Coalgebraic Representation Theory
of Fractals

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Abstract

We develop a *representation theory* in which a point of a fractal specified by *metric* means (by a variant of an *iterated function system, IFS*) is represented by a suitable equivalence class of infinite streams of symbols. The framework is categorical: symbolic representatives carry a final coalgebra; an IFS-like metric specification of a fractal is an algebra for the same functor. Relating the two there canonically arises a *representation map*, much like in America and Rutten's use of metric enrichment in denotational semantics. A distinctive feature of our framework is that the canonical representation map is bijective. In the technical development, *gluing* of shapes in a fractal specification is a major challenge. On the metric side we introduce the notion of *injective IFS* to be used in place of conventional IFSs. On the symbolic side we employ Leinster's presheaf framework that uniformly addresses necessary identification of streams—such as $.0111\ldots = .1000\ldots$ in the binary expansion of real numbers. Our leading example is the unit interval $\mathbb{I} = [0, 1]$.

Keywords: Fractal, Coalgebra, Category Theory, Denotational Semantics, Real Number Computation

1 Introduction

A *fractal* is described by Mandelbrot [19] as "a rough or fragmented geometric shape that can be split into parts, each of which is (at least approximately) a reduced-size copy of the whole." Fractals have fascinated general audiences through their aesthetic merits; they have also found engineering applications e.g. in computer graphics [21].

However, fractals (in the above broader sense of Mandelbrot's) are not restricted to queer shapes like the Koch snowflake, the Sierpiński triangle or the British coastline. Basic shapes like the (closed) unit interval $\mathbb{I} = [0, 1]$ are also examples; the unit interval is the union of two shrunk copies of itself, namely $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$.

Now it is not hard to see a connection between this fractal view of $\mathbb{I} = [0, 1]$ and the binary representation of real numbers: given $x \in \mathbb{I}$, if x lies in the first copy $[0, \frac{1}{2}]$ then we take 0 as the first digit; if $x \in [\frac{1}{2}, 1]$ then we take 1. Continuing this way we obtain an (infinite) stream over $2 = \{0, 1\}$. From this example we derive a general principle: fractal structure of a certain shape enables us to symbolically represent the shape, using infinite streams. The current paper is all about making this principle formal.

We are interested in the kind of a "fractal" that is introduced by an *iterated function system (IFS)* [13], as its unique *attractor*—a compact and non-empty fixed point. [2] An IFS is defined on a complete metric space, and the existence of an attractor hinges on the metric structure. We shall thus refer to a fractal of this kind as a *metric* fractal.

Our initial observation is that the set of streams—which shall represent such a metric fractal—carries a *final coalgebra*. A final coalgebra can be seen as a "fractal" in a broad sense, too, being a fixed point of the relevant functor.

Our goal is a canonical bijective correspondence between this *coalgebraic* fractal and a metric fractal. The former consists of streams hence is of *symbolic* nature; a stream is a standard way to represent infinitary data in computer science, like for exact computation with real numbers in effective analysis [26,7]. Hence one way of the envisaged correspondence—from a metric fractal to a coalgebraic one—carries a point in a metric space to something more familiar and easier to handle. It is like in *representation theory* in (mainstream) algebra where an element of some exotic group is mapped to a matrix which is easier to study. With this intuition, we shall call the map a *representation map*. [3] The other way of the correspondence carries a symbolic entity to a mathematical entity that is denoted by it; we call it a *denotation map* like in "denotational semantics."

$$\boxed{\text{metric fractal}} \;\begin{array}{c} \xrightarrow{\text{representation map}} \\[-2pt] \xleftarrow[\text{denotation map}]{\cong} \end{array}\; \boxed{\begin{array}{c}\text{coalgebraic fractal} \\ \text{(of symbolic nature)}\end{array}} \qquad (1)$$

Towards our goal, however, *gluing* of shapes emerges as a major technical challenge. For the coalgebraic fractal side, Leinster's presheaf framework [16,17] uniformly addresses necessary modding of streams. For the metric side, we discard IFSs as means to specify fractals because they are not capable of carrying explicit information on gluing of shapes. Instead we introduce the notion of *injective IFS (IIFS)* which is also based on presheaves.

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[2] Another common kind of "fractal" is introduced by a recurrence relation (like $f_c(z) = z^2 + c$) where the fractal is the *Mandelbrot set*, the *Julia set*, etc.

[3] The word "representation" is used in so much divergent meanings in different fields; it can also mean our "denotation map." We believe there is a good reason for our choice of terminologies.

Most proofs as well as some constructions are omitted due to the lack of space. They are found in the appendix of the extended version [10] of this article.

Related Work

For the purpose of exact computation with real numbers [26,7], it is standard to employ a "representation" of real numbers which is a surjective map $r : A \twoheadrightarrow \mathbb{I}$ with $A \subseteq \mathbb{N}^{\omega}$ (see [15]). This "representation" goes in the opposite direction compared to ours, carrying symbolic representatives to points in $\mathbb{I}$.
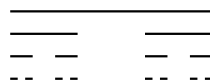
A more important difference between the "representations" in [15] and ours is that the former are not necessarily bijective. An example is in the binary expansion of real numbers where two streams $1000\ldots$ and $0111\ldots$ "represent" the same point $\frac{1}{2} \in \mathbb{I}$. In our current framework this correspondence is forced to be bijective: we suitably mod out the streams and take the equivalence classes, not the streams themselves, as representatives. We believe our establishment of such bijective correspondences is at least of a mathematical interest. We are also keen to pursue its computational use by formalizing the notion of *computation* on top of our representation theory, towards *general theory of exact computation over fractals*.

Many authors have studied fractals from the *domain theory* viewpoint: see e.g. [12,6,23]. Roughly speaking, an IFS on a complete metric space X induces a continuous map on the dcpo $\mathcal{U}X$ of non-empty compact subsets of X equipped with the reverse inclusion order; this allows one to approximate the attractor by the Kleene fixed point theorem. Our results here are distinguished in two aspects. Firstly, in the domain-theoretic literature a bijective representation is not the main concern. Secondly, it is the idea of reasoning about infinitary processes purely in categorical terms—avoiding use of order-theoretic or metric arguments—that has put forward the theory of coalgebra and coinduction [22]. Although metric structure is indispensable, we stick to this motivation which is eminent e.g. in our proof of uniqueness of attractors (Thm. 5.6).

Categorical/coalgebraic properties of a (version of) real line have been studied e.g. in [20,8]; this work brings a fractal viewpoint to this topic, following Freyd's observation (see §3.1). Conversely, canonical metric structure on a final coalgebra has been studied e.g. in [3,1].

2 Leading example I: the Cantor set

To describe our goal more concretely, we start with the simple "gluing-free" example of the Cantor set C. It is obtained by repeatedly removing the "middle thirds" from the unit interval $\mathbb{I}$, as shown on the right. The Cantor set is a "fractal" in an obvious way: it is the same as the disjoint union of two copies of itself, each shrunk by the factor 1/3.

In this paper we are interested in *iterated function systems (IFSs)* [13] as a standard way for specifying fractals. For the Cantor set we can use the following

IFS on the complete metric space $\mathbb{I}$, the unit interval. It consists of two functions

$$\varphi_0(x) = x/3 \ , \quad \varphi_1(x) = (2+x)/3 \ . \tag{2}$$

This IFS yields the Cantor set C as its unique *attractor*: $C = \varphi_0(C) \cup \varphi_1(C)$. This is how we obtain C by *metric* means, as a subset of the complete metric space $\mathbb{I}$.

It is well-known (and easy to observe) that the Cantor set C allows *symbolic representation* via (infinite) binary streams. Given a binary stream $\sigma = a_0 a_1 \ldots \in 2^\omega$—where $2 = \{0,1\}$— we assign a point $[\![\sigma]\!] \in \mathbb{I}$. It goes in the following way: if $a_0 = 0$ then $[\![\sigma]\!]$ lies in $[0, \frac{1}{3}]$; if $a_0 = 1$ then $[\![\sigma]\!]$ lies in $[\frac{2}{3}, 1]$; continuing the same with $a_1, a_2, \ldots$ determines a point $[\![\sigma]\!] \in \mathbb{I}$. It is intuitively clear that this assignment $[\![\_]\!] : 2^\omega \to \mathbb{I}$—we shall call this the *denotation map*—restricts to $2^\omega \stackrel{\cong}{\Rightarrow} C$, and that the restriction is bijective. Its inverse $C \stackrel{\cong}{\Rightarrow} 2^\omega$ we shall call the *representation* map.

Our first fundamental observation is that the set 2^ω of *representatives* carries the final coalgebra for the functor $2 \cdot (\_)$: **Sets** → **Sets**, as on the right (cf. [22]). This is not a mere coincidence: the two-element set 2 in the functor reflects the fact that the Cantor set is *two* copies of itself combined together. Therefore one can think of the functor $2 \cdot (\_)$ to be the *combinatorial specification* of the Cantor set; and then the *symbolic Cantor set* 2^ω arises as its final coalgebra. More generally, if a fractal is described as the *disjoint* union of n copies of itself, the set of its symbolic representatives is given by the final coalgebra $\mathsf{n}^\omega \stackrel{\cong}{\Rightarrow} \mathsf{n} \cdot \mathsf{n}^\omega$, where n is an n-element set.

We can press this coalgebraic/categorical view on fractals further, so as to accommodate the metric way of defining fractals (via IFSs). Here comes our second fundamental observation: the IFS (2) can be put together to form an *algebra* for the functor $2 \cdot (\_)$—the same functor that we used for expressing the combinatorial specification of the Cantor set—as above on the right.

Our third crucial observation is: the denotation map $[\![\_]\!] : 2^\omega \to \mathbb{I}$—which we have introduced only informally—is characterized categorically using the diagram below left, bridging the symbolic fractal ι and the IFS χ. More specifically, the above informal description of $[\![\_]\!]$ is equivalent to the condition that the map $[\![\_]\!]$ makes the diagram commute, when put in place of $[\![\_]\!]_\chi$ in the diagram.

$$\tag{3}$$

For example, let us spell out the commutativity condition for the stream $10111 \ldots \in 2^\omega$. By the commutativity we have $[\![10111 \ldots]\!]_\chi = \frac{2 + [\![0111 \ldots]\!]_\chi}{3}$; it lies in the right shrunk copy of C, more specifically as the point represented by $0111 \ldots$ in that shrunk copy (see above right). This is how we informally introduced the map $[\![\_]\!]$.

It is intuitively clear that our informal description of $[\![\_]\!]$ determines the function $2^\omega \to \mathbb{I}$ uniquely; therefore it must be that there exists a unique $[\![\_]\!]_\chi$ that makes the diagram (3) commute. But how can we prove this?

The key is that the diagram (3) resembles a familiar one in denotational semantics, namely that of *initial algebra-final coalgebra (IA-FC) coincidence*, in the sense that a final coalgebra ι plays a role of an initial algebra. The IA-FC coincidence is used as an important tool for providing denotational semantics for datatype constructors with mixed variance (such as the function type $(-) \Rightarrow (+)$), see e.g. [9]. [4] The IA-FC coincidence occurs, intuitively, when the base category is enriched with some structure that allows "approximation of infinitary data by streams of finitary data." Examples of such structure are cpos [24] and complete metric spaces [2].

In the current setting we do not have the IA-FC coincidence as it is; it is hard to find a common base category for the coalgebra ι (which does not have intrinsic metric structure [5]) and the algebra χ (whose metric structure is crucial). Still the IA-FC coincidence works as a good guideline: simulating the proof of the coincidence in a metric-enriched setting (e.g. in [2]), we can prove the following.

Theorem 2.1 *There exists a unique $[\![\_]\!]_\chi$ that makes the diagram in (3) commute.*

Proof (Sketch) The homset $\mathbf{Sets}(2^\omega, \mathbb{I})$ is a complete metric space by $d(f, g) = \sup_\sigma d(f\sigma, g\sigma)$. On that set, the map $\Phi : f \mapsto \chi \circ (2 \cdot f) \circ \iota$ is a contracting map. By the Banach fixed point theorem (see e.g. [14]) Φ has a unique fixed point. $\square$

For our story that the final coalgebra 2^ω provides a symbolic representation of the Cantor set via $[\![\_]\!]_\chi$ in (3), we still have to show: 1) the unique $[\![\_]\!]_\chi$ is injective as a function; 2) its image $\mathrm{Im}\,[\![\_]\!]_\chi$ coincides with the Cantor set $C \subseteq \mathbb{I}$. Both facts are obvious for the map $[\![\_]\!]$ that we informally introduced. We shall prove these (and Thm. 2.1) in a far more general form, later in §5.

We now summarize the arguments so far, by presenting a general scenario.

Scenario 2.2 • The combinatorial specification for a "fractal" determines a functor $\mathsf{n} \cdot (\_) : \mathbf{Sets} \to \mathbf{Sets}$. The *final coalgebra* $\mathsf{n}^\omega \stackrel{\cong}{\Rightarrow} \mathsf{n} \cdot \mathsf{n}^\omega$ for this functor is the set of symbolic representatives for the fractal; we call it a *coalgebraic fractal*.

• A fractal is more standardly introduced via an IFS on a complete metric space X as its unique attractor—a *metric* fractal. Such an IFS is identified with an *algebra* $\chi : \mathsf{n} \cdot X \to X$ for the same functor.

• Relating the two "fractals" there exists a unique map $[\![\_]\!]_\chi$ which makes the diagram on the right commute. This is the *denotation* map.

$$
\begin{array}{ccc}
\mathsf{n} \cdot \mathsf{n}^\omega & \xrightarrow{\mathsf{n} \cdot [\![\_]\!]_\chi} & \mathsf{n} \cdot X \\
{\scriptstyle\cong}\big\uparrow{\scriptstyle\iota} & & \big\downarrow{\scriptstyle\chi} \\
\mathsf{n}^\omega & \xrightarrow[{[\![\_]\!]_\chi}]{} & X
\end{array}
$$

• The denotation map $[\![\_]\!]_\chi$ is injective, and its image $\mathrm{Im}\,[\![\_]\!]_\chi$ coincides with the metric fractal (i.e. the attractor) specified by the IFS χ. We take its codomain restriction $\mathsf{n}^\omega \stackrel{\cong}{\Rightarrow} \mathrm{Im}\,[\![\_]\!]_\chi$ and then its inverse $\mathrm{Im}\,[\![\_]\!]_\chi \stackrel{\cong}{\Rightarrow} \mathsf{n}^\omega$; the last is the *representation* map carrying a point of the fractal to its symbolic representative.

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[4] The IA-FC coincidence has been also applied to trace semantics for coalgebras; see [11].

[5] For this example the set 2^ω carries standard metric structure by: $d(\sigma, \sigma') = 2^{-n}$ where n is the length of the longest common prefix. However it is not clear how this generalizes when gluing is present.
Another possible view of the diagram (3) is that the codomain algebra χ is a *corecursive algebra* [5]. In fact the proof of Theorem 2.1 can rely on the corecursiveness, instead of metric. Such adaptation of the general theory in the remainder of the paper is left as future work.

This is the (so simple!) story that we wish to convey using the rest of the paper. However there is an important piece missing so far: *gluing* of shapes, in other words *overlaps* of images of functions in an IFS. When gluing is present, by literally following the scenario we are led to a non-injective $[\![\_]\!]_\chi$. Fortunately the main line of the scenario survives with suitable modification and generalization, to which the rest of the paper is devoted. In §3 we shall exhibit the problem of gluing using the unit interval $\mathbb{I} = [0, 1]$. For the coalgebraic fractal side we employ Leinster's presheaf framework [16] (§3). For the metric side we will introduce a central technical notion of *injective IFS (IIFS)* (§4). An IIFS is a variant of an IFS that is equipped with explicit information on how images overlap; this information is expressed with the help of presheaves too.

3   Coalgebraic Fractal

In this section we review Leinster's framework [16, 17]. [6]   It is used to obtain a *coalgebraic fractal*, which is the set of symbolic representatives induced by a combinatorial specification, such as $2^\omega \overset{\cong}{\Rightarrow} 2 \cdot 2^\omega$ in §2. Leinster's framework suitably addresses *gluing* of shapes; the corresponding mathematical shift is to move from **Sets** to a *presheaf category* $\mathbf{Sets}^\mathbb{A}$.

3.1   Presheaf and Module

It was first noticed by Freyd that the (closed) unit interval $\mathbb{I} = [0, 1]$ can be characterized as a final coalgebra (see [16]).   This observation motivated Leinster's work [16, 17], whose treatment of gluing by presheaves has inspired the current work.

We have noted in §1 that $\mathbb{I}$ is a "fractal," being the union of two shrunk copies of itself. This stimulated us to denote a point $x \in \mathbb{I}$ by a binary stream $\sigma \in 2^\omega$, like we did for the Cantor set C. However $\mathbb{I}$ is topologically distinguishable from the Cantor set; so there must be some difference in the two denotation schemes. What is it?

The difference is, when forming $\mathbb{I}$ as a union of $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$, there occurs *gluing* of the shrunk copies that identifies two points ($\frac{1}{2}$ in each copy). It is due to this gluing that the standard denotation of $x \in \mathbb{I}$ by a binary stream is not unique—both streams $0111\ldots$ and $1000\ldots$ can denote $\frac{1}{2}$.

Therefore if we aim at a representation of $\mathbb{I}$—that is, a mapping from $\mathbb{I}$ to the set of symbolic, stream-like representatives—such a representative cannot simply be a stream, but a certain equivalence class of streams. It is the framework in [16] that describes such modding of streams in a uniform, categorical manner. In the framework the necessary equivalence relation over streams is categorically induced by *presheaves* and *modules*.

First we fix a category $\mathbb{A}$ of *types*. For $\mathbb{I}$ we take the two-object category $\mathbb{A}_\mathbb{I}$ below

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[6] Some notations and terminologies are modified for the better fit to the current context.

left. With this $\mathbb{A}_{\mathbb{I}}$ we refine the description of $\mathbb{I}$ into the presheaf [7] $P_{\mathbb{I}} : \mathbb{A}_{\mathbb{I}} \to \textbf{Sets}$ below right; it singles out the two points $0, 1 \in \mathbb{I}$ on which gluing possibly occurs.

$$\mathbb{A}_{\mathbb{I}} = \left(\; 0 \underset{r}{\overset{l}{\rightrightarrows}} 1 \; \right) \qquad \left(\; 0 \underset{r}{\overset{l}{\rightrightarrows}} 1 \; \right) \; \overset{P_{\mathbb{I}}}{\longmapsto} \; \left(\; \{*\} \underset{1}{\overset{0}{\rightrightarrows}} \mathbb{I} \; \right) \qquad (4)$$

Our next step is to mathematically express a *combinatorial specification* of a fractal—like "the Cantor set C is the disjoint union of two copies of C." For C it was done simply by the set $2 = \{0, 1\}$. Its appropriate generalization—now that we must handle gluing—is given in the form of a functor $M : \mathbb{A}^{\mathrm{op}} \times \mathbb{A} \to \textbf{Sets}$; a functor of such a type is called a *module*.

A module is also called a *bimodule*, a *profunctor* or a *distributor*; it is usually denoted by $M : \mathbb{A} \nrightarrow \mathbb{A}$. With modules over rings in our mind, we can think of a (categorical) module $M : \mathbb{A}^{\mathrm{op}} \times \mathbb{A} \to \textbf{Sets}$ as a "family of sets with left and right $\mathbb{A}$-actions." See e.g. [4] for full-fledged expositions on modules. We shall define a *combinatorial specification* to be a pair $(\mathbb{A}, M)$ of a category $\mathbb{A}$ and a module $M : \mathbb{A}^{\mathrm{op}} \times \mathbb{A} \to \textbf{Sets}$; a formal definition is deferred to Def. 3.7 since it needs some preparatory notions.

A module M that we will be employing is such that $M(a, b)$ is a finite set, for all $a, b \in \mathbb{A}$. Intuitively, the finite set $M(a, b)$ represents the "multitude" in the combinatorial specification: the "outcome" space of type b has, inside it, $|M(a, b)|$ copies of the "ingredient" space of type a. A module allows different multitudes $|M(a, b)|$ for different $a, b \in \mathbb{A}$; moreover its action on arrows in $\mathbb{A}$ is how we express gluing that occurs in a combinatorial specification, as we see shortly. We note that when there is no gluing we take $\mathbb{A}$ to be the terminal category $\mathbf{1}$, in which case we can identify the finite set $M(*, *)$ with the finite set/number n in §2.

Example 3.1 [The module $M_{\mathbb{I}}$ for the unit interval] The combinatorial specification for the unit interval $\mathbb{I}$ is informally: $\mathbb{I}$ is the union of two copies of itself, identifying two points, one from each copy. It is made formal as the module $M_{\mathbb{I}}$ displayed below left; it is taken from [16]. The display is according to the legend below right. For example, $M_{\mathbb{I}}(1, 1)$—the two-element set $\{\bullet\!\!-\!\!\bullet, -\!\!\bullet\!\!\bullet\}$ whose elements we named suggestively—says that $\mathbb{I}$ is the same as *two* copies of $\mathbb{I}$. The set $M_{\mathbb{I}}(0, 1)$ has three elements which are again suggestively named. In the rest of this example we often abbreviate $M_{\mathbb{I}}$ by M.

$$M_{\mathbb{I}} = \begin{pmatrix} \{\bullet\} \underset{1}{\overset{0}{\rightrightarrows}} \{\bullet\!\!-, -\!\!\bullet\!\!-, -\!\!\bullet\} \\ \uparrow\uparrow \qquad\qquad \inf\uparrow\uparrow\sup \\ \emptyset \rightrightarrows \{\bullet\!\!-\!\!\bullet, -\!\!\bullet\!\!\bullet\} \end{pmatrix} \qquad \begin{array}{ccc} & \overset{M(0,l)=l\cdot\_}{} & \\ M(0,0) & \xrightarrow{} & M(0,1) \\ {\scriptstyle M(l,0)}\uparrow\uparrow{\scriptstyle M(r,0)} & {\scriptstyle M(0,r)} & {\scriptstyle M(l,1)}\uparrow\uparrow{\scriptstyle M(r,1)=\_\cdot r} \\ M(1,0) & \xrightarrow[M(1,r)]{M(1,l)} & M(1,1) \end{array} \qquad (5)$$

The functions $M(l, 0)$ and $M(l, 1)$ shall be both denoted by $\_\cdot l$; similarly $\_\cdot r$ denotes the functions $M(r, 0)$ and $M(r, 1)$. [8] The functions $\_\cdot l$ are called *right* l-*actions* in

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[7] A *presheaf* over a category $\mathbb{A}$ is a functor $P : \mathbb{A} \to \textbf{Sets}$. See e.g. [18].

[8] This is compliant to the notational convention for modules over rings. Given two successive arrows $\cdot \xrightarrow{f} \cdot \xrightarrow{g} \cdot$ in $\mathbb{A}$, due to the contravariance of M in its first argument, we have $M(g \circ f, 0) = M(f, 0) \circ M(g, 0)$ hence $\_\cdot (g \circ f) = (\_\cdot g) \cdot f$.

M; similarly for r. In the example of $M_{\mathbb{I}}$, a right r-action $\_ \cdot r : M(1,1) \to M(0,1)$ is defined by $\bullet\!\!-\!\!\bullet \cdot r = -\!\bullet$ and $-\!\!\bullet\!\!\bullet \cdot r = -\!\!\bullet$, explaining our notation sup in (5). The intuition for this right r-action is as follows: $\mathbb{I}$ is the union of two $(= |M_{\mathbb{I}}(1,1)|)$ shrunk copies of $\mathbb{I}$, but in each shrunk copy (i.e. in an "ingredient" unit interval) lies the singleton $\{*\}$ embedded along r, specifically on its right end. The function $\_ \cdot r$ specifies how these "ingredient" singletons (one in each of $\bullet\!\!-\!\!\bullet$ and $-\!\!\bullet\!\!\bullet$) lie in the "outcome" $\mathbb{I}$.

Let us turn to *left* l- and r-actions such as $l \cdot \_ : M(0,0) \to M(0,1)$. In the example $M_{\mathbb{I}}$, we have $l \cdot \bullet = \bullet\!\!-$ and $r \cdot \bullet = -\!\!\bullet$, explaining our notations 0 and 1 in (5). The intuition is as follows: $\bullet \in M(0,0)$ represents the only way in which the "ingredient" type-0 space (i.e. the singleton $\{*\}$) is used in composing up the "outcome" type-0 space; but the latter is embedded via arrows l and r in the "outcome" type-1 space. Hence the "ingredient" type-0 space appears in the "outcome" type-1 space via l and r; the left action $l \cdot \_ : M(0,0) \to M(0,1)$ tells how this happens.

Finally, gluing in the combinatorial specification of $\mathbb{I}$ is hinted in the equality: $-\!\!\bullet\!\!\bullet \cdot l = \bullet\!\!-\!\!\bullet \cdot r = -\!\bullet$. Note that, although the notations like $\bullet\!\!-\!\!\bullet$ come from $\mathbb{I}$ that is "continuous," the module $M_{\mathbb{I}}$ is purely "discrete" or "combinatorial": it is a bunch of finite sets and functions between them.

The following additional notational convention hopefully provides further intuition. We shall denote a module element $m \in M(b,a)$ by $m : b \nrightarrow a$. Given $\mathbb{A}$-arrows $f : b' \to b$ and $g : a \to a'$, we denote their left- and right-actions

$$g \cdot m = M(b,g)(m) \quad \text{by} \quad b \overset{m}{\nrightarrow} a \overset{g}{\to} a' \,, \quad \text{and}$$
$$m \cdot f = M(f,a)(m) \quad \text{by} \quad b' \overset{f}{\to} b \overset{m}{\nrightarrow} a \,.$$

These notations $g \cdot m$ and $m \cdot f$ resemble that for composition of arrows in a category. In the sequel we sometimes suppress $\cdot$ in the left- and right-actions, e.g. mf for $m \cdot f$.

Remark 3.2 In [16,17] what we have called a combinatorial specification is called a *self-similarity system*; and the induced coalgebraic fractal (i.e. a final coalgebra) is called the *solution* of the self-similarity system. Their symbolic/combinatorial nature is not emphasized there.

The directions of further developments are different, too. In the current paper we focus on a *metric* extension, relating a coalgebraic fractal with a metric fractal induced by an IFS-like specification. In contrast, Leinster pursues mostly a *topological* extension: he endows a coalgebraic fractal with canonical topological structure. Based on this, in [17] Leinster presents *recognition theorems*: they tell if a given topological space is a solution of a certain self-similarity system or not.

3.2 Tensor Product

We have replaced a finite set n (like 2 for the Cantor set) by a module $M : \mathbb{A}^{\mathrm{op}} \times \mathbb{A} \to$ **Sets** to cope with gluing; now we shall upgrade the functor $n \cdot (\_) : \mathbf{Sets} \to \mathbf{Sets}$ accordingly, into $M \otimes (\_) : \mathbf{Sets}^{\mathbb{A}} \to \mathbf{Sets}^{\mathbb{A}}$ following [16]. Here $\otimes$ is an operation

of *tensor product*, a standard construction for modules. It is usually defined via *coends* (see e.g. [4]), but we would rather describe it concretely.

Definition 3.3 [Tensor product] Given a module $M : \mathbb{A}^{\mathrm{op}} \times \mathbb{A} \to \mathbf{Sets}$ and a presheaf $P : \mathbb{A} \to \mathbf{Sets}$, the *tensor product* $M \otimes P : \mathbb{A} \to \mathbf{Sets}$ is defined by: $(M \otimes P)a = \left(\coprod_{b \in \mathbb{A}} M(b, a) \cdot Pb \right) /\!\sim$, where the equivalence relation $\sim$ is described below.

The set $M(b, a) \cdot Pb$ appearing in the definition describes copies of Pb—one for each module element $m : b \twoheadrightarrow a$—summed up altogether. Hence an element of $(M \otimes P)a$ can be written in the form $[(b \overset{m}{\twoheadrightarrow} a, x \in Pb)]$—the pair (m, x) modded out modulo $\sim$—for some $b \in \mathbb{A}$. We denote this element by $m \otimes x$ (like for modules over rings), where the "mediating" object $b \in \mathbb{A}$ is implicit in m's type $m : b \twoheadrightarrow a$.

Let us now describe the equivalence $\sim$ in Def. 3.3, that is, describe when we have $m \otimes x = m' \otimes x'$. It is about different choices of the mediating object $b \in \mathbb{A}$ which we want to ignore. Recall that $M(b, a)$ is contravariant in $b \in A$ while Pb is covariant in b. Assume that we have an arrow $f : b \to b'$ in $\mathbb{A}$, a module element $m : b' \twoheadrightarrow a$ and $x \in Pb$. Using these three building blocks we can form two elements of $\coprod_b M(b, a) \cdot Pb$, namely (mf, x) and (m, fx).[9] The equivalence $\sim$ in Def. 3.3 identifies these two: it is the equivalence generated by $(mf, x) \sim (m, fx)$. Therefore we have $mf \otimes x = m \otimes fx$ as elements of $(M \otimes P)a$, an equality familiar in modules over rings.

Example 3.4 Let us calculate the tensor product $M_{\mathbb{I}} \otimes P_{\mathbb{I}}$, with $M_{\mathbb{I}}$ from Expl. 3.1 and $P_{\mathbb{I}}$ from (4). First we see from Def. 3.3 that an element of $(M_{\mathbb{I}} \otimes P_{\mathbb{I}})1$ can be written in either of the following forms: $\bullet\!\!-\!\!-\otimes *,\ \to\!\!\bullet\!\!-\otimes *,\ -\!\!\bullet\otimes *$, and $\bullet\!\!\bullet\!\!- \otimes x,\ \to\!\!\bullet\!\!\bullet \otimes x$ for each $x \in \mathbb{I}$. Now the identifications caused by $\sim$ are the following three, the first one of which is derived by the calculation further below.

$$\bullet\!\!-\!\!- \otimes * = \bullet\!\!\bullet\!\!- \otimes 0\ ,\quad \bullet\!\!\bullet\!\!- \otimes 1 = \to\!\!\bullet\!\!- \otimes * = -\!\!\bullet\!\!\bullet \otimes 0\ ,\quad \text{and}\quad -\!\!\bullet\!\!\bullet \otimes 1 = -\!\!\bullet \otimes *\ ;$$

$$\bullet\!\!-\!\!- \otimes * = (0 \overset{\bullet\!\!-\!\!-}{\twoheadrightarrow} 1) \otimes * = (0 \overset{!}{\to} 1 \overset{\bullet\!\!\bullet\!\!-}{\twoheadrightarrow} 1) \otimes * \overset{(\dagger)}{=} \bullet\!\!\bullet\!\!- \otimes (! \cdot *) = \bullet\!\!\bullet\!\!- \otimes 0\ .$$

The equality (†) is the general equality $mf \otimes x = m \otimes fx$ described above. Therefore the set $(M_{\mathbb{I}} \otimes P_{\mathbb{I}})1$ looks like:

It is the union of two copies of $\mathbb{I}$, with the element $1 \in \mathbb{I}$ in the left copy identified with the element 0 in the right copy via the mediating element $\to\!\!\bullet \otimes *$. As a set this is isomorphic to the interval $[0, 2]$, hence to $\mathbb{I} = [0, 1]$. It is easy to see that $(M_{\mathbb{I}} \otimes P_{\mathbb{I}})0 \cong \{*\}$ and that $M_{\mathbb{I}} \otimes P_{\mathbb{I}} \cong P_{\mathbb{I}}$, too. In particular, $P_{\mathbb{I}}$ is a fixed point of $M_{\mathbb{I}} \otimes (\_)$.

---

[9] Recall $mf = (b \overset{f}{\to} b' \overset{m}{\twoheadrightarrow} a)$ is short for $M(f, a)(m)$; similarly we let fx denote $(Pf)x$. These notations are customary.

3.3 Coalgebraic Fractal

We have seen the functor $M \otimes (\_) : \mathbf{Sets}^{\mathbb{A}} \to \mathbf{Sets}^{\mathbb{A}}$ express the combinatorial specification of a fractal, just like $2 \cdot (\_) : \mathbf{Sets} \to \mathbf{Sets}$ for the Cantor set (§2). The next piece in Scenario 2.2 is the set of symbolic representatives obtained as a final coalgebra, like the symbolic Cantor set $2^{\omega} \xrightarrow{\cong} 2 \cdot 2^{\omega}$. Leinster [16] showed that the basic scenario carries over even in presence of gluing—but with a slight additional technicality, namely *non-degeneracy*.

The first observation is that for the functor $M_{\mathbb{I}} \otimes (\_) : \mathbf{Sets}^{\mathbb{A}_{\mathbb{I}}} \to \mathbf{Sets}^{\mathbb{A}_{\mathbb{I}}}$ in §3.2, the final coalgebra is carried by the presheaf P_{deg} on the right. This does not seem to yield any useful representation of $\mathbb{I}$. The trouble here is that gluing worked too much, giving rise to a "degenerate" solution P_{deg}. We need a way to regulate gluing, so that two points are identified only when they really need to be. The *non-degeneracy* requirement is introduced in [16] for that purpose.

$$(\; 0 \underset{r}{\overset{l}{\rightrightarrows}} 1 \;)$$
$$\Big\downarrow P_{\mathrm{deg}}$$
$$(\; \{*\} \underset{!}{\overset{!}{\rightrightarrows}} \{*\} \;)$$

Definition 3.5 [Non-Degeneracy] A presheaf $P : \mathbb{A} \to \mathbf{Sets}$ is said to be *non-degenerate* if it satisfies the following two conditions.

- Assume that two elements $x \in Pa$ and $x' \in Pa'$ are identified by arrows $f : a \to b$ and $f' : a' \to b$, that is, $fx = f'x'$ as an element of Pb. Then there exist $c \in \mathbb{A}$, $z \in Pc$, arrows $g : c \to a$ and $g' : c \to a'$ such that $x = gz$, $x' = g'z$ and $fg = f'g'$.

- Assume that $f, f' : a \rightrightarrows b$ are arrows in $\mathbb{A}$, and that $x \in Pa$ satisfies $fx = f'x$. Then there exist $c \in \mathbb{A}$, $z \in Pc$ and $g : c \to a$ such that $x = gz$ and $fg = f'g$.

The full subcategory of $\mathbf{Sets}^{\mathbb{A}}$ with non-degenerate presheaves as objects is denoted by $[\mathbb{A}, \mathbf{Sets}]_{\mathrm{ND}}$.

The two conditions are best depicted in the category $\mathrm{el}(P)$ of *elements* of P:[10]

$$
\begin{array}{cc}
\underset{\exists g}{\swarrow}\;\exists(c,z)\;\underset{\exists g'}{\searrow} & \exists(c,z) \\
(a,x) \qquad (a',x') & \exists g \downarrow \\
\underset{f}{\searrow}\;(b,y)\;\underset{f'}{\swarrow} & (a,x) \\
 & f' \downdownarrows f \\
 & (b,y)
\end{array}
$$

What the conditions say is, intuitively: if two elements x and x' are ever to be identified (like in $fx = f'x'$), then this identification is "forced" by equality of arrows in $\mathbb{A}$.[11] Hence a presheaf $P : \mathbb{A} \to \mathbf{Sets}$ is non-degenerate if P has "no unforced equalities." When $\mathbb{A} = \mathbb{A}_{\mathbb{I}}$ in (4), non-degeneracy is reduced to the following simple condition. This observation is due to [16].

Lemma 3.6 *A presheaf $P : \mathbb{A}_{\mathbb{I}} \to \mathbf{Sets}$ is non-degenerate if and only if: 1) both functions $P\mathsf{l}, P\mathsf{r} : P0 \rightrightarrows P1$ are injective; and 2) their images are disjoint.* $\qquad\square$

---

[10] An object of $\mathrm{el}(P)$ is a pair (a, x) of $a \in \mathbb{A}$ and $x \in Pa$; an arrow $f : (a, x) \to (b, y)$ in $\mathrm{el}(P)$ is an arrow $f : a \to b$ in $\mathbb{A}$ such that $(Pf)x = y$. See e.g. [18].

[11] The non-degeneracy condition can be rephrased as a weak form of *flatness* of a presheaf P, or weak *cofilteredness* of the category $\mathrm{el}(P)$ of elements. See [16].

The above presheaf P_{deg} clearly violates the condition; $P_{\mathbb{I}}$ in (4) does not. In fact it is shown in [17] that $P_{\mathbb{I}}$ carries the final coalgebra for $M_{\mathbb{I}} \otimes (\_) : [\mathbb{A}_{\mathbb{I}}, \mathbf{Sets}]_{\text{ND}} \to [\mathbb{A}_{\mathbb{I}}, \mathbf{Sets}]_{\text{ND}}$, the functor $M_{\mathbb{I}} \otimes (\_)$ now restricted to the category of non-degenerate presheaves. Therefore we shall think of the final *non-degenerate* presheaf coalgebra as the set of symbolic representatives.

To do that, however, we have to convince ourselves that a final non-degenerate coalgebra is of symbolic character, such as a set of streams modulo some equivalence. This was obvious when there was no gluing (§2). Fortunately it is also the case in presence of gluing, too, thanks to Leinster's concrete "symbolic" construction [16] of a final non-degenerate coalgebra by streams modulo an equivalence.

We defer the construction to Appendix A.1 of [10], providing only hints here. Those streams which reside in the final non-degenerate coalgebra are infinite sequences of module elements with matching types; two such streams are modded out when they are *connected* via arrows in $\mathbb{A}$. For $\mathbb{A}_{\mathbb{I}}$ and $M_{\mathbb{I}}$ in §3.1, one of such streams is $1 \xleftarrow{\bullet\!\!\bullet} 1 \xleftarrow{\bullet\!\!\bullet} 1 \xleftarrow{\bullet\!\!\bullet} \cdots$ which we can think of as $100\ldots \in 2^{\omega}$. Through the connectedness via $\mathbb{A}_{\mathbb{I}}$-arrows there arises an equivalence relation on such streams; it corresponds to the standard modding in the binary expansion code, such as $1000\ldots = 0111\ldots$.

We are ready to make the technical definition which we postponed in §3.1.

Definition 3.7 A *combinatorial specification* (of a fractal) is a pair $(\mathbb{A}, M)$ of a small category $\mathbb{A}$ and a finite non-degenerate module $M : \mathbb{A}^{\text{op}} \times \mathbb{A} \to \mathbf{Sets}$. Here a module M is said to be *finite* if for each $a \in \mathbb{A}$, there are only finitely many module elements $b \xrightarrow{m} a$ with varying b; M is *non-degenerate* if for each $b \in \mathbb{A}$, the presheaf $M(b, \_) : \mathbb{A} \to \mathbf{Sets}$ is non-degenerate (Def. 3.5).

It is proved in [16] that, given a combinatorial specification $(\mathbb{A}, M)$, the functor $M \otimes (\_)$ preserves non-degeneracy of presheaves. Hence the functor $M \otimes (\_)$ restricts to an endofunctor on the category $[\mathbb{A}, \mathbf{Sets}]_{\text{ND}}$ of non-degenerate presheaves.

Finally, the following is our notion of "symbolic" fractal.

Definition 3.8 [Coalgebraic fractal] Let $(\mathbb{A}, M)$ be a combinatorial specification. The *coalgebraic fractal* induced by $(\mathbb{A}, M)$ is the (carrier of the) final coalgebra for the functor $M \otimes (\_) : [\mathbb{A}, \mathbf{Sets}]_{\text{ND}} \to [\mathbb{A}, \mathbf{Sets}]_{\text{ND}}$. We denote this final non-degenerate coalgebra by $\iota : I \overset{\cong}{\Rightarrow} M \otimes I$; then the carrier presheaf I consists of suitable equivalence classes of streams, due to the construction in [16].

4 Injective IFS

4.1 Motivation

For our aim of a bijective representation-denotation correspondence (1), the conventional notion of IFS as it is is not a satisfactory way of specifying a fractal. It does not provide explicit treatment of overlaps of images; this may lead to a non-injective

denotation map $\llbracket \_ \rrbracket_\chi$ in (3). An example is the unit interval $\mathbb{I}$ and the IFS

$$\varphi_0(x) = x/2 \ , \quad \varphi_1(x) = (1+x)/2 \tag{6}$$

which result in $\llbracket 011\ldots \rrbracket_\chi = \llbracket 100\ldots \rrbracket_\chi = \frac{1}{2}$ (cf. §3.1). Roughly speaking, $\llbracket \_ \rrbracket_\chi$ is not injective because the IFS χ is not injective.

Hence we cannot start from an IFS $\{\varphi_i : X \to X\}_{i\in[0,n-1]}$ and crudely bundle them up as an algebra $[\varphi_i] : \mathsf{n} \cdot X \to X$. Instead, the solution we propose is to start from an algebra for $M \otimes (\_)$ (as shown on the right) whose algebraic structure χ is injective. Such an algebra that we will use instead of an IFS shall be called an *injective IFS*, or *IIFS* in short.

$$\begin{array}{c} M \otimes X \\ \downarrow{\scriptstyle \chi} \\ X \end{array}$$

We can look at an IIFS as a variant of an IFS, where the gluing structure is made explicit with the help of the categorical machinery ($\mathbb{A}$ and M). The opposite view is that it is a combinatorial specification $(\mathbb{A}, M)$ of a fractal, augmented with the information on how the symbolic fractal is to be "realized" in a complete metric space.

Let us briefly elaborate on *injectivity* of an IIFS. The algebraic structure χ being injective means that "$(\mathbb{A}, M)$ has successfully modded out points in $M \otimes X$." That is, using the equivalence relation $\sim$ in $M \otimes X$ induced by $(\mathbb{A}, M)$ (Def. 3.3), two points in an overlap of images in an IFS—such as $1 \overset{\varphi_0}{\mapsto} \frac{1}{2}$ and $0 \overset{\varphi_1}{\mapsto} \frac{1}{2}$ in (6)—have got "already identified" in the domain $M \otimes X$ of χ.

4.2   Metric Preliminaries

As a *metric* way of specifying a fractal, complete metric structure is indispensable for an IIFS. Before introducing IIFSs formally, we need some metric notions. We denote by $\mathbf{CMet}_1^{\mathrm{TB}}$ the category of 1-bounded [12] totally bounded [13] complete metric spaces and non-expansive functions between them.   Total boundedness is a technical condition that we need later in the proof of Prop. 4.2; imposing it is justified because all the fractals of our interest are compact, hence are totally bounded. A function $f : X \to Y$ is *non-expansive* if $d_Y(fx, fx') \leq d_X(x, x')$ for each $x, x' \in X$; f is *contracting* if there is a number $\delta \in [0, 1)$ such that $d_Y(fx, fx') \leq \delta \cdot d_X(x, x')$ for any $x, x' \in X$.

A *pseudometric* d is like a metric but $d(x, x') = 0$ need not imply $x = x'$. By $\mathbf{CPMet}_1^{\mathrm{TB}}$ we denote the category of 1-bounded and totally bounded complete *pseudo*-metric spaces and non-expansive functions.

The tensor product construction (Def. 3.3) also applies to "presheaves" with extra structure, like a *topological* version which is exploited in [16]. Here we use a *metric* version: given a functor $P : \mathbb{A} \to \mathbf{CMet}_1^{\mathrm{TB}}$, we shall define a functor $M \otimes P$. For that we shall "metrize" the coproduct and quotient operations in Def. 3.3. The category $\mathbf{CMet}_1^{\mathrm{TB}}$ has a coproduct, which is a set-theoretic coproduct $\coprod_i X_i$

---

[12] 1-boundedness is not an essential requirement: forcing it by $d'(x, y) := \min\{1, d(x, y)\}$ does not change the convergence properties. Assuming it makes some proofs simpler.
[13] See e.g. [14] for the relevant metric notions. Total boundedness together with completeness is equivalent to compactness.

equipped with the obvious metric: $d(x, x') = d_{X_i}(x, x')$ if x and x' are in the same summand X_i; $d(x, x') = 1$ otherwise. Problematic are coequalizers—i.e. taking quotients—which we do use in Def. 3.3. There is a standard way to define a metric on a quotient space, but it in general only yields a pseudometric—$d(x, x') = 0$ need not imply $x = x'$.

Definition 4.1 [Quotient pseudometric] Let (X, d) be a metric space, and $\sim$ be an equivalence relation on X. A *path* from $x \in X$ to x' is a finite sequence of points $x_0, x_1, \ldots, x_{2n+1}$ with $x = x_0$ and $x' = x_{2n+1}$, such that: $x_1 \sim x_2, x_3 \sim x_4, \ldots, x_{2n-1} \sim x_{2n}$. The *length* $\ell(x_0, \ldots, x_{2n+1})$ of such a path is defined to be the sum $d(x_0, x_1) + d(x_2, x_3) + \cdots + d(x_{2n}, x_{2n+1})$. Then we define a pseudometric on $X/\sim$ to be the infimum of the length of such paths (or 1 if it exceeds 1):

$$d([x], [x']) = \min\big\{ 1, \inf\{\ell(x_0, \ldots, x_{2n+1}) \mid x_0, \ldots, x_{2n+1} \text{ is a path from } x \text{ to } x'\} \big\} .$$

Intuitively: the quotient pseudometric is the distance to go from x to x', where we are allowed to make a finite number of "leaps" along $\sim$.

Proposition 4.2 *The construction indeed yields a 1-bounded pseudometric on $X/\sim$. Moreover, if X is totally bounded and complete, then the pseudometric is also totally bounded and* complete: *any Cauchy sequence has a limit (which is by the way not necessarily unique).* □

The proof is found in [10, Appendix A.2]. Using the coproduct and quotient (pseudo)metrics that we have described, we can define a "metric" version of tensor products. In this paper, when it is employed, it always involves a *discount factor* $\delta \in [0, 1)$.

Definition 4.3 [Metric tensor $M \otimes \delta X$] Given a functor $X : \mathbb{A} \to \mathbf{CMet}_1^{\mathrm{TB}}$ and a number $\delta \in [0, 1)$, we define $\delta X : \mathbb{A} \to \mathbf{CMet}_1^{\mathrm{TB}}$ to have: 1) the same underlying sets and functions as X, that is, $U((\delta X)a) = U(Xa)$ and $U((\delta X)f) = U(Xf)$, but; 2) with the metric on each space discounted by δ, i.e. $d_{(\delta X)a}(x, x') = \delta \cdot d_{Xa}(x, x')$.

When we are further given a module $M : \mathbb{A}^{\mathrm{op}} \times \mathbb{A} \to \mathbf{Sets}$, we define the functor $M \otimes \delta X : \mathbb{A} \to \mathbf{CPMet}_1^{\mathrm{TB}}$ by $(M \otimes \delta X)a = \big(\coprod_{b \in \mathbb{A}} M(b, a) \cdot (\delta X)b \big) /\sim$, that is, the coproduct $\coprod_{b \in \mathbb{A}} M(b, a) \cdot (\delta X)b$ equipped with the coproduct metric, quotiented by the same equivalence $\sim$ as in Def. 3.3. It is straightforward that this $M \otimes \delta X$ indeed determines a functor $\mathbb{A} \to \mathbf{CPMet}_1^{\mathrm{TB}}$.

To summarize: $M \otimes \delta X$ has the same underlying sets as the presheaf $M \otimes X$; those sets are equipped with pseudometrics that are essentially X's metric, shrunk by δ. In our applications the induced pseudometrics are in fact shown to be metrics (Lem. 4.7).

Example 4.4 We can readily show that the metric space $(M \otimes \frac{1}{2}P_{\mathbb{I}})\mathbf{1}$ is isometric to the unit interval $\mathbb{I}$ equipped with the standard metric. Recall the picture of $(M \otimes P_{\mathbb{I}})\mathbf{1}$ in Expl. 3.4; each of its two line segments is $\mathbb{I}$, now shrunk by the discount factor $\frac{1}{2}$.

4.3   Formal Definition

Definition 4.5 [Injective IFS] Let $(\mathbb{A}, M)$ be a combinatorial specification and $\delta \in [0, 1)$ be a fixed number which we call a *discount factor*. An *injective IFS* (*IIFS* in short), over $(\mathbb{A}, M)$ and $\delta \in [0, 1)$, is a pair (X, χ) such that

- $X : \mathbb{A} \to \mathbf{CMet}_1^{\mathrm{TB}}$ is a functor which is *non-degenerate*, meaning that its composite $\mathbb{A} \xrightarrow{X} \mathbf{CMet}_1^{\mathrm{TB}} \xrightarrow{U} \mathbf{Sets}$ with the forgetful functor is non-degenerate;[14]

- χ is a natural transformation $\chi : M \otimes \delta X \to X$ between functors of the type $\mathbb{A} \to \mathbf{CMet}_1^{\mathrm{TB}}$. The functor $M \otimes \delta X$ here is defined by a metric tensor (Def. 4.3).

It is subject to the following further conditions:

(i)   each component $\chi_a : (M \otimes \delta X)a \to Xa$ is an injective function;

(ii)   the space $Xa \in \mathbf{CMet}_1^{\mathrm{TB}}$ and the set Ia (Def. 3.8) are non-empty for each $a \in \mathbb{A}$;

(iii)   for each arrow $f : b \to a$ in $\mathbb{A}$ and $y \in Xb$, if fy (i.e. $(Xf)y$) belongs to $\mathrm{Im}\,\chi_a$, then y belongs to $\mathrm{Im}\,\chi_b$. That is, $(Xf)^{-1}(\mathrm{Im}\,\chi_a) \subseteq \mathrm{Im}\,\chi_b$.

Hence an IIFS is an *algebra* $M \otimes \delta X \to X$ with additional conditions. Some explanations are in order. Cond. (i) is so that we have an injective denotation map $[\![\_]\!]_\chi$, as explained in §4.1. Cond. (ii) is a natural one and backed up by a result in Leinster's original work [17, Lem. 4.2]. The last Cond. (iii) might look technical but is nevertheless natural and important. Its informal reading is: a part of a fractal, upon which gluing occurs, itself has structure as a fractal. Let us study examples.

Example 4.6 [The unit interval $\mathbb{I}$] The standard IFS that induces $\mathbb{I}$ is the one in (6) on the complex plane $\mathbb{C}$. To make its gluing structure explicit we first focus on the subspace on $\mathbb{C}$ which is relevant, namely $\mathbb{I}$ itself. Then the data $\mathbb{A}_\mathbb{I}$ and $M_\mathbb{I}$ in §3.1 naturally arises to describe the combinatorial gluing structure.

We take $P_\mathbb{I}$ in (4) as the carrier $X_\mathbb{I}$ of the aimed IIFS. In Expl. 4.4 we observed an isomorphism $M_\mathbb{I} \otimes (\frac{1}{2} X_\mathbb{I}) \xrightarrow{\cong} X_\mathbb{I}$. We take this isomorphism to be the algebra structure χ of the aimed IIFS. It is straightforward to see that this (X, χ) satisfies the conditions in Def. 4.5.

The previous example is a peculiar one, where the fractal to be defined (namely $\mathbb{I}$) coincides with the whole domain $X_\mathbb{I}1$ of the IIFS, and χ is an isomorphism. This is not the case with e.g. the Cantor set $C \subsetneq \mathbb{I}$ (§2). Another example of an IIFS is the one for the *Gray code* coming later in Expl. 5.4. In [17] there are many other examples of "fractals" that can be described by a combinatorial specification $(\mathbb{A}, M)$. Among them are the Sierpiński triangle, the square $\mathbb{I} \times \mathbb{I}$, the n-dimensional simplex Δ^n and the unit circle S^1. For many of them we can write down IIFSs, too.

In forming the tensor $M \otimes \delta X$ (Def. 4.3) we took a quotient of a metric space; this in general results in a *pseudo*-metric (Prop. 4.2). In an IIFS we have the following, because $(M \otimes \delta X)a$ has a non-expansive injection χ_a into a metric space Xa.

---

[14] This is the same as saying that $X : \mathbb{A} \to \mathbf{Sets}$ is a non-degenerate presheaf in which Xa (for each $a \in \mathbb{A}$) and Xf (for each arrow f in $\mathbb{A}$) have suitable metric structure.

Lemma 4.7 *Let $\chi : M \otimes \delta X \to X$ be an IIFS. Then the space $(M \otimes \delta X)a$ is a (proper) metric space, for each $a \in \mathbb{A}$.*   □

Remark 4.8 An IIFS is a variant of an IFS; it is also a combinatorial specification $(\mathbb{A}, M)$ together with information on how to realize it in a complete metric space. Yet another way to look at it is as follows: the existence of an IIFS is a "sanity check" for a combinatorial specification $(\mathbb{A}, M)$.

First recall the principle from §1: it is a metric shape's fractal structure that enables representation of its points by stream-like representatives. That is, in more technical terms, identification of a "suitable" $(\mathbb{A}, M)$ gives rise to the set I of symbolic representatives (Def. 3.8) and a representation map $[\![\_]\!]_\chi^{-1}$ (§5 later). Such $(\mathbb{A}, M)$ is often not hard to come up with (like Expl. 3.1 for $\mathbb{I}$) but it is not precise what it means for $(\mathbb{A}, M)$ to be "suitable." The notion of IIFS formalizes this very point: if we find a "witness" χ (which is based on $(\mathbb{A}, M)$) which satisfies the conditions in Def. 4.5 (some of them are subtle), then our results in §5 ensures that $(\mathbb{A}, M)$ is "suitable" fractal structure that indeed results in symbolic representation.

Remark 4.9 Aside from the conceptual similarity between IFSs and IIFSs, we are yet to establish any technical relationship between them. In particular, we are not sure there is any general translation of an IFS into an IIFS, nor that there is a canonical (bijective) representation for an arbitrary IFS-based fractal. Detecting overlapping structure in an IFS and organizing it as $(\mathbb{A}, M)$ seems hard for some IFS-based fractals such as a fern. An observation [17, Expl. 2.11] can lead to such a translation which, however, works only for a limited class of IFSs. A related issue which draws our interest—and is left as future work—is a characterization of those metric spaces which arise as fractals induced by IIFSs. This question is a metric version of Leinster's (topological) *recognition theorem* [17, Thm. 3.1].

5 Representation Theory

In this section we technically develop the rest of Scenario 2.2. Specifically, we prove that: 1) an IIFS χ has a unique *attractor*—which we consider as a *metric* fractal—like an IFS does, and; 2) it has a canonical *representation* by the coalgebraic fractal (§3), the latter being induced by the same combinatorial specification $(\mathbb{A}, M)$ on which the IIFS χ is based. Most proofs here are deferred to the appendix of [10].

5.1 The Denotation Map $[\![\_]\!]_\chi$

First we present a result which is crucial in the sequel. Its proof goes much like the one for Thm. 2.1, using the Banach fixed point theorem; see Appendix A.3 of [10]. Notations:

$$
\begin{array}{ccc}
M \otimes C & \xrightarrow{M \otimes \widehat{\gamma}} & M \otimes X \\
\gamma \uparrow & & \downarrow \chi \\
C & \dashrightarrow{\widehat{\gamma}} & X
\end{array}
$$

given an IIFS $\chi : M \otimes \delta X \to X$, by writing $\chi : M \otimes X \to X$ (without δ) we mean forgetting the metric structure; for example $M \otimes X$ in the latter denotes the presheaf $U(M \otimes \delta X)$, which is the same as $M \otimes UX$ by Def. 4.3.

Theorem 5.1 *Let $\chi : M \otimes \delta X \to X$ be an IIFS, and $\gamma : C \to M \otimes C$ be a coalgebra with $C \in [\mathbb{A}, \mathbf{Sets}]_{\mathrm{ND}}$.* [15] *Assume further that there exists at least one natural transformation from C to X (more precisely to UX). Then there exists a unique arrow $\widehat{\gamma}$ that makes the diagram above on the right commute.* □

We obtain—much like in §2—the *denotation map* $[\![\_]\!]_\chi$ that goes from a coalgebraic fractal $\iota : I \to M \otimes I$ to a metric fractal. We use the previous theorem; the condition of existence of a natural transformation from

$$\begin{array}{ccc} M \otimes I & \xrightarrow{\ M \otimes [\![\_]\!]_\chi\ } & M \otimes X \\ \iota \uparrow \cong & & \downarrow \chi \\ I & \dashrightarrow[\ [\![\_]\!]_\chi\ ]{} & X \end{array}$$

I to UX is shown by investigation of Leinster's construction of I [10, Appendix A.1].

Theorem 5.2 *Let (X, χ) be an IIFS over a combinatorial specification $(\mathbb{A}, M)$, and $\iota : I \to M \otimes I$ be the coalgebraic fractal induced by $(\mathbb{A}, M)$ (Def. 3.8). There exists a natural transformation $[\![\_]\!]_\chi : I \to X$. It makes the diagram in the above right commute; moreover by Thm. 5.1 it is the unique such.* □

In Scenario 2.2 the image $\mathrm{Im}\,[\![\_]\!]_\chi$ of the map $[\![\_]\!]_\chi$ thus obtained is identified with the metric fractal; and its codomain restriction $I \xrightarrow{\cong} \mathrm{Im}\,[\![\_]\!]_\chi$ gives a bijective correspondence between coalgebraic and metric fractals. For this intuition to be valid we need the following result; this is one of the main technical results of this paper. Its proof makes essential use of Cond. (iii) in Def. 4.5.

Proposition 5.3 *The denotational map $[\![\_]\!]_\chi$ defined in Thm. 5.2 is a mono, i.e. its components $([\![\_]\!]_\chi)_a$ are all injective functions.* □

Example 5.4 [The Gray code] The *Gray code* is another way of representing real numbers in $\mathbb{I} = [0, 1]$ by binary streams $\sigma \in 2^\omega$, other than the standard binary expansion code (see Fig. 1). Its feature is: two binary streams that denote the same point in $\mathbb{I}$ differ in only one digit, in contrast to the binary expansion where $[\![011\ldots]\!] = [\![100\ldots]\!]$. One can also claim superiority of the Gray code in domain-theoretic terms: see [25], where computability of real number functions via a variant of the Gray code is discussed.
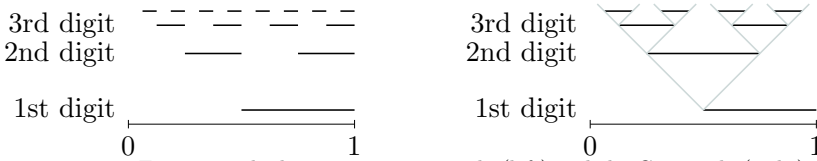

Figure 1. The binary expansion code (left) and the Gray code (right)

We claim that the two codings arise from two different views on $\mathbb{I}$ as a fractal; they both can arise as $[\![\_]\!]_\chi$ in our framework, but with different IIFSs χ. In fact, what we have called "the IIFS for the unit interval" in Expl. 3.1 and 4.6 is more precisely for the binary expansion code of the unit interval.

An IIFS χ_{G} for the Gray code is defined as follows. We take $\mathbb{A}_{\mathrm{G}} = \mathbb{A}_{\mathbb{I}}$; a module M_{G} is the same as $M_{\mathbb{I}}$ except that we define $\multimap\!\!\bullet \cdot \mathsf{l} = \longrightarrow\!\!\bullet$ and $\multimap\!\!\bullet \cdot \mathsf{r} = \bullet\!\!\multimap$. The intuition here is that, in forming $\mathbb{I}$ as the union of its two copies, the second copy

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[15] Recall that γ involves no metric structure.

is turned around. We define $X_{\mathrm{G}} = X_{\mathbb{I}}$ (cf. Expl. 4.6) and $\chi_{\mathrm{G}} : M_{\mathrm{G}} \otimes X_{\mathrm{G}} \to X_{\mathrm{G}}$ by: $\bullet\!\!-\!\!\bullet \otimes x \mapsto \frac{1}{2}x$, and $-\!\!\bullet\!\!\bullet \otimes x \mapsto 1 - \frac{1}{2}x$. This IIFS χ_{G} induces the Gray code via the denotation map $\llbracket \_ \rrbracket_{\chi_{\mathrm{G}}}$.

Other real number representations that can be accommodated in a similar way include: the (standard) decimal one, the signed digit one [7], and so on. It is our future work to discuss their comparison in terms of the IIFSs that induce them.

5.2 Uniqueness of an Attractor

Finally we shall present some results that justify our identification of the image $\mathrm{Im}\,\llbracket \_ \rrbracket_{\chi}$ (Thm. 5.2) with the *metric fractal* induced by the IIFS. More specifically, an IIFS has a unique *attractor* as an IFS does; and it coincides with $\mathrm{Im}\,\llbracket \_ \rrbracket_{\chi}$.

Definition 5.5 [Attractor] An *attractor* of an IIFS (X, χ) is a non-degenerate presheaf $S : \mathbb{A} \to \mathbf{Sets}$ and a natural transformation $\varepsilon : S \to X$ such that:

(i) each component $\varepsilon_a : Sa \hookrightarrow Xa$ is an injection—hence S is a subobject of X. Moreover the image of ε_a is a closed subset of the metric space Xa;

(ii) the set Sa is non-empty for each $a \in \mathbb{A}$;

(iii) there exists a natural isomorphism $\sigma : S \overset{\cong}{\to} M \otimes S$ that makes the diagram on the right commute.

$$\begin{array}{ccc} M \otimes S & \xrightarrow{M \otimes \varepsilon} & M \otimes X \\ \sigma \uparrow \cong & & \downarrow \chi \\ S & \xrightarrow{\ \varepsilon\ } & X \end{array}$$

Cond. (iii) says that S is a *fixed point* of $M \otimes (\_)$, i.e. of the combinatorial specification of the fractal. Conventionally, an attractor for an IFS is defined to be the unique non-empty *compact* fixed point; the corresponding restrictions can be found in Cond. (i) and (ii).

Theorem 5.6 *The coalgebraic fractal $\iota : I \to M \otimes I$ with its embedding $\llbracket \_ \rrbracket_{\chi}$ (Thm. 5.2) is an attractor. Moreover, it is a unique one up to a canonical isomorphism.* $\square$

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References

[1] Adámek, J., *Final coalgebras are ideal completions of initial algebras*, Journ. of Logic and Computation **12(2)** (2002), pp. 217–242.

[2] America, P. and J. J. M. M. Rutten, *Solving reflexive domain equations in a category of complete metric spaces*, Journ. Comp. Syst. Sci **39(3)** (1989), pp. 343–375.

[3] Barr, M., *Terminal coalgebras in well-founded set theory*, Theor. Comp. Sci. **114(2)** (1993), pp. 299–315, corrigendum in *Theor. Comp. Sci.* 124:189–192, 1994.

[4] Bénabou, J., *Distributors at work*, Lecture notes taken by T. Streicher (2000), www.mathematik.tu-darmstadt.de/~streicher/FIBR/DiWo.pdf.gz.

[5]   Capretta, V., T. Uustalu and V. Vene, *Corecursive algebras: A study of general structured corecursion*, in: M. V. M. Oliveira and J. Woodcock, editors, *SBMF*, Lecture Notes in Computer Science **5902** (2009), pp. 84–100.

[6]   Edalat, A., *Dynamical systems, measures and fractals via domain theory*, Inf. & Comp. **120** (1995), pp. 32–48.

[7]   Edalat, A. and R. Heckmann, *Computing with real numbers.*, in: G. Barthe, P. Dybjer, L. Pinto and J. Saraiva, editors, *Applied Semantics, International Summer School, APPSEM 2000, Caminha, Portugal, September 9–15, 2000, Advanced Lectures*, Lect. Notes Comp. Sci. **2395** (2002), pp. 193–267.

[8]   Escardó, M. H. and A. K. Simpson, *A universal characterization of the closed Euclidean interval*, in: *Logic in Computer Science* (2001), pp. 115–128.

[9]   Fiore, M. P., "Axiomatic Domain Theory in Categories of Partial Maps," Distinguished Dissertations in Computer Science, Cambridge Univ. Press, 1996.

[10] Hasuo, I., B. Jacobs and M. Niqui, *Coalgebraic representation theory of fractals*, Including an appendix, http://www.kurims.kyoto-u.ac.jp/~ichiro/ (2010).

[11] Hasuo, I., B. Jacobs and A. Sokolova, *Generic trace semantics via coinduction*, Logical Methods in Comp. Sci. **3** (2007).

[12] Hayashi, S., *Self-similar sets as Tarski's fixed points*, Publ. Res. Inst. Math. Sci. **21** (1985), pp. 1059–1066.

[13] Hutchinson, J., *Fractals and self-similarity*, Indiana Univ. Math. J. **30** (1981), pp. 713–747.

[14] Khamsi, M. A. and W. A. Kirk, "An Introduction to Metric Spaces and Fixed Point Theory," Wiley-Interscience, 2001.

[15] Kreitz, C. and K. Weihrauch, *Theory of representations*, Theor. Comp. Sci. **38** (1985), pp. 35–53.

[16] Leinster, T., *A general theory of self-similarity I* (2004), arXiv:math/0411344v1.

[17] Leinster, T., *A general theory of self-similarity II: Recognition* (2004), arXiv:math/0411345.

[18] Mac Lane, S. and I. Moerdijk, "Sheaves in Geometry and Logic. A First Introduction to Topos Theory," Springer, New York, 1992.

[19] Mandelbrot, B., "The Fractal Geometry of Nature," W.H. Freeman and Company, 1982.

[20] Pavlović, D. and V. Pratt, *The continuum as a final coalgebra*, Theor. Comp. Sci. **280 (1-2)** (2002), pp. 105–122.

[21] Peruggia, M., "Discrete Iterated Function Systems," A.K. Peters, 1993.

[22] Rutten, J. J. M. M., *Universal coalgebra: a theory of systems*, Theor. Comp. Sci. **249** (2000), pp. 3–80.

[23] Scriven, A., *A functional algorithm for exact real integration with invariant measures*, Elect. Notes in Theor. Comp. Sci. **218** (2008), pp. 337–353.

[24] Smyth, M. B. and G. D. Plotkin, *The category theoretic solution of recursive domain equations*, SIAM Journ. Comput. **11** (1982), pp. 761–783.

[25] Tsuiki, H., *Real number computation through Gray code embedding*, Theor. Comput. Sci. **284** (2002), pp. 467–485.

[26] Weihrauch, K., *Type 2 recursion theory*, Theor. Comp. Sci. **38** (1985), pp. 17–33.