Available online at www.sciencedirect.com
ScienceDirect
Electronic Notes in Theoretical Computer Science 202 (2008) 279–293
Electronic Notes in Theoretical Computer Science
www.elsevier.com/locate/entcs

Integral of Fine Computable functions and Walsh Fourier series

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Abstract

We define the effective integrability of Fine-computable functions and effectivize some fundamental limit theorems in the theory of Lebesgue integral such as Bounded Convergence Theorem and Dominated Convergence Theorem. It is also proved that the Walsh-Fourier coefficients of an effectively integrable Fine-computable function form an $\mathbb{E}$-computable sequence of reals and converge effectively to zero. The latter fact is the effectivization of Walsh-Riemann-Lebesgue Theorem. The article is closed with the effective version of Dirichlet's test.

Keywords: Fine-computable function, Fine convergence, Walsh Fourier series, effective integrability.

1 Introduction

In this article, we make an introductory step to reconstruct effectively the theory of Walsh-Fourier series ([3], [12]). Although Walsh-Fourier series and Haar wavelets have become important tools in digital processing nowadays, it seems that Haar, Rademacher, Walsh, Fine and others had already investigated these subjects in the middle of the twentieth century from mathematical interest. The theory of Walsh-Fourier series is treated similarly to that of Fourier series by replacing trigonometric functions with Walsh functions.

Let $S_n(f)$ be the nth partial sum of the Walsh-Fourier series of a function f. A major problem concerning $S_n(f)$ is to find a sufficient condition for the convergence of $\{S_n(f)\}$ to f. Many types of convergence, such as pointwise convergence, uniform

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[4] This work has been supported in part by Scientific Foundations of JSPS No. 6340028.

doi:10.1016/j.entcs.2008.03.021

convergence, almost everywhere convergence and L^p-convergence, are treated in [3] and [12].

From the standpoint of computable analysis, it is more appropriate if the pointwise convergence of $\{S_n(f)\}$ to f would be replaced by some kind of effective convergence, which is stronger than pointwise convergence. We have adopted "effective Fine-convergence" in [10] for Fine-computable functions.

Our present objective is to effectivize Dirichlet's test for Fine-computable functions with respect to effective Fine-convergence. For this purpose, we need to reformulate integration theory in an effective way and prove effective versions of some fundamental theorems such as Bounded Convergence Theorem and Dominated Convergence Theorem of Lebesgue (Theorems 3.9, 3.13). These are treated in Section 3. A Fine-computable function is Fine-continuous, and hence is $\mathbb{E}$-continuous at dyadic irrationals. This means that such a function is measurable, and so the classical theorems hold for it. Therefore, the effectivization of integration theory is reduced essentially to the replacement of "convergence" by "effective convergence".

In Section 4, we prove $\mathbb{E}$-computability of the indefinite integral and the second mean value theorem (Theorem 4.4) for an effectively integrable Fine-computable function (Theorem 5.12).

In Section 5, we prove the effectivizations of Walsh-Riemann-Lebesgue Theorem (Theorem 5.7) and Dirichlet's Test.

For the reader's convenience, we review some basics of Fine metric, Fine-computable functions and Fine-convergence, and some fundamental theorems of integration.

We assume the knowledge of computability of the real number sequences and the real function sequences with respect to the Euclidean topology. See [11] for details.

2 Preliminaries

The *Fine-metric* on $[0,1)$ was introduced in [2]. It is defined by

$$d_F(x,y) = \textstyle\sum_{k=1}^{\infty} |\sigma_k - \tau_k| 2^{-k}, \tag{1}$$

where, $\sigma_1\sigma_2\cdots$ and $\tau_1\tau_1\cdots$ are dyadic expansions of x and y respectively with infinitely many $0'$s.

A left-closed right-open interval with dyadic rational end points is called a *dyadic interval*. It is easy to see that a dyadic interval is open with respect to the Fine metric.

We use the following notations for special dyadic intervals.

$$I(n,k) = [k\,2^{-n}, (k+1)2^{-n}),\ 0 \leqslant k \leqslant 2^n - 1,$$
$$J(x,n) = \text{such } I(n,k) \text{ that includes } x.$$

We call $I(n,k)$ a *fundamental dyadic interval* (*of order* n) and $J(x,n)$ a *dyadic neighborhood of* x (*of order* n).

The topology generated by $\{J(x,n) \,|\, x \in [0,1), n = 1,2,3,\cdots\}$ is equivalent to that induced by the Fine metric. We call this topological space the *Fine space*. We put "Fine-" to the topological notions with respect to this topology. For topological notions with respect to the usual Euclidean metric, we put prefix "$\mathbb{E}$-".

We cite the following lemma from [10] concerning $I(n,k)$ and $J(x,n)$.

Lemma 2.1 ([10]) *The following three conditions are equivalent for any $x, y \in [0,1)$ and any positive integer n.*

(i) $y \in J(x,n)$.

(ii) $x \in J(y,n)$.

(iii) $J(x,n) = J(y,n)$.

A sequence of dyadic rationals $\{r_n\}$ in $[0,1)$ is called *recursive* if there exist recursive functions $\alpha(n)$ and $\beta(n)$ which satisfy $r_n = \alpha(n)2^{-\beta(n)}$. A double sequence $\{x_{m,n}\}$ in $[0,1)$ is said to *Fine-converge effectively* to a sequence $\{x_m\}$ from $[0,1)$ if there exists a recursive function $\alpha(m,k)$ such that, for all m, k, $x_{m,n} \in J(x_m, k)$ for all $n \geqslant \alpha(m,k)$.

A sequence $\{x_m\}$ in $[0,1)$ is said to be *Fine-computable* if there exists a recursive sequence of dyadic rationals $\{r_{m,n}\}$ which Fine-converges effectively to $\{x_m\}$. For a real number $x \in [0,1)$, Fine-computability is equivalent to $\mathbb{E}$-computability ([1]). In such a case, we put neither "Fine-" nor "$\mathbb{E}$-".

Definition 2.2 (*Uniformly Fine-computable sequence of functions,* [6]) A sequence of functions $\{f_n\}$ is said to be uniformly Fine-computable if (i) and (ii) below hold.

(i) (Sequential Fine-computability) The double sequence $\{f_n(x_m)\}$ is $\mathbb{E}$-computable for any Fine-computable sequence $\{x_m\}$.

(ii) (Effectively uniform Fine-continuity) There exists a recursive function $\alpha(n,k)$ such that, for all n, k and all $x, y \in [0,1)$, $y \in J(x, \alpha(n,k))$ implies $|f_n(x) - f_n(y)| < 2^{-k}$.

The Fine-computability of a single function f is defined by that of the sequence $\{f, f, \ldots\}$.

Notice that the computability of the sequence $\{f_n(x_m)\}$ in (i) is $\mathbb{E}$-computability.

Throughout this article, we fix an effective enumeration of all dyadic rationals in $[0,1)$ and denote it with $\{e_i\}$.

Definition 2.3 (*Effectively uniform convergence of functions,* [6]) A double sequence of functions $\{g_{m,n}\}$ is said to converge effectively uniformly to a sequence of functions $\{f_m\}$ if there exists a recursive function $\alpha(m,k)$ such that, for all m, n and k,

$$n \geqslant \alpha(m,k) \ \text{ implies } \ |g_{m,n}(x) - f_m(x)| < 2^{-k} \ \textit{ for all } \ x.$$

Theorem 2.4 ([6]) *If a uniformly Fine-computable sequence of functions $\{f_n\}$ Fine-converges effectively uniformly to a function f, then f is also uniformly Fine-computable.*

We can treat weakened notions of computability and convergence as follows.

Definition 2.5 (*Locally uniformly Fine-computable sequence of functions,* [7]) A sequence of functions $\{f_n\}$ is said to be locally uniformly Fine-computable if the following (i) and (ii) hold.

(i) $\{f_n\}$ is sequentially Fine-computable.

(ii) (Effectively locally uniform Fine-continuity) There exist recursive functions $\alpha(n,i,k)$ and $\beta(n,i)$ which satisfy the following (ii-a) and (ii-b).

(ii-a) For all i, n and k, $|f_n(x) - f_n(y)| < 2^{-k}$ if $x, y \in J(e_i, \beta(n,i))$ and $y \in J(x, \alpha(n,i,k))$.

(ii-b) $\bigcup_{i=1}^{\infty} J(e_i, \beta(n,i)) = [0,1)$ for each n.

Definition 2.6 (*Effectively locally uniform Fine-convergence,* [7]) A double sequence of functions $\{g_{m,n}\}$ is said to Fine-converge effectively locally uniformly to a sequence of functions $\{f_m\}$ if there exist recursive functions $\alpha(m,i)$ and $\beta(m,i,k)$ such that

(a) $|g_{m,n}(x) - f_m(x)| < 2^{-k}$ for $x \in J(e_i, \alpha(m,i))$ and $n \geq \beta(m,i,k)$,

(b) $\cup_{i=1}^{\infty} J(e_i, \alpha(m,i)) = [0,1)$.

Theorem 2.7 ([7]) *If a locally uniformly Fine-computable sequence of functions $\{f_n\}$ Fine-converges effectively locally uniformly to f, then f is locally uniformly Fine-computable.*

Definition 2.8 (*Fine-computable sequence of functions*) A sequence of functions $\{f_n\}$ is said to be Fine-computable if it satisfies the following.

(i) $\{f_n\}$ is sequentially Fine-computable.

(ii) (Effective Fine-Continuity) There exists a recursive function $\alpha(n,k,i)$ such that

(ii-a) $x \in J(e_i, \alpha(n,k,i))$ implies $|f_n(x) - f_n(e_i)| < 2^{-k}$,

(ii-b) $\bigcup_{i=1}^{\infty} J(e_i, \alpha(n,k,i)) = [0,1)$ for each n, k.

Definition 2.9 (*Effective Fine-convergence of functions*) We say that a double sequence of functions $\{g_{m,n}\}$ Fine-converges effectively to a sequence of functions $\{f_m\}$ if there exist recursive functions $\alpha(m,k,i)$ and $\beta(m,k,i)$, which satisfy

(a) $x \in J(e_i, \alpha(m,k,i))$ and $n \geqslant \beta(m,k,i)$ imply $|g_{m,n}(x) - f_m(x)| < 2^{-k}$,

(b) $\bigcup_{i=1}^{\infty} J(e_i, \alpha(m,k,i)) = [0,1)$ for each m and k.

Definition 2.10 (*Computable sequence of dyadic step functions,* [6]) A sequence of functions $\{\varphi_n\}$ is called a computable sequence of dyadic step functions if there exist a recursive function $\alpha(n)$ and a $\mathbb{E}$-computable sequence of reals $\{c_{n,j}\}$ ($0 \leqslant j < 2^{\alpha(n)}$, $n = 1, 2, \ldots$) such that

$$\varphi_n(x) = \textstyle\sum_{j=0}^{2^{\alpha(n)}-1} c_{n,j} \chi_{I(\alpha(n),j)}(x),$$

where χ_A denotes the indicator (characteristic) function of A.

Proposition 2.11 *Let f be a Fine-computable function. Define a computable sequence of dyadic step functions $\{\varphi_n\}$ by*

$$\varphi_n(x) = \textstyle\sum_{j=0}^{2^n-1} f(j2^{-n}) \chi_{I(n,j)}(x). \tag{2}$$

Then $\{\varphi_n\}$ Fine-converges effectively to f.

Remark 2.12 If f is uniformly Fine-computable or locally uniformly Fine-computable, then the convergence can be replaced by the effectively uniform Fine-convergence or the effectively locally uniform Fine-convergence respectively([7,6]).

Theorem 2.13 ([10]) *If a Fine-computable sequence of functions $\{f_n\}$ Fine-converges effectively to f, then f is Fine-computable.*

Now, we review the theory of Lebesgue integral for functions on $[0, 1)$. In the following, we will say simply "measurable" or "integrable" instead of "Lebesgue measurable" or "Lebesgue integrable" respectively.

A function $\varphi(x)$ is called a simple function if it is represented as a finite linear combination of indicator functions of some measurable sets, that is, if $\varphi(x) = \sum_{i=0}^{n-1} a_i \chi_{E_i}(x)$, where a_i's are real numbers and E_i's are mutually disjoint measurable sets satisfying $\bigcup_{i=0}^{n-1} E_i = [0, 1)$. The integral $\int_0^1 \varphi dx$ is defined by $\sum_{i=0}^{n-1} a_i |E_i|$, where $|E_i|$ is the Lebesgue measure of the set E_i.

For a bounded measurable function f, there exists a sequence of simple functions $\{\varphi_n\}$ which converges pointwise to f. In this case, $\int_0^1 \varphi_n dx$ converges and we denote this limit as $\int_0^1 f dx$. It holds that, if $\{\psi_n\}$ is another approximating sequence of simple functions of f, then $\lim_{n\to\infty} \int_0^1 \varphi_n dx = \lim_{n\to\infty} \int_0^1 \psi_n dx$ and hence the above definition is sound.

For a positive function f, we say that f is integrable if the limit $\lim_{n\to\infty} \int_0^1 f \wedge 2^n dx$ exists, and we denote this limit as $\int_0^1 f dx$, where $(f \wedge 2^n)(x) = \min\{f(x), 2^n\}$. A general measurable function f is called integrable if $f^+ = f \vee 0$ and $f^- = (-f) \vee 0$ are both integrable. We define $\int_0^1 f dx = \int_0^1 f^+ dx - \int_0^1 f^- dx$.

For the reader's convenience, we cite two fundamental theorems from [4] and [5].

Theorem 2.14 *A bounded function f is Riemann integrable if and only if the Lebesgue measure of the set of all $\mathbb{E}$-discontinuous points is zero. In this case, f is also Lebesgue integrable and the both integrals have the same value.*

From Theorem 2.14, a bounded Fine-computable function is Riemann integrable and also Lebesgue integrable.

Theorem 2.15 (Bounded convergence theorem) *Let $\{f_n\}$ be a uniformly bounded sequence of measurable functions which converges pointwise to a function f. Then $\lim_{n\to\infty} \int_0^1 f_n dx = \int_0^1 f dx$. (Uniformly boundedness means that there exists a constant M such that $|f_n(x)| \leqslant M$ for all n and x.)*

Theorem 2.16 (Dominated convergence theorem) *Let $\{f_n\}$ be a sequence of integrable functions which converges pointwise to a function f. Suppose further that there exists an integrable function g such that $|f_n(x)| \leqslant g(x)$ for all n and x. Then, f is integrable and $\lim_{n\to\infty} \int_0^1 f_n dx = \int_0^1 f dx$.*

3 Effective integrability of Fine-computable functions

In this section, we discuss the effective computability of integrals for Fine-computable functions on the Fine space. The main objective is the effectivization of Theorems 2.15 and 2.16. A Fine-continuous function is $\mathbb{E}$-continuous at every dyadic irrational, and so the Lebesgue measure of the set of all $\mathbb{E}$-discontinuous points is zero, and hence Theorems 2.15 and 2.16 are valid for Fine-computable functions. Therefore the proofs of effectivizations of these theorems are reduced to effective convergence. Since the Fine space does not include the point 1, we write the integral of f on $[0, 1)$ as $\int_{[0,1)} f(x)dx$ rather than $\int_0^1 f(x)dx$ or $\int_0^{1-0} f(x)dx$.

In classical calculus, integration is defined first for bounded functions, next for nonnegative functions and finally for general functions as in Section 2. So, we define effective integrability of Fine-computable functions first for bounded Fine-computable functions, next for nonnegative Fine-computable functions and finally for Fine-computable functions. We note that $f^+ = f \vee 0$ and $f^- = (-f) \vee 0$ are Fine-computable if f is Fine-computable.

Definition 3.1 (*Effective integrability of a function*)

(i) A bounded Fine-computable function f is called effectively integrable if $\int_{[0,1)} f(x)dx$ is a computable number.

(ii) A nonnegative Fine-computable function f is said to be effectively integrable if it is integrable and $\int_{[0,1)} f(x)dx$ is a computable number.

(iii) A Fine-computable function f is called effectively integrable if it is integrable and $\int_{[0,1)} f^+(x)dx$ and $\int_{[0,1)} f^-(x)dx$ are computable numbers.

We also need these definitions for a sequence of functions.

Definition 3.2 (*Effective integrability of a sequence of functions*)

(i) A sequence of bounded Fine-computable functions $\{f_n\}$ is said to be effectively integrable if each f_n is integrable and the sequence $\{\int_{[0,1)} f_n(x)dx\}$ forms an $\mathbb{E}$-computable sequence of reals.

(ii) A sequence of nonnegative Fine-computable functions is said to be effectively integrable if each f_n is integrable and $\{\int_{[0,1)} f_n(x)dx\}$ is an $\mathbb{E}$-computable sequence of reals.

(iii) A sequence of Fine-computable functions $\{f_n\}$ is called effectively integrable if each f_n is integrable and $\{\int_{[0,1)} f_n^+(x)dx\}$ and $\{\int_{[0,1)} f_n^-(x)dx\}$ are $\mathbb{E}$-computable sequences of real numbers.

Definition 3.3 (i) Let E be a finite union of dyadic intervals. Then, a Fine-computable function f is said to be effectively integrable on E if f is integrable on E and $\int_E f(x)dx = \int_{[0,1)} \chi_E(x)f(x)dx$ is a computable number.

(ii) Suppose that $\{E_m\}$ is a computable sequence of finite unions of dyadic intervals, that is, there exists a recursive function $\alpha(m)$ and recursive sequences of dyadic rationals $\{a(m, i)\}$ and $\{b(m, i)\}$ such that $E_m = \bigcup_{i=1}^{\alpha(m)} [a(m, i), b(m, i))$. Then, a Fine-computable function f is said to be effectively integrable on $\{E_m\}$ if f is integrable on each E_m and $\{\int_{E_m} f(x)dx\}$ is an $\mathbb{E}$-computable sequence of reals.

(iii) Effective integrability of a sequence of Fine-computable functions on E and $\{E_n\}$ are defined similarly.

For a computable dyadic step function φ of the form $\varphi(x) = \sum_{i=0}^{2^k-1} c_i \chi_{[i2^{-k},(i+1)2^{-k})}(x)$, its integral $\int_{[0,1)} \varphi(x)dx$ is equal to $2^{-k}\sum_{i=0}^{2^k-1} c_i$. $\{\int_{[0,1)} \varphi_n(x)dx\}$ is hence an $\mathbb{E}$-computable sequence of reals if $\{\varphi_n\}$ is a computable sequence of dyadic step functions due to Definition 2.10.

For a uniformly Fine-computable function, the following theorem is essentially proved in the proof of Proposition 4.5 in [8].

Theorem 3.4 *A uniformly Fine-computable function is effectively integrable.*

The proof goes as follows. Let f be uniformly Fine-computable and let $\{\varphi_n\}$ be an approximating computable sequence of dyadic step functions defined by Equation (2). Then f is bounded and integrable. In addition, $\int_{[0,1)} \varphi_n(x)dx$ $\mathbb{E}$-converges effectively uniformly to $\int_{[0,1)} f(x)dx$, since

$$|\int_{[0,1)} \varphi_n(x)dx - \int_{[0,1)} f(x)dx| \leqslant \int_{[0,1)} |\varphi_n(x) - f(x)|dx \leqslant \sup_{x\in[0,1)} |\varphi_n(x) - f(x)|.$$

Therefore $\int_{[0,1)} f(x)dx$ is computable.

For a locally uniformly Fine-computable function, we have the following counter-example.

Example 3.5 (Brattka [1]) Let α be an injective recursive function whose range is not recursive. Define

$$\varphi(x) = 2^k 2^{-\alpha(k)} \text{ if } 1 - 2^{-(k-1)} \leqslant x < 1 - 2^{-k}, k = 1, 2, \ldots.$$

Then φ is locally uniformly Fine-computable but $\int_{[0,1)} \varphi(x)dx = \sum_{k=1}^{\infty} 2^{-\alpha(k)}$ is not computable.

Let us further note the following. Define

$$\varphi_n(x) = \begin{cases} 2^k 2^{-\alpha(k)} & \text{if } 1 - 2^{-(k-1)} \leqslant x < 1 - 2^{-k}, k = 1, 2, \ldots, n \\ 0 & \text{if } x \geqslant 1 - 2^{-n} \end{cases}$$

Then $\{\varphi_n\}$ is effectively integrable.

Classically, $\{\int_{[0,1)} \varphi_n\}$ $\mathbb{E}$-converges to $\int_{[0,1)} \varphi(x)dx$, but the convergence is not effective.

This counter-example shows that the requirement on the computability of the integral is not redundant in the definition of effective integrability of a Fine-computable function.

The next example shows that the computability of $\int_{[0,1)} f(x)dx$ is generally not sufficient for effective integrability.

Example 3.6 Let α be an injective recursive function whose range is not recursive. Put

$$\varphi(x) = \begin{cases} 2^k 2^{-\alpha(k)} \text{ if } 1 - 2^{-(2k-2)} \leqslant x < 1 - 2^{-(2k-1)} \\ -2^k 2^{-\alpha(k)} \text{ if } 1 - 2^{-(2k-1)} \leqslant x < 1 - 2^{-2k} \end{cases} \quad (k = 1, 2, \ldots).$$

Then φ^+ and φ^- are not effectively integrable but $\int_{[0,1)} \varphi(x)dx = 0$.

We need the following lemmas.

Lemma 3.7 (Monotone convergence [11]) *Let $\{x_{n,k}\}$ be an $\mathbb{E}$-computable sequence of reals which $\mathbb{E}$-converges monotonically to $\{x_n\}$ as k tends to infinity for each n. Then $\{x_n\}$ is $\mathbb{E}$-computable if and only if the $\mathbb{E}$-convergence is effective.*

Lemma 3.8 *$|A|$ will denote the Lebesgue measure of a set A.*

Let $\{[a_k, b_k)\}$ be a recursive sequence of dyadic intervals, that is, $\{a_k\}$ and $\{b_k\}$ are recursive sequence of dyadic rationals. If we define $E_n = \bigcup_{k=1}^{n}[a_k, b_k)$, then $\{|E_n|\}$ is an $\mathbb{E}$-computable sequence of reals. Assume that $\{E_n\}$ converges to $[0,1)$, i.e. $\bigcup_{k=1}^{\infty}[a_k, b_k) = [0,1)$. Then $\{|E_n|\}$ $\mathbb{E}$-converges effectively, i.e., there exists a recursive function $\alpha(p)$ such that $|E_n| > 1 - 2^{-p}$ (or $|E_n^C| < 2^{-p}$) for $n \geqslant \alpha(p)$, where A^C denotes the complement of a set A.

Proof For a dyadic interval $[a,b)$, $|[a,b)| = b - a$. E_n can be represented as the union of finite mutually disjoint dyadic intervals whose ends-points are determined effectively from a_k's and b_k's. Therefore, $\{|E_n|\}$ is an $\mathbb{E}$-computable sequence of reals and $\mathbb{E}$-converges monotonically to 1. □

Theorem 3.9 (Effective bounded convergence theorem) *Let $\{g_n\}$ be a bounded Fine-computable sequence of functions which is effectively integrable and Fine-converges effectively to f. Then, f is Fine-computable and $\{\int_{[0,1)} g_n(x)dx\}$ $\mathbb{E}$-converges effectively to $\int_{[0,1)} f(x)dx$. As a consequence, f is integrable.*

Proof Suppose that $\{g_n\}$ Fine-converges effectively to f with respect to $\alpha(i,k)$ and $\beta(i,k)$ and, for some integer M, $|g_n(x)| \leqslant M$. Then Theorem 2.13 yields that f is Fine-computable. Since $\{g_n(x)\}$ $\mathbb{E}$-converges to $f(x)$, $|f(x)| \leqslant M$ holds, and $\{\int_{[0,1)} g_n(x)dx\}$ $\mathbb{E}$-converges to $\int_{[0,1)} f(x)dx$ by virtue of Theorem 2.15.

We denote $\bigcup_{i=1}^{m} J(e_i, \alpha(i,k))$ with $E_{k,m}$. By Definition 2.9, $\bigcup_{m=1}^{\infty} E_{k,m} = [0,1)$. So, for each k, we can find effectively an $m = m(k)$ such that $|E_{k,m}| > 1 - 1/(2^{(k+2)}M)$ from Lemma 3.8. If we take $n \geq \delta(k) = \max\{\alpha(1,k+1), \alpha(2,k+1), \ldots, \alpha(m(k), k+1)\}$, then

$$\int_{[0,1)} |g_n(x) - f(x)|dx \leqslant \int_{E_{k,m}} |g_n(x) - f(x)|dx + \int_{E_{k,m}^C} |g_n(x)| + \int_{E_{k,m}^C} |f(x)|dx$$
$$< 2^{-(k+1)} + 2^{-(k+2)} + 2^{-(k+2)} = 2^{-k}.$$
□

If f is a bounded Fine-computable function, then the sequence of dyadic step functions $\{\varphi_n\}$ defined by Equation 2 is also bounded. So, the assumption of Theorem 3.9 holds for f and $\{\varphi_n\}$, and we obtain the following theorem.

Theorem 3.10 *A bounded Fine-computable function is effectively integrable.*

From the definition of Lebesgue integral and Lemma 3.7, we obtain the following proposition.

Let us here note that, if f is Fine-computable, then $\{f \wedge 2^n\}$ is a Fine-computable sequence of functions.

Proposition 3.11 *Let f be a nonnegative integrable Fine-computable function. Then f is effectively integrable if and only $\{\int_{[0,1)} f \wedge 2^n\}$ $\mathbb{E}$-converges effectively to $\{\int_{[0,1)} f(x)dx\}$.*

Proposition 3.12 *Let f be an effectively integrable Fine-computable function and let I_n be a sequence of dyadic intervals such that $\bigcup_{n=1}^{\infty} I_n = [0,1)$. Put $E_n = \bigcup_{i=0}^{n} I_i$. Then, $\int_{E_n} f(x)dx$ $\mathbb{E}$-converges effectively to $\int_{[0,1)} f(x)dx$, or equivalently, $\int_{E_n^C} f(x)dx$ $\mathbb{E}$-converges effective to zero.*

Proof. Since $|\int_E f(x)dx| \leqslant \int_E f^+(x)dx + \int_E f^-(x)dx$, it is sufficient to prove the case where f is nonnegative. Put $f_n = f \wedge 2^n$. Then $\int_{[0,1)} f_n(x)dx$ $\mathbb{E}$-converges effectively to $\int_{[0,1)} f(x)dx$ due to Proposition 3.11. Hence, there exists a recursive function $\beta(k)$ such that $n \geq \beta(k)$ implies $0 \leqslant \int_{[0,1)} f(x)dx - \int_{[0,1)} f_n(x)dx < 2^{-k}$. In particular, we get

$$0 \leqslant \int_{[0,1)} f(x)dx - \int_{[0,1)} f_{\beta(k+1)}(x)dx < 2^{-(k+1)}.$$

By virtue of Lemma 3.8, there exists a recursive function $\delta(k)$ such that $m \geq \delta(k)$ implies $|E_m^C| < 2^{-k}$. If we take $m \geq \delta(\beta(k+1)+k+1)$, then

$$\begin{aligned}
&\int_{[0,1)} f(x)dx - \int_{E_m} f(x)dx = \int_{E_m^C} f(x)dx \\
&\leqslant \int_{E_m^C} (f(x) - f_{\beta(k+1)}(x))dx + \int_{E_m^C} f_{\beta(k+1)}(x)dx \\
&\leqslant \int_{[0,1)} (f(x) - f_{\beta(k+1)}(x))dx + \int_{E_m^C} f_{\beta(k+1)}(x)dx \\
&< 2^{-(k+1)} + 2^{\beta(k+1)} 2^{-(\beta(k+1)+k+1)} = 2^{-k}.
\end{aligned}$$

□

Theorem 3.13 (Effective dominated convergence theorem) *Let $\{g_n\}$ be an effectively integrable Fine-computable sequence which Fine-converges effectively to f. Suppose that there exists an effectively integrable Fine-computable function h such that $|g_n(x)| \leqslant h(x)$. Then, $\{\int_{[0,1)} g_n(x)dx\}$ $\mathbb{E}$-converges effectively to $\int_{[0,1)} f(x)dx$.*

Proof. From Theorem 2.16, f is integrable and $\{\int_{[0,1)} g_n(x)dx\}$ $\mathbb{E}$-converges to $\int_{[0,1)} f(x)dx$. It also holds that $|f(x)| \leqslant h(x)$.

Suppose that $\{g_n\}$ Fine-converges effectively to f with respect to $\alpha(k,i)$ and $\beta(k,i)$. Then,

$$x \in J(e_i, \alpha(k,i)) \text{ and } n \geqslant \beta(k,i) \text{ imply } |g_n(x) - f(x)| < 2^{-k},$$

$$\bigcup_{i=1}^{\infty} J(e_i, \alpha(k,i)) = [0,1) \text{ for each } k.$$

Put $I_i = J(e_i, \alpha(k+1,i))$ and $E_m = \bigcup_{i=1}^{m} I_i$. From Proposition 3.12, we can obtain a recursive function $\delta(k)$ which satisfies that $\int_{E_m^C} h(x)dx < 2^{-k}$ for $m \geqslant \delta(k)$.

Suppose that $n \geqslant \max\{\beta(k+1,1), \ldots, \beta(k+1, \delta(k+2))\}$. Then

$$\begin{aligned}
&|\int_{[0,1)} g_n(x)dx - \int_{[0,1)} f(x)dx| \\
&\leqslant \int_{E_{\delta(k+2)}} |g_n(x) - f(x)|dx + \int_{(E_{\delta(k+2)})^C} |g_n(x)|dx + \int_{(E_{\delta(k+2)})^C} |f(x)|dx \\
&\leqslant 2^{-(k+1)} + 2\, 2^{-(k+2)} = 2^{-k}
\end{aligned}$$

□

We have stated and proved the theorems and the propositions for a single Fine-computable function up to now. We can easily extend them to the case of a Fine-computable sequence of functions.

The sequentializations of Theorems 3.9, 3.10 and 3.13 can be stated as follows.

Theorem 3.14 (Sequential effective bounded convergence theorem) *Let $\{f_n\}$ be a Fine-computable sequence and let $\{g_{n,m}\}$ be a bounded Fine-computable sequence of functions which is effectively integrable and Fine-converges effectively to $\{f_n\}$. Assume also that there exists an $\mathbb{E}$-computable sequence of reals $\{M_n\}$ such that $|g_{n,m}(x)| \leqslant M_n$. Then, $\{\int_{[0,1)} g_{n,m}(x)dx\}$ $\mathbb{E}$-converges effectively to $\{\int_{[0,1)} f_n(x)dx\}$.*

Theorem 3.15 *Let $\{f_n\}$ be Fine-computable and effectively bounded, that is, there exists an $\mathbb{E}$-computable sequence of reals $\{M_n\}$ such that $|f_n(x)| \leqslant M_n$. Then $\{f_n\}$ is effectively integrable.*

Theorem 3.16 (Sequential effective dominated convergence theorem) *Let $\{g_{m,n}\}$ be an effectively integrable Fine-computable sequence which Fine-converges effectively to $\{f_m\}$. Suppose that there exists an effectively integrable Fine-computable sequence $\{h_m\}$ such that $|g_{m,n}(x)| \leqslant h_m(x)$. Then, $\{\int_{[0,1)} g_{m,n}(x)dx\}$ $\mathbb{E}$-converges effectively to $\int_{[0,1)} f_m(x)dx$.*

4 Indefinite integral and mean value theorem

In this section, we consider $\mathbb{E}$-computability of the indefinite integral $\int_0^x f(x)dx$ ($x \in [0,1]$) and the second mean value theorem for a Fine-computable and effectively integrable function f.

We can prove The following fundamental fact by effectivising the classical proof.

Theorem 4.1 *Suppose $\{f_n\}$ is a Fine-computable and effectively integrable sequence of functions. Define $F_n(x) = \int_0^x f_n(x)dx$. Then $\{F_n\}$ is a uniformly $\mathbb{E}$-computable sequence of functions on $[0,1]$.*

Theorem 4.2 (Effective intermediate value theorem, Theorem 8 in Section 0.6 of [11]) *Let $[a,b]$ be an interval with computable endpoints, and let f be an $\mathbb{E}$-computable function on $[a,b]$ such that $f(a) < f(b)$. Let s be a computable real with $f(a) < s < f(b)$. Then there exists a computable point c in (a,b) such that $f(c) = s$.*

It is pointed out in [11] that the sequential version of Theorem 4.2 does not hold.

We can prove the following variation of Theorem 4.2.

Theorem 4.2′ *Let $[a,b]$ be an interval with rational endpoints, and let f be $\mathbb{E}$-computable on $[a,b]$. Put $m = \min_{x\in[a,b]} f(x)$ and $M = \max_{x\in[a,b]} f(x)$ and assume $m < M$. For a computable real number s with $m < s < M$, there exists a computable point c in (a,b) such that $f(c) = s$.*

Proof. Define

$m_n = \min_{0\leqslant i\leqslant n-1}\{f(a+i(b-a)/n)\}$ and $M_n = \max_{0\leqslant i\leqslant n-1}\{f(a+i(b-a)/n)\}$.

Then $\{m_n\}$ is $\mathbb{E}$-computable and $\mathbb{E}$-converges effectively to m, and $\{M_n\}$ is also $\mathbb{E}$-computable and $\mathbb{E}$-converges effectively to M ([11]).

Suppose s is a computable real number such that $m < s < M$. Then one can find effectively n_1 and n_2 such that $m_{n_1} < s < M_{n_2}$. For such n_1 and n_2, we can find effectively $i_1 < n_1$ and $i_2 < n_2$ satisfying the following conditions. If we put $x_{n_1} = a + i_1(b-a)/n_1$ and $y_{n_2} = a + i_2(b-a)/n_2$, then $s > f(x_{n_1})\,(\geqslant m_{n_1})$ and $s < f(y_{n_2}) \leqslant (M_{n_2})$ hold. If we apply Theorem 4.2 to the interval $[x_{n_1}, y_{n_2}]$ (or $[y_{n_2}, x_{n_1}]$), we obtain the desired c. □

Since a Fine-computable function may be $\mathbb{E}$-discontinuous, the (first) mean value theorem does not hold. On the other hand, the second mean value theorem applies also to some class of $\mathbb{E}$-discontinuous functions. For the proof of an effectivization of this theorem for fine-computable functions, we need the following proposition, which can be proved easily following the classical proof.

Proposition 4.3 *Let f be Fine-computable and effectively integrable, and let g be bounded and Fine-computable. Then fg is also effectively integrable.*

We can prove Effective intermediate value theorem by modifying the classical proof.

Theorem 4.4 (Effective second mean value theorem) *Let f be Fine-computable and effectively integrable. Suppose that a and b are dyadic rationals satisfying $0 \leqslant a < b < 1$.*

(i) *Let g be Fine-computable, nonnegative and strictly decreasing. Then, there exists a computable point $c \in [a, b]$ which satisfies*

$$\int_a^b g(t)f(t)dt = g(a)\int_a^c f(t)dt. \tag{3}$$

(ii) *If g is Fine-computable and strictly monotone, then there exists a computable point $c \in [a, b]$ which satisfies*

$$\int_a^b g(t)f(t)dt = g(a)\int_a^c f(t)dx + g(b)\int_c^b f(t)dt. \tag{4}$$

5 Effective Fine-convergence of Walsh-Fourier series

The system of Walsh functions $\{w_n\}$ is defined on $[0, 1)$ by

$$w_n(x) = (-1)^{\sum_{i=0}^{k} \sigma_{i+1} n_{-i}}, \tag{5}$$

where, $\sigma_1\sigma_2\cdots$ is the dyadic expansion of x with infinitely many $0'$s and $n = n_0 + n_{-1}2 + \cdots + n_{-k}2^k$ is the dyadic expansion of a positive integer n.

It can be easily shown that $\{w_n\}$ is a Fine-computable sequence of functions, and that, if f is Fine-computable and effectively integrable, then so is the sequence $\{fw_n\}$.

Theorem 5.1 (Computability of Walsh-Fourier coefficients) *If f is Fine-computable and effectively integrable, then the sequence of Walsh-Fourier coefficients $\{\int_{[0,1)} f(x)w_n(x)dx\}_{n=0}^{\infty}$ is an $\mathbb{E}$-computable sequence of reals.*

Proof. Put $f_n(x) = (f(x) \wedge 2^n) \vee (-2^n)$.

The sequence of Fine-computable functions $\{f_n w_m\}$ satisfies the assumption of Theorem 3.15. So $\{\int_{[0,1)} f_n(x)w_m(x)dx\}$ is an $\mathbb{E}$-computable (double) sequence of reals.

Then, $\{g_{m,n}\} = \{f_n w_m\}$ satisfies the assumption of Theorem 3.16 with $h_m(x) = f(x)$ and Fine-converges effectively to $f w_m$. So, $\{\int_{[0,1)} f_n(x)w_m(x)dx\}$ $\mathbb{E}$-converges effectively to $\{\int_{[0,1)} f(x)w_m(x)dx\}$ and hence the latter is an $\mathbb{E}$-computable sequence of reals. □

Definition 5.2 The partial sum $S_n(f)$ and modified Dirichlet kernel $D_n(x,t)$ are defined by

$$S_n(f)(x) = \textstyle\sum_{i=0}^{n-1} c_i w_i(x),\ D_n(x,t) = \sum_{i=0}^{n-1} w_i(x)w_i(t),$$

where $\{c_i\}$ is the Walsh-Fourier coefficients of f, i.e. $c_i = \int_{[0,1)} f(t)w_i(t)dt$.

It is well known that

$$S_n(f)(x) = \int_{[0,1)} f(t)D_n(x,t)dt. \tag{6}$$

Remark 5.3 In the theory of classical Walsh-Fourier series, $D_n(x \oplus t)$ is usually used instead of $D_n(x,t)$, where $D_n(x) = D_n(x,0) = D_n(0,x)$ ([3], [12]). Since the *dyadic sum* $x \oplus t$ is not defined for all x and t in $[0,1)$, we do not use the dyadic sum $x \oplus t$.

Lemma 5.4 (Paley) ([3])

$$D_{2^n}(x,t) = \begin{cases} 2^n & \text{if } \ t \in J(x,n) \\ 0 & \text{otherwise} \end{cases}.$$

Notice that $I(m,k) = [k2^{-m},(k+1)2^{-m}) = I(m+1,2k) \cup I(m+1,2k+2)$.

Lemma 5.5 *If φ is constant on $I(m,k)$ for some k and $2^m \leqslant n$, then* $\int_{I(m,k)} \varphi(x)w_n(x)dx = 0$.

Proof First we note that $|w_n(x)| = 1$.

It is sufficient to prove in the case $2^m \leqslant n < 2^{m+1}$. In this case, $w_n(x)$ is constant on $I(m+1,i)$ for each i $(0 \leqslant i < 2^{m+1})$. Moreover, the sine of $w_n(x)$ on $I(m+1,2k)$ is opposite to that on $I(m+1,2k+1)$ for each j $(0 \leqslant j < 2^m)$. So lemma follows. □

We can prove the following Theorems in a manner similar to the proof of Proposition 4.5 in [6]. The Fine-convergence of $\{S_{2^n} f\}$ can be proved similarly to the proof of Proposition 4.5 in [6] using the Paley's lemma.

Theorem 5.6 *If f is Fine-computable and effectively integrable, then $S_{2^n} f$ Fine-converges effectively to f.*

Proof. Recall that

$$S_{2^n} f(x) = \int_{[0,1)} f(t)D_{2^n}(x,t)dt = \int_{J(x,n)} f(t)D_{2^n}(x,t)dt.$$

Now, from Paley's Lemma,

$$S_{2^n}f(x) - f(x) = \int_{J(x,n)}(f(t)D_{2^n}(x,t) - 2^n f(x))dt = 2^n \int_{J(x,n)}(f(t) - f(x))dt.$$

Suppose that f is Fine-continuous with respect to $\gamma(k,i)$. If $x \in J(e_i, \gamma(k+1,i))$ and $n \geqslant \gamma(k+1,i)$, then $t \in J(e_i, \gamma(k+1,i))$ for $t \in J(x,n)$. In this case, we obtain

$$|f(t) - f(x)| \leqslant |f(t) - f(e_i)| + |f(e_i) - f(x)| < 2^{-k}.$$

Hence, we obtain $|S_{2^n}f(x) - f(x)| < 2^{-k}$. If we define $\alpha(k,i) = \gamma(k+1,i)$ and $\beta(k,i) = \gamma(k+1,i)$, then $S_{2^n}(f)$ Fine-converges effectively to f with respect to α and β. □

The effective version of the Walsh-Riemann-Lebesgue theorem ([12]) can be stated and proved as follows.

Theorem 5.7 (Effective Walsh-Riemann-Lebesgue theorem) *If f is Fine-computable and effectively integrable, then its Walsh-Fourier coefficients $\{c_n\}$ $\mathbb{E}$-converges effectively to zero.*

Proof. (i) First, we assume that f is bounded. Let $\{\varphi_m\}$ be the approximating sequence of dyadic step functions defined by Equation (2) and put $d_{m,n} = \int_{[0,1)} \varphi_m(x)w_n(x)dx$. Then $d_{m,n} = 0$ if $n \geqslant 2^m$ by Lemma 5.5, since φ_m is constant on each $I(m,k)$.

On the other hand $\{d_{m,n}\}$ is $\mathbb{E}$-computable by Theorem 3.15 and

$$|d_{m,n} - c_n| = |\int_{[0,1)}(\varphi_m(x) - f(x))w_n(x)dx| \leqslant \int_{[0,1)} |\varphi_m(x) - f(x)|dx.$$

The right-hand side $\mathbb{E}$-converges effectively to zero by Theorem 3.9. So, $\{d_{m,n}\}$ $\mathbb{E}$-converges effectively to $\{c_n\}$ as m tends to infinity uniformly in n. This means that there exists a recursive function γ such $m \geqslant \gamma(k)$ implies $|d_{m,n} - c_n| < 2^{-k}$ for all n.

Suppose that $n \geqslant \gamma(k)$, then we obtain that $d_{\gamma(k),n} = 0$ and hence $|c_n| < 2^{-k}$. This proves that $\{c_n\}$ $\mathbb{E}$-converges effectively to zero.

(ii) For a general f, $f = f^+ - f^-$ and $c_n = \int_0^1 f^+(x)w_n(x)dx - \int_0^1 f^-(x)w_n(x)dx$. Therefore, it is sufficient to prove the case where f is nonnegative.

Put $f_\ell = f \wedge 2^\ell$ and $c_{\ell,n} = \int_{[0,1)} f_\ell(x)w_n(x)dx$. $\{f_\ell w_n\}$ is Fine-computable and effectively integrable as a double sequence of functions and $|f_\ell w_n| \leqslant 1$. $\{c_{\ell,n}\}$ is $\mathbb{E}$-computable by an extended version of Theorem 3.15.

Notice that the proof of (i) can be modified for effectively bounded sequence of functions $\{f_\ell\}$. This means that there exists a recursive function $\delta(\ell,k)$ such that $n \geqslant \delta(\ell,k)$ implies $|c_{\ell,n}| < 2^{-k}$. Similarly to (i), we obtain

$$|c_{\ell,n} - c_n| = |\int_{[0,1)}(f_\ell(x) - f(x))w_n(x)dx| \leqslant \int_{[0,1)}(f(x) - f_\ell(x))dx.$$

The right-hand side $\mathbb{E}$-converges effectively to zero by Proposition 3.11. So, $\{c_{\ell,n}\}$ $\mathbb{E}$-converges effectively to $\{c_n\}$ as ℓ tends to infinity uniformly in n. Let β be the modulus of this convergence. Then it holds that $\ell \geqslant \beta(k)$ implies $|c_{\ell,n} - c_n| < 2^{-k}$ for all n.

If $n \geqslant \delta(\beta(k+1), k+1)$, then

$$|c_n| \leqslant |c_{\beta(k+1),n} - c_n| + |c_{\beta(k+1),n}| < 2^{-(k+1)} + 2^{-(k+1)} = 2^{-k}.$$

This proves that $\{c_n\}$ $\mathbb{E}$-converges effectively to zero with respect to $\alpha(k) = \delta(\beta(k+1), k+1)$. □

To prove an effective version of Dirichlet's test, we need the following two lemmas. The second one is the effectivization of the fundamental lemma which is used in proving pointwise convergence of the partial sums $S_n(f)$ to f (cf. [12]).

Lemma 5.8 ([12]) *$D_n(x,t) = w_n(x)w_n(t)\sum_{j=0}^{N} n_{-j}\phi_j(x)\phi_j(t)D_{2^j}(x,t)$, where $\phi_j(x)$ is the j-th Radmacher function and $n = n_0 + n_1 2 + \cdots + n_N 2^N$ is the dyadic expansion of n.*

Prior to the next lemma, let us make the following remark. In the classical case, one can use $w_j(x \oplus t)$ instead of $w_j(x)w_j(t)$, and this fact leads us to the desired conclusion quickly. Here, however, we cannot use $w_j(x \oplus t)$, and hence we need some elaborate work.

In a similar way to the proof of Theorem 5.7, we can prove the following key lemma.

Lemma 5.9 (Key lemma) *If f is Fine-computable and effectively integrable, then $F_{M,n}(x) = \int_{[0,1)\setminus J(x,M)} f(t)D_n(x,t)dt$ Fine-converges effectively to zero as n tends to infinity, effectively in M uniformly in x. This means that there exists a recursive function $\alpha(M,k)$ which satisfies that $n \geqslant \alpha(M,k)$ implies $|F_{M,n}(x)| < 2^{-k}$.*

Before we treat the final objective, the effectivization of the Dirichlet's test, we study the computability of the variation of a Fine-computable function.

Zheng, Rettinger and Braunmühl investigated functions of bounded variation and Jordan decomposability ([15]). They showed an example that is effectively absolutely continuous but not effectively Jordan decomposable.

Subsequently $V_0^x(f)$ denotes the variation of f in $[0,x]$ $(0 \leqslant x < 1)$. $V_0^1(x)$ is defined to be $\sup_{0\leqslant x<1} V_0^x(f)$.

The following example is a modification of Proposition 4.2 in [10].

Example 5.10 Let α be an injective recursive function whose range is not recursive. Define

$$f(x) = e^{-\alpha(n)} \text{ if } \tfrac{1}{2} - 2^{-n} \leqslant x < \tfrac{1}{2} - 2^{-(n+1)} \ (n = 1,2,\ldots).$$

Then $V_0^x(f) = \sum_{n=1}^{\infty} e^{-\alpha(n)}$ for $x \geqslant \frac{1}{2}$, and $\sum_{n=1}^{\infty} e^{-\alpha(n)}$ is not computable.

According to Example 5.10, sequential computability of the variation fails. However, we can prove easily effective Fine-continuity of $V_0^x(f)$ if it is finite.

Definition 5.11 (*Jordan decomposability*) A Fine-computable function is said to be effectively Jordan decomposable if there exist monotone increasing Fine-computable functions ψ_1 and ψ_2 such that $f = \psi_1 - \psi_2$.

Theorem 5.12 (Effective Dirichlet's test) *Let f be Fine-computable, effectively integrable and effectively Jordan decomposable. Then $\{S_n(f)\}$ Fine-converges effectively to f.*

Keeping in mind that the following estimates hold (classically, [12]);

$$\sup_{x,y\in[0,1),n} |\int_{[0,y)} D_n(x,t)dt| \leqslant 2,$$
$$\int_{J(x,M)}(f(t)-f(x))D_n(x,t)dt \leqslant 4\sup_{t\in J(x,M)}|f(t)-f(x)|,$$

we can prove the above theorem.

In Theorems 5.6 and 5.12, we can replace "Fine-convergence" to "uniform Fine-convergence" if f is uniformly Fine-computable, and to "locally uniform Fine-convergence" if f is locally uniformly Fine-computable.

Theorems 5.6, 5.7 and Lemmas 5.4, 5.9 are effectivizations of corresponding classical Theorems and Lemmas. So the following classical version of Theorem 5.12 holds. (See [10] for terminologies.)

Theorem 5.13 *If f is Fine-continuous, integrable and of bounded variation, then $S_n(f)$ Fine-converges to f.*

It is pointed out in [10] that Fine-convergence is weaker than locally uniform Fine-convergence and stronger than pointwise convergence. For a sequence of Fine-continuous functions, Fine-convergence is equivalent to continuous convergence.

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