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Coinductive Predicates and Final Sequences in a Fibration

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Abstract

Coinductive predicates express persisting "safety" specifications of transition systems. Previous observations by Hermida and Jacobs identify coinductive predicates as suitable final coalgebras in a *fibration*—a categorical abstraction of predicate logic. In this paper we follow the spirit of a seminal work by Worrell and study final sequences in a fibration. Our main contribution is to identify some categorical "size restriction" axioms that guarantee stabilization of final sequences after ω steps. In its course we develop a relevant categorical infrastructure that relates fibrations and locally presentable categories, a combination that does not seem to be studied a lot. The genericity of our fibrational framework can be exploited for: binary relations (i.e. the logic of "binary predicates") for which a coinductive predicate is bisimilarity; constructive logics (where interests are growing in coinductive predicates); and logics for name-passing processes.

Keywords: coalgebra; (co)recursive predicate; modal logic; fibration; locally presentable category

1 Introduction

Coinductive predicates postulate properties of state-based dynamic systems that persist after a succession of transitions. In computer science, *safety properties* of nonterminating, reactive systems are examples of paramount importance. This has led to an extensive study of specification languages in the form of fixed point logics and model-checking algorithms.

In this paper we follow [28,29] (further extended in [5,20]; see also [34, Chap. 6]) and take a categorical view on coinductive predicates. Here *coalgebras* represent transition systems; a *fibration* is a "predicate logic"; and a coinductive predicate is identified as a suitable coalgebra in a fibration. Our contribution is the study of *final sequences*—an iterative construction of final coalgebras that is studied notably in [2,46]—in such a fibrational setting.

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Coalgebras have been successfully used as a categorical abstraction of transition systems (see e.g. [34,43]): by varying base categories and functors, coalgebras bring general results that work for a variety of systems at once. Fixed point logics (or modal logics in general), too, have been actively studied coalgebraically: coalgebraic modal logic is a prolific research field (see [12]); their base category is typically **Sets** but works like [36] go beyond and use presheaf categories for processes in name-passing calculi; and literature including [11, 13, 45] studies coalgebraic fixed point logics.

Unlike most of these works, we follow [28, 29] and parametrize the underlying "predicate logic" too with the categorical notion of *fibration*. The conventional setting of classical logic is represented by the fibration $\begin{smallmatrix}\mathbf{Pred}\\ \downarrow\\ \mathbf{Sets}\end{smallmatrix}$ (see Appendix A.3 of the extended version [24] for an introduction to fibrations).

However there are various other "logics" modeled as fibrations, and hence the fibrational language provides a uniform treatment of these different settings. An example is binary relations (instead of unary predicates) that form a fibration $\begin{smallmatrix}\mathbf{Rel}\\ \downarrow\\ \mathbf{Sets}\end{smallmatrix}$ (see Appendix A.3 in [24]). In this case coinductive predicates are bisimilarity (see the table, and Example 5.12 later).

fibration $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{smallmatrix}$	$\begin{smallmatrix}\mathbf{Pred}\\ \downarrow\\ \mathbf{Sets}\end{smallmatrix}$	$\begin{smallmatrix}\mathbf{Rel}\\ \downarrow\\ \mathbf{Sets}\end{smallmatrix}$
coalgebra	invariant	bisimulation
final coalgebra	coindutive predicate	bisimilarity

Another example is predicates in constructive logics. They are modeled by the subobject fibration of a topos. In fact, coinductive predicates in constructive logics are an emerging research topic: coinduction is supported in the theorem prover Coq (based on the constructive *calculus of constructions*), see e.g. [6]; and, working in Coq, some interesting differences between classically equivalent (co)inductive predicates have been studied e.g. in [41].

Yet another example is modal logics for processes in various name-passing calculi. They are best modeled by the subobject fibration of a suitable (pre)sheaf category like $\mathbf{Sets}^{\mathbf{I}}$ and $\mathbf{Sets}^{\mathbf{F}}$.

1.1 Coinductive Predicates and Their Construction, Conventionally

In order to illustrate our technical contributions (§3) we first present a special case, with classical logic and Kripke models. We first introduce syntax.

Definition 1.1 (Rudimentary logic $\mathsf{R}\nu$) This fragment of the μ-calculus allows only one greatest fixed-point operator at the outermost position.

$$\mathsf{R}\nu_u \ni \alpha ::= a \mid \overline{a} \mid \Box u \mid \Diamond u \mid \alpha \wedge \alpha \mid \alpha \vee \alpha \ ; \qquad \mathsf{R}\nu \ni \beta ::= \nu u.\, \alpha \ . \qquad (1)$$

Here a belongs to the set AP of *atomic propositions*; $\overline{a}$ stands for the negation of a; and u is the only fixed-point variable (with possibly multiple occurrences).

An $\mathsf{R}\nu$-formula can be thought of as a recursive definition of a coinductive predicate. Later we will model such a "definition" categorically as a predicate lifting.

A specification expressible in Rν is (may-) deadlock freedom ("there is an infinite path"). It is expressed by $\nu u. \Diamond u$ and is our recurring example.

An Rν-formula is interpreted in Kripke models. Let $c = (X, \to, V)$ be a Kripke model, where X is a state space, $\to \subseteq X \times X$ is a transition relation and $V : X \to \mathcal{P}(\mathsf{AP})$ is a valuation. The conventional interpretation $[\nu u.\alpha]_c$ of Rν-formulas in the Kripke model c is given as follows (see e.g. [9]). Firstly, we interpret $\alpha \in \mathsf{R}\nu_u$ as a function $[\alpha]_c : \mathcal{P}X \to \mathcal{P}X$. Concretely:

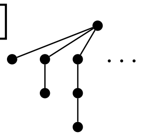
$$\begin{array}{ll}
[a]_c(P) = \{x \mid a \in V(x)\} & [\overline{a}]_c(P) = \{x \mid a \notin V(x)\} \\
[\Box u]_c(P) = \{x \mid \forall y \in X. (x \to y \text{ implies } y \in P)\} & [\Diamond u]_c(P) = \{x \mid \exists y \in X. (x \to y \text{ and } y \in P)\} \\
[\alpha \wedge \alpha']_c(P) = [\alpha]_P \cap [\alpha']_P & [\alpha \vee \alpha']_c(P) = [\alpha]_P \cup [\alpha']_P
\end{array}$$

This function $[\alpha]_c$ is easily seen to be monotone, since u occurs only positively in α. Finally we define $[\nu u.\alpha]_c \subseteq X$ to be the greatest fixed point of the monotone function $[\alpha]_c : \mathcal{P}X \to \mathcal{P}X$.

The Knaster-Tarski theorem guarantees the existence of such a greatest fixed point $[\nu u.\alpha]_c$ in a complete lattice $\mathcal{P}X$. However its proof is highly nonconstructive. In contrast, a well-known construction [14] by Cousot and Cousot computes $[\nu u.\alpha]_c$ as the limit of the following descending chain (see also [9]). Here $\top$ denotes the subset $X \subseteq X$.

$$\top \;\geq\; [\alpha]_c \top \;\geq\; [\alpha]_c^2 \top \;\geq\; \cdots \tag{2}$$

$\boxed{c_1}$



An issue now is the length of the chain. If $[\alpha]_c$ preserves limits $\bigwedge$ (which is the case with $\alpha \equiv \Box u$), clearly ω steps are enough and yields $\bigwedge_{i\in\omega}([\alpha]_c^i \top)$ as the greatest fixed point. This is not the case with $\alpha \equiv \Diamond u$. Indeed, for the Kripke model c_1 on the right $[\nu u. \Diamond u]_{c_1} \neq \bigwedge_{i\in\omega}([\Diamond u]_{c_1}^i \top)$: there is no infinite path from the root; but it satisfies $[\Diamond u]_{c_1}^i \top$ ('there is a path of length $\geq i$') for each i.

Yet the chain (2) eventually stabilizes, bounded by the size of the poset $\mathcal{P}X$. Therefore the calculation of $[\nu u.\alpha]_c$ is, in general, via *transfinite* induction. This is what we call a *state space bound* for (2).

Besides a state space bound, another (possibly better and seemingly less known) bound can be obtained from a *behavioral view*. One realizes that not only the size of the state space X but also the *branching degree* can be used to bound the length of the chain (2). For example, a result similar to [26, Thm. 2.1] states that the chain stabilizes after ω steps if the Kripke model c is *finitely branching*. This holds however large the state space X is; and also for any Rν-formula $\nu u.\alpha$. Notice that the model c_1 (depicted above) is not finitely branching.

1.2 Final Sequences in a Fibration

This paper is about putting the observations in §1.1 in general categorical terms. Our starting observation is that the chain (2) resembles a *final sequence*, a classic construction of a final coalgebra.

In the theory of coalgebra a *final F-coalgebra* is of prominent importance since it is a fully abstract domain with respect to the *F-behavioral equivalence*. Therefore a natural question is if a final F-coalgebra exists; the well-known Lambek lemma prohibits e.g. a final $\mathcal{P}$-coalgebra. What matters is the *size* of F: when it is suitably bounded, it is known that a final coalgebra can be constructed via the following *final F-sequence*.

$$1 \xleftarrow{\;!\;} F1 \xleftarrow{\;F!\;} \cdots \xleftarrow{\;F^{i-1}!\;} F^i 1 \xleftarrow{\;F^i!\;} \cdots \tag{3}$$

Here 1 is a final object in $\mathbb{C}$, and ! is the unique arrow. In particular, if F is *finitary*, a final coalgebra arises as a suitable quotient of the ω-limit of the final sequence (3). This construction in **Sets** is worked out in [46]; it is further extended to locally presentable categories (those are categories suited for speaking of "size") with additional assumptions in [2].

Turning back to coinductive predicates, indeed, the fibrational view [28,29] identifies coinductive predicates as final coalgebras in a fibration. This leads us to scrutinize final sequences in a fibration. Our main result (Thm. 3.7) is a categorical generalization of the behavioral ω-bound (§1.1)—more precisely we axiomatize categorical "size restrictions" for that bound to hold.

The conditions are formulated in the language of locally presentable categories (see e.g. [4]; also Appendix A.2 of [24]); and the combination of fibrations and locally presentable categories does not seem to have been studied a lot (an exception is [39, §5.3]). We therefore develop a relevant categorical infrastructure (§5.1). Our results there include a sufficient condition for the total category $\mathrm{Sub}(\mathbb{C})$ of a subobject fibration to be locally (finitely) presentable, and the same for a family fibration $\mathrm{Fam}(\Omega)$ too. Via these results, in §5.2 we list some concrete examples of fibrations to which our results in §3 on the behavioral bounds apply. They include: $\begin{smallmatrix}\mathbf{Pred}\\ \downarrow\\ \mathbf{Sets}\end{smallmatrix}$ (classical logic); $\begin{smallmatrix}\mathbf{Rel}\\ \downarrow\\ \mathbf{Sets}\end{smallmatrix}$ (for bisimulation and bisimilarity); $\begin{smallmatrix}\mathrm{Sub}(\mathbb{C})\\ \downarrow\\ \mathbb{C}\end{smallmatrix}$ for $\mathbb{C}$ that is locally finitely presentable and locally Cartesian closed (a topos is a special case); and $\begin{smallmatrix}\mathrm{Fam}(\Omega)\\ \downarrow\\ \mathbf{Sets}\end{smallmatrix}$ for a well-founded algebraic lattice Ω.

1.3 Summary and Future Work

To summarize, our contributions are: 1) combination of the mathematical observations in [28,29] and [34, Chap. 6] for a general formulation of coinductive predicates; 2) categorical behavioral bounds for final sequences that approximate coinductive predicates; and 3) a categorical infrastructure that relates fibrations and locally presentable categories.

While our focus is on coinductive predicates, inductive ones are just as important in system verification; so are their combinations. Such mixture of induction and coinduction is studied fibrationally in [27], but over mixed inductive and coinductive data types, and not over a coalgebra. We have obtained some preliminary fibrational observations in this direction.

Search for useful coinduction proof principles is an active research topic (see e.g. [8, 30]). We are interested in the questions of whether these principles are sound in a general fibrational setting, and what novel proof principles a fibrational view can lead to.

Coalgebraic modal logic is more and more often introduced based on a Stone-like duality (see e.g. [36]). Fibrational presentation of such dualities will combine the benefits of duality-based modal logics and the current results. We are also interested in the relationship to *coalgebraic infinite traces* [10, 32].

Kozen's *metric coinduction* [37] is a construction of coinductive predicates by the Banach fixed point theorem and is an alternative to the current paper's order-theoretic one. Its fibrational formulation is an interesting future topic.

Practical applications of our categorical behavioral bounds shall be pursued, too. Our results' precursor—the bounds for the final sequences in **Sets** [2,46]—have been used to bound execution of some algorithms e.g. for state minimization [3, 15, 16]. We aim at similar use. Finally, *games* are an extremely useful tool in fixed point logics (also in their coalgebraic generalization, see [11, 13, 45]; also [38]). We plan to investigate the use of games in the current (even more general) fibrational setting.

Organization of the Paper

In §2 we identify coinductive predicates as final coalgebras in a fibration, following the ideas of [28, 29, 34]. The main technical results are in §3, where we axiomatize size restrictions on fibrations and functors for a final sequence to stabilize after ω steps. These results are reorganized in §4 as a fibration of invariants. §5 is devoted to examples: first we develop a necessary categorical infrastructure then we discuss concrete examples.

The extended version [24] of this paper comes with two appendices. In Appendix A we present minimal introductions to the theories of coalgebras, locally presentable categories and fibrations—the three topics that our technical developments rely on. Most proofs are deferred to Appendix B there.

2 Coinductive Predicates as Final Coalgebras

In this section we follow the ideas in [28, 29, 34] and characterize coinductive predicates in various settings (for different behavior types, and in various underlying logics) in the language of fibration. An introduction to fibration is e.g. in [31]; see also Appendix A.3 in [24]. In this paper for simplicity we focus on poset fibrations. It should however not be hard to move to general fibrations.

Definition 2.1 (Fibration) We refer to poset fibrations (where each fiber is a poset rather than a category) simply as *fibrations*.

Definition 2.2 (Predicate lifting) Let $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{smallmatrix}$ be a fibration and F be an endofunctor on $\mathbb{C}$. A *predicate lifting* of F along p is a functor $\varphi : \mathbb{P} \to \mathbb{P}$ such that (φ, F) is an endomap of fibrations. This means: that the diagram on the right commutes; and that φ

$$\begin{array}{ccc} \mathbb{P} & \xrightarrow{\varphi} & \mathbb{P} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ \mathbb{C} & \xrightarrow{F} & \mathbb{C} \end{array} \qquad (4)$$

preserves Cartesian arrows, that is, $\varphi(f^*Q) = (Ff)^*(\varphi Q)$. See below.

$$
\begin{array}{c} \mathbb{P} \\ \downarrow p \\ \mathbb{C} \end{array}
\qquad
\begin{tikzcd} f^*Q \arrow[r, "\overline{f}Q"] & Q \\ X \arrow[r, "f"] & Y \end{tikzcd}
\qquad
\begin{tikzcd} \varphi(f^*Q) \arrow[r, "\varphi(\overline{f}Q)"] & \varphi Q \\ (Ff)^*(\varphi Q) \arrow[ur, "\overline{Ff}(\varphi Q)"'] & \\ FX \arrow[r, "Ff"] & FY \end{tikzcd}
\tag{5}
$$

In the prototype example $\begin{array}{c}\mathbf{Pred}\\ \downarrow\\ \mathbf{Sets}\end{array}$, the above definition coincides (see [34]) with the one used in coalgebraic modal logic (see e.g. [12])—presented as a (monotone) natural transformation $2^{(\_)} \overset{\varphi}{\Rightarrow} 2^{F(\_)} : \mathbf{Sets}^{\mathrm{op}} \to \mathbf{Sets}$.

We think of predicate liftings as (co)recursive definitions of coinductive predicates (see Example 2.4). On top of it, we identify coinductive predicates (and invariants) as coalgebras in a fiber.

Definition 2.3 (Invariant, coinductive predicate) Let φ be a predicate lifting of F along $\begin{array}{c}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{array}$; and $X \xrightarrow{c} FX$ be a coalgebra in $\mathbb{C}$. They together induce an endofunctor (a monotone function) on the fiber $\mathbb{P}_X$, namely $\mathbb{P}_X \xrightarrow{\varphi} \mathbb{P}_{FX} \xrightarrow{c^*} \mathbb{P}_X$, where φ restricts to $\mathbb{P}_X \to \mathbb{P}_{FX}$ because of (4).

(i) A *φ-invariant* in c is a $(c^* \circ \varphi)$-coalgebra in $\mathbb{P}_X$, that is, an object $P \in \mathbb{P}_X$ such that $P \leq c^*(\varphi P)$ in $\mathbb{P}_X$.

(ii) The *φ-coinductive predicate* in c is the final $(c^* \circ \varphi)$-coalgebra (if it exists). Its carrier shall be denoted by $[\![\nu\varphi]\!]_c$. It is therefore the largest φ-invariant in c; Lambek's lemma yields that $[\![\nu\varphi]\!]_c = (c^* \circ \varphi)([\![\nu\varphi]\!]_c)$.

Example 2.4 ($\mathsf{R}\nu$) The conventional interpretation $[\nu u.\alpha]_c$ (described in §1.1) of $\mathsf{R}\nu$-formulas is a special case of Def. 2.3. Indeed, let us work in the fibration $\begin{array}{c}\mathbf{Pred}\\ \downarrow\\ \mathbf{Sets}\end{array}$, and with the endofunctor $F_{\mathrm{K}} = \mathcal{P}(\mathsf{AP}) \times \mathcal{P}(\_)$ on $\mathbf{Sets}$. An F_{K}-coalgebra $X \xrightarrow{c} \mathcal{P}(\mathsf{AP}) \times \mathcal{P}X$ is precisely a Kripke model: c combines a valuation $X \to \mathcal{P}(\mathsf{AP})$ and the map $X \to \mathcal{P}X$ that carries a state to the set of its successors. To each formula $\alpha \in \mathsf{R}\nu_u$ we associate a predicate lifting φ_α of F_{K}. This is done inductively as follows.

$$
\begin{array}{ll}
\varphi_a(U \subseteq X) = \big(\{V \in F_{\mathrm{K}}X \mid a \in \pi_1(V)\} \subseteq F_{\mathrm{K}}X\big) & \varphi_{\overline{a}}(U \subseteq X) = \big(\{V \mid a \notin \pi_1(V)\} \subseteq F_{\mathrm{K}}X\big) \\
\varphi_{\Box u}(U \subseteq X) = \big(\{V \mid \pi_2(V) \subseteq U\} \subseteq F_{\mathrm{K}}X\big) & \varphi_{\Diamond u}(U \subseteq X) = \big(\{V \mid \exists x \in U.\, x \in \pi_2(V)\} \subseteq F_{\mathrm{K}}X\big) \\
\varphi_{\alpha\wedge\alpha'}(U \subseteq X) = \big((\varphi_\alpha U \cap \varphi_{\alpha'} U) \subseteq F_{\mathrm{K}}X\big) & \varphi_{\alpha\vee\alpha'}(U \subseteq X) = \big((\varphi_\alpha U \cup \varphi_{\alpha'} U) \subseteq F_{\mathrm{K}}X\big)
\end{array}
\tag{6}
$$

In the above, π_1 and π_2 denote the projections from $F_{\mathrm{K}}X = \mathcal{P}(\mathsf{AP}) \times \mathcal{P}X$. Then it is easily seen by induction that $[\![\nu\varphi_\alpha]\!]_c$ in Def. 2.3 coincides with the conventional interpretation $[\nu u.\alpha]_c$ described in §1.1.

In fact, the predicate liftings φ_α in (6) are the ones commonly used in coalgebraic modal logic (where they are presented as natural transformations). We point out that the same definition of φ_α—they are written in the internal language of toposes—works for the subobject fibration $\begin{array}{c}\mathrm{Sub}(\mathbb{C})\\ \downarrow\\ \mathbb{C}\end{array}$ of any topos $\mathbb{C}$. Therefore the categorical definition of coinductive predicates (Def. 2.3) allows us to interpret the language $\mathsf{R}\nu$ in constructive underlying logics. Suitable completeness of $\mathbb{C}$ ensures that a final $(c^* \circ \varphi)$-coalgebra in Def. 2.3 exists.

Proposition 2.5 *Let φ be a predicate lifting of F along $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{smallmatrix}$; $X \xrightarrow{c} FX$ be a coalgebra in $\mathbb{C}$; and $P \in \mathbb{P}_X$. We have $P \leq [\![\nu\varphi]\!]_c$ if and only if there exists a φ-invariant Q such that $P \leq Q$.* □

The proposition is trivial but potentially useful. It says that an invariant can be used as a "witness" for a coinductive predicate. This is how bisimilarity is commonly established; and it can be used e.g. in [1, §6] as an alternative to the metric coinduction principle used there. [1]

Remark 2.6 The coalgebraic modal logic literature exploits the fact that there can be many predicate liftings (in the form of natural transformations) of the same functor F. Different predicate liftings correspond to different modalities (such as $\Box$ vs. $\Diamond$ for the same functor $\mathcal{P}$). This view of predicate liftings is also the current paper's (see Example 2.4).

In contrast, in fibrational studies like [5, 20, 28, 29], use of predicate liftings has focused on the validity of the *(co)induction proof principle*. For such purposes it is necessary to choose a predicate lifting φ that is "comprehensive enough," covering all the possible F-behaviors. In fact, it is common in these studies that "the" predicate lifting, denoted by $\mathrm{Pred}(F)$, is assigned to a functor F. An exception is [33].

3 Final Sequences in a Fibration

Here we present our main technical result (Thm. 3.7). It generalizes known behavioral ω-bounds (like [26, Thm. 2.1]; see §1.1); and claims that the chain (2) for a coinductive predicate stabilizes after ω steps, assuming that the behavior type functor F and the underlying logic $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{smallmatrix}$ are "finitary" in a suitable sense (but no size restriction on φ).

3.1 Size Restrictions on a Fibration

We axiomatize finitariness conditions in the language of locally presentable categories (see Appendix A.2 in [24] for a minimal introduction). Singling out these conditions lies at the heart of our technical contribution.

Definition 3.1 (LFP category) A category $\mathbb{C}$ is *locally finitely presentable (LFP)* if it is cocomplete and it has a (small) set $\mathbb{F}$ of finitely presentable (FP) objects such that every object is a directed colimit of objects in $\mathbb{F}$.

[1] To be precise: only if we take PE in [1] as an atomic proposition (and that is essentially what is done in the proofs in [1, §6]). Our future work on nested μ's and ν's will more adequately address the situation.

Definition 3.2 (Finitely determined fibration) A (poset) fibration $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{smallmatrix}$ is *finitely determined* if it satisfies the following.

(i) $\mathbb{C}$ is LFP, with a set $\mathbb{F}$ of FP objects (as in Def. 3.1).

(ii) $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{smallmatrix}$ has fiberwise limits and colimits.

(iii) For arbitrary $X \in \mathbb{C}$, let $(X_I)_{I\in\mathbb{I}}$ be the canonical diagram for X with respect to $\mathbb{F}$ (i.e. $\mathbb{I} = (\mathbb{F} \downarrow X)$), with a colimiting cocone $(X_I \stackrel{\kappa_I}{\to} X)_{I\in\mathbb{I}}$. Then for any $P, Q \in \mathbb{P}_X$,

$$P \leq Q \quad\Longleftrightarrow\quad \kappa_I^* P \leq \kappa_I^* Q \text{ in } \mathbb{P}_{X_I} \text{ for each } I \in \mathbb{I}.$$

The intuition of Cond. iii) is that a predicate $P \in \mathbb{P}_X$ (over arbitrary $X \in \mathbb{C}$) is determined by its restrictions $(\kappa_I^* P)_{I\in\mathbb{I}}$ to FP objects X_I. One convenient sufficient condition for Cond. iii) is that the total category $\mathbb{P}$ is itself LFP, with its FP objects above the FP objects in $\mathbb{C}$ (Cor. 5.3). We note that Cond. i) guarantees, since LFP implies completeness, an (ω^{op}-)limit $F^\omega 1$ of the final F-sequence (3). However this does not mean (nor we need for later) that $F^\omega 1$ carries a final F-coalgebra (it fails for $F = \mathcal{P}_\omega$; see [46]).

Definition 3.3 (Well-founded fibration) A *well-founded fibration* is a finitely determined fibration that further satisfies:

(iv) If $X \in \mathbb{F}$ (hence FP), the fiber $\mathbb{P}_X$ is such that: the category $\mathbb{P}_X^{\mathrm{op}}$ consists solely of FP objects.

Since $\mathbb{P}_X$ is complete, this is equivalent to: there is no (ω^{op}-)chain $P_0 > P_1 > \cdots$ in $\mathbb{P}_X$ that is strictly descending.

We note that the following stronger variant of the condition

(iv') For *any* $X \in \mathbb{C}$, there is no strictly descending ω^{op}-chain in $\mathbb{P}_X$

rarely holds (it fails in $\begin{smallmatrix}\mathbf{Pred}\\ \downarrow\\ \mathbf{Sets}\end{smallmatrix}$). The original Cond. iv) holds in many examples (as we will see later in §5) thanks to the restriction that X is FP.

The following trivial fact is written down for the record.

Lemma 3.4 *A finitely determined fibration* $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{smallmatrix}$ *is well-founded if* $\mathbb{P}_X$ *is a finite category for each* $X \in \mathbb{F}$. □

3.2 Final Sequences in a Fibration

The following result from [31, Prop. 9.2.1] is crucial in our development.

Lemma 3.5 *Let* $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{smallmatrix}$ *be a fibration, with* $\mathbb{C}$ *being complete. Then* p *has fiberwise limits if and only if* $\mathbb{P}$ *is complete and* $p : \mathbb{P} \to \mathbb{C}$ *preserves limits. If this is the case,* *a limit of a small diagram $(P_I)_{I\in\mathbb{I}}$ in $\mathbb{P}$ can be given by*

$$\textstyle\bigwedge_{I\in\mathbb{I}}(\pi_I^* P_I) \qquad \text{over } \mathrm{Lim}_{I\in\mathbb{I}}\, X_I\,.$$

Here $X_I := pP_I$; $(\mathrm{Lim}_{I\in\mathbb{I}}\, X_I \xrightarrow{\pi_I} X_I)_{I\in\mathbb{I}}$ is a limiting cone in $\mathbb{C}$; and $\bigwedge_{I\in\mathbb{I}}$ denotes the limit in the fiber $\mathbb{P}_{\mathrm{Lim}_I X_I}$. □

Fig. 1 presents two sequences. Here we assume that $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{smallmatrix}$ is finitely determined (Def. 3.2) and that φ is a predicate lifting of F. In the bottom diagram (in $\mathbb{C}$), the

$$\boxed{\mathbb{P}}\quad
\begin{tikzcd}
 & & & & \varphi^{\omega}\top_1 \arrow[llll, bend right] \arrow[lll, bend right] \arrow[ld, bend right] \\
\top_1 & \varphi\top_1 \arrow[l] & \cdots \arrow[l] & \varphi^{i}\top_1 \arrow[l] & \cdots \arrow[l] \\
 & & & & \varphi^{\omega+1}\top_1 \arrow[llll, bend left] \arrow[lll, bend left] \arrow[lu, bend left] \arrow[uu, dashed, bend right, "b'"']
\end{tikzcd}$$

$$\boxed{\mathbb{C}}\quad
\begin{tikzcd}
 & & & & F^{\omega}1 \arrow[llll, bend right] \arrow[lll, bend right] \arrow[ld, bend right, "\pi_i"] \\
1 & F1 \arrow[l, "!"] & \cdots \arrow[l] & F^{i}1 \arrow[l, "F^{i-1}!"'] & \cdots \arrow[l, "F^{i}!"'] \\
 & & & & F^{\omega+1}1 \arrow[llll, bend left] \arrow[lll, bend left] \arrow[lu, bend left, "F\pi_{i-1}"] \arrow[uu, dashed, bend right, "b"']
\end{tikzcd}$$

Figure 1. Final sequences in a fibration

object $1 \in \mathbb{C}$ is a final one (it exists since LFP implies completeness); $F1 \xrightarrow{!} 1$ is the unique map; $F^{\omega+1}1 := F(F^{\omega}1)$; and b is a unique mediating arrow to the limit $F^{\omega}1$. In the top diagram (in $\mathbb{P}$), the object $\top_1$ is the final object in the fiber $\mathbb{P}_1$; by Lem. 3.5 this is precisely a final object in the total category $\mathbb{P}$. Hence this diagram is nothing but a final sequence for the functor φ in $\mathbb{P}$. A limit $\varphi^{\omega}\top_1$ of this final sequence exists, again by Lem. 3.5, and moreover it can be chosen above $F^{\omega}1$. We define $\varphi^{\omega+1}\top_1 := \varphi(\varphi^{\omega}\top_1)$.

Lemma 3.6 (Key lemma) *Let $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{smallmatrix}$ be a well-founded fibration; $F : \mathbb{C} \to \mathbb{C}$ be finitary; and φ be a predicate lifting of F. Then the final φ-sequence stabilizes after ω steps. More precisely: in Fig. 1, we have $\varphi^{\omega+1}\top_1 = b^*(\varphi^{\omega}\top_1)$.*

The object $\varphi^{\omega}\top_1$ is a "prototype" of φ-coinductive predicates in various coalgebras. This is one content of the following main theorem.

It is standard that a coalgebra $X \xrightarrow{c} FX$ in $\mathbb{C}$ induces a cone over the final F-sequence, and hence a mediating arrow $X \to F^{\omega}1$ (see below). Concretely, $c_i : X \to F^i 1$ is defined inductively by: $X \xrightarrow{c_0} 1$ is $!$; and c_{i+1} is the composite $X \xrightarrow{c} FX \xrightarrow{Fc_i} F^{i+1}1$. The induced arrow to the limit $F^{\omega}1$ is denoted by c_{ω}.

$$\begin{tikzcd}
 & & & & F^{\omega}1 \arrow[llll, bend right] \arrow[lll, bend right] \arrow[ld, bend right, "\pi_i"'] \\
1 & F1 \arrow[l, "!"] & \cdots \arrow[l] & F^{i}1 \arrow[l] & \cdots \arrow[l] \\
 & & & & X \arrow[llll, bend left] \arrow[lll, bend left] \arrow[lu, bend left, "c_i"] \arrow[uu, dashed, bend right, "c_\omega"']
\end{tikzcd} \qquad (7)$$

Theorem 3.7 (Main result) *Let $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{smallmatrix}$ be a well-founded fibration; $F : \mathbb{C} \to \mathbb{C}$ be a finitary functor; φ be a predicate lifting of F along p; and $X \xrightarrow{c} FX$ be a coalgebra* *in* $\mathbb{C}$.

(i) *The* φ*-coinductive predicate* $[\![\nu\varphi]\!]_c$ *in* c *(Def. 2.3) exists. It is obtained by the following reindexing of* $\varphi^\omega\top_1$*, where* c_ω *is the mediating map in (7).*

$$[\![\nu\varphi]\!]_c \;=\; c_\omega^*(\varphi^\omega\top_1) \tag{8}$$

(ii) *Moreover, the predicate* $[\![\nu\varphi]\!]_c$ *is the limit of the following* ω^{op}*-chain in the fiber* $\mathbb{P}_X$

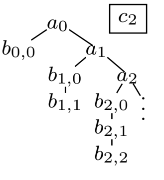
$$\top_X \;\geq\; (c^*\circ\varphi)(\top_X) \;\geq\; (c^*\circ\varphi)^2(\top_X) \;\geq\; \cdots\;, \tag{9}$$

that stabilizes after ω *steps. That is,* $[\![\nu\varphi]\!]_c = \bigwedge_{i\in\omega}(c^*\circ\varphi)^i(\top_X)$. □

Example 3.8 **($\mathsf{R}\nu$)** We continue Example 2.4 and derive from Thm. 3.7 the behavioral bound result described in §1.1: the chain (2) stabilizes after ω steps, for each $\alpha\in\mathsf{R}\nu_u$ and each *finitely branching* Kripke model c.

Indeed, the latter is the same thing as a coalgebra $X\overset{c}{\to}F_{\mathrm{fbK}}X$, where $F_{\mathrm{fbK}} = \mathcal{P}(\mathsf{AP})\times\mathcal{P}_\omega(\_)$. Compared to F_{K} in Example 2.4 the powerset functor is restricted from $\mathcal{P}$ to $\mathcal{P}_\omega$; this makes F_{fbK} a finitary functor. Still the same definition of φ_α defines a predicate lifting of F_{fbK}. Thm. 3.7.ii can then be applied to the fibration $\begin{smallmatrix}\mathbf{Pred}\\ \downarrow\\ \mathbf{Sets}\end{smallmatrix}$ (easily seen to be well-founded, Example 5.11), the finitary functor F_{fbK} and the predicate lifting φ_α for each α. It is not hard to see that the function $[\alpha]_c : \mathcal{P}X\to\mathcal{P}X$ in §1.1 coincides with $c^*\circ\varphi_\alpha : \mathbf{Pred}_X\to\mathbf{Pred}_X$ (note that $\mathbf{Pred}_X\cong 2^X\cong\mathcal{P}X$); thus the chain (2) coincides with (9) that stabilizes after ω steps by Thm. 3.7.

Remark 3.9 The ω-bound of the length of the chain (9) is sharp. A (counter)example is given in the setting of Example 3.8, by the predicate lifting $\varphi_{\Diamond u}$ and the coalgebra (i.e. Kripke structure) c_2 on the right. There $b_{i,i}$ has no successors. Indeed, while $[\![\nu\varphi_{\Diamond u}]\!]_{c_2}$ is $\{a_i\mid i\in\omega\}$, its i-th approximant $((c_2)_i^*\circ\varphi_{\Diamond u}^i)(\top_X)$ in (9) contains $b_{i,0}$ too.



Remark 3.10 It is notable that Thm. 3.7 imposes no size restrictions on $\varphi:\mathbb{P}\to\mathbb{P}$. Being a predicate lifting is enough.

Final F-sequences are commonly used for the construction of a final F-coalgebra. It is not always the case, however, that the limit $F^\omega 1$ is itself the carrier of a final coalgebra (even for finitary F; see [46, §5]). One obtains a final coalgebra either by: 1) quotienting $F^\omega 1$ by the behavioral equivalence (see e.g. [42]); or 2) continuing the final sequence till $\omega+\omega$ steps. The latter construction is worked out in [46] (in **Sets**) and in [2] (in LFP $\mathbb{C}$ with additional assumptions). Its relevance to the current work is yet to be investigated.

Coalgebra morphisms are compatible with coinductive predicates. This fact, like Prop. 2.5, is potentially useful in establishing coinductive predicates.

Proposition 3.11 *Let* $f : X\to Y$ *be a coalgebra morphism from* $X\overset{c}{\to}FY$ *to* $Y\overset{d}{\to}FY$*. In the setting of Lem. 3.6 and Thm. 3.7:*

(i) *If $Q \in \mathbb{P}_Y$ is a φ-invariant in d, so is $f^*Q \in \mathbb{P}_X$ in c.*

(ii) *We have $[\![\nu\varphi]\!]_c = f^*\big([\![\nu\varphi]\!]_d\big)$.* $\square$

Remark 3.12 The current paper focuses on *finitely presentable* objects, *finitary* functors, etc.—i.e. the ω-presentable setting (see [4, §1.B]). This is for the simplicity of presentation: the results, as usual (as e.g. in [36]), can be easily generalized to the λ-presentable setting for an arbitrary regular cardinal λ. In such an extended setting we obtain a behavioral λ-bound.

4 A Fibration of Invariants

We organize the above observations in a more abstract fibered setting. The technical results are mostly standard; see e.g. [28, 29] and [34, Chap.6].

We write $\mathbf{Coalg}(F)$ for the category of F-coalgebras.

Proposition 4.1 *Let φ be a predicate lifting of F along $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{smallmatrix}$. Then the fibration $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{smallmatrix}$ is lifted to a fibration $\begin{smallmatrix}\mathbf{Coalg}(\varphi)\\ \downarrow \overline{p}\\ \mathbf{Coalg}(F)\end{smallmatrix}$, with two forgetful functors forming a map of fibrations from the latter to the former.* $\square$

The next observation explains the current section's title.

Proposition 4.2 *Let $\begin{smallmatrix}\mathbf{Coalg}(\varphi)\\ \downarrow \overline{p}\\ \mathbf{Coalg}(F)\end{smallmatrix}$ be the lifted fibration in Prop. 4.1. For each coalgebra $X \xrightarrow{c} FX$, the fiber over c coincides with the poset of φ-invariants in c. That is:*
$$\begin{tikzcd} \mathbf{Coalg}(\varphi)_{X \xrightarrow{c} FX} \arrow[rr, "\cong"] \arrow[dr] & & \mathbf{Coalg}(c^* \circ \varphi) \arrow[dl] \\ & \mathbb{P}_X & \end{tikzcd}.$$
$\square$

Therefore Thm. 3.7.i) and Prop. 3.11.ii) state the fibration $\begin{smallmatrix}\mathbf{Coalg}(\varphi)\\ \downarrow \overline{p}\\ \mathbf{Coalg}(F)\end{smallmatrix}$ has fiberwise final objects. (At least part of) this statement itself is shown quite easily using the Knaster-Tarski theorem (each fiber is a complete lattice). Our contribution is its concrete construction as an ω^{op}-limit (Thm. 3.7.ii).

The following is an immediate consequence of Lem. 3.5.

Corollary 4.3 *Let φ be a predicate lifting of F along $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{smallmatrix}$; and assume that a final F-coalgebra exists. The following are equivalent.*

(i) *The coinductive predicate $[\![\nu\varphi]\!]_c$ exists for each coalgebra $c : X \to FX$. Moreover they are preserved by reindexing (along coalgebra morphisms).*

(ii) *There exists a final φ-coalgebra that is above a final F-coalgebra.* $\square$

5 Examples of Fibrations

5.1 Examples at Large

Here are several results that ensure a fibration to be finitely determined or well-founded, and hence enable us to apply Thm. 3.7. Some of them are well-known; others—especially those which relate fibrations and locally (finitely) presentable categories, including Lem. 5.4 and Cor. 5.7—seem to be new.

Lemma 5.1 [31, Prop. 5.4.7] *An (elementary) topos is a locally Cartesian closed category (LCCC).* □

The following results provide sufficient conditions for a fibration to be finitely determined (Def. 3.2). Recall that a full subcategory $\mathbb{F}$ of $\mathbb{P}$ is said to be *dense* if each object $P \in \mathbb{P}$ is a colimit of a diagram in $\mathbb{F}$.

Lemma 5.2 *Let* $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{smallmatrix}$ *be a fibration with fiberwise limits and colimits. Assume further that $\mathbb{C}$ is LFP with a set $\mathbb{F}_{\mathbb{C}}$ of FP objects (as in Def. 3.1). If the total category $\mathbb{P}$ has a dense subcategory $\mathbb{F}_{\mathbb{P}}$ such that every $R \in \mathbb{F}_{\mathbb{P}}$ is above $\mathbb{F}_{\mathbb{C}}$ (i.e. $pR \in \mathbb{F}_{\mathbb{C}}$), then p is finitely determined.* □

Corollary 5.3 *Let* $\begin{smallmatrix}\mathbb{P}\\ \downarrow p\\ \mathbb{C}\end{smallmatrix}$ *be a fibration with fiberwise limits and colimits, where $\mathbb{C}$ is LFP with a set $\mathbb{F}_{\mathbb{C}}$ of FP objects (in Def. 3.1). If the total category $\mathbb{P}$ is also LFP, with a set $\mathbb{F}_{\mathbb{P}}$ of FP objects (as in Def. 3.1) chosen so that every $R \in \mathbb{F}_{\mathbb{P}}$ is above $\mathbb{F}_{\mathbb{C}}$, then p is finitely determined.* □

The following is one of the results that are less trivial.

Lemma 5.4 *Let $\mathbb{C}$ be an LFP category with $\mathbb{F}$ being a set of FP objects (as in Def. 3.1). Assume that $\mathbb{C}$ is at the same time an LCCC. Then the total category* $\mathrm{Sub}(\mathbb{C})$ *of the subobject fibration is LFP: the set* $\mathbb{F}_{\mathrm{Sub}(\mathbb{C})} := \{\, (P \rightarrowtail X) \mid P, X \in \mathbb{F}\}$ *consists of FP objects in* $\mathrm{Sub}(\mathbb{C})$*; and every object* $(Q \rightarrowtail Y) \in \mathrm{Sub}(\mathbb{C})$ *is a colimit of a directed diagram in* $\mathbb{F}_{\mathrm{Sub}(\mathbb{C})}$. □

It follows from Lem. 5.1, 5.4, and Cor. 5.3 that the internal logic of a topos that is LFP is finitely determined.

Corollary 5.5 *Let $\mathbb{C}$ be LFP and at the same time a topos (or more generally an LCCC). Then the subobject fibration* $\begin{smallmatrix}\mathrm{Sub}(\mathbb{C})\\ \downarrow\\ \mathbb{C}\end{smallmatrix}$ *is finitely determined.* □

We turn to the family fibration $\begin{smallmatrix}\mathrm{Fam}(\Omega)\\ \downarrow\\ \mathbf{Sets}\end{smallmatrix}$ over a poset Ω (see Appendix A.3 in [24]).

Lemma 5.6 *Let Ω be an* algebraic lattice, *i.e. a complete lattice in which each element is a join of compact elements. (Equivalently, Ω is LFP when considered as* *a category.) Then the set*

$$\mathbb{F}_{\mathrm{Fam}(\Omega)} := \{\, f : X \to \Omega \mid X \textit{ is finite; for each } x \in X,\ f(x) \textit{ is compact in } \Omega \,\} \tag{10}$$

consists of finitely generated objects and is dense in $\mathrm{Fam}(\Omega)$*. Therefore by Lem. 5.2,* $\begin{smallmatrix}\mathrm{Fam}(\Omega)\\ \downarrow\\ \mathbf{Sets}\end{smallmatrix}$ *is finitely determined.* □

It is known that the existence of a dense set of FG objects (like $\mathbb{F}_{\mathrm{Fam}(\Omega)}$ in Lem. 5.6) ensures the category to be locally λ-presentable. This is however for some regular cardinal λ that is possibly bigger than ω. See [4, Thm. 1.70].

Corollary 5.7 *Let* Ω *be an algebraic lattice. Then the total category* $\mathrm{Fam}(\Omega)$ *of* $\begin{smallmatrix}\mathrm{Fam}(\Omega)\\ \downarrow\\ \mathbf{Sets}\end{smallmatrix}$ *is locally presentable.* □

We turn to the notion of well-founded fibration (Def. 3.3; see also Lem. 3.4).

Example 5.8 (Presheaf categories) Let $\mathbb{A}$ be small. The presheaf category $\mathbf{Sets}^{\mathbb{A}}$ is LFP: the set $\mathbb{F}$ of finite colimits of representable presheaves $\mathbf{y}A$, where $\mathbf{y}A = \mathbb{A}(A, \_\,)$, satisfies the conditions of Def. 3.1.

The coming results are less trivial, too.

Lemma 5.9 *Let* $\mathbb{A}$ *be small. For any* $X \in \mathbb{A}$, $\mathrm{Sub}(\mathbf{y}X)$ *is finite if and only if the subset* $\{\mathrm{Im}(\mathbf{y}A \xrightarrow{\mathbf{y}f} \mathbf{y}X) \mid A \in \mathbb{A}, f\colon X \to A\} \subseteq \mathrm{Sub}(\mathbf{y}X)$ *is finite.*

As a special case, if every arrow f *with domain* $X \in \mathbb{A}$ *factors* $f = m \circ e$ *as a split mono* m *followed by an epi* e*, then* $\mathrm{Sub}(\mathbf{y}X)$ *is finite if and only if* $\mathrm{Quot}(X)$ *is finite. Here* $\mathrm{Quot}(X)$ *denotes the set of quotient objects of* X. □

Corollary 5.10 *If one of the conditions in Lem. 5.9 holds, the fibration* $\begin{smallmatrix}\mathrm{Sub}(\mathbf{Sets}^{\mathbb{A}})\\ \downarrow\\ \mathbf{Sets}^{\mathbb{A}}\end{smallmatrix}$ *is well-founded.* □

5.2 Concrete Examples

Example 5.11 (Pred) The fibration $\begin{smallmatrix}\mathbf{Pred}\\ \downarrow\\ \mathbf{Sets}\end{smallmatrix}$ for the conventional setting of classical logic is easily seen to be well-founded. In particular, $\mathbf{Pred}_X \cong \mathcal{P}X$ is finite if X is FP (i.e. finite). Therefore to any finitary F and any predicate lifting φ, the results in §3 apply.

The (interpretations of the) formulas in $\mathsf{R}\nu$ (see Example 3.8) are examples of coinductive predicates in $\begin{smallmatrix}\mathbf{Pred}\\ \downarrow\\ \mathbf{Sets}\end{smallmatrix}$. Besides them, the study of coalgebraic modal logic has identified many predicate liftings for many functors F (probabilistic systems, neighborhood frames, strategy frames, weighted systems, etc.; see e.g. [12] and the references therein). These “modalities” all define coinductive predicates, to which the results in §3 may apply.

Example 5.12 (Rel) The fibration $\begin{array}{c}\mathbf{Rel}\\ \downarrow\\ \mathbf{Sets}\end{array}$ can be introduced from $\begin{array}{c}\mathbf{Pred}\\ \downarrow\\ \mathbf{Sets}\end{array}$ via change-of-base; concretely, an object of **Rel** is a pair (X,R) of a set X and a relation $R \subseteq X \times X$; an arrow $f : (X,R) \to (Y,S)$ is a function $f : X \to Y$ such that xRx' implies $f(x)Sf(x')$. See [31, p. 14].

This fibration is also easily seen to be well-founded; therefore to any finitary F the results in §3 apply. A predicate lifting φ along $\begin{array}{c}\mathbf{Rel}\\ \downarrow\\ \mathbf{Sets}\end{array}$ is more commonly called a *relation lifting* [29]; by choosing a suitable φ (a "sufficiently comprehensive" one) like in [29], a φ-invariant is precisely a bisimulation relation, and the φ-coinductive predicate is bisimilarity. We expect that the ω-behavioral bound in Thm. 3.7 can be used to bound execution of bisimilarity checking algorithms by partition refinement (for many different functors F).

In the following example, one can think of Ω as a Heyting algebra, and then the underlying logic becomes constructive.

Example 5.13 ($\mathrm{Fam}(\Omega)$) Let Ω be an algebraic lattice that has no strictly descending (ω^{op}-)chains. Then the family fibration $\begin{array}{c}\mathrm{Fam}(\Omega)\\ \downarrow\\ \mathbf{Sets}\end{array}$ is well-founded (see Lem. 5.6). Therefore to any finitary F the results in §3 apply. It is not hard to interpret the language $\mathsf{R}\nu$ in this setting, by defining predicate liftings similar to (6). This gives examples of coinductive predicates in $\begin{array}{c}\mathrm{Fam}(\Omega)\\ \downarrow\\ \mathbf{Sets}\end{array}$.

Presheaf Examples

Let $\mathbf{F}$ be the category of natural numbers as finite sets (i.e. $n = \{0,1,\ldots,n-1\}$) and all functions between them; $\mathbf{F}_+$ be its full subcategory of nonzero natural numbers; and $\mathbf{I}$ be the category of natural numbers and injective functions. Coalgebras in the presheaf categories $\mathbf{Sets}^{\mathbf{F}}$, $\mathbf{Sets}^{\mathbf{F}_+}$ and $\mathbf{Sets}^{\mathbf{I}}$ are commonly used for modeling processes in various name-passing calculi. For the π-calculus $\mathbf{Sets}^{\mathbf{I}}$ has been found appropriate (see e.g. [17,18]); while for the fusion calculus we do need non-injective functions in $\mathbf{F}$ or $\mathbf{F}_+$ (see [40,44]).

Inspired by [36], we are interested in coinductive predicates for such processes. They are naturally modeled in the subobject fibration of a presheaf category. Here we find a distinction: the subobject fibrations of $\mathbf{Sets}^{\mathbf{F}}$ and $\mathbf{Sets}^{\mathbf{F}_+}$ are well-founded; but that of $\mathbf{Sets}^{\mathbf{I}}$ is not. In view of Cor. 5.5 and Example 5.8, the only condition to check is Cond. iv) in Def. 3.3.

Example 5.14 ($\mathrm{Sub}(\mathbf{Sets}^{\mathbf{F}})$, $\mathrm{Sub}(\mathbf{Sets}^{\mathbf{F}_+})$) The subobject fibration $\begin{array}{c}\mathrm{Sub}(\mathbf{Sets}^{\mathbf{F}_+})\\ \downarrow\\ \mathbf{Sets}^{\mathbf{F}_+}\end{array}$ is well-founded: this is shown by Cor. 5.10. An important fact here is that in **Sets** a mono with a nonempty domain splits.

The subobject fibration $\begin{array}{c}\mathrm{Sub}(\mathbf{Sets}^{\mathbf{F}})\\ \downarrow\\ \mathbf{Sets}^{\mathbf{F}}\end{array}$ is well-founded, too. To show that $\mathrm{Sub}(\mathbf{y}0)$ is finite, we appeal to the first half of Lem. 5.9: we observe that the set $\{\mathrm{Im}\,\mathbf{y}f \mid n \in \mathbf{F}, f\colon 0 \to n\}$ is equal to the two-element set $\big\{\mathrm{Im}(\mathbf{y}(0 \xrightarrow{\mathrm{id}_0} 0)), \mathrm{Im}(\mathbf{y}(0 \xrightarrow{!} 1))\big\}$ since $0 \xrightarrow{!} n$ and $0 \xrightarrow{!} m$ factor through each other, for each $n, m \geq 1$.

We turn to functors F and φ. In modeling processes of name-passing calculi as coalgebras in these categories, one typically uses endofunctors F that are constructed from the following building blocks. Let $\mathbf{N} \in \{\mathbf{F}, \mathbf{F}_+, \mathbf{I}\}$.

- Constant functors, binary sum $+$, binary product $\times$, and exponentials $(\_)^X$. These are much like for polynomial functors on **Sets**. An important example of the first is the *name* presheaf $\mathcal{N} = \mathrm{Hom}(1, \_) \in \mathbf{Sets}^{\mathbf{N}}$.
- The *abstraction* functor $\delta : \mathbf{Sets}^{\mathbf{N}} \to \mathbf{Sets}^{\mathbf{N}}$ given by $\delta X = X(\_ + 1)$.
- The free semilattice functor $\mathcal{P}_{\mathrm{f}}$ for finite branching. This captures Kuratowski finiteness and suitable in $\mathbf{Sets}^{\mathbf{I}}$. See e.g. [17, 44].
- In $\mathbf{Sets}^{\mathbf{F}}$ and $\mathbf{Sets}^{\mathbf{F}_+}$, another choice of a "finite powerset functor" $\widetilde{K}$ is more appropriate. See [40]; also [44, p. 4].

All such functors are known to be finitary (see e.g. [40]).

Coinductive predicates in this setting can be introduced much like $\mathsf{R}\nu$ in Example 2.4 (note that $\mathbf{Sets}^{\mathbf{N}}$ is a topos), for properties like deadlock freedom. Such a language can be extended further through the modalities proposed in [36]: they correspond to constructions specific to presheaves and include the modality $\langle \overline{a}(b) \rangle$ for a binding 'input' operation. More examples will be worked out in our future paper.

Example 5.15 **(**$\mathrm{Sub}(\mathbf{Sets}^{\omega}), \mathrm{Sub}(\mathbf{Sets}^{\mathbf{I}})$**)** Consider the presheaf category $\mathbf{Sets}^{\omega}$ over the ordinal ω as a poset. The fibration $\begin{smallmatrix}\mathrm{Sub}(\mathbf{Sets}^{\omega})\\ \downarrow \\ \mathbf{Sets}^{\omega}\end{smallmatrix}$ is finitely determined but not well-founded. It fails to satisfy Cond. iv) in Def. 3.3: let $P_n : \omega \to \mathbf{Sets}$ be the family of presheaves defined by

$$P_n(m) := \quad \big(\, 0 \text{ if } m < n; \quad 1 \text{ if } n \leq m \,\big)$$

for each $n \in \omega$. Then $P_0 > P_1 > \cdots$ is a strictly descending chain in $\mathrm{Sub}(\mathbf{y}0)$. The same counterexample works for $\mathrm{Sub}(\mathbf{Sets}^{\mathbf{I}})$.

In contrast, the subobject fibration for $\mathbf{Sets}^{\omega^{\mathrm{op}}}$ is well-founded by Lem. 5.9.

Remark 5.16 Well-foundedness fails in $\mathrm{Sub}(\mathbf{Sets}^{\omega})$, $\mathrm{Sub}(\mathbf{Sets}^{\mathbf{I}})$, and in $\mathrm{Fam}(\Omega)$ for Ω that does have a strictly descending ω^{op}-chain. This means the logics modeled by the fibrations are inherently "big." Still, extensions of our results in §3 are possible from finitary (i.e. ω-presentable) to the λ-presentable setting for bigger λ, so that they apply to the (current) nonexamples.

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