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# Supermetrics over Apartness Lattice-Ordered Semigroup<sup>1</sup>

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## Abstract

We define the notion of supermetrics over apartness lattice-ordered semigroups, present the relationship between supermetrics and additive functions, and prove several conditions of metric completeness.

*Keywords:* set-set apartness, lattice-ordered semigroup, metrics, completion.

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## 1 Introduction

According to the set-set apartness relation introduced in [12],  $A$  and  $B$  are apart if their intersection is an empty set. We considering a weaker form of (set-set) apartness where two sets  $A$  and  $B$  have common elements, but these elements are not considered when we define the apartness relation between  $A$  and  $B$ .

Let  $\mathcal{C}$  be a family of subsets of a nonempty set  $X$  satisfying the following axioms:

$$\begin{aligned}A \cup B &\in \mathcal{C}, \text{ for all } A, B \in \mathcal{C}; \\ A \setminus B &\in \mathcal{C}, \text{ for all } A, B \in \mathcal{C}.\end{aligned}$$

From these axioms it follows that  $A \cap B \in \mathcal{C}$ , and  $A \triangle B \in \mathcal{C}$  for all  $A, B \in \mathcal{C}$ ; moreover  $\emptyset \in \mathcal{C}$ . Therefore  $(\mathcal{C}, \triangle, \emptyset)$  is a monoid, and  $(\mathcal{C}, \cup, \cap, \emptyset)$  is a lattice with the least element  $\emptyset$ . Additionally, we have  $A \triangle B = (A \cup B) \triangle (A \cap B)$  for all  $A, B \in \mathcal{C}$ .

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These facts inspire us to define a new structure called apartness lattice-ordered semigroup  $(S, \cdot, \vee, \wedge, u)$ . In such a structure we impose the distributivity of  $\cdot$  with respect to  $\vee$  and  $\wedge$ . The distributivity properties are not satisfied by the above example where we only have  $A\Delta(B \cup C) \subseteq (A\Delta B) \cup (A\Delta C)$  and  $A\Delta(B \cap C) \supseteq (A\Delta B) \cap (A\Delta C)$ . However we can find such properties in the structure of natural number set and in divisibility theory, to mention only few of the examples presented in Section 2.

Let  $\mu : \mathcal{C} \rightarrow \mathbb{R}_+$  be an additive function on  $\mathcal{C}$ . Then the function  $d : \mathcal{C} \times \mathcal{C} \rightarrow \mathbb{R}_+$  defined by  $d(A, B) = \mu(A\Delta B)$  is a pseudo-metric on  $\mathcal{C}$ .  $\mu$  is a metric whenever  $\mu(A) = 0$  iff  $A = \emptyset$ . Since  $\mu$  is an additive function, then it is (increasing) monotone. Moreover, we have  $d(A, B) = \mu(A \cup B) - \mu(A \cap B)$ . Considering the structure of apartness lattice-ordered semigroup  $(S, \cdot, \vee, \wedge, u)$  and an increasing monotone function  $f : S \rightarrow \mathbb{R}$ , it is possible to define a function  $d : S \times S \rightarrow \mathbb{R}_+$  by  $d(x, y) = f(x \vee y) - f(x \wedge y)$ . This function  $d$  has several nice properties, and we call it a supermetric. We study some of these properties, including several conditions related to metric completeness.

The structure of the paper is as follows. Section 2 presents the apartness lattice-ordered semigroups. Section 3 defines the supermetrics over this structure, and it provides the main novel concept of the paper. In Section 4 we present the relationship between supermetrics and additive functions. Some results related to the complete apartness lattice-ordered semigroups and to their completions are presented in Sections 5 and 6.

## 2 Apartness lattice-ordered semigroups

**Definition 2.1** An apartness lattice-ordered semigroup (alo-semigroup) is a particular lattice-ordered semigroup system  $(S, \cdot, \vee, \wedge, u)$ , where

1.  $(S, \cdot, u)$  is a semigroup with unit  $u$  (monoid);
2.  $(S, \vee, \wedge, u)$  is a lattice with the least element  $u$ ; if  $S$  has a zero  $z$ ,  $z \neq u$  and  $z$  is the greatest element;
3. for every  $a, b, c \in S$  we have
  - (i)  $ab = (a \vee b)(a \wedge b)$
  - (ii)  $a(b \vee c) = ab \vee ac$
  - (iii)  $a(b \wedge c) = ab \wedge ac$

The usual order in an alo-semigroup  $(S, \cdot, \vee, \wedge, u)$  is the order induced by its lattice:  

$$x \leq y \text{ iff } x = x \wedge y.$$

Let  $S$  be an alo-semigroup and  $R \subseteq S$ .  $R$  is a sub alo-semigroup of  $S$  if  $R$  is an alo-semigroup with respect to the operations of  $S$  restricted to  $R$ , and if  $R$  (as a lattice) is a sublattice of  $S$ .

We give some examples of alo-semigroups:

1.  $S = \{f \mid f : [0, 1] \rightarrow [0, 1]\}$ , where  $(f \wedge g)(x) = \max(f(x), g(x))$ ,  $(f \vee g)(x) =$

$\min(f(x), g(x))$  and  $(fg)(x) = f(x)g(x)$  for every  $x \in [0, 1]$ .  $u$  and  $z$  are the constant functions 1 and 0.

2. All the distributive lattices with  $z$  and  $u < z$  where  $ab = a \vee b$ .
3. Let  $\mathcal{C}$  be a family of subsets of  $X$  closed under finite union and intersection, and containing  $X$ . Then  $(\mathcal{C}, \cup, \cap, \cup, \emptyset)$  is an alo-semigroup where  $z = X$ .
4.  $(\mathbb{N}^*, \cdot, \vee, \wedge, 1)$ , where  $\cdot$  is the usual multiplication over the natural numbers,  $x \vee y = \text{lcm}(x, y)$  and  $x \wedge y = \text{gcd}(x, y)$ . In this case  $x \leq y$  whenever  $y$  is a multiple of  $x$ . Hence, the order induced by the subjacent lattice of this alo-semigroup is the same with the "natural" preorder induced by the semigroup operation (i.e.  $x \sqsubseteq y$  iff there is  $t \in S$  such that  $y = xt$ ).
5. The real numbers  $x \geq 1$  together with a special symbol  $\infty$ , considering the usual multiplication and order over the real numbers, and  $z = \infty$ .
6.  $([0, 1], \cdot, \preceq)$ , considering the usual multiplication over the real numbers, and  $x \preceq y \Leftrightarrow y \leq x$ .
7. Let  $(S, \cdot, \vee, \wedge, u)$  be an alo-semigroup, and  $X \neq \emptyset$ . We define  $S^X = \{f \mid f : X \rightarrow S\}$  together with the following operations over  $S^X$ :  $\forall f, g \in S^X$ ,  $(f \cdot g)(x) = f(x) \cdot g(x)$ ,  $(f \vee g)(x) = f(x) \vee g(x)$ ,  $(f \wedge g)(x) = f(x) \wedge g(x)$ , and  $u : X \rightarrow S$ ,  $u(x) = u$ ,  $\forall x \in X$ . We can define  $z : X \rightarrow S$  by  $z(x) = z$ , where  $z$  is zero in  $S$ . Then  $(S^X, \cdot, \vee, \wedge, u)$  is an alo-semigroup. As a particular case, if  $X = \mathbb{N}$ , then  $S^{\mathbb{N}} = \{(x_n) \mid (x_n) \text{ is a sequence of elements of } S\}$ . The structure  $(S^{\mathbb{N}}, \cdot, \vee, \wedge, u)$  is an alo-semigroup. Moreover, for  $X = \{1, 2, \dots, n\}$  we get  $S^X = S^n$ . It results that  $(S^n, \cdot, \vee, \wedge, u)$  is an alo-semigroup.

For each  $f \in S^X$ , the support of  $f$  is  $\text{supp}(f) = \{x \in X \mid f(x) \neq u\}$ . Let  $E(X, S) = \{f \in S^X \mid \text{supp}(f) \text{ is finite}\}$ . Since for all  $f, g \in E(X, S)$  we have  $f \cdot g, f \vee g, f \wedge g \in E(X, S)$ , then  $E(X, S)$  is a sub alo-semigroup.

### 3 Supermetrics over alo-semigroups

Let  $(S, \cdot, \vee, \wedge, u)$  be an alo-semigroup. We define a specific metric over  $S$  having similar properties to a module function over  $\mathbb{R}$ .

**Definition 3.1** A function  $d : S \times S \rightarrow [0, \infty)$  is a supermetric over  $S$  iff it satisfies the following conditions:

- (i)  $d(xw, yw) = d(x \vee y, u) - d(x \wedge y, u)$ ,  $\forall x, y, w \in S$ , (PSM)
- (ii) if  $d(x, y) = 0$ , then  $x = y$ .

If  $d : S \times S \rightarrow [0, \infty)$  satisfies only the first condition, then  $d$  is called a pseudo-supermetric (this is the reason why we denote the first condition by PSM). We present some elementary properties of a pseudo-supermetric.

**Proposition 3.2** If  $d : S \times S \rightarrow [0, \infty)$  is a pseudo-supermetric, then we have:

- (i)  $d(xw, yw) = d(x, y)$ , for all  $x, y, w \in S$ ;
- (ii)  $d(xy, y) = d(x, u)$ , for all  $x, y \in S$ ;

- (iii)  $d(x, y) = d(y, x)$ , for all  $x, y \in S$ ;
- (iv)  $d(xy, u) = d(x, u) + d(y, u)$ , for all  $x, y \in S$ ;
- (v)  $d(x \vee y, x \wedge y) = d(x, y)$ , for all  $x, y \in S$ ;
- (vi)  $d(x, y) = d(x \vee y, y) + d(x \wedge y, y)$ , for all  $x, y \in S$ ;
- (vii)  $x < y$  implies  $d(x, u) \leq d(y, u)$ ;  
moreover, if  $d$  is a supermetric, then  $x < y$  implies  $d(x, u) < d(y, u)$ ;
- (viii)  $E(S) \subseteq \{x \in S \mid d(x, u) = 0\}$ ; if  $d$  is a supermetric, then  $E(S) = \{u\}$ .

**Proposition 3.3** *If  $d : S \times S \rightarrow [0, \infty)$  is a pseudo-supermetric over  $S$ , then we have  $d(x, u) + d(y, u) = d(x \vee y, u) + d(x \wedge y, u)$ , and  $d(x, x) = 0$  for all  $x, y \in S$ .*

**Proof.** Let  $x, y \in S$ . According to axiom 3(i) and Proposition 3.2(iv), we get  $d(x, u) + d(y, u) = d(xy, u) = d((x \vee y)(x \wedge y), u) = d(x \vee y, u) + d(x \wedge y, u)$ .  $\square$

**Corollary 3.4** *If  $d : S \times S \rightarrow [0, \infty)$  is a pseudo-supermetric over  $S$ , then we have  $d(x, y) = d(x, u) + d(y, u) - 2d(x \wedge y, u) = 2d(x \vee y, u) - d(x, u) - d(y, u)$ , for all  $x, y \in S$ .*

It is easy to see that each pseudo-supermetric induces a valuation over alo-semigroup  $S$ . Recall that if  $(L, \vee, \wedge)$  is a lattice, then  $v : (L, \vee, \wedge) \rightarrow (R, +)$  is a valuation if  $v(a \vee b) + v(a \wedge b) = v(a) + v(b)$ , for all  $a, b \in L$ . Moreover, if  $v$  is strictly increasing, then  $(L, \vee, \wedge)$  is modular [2]. Therefore, if we have a supermetric over an alo-semigroup, then the underlying lattice is modular. Moreover, we can prove that an alo-semigroup having a supermetric is a distributive lattice.

**Proposition 3.5** *If we have a supermetric  $d$  over  $(S, \cdot, \vee, \wedge, u)$ , then the underlying semigroup  $(S, \cdot)$  is cancellative (i.e.,  $xa = ya$  implies  $x = y$ ).*

**Proof.** If  $xa = ya$ , then  $d(xa, ya) = 0$ . However  $d(xa, ya) = d(x, y)$ . Since  $d(x, y) = 0$ , then  $x = y$ .  $\square$

**Corollary 3.6** *If we have a finite alo-semigroup with a supermetric, then its underlying semigroup is a commutative group.*

**Theorem 3.7** *If we have a supermetric  $d$  over  $S$ , then the underlying lattice  $(S, \vee, \wedge)$  is distributive.*

**Proof.** According to an equivalent formulation, a lattice  $S$  is distributive if and only if  $x \vee a = y \vee a$  and  $x \wedge a = y \wedge a$  implies  $x = y$ , for all elements  $x, y, a \in S$ . According to axiom 3(i), if  $x \vee a = y \vee a$  and  $x \wedge a = y \wedge a$ , then  $xa = (x \vee a) \cdot (x \wedge a) = (y \vee a) \cdot (y \wedge a) = ya$ . We get  $x = y$  by Proposition 3.5.  $\square$

**Proposition 3.8** *If  $d : S \times S \rightarrow [0, \infty)$  is a pseudo-supermetric, then for all  $x, y, w \in S$  we have*

- (i)  $d(xw \vee yw, u) \leq d(x \vee w, u) + d(w \vee y, u)$ , and
- (ii)  $d(xw \wedge yw, u) \geq d(x \wedge w, u) + d(w \wedge y, u)$ .

**Proposition 3.9** *Every pseudo-supermetric is a pseudo-metric, and every supermetric is a metric.*

**Proof.** Let  $d$  be a pseudo-supermetric over  $S$ . Then PSM implies  $d(x, x) = 0$  for all  $x \in S$ . Also,  $d(x, y) = d(y, x)$  for all  $x, y \in S$ . Let  $x, y, w \in S$ . By Proposition 3.8(i) we have:  $d(xw \vee yw, u) \leq d(x \vee w, u) + d(w \vee y, u)$ . By Proposition 3.2(iv) and axiom 3(ii), we get  $d(x \vee y, u) + d(w, u) \leq d(x \vee w, u) + d(w \vee y, u)$ . Multiplying by 2, we get an equivalent inequality  $2d(x \vee y, u) - d(x, u) - d(y, u) \leq [2d(x \vee w, u) - d(x, u) - d(w, u)] + [2d(y \vee w, u) - d(y, u) - d(w, u)]$ . According to Corollary 3.4, we get  $d(x, y) \leq d(x, w) + d(w, y)$ . Therefore,  $d$  is a pseudo-metric on  $S$ . Moreover, if  $d$  is a supermetric, then  $d(x, y) = 0$  implies  $x = y$ , and so  $d$  is a metric.  $\square$

**Proposition 3.10** *If  $d : S \times S \rightarrow [0, \infty)$  is a pseudo-supermetric, then*

$$\max\{d(x \vee w, y \vee w), d(x \wedge w, y \wedge w)\} \leq d(x, y) \text{ for all } x, y, w \in S.$$

Proposition 3.10 implies that the lattice operations are uniformly continuous [10]. On the other hand, every pseudo-supermetric is invariant with respect to the semigroup operation, and so this operation is also uniformly continuous. Consequently, any alo-semigroup with a pseudo-supermetric is a uniform alo-semigroup. If  $d$  is a supermetric over an alo-semigroup  $S$ , the continuity of the lattice operations ensure the compatibility among the lattice order and the convergence generated by  $d$ .

**Proposition 3.11** *If  $d$  is a supermetric over  $S$ , and  $\{a_n\}, \{b_n\} \subset S$  are two sequences such that  $a_n \leq b_n$  for all  $n \in \mathbb{N}$ , and  $a_n \xrightarrow{d} a$  and  $b_n \xrightarrow{d} b$ , then  $a \leq b$ .*

**Proof.** Since  $a_n \leq b_n$  for all  $n \in \mathbb{N}$ , then we have  $a_n = a_n \wedge b_n$  for all  $n \in \mathbb{N}$ . Since  $a_n \xrightarrow{d} a$ ,  $b_n \xrightarrow{d} b$  and the function  $(x, y) \rightarrow x \wedge y$  is continuous, then we have that  $a_n \wedge b_n \xrightarrow{d} a \wedge b$ . On the other hand,  $a_n \wedge b_n = a_n \xrightarrow{d} a$ . Since  $d$  is a metric, then the limit of a sequence is unique, and so  $a = a \wedge b$ . Therefore  $a \leq b$ .  $\square$

**Corollary 3.12** *If  $d$  is a supermetric over  $S$ , then every interval  $[a, b]$  is a  $d$ -closed set for all  $a, b \in S$  with  $a \leq b$ .*

**Proposition 3.13** *If  $(x_n) \subset S$  is  $d$ -Cauchy, then  $\{d(x_n \vee x, u)\}$  and  $\{d(x_n \wedge x, u)\}$  are convergent for all  $x \in S$ .*

**Proof.** Since  $|d(x_n \vee x, u) - d(x_m \vee x, u)| \leq d(x_n \vee x, x_m \vee x) \leq d(x_n, x_m)$  and  $|d(x_n \wedge x, u) - d(x_m \wedge x, u)| \leq d(x_n \wedge x, x_m \wedge x) \leq d(x_n, x_m)$  for all  $n, m \in \mathbb{N}$ , it follows that  $\{d(x_n \vee x, u)\}$  and  $\{d(x_n \wedge x, u)\}$  are Cauchy, and so convergent for all  $x \in S$ .  $\square$

## 4 Additive functions over alo-semigroups

In this section we present a one-to-one correspondence between supermetrics over alo-semigroups and strict increasing additive functions. Let  $(S, \cdot, \vee, \wedge, u)$  be an alo-semigroup. A function  $f : S \rightarrow \mathbb{R}$  is called additive if  $f(xy) = f(x) + f(y)$  for all  $x, y \in S$ . It is obvious that  $f(u) = 0$  for every additive function  $f$ . Therefore, if

$f : S \rightarrow \mathbb{R}$  is additive and increasing, then  $f$  is positive. Let  $\mathcal{PSM}$  be the set of all pseudo-supermetrics over  $S$ ,  $\mathcal{SM}$  be the set of all supermetrics over  $S$ ,  $\mathcal{F}$  be the set of all increasing additive functions over  $S$ , and  $\mathcal{F}_s$  the set of all strict increasing additive functions over  $S$ .

Since  $\lambda f_1$  and  $f_1 + f_2$  are in  $\mathcal{F}$  for all  $\lambda > 0$  and  $f_1, f_2 \in \mathcal{F}$ , then  $\mathcal{F}$  is a cone [2]. In a similar way, we get that  $\mathcal{F}_s, \mathcal{PSM}$  and  $\mathcal{SM}$  are cones. Let  $f : S \rightarrow [0, \infty)$  and  $d_f : S \times S \rightarrow [0, \infty)$  defined by  $d_f(x, y) = f(x \vee y) - f(x \wedge y)$ .

**Proposition 4.1** *If  $f \in \mathcal{F}$ , then  $d_f \in \mathcal{PSM}$  and  $f(x) = d_f(x, u)$  for all  $x \in S$ . Moreover, if  $x, y \in S$  are comparable, then  $d_f(x, y) = |f(x) - f(y)|$ .*

**Proof.** Since  $f$  is increasing, we have  $d_f(x, y) \geq 0$ . Let  $x, y, w \in S$ . Then  $d_f(xw, yw) = f(xw \vee yw) - f(xw \wedge yw) = f((x \vee y)w) - f((x \wedge y)w) = f(x \vee y) + f(w) - f(x \wedge y) - f(w) = d_f(x, y)$ . On the other hand, we have  $d_f(x, u) = f(x \vee u) - f(x \wedge u) = f(x) - f(u) = f(x)$ . Therefore  $d_f(xw, yw) = d_f(x, y) = d_f(x \vee y, u) - d_f(x \wedge y, u)$ , for all  $x, y, w \in S$ , i.e.,  $d_f$  satisfies PSM, and so it is a pseudo-supermetric.  $\square$

**Theorem 4.2**  $\mathcal{F} \simeq \mathcal{PSM}$ , i.e. there is a linear bijective function  $\Phi : \mathcal{F} \rightarrow \mathcal{PSM}$ .

**Proof.** Let  $\Phi : \mathcal{F} \rightarrow \mathcal{PSM}$  defined by  $\Phi(f) = d_f$  for all  $f \in \mathcal{F}$ . According to Proposition 4.1,  $\Phi$  is well-defined. If  $\Phi(f_1) = \Phi(f_2)$ , then  $f_1(x) = d_{f_1}(x, u) = d_{f_2}(x, u) = f_2(x)$  for all  $x \in S$ . Thus  $\Phi$  is one-to-one. Let  $d \in \mathcal{PSM}$  and let  $f : S \rightarrow [0, \infty)$  defined by  $f(x) = d(x, u)$  for all  $x \in S$ .  $f \in \mathcal{F}$  by Proposition 3.2. Also  $d_f(x, y) = f(x \vee y) - f(x \wedge y) = d(x \vee y, u) - d(x \wedge y, u) = d(x, y)$  for all  $x, y \in S$ . Hence  $\Phi(f) = d$ , and  $\Phi$  is onto.

Let  $\lambda > 0$  and  $f_1, f_2 \in \mathcal{F}$ . Since  $d_{f_1+f_2} = d_{f_1} + d_{f_2}$  and  $d_{\lambda f} = \lambda d_f$ , then  $\Phi$  is linear (i.e. it is additive and positive homogeneously).  $\square$

We can prove similarly the following result:

**Theorem 4.3**  $\mathcal{F}_s \simeq \mathcal{SM}$ .

As a consequence, we can reformulate Proposition 3.5 and Theorem 3.7.

**Theorem 4.4** *If  $\mathcal{F}_s \neq \emptyset$ , then the semigroup  $(S, \cdot)$  is cancellative, and the lattice  $(S, \vee, \wedge)$  is distributive.*

Let  $(\mathbb{N}^*, \cdot, \vee, \wedge, 1)$ , where  $\cdot$  is the usual multiplication,  $x \vee y = \text{lcm}(x, y)$  and  $x \wedge y = \text{gcd}(x, y)$ . Since the function  $f : \mathbb{N}^* \rightarrow [0, \infty)$ ,  $f(n) = \ln(n)$  is additive and strict increasing, it follows that  $d_f(n, m) = \ln \frac{\text{lcm}(n, m)}{\text{gcd}(n, m)}$  is a supermetric on  $\mathbb{N}^*$ , and the lattice  $(\mathbb{N}^*, \vee, \wedge)$  is distributive.

**Proposition 4.5** *If  $f \in \mathcal{F}$ , then  $f$  is  $d_f$ -uniformly continuous.*

**Proof.** Since  $x \wedge y \leq x$ ,  $y \leq x \vee y$  and  $f$  is increasing, we get the inequality  $|f(x) - f(y)| \leq d_f(x, y)$  for all  $x, y \in S$ .  $\square$

**Proposition 4.6** *If  $\mathcal{F}_s \neq \emptyset$ , then  $x^2 \wedge y^2 \leq xy \leq x^2 \vee y^2$  for all  $x, y \in S$ .*

**Proof.** Let  $x, y \in S$ . Since  $\mathcal{F}_s \neq \emptyset$ , the lattice  $(\mathbb{N}^*, \vee, \wedge)$  is distributive, and then  $(x^2 \vee y^2) \wedge xy = (x^2 \wedge xy) \vee (y^2 \wedge xy) = x(x \wedge y) \vee y(y \wedge x) = (x \vee y)(x \wedge y) = xy$ . Then  $xy \leq x^2 \vee y^2$ . Analogous, we prove  $x^2 \wedge y^2 \leq xy$ .  $\square$

**Proposition 4.7** *If  $\mathcal{F}_s \neq \emptyset$ , then  $(x \wedge y)^n = x^n \wedge y^n$  and  $(x \vee y)^n = x^n \vee y^n$  for all  $x, y \in S$  and  $n \in \mathbb{N}^*$ .*

## 5 Complete alo-semigroups

Let  $(S, \cdot, \vee, \wedge, u)$  be an alo-semigroup and  $f \in \mathcal{F}_s$ .

We consider the following assertions:

**Conditional Complete condition:**

(CC) Every non-empty upper bounded subset of  $S$  has a supremum.

**$\sigma$ -Complete condition:**

( $\sigma$ C) Every ascending sequence of  $S$  has a supremum, and every descending sequence of  $S$  has an infimum.

**Metric Complete condition:**

(MC) Every  $d_f$ -Cauchy sequence is  $d_f$ -convergent.

**Ascending Sequences Convergence condition:**

(ASC) If  $\{x_n\} \subseteq S$  is an upper bounded ascending sequence, then there is  $x = \bigvee_n x_n$  and  $x_n \xrightarrow{d_f} x$ .

**Cantor condition:**

(CA) If  $\{[a_n, b_n]\}$  is a descending sequence of intervals of  $S$  such that  $d_f(a_n, b_n) \rightarrow 0$ , then there exists  $c \in S$  such that  $\bigcap_n [a_n, b_n] = \{c\}$ .

**Bounded to Compact condition:**

(B2C) If  $A \subset S$  is  $d_f$ -closed and upper bounded, then  $A$  is  $d_f$ -compact.

**Sequential Scott Continuity condition:**

(SSC) If  $\{x_n\} \subseteq S$  is an ascending sequence such that there is  $x = \bigvee_n x_n$ , then  $f(x) = \bigvee_n f(x_n)$ .

**Scott Continuity condition:**

(SC) If  $D$  is a directed subset of  $S$  such that there is  $x = \bigvee D$ , then  $f(x) = \bigvee f(D)$ .

**Hard Scott Continuity condition:**

(HSC) For every subset  $A$  of  $S$  such that there is  $x = \bigvee A$ , then  $f(x) = \bigvee f(A)$ .

**Directed Bounded condition:**

(DB) For every directed subset  $D$  of  $S$ , there is an ascending sequence of elements of  $D$  which has the same upper bounds as  $D$ .

By [2] (Chap.V, Th.15), we have that (MC) implies (CC) & (ASC). By the same theorem, we also have that ( $\sigma$ C) & (ASC) implies (MC). However, if  $\mathcal{F} \neq \emptyset$ , then ( $\sigma$ C) is false. Indeed, if  $f \in \mathcal{F}$  and ( $\sigma$ C) is true, then the sequence  $\{x^n\}$  is ascending

for any  $x \in S$ ,  $x \neq u$ . Then, according to  $(\sigma C)$ , there exists  $y \in S$  such that  $x^n \leq y$  for all  $n \in \mathbb{N}$ . Therefore,  $nf(x) = f(x^n) \leq f(y)$  for all  $n \in \mathbb{N}$ ; contradiction.

It is obviously that  $(HSC)$  implies  $(SC)$  and  $(SC)$  implies  $(SSC)$ .

The following theorems express sufficient conditions for  $(MC)$ .

**Theorem 5.1** *Let  $f \in \mathcal{F}_s$ . Then we have:*

- (i)  $(CC) \ \& \ (SSC)$  implies  $(ASC)$ ;
- (ii)  $(ASC)$  implies  $(SSC)$ ;
- (iii)  $(SSC) \ \& \ (DB)$  implies  $(SC)$ .

We do not use the hypothesis that  $D$  is a directed set. in the proof of “ $(SSC) \ \& \ (DB)$  implies  $(SC)$ ”. If we consider another condition, then we get a new result.

**Hard Bounded condition:**

(HB) For every subset  $D$  of  $S$ , there is an ascending sequence of elements of  $D$  which has the same upper bounds as  $D$ .

**Theorem 5.2**  $(SSC) \ \& \ (HB)$  implies  $(HSC)$ .

**Theorem 5.3** *Let  $f \in \mathcal{F}_s$  and  $A \subset S$  such that there is  $x = \vee A$ . Then  $f(x) = \vee f(A)$  if only if  $x \in cl_{d_f} A$ .*

**Theorem 5.4**  $(ASC)$  implies  $(CA)$ .

**Theorem 5.5**  $(CC) \ \& \ (HSC)$  implies  $(B2C)$ .

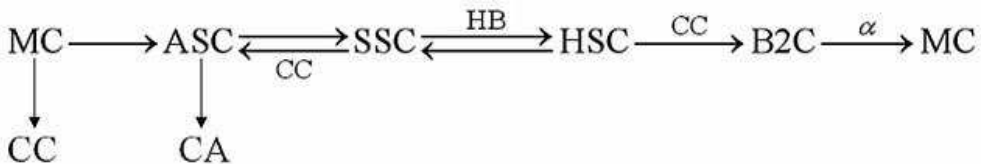
**Theorem 5.6** *If  $(B2C)$  and  $f^{-1}([0, \alpha])$  is upper bounded for all  $\alpha > 0$ , then  $(MC)$ .*

**Corollary 5.7** *If  $(CC)$ ,  $(HSC)$ , and  $f^{-1}([0, \alpha])$  is upper bounded for all  $\alpha > 0$ , then  $(MC)$ .*

We denote by  $\alpha$  the condition used in the previous theorem:

$(\alpha)$   $f^{-1}([0, \alpha])$  is upper bounded, for all  $\alpha > 0$ .

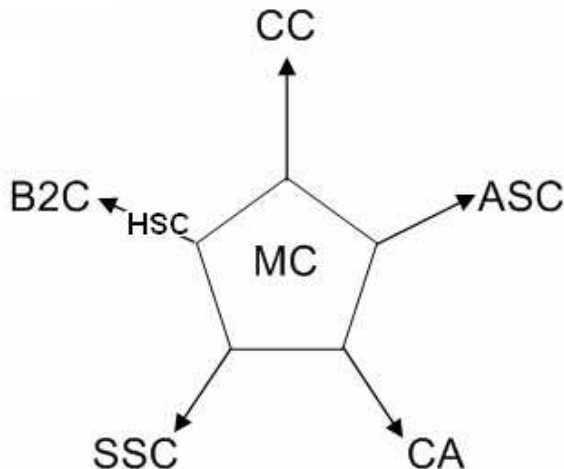
The results of this section can be summarized in the following diagram:



## 6 Completion of an alo-semigroup

If the metric space  $(S, d_f)$  is complete (condition MC is satisfied), we have several metric and algebraic properties similar to those of  $\mathbb{R}$ . They are graphically represented in the following diagram





An important property is (B2C); it is similar to a property which characterizes the finite-dimensional normed spaces. In our framework, the boundedness is given by the lattice order. On the other hand, we can get more results if we work with an alo-semigroup satisfying (MC). Starting from an arbitrary alo-semigroup  $S$  together with an increasing additive function  $f$  over  $S$ , and using a standard procedure of completion for metric spaces, we can build an alo-semigroup  $S_1$  and a strictly increasing additive function  $f_1$  over  $S_1$  such that the metric space  $(S_1, d_{f_1})$  satisfy (MC), and  $S$  is  $d_{f_1}$ -dense in  $S_1$ . If  $(S, \cdot, \vee, \wedge, u)$  is an alo-semigroup, and  $f \in \mathcal{F}_s$ , then the pair  $(S, f)$  is called an additive alo-semigroup.

**Definition 6.1** Let  $(S, f)$  be an additive alo-semigroup. An additive alo-semigroup  $(S_1, f_1)$  is called the completion of  $(S, f)$  iff

- (i)  $(S_1, f_1)$  satisfies (MC),
- (ii) there is an isomorphism of alo-semigroups  $\phi : S \rightarrow \phi(S) \subseteq S_1$ ,
- (iii)  $\phi(S)$  is  $d_{f_1}$ -dense in  $S_1$ .

**Theorem 6.2** Let  $(S, f)$  be an additive alo-semigroup. Then there is a completion  $(S_1, f_1)$  of  $(S, f)$ . Moreover, any two completions of  $(S, f)$  are isomorphic.

Let  $(S, \cdot, \vee, \wedge, u)$  be an alo-semigroup, and  $g : S \rightarrow S$  a function.  $x \in S$  is a fixed point for  $g$  if  $g(x) = x$ . We suppose that the conditions (CC) and (HB) are satisfied, and let  $\mathcal{F}'_s$  be the subset of  $\mathcal{F}_s$  which satisfy the conditions (SSC) and  $(\alpha)$ . Then, for each  $f \in \mathcal{F}'_s$ , the space  $(S, d_f)$  satisfies (MC). In fact, we can renounce to these four conditions if we work with the complete alo-semigroup of  $(S, f)$ . Using the Banach fixed-point principle, we get the following result:

**Theorem 6.3** If there are  $n, m \in \mathbb{N}^*$  such that  $m < n$  and  $[xy]^m [g(x)g(y)]^n \leq [x \vee y]^{2m} [g(x) \wedge g(y)]^{2n}$  for all  $x, y \in S$ , then  $x$  is a unique fixed point of  $g$ . Moreover, for any  $x_0 \in S$  and  $f \in \mathcal{F}'_s$ , the sequence  $x_n = g(x_{n-1})$  satisfies  $x_n \xrightarrow{d_f} x$ .

## 7 Conclusion

Paper [5] is devoted to the algebraic properties of a similar algebra with two underlying structures (monoid and lattice), establishing some interesting connections between these structures. Paper [6] is mainly devoted to the topological properties of the apartness lattice-ordered semigroups; we introduce and study the uniform apartness lattice-ordered semigroups, discuss about various quasi-metrics, and present several separation properties.

In this paper we define a supermetric over the set underlying an alo-semigroup. The properties which the distance function needs to satisfy are expressed in terms of the semigroup and lattice operations. Several results of these supermetrics are proved. We define the notion of a pseudo-supermetric over an alo-semigroup, and we characterize the supermetrics as the strict, increasing additive functions from such a structure to the reals. We investigate various notions of completions of our structures with respect to the introduced supermetrics, and we show that each structure endowed with a strict, increasing additive function (which corresponds to a unique supermetric) has a completion, and any two such completions are isomorphic.

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