



# Twenty years of continuous multiobjective optimization in the twenty-first century

Gabriele Eichfelder\*

*Institute of Mathematics, TU Ilmenau, Po 10 05 65, 98684 Ilmenau, Germany*

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## ABSTRACT

The survey highlights some of the research topics which have attracted attention in the last two decades within the area of mathematical optimization of multiple objective functions. We give insights into topics where a huge progress can be seen within the last years. We give short introductions to the specific sub-fields as well as some selected references for further reading. Primarily, the survey covers the progress in the development of algorithms. In particular, we discuss publicly available solvers and approaches for new problem classes such as non-convex and mixed integer problems. Moreover, bilevel optimization problems and the handling of uncertainties by robust approaches and their relation to set optimization are presented. In addition, we discuss why numerical approaches which do not use scalarization techniques are of interest.

## 1. Introduction and basic recent literature

Multiobjective optimization, i.e., the optimization of multiple objectives at the same time, has been an active area of research since the early works of Edgeworth and Pareto. Edgeworth introduced an optimality notion for such problems in his book in 1881 (Edgeworth, 1881) and Pareto in his book on political economy in 1906 (Pareto, 1971). In mathematics, this branch of optimization started with a paper by Kuhn and Tucker (Kuhn and Tucker, 1951). Since about the end of the 1960s, research has intensively been made in this area. For a short historical overview see Eichfelder and Jahn (2016).

Multiobjective optimization can be seen as a special case of vector optimization: if one collects multiple, for instance  $m$ , scalar-valued objective functions in a vector, then one can consider the objective function of the optimization problem to be a vector-valued map. The image space is then the  $m$ -dimensional linear space of real numbers. Moreover, one needs a partial ordering in the image space which is often assumed to be the componentwise—called natural—ordering. If one allows more general linear spaces as image space of the vector-valued objective function of the optimization problem, then the name vector optimization is more suitable for these problems. For a good introduction to optimization with vector-valued maps as well as many results on a characterization of the optimal solutions of such problems we refer to the book by Jahn (2011).

In this paper, we focus on multiobjective optimization problems, i.e., problems with a finite dimensional image space. Moreover, we concentrate on continuous problems (including mixed integer problems). Also in the field of discrete multiobjective optimization many important new

results, algorithms, and applications have been proposed in the last two decades, but we do not try to cover those. We give examples of topics of interest and progress in the field of continuous multiobjective optimization in the last twenty years. We select some topics, which can, of course, only be a personal view on this field. The literature on multiobjective optimization is exploding, and such a survey cannot cover all aspects of interest. Instead, we hope that this selection raises the reader's interest to explore the presented or other modern aspects of multiobjective optimization further.

Before we start to discuss some specific topics, we would like to highlight some of the most fundamental books which have appeared in the last twenty years on the topic. We already mentioned the book by Jahn (2011) with the first edition from 2004 and the second edition from 2011. The book discusses vector optimization in a general setting, and applications to vector approximation, cooperative game theory, and multiobjective optimization are described. The theory is extended in the second edition by a chapter on set optimization, i.e., on optimization problems with a set-valued objective function, which will be a topic of our Section 8. Also in a general setting, Göpfert, Riahi, Tammer and Zălinescu study, in their book (Göpfert et al., 2003) from 2003, solution concepts, optimality conditions, scalarizations, duality, and applications of vector optimization problems. A book authored by Ehrgott (2005), which is written in the final dimensional setting and is a perfect introduction to the topic, appeared in 2000 and has now its second edition in 2005. The book provides the necessary mathematical foundation of multiobjective optimization and covers topics such as the most famous scalarization techniques or approaches for linear multiobjective problems. In addition, combinatorial problems as the knapsack problem are

\* Corresponding author.

E-mail address: [gabriele.eichfelder@tu-ilmenau.de](mailto:gabriele.eichfelder@tu-ilmenau.de)

discussed. More recently, for linear multiobjective optimization problems there is a book by [Luc, 2016](#). Another book written in the finite dimensional setting, but which focuses on nonlinear problems and additionally covers interactive methods for decision making in multiobjective optimization, is due to [Miettinen, 1999](#).

In the last two decades, next to the above introductory books, several valuable books on more specific topics in vector optimization appeared. Many of those are part of the book series entitled *Vector Optimization* with the series editor Jahn, which started in 2008 with the first book about adaptive scalarization methods ([Eichfelder, 2008](#)). The series contains publications in various sub-fields of optimization with vector-valued objective functions. So far, ten books have been published in this series. For instance, in the book by [Löhne \(2011\)](#) an interesting new solution concept for vector optimization problems is introduced which allows to define an appropriate concept of the infimum with a high degree of analogy to corresponding results in single-objective (i.e., scalar-valued) optimization.

Moreover, we would like to specifically mention the book on recent developments in vector optimization which is edited by [Ansari and Yao \(2012\)](#) and which contains a wide range of topics such as duality results, new ordering structures, specific classes of multiobjective optimization problems such as fractional problems, and vector variational principles.

Another collection of recent results on multiobjective optimization, and most of all the related field of multicriteria decision making, is [Greco et al. \(2016\)](#) which is edited by Greco, Ehrgott and Figueira and appeared in 2016 (after a first edition in 2005). The two volumes include surveys on the foundations and on many areas of current research as well as new applications. The main focus of the book is on multicriteria decision making, as the title says, which covers the aspect of finally selecting one of the optimal solutions of a multiobjective optimization problem in a decision making process. Such a selection is for instance reached by interactive methods or preference modeling. However, decision making is not the topic of this survey. Instead, we aim at finding all optimal solutions of the multiobjective optimization problem, which are, from a mathematical point of view, non-comparable.

The book [Branke et al., 2008](#) edited by Branke, Deb, Miettinen, and Slowinski from 2008 covers interactive methods and evolutionary algorithms to solve multiobjective optimization problems. In the area of evolutionary approaches an intense progress has been seen since the fundamental book by [Deb \(2001\)](#), but these approaches will also not be covered within this survey.

Finally, we would like to point out that since 2004 there has been a series of Dagstuhl meetings, with the first meeting organized by Branke, Deb, Miettinen, and Steuer, which bring together people from multicriteria decision making, multiobjective optimization, and evolutionary algorithms. Until now, seven meetings have taken place which all covered topics of recent research interest as Scalability in Multiobjective Optimization ([Dag, 0000](#)), Personalized Multiobjective Optimization, or Hybrid and Robust Approaches. A link to the websites of the earlier meetings can be found on the website of the most recent meeting ([Dag, 0000](#)).

In [Section 2](#), we briefly recall the basic concepts of multiobjective optimization for clarifying our notation. In [Section 3](#), we present some useful and publicly available solvers for certain classes of multiobjective optimization problems. Scalarization is a widely used tool in multiobjective optimization and we recall the basics in [Section 4](#). We discuss why the study of non-scalarization based methods is of interest. An important class of multiobjective optimization problems for which non-scalarization based methods might be more suitable are those of non-convex problems, which includes, in particular, mixed integer multiobjective optimization. These problem classes are the topic of [Section 5](#). A further challenging problem class is bilevel optimization. We discuss the developments in this field in [Section 6](#). These problems have a relation to set-valued optimization. This is also the case for the problems discussed in [Section 7](#): robust multiobjective optimization problems for

handling uncertainties. Finally, we conclude the survey in [Section 8](#) with some comments on other ordering structures, which appeared more frequently in the literature in the last decade, and on developments in set optimization.

## 2. Basic notation

In the following, we briefly define the basic concepts, mainly for having a unified notation. For a more thorough introduction we refer to one of the books mentioned in the previous section. In multiobjective optimization one studies optimization problems formally defined by

$$\begin{aligned} \min f(x) &= (f_1(x), \dots, f_m(x))^T \\ \text{subject to the constraint} & \\ x &\in M, \end{aligned} \quad \text{MOP}$$

with a vector-valued objective function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  ( $m, n \in \mathbb{N}$ ,  $m \geq 2$ ) and a nonempty set of feasible points  $M \subseteq \mathbb{R}^n$ .

The most common optimality notion is the one of efficiency, also called Pareto optimality or Edgeworth-Pareto optimality.

**Definition 2.1.** A point  $\bar{x} \in M$  is called an efficient solution or efficient point or efficient for (MOP), and  $f(\bar{x})$  is called a nondominated point, if there is no  $x \in M$  with

$$\begin{aligned} f_i(x) &\leq f_i(\bar{x}) && \text{for all } i \in \{1, \dots, m\} \text{ and} \\ f_j(x) &< f_j(\bar{x}) && \text{for at least one } j \in \{1, \dots, m\}. \end{aligned}$$

By using a set notation one obtains that  $\bar{x} \in M$  is efficient for (MOP) if and only if

$$(\{f(\bar{x})\} - \mathbb{R}_+^m) \cap f(M) = \{f(\bar{x})\},$$

with  $f(M) = \{f(x) \in \mathbb{R}^m \mid x \in M\}$  and  $\mathbb{R}_+^m = \{y \in \mathbb{R}^m \mid y_i \geq 0, i = 1, \dots, m\}$ . The set of all efficient solutions is named efficient set, and the set of all nondominated points is the nondominated set for (MOP) denoted by  $\mathcal{N} \subseteq f(M)$ .

For an arbitrary set  $S \subseteq \mathbb{R}^m$ , we say that a point  $y^1 \in S$  dominates  $y^2 \in S$ , and that  $y^2$  is dominated by  $y^1$ , if  $y^1 \neq y^2$  and  $y_i^1 \leq y_i^2$  for all  $i \in \{1, \dots, m\}$ . The set of all not dominated points within  $S$  is called the set of nondominated points or the nondominated set of  $S$ .

When one speaks of solving the problem (MOP) one typically aims to find all efficient solutions, i.e., the efficient set, or all nondominated points, i.e., the nondominated set for (MOP). If the set  $M$  is not finite, in general an infinite number of efficient solutions exist. Especially for nonlinear problems, in general not all efficient points can be calculated with numerical methods. In that case one aims to find an approximation (or representation) of the efficient (or nondominated) set or to find a covering (or enclosure). For a definition and a classification of such concepts we refer to the survey [Ruzika and Wiecek, 2005](#). Within such methods, often inefficient solutions are generated which are just weakly efficient solutions. For that reason we briefly recall their definition.

**Definition 2.2.** A point  $\bar{x} \in M$  is called a weakly efficient point or weakly efficient solution or weakly efficient for (MOP), and  $f(\bar{x})$  is called a weakly nondominated point, if there is no  $x \in M$  with

$$f_i(x) < f_i(\bar{x}) \quad \text{for all } i \in \{1, \dots, m\}.$$

By using a set notation, one obtains that  $\bar{x} \in M$  is weakly efficient for (MOP) if and only if

$$(\{f(\bar{x})\} - \text{int}(\mathbb{R}_+^m)) \cap f(M) = \emptyset.$$

With  $\text{int}(\mathbb{R}_+^m) = \{y \in \mathbb{R}^m \mid y_i > 0, i = 1, \dots, m\}$  we denote the interior of the cone  $\mathbb{R}_+^m$ . Any efficient point for (MOP) is weakly efficient for (MOP). From an application point of view one would in general only be interested in a weakly efficient point which is at the same time efficient, as otherwise one could improve the value of one objective function without deteriorating the others. Still, this concept is important for characterizing the output of numerical algorithms or for theoretical results such as optimality conditions.

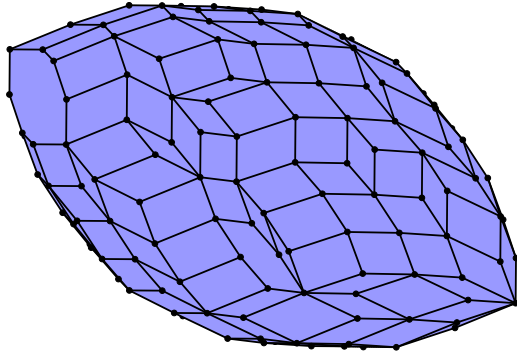


Fig. 1. The nondominated set of a triobjective optimization problem as a result of a run of Bensolve for a multiobjective optimization problem with three objective functions (provided by D. Dörfler, FSU Jena).

### 3. Publicly available solvers

A huge progress can be seen in solving problems of the form (MOP) numerically, and many algorithms for specific classes of problems as non-convex problems are discussed in the literature. We will look into details of some of them later. However, only for a few classes of problems, implementations of solvers exist which are freely available. The International Society on Multiple Criteria Decision Making (MCDM) lists (MCD, 0000) several freely available numerical algorithms for decision making and for multiobjective optimization. In the following we present some of them.

Linear multiobjective optimization problems with many variables and several objective functions can be handled with Bensolve (Bensolve). That is, problems of the form

$$\begin{aligned} \min_{x \in \mathbb{R}^n} & \begin{pmatrix} (c^1)^T x \\ \vdots \\ (c^m)^T x \end{pmatrix} \\ \text{s.t.} \quad & b^1 \leq Bx \leq b^2 \\ & \ell b \leq x \leq ub \end{aligned}$$

with appropriate vectors  $b^1, b^2$ , a matrix  $B$ , and vectors  $c^1, \dots, c^m, \ell b, ub \in \mathbb{R}^n$  can be solved, i.e., all nondominated points can be calculated. For two vectors  $w, z \in \mathbb{R}^k$ , the inequality  $w \leq z$  has to be read componentwise as  $w_i \leq z_i$ ,  $i = 1, \dots, k$ . The algorithm is based on what is known as Benson's algorithm and its extensions, see Löhne and Weißing, 2017. According to Dörfler et al. (2021) some of the basic ideas can be traced back to earlier works in different areas of research, as in global optimization or in convex optimization. The algorithm available on Bensolve was first implemented in Matlab, and from version 2.0.0 it is written in the C programming language. For problems with  $m = 2$  and  $m = 3$ , visualizations of the nondominated set are generated, see Fig. 1 for such an output.

For this class of problems another solver is given on GitHub by Csirmaz (2021); Cr2 (0000). The solver also makes use of Benson's basic idea and the author states that the algorithm is specifically appropriate for problems with ten or more objective functions.

With FLO (FLO, 0000) there is a Matlab based implementation by Günther of a deterministic solver for planar multiobjective location problems, based on, among others, Alzorba et al. (2015) by Alzorba, Günther, Popovici, and Tammer. One of the problem classes which can be solved with FLO are multiobjective single-facility location problems which consist in minimizing the distances between a new facility  $x \in \mathbb{R}^2$  and all given attraction facilities  $a^1, \dots, a^m \in \mathbb{R}^2$  simultaneously. For the Manhattan metric one obtains a problem formulation with  $m$  objective functions of the type

$$f_i(x) = |x_1 - a_1^i| + |x_2 - a_2^i|, \quad i = 1, \dots, m,$$

but other metrics can also be used. The solver determines all nondominated points.

For more general constrained nonlinear multiobjective optimization problems with a small number of objective functions, approximations of the set of nondominated points for (MOP) can be generated with the Matlab based implementation ASMO (ASM, 0000) provided by Warnow, which is based on the algorithm from Eichfelder, 2008. For  $m = 2$ , nearly equidistant approximations of the set of nondominated points can be generated for smooth nonlinear problems, i.e., for problems with smooth objective and constraint functions. The method uses sensitivity information and a scalarization approach. As the scalarization problems have to be solved to global optimality, and as a local solver is used within the implementation, the algorithm is in particular suited for convex optimization problems.

Another solver for constrained nonlinear multiobjective optimization problems named MOSQP is provided by Vaz based on his joint work with Fliege (Fliege and Vaz, 2016). The idea behind the algorithm is the extension of single-objective sequential quadratic programming to the multiobjective setting. The extension is done in such a way that not just a single efficient solution is calculated but an approximation of the nondominated set. A Matlab based implementation is provided (MOS, 0000) together with results of comparisons on benchmark problems with, for instance, the evolutionary algorithm NSGA-II. One can contact the authors for sets of benchmark problems.

MOSQP makes use of derivative information of the objective functions and constraints to build quadratic models. Another well-known solver for constrained nonlinear multiobjective optimization problems, named Direct MultiSearch (DMS), aims to avoid exactly this. An algorithmic implementation with Matlab can be obtained by sending an email (DMS, 0000) and is freely available for academic use. The algorithm is based on Custodio et al. (2011) by Custodio, Madeira, Vaz and Vicente, which extends the idea of multisearch developed for single-objective optimization to multiobjective optimization.

In addition, there is a solver for linear or integer linear (but not mixed integer) multiobjective optimization named PolySCIP which is available on Pol (0000) by Schenker and Strunk, with a last update from 2017. The implementation is based on Borndörfer et al. (2016) by Borndörfer, Schenker, Skutella, and Strunk. PolySCIP only finds the so-called supported solutions. Supported solutions are those which can be found by the weighted sum scalarization, see Subsection 5.3. For non-convex problems as integer problems, the supported solutions in general only form a subset of the set of efficient solutions.

A solver for determining an enclosure of the nondominated set of a multiobjective mixed integer convex optimization problem based on a branch-and-bound method can be downloaded from MOM (0000) and is based on De Santis et al., 2020 by De Santis, Eichfelder, Niebling and Rocktäschel. We discuss this problem class and the approach from De Santis et al., 2020 in more detail in Section 5. Moreover, in Section 6 we present multiobjective bilevel optimization problems, for which an implementation of the algorithm from Eichfelder (2010), which determines an approximation of the nondominated set, can be downloaded from MOB (0000).

Finally, for strictly convex quadratic multiobjective problems with linear constraints, reformulations as parametric linear complementarity problems can be used, for which at least rudimentary implementations exist as detailed by Jayasekara, Adelgren and Wiecek in Jayasekara et al. (2020).

### 4. Direct approaches avoiding scalarization

A widely used approach to solve multiobjective optimization problems are scalarizations. Scalarization means that one formulates a parameter-dependent single-objective optimization problem to the multiobjective optimization problem. A huge progress can be seen for scalarization approaches in the last decades, for instance in procedures for choosing the parameters for those, and in extending the techniques for

a smart parameter control to problems with three or more objective functions, see for instance the recent publications [Burachik et al., 2014; 2017; 2021](#) by Burachik, Kaya and Rizvi.

However, in the last decade an increasing interest in numerical methods without such scalarizations can be observed. While some of these methods still use scalarizations on subproblems, as for instance for finding new candidate points for weakly nondominated points in subregions of the image space, they avoid to first scalarize the overall problem and then to apply standard solvers from single-objective optimization. In this section, we discuss such approaches and collect arguments against a scalarization. But first we recall the basic ideas of scalarization approaches.

#### 4.1. Basics on scalarizations

For a scalarization one formulates a parameter-dependent single-objective optimization problem  $P(\omega)$  for a set of parameters  $\Omega$  to the multiobjective optimization problem (MOP) and then aims

- (i) to obtain an efficient solution for (MOP) as optimal solution of  $P(\omega)$  for any choice of the parameter  $\omega \in \Omega$ , and
- (ii) to find all efficient solutions for (MOP) as optimal solutions of  $P(\omega)$  by varying the parameter  $\omega \in \Omega$  appropriately.

Well known scalarization approaches are the weighted sum or the  $\varepsilon$ -constraint scalarization. For the weighted sum method one solves

$$\min_{x \in M} \sum_{i=1}^m w_i f_i(x), \quad P_{WS}(w)$$

with weight set  $\Omega = W^1 := \{w \in \mathbb{R}_+^m \mid \sum_{i=1}^m w_i = 1\}$  or  $\Omega = W^2 := \{w \in \text{int}(\mathbb{R}_+^m) \mid \sum_{i=1}^m w_i = 1\}$ . The weighted-sum scalarization is named linear scalarization, as the linear function  $y \mapsto w^\top y$  is applied to the objective function vector  $f(x)$ .

For  $w \in W^2$ , any optimal solution  $\bar{x}$  of  $(P_{WS}(w))$  is efficient for (MOP). If  $w \in W^1$ , an optimal solution of  $(P_{WS}(w))$  is only guaranteed to be weakly efficient for (MOP). For these results see, for instance, Chapter 3 in [Ehrgott \(2005\)](#). On the other hand, aim (ii) is known to only hold for  $(P_{WS}(w))$  for convex problems, i.e., if  $f_i$ ,  $i = 1, \dots, m$  are convex functions and the set  $M$  is convex. Under these convexity assumptions one can find all efficient solutions of (MOP) by allowing  $w \in W^1$ , but not if one restricts the parameter set to  $W^2$ , cf. [Ehrgott \(2005\)](#). Thus there is a gap between the necessary and the sufficient conditions even in the convex case.

Another well-known scalarization is the  $\varepsilon$ -constraint scalarization

$$\min \{f_m(x) \mid f_i(x) \leq \varepsilon_i, i = 1, \dots, m-1, x \in M\} \quad P_C(\varepsilon)$$

with upper bounds  $\varepsilon \in \Omega \subseteq \mathbb{R}^{m-1}$ . We pick here the objective function  $f_m$  to be minimized without loss of generality. The  $\varepsilon$ -constraint scalarization belongs to the class of nonlinear scalarizations as it is derived from solving an approximation problem with a parametric norm, or, alternatively, a nonlinear separation functional on the set  $f(M)$ . For more details we refer to [Jahn \(2011\); Tammer and Weidner \(2020\)](#). For this scalarization, a difficult question is how to restrict the parameter set  $\Omega$ , especially for the case  $m \geq 3$ . Moreover, any optimal solution of  $(P_C(\varepsilon))$  is only guaranteed to be weakly efficient for (MOP) unless it is unique, [Ehrgott \(2005\)](#).

Many more of such scalarizations are known, and many of them can be seen as a special case of minimizing the very general Tammer-Weidner-functional, which is again related to the scalarization by Pascoletti and Serafini, c.f. [Pascoletti and Serafini \(1984\)](#). See [Eichfelder \(2009\)](#) for such relations and the recent paper ([Bouza et al., 2019](#)) by Bouza, Quintana and Tammer on a unified characterization of nonlinear scalarization functionals. Just recently, an extensive book on the Tammer-Weidner-functional appeared which presents all its useful properties and application fields, see ([Tammer, Weidner, 2020](#)) .

The above mentioned gap for the weighted sum method between (i) and (ii) already shows a drawback of scalarization approaches. Nevertheless, they are widely used in multiobjective optimization. The advantage is that for the scalarization, i.e., for the problem  $P(\omega)$ , one can use all theoretical results and numerical algorithms which have been developed for the single-objective case. For instance, if the objective functions of (MOP) are linear and the set  $M$  is defined by linear constraints and some of the variables  $x_j$ ,  $j \in J \subseteq \{1, \dots, n\}$  are additionally assumed to be integer, then the weighted sum approach reduces the problem to a linear mixed integer single-objective optimization problem for which powerful solvers are available. As a drawback, one has to check carefully whether the optimal solutions of the family of scalarization problems  $\{P(\omega) \mid \omega \in \Omega\}$  indeed represent the efficient set. For instance, in the mixed-integer case a weighted sum approach is not suitable due to the intrinsic non-convexity of the problem. With the weighted sum method one is only able to find the supported solutions, which is s topic of [Subsection 5.3](#).

Only in some rare cases one is satisfied with finding only one efficient solution, i.e., with aim (i) from above. An example are problems which are so challenging that one has to accept the fact that already calculating just one efficient solution has to be enough. Such a situation occurs if multiple expensive simulation based objective functions appear, cf. [Prinz et al., 2021](#) by Prinz, Thomann, Eichfelder, Boeck and Schumacher.

However, even then, when solving just one scalarization problem, just by choosing a certain scalarization it might happen that one gets unexpected or even unwanted efficient solutions. For instance, the weighted sum scalarization applied to a non-convex multiobjective optimization problem might deliver even for equal weights  $w_i$  just an individual minimizer, i.e., some  $x \in \text{argmin}\{f_i(x) \mid x \in M\}$  for some  $i \in \{1, \dots, m\}$ . To see this, take  $M = \{x \in \mathbb{R}_+^2 \mid \|x\|_2 \geq 1\}$  and  $f(x) = x$ . In this example, with the weighted sum method only the efficient points  $(1, 0)^\top$  and  $(0, 1)^\top$  can be found, no matter which weights are chosen. Obtaining one of these individual minimizers might be unexpected or even not wanted, as one might believe that one gets, for equal weights  $w_i$ , a solution with a balanced trade-off between the objective functions. Here, such a balanced solution would be  $x = (1/\sqrt{2}, 1/\sqrt{2})^\top$ . For other nonlinear scalarizations a careful choice of the parameter  $\omega \in \Omega$  can be important. Otherwise, the scalarization problem might not be solvable despite the original multiobjective optimization problem having efficient solutions. An example for a scalarization problem which might not be solvable despite there exists an efficient solution is  $(P_C(\varepsilon))$  with a parameter  $\varepsilon_1$  with  $\varepsilon_1 < \inf_{x \in M} f_1(x)$ .

#### 4.2. Directly extending single-objective methods

Thus, even if one is satisfied with only one efficient solution, scalarizations might not be the best choice. Another argument against using scalarizations is that the scalarization might destroy the structure of the optimization problem. For instance, in [Thomann and Eichfelder \(2019\)](#), the authors study unconstrained multiobjective optimization problems which are called heterogeneous: one of the objective functions, e.g.  $f_1$ , is assumed to be an expensive function, i.e., each function evaluation  $f_1(x)$  is very time consuming, for instance a simulation run, while the other objective functions  $f_2, \dots, f_m$  are analytically given and easy to evaluate. Then one should of course make use of the heterogeneous structure within an algorithm. But the weighted sum scalarization would not allow that and one would end up with only one expensive objective function  $w^\top f(\cdot)$ . Similarly, the  $\varepsilon$ -constraint scalarization adds constraints and thus destroys the simplicity of having an unconstrained optimization problem. Thus, in [Thomann and Eichfelder \(2019\)](#) the authors develop a tailored trust-region method to make use of the heterogeneity.

Non-scalarization-based approaches might also have advantages for problems without such a heterogeneous structure but with purely expensive functions. In [Ryu and Kim \(2014\)](#), and in [Prinz et al., 2021](#), the authors work with model functions for expensive objective functions



and with model-based trust-region methods. The approach proposed in [Prinz et al., 2021](#) guarantees to find, from some starting point  $x^0$ , an efficient solution  $\bar{x}$  for (MOP), or at least a point satisfying the necessary optimality conditions for that, see [Definition 4.1](#). What is more, this point  $\bar{x}$  satisfies  $f_i(\bar{x}) \leq f_i(x^0)$  for all  $i = 1, \dots, m$ . A good starting point, i.e., a meaningful guess for a good solution, is often available in applications. The important aspect is that it is guaranteed that in each successful iteration the new iterate  $x^{k+1}$  satisfies  $f_i(x^{k+1}) < f_i(x^k)$  for all  $i = 1, \dots, m$ . As a consequence, even in case the algorithm has to be stopped during the run as the evaluations of the functions are too costly, one has already a guaranteed improvement.

Many well-known techniques from single-objective optimization as descent methods, Newton's method, trust-region methods, and SQP methods for constrained optimization problems have successfully been extended to solve multiobjective optimization problems without an intermediate scalarization step. For instance, one of the earlier papers is on a steepest descent method for multiobjective optimization, see [Drummond and Svaiter \(2005\)](#). A year later, Fliege has proposed an interior point method based approach, see [Fliege \(2006\)](#). Fliege, Drummond and Svaiter have extended Newton's method in [Fliege et al. \(2009\)](#). Moreover, we have already mentioned in [Section 3](#) an extension of sequential quadratic programming, see [Fliege and Vaz \(2016\)](#). In addition, just recently a descent method for non-smooth but Lipschitz multiobjective optimization problems has been proposed by [Gebken and Peitz \(2021\)](#), which shows the ongoing interest in such methods.

As the basic techniques are all methods from convex, i.e., non-global single-objective optimization, which only guarantee to find locally optimal solutions, the extensions to multiple objectives only allow to find locally efficient solutions — unless the problems are convex. Recall that a point  $\bar{x} \in \mathbb{R}^n$  is called a locally (weakly) efficient solution for (MOP) if there exists a neighborhood  $U \subseteq \mathbb{R}^n$  of  $\bar{x}$  such that  $\bar{x}$  is (weakly) efficient for (MOP) restricted to  $M \cap U$ . In [Section 5](#) we discuss methods which guarantee to find globally efficient solutions for non-convex problems.

#### 4.3. Descent directions and proximity measures

Most of the extensions above (except the SQP extension) are for unconstrained optimization problems. In single-objective optimization one aims to find a stationary point, i.e., a point at which the gradient vanishes. The well-known concept of stationarity from single-objective optimization transfers as defined next, cf. [Smale \(1975\)](#). The transferred stationarity concept is used in a work on steepest descent methods for multiobjective optimization by [Fliege and Svaiter \(2000\)](#).

**Definition 4.1.** Let  $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $i = 1, \dots, m$ , be continuously differentiable and  $M = \mathbb{R}^n$ . A point  $\bar{x} \in \mathbb{R}^n$  is called Pareto critical for (MOP) if for every  $d \in \mathbb{R}^n$  there exists an index  $j \in \{1, 2, \dots, m\}$  such that  $\nabla f_j(\bar{x})^\top d \geq 0$  holds.

Obviously, for  $m = 1$  the definition reduces to  $\nabla f(\bar{x}) = 0$ . Pareto criticality is indeed a necessary condition for local weak efficiency, see, for example, [Fliege and Svaiter \(2000\)](#): If  $\bar{x} \in \mathbb{R}^n$  is locally weakly efficient for (MOP), then it is Pareto critical for (MOP).

However, in contrast to the single-objective case, even if each  $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $i = 1, \dots, m$  is convex, Pareto criticality does not imply efficiency for (MOP) with  $M = \mathbb{R}^n$ . To see this take the simple example from the work by Fliege, Drummond and Svaiter, [Fliege et al. \(2009\)](#):  $f : \mathbb{R} \rightarrow \mathbb{R}^2$  with  $f(x) = (1, x)^\top$ . Then all  $x \in \mathbb{R}$  are Pareto critical but no  $x$  is efficient. Instead one needs strict convexity of all functions  $f_i$  such that Pareto criticality implies efficiency for (MOP).

To prove convergence results and to measure somehow the distance to Pareto criticality, similar to  $\|\nabla f(x)\|_2$  in the single-objective case, the function  $\omega$  defined in [Lemma 4.2](#) is a helpful tool, cf. [Drummond and Svaiter \(2005\)](#).

**Lemma 4.2.** Let  $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $i = 1, \dots, m$ , be continuously differentiable functions and  $M = \mathbb{R}^n$ . For the function  $\omega : \mathbb{R}^n \rightarrow \mathbb{R}$  defined by

$$\omega(x) := - \min_{\|d\|_2 \leq 1} \max_{i=1, \dots, m} \nabla f_i(x)^\top d \quad (1)$$

the following statements hold.

1. The map  $x \mapsto \omega(x)$  is continuous.
2. It holds  $\omega(x) \geq 0$  for all  $x \in \mathbb{R}^n$ .
3. A point  $x \in \mathbb{R}^n$  is Pareto critical for (MOP) if and only if it holds  $\omega(x) = 0$

Recall that a descent direction for (MOP) at a point  $x$  is a vector  $d_\omega \in \mathbb{R}^n$  such that there exists a scalar  $t_0 > 0$  with  $f_i(x + t d_\omega) < f_i(x)$  for all  $t \in (0, t_0]$  and for all  $i \in \{1, \dots, m\}$ . A sufficient condition for such a descent direction is  $\nabla f_i(x)^\top d_\omega < 0$  for all  $i = 1, \dots, m$ , cf. [Thomann and Eichfelder, 2019, Lemma 2.5](#). If  $x \in \mathbb{R}^n$  and  $d_\omega \in \mathbb{R}^n$  denotes a solution of the optimization problem in (1) defining  $\omega(x)$  with  $\omega(x) > 0$ , then  $d_\omega$  is a descent direction for (MOP) at the point  $x$ . As a consequence, the point  $x$  is not Pareto critical for (MOP).

Similarly, for the Newton method, Fliege, Drummond and Svaiter propose in [Fliege et al. \(2009\)](#) to solve the problem

$$\min_{d \in \mathbb{R}^n} \max_{i=1, \dots, m} \nabla f_i(x)^\top d + \frac{1}{2} d^\top \nabla^2 f_i(x) d$$

for obtaining a direction for calculating the next iterate. Here,  $\nabla^2 f_i(x)$  denotes the Hessian of  $f_i$  in  $x$ . For any  $x \in \mathbb{R}^n$ , the optimal solution  $d$  is a Newton direction.

In algorithms for single-objective unconstrained optimization, the norm of the gradient of the objective function at some iteration point  $x^k$  is often used as stopping criterion for the algorithm. That is, if the norm of the gradient at some  $x^k$  is less than some positive constant  $\epsilon$ , the iteration point  $x^k$  is accepted as "almost stationary". On the other hand, large values of the norm of the gradient are interpreted as being far away from having found a stationary point. Similarly, the value  $\omega(x)$  for some  $x \in \mathbb{R}^n$  as defined in [Lemma 4.2](#) can be interpreted as a measure for how close some point  $x$  is to Pareto criticality. The function  $\omega(x)$  from [Lemma 4.2](#) is only suitable for unconstrained optimization problems. Similar concepts for constrained optimization problems are discussed in the literature. They are for instance used by Deb, Abouhawwash and Dutta in [Deb et al. \(2015\)](#) for evolutionary algorithms as an additional selection or stopping criterion: based on the selection criterion, candidates from the current set of iterates (named population) are selected for proceeding to the next iteration. An important property of such functions as  $\omega$  from [Lemma 4.2](#) is that they are continuous, as discussed in [Deb et al. \(2015\)](#). We recall the definition of a proximity measure according to [Eichfelder and Warnow \(2021b\)](#):

**Definition 4.3.** A function  $\omega : \mathbb{R}^n \rightarrow \mathbb{R}$  is called a proximity measure for (MOP) if for every efficient solution  $\bar{x} \in M$  for (MOP) in which the Abadie constraint qualification (CQ) holds and every sequence  $(x^k)_{k \in \mathbb{N}} \subseteq \mathbb{R}^n$  with  $\lim_{k \rightarrow \infty} x^k = \bar{x}$  the following three properties are satisfied:

- (PM1)  $\omega(x) \geq 0$  for all  $x \in \mathbb{R}^n$ ,
- (PM2)  $\omega(\bar{x}) = 0$ ,
- (PM3)  $\lim_{k \rightarrow \infty} \omega(x^k) = \omega(\bar{x})$ .

Next to proposals for proximity measures based on a scalarize-first approach, measures which directly use the multiobjective Karush-Kuhn-Tucker (KKT) optimality conditions for constrained multiobjective optimization problems have been developed. For presenting those, we assume in the following that the feasible set of (MOP) is nonempty and defined by

$$M := \{x \in \mathbb{R}^n \mid g_j(x) \leq 0, j = 1, \dots, p\} \quad (2)$$

with  $g_j : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $j = 1, \dots, p$ . We give KKT conditions using the Abadie CQ. Such necessary optimality conditions can be found in the basic literature, see for instance [\(Jahn, 2011, Theorem 7.8\)](#). They can be based on various constraint qualifications.

**Theorem 4.4.** Let  $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$  and  $g_j : \mathbb{R}^n \rightarrow \mathbb{R}$  be continuously differentiable functions for all  $i = 1, \dots, m$  and all  $j = 1, \dots, p$  and let  $M$  be given as in (2). Let  $\bar{x} \in M$  be a weakly efficient solution for (MOP) and let the Abadie CQ hold at  $\bar{x}$ . Then there exist Lagrange multipliers  $\bar{\eta} \in \mathbb{R}_+^m$  and  $\bar{\lambda} \in \mathbb{R}_+^p$  such that  $(\bar{x}, \bar{\eta}, \bar{\lambda})$  is a KKT point, i.e., such that it holds

$$\begin{aligned} \sum_{i=1}^m \bar{\eta}_i \nabla f_i(\bar{x}) + \sum_{j=1}^p \bar{\lambda}_j \nabla g_j(\bar{x}) &= 0, \\ g(\bar{x}) &\leq 0, \quad \bar{\lambda}^\top g(\bar{x}) = 0, \\ \sum_{i=1}^m \bar{\eta}_i &= 1. \end{aligned}$$

Note that no convexity assumption is required for the optimality conditions in Theorem 4.4. They have nothing to do with a weighted sum scalarization. In the following, we present a proximity measure, introduced in Eichfelder and Warnow (2021b), which is easy to calculate as it requires to solve only a linear optimization problem.

**Theorem 4.5.** Let  $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$  and  $g_j : \mathbb{R}^n \rightarrow \mathbb{R}$  be continuously differentiable functions for all  $i = 1, \dots, m$  and all  $j = 1, \dots, p$  and let  $M$  be given as in (2). The function  $\omega_s : \mathbb{R}^n \rightarrow \mathbb{R}$  defined by

$$\omega_s(x) := \min \left\{ \varepsilon \in \mathbb{R}_+ \mid \begin{aligned} &\left\| \sum_{i=1}^m \eta_i \nabla f_i(x) + \sum_{j=1}^p \lambda_j \nabla g_j(x) \right\|_\infty \leq \varepsilon, \\ &\sum_{j=1}^p \lambda_j g_j(x) \geq -\varepsilon, \quad \sum_{i=1}^m \eta_i = 1, \\ &g_j(x) \leq \varepsilon \text{ for all } j \in \{1, \dots, p\}, \\ &\eta \in \mathbb{R}_+^m, \lambda \in \mathbb{R}_+^p \end{aligned} \right\}$$

for all  $x \in \mathbb{R}^n$  is a proximity measure. Moreover, for any  $x \in \mathbb{R}^n$  it holds  $\omega_s(x) = 0$  if and only if there exist  $\eta \in \mathbb{R}_+^m$  and  $\lambda \in \mathbb{R}_+^p$  such that  $(x, \eta, \lambda)$  is a KKT point.

Proximity measures can be used within numerical procedures to obtain a termination criterion, or, for evolutionary algorithms, as a selection criterion. Note that these proximity measures are based on local information, i.e., on the gradients in the current iterate  $x$ . Thus they should either be used for convex optimization problems, or it should be taken into account that they do not measure a proximity to a globally efficient solution.

#### 4.4. Guarantee of quality

Numerical algorithms in multiobjective optimization in general aim to find a representation or approximation of the efficient or the nondominated set. Representations and approximations often consist of a finite number of points, which are either a subset of the nondominated set, i.e., representations, or are close to the nondominated set, i.e., approximations. Moreover, approximations can be based on linear or quadratic functions, and can be a part of a sandwich-type approximation which constructs inner and outer approximations for the nondominated set. An example for that are enclosures as discussed in Subsection 5.4. A classification of such concepts can be found in Ruzika and Wiecek, 2005. There are non-scalarization based solution approaches which allow to guarantee a certain quality of the generated approximation, like an enclosure with a certain width, after a finite number of iterations.

In the next section on non-convex multiobjective optimization we give more details on such enclosures. Some of the mentioned procedures give further examples for advantages of non-scalarization based methods. Also for convex multiobjective optimization such enclosure-producing methods exist, as Benson's algorithm extended to convex problems by Ehrgott, Shao and Schöbel in Ehrgott et al. (2011), and by Löhne, Rudloff and Ulus in Löhne et al. (2014). The method delivers polyhedral inner and outer approximations of the nondominated set. The box coverage algorithm in Eichfelder and Warnow (2020) produces an enclosure which consists of the union of  $m$ -dimensional intervals, i.e., of boxes. The authors show that this enclosure contains the nondominated set. Moreover, it can be guaranteed that the enclosure has a certain width after a finite number of iterations, and that a side length

of all boxes is halved in each iteration. See Fig. 6 for such enclosures. We discuss concepts of box-coverages and enclosures in the next section on non-convex problems.

### 5. Non-convex multiobjective optimization

Solving non-convex single-objective optimization problems to global optimality or at least with some guaranteed proximity to the optimal value is known to be a challenge in mathematical optimization. This challenge of finding globally optimal solutions transfers to non-convex multiobjective optimization. The source of non-convexity can be that either one or several of the objective functions are not convex, or that the feasible set is not convex. A reason for the latter can be that some of the variables are required to take only integer values. Then one speaks of mixed integer multiobjective optimization.

For non-convex problems a possible approach are scalarization methods, see, for instance, the method by Burachik, Kaya and Rizvi, Burachik et al. (2021). However, then the parameter dependent single-objective subproblems are non-convex and have to be solved to global optimality. Solving single-objective subproblems to global optimality has to be done iteratively for many choices of the parameters of the scalarization. Let us assume that the single-objective subproblems are solved by a branch-and-bound method, which is often used for non-convex optimization. Then, the steps of the branch-and-bound method have to be repeated for each single-objective subproblem, despite the fact that they are typically quite similar. Instead, a better approach might be to directly apply a branch-and-bound procedure for the multiobjective optimization problem as a whole and without the detour of scalarization. We concentrate on such direct methods in this section, as they have experienced a significant progress in the last two decades.

For the following, we concentrate on multiobjective optimization problems of the form

$$\begin{aligned} \min_{x \in \mathbb{R}^n} f(x) &= (f_1(x), \dots, f_m(x))^\top \\ \text{s.t. } g_j(x) &\leq 0, \quad j = 1, \dots, p, \\ x_l &\in \mathbb{Z}, \quad l \in B \end{aligned} \quad (\text{MOP-NC})$$

with objective functions  $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $i = 1, \dots, m$ , constraint functions  $g_j : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $j = 1, \dots, p$ , and an index set  $B \subseteq \{1, \dots, n\}$ . As before, let  $M$  denote the feasible set, which is assumed to be nonempty. If  $B \neq \emptyset$  or if one of the functions  $f_i$  or  $g_j$  is non-convex, then (MOP-NC) is a non-convex optimization problem.

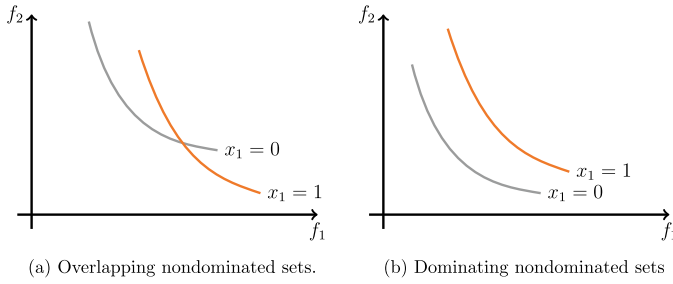
In the literature, one often finds that the notion mixed integer multiobjective optimization is used for problems of this type with  $B \neq \emptyset$  for affine-linear objective and constraint functions. In this section, we present approaches for  $B \neq \emptyset$  with linear but also with nonlinear functions  $f_i$  and  $g_j$ . Moreover, we discuss problems without an integrality constraint, i.e., with  $B = \emptyset$ , but with non-convex objective functions. For a recent book on non-convex multiobjective optimization with  $B = \emptyset$  see Pardalos et al. (2017) by Pardalos, Žilinskas and Žilinskas. Problems with  $B = \{1, \dots, n\}$ , i.e. purely integer problems, are not in the focus of this paper.

#### 5.1. Differences to the single-objective case

First, we point out some differences one can observe between the single-objective and the multiobjective case.

For a start, we assume that all functions are convex and that there is only one integer variable, e.g.,  $B = \{1\}$ , and that  $x_1$  is binary, i.e., it can only take the values  $x_1 = 0$  or  $x_1 = 1$ . For  $m = 1$  and convex functions  $f_i$  and  $g_j$  one has to solve only two convex single-objective optimization problems, one for each integer assignment, i.e., one for each fixing of the integer variable. The smaller optimal value of the two problems is equal to the optimal value of the mixed integer optimization problem.

For  $m \geq 2$ , i.e., for two or more objective functions, already this simple setting is much more challenging. Solving the problems with fixed values for  $x_1$  would mean to determine the whole nondominated set of



**Fig. 2.** Illustration of convex mixed integer biobjective optimization with a binary variable  $x_1$ . The curves show the nondominated sets of the subproblems obtained by fixing the binary variable.

a convex multiobjective optimization problem, which is in general infinite. Then, after computing two sets of nondominated points one has to compare them and to determine the “smallest” values. Situations as illustrated with Fig. 2 can occur. In Fig. 2(a) large parts of the individual nondominated sets for a fixing of  $x_1 = 0$  or  $x_1 = 1$  have been calculated which finally do not contribute to the overall nondominated set. On the other hand, this illustration shows that several integer assignments can contribute to the overall solution set, which is not typical in mixed integer single-objective optimization. In Fig. 2(b) it is even worse and no efficient solution to  $x_1 = 1$  contributes to the overall efficient set at all. For a similar discussion see the recent paper (Cabrera-Guerrero et al., 2021) by Cabrera-Guerrero, Ehrgott, Mason, and Raith. The authors propose an algorithm for biobjective mixed-binary problems which makes use of the idea of comparing nondominated sets to integer assignments by using bounds for the sets.

Thus, problems which are known to be simple to solve in the single-objective setting might no longer be simple in the multiobjective setting. To give another example, solving (MOP – NC) without constraints,  $m = 1$ , and a strictly convex quadratic objective function can be done analytically. But solving (MOP – NC) with  $m = 2$  and strictly convex quadratic objective functions is in general already NP-hard. The fact that solving these problems can no longer be done analytically has an influence on the development of solution methods for (mixed) integer convex quadratic multiobjective optimization problems as in De and Eichfelder (2021).

It is well known that, unless the feasible set of the multiobjective optimization problem is convex and the objective functions are strictly convex, cf. Burachik et al. (2017) by Burachik, Kaya and Rizvi, there can be weakly efficient solutions which are not efficient. From an application point of view, these points are often not of interest. However, scalarizations can often only guarantee to find weakly efficient solutions. Strictly convex objective functions are not sufficient to guarantee that any weakly efficient solution is efficient. For instance, take an integer biobjective optimization problem without inequality constraints with  $n = 1$  and  $B = \{1\}$  and with  $f_1(x) = (x - 1.5)^2$  and  $f_2(x) = (x - 1)^2$ . Then  $x = 1$  and  $x = 2$  are weakly efficient but only  $x = 1$  is efficient, as  $f(1) = (0.25, 0)^T$  and  $f(2) = (0.25, 1)^T$ .

Next to the above listed examples, also other ideas from the single-objective setting cannot be transferred to problems with multiple objectives: it was shown by Eichfelder and Groetzner (2021), after a first examination by Bai and Guo (2014), that lifting procedures which use completely positive relaxations for mixed binary non-convex quadratic problems fail for two or more objective functions in the sense that they can only find supported efficient solutions.

## 5.2. Decision space and criterion space methods

For non-convex optimization problems of the type (MOP – NC) with  $B \neq \emptyset$ , i.e. for (mixed) integer problems, the existing algorithms are mainly for affine-linear objective and constraint functions, see for instance the review in Belotti et al. (2016) by Belotti, Soylu and Wiecek

for biobjective problems. Two of the few exceptions are the procedure by de Santis, Eichfelder, Niebling and Rocktäschel (De Santis et al., 2020), for multiobjective mixed integer convex optimization problems, and by Cabrera-Guerrero, Ehrgott, Mason and Raith (Cabrera-Guerrero et al. (2021)) for biobjective mixed binary convex optimization problems. The known algorithms for multiobjective mixed integer problems can be divided into two main classes: decision space search algorithms, i.e., approaches that work in the space of feasible points  $\mathbb{R}^n$ , and criterion space search algorithms, i.e., methods that work in the space of objective function values  $\mathbb{R}^m$ .

Among the decision space search algorithms, the method proposed by Mavrotas and Diakoulaki (1998) is the first branch-and-bound algorithm for solving mixed binary linear multiobjective problems. Also the method presented recently in De Santis et al., 2020 for solving convex mixed integer multiobjective problems is such a decision space method. An implementation of the algorithm from De Santis et al., 2020 is provided in MOM (0000). The drawback of such methods is that in general the dimension of the decision space is much larger than the dimension of the criterion space. On the other hand, the decision space allows a more direct treatment of the integer variables and to transfer some of the techniques from single-objective optimization more directly.

Criterion space search algorithms find nondominated points by solving a sequence of single-objective optimization problems, for instance on the search zones as explained below. Once a nondominated point is computed, dominated parts of the criterion space are removed and the algorithm continues the search for new nondominated points. Criterion space search algorithms for linear integer biobjective and triobjective problems are given by Boland, Charkhgard and Savelsbergh in Boland et al. (2015a, 2016) and for linear mixed integer problems in Boland et al. (2015b), and more recently by Perini, Boland, Pecen and Savelsbergh in Perini et al. (2020). The method from Cabrera-Guerrero et al. (2021) for mixed integer convex biobjective optimization problems is also a procedure with bounds in the image space. And recently, Eichfelder and Warnow propose in Eichfelder and Warnow, 2021a a criterion space search algorithm for mixed integer convex multiobjective optimization problems.

In Stidsen et al. (2018), Stidsen and Andersen give a good overview on the literature and a discussion on the two different classes, decision space and criterion space methods. Moreover, the authors promote the use a hybrid approach and propose such an algorithm for linear mixed integer biobjective optimization problems.

## 5.3. Supported solutions

In this subsection, we would like to point out that some methods for non-convex multiobjective optimization only generate what is called supported solutions. A feasible point  $\bar{x} \in M$  of the problem (MOP – NC) is called supported if there exist weights  $w \in \mathbb{R}_+^m$  with  $\sum_{i=1}^m w_i = 1$  such that for every  $x \in M$  it holds that  $w^T f(\bar{x}) \leq w^T f(x)$ , i.e.,  $\bar{x}$  is an optimal solution of the weighted sum scalarization

$$\min\{w^T f(x) \mid x \in M\}.$$

For instance, in Bai and Guo (2014) only supported solutions are generated by a relaxation of the original optimization problem with the cone of doubly nonnegative matrices for specific problems (MOP – NC) with  $B = \emptyset$ . This is examined in more detail for  $B = \emptyset$  and for  $B \neq \emptyset$  in Eichfelder and Groetzner (2021).

The method named Poly SCIP, Pol (0000), is for linear problems with purely integer variables, i.e.,  $B = \{1, \dots, n\}$  (or purely continuous, i.e.,  $B = \emptyset$ ). The method finds all nondominated points for two and three objective functions. For  $m \geq 4$  the method is only able to find those nondominated points which are supported points. Przybylski, Gandibleux and Ehrgott examine in Przybylski et al. (2010) linear integer multiobjective optimization problems, but only a subset of the supported solutions are calculated, the so called extreme supported solutions. Özpeynirci and Köksalan present in Özpeynirci and Köksalan (2010) an

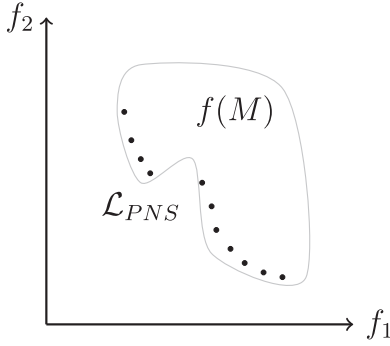


Fig. 3. Example for a set  $\mathcal{L}_{PNS}$ , cf. Niebling and Eichfelder (2019).

exact algorithm to find all extreme supported solutions for a linear mixed integer multiobjective optimization problem. A more recent work on this topic by Przybylski, Klamroth and Lacour contains a good literature review on such approaches, cf. Przybylski et al., 2019.

#### 5.4. Enclosure of the nondominated set

Numerical methods in non-convex single-objective optimization often generate converging sequences of lower and upper bounds on the optimal value. In single-objective optimization, the largest lower bound and the smallest upper bound are unique. As a consequence, the distance between the bounds, which are scalars, can easily be calculated as their difference. In multiobjective optimization this is no longer possible. Upper bounds are often derived from images of feasible points under the objective function. For  $m \geq 2$  these are vectors in a partially ordered space and no longer a unique smallest upper bound might exist. Thus one collects these images of feasible points under the objective function and determines the nondominated points among them. In the following, this set of points will be denoted the provisionally nondominated set  $\mathcal{L}_{PNS}$ , as they are not yet nondominated for the multiobjective optimization problem but only nondominated among the points found so far. The set  $\mathcal{L}_{PNS}$  is also called incumbent set in some of the literature on (mixed) integer multiobjective optimization. An illustration is given in Fig. 3.

Hence, any feasible point  $x$  of  $(MOP - NC)$  gives with  $f(x)$  an upper bound for the nondominated points for  $(MOP - NC)$ : It gives an upper point in the sense that there is no efficient point  $\bar{x}$  for  $(MOP - NC)$  with  $f(\bar{x}) \in \{f(x)\} + \mathbb{R}_+^m \setminus \{0\}$ . Thus, if  $\mathcal{L}_{PNS}$  denotes a finite set of points  $f(x)$  with  $x$  feasible for  $(MOP - NC)$ , then the nondominated set  $\mathcal{N}$  of  $f(M)$  is a subset of

$$\mathcal{L}_{PNS} \cup \left( \mathbb{R}^m \setminus \bigcup_{y \in \mathcal{L}_{PNS}} (\{y\} + \mathbb{R}_+^m) \right). \quad (3)$$

For an illustration of a set  $\mathcal{L}_{PNS}$  for a problem  $(MOP - NC)$  with  $m = 2$ ,  $B = \emptyset$  and  $f_i$  non-convex see the dots in Fig. 3. The set from (3) is marked in Fig. 4 in gray.

By using the concept of local upper bounds from Klamroth, Lacour, and Vanderpooten, Klamroth et al. (2015), the second set in the union in (3) (restricted to a superset  $\hat{Z}$  of  $f(M)$ ) can be expressed with the help of a finite set  $U$  as  $U - \text{int}(\mathbb{R}_+^m)$ . The usefulness of such lower and upper bound sets for multiobjective combinatorial optimization problems has already been recognized earlier by Ehrgott and Gandibleux (2007). For the definition below based on Klamroth et al. (2015), a set  $\mathcal{L}$  is called stable if for any two points  $y^1, y^2 \in \mathcal{L}$  it either holds  $y^1 = y^2$  or  $y^1 \not\leq y^2$ . For  $\mathcal{L}$  we will later take the set  $\mathcal{L}_{PNS}$ . Moreover, a box  $\hat{Z}$  is required with  $f(M) \subseteq \text{int}(\hat{Z})$ . We denote its interior by  $Z$ , i.e.,  $Z := \text{int}(\hat{Z})$ .

**Definition 5.1.** Let  $\mathcal{L}$  be a finite and stable set of points in  $Z$ . A finite set  $U(\mathcal{L}) \subseteq \hat{Z}$  is called a local upper bound set with respect to  $\mathcal{L}$  if for

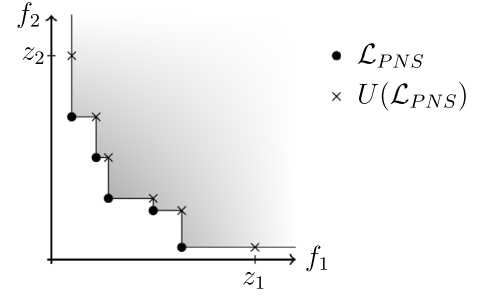


Fig. 4. Set  $\mathcal{L}_{PNS}$  marked with dots and local upper bound set  $U(\mathcal{L}_{PNS})$  marked with crosses, cf. Klamroth et al. (2015); Niebling and Eichfelder (2019).

the set  $S = S(\mathcal{L})$  with

$$S(\mathcal{L}) = Z \setminus \bigcup_{y \in \mathcal{L}} (\{y\} + \mathbb{R}_+^m)$$

it holds

1.  $\forall z \in S(\mathcal{L}) \exists p \in U(\mathcal{L}) : z < p$
2.  $\forall z \in Z \setminus S(\mathcal{L}) \forall u \in U(\mathcal{L}) : z \not\leq u$  and
3.  $\forall u^1, u^2 \in U(\mathcal{L}) : u^1 \not\leq u^2 \text{ or } u^1 = u^2$ .

The set  $S(\mathcal{L})$  in the definition above is called search region. For the search zones  $C(u)$  defined by  $C(u) := (\{u\} - \text{int}(\mathbb{R}_+^m)) \cap Z$  for  $u \in U(\mathcal{L})$ , it holds

$$S(\mathcal{L}) = \bigcup_{u \in U(\mathcal{L})} C(u) = (U(\mathcal{L}) - \text{int}(\mathbb{R}_+^m)) \cap Z$$

and

$$\mathcal{N} \subseteq \mathcal{L}_{PNS} \cup S(\mathcal{L}_{PNS}) \subseteq U(\mathcal{L}_{PNS}) - \mathbb{R}_+^m.$$

For an illustration of a local upper bound set see Fig. 4. There,  $(z_1, z_2)$  is related to the box  $\hat{Z}$ .

While the local upper bound set is easy to compute for  $m = 2$ , as can be seen in Fig. 4, it gets much more complicated in higher dimensions. But the authors in Klamroth et al. (2015), and more recently Dächert, Klamroth and Lacour in Dächert et al. (2017), give algorithms which allow a fast computation. The local upper bounds can be used for a calculation of the hypervolume indicator, see Lacour, Klamroth and Fonseca, Lacour et al. (2017). The hypervolume indicator is used for comparing the quality of discrete approximations or representations of the nondominated set. It measures the volume of the part of the criterion space dominated by the points of the approximation and bounded by some reference point.

The local upper bounds can well be used within algorithms. For instance, the search zones are used for improving the set  $\mathcal{L}_{PNS}$  by solving a scalarization within a search zone. A further example is the algorithm from Niebling and Eichfelder (2019) which is for problems of the type  $(MOP - NC)$  with  $B = \emptyset$ , convex constraints, and non-convex objective functions. The algorithm generates, for a predefined  $\varepsilon > 0$ , a list  $\mathcal{L}_{PNS}$  such that for the related set of local upper bounds  $U = U(\mathcal{L}_{PNS})$  it holds that all nondominated points for  $(MOP - NC)$  are contained in the set

$$T := \left( \bigcup_{\bar{p} \in U} \{\bar{p}\} - (\mathbb{R}_+^m) \right) \setminus \left( \bigcup_{\bar{p} \in U} \left\{ \bar{p} - \frac{\varepsilon}{2} e \right\} - \text{int}((\mathbb{R}_+^m)) \right). \quad (4)$$

In (4), the vector  $e$  denotes the vector of ones (also known as all-ones vector). An illustration of such a set  $T$  is shown in Fig. 5.

The algorithm is a branch-and-bound method with subdivisions in the decision space. For applying the algorithm, a box  $X^0$  has to be known with  $M \subseteq X^0$ . The basic idea of the algorithm goes back to a paper from Fernández and Tóth, Fernández and Tóth (2009), in which interval arithmetic and lower bounds by ideal points are used. The ideal point of  $(MOP - NC)$  is the point  $a \in \mathbb{R}^m$  with  $a_i := \min_{x \in M} f_i(x)$ ,  $i = 1, \dots, m$ . In Niebling and Eichfelder (2019), lower bounds generated with the help



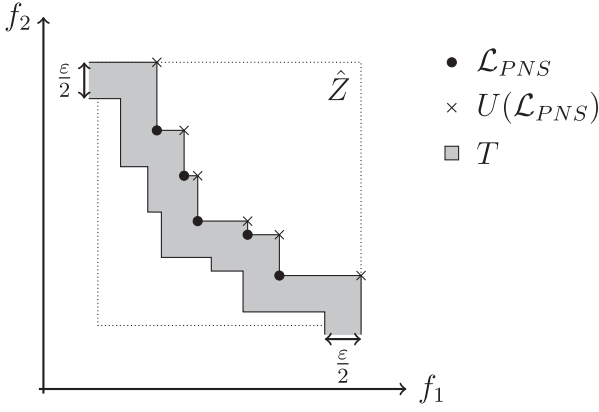


Fig. 5. Set  $T$  from (4) depending on  $\frac{\epsilon}{2}$ , cf. Niebling and Eichfelder (2019).

of convex relaxations and outer approximations are used. Those deliver lower bounds in such a way that the above property in (4) is guaranteed after a finite number of iterations. The local upper bounds are used in a similar way for convex mixed integer multiobjective optimization within a decision space branch-and-bound algorithm in De Santis et al., 2020.

For non-convex single-objective optimization the generated lower bounds are used in general more directly than in Niebling and Eichfelder (2019) where they are only used for discarding tests. Using the lower bounds as in the single-objective setting was transferred recently to the multiobjective setting by Eichfelder, Kirst, Meng and Stein in Eichfelder et al. (2021a). As a result, the authors obtain a box-based coverage of the nondominated set, called enclosure in Eichfelder et al. (2021a). Recall that we denote the nondominated set of  $f(M)$  by  $\mathcal{N}$ . We use for a box with bounds  $\ell, u \in \mathbb{R}^m$  the notation

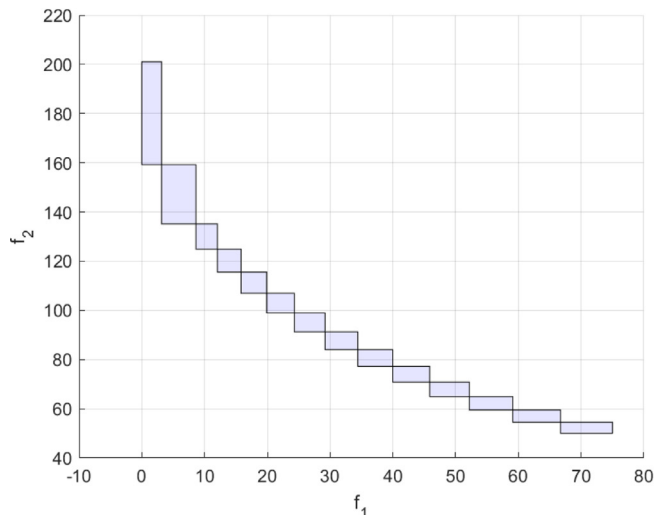
$$[\ell, u] := \{y \in (\mathbb{R})^m \mid \ell_i \leq y_i \leq u_i \text{ for all } i = 1, \dots, m\}.$$

**Definition 5.2.** Let  $L, U \subseteq \mathbb{R}^m$  be two finite sets with

$$\mathcal{N} \subseteq L + \mathbb{R}_+^m \text{ and } \mathcal{N} \subseteq U - \mathbb{R}_+^m.$$

Then  $L$  is called lower bound set,  $U$  is called upper bound set, and the set  $A$  which is given as

$$A = A(L, U) := (L + \mathbb{R}_+^m) \cap (U - \mathbb{R}_+^m) = \bigcup_{\ell \in L} \bigcup_{\substack{u \in U, \\ \ell \leq u}} [\ell, u]$$



is called approximation or enclosure of the nondominated set  $\mathcal{N}$  given  $L$  and  $U$ .

In Eichfelder et al. (2021a) it is proposed to use for  $U$  the set of local upper bounds to a stable set  $\mathcal{L} = \mathcal{L}_{PNS}$  of images of feasible points. The lower bounds are generated by ideal points, for which relaxations of  $f_i$  and of  $M$  as well as subproblems, e.g. over  $M \cap X^k$  with some subbox  $X^k$ , are used.

To measure the width of this enclosure, similar to the difference between the upper and the lower bound on the optimal value in single-objective global optimization, the following is used: The width  $w(L, U)$  is defined as the optimal value of the problem

$$\max_{y, t} \{ \| (y + te) - y \|_2 / \sqrt{m} \mid t \geq 0, y, y + te \in A(L, U) \}.$$

The authors in Eichfelder et al. (2021a) show that the width  $w(L, U)$  of  $A(L, U)$  coincides with

$$\max_{\ell, u} \{ s(\ell, u) \mid (\ell, u) \in L \times U, \ell \leq u \}$$

where the length of a shortest edge  $s(\ell, u)$  of a box  $[\ell, u]$  is used which is defined by

$$s(\ell, u) := \min_{j=1, \dots, m} (u_j - \ell_j).$$

Moreover, the authors in Eichfelder et al. (2021a) show that if  $w(L, U) < \epsilon$  holds for some  $\epsilon > 0$ , then all points from the set  $A(L, U) \cap f(M)$  are at least  $\epsilon$ -nondominated for  $(MOP - NC)$ .

For  $\epsilon > 0$ , a point  $\bar{x} \in M$  is called  $\epsilon$ -efficient for  $(MOP - NC)$  if there exists no  $x \in M$  such that  $f(x) \leq f(\bar{x}) - \epsilon e$  and  $f(x) \neq f(\bar{x}) - \epsilon e$  hold. In that case the vector  $f(\bar{x})$  is called  $\epsilon$ -nondominated for  $(MOP - NC)$ . For a source of this definition see, for instance, Loridan (1984) with the specific choice  $\epsilon e \in \mathbb{R}_+^m$ . In Fig. 6 we illustrate the concept of an enclosure for a convex biobjective and for a convex triobjective problem, respectively.

## 6. Multiobjective bilevel optimization

Bilevel optimization is another class of problems that, already in the single-objective case, are very challenging from both the theoretical and numerical point of view. In bilevel optimization two optimization problems appear, one on the so-called upper level and one on the lower level. The optimization variable of the upper level is a parameter for the lower-level problem, and the optimal solutions of the lower-level problem influence the objective function value on the upper level. For a good introduction to the topic we refer to the book by Dempe, 2012. Due

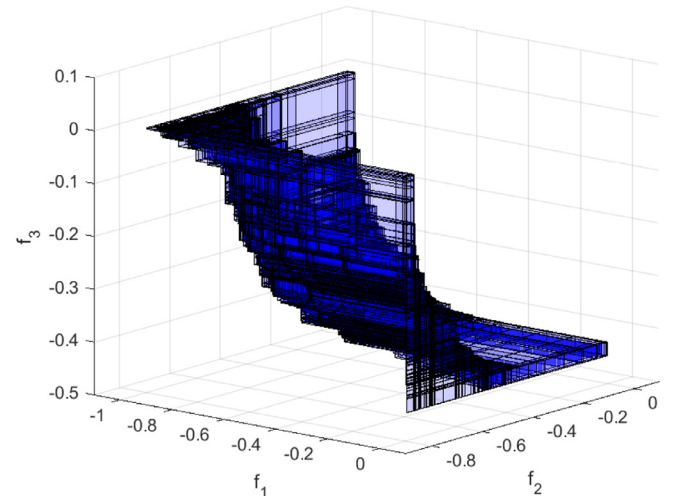


Fig. 6. Enclosure for a (a) biobjective and a (b) triobjective problem.

to several applications, for instance in the energy market, this problem class is of recent interest.

A simply structured bilevel optimization problem with just one objective function on each level and without further constraints is the following,

$$\begin{aligned} & \min_{x_u \in \mathbb{R}^{n_u}} f_u(x_\ell, x_u) \\ \text{s.t. } & x_\ell \in \operatorname{argmin}_{z \in \mathbb{R}^{n_\ell}} \{f_\ell(z, x_u)\}, \end{aligned} \quad (5)$$

where  $f_u, f_\ell : \mathbb{R}^{n_\ell} \times \mathbb{R}^{n_u} \rightarrow \mathbb{R}$  ( $n_\ell, n_u \in \mathbb{N}$ ). Here, we assume that the lower-level problem has a minimal solution for any  $x_u \in \mathbb{R}^{n_u}$ . Already in this simple setting, one of the major difficulties of this problem class can appear: if the lower-level problem has a non-unique optimal solution, then the bilevel problem is not well defined. For some  $x_u$  from the upper level and a set of optimal solutions  $X_\ell(x_u)$  to that  $x_u$  for the lower level, it is not clear for which  $x_\ell \in X_\ell(x_u)$  the upper-level objective function  $f_u$  should be evaluated. In approaches for bilevel optimization one often just assumes that the lower-level problem has a unique optimal solution, which is for instance the case for strictly convex functions  $f_\ell(\cdot, x_u)$ . If there is a multiobjective optimization problem with two or more objective functions on the lower level, this can no longer be assumed, since these problems have in general an infinite set of efficient solutions, even in case all functions are strictly convex. Already in this simple setting, one of the major difficulties of this problem class can appear: if the lower-level problem has a non-unique optimal solution, then the bilevel problem is not well defined. For some  $x_u$  from the upper level and a set of optimal solutions  $X_\ell(x_u)$  to that  $x_u$  for the lower level, it is not clear for which  $x_\ell \in X_\ell(x_u)$  the upper-level objective function  $f_u$  should be evaluated. In approaches for bilevel optimization one often just assumes that the lower-level problem has a unique optimal solution, which is for instance the case for strictly convex functions  $f_\ell(\cdot, x_u)$ . If there is a multiobjective optimization problem with two or more objective functions on the lower level, this can no longer be assumed, since these problems have in general an infinite set of efficient solutions, even in case all functions are strictly convex.

If the lower-level problem has non-unique optimal solutions, then a common approach is the optimistic approach: the upper level function is not only minimized with respect to  $x_u$  but also with respect to  $x_\ell$ . Next to the optimistic approach, pessimistic formulations are discussed in the literature for which the upper level function is maximized w.r.t.  $x_\ell$ .

Before we discuss the optimistic approach for the multiobjective setting, we would like to mention that bilevel optimization problems have already raised the interest in the multiobjective community with respect to another research question: Is it possible to reformulate a bilevel optimization problem equivalently as a biobjective optimization problem? It can easily be seen that the biobjective optimization problem with objective functions  $f_u$  and  $f_\ell$  does not fulfill that. However, there are more complex reformulations, which require for instance the study of multiobjective optimization problems with a vector-valued objective function defined by  $F : \mathbb{R}^{n_\ell+n_u} \rightarrow \mathbb{R}^{n_u+3}$

$$F(x_\ell, x_u) = (x_u, f_u(x_\ell, x_u), f_\ell(x_\ell, x_u), \|\nabla_{x_\ell} f_\ell(x_\ell, x_u)\|_2)$$

and a specific partial ordering. A first discussion of this topic is due to (Fliege, Vicente, 2006) from 2006. In a recent survey by Ruuska, Miet-

inen and Wiecek, (Ruuska et al. (2012)), this topic is thoroughly examined together with an excellent literature survey on other contributions to this topic. It is still an open research question whether these reformulations can be used for developing numerical algorithms for bilevel problems with more than one or two variables on the upper level.

If one allows multiple objectives, a more general lower-level problem than the one in (5) might read as

$$\min_{x \in \mathbb{R}^{n_1}} \{f(x, y) \mid (x, y) \in G\} \quad (6)$$

with the parameter  $y \in \mathbb{R}^{n_2}$  from the upper level, a vector-valued function  $f : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \rightarrow \mathbb{R}^{m_1}$ , and a nonempty feasible set  $G \subseteq \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$  ( $n_1, n_2, m_1 \in \mathbb{N}$ ). For the upper-level variable  $y \in \mathbb{R}^{n_2}$  we define

$$\Psi(y) := \operatorname{argmin}_x \{f(x, y) \mid (x, y) \in G\} \subseteq \mathbb{R}^{n_1}.$$

For  $f$  vector-valued, it has to be clarified what a minimal solution is, i.e., how the set  $\Psi(y)$  is defined: whether it is the set of efficient solutions, or the set of weakly efficient solutions, or something different (as the set of properly efficient solutions which is a subset of the set of efficient solutions). For the following, we define them to be the efficient solutions of (6).

The optimization problem on the upper level might then be given as

$$\min_{y \in \mathbb{R}^{n_2}} \{F(x, y) \mid x \in \Psi(y), y \in \tilde{G}\},$$

with a vector-valued function  $F : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \rightarrow \mathbb{R}^{m_2}$ ,  $m_2 \in \mathbb{N}$ , and a compact set  $\tilde{G} \subseteq \mathbb{R}^{n_2}$ . As discussed above, the upper-level problem is not well-defined without saying which of the efficient solutions of the lower-level problem should be chosen for the evaluation of the upper level function. An often chosen approach is the optimistic approach, i.e., one solves

$$\min_{x \in \mathbb{R}^{n_1}, y \in \mathbb{R}^{n_2}} \{F(x, y) \mid x \in \Psi(y), y \in \tilde{G}\}.$$

Another possible approach for a non-singleton set  $\Psi(y)$  is to study all efficient solutions of (6) at the same time and thus the set-valued map

$$y \mapsto \{F(x, y) \mid x \in \Psi(y)\} \subseteq \mathbb{R}^{m_2}.$$

Examinations on such an approach can be found in a recent PhD thesis by Pilecka, 2016. The use of set relations, which compare sets as a whole, then allows to study optimal solutions of a bilevel problem without explicitly choosing one of the lower level efficient solutions. Possible set relations are, for instance, the lower-less order relation defined by

$$\begin{aligned} A \leq_\ell B & \quad :\Leftrightarrow \quad B \subseteq A + \mathbb{R}_+^{m_2} \\ & \quad \Leftrightarrow \quad \forall b \in B \exists a \in A : a_i \leq b_i \quad \forall i \in \{1, \dots, m_2\} \end{aligned}$$

for two nonempty sets  $A, B \subseteq \mathbb{R}^{m_2}$ . Another widely used set relation is called the upper-less order relation defined by

$$\begin{aligned} A \leq_u B & \quad :\Leftrightarrow \quad A \subseteq B - \mathbb{R}_+^{m_2} \\ & \quad \Leftrightarrow \quad \forall a \in A \exists b \in B : a_i \leq b_i \quad \forall i \in \{1, \dots, m_2\} \end{aligned} \quad (7)$$

for two nonempty sets  $A, B \subseteq \mathbb{R}^{m_2}$ . For an illustration of both concepts see Fig. 7. For more details on set optimization and on solution concepts based on such set relations we refer to the book by Tammer, Khan and Zălinescu, Khan et al. (2015). The set relations then allow to model optimistic or pessimistic situations as well, see Pilecka, 2016.

Solving bilevel optimization problems numerically is known to be a challenge, but this is even more the case for multiple objectives on the lower level. To see this, observe that in the single-objective case ( $m_1 = 1$ ) the lower level problem can be reformulated by using the minimal value function

$$w(y) := \min_x \{f(x, y) \mid (x, y) \in G\}$$

and by replacing the upper level constraint  $x \in \Psi(y)$  by

$$f(x, y) \leq w(y), \quad (x, y) \in G. \quad (8)$$

Thus, smoothness of this minimal value function is important. If the function  $f$  on the lower level maps to  $\mathbb{R}^{m_1}$  with  $m_1 \geq 2$ , then the values

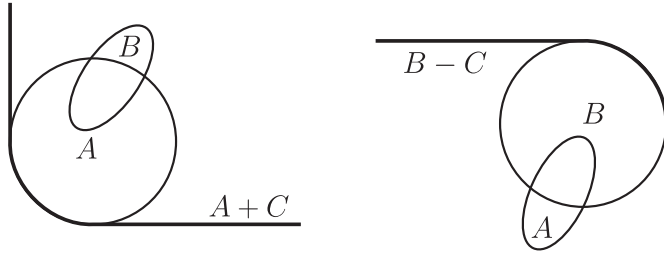


Fig. 7. Lower-less (left) and upper-less (right) order relation for two sets  $A, B \subseteq \mathbb{R}^2$  and ordering cone  $C := \mathbb{R}_+^2$ .

of the minimal value function  $w(y)$  are sets of nondominated points (or of weakly nondominated points, dependent on the choice of  $\Psi(y)$ ), and (8) has to be replaced by the more complicated formulation

$$f(x, y) \in w(y) - \mathbb{R}_+^m, \quad (x, y) \in G.$$

As multiobjective bilevel problems are important within many application areas, many researchers have tried to contribute to solvers for these problems. For linear multiobjective bilevel problems procedures have already been proposed more than twenty years ago, see for instance the paper by (Nishizaki, Sakawa, 1999).

Much fewer papers are dealing with nonlinear multiobjective bilevel problems. A first interactive method has been proposed in 1997 by Shi and Xia (1997). Other contributions which appeared in the last ten years use scalarizations, as by Sinha, Malo and Deb in Sinha et al. (2018). However, in the case the scalarization does not characterize the set of (weakly) efficient solutions of the lower level problem exactly, this might be a problem and imply a relaxation or restriction of the overall problem. What is more, as discussed by Dempe and Mehlitz (2020), it can be a delicate issue if just locally optimal solutions of the scalarization are investigated. Further approaches are the characterization of the upper-level feasible set as solution set of specific multiobjective optimization problems, as proposed by Eichfelder (2010); Eichfelder, 2008. Bonnel and Morgan consider in Bonnel and Morgan (2006) a semivectorial bilevel optimization problem and propose a solution method based on a penalty approach. In Gebhardt and Jahn (2009), Gebhardt and Jahn propose to use a multiobjective search algorithm with a subdivision technique. Approaches for solving multiobjective bilevel optimization problems by using evolutionary algorithms in combination with local solvers can be found, for instance, in Deb and Sinha (2010). In 2020, a survey on multiobjective bilevel optimization by Eichfelder appeared which summarizes the progress in numerical solvers in more detail, see Eichfelder (2020). Moreover, just recently, an implementation of the algorithm from Eichfelder (2010) was made publicly available MOB (0000).

However, all existing approaches are limited in the sense that they only apply to very specific problem classes and can, for instance, only be applied for a small number of objective functions on the levels and/or a small number of variables, specifically on the upper level. Thus more research is needed.

## 7. Robustness in uncertain multiobjective optimization

In many real-world applications there are uncertainties in the optimization problems under examination which should be taken into account when solving the problem. One way to deal with uncertainties is robust optimization. Then, the aim is to find solutions which remain feasible and of good quality for all possible scenarios, i.e., for all possible realizations of the uncertain data. The uncertainties may be modeled by some uncertainty set  $\mathcal{U} \subseteq \mathbb{R}^k$ . For each parameter  $\xi \in \mathcal{U}$ , called scenario, one studies the parametric optimization problem

$$\min_{x \in M(\xi)} f(x, \xi) \quad \mathcal{P}(x)$$

with objective function  $f : \mathbb{R}^n \times \mathbb{R}^k \rightarrow \mathbb{R}^m$  and nonempty feasible set  $M(\xi) \subseteq \mathbb{R}^n$  for each scenario  $\xi \in \mathcal{U}$ . The uncertain optimization problem  $\mathcal{P}(\mathcal{U})$  is given as the family  $(\mathcal{P}(\xi), \xi \in \mathcal{U})$ . For being feasible one typically requires that some  $x \in \mathbb{R}^n$  is feasible for all scenarios  $\xi \in \mathcal{U}$ , i.e., one requires

$$x \in M := \bigcap_{\xi \in \mathcal{U}} M(\xi). \quad (9)$$

Then, for  $m = 1$ , following the classical approach of minmax-robustness, one has to solve the optimization problem

$$\min_{x \in M} \max_{\xi \in \mathcal{U}} f(x, \xi).$$

In the last ten years, many authors have examined how this can be extended to meaningful definitions for problems with  $m \geq 2$  objective functions. One can observe an increasing interest in the definition of suitable optimality notions and solution approaches.

### 7.1. Objective-wise worst case

A first approach to extend minmax-robustness is due to Kuroiwa and Lee (2012). Doolittle, Kerivin and Wiecek, Doolittle et al. (2018), follow a similar approach. They replace each objective function of the uncertain multiobjective optimization problem by the worst case of the respective objective function values, i.e., they study the deterministic multiobjective optimization problem

$$\min_{x \in M} f_{\mathcal{U}}^{OW}(x) \quad \text{with} \quad f_{\mathcal{U}}^{OW}(x) := \begin{pmatrix} \max_{\xi \in \mathcal{U}} f_1(x, \xi) \\ \vdots \\ \max_{\xi \in \mathcal{U}} f_m(x, \xi) \end{pmatrix}, \quad (10)$$

where  $OW$  stands for objective-wise. Then, the robust (weakly) efficient solutions for  $\mathcal{P}(\mathcal{U})$  w.r.t. the objective-wise approach are the (weakly) efficient solutions for (10).

A similar concept was studied by Fliege and Werner (2013) in the context of portfolio selection. The above reformulation is motivated by applying first an  $\varepsilon$ -constraint scalarization to the problems  $(\mathcal{P}(x))$ . Then one obtains a formulation as

$$\min_{x \in M(\xi)} \{f_m(x, \xi) \mid f_i(x, \xi) \leq \varepsilon_i, \forall i = 1, \dots, m-1\},$$

for some  $\varepsilon \in \mathbb{R}^{m-1}$ . As a next step, when applying the same idea as stated in (9) for feasibility, a point  $x$  is feasible for all  $\xi \in \mathcal{U}$  if it holds  $x \in M$  and

$$\max_{\xi \in \mathcal{U}} f_i(x, \xi) \leq \varepsilon_i \quad \text{for all } i = 1, \dots, m-1.$$

The minmax-robustness approach for this single-objective optimization problem results in

$$\begin{aligned} & \min_{x \in M} \max_{\xi \in \mathcal{U}} f_m(x, \xi) \\ \text{s.t.} \quad & \max_{\xi \in \mathcal{U}} f_i(x, \xi) \leq \varepsilon_i, \quad \text{for all } i = 1, \dots, m-1. \end{aligned}$$

The above problem is exactly an  $\varepsilon$ -constraint scalarization of (10).

Next to this motivation, a further advantage of the formulation in (10) is that the problems are in general easier to solve than those being a result from the approach in the next subsection. We illustrate the concept using the objective-wise worst case with an example:

**Example 7.1.** Let  $m = 2$ ,  $n = k = 1$  and assume we have as uncertainty set the interval  $\mathcal{U} = [0, 1]$ . Moreover, assume that there are only two feasible points, i.e.,  $M = \{x^1, x^2\} \subseteq \mathbb{R}$  and that  $f : M \times \mathcal{U} \rightarrow \mathbb{R}^2$  is defined by

$$f(x^1, \xi) = \xi \begin{pmatrix} 5 \\ 2 \end{pmatrix} + (1 - \xi) \begin{pmatrix} 2 \\ 5 \end{pmatrix} \quad \text{and} \quad f(x^2, \xi) = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \quad \text{for all } \xi \in \mathcal{U}.$$

Then for the objective function in (10) we have

$$f_{\mathcal{U}}^{OW}(x^1) = \begin{pmatrix} 5 \\ 5 \end{pmatrix}, \quad f_{\mathcal{U}}^{OW}(x^2) = \begin{pmatrix} 4 \\ 4 \end{pmatrix}.$$

and thus  $x^2$  is the only efficient solution for (10) and thus the only robust efficient solution w.r.t. the objective-wise worst case approach for  $\mathcal{P}(\mathcal{U})$ .

In the above example we have a specific situation: the uncertainty has only an impact for  $x^1$ . For  $x^1$  we have for many scenarios  $\xi$  smaller objective function values for each objective function than for  $x^2$ . For the remaining scenarios  $\xi$  we have that  $f(x^1, \xi)$  and  $f(x^2, \xi)$  are not comparable w.r.t. the componentwise ordering. We never have that  $f(x^2, \xi) \leq f(x^1, \xi)$ , but still  $x^1$  is not a robust efficient solution w.r.t. the objective-wise worst case approach.

The reason for that is that with this approach we do not model that one scenario will finally happen for the overall problem. Instead we model that one scenario  $\xi^1$  can lead to the worst case of objective function  $f_1$  and another  $\xi^2$  to the worst case for objective function  $f_2$ . In practice, one might instead want to model the worst-case for the vector-valued function  $f$  directly and to keep the interwoven structure of the objective functions  $f_i$ ,  $i = 1, \dots, m$ , connected by the parameter  $\xi$ .

Moreover, such an objective-wise approach can only be studied if one has the componentwise ordering in the image space, i.e., with  $y \leq z$  if and only if  $y_i \leq z_i$  for all  $i = 1, \dots, m$ . The objective-wise approach cannot be transferred to more general ordering structures as those briefly mentioned in Section 8. Thus, other robustness notions are discussed in the literature which use a set based approach, see the next subsection.

## 7.2. Set-valued based worst case

We call the approach in this section set-valued approach as it makes use of the set-valued function  $f_{\mathcal{U}} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ ,

$$f_{\mathcal{U}}(x) := \{f(x, \xi) \in \mathbb{R}^m \mid \xi \in \mathcal{U}\}.$$

Based on this function, Ehrgott, Ide and Schöbel define in Ehrgott et al. (2014) the following optimality notions for  $\mathcal{P}(\mathcal{U})$ :

**Definition 7.2.** A point  $\bar{x} \in M$  with  $M$  as in (9) is denoted

1. robust weakly efficient (rwe) for  $\mathcal{P}(\mathcal{U})$  if there is no  $x \in M \setminus \{\bar{x}\}$  such that

$$f_{\mathcal{U}}(x) \subseteq f_{\mathcal{U}}(\bar{x}) - \text{int}(\mathbb{R}_+^m).$$

2. robust efficient (re) for  $\mathcal{P}(\mathcal{U})$  if there is no  $x \in M \setminus \{\bar{x}\}$  such that

$$f_{\mathcal{U}}(x) \subseteq f_{\mathcal{U}}(\bar{x}) - (\mathbb{R}_+^m \setminus \{0\}).$$

3. robust strictly efficient (rse) for  $\mathcal{P}(\mathcal{U})$  if there is no  $x \in M \setminus \{\bar{x}\}$  such that

$$f_{\mathcal{U}}(x) \subseteq f_{\mathcal{U}}(\bar{x}) - \mathbb{R}_+^m. \quad (11)$$

Note that with (7) the condition in (11) can equivalently be written as

$$f_{\mathcal{U}}(x) \leq_u f_{\mathcal{U}}(\bar{x}).$$

In the following we use the abbreviations in the brackets in Definition 7.2. It holds that rse implies re which again implies rwe for  $\mathcal{P}(\mathcal{U})$ . The authors in Ehrgott et al. (2014) also compare their approach with the objective-wise worst case approach from above. According to their Theorem 4.11, in the case  $\max_{\xi \in \mathcal{U}} f_i(x, \xi)$  exists for all  $i = 1, \dots, m$  and all  $x \in M$ , then any weakly efficient solution for (10) is rwe for  $\mathcal{P}(\mathcal{U})$ .

**Example 7.3.** We determine the rwe, re and rse solutions to the problem in Example 7.1. The point  $x^2$  is efficient and hence weakly efficient for (10). Thus, by the above cited theorem it is rwe for  $\mathcal{P}(\mathcal{U})$ . By applying the definition above it is easy to verify that  $x^2$  is rse and re for  $\mathcal{P}(\mathcal{U})$ , and  $x^1$  is rse, re and thus rwe for  $\mathcal{P}(\mathcal{U})$ .

The example shows that we can find more robust efficient solutions with the set-valued approach. Moreover, in general many feasible points are rse and re for  $\mathcal{P}(\mathcal{U})$  as now sets are compared by using the upper-less

order relation defined in (7), and variants of it, and sets are often not comparable.

Robust multiobjective optimization with the set-valued approach has an obvious relation to set optimization, which is widely used in the literature for further theoretical examinations and for first numerical approaches. For instance, this is studied in the specific case of decision uncertainty by Eichfelder, Krüger and Schöbel in Eichfelder et al. (2017), i.e., for the case that one has for the problem  $(\mathcal{P}(x))$  objective functions of the form  $f_i(x, \xi) = \tilde{f}_i(x + \xi)$ , with functions  $\tilde{f}_i : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $i = 1, \dots, m$  and  $\mathcal{U} \subseteq \mathbb{R}^n$ . The uncertainty is thus in the realization of the variable  $x$  and we obtain the set-valued function

$$f_{\mathcal{U}}(x) = \{\tilde{f}(x + \xi) \in \mathbb{R}^m \mid \xi \in \mathcal{U}\}.$$

For this type of problem, numerical approaches based on comparing the sets with a decision space branch-and-bound procedure have been proposed by Eichfelder, Niebling and Rocktäschel, see Eichfelder et al. (2020).

Multiobjective optimization problems with decision uncertainty arise in applications whenever exactly computed optimal solutions cannot be realized in practice with such an exactness. To give an example, for a horticulture problem with economic and environmental goals uncertainties in the mixing process have been modeled and robust solutions have been calculated by Krüger, Castellani, Geldermann and Schöbel in Krüger et al. (2018).

The relation of robust optimization for uncertain multiobjective optimization to set optimization is intensively studied by Klamroth, Köbis, Schöbel, and Tammer in Klamroth et al. (2017). As various set relations exist, one can derive various notions of robustness when replacing the upper-less order relation by one of those. A recent survey by Ide and Schöbel (2016) gives a good overview on the variety of concepts used in robust uncertain multiobjective optimization. For a more recent paper with a survey and with another new concept called multi-scenario efficiency see Botte and Schöbel (2019).

In some specific situations one has that the uncertainties of the individual objective functions are independent of each other, called objective-wise uncertainty. Then one studies instead of  $(\mathcal{P}(x))$  a multi-objective optimization problem with objective function  $f : \mathbb{R}^n \times \mathbb{R}^k \rightarrow \mathbb{R}^m$ ,

$$f(x, \xi) = \begin{pmatrix} f_1(x, \xi_1) \\ \vdots \\ f_m(x, \xi_k) \end{pmatrix}$$

with  $\xi_i \in \mathcal{U}_i \subseteq \mathbb{R}^{k_i}$ ,  $i = 1, \dots, m$ ,  $\sum_{i=1}^m k_i = k$ , and  $\mathcal{U} = \mathcal{U}_1 \times \dots \times \mathcal{U}_m$ , see for instance (Ehrgott et al., 2014, Def. 5.1) by Ehrgott, Ide and Schöbel. According to their Theorem 4.4, if  $\max_{\xi_i \in \mathcal{U}_i} f_i(x, \xi_i)$  exists for all  $i = 1, \dots, m$  and all  $x \in M$ , then a point  $\bar{x}$  is an efficient solution for (10) if and only if it is re for  $\mathcal{P}(\mathcal{U})$ .

Robust solutions are in some sense expensive: for being robust to all possible scenarios one might decide for a solution, as for  $x^2$  in Example 7.1, despite for many scenarios much better function values would have been possible (for instance for  $\xi = 0.5$ ,  $x^1$  would have been the better choice). Consequently, the arising gaps of the objective function values of the robust solutions and of the solutions for the individual scenarios are of interest and subject to research, see, for instance, Krüger, Schöbel and Wiecek (Krüger et al., 2017). Similarly, in Groetzner and Werner (2021), the authors aim to minimize the regret of not having chosen the optimal decision. In their paper (Groetzner and Werner, 2021) they study the objective-wise worst case approach. The extensions from the concepts of the single-objective case to the multi-objective case are often not straightforward and finally complicated to calculate numerically. Thus, more progress has to be made before robust solutions can be calculated numerically, and with an appropriate trade-off between robustness and a possible regret.



## 8. Further topics such as extension of concept

The sections in this survey intend to give the basic ideas of some of the fields in continuous multiobjective optimization which have seen a huge progress or which have raised ongoing interest in the last twenty years. Of course, there are many more topics for which important and new results have been gained and where the area of multiobjective optimization is developed further.

To list some of the further topics, there is the study of multiobjective optimal control problems and how these can be solved by means of model predictive control, see [Grüne and Stieler \(2019\)](#). A reason for the increasing interest in this type of problem is that there are important applications as in the energy market, see [Sauerteig and Worthmann \(2020\)](#). Also for multiobjective optimal control problems scalarizations are a possible approach, as examined for instance by Kaya and Maurer in [Kaya and Maurer \(2014\)](#) and by Logist, Houska, Diehl and Van Impe in [Logist et al. \(2010\)](#).

A further topic of ongoing interest is to try to include more information on the application problem in the formulation of the multiobjective optimization problem by introducing more general ordering structures than partial orderings. So far in this paper, for defining an efficient solution for (MOP) we use the so called natural ordering, i.e., the componentwise ordering in the criterion space: for  $y, z \in \mathbb{R}^m$  we use

$$y \leq z \Leftrightarrow y_i \leq z_i \text{ for all } i = 1, \dots, m.$$

The above concept implies that a feasible point  $x$  is preferred to some feasible point  $w$  if all objective functions are satisfied at least equally well, i.e.,  $f_i(x) \leq f_i(w)$ ,  $i = 1, \dots, m$ , and at least one has a smaller value, i.e.,  $f(x) \neq f(w)$ . As a consequence, it holds for an efficient solutions that there is no feasible point such that one can improve the value of any of the objective functions without deteriorating at least one of the others.

However, one can take additional knowledge on the preferences on a possible trade-off between the objective functions into account which has already been done in vector optimization for a long time. For instance, one might not be willing to accept a slight improvement for one function value if this implies a huge worsening of another objective function value. Then such information on the preferences for an allowed trade-off can be modeled mathematically by an ordering cone, see [Hunt and Wiecek \(2003\)](#). In that case, one needs to define efficiency by using a more general (antisymmetric) partial ordering  $\leq$ . We use here the notion of a partial ordering in a linear space according to the book [Jahn \(2011\)](#). Any such antisymmetric partial ordering can be represented by a pointed convex cone  $K \subseteq \mathbb{R}^m$ , then called ordering cone, by

$$y \leq_K z \Leftrightarrow z - y \in K.$$

Recall that a nonempty set  $K \subseteq \mathbb{R}^m$  is a convex cone if  $\lambda(x + y) \in K$  for all  $\lambda \geq 0$ ,  $x, y \in K$ , and  $K$  is a pointed convex cone if additionally  $K \cap (-K) = \{0\}$ . Then a point  $\bar{x} \in M$  is an efficient solution for  $\min_{x \in M} f(x)$  w.r.t. a partial ordering defined by  $K$  if

$$(\{f(\bar{x})\} - K) \cap f(M) = \{f(\bar{x})\}.$$

Thus one can model problems with more general ordering cones than the cone  $\mathbb{R}_+^m$ . Such a modeling with more general ordering cones is a topic of [Wiecek \(2007\)](#), in which Wiecek also discusses the concept of variable domination which goes back to [Yu \(1974\)](#). Variable ordering structures can be defined in several ways. For instance, for a set-valued map  $D : \mathbb{R}^m \rightarrow 2^{\mathbb{R}^m}$  with  $D(y)$  a pointed convex cone for all  $y \in \mathbb{R}^m$  one can define for  $y, z \in \mathbb{R}^m$

$$y \leq_D z \Leftrightarrow z - y \in D(y) \Leftrightarrow z \in \{y\} + D(y).$$

The binary relation  $\leq_D$  is in general reflexive, but not transitive, not antisymmetric, and not compatible with addition or nonnegative scalar multiplication, see Lemma 1.10 in the book [Eichfelder \(2014b\)](#) on this topic. Such non-transitive binary relations are important for modeling

applications when the set of less preferred directions in the image space, so far  $\mathbb{R}_+^m$  or  $K$ , depends on  $y$ , i.e., it is  $D(y)$ . For instance, in intensity modulated radiotherapy a slight worsening of the dose value delivered to one organ, in the case one is still much below some bound, can be accepted for a significant improvement in another organ. But there are regions of values for the dose where a slight worsening of the value would no longer be acceptable as it would imply a strong impact on that organ, cf. [Eichfelder \(2014b\)](#). For modeling such a situation, bounded set  $D(y)$ , instead of cones, are required. In the literature, a huge progress on theoretical results for variable ordering structures can be seen. First numerical algorithms, as in [Hirsch et al. \(2011\)](#) by Hirsch, Shukla and Schmeck and in [Eichfelder \(2014a\)](#) by Eichfelder, have already been proposed.

Next to extensions to more general ordering concepts, multiobjective optimization was extended already from its first days from  $\mathbb{R}^m$  as criterion space to general linear spaces  $Y$  as image space, see for instance the book [Jahn \(2011\)](#). Then another issue can be that many theoretical results use the topological interior of the ordering cone. For instance, above we define the weakly efficient elements by using the interior of  $\mathbb{R}_+^m$ . But in many topological linear spaces ordering cones are of interest which have an empty interior, then called non-solid. Theoretical results for such problems can be found, for instance, in the paper [Durea et al., 2010](#) by Durea, Dutta and Tammer.

In addition, in the last twenty years the extension to set-valued problems with objective map  $F : M \rightarrow 2^Y$  for some feasible set  $M$  and a linear space  $Y$  have become more important, i.e., the extension to optimization problems

$$\min_{x \in M} F(x). \quad (\text{SOP})$$

[Section 6](#) and [Section 7](#) give examples where such set-valued optimization problems, called set optimization problems, play an important role.

A first optimality notion for (SOP) is obtained by directly transferring the concepts from multiobjective optimization. It is called the vector approach. We demonstrate this here for  $Y = \mathbb{R}^m$  and ordering cone  $\mathbb{R}_+^m$ . For that, define a set  $S$  by

$$S = \bigcup_{x \in M} F(x)$$

and determine a nondominated element  $\bar{y}$  of  $S$ , i.e.,  $\bar{y} \in S$  with  $(\{\bar{y}\} - \mathbb{R}_+^m) \cap S = \{\bar{y}\}$ . Then any pair  $(\bar{x}, \bar{y})$  with  $\bar{y} \in F(\bar{x})$ ,  $\bar{x} \in M$ , is called a minimizer for (SOP), cf. ([Jahn, 2011, Definition 14.2](#)). We illustrate this concept with an example.

**Example 8.1.** Let  $M = \{1, 2, 3\}$  and  $F : M \rightarrow 2^{\mathbb{R}^2}$  be defined by

$$\begin{aligned} F(1) &= \text{conv}\{(1, 1)^\top, (0.5, 3)^\top\}, \\ F(2) &= \text{conv}\{(1, 1)^\top, (3, 2)^\top\}, \\ F(3) &= \text{conv}\{(1, 1)^\top, (3, 0.5)^\top\}. \end{aligned}$$

Then for  $\bar{y} := (1, 1)^\top$  all pairs  $(x, \bar{y})$  with  $x \in M$ , and all pairs  $(3, y)$  with  $y \in F(3)$ , are minimizers for (SOP).

However, as discussed in [Section 6](#), there are concepts as the lower-less order relation which allow to compare the sets  $F(x)$  for  $x \in M$  as a whole, and not represented by one element  $\bar{y}$ . Then one can define that  $\bar{x} \in M$  is a minimal solution for (SOP) w.r.t. the lower-less order relation if it holds

$$x \in M, F(x) \leq_\ell F(\bar{x}) \Rightarrow F(\bar{x}) \leq_\ell F(x),$$

cf. ([Jahn, 2011, Definition 14.5](#)). The above approach is called the set approach. We illustrate it on the set optimization problem from [Example 8.1](#).

**Example 8.2.** We consider again the set  $M$  and the set-valued map  $F$  from [Example 8.1](#). The points  $x = 1$  and  $x = 3$  are minimal solutions for (SOP) w.r.t. the lower-less order relation, but not  $x = 2$ , as  $F(1) \leq_\ell F(2)$ , but  $F(2) \not\leq_\ell F(1)$ .

Set-optimization with the set approach has been in the focus of the research in set optimization for about twenty years as it is considered to be more practically relevant than the vector approach. Also for this set approach for set optimization, variable ordering structures have been examined, see for instance Köbis, 2017 and Eichfelder and Pilecka, 2016. Moreover, scalarizations for set optimization problems have been proposed, even if the ordering cones have an empty interior, see, for instance, Gutiérrez, Jiménez, Miglierina and Molho, Gutiérrez et al. (2015). In addition, relations between the optimal solution sets of (finite-dimensional) multiobjective optimization problems and set-optimization problems w.r.t. the set approach have been explored by Eichfelder and Rocktäschel (2021) based on support functionals for convex-valued maps, and for nonconvex problems by Eichfelder, Quintana and Rocktäschel in Eichfelder et al., 2021b.

To sum up this survey, a huge progress can be seen in multiobjective optimization, and many new approaches, fields, and applications have been explored in the last twenty years. However, there are still many open research questions, and significant improvements in solution procedures for many problem classes are required. We suggest that it might be a promising approach to develop algorithms which are not based on a scalarization. Moreover, it is important that the developed algorithms are provided to the public such that multiobjective optimization techniques are used more widely for solving application problems in practice.

### Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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