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On the Computational Complexity of the Helly Number in the P_3 and Related Convexities [1]

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Abstract

Given a graph G, the P_3-*convex hull* (resp. P_3^*-*convex hull*) of a set $C \subseteq V(G)$ is obtained by iteratively adding to C vertices with at least two neighbors inside C (resp. at least two non-adjacent neighbors inside C). A P_3-*Helly-independent* (resp. P_3^*-Helly-independent) of a graph G is a set $S \subseteq V(G)$ such that the intersection of the P_3-convex hulls (P_3^*-*convex hulls*) of $S \setminus \{v\}$ ($\forall v \in S$) is empty. We denote by P_3-Helly number (resp. P_3^*-Helly number) the size of a maximum P_3-Helly-independent (resp. P_3^*-Helly-independent). The edge counterparts of these two P_3-Helly-independents follow the same restrictions applied to its edges. The VP3HI (resp. VSP3HI, EP3HI, and ESP3HI) problem aims to determine the P_3-Helly number (resp. P_3^*-Helly number, edge P_3-Helly number, and edge P_3^*-Helly number) of a graph. We establish the computational complexities of VP3HI, VSP3HI, EP3HI, and ESP3HI for a collection of graph classes, including bipartite graphs, split graphs, and join of graphs.

Keywords: algorithms and computational complexity; cliques, dominating and independent sets; P_3-Helly-independent.

1 Introduction

A natural application of the study of Helly properties on the P_3-convexities is to verify the safety of networks against a set of cascading failure errors. Figure 1 depicts a grid network. Let $S = \{a, b, c, d\}$ be a set of possible defective stations of

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Graph Class	vp3hi	ep3hi	vsp3hi	esp3hi
Bipartite	$\mathcal{NP}$-hard[Thm 3.1]	$\mathcal{NP}$-hard[Thm 3.3]	$\mathcal{NP}$-hard[Cor 3.4]	$\mathcal{NP}$-hard[Cor 3.6]
Line(BinaryStar$I$)	$\mathcal{NP}$-hard[Thm 3.11]	$\mathcal{NP}$-hard[Thm 3.11]	$\mathcal{P}$[Thm 3.12]	Open
Line(Hamiltonian)	$\mathcal{NP}$-hard[Cor 5.6]	$\mathcal{P}$[Rem 4.5]	$\mathcal{NP}$-hard[Thm 4.6]	Open
Split	$\mathcal{NP}$-hard[Thm 4.1]	$\mathcal{P}$[Cor 4.2]	$\mathcal{P}$[Thm 4.3]	$\mathcal{P}$[Thm 4.4]
Join of two graphs	$\mathcal{P}$[Cor 5.1]	$\mathcal{NP}$-hard[Thm 5.2]	$\mathcal{NP}$-hard[Thm 5.3]	$\mathcal{NP}$-hard[Thm 5.4]
Join ($clique \wedge G$)	$\mathcal{P}$[Thm 5.8]	$\mathcal{NP}$-hard[Thm 5.8]	$\mathcal{P}$[Thm 5.8]	$\mathcal{P}$[Thm 5.8]
$4K_1$-free	$\mathcal{P}$[Cor 5.5]	$\mathcal{P}$[Cor 5.5]	$\mathcal{NP}$-hard[Cor 5.5]	Open
$(q, q-4)$ fixed q	$\mathcal{P}$[[1]]	$\mathcal{P}$[[1]]	$\mathcal{P}$[[1]]	$\mathcal{P}$[[1]]

Table 1
Complexities of vp3hi, vsp3hi, ep3hi, and esp3hi.

the network and S' be the four proper subsets of S with size $|S|-1$. A cascading failure in this network affects every station that shares information with other two defective stations. Each cascading failure can be seen as a layer of the P_3-convex hull applied to a set in S'. If S is a P_3-Helly-independent set, then no station is

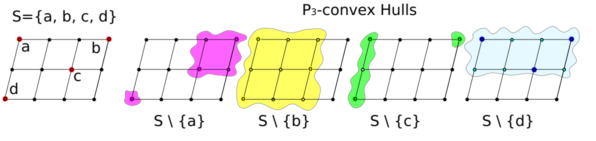

Fig. 1. A set of cascading failure errors in a grid network.

affected by all four cascading failure errors given by the sets of S'. Therefore, we may design the stations so they continue to work in a safe mode considering they are not affected by a total critical failure.

Different convexities on graphs are studied extensively [2,4,5] due to their wide number of applications, including distributed systems [14], social networks, and marketing strategies [9]. Moreover, the Helly property on graphs has been studied quite intensively in the past [3]. The problem we address considers the Helly property on the P_3-convexity of a graph. This paper is the first systematic study of the computational complexities of P_3-Helly-independent problems. We present graph classes for which the computational complexities of vp3hi, ep3hi, vsp3hi completely diverge in regard of the $\mathcal{P}$ versus $\mathcal{NP}$-hard dichotomy (See Table 1). We also establish relations between the Helly number parameters in the P_3 and related convexities with known graph parameters for an arbitrary graph G: size of a maximum independent set $\alpha(G)$, minimum dominating set $\gamma(G)$, minimum independent dominating set $\iota(G)$, maximum induced matching $\beta^*(G)$, and maximum clique $\omega(G)$.

This paper is organized as follows. Section 2 contains notations, restrictions, and properties of P_3-Helly-independent problems used in our proofs. Sections 3, 4, and 5 contain 25 results on the computational complexities of all four vp3hi, ep3hi, vsp3hi, and esp3hi problems for subclasses of bipartite graphs, split graphs, and join of graphs, respectively (See Table 1 for a detailed description). We relate the parameter $\alpha(G)$ for an arbitrary graph G with h_{P_3} for a subclass of bipartite graphs (Section 3) and a subclass of split graphs (Section 4); $\gamma(G)$ for an arbitrary graph G with h'_{P_3} for a subclass of bipartite graphs (Section 3); and $\beta^*(G)$ and $\omega(G)$ for an arbitrary graph G with h'_{P_3} and $h_{P_3^*}$ for subclasses of join of graphs (Section 5).

2 P_3-Helly-independent problems

Throughout this paper we only consider simple graphs. Given a graph G, the P_3-*convex hull* (resp. P_3^*-*convex hull*) of a set $C \subseteq V(G)$ is obtained by iteratively adding to C vertices with at least two neighbors inside C (resp. at least two non-adjacent neighbors inside C). A set of vertices $S \subseteq V(G)$ of a graph G is a P_3-*Helly-independent* (resp. P_3^*-Helly-independent) if and only if **(restriction (i))** the intersection of the P_3-convex hulls (resp. P_3^*-convex hulls) of $S \setminus \{v\}$ (for all $v \in S$) is empty, i.e., $\bigcap_{\forall v \in S}$ (P_3-convex hull of $S \setminus \{v\}) = \emptyset$ [resp. $\bigcap_{\forall v \in S}$ (P_3^*-convex hull of $S \setminus \{v\}) = \emptyset$]. For the sake of convenience, we also consider a weaker restriction: if S is a P_3-Helly-independent set (P_3^*-Helly-independent set), then **(restriction (ii))** $\forall v \in S$, the P_3-convex hull (resp. P_3^*-convex hull) of $S \setminus \{v\}$ does not contain v.

Hereinafter, we denote **restriction (i)** for P_3-Helly-independent set (resp. P_3^*-Helly-independent set) by **vertex restriction 1** (resp. **star restriction 1**) and **restriction (ii)** for P_3-Helly-independent set (resp. P_3^*-Helly-independent set) by **vertex restriction 2** (resp. **star restriction 2**). We denote by P_3-*Helly number* (P_3^*-*Helly number*) the size of a maximum P_3-*Helly-independent set* (resp. P_3^*-*Helly-independent set*). The edge counterparts of these two P_3-Helly-independent problems follow the same restrictions applied to its edges. The VP3HI (resp. VSP3HI, EP3HI, and ESP3HI) problem aims to determine the P_3-Helly number h_{P_3} (resp. P_3^*-Helly number $h_{P_3^*}$, edge P_3-Helly number h'_{P_3}, and edge P_3^*-Helly number $h'_{P_3^*}$) of a graph.

Note that the edge restrictions of EP3HI (resp. ESP3HI) for a graph G are directly related to the vertex restrictions of VP3HI (resp. ESP3HI) for $L(G)$, the line graph of G. Therefore, $h'_{P_3}(G) = h_{P_3}(L(G))$ (resp. $h'_{P_3^*}(G) = h_{P_3^*}(L(G))$). Throughout the text we only consider simple connected graphs, since h_{P_3} (resp. $h_{P_3^*}$, h'_{P_3}, and $h'_{P_3^*}$) of a disconnected graph G is the sum of the parameters on its connected components.

Remark 2.1 The computational complexity of EP3HI (resp. ESP3HI) for a graph class $\mathcal{C}$ is the same of VP3HI (resp. VSP3HI) for a graph class $\mathcal{C}'$ consisting of the line graphs of graphs of $\mathcal{C}$.

When we say that a configuration is forbidden, we are considering not only the subgraph, but also the elements of the subgraph used in the P_3-Helly-independent problems. There are some trivial **forbidden configurations** displayed in Figure 2 (the selected elements are represented by the red/bold color). In **(a.1)** v_1 disrespects **vertex restriction 1**; **(a.2)** v_2 disrespects **vertex restriction 2**; **(a.3)** v_3 disrespects **vertex restriction 1** and; **(a.4)** both v_4 and v_5 disrespect **vertex restriction 1**. In **(b.1)** v_2 disrespects **star restriction 2** and; **(b.2)** no matter if v_4 is picked, it disrespects **star restriction 1** in all situations. In **(c.1)** e_1 disrespects **edge restriction 2**; **(c.2)** e_2 disrespects **edge restriction 1**; **(c.3)** so does e_3 and; **(c.4)** e_4 disrespects **edge restriction 2**. In **(d.1)** e_1 disrespects **star edge restriction 2**; **(d.2)** e_2 disrespects **star edge restriction 1** and; **(d.3)** e_3 disrespects **star edge restriction 2**. Note that the forbidden configurations for P_3^*-Helly-independent sets are the only ones that require induced graphs, i.e., the

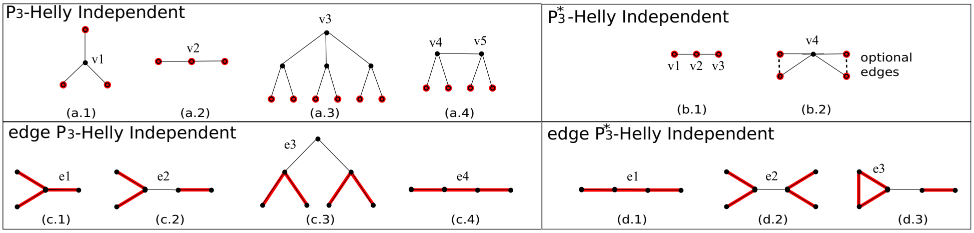

Fig. 2. Forbidden configurations for a P_3 (resp. P_3^*, edge P_3, and edge P_3^*)-Helly-independent sets.

addition of edges to the other forbidden configurations results in another forbidden configuration.

The edge P_3^*-Helly-independent sets have interesting properties: **(Property ESP3HI 1)** a set without the forbidden configurations (d.1), (d.2) and (d.3) is edge P_3^*-Helly-independent; **(Property ESP3HI 2)** An edge P_3^*-Helly-independent set M has $G[M]$ as the union of star graphs and triangles. **(Property ESP3HI 3)** if $G \neq K_3$, then there is a maximum edge P_3^*-Helly-independent set without triangles; **(Property ESP3HI 4)** $|V(G)| - \gamma(G) \geq h'_{P_3^*}(G) \geq |V(G)| - \iota(G)$ and; **(Property ESP3HI 5)** $h'_{P_3^*}$ is closed under induced subgraphs. We also have properties on bounds of the Helly number on P_3-Helly-independent problems of a graph G: **(Property Bound 1)** $\beta^*(G) \leq h'_{P_3}(G)$; **(Property Bound 2)** if G is has no maximal cliques of size two, $h'_{P_3}(G) = \max\{2, \beta^*(G)\}$; **(Property Bound 3)** $h'_{P_3}(G) \leq 14\beta^*(G)$. **(Property Bound 4)** if $h_{P_3}(G)$ (resp.. $h'_{P_3}(G)$, $h_{P_3^*}(G)$, or $h'_{P_3^*}(G)$) is a constant, then VP3HI (resp. EP3HI, VSP3HI, or ESP3HI) is in $\mathcal{P}$. **(Property Bound 5)** $h_{P_3}(G) \leq 2\alpha(G)$; **(Property Bound 6)** $h'_{P_3}(G) \leq 2\alpha(G)$ and; **(Property Bound 7)** $\alpha(G^2) \leq h_{P_3}(G)$ (where G^2 is the square graph of G).

3 Bipartite Graphs

Let G' be the graph obtained from a graph G by adding a true twin vertex for each of its degree one vertices. Note that G' has minimum degree at least two, and that $\alpha(G) = \alpha(G')$, and $\gamma(G) = \gamma(G')$. Hereinafter, consider only graphs with the addition of a true twin vertex for each degree one vertex.

Theorem 3.1 VP3HI *is $\mathcal{NP}$-hard for bipartite graphs.*

Proof. We show a polynomial time transformation from $\mathcal{NP}$-hard MIS problem [7] to VP3HI for bipartite graphs. The construction is similar to the hardness proof of MIS for square of bipartite graphs [6]. Consider a graph G with $\alpha(G) \geq 5$, a modified instance of the MIS problem. Let H be the graph with: $V(H) = V(G) \cup E(G) \cup \{u_1, u_2, u_3, u_4\}$; $\forall uv \in E(G)$, add edges $\{u\ uv, v\ uv\}$ in $E(H)$; add all edges between u_1 and $E(G)$-vertices of H; add all edges between u_2 and $E(G)$-vertices of H and; add edges $\{u_1u_3, u_2u_4\}$. There is a P_3-Helly-independent set of H with size $\alpha(G)+2$ using u_3, u_4 (which have distance three to all $V(G)$-vertices) and the $V(G)$-vertices of a maximum independent set of G, which have distance four among them in H.

We have at most four vertices of $\{u_1, u_2, u_3, u_4\} \cup E(G)$ in any maximum P_3-Helly-independent S of H or we have a forbidden configuration (a.1) or (a.2). Since

$h_{P_3}(H) \geq \alpha(G) + 2 \geq 7$, there are at least three $V(G)$-vertices in S. There are no three $E(G)$-vertices in S or one of them implies a **vertex restriction 2** contradiction. Moreover, u_1 or u_2 does not belong to S with three other $V(G)$-vertices in it. Otherwise, two of these $V(G)$-vertices in S are incident to a same $E(G)$-vertex and we have a forbidden configuration (a.1) or two of the $V(G)$-vertices in S include all $E(G)$-vertices in the P_3-convex-hull, which implies a **vertex-restriction 2** contradiction with the third $V(G)$-vertex in S, since the degrees of $V(G)$-vertices are at least two.

The remaining cases occurs when we have two or one $E(G)$-vertices in S. In the first case, these two vertices include all $E(G)$-vertices in the P_3-convex hull, implying a **vertex restriction 2** contradiction. In the second case, if u_3 and u_4 are not in S, since $h_{P_3}(H) \geq \alpha(G) + 2$, we have two $V(G)$-vertices in S incident to a same $E(G)$-vertex, these vertices include all $E(G)$-vertices in the P_3-convex hull, implying a **vertex restriction 2** contradiction. Otherwise, consider a vertex u_3 or u_4 in S. In any case, it includes u_1 or u_2 in the P_3-convex hull and, in the next iterations, all $E(G)$-vertices, implying a **vertex restriction 2** contradiction.

Thus, S must be composed by u_3, u_4, and $V(G)$-vertices. For the sake of contradiction assume $h_{P_3}(H) \geq \alpha(G) + 3$. There are: (i) two vertices u_3 and u_4 and one pair of $V(G)$-vertices within distance two in S; (ii) one vertex u_3 or u_4 and two pairs of $V(G)$-vertices within distance two in S incident to distinct $E(G)$-vertices (because of forbidden configuration (a.1)) or; (iii) three pairs of $V(G)$-vertices within distance two in S incident to distinct $E(G)$-vertices. In all cases, all $E(G)$-vertices are included in the P_3-convex hull, which implies a **vertex restriction 2** contradiction in another $V(G)$-vertex of S. Therefore, $h_{P_3}(H) = \alpha(G) + 2$. □

Remark 3.2 The subdivision $SUB(G)$ of a graph G is a bipartite graph and it has $\iota(G) = \gamma(G)$ [17]. Moreover, Ko and Shepherd [10] proved that $\beta^*(G) + \gamma(SUB(G)) = \beta^*(SUB(G)) + \gamma(G) = |V(G)|$.

Theorem 3.3 EP3HI *is $\mathcal{NP}$-hard for bipartite graphs (so is* VP3HI *for line graphs of bipartite graphs)*

Proof. Consider a modified instance graph G of the $\mathcal{NP}$-hard problem MDS [7] with $\gamma(G) \leq n - 7$ and H be the graph with: $V(H) = V(G) \cup E(G) \cup \{u_1, u_2\}$; $\forall uv \in E(G)$, add edges $\{u\ uv, v\ uv\}$ in $E(H)$; add all edges between u_1 and $E(G)$-vertices and; all edges between u_2 and $V(G)$-vertices. Let M be a maximum edge P_3-Helly-independent set of H with $h'_{P_3}(H) \geq 7$. There are no two or more edges incidents to u_1 or u_2 in M. Otherwise, these edges include all edges incident to u_1 (resp. u_2) in the edge P_3-convex hull, which includes all edges incident to endpoints of edges in M that have endpoints in $V(G)$-vertices and $E(G)$-vertices (denoted by *middle edges*) and we have an **edge restriction 2** contradiction. Moreover, there is no edge incident to u_1 or u_2, or two middle edges incident to a same $V(G)$-vertex (resp. $E(G)$-vertex) in M. Otherwise, it includes another edge of u_1 (resp. u_2) in the edge P_3-convex hull, and it includes all edges of H in the next iterations. Therefore, it also implies an **edge restriction 2** contradiction in a middle edge of M.

Thus, there are only middle edges in M, and these edges do not share a common vertex in a $V(G)$-vertex (resp. $E(G)$-vertex). Additionally, there is no edge incident to both endpoints of two selected edges. Otherwise, these edges include an edge incident to u_1 (resp. u_2) in the edge P_3-convex hull and we are dealing with the previous case. Therefore, the edges in M are an induced matching of H, and this induced matching is maximum, since if we select an edge incident to u_1 (resp. u_2) to a maximum induced matching, we have no edges incident to the $E(G)$-vertex (resp. $V(G)$-vertex) in its other endpoint and no edges incident to the other endpoints of these edges. Thus, we may exchange, for instance, the edges $\{u_1\ de\}$ or the edge $\{u_2\ a\}$ for the middle edges $\{d\ de\}$ and $\{a\ ab\}$ (for any ab in $E(G)$-vertex). By Remark 3.2, $\gamma(G)+\beta^*(SUB(G)) = |V(G)|$. Therefore, determine $\gamma(G)$ is equivalent to determine $\beta^*(SUB(G)) = \beta^*(H) = h'_{P_3}(H)$. □

Since a triangle-free graph G has all its P_3's induced, if VP3HI is $\mathcal{NP}$-hard (resp. in $\mathcal{P}$), so is VSP3HI.

Corollary 3.4 VSP3HI *is $\mathcal{NP}$-hard for bipartite graphs.*

Remark 3.5 By **Property ESP3HI 4**, $|V(G)|-\gamma(G) \geq h'_{P_3^*}(G) \geq |V(G)|-\iota(G)$. Hence, for any graph class $\mathcal{C}$ with graphs G where $\iota(G) = \gamma(G)$, the computational complexity of MDS in $\mathcal{C}$ is the same as ESP3HI.

The next corollary follows from Remarks 3.5, 3.2 and MIM (MAXIMUM INDUCED MATCHING) is $\mathcal{NP}$-hard [7], which implies MDS is $\mathcal{NP}$-hard for bipartite graphs (particularly, the subdivision graphs of the MIM's instances).

Corollary 3.6 ESP3HI *is $\mathcal{NP}$-hard for bipartite graphs (so is* VSP3HI *for line graphs of bipartite graphs).*

Construction 3.7 A bipartite graph H (denoted by *Binary Star I or II graphs*) is obtained from a bipartite graph $G = (X, Y, E)$ as follows. $V(H) = V(G) \cup \{u_1, u_2\}$ (we denote u_1 and u_2 by *universal star vertices*); **(i)** Binary Star I graphs: add all edges between u_1 and X-vertices, all edges between u_2 and Y-vertices and add a desired number of degree one vertices incident to u_1 or to u_2; **(ii)** Binary Star II graphs: add all edges between u_1 and u_2 to X-vertices and add a desired number of degree one vertices incident to u_1 or to u_2;.

The class of Binary Star I of Construction 3.7 contains the graphs H of Theorem 3.3, and the class of Binary Star II contains the graphs H of Theorem 3.1. Corollary 3.8 follows by Theorems 3.1 and 3.3 and the fact that VP3HI and VSP3HI have the same computational complexity on triangle-free graphs. Moreover, Corollary 3.9 follows by **Property Bound 2**, since if G has no degree two vertex, then $L(G)$ has no maximal clique of size two. Finally, Corollary 3.10 follows from Corollary 3.9. and the fact that $\beta^*(L(G)) = P_3\text{-}part(G)$ [13].

Corollary 3.8 VP3HI *and* VSP3HI *are $\mathcal{NP}$-hard for Binary Star II graphs and so is* EP3HI *for Binary Star I.*

Corollary 3.9 *A line graph $L(G)$ of a graph G with no degree two vertex has $h'_{P_3}(L(G)) = \beta^*(L(G))$.*

The P_3-PARTITION aims to obtain P_3-*part*, the minimum number of vertex disjoint P_3's to cover its vertices.

Corollary 3.10 *If* P_3-PARTITION *for a graph class $\mathcal{C}$ with no degree two vertex or* MIM *for $\mathcal{C}'$, composed by the line graphs of graphs of $\mathcal{C}$ is $\mathcal{NP}$-hard (resp. $\mathcal{P}$), then so is* EP3HI.

Theorem 3.11 VP3HI *and* EP3HI *are $\mathcal{NP}$-hard for line graphs of Binary Star I graphs.*

Proof. The VP3HI proof follows from Remark 2.1 and by Corollary 3.8. The P_3-PARTITION is $\mathcal{NP}$-hard for an instance bipartite graph G with $\delta(G) \geq 2$ [12]. Let G' be the Binary Star I graph obtained from G by adding the two universal star vertices with two degree one vertices incident in each. Clearly, G' has no degree two vertex and P_3-$part(G') = P_3$-$part(G) + 2$. Now, the EP3HI result follows from Corollary 3.10. □

Theorem 3.12 ESP3HI *is in $\mathcal{P}$ for Binary Star I (so is* VSP3HI *for line graphs of Binary Sar I graphs).*

Proof. Let H be a graph obtained from Construction 3.7. Consider any edge P_3^*-Helly-independent M of H. By **Property ESP3HI 1**, it is possible to pick all edges incident to the two star vertices to be part of M, since there is no forbidden configurations (d.1), (d.2), and (d.3). On the other hand, H has no universal vertex, and by **Property ESP3HI 4**, this is the best possible. Thus, $h'_{P_3^*}(H) = h_{P_3^*}(L(H)) = |V(G)|$. □

4 Split Graphs

Let $G = (I \cup K, E)$ be a split graph for which I is an independent set and K induces a clique. We require two properties: **(Property SPLIT 1)** $\alpha(G^2) \leq h_{P_3}(G) \leq \alpha(G^2)+1$ and; **(Property SPLIT 2)** a maximum P_3-Helly-independent set S with a vertex of K, or two vertices of I within distance two, has at most two vertices of I in S with degree larger than one.

Theorem 4.1 VP3HI *is $\mathcal{NP}$-hard for split graphs.*

Proof. We show a polynomial time transformation from the $\mathcal{NP}$-hard MIS problem for a graph G [7] with $\alpha(G) \geq 4$ to VP3HI for a split graph H (with $h_{P_3}(H) \geq 4$) described as follows. $V(H) = V(G) \cup V(G) \cup E(G)$ (we denote the second $V(G)$ by $V_2(G)$). The $(E(G) \cup V_2(G))$-vertices form a clique in H. There are edges between a $V(G)$-vertex v and a $E(G)$-vertex e if v is one of the endpoints of e in G and between a vertex $V(G)$-vertex v and a $V_2(G)$-vertex v_2 if they represent the same vertex in G. Note that if $uv \in E(G)$, then u and v have distance two in H. Otherwise, if $uv \notin E(G)$, then they have distance three in H.

By construction, H has minimum degree at least two. Thus, by **Property SPLIT 2**, since $h_{P_3}(H) \geq 4$, we cannot have a vertex of the clique or two vertices within distance two in the independent set in a maximum P_3-Helly-independent set

S of H. Therefore, S has only vertices of the independent set of H which have distance at least three among them, i.e., $h_{P_3}(H) = \alpha(H^2) = \alpha(G)$.

Consider a maximum independent S set of G with $\alpha(G)$ vertices. By **Property Bound 7**, it is possible to pick the same $V(G)$-vertices related to the vertices of S to belong to a P_3-Helly-independent set of H with $\alpha(H^2) = \alpha(G)$ vertices. Conversely, consider a maximum P_3-Helly-independent set S' of H with $h_{P_3}(H)$ vertices. Since all vertices in S' have distance at least three among them in H, the related $V(G)$-vertices in G have distance at least two among them in G. Therefore, $h_{P_3}(H) = \alpha(G)$. $\square$

Let G be a split graph. It is possible to verify that the edge P_3-convex hull of any set with at least two edges of G is $E(G)$. Therefore, we have $h'_{P_3}(G) \leq 2$ or we have an **edge restriction 2** contradiction. The next corollary follows by **Property Bound 4**, since $h'_{P_3}(G)$ is constant:

Corollary 4.2 EP3HI *is in $\mathcal{P}$ for split graphs (so is* VP3HI *for line graphs of split graphs).*

Theorem 4.3 VSP3HI *is in $\mathcal{P}$ for split graphs.*

Proof. Let S be a maximum P_3^*-Helly-independent set of a split graph $G = (I \cup K, E)$. For the sake of contradiction, assume $h_{P_3^*}(G) \geq \omega(G) + 2$. Consider the case we have two pairs of distinct vertices of I with distance two in S. No matter if there are edges between these two pair of vertices and their shared neighborhood in K, these vertices and their shared neighborhood imply a **star restriction 1** contradiction. Thus, we have at most one pair of vertices with distance two of I in S, the other vertices of I in S are adjacent to distinct vertices of the clique. Therefore, $h_{P_3^*}(G) \geq \omega(G) + 2$ implies at least one vertex of the clique in S.

Consider now the case in which there is no pair of vertices with distance two of I in S. There are at least two vertices of the clique in S, and there is at most one vertex of I in S adjacent to a vertex of K in S. Otherwise, if there are two vertices of I in S adjacent to a vertex K in S, they must be adjacent to all vertices of K in S to avoid a forbidden configuration (b.1), which also force a forbidden configuration (b.1). Therefore, the others vertices of I in S and K in S must not be adjacent. Now, each vertex v_i of I in S is related to a distinct vertex w_i of K, where w_i must not be in S. This implies $|S| \leq \omega(G) + 1$, a contradiction.

Finally, consider the case we have one pair of vertices with distance two of I in S. Note that there is at least one vertex of K in S, there is no vertex I in S adjacent to a vertex K in S, and there are at least $\omega(G) + 1$ of such vertices (discounting the pair of vertices of I in S). Since each vertex of I in S is related to a distinct vertex of K in S, this implies that there is a vertex of I in S adjacent to a vertex of K in S. However, to avoid a **star restriction 1** contradiction, one of the pair of vertices of distance two must be adjacent to this vertex of K, which also implies forbidden configuration (b.1).

Therefore, $\omega(G) \leq h_{P_3^*}(G) \leq \omega(G) + 1$, which can be determined in polynomial time for a split graph G. $\square$

Theorem 4.4 ESP3HI *is in $\mathcal{P}$ for split graphs (so is* VSP3HI *for line graphs of split graphs).*

Proof. Let M be a triangle-free edge P_3^*-Helly-independent set of a split graph G and v be a maximum degree vertex of G. Clearly, v is part of the clique. It is possible to pick the edges incident in v as a possible set of M. By **Properties ESP3HI 2** and **ESP3HI 3**, $G[M]$ is a union of star graphs with no edges between centers with more than two leaves. There are no two centers of stars with more than two leaves in vertices of the clique, or we have a forbidden configuration (d.2). Thus, there is only one vertex v of the clique with more than one edge of M incident to it. This implies all edges of M not incident to v are a matching between vertices of the clique and the independent set. However, for each edge of this matching, there is a missing edge between v and one of the endpoints of the edge or else there is a forbidden configuration (d.1).

Therefore, we need $|V(G)| - \Delta(G)$ (the degree of v) stars to partition the vertices of G with non-adjacent centers of the stars with at least two leaves. By **Property ESP3HI 4**, each star increases the difference between $h'_{P_3^*}(G)$ and $V(G)$ in one unity. I.e., $|M| = |V(G)| - (|V(G)| - \Delta(G)) = \Delta(G)$. By **Property ESP3HI 3**, all maximum edge P_3^*-Helly-independent set with triangles M' of G has $|M'| \leq |M|$.□

Remark 4.5 Note that Corollary 3.10 and the fact that MIM is in $\mathcal{P}$ for line graphs of Hamiltonian graphs [11] imply that EP3HI is in $\mathcal{P}$ for line graphs of Hamiltonian graphs with $\delta \geq 3$.

Theorem 4.6 ESP3HI *is $\mathcal{NP}$-hard for Hamiltonian graphs (so is* VSP3HI *for line of Hamiltonian graphs).*

Proof. This proof is based in the $\mathcal{NP}$-completeness proof of MIDS (MINIMUM INDEPENDENT DOMINATING SET) for $2P_3$-free graphs [16]. Let $J = (U, C)$ be an instance of the well-known $\mathcal{NP}$-complete problem 3-SAT [7] with $|U| = p$ and $|C| = q$. We construct a graph G from an instance J as follows: for each variable $u \in U$ we add a P_2 with labels u and $\overline{u}$ to G; for each clause $c \in C$ we add a vertex with label c to G; we add all edges between clause vertices and; for each clause $c \in C$ (e.g., $c = (x, \overline{y}, z)$ we add edges between the literals and c (e.g., the edges xc, $\overline{y}c$, zc). We assume there is no clause connect to both literals of a variable, since this clause would be always truth and it could be removed from C, and that $|U| \geq 2$ (since verify a truth assignment for clauses with only one variable is trivial). Now, duplicate (i.e., add true twins) for each clause vertex $2p$ times.

We claim that a maximum edge P_3^*-Helly-independent set M of G has size $|V(G)| - p$ and it uses the same vertices of a minimum independent dominating set ID of G as the center of its stars if and only if $\iota(G) = p$. For the sake of contradiction assume there is a triangle in M. Note that any triangle of G has at least two vertices in the clique. By Forbidden configuration (d.3), all vertices adjacent to this triangle are isolated stars in M. Moreover, we need to cover the remaining P_2 vertices with other p stars centered in these vertices. Therefore, $|M| \leq |V(G)| - q - p$, a contradiction with $h'_{P_3^*}(G) = |V(G)| - p$. Now, assume M has a center of star in a clause clique vertex. Their leaves in the P_2's vertices have not their variable counterpart incident to this center of star. Therefore, it requires at least p more center of stars, i.e., $|M| \leq |V(G)| - p - 1$, also a contradiction.

Now, we show that at least p non-adjacent centers of stars are needed to partition the vertices of G, and these vertices are related to a truth assignment of C. Since there is no star with center in the clique, all the stars have centers in the P_2 vertices and we need at least p of them to cover all P_2's vertices. If we use a vertex of the P_2 as a center of star with high degree, and the other endpoint of the P_2 as a center of star with degree one, we still need at least another $p - 1$ vertices of the P_2's to partition the vertices of G into non-adjacent stars, a contradiction with $h'_{P_3^*}(G) = |V(G)| - p$. Consider a truth assignment of C. Pick a set of vertices M related to this truth assignment as centers of stars. Each star covers the variable counterpart of the literal and, since it is a truth assignment, all clause vertices are reached by a center of star. These p centers of stars are non-adjacent. Thus, M is a maximum edge P_3^*-Helly-independent of G with size $|M| = h'_{P_3^*}(G) = |V(G)| - p$. Conversely, consider any maximum edge P_3^*-Helly-independent M of G with size $|M| = h'_{P_3^*}(G) = |V(G)| - p$. These p centers of stars need to be in P_2's vertices, one in each P_2. Moreover, since they reach all clause vertices, the assignment using the related literals of vertices in M with true value, is a truth assignment of J. □

5 Join Graphs

The join graph $G = G1 \wedge G_2$ has $h_{P_3}(G) \leq 2$. Thus, Corollary 5.1 follows by **Property Bound 4**.

Corollary 5.1 VP3HI *is polynomially solvable for $G = G_1 \wedge G_2$, the join of two graphs G_1 and G_2.*

Theorem 5.2 EP3HI *is $\mathcal{NP}$-hard for join of two graphs (so is* VP3HI *for line graphs of join of two graphs).*

Proof. Consider a graph G with $\beta^*(G) \geq 6$, an instance of the $\mathcal{NP}$-hard problem MIM problem [15]. Let $H = G \wedge \{v\}$ and M be a maximum edge P_3-Helly-independent set of H. There is no edge incident to v in M. Otherwise, since $\beta^*(G) \geq 6$, there are at least three other edges in M, and this implies an **edge restriction 2** contradiction with one of these three edges. There are also no two edges incident to a same vertex w of G in M or the edge vw is included in the next iteration of the edge P_3-convex hull, and we are dealing with the previous case. Moreover, there are no edge outside M sharing both endpoints with two edges of M or we include an edge incident to v in the edge P_3-convex hull of M and we are again dealing with the previous cases. Therefore, M is an induced matching, i.e., $h'_{P_3}(H) = \beta^*(G)$. □

Theorem 5.3 VSP3HI *is $\mathcal{NP}$-hard for join of graphs.*

Proof. Consider a graph G instance of the $\mathcal{NP}$-hard problem MAXIMUM CLIQUE [7]. Let H be the graph with $|V(G)| + 1$ copies of G with all possible edges between vertices of different copies and S^* be a maximum P_3^*-Helly-independent set of H. Consider a set S' with the vertices of a maximum clique of G for each copy of G in H. Thus, $|S'| = (|V(G)| + 1)\omega(G)$ is P_3^*-Helly-independent and $|S^*| \geq |S'|$. For the sake of contradiction, assume we pick two non adjacent vertices of a copy of G in H to be part of S^*. We may not pick any other vertex of other copy of G to be part of S^* or this vertex implies a **star restriction 2** contradiction. Thus, $|S^*| \leq |V(G)|$, a contradiction with the fact that S^* is maximum. Therefore, we may only pick vertices of a maximum clique in each copy of G to be part of S^*, and only one maximum clique for copy. I.e., $h_{P_3^*}(H) = (|V(G)| + 1)\omega(G)$. □

Theorem 5.4 ESP3HI *is $\mathcal{NP}$-hard for join of graphs (so is* VSP3HI *for line graphs of join of graphs).*

Proof. Let M be a triangle-free edge P_3^*-Helly-independent set of $G = G_1 \wedge G_2$. We claim that $h'_{P_3^*}(G) = \max = \max\{|V(G_1)| + h'_{P_3^*}(G_2), |V(G_2)| + h'_{P_3^*}(G_1)\}$. There is at least one vertex of G_1 or G_2 which is a star with at least two leaves in M. Otherwise, M is a matching between vertices of G_1 and G_2, and $h'_{P_3^*}(G) \leq \frac{|V(G_1)|+|V(G_2)|}{2}$, which is smaller than max, because of **Property ESP3HI 4** and the fact that $\iota(H) \geq \frac{|V(H)|}{2}$ for any H, $h'_{P_3^*}(G_2) \geq \frac{|V(G_2)|}{2}$ and $h'_{P_3^*}(G_1) \geq \frac{|V(G_1)|}{2}$, a contradiction.

Note that there is an M with $|M| = |V(G_1)| + h'_{P_3^*}(G_2)$ (resp. $|M| = |V(G_2)| + h'_{P_3^*}(G_1)$), since we may pick a maximum edge P_3^*-Helly-independent set of G_1 (resp. G_2) and add to one of the center of the stars of G_1 (resp. G_2) with more than one leaf all the edges incidents to vertices of G_2 (resp. G_1). Therefore, $h'_{P_3^*}(G) \geq \max$. Moreover, there are no centers of stars with more than two leaves in both G_1 or G_2, or we have a forbidden configuration (d.2). For the sake of contradiction assume $h'_{P_3^*}(G) > \max$. W.l.o.g., let $|V(G_1)| + h'_{P_3^*}(G_2) \geq |V(G_2)| + h'_{P_3^*}(G_1)$. Thus, we use $h'_{P_3^*}(G_2)$ edges incident only to vertices of G_2 and we need at least $|V(G_1)| + 1$ edges incidents to vertices of G_1. However, it implies that at least one vertex of G_1 is a star with more than two leaves, a contradiction. By **Property ESP3HI 3**, all edge P_3^*-Helly-independent set with triangles M' of G has $|M'| \leq |M|$. So, $h'_{P_3^*} = \max$.

Consider any graph class for which ESP3HI is $\mathcal{NP}$-hard (e.g., Corollary 3.6). The hardness result follows directly from the graph class obtained by the join of an instance G' of the previous $\mathcal{NP}$-hard case with itself, $G = G' \wedge G'$, for which $h'_{P_3^*}(G) = max\{|V(G')| + h'_{P_3^*}(G'), |V(G')| + h'_{P_3^*}(G')\}$. □

The join of $4K_1$-free graphs continues to be $4K_1$-free. The first result of Corollary 5.5 follows from Theorem 5.3 and the fact that MAXIMUM CLIQUE is $\mathcal{NP}$-hard for $4K_1$-free graphs [8]. The second result follows because $\alpha(G) \leq 4$ for $4K_1$-free graphs, **Property Bounds 4**, **5**, and **6**. Moreover, Corollary 5.6 follows from the proof of Theorem 5.2, which holds to a Hamiltonian graph obtained by the addition of universal vertices.

Corollary 5.5 VSP3HI *is $\mathcal{NP}$-hard for $4K_1$-free graphs whereas* VP3HI *and* EP3HI *are in $\mathcal{P}$.*

Corollary 5.6 EP3HI *is $\mathcal{NP}$-hard for Hamiltonian graphs (so is* VP3HI *for line graphs of Hamiltonian graphs).*

Construction 5.7 Construct a graph H, denoted by $(clique \wedge G)$ from a graph G as follows. Add a copy of G to H; add a clique of size $|V(G)|$ to H and; add a universal vertex u.

Theorem 5.8 VP3HI*,* VSP3HI*, and* ESP3HI *are in $\mathcal{P}$ for a clique $\wedge$ G graph H, whereas* EP3HI *is $\mathcal{NP}$-hard.*

Proof. By Corollary 5.1, VP3HI is in $\mathcal{P}$ for join of graphs. Consider a maximum edge P_3^*-Helly-independent set M of H. If we pick u as a center of a star, we have $|M| = |V(H)| - 1$, which is best possible by **Property ESP3HI 1**. Consider now a maximum P_3^*-Helly-independent set S of H. There are no two vertices of the large clique and two vertices of G in S or we have a forbidden configuration (b.2). Besides, u cannot be part of S if there are vertices of the large clique and of G in S or we have a forbidden configuration (b.1). Thus, there are at most $|V(G)| + 1$ vertices in S. Such S can be pick as the vertices of the large clique and u. Consider a maximum edge P_3-Helly-independent set M' of H. If we pick more than one edge of the clique to M', the edge P_3-convex hull is $E(H)$ and we have an **edge restriction 2**. Therefore, $h'_{P_3}(clique \wedge G) \leq h'_{P_3}(G \wedge \{u\}) + 1$. There is such M' with one edge of the clique and a maximum edge P_3-Helly-independent M' of $G \wedge \{u\}$ (which is an induced matching). The hardness result follows from Theorem 5.2 on $G \wedge \{u\}$.□

6 Final Remarks

This paper contains results on the Helly number of P_3 and related convexities. In order to present a comparative study of the computational complexities of VP3HI, EP3HI, and VSP3HI (See Table 1), we devote our efforts to subclasses of bipartite graphs, split graphs, and join of graphs. The ESP3HI problem started as a support to the proofs of VSP3HI for line graphs. Edge P_3^*-Helly-independent sets have only three types of minimal forbidden configurations, unlike the other P_3-Helly-independent sets, which have infinity types. As a consequence, we establish a relation between ESP3HI and a variation of STAR PARTITION problem.

By **Property ESP3HI 4**, $|V(G)| - \gamma(G) \geq h'_{P_3^*}(G) \geq |V(G)| - \iota(G)$. Like pieces of a puzzle starting to fit together, there exists a result of [17] that characterizes, by forbidden subgraphs, the class of graphs G for which $\iota(H) = \gamma(H)$ for any induced subgraph H of G. Surprisingly, they are exactly the same subgraphs in the forbidden configuration (d.2). Thus, for a triangle-free graph G with $\iota(G) = \gamma(G)$, there is no forbidden configuration (d.2). Moreover, $h'_{P_3^*}$ is closed by induced subgraph (**Property ESP3HI 5**). Therefore, for any induced subgraph H of a triangle-free graph G with $\iota(G) = \gamma(G)$, $h'_{P_3^*}(H) = |V(H)| - \gamma(H)$.

As future work, we aim to establish the missing computational complexities of ESP3HI in Table 1. We also have partial results on the computational complexities of all four P_3-Helly-independent problems in planar graphs, chordal graphs, interval graphs, and grid graphs.

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