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Discretized Fractional Calculus with a Series of Chebyshev Polynomial

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Abstract

In this paper, we tried to evaluate the fractional derivatives by using the Chebyshev series expansion. We discuss the indefinite quadrature rule to estimate the fractional derivatives of Riemann-Liouville type.

Keywords: Fractional derivative, Chebyshev series expansion, Quadrature formula

1 Introduction

When $0 < \alpha < 1$, the definition of Riemann-Liouville type fractional derivative is written as

(1) $$D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} D \int_0^t (t-\tau)^{-\alpha} f(\tau) d\tau.$$

We can rewrite the above equation with integration by parts as follows,

(2) $$D^\alpha f(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)} f(0) + \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} Df(\tau) d\tau,$$

where Df means the first order derivative of f.

We try to compute the above integration as a derivative of $f(x)$ of order α.

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2 Indefinite quadrature formula

For the finite integration of the product of a sufficient smooth function and a special function, Hasegawa et al.[1][2] proposed to estimate the approximation of the infinite integration when the special function has the type of $|t-c|^\alpha$.

The integration

$$Q(x,y,c)=\int_x^y |t-c|^\alpha f(t)dt,\quad -1\le x,y,c\le 1,\quad x\le y,\quad \alpha>-1 \tag{3}$$

is considered. We denote $k-th$ Chebyshev polynomial as $T_k(t)$ and write

$$f(t)\sim p_N(t)=\sum_{k=0}^{N}{}'' a_k^N T_k(t)=\frac{1}{2}a_0+\sum_{k=1}^{N-1}a_k^N T_k(t)+\frac{1}{2}a_N^N T_N(t) \tag{4}$$

This gives the following approximate integration

$$Q(x,y,c)=\int_x^y |t-c|^\alpha p_N(t)dt. \tag{5}$$

Here, we introduce the following N-th polynomial

$$F_N(t)=\sum_{k=1}^{N}(b_{k-1}-b_{k+1})\frac{T_k(t)}{2k}, \tag{6}$$

and set

$$G_N(t;c)=|t-c|^\alpha(t-c)\{F_N(t)-F_N(c)+\frac{p_N(c)}{\alpha+1}\}. \tag{7}$$

From the property of Chebyshev polynomial

$$2\int T_k(t)dt=\frac{T_{k+1}(t)}{k+1}-\frac{T_{k-1}(t)}{k-1}+C,\quad (k\ge 2),$$

we have

$$F_N'(t)=\frac{b_0}{2}T_1'(t)+\frac{b_1}{2\cdot 2}T_2'(t)+b_2T_2(t)+\cdots+b_{N-1}T_{N-1}(t)-\frac{b_N}{2(N-1)}T_{N-1}'(t) -\frac{b_{N+1}}{2N}T_N'(t). \tag{8}$$

Therefore from $T_0(t)=1,\ T_1(t)=t,$ and setting $b_N=0,\ b_{N+1}=0$, we can get

$$F_N'(t)=\frac{1}{2}b_0+\sum_{k=1}^{N-1}b_kT_k(t) \tag{9}$$

Furthermore we have the relation

$$2tT_k(t)=T_{k+1}(t)+T_{k-1}(t),$$

so by putting $b_{-1}=b_1$, the following relation holds

$$2(t-c)F_N'(t)=\sum_{k=0}^{N}(b_{k+1}-2cb_k+b_{k-1})T_k(t). \tag{10}$$

If we determine the coefficients b_k by the following way

(11) $$b_{k+1}(1-\frac{\alpha+1}{k})-2cb_k+b_{k-1}(1+\frac{\alpha+1}{k})=2a_k^N,\quad k=N,N-1,\cdots,1$$

where $b_N=0, b_{N+1}=0$, we can find the next relation

(12) $$G_N'(t;c)=|t-c|^\alpha p_N(t)$$

Consequently the approximate value of the integration is estimated as follows

(13) $$Q_N(x,y,c)=G_N(y;c)-G_N(x;c).$$

3 Descretize Fractional Derivatives

We write the principal part of the integration (2) in Sec.1 as follows,

(14) $$\tilde{D}^q f(s)=f(0)s^{-q}+\int_0^s f'(t)(s-t)^{-q}dt,$$

and then we apply the result of Sec.2 to this integration.

At the beginning, we expand the function $f(s)$ in Chebyshev series as follows;

(15) $$f(s)\sim f_n(s)=\sum_{k=0}^{n}{}'' a_kT_k(2s-1),$$

$$where \quad a_k=\frac{\tau_k}{n}\sum_{l=0}^{n}{}'' f(2cos\frac{\pi l}{n}-1)cos\frac{\pi kl}{n}$$

$$0\le k\le n \ \ and\ \tau_0=\tau_n=1,\tau_k=2(1\le k\le n-1),$$

and the double prime denotes that the first and the last terms of summation are halved.

According to the preceding result, for the integration

(16) $$\int_0^s (s-t)^{-q}f_n'(t)dt,$$

we can find some polynomial $F_{n-1}(s)$ with order $n-1$, such that

(17) $$\int_x^s (s-t)^{-q}(f_n'(s)-f_n'(t))dt=(s-x)^{1-q}\{F_{n-1}(s)-F_{n-1}(x)\}.$$

That is, the following equation is satisfied

(18) $$f_n'(x)-f_n'(s)=(x-s)F_{n-1}'(x)+(1-q)(F_{n-1}(x)-F_{n-1}(s)).$$

Then, by setting $x=0$, we can get the formula for estimating the value of the derivative of order q

(19) $$\tilde{D}^q f_n(s)=f(0)s^{-q}+\int_0^s f_n'(s)(s-t)^{-q}dt-(s-x)^{1-q}\{F_{n-1}(s)-F_{n-1}(0)\}$$

(20) $$=f(0)s^{-q}+(s)^{1-q}\{\frac{f_n'(s)}{1-q}-F_{n-1}(s)+F_{n-1}(0)\}.$$

In order to determine the unknown function $F_{n-1}(x)$, we set

$$F'_{n-1}(x) = \sum_{k=0}^{n-2} b_k U_k(2x-1), \tag{21}$$

where $U_k(x)$ is a Chebyshev polynomial of the second kind for which obeys the recurrence formula

$$2xU_k(x) = U_{k+1}(x) + U_{k-1}(x), \quad U_{-1}(0) = 0.$$

Accordingly we have

$$F_{n-1}(x) = \sum_{k=1}^{n-1} \{\frac{b_{k-1}}{4} - \frac{b_{k+1}}{4(k+2)}\} U_k(2x-1), \tag{22}$$

and

$$f'_n(x) = 2\sum_{k=0}^{n-1} (k+1) a_{k+1} U_k(2x-1). \tag{23}$$

Substituting these relations to the equation (18), we can find the next equation on the coefficients b_k,

$$b_{k-1} = \left(2(2s-1)b_k - (1 - \frac{1-q}{k+2})b_{k+1} + 8(k+1)a_{k+1}\right) / (1 + \frac{1-q}{k}) \tag{24}$$

$$k = n-1, n-2, \cdots, 1,$$

where $b_n = b_{n-1} = 0$. Then we can estimate the right-hand side of (20).

Finally we can summarize the process of the estimation of $D^q f(s)$ as follows.

Step 1: $f(x)$;given
Step 2: Compute coefficients of expansion into the Chebyshev series; $\{a_k\}$
Step 3: Compute coefficients of expansion into the Chebyshev series; $\{b_k\}$
Step 4: Compute the value of $f'_n(s) = 2\sum_{k=0}^{n-1}(k+1)a_{k+1}U_k(2s-1)$
Step 5: Compute the value of

$$F(x) = \sum_{k=1}^{n-1} \left(\frac{b_{k-1}}{4k} - \frac{b_{k+1}}{4(k+2)}\right) U_k(2x-1)$$

at $x = 0$ and $x = s$
Step 6: Compute the value of $D^q f(s) = f(0)s^{-q} + \left(\frac{f'_n(s)}{1-q} - F(s) + F(0)\right) s^{1-q}$

4 Illustrate Fractional Derivatives

Here we show an example of the function $f(x) = x^k$. The q-th derivative is estimated as follows;

$$D^q s^k = \frac{1}{\Gamma(1-q)} \frac{d}{ds} \int_0^s t^k (s-t)^{-q} dt$$
$$= \frac{1}{\Gamma(1-q)} \frac{d}{ds} \{ \frac{\Gamma(k+1)\Gamma(1-q)}{\Gamma(k+2-q)} s^{k-q+1} \}$$
$$= \frac{\Gamma(k+1)}{\Gamma(k+1-q)} s^{k-q}$$

where we use the relation $B(k+1, 1-q) = \int_0^1 t^k (1-t)^{-q} dt$.

Table 1
In case of $f(x) = x^7$

q-differential	value at x=0.4	error
0.3	0.00509416	-1.46584×10^{-16}
0.5	0.0123669	-9.19403×10^{-17}
0.7	0.0368992	-6.93889×10^{-17}

Table 2
In case of $f(x) = x^5$

q-differential	x	value	error
0.3	0.4	0.0289503	-1.11022×10^{-16}
0.5	0.4	0.0657914	-3.60822×10^{-16}
0.5	0.7	0.816283	2.22045×10^{-16}
0.7	0.4	0.183343	-9.99201×10^{-16}

In Table 1 we show the case of $k = 7$ and $q = 0.3, 0.5, 0.7$, and the case of $k = 5$ is in Table 2. In these cases the numerical results coincide with the exact values in very small difference. Then these results are showing that our expansion method is positively working accurate.

Furthermore we tried to treat some examples due to Oldham and Spanier [5], when the order of derivative is 1/2.

(i)

$$f(x) = exp(x)$$
$$D^{1/2} f(x) = \frac{1}{\sqrt{\pi x}} + exp(x) Erf(\sqrt{x})$$

(ii)

$$f(x) = \sqrt{1+x}$$
$$D^{1/2} f(x) = \frac{1}{\sqrt{\pi x}} + \frac{arctan(\sqrt{x})}{\sqrt{\pi x}}$$

Table 3
In cases of above $f(x)$

f(x)	q-differential	value at x=0.4	error
(1)	0.5	1.83028	-1.64979×10^{-13}
(2)	0.5	0.183343	-9.99201×10^{-16}

For the calculation of the expansion coefficients we set $n = 10$ as an expansion order of the Chebyshev series in these cases. In order to compare with the exact values we examined simple cases which are known their analytic solutions.

At the begining, we made the direct application of Hasegawa-Torii algorithm to compute the integration part. On the series expansion for the first derivative $F'(x)$, we take Prof. Sugiura's suggestion. The author would like to express the sincere thanks to him.

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