *limit*, a measure of the extent to which its constituent processes are 'compatible' with each other. Intuitively we can think of systems as sets of states, and processes as functions which transform states of their domain into states of their codomain. We can then define a 'state of the diagram' Ψ to be a choice of state $\Psi_S \in S$ for each system S in the diagram, or equivalently a state of the 'direct sum' of all the systems in the diagram. From the perspective of quantum mechanics, this is similar to treating each system as a superselection sector. Then a state Ψ is a *limit state* if, for all processes $f : S \longrightarrow T$ in the diagram and all systems S and T, we have $f(\Psi_S) = \Psi_T$. It can be thought of as a state of the diagram as a whole which is 'unchanged' by the processes in the diagram. The more processes the diagram contains, and the more 'diverse' these processes are, the more difficult the limit condition will be to satisfy. It is quite possible for a nontrivial diagram to have no limit states at all.

The limit L of the diagram is the set of all of its limit states. There are obvious functions $l_S : L \longrightarrow S$ for each S, sending a limit state Ψ to the state $\Psi_S \in S$. These functions are called the *limit maps*. We require that this limit L is *itself* a system in our category of processes, and that each limit map $l_S : L \longrightarrow S$ exists as a process in the category. One potential problem is that a particular category of processes might not have available an interpretation of systems in terms of sets of states, and even if it did, we would not necessarily want to rely on it. In this case we turn to category theory, where limits are fundamental constructions with an abstract definition: the limit L is a system equipped with limit map processes $l_S : L \longrightarrow S$ for each system S in the diagram, satisfying $l_S; f = l_T$ for all processes $f : S \longrightarrow T$ in the diagram, and also satisfying a universal property which expresses our desire for every 'limit state' to be included in L.

With the definition of limit in hand, consider our example diagram above. Suppose that we are given the limit system L, but only the limit maps $l_A : L \longrightarrow A$, $l_C : L \longrightarrow C$ and $l_E : L \longrightarrow E$. From the properties of the limit maps the missing ones can be reconstructed; for example, $l_B = l_A; g$. This means intuitively that every limit state Ψ is completely defined by its values at the systems A, C and E, and so L is somehow a subsystem of the union of these. If a set of systems in a diagram has this property, we call it a *set of sources*. We can then interpret the limit maps l_A, l_C and l_E as telling us how L is 'partitioned' between the members of our chosen set of sources.

Our final requirement is that this partitioning is compatible with the †-functor on our category of processes, which allows us to formally 'reverse' processes. Consider the process $l_A; l_A^\dagger$: this evolves a state of L into a state of A, and then evolves this back again into a state of L. So, applying this to a particular limit state Ψ, we interpret $(l_A; l_A^\dagger)(\Psi)$ as that part of Ψ which arises from A. If each state in L is distributed in a well-behaved way between the systems A, C and E, then we might expect that

$$l_A; l_A^\dagger + l_C; l_C^\dagger + l_E; l_E^\dagger = \mathrm{id}_L. \qquad (3)$$

We call this the *normalization condition*. Intuitively, this says that if we take the superposition of three processes — finding for each limit state that part of it which arises from A, C and E respectively — we obtain the limit state itself. This makes sense, since we described above how each limit state is *determined* by its values at A, C and E.

We described a particular set of sources, but there are others: we could have chosen B, C, D and E, or all systems in the diagram together. If we can find limit maps satisfying the normalization condition for a particular set of sources, we call this a *†-limit*. If we can do this for every diagram, and for every set of sources, we say that our category has *all †-limits*.

Categories with all †-limits have many interesting properties, which we explore throughout this paper. One useful property is that a category can have at most a single superposition rule (the '+' operation) such that all †-limits exist! So the superposition rule is more like a *property* of the †-limits than a structure on the underlying category, and we do not need to specify it explicitly.

We can now state our main result. Suppose we have a category of processes which has a †-functor, compound systems and all †-limits, such that the 'trivial system' I — the unit for constructing compound systems — is 'simple', meaning that the only smaller system is the empty system. Then we can show that 'quantum amplitudes' in this category take values in an involutive field with characteristic 0, and with orderable fixed field. We interpret this field as analogous to $\mathbb{C}$, the involution as analogous to complex conjugation, and the orderable fixed field as analogous to $\mathbb{R}$.

Furthermore, suppose that the results of measurements in our theory are valued in this orderable fixed field. Then if every bounded sequence of measurement results has a least upper bound, and these least upper bounds are preserved when we add a constant to our measurement results, it follows that our involutive field is $\mathbb{C}$ itself, the orderable fixed field is $\mathbb{R}$, and that the order is the familiar order on the real numbers.

2 †-Functors, †-categories and †-limits

2.1 The †-functor

Of all the categorical structures that we will make use of, the most fundamental is the †-functor, first make explicit in the context of categorical quantum mechanics by Abramsky and Coecke [1,2]. As described in the introduction, it is motivated by the process of taking the *adjoint* of a linear map between two Hilbert spaces: for any bounded linear map of Hilbert spaces $f : H \longrightarrow J$, the adjoint $f^\dagger : J \longrightarrow H$ is the unique map satisfying

$$\langle f(\phi), \psi\rangle_J = \langle \phi, f^\dagger(\psi)\rangle_H \tag{4}$$

for all $\phi \in H$ and $\psi \in J$, where the angle brackets represent the inner products for each space.

Abstractly, we define a *†-functor* as a contravariant functor from a category to itself, which is the identity on objects, and which satisfies $\dagger \circ \dagger = \mathrm{id}_{\mathbf{C}}$. A *†-category* is a category equipped with a particular choice of †-functor. These are sometimes known instead as *Hermitian categories* or *∗-categories*, but we prefer the '†' notation, since it is snappier and more flexible than 'Hermitian', and the symbol '∗' is also used to denote duals for objects in a monoidal category. Although it is often uninformative to name something after the symbol that denotes it, this is outweighed by the convenience of having a straightforward naming convention [6] for '†-versions' of many familiar constructions, such as †-biproducts, †-equalizers, †-kernels, †-limits, †-subobjects and so on, which we will encounter below.

We write the action of a †-functor on a morphism $f : A \longrightarrow B$ as $f^\dagger : B \longrightarrow A$, and we refer to the morphism $f^\dagger$ as the *adjoint* of f. We also make the following straightforward definitions, taken from the vocabulary of functional analysis: a morphism is *unitary* if its adjoint is its inverse, an *isometry* if its adjoint is its retraction, and is *self-adjoint* if it equals its adjoint. If a morphism $f : A \longrightarrow B$ is an isometry, we also say that A is a *†-subobject* of B. If two objects in a †-category have a unitary morphism going between them, we say that they are *unitarily isomorphic*; if every pair of isomorphic objects are unitarily isomorphic, then the category is a *unitary †-category.* Many important †-categories are unitary; for example, the †-category of Hilbert spaces with †-functor given by adjoint, the †-category of manifolds and cobordisms with †-functor given by the opposite cobordism, or any 2-Hilbert space [3].

2.2 Constructing †-limits

We already defined †-limits informally in the introduction, but since they are the central new construction of this paper we describe them again here.

A *diagram with sources* is a diagram $F : \mathbf{J} \longrightarrow \mathbf{C}$ along with a subset $\Omega \subseteq \mathrm{Ob}(\mathbf{J})$ of source objects, such that each object in $\mathbf{J}$ has a morphism to it from some source object. If $\mathbf{C}$ is a †-category which is has a superposition rule '+' on the hom-sets — or technically, which is *enriched in commutative monoids* — then a *†-limit* for a diagram with sources is a limit for the underlying diagram, such that the normalization condition

$$\sum_{S \in \Omega} l_S ; l_S^\dagger = \mathrm{id}_L \tag{5}$$

holds, where each $l_S : L \longrightarrow F(S)$ is a cone map from the limit object L. It is straightforward to show that a †-limit for a diagram with sources is unique up to unique unitary isomorphism. In general, the cone maps for a †-limit depend significantly on the choice of source objects $\Omega \subseteq \mathrm{Ob}(\mathbf{J})$.

†-Products and †-equalizers

We will make substantial use of two particularly important types of †-limit. The first type is the *†-product*, which is the †-limit of a discrete diagram, for which every object is (necessarily) a source object. A useful result is that these †-products are exactly *†-biproducts*, which are well-known generalizations of the concept of 'orthogonal direct sum': for two objects A and B, their †-biproduct is an object

$A \oplus B$ equipped with *injection morphisms* $i_A : A \longrightarrow A \oplus B$ and $i_B : B \longrightarrow A \oplus B$ satisfying the following equations:

$$\begin{gathered} i_A^\dagger ; i_A + i_B^\dagger ; i_B = \mathrm{id}_{A \oplus B} \\ \begin{aligned} i_A ; i_A^\dagger &= \mathrm{id}_A & \qquad i_B ; i_B^\dagger &= \mathrm{id}_B \\ i_A ; i_B^\dagger &= 0_{A,B} & \qquad i_B ; i_A^\dagger &= 0_{B,A} \end{aligned} \end{gathered} \tag{6}$$

The adjoints to the injection morphisms are called the *projection morphisms*. This definition of †-biproduct generalizes in an obvious way to any finite list of objects.

Lemma 2.1 *The †-limit of a discrete diagram (that is, a †-product) is the †-biproduct of the objects of the diagram, and the cone maps are the †-biproduct projections.*

Proof. We prove our lemma for the case of a discrete diagram with two objects; the extension to any finite discrete diagram of objects is straightforward. Consider the †-limit of the diagram consisting of two objects, A and B. The †-limit is a limit object L, equipped with morphisms l_A and l_B which satisfy

$$l_A ; l_A^\dagger + l_B ; l_B^\dagger = \mathrm{id}_L. \tag{7}$$

Since L is the limit, there is a unique map $\langle 0_{B,A}, \mathrm{id}_B \rangle : B \longrightarrow L$ with $\langle 0_{B,A}, \mathrm{id}_B \rangle ; l_A = 0_{B,A}$ and $\langle 0_{B,A}, \mathrm{id}_B \rangle ; l_B = \mathrm{id}_B$, where $0_{B,A} : B \longrightarrow A$ is the unit for the enrichment in commutative monoids. Precomposing (7) with this map we obtain $l_B^\dagger = \langle 0_{B,A}, \mathrm{id}_B \rangle$, and so we have

$$l_B^\dagger ; l_A = 0_{B,A}, \qquad\qquad l_B^\dagger ; l_B = \mathrm{id}_B. \tag{8}$$

Similarly we can show that $l_A^\dagger = \langle \mathrm{id}_A, 0_{A,B} \rangle : A \longrightarrow L$, which leads to the equations

$$l_A^\dagger ; l_B = 0_{A,B}, \qquad\qquad l_A^\dagger ; l_A = \mathrm{id}_A. \tag{9}$$

Altogether, these equations witness the fact that L is the †-biproduct of A and B, with projections l_A, l_B and injections $l_A^\dagger$, $l_B^\dagger$. □

In a category with biproducts there is a ‘superposition rule’ — more technically referred to as an enrichment in commutative monoids — which can be defined in the following way:

$$\begin{tikzcd} A \arrow[r, "f+g"] \arrow[d, "\Delta_A"'] & B \\ A \oplus A \arrow[r, "f \oplus g"'] & B \oplus B \arrow[u, "\nabla_A"'] \end{tikzcd} \tag{10}$$

The diagonal Δ_A and the codiagonal ∇_A are adjoint to each other, as demonstrated by the following lemma.

Lemma 2.2 *For any †-biproduct $A \oplus A$, the diagonal $\Delta_A : A \longrightarrow A \oplus A$ and codiagonal $\nabla_A : A \oplus A \longrightarrow A$ satisfy $(\Delta_A)^\dagger = \nabla_A$.*

Proof. We see that $\mathrm{id}_A = (\mathrm{id}_A)^\dagger = (p_i \circ \Delta_A)^\dagger = (\Delta_A)^\dagger \circ p_i^\dagger$, where $i \in \{1,2\}$ and p_i is a projector onto one of the factors of the biproduct. But $\mathrm{id}_A = \nabla_A \circ p_i^\dagger$ for all i is the defining equation for the codiagonal, and so $(\Delta_A)^\dagger = \nabla_A$. □

From this lemma, and from the definition of $f+g$ given by equation (10), follows that the commutative monoid structure is compatible with the action of the †-functor, satisfying

$$(f+g)^\dagger = f^\dagger + g^\dagger \tag{11}$$

for all parallel morphisms f and g.

In a category with biproducts we have a matrix calculus available to us: a morphism $f : \bigoplus_i A_i \longrightarrow \bigoplus_i B_i$ corresponds to a matrix of morphisms $f_{i,j} : A_i \longrightarrow B_j$, and composition of morphisms is given by matrix multiplication. In any †-category with †-biproducts, it can be shown that the adjoint of a matrix has the following form:

$$\begin{pmatrix} f & g & \cdots & x \\ h & j & & \\ \vdots & & \ddots & \\ y & & & z \end{pmatrix}^\dagger = \begin{pmatrix} f^\dagger & h^\dagger & \cdots & y^\dagger \\ g^\dagger & j^\dagger & & \\ \vdots & & \ddots & \\ x^\dagger & & & z^\dagger \end{pmatrix} \tag{12}$$

This is just the familiar matrix conjugate-transpose operation, with the 'conjugate' of each entry in the matrix being its adjoint.

Our second important type of †-limit is of a diagram consisting of two parallel arrows $f, g : A \longrightarrow B$, for which we choose A to be the only source object. We call the †-limit $l_A : L \longrightarrow A$ of such a diagram a *†-equalizer*; the normalization condition states that $l_A; l_A^\dagger = \mathrm{id}_L$. We note that a †-equalizer is precisely an equalizer in the usual sense, which happens to also be an isometry. As a natural extension of this terminology, we also define a *†-kernel* to be a kernel which is also an isometry. This extra isometry condition is a natural one to consider in a †-category, since equalizers are always monic, and the isometry condition can be considered strengthening of this. This research programme was born out of a study of the properties of †-categories with †-equalizers, and I am grateful to Peter Selinger for suggesting them as a construction.

The category **Hilb** has all †-limits, and so in particular has both †-biproducts and †-equalizers: the †-biproduct of a finite list of Hilbert spaces is given by their direct sum, and for some parallel pair of linear maps $f, g : A \longrightarrow B$, their †-equalizer is given by an isometry that is surjective on the largest subspace of A on which the two linear maps agree.

Uniqueness of the superposition rule

Because of the normalization condition (5), it seems that the definition of †-limits depends on the choice of the superposition rule '+', which we refer to as the *enrichment in commutative monoids*. This is true, but can be easily overcome, thanks to the following fact: if by some enrichment in commutative monoids a †-category at least has †-limits of discrete diagrams, then the category in fact admits a *unique* enrichment in commutative monoids. So in particular, if a †-category admits an enrichment in commutative monoids such that it has all finite †-limits, then that enrichment is determined uniquely. This follows from lemma 2.1 along with the following well-known result, for which we omit the proof.

Lemma 2.3 *Suppose that a category has a zero object and all finite biproducts. Then it has a unique enrichment in commutative monoids.*

2.3 Properties of †-categories with †-limits

The existence of †-limits in a †-category guarantees some interesting properties, which we now investigate.

Nondegeneracy

The first property we will examine is *nondegeneracy*, also called *positivity* by some authors [4, Definition 8.9]. In a †-category with a zero object, the †-functor is nondegenerate if $f; f^\dagger = 0$ implies $f = 0$ for all morphisms f. We show now that this property is closely linked to the existence of †-equalizers.

Lemma 2.4 (Nondegeneracy) *In a †-category with a zero object and finite †-equalizers, the †-functor is nondegenerate.*

Proof. Let $f : A \longrightarrow B$ be an arbitrary morphism satisfying $f; f^\dagger = 0_{A,A}$. Then f must factor through the †-kernel of $f^\dagger$ as indicated by the following commuting diagram, where the factorizing morphism is denoted $\tilde{f}$, and (K, k) forms the †-kernel of $f^\dagger$:

$$\begin{tikzcd} & I \arrow[dl, bend right, "\tilde{f}"'] \arrow[dr, bend left, "f"] & & \\ K \arrow[rr, hook, "k"] & & B \arrow[r, shift left, "f^\dagger"] \arrow[r, shift right, "0_{B,A}"'] & A \end{tikzcd} \tag{13}$$

By definition we have $k; f^\dagger = 0_{K,A}$, and we apply the †-functor to obtain $f; k^\dagger = 0_{A,K}$. Also, since (K, k) is a †-kernel, k is an isometry, which means $k; k^\dagger = \mathrm{id}_K$. We can now demonstrate that f is zero:

$$f = \tilde{f}; k = \tilde{f}; k; k^\dagger; k = f; k^\dagger; k = 0_{A,K}; k = 0_{A,B}.$$

□

An important feature of this proof, which will recur in other proofs throughout this paper, is that although the †-functor is used sparingly, it is used crucially: in this case, to translate $k; f^\dagger = 0_{K,A}$ into $f; k^\dagger = 0_{A,K}$.

Cancellability

We now study various cancellability properties satisfied by the additive structure on the elements of the hom-sets. Say that a commutative monoid is *cancellable* if, for any three elements a, b, c in the monoid, $a + c = b + c \Rightarrow a = b$. We are motivated to study this condition since, in particular, it is satisfied by the addition of linear maps between Hilbert spaces. We now show that it follows as a consequence of having finite †-limits.

Lemma 2.5 (Cancellable addition) *In a †-category with all finite †-limits, hom-set addition is cancellable; that is, for arbitrary f, g, h in the same hom-set,*

$$f + h = g + h \Rightarrow f = g. \tag{14}$$

Proof. Let $f, g, h : A \longrightarrow B$ be morphisms satisfying the equation $f + h = g + h$. Then we can form the following commuting diagram, consisting of a †-equalizer (E, e) for the parallel pair $(f\ \ h)$ and $(g\ \ h)$ along with two cones (A, i_2) and (A, Δ_A):

$$\begin{tikzcd}
& A \arrow[dl, bend right, "\tilde{i}_2"'] \arrow[dr, bend left, "i_2 = \left(\begin{smallmatrix} 0_{A,A} \\ \mathrm{id}_A \end{smallmatrix}\right)"] & & \\
E \arrow[rr, hook, "e = \left(\begin{smallmatrix} e_1 \\ e_2 \end{smallmatrix}\right)"] & & A \oplus A \arrow[r, shift left, "(f\ \ h)"] \arrow[r, shift right, "(g\ \ h)"'] & B \\
& A \arrow[ul, bend left, "\widetilde{\Delta}_A"] \arrow[ur, bend right, "\Delta_A = \left(\begin{smallmatrix} \mathrm{id}_A \\ \mathrm{id}_A \end{smallmatrix}\right)"'] & &
\end{tikzcd} \tag{15}$$

The morphism i_2 is the injection of the second factor into the †-biproduct, and the morphism Δ_A is the diagonal for the †-biproduct. Since i_2 and Δ_A are cones they must factorise uniquely through e, and we denote these factorisations by $\tilde{i}_2$ and $\widetilde{\Delta}_A$ respectively. The condition that e is an isometry gives the equation

$$e_1; e_1^\dagger + e_2; e_2^\dagger = \mathrm{id}_E. \tag{16}$$

Precomposing with $\tilde{i}_2$ gives $e_2^\dagger = \tilde{i}_2$, and postcomposing this with e_1 and e_2 respectively gives

$$e_2^\dagger; e_1 = 0_{A,A}, \tag{17}$$
$$e_2^\dagger; e_2 = \mathrm{id}_A. \tag{18}$$

Similarly, precomposing (16) with $\widetilde{\Delta}_A$ gives us $e_1^\dagger + e_2^\dagger = \widetilde{\Delta}_A$, and postcomposing with with e_1 and e_2 respectively gives

$$e_1^\dagger; e_1 + e_2^\dagger; e_1 = \mathrm{id}_A, \tag{19}$$
$$e_1^\dagger; e_2 + e_2^\dagger; e_2 = \mathrm{id}_A. \tag{20}$$

We will show that $i_1 = \left(\begin{smallmatrix} \mathrm{id}_A \\ 0_{A,A} \end{smallmatrix}\right) : A \longrightarrow A \oplus A$ is a cone for the parallel pair, which directly leads to the required conclusion $f = g$. We must find a factorising morphism $c : A \longrightarrow E$ which gives i_1 upon composition with $e : E \longrightarrow A \oplus A$. We choose $c = e_1^\dagger$, and so we must show that $e_1^\dagger; e_1 = \mathrm{id}_A$ and $e_1^\dagger; e_2 = 0_{A,A}$. The first of these is obtained by applying equation (17) to equation (19), and the second by applying the †-functor to equation (17). □

An important observation is that it seems to be impossible to avoid the use of the †-functor for the final stage of this proof. Without it, the strongest equation that we can easily derive for the endomorphism $e_1^\dagger; e_2$ is

$$e_1^\dagger; e_2 + \mathrm{id}_A = \mathrm{id}_A, \tag{21}$$

obtained by combining equations (18) and (20). Of course, without the cancellability property that we are trying to prove, this is not enough to establish that $e_1^\dagger; e_2 = 0_{A,A}$.

We now investigate another form of cancellability. In a category enriched in commutative monoids, for any natural number n and any morphism f, we define the *n-fold sum* of f to be $n \cdot f := f + f + \cdots + f$, where we sum over a total of n copies of f. We can then prove the following lemma.

Lemma 2.6 *In a †-category with †-limits, for any f, g in the same hom-set, if there exists a nonzero n with $n \cdot f = n \cdot g$, then $f = g$.*

Proof. Consider the following commutative diagram, where $f, g : A \longrightarrow B$ are morphisms satisfying $n \cdot f = n \cdot g$:

$$\begin{tikzcd}
E \arrow[dr, bend left, "e"] & & \\
& A^{\oplus n} \arrow[r, shift left, "(f\ f\ \cdots\ f)"] \arrow[r, shift right, "(g\ g\ \cdots\ g)"'] & B \\
A \arrow[uu, "\widetilde{\Delta}"] \arrow[ur, bend right, "\Delta"'] & &
\end{tikzcd} \tag{22}$$

The diagonal morphism $\Delta : A \longrightarrow A^{\oplus n}$ is a cone for the parallel pair, and so it factors uniquely through the †-equalizer $e : E \longrightarrow A^{\oplus n}$ as $\widetilde{\Delta} : A \longrightarrow E$. Let $p_i : A^{\oplus n} \longrightarrow A$ be the projection onto the ith factor of the †-biproduct, and define $e_i := e; p_i : E \longrightarrow A$ as the ith element of the †-equalizer morphism $e : E \longrightarrow A^{\oplus n}$. We have $\Delta = \widetilde{\Delta}; e = \widetilde{\Delta}; e; e^\dagger; e = \Delta; e^\dagger; e$, and by postcomposing with p_1 we obtain $\mathrm{id}_A = \sum_{i \in N} e_i^\dagger; e_1$ where N is a set with n elements. Taking the adjoint of this gives $\mathrm{id}_A = \sum_{i \in N} e_1^\dagger; e_i$. Since e is a cone we have $\sum_{i \in N}(e_i; f) = \sum_{i \in N}(e_i; g)$, and by precomposing with $e_1^\dagger$ and reorganising we obtain $(\sum_{i \in N} e_1^\dagger; e_i); f = (\sum_{i \in N} e_1^\dagger; e_i); g$. We have already shown that $\sum_{i \in N} e_1^\dagger; e_i = \mathrm{id}_A$, and so we obtain $f = g$. □

Finally we show that the n-fold sum operation has an inverse for any positive n. It follows from this that we can construct *fractions* of morphisms.

Lemma 2.7 *In a †-category with all finite †-limits, for each object A and each nonzero natural number n, there exists a unique morphism $\frac{\mathrm{id}_A}{n} : A \longrightarrow A$ with $n \cdot \frac{\mathrm{id}_A}{n} = \mathrm{id}_A$.*

Proof. Consider the equalizer diagram consisting of the projection maps $p_i : A^{\oplus n} \longrightarrow A$. Let $e : E \longrightarrow A^{\oplus n}$ be their †-equalizer, and let $\Delta_A^n : A \longrightarrow A^{\oplus n}$ be the n-fold diagonal map, which is also an equalizer. Then there is a unique map

$m : A \longrightarrow E$ mediating between these equalizers.

$$\begin{tikzcd}
E \arrow[dr, bend left, "e"] & & \\
 & A^{\oplus n} \arrow[r, shift left, "p_1"] \arrow[r, phantom, "\cdots"] \arrow[r, shift right, "p_n"'] & A \\
A \arrow[uu, "\widetilde{\Delta}^n_A"] \arrow[ur, bend right, "\Delta^n_A"'] & &
\end{tikzcd} \tag{23}$$

Let $e_i : E \longrightarrow A$ be the ith component of the †-equalizer e, defined by $e_i = e; p_i$. Since e is an equalizer for the morphisms p_i, each of these components e_i are equal. Then $\mathrm{id}_A = \Delta^n_A; p_1 = \widetilde{\Delta}^n_A; e_1 = \widetilde{\Delta}^n_A; e; e^\dagger; e_1 = \Delta^n_A; e^\dagger; e_1 = \sum_i e_i^\dagger; e_1 = \sum_i e_1^\dagger; e_1 = n \cdot e_1^\dagger; e_1$, and we can define $\frac{\mathrm{id}_A}{n} := e_1^\dagger; e_1$. It follows from lemma 2.6 that this morphism is the unique one with the necessary property. □

Exchange lemma

The final property that we prove is an 'exchange lemma', which identifies a restriction on the algebra of morphism composition in the presence of †-limits. It can be seen as a stronger form of the nondegeneracy property demonstrated in lemma 2.4. We will use this exchange lemma in an essential way in the next section, to prove that our generalised real numbers admit a total order.

Lemma 2.8 (Exchange) *In a †-category with all finite †-limits, for any parallel morphisms f and g,*

$$f^\dagger; f + g^\dagger; g = f^\dagger; g + g^\dagger; f \;\Rightarrow\; f = g. \tag{24}$$

Proof. Let $f, g : A \longrightarrow B$ be morphisms satisfying $f^\dagger; f + g^\dagger; g = f^\dagger; g + g^\dagger; f$. As might be expected from the earlier lemmas, our proof strategy is to construct a †-equalizer diagram, which in this case consists of the parallel pair $(f \;\; g)$ and $(g \;\; f)$. We next deduce the existence of certain cones, (B, p) and (B, q), which factorise through the †-equalizer (E, e) via $\tilde{p}$ and $\tilde{q}$ respectively:

$$\begin{tikzcd}
 & B \arrow[dl, bend right, "\tilde{p}"'] \arrow[dr, bend left, "p = \begin{pmatrix} f^\dagger \\ g^\dagger \end{pmatrix}"] & & \\
E \arrow[rr, hook, "e = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}"] & & A \oplus A \arrow[r, shift left, "(f \;\; g)"] \arrow[r, shift right, "(g \;\; f)"'] & B \\
 & B \arrow[ul, bend left, "\tilde{q}"] \arrow[ur, bend right, "q = \begin{pmatrix} g^\dagger \\ f^\dagger \end{pmatrix}"'] & &
\end{tikzcd} \tag{25}$$

Since $e : E \longrightarrow A \oplus A$ is a †-equalizer we have $p^\dagger = e^\dagger; \tilde{p}^\dagger = e^\dagger; e; e^\dagger; \tilde{p}^\dagger = e^\dagger; e; p^\dagger$, and similarly $q^\dagger = e^\dagger; e; q^\dagger$. The equalising morphism e is a cone, and given that $(f \;\; g) = p^\dagger$ and $(g \;\; f) = q^\dagger$, we obtain $e; p^\dagger = e; q^\dagger$. It is then straightforward to see that $(f \;\; g) = p^\dagger = e^\dagger; e; p^\dagger = e^\dagger; e; q^\dagger = q^\dagger = (g \;\; f)$, and therefore that $f = g$ as required. □

We call this the 'exchange lemma' since, passing from one side of the main equation to the other, the morphisms f and g exchange positions.

2.4 Existence theorem for †-limits

We now develop some lemmas aiming towards an existence theorem for †-limits in terms of †-equalizers and †-biproducts.

Lemma 2.9 *If a †-category has binary †-equalizers and binary †-biproducts, then it has all finite †-equalizers.*

Proof. Let $f_i : A \longrightarrow B$ be a set of parallel arrows indexed by $i \in I$, a finite set. Then we can construct the I-fold †-biproduct $B^{\oplus I}$, and define a column vector $F : A \longrightarrow B^{\oplus I}$ as the unique morphism with the property that $F; p_i = f_i$, where $p_i : B^{\oplus I} \longrightarrow B$ is the projection onto the ith factor. Let $\Delta : B \longrightarrow B^{\oplus I}$ be the diagonal map, and construct the following †-equalizer:

$$\begin{tikzcd} E \arrow[r, "e"] & A \arrow[r, shift left, "F"] \arrow[r, shift right, "f_1;\Delta"'] & B^{\oplus I} \end{tikzcd} \tag{26}$$

Postcomposing with $p_i : B^{\oplus I} \longrightarrow B$ we obtain $e; F; p_i = e; f_1; \Delta; p_i$, which simplifies to $e; f_i = e; f_1$. It follows that $e; f_i = e; f_j$ for all $i, j \in I$, and so $e : E \longrightarrow A$ is a cone for the morphisms $f_i : A \longrightarrow B$. Now let $x : X \longrightarrow A$ be any map such that $x; f_i = x; f_j$ for all $i, j \in I$. Then x is also a cone for the morphisms F and $f_1; \Delta$, and so factorises uniquely through $e : E \longrightarrow A$. It follows that the morphism e is the †-equalizer of the morphisms $f_i : A \longrightarrow B$. □

Lemma 2.10 *In a unitary †-category with binary †-equalizers and binary †-biproducts, for each object A and each natural number n, there is an isomorphism $r : A \longrightarrow A$ with $r; r^\dagger = n \cdot \mathrm{id}_A$.*

Proof. Write $p_i : A^{\oplus n} \longrightarrow A$ for the projection of the †-biproduct onto its ith factor, and consider all these maps together as forming an equalizer diagram:

$$\begin{tikzcd} E \arrow[dr, bend left, "e"] \arrow[dd, dashed, bend right=60, "u"'] & & \\ & A^{\oplus n} \arrow[r, shift left=2, "p_1"] \arrow[r, phantom, "\cdots"] \arrow[r, shift right=2, "p_n"'] & A \\ A \arrow[uu, "m"] \arrow[ur, bend right, "\Delta"'] & & \end{tikzcd} \tag{27}$$

The n-fold diagonal map $\Delta : A \longrightarrow A^{\oplus n}$ is an equalizer for these maps, since given any $x : X \longrightarrow A^{\oplus n}$ with $x; p_i = x; p_j$ for all valid i and j, x factors uniquely through Δ as $x; p_1; \Delta$. By lemma 2.9 we can construct the †-equalizer of the maps p_i, which we denote by $e : E \longrightarrow A^{\oplus n}$. Since e and Δ are both equalizers, there is a unique isomorphism $m : A \longrightarrow E$ with $m; e = \Delta$; and since A and E are isomorphic, by unitarity of the †-category, there exists some unitary morphism $u : E \longrightarrow A$.

Defining an endomorphism $r := m; u : A \longrightarrow A$, we see that

$$r; r^\dagger = m; u; u^\dagger; m^\dagger = m; m^\dagger = m; e; e^\dagger; m^\dagger = \Delta; \nabla = n \cdot \mathrm{id}_A, \tag{28}$$

where in the fourth expression we have inserted the identity in the form $\mathrm{id}_E = e; e^\dagger$. Since both m and u are isomorphisms it follows that $r : A \longrightarrow A$ is also an isomorphism. □

We now introduce a useful new construction.

Definition 2.11 *In a †-category, given a finite family of isometries $x_i : X_i \longrightarrow A$, their* †-intersection *is a pullback (P, π_i) such that each map $\pi_i : P \longrightarrow X_i$ is an isometry.*

The notion of †-intersection is a geometrical one. Given a family of isometries representing subobjects of a given object, it is interesting to ask whether there exists an isometry that represents the intersection of all the subobjects. Of course, this intersection could be zero. We now give an existence theorem for †-intersections.

Lemma 2.12 *If a unitary †-category has all binary †-equalizers and binary †-biproducts then it has all finite †-intersections.*

Proof. Let $x_i : X_i \hookrightarrow A$ be our family of isometries in a unitary †-category $\mathbf{C}$, indexed by a finite set J. We construct the †-biproduct $\bigoplus_{i \in J} X_i$, with canonical projections $p_i : \bigoplus_{i \in J} X_i \longrightarrow X_i$. Considering our family of isometries as a diagram in $\mathbf{C}$, we can construct its †-pullback by forming the †-equalizer $e : E \longrightarrow \bigoplus_{i \in J} X_i$ of the morphisms $p_i; x_i : \bigoplus_{i \in J} X_i \longrightarrow A$. The cone maps of the †-limit are then given by $e; p_i : E \longrightarrow X_i$. It is straightforward to check that they form a limit, and the normalization condition is satisfied since $\sum_{i \in J} e; p_i; p_i^\dagger; e^\dagger = e; (\sum_{i \in J} p_i; p_i^\dagger); e^\dagger = e; e^\dagger = \mathrm{id}_E$.

Any of the composites $e; p_i; x_i : E \longrightarrow A$, all of which are equal, intuitively represents the intersection of the isometries $x_i : X_i \longrightarrow A$. However, these composites are not isometries in general; we must add a normalization factor. We construct the †-intersection of the morphisms x_i as $s := r; e; p_i; x_i : E \longrightarrow A$, for any choice of $i \in J$, where $r : E \longrightarrow E$ is an isomorphism satisfying $r; r^\dagger = |J| \cdot \mathrm{id}_E$ as described by lemma 2.10, and $|J|$ is the number of elements of J. Our morphism s does indeed factor through the projections of a pullback in the necessary way, since we have already shown that the morphisms $e; p_i$ form the projections of a †-pullback, and since limits are preserved by isomorphisms, so do the morphisms $r; e; p_i$. To show that s is an isometry is to show that $s; s^\dagger = \mathrm{id}_E$, and by lemma 2.6, it suffices to show that $|J| \cdot (s; s^\dagger) = |J| \cdot \mathrm{id}_E$:

$$\begin{aligned}
|J| \cdot (s; s^\dagger) &= \sum_{i \in J} s; s^\dagger \\
&= \sum_{i \in J} ((r; e; p_i; x_i); (r; e; p_i; x_i)^\dagger) \\
&= \sum_{i \in J} \left(r\ ; e; p_i; x_i; x_i^\dagger; p_i^\dagger; e^\dagger; r^\dagger\right) \\
&= r; e; \left(\sum_{i \in J} p_i; p_i^\dagger\right); e^\dagger; r^\dagger \\
&= r; e; e^\dagger; r^\dagger \\
&= r; r^\dagger = |J| \cdot \mathrm{id}_E.
\end{aligned} \tag{29}$$

This completes the proof. □

Theorem 2.13 (Existence theorem for †-limits.) *A unitary †-category has finite †-limits iff it has a zero object, binary †-equalizers and binary †-biproducts.*

Proof. If a †-category has finite †-limits then it has these three constructions; the zero object is the †-limit of the empty diagram, binary †-equalizers are manifestly †-limits, and binary †-biproducts are †-limits by lemma 2.1.

Conversely, consider a unitary †-category $\mathbf{C}$ with a zero object, binary †-equalizers and binary †-biproducts. By lemma 2.9 such a category actually has all finite †-equalizers, and it is straightforward to obtain all finite †-biproducts from binary †-biproducts. Since finite biproducts exist the category is enriched in commutative monoids, and so the notion of a †-limit is well-defined. Consider a diagram $F : \mathbf{J} \longrightarrow \mathbf{C}$, with a chosen set of sources $\Omega \subseteq \mathrm{Ob}(\mathbf{J})$. We will show that this has a †-limit.

If Ω is empty then $\mathbf{J}$ must also be empty, and the †-limit of F is given by the zero object in $\mathbf{C}$. Otherwise, form the †-biproduct in $\mathbf{C}$ of the images $F(S)$ of the source objects, for all $S \in \Omega$. We denote this †-biproduct by $\bigoplus_{F(\Omega)}$, and write the projections onto the factors as $p_S : \bigoplus_{F(\Omega)} \longrightarrow F(S)$ for all $S \in \Omega$.

For each $T \in \mathrm{Ob}(\mathbf{J})$, denote by A_T the set of arrows in $\mathbf{J}$ which go from a source object to T; also, for each arrow $f \in A_T$, denote its domain source object by $\sigma(f) \in \Omega$, so we have $f : \sigma(f) \longrightarrow T$. For each $f \in A_T$, we can construct a morphism $[f] : \bigoplus_{F(\Omega)} \longrightarrow F(T)$ as the following composite:

$$[f] := \bigoplus_{F(\Omega)} \xrightarrow{p_{\sigma(f)}} F(\sigma(f)) \xrightarrow{F(f)} F(T) \tag{30}$$

Let $e_T : E_T \longrightarrow \bigoplus_{F(S)}$ be the †-equalizer in $\mathbf{C}$ of the arrows $[f]$ for all $f \in A_T$.

Our candidate for the †-limit is the †-intersection of the isometries e_T, over all objects $T \in \mathrm{Ob}(\mathbf{J})$. We denote this †-intersection by $\pi_T; e_T : P \longrightarrow \bigoplus_{F(\Omega)}$, which has the same value for any $T \in \Omega$; the morphisms $\pi_T : P \longrightarrow E_T$ are a family of isometric pullback projections, which are guaranteed to exist by lemma 2.12. The †-limit maps to the source objects are $l_S := \pi_T; e_T; p_S : P \longrightarrow F(S)$ for any

$T \in \mathrm{Ob}(\mathbf{J})$, and for all $S \in \Omega$. We must show that these maps form a universal, normalized cone for the diagram.

We first show that the maps $l_s : P \longrightarrow F(s)$ satisfy the normalization condition (5):

$$\begin{aligned}\sum_{S\in\Omega} l_S; l_S^\dagger &= \sum_{S\in\Omega} \pi_T; e_T; p_S; (\pi_T; e_T; p_S)^\dagger \\ &= \sum_{S\in\Omega} \pi_T; e_T; p_S; p_S^\dagger; e_T^\dagger; \pi_T^\dagger \\ &= \pi_T; e_T; \left(\sum_{S\in\Omega} p_S; p_S^\dagger\right); e_T^\dagger; \pi_T^\dagger \\ &= \pi_T; e_T; e_T^\dagger; \pi_T^\dagger = \pi_T; \pi_T^\dagger = \mathrm{id}_E. \end{aligned} \tag{31}$$

To establish that the morphisms l_S define a cone, we must show that the equation $l_{\sigma(f)}; F(f) = l_{\sigma(g)}; F(g)$ is satisfied for all $T \in \mathrm{Ob}(\mathbf{J})$ and all $f, g \in A_T$. By the definition of the cone maps $l_{\sigma(f)}; F(f) = \pi_T; e_T; [f]$, and since $e_T; [f] = e_T; [g]$ we see that the cone property holds. To establish the universal property, consider a cone of morphisms $x_S : X \longrightarrow F(S)$ for all $S \in \Omega$; the cone property is that for all $T \in \mathrm{Ob}(\mathbf{J})$ and all $f, g \in A_T$, we have $x_{\sigma(f)}; F(f) = x_{\sigma(g)}; F(g)$. Let $\widetilde{x} : X \longrightarrow \bigoplus_{F(S)}$ be the unique morphism such that $\widetilde{x}; p_S = x_S$ for all $S \in \Omega$. Then by the cone property, for all $T \in \mathrm{Ob}(\mathbf{J})$ and all $f, g \in A_T$ we have $\widetilde{x}; [f] = \widetilde{x}; [g]$, and so for all $T \in \mathrm{Ob}(\mathbf{J})$ there is a unique morphism $\chi_T : X \longrightarrow E_T$ with $\widetilde{x} = \chi_T; e_T$. Since (P, π_T) form a pullback of the morphisms e_T, there must in turn be a unique morphism $\widetilde{\chi} : X \longrightarrow P$ such that $\widetilde{\chi}; \pi_T = \chi_T$. Since each e_T has a retraction, $\widetilde{\chi}$ is also the unique morphism with the property that $\widetilde{\chi}; \pi_T; e_T = \chi_T; e_T = \widetilde{x}$. It follows that $\widetilde{\chi}$ is the unique morphism with $\widetilde{\chi}; \pi_T; e_T; p_S = \widetilde{x}; p_S$ for all $S \in \Omega$, and so it is also the unique morphism with $\widetilde{\chi}; l_S = x_S$. So $(P; l_S, S \in \Omega)$ indeed gives a †-limit for the diagram $F : \mathbf{J} \longrightarrow \mathbf{C}$ with Ω the set of sources. □

3 Embedding the scalars into a field

Our main theorem of this section is stated most naturally in a *monoidal †-category.* Conventionally, this means a monoidal category which is also a †-category, such that the unit and associator natural isomorphisms are unitary. While this gives the category nicer properties as a whole, we will not need to use them. So, for our purposes, a monoidal †-category can be simply taken to mean a monoidal category which is also a †-category.

In any monoidal category, we define the *scalars* to be the hom-set $\mathrm{Hom}(I, I)$. This will have a certain amount of extra structure, depending on the properties of the ambient category. At the very least, as is well-known, it is a commutative monoid, where monoid multiplication is given by morphism composition.

Our main result concerns the scalars in a monoidal †-category with finite †-limits, which have the structure of a semiring with involution. We will prove the following theorem:

Theorem 3.1 *In a nontrivial monoidal †-category with simple tensor unit, and with all finite †-limits, the involutive semiring of scalars has an involution-preserving embedding into an involutive field with characteristic 0 and orderable fixed field.*

The proof of this theorem will be given piece-by-piece throughout this section. Just to be clear, by 'field' we mean a classical algebraic field: a commutative ring with multiplicative inverses for every nonzero element. By 'characteristic 0' we mean that no finite sum of the form $1 + 1 + \cdots + 1$ gives zero. By 'involutive semiring' and 'involutive field' we mean a structure equipped with an order-2 automorphism that respects addition and multiplication, and by 'fixed field' we mean the subfield on which the automorphism acts trivially.

The connection between this theorem and the complex numbers is given by the following well-known characterisation of the subfields of the complex numbers.

Theorem 3.2 *The subfields of the complex numbers are precisely the fields of characteristic 0 which are at most of continuum cardinality.*

It follows immediately that, if we have a monoidal †-category satisfying the conditions of theorem 3.1 for which the scalars are at most continuum cardinality, they must embed as a semiring into the complex numbers. However, this embedding will not necessarily take the involution on the semiring into complex conjugation; for this we require an extra completeness condition which we consider in the next section.

In addition to the finite †-limits which we studied in the previous section, theorem 3.1 requires two extra conditions: nontriviality, and that the monoidal unit object is simple, meaning that it lacks proper subobjects. Both are natural, in the sense that they prevent the theorem from being 'obviously' false. A field is required to have $0 \neq 1$, and this translates to the condition that our category is nontrivial. Also, if we had a monoidal †-category satisfying the conditions of the theorem, we could take the cartesian product of this category with itself; this has an obvious monoidal structure for which the monoidal unit *does* have proper subobjects, the scalars being pairs of scalars in the original category. Such a semiring can never embed into a field, since it contains *zero divisors*, nonzero elements a and b which satisfy $ab = 0$. Requiring the monoidal unit to lack proper subobjects blocks this obvious source of counterexamples.

The scalars as a semiring

We begin by recalling the well-known fact that the scalars in a monoidal category form a commutative monoid. In fact, this commutativity property is the only reason that we prove theorem 3.1 for the scalars in a monoidal category; it would hold for any commutative endomorphism monoid on an object without proper †-subobjects. If the monoidal category also has biproducts, the scalars form a commutative semiring. A semiring, sometimes called a *rig*, is a structure similar to a ring, but is not required to have have additive inverses for all elements. In this paper a ring always has a multiplicative unit, and the zero element satisfies $0x = x0 = 0$ for all elements x in the ring.

We now consider the extra structure given by the †-functor and †-biproducts. The †-functor gives us an involution on the scalars, sending $a : I \longrightarrow I$ to $a^\dagger : I \longrightarrow I$. This involution is order-reversing for multiplication, due to the contravariance of the †-functor, and distributes over addition as explained in the discussion around equation (11). This gives the scalars the structure of an *involutive semiring.*

Embedding into a field

To achieve our goal of embedding the scalars into a field, it is clear that additive cancellability is a necessary property. We demonstrated this for all hom-sets in †-categories with finite †-biproducts and finite †-equalizers in lemma 2.5. Another property which is clearly necessary is cancellable multiplication.

Definition 3.3 *A commutative semiring has* cancellable multiplication *when, for any three elements a, b, c in the semiring, $ac = bc$, $c \neq 0 \Rightarrow a = b$.*

We now show that the scalars have this property in any category of the type which we are considering. The condition that the monoidal unit has no proper †-subobjects is clearly crucial here, but this is far from the only role played by this condition in proving the theorem.

Lemma 3.4 *In a monoidal †-category with simple tensor unit, a zero object and finite †-equalizers, the scalars have cancellable multiplication.*

Proof. Suppose that the scalars did not have cancellable multiplication. Then there would exist scalars a, b, c with $c \neq 0$, such that $a \neq b$ but $ac = bc$. We consider the following commuting diagram:

$$\begin{tikzcd} & I \arrow[dl, bend right, "\tilde{c}"'] \arrow[d, bend left, "c"] & \\ E \arrow[r, hook, "e"] & I \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & I \end{tikzcd} \tag{32}$$

The †-equalizer morphism $e : E \longrightarrow I$ gives a †-subobject of I. It is not zero, since c factors through it and $c \neq 0$; also, since $a \neq b$, it cannot be an isomorphism. It follows that I has a proper †-subobject, but this contradicts our hypothesis. It follows that the scalars have cancellable multiplication. □

As a first step towards embedding the scalars into a field, we first embed them into a ring. Given our semiring S of scalars, we can construct its *difference ring* $D(S)$. Elements of $D(S)$ are equivalence classes of ordered pairs (a, b) of elements of S, which we write using the suggestive notation $a - b$. The equivalence relation is given by

$$a - b \sim c - d \quad \text{iff} \quad a + d = c + b. \tag{33}$$

It is a standard exercise to show that this is symmetric, transitive and reflexive, for which we rely on the fact that the scalars have cancellable addition. Addition and multiplication are defined on representatives of the equivalence classes in the familiar algebraic way:

$$(a-b)+(c-d)=(a+c)-(b+d) \tag{34}$$
$$(a-b)(c-d)=(ac+bd)-(ad+bc) \tag{35}$$

These are well-defined on equivalence classes.

We see that the scalars in our category embed into their difference semiring, under the obvious mapping $a \mapsto a-0$. For two elements to be sent to the same element of the difference ring would mean that $a-0 \sim b-0$, but applying the definition of the equivalence relation then gives $a=b$, so the mapping is faithful.

As we will see, the difference ring embeds into a field if and only if it has cancellable multiplication. From definition 3.3, this condition is

$$(a-b)(c-d) \sim (a-b)(e-f),\ a-b \nsim 0 \Rightarrow c-d \sim e-f \tag{36}$$

for all choices of elements $a,b,c,d,e,f \in S$. Using the definition of the equivalence relation to write this directly in terms of the elements of the underlying semiring, we obtain

$$a(c+f)+b(d+e)=a(d+e)+b(c+f),\, a \neq b \Rightarrow c+f=d+e. \tag{37}$$

Defining $A := c+f$ and $B := d+e$, this reduces to the condition

$$aA+bB=aB+bA,\, a \neq b \Rightarrow A=B. \tag{38}$$

We now show that this holds in any category of the type we are working with. In some ways, this condition resembles that of the exchange lemma 2.8, but it is logically independent from it.

Lemma 3.5 *In a monoidal †-category with all finite †-limits, for which the monoidal unit is simple, any choice of scalars $A,B,a,b : I \longrightarrow I$ satisfies the implication*

$$aA+bB=aB+bA,\, a \neq b \Rightarrow A=B.$$

Proof. We have already shown that the scalars in such a category are commutative and have cancellable addition and multiplication, and we will use these properties throughout. Let A,B,a,b be scalars satisfying $aA+bB=aB+bA$ and $a \neq b$. If $a=0$ then $bB=bA$, and cancelling the nonzero b we obtain $B=A$; the case $b=0$ is similar. Conversely, if $A=0$ then $bB=aB$, and $B=A=0$ is the only possibility, or B would cancel contradicting our assumption that $a \neq b$; the case $B=0$ is similar. In each of these cases, therefore, the implication holds.

We now consider the case in which none of the four scalars are zero. We construct the following commutative diagram where (E,e) is a †-equalizer for the parallel pair

$(A\ \ B)$ and $(B\ \ A)$, and (I, p) and (I, q) are cones:

$$\begin{tikzcd}
& I \arrow[dl, bend right, "\tilde{p}"'] \arrow[dr, bend left, "p = \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)"] & & \\
E \arrow[rr, hook, "e = \left(\begin{smallmatrix} e_1 \\ e_2 \end{smallmatrix}\right)"] & & I \oplus I \arrow[r, shift left, "(A\ \ B)"] \arrow[r, shift right, "(B\ \ A)"'] & I \\
& I \arrow[ul, bend left, "\tilde{q}"] \arrow[ur, bend right, "q = \left(\begin{smallmatrix} b \\ a \end{smallmatrix}\right)"'] & &
\end{tikzcd} \tag{39}$$

For each cone, we denote the unique factorisation through the equalizer with a tilde. Using the matrix calculus and the †-equalizer equation $e; e^\dagger = \mathrm{id}_E$ we see that $p = p; e^\dagger; e$ and $q = q; e^\dagger; e$, and writing these out in components, we obtain the following:

$$a = a; e_1^\dagger; e_1 + b; e_2^\dagger; e_1 \tag{40}$$
$$b = a; e_1^\dagger; e_2 + b; e_2^\dagger; e_2 \tag{41}$$
$$b = b; e_1^\dagger; e_1 + a; e_2^\dagger; e_1 \tag{42}$$
$$a = b; e_1^\dagger; e_2 + a; e_2^\dagger; e_2 \tag{43}$$

The first two equations come from the components of p, and the second two from the components of q.

Multiplying equation (40) by b and (42) by a and equating the right-hand sides, this gives

$$ba; e_1^\dagger; e_1 + b^2; e_2^\dagger; e_1 = ab; e_1^\dagger; e_1 + a^2; e_2^\dagger; e_1. \tag{44}$$

We apply commutativity and additive cancellability to obtain

$$b^2; e_2^\dagger; e_1 = a^2; e_2^\dagger; e_1. \tag{45}$$

We note that the quantity $e_2^\dagger; e_1$ is a scalar. Either it is zero, or it is nonzero and it can be cancelled to give $a^2 = b^2$. We will consider these cases separately. First we assume that $e_2^\dagger; e_1 \neq 0_{I,I}$ and $a^2 = b^2$. Defining $c := a + b$, we see that

$$ca = a^2 + ba = b^2 + ab = cb. \tag{46}$$

So $ca = cb$, and if $c \neq 0$ it will cancel from both sides to give $a = b$. However, by assumption $a \neq b$, and so we must have $c = 0$ and $a + b = 0$. Returning to our equation $aA + bB = aB + bA$ and adding $b(A + B)$ to both sides, we obtain

$$\begin{aligned} aA + bB + b(A + B) &= aB + bA + b(A + B) \\ \Rightarrow (a + b)A + 2bB &= (a + b)B + 2bA \\ \Rightarrow 2bB &= 2bA. \end{aligned} \tag{47}$$

Since $2 : I \longrightarrow I$ is equal to $\Delta_I^2; \nabla_I^2 = \Delta_I^2; (\Delta_I^2)^\dagger$, by lemma 2.4 it is nonzero, and so it can be cancelled from both sides. By assumption $b \neq 0$, and so it can be cancelled as well. This gives $B = A$ as required. The only unresolved case is $e_2^\dagger; e_1 = 0$.

Alternatively, we could have multiplied equation (41) by a and equation (43) by b and equated the right-hand sides. This leads to a similar conclusion that either $e_1^\dagger; e_2 \neq 0_{I,I}$ and $A = B$, or $e_1^\dagger; e_2 = 0_{I,I}$ and the theorem is not immediately resolved. Since this line of argument is independent from the previous one, the only remaining case to consider is that $e_2^\dagger; e_1 = e_1^\dagger; e_2 = 0_{I,I}$.

We have not yet used the fact that the equalizer $e : E \longrightarrow I \oplus I$ is a cone, which is asserted by the following equation:

$$e_1; A + e_2; B = e_1; B + e_2; A. \tag{48}$$

Composing on the left with $e_2^\dagger$, we obtain

$$e_2^\dagger; e_1; A + e_2^\dagger; e_2; B = e_2^\dagger; e_1; B + e_2^\dagger; e_2; A. \tag{49}$$

Applying $e_2^\dagger; e_1 = 0$, this gives

$$e_2^\dagger; e_2; B = e_2^\dagger; e_2; A. \tag{50}$$

To deal with this we need to know the value of the scalar $e_2^\dagger; e_2$. We observe that $\Delta_I = \binom{\mathrm{id}_I}{\mathrm{id}_I} : I \longrightarrow I \oplus I$ is a cone, and so there exists some $\widetilde{\Delta}_I : I \longrightarrow E$ satisfying $\Delta_I; e = \Delta_I$. Using the †-equalizer equation $e; e^\dagger = \mathrm{id}_E$ we obtain $\Delta_I = \Delta_I; e^\dagger = e_1^\dagger + e_2^\dagger$. Postcomposing with e_2 gives the equation

$$\widetilde{\Delta}_I; e_2 = 1 = e_1^\dagger; e_2 + e_2^\dagger; e_2. \tag{51}$$

Applying the assumption that $e_1^\dagger; e_2 = 0$, this gives $e_2^\dagger; e_2 = 1$. Equation (50) then gives $B = A$ as needed, which completes the proof. $\square$

For any nontrivial commutative ring R with cancellable multiplication, we can obtain its *quotient field* $Q(R)$ into which R embeds. Elements of $Q(R)$ are equivalence classes of pairs (s, t) of elements of R with $t \neq 0$. We write these pairs in the form $\frac{s}{t}$, to resemble a fraction. The equivalence relation is given by

$$\frac{s}{t} \sim \frac{u}{v} \quad \text{iff} \quad sv = ut. \tag{52}$$

This is symmetric, transitive and reflexive, as required. We rely on the cancellable multiplication to demonstrate transitivity. Multiplication and addition are defined on representatives of the equivalence classes as if they were conventional fractions:

$$\frac{s}{t} \cdot \frac{u}{v} = \frac{su}{tv} \tag{53}$$

$$\frac{s}{t} + \frac{u}{v} = \frac{sv + ut}{tv} \tag{54}$$

These operations are well-defined on the equivalence classes. Furthermore, the ring R embeds into $Q(R)$ under the mapping $r \mapsto \frac{r}{1}$, and this is faithful since $\frac{r}{1} \sim \frac{s}{1} \Rightarrow r = s$. It is straightforward to see that this embedding preserves multiplication and addition.

We require the commutative ring R to be nontrivial, satisfying $0 \neq 1$, since a field must satisfy this by definition. This leads to the requirement that the monoidal category from which we obtain our scalars must be nontrivial, having more than one morphism. We must require this explicitly, since the one-morphism category otherwise satisfies all of our conditions: it is a monoidal †-category with all finite †-limits, for which the monoidal unit object has no proper †-subobjects. Altogether, for a nontrivial monoidal †-category with all finite †-limits, in which the monoidal unit has no proper †-subobjects, we have shown that the commutative semiring S of scalars embeds into the commutative difference ring $D(S)$; that this ring has cancellable multiplication; and that any ring R with cancellable multiplication embeds into its quotient field $Q(R)$. It follows that the semiring S embeds into $Q(D(S))$, and so the scalars in our monoidal category embed into a field.

We next show that the semiring S of scalars has characteristic 0. Since we have shown that this semiring embeds into the field $Q(D(S))$, it follows that this field must also have characteristic 0.

Lemma 3.6 *In a nontrivial monoidal †-category with finite †-biproducts and †-equalizers, for which the monoidal unit object has no †-subobjects, the scalars have characteristic 0.*

Proof. Suppose that scalar addition is not of characteristic 0. Then there exists some nonzero scalar $a : I \longrightarrow I$, and positive natural number n, such that

$$a + \cdots + a = 0 \tag{55}$$

where the sum contains n copies of a. This sum is equal to $n \cdot a$, where $n : I \longrightarrow I$ is a scalar given by $\Delta_I^n ; \nabla_I^n$, for Δ_I^n the n-fold codiagonal of I and ∇_I^n the n-fold diagonal. From the †-biproduct property it follows that $\nabla_I^n = (\Delta_I^n)^\dagger$ by lemma 2.2, and from the †-equalizer property it follows in turn that $n = \Delta_I^n ; (\Delta_I^n)^\dagger \neq 0$ by lemma 2.4. However, by lemma 3.3, the product of two nonzero scalars cannot be zero. We conclude that our original assumption was wrong, and that scalar addition is of characteristic 0. □

Involution and ordering

The action of the †-functor gives the scalars the structure of an *involutive* semiring, equipping it with an involution that respects semiring addition and multiplication: we have $(a + b)^\dagger = a^\dagger + b^\dagger$ by lemma 2.2, and $(ab)^\dagger = a^\dagger b^\dagger$ by functoriality. An involution is usually required to be order-reversing for multiplication, which is satisfied in a natural way since the †-functor is contravariant, but we can neglect this here as the scalars are commutative. The self-adjoint scalars are those scalars satisfying $a = a^\dagger$. These self-adjoint scalars are closed under multiplication and addition, and so form a subsemiring. It is easy to see that the field $Q(D(S))$ into which the scalars S embed inherits the involution, and so is an involutive field. The self-adjoint elements of $Q(D(S))$ also form a field, and the self-adjoint scalars embed into this field.

We now demonstrate that the self-adjoint scalars admit an *order*. An order on a semiring is a reflexive total order on the underlying set, such that the following conditions hold:

$$a \le b \;\Rightarrow\; a + c \le b + c \tag{56}$$

$$0 \le a, 0 \le b \;\Rightarrow\; 0 \le ab \tag{57}$$

We will not work directly with these conditions. Instead, we will take advantage of the fact that our scalars embed into a field, and use the following classical theorem on orders for fields (for a proof, see [5, Theorem 3.3.3].)

Theorem 3.7 *A field admits an order if and only if a finite sum of squares of nonzero elements is never zero.*

We will use this theorem to show that the self-adjoint elements of the field $Q(D(S))$ admits an order. It then follows straightforwardly that the semiring of self-adjoint elements of S admits an order, through its involution-preserving embedding into $Q(D(S))$. However, we emphasise that there is no guarantee that this order will be unique, or that there will be a canonical choice of order.

We actually prove a more general theorem, on sums of *squared norms* of elements of $Q(D(S))$.

Definition 3.8 *For a field with involution $a \mapsto a^\dagger$, the* squared norm *of a is $aa^\dagger$.*

Lemma 3.9 *Let S be the semiring of scalars in a nontrivial monoidal †-category with simple tensor unit, and with all finite †-limits. Then a finite sum of squared norms of nonzero elements of the field $Q(D(S))$ is never zero.*

Proof. We must show that, given any finite sum satisfying

$$a_1 {a_1}^\dagger + a_2 {a_2}^\dagger + \cdots + a_N {a_N}^\dagger = 0 \tag{58}$$

where each a_i is an element of $Q(D(S))$, each a_i is actually zero. By construction, each a_i is a formal quotient b_i/c_i of some pair of elements b_i, c_i in $D(S)$. Writing the sum in terms of these quotients, and multiplying through by each denominator, we obtain another sum in the form of (58) in which each term is a squared norm of an element of $Q(D(S))$ with trivial denominator; in other words, an element of $D(S)$. Writing these elements as formal ordered pairs $d_i - e_i$, where d_i, e_i are elements of S, we obtain the sum

$$(d_1 - e_1)(d_1 - e_1)^\dagger + (d_2 - e_2)(d_2 - e_2)^\dagger + \cdots + (d_N - e_N)(d_N - e_N)^\dagger = 0. \tag{59}$$

We define the morphism $d : I \longrightarrow I^{\oplus N}$ to be the column vector with components $(d_1, d_2, \ldots, d_N)$, and the morphism $e : I \longrightarrow I^{\oplus N}$ to be the column vector with components $(e_1, e_2, \ldots, e_N)$. By matrix multiplication, we see that equation (59) is precisely equivalent to the equation $d; d^\dagger + e; e^\dagger = d; e^\dagger + e; d^\dagger$. We can now apply the exchange lemma 2.8 to conclude that $d = e$, and so $e_i = d_i$ for all i. It follows that each of the original $a_i = \frac{d_i - e_i}{\text{denom}}$ was zero, and that the sum of squared norms was in fact a sum of zeros, which proves the lemma. □

From this lemma we see that a sum of squares of nonzero self-adjoint elements of the field $Q(D(S))$ is nonzero. So by theorem 3.7 the self-adjoint elements of $Q(D(S))$ admit an ordering, and in general they will admit many different orderings. By extension, the self-adjoint elements of the scalar semiring S also admit an ordering, since they embed into the self-adjoint elements of $Q(D(S))$. This concludes the proof of the main theorem.

4 Completing the scalars

We have shown that, in a monoidal †-category with all finite †-limits that satisfies the conditions of the previous section, the scalars share many properties with the complex numbers. In particular, the self-adjoint scalars will admit a total order, just as the real numbers do. In fact, the order on the real numbers is *Dedekind-complete*: every subset with an upper bound has a least upper bound, and every subset with a lower bound has an greatest lower bound. This property will form a crucial part of our final axiomatization of the complex numbers. In the following, we freely make use of the symbols $<$, $\leq$, $>$ and $\geq$ to denote relationships between the elements of the total order, with their obvious meanings. Also, if X represents some set of elements, then we write $X + a$ to represent the set $\{x + a | x \in X\}$.

Lemma 4.1 *Suppose a commutative semiring contains the positive rational numbers and is additively cancellable, multiplicatively cancellable, totally-ordered, and Dedekind-complete such that suprema and infima are preserved by addition. Then it has the following properties:*

(i) (Means.) *For any pair of elements $a < b$ we can construct their 'mean' as $\frac{1}{2}(a+b)$, which satisfies $a < \frac{1}{2}(a+b) < b$.*

(ii) (Partial subtraction.) *For any pair of positive elements a and b with $a < b$, there exists an element c with $c + a = b$.*

(iii) (No positive infinitesimals.) *For any positive element a, there exists a natural number n such that $an > 1$.*

(iv) (No positive infinite elements.) *For any positive element a, there exists a natural number n such that $a < n$.*

(v) (Dense positive rationals.) *For any two unequal positive elements, there is a rational number between them.*

(vi) (Real numbers.) *The semiring is isomorphic to either the semiring $\mathbb{R}^{\geq 0}$ of nonnegative real numbers, or the field $\mathbb{R}$ of all real numbers.*

Proof. We prove these properties sequentially, at times using lower-numbered properties to aid the proof of higher-numbered ones.

(*i*) (Means.) Since $a < b$ we have $a + a = 2a < a + b$, and multiplying by the fraction $\frac{1}{2}$, we obtain $a < \frac{1}{2}(a+b)$. Similarly, we can also show that $\frac{1}{2}(a+b) < b$.

(*ii*) (Partial subtraction.) For a pair of elements $a, b \in L$ with $0 < a < b$, consider the following sets:

$$J = \{x \in L, x + a > b\} \tag{60}$$
$$K = \{x \in L, x + a < b\} \tag{61}$$

The set J has a lower bound 0 and the set K has an upper bound b, so the greatest lower bound $\bigwedge(J)$ and greatest upper bound $\bigvee(K)$ both exist by Dedekind-completeness. If $\bigvee(J)+a = b$ or $\bigwedge(K)+a = b$ then we have discovered c and we are done, so suppose that neither hold. Suppose that $\bigwedge(J) + a < b$: then $\bigwedge(J + a) < b$ by the preservation of infima by addition, but this is not possible, since b would then serve as a greater lower bound. Similarly, we can rule out $\bigvee(K) + a > b$. The only remaining situation is that in which $\bigvee(K) + a < b < \bigwedge(J) + a$, from which it follows by additive cancellability that $\bigvee(K) < \bigwedge(J)$. Construct the mean of $\bigvee(K)$ and $\bigwedge(J)$ as $m := \frac{1}{2}(\bigvee(K) + \bigwedge(J))$; then by property (i),

$$\bigvee(K) < m < \bigwedge(J). \tag{62}$$

Consider the value of $m+a$. Suppose that $m+a < b$; then $m \in K$ and so $m \leq \bigvee(K)$, but this contradicts equation (62). Similarly, suppose that $m + a > b$; then $m \in J$ and so $m \geq \bigwedge(K)$, and this again leads to a contradiction. The only remaining possibility is that $m + a = b$, and so we are done.

(*iii*) (No infinitesimals.) Consider the set

$$I = \{x \in L, x > 0, \forall n \in \mathbb{N}\ nx < 1\}, \tag{63}$$

the elements of which we call the infinitesimals. Suppose the set I is not empty; since the element 1 serves as an upper bound, the supremum $\bigvee(I)$ must therefore exist, and will satisfy $\bigvee(I) > 0$ since it is certainly greater than each positive infinitesimal. Suppose $\bigvee(I)$ is not itself an infinitesimal; then there exists some $m \in \mathbb{N}$ with $m \bigvee(I) > 1$, and multiplying by the rational number $\frac{1}{m}$ it follows that $\bigvee(I) > \frac{1}{m}$. But then $\frac{1}{m}$ serves as a lower upper bound to the infinitesimals than $\bigvee(I)$; this gives a contradiction, and so $\bigvee(I)$ must be an infinitesimal. Since $\bigvee(I) > 0$ it follows that $2 \bigvee(I) > \bigvee(I)$; the quantity $2 \bigvee(I)$ is therefore not an infinitesimal, and there must exist some $p \in \mathbb{N}$ with $2p \bigvee(I) > 1$. But since $2p$ is a natural number, $\bigvee(I)$ is not infinitesimal, and so we have a contradiction. It follows that the set I is empty.

(*iv*) (No positive infinite elements.) This property is proved in a similar way to property (iii). Define the set

$$H = \{x \in L, \forall n \in \mathbb{N}\ x > n\}, \tag{64}$$

containing the infinite elements, and assume that it is not empty. Clearly this set has a positive lower bound given by any natural number, so by Dedekind-completeness it must have a positive greatest upper bound $\bigwedge(H)$. Since $\frac{1}{2}\bigwedge(H) < \bigwedge(H)$ it follows that $\frac{1}{2}\bigwedge(H)$ is not an infinite element, and so there exists some $n \in \mathbb{N}$

with $\frac{1}{2}\bigwedge(H) < n$; from this we see that $\bigwedge(H) < 2n$, and so $\bigwedge(H)$ itself is not an infinite element. But then $2n$ is a greater lower bound for the elements of H, which contradicts the definition of $\bigwedge(H)$. The only remaining possibility is that the set H is empty.

(*v*) (Dense positive rationals.) Let $a, b \in L$ be two unequal positive elements without a rational number between them. Without loss of generality, assume $a < b$. By property ii there exists a positive element $c \in L$ with $a+c = b$, and by property iii there exists some natural number $n \in L$ with $nc > 1$. It follows that $nb = na+nc > na+1$. Write $p \in L$ for the smallest natural number greater than na, which exists by property iv; it satisfies $na + 1 > p > na$. Then $nb > na + 1 > p > na$. Multiplying by the rational $\frac{1}{n}$ we obtain $b > \frac{p}{n} > a$, and we have proved the property.

(*vi*) (Real numbers.) For any positive element a, define the set $\mathbb{Q}^+_{<a}$ to consist of the positive rational numbers strictly less than a. From property iii there are no infinitesimals an $\mathbb{Q}^+_{<a}$ is not empty; also, since it has an upper bound a it has a least upper bound $\bigvee(\mathbb{Q}^+_{<a})$. Suppose $\bigvee(\mathbb{Q}^+_{<a}) < a$; then by property v there exists some rational element r satisfying $\bigvee(\mathbb{Q}^+_{<a}) < r < a$. But this contradicts the definition of $\bigvee(\mathbb{Q}^+_{<a})$, and we conclude that $\bigvee(\mathbb{Q}^+_{<a}) = a$. We immediately obtain an isomorphism between the nonnegative elements of L and the positive real numbers $\mathbb{R}^{\geq 0}$, since any positive real number is the supremum of the positive rationals below it.

Suppose that the nonnegative elements do not comprise the entire semiring; then there exists some $b \in L$ with $b < 0$. Then $b^2 > 0$, and identifying b^2 with a real number, we can find a positive element $c \in L$ with $c^2 = b^2$, and a positive element $\frac{1}{c} \in L$ which is the reciprocal of c. Then defining $x = \frac{b}{c} + 1$, we see that

$$x^2 = \left(\tfrac{b}{c} + 1\right)^2 = \left(\tfrac{b^2}{c^2} + 1 + 2\tfrac{b}{c}\right) = 2 + 2\tfrac{b}{c} = 2\left(1 + \tfrac{b}{c}\right) = 2x. \tag{65}$$

Suppose that $x \neq 0$; from the multiplicative cancellability property this implies that $x = \frac{b}{c} + 1 = 2$, and therefore that $b = c$. But this is not possible, since $b < 0$ and $c > 0$. We conclude that $x = 0$, and therefore that $\frac{b}{c} + 1 = 0$ and $\frac{b}{c} = -1$. It follows that the semiring is in fact a ring, and that the negative elements are in bijection with the positive elements under multiplication by -1. We therefore obtain an isomorphism between the entire ring and the real numbers $\mathbb{R}$ by the method described in the previous paragraph. $\square$

Theorem 4.2 *In a monoidal †-category with simple tensor unit, which has all finite †-limits, and for which the self-adjoint scalars have addition-compatible Dedekind-completeness, the scalars have an involution-preserving embedding into the complex numbers. In particular, the scalars can be identified with either* $\mathbb{R}^{\geq 0}$ *or* $\mathbb{R}$ *with trivial involution, or* $\mathbb{C}$ *with complex conjugation as involution.*

Proof. Writing S for the semiring of scalars, we write $L \subseteq S$ for the subsemiring of self-adjoint scalars. This semiring is commutative, contains the positive rational numbers by lemma 2.7, is additively cancellable by lemma 2.5, is multiplicatively cancellable by lemma 3.4, admits a total ordering by theorem 3.1, and in fact admits an addition-compatible Dedekind-complete ordering by hypothesis. Lemma 4.1

therefore applies and L is either $\mathbb{R}^{\geq 0}$ or $\mathbb{R}$, the latter being the smallest field into which L embeds. It follows that $Q(D(L)) = D(L) = \mathbb{R}$, where $Q(-)$ and $D(-)$ construct the smallest field containing a particular ring and and smallest ring containing a particular semiring respectively, in the manner described in section 3. We also observe that the self-adjoint elements of $D(S)$ are precisely $D(L)$, and the self-adjoint elements of $Q(D(S))$ are precisely $Q(D(L))$, which is straightforward to demonstrate; as a consequence, we can identify the self-adjoint elements of $Q(D(S))$ with $\mathbb{R}$, and we will use this identification freely in the rest of the proof.

We will demonstrate an involution-preserving embedding of $Q(D(S))$ into the complex numbers. Since L is either $\mathbb{R}^{\geq 0}$ or $\mathbb{R}$, then $Q(D(L)) = D(L) = \mathbb{R}$. Suppose that the involution on the scalars is trivial; then $L = S$, and $Q(D(S)) = Q(D(L)) = \mathbb{R} \subset \mathbb{C}$, so the theorem holds. Otherwise, let $x \in Q(D(S))$ be an element of our field such that $x^\dagger \neq x$; then $y := x - x^\dagger$ is a nonzero element satisfying $y^\dagger = -y$, and $y^\dagger y \in F$ is a nonzero real number. Suppose that $y^\dagger y < 0$; then $-y^\dagger y$ is a positive real number with a positive root $r \in F$ satisfying $r^\dagger r + y^\dagger y = 0$. But by lemma 3.9 this cannot be the case, and we conclude that $y^\dagger y > 0$. Let $s \in F$ be the positive root of $y^\dagger y$ satisfying $s^2 = y^\dagger y$, and define $j = y/s$. Then $j^\dagger = y^\dagger/s^\dagger = -y/s = -j$ and $j^2 = y^2/s^2 = -y^\dagger y/s^2 = -1$, and j satisfies the properties that we expect of $i \in \mathbb{C}$. With this in mind, for all elements $z \in Q(D(S))$, we define $\mathrm{Re}(z) = \frac{1}{2}(z + z^\dagger)$ and $\mathrm{Im}(z) = \frac{1}{2j}(z - z^\dagger)$. We then define a field homomorphism $\sigma : Q(D(S)) \longrightarrow \mathbb{C}$ as $\sigma(z) = \mathrm{Re}(z) + i\mathrm{Im}(z)$. This is clearly involution-preserving, and it is straightforward to show that it is in fact an isomorphism of fields.

Finally we will show that if the involution on the scalars is nontrivial, then the scalar semiring is actually isomorphic to $\mathbb{C}$, with involution given by complex conjugation. We have demonstrated the existence of an involution-preserving embedding of S into $\mathbb{C}$; this embedding contains the nonnegative reals, and nontriviality of the involution implies that there is at least one point off the real line. However, from a consideration of the geometry of the complex plane, it is straightforward to show that any such semiring must in fact be the entire complex plane. □

Acknowledgement

I am grateful to Samson Abramsky, Chris Isham, Zurab Janelidze and Paul Taylor for useful discussions. Special thanks also go to Kevin Buzzard, Chris Heunen and Peter Selinger for detailed comments on an earlier draft.

I have used Paul Taylor's diagrams package, and I am grateful for financial support from the EPSRC and the ONR.

References

[1] Samson Abramsky and Bob Coecke. A categorical semantics of quantum protocols. *Proceedings of the 19th Annual IEEE Symposium on Logic in Computer Science*, pages 415–425, 2004. IEEE Computer Science Press.

[2] Samson Abramsky and Bob Coecke. *Handbook of Quantum Logic and Quantum Structures*, volume 2, chapter Categorical Quantum Mechanics. Elsevier, 2008.

[3] John Baez. Higher-dimensional algebra II: 2-Hilbert spaces. *Advances in Mathematics*, 127:125–189, 1997.

[4] Hans Halvorson and Michael Müger. Algebraic quantum field theory. *Handbook of the Philosophy of Physics*. (to appear).

[5] David Marker. *Model Theory: An Introduction*. Springer, 2002.

[6] Peter Selinger. Idempotents in dagger categories. In *Proceedings of the 4th International Workshop on Quantum Programming Languages*, July 2006.