



Approximation Simply Characterized

Paweł Waszkiewicz

*Institute of Computer Science
Jagiellonian University
Kraków, Poland*

Abstract

We provide a method for checking if a given auxiliary relation on a poset is the approximation relation on a domain.

Keywords: (continuous) domain, way-below relation, lower semicontinuous function

1 Motivation

If (P, \sqsubseteq) is a partial order, then the *approximation* (called also the *way-below* relation) is defined for $x, y \in P$ as $x \ll y$ if and only if for any directed subset D of P , if $y \sqsubseteq \bigvee^\uparrow D$, then $x \sqsubseteq d$ for some $d \in D$ ($\bigvee^\uparrow D$ denotes the supremum of D). The goal of this short paper is to give a simple method of recognizing the approximation relation in various concrete partial orders. Our results can be seen as a direct generalizations of these in [2] as much as we carry the proofs not only in the powerset lattice but in more general partial orders. It is often the case that approximation is characterized in terms of some interpolating family of elements, for example: if X is a set, then $A \ll B$ in the powerset $\mathcal{P}(X)$ iff $A \subseteq F \subseteq B$ for some finite subset F of X . In this case finite subsets of X form the interpolating family and, moreover, they are a *basis* of $\mathcal{P}(X)$. (A subset B of a poset P is a basis of P if and only if every $x \in P$ arises as the directed supremum of the set $\downarrow x \cap B$,

¹ Email: pqw@ii.uj.edu.pl

where $\downarrow x = \{y \in P \mid y \ll x\}$.) In most cases, however, the interpolating family cannot serve as a basis, since its elements are of different type than the elements of the poset under investigation. For instance, if X is a locally compact topological space, then $x \ll y$ in the lattice of open sets $\mathcal{O}(X)$ if and only if there is a compact subset q of X such that $x \subseteq q \subseteq y$. Our main results, in Section 2, allow to recognize when it is the case that existence of an auxiliary interpolating family characterizes the approximation relation. We claim that the methods proposed in this paper can simplify the task of proving that certain posets are domains. In Section 3 we give various examples supporting our claim.

We assume basic familiarity with the theory of domains. (A *domain* is a poset which has a basis in the sense defined above.) Our main reference to the subject is [1] and we adhere to the notation and terminology used in the book.

2 Main results

In the paper P will denote a poset, $H, T \subseteq P$ will be nonempty with H closed under directed suprema, and \prec will be a binary relation on P satisfying:

- (i) \prec refines \sqsubseteq ,
- (ii) $a \sqsubseteq b \prec c \sqsubseteq d$ implies $a \prec d$, for all $a, b, c, d \in P$,
- (iii) for all $t \in T$, $h \in H$, $t \prec h$ implies $t \prec h' \prec t' \prec h$ for some $t' \in T$ and $h' \in H$.

Note that \prec must be transitive by (i) and (ii). We will often call the property (iii) “interpolation”.

Theorem 2.1 *Axioms (1)-(2):*

- (1) $h = \bigvee^{\uparrow} \{t \in T \mid t \prec h\}$,
- (2) $t \prec \bigvee^{\uparrow} \{h_i \in H \mid i \in I\}$ implies $t \prec h_j$ for some $j \in I$

are equivalent to (a)-(b):

- (a) $h = \bigvee^{\uparrow} \{h' \in H \mid h' \ll_H h\}$ (i.e. H is a domain in the relative order),
- (b) $h \ll_H h'$ iff $h \prec t \prec h'$ for some $t \in T$.

Proof. Assume (1)-(2). For (b), suppose that $h \prec t \prec h'$ for some $t \in T$ and $h' \sqsubseteq \bigvee^{\uparrow} h_i$, $i \in I$. By (iii), $t \prec \bigvee^{\uparrow} h_i$ and thus $t \prec h_j$ by (3), which implies $h \prec h_j$. This proves $h \ll_H h'$. Conversely, assume $h_1 \ll_H h_2$. By (1), $h_2 = \bigvee^{\uparrow} \{t \in T \mid t \prec h_2\}$. By interpolation, $h_2 = \bigvee^{\uparrow} \{h \in H \mid \exists t \in T (h \prec t \prec h_2)\}$. By assumption, $h_1 \sqsubseteq h \prec t \prec h_2$ for some $h \in H$ and $t \in T$, which by (ii) gives $h_1 \prec t \prec h_2$, as required. Note that (a) follows immediately from (1)

and (b).

Conversely, assume (a)-(b). Part (1) is clear (use interpolation). To see (2), suppose $t \prec \bigvee^\uparrow h_i$. Since H is closed under directed suprema, we can use interpolation to get $t \prec h' \prec t' \prec \bigvee^\uparrow h_i$. Hence $h' \ll_H \bigvee^\uparrow h_i$, by (b). Therefore, there exists $j \in I$ such that $t \prec h' \sqsubseteq h_j$. By (ii), $t \prec h_j$ and (2) is now proved. \square

When H is a domain, denote by \hat{H} the domain of Scott-open filters on H (the Lawson dual of H).

Theorem 2.2 *Assume that H is a domain in the relative order from P and that T is closed under filtered infima. If we have for all $h \in H, t \in T$:*

- (3) $h \succ \bigwedge t_i$ implies $h \succ t_j$ for some $j \in I$,
- (4) $h_1, h_2, \succ t$ implies $h_1, h_2 \sqsupseteq h \succ t$ for some $h \in H$,
- (5) $t = \bigwedge_\downarrow \{h \in H \mid h \succ t\}$.

then:

- (c) \hat{H} is isomorphic to T , the inverse isomorphism being

$$T \ni t \mapsto \{h \in H \mid h \succ t\} \in \hat{H},$$

- (d) \hat{T} is isomorphic to H , the inverse isomorphism being

$$H \ni h \mapsto \{t \in T \mid t \prec h\} \in \hat{T},$$

- (e) T is a domain in the dual to the relative order from P with the way-below characterized as follows: $t \ll_T t'$ iff $t' \prec h \prec t$ for some $h \in H$.

Proof. For (c), let $F \in \hat{H}$ and $h \in F$ (F is nonempty as a filter). Consider $G_F := \{t \in T \mid h_1 \succ t \succ h_2 \text{ for } h_1, h_2 \in F\}$. Since F is Scott-open in a domain H , there exists $h' \in F$ with $h' \ll_H h$. Thus by (b), $h' \prec t \prec h$ for some $t \in T$, which proves that G_F is nonempty. Now, if $h_i \in F$, $i = 1, 2$, then $h_i \gg_H h_3 \succ t \succ h_4$ for some $t \in T$ and $h_j \in F$, $j = 3, 4$. This implies that G_F is filtered and hence has an infimum, call it t_F . Therefore, $h \succ t_F$ by \ll -roundness of F and (b). We have proved that $h \in F$ implies $h \succ t_F$. To show the reverse implication, assume $h' \succ t_F$ for $h' \in F$. By (3), we have $h' \succ t$ for some $t \in G_F$. Therefore, $h' \succ t \succ h_2$ for some $h_2 \in H$ by definition of G_F . But this proves that $h' \in F$, as F is upper. We have shown that $h \in F$ iff $h \succ t_F$. We conclude that the map $\hat{H} \ni F \mapsto t_F$ is injective. Clearly, it is also monotone. To see surjectivity, fix $t \in T$ and pick some filtered collection $\{t_i\}, i \in I$ with infimum t (it is possible by (4),(5) and interpolation). Let $F_t := \{h \in H \mid h \succ t\}$. Surely, $F_t \neq \emptyset$ by (5). Fix a particular $h \in F_t$. Hence

$h \succ \bigwedge_{\downarrow} t_i$ and by (3), $h \succ t_j$ for some $j \in I$. Let $\{h_k\}, k \in K$ be the directed subset of H -approximants of h . Since $h = \bigvee^{\uparrow} h_k$, by (2) we have $h_k \succ t_j$ for some $k \in K$. Therefore, $t = \bigwedge_{\downarrow} t_i \subseteq t_j \prec h_k$. That is, $h_k \in F_t$, which proves that F_t is Scott-open. Clearly, it is also upper. Next, if h_1, h_2 in F_t , then by (4) and interpolation, there is some $h_3 \in F_t$ with $h_1, h_2 \succ h_3$. We have shown that F_t is a filter, and hence the mapping is surjective. To see that the inverse mapping is well-defined, it remains to demonstrate that the filter F_t is defined in a unique way, that is, that different t 's give rise to distinct F_t 's. But this is clear from the uniqueness of infima. Therefore, \hat{H} is isomorphic to T . Part (d) is now easy as it follows from Lawson-duality that $\hat{T} \cong \hat{\hat{H}} \cong H$. The isomorphism $H \cong \hat{\hat{H}}$ is concretely given as $H \ni h \mapsto \{F \in \hat{H} \mid h \in F\} \in \hat{\hat{H}}$. Using the equivalence $h \in F_t$ iff $h \succ t$ proved above, we can restate the isomorphism above as $H \ni h \mapsto \{t \in T \mid t \prec h\} \in \hat{T}$. Lastly, for (e), note that T is a domain, because it is isomorphic to $\hat{\hat{H}}$, which is a domain since H is. Suppose now that $t, t' \in T$ with $t \ll t'$. This is however equivalent to $F_t \ll F_{t'}$ in \hat{H} . By Theorem II-1.17 of [1], the last assertion is in turn equivalent to $F_t \subseteq \uparrow h$ for some $h \in F_{t'}$ (that is, $t \sqsupseteq h \succ t'$). Using interpolation, this implies (and is implied) by $t \succ h \succ t'$, as required. \square

3 Corollaries

In the sequel, we will refer to numbering of properties and names of sets as used in Theorems 2.1 and 2.2.

Example 3.1 Setting $P = (\mathcal{P}(X), \subseteq)$ for any set X , $H = \mathcal{P}(X)$, $T = \mathcal{P}_{fin}(X)$ and $\prec = \subseteq$, we get by Theorem 2.1 that

$$A \ll B \text{ iff } \exists F \in \mathcal{P}_{fin}(X). A \subseteq F \subseteq B,$$

for all $A, B \in \mathcal{P}(X)$.

Example 3.2 Let $P = (D, \sqsubseteq)$ be a domain. Let $H = D$, $\prec = \ll$ and T be a basis of D . Then Theorem 2.1 amounts to a statement that B is dense in the way-below relation. On the other hand, if we choose T to be a subset dense in the way-below relation, then the Theorem gives that T is a basis for D .

Example 3.3 Let H be the collection of nonempty open sets of a topological space X , which is T_0 , locally compact, well-filtered (see Theorem II-1.21 of [1] for the definition) and such that the intersection of two nonempty compact saturated sets is compact. Let $P = (\mathcal{P}(X), \subseteq)$, $\prec = \subseteq$ and let T be the collection of all nonempty compact saturated sets. Then (i)-(ii), (2) are trivial; (iii) and (1) follow from local compactness. Theorem 2.1 implies that $(\mathcal{O}(X), \subseteq)$

is a continuous lattice with the way-below: $O_1 \ll O_2$ iff $O_1 \subseteq C \subseteq O_2$ for some compact saturated subset C of X . Now, (3) is exactly well-filteredness; (5) holds because the intersection of saturated compacta is compact and again by well-filteredness of X . Theorem 2.2 then implies that T is a domain with $C_1 \ll C_2$ iff $C_2 \subseteq O \subseteq C_1$ for some $O \in H$ and, more importantly, that T is isomorphic to the domain of Scott-open filters on $\mathcal{O}(X)$. The latter statement is nothing but the Hofmann-Mislove Theorem for locally compact sober spaces.

Example 3.4 Let P be the poset of all nonempty, convex, compact subsets of \mathbb{R}^n ordered by the inverse inclusion. Set $H = P$, $\prec = \supseteq$ and $T \subseteq H$ be the collection of all polyhedra with rational coordinates. Then the hypothesis of Theorem 2.1 is satisfied and hence P is a domain with the way-below relation $K \ll L$ iff $L \subseteq R \subseteq K$ for some polyhedron $R \in T$. In particular, we conclude that the set of all compact intervals with rational endpoints is a basis in \mathbb{IR} .

Example 3.5 This result appears in [1] as a part of Proposition I-1.22.1. Let X be a compact Hausdorff space. By $\text{LSC}(X, \mathbb{R}^*)$ we denote the complete lattice of all lower semicontinuous functions of type $X \rightarrow \mathbb{R}^*$, where \mathbb{R}^* is the extended real line. For any function $f: X \rightarrow \mathbb{R}^*$, its upper graph is a set $G_f = \{(x, r) \in X \times \mathbb{R}^* \mid r < f(x)\}$. Note that it is an open subset of $X \times \mathbb{R}^*$ iff $f \in \text{LSC}(X, \mathbb{R}^*)$. Let $P = (\mathcal{P}(X \times \mathbb{R}^*), \subseteq)$, choose H to be the collection of upper graphs of functions from $\text{LSC}(X, \mathbb{R}^*)$ and let T be the family of all closures of elements of H . Take $\prec = \subseteq$. Properties (i)-(ii) are trivial.

Note that the three posets: $\text{LSC}(X, \mathbb{R}^*)$, graphs of lower semicontinuous maps under inclusion, and open subsets of $X \times \mathbb{R}^*$ with \mathbb{R}^* -components of the form $[-\infty, r)$ for $r \in \mathbb{R}^*$, are all isomorphic. Therefore, (iii) can be easily deduced from normality of $X \times \mathbb{R}^*$. Part (2) follows from compactness of $X \times \mathbb{R}^*$. Part (1) is a consequence of both normality and compactness of the space. By Theorem 2.1, we can then conclude that $\text{LSC}(X, \mathbb{R}^*)$ is a continuous lattice with $f \ll g$ iff $\text{cl}(G_f) \subseteq G_g$.

References

- [1] Gierz, G.K., Hoffmann, K.H., Keimel, K., Lawson, J.D., Mislove, M.W. and Scott, D.S., *Continuous Lattices and Domains*, Cambridge University Press, Encyclopedia of Mathematics and Its Applications, v. 93, 2003.
- [2] Norberg, T. *A Note on Continuous Collection of Sets*,
URL: <http://www.math.chalmers.se/~tommy/nccs.ps>