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# Orbit Complexity and Entropy for Group Endomorphisms (Extended Abstract)

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**Abstract**

We consider the pointwise inequality (orbit complexity $\leq$ topological entropy), known in the case of computable maps and computable metric spaces, for endomorphisms of locally compact groups with an arbitrary upper semicomputable distance. Weaker conditions on the effectiveness of the product and metric neighbourhoods are observed which, in $\mathbb{R}^n$, are transferred to a norm-induced metric and used to prove a version of the inequality on locally compact abelian groups.

*Keywords:* Orbit complexity, topological entropy, effective uniform equivalence

## 1 Introduction

Let $(X, d)$ be a metric space and $T : X \to X$ a uniformly continuous map. The topological entropy $h_d(T) \in [0, \infty]$ is a well-known quantity in dynamical systems (dependent on the uniformity induced by $d$) which in a certain sense measures how difficult initial segments of orbits are to specify. To discuss the complicatedness of an individual orbit quantitatively, there are many possible indicators, including Brin-Katok local entropy with respect to an invariant measure, and dimension-like characteristics of the orbit closure, such

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as topological entropy, Hausdorff dimension or box dimension. There is also the orbit complexity $K^{\text{sup}}$ of Brudno [3], which perhaps should be called 'algorithmic entropy', though it is quite coarse by the standards of algorithmic information theory. On compact spaces, complexity and entropy have various relations, including $\sup_{x\in X} K^{\text{sup}}(x,T) = h_d(T)$. Motivated as an extension to (metrizable separable) noncompact phase spaces $X$, an equivalent definition of orbit complexity $\bar{S}$ was proposed by Galatolo ([6]), with respect to a given numbering $\nu :\subseteq \mathbb{N} \to A$ of a dense subset $A \subseteq X$. This highlighted the use of assumptions from computable analysis to prove pointwise results about complexity and entropy. At the same time, many of Galatolo's results apply in less effective situations, and since complexity depends only on the uniform class of the metric, this leads us to hope for information about intuitively noncomputable noncompact systems also. The notable exception is the inequality $\bar{S}(x, T, \nu, (X, d)) \leq h_d(T)$, whose general proof requires an effective means of separating points. In this work we observe how one can proceed in the direction of this inequality when $(X, d, \nu)$ has an effectively separable semicomputable metric structure and $T$ is an endomorphism of a compactly generated locally compact abelian group; such groups are known to have the form $\mathbb{R}^a \times \mathbb{Z}^b \times F$ where $F$ is the maximal compact subgroup. The main item of interest is the trick used to reduce 'calculation' of $\bar{S}$ for linear $S : (\mathbb{R}^a, d) \to (\mathbb{R}^a, d)$ and a translation-invariant metric $d$ to the corresponding calculation for a norm - see Section 4, where an appropriate class of $\nu$ is introduced and linear algebra set in these terms. For definiteness, we should state the final result. Orbit complexity, entropy & relevant inequalities are reviewed in Section 3. Basic definitions and notation are given in Section 2.

**Theorem 1.1** *Let $G$ be a locally compact compactly generated abelian group with invariant metric $d$, $\nu :\subseteq \mathbb{N} \to G$ effectively separable and semicomputable such that $+ : G \times G \to G$ is approximable. If there exists a $\nu$-computable sequence dense in the maximal compact subgroup $F$, then any continuous group homomorphism $T : G \to G$ which is approximable with respect to $\nu$, $d$ has $\bar{S}(x, T, \nu, (G, d)) \leq h_d(T)$ for all $x \in G$.*

Throughout we work in the framework of classical mathematics, including the Axiom of Choice, and use the theory of recursive functions $\subseteq \mathbb{N} \to \mathbb{N}$ in this context. Although Church's thesis is used freely, *recursively enumerable* (r.e.), *partial recursive* (p.r.), *total recursive* (t.r.) are preferred to "computably enumerable" (c.e.), etc. to avoid confusion with computable analysis concepts.

## 2 Notation

Throughout, $f :\subseteq X \to Y$ denotes a partial function with domain $\emptyset \subseteq \operatorname{dom} f \subseteq X$; $f(x)$ is defined ($f(x)\downarrow$) for $x \in \operatorname{dom} f$ and undefined ($f(x)\uparrow$) for $x \in X \setminus \operatorname{dom} f$, while if $A \subseteq Y$ then $f^{-1}A := \{x \in \operatorname{dom} f | f(x) \in A\}$. $f : X \to Y$ always denotes a total function. We often identify $\mathbb{N}$ and $\{0,1\}^*$ via the lexicographical ordering $\lambda, 0, 1, 00, \ldots$, where $\lambda$ is the empty string, and use as a volume function the binary length $\mathrm{lh}_2(a) = \lfloor \log_2(a+1) \rfloor$. $\langle \cdot \rangle$ is used to denote standard tupling functions of various arities.

Given a space $X$, for convenience we abuse the notation of Galatolo by calling a partial function $\nu :\subseteq \mathbb{N} \to X$ an *interpretation* of $X$, i.e. a *numbering* of some (possibly empty) countable subset $A := \nu(\operatorname{dom}\nu)$. Usually, however, $\nu$ will at least be dense. Many-one reducibility is defined as usual by: $\nu \le \lambda$ iff there exists p.r. $f :\subseteq \mathbb{N} \to \mathbb{N}$ such that $\operatorname{dom}\nu \subseteq f^{-1}\operatorname{dom}\lambda$ and $\nu|_{\operatorname{dom}\nu} = (\lambda \circ f)|_{\operatorname{dom}\nu}$. Similarly we denote $\nu \equiv \lambda \iff (\nu \le \lambda) \wedge (\lambda \le \nu)$. For a metric $d$ on $X$, recalling a sequence $(x_n)_1^\infty \subseteq X$ is (strictly) normed if $d(x_m, x_n) < 2^{-\min\{m,n\}}$ for all $m, n \ge 1$, define the derived interpretation $\mathcal{N}\nu = \mathcal{N}_d\nu$ by $\mathcal{N}_d\nu(p) = \lim_{n\to\infty}(\nu \circ \phi_p)(n)$ and $\operatorname{dom}\mathcal{N}_d\nu = \{p \in \mathbb{N} | \mathbb{N} \subseteq \phi_p^{-1}\operatorname{dom}\nu$ and $((\nu \circ \phi_p)(n))_1^\infty$ normed & convergent$\}$, for some fixed (total) acceptable numbering $\phi_0, \phi_1, \ldots$ of the p.r. functions $\mathbb{N} \to \mathbb{N}$. We call any $x \in X_{\mathrm{C}} = X_{\mathrm{C},\nu} := \mathcal{N}_d\nu(\operatorname{dom}\mathcal{N}_d\nu)$ *computable*. In this way a standard numbering $\nu_{\mathbb{R}_c}$ of the computable reals $\mathbb{R}_c$ is obtained from a standard (total) numbering $I_{\mathbb{D}}$ of the dyadic rationals $\mathbb{D} := \{m.2^n | m, n \in \mathbb{Z}\} \subseteq (\mathbb{R}, |\cdot|)$. We also denote left- (right-) computable reals by $\mathbb{R}_{\mathrm{lc}}$ ($\mathbb{R}_{\mathrm{rc}}$).

If $\nu$, $\lambda$ are interpretations of $X$, $Y$, recall a map $f : \nu(\operatorname{dom}\nu) \to \lambda(\operatorname{dom}\lambda)$ is usually called $(\nu,\lambda)$-effective if $f \circ \nu \le \lambda$ (with the usual composition for partial maps). We will abuse this notation by calling instead total $T : X \to Y$ $(\nu,\lambda)$-*effective* if $T|_{\nu(\operatorname{dom}\nu)}$ has the above property - this should not cause confusion as almost all maps mentioned in this paper will be total. If $(X, d, \nu)$, $(Y, d', \lambda)$ are understood, we will call a map $f : X \to Y$ *approximable* if some p.r. $F :\subseteq \mathbb{N} \times \mathbb{Q}^+ \to \mathbb{N}$ has $(a, \eta) \in (\nu \circ F)^{-1}B(f\nu(a); \eta)$ for all $a \in \operatorname{dom}\nu$, $\eta \in \mathbb{Q}^+$, and $T : X \to X$ *effectively iterable* if some p.r. $F :\subseteq \mathbb{N}^2 \times \mathbb{Q}^+ \to \mathbb{N}$ has $F(a, j, \eta) \in \nu^{-1}B(T^j\nu(a); \eta)$ for all $a \in \operatorname{dom}\nu$, $j \in \mathbb{N}$ $\eta \in \mathbb{Q}^+$. In this terminology, $f$ is approximable iff $f \circ \nu \le \mathcal{N}\lambda$ iff it is $(\nu, \mathcal{N}\lambda)$-effective. If $f$ is approximable and eff. uniformly continuous (with respect to the metrics $d$, $d'$), it will be $(\mathcal{N}\nu, \mathcal{N}\lambda)$-effective, hence effectively iterable if $(X, d, \nu) = (Y, d', \lambda)$.

A triple $(X, d, \nu)$ is *effectively separable* if some r.e. $A \subseteq \operatorname{dom}\nu$ has dense image, or *semicomputable* (s.c.) if some p.r. $f :\subseteq \mathbb{N}^3 \to \mathbb{Q}^+$ has $(\operatorname{dom}\nu)^2 \times \mathbb{N} \subseteq \operatorname{dom} f$ and $(f(a, b, k))_{k=0}^\infty$ strictly decreasing with limit $d(\nu(a), \nu(b))$ whenever $a, b \in \operatorname{dom}\nu$. Equivalently, we could require some p.r. $F :\subseteq \mathbb{N} \times \mathbb{Q}^+ \times \mathbb{N} \to \mathbb{N}$ enumerates, up to $\operatorname{dom}\nu$, all $\nu$-names in ideal balls, i.e. $\operatorname{dom}\nu \times \mathbb{Q}^+ \times \mathbb{N} \subseteq$

$\operatorname{dom} F$ and $F(a, \epsilon, \mathbb{N}) \cap \operatorname{dom} \nu = \nu^{-1} B(\nu(a); \epsilon)$ for all $a \in \operatorname{dom} \nu$, $\epsilon \in \mathbb{Q}^+$. Here, let $A_{a,\epsilon} := F(a, \epsilon, \mathbb{N}) \setminus \operatorname{dom} \nu$. Such $(X, d, \nu)$ (when $\nu$ is dense and total) are also called "computable metric spaces" in the literature; we use this term only when distances are also approximable from below. If $(X, d, \nu)$ is s.c., so are $(X, d, \mathcal{N}\nu)$ and $(X, d, \lambda)$ for any $\lambda :\subseteq \mathbb{N} \to X$ with $\lambda \leq \nu$.

## 3 Complexity, entropy & group quotients

Recall the *Kolmogorov complexity* of $n \in \mathbb{N}$ with respect to a p.r. function $f :\subseteq \mathbb{N} \to \mathbb{N}$ is the length of a shortest input which produces the output $n$,

$$K_f(n) = \min\{\mathrm{lh}_2(m) | m \in \operatorname{dom} f \wedge f(m) = n\},$$

with $\min \emptyset = +\infty$, and that there exists an *additively optimal* $D$, such that:

$$(\forall \text{p.r. } g)(\exists c_{D,g} \in \mathbb{N})(\forall n)\, (K_D(n) \leq K_g(n) + c_{D,g})\,;$$

for example $\exists c \forall n (K_D(n) \leq \mathrm{lh}_2(n) + c)$ (see e.g. [4] or [10]). Consider now a metric space $(X, d)$ equipped with an interpretation $\nu :\subseteq \mathbb{N} \to X$, and for p.r. $f :\subseteq \mathbb{N} \to \mathbb{N}$, $W : \mathbb{N}^* \to \mathbb{N}$ injective and effective (with respect to a standard numbering, see e.g. [13] or [8]) and $(x_i)_0^\infty \subseteq X$, similarly define

$$\mathcal{F}^\nu_{\epsilon,W,f}((x_i)_0^{m-1}) = \min\{K_f(W(n_0, \ldots, n_{m-1})) | d(\nu(n_i), x_i) < \epsilon \text{ for } 0 \leq i \leq m-1\}$$

for all $m \geq 1$, $\epsilon > 0$; again if $f$ is additively optimal, then for each $W$, $W'$, $f'$ there exists $c \in \mathbb{N}$ such that

$$(\forall m \geq 1)(\forall x_0, \ldots, x_{m-1} \in X) \mathcal{F}^\nu_{\epsilon,W,f}((x_i)_0^{m-1}) \leq \mathcal{F}^\nu_{\epsilon,W',f'}((x_i)_0^{m-1}) + c$$

(a proof of this and equivalence with the slightly different coding of points in [6] are omitted). In particular, fixing $f$, $W$ (and $K = K_f$) will not change

$$\bar{S}_\epsilon((x_i)_0^\infty, (X, d, \nu)) := \limsup_{m \to \infty} \frac{1}{m} \mathcal{F}^\nu_{\epsilon,W,f}((x_i)_0^{m-1}).$$

Although taking such a growth rate destroys many interesting properties of $K$ (moreover, a divisor $g(n) := n$ with $\log_2 \ldots \log_2 = \mathbf{o}\,(g)$ means we can replace $K$ with other complexities [11]), it is well-motivated from the viewpoint of comparison with established entropies in ergodic theory: for $T : X \to X$ and $x \in X$, $\bar{S}_\epsilon(x, T, (X, d, \nu)) := \bar{S}_\epsilon((T^i x)_0^\infty, (X, d, \nu))$ is an optimal upper bound on the average information (bits) per iterate needed to specify long initial segments of $\xi = (T^i x)_{i=0}^\infty$ to within $\epsilon$; compare with $h_d(T)$ below. Noting $\bar{S}_\epsilon$ is nonincreasing in $\epsilon$ and taking $\bar{S}(x, T, (X, d, \nu)) := \lim_{\epsilon \searrow 0} \bar{S}_\epsilon(x, T, (X, d, \nu))$, similarly $\bar{S}(\xi, (X, d, \nu))$, we obtain a quantity called *orbit complexity at* $x$ [6] when $\nu$ is dense, dependent only on the uniformity (see Lemma 3.2(vi)).

Several relations between complexity and various entropies are worth mentioning, even though we shall only deal directly with $h_d(T)$. Firstly, if $(X, d)$

is compact, for continuous $T$ there always exist Borel probability measures $\mu$ which are $T$-invariant and ergodic (e.g. [12]). It is known ([3])

(1) $$K^{\sup}(x,T) = h_\mu(T) \ \mu\text{-a.e. for any ergodic } T\text{-invariant } \mu,$$

where $h_\mu(T)$ is the measure-theoretic entropy (e.g. [12]), and moreover

(2) $$\sup_{x\in X} K^{\sup}(x,T) = h_d(T).$$

Thus $K^{\sup}$ (which is topologically defined) simultaneously carries all the traditional ergodic-theoretic information about invariant measures. If also $X$ is a computable metric space, one of the results of [6] states $\bar{S}(\cdot,\nu) = K^{\sup}$, so $\bar{S}(\cdot,\nu)$ shares these properties. More generally (by a direct proof not discussed here) the lower Brin-Katok local entropy $h^-_\mu(T,x)$ bounds $\bar{S}(\cdot,\nu)$ below $\mu$-almost everywhere; since this coincides[2] $\mu$-a.e. with $h_\mu(T)$, and $\bar{S} \le K^{\sup}$ ([6, Thm 10]), we obtain from the variational principle[12] $\sup_\mu h_\mu(T) = h_d(T)$ that (1) and (2) hold for any dense $\nu$. For an endomorphism of a locally compact group, Haar measure may not be ergodic or invariant, but it satisfies $(\forall x) h^-_\mu(T,x) = h_d(T)$ [1], so to get analogues of (1) and (2) it suffices to prove the bound $\bar{S} \le h_d(T)$. We now proceed to stating elementary properties of $\bar{S}$, but first it is useful to have a notion of complexity independent of $T$[5]:

**Definition 3.1** Given interpretation $\nu :\subseteq \mathbb{N} \to X$, the *complexity of a point $x \in X$ with respect to $\nu$* is $\bar{C}(x,\nu,(X,d)) = \limsup_{m\to\infty} \frac{1}{m}\mathcal{C}^\nu_m(x)$ where, for each $m \in \mathbb{N}$, $\mathcal{C}^\nu_m(x) = \min\{K(a) | a \in \nu^{-1}B_d(x;2^{-m})\}$.

**Lemma 3.2** *For metric space $(X,d)$, $(x_i)_0^\infty \subseteq X$, $x \in X$ and interpretations $\nu, \lambda :\subseteq \mathbb{N} \to X$:*

*(i) If $(x_i)_0^\infty \subseteq Y \subseteq X$ and $\nu(\operatorname{dom}\nu) \subseteq Y$, then $\bar{C}(x_0,\nu,(X,d)) = \bar{C}(x_0,\nu,(Y, d_Y))$ and $\bar{S}_\epsilon((x_i)_0^\infty,\nu,(X,d)) = \bar{S}_\epsilon((x_i)_0^\infty,\nu,(Y,d_Y))$ for all $\epsilon > 0$.*

*(ii) If $\nu \le \lambda$ then $\bar{C}(x,\lambda,X) \le \bar{C}(x,\nu,X)$ and $\bar{S}_\epsilon((x_i)_0^\infty,\lambda,X) \le \bar{S}_\epsilon((x_i)_0^\infty,\nu,X)$.*

*(iv) $\bar{C}(x,\nu,X) = \bar{C}(x,\mathcal{N}\nu,X)$ and $\bar{S}_{\epsilon+\eta}((x_i)_0^\infty,\nu,X) \le \bar{S}_\epsilon((x_i)_0^\infty,\mathcal{N}\nu,X)$.*

*(v) $\bar{S}_\epsilon\left(((x_j^{(1)},x_j^{(2)}))_{j=0}^\infty,\nu_1\times\nu_2,(X_1\times X_2,d)\right) \le \sum_i \bar{S}_\epsilon((x_j^{(i)})_{j=0}^\infty,\nu_i,(X_i,d_i))$ and $\bar{C}((x_1,x_2),\nu_1\times\nu_2,(X_1\times X_2,d)) \le \sum_i \bar{C}(x_i,\nu_i,(X_i,d_i))$, where $d((x_1,x_2),(y_1,y_2)) := \max_i d_i(x_i,y_i)$.*

*(vi) If $\Psi : X \to Y$ is uniformly continuous, for all $\epsilon > 0$ there exists $\delta > 0$ such that $\bar{S}_\epsilon((\Psi x_i)_0^\infty,\Psi\circ\nu,Y) \le \bar{S}_\delta((x_n)_0^\infty,\nu,X)$ (hence $\bar{S}$ is invariant under uniformly equivalent metrics). If $\Psi$ is $\alpha$-Hölder continuous ($0 < \alpha \le 1$), $\bar{C}(\Psi x,\Psi\circ\nu,Y) \le \frac{1}{\alpha}\bar{C}(x,\nu,X)$.*

*(vii) If $x \in \nu(\operatorname{dom}\nu)$ then $\bar{C}(x,\nu,(X,d)) = 0$, and $\bar{S}(x,T,\nu,(X,d)) = 0$ if $T : X \to X$ is effectively iterable.*

*(x) If $T : X \to X$ is effectively iterable and Lipschitz with constant $C \ge 1$,*

$\bar{S}(x, T, \nu, (X, d)) \leq \log_2 C.\bar{C}(x, \nu, (X, d))$ *for all* $x \in X$.

*(xi) For any* $k \geq 1$ *and* $0 \leq r \leq k-1$,

$$\bar{S}_\epsilon((x_i)_0^\infty, \nu, X) = \limsup_{m\to\infty} \frac{\mathcal{F}_\epsilon^\nu((x_i)_0^{km+r-1})}{km+r},$$

$$\bar{C}(x, \nu, (X, d)) = \limsup_{n\to\infty} \frac{\mathcal{C}_{kn+r}^\nu(x)}{kn+r},$$

$$\text{and } \bar{S}_\epsilon((x_{ik+j})_{i=0}^\infty, \nu, (X, d)) \leq k\bar{S}_\epsilon((x_i)_0^\infty, \nu, (X, d)).$$

*(xii) If* $\epsilon, \eta > 0$ *and* $T$ *is* $\nu$*-approximable and uniformly continuous, then*

$$\bar{S}_{\epsilon+\eta}(x, T, \nu, (X, d)) \leq \frac{1}{k}\bar{S}_\epsilon(x, T^k, \nu, (X, d_{k,T})),$$

*hence* $\bar{S}(x, T^k, \nu, (X, d_{k,T})) = k\bar{S}(x, T, \nu, (X, d))$.

Here $d_{k,T}(x, y) := \max_{0\leq i\leq k-1} d(T^i x, T^i y)$ $(k \geq 1)$ are metrics uniformly equivalent to $d$ (eff. unif. equivalent if $T$ is eff. unif. continuous) which can be used also to give a definition of topological entropy (e.g. [12]). Namely, for compact $K \subseteq X$ a set $Y \subseteq K$ is a $(n, \epsilon)$-*spanning subset* if the closed $d_{n,T}$-balls $\bar{B}_{d_{n,T}}(y; \epsilon)$ $(= \cap_{i=0}^{n-1} T^{-i}\overline{B}(T^i y; \epsilon))$ $(y \in Y)$ cover $K$; obviously such $Y$ can be chosen to be of minimal (finite) cardinality $S'_d(K, T, \epsilon, n) := |Y|$. One then defines $h_d(K, T) := \lim_{\epsilon\searrow 0} \limsup_{n\to\infty} \frac{1}{n} \log_2 S'_d(K, T, \epsilon, n)$ and the *topological entropy* $h_d(T) := \sup_K h_d(K, T)$, depending only on the uniform structure, so denoted $h_{\text{top}}(T)$ in the compact case. Here $\limsup_{n\to\infty} \frac{1}{n} \log_2 S'_d(K, T, \epsilon, n)$ may be interpreted as an optimal upper bound for the information necessary to "distinguish up to $\epsilon$" a long initial segment of an arbitrary forward orbit $(T^i x)_0^\infty$ starting in $K$.

Now we review the situation for quotients of metric groups. Namely, let $G$ be a metrizable topological group, $H$ a closed subgroup, choose a metric $d$ on $G$ invariant under the right multiplications $R_h (h \in H)$, and denote by $\tilde{d}$ the corresponding metric on the left-coset space $G/H$: $\tilde{d}(xH, yH) = \inf_{h\in H} d(x, yh)$. This makes the projection $\pi : G \to G/H, x \mapsto xH$ uniformly continuous, and any uniformly continuous $T : G \to G$ with the property

$$T(xH) \subseteq (Tx)H \text{ for all } x \in G, \tag{3}$$

projects naturally to a uniformly continuous factor $S : G/H \to G/H, xH \mapsto (Tx)H$, i.e. $S \circ \pi = \pi \circ T$. When also $G$ is compact and

$$\tau(h) := (Tx)^{-1}T(xh) \in H \text{ is independent of } x \in G, \tag{4}$$

it is known that ([1, Theorem 19])

$$h_{\text{top}}(T) = h_{\text{top}}(S) + h_{\text{top}}(\tau);$$

in particular this applies with $\tau = T|_H$ when $T$ is a continuous group homomorphism and $T(H) \subseteq H$. In fact, the following statement holds (the proof is omitted); note for purposes of application that continuous homomorphisms are always uniformly continuous with respect to the left uniformity.

**Proposition 3.3** *Assume $G$ is locally compact & metrizable, $d$ is a left-invariant metric, $T$ (uniformly continuous with respect to $d$) satisfies (3) and (4), and $H$ is compact. Then*

$$d^H(x,y) := \sup_{h\in H} d(xh, yh)$$

*is a uniformly equivalent metric invariant under $L_g(g \in G)$ and $R_h(h \in H)$, $\pi : (G,d) \to (G/H, \tilde{d^H})$ and $S : (G/H, \tilde{d^H}) \to (G/H, \tilde{d^H})$ are uniformly continuous, and*

$$h_d(T) = h_{\tilde{d^H}}(S) + h_{\mathrm{top}}(\tau)$$

In one direction at least, there is an analogous pointwise result for complexity, which we now give:

**Theorem 3.4** *Suppose $G$ is a metrizable separable topological group, $H$ a compact subgroup, $d$ a metric on $G$ invariant under $L_g$ $(g \in G)$ and $R_h$ $(h \in H)$, and the group operation is $\nu$-approximable for some dense $\nu :\subseteq \mathbb{N} \to G$. For any $\nu$-approximable and uniformly continuous map $T : G \to G$ satisfying (3) and (4), and any $x \in G$,*

$$\bar{S}(x, T, \nu, (G,d)) \leq \bar{S}(xH, S, \pi\circ\nu, (G/H, \tilde{d})) + h_{\mathrm{top}}(\tau).$$

**Proof (Sketch)** One checks from the conditions on $T$ that $\tau$ is a continuous endomorphism of $H$. By the use of uniform equivalence of $\widetilde{d_{n,T}}$, $\tilde{d}$ and Lemma 3.2(xi),(xii), it is enough to prove

$$(5)\bar{S}_\theta(x, T^n, \nu, (G, d_{n,T})) \leq \bar{S}(xH, S^n, \pi\circ\nu, (G/H, \widetilde{d_{n,T}})) + \log_2 S'_{d|_H}(H, \tau, \epsilon, n)$$

for large $n$ and arbitrary small $\theta$, $\epsilon$. In line with this, we claim an $\epsilon$-approximation $(n_j)_0^{k-1}$ of $(S^{jn}(xH))_{j=0}^{k-1}$ with respect to $\pi\circ\nu$, $\widetilde{d_{n,T}}$ and a suitable "coding" $(\gamma_j)_0^{k-1} \subseteq \{1,\ldots,|E|\}$ of the trajectory $(T^{jn}x)_0^{k-1}$ with respect to a fixed $(n,\epsilon)$-spanning subset $E \subseteq H$ (with respect to $\tau$, $d|_H$) are enough to determine a $4\epsilon$-approximation of $(T^{jn}x)_0^{k-1}$ with respect to $\nu$ and $d_{n,T}$. Namely, choosing $\gamma_j$ based on the error approximating $T^{jn}x$ by $\nu(n_j)$ and using left-invariance of $d$, we obtain a $2\epsilon$-approximation to $T^{jn}x$ in the form $\nu(n_j).a_{\gamma_j}$ where $(a_i)_1^{|E|} = E$, and then use uniform equivalence of $d$, $d_{n,T}$ to check these products can be approximated uniformly in $x = \nu(n_j)$. Since $\nu$-approximations to $a_i$ may be made independently of $k$, in the limit the desired inequality holds. □

## 4 Weak semicomputability

Given $(X, d_1, \nu)$ and $f : X \to X$ which is $\nu$-approximable with respect to $d_1$, if another metric $d_2$ is such that $\mathrm{id} : (X, d_1) \to (X, d_2)$ is effectively uniformly continuous, it is easy to check $f$ is $\nu$-approximable with respect to $d_2$ (or note $\mathcal{N}_{d_1}\nu \leq \mathcal{N}_{d_2}\nu$ directly). In this context the following observation is interesting:

**Proposition 4.1** *For any (real or complex) normed space $(X, \|\cdot\|)$, for any translation-invariant metric $d$ inducing the topology of $X$, the map $\mathrm{Id} : (X, d) \to (X, \|\cdot\|)$ is effectively uniformly continuous.*

**Proof.** Denote $U_k := B_d(0; 2^{-k})$ $(k \geq 1)$ and note these have the property $(2U_{k+1} \subseteq)\ U_{k+1} + U_{k+1} \subseteq U_k$. Certainly there exists $N$ such that $U_N \subseteq B_{\|\cdot\|}(0; 1)$ (by topological equivalence of $d$ and $\|\cdot\|$), so given $\mathbb{Q}^+ \ni \epsilon < 1$, pick $k = \lfloor \log_2 \left(\frac{1}{\epsilon}\right) \rfloor + N + 1$. We then have $U_{k+1} \subseteq 2^{-1}U_k \subseteq \ldots \subseteq 2^{-(k-N+1)}U_N \subseteq B_{\|\cdot\|}(0; \epsilon)$, using the scaling property of $\|\cdot\|$. □

Since some algorithms need to be able to recognise nearby points, we would like to obtain a form of semicomputability invariant under such changes of metric. The following is the weakest of several obvious definitions, and rather ill-formed; note that $U := \cup_{n \in \mathrm{dom}\,\nu} \cup_{k \in \mathbb{N}} U_k^{(n)}$ may be a (dense if $\nu$ is) proper open subset of $X$. However in our algebraic setting it is strong enough to get somewhere, as the properties immediately following the definition show.

**Definition 4.2** In a metric space $(X, d)$, an interpretation $\nu :\subseteq \mathbb{N} \to X$ is *weakly semicomputable* if there exist open neighbourhoods $U_k^{(n)} \subseteq B_d(\nu(n); 2^{-k})$ of $\nu(n)$ $(n \in \mathrm{dom}\,\nu, k \in \mathbb{N})$ and some p.r. $F :\subseteq \mathbb{N}^3 \to \mathbb{N}$ with $\mathrm{dom}\,\nu \times \mathbb{N}^2 \subseteq \mathrm{dom}\,F$ such that

$$(\forall n \in \mathrm{dom}\,\nu)(\forall k \in \mathbb{N})(\exists A_{n,k} \subseteq \mathbb{N} \setminus \mathrm{dom}\,\nu)\{F(n, k, l) | l \in \mathbb{N}\} = A_{n,k} \dot{\cup} \nu^{-1}U_k^{(n)}$$

**Proposition 4.3** *(i) For an effectively separable weakly semicomputable interpretation $\nu$ of a topological group $X$ with right-invariant metric $d$, if the product is $\nu$-approximable then so is the inverse, and identity $e \in X_C$.*

*(ii) Given $\nu :\subseteq \mathbb{N} \to X$, if homeomorphism $\Psi : (X, d) \to (\Psi(X), d')$ is effectively uniformly continuous (in the forward direction) and $(X, d, \nu)$ is weakly semicomputable, then so is $(\Psi(X), d', \Psi \circ \nu)$.*

*(iii) If $(X, \|\cdot\|)$ is a real normed space, $\nu :\subseteq \mathbb{N} \to X$ is effectively separable and weakly semicomputable, and $+ : X \times X \to X$ is $\nu$-approximable, then $\frac{1}{2}\mathrm{Id} : X \to X$ is $(\mu, \mu)$-effective, where $\mu = \mathcal{N}_{\|\cdot\|}\nu$.*

*(iv) In metric space $(X, d)$, if $\nu, \lambda :\subseteq \mathbb{N} \to X$ have $\lambda \leq \nu$ and $\nu$ is weakly semicomputable, then so is $\lambda$.*

Properties (ii) & (iv) are obvious from the definitions, or at least have rather canonical methods of proof. So we give proofs only for (i) & (iii).

**Proof of (i)** Assume without loss of generality $|X| > 1$, pick $a \in \nu^{-1}(X \setminus \{e\})$ arbitrarily, let $C \subseteq \operatorname{dom}\nu$ be an r.e. set with dense image, and let p.r. $P :\subseteq \mathbb{N}^2 \times \mathbb{Q}^+ \to \mathbb{N}$ witness approximability of the group operation. To see $e \in X_{\mathrm{C}}$, consider an algorithm which, on input $i \in \mathbb{N}$, dovetails calculating and searching for $P(n, a, 2^{-k})$ in an enumeration of $A_{\dots}\dot{\cup}\nu^{-1}U^{(a)}_{i+1}$ over all $k \geq i+1$, $n \in C$, halting and outputting $n$ if found. By density of $\nu(C)$, there exists some $n \in C$ with $\nu(n) \in U^{(a)}_{i+1}\nu(a)^{-1}$, and then $(\exists k \geq i+1)(\nu \circ P)(n, a, 2^{-k}) \in U^{(a)}_{i+1}$, so the algorithm must halt. Conversely, any output $n$ has $n \in C \subseteq \operatorname{dom}\nu \Rightarrow P(n, a, 2^{-k}) \in \nu^{-1}U^{(a)}_{i+1}$, so $d(\nu(n), e) = d(\nu(n)\nu(a), \nu(a)) < 2^{-k} + 2^{-i-1} \leq 2^{-i}$, and the output correctly provides an approximation to $e$ within $2^{-i}$.

So let t.r. $f : \mathbb{N} \to \mathbb{N}$ have $(\forall n)(f(n) \in \nu^{-1}B(e; 2^{-n}))$, and consider an algorithm which on input $a, i \in \mathbb{N}$ dovetails computing and searching for $c := P(b, a, 2^{-k})$ in an enumeration of $A_{\dots}\dot{\cup}\nu^{-1}U^{(f(i+2))}_{i+2}$ over all $k \geq i+2$, $b \in C$, halting and outputting $b$ if a match is found. Certainly each output $b$ has $b \in \operatorname{dom}\nu$, so if $a \in \operatorname{dom}\nu$ then $d(\nu(b), \nu(a)^{-1}) = d(\nu(b)\nu(a), e) < d(\nu(b)\nu(a), \nu(c)) + 2^{-i-2} + 2^{-i-2} < 2^{-i}$. Conversely, some $b \in C$ must have $\nu(b) \in U^{(f(i+2))}_{i+2}\nu(a)^{-1}$, and then $(\exists k \geq i+2)(\nu \circ P)(b, a, 2^{-k}) \in U^{(f(i+2))}_{i+2}$, so the algorithm must halt, showing $\cdot^{-1} : X \to X$ is approximable. □

**Proof of (iii) (Sketch)** By an obvious algorithm one can effectively find $b$ such that $2\nu(b)$ is $2^{-k+1}$-close to $\nu(a)$, and the scaling property of $\|\cdot\|$ guarantees an approximation to $\frac{1}{2}\nu(a)$. On the other hand, $\frac{1}{2}\mathrm{Id}$ is an open mapping, so we will eventually find $b \in \nu^{-1}(\frac{1}{2}U^{(a)}_k)$, and the algorithm (demonstrating $\frac{1}{2}\mathrm{Id}$ is approximable) must halt. Since $\frac{1}{2}\mathrm{Id}$ is a bounded linear operator, the $\mu$-effectivity follows. □

**Remark 4.4** The converse of Proposition 4.1 is not true, since it is easy to construct an invariant metric $d$ on $\mathbb{R}^d$ for which $\mathrm{Id} : (\mathbb{R}^d, \|\cdot\|) \to (\mathbb{R}^d, d)$ is not eff. unif. continuous for any norm $\|\cdot\|$; let $U_k = B_{\|\cdot\|}(0; 2^{-m_k})$ where strictly increasing $(m_k)_1^\infty \subseteq \mathbb{N}$ is not bounded by any t.r. function, note these still have $(\forall k)(U_{k+1} + U_{k+1} \subseteq U_k)$ and $\cap_k U_k = \{0\}$, and apply a standard construction [9, pg 68] to get an invariant metric $d$ with $U_{k+3} \subseteq B_d(0; 2^{-k}) \subseteq U_k$ for all $k$.

With this notion established, we elaborate on the linear algebra situation under the assumptions of (iii) above. Since vector addition $+$ and negation $- : X \to X$ are trivially eff. uniformly continuous (for any fixed bi-invariant metric), for scalar multiplication $k : \mathbb{R} \times X \to X, (\alpha, x) \mapsto \alpha x$ we find $k_{\mathbb{D}} := k|_{\mathbb{D}\times X}$ is $(I_{\mathbb{D}} \times \nu, \mathcal{N}_{\|\cdot\|}\nu)$-effective. The proof of the next lemma is omitted.

**Lemma 4.5** *For d-dimensional real normed space $(X, \|\cdot\|)$, weakly s.c. $\nu$ such that $+$, $k_{\mathbb{D}}$ are approximable, and linearly independent $v_1, \ldots, v_m \in X_C$:*

*(i) If $V = \mathrm{span}_{\mathbb{R}}\{v_1, \ldots, v_m\}$, $\emptyset \neq J \subseteq \{1, \ldots, m\}$ and $V_J = \mathrm{span}_{\mathbb{R}}\{v_i | i \in J\}$, the linear map $P^V_{V_J} : V \to X, \sum_{i=1}^m \alpha_i v_i \mapsto \sum_{i\in J} \alpha_i v_i$ is $\nu$-approximable as a partial function $\subseteq X \to X$. For $m = d$, $P_{V_J} = P^X_{V_J}$ is a $(\mu, \mu)$-effective total linear map for $\mu := \mathcal{N}_{\|\cdot\|}\nu$. For $J = \{j\}$, the functional $p^V_{v_j} :\subseteq X \to \mathbb{R}, \sum_{i=1}^m \alpha_i v_i \mapsto \alpha_j$ is $(\nu, I_{\mathbb{D}})$-approximable with respect to $|\cdot|$, and a $(\mu, \nu_{\mathbb{R}_c})$-effective total function if $m = d$.*

*(ii) If $u = \sum_{i=1}^m \alpha_i v_i \in X_C$ for some $(\alpha_i)_1^m \subseteq \mathbb{R}$ then $(\alpha_i)_1^m \subseteq \mathbb{R}_c$. If $m = d$, then $X_C = \mathrm{span}_{\mathbb{R}_c}\{v_1, \ldots, v_d\}$, and any linear $S : X \to X$ with $S(X_C) \subseteq X_C$ has $[S]_{(v_1,\ldots,v_d)} \in M_{d\times d}(\mathbb{R}_c)$.*

Actually, it may be more helpful to consider the latter statement in a slightly different form. Denote by $\mathcal{D}$ the class of invariant metrics inducing the topology of $X$, and by $\mathcal{A}$ the class of interpretations $\nu :\subseteq \mathbb{N} \to X$ such that some $d \in \mathcal{D}$ makes $(X, d, \nu)$ eff. separable & weakly s.c. with $+$ approximable. Without loss of generality we assume $d$ is induced by a fixed norm $\|\cdot\|$.

**Corollary 4.6** *For any basis $v_1, \ldots, v_d$,*

$$\lambda :\subseteq \mathbb{N} \to X, \langle q_1, \ldots, q_d\rangle \mapsto \sum_{i=1}^{d} \nu_{\mathbb{R}_c}(q_i).v_i$$

*is weakly s.c., eff. separable and makes vector addition effective. Conversely, for any $\nu \in \mathcal{A}$, basis $v_1, \ldots, v_d \in X_{C,\nu}$ and $\lambda$ as above we have $\mathcal{N}_{\|\cdot\|}\nu \equiv \lambda$.*

So we might say we have identified the "general form" of interpretations $\nu, \nu' \in \mathcal{A}$ of the topological group $(X, +)$, up to equivalence $\mathcal{N}\nu \equiv \mathcal{N}\nu'$. In fact, assuming $\nu$ eff. separable with $+$ approximable, a basis $v_1, \ldots, v_d \in X_{\mathrm{C},\nu}$, and results & notation from [8] including the normed limit operator $N :\subseteq X^{\mathbb{N}} \to X$, we find $\nu$ is weakly s.c. iff the Cauchy representation $\rho$ induced by $\nu$, $\|\cdot\|$ lies in the minimal class of representations making effective the structure $\mathcal{S} = (X, v_1, \ldots, v_d, +, -, \frac{1}{2}\mathrm{Id}, N)$ (which is not r-effectively categorical). For current purposes, though, we use only interpretations, for which we list a few more properties: $\nu \in \mathcal{A} \Rightarrow \mathcal{N}_{\|\cdot\|}\nu \in \mathcal{A}$, while a general $d \in \mathcal{D}$ witnessing $\nu \in \mathcal{A}$ may be replaced with a norm $\|\cdot\|$ such that $(X, \|\cdot\|, \nu)$ is semicomputable (this is essentially part of the proof of Corollary 4.6, which was omitted). On the other hand, it is easy to see examples of weakly s.c. but non-s.c. interpretations:

**Example 4.7** In $(\mathbb{R}, \|\cdot\|_2) = (\mathbb{C}, |\cdot|)$, let $v_1 = 1$, $v_2 = e^{\mathbf{i}\theta}$ for some $\theta \in (0, \frac{\pi}{2})$ with $\theta \in \mathbb{R}_{\mathrm{lc}} \setminus \mathbb{R}_{\mathrm{c}}$, and $\lambda :\subseteq \mathbb{N} \to \mathbb{C}, \langle q_1, q_2\rangle \mapsto \nu_{\mathbb{R}_c}(q_1).v_1 + \nu_{\mathbb{R}_c}(q_2).v_2$. Since $\cos|_{(0,\frac{\pi}{2})}$ is strictly decreasing and computable, $\cos\theta \in \mathbb{R}_{\mathrm{rc}} \setminus \mathbb{R}_{\mathrm{lc}}$, hence

$|v_1 - v_2|^2 = 2(1 - \cos\theta) \in \mathbb{R}_{\mathrm{lc}} \setminus \mathbb{R}_{\mathrm{rc}}$, so $(\mathbb{R}^2, \|\cdot\|_2, \lambda)$ is not semicomputable, but is weakly s.c., eff. separable and makes addition effective.

Now we extend the above considerations to $\mathbb{C}^d$. Namely, extend $\nu :\subseteq \mathbb{N} \to \mathbb{R}^d$ to $\hat{\nu} :\subseteq \mathbb{N} \to \mathbb{C}^d$ by $\mathrm{dom}\,\hat{\nu} = \{\langle a, b\rangle | a, b \in \mathrm{dom}\,\nu\}$ and $\hat{\nu}(\langle a, b\rangle) = \nu(a) + \mathbf{i}\nu(b)$. Considering $\mathbb{R}^d \subseteq \mathbb{C}^d$ and taking a metric $d$ on $\mathbb{R}^d$ to $\hat{d}(x, y) := \max\{d(\Re x, \Re y), d(\Im x, \Im y)\}$, if $0 \in \nu(\mathrm{dom}\,\nu)$ we get

$$\bar{C}(x, \nu, (\mathbb{R}^d, d)) = \bar{C}(x, \hat{\nu}, (\mathbb{C}^d, \hat{d})),\ \bar{S}_\epsilon((x_j)_0^\infty, \nu, (\mathbb{R}^d, d)) = \bar{S}_\epsilon((x_j)_0^\infty, \hat{\nu}, (\mathbb{C}^d, \hat{d}))$$

for any $x \in \mathbb{R}^d$, $(x_j)_0^\infty \subseteq \mathbb{R}^d$, $\epsilon > 0$.

One also checks the following assumptions of approximability (with respect to $\nu$, $d$) on real operations $+$, (unary) $-$, $k_{\mathbb{D}}$ are sufficient for corresponding complex operations to be approximable with respect to $\hat{\nu}$, $\hat{d}$: $+$ for $+$; $-$ for $\mathbf{i}.\mathrm{Id}$; $-$ and $k_{\mathbb{D}}$ for complex scalar multiplication $\hat{k}_{\mathbb{D}} : (\mathbb{D}+\mathbf{i}\mathbb{D})\times\mathbb{C}^d \to \mathbb{C}^d$; and $-$ for conjugation $\mathrm{conj} : \mathbb{C}^d \to \mathbb{C}^d$. If linear $T : \mathbb{R}^d \to \mathbb{R}^d$ is approximable, clearly the complexification $\hat{T} : \mathbb{C}^d \to \mathbb{C}^d, x_1 + \mathbf{i}x_2 \mapsto (Tx_1) + \mathbf{i}(Tx_2)$ is approximable, and if $d \in \mathcal{D}$ witnesses $\nu \in \mathcal{A}$, $T$ will be $(\mu, \mu)$-effective as before, so $\hat{T}$ is $\hat{\nu}$-approximable with respect to $\hat{d}'$ for any norm-induced metric $d'$ on $\mathbb{R}^d$, and also $(a_{i,j})_{i,j} = [T]_{(v_1,\ldots,v_d)} \in M_{d\times d}(\mathbb{R}_{\mathrm{c}})$ for any basis $v_1, \ldots, v_d \in X_{\mathrm{C},\nu}$. Considering the generalised eigenspace $S_\lambda$ for eigenvalue $\lambda$ (for $\lambda \in \mathbb{C}$ recall $S_\lambda = \mathbb{R}^d \cap (\hat{S}_\lambda \oplus \hat{S}_{\bar{\lambda}})$, where $\hat{S}_a := \cup_{k\in\mathbb{N}} \ker(\hat{T} - a.\mathrm{Id})^k$), the next two results (well-known in some form) show computable bases for $\hat{S}_\lambda$ exist for all eigenvalues $\lambda$.

**Proposition 4.8** *If $A \in M_{d\times d}(\mathbb{R}_{\mathrm{c}})$ has an eigenvalue $\lambda \in \mathbb{R}_{\mathrm{c}}(\mathbf{i})$, the generalised eigenspace $\hat{S}_\lambda$ of $\hat{T} : \mathbb{C}^d \to \mathbb{C}^d, x \mapsto Ax$ has a basis in $\mathbb{R}_{\mathrm{c}}(\mathbf{i})^d$, and the (real) generalised eigenspace $S_\lambda$ of $T : \mathbb{R}^d \to \mathbb{R}^d, x \mapsto Ax$ has a basis in $\mathbb{R}_{\mathrm{c}}^d$.*

**Proposition 4.9** *$\mathbb{R}_{\mathrm{c}}(\mathbf{i})$ is algebraically closed, i.e. every $f(X) \in \mathbb{R}_{\mathrm{c}}(\mathbf{i})[X]$ splits over $\mathbb{R}_{\mathrm{c}}(\mathbf{i})$.*

Finally, we need the following simple estimate.

**Lemma 4.10** *For any norm $\|\cdot\|$ on $X$ and dense $\nu :\subseteq \mathbb{N} \to X$ such that $+$, $k_{\mathbb{D}}$ are approximable, we have $\bar{C}(y, \nu, (X, \|\cdot\|)) \le d$ for all $y \in X$.*

This can be proven using (e.g.) binary expansions of coefficients with respect to an arbitrary basis in $\nu(\mathrm{dom}\,\nu)$. A similar bound can be obtained for a left-invariant Riemannian metric on a Lie group, with assumptions similar to Theorem 1.1, though the proof is longer and more technical than that of the above lemma. When $X$ is a computable metric space, in general ([7]) we have $(\forall x)(\bar{C}(x, \nu, (X, d)) \le \overline{\dim}_{\mathrm{b}}(X, d))$ where $\overline{\dim}_{\mathrm{b}}$ is the upper box dimension, but in the semicomputable case this is not clear. For general $\nu$, this is not true at all; for any $C > 0$, separable metric space $X$ and nowhere dense subset $A$

one can construct dense $I : \mathbb{N} \to X$ such that $(\forall x \in A)(\bar{C}(x, I, (X, d)) \geq C)$. From the current bound, we note we can now prove the theorem of this section:

**Theorem 4.11** *Considering $(\mathbb{R}^d, +)$, if $\nu \in \mathcal{A}$ is witnessed by $d \in \mathcal{D}$ and linear map $T : \mathbb{R}^d \to \mathbb{R}^d$ with distinct eigenvalues $\lambda_1, \ldots, \lambda_r \in \mathbb{C}$ is approximable with respect to $\nu$, $d$, then*

$$\bar{S}(x, T, \nu, (\mathbb{R}^d, d)) \leq \sum_{j \in \Gamma} \log_2 \left|\lambda_{i_j}\right| \dim S_{\lambda_{i_j}}, \tag{6}$$

*where $(\lambda_{i_j})_{j \in \Gamma}$ are the eigenvalues with $\Im \lambda_{i_j} \geq 0$ and $\left|\lambda_{i_j}\right| \geq 1$.*

By the well-known formula for topological entropy of a linear map[1], the upper bound here is just $h_d(T)$. We also need one more technical lemma.

**Lemma 4.12** *In a metrizable topological group $G$ with right-invariant metric $d$ and effectively separable weakly semicomputable interpretation $\nu :\subseteq \mathbb{N} \to G$, if the product $\cdot : G \times G \to G$ is approximable and the connected component $G_0$ of the identity is open then each restriction of $\nu$ to a connected component is effectively separable.*

Now we can prove Theorem 1.1, although we will denote the objects in that statement by $G'$, $d'$, $\nu'$, $F'$, $T'$; since by [9, Thm 9.8] $F'$ is compact and there is a homeomorphism & group isomorphism $\Psi : G' \to \mathbb{R}^a \times \mathbb{Z}^b \times F' =: G$ for some $a, d \in \mathbb{N}$, we can use $d(x, y) := d'(\Psi^{-1}x, \Psi^{-1}y)$, $T := \Psi \circ T' \circ \Psi^{-1}$ and $\nu := \Psi \circ \nu'$ (these plainly give the same entropy and complexity).

**Proof of Theorem 1.1 (Sketch)** By Theorem 3.4 and Proposition 3.3 applied to $H := \{(0, 0)\} \times F'$, it is sufficient to show $\bar{S}(\pi z, S, \pi \circ \nu, (G/H, \tilde{d})) \leq h_{\tilde{d}}(S)$ for all $z \in G$. Noting the connected component $V \cong \mathbb{R}^a$ of the identity in $G/H$ has $S(V) \subseteq V$ (from continuity and $S(0) = 0$) and $L := S|_V$ linear, and that $h_{d_V}(L) \leq h_{\tilde{d}}(S)$ (for $d_V := \tilde{d}|_{V \times V}$), we plan to take a decomposition $G/H = V \oplus W$ for which all $w \in W$ have $\bar{S}(w, S, \pi \circ \nu, (G/H, \tilde{d})) = 0$, and try to bound $\bar{S}(v, L, \pi \circ \nu, (G/H, \tilde{d}))$ ($v \in V$) using Theorem 4.11 (note here we are using invariance of the metric $\tilde{d}$ to ensure Lemma 3.2(vi) applies to $+$). For this purpose it is convenient to use the obvious homeomorphism & group isomorphism $\Phi : G/H \to \mathbb{R}^a \times \mathbb{Z}^b$, pick arb. $u_j \in \pi\nu(\operatorname{dom} \nu)$ corresponding to $\mathbb{R}^a \times \{(\delta_{i,j})_{i=1}^b\}$, and consider $W := \operatorname{span}_{\mathbb{Z}}\{u_1, \ldots, u_b\}$. We also use an obvious extension $Y \cong \mathbb{R}^a \times \mathbb{R}^b$ of $G/H$ to fix a norm-induced metric $\hat{d}$; then $\mathrm{id} : (G/H, \tilde{d}) \to (G/H, \hat{d})$ is eff. unif. continuous (from Proposition 4.1 and openness of $V$) and so is $S : (G/H, \hat{d}) \to (G/H, \hat{d})$ (checked directly).

From these, approximability of $S$ (with respect to $\tilde{d}$), $u_j \in \pi\nu(\operatorname{dom} \nu)$ and Lemma 3.2(vii) one gets $\bar{S}(u_j, S, \pi \circ \nu, (G/H, \tilde{d})) = 0$. On the other hand, since $V$ is an open subgroup one can check the restriction of $\pi \circ \nu$ to $V$ has $+$, $L$ approximable with respect to $d_V$, and by Proposition 4.3(iv) and Lemma 4.12, a sufficient condition to apply Theorem 4.11 is that $(G/H, \tilde{d}, \pi \circ \nu)$ be (effectively separable and) weakly semicomputable. This is the reason for requiring some t.r. $f : \mathbb{N} \to \mathbb{N}$ have $(\nu \circ f)(\mathbb{N})$ dense in $H$. Assuming also p.r. $h$, $P$ witnessing the $(\mathcal{N}\nu, \mathcal{N}\nu)$-, $(\mathcal{N}\nu \times \mathcal{N}\nu, \mathcal{N}\nu)$-effectivity of $- : G \to G$ and $+ : G \times G \to G$, and noting $\nu$ and $\theta := \mathcal{N}_d\nu$ are semicomputable with respect to $d$, we consider the following algorithm:

**Algorithm 1** *On input $n \in \mathbb{N}$, $\eta \in \mathbb{Q}^+$, dovetail calculation of $z := P(\langle n, (h \circ f)(i)\rangle)$ and enumerations of $A_{z,\eta} \dot\cup \theta^{-1} B_d(\theta(z); \eta)$ over all $i \in \mathbb{N}$.*

For any $a \in \theta^{-1}\pi^{-1}B_{\tilde{d}}((\pi \circ \theta)(n); \eta)$, there exist $g \in H$ and $\xi \in (0, \eta)$ such that $\theta(a) + g \in B_d(\theta(n); \xi)$, and then $\exists i \in \mathbb{N}$ such that $d(g, (\theta \circ f)(i)) < \eta - \xi$, so

$$\begin{aligned} d\left(\theta(a), \theta(n) - (\theta \circ f)(i)\right) &\le d\left(\theta(a), \theta(n) - g\right) + d\left(\theta(n) - g, \theta(n) - (\theta \circ f)(i)\right) \\ &= d\left(\theta(a) + g, \theta(n)\right) + d\left(g, (\theta \circ f)(i)\right) < \eta, \end{aligned}$$

and $a$ must appear in the output. Conversely, any output $a \in \mathbb{N}$ either has $a \in A_{\dots} \subseteq \mathbb{N} \setminus \operatorname{dom}\theta$ or $a \in \theta^{-1}B_d(\theta(n) - (\theta \circ f)(i); \eta)$ for some $i \in \mathbb{N}$, in which case

$$\tilde{d}((\pi \circ \theta)(a), (\pi \circ \theta)(n)) \le d(\theta(a), \theta(n) - (\theta \circ f)(i)) < \eta,$$

and the output is correct. This shows $\pi \circ \theta$ is semicomputable with respect to $\tilde{d}$, hence $\pi \circ \nu \le \pi \circ \theta$ is also, and the proof is finished. □

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