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On Subset Families That Form a Continuous Lattice

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Abstract

It is well known that continuous lattices and algebraic lattices can be respectively represented by the family of all fixed points of the projection operator and the closure operator preserving sups of directed sets on the power set of a set X. Similar to the algebraic $\bigcap$-structure as the concrete representation of algebraic lattices, can we have a the concrete representation of continuous lattices by families of sets? We give a positive answer to this in this paper. Also, as a special type of continuous lattice, a concrete representation of completely distributive complete lattices by families of sets is obtained.

Keywords: continuous lattices, completely distributive, $\mathbb{C}$-sets, $\mathbb{A}$-sets, $\mathbb{C}$-$\bigcup$-semiring.

1 Introduction

In Lattice Theory, the representations of special type of lattices are always important research topics. Many approaches, such as in terms of families of subsets of a set, topological spaces, formal concepts and information systems [4,5,9,12,13,15], have been used for representing special type of lattices. The most intuitive one is by means of families of subsets of a set.

In [2], Buchi showed that a lattice is complete if and only if it is isomorphic to a topped $\bigcap$-structure (intersection structure). Reney proposed the concept of the ring of sets and revealed that the completely distributive algebraic lattice (completely supercontinuous lattice) are one-to-one correspondent to the ring of sets in [13]. After then, Deng [5] suggested the conclusion that a complete lattice is completely distributive if and only if it is isomorphic to a complete $\bigcup$-semiring with condition

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($*$). Also in [3], the author proved that a lattice is algebraic if and only if it is isomorphic to an algebraic $\bigcap$-structure. As a generalization of algebraic $\bigcap$-structure, the notion of F-algebraic $\bigcap$-structure by Guo and Li in [9]. It has been showed that a dcpo is an algebraic domain if and only if it is isomorphic to an F-algebraic $\bigcap$-structure.

As a more general result, it is proved in [8] that a lattice L is continuous iff it is isomorphic to the family of all fixed points of a projection operator preserving directed sups. Thus the family of all fixed points of a projection operator preserving directed sups can be viewed as a representation of continuous lattices, but it is difficult to justify whether a family of sets satisfy the conditions. In this paper, our purpose is to obtain a concrete representation of continuous lattices, by means of families of subsets of a set, similar to the algebraic $\bigcap$-structure as the concrete representation of the algebraic lattice.

The rest of this paper is structured as follows. In Section 2, we recall some basic concepts in lattice theory which will be frequently used in this paper. In Section 3, we obtain the representations of continuous lattices (algebraic lattices, completely distributive lattices, respectively). In Section 4, we consider a special family of sets: the Scott topology on a poset. Then we can get some equivalent conditions for a poset to be continuous.

2 Preliminaries

A partially ordered set (poset) is a nonempty set P equipped with a reflexive, transitive and antisymmetric relation $\leq$. Let P be a poset and $X \subseteq P$, we use the symbol $\sup X = \bigvee X$ to denote the least upper bound of X, and $\inf X = \bigwedge X$ for the greatest lower bound of X. We denote $\uparrow X = \{x \in P : \exists y \in X, s.t. y \leq x\}$, $\downarrow X = \{x \in P : \exists y \in X, s.t. x \leq y\}$, $X^l = \{x \in P : \forall y \in X, x \leqslant y\}$ and $X^u = \{x \in P : \forall y \in X, y \leq x\}$.

A subset D of P is said to be directed provided it is nonempty and every finite subset of D has an upper bound in D. Dually, we call a nonempty subset F of P filtered if every finite subset of F has a lower bound in F.

Definition 2.1 *[8] Let P be a poset. We say that x is way below y, in symbols $x \ll y$, iff for all directed subsets $D \subseteq P$ for which $\sup D$ exists, the relation $y \leq \sup D$ implies the existence of an element $d \in D$ with $x \leq d$. An element satisfying $x \ll x$ is said to be compact, the set of all compact elements is denoted by $K(P)$.*

For each $x \in P$, we simply write $\mathord{\downarrow\!\!\!\downarrow} x = \{y \in P \mid y \ll x\}$, $\mathord{\uparrow\!\!\!\uparrow} x = \{y \in P \mid x \ll y\}$. It is easy to see that $\mathord{\downarrow\!\!\!\downarrow} x$ is a lower subset of P and $\mathord{\uparrow\!\!\!\uparrow} x$ is an upper subset of P for each $x \in P$.

Definition 2.2 *[8] 1) A poset P is called continuous if for all $x \in P$, the set $\mathord{\downarrow\!\!\!\downarrow} x = \{y \in P \mid y \ll x\}$ is directed and $x = \sup \mathord{\downarrow\!\!\!\downarrow} x$.*
2) A dcpo which is continuous as a poset is called a domain.
3) A domain which is a complete lattice is called a continuous lattice.

Let L be a complete lattice and $x \in L$. Then $\Downarrow x$ is automatically directed because $a \ll x$ and $b \ll x$ imply that $a \vee b \ll x$. Then, to prove a complete lattice L is continuous, it is sufficient to show that there exists a subset $A_x \subseteq \Downarrow x$ such that $\sup A_x = x$ for each $x \in L$.

Proposition 2.3 *[8] In a continuous poset, the way below relation satisfies the Interpolation Property:*

$$x \ll z \text{ implies } (\exists\, y)\ x \ll y \ll z$$

.

Definition 2.4 *[8] A complete lattice L is called algebraic iff it satisfies the axiom of Compact Approximation: for all $x \in L$ the set $\downarrow x \bigcap K(L)$ is directed and $x = \sup(\downarrow x \bigcap K(L))$.*

It is obvious that every algebraic lattice is a continuous lattice. Similarly to the case of continuous lattice, $a, b \in K(L)$ implies that $a \bigvee b \in K(L)$ if L is a complete lattice. To prove a complete lattice L is an algebraic lattice, it is sufficient to find a subset $A_x \subseteq \downarrow x \bigcap K(L)$ such that $\sup A_x = x$ for each $x \in L$.

Definition 2.5 *[3] A nonempty family $\mathcal{L}$ of subsets of a set X is said to be an algebraic $\bigcap$-structure if*

(i) $\bigcap\limits_{i\in I} A_i \in \mathcal{L}$ *for any nonempty family* $\{A_i\}_{i\in I} \subseteq \mathcal{L}$,

(ii) $\bigcup\limits_{i\in I} A_i \in \mathcal{L}$ *for any directed family* $\{A_i\}_{i\in I} \subseteq \mathcal{L}$.

A well-known result on algebraic lattice is that a poset P is an algebraic lattice if and only if it is isomorphic to an algebraic $\bigcap$-structure.

Definition 2.6 *[8] A lattice L is called completely distributive iff it is complete and for any family $\{x_{j,k} : j \in J, k \in K\}$ in L the identity*

$$\bigwedge_{j\in J} \bigvee_{k\in K} x_{j,k} = \bigvee_{f\in M} \bigwedge_{j\in J} x_{j,f(j)}$$

holds, where M denoted the set of choice functions defined on J with values $f(j) \in K(j)$.

Completely distributive lattices can be equivalent characterized in the following way.

Definition 2.7 *[8] Let L be a complete lattice. We say that x is completely way below y, in symbols $x \lhd y$, iff for all subsets $A \subseteq L$, the relation $y \leq \sup A$ implies the existence of an element $a \in A$ with $x \leq a$.*

It is obvious that $y \lhd x$ implies that $y \ll x$. For each $x \in L$, we simply write $\Downarrow x = \{y \in L : y \lhd x\}$.

Lemma 2.8 *A complete lattice L is completely distributive iff $x = \sup \Downarrow x$ for each $x \in L$.*

From Lemma 2.1, one can see that a completely distributive lattice is a continuous lattice.

3 Families of sets with property $\mathbb{C}$

In this section, we give the representations of continuous lattices, algebraic lattices and completely distributive complete lattices. For a continuous lattice L, there exists an order isomorphism between L and $(\mathcal{A}_L, \subseteq)$, where $\mathcal{A}_L = \{\twoheaddownarrow x : x \in L\}$. Then the rest work is to find some properties of $\mathcal{A}_L$ to ensure it is continuous. The same problems will be considered for algebraic lattice M and a completely distributive complete lattice N, where $\mathcal{A}_M = \{\downarrow x \bigcap K(M) : x \in M\}$ and $\mathcal{A}_N = \{\Downarrow x : x \in N\}$.

3.1 Continuous lattices and $\mathbb{C}$-sets

In this subsection, we introduce the conception of $\mathbb{C}$-sets and prove that a poset is a continuous lattice if and only if it is isomorphism to a $\mathbb{C}$-sets.

If a nonempty family $\mathcal{A}$ of sets ordered by inclusion is a complete lattice, we denote $K_x = \inf\{A \in \mathcal{A} : x \in A\}$ for each $x \in \bigcup_{A\in\mathcal{A}} A$.

(Property $\mathbb{C}$):

$$A = \bigcup_{x\in A} K_x$$

for each $A \in \mathcal{A}$.

Definition 3.1 *Suppose a nonempty family $\mathcal{A}$ of sets is closed under directed unions and $(\mathcal{A}, \subseteq)$ is a complete lattice, then $\mathcal{A}$ is said to be a $\mathbb{C}$-sets iff it satisfies Property $\mathbb{C}$.*

Example 3.2 (i) For a nonempty family $\mathcal{A}$ of sets which is a complete lattice under inclusion, Property $\mathbb{C}$ is equal to the following property: If $x \in A \in \mathcal{A}$, then there exists $y \in A$ such that $x \in K_y$.

(ii) A ring of sets is a $\mathbb{C}$-sets, where a ring of sets means a nonempty family of sets which is closed under arbitrary unions and joins.

(iii) A topped algebraic $\bigcap$-structure $\mathcal{A}$ is always a $\mathbb{C}$-sets. If $A_i \in \mathcal{A}$ for each $i \in I$, then $\inf\{A_i : i \in I\} = \bigcap\{A_i : i \in I\}$. Thus $K_x = \inf\{A \in \mathcal{A} : x \in A\} = \bigcap\{A \in \mathcal{A} : x \in A\}$ implies $x \in K_x$ for each $x \in A \in \mathcal{A}$. That is, if $x \in A \in \mathcal{A}$, then $x \in K_x \subseteq A$.

(iv) For a $\mathbb{C}$-sets $\mathcal{A}$ and $x \in A \in \mathcal{A}$, we always have $x \in \bigcap \mathcal{A}_x$, but $x \notin K_x$ is possible. For example, just let $\mathcal{A} = \{\phi, \{a, b\}, \{a, c\}, \{a, b, c\}\}$, then $K_a = \phi$.

Lemma 3.3 *If $\mathcal{A}$ is a nonempty family of sets which is closed under directed union and $(\mathcal{A}, \subseteq)$ is a complete lattice, then $K_x \ll A$ in $(\mathcal{A}, \subseteq)$ for each $x \in \bigcup_{A\in\mathcal{A}} A$. As a special case, if $x \in K_x$, then $K_x \ll K_x$. That is to say, K_x is a compact element in $(\mathcal{A}, \subseteq)$.*

Proof. Suppose that $\{A_i : A_i \in \mathcal{A}$ for each $i \in I\}$ is a directed subfamily of $\mathcal{A}$ such that $A \subseteq \sup\{A_i : i \in I\} = \bigcup\limits_{i \in I} A_i$, then there exists $i \in I$ such that $x \in A_i$. Since $K_x = \inf\{A \in \mathcal{A} : x \in A\} \subseteq A_i$, we conclude that $K_x \ll A$. □

Theorem 3.4 *If $\mathcal{A}$ is a $\mathbb{C}$-sets, then $(\mathcal{A}, \subseteq)$ is a continuous lattice.*

Proof. From Definition 3.1 we know that $(\mathcal{A}, \subseteq)$ is a complete lattice. For each $A \in \mathcal{A}$, if $A = \phi$, then $\twoheaddownarrow A = \{\phi\}$. Now we suppose $A \neq \phi$. For each $x \in A$ by Property $\mathbb{C}$ and Lemma 3.3 we know that there exists an element $y \in A$ such that $x \in K_y \subseteq \bigcup\limits_{a \in A} K_a \subseteq \sup\{K_a : a \in A\} \subseteq \sup \twoheaddownarrow A \subseteq A$. Thus, $\bigcup\{K_a : a \in A\} = \sup\{K_a : a \in A\} = \sup \twoheaddownarrow A = A$. For a complete lattice $(\mathcal{A}, \subseteq)$, since $\twoheaddownarrow A$ is directed, $(\mathcal{A}, \subseteq)$ is a continuous lattice. □

Proposition 3.5 *Let L be a continuous lattice and $\mathcal{A}_L = \{\twoheaddownarrow x : x \in L\}$. Then $\mathcal{A}_L$ is a $\mathbb{C}$-sets.*

Proof. Obviously $\mathcal{A}_L$ is a nonempty family of subsets of L.

$\mathcal{A}_L$ is closed under directed unions: Notice that in a continuous lattice L, $x \leqslant y \Leftrightarrow \twoheaddownarrow x \subseteq \twoheaddownarrow y$. Suppose $\{\twoheaddownarrow x_i : i \in I\}$ is directed in $(\mathcal{A}_L, \subseteq)$, then $\twoheaddownarrow x_i \subseteq \twoheaddownarrow \sup\{x_i : i \in I\}$ for each $i \in I$, thus $\bigcup\{\twoheaddownarrow x_i : i \in I\} \subseteq \twoheaddownarrow \sup\{x_i : i \in I\}$. Conversely for each $y \in \twoheaddownarrow \sup\{x_i : i \in I\}$, by the Interpolation Property of continuous lattice, there exists an element $z \in L$ such that $y \ll z \ll \sup\{x_i : i \in I\}$. Since $\{x_i : i \in I\}$ is directed, there exists $i \in I$ such that $z \leqslant x_i$. Thus we have $y \ll x_i$, i.e., $y \in \bigcup\{\twoheaddownarrow x_i : i \in I\}$. Therefore, $\bigcup\{\twoheaddownarrow x_i : i \in I\} = \twoheaddownarrow \sup\{x_i : i \in I\} \in \mathcal{A}_L$. This shows $\mathcal{A}_L$ is closed under directed unions.

$(\mathcal{A}_L, \subseteq)$ is a complete lattice: Since L is a complete lattice, $\sup L$ is the largest element in L. Thus $\twoheaddownarrow \sup L$ is the largest element in $(\mathcal{A}_L, \subseteq)$. Suppose $x_i \in L$ for each $i \in I \neq \phi$. Then $\twoheaddownarrow \inf\{x_i : i \in I\} \subseteq \twoheaddownarrow x_i$ for each $i \in I$. If there exists $\twoheaddownarrow y \in \mathcal{A}_L$ such that $\twoheaddownarrow y \subseteq \bigcap\limits_{i \in I} \twoheaddownarrow x_i$, then $\twoheaddownarrow y \subseteq \twoheaddownarrow x_i$ for each $i \in I$, hence $y \leqslant x_i$, thus $y \leqslant \inf\{x_i : i \in I\}$, i.e., $\twoheaddownarrow y \subseteq \twoheaddownarrow \inf\{x_i : i \in I\}$. This implies that in $(\mathcal{A}_L, \subseteq)$, $\inf\{\twoheaddownarrow x_i : i \in I\} = \twoheaddownarrow \inf\{x_i : i \in I\}$.

Property $\mathbb{C}$: Suppose $x \in \twoheaddownarrow y \in \mathcal{A}_L$, then $x \ll y$ in L. From the Interpolation Property of continuous lattice there exists an element $z \in L$ such that $x \ll z \ll y$, whence $x \in \twoheaddownarrow z \subseteq \downarrow z \subseteq \bigcap\{\twoheaddownarrow a : z \ll a\}$. Notice $K_z = \inf\{\twoheaddownarrow a : z \ll a\}$ and $\twoheaddownarrow z \in \mathcal{A}_L$, we have $x \in \twoheaddownarrow z \subseteq K_z$. □

Lemma 3.6 *Let L be a continuous lattice and $\mathcal{A}_L = \{\twoheaddownarrow x : x \in L\}$. Then $\mathcal{A}_L$ is isomorphic to L.*

Proof. It is obvious that the map $\varphi : x \mapsto \twoheaddownarrow x$ is an isomorphism of L onto $\mathcal{A}_L$. Because L is a continuous lattice, $x \leqslant y$ in L is equal to $\twoheaddownarrow x \subseteq \twoheaddownarrow y$ in $\mathcal{A}_L$, therefore φ is an order-isomorphism. □

Therefore we have the following representation of continuous lattices by means of families of sets.

Theorem 3.7 *A poset is a continuous lattice if and only if it is isomorphic to a $\mathbb{C}$-sets.*

Proof. It is straightforward from Theorem 3.4, Proposition 3.5 and Lemma 3.6. □

3.2 Algebraic lattices and $\mathbb{A}$-sets

As is well known, topped algebraic $\bigcap$-structure is a representation of algebraic lattices by means of families of sets. In this subsection, we introduce the concept of $\mathbb{A}$-sets, as a generalization of topped algebraic $\bigcap$-structure. It is showed that the $\mathbb{A}$-sets is also a representation of algebraic lattices.

Definition 3.8 *A set family $\mathcal{A}$ is called an $\mathbb{A}$-sets if $x \in K_x$ for each $x \in \bigcup_{A\in\mathcal{A}} A$.*

Example 3.9 (i) Suppose that $\mathcal{A}$ is a nonempty family of sets and $(\mathcal{A}, \subseteq)$ is a complete lattice, then the property $x \in K_x$ for each $x \in \bigcup_{A\in\mathcal{A}} A$ implies that $\mathcal{A}$ satisfy the property $\mathbb{C}$.

(ii) An $\mathbb{A}$-sets is always a $\mathbb{C}$-sets, then it is a continuous lattice under inclusion order. As what has been pointed out in Example 3.2 (4), a $\mathbb{C}$-sets may not be an $\mathbb{A}$-sets.

(iii) A topped algebraic $\bigcap$-structure $\mathcal{A}$ is always an $\mathbb{A}$-sets. $\mathcal{A}$ must be a $\mathbb{C}$-sets by Example 3.2 (3). For each $x \in A \in \mathcal{A}$, $K_x = \inf\{A \in \mathcal{A} : x \in A\} = \bigcap\{A \in \mathcal{A} : x \in A\}$, then $x \in K_x$.

Lemma 3.10 *An $\mathbb{A}$-sets is always an algebraic lattice under the inclusion order.*

Proof. Suppose $\mathcal{A}$ is an $\mathbb{A}$-sets. By Definition 3.8 and Theorem 3.4, $(\mathcal{A}, \subseteq)$ is a complete lattice. For each $A \in \mathbb{A}$, we know that $A = \bigcup_{x\in A} K_x$ and each K_x is a compact element in $(\mathcal{A}, \subseteq)$, then $(\mathcal{A}, \subseteq)$ is an algebraic lattice.

Notice that to prove a complete lattice M is an algebraic lattice, it is enough to show the existence of a nonempty subset $M(x) \subseteq {\downarrow}x \bigcap K(M)$ such that $\sup M(x) = x$ for each $x \in P$. □

More over, we get the following theorem using our method, it is a representation of algebraic lattices by families of sets.

Theorem 3.11 *Let P be a poset. The following statements are always equivalent:*

(i) *P is an algebraic lattice.*

(ii) *P is isomorphism to an $\mathbb{A}$-sets.*

(iii) *P is isomorphism to a topped algebraic $\bigcap$-structure.*

Proof. (2)⇒ (1): By Lamma 3.10.

(3)⇒ (2): By Example 3.9 (2).

(1)⇒ (3): Let P be an algebraic lattice, we denote $\mathcal{A} = \{{\downarrow}x \bigcap K(P) : x \in P\}$, where $K(P)$ denote the set of all compact elements of P. Then, it is not difficult to check that $\mathcal{A}$ is a topped algebraic $\bigcap$-structure. Obvious, P is order isomorphism to $(\mathcal{A}, \subseteq)$. □

3.3 Completely distributive complete lattices and $\mathbb{C}$-$\bigcup$-semiring

In [5], Deng obtained the result that a complete lattice is completely distributive if and only if it is isomorphic to a complete $\bigcup$-semiring with condition $(*)$ which will be given below. In this subsection, we will show that a poset is a completely distributive complete lattice if and only if it is isomorphism to a complete $\bigcup$-semiring with Property $\mathbb{C}$.

Definition 3.12 *[5] Let X be a nonempty set and $\mathcal{P}(X)$ be the power set of X.*

(i) *A complete $\bigcup$-semiring is a family $\mathcal{A} \subseteq \mathcal{P}(X)$ which is closed under arbitrary unions.*

(ii) *For any $M \subseteq X$, denote $M^0 = \bigcup\{N \in \mathcal{A} : N \subseteq M\}$.*

(iii) *For any $x \in X$, denote $M_x = M^0$ where $M = \bigcap\{N \in \mathcal{A} : x \in N\}$.*

A complete $\bigcup$-semiring ordered by inclusion is a complete lattice. Let $\rho(X)$ be the set of those complete $\bigcup$-semirings satisfying the condition $(*)$: $N \subseteq \bigcup_{x \in N} M_x$ for each $N \in \mathcal{A}$.

Theorem 3.13 *[5] A complete lattice is completely distributive iff it is isomorphic to some $\mathcal{A} \in \rho(X)$.*

By Lemma 2.8, to prove a complete lattice L is completely distributive, it is sufficient to check $x = \sup \Downarrow x$ for each $x \in L$.

Definition 3.14 *A complete $\bigcup$-semiring is called a $\mathbb{C}$-$\bigcup$-semiring iff it satisfies Property $\mathbb{C}$.*

Lemma 3.15 *Let N be a completely distributive complete lattice. Then the family of sets $\mathcal{A}_N = \{\Downarrow x : x \in N\}$ is a $\mathbb{C}$-$\bigcup$-semiring and N is order isomorphic to $(\mathcal{A}_N, \subseteq)$*

Proof. The isomorphism can be naturally established by Lemma 2.8.

Suppose $x_i \in N$ for each $i \in I$. Then $\bigcup_{i\in I} \Downarrow x_i = \Downarrow \sup\{x_i : i \in I\}$: $\bigcup_{i\in I} \Downarrow x_i \subseteq \Downarrow \sup\{x_i : i \in I\}$ holds because $y \lhd x_i \leq \sup\{x_i : i \in I\}$ implies that $y \lhd \sup\{x_i : i \in I\}$. Suppose that $y \lhd \sup\{x_i : i \in I\} = \sup \bigcup_{i\in I} \Downarrow x_i$. Then there exists $i \in I$ such that $y \in \Downarrow x_i$. This means that $\bigcup_{i\in I} \Downarrow x_i \supseteq \Downarrow \sup\{x_i : i \in I\}$. So $\mathcal{A}_N$ is a complete $\bigcup$-semiring.

For Property $\mathbb{C}$, suppose $x, y \in N$ with $y \in \Downarrow x$. Since N is completely distributive, $x = \sup \Downarrow x = \sup \bigcup\{\Downarrow z : z \lhd x\}$. Thus there exists $z \in N$ such that $y \lhd z \lhd x$, i.e., $y \in \Downarrow z$ and $z \in \Downarrow x$. Notice that $\Downarrow z \subseteq \downarrow z \subseteq \bigcap\{\Downarrow a : z \lhd a\}$, we have $\Downarrow z \subseteq K_z$. This shows $x \in \Downarrow z \subseteq K_z$. So the family of sets $\mathcal{A}_N = \{\Downarrow x : x \in N\}$ satisfy Property $\mathbb{C}$. $\square$

Lemma 3.16 *Let $\mathcal{A}$ be a $\mathbb{C}$-$\bigcup$-semiring. Then $(\mathcal{A}, \subseteq)$ is a completely distributive complete lattice.*

Proof. By Definition 3.12, 3.14 and Lemma 2.8, we only need to check $K_x \lhd A$ for each $x \in A \in \mathcal{A}$.

Suppose that $A_i \in \mathcal{A}$ for each $i \in I$ and $A \subseteq \sup\{A_i : i \in I\} = \bigcup_{i\in I} A_i$. $x \in A$ implies there exists $i_0 \in I$ such that $x \in A_{i_0}$, then $K_x = \inf\{B \in \mathcal{A} : x \in B\} \subseteq$

$\bigcap\{B \in \mathcal{A} : x \in B\} \subseteq A_{i_0}$. □

Now, it is the positive to give the following Theorem 3.17, which can be viewed as a representation of completely distributive complete lattices.

Theorem 3.17 *A complete lattice is completely distributive if and only if it is isomorphic to some* $\mathbb{C}$-$\bigcup$*-semiring.*

Example 3.18 Let P be a poset and denote $\mathcal{U}(P) = \{A \subseteq P : A = \uparrow A\}$. It is obvious that $\mathcal{U}(P)$ is a complete $\bigcup$-semiring. For each $x \in P$, it is easy to get $K_x = \uparrow x$. Thus the family $\mathcal{U}(P)$ satisfies Property $\mathbb{C}$. Therefore, it is a $\mathbb{C}$-$\bigcup$-semiring. It follows from Theorem 3.17 that $\mathcal{U}(P)$ is a completely distributive complete lattice.

Contrasting Property $\mathbb{C}$ with condition $(*)$. Actually, a complete $\bigcup$-semiring $\mathcal{A}$ satisfies the condition $(*)$ iff it satisfies Property $\mathbb{C}$.

For each $x \in N \in \mathcal{A}$, denote $M_x = M^0 = \bigcup\{N \in \mathcal{A} : N \subsetneq M\}$, where $M = \bigcap\{N \in \mathcal{A} : x \in N\}$. Since $\mathcal{A}$ is closed under arbitrary unions, $M_x = M^0 = \inf\{N \in \mathcal{A} : x \in N\} = \mathcal{A}_x$. By Definition 3.12, we have $N \supseteq \bigcup_{x \in N} M_x$ for each $N \in \mathcal{A}$.

From Lemma 3.16 we know a $\mathbb{C}$-$\bigcup$-semiring is always completely distributive. Actually, if a complete $\bigcup$-semiring is completely distributive, then it is a $\mathbb{C}$-$\bigcup$-semiring.

Theorem 3.19 *A complete* $\bigcup$*-semiring is completely distributive if and only if it is a* $\mathbb{C}$-$\bigcup$*-semiring.*

Proof. We only need to prove that every completely distributive $\bigcup$-semiring satisfies Property $\mathbb{C}$.

Suppose that a complete $\bigcup$-semiring $\mathcal{A}$ is completely distributive. For each $A \in \mathcal{A}$, let $J = \{a : a \in A\}$, $K(a) = \{U : a \in U \in \mathcal{A}\}$ and $x_{a,U} = U$. It is obvious that $\{x_{j,k} : j \in J, k \in K(j)\}$ is a nonempty subset of $\mathcal{A}$. By Definition 2.6, we have

$$\bigvee_{j \in J} \bigwedge_{k \in K(j)} x_{j,k} = \bigwedge_{f \in M} \bigvee_{j \in J} x_{j,f(j)}.$$

where M denoted the set of choice functions defined on J with values $f(j) \in K(j)$.

On the left side,

$$\bigvee_{j \in J} \bigwedge_{k \in K(j)} x_{j,k} = \bigvee_{a \in A} \bigwedge_{U \in \mathcal{A}: a \in U} x_{a,U} = \bigvee_{a \in A} \bigwedge_{U \in \mathcal{A}: a \in U} U = \bigvee_{a \in A} K_a = \bigcup_{a \in A} K_a.$$

On the right side,

$$\bigwedge_{f \in M} \bigvee_{j \in J} x_{j,f(j)} = \bigwedge_{f \in M} \bigcup_{a \in A} x_{a,f(a)}.$$

Next we will prove that $\bigwedge\limits_{f \in M} \bigcup\limits_{a \in A} x_{a,f(a)} = A$.

If we choose $f_0 : J \to \bigcup_{j\in J} K(j)$ by $f_0(a) \equiv A$, then $f_0 \in M$, and

$$\bigwedge_{f\in M} \bigcup_{a\in A} x_{a,f(a)} \subseteq \bigcup_{a\in A} x_{a,f_0(a)} = \bigcup_{a\in A} A = A.$$

On the other hand, since $f(j) \in K(j)$ for each $f \in M$, $f(a) \in \{U : a \in U \in \mathcal{A}\}$. So $a \in f(a)$. Thus $\bigcup\limits_{a\in A} x_{a,f(a)} = \bigcup\limits_{a\in A} f(a) \supseteq A$. Since $A \in \mathcal{A}$, we have $A \subseteq \bigwedge\limits_{f\in M} \bigcup\limits_{a\in A} x_{a,f(a)}$.

This shows $\bigcup\limits_{a\in A} K_a = A$ for each $A \in \mathcal{A}$. That is to say, $\mathcal{A}$ satisfies Property $\mathbb{C}$.□

4 Topology with property $\mathbb{C}$

Let (X, τ) be a topological space. Then the nonempty family τ of sets ordered by inclusion is closed under arbitrary unions and finite intersections, therefore it is a complete $\bigcup$-semiring. By Theorem 3.17, the family of sets τ satisfies Property $\mathbb{C}$ if and only if it is a completely distributive complete lattice.

For each $x \in X$, we denote $\tau_x = \inf\{A \in \tau : x \in A\} = int \bigcap\{A \in \tau : x \in A\}$. Property $\mathbb{C}$ may not be satisfied for a general topological space (X, τ). See the following example:

Example 4.1 Let $X = [0, 1]$ and τ be the topology generated by the base $\{[0, a) : a \in [0, 1]\} \bigcup X$. Then $\tau_x = \phi$ for every $x \in [0, 1)$ and $\tau_1 = X$, so τ does not satisfy Property $\mathbb{C}$.

Proposition 4.2 *Let (X, τ) be a topological space with Property $\mathbb{C}$, then the family $\{\tau_x : x \in X\}$ is a base for the topological space (X, τ).*

Proof. Since $\tau_x = int \bigcap\{A \in \tau : x \in A\}$, $\{\tau_x : x \in X\} \subseteq \tau$. Suppose $y \in U \in \tau$, $y \in U = \bigcup_{x\in U} \tau_x$. Then there exists an element $x \in U$ such that $y \in \tau_x \subseteq U$. □

Definition 4.3 *[10] A c-space is any topological space X such that, for every $x \in X$, for every open neighbourhood U of x, there is a point $y \in U$ such that $x \in int(\uparrow y)$.*

Theorem 4.4 *Let (X, τ) be a topological space. The following statements are equivalent.*

(i) *(X, τ) is a c-space.*

(ii) *$(\tau, \subseteq)$ is a completely distributive lattice.*

(iii) *τ satisfies Property $\mathbb{C}$.*

Proof. Note that $int(\uparrow x) = int \bigcap\{A \in \tau : x \in A\} = \tau_x$ hold for any $x \in X$. Thus (X, τ) is a c-space iff it satisfies Property $\mathbb{C}$. □

Now we consider the Scott topology $\sigma(P)$ of a poset P.

Theorem 4.5 *Let P be a poset, the following statements are equivalent:*

(i) *P is a continuous poset.*

(ii) *$\sigma(P)$ is a $\mathbb{C}$-sets.*

(iii) *$\sigma(P)$ is completely distributive.*

(iv) *$(P, \sigma(P))$ is a c-space.*

Proof. (1) implies (2): Suppose $x \in U \in \sigma(P)$, $\sigma(P)_x = int \bigcap\{A \in \sigma(P) : x \in A\}$. Since every Scott open set is an upper set in P, then $\sigma(P)_x = int \bigcap\{A \in \sigma(P) : x \in A\} \supseteq int{\uparrow}x = \mathop{\uparrow\!\!\!\uparrow} x$. So we conclude that $\mathop{\uparrow\!\!\!\uparrow} x \subseteq \sigma(P)_x \subseteq U$ for each $x \in U$. Since P is a continuous poset, $x \in U \in \sigma(P)$ implies that there exists an element $y \in U$ such that $y \ll x$. Then $x \in \mathop{\uparrow\!\!\!\uparrow} y \subseteq \sigma(P)_y \subseteq U$. Thus the family of sets $\sigma(P)$ satisfy Property $\mathbb{C}$.

(2) implies (3): By Lemma 3.16.

(3) implies (1): By Theorem 4.9 in [16]. □

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