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Comparing Topological Models for Concurrency

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Abstract

Several categories of models for concurrency involving topology have been put forward in each of which a notion of fundamental category is defined. One of them, the category of pospaces, is canonically included in almost all the others. Given a pospace $\overrightarrow{X}$ and $i(\overrightarrow{X})$, the image of $\overrightarrow{X}$ by the inclusion i of PoTop in some of the other category in which the fundamental category is defined, it is then natural to ask how the fundamental categories of $\overrightarrow{X}$ and $i(\overrightarrow{X})$ are related. The answer to this question is one of the purposes along of this article.
We introduce a general framework for categories in which a reasonable notion of fundamental categories can be defined.

Keywords: directed paths, directed homotopy, fundamental category, models for concurrency, topologically concrete category

1 Introduction

The original motivation for studying topological models of concurrency is the notion of progress graph introduced in 1968 by *Dijkstra* in his article [3]. The idea is that a PV programm, which consists on a finite set of processes each of which performing lock and release on semaphores can be represented by a geometrical shape equiped with an order relation. Let us examine the following PV programs:

$$P_cV_c|P_cV_c \quad P_aP_bV_bV_a|P_bP_aV_aV_b \text{ and } P_cP_aP_bV_bV_aV_c|P_cP_bP_aV_aV_bV_c$$

Each of these programs have 2 processes. The letters a, b, c denote semaphores of arity 1 i.e. that each of them can be simultaneously used by, at most, 1 process. If the next instruction to be performed by the process Π is P_a then it tries to "take" the semaphore a, then either a is "free" (so Π can take it) or it is not (because it has

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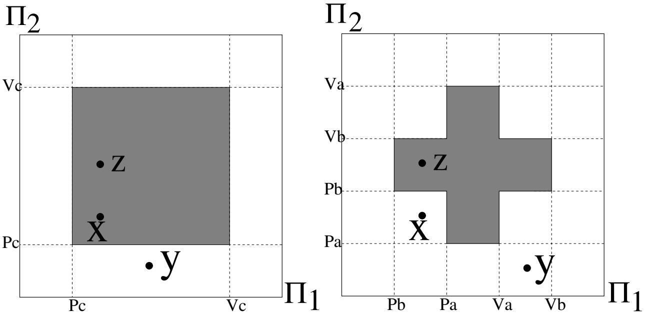
already been taken by another process). In the first case the process Π can perform P_a and goes to the next instruction, in the other one Π has to wait till the process which holds a releases it. The process that holds a can release it performing the instruction V_a. Then the collection of states can be represented as follows



The left hand picture is associated to the first and third program while the right hand one is associated to the second. For example, consider the point x on the first figure, it represents a state in which both processes have taken a which is impossible. Such points form region of the forbidden states. On the second picture, y is a state in which Π_1 has already performed $P_b P_a V_a$ while the Π_2 has not even execute its first instruction. We would like to distinguish these shapes. A careful examination of the third program shows that it has the same behaviour as the first one, but this fact becomes immediate when we observe their geometric models provided we have a theorem such as "equivalent geometric models implies same behaviour". We have moved the analysis of PV programs to the analysis oh their geometric models. From an Euclidean point of view, the models of our example are different, but the classification up to isometry is way too strong. On the other hand, the classification up to homotopy equivalence as in classical algebraic topology is too loose since it does not distinguish these geometric shapes. Then we equip our models with the partial order induced by the one of $\mathbb{R}^2$ and observe that the second one has a local maximum while the first one does not. This remark motivates the introduction of the notions of pospaces ([16]) and directed algebraic topology ([12],[8],[6]).

Now, we briefly recall some definitions, a general reference for the topological approach to concurrency is [12]. The category of *Hausdorff* spaces is denoted Haus. A **pospace** is a pair $(X, \leq_X)$ where X is a topological space and $\leq_X$ a partial order relation on $|X|$ (the underlying set of X) whose graph is closed in $X \times X$ See [16]. Together with the increasing continuous maps, they form a category denoted PoTop. Weakening the notion of pospace asking $\leq_X$ just be reflexive we obtain the **related spaces** which also form a category, denoted RTop, together with continuous maps such that $\forall x, x' \in X$ if x and x' are related then so are $f(x)$ and $f(x')$. For technical convenience, we also require that the underlying topological space of an object of RTop be *Hausdorff*. Originally, I have introduced them as a technical tool to prove that PoTop is cocomplete.

A **directed space**, see [10] and [9], is a pair (X, dX) where X is a *Hausdorff* topological space and dX is a family of paths on X containing all the constant paths, stable under concatenation and satisfying $\forall \theta : [0,1] \longrightarrow [0,1]$ continuous

and increasing, $\forall \gamma \in dX\ \gamma \circ \theta \in dX$.[2] Together with continuous maps f satisfying $\forall \gamma \in dX\ f \circ \gamma \in dY$, they form a category denoted dTop.
A **local pospace** is a topological space X together with an open covering V_i and a family of partial order $\leq_i$ on V_i such that $\forall i\ (V_i, \leq_i)$ is a pospace. The morphisms from $(X, V_i, \leq_i)$ to $(Y, W_j, \leq'_j)$ are the continuous maps from X to Y such that $\forall x \in X\ \forall j$ such that $f(x) \in W_j$, $\exists U \subseteq V_i$ (for some i) a neighborhood of x such that f induces a dimap from $(U, \leq_i |_U)$ to $(W_j, \leq'_j)$. Then we have a category denoted LPoTop. See [4].
Roughly speaking, the machinery we will introduce can be applied to any category whose objects are made of a (*Hausdorff*) topological space X equiped with some structure compatible with respect to the topology of X. In fact, the category of Flows introduced by *Philippe Gaucher* (see [7]) is the only one category of models for concurrency which is not topologically concrete (see definition 3.1) that I know of.

2 Category with paths

In classical algebraic topology, the unit segment $[0, 1]$ plays a crucial role. This is also the case in PoTop, LPoTop, dTop or RTop provided that it is equiped with the suitable structure, that is to say a structure that makes it directed. The notion of category with paths is based on this fact. Of course, in classical algabraic topology, the idea of using $[0, 1]$ as an elementary brick is not new and appears, for example, in the notion of path object, see [1] or [14].
Let C be a category with a terminal object, such an object is unique up to isomorphism, let us choose one of them and denote it $*$. A **point** of an object X of C is a morphism $p \in \mathsf{C}[*, X]$. In particular, given a point p of X and an object A of C, there is a unique morphism $f \in \mathsf{C}[A, X]$ such that $f = p \circ \zeta_A$ where ζ_A is the unique element of $\mathsf{C}[A, *]$. The morphism f we have described is called the **constant** morphism of value p from A to X. Thus, a morphism is said constant when it factorizes through the terminal object. The intention behind this definition becomes clear when it is particularized to Set. We also require and choose an object $\mathbb{I}$ of C, that will be called the **generic path** and two morphisms $s, t \in \mathsf{C}[*, \mathbb{I}]$ so that for any $\phi \in \mathsf{C}[\mathbb{I}, \mathbb{I}]$ isomorphism, we have

$$\Big((\phi \circ s = s \text{ and } \phi \circ t = t) \text{ or } (\phi \circ s = t \text{ and } \phi \circ t = s)\Big) \text{ i.e. } \{\phi \circ s, \phi \circ t\} = \{s, t\}$$

and so that $\forall n \in \mathbb{N}$, the following diagram

$$\underbrace{* \xrightarrow{s} \mathbb{I} \xleftarrow{t} * \xrightarrow{s} \mathbb{I} \xleftarrow{t} * \xrightarrow{s} \mathbb{I} \ \cdots\ \mathbb{I} \xleftarrow{t} * \xrightarrow{s} \mathbb{I} \xleftarrow{t} * \xrightarrow{s} \mathbb{I} \xleftarrow{t} *}_{n \text{ copies of } \mathbb{I}}$$

[2] in the original definition it is not required that the underling topological space be *Hausdorff*.

has a colimit in C. This colimit, unique up to isomorphism in C, is denoted $n \cdot \mathbb{I}$ together with

$$
\begin{tikzcd}[column sep=small]
 & & & & & & n\cdot\mathbb{I} & & & & & & \\
 & \mathbb{I} \arrow[urrrrr, dotted, "i_1^{(n)}"] & & \mathbb{I} \arrow[urrr, dotted, "i_2^{(n)}"] & & \mathbb{I} \arrow[ur, dotted, "i_3^{(n)}"] & \dots & \mathbb{I} \arrow[ul, dotted, "i_{n-2}^{(n)}"'] & & \mathbb{I} \arrow[ulll, dotted, "i_{n-1}^{(n)}"'] & & \mathbb{I} \arrow[ulllll, dotted, "i_n^{(n)}"'] & \\
* \arrow[ur, "s"'] & & * \arrow[ul, "t"] \arrow[ur, "s"'] & & * \arrow[ul, "t"] \arrow[ur, "s"'] & & & & * \arrow[ul, "t"] \arrow[ur, "s"'] & & * \arrow[ul, "t"] \arrow[ur, "s"'] & & * \arrow[ul, "t"]
\end{tikzcd}
$$

n copies of $\mathbb{I}$

Note that, in the preceeding diagram, the arrows from $*$ have been omitted, indeed, they have to make the diagram commutative so they are implicitly determined. The stability of $\{s, t\}$ under any automorphism of $\mathbb{I}$ is the categorical way to say that s and t are the extremities of $\mathbb{I}$. Besides, an automorphism that exchanges s and t can be thought of as a "time reversal". Notice that for any isomorphism $\phi \in \mathsf{C}[A, \mathbb{I}]$, A together with $\phi^{-1} \circ s$ and $\phi^{-1} \circ t$ can be taken as a generic object. Indeed, let $\psi \in \mathsf{C}[A, A]$ be an isomorphism, since $\phi \circ \psi \circ \phi^{-1}$ is an automorphism of $\mathbb{I}$, we have $\{\phi \circ \psi \circ \phi^{-1} \circ s, \phi \circ \psi \circ \phi^{-1} \circ t\} = \{s, t\}$ i.e. $\{\psi \circ \phi^{-1} \circ s, \psi \circ \phi^{-1} \circ t\} = \{\phi^{-1} \circ s, \phi^{-1} \circ t\}$. Moreover, given isomorphisms $\phi_1, \phi_2 \in \mathsf{C}[A, \mathbb{I}]$, $\phi_1^{-1} \circ \phi_2$ is an automorphism of A, hence $\{(\phi_1^{-1} \circ \phi_2) \circ \phi_2^{-1} \circ s, (\phi_1^{-1} \circ \phi_2) \circ \phi_2^{-1} \circ t\} = \{\phi_2^{-1} \circ s, \phi_2^{-1} \circ t\}$, so $\{\phi_1^{-1} \circ s, \phi_1^{-1} \circ t\} = \{\phi_2^{-1} \circ s, \phi_2^{-1} \circ t\}$. Hence, up to a "time reversal", $\underline{s \text{ and } t \text{ are entirely determined by the choice of } \mathbb{I}}$.
In fact, we cannot take any object of C as a generic path. For example, in Top, the Euclidean circle S^1 cannot be taken as a generic path object. Indeed, for any points x and y of S^1, there is an automorphism of S^1, for example a rotation, that respectively sends x and y onto x' and y' so that $\{x, y\} \cap \{x', y'\} = \emptyset$. On the other hand, any automorphism of [0,1] induces a 1-1 mapping from $\{0, 1\}$ to $\{0, 1\}$. The reason is that $\{0, 1\}$ is the boundary of $[0, 1]$.
We say that $\mathbb{I}$ provides a notion of **direction** to C when $\phi \circ s = s$ and $\phi \circ t = t$ for any automorphism ϕ of $\mathbb{I}$, otherwise, we say $\mathbb{I}$ provides a notion of **connection** to C.
The second hypothesis enables us to define a concatenation which is <u>strict</u> instead of up to isomorphism. To this end, we <u>choose</u> for each $n \in \mathbb{N}$ a cocone $(n \cdot \mathbb{I}, i_1^{(n)}, ..., i_n^{(n)})$ that represents the colimit of the diagram

$$
\begin{tikzcd}[column sep=small]
 & \mathbb{I} & & \mathbb{I} & & \mathbb{I} & \dots & \mathbb{I} & & \mathbb{I} & & \mathbb{I} & \\
* \arrow[ur, "s"'] & & * \arrow[ul, "t"] \arrow[ur, "s"'] & & * \arrow[ul, "t"] \arrow[ur, "s"'] & & & & * \arrow[ul, "t"] \arrow[ur, "s"'] & & * \arrow[ul, "t"] \arrow[ur, "s"'] & & * \arrow[ul, "t"]
\end{tikzcd}
$$

n copies of $\mathbb{I}$

In the rest of the paper, we will refer to the preceeding diagram as V_n. Moreover, for $n := 0$ we can suppose that $0 \cdot \mathbb{I} := *$ and for $n := 1$ that $1 \cdot \mathbb{I} := \mathbb{I}$ and $i_1^1 := id_{\mathbb{I}}$. By induction over $n \in \mathbb{N}$ choose the cocones $(n \cdot \mathbb{I}, i_1^{(n)}, ..., i_n^{(n)})$ so that if $n \cdot \mathbb{I} \cong p \cdot \mathbb{I}$ in C then $n \cdot \mathbb{I} = p \cdot \mathbb{I}$.

Lemma 2.1 (Monoid of paths) *Setting for all $n, p \in \mathbb{N}$ $(n\cdot\mathbb{I})+(p\cdot\mathbb{I}) := (n+p)\cdot\mathbb{I}$, we turn $\{n\cdot\mathbb{I}|n \in \mathbb{N}\}$ into a commutative monoid whose unit is $0\cdot\mathbb{I}$ i.e. $*$. Further, we have a morphism of monoids from $(\mathbb{N}, +, 0)$ onto $(\{n\cdot\mathbb{I}/n \in \mathbb{N}\}, +, *)$. Furthermore, if there are $n, p \in \mathbb{N}$ such that $n\cdot\mathbb{I} \neq p\cdot\mathbb{I}$ then $\{n\cdot\mathbb{I}/n \in \mathbb{N}\}$ is finite. Otherwise it is isomorphic to $\mathbb{N}$.*

Proof. Left to the reader. $\square$

We can always take $\mathbb{I} := *$ as a generic path, making the monoid of paths trivial. For any $n \in \mathbb{N}$, we define $s^{(n)} := i_1^{(n)} \circ s$ and $t^{(n)} := i_n^{(n)} \circ t$. Then, using the universal property of colimits, for any pair of integers (n, p) we uniquely define $g_n^{(n+p)} \in \mathsf{C}[n\cdot\mathbb{I}, (n+p)\cdot\mathbb{I}]$ and $d_p^{(n+p)} \in \mathsf{C}[p\cdot\mathbb{I}, (n+p)\cdot\mathbb{I}]$ so that

$$\begin{cases} g_n^{(n+p)} \circ i_k^{(n)} = i_k^{(n+p)} \text{ for every } k \in \{1, ..., n\} \\ d_p^{(n+p)} \circ i_k^{(p)} = i_{n+k}^{(n+p)} \text{ for every } k \in \{1, ..., p\} \end{cases}$$

In particular, $g_0^{(n)} = i_1^{(n)} \circ s = s^{(n)}$, $d_0^{(n)} = i_n^{(n)} \circ t = t^{(n)}$ and $g_n^{(n)} = d_n^{(n)} = id_{n\cdot\mathbb{I}}$. Furthermore:

Proposition 2.2 *For all $n, p \in \mathbb{N}$*

$$\begin{tikzcd} & (n+p)\cdot\mathbb{I} & \\ n\cdot\mathbb{I} \arrow[ur, "g_n^{(n+p)}"] & & p\cdot\mathbb{I} \arrow[ul, "d_p^{(n+p)}"'] \\ & * \arrow[ul, "t^{(n)}"] \arrow[ur, "s^{(p)}"'] & \end{tikzcd}$$

is a push-out in C. Moreover, if $\alpha \in \mathsf{C}[n\cdot\mathbb{I}, X]$ and $\beta \in \mathsf{C}[p\cdot\mathbb{I}, X]$ satisfy $\alpha \circ t^{(n)} = \beta \circ s^{(p)}$ then the unique morphism $h \in \mathsf{C}[(n+p)\cdot\mathbb{I}]$ such that

$$\begin{tikzcd} & X & \\ & (n+p)\cdot\mathbb{I} \arrow[u, "h"] & \\ n\cdot\mathbb{I} \arrow[ur, "g_n^{(n+p)}"] \arrow[uur, bend left, "\alpha"] & & p\cdot\mathbb{I} \arrow[ul, "d_p^{(n+p)}"'] \arrow[uul, bend right, "\beta"'] \\ & * \arrow[ul, "t^{(n)}"] \arrow[ur, "s^{(p)}"'] & \end{tikzcd}$$

is also the unique h such that

$$\begin{cases} \alpha \circ i_k^{(n)} = h \circ i_k^{(n+p)} \text{ for every } k \in \{1, ..., n\} \\ \beta \circ i_k^{(n)} = h \circ i_{n+k}^{(n+p)} \text{ for every } k \in \{1, ..., p\} \end{cases}$$

Proof. The proof is entirely contained in the following commutative diagram

$$\begin{tikzcd}
 & & & & & & & (n+p)\cdot\mathbb{I} & & & & & & & \\
 & & & & & n\cdot\mathbb{I} \arrow[urr, dashed, "g_n^{(n+p)}"] & & & & p\cdot\mathbb{I} \arrow[ull, dashed, "d_p^{(n+p)}"'] & & & & & \\
\cdots & \mathbb{I} \arrow[urrrr, dotted] & & \mathbb{I} \arrow[urr, dotted] & & \mathbb{I} \arrow[u, dotted, "i_n^{(n)}"] & & & & \mathbb{I} \arrow[u, dotted, "i_1^{(p)}"'] & & \mathbb{I} \arrow[ull, dotted] & & \mathbb{I} \arrow[ullll, dotted] & \cdots \\
 & & * \arrow[ul, "t"] \arrow[ur, "s"'] & & * \arrow[ul, "t"] \arrow[ur, "s"'] & & & * \arrow[ull, "t"] \arrow[urr, "s"'] \arrow[uull, "t^{(n)}"'] \arrow[uurr, "s^{(p)}"] & & & * \arrow[ul, "t"] \arrow[ur, "s"'] & & * \arrow[ul, "t"] \arrow[ur, "s"'] & &
\end{tikzcd}$$

More precisely, $(\alpha \circ i_1^{(n)}, ..., \alpha \circ i_n^{(n)}, \beta \circ i_1^{(p)}, ..., \beta \circ i_p^{(p)})$ is a cocone whose basis is

$$* \xrightarrow{s} \mathbb{I} \xleftarrow{t} * \xrightarrow{s} \mathbb{I} \xleftarrow{t} * \xrightarrow{s} \mathbb{I} \quad \cdots \quad \mathbb{I} \xleftarrow{t} * \xrightarrow{s} \mathbb{I} \xleftarrow{t} * \xrightarrow{s} \mathbb{I} \xleftarrow{t} *$$

so we have a unique $h \in \mathsf{C}[(n+p)\cdot\mathbb{I}, X]$ such that

$$\begin{cases} \alpha \circ i_k^{(n)} = h \circ i_k^{(n+p)} = h \circ g_n^{(n+p)} \circ i_k^{(n)} \text{ for every } k \in \{1, ..., n\} \\ \beta \circ i_k^{(n)} = h \circ i_{n+k}^{(n+p)} = h \circ d_p^{(n+p)} \circ i_k^{(p)} \text{ for every } k \in \{1, ..., p\} \end{cases}$$

Applying the uniqueness part of the universal property of colmimits $(n\cdot\mathbb{I}, i_1^{(n)}, ..., i_n^{(n)})$ and $(p \cdot \mathbb{I}, i_1^{(p)}, ..., i_p^{(p)})$ we have $\alpha = h \circ g_n^{(n+p)}$ and $\beta = h \circ d_p^{(n+p)}$. If $h' \in \mathsf{C}[(n+p)\cdot\mathbb{I}, X]$ satisfy $\alpha = h \circ g_n^{(n+p)}$ and $\beta = h \circ d_p^{(n+p)}$, then necessarily, applying the uniqueness part of the uniqueness of the universal property of the colimit $((n+p)\cdot\mathbb{I}, i_1^{(n+p)}, ..., i_{n+p}^{(n+p)})$ we have $h = h'$. □

Definition 2.3 A **Category with paths** is given by:

(i) a category C with a terminal object and $*$ a distinguished representative of it..

(ii) a diagram $* \overset{s}{\underset{t}{\rightrightarrows}} \mathbb{I}$ such that for any isomorphism $\phi \in \mathsf{C}[\mathbb{I}, \mathbb{I}]$ we have $\{\phi \circ s, \phi \circ t\} = \{s, t\}$.

(iii) For all $n \in \mathbb{N}$, V_n has a colimit in C and we have a distinguished colimiting cocone $(n\cdot\mathbb{I}, i_1^{(n)}, ..., i_n^{(n)})$ the colimit of the diagram V_n so that $0\cdot\mathbb{I} = *$, $1\cdot\mathbb{I} = \mathbb{I}$, $i_1^{(1)} = id_{\mathbb{I}}$ and that $\forall n, p \in \mathbb{N}$ if $n \cdot \mathbb{I} \cong p \cdot \mathbb{I}$ in C then $n \cdot \mathbb{I} = p \cdot \mathbb{I}$.

When the context is clear, we will just refer to the structure of category with paths of C as C, letting implicit the rest of the data. However, the distinguished terminal and cocones are part of the structure. Given an object X of C, we can define a <u>path</u> on X as an element (γ, n) of $\bigcup_{n\in\mathbb{N}} \mathsf{C}[n \cdot \mathbb{I}, X] \times \{n\}$ and the source and the target of $(\gamma, n) \in \mathsf{C}[n \cdot \mathbb{I}, X] \times \{n\}$ respectively as $\gamma \circ s^{(n)}$ and $\gamma \circ t^{(n)}$. Refering to the definition of constant morphism, any point is a constant path, this remark enable us to treat paths defined on $0 \cdot \mathbb{I}$ as any other. In fact, the constant paths i.e. those that are defined on $0 \cdot \mathbb{I} = *$ will be the identities of the category of paths of X that we will defined later. Given $(\gamma, p) \in \mathsf{C}[p \cdot \mathbb{I}, X] \times \{p\}$ and $(\delta, n) \in \mathsf{C}[n \cdot \mathbb{I}, X] \times \{n\}$ two paths on X so that $src(\gamma) = tgt(\delta)$ we define the **concatenation** of (δ, n) followed by (γ, p), denoted $(\gamma \cdot \delta, n + p)$, by means of the universal property of the push-out depicted on the figure below. An immediate corollary of Proposition 2.2 is that the concatenation we have just defined is "strictly" associative, i.e. not only up to isomorphism.

$$
\begin{tikzcd}
& X & \\
& (p+n)\cdot\mathbb{I} \arrow[u, "\gamma\cdot\delta"] & \\
p\cdot\mathbb{I} \arrow[uur, bend left, "\gamma"] \arrow[ur, dashed] & \text{push out} & n\cdot\mathbb{I} \arrow[uul, bend right, "\delta"'] \arrow[ul, dashed] \\
& * \arrow[ul, "s^{(p)}"] \arrow[ur, "t^{(n)}"'] &
\end{tikzcd}
$$

Remark 2.4 Let $(\gamma, n) \in \mathsf{C}[n \cdot \mathbb{I}, X] \times \{n\}$, we have $\gamma = (\gamma \circ i_n^{(n)}) \cdot \ldots \cdot (\gamma \circ i_1^{(n)})$. The second component $\{n\}$ cannot be omitted, indeed, by definition of a category with paths, if $\mathbb{I} + \mathbb{I} \cong \mathbb{I}$ then $\forall n \in \mathbb{N}$ we have $n \cdot \mathbb{I} = 1 \cdot \mathbb{I} = \mathbb{I}$. But, as in the notion of *Moore* paths, we wish to have, with each path, an information about how many "elementary" paths it is made of. In some categories, as RTop, if $n \neq p$, we have $n \cdot \mathbb{I} \ncong p \cdot \mathbb{I}$, so the source of γ as a morphism of C i.e. $n \cdot \mathbb{I}$ contains this information. In most of the others cases, as in Top, we have $\mathbb{I} + \mathbb{I} \cong \mathbb{I}$, so this information has to be kept as a "extra data". Once again, the advantage is that we have a strict concatenation. For the sake of simplicity, we will consider that $\gamma \in \mathsf{C}[n \cdot \mathbb{I}, X]$ really means $(\gamma, n) \in \mathsf{C}[n \cdot \mathbb{I}, X] \times \{n\}$.

Now we can describe the category of paths on X denoted $\Gamma(X)$. The objects of $\Gamma(X)$ are the points of X i.e. $Ob(\Gamma(X)) = \mathsf{C}[*, X]$. Then, given two points x and y of X, $\Gamma(X)[x, y] := \left\{\gamma \in \bigcup_{n \in \mathbb{N}} \mathsf{C}[n \cdot \mathbb{I}, X] \Big/ src(\gamma) = x \text{ et } tgt(\gamma) = y\right\}$. The concatenation is defined as above and we check that we have a category whose identities are the points $x : * \to X$ which can be seen as a path on X since $* = 0 \cdot \mathbb{I}$. The preceeding construction is functorial

Proposition 2.5 *There is a functor $\Gamma : \mathsf{C} \longrightarrow \mathsf{Cat}$ which associates to any object X of C its category of paths $\Gamma(X)$. In particular, if $f \in \mathsf{C}[X, Y]$ then we have a functor $\Gamma(f) : \Gamma(X) \longrightarrow \Gamma(Y)$ given by:*

(i) *For all point x of X, $\Big(\Gamma(f)\Big)(x) := f \circ x$.*

(ii) *For all $\gamma \in \mathsf{C}[n \cdot \mathbb{I}, X]$ is a path from x_1 to x_2, i.e. $\gamma \in \Big(\Gamma(X)\Big)[x_1, x_2]$,*

$\Big(\Gamma(f)\Big)(\gamma) := f \circ \gamma.$

Proof. By proposition 2.2 we have

$$f \circ (\gamma \cdot \delta) = (f \circ \gamma) \cdot (f \circ \delta)$$

which proves that $\Gamma(f)$ is actually a functor from $\Gamma(X)$ to $\Gamma(Y)$.
□

Remark 2.6 If the category with paths $\Big(\mathsf{C}, * \overset{s}{\underset{t}{\rightrightarrows}} \mathbb{I}\Big)$ has an automorphism ϕ of $\mathbb{I}$ such that $\phi \circ s = t$ and $\phi \circ t = s$ (i.e. a **time reversal**) then for all points x_1, x_2 of an object X of C, $\gamma \in \Gamma(X)[x_1, x_2] \longmapsto \gamma \circ \phi \in \Gamma(X)[x_2, x_1]$ is a bijection. It suffices to note that $\gamma \circ \phi \circ s^{(n)} = \gamma \circ t^{(n)}$ and $\gamma \circ \phi \circ t^{(n)} = \gamma \circ s^{(n)}$ and that the inverse mapping is $\gamma \in \Gamma(X)[x_2, x_1] \longmapsto \gamma \circ \phi^{-1} \in \Gamma(X)[x_1, x_2]$.

Given a category $\mathcal{A}$, a **congruence** on $\mathcal{A}$ is family of equivalence relations $\sim_{a_1,a_2}$ on $\mathcal{A}[a_1, a_2]$ where $(a_1, a_2) \in Ob(\mathcal{A}) \times Ob(\mathcal{A})$ such that

$$\begin{tikzcd} a_1 \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\alpha'"'] \arrow[r, phantom, "\wr"] & a_2 \arrow[r, bend left, "\beta"] \arrow[r, bend right, "\beta'"'] \arrow[r, phantom, "\wr"] & a_3 \end{tikzcd} \Longrightarrow \begin{tikzcd} a_1 \arrow[r, bend left, "\beta\circ\alpha"] \arrow[r, bend right, "\beta'\circ\alpha'"'] \arrow[r, phantom, "\wr"] & a_3 \end{tikzcd}$$

Now, we wish to have axioms that have to be satisfied by any reasonable notion of homotopy. To any object X of C, one associates a congruence over $\Gamma(X)$ denoted $\sim_X$, so we have a mapping $\Big(X \in Ob(\mathsf{C}) \longmapsto \sim_X \text{ a congruence over } \Gamma(X)\Big)$. Finally, we ask the **Homotopy Congruence Property** or **HCP** be satisfied, which means that

$$\boxed{\begin{array}{c} \forall X, Y \in Ob(\mathsf{C})\ \forall f \in \mathsf{C}[X, Y]\ \forall x_1, x_2 \in Ob(\Gamma(X))\ \forall \gamma, \delta \in \Gamma(X)[x_1, x_2] \\ \gamma \sim_X \delta \implies f \circ \gamma \sim_Y f \circ \delta. \end{array}}$$

and

$$\boxed{\begin{array}{c} \forall X \in Ob(\mathsf{C})\ \forall x \in Ob(\Gamma(X))\ \forall \gamma, \delta \in \Gamma(X)[x, x], \\ \text{if } \gamma \text{ and } \delta \text{ are constant with the same value } x \text{ then } \gamma \sim_X \delta \end{array}}$$

Let us make clear the meaning of the second axiom, by definition of a constant morphism, γ is constant with value p implies that $p \in \mathsf{C}[*, X]$ and then p can be seen as a path since $0 \cdot \mathbb{I} \cong *$. This is the reason why we would like to identify any point p with any constant path with value p. Note that, in this case, $x_1 = x_2 = x$.
Then given an object X of C, we define $\overrightarrow{\pi_1}(\overrightarrow{X}) := \Gamma(X)/_{\sim_X}$ thus defining the **fundamental category** of X. By HCP, the mapping $X \in Ob(\mathsf{C}) \longmapsto \overrightarrow{\pi_1}(\overrightarrow{X}) \in Ob(\mathsf{Cat})$ induces a functor form C to Cat. Indeed, the HCP makes the following definition sound: Any object X of C is sent to the quotient $\overrightarrow{\pi_1}(X) := \Gamma(X)/_{\sim_X}$. Any morphism $f \in \mathsf{C}[X, Y]$ is sent to the functor $\overrightarrow{\pi_1}(f) \in \mathsf{Cat}\Big[\Gamma(X)/_{\sim_X}, \Gamma(X)/_{\sim_X}\Big]$ which

sends any point $(* \to X)$ to the point $f \circ (* \to X)$ and any $\sim_X$-equivalence class $[\, n \cdot \mathbb{I} \xrightarrow{\alpha} X \,]_{\sim_X}$ to the $\sim_Y$-equivalence class $\left[f \circ (n \cdot \mathbb{I} \xrightarrow{\alpha} X) \right]_{\sim_Y}$.
While the definition of Γ can been made under very weak hypothesis, the HCP is an extremly strong requirement since it provides a "simultaneous choice" of a congruence for each object of C. In all the "concrete" cases (we will give a formal meaning to "concrete" later) these congruences come from a canonical idea of directed homotopy. A mapping $\Big(X \in Ob(\mathsf{C}) \longmapsto \sim_X \;$ a congruence on $\Gamma(X) \Big)$ which satisfies the HCP is called a **notion of homotopy** over C.

Remark 2.7 Suppose that $\mathbb{I} \neq *$. Let α be a constant path with value x, γ be a path whose source is x and δ be a path whose end is x then we have

$$\left.\begin{array}{l} x \sim_X \alpha \\ \gamma \sim_X \gamma \end{array}\right\} \implies \gamma = \gamma \cdot x \sim_X \gamma \cdot \alpha \;\text{ and }\; \begin{array}{l} x \sim_X \alpha \\ \delta \sim_X \delta \end{array} \implies \delta = x \cdot \delta \sim_X \alpha \cdot \delta$$

In other words, the paths $\alpha \in \mathsf{C}[0 \cdot \mathbb{I}, X]$ can be ignored. As a consequence, we can give another definition of the fundamental category of X taking X as the set of objects and $\overrightarrow{\pi}_1(X)[x,y]$ the set of $\sim_X$-equivalence classes $[\gamma]_{\sim_X}$ where $\gamma \in \mathsf{C}[n \cdot \mathbb{I}, X]$ with $\underline{n \neq 0}$. This will be useful when we deal with concrete categories.

Proposition 2.8 (Constant paths) *Given an object X of C, a point x of X i.e. $x \in \mathsf{C}[*, X]$ and $n \in \mathbb{N}$, we set c_x^n for the unique morphism of $\mathsf{C}[n \cdot \mathbb{I}, X]$ constant with value x. If $f \in \mathsf{C}[X,Y]$ then $f \circ c_x^n = c_{f \circ x}^n$, if $n, p \in \mathbb{N}$ and x a point of X then $c_x^n \cdot c_x^p = c_x^{n+p}$ and $c_x^{(0)} = id_x$ in $\Gamma(X)$. The relation on paths of X defined by $\alpha \sim_X \beta$ iff there exists a finite sequence $x_n, ..., x_0$ of points of X, where $n \in \mathbb{N}$ and $1 \leq n$, a finite sequence $\gamma_n, ..., \gamma_1$ of paths on X so that for all $k \in \{1, ..., n\}$ the source and the target of γ_k are respectively x_{k-1} and x_k and*

$$\begin{cases} \alpha = t_n \cdot \gamma_n \cdot ... t_1 \cdot \gamma_1 \cdot t_0 \\ \beta = t'_n \cdot \gamma_n \cdot ... t'_1 \cdot \gamma_1 \cdot t'_0 \end{cases}$$

where t_k and t'_k are constant with value x_k for $k \in \{0, ..., n\}$, satisfies the HCP.

The notion of homotopy provided by Proposition 2.8 amounts to "remove the pauses". Note that if all the constant paths $t_0, \ldots, t_n$ are defined on $0 \cdot \mathbb{I} = *$ then $\alpha = t_n \cdot \gamma_n \cdot ... \cdot t_1 \cdot \gamma_1 \cdot t_0 = \gamma_n \cdot ... \cdot \gamma_1$

Proposition 2.9 (Reparametrization) *Given an object X of C, the relation over paths of X defined by $\alpha \sim_X \beta$ iff there exists a finite sequence $x_n, ..., x_0$ of points of X, where $n \in \mathbb{N}$ and $1 \leq n$, a finite sequence $\gamma_n, ..., \gamma_1$ of paths on X so that for all*

$k \in \{1, ..., n\}$ the begin and the end of γ_k are respectively x_{k-1} and x_k and

$$\begin{cases} \alpha = t_n \cdot \gamma_n \cdot ... \cdot t_1 \cdot \gamma_1 \cdot t_0 \\ \beta = t'_n \cdot \gamma'_n \cdot ... \cdot t'_1 \cdot \gamma'_1 \cdot t'_0 \end{cases}$$

where for all $k \in \{0, ..., n\}$

(i) *t_k and t'_k are constant with value x_k*

(ii) *$\gamma'_k = \gamma_k \circ \phi_k$ where ϕ_k is an automorphism.*

satisfies the HCP.

Proposition 2.10 (Lattice of notions of homotopy) *The collection of notions of homotopy over the category with paths C whose generic path is $* \overset{s}{\underset{t}{\rightrightarrows}} \mathbb{I}$ is a complete lattice ordered by inclusion. Its least element is the notion of homotopy described in Proposition 2.8 and its greatest one identifies two paths exactly when they have the same source and the same target.*

The two extreme notions of homotopy given by proposition 2.10 are not very interesting and they do not really reflect what we have in mind when we think of homotopy. Up to some additional hypothesis about C, we are able to give many non trivial examples.

3 Topologically concrete catgeories

Almost all the interesting models of concurrency involving topology have objects which are built over topological spaces. We take advantage of the fact to define notions of homotopy that look like the usual one. Inspired by the usual definition of concrete category (see [15] or [17]) we have

Definition 3.1 A **topologically concrete category** is a category C equiped with a faithful functor U whose codomain is a reflective sub-category of Top and which has a left adjoint denoted F. If T is the codomain of U, we say that C is topologically concrete over T.

We recall that, in particular, Haus is a reflective sub-category of Top and thus, PoTop, RTop, LPoTop are examples of topologically concrete categories over Haus. dTop is topologically concrete over Top or Haus depending on the definition of objects of dTop we have chosen. Our aim is to equip a topologically concrete category with a suitable structure of category with paths.

Definition 3.2 [Compatibility] Let C be a topologiquement concrete category with $U \dashv F$. We also suppose that C is equiped with a structure of category with paths $\mathbb{I}'$, s', t' whose distinguished cocones are $(n \cdot \mathbb{I}', i_1'^{(n)}, ..., i_n'^{(n)})$ for $n \in \mathbb{N}$. Finally, suppose that U preserves the structures of a category with paths i.e. T is also equiped with a structure of a category with paths $\mathbb{I}$, s, t whose distinguished cocones are $(n \cdot \mathbb{I}, i_1^{(n)}, ..., i_n^{(n)})$ for $n \in \mathbb{N}$ and that

(i) $\forall n \in \mathbb{N}\ U(n \cdot \mathbb{I}') = n \cdot \mathbb{I}$ so in particular $U(\mathbb{I}') = \mathbb{I}$ and $U(*') = *$.

(ii) $U(s') = s$ and $U(t') = t$.

(iii) $\forall n \in \mathbb{N}\ \forall k \in \{1, ..., n\}\ U(i_k'^{(n)}) = i_k^{(n)}$.

We summarize this data by saying that C is a **topologically concrete category with paths** or **TCCP** for short.

Lemma 3.3 *$\forall n \in \mathbb{N}\ \forall k \in \{1, ..., n\}\ U(s'^{(n)}) = s^{(n)}$ and $U(t'^{(n)}) = t^{(n)}$.*

Proof. It suffices to remark that, by definition, $s'^{(n)} = i_1'^{(n)} \circ s'$ and $t'^{(n)} = i_n'^{(n)} \circ t'$. The result follows since $U(s') = s,\ U(t') = t,\ U(i_1'^{(n)}) = i_1^{(n)}$ and $U(i_n'^{(n)}) = i_n^{(n)}$. □

Remark that since U has a left adjoint, U preserve (up to isomorphism) the terminal object of C which is the limit of the empty functor. However, the hypothesis $U(*') = *$ is stronger since it forces this preservation to be strict.

Definition 3.4 Let C be a TCCP (over T), an object D of C is called **domain for dihomotopy** (in C) if $U(D) = [0,1] \times [0,1]$ (Cartesian product in T) and if $\forall (x,y) \leq (x',y') \in [0,1] \times [0,1]\ \exists n \in \mathbb{N}\ \exists \gamma \in \mathsf{C}[n \cdot \mathbb{I}, D]$ such that

$$\Big(U(\gamma)\Big)(0) = (x,y) \text{ and } \Big(U(\gamma)\Big)(1) = (x',y').$$

The we chose a collection $\mathcal{D}$ of domains for dihomotopy whose elements are, by definition, the **acceptable domains for dihomotopy**.

Definition 3.5 Let C be a TCCP (over T). Let X be an object of C. Let $\gamma \in \mathsf{C}[n \cdot \mathbb{I}', X]$ and $\delta \in \mathsf{C}[p \cdot \mathbb{I}', X]$ with $n, p \in \mathbb{N}$ i.e. two paths on the object X of C. We call **concrete dihomotopy** in C from γ to δ a morphism $H \in \mathsf{C}[D, X]$, where D is an acceptable domain for dihomotopy in C, such that $U(H)$ be a classical homotopy from $U(\gamma)$ to $U(\delta)$ (with fixed end points).

In fact, definition 3.5 amounts to restrict the collection of homotopies to those which are in the image of U. This limitation is very strong. Also remark that, in general, paths whose domain is $0 \cdot \mathbb{I}$ cannot be "concretely" homotopic to paths whose domain is $n \cdot \mathbb{I}$ for some $n \neq \mathbb{N}$. This is a pathology removed by the remark 2.7.

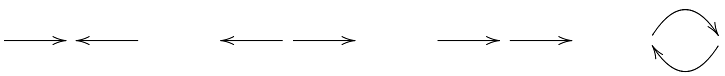
Lemma 3.6 *Let C be a TCCP over T. If there is a concrete dihomotopy from γ to δ then γ and δ have the same source and the same target.*

Proof. Let $\alpha \in \mathsf{C}[n \cdot \mathbb{I}', X]$ and $\beta \in \mathsf{C}[p \cdot \mathbb{I}', X]$ be, then we have $U(\alpha) \in \mathsf{T}[n \cdot \mathbb{I}, U(X)]$ and $U(\beta) \in \mathsf{T}[p \cdot \mathbb{I}, U(X)]$. By hypothesis, we have a classical homotopy from $U(\alpha)$ to $U(\beta)$, hence $(U(\alpha))(0) = (U(\beta))(0)$ i.e. $(U(\alpha)) \circ t^{(n)} = (U(\beta)) \circ s^{(p)}$ or $U(\alpha \circ t'^{(n)}) = U(\alpha \circ s'^{(p)})$ by lemma 3.3. Then, since U is faithful, we have $\alpha \circ t'^{(n)} = \alpha \circ s'^{(p)}$, in other words α and β have the same source. □

Lemma 3.7 *Let* C *be a TCCP, then* U *preserves the constant morphisms and* U *is the functor associated to* C *with the notation of Definition 3.1.*

Proof. Let $f \in \mathsf{C}[X, Y]$ be constant (i.e. which factorizes in C through the terminal object). So we can write f as $f = f' \circ \zeta_X$ where ζ_X is the unique morphism from X to $*$. As $U(*) = *'$, we have $U(f) = U(f') \circ U(\zeta_X)$ hence $U(f)$ is constant. □

Next result requires the notion of **zigzag** which is defined as follows: Given a graph (V, A, s, t) where V and A are the sets of vertices and arrows of the graph and $\forall a \in A$ $s(a)$ and $t(a)$ are the source and target of a . A zigzag between to vertices v_1 and v_2 is a finite sequence $a_1, \ldots, a_{n-1}$ of arrows such that $v_1 \in \{s(a_1), t(a_1)\}$, $v_2 \in \{s(a_{n-1}), t(a_{n-1})\}$ and $\forall k \in \{1, \ldots, n-1\}$ $\{s(a_k), t(a_k)\} \cap \{s(a_{k+1}), t(a_{k+1})\} \neq \emptyset$. Given two consecutive arrows of a zigzag w_k, w_{k+1} we have one of the four following cases



Clearly, the relation $\{(v_1, v_2) \in V \times V |$ there is a zigzag between v_1 and v_2 $\}$ is an equivalence relation on V. In what follows, the vertices of the graph are paths and the arrows are the concrete dihomotopies.

Proposition 3.8 *Let* C *be a TCCP. We write dihomotopy for concrete dihomotopy in* C*. We* <u>*suppose*</u> *that for every object* X *of* C *we have the following properties*

(i) *(Left identities) For all* $\gamma \in \mathsf{C}[n \cdot \mathbb{I}', X]$ *where* $n \neq 0$ *and* $\alpha \in \mathsf{C}[p \cdot \mathbb{I}', X]$ *such that* α *is constant with value* $\gamma \circ t^{(n)}$ *there is a zigzag of dihomotopies between* $\alpha \cdot \gamma$ *and* γ.

(ii) *(Right identities) For all* $\gamma \in \mathsf{C}[n \cdot \mathbb{I}', X]$ *where* $n \neq 0$ *and* $\alpha \in \mathsf{C}[p \cdot \mathbb{I}', X]$ *such that* α *is constant with value* $\gamma \circ s^{(n)}$ *there is a zigzag of dihomotopies between* $\gamma \cdot \alpha$ *and* γ.

(iii) *(Congruence) If there is a dihomotopy from* α *to* α'*, another one from* β *to* β' *such that the source of* β *is the target of* α *then there is a zigzag of dihomotopies between* $\beta \cdot \alpha$ *and* $\beta' \cdot \alpha'$.

(iv) *(Compatibility) If* $\gamma \in \mathsf{C}[n \cdot \mathbb{I}', X]$ *and* $\delta \in \mathsf{C}[p \cdot \mathbb{I}', X]$ *for* $n, p \neq 0$ *and* $U(\gamma) = U(\delta)$ *then such that there is a zigzag of dihomotopies between* γ *and* δ.

Then the transitive closure of

$$\left\{(\gamma, \delta) \middle/ \text{ there is a dihomotopy from } \gamma \text{ to } \delta \text{ or from } \delta \text{ to } \gamma \right\}$$

i.e. the relation

$$\left\{(\gamma, \delta) \middle/ \text{ there is a zigzag of dihomotopies between } \gamma \text{ and } \delta \right\}$$

defines a notion of dihomotopy over C *i.e. the family* $(\ \sim_X\ )_{X \in Ob(\mathsf{C})}$ *satisfies the HCP.*

Proof. Every $\sim_X$ is obviously an equivalence relation. The axiom (iii) implies that it is a congruence. The first part of the HCP is satisfied since every $f \in \mathsf{C}[X, Y]$ and every zigzag of dihomotopies $w_1, \dots, w_n$ induce a zigzag of dihomotopies $f \circ w_1, \dots, f \circ w_n$. The axioms (i) and (ii) give the second one. □

Remark 3.9 The axiom (iv) is not required by the proof of Proposition 3.8, it is just a "reasonable" requirement.

It remains to check that the machinery we have developed (proposition 3.8) applies to Top, PoTop, dTop, LPoTop etc.

4 Applications

We give several examples of a category with paths, some of them are concrete but not all. First, we notice that for any category with a terminal object $*$, we have a structure of category with paths setting $\mathbb{I} := *$. Of course, in this case, we also have $s = t$ and for any object X of C, Γ_X is just the discrete category whose objects are the points of X. This structure will be refered to as the trivial one.

4.1 Set

Up to isomorphism, the only non trivial generic path of Set is $\{0, 1\}$. Indeed, if P is a set containing at least 3 elements, for all $\{s, t\} \subseteq P$ we have a bijection ϕ from P to P such that $\phi(\{s, t\}) \neq \{s, t\}$. It follows that any generic path on Set has at most two elements. Let us suppose that $\mathbb{I} := \{0, 1\}$. Clearly, $n \cdot \mathbb{I} = \{0, \dots, n\}$. Let X be a set and $a, b \in X$, the paths from a to b are the sequences $x \in X^{\{0,\dots,n\}}$ such that $x_0 = a$ and $x_n = b$. Concatenation of $x \in X^{\{0,\dots,n\}}$ followed by $y \in X^{\{0,\dots,p\}}$ is $z \in X^{\{0,\dots,n+p-1\}}$ where $z_k = x_k$ if $0 \leq k \leq n$ and $z_k = y_{k-n-1}$ if $n < n + p - 2$. For example $(3, 4, 5) \cdot (1, 2, 3) = (1, 2, 3, 4, 5)$, 3 is not repeated. The next assertion shows the strength of the HCP: the only notions of homotopies are the extreme ones. It means that if $\sim_X$ is a notion of homotopy then we have either

(i) for every set X and $\forall x \in X^{\{0,\dots,n\}}$ $\forall y \in X^{\{0,\dots,p\}}$ $x \sim_X y$ iff $\forall i \in \{0, \dots, n\}$ $\forall j \in \{0, \dots, p\}$ $x_i = y_j$
or
(ii) for every set X and $\forall x \in X^{\{0,\dots,n\}}$ $\forall y \in X^{\{0,\dots,p\}}$ $x \sim_X y$

4.2 Cat

We choose the generic path $\mathbb{I} := (0 \to 1)$ which can be seen as the poset $\{0 < 1\}$. A point of an object X of Cat is just an object of X. A path on X is just a morphism of X. The only automorphism of $\mathbb{I}$ is the identity hence there is no time reversal. Further, $\Gamma(X)$ is the free category generated by the underlying graph of X. In other words, if U is the forgetful functor from Cat to Grph and F its left adjoint then $\Gamma(X) := F \circ U(X)$. For any small category X, and any paths (i.e. composable sequence of X) $\alpha_n, \dots, \alpha_0$ and $\beta_p, \dots, \beta_0$ with the same source and target, put $\alpha_n, \dots, \alpha_0 \sim_X \beta_p, \dots, \beta_0$ iff their composites agree in X. This provides a notion of homotopy and the fundamental category of X (i.e. $\Gamma(X)/_{\sim_X}$) is X. Up to ismorphism, $n \cdot \mathbb{I}$ is the poset $\{0 < ... < n\}$. Note that if the generic path is $\{0 \leftrightarrow 1\}$ i.e. the equivalence relation on $\{0,1\}$ that identifies 0 and 1 then we have a time reversal.

4.3 2-Cat

Let us be loose about what a small 2-category is and just say that it is a small category with 2-arrows between arrows (the usual ones) with the same source and target. The idea is pictured by
$$\begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\eta"] & y \end{tikzcd}$$
Given a small 2-category X, we set $\Gamma_2(X) := \Gamma(UX)$ where Γ is the functor defined in the example of Cat and UX the underlying small category of X (there is an obvious forgetful functor from 2-Cat to Cat). The the congruence $\sim_X$ over $\Gamma(X)$ is generated by the relation that identifies two morphisms $\alpha_n, \ldots, \alpha_0$ and $\beta_p, \ldots, \beta_0$ of $\Gamma(X)$ when there is a 2-morphism from the composite of α to the one of β. Note that if for all morphisms α, β of X, there is a 2-arrow from α to β iff there is a 2-arrow from β to α, then the relation $\sim_X$ is an equivalence relation and we do not need to say "generated by".

4.4 Top *and* Haus

We choose the generic path $\mathbb{I} := [0,1]$, s and t send $* := \{0\}$ to 0 respectively 1. The map $t \in [0,1] \longmapsto (1-t) \in [0,1]$ is time reversal.
In order to obtain a category with paths, we set $0 \cdot \mathbb{I} = \{0\}$, $1 \cdot \mathbb{I} = [0,1]$ and for $n \in \mathbb{N}$ $n \geq 2$, $n \cdot \mathbb{I} := [0,1]$. We also set for $n \in \mathbb{N}\backslash\{0\}$ and $k \in \{1,...,n\}$ $i_k^{(n)} : x \in \mathbb{I} = [0,1] \longmapsto \frac{(k-1)+x}{n} \in n \cdot \mathbb{I} = [0,1]$. Then we have:

Lemma 4.1 *For all $n \in \mathbb{N}$ $(n \cdot \mathbb{I} = \mathbb{I}, i_1^{(n)}, ..., i_n^{(n)})$ is colimit representation of the diagram*

$$\underbrace{\begin{tikzcd}[column sep=tiny] & \mathbb{I} & & \mathbb{I} & & \mathbb{I} & \cdots & \mathbb{I} & & \mathbb{I} & & \mathbb{I} & \\ * \arrow[ur, "s"'] & & * \arrow[ul, "t"] \arrow[ur, "s"'] & & * \arrow[ul, "t"] \arrow[ur, "s"'] & & & & * \arrow[ul, "t"] \arrow[ur, "s"'] & & * \arrow[ul, "t"] \arrow[ur, "s"'] & & * \arrow[ul, "t"] \end{tikzcd}}_{n \text{ copies de } \mathbb{I}}$$

Moreover, given $\gamma_1 : n \cdot \mathbb{I} = [0,1] \longrightarrow X$, $\gamma_2 : m \cdot \mathbb{I} = [0,1] \longrightarrow X$ and $\gamma_3 : p \cdot \mathbb{I} = [0,1] \longrightarrow X$ so that $\gamma_1(1) = \gamma_2(0)$ and $\gamma_2(1) = \gamma_3(0)$, we have $\gamma_3 \cdot (\gamma_2 \cdot \gamma_1) = (\gamma_3 \cdot \gamma_2) \cdot \gamma_1$ where $\cdot$ is the barycentric concatenation. The equality is strict, of course, the definition of $\cdot$ depends on n, m and p.

Proof. We check that
$\forall x \in \left[0, \frac{n}{n+m+p}\right]$ $\gamma_3 \cdot (\gamma_2 \cdot \gamma_1)(x) = (\gamma_3 \cdot \gamma_2) \cdot \gamma_1(x) = \gamma_1\left(\frac{n+m+p}{n}x\right)$
$\forall x \in \left[\frac{n}{n+m+p}, \frac{n+m}{n+m+p}\right]$ $\gamma_3 \cdot (\gamma_2 \cdot \gamma_1)(x) = (\gamma_3 \cdot \gamma_2) \cdot \gamma_1(x) = \gamma_2\left(\frac{n+m+p}{m}x - \frac{n}{m}\right)$
$\forall x \in \left[\frac{n+m}{n+m+p}, 1\right]$ $\gamma_3 \cdot (\gamma_2 \cdot \gamma_1)(x) = (\gamma_3 \cdot \gamma_2) \cdot \gamma_1(x) = \gamma_3\left(\frac{n+m+p}{p}x - \frac{n+m}{p}\right)$
$\square$

Lemma 4.1 provides a structure of category with paths over Top. It is defined for couples (n, γ) where $n \in \mathbb{N}$ and γ a continuous map from $[0, 1]$ to X, in other words, with respect to $\cdot$, (n, γ) and (p, γ) are different when $n \neq p$.
Let X be an object of Top, $\Gamma(X)$ is the category of *Moore* paths of X. In particular, we have an isomorphic category setting $n \cdot \mathbb{I} := [0, n]$ and $i_k^{(n)} : x \in [0, 1] \longmapsto x + k - 1 \in [0, n]$. The relation $\sim_X$ over $\Gamma(X)$ is the classical homotopy relation, we check that it provides a notion of homotopy and that the fundamental category of X is just its (classical) fundamental groupoid ([13]). The structure of category of paths that we have defined over Top also provides such a structure over Haus.

4.5 PoTop

The generic path is $\mathbb{I} := \overrightarrow{[0, 1]}$ the closed unit segment with classical topology and order, s, t are defined as in the example of Top. Any automorphism ϕ of $\mathbb{I}$ satisfies $\phi(0) = 0$ and $\phi(1) = 1$, there is no time reversal. We set $0 \cdot \mathbb{I} = \{0\}$, $1 \cdot \mathbb{I} = \overrightarrow{[0, 1]}$ and for $n \in \mathbb{N}$ $n \geq 2$, $n \cdot \mathbb{I} := \overrightarrow{[0, 1]}$. We also set for $n \in \mathbb{N} \backslash \{0\}$ and $k \in \{1, ..., n\}$ $i_k^{(n)} : x \in \mathbb{I} = \overrightarrow{[0, 1]} \longmapsto \frac{(k-1)+x}{n} \in n \cdot \mathbb{I} = \overrightarrow{[0, 1]}$. Moreover, Lemma 4.1 can be adapted to PoTop without changes, providing it with structure of a category with paths.
Moreover, the forgetful functor U : PoTop $\longrightarrow$ Haus is faithful and has a left adjoint since Haus is a reflective sub-category of Top, PoTop is topologically concrete over Haus.
Let $\overrightarrow{X}$ be an object of PoTop. The only acceptable domain for dihomotopy is $\overrightarrow{[0, 1]} \times \overrightarrow{[0, 1]}$ (see Definition 3.4), we apply proposition 3.8 to have the notion of (concrete) dihomotopy (see definition 3.5). Then, the relation $\sim_X$ we put over $\Gamma(X)$ is the usual notion of dihomotopy. It follows that $\overrightarrow{\pi_1}(\overrightarrow{X})$ is the usual fundamental category of $\overrightarrow{X}$ (see [12], [8], [6] or [5]).

4.6 RTop

A similar construction proves that RTop is a TCCP over Haus. We have an obvious inclusion functor i from PoTop to RTop which satifies $\overrightarrow{\pi_1}(\overrightarrow{X}) = \overrightarrow{\pi}_1(X, \leq_X)$ where the fundamental categories on both sides of the equality are respectively determined in PoTop and RTop (see [11]).

4.7 dTop

As suggested by *Marco Grandis* in [10], we take as generic path

$$\mathbb{I} := ([0, 1], \{\text{continuous increasing mappings from } [0, 1] \text{ to } [0, 1]\})$$

and s, t as in the preceeding examples. We note that there is no time reversal. This category is concrete over Haus (assuming that the underlying topological space of a directed space has to be Hausdorff), the concrete dihomotopy from $\alpha \in dX$ to

$\beta \in dX$ is a morphism of $\mathsf{dTop}[\mathbb{I} \times \mathbb{I}, (X, dX)]$ whose underlying map is a classical homotopy from α to β. Proposition 3.8 can be applied: the relation $\sim_X$ over $\Gamma(X)$ that it provides as well as the fundamental category it leads to correspond to the directed homotopy respectively the fundamental category of a directed space defined by *M. Grandis* in [10].

Any pospace $\overrightarrow{X}$ can be seen as a directed space (X, dX) where

$$dX := \mathsf{PoTop}[\overrightarrow{[0,1]}, \overrightarrow{X}] \ ;$$

this remark induces a kind of "inclusion functor" denoted i from PoTop to dTop, with the preceeding notation, we have $\overrightarrow{\pi_1}(\overrightarrow{X}) = \overrightarrow{\pi}_1(X, dX)$ where the fundamental categories on both sides of the equality are respectively determined in PoTop and dTop (see [11]). Moreover the functor i has a left adjoint, the proof of this fact use, as a technical intermediate, the category RTop. More precisely, the inclusion functor from PoTop to RTop has a left adjoint, thus PoTop is a reflective sub-category of RTop and we deduce the cocompleteness of PoTop from the one of RTop, indeed, it is a general fact that any reflective sub-category of a cocomplete category is cocomplete itself (see [2]). Besides, we also have an "inclusion" functor from RTop to dTop applying the same construction as for the "inclusion" of PoTop in RTop. This inclusion also has a left adjoint. We conclude by composing the adjunctions. All the details can be found in [11].

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