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Pixel Geometry

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Abstract

An alternative approach to studying the denotational semantics of programming languages is suggested and studied through an example borrowed from geometry. In contrast to the prevailing Scott-Strachey approach using continuous mathematics a discrete alternative is suggested having more in common with data structuring than T_0 separable topology.

1 Introduction

In the denotational approach to programming language semantics inspired by Dana Scott and Christopher Strachey [5] each program or part thereof is assigned a value in a Scott topology $(S,\ \tau \subseteq 2^S)$. As non-trivial programming languages are inherently undecidable it is unsurprising that the separability of a Scott topology is so weak as to be T_0. This induces a partial ordering $\forall\ x,\ y \in S\ .\ x \sqsubseteq y \ \Leftrightarrow\ \forall\ O \in \tau\ .\ x \in O \ \Rightarrow\ y \in O$. This information ordering captures well our intuition in functional programming that at some stage in the evaluation of an expression our semantics is x and after a bit more computation we will know more and have produced y. For example, we can use this relation to describe the sequence of evaluation steps $\bot \sqsubseteq 3 :: \bot \sqsubseteq 3 :: (5 :: \bot) \sqsubseteq 3 :: (5 :: [\,]) = [3,\ 5]$ in the computation of a finite SML list [1]. The symbol $\bot$ denoting the totally unknown value is the most partial and hence least member of a Scott topology. What is taken for granted in such an approach to denotational semantics using continuous complete lattices is that each partial object can always be completed, that is, each member of the topology has a maximal element above it. This approach to denotational semantics thus assigns to each program an approximation in a weak (i.e. non-Hausdorff) topology to some item in the Hausdorff sub space of complete (i.e. maximal) items. In other words, an admitedly provocative interpretation

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of denotational semantics is that it is a *weak* form of mathematics. In this paper we propose the thesis that as denotational semantics can be formed to be a model of successive approximations to Hausdorff mathematics then it must also be possible to establish a discrete theory of finite entities without reference to Hausdorff separability. In other words, as denotational semantics approximates Hausdorff mathematics then why not turn things around and define Hausdorff mathematics to be, in some sense, the completion of a *discrete denotational semantics*. To investigate the possibility of such a semantics we explore how a classic example of Hausdorff mathematics could be replaced by a theory which can be fully realised by a computer.

2 Pixels

In the real plane $\Re^2$ there exists a unique straight line between any two distinct points. However, such a line consists of an infinite set of zero width points which could not possibly be realised upon a computer screen as it would need both infinite precision arithmetic and an infinite amount of time to draw the line. The programmer thus approximates the notion of a straight line by a finite set of so-called *pixels* of sufficiently small size and large number which cover the path of the straight line in such a way as to create the illusion of a straight line. By analogy with denotational semantics we see that the pixels

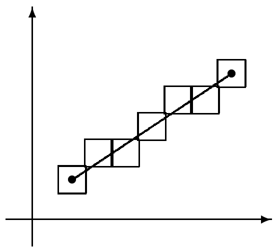
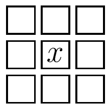


Fig. 1. covering a straight line with pixels

are an approximation to the straight line, one which can get ever closer to the real thing as we use ever smaller and more numerous pixels. It would seem that pixels cannot exist without the straight line itself, as to place each pixel we would first need to compute the path of the straight line. As with denotational semantics it would seem that our pixels can only be a poor imitation of the real thing which is a straight line in the real plane. To challenge this view of pixels as being second class citizens this paper endeavours to construct a notion of *straight line* solely in terms of the language of pixels and without reference to the supposed 'real thing' that is a straight line in the real plane. If this can be done for pixels then why not for other partial concepts inherent to programming language semantics? We cannot regard a pixel to be an approximation to a point as there are an uncountable number of such points in the real plane $\Re^2$. The concept of a *point* is an infinite one thus having to be barred from any discussion of pixels which is necessarily discrete. By virtue of the fact that points are infinitely small a point can be placed between any two distinct points, a property which cannot hold in general for pixels. Instead we need a means of asserting that one pixel may 'touch' another.

Definition 1 *A* **tolerance** *is a triple* $(S,\ \sim\subseteq S^2,\ d : \sim\rightarrow \omega)$ *consisting of a set* S*, of so-called 'pixels', a reflexive symmetric binary 'tolerance relation'* $\sim$ *to specify which pixels touch, and a symmetric function* d *such that* $d(x,y) = 0 \ \Leftrightarrow\ x = y$ *to tell us how loose is each touching. As a shorthand we write* $x \overset{a}{\sim} y$ *if* $x \sim y$ *and* $d(x,y) = a$*.*

Definition 2 *The* **pixel plane** *is the tolerance* $(\mathbb{Z}^2, \sim, d)$ *having the following distinct tolerable pairs. For all* $a, b \in \mathbb{Z}$*,* $(a,b) \overset{2}{\sim} (a{+}1, b)$*,* $(a,b) \overset{2}{\sim} (a, b{+}1)$*,* $(a,b) \overset{3}{\sim} (a+1, b+1)$*, and* $(a,b) \overset{3}{\sim} (a+1, b-1)$*.*



A pixel x and its 8 tolerable neighbours

Fig. 2. the pixel plane

Being weaker than equivalence relations, tolerance relations can be applied to the study of vagueness where touching is a challenging problem in artificial intelligence [2]. Our definition of a tolerance is just an undirected graph in which each edge has an associated *closeness* value. In digital topology [3] undirected graphs are used, but without a closeness value. The role of the closeness value in tolerances will become clear in Definition 8.

There are three objectives in this paper. Firstly, to give an axiomatisation for the notion of a *pixel geometry* and derive some of its key properties. Secondly, to study the pixel plane as an affine pixel geometry defined using tolerances. Thirdly, to provide a definition for a 'straight line' between two pixels where no such obvious definition exists.

3 Axiomatising pixel geometry

An *incidence geometry*, as defined by Prenowitz [4], is a collection of so-called 'points', and sets of points termed 'lines' and 'planes' satisfying a given set of axioms (see Section 4). The axioms specify the incidence, that is, the intersection between lines and planes. A pixel geometry will be defined using the following axioms.

Axiom P 1 *A line is a set of pixels, containing at least two pixels.*

Axiom P 2 *Two distinct pixels are contained in at most one line.*

Axiom P 3 *A plane is a set of pixels, containing at least three pixels which do not belong to the same line.*

Axiom P 4 *Two distinct intersecting lines are contained in one and only one plane.*

Axiom P 5 *If a plane contains two distinct pixels of a line, it contains all the pixels of the line.*

Axiom P 6 *Two planes containing a pixel contain a line containing that pixel.*

Axiom P 7 *A line and a pixel not in the line are contained in at most one plane.*

Definition 3 *A* **pixel geometry** *is a collection of entities termed 'pixels', 'lines' and 'planes' satisfying axioms P1 to P7*

These are the axioms for an incidence geometry of up to three dimensions [4] modified to distinguish the notion of a pixel from that of a point. With pixels we cannot assume as is done with points that there is always a unique line between two distinct pixels, only that there is at most one. For an example of this see Fig. 5. Many properties of point geometry also hold in pixel geometry. Theorems 1-4 give the reader a sample of such properties and how their proofs are constructed.

Theorem 1 *Two distinct lines in a pixel geometry have at most one pixel in common.*

Proof: Suppose that L and M are lines having distinct pixels in common. Then by P2, $L = M$. □

In subsequent work we use the notation xy (equivalently yx) for the unique line containing distinct pixels x and y if it exists.

Theorem 2 *If pixels x and y are collinear and $z \notin xy$ then x, y, and z are distinct and non collinear.*

Proof: By definition of xy, $x \neq y$. Suppose $z = x$. Then since x is in xy so is z, but this contradicts the hypothesis. The supposition $z = x$ is therefore false, and so $z \neq x$. Similarly, $z \neq y$. Thus x, y, and z are distinct.

Now suppose x, y, and z are collinear. Then, by definition, there is a line L containing x, y, and z. Since L contains x and y, and as $x \neq y$ it follows by Theorem 1 that $L = xy$. But z is in L; hence z is in xy, contradicting the hypothesis. The supposition is therefore false, and so x, y, and z are non collinear. □

Lines L and M are said to be **parallel** (written $L \| M$) if they are contained in the same plane and have no common pixel.

Theorem 3 *If two lines are parallel then there exists a unique plane containing them both.*

Proof: Suppose that L and M are parallel lines. Then by the definition of parallel lines there exists a plane P containing both lines. Suppose that P' is also a plane containing L and M. By P1 there exists a pixel $x \in M$. Thus x and L are each contained in planes P and P'. But $x \notin L$ as $L \cap M = \phi$, and so by P7, $P = P'$. □

Theorem 4 *If two distinct planes intersect, their intersection is a line.*

Proof: Suppose that distinct planes P and P' have a pixel x in common. Then by P6 both planes contain a line L containing x.

Suppose by way of contradiction that $P \cap P' \neq L$. Then we can choose a pixel z which lies in both P and P' but not in L. But by P7, z and L can lie in at most one plane. Thus $P = P'$, a contradiction. $\square$

As distinct pixels may be non collinear we have theorems such as the following which would be vacuous in point geometry.

Theorem 5 *Two distinct non collinear pixels are contained in at most one plane.*

Proof: Suppose that x and y are distinct non collinear pixels contained in each of distinct planes P and P'. Then by Theorem 4 there is a line L such that $L = P \cap P'$. Thus $x \in L$ and $y \in L$, a contradiction as x and y are non collinear by supposition. $\square$

As we cannot assume that there is a line containing any two distinct pixels neither can we assume that a line and a pixel not on the line are contained in a plane. As with their counterpart axioms for point geometry (see Section 4), P1-P7 are too weak for any useful application. By adding additional axioms geometry can be made affine, Euclidean, ordered, projective, elliptic, etc. As our concept of a pixel is a solid entity in three dimensional Euclidean style space we need to add a version of Euclid's postulate which asserts that (in point geometry) for any given line and point not on a line there is a unique line containing that point parallel to the first line. To be adopted such a postulate would first have to take account of the fact that in pixel geometry a pixel and a line not containing the pixel are not necessarily contained in a plane. To do this we study transverse lines and planes. Lines L and M are said to be **transverse** (denoted $L \curlywedge M$) if $L \cap M \neq \phi$ and $L \neq M$. A line L and a plane P are said to be **transverse** (denoted $L \curlywedge P$) if $L \cap P \neq \phi$ and $L \nsubseteq P$. Now we introduce a concept allowing us to study a degenerate form of parallelism in pixel geometry.

Definition 4 *Lines L and M are* **cotransverse** *(denoted $L|M$) if they are disjoint, and if for each plane P, $L \curlywedge P$ if and only if $M \curlywedge P$.*

Axiom P 8 *If x is a pixel not in a line L then there exists a unique line M such that $x \in M$ and either $L\|M$ or $L|M$.*

Definition 5 *A pixel geometry is* **affine** *if it satisfies axiom P8*

In an affine point geometry the relation $L = M \vee L\|M$ is an equivalence, allowing us to associate the notion of a 'direction' to each line. Theorems 6-7 demonstrate that in affine pixel geometry the relation $L = M \vee L|M$ can serve the same purpose.

Theorem 6 *Parallel lines in an affine pixel geometry are cotransverse.*

Proof: Let L and M be parallel lines in an affine pixel geometry. Then $L \cap M = \phi$ and there exists a plane P containing both L and M. First we show that any plane transverse to L is transverse to M. Suppose that P' is a plane such that $L \pitchfork P'$. Then by P5 there is a unique pixel x such that $\{x\} = L \cap P'$. Then $P \cap P' \neq \phi$ as $L \subseteq P$. Thus by Theorem 4, there exists a line K such that $K = P \cap P'$. Thus $x \in K$ as $x \in P$ and $x \in P'$. Suppose, by way of contradiction, that $K \cap M = \phi$. Then $K \| M$ as $K \subseteq P$ and $M \subseteq P$. Thus, $x \notin M$ (as $L \cap M = \phi$), $x \in L$, $x \in K$, $L \| M$, and $K \| M$. Thus by P8, $L = K$, a contradiction, as $L \pitchfork P'$ and $K \subseteq P'$. Thus $K \cap M \neq \phi$. Thus $M \cap P' \neq \phi$ as $K \subseteq P'$. Also, $K \neq M$ as $x \in K$ and $x \notin M$. Thus $K \nsubseteq P'$ as $K = P \cap P'$ and $M \subseteq P$. Thus $M \pitchfork P'$. Similarly we can show that each plane transverse to M is transverse to L. $\square$

Theorem 6 tells us that in an affine pixel geometry 'cotransverse' is a degenerate notion of parallelism. If two lines are cotransverse but not parallel then any plane containing one line is disjoint from the other. This does not seem unreasonable for a degenerate notion of parallelism in a geometry where there is not always a plane containing a given line and a given pixel not in it.

Theorem 7 *For lines L, M, and N in an affine pixel geometry, if $L|M$, $M|N$, and $L \neq N$ then $L|N$.*

Proof: Suppose that for a plane P, $L \pitchfork P$. Then $M \pitchfork P$ as $L|M$, and so $P \pitchfork N$ as $M|N$. Similarly we can show that each plane transverse to N is transverse to L. To complete the proof that $L|N$ we need to prove that $L \cap N = \phi$. Suppose, by way of contradiction, that $L \cap N \neq \phi$. Then $L \pitchfork N$ as $L \neq N$. Thus there exists a pixel x such that $\{x\} = L \cap N$. $x \notin M$ as $x \in L$ and $L \cap M = \phi$ (as $L|M$). Thus, $x \notin M$, $x \in L$, $x \in N$, $L|M$, and $N|M$. Thus by P8, $L = N$, a contradiction of the hypothesis $L \neq N$. Thus $L \cap N = \phi$, and so $L|N$. $\square$

Definition 6 *Lines L and M are* **codirectional** *if $L = M$ or $L|M$.*

Theorem 7 tells us that codirectionality is an equivalence relation in affine pixel geometry. Each equivalence class is referred to as a *direction*. The pixel plane (see Fig. 2) is an affine pixel geometry having four directions, namely the horizontal, vertical, updward diagonal, and downard diagonal lines. Note that this number four can be computed as half the number of edges and corners of a pixel. Similarly there are thirteen directions in the extension of the pixel plane to three dimensions where each pixel is a cube. This number is computed as half the total number of faces, edges and corners of a cube.

Now we consider a notion which, while vacuous in point geometry, is useful to pixel geometry in understanding planes. We say that pixels x and y are **bilinear** if they are non collinear and there exist transverse lines L and M such that $x \in L$ and $y \in M$.

Theorem 8 *Bilinear pixels are contained in a unique plane.*

Proof: Suppose that pixels x and y are bilinear. Then we can choose lines L and M such that $x \in L$, $y \in M$, and $L \cap M \neq \phi$. $L \neq M$ as x and y are non collinear. Thus by P4, there exists a unique plane P containing L and M. Thus there is a plane containing x and y. Suppose by way of contradiction that there exists a plane P' containing x and y such that $P \neq P'$. Then by Theorem 4 there exists a line K such that $K = P \cap P'$. Thus x and y are in K as they are both in each of P and P', but this is a contradiction of the hypothesis that x and y are non collinear. Thus x and y are contained in a unique plane. □

We can extend the definition of bilinearity to lines. Lines L and L' are said to be **bilinear** if they are disjoint, there exist distinct pixels $x, y \in L$ and $x', y' \in L'$ such that xx' exists, yy' exists, and x & y' are bilinear.

Theorem 9 *Bilinear lines are parallel.*

Proof: Suppose that L and L' are bilinear lines. Then there exist distinct pixels $x, y \in L$, $x', y' \in L'$ such that xx' exists, yy' exists, and x & y' are bilinear. x, y, and y' are non collinear as $L \cap L' = \phi$. Thus L and yy' are transverse lines, and thus by P4 are contained in a plane P. x, x', and y' are non collinear as $L \cap L' = \phi$. Thus L' and xx' are transverse lines, and thus by P4 are contained in a plane P'. Thus P and P' each contain x and y'. Suppose, by way of contradiction, that $P \neq P'$. Then by Theorem 4, there exists a line M such that $M = P \cap P'$, and so $x, y' \in M$. But this is a contradiction of the hypothesis that x and y' are bilinear. Thus $P = P'$. □

4 Points versus Pixels

As the motivation for this paper is to explore the possibility of a discrete denotational semantics for programming languages it is important to establish that the Hausdorff notion of 'point' as studied here in point geometry is nothing more than a special kind of pixel. To do this we show that each point geometry is a pixel geometry. First recall the following axioms for a point geometry taken from Prenowitz [4].

Axiom I1 *A line is a set of points, containing at least two points.*

Axiom I2 *Two distinct points are contained in one and only one line.*

Axiom I3 *A plane is a set of points, containing at least three points which do not belong to the same line.*

Axiom I4 *Three distinct points which do not belong to the same line are contained in one and only one plane.*

Axiom I5 *If a plane contains two distinct points of a line, it contains all the points of the line.*

Axiom I6 *If two planes have one point in common, they have a second point in common.*

Theorem 10 *Each point geometry is a pixel geometry.*

Proof: The proof consists of showing that axioms P1 to P7 hold. In what follows interpret the word 'pixel' as being 'point'.

P1 holds as it is implied by I1.

P2 holds as it is implied by I2.

P3 holds as it is implied by I3.

Now we prove P4. Suppose that L and M are distinct intersecting lines. Then we can choose a pixel $x \in L \cap M$. Then by I1 we can choose points $y \in L$ and $z \in M$ each distinct from x. Suppose by way of contradiction that x, y, and z are collinear. Then by I2, the line L containing x and y is the same line as M containing x and z, a contradiction of our hypothesis that $L \neq M$. Also, $y \neq z$ as $L \neq M$. Thus x, y, and z are distinct non collinear points, and as such by I4 are contained in one and only one plane. Thus P4 holds.

P5 holds as it is implied by I5.

Now we prove P6. Suppose two planes P and P' each contain a point x. Then by I6 there is a distinct point $y \in P \cap P'$. By I2 there is a unique line L containing both x and y. Thus by I5, $L \subseteq P$ as $x \in P$ and $y \in P$. Similarly by I5, $L \subseteq P'$ as $x \in P'$ and $y \in P'$. Thus $L \subseteq P \cap P'$. Thus P6 holds.

Now we prove P7. Suppose that x is a point not on a line L and that there is a plane P containing both x and L. By I1, L contains two distinct points y and z, neither of which can equal x as $x \notin L$. Thus x, y, and z are distinct points. Neither can they be collinear as $x \notin L$ and by I2, L is the only line containing y and z. Thus by I4, x, y, and z are contained in a unique plane. Thus P7 holds. □

Theorem 11 *A pixel geometry such that there is a line containing any two distinct pixels is a point geometry.*

Proof: The proof consists of showing that axioms I1 to I6 hold. In what follows interpret the word 'point' as being 'pixel'.

I1 holds as it is implied by P1.

Now we prove I2. Suppose that x and y are distinct pixels. Then by the hypothesis of the theorem there exists a line containing x and y. Thus by P2 this line is unique. Thus I2 holds.

I3 holds as it is implied by P3.

Now we prove I4. Suppose x, y, and z are distinct pixels which do not belong to the same line. Then by I2 (which we have already proved) there is a unique line L containing x and z. Similarly by I2 there is a unique line M containing y and z. $L \neq M$ as x, y, and z are non collinear. Thus by P4 there is a unique plane containing L and M, and so there is a plane containing x, y, and z. This plane must be unique as any plane containing x, y, and z must by P5 contain L and M which by P4 are contained in a unique plane.

I5 holds as it is implied by P5.

Now we prove I6. Suppose that planes P and P' have a pixel x in common. Then by P6, P and P' both contain a line L which contains x. By P1 there exists a pixel $y \in L$ distinct from x. Thus P and P' both contain x and y. Thus I6 holds. □

Theorems 10 and 11 demonstrate that the notion of 'pixel' in pixel geometry is a generalisation of the notion of 'point' in point geometry. Pixels are points which are not necessarily connected by lines. Axioms P1-P6 are derived by relaxing I1-I6 so as to allow some distinct points to be not necessarily connected by a line. For example, there is no obvious way in which a unique notion of *straight line* could be drawn to contain the squares (i, j) and $(i+2, j+1)$ of the pixel plane. If we took $(i, j) \sim (i+1, j+1) \sim (i+2, j+1)$ to be our line then why would this be any better than $(i, j) \sim (i+1, j) \sim (i+2, j+1)$? Note that to prove Theorem 11 we do not need P7, and so this axiom is not part of the relaxation. In relaxing axioms I1-I6 we lose the natural and desirable property that a pixel not on a line are both contained in at most one plane, and so this is reintroduced as axiom P7.

5 The Pixel Plane as a Pixel Geometry

This section explains how tolerances can be used to define a model in pixel geometry for the pixel plane (see Fig. 2). This involves defining each of the terms 'pixel', 'line', and 'plane' as constructions in a tolerance which satisfy axioms P1 to P7. The first of these three constructions is simple, for each tolerance $(S, \sim, d)$ a **pixel** is defined to be a member of S. Next we consider how to define lines.

Definition 7 *A* **path** *p from $x \in S$ to $y \in S$ in a tolerance $(S, \sim, d)$ is a finite sequence $p = \langle p_1 \ , \ p_2 \ , \ \ldots \ , \ p_n \rangle$ of pixels such that $x = p_1 \overset{a_1}{\sim} p_2 \overset{a_2}{\sim} \ \ldots \ \overset{a_{n-1}}{\sim} p_n = y$. The* **length** *of p, denoted $\sharp p$ is $\sum_{i<n} a_i$. p is a* **shortest path** *if there is no other path from x to y of smaller length. p is* **straight** *if $x \neq y$ and p is the only shortest path from x to y.*

In the pixel plane of Fig. 2 the straight paths are the horizontals, verticals, and diagonals. The plan is that each pixel of a straight path from x to y should be contained in the line containing x and y. Before this can be achieved a problem has to be resolved. In Fig. 3 $\langle x, \ y, \ u \rangle$ and $\langle x, \ y, \ z \rangle$ are both unique shortest paths containing x and y, and so to conform with P2 we would have to agree that x, y, u, and z are all contained in the same line. Yet there

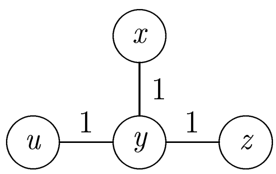
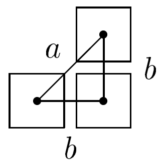


Fig. 3. a non deterministic tolerance

is no straight path containing all four pixels, and so we must restrict ourselves to only tolerances in which the concept of a straight path can be equated with the notion of a line from pixel geometry.

Definition 8 *A tolerance is* **deterministic** *if for each straight path* $\langle p_1, \ldots, p_n \rangle$ *there exists at most one pixel* x *such that* $\langle p_1, \ldots, p_n, x \rangle$ *is a straight path.*

Definition 8 asserts that in accordance with out intuitive understanding of the word 'straight' there should be at most one way of making each straight path one pixel longer, and so exclude tolerances such as in Fig. 3 from further consideration. In order to ensure that the pixel plane is a determinisitic tolerance we need to choose the closeness values a and b between pixels (preferably the smallest) as in Fig. 4. $a = 3$ and $b = 2$ is thus a satisfactory choice.



$$b < a < b + b$$

Fig. 4. closeness values for the pixel plane

Definition 9 *For pixels* x *and* y *in a deterministic tolerance such that there is a straight path from* x *to* y*, the* **line** *containing* x *and* y *(denoted* xy*) is the set of all pixels* z *such that there exists a straight path containing* x*,* y*, and* z*.*

Lines are seen to satisfy P1 as each line is defined so as to contain at least two pixels. We say that x and y are **collinear** if they are distinct and lie upon a line. If x and y are collinear then for each z on a line containing x and y there is in a deterministic tolerance a straight path from x (or y) to z containing y (or x). Thus there can only be one line containing collinear pixels, thus P2 holds. The next task is to define the notion of 'plane' for a deterministic tolerance. Here the set S is defined to be the one and only plane, clearly satisfying axioms P3-P7. And thus the pixel plane $(\mathbb{Z}^2, \sim, d)$ is indeed a pixel geometry. It is also an affine pixel geometry as for each line and a pixel not in it there is a unique parallel line containing the pixel.

6 Creating straight lines where none exist

In a pixel geometry there are straight lines, but not as many as we would like. Between two distinct pixels in a tolerance model of an affine pixel geometry it is often the case that there is more than one path of equal shortest length either of which could be a candidate for the notion of a straight line. In Fig. 1 we argued that a straight line of points in the real plane $\Re^2$ could be covered by an appropriate set of pixels. But in pixel geometry we need to define such a set without having the real line itself. And so in this section we look at the set of all shortest paths in a tolerance model for an affine pixel geometry such as the pixel plane. Our aim is to define a notion of 'straight line' between any two distinct pixels which coincides with 'line' as in pixel geometry when such a line exists, otherwise the best construction that can reasonably be obtained.

Definition 10 *For each path* $p = \langle p_1, \ \dots \ p_n \rangle$ *in a tolerance let* $\$p$ *denote the number* n *of pixels in* p *. A tolerance* $(S, \sim, d)$ *is said to be* **connected** *if there is a path between any two pixels, and if whenever shortest paths* p *&* q *are such that* $p_1 = q_1$ *and* $p_{\$p} = q_{\$q}$ *then* $\$p = \$q$ *.*

The pixel plane is an example of a connected tolerance, as we have been careful to ensure that the diagonal closeness value (i.e. a in Fig. 4) is strictly less than the sum of the horizontal or vertical closeness values (i.e. b in Fig. 4). Thus (as can be seen in Fig. 5) if a path is shortest then we can determine the number of either horizontal or vertical b values, and also the number of a values. Thus given two distinct pixels in the pixel plane we can determine that number which is the length of any shortest path between them.

Definition 11 *The* **hop-distance** *between pixels x and y in a connected tolerance denoted* $|x - y|$ *is the length of a shortest path from x to y.*

It can easily be shown that hop-distance is a metric, that is, for all pixels x, y, and z, $|x-y| = 0 \Leftrightarrow x = y$, $|x-y| = |y-x|$, and, $|x-z| \leq |x-y|+|y-z|$.

Let $\mathbb{A}^x_y$ denote the set of all shortest paths from x to y in a connected tolerance. Our first attempt at defining the notion of a straight line of pixels is to consider this set. Fig. 5 is a segment of the pixel plane containing three of the many possible shortest paths from x to y. What is noticeable is that while the path marked out using a $\bigcirc$ appears to be a most direct path it nonetheless has the same length of 39 as each of the paths marked out using a $\bullet$ and a $\diamondsuit$.

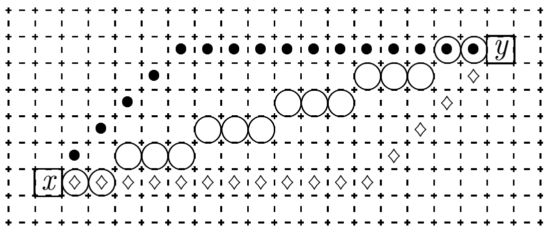


Fig. 5. a segment of the pixel plane

Thus $\mathbb{A}^x_y$ is too crude a candidate for our required notion of a straight line of pixels, and so we look for a subset which will be more acceptable.

Definition 12 *A tolerance* $(S, \sim, d)$ *is* **bounded** *if for each* $x \in S$ *the neighbourhood set* $\widetilde{x} = \{y \in S \mid x \sim y\}$ *is finite.*

Clearly the the pixel plane is bounded as each $\widetilde{x}$ has 9 members.

Theorem 12 *For each* $\langle p_1, \ \dots \ , p_n \rangle \in \mathbb{A}^x_y$ *and for each* $1 \leq i \leq n$ *,*

$\langle p_1, \ \ldots \ , \ p_i \rangle \ \in \ \mathbb{A}^x_{p_i} \qquad and \ \langle p_i, \ \ldots \ , \ p_n \rangle \ \in \ \mathbb{A}^{p_i}_y$.

Proof. Suppose by way of contradiction that $\langle p_1, \ \ldots \ , \ p_i \rangle \ \notin \ \mathbb{A}^x_{p_i}$, then there exists a shorter path $\langle q_1, \ \ldots \ , \ q_j \rangle \ \in \ \mathbb{A}^x_{p_i}$. Thus $\langle q_1, \ \ldots \ , \ q_j \ , \ p_{i+1}, \ \ldots \ , \ p_n \rangle$ is a path from x to y of length less than that of p, a contradiction. Thus $\langle p_1 \ , \ \ldots \ , \ p_i \rangle \ \in \ \mathbb{A}^x_{p_i}$. By a similar argument we can prove that $\langle p_i, \ \ldots \ , \ p_n \rangle \ \in \ \mathbb{A}^{p_i}_y$. □

Definition 13 *For each non empty set A of shortest paths between members x and y of a bounded connected tolerance the* **variance** *of A is,*

$$\mathbf{V}[A] \quad = \quad max\{ \ | \ p_i - q_i \ | \ \ | \ \ p \ , \ q \ \in \ A \ \ \wedge \ \ 1 \ \le \ i \ \le \ \$p \ \}$$

Theorem 13 *For all members x and y in a bounded connected tolerance,* $\mathbf{V}[\mathbb{A}^x_y] \ \le \ | \ x - y \ |$.

Proof. We can choose $p, \ q \ \in \ \mathbb{A}^x_y$ and $1 \le i \le \$p$ such that $\mathbf{V}[\mathbb{A}^x_y] \ = \ | \ p_i - q_i \ |$. Thus by Theorem 12 $| \ x - p_i \ | \ + \ | \ p_i - y \ | \ = \ | \ x - y \ | \ = \ | \ x - q_i \ | \ + \ | \ q_i - y \ |$. Thus,

$$\begin{aligned}
\mathbf{V}[\mathbb{A}^x_y] \quad &= \quad | \ p_i - \ q_i \ | \\
&= \quad \tfrac{1}{2} \ \times \ (\ | \ p_i - q_i \ | \ + \ | \ p_i - q_i \ | \ ) \\
&\le \quad \tfrac{1}{2} \ \times \ (\ (\ | \ p_i - x \ | \ + \ | \ x - q_i \ | \ ) \ + \ (\ | \ p_i - y \ | \ + \ | \ y - q_i \ | \ ) \ ) \\
&= \quad \tfrac{1}{2} \ \times \ (\ (\ | \ x - p_i \ | \ + \ | \ p_i - y \ | \ ) \ + \ (\ | \ x - q_i \ | \ + \ | \ q_i - y \ | \ ) \ ) \\
&= \quad \tfrac{1}{2} \ \times \ (\ | \ x - y \ | \ + \ | \ x - y \ | \ ) \\
&= \quad | \ x - y \ |
\end{aligned}$$

□

Definition 13 is well defined as in a bounded connected tolerance there are only a finite number of paths of a given length from a given member. Variance is a measure of the extent to which two shortest paths can vary, a concept we plan to use to identify a most direct path such as the one studied in Figure 5. Each self variance $\mathbf{V}[\mathbb{A}^x_x] \ = \ 0$, and if $\mathbf{V}[\mathbb{A}^x_y] \ = \ 0$ then there is a unique shortest path between x and y. We can picture the set $\mathbb{A}^x_y$ of all shortest paths as in Fig. 6.

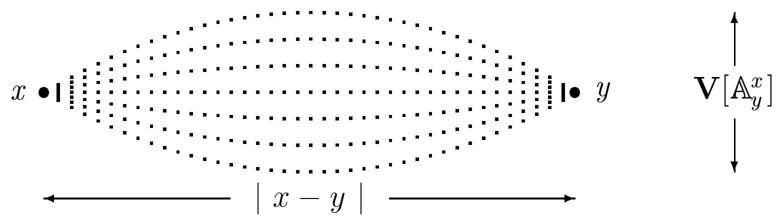


Fig. 6. the variance of $\mathbb{A}^x_y$.

Any two shortest paths from x to y may diverge, but will sooner or later converge. The notion of a 'shortest path' is thus *vague* in the sense that the extent of possible variations between shortest paths can be measured by $\mathbf{V}[\mathbb{A}^x_y]$. Clearly the greater the hop-distance the more likely it is that the variance will rise as well, and so it is the ratio of hop-distance to variance which tells us how vague is the notion of 'shortest path'. For each x and y in the pixel plane if $|x - y| \geq 5$ then $\mathbf{V}[\mathbb{A}^x_y] \leq 0.4 \times |x - y|$. Thus the variance in the pixel plane is at worst almost half the hop-distance, clear evidence that $\mathbb{A}^x_y$ is far too crude to be employed as a notion of a straight line of pixels. In order to select a more useful subset of $\mathbb{A}^x_y$ it is first necessary to examine the structure of tolerances in an affine pixel geometry. We have already established that such geometries have a notion of *direction*. Our plan is now to consider each shortest path and the directions it uses. The following definitions assume a bounded deterministic connected tolerance affine pixel geometry.

Definition 14 *The* **concatenation** *of paths p and q such that $p_{\$p} = q_1$ (denoted $p@q$) is the path $\langle p_1, \ldots, p_{\$p}, q_2, \ldots q_{\$q}\rangle$. A path is a* **link** *if it is a line. A* **chain** *is a concatenation of links. For each path p the* **size** *of p (denoted $\overline{p}$) is the smallest number of links needed to make a concatenation for p.*

Thus each path p has a unique concatenation $p^1@p^2 \ldots @p^{\overline{p}}$ of links such that no two successive links have the same direction. In studying links we assume that there is a small, probably finite, number of directions such as in the pixel plane which has four.

Definition 15 *For pixels x and y the* **link-distance** *is,*

$$\lceil x - y \rceil = max \{ \overline{p} \mid p \in \mathbb{A}^x_y \}$$

Note that link-distance is less than or equal to hop-distance.

Definition 16 *For pixels x and y, $\mathbb{B}^x_y = \left\{ p \in \mathbb{A}^x_y \mid \overline{p} = \lceil x - y \rceil \right\}$.*

The set $\mathbb{B}^x_y$ of all shortest paths from x to y of maximum link-distance is introduced to help us establish a tighter notion of *straight line* between pixels x and y than that provided by $\mathbb{A}^x_y$. The next theorem captures an intuitively appealing property that any pair of straight lines between two pixels should have a minimal variance.

Theorem 14 *For members x and y of the pixel plane $\mathbf{V}[\mathbb{B}^x_y] \leq 2$.*

Proof. Suppose by way of contradiction that there exists x and y in the pixel plane such that $\mathbf{V}[\mathbb{B}^x_y] > 2$. Then we can choose $p, q \in \mathbb{B}^x_y$ and i such that $|p_i - q_i| > 2$. Then we can choose a pixel $z \in \mathbb{A}^{p_i}_{q_i}$ next to p_i which can re-route p through z to give a path of the same length from x to y and is such that $|z - q_i| < |p_i - q_i|$. But this is a contradiction as it implies that whenever $\mathbf{V}[\mathbb{B}^x_y] > 2$ we can further reduce it. □

Although $\mathbb{B}^x_y$ may seem like the last necessary refinement to our concept of what a straight line should be in a tolerance there is a need to 'balance' the lengths of links in a chain so that no link is unnecessarily too short nor too long. For example in Fig. 7

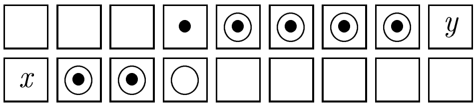


Fig. 7. an almost straight line of pixels

the path from x to y marked out with a $\bullet$ hardly seems as symmetric as the one marked out with a $\bigcirc$. We now refine $\mathbb{B}^x_y$ which is our current notion of a *straight line* to make the shorter links as long as possible and the longer links as short as possible.

Definition 17 *For each path p in a bounded connected affine tolerance pixel geometry the* **link-difference** *is,*

$$\lceil p \rceil \quad = \quad max\left\{ \sharp p^i \mid 1 \leq i \leq \overline{p} \right\} \quad - \quad min\left\{ \sharp p^i \mid 1 \leq i \leq \overline{p} \right\}$$

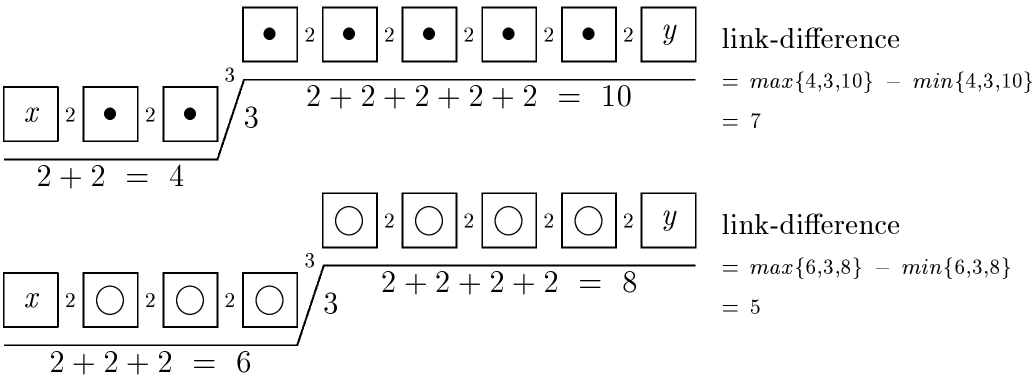


Fig. 8. computing link-differences for Fig. 7

Definition 18 *For each bounded connected affine tolerance pixel geometry,*

$$\mathbb{C}^x_y \quad = \quad \left\{ \ p \ \in \ \mathbb{B}^x_y \ \mid \ \lceil p \rceil \ = \ min\left\{ \ \lceil q \rceil \mid q \ \in \ \mathbb{B}^x_y \ \right\} \ \right\}$$

In Fig. 7 the path marked with a $\bullet$ is now to be excluded from our further consideration as a straight path as its link-difference is 7 which is greater than the minimum link-difference of 5 which is that of the path marked out with a $\bigcirc$. In Fig. 5 the path marked out with a $\bigcirc$ is (in this example) the only member of $\mathbb{C}^x_y$ and has a link-difference of 1 . Thus a construction has been established which for the pixel plane we can surmise determines a set of pixels corresponding to those which would cover a staright line of points as in Fig. 1.

7 Conclusions and further work

In this paper pixels have been considered without reference to the Hausdorff notion of a 'point' such as is found in the real plane $\Re^2$. To demonstrate that pixels can be studied independently of points the traditional procedure of axiomatising an incidence geometry of points [4] has been followed to create an incidence geometry of pixels. This axiomatisation has been developed into the affine world where the notion of cotransverse lines is seen to capture the notion of 'direction' which in a geometry of points is defined in terms of parallelism. Tolerances have been studied as possible models for pixel geometry. The problem of how to model a straight line between two pixels for which no such line exists has been addressed for affine pixel geometry. A case has thus been made for a discrete approach to mathematical structures such as geometry traditionally regarded as being Hausdorff separable as this term is understood in point set topology. Pixel geometry itself can be further developed for possible applications in computing areas such as digital topology [3]. The resolution (i.e. the size) of pixels has not been considered in this paper, but is an interesting subject for future research. The notion of a straight line as found in point geometry is approached in this work as being a discrete data structure of pixels glued together by a tolerance relation. This example suggests that a new style of denotational semantics involving the structuring of discrete entities without the notions of continuity and approximation inherent to the Scott-Strachey approach may indeed be possible.

References

[1] L.C. Paulson. *ML for the working programmer*, Cambridge University Press 1991.

[2] Felix Hovsepian. *A Metalogical Analysis of Vagueness: An Exploratory Study into the Geometry of Logic*, doctoral thesis, University of Warwick, 1992.

[3] T.Y. Kong & A. Rosenfeld. *Digital topology: introduction and survey*, Computer Vision, Graphics, and image processing, Vol.48 (1989), pp.357-393.

[4] W.P. Prenowitz & M. Jordan, *Basic Concepts of Geometry*, Blaisdell Publishing Company, 1965.

[5] Joseph E. Stoy. *Denotational Semantics: The Scott-Strachey Approach to Programming language Theory.* The MIT Press, 1977.