Available online at www.sciencedirect.com
ScienceDirect
Electronic Notes in Theoretical Computer Science 346 (2019) 625–643
Electronic Notes in Theoretical Computer Science
www.elsevier.com/locate/entcs

Two Problems on Interval Counting

Lívia Salgado Medeiros,[1] Fabiano de Souza Oliveira[2]

Universidade do Estado do Rio de Janeiro
Rio de Janeiro, Brazil

Jayme Luiz Szwarcfiter[3]

Universidade Federal do Rio de Janeiro *Universidade do Estado do Rio de Janeiro*
Rio de Janeiro, Brazil

Abstract

Let $\mathcal{F}$ be a family of intervals on the real line. An interval graph is the intersection graph of $\mathcal{F}$. An interval order is a partial order $(\mathcal{F}, \prec)$ such that for all $I_1, I_2 \in \mathcal{F}$, $I_1 \prec I_2$ if and only if I_1 lies entirely at the left of I_2. Such a family $\mathcal{F}$ is called a model of the graph (order). The interval count of a given graph (resp. order) is the smallest number of interval lengths needed in any model of this graph (resp. order). The first problem we consider is related to the classes of graphs and orders which can be represented with two interval lengths, regarding to the inclusion hierarchy among such classes. The second problem is an extremal problem which consists of determining the smallest graph or order which has interval count at least k. In particular, we study a conjecture by Fishburn on this extremal problem, verifying its validity when such a conjecture is constrained to the classes of trivially perfect orders and split orders.

Keywords: Extremal problems. Interval count. Interval graphs. Interval orders. Split graphs. Trivially perfect graphs.

1 Introduction

In graph theory, a well known graph class is that of interval graphs, which is strictly related to interval orders, also a well known topic in order theory. Interval graphs initially appeared in the areas of genetics and combinatorics [1]. A graph is an *interval graph* if its vertex set is a family of intervals on the real line, called a *model*, in which two distinct intervals are adjacent in the graph if such intervals intersect. An *interval order* is a partial order on a family of intervals on the real line in which the precedence relation correspond that of the intervals, that is, if the interval I_a precedes the interval I_b in the order, then I_a is entirely at the left of I_b.

[1] Email: liviasmedeiros@gmail.com

[2] Partially supported by FAPERJ. Email: fabiano.oliveira@ime.uerj.br

[3] Partially supported by CNPq (Brazil). Email: jayme@nce.ufrj.br

https://doi.org/10.1016/j.entcs.2019.08.055
1571-0661/

Ronald Graham raised the question of how many distinct interval lengths are required to any model of a given interval graph (cf. [2,8]). In other words, he suggested the problem of determining a model of a given interval graph having the smallest number of distinct interval lengths, which is called the *interval count* problem. A survey on the interval count parameter appeared in [1]. The problem of deciding efficiently whether a graph or order admits a model using a unique length is solved, since it is equivalent to the problem of recognizing unit interval graphs and orders (see [1]). However, deciding efficiently whether a graph or order admits a model using at most two lengths is open. Graham conjectured that, for every graph G, $IC(G) \leq k+1$ if $IC(G\backslash\{u\}) = k$ for some vertex $u \in V(G)$. Leibowitz, Assmann and Peck [2] proved it for $k = 1$ and provided a counterexample to such a conjecture for all $k \geq 2$, by presenting a graph G for each $k \geq 2$ such that, for a special vertex $u \in V(G)$, $IC(G\backslash\{u\}) = k$ and $IC(G) = k + 2$. Trotter [3] conjectured that the removal of a vertex can arbitrarily decrease the interval count (by a non-constant value), this is still open.

Although the study of the interval count parameter dates back the seventies, and despite all the efforts, most of the suggested problems of that time remain open until now. Klavík, in his excellent recent thesis on the subject, wrote:

> *The classes* k-LengthINT [graphs having interval count k] *were introduced by Graham as a natural hierarchy between unit interval graphs and interval graphs (...). Even after decades of research, the only results known are curiosities that illustrate the incredibly complex structure of such a class, very different from the case of unit interval graphs.* ([9], page 36)

In this work, we present new results on two problems. First, we study subclasses of graphs and orders having interval count at most two. An $\{a, b\}$*-model* is a model in which each interval has length a or b. The class of graphs which admit an $\{a, b\}$-model is denoted by $\textsc{Len}(a, b)$. Skrien [4] provided a characterization for $\textsc{Len}(0, 1)$. Rautenbach and Szwarcfiter [5] described a characterization and a linear-time algorithm to recognize graphs of $\textsc{Len}(0, 1)$. Boyadzhiyska, Isaak and Trenk [6] presented a characterization both for interval orders which admit a $\{0, 1\}$-model and for those which admit a $\{1, 2\}$-model. In [6], the question regarding the inclusion relation among these two classes was not considered. That motivated us to study the inclusion relation among the classes $\textsc{Len}(a', b')$ and $\textsc{Len}(a, b)$ for all $0 \leq a' < b'$, $0 \leq a < b$. Our results are described in Section 3. Except for the trivial case when $\textsc{Len}(a', b') = \textsc{Len}(a, b)$ which occurs when the lengths are proportional ($\frac{a'}{b'} = \frac{a}{b}$), $\textsc{Len}(a', b')$ surprisingly neither contains nor is contained in $\textsc{Len}(a, b)$.

Regarding the interval count parameter, more generally, it is not known how to efficiently decide whether the interval count of a graph or order is at most k, for all $k \geq 2$ (see [1]). Fishburn [7] studied the problem of determining the smallest order (in number of elements) and graph (in number of vertices) which requires k or more distinct interval lengths for their models. More specifically, this problem consists of determining the value of the function $\sigma_{\mathcal{C}}(k)$ (resp. $\tilde{\sigma}_{\mathcal{C}}(k)$) defined as the smallest number of intervals for which there is an order (resp. a graph) belonging to class $\mathcal{C}$ which has interval count at least k. Fishburn [7] formulated a conjecture to the value of $\sigma_{\mathcal{C}}(k)$ for $\mathcal{C}$ corresponding to general orders. The conjecture by Fishburn [7] is that $\sigma_{\mathcal{C}}(k) = 3k - 2$ when $\mathcal{C}$ is the class of all interval orders. We present the exact values of $\sigma_{\mathcal{C}}(k)$ and $\tilde{\sigma}_{\mathcal{C}}(k)$ for for $\mathcal{C}$ being the classes of trivially perfect and split orders and graphs. These results are presented in Section 4.

The paper is organized as follows. Section 2 describes the notation, Section 3 presents the results related to the inclusion hierarchy among the classes $\textsc{Len}(a, b)$ for all $0 \leq a < b$, Section 4 describes the study on Fishburn's conjecture, and Section 5 consists of concluding remarks. The missing proofs are found in Appendix.

2 Preliminary

Let $I = [\ell(I), r(I)]$ be a closed interval of the real line, where $\ell(I)$ and $r(I)$ denote respectively the *left* and *right extreme point* of I. The *interval graph* of a family $\mathcal{R}$ of such closed intervals is the graph G such that $V(G) = \mathcal{R}$ and, for all distinct $I, J \in V(G)$, $(I, J) \in E(G)$ if and only if $I \cap J \neq \emptyset$. We call $\mathcal{R}$ a *model* of G. An *order* $P = (X, \prec)$ is a binary relation $\prec$ on the set X which is irreflexive and transitive. Moreover, P is an *interval order* if there is a model $\mathcal{R} = \{I_x \mid x \in X\}$ such that, for all $x, y \in X$, $x \prec y$ if and only if $r(x) < \ell(y)$. We say that a graph G *agrees* with an order P if there is a same model corresponding to both G and P. For the sake of convenience, when there is a model $\mathcal{R}$ of a graph G (resp. order $P = (X, \prec)$), an interval $I_x \in \mathcal{R}$ and the corresponding vertex $x \in V(G)$ (resp. element $x \in X$) can be used interchangeably. The minimum number $IC(G)$ (resp. $IC(P)$) of distinct lengths of intervals required in a model of G (resp. P) is called *interval count* of G (resp. P). Regarding the determination of the interval count, graphs and orders are assumed to be free of twins, since intervals corresponding to two twin vertices/elements can coincide.

An interval graph G is *trivially perfect* (in class $\textsc{tp}$) if G is P_4-free. A graph G is an *extended bull* if it is isomorphic to the graph obtained from a path $x_1, y_1, \ldots, y_n, x_2$ by adding a vertex z such that z is adjacent to vertices $y_1, \ldots, y_n$ and is adjacent neither to x_1, nor to x_2. An interval graph G is a *split* graph (in class $\textsc{split}$) if there exists a partition (K, I) of $V(G)$ in which K is a clique and I is an independent set. An order P is said to be in a class $\mathcal{C}$ if the graph which agrees with P is in class $\mathcal{C}$. A vertex $v \in V(G)$ is called *simplicial* if $N[v]$ induces a clique in G. A *canonical model* $\mathcal{R}$ of a split graph G with partition $(K(G), I(G))$ is defined as follows. First, assume that the intervals in $I(G)$ are labeled as $1, 2, \ldots, |I(G)|$ from left to right in $\mathcal{R}$. Then, each interval $k \in K(G)$ is labeled (i, j) where i is the rightmost interval of $I(G)$ which k succeeds and j is the leftmost interval that k precedes. In addition, i (resp. j) is labeled as "-" when undefined, that is, in case that k does not succeed (resp. precede) any interval of $I(G)$. Furthermore, we will say that an interval belongs to $K_1(\mathcal{R})$, $K_2(\mathcal{R})$, or $K_3(\mathcal{R})$ if its canonical label is, respectively, $(-, j)$, $(i, -)$, or (i, j).

Let $P = (X, \prec)$. Define the *nesting relation* $\subset_A(P) = (X, \subset_A)$ as, for all $x, y \in X$, $x \subset_A y$ if there are $a, b \in X$ such that $a \prec x \prec b$ and y is incomparable to a, x, and b. We will say that x is *nested* in y, and y *nests* x. Note that in any possible model $\{I_v \mid v \in X\}$ of P, $|I_x| < |I_y|$. The *nesting depth* is defined as the largest value of k such that there exists $x_1 \subset_A x_2 \subset_A \ldots \subset_A x_k$.

3 On classes of interval count two

We show that $\textsc{Len}(a', b') \not\subseteq \textsc{Len}(a, b)$ if and only if $\frac{a'}{b'} \neq \frac{a}{b}$ for all $0 \leq a' < b'$ and $0 \leq a < b$. First, the classes $\textsc{Len}(0, b')$, for all $b' > 0$, will be compared to the classes $\textsc{Len}(a, b)$ for all $0 \leq a < b$ (Corollary 3.2 and Theorem 3.3). Then, we present a comparison among the classes $\textsc{Len}(a', b')$, for all $0 < a' < b'$, and $\textsc{Len}(a, b)$ for all $0 < a < b$ (Theorem 3.4).

Lemma 3.1 *Let $\mathcal{R}$ be an $\{a, b\}$-model of an order P. There exists an $\{ak, bk\}$-model of P for all $k > 0$.*

Proof. Let $\mathcal{R}'$ be the model obtained from $\mathcal{R}$ by multiplying each interval extreme point by $k > 0$. Note that $r(x) < \ell(y)$ in $\mathcal{R}$ $\iff$ $kr(x) < k\ell(y)$ $\iff$ $r(x) < \ell(y)$ in $\mathcal{R}'$. Furthermore, $r(x) - \ell(x) = a \iff kr(x) - k\ell(x) = ak$ and $r(x) - \ell(x) = b \iff kr(x) - k\ell(x) = bk$. Therefore, $\mathcal{R}'$ is a $\{ak, bk\}$-model of P. □

Corollary 3.2 $\textsc{Len}(0, b') = \textsc{Len}(0, b)$, *for all* $0 < b'$ *and* $0 < b$.

Proof. From Lemma 3.1, if there is a $\{0, b'\}$-model of a graph G, then there is a $\{0, b'(\frac{b}{b'})\}$-model of G. □

Theorem 3.3 $\textsc{Len}(0, k) \not\subseteq \textsc{Len}(a, b)$ *and* $\textsc{Len}(a, b) \not\subseteq \textsc{Len}(0, k)$, *for all* $k > 0$ *and* $0 < a < b$.

Proof. First, suppose $a \geq 1$. Consider the order corresponding to the $\{0, k\}$-model schemed in Figure 1 (a). Let G be the corresponding interval graph. In this figure, there are $b+2$ intervals of length 0 plus an interval of length k. Note that b intervals of those having length 0 are nested in the universal interval and, therefore, under the hypothesis of the existence of an $\{a, b\}$-model $\mathcal{R}$ of G, such b nested intervals must be of length a and the universal vertex of length b. On the other hand, the universal interval must have length greater than ba, which is at least b since $a \geq 1$, a contradiction. Therefore, $\textsc{Len}(0, k) \not\subseteq \textsc{Len}(a, b)$ for $a \geq 1$. Next, let $0 < a < 1$. From Lemma 3.1, $\textsc{Len}(a, b) = \textsc{Len}(1, b/a)$. Note that $b/a > 1$. Using the first part of the argumentation, we conclude that indeed $\textsc{Len}(0, k) \not\subseteq \textsc{Len}(a, b)$, in any case.

Fig. 1. (a) G admits a $\{0, k\}$-model but not an $\{a, b\}$-model, and (b) G admits an $\{a, b\}$-model but not a $\{0, k\}$-model.

Consider now G as the corresponding graph of the $\{a, b\}$-model depicted in Figure 1 (b). This model has a P_5 as induced subgraph, so the model is unique up to reversal (unique with respect to the order of maximal cliques, read in the model from left to right). The central vertex v of the P_5 is nested in the vertex w, universal to the P_5. This implies $|I_v| = a$ and $|I_w| = b$. Since v is not simplicial, it can not have length 0. Therefore, there does not exist a $\{0, k\}$-model of G, following that $\textsc{Len}(a, b) \nsubseteq \textsc{Len}(0, k)$. □

Theorem 3.4 $\textsc{Len}(a', b') \nsubseteq \textsc{Len}(a, b)$ *for all rational* $0 < a' < b'$, $0 < a < b$ *such that* $\frac{b'}{a'} \neq \frac{b}{a}$.

Proof. First, assume a', b', a, b are all natural numbers. Furthermore, assume $\frac{b'}{a'} < \frac{b}{a}$. The proof of $\textsc{Len}(a', b') \nsubseteq \textsc{Len}(a, b)$ is delineated as follows. We build an $\{a', b'\}$-model $\mathcal{R}$ associated to the values a', b', a, b. From such a model, it follows that $G \in \textsc{Len}(a', b')$, where G is the interval graph of $\mathcal{R}$. Then, we prove that $G \notin \textsc{Len}(a, b)$, following the result. Let $\mathcal{R}$ be the model schemed in Figure 2 (i). In the scheme, the intervals are placed in such a way:

- $r(x_i) - \ell(x_i) = a'$, for all $0 \leq i \leq b + 1$;
- $r(y_i) - \ell(y_i) = b'$, for all $1 \leq i \leq a$;
- $\ell(x_{i+1}) = r(x_i)$, for all $0 \leq i \leq b$;
- $\ell(y_1) = r(x_0) + \epsilon$; $\ell(y_{i+1}) = r(y_i) + \epsilon$ for all $1 \leq i < a$, for any $0 < \epsilon < \frac{ba'-ab'}{a}$.

Note that $r(y_a) = r(x_0) + ab' + a\epsilon < r(x_0) + ab' + a(\frac{ba'-ab'}{a}) = r(x_0) + ba' = \ell(x_{b+1})$. Therefore, the y_i-intervals all lie between $\ell(x_1)$ and $r(x_b)$, indeed as the scheme suggests. In $\mathcal{R}$, there are also more intervals than those explicitly represented. Each interval drawn as double and triple bars will denote that there are more intervals associated with it. If w is a double bar interval, then we shall add five more intervals to the model such that w and those five intervals consist of the submodel represented in the upper part of Figure 2 (ii). The actual left and right extreme points of those additional intervals are omitted because they can be chosen arbitrarily as long as the intervals form a submodel isomorphic to that of the figure and their lengths are the prescribed ones. If w is a triple bar interval, then we shall add three more intervals associated to w to form the submodel given by the lower part of Figure 2 (ii). This completes the description of $\mathcal{R}$.

Let G be the interval graph corresponding to $\mathcal{R}$. Suppose there exists an $\{a, b\}$-model $\mathcal{R}'$ of G. The following properties must hold in $\mathcal{R}'$:

- since y_i is the center of a claw, it must have length b, for all $1 \leq i \leq a$;
- since x_i is adjacent to the center of a P_5, but not to the other vertices of the P_5, it must be nested in any model of G and thus have length a, for all $1 \leq i \leq b$;
- $r(x_b) - \ell(x_1) \leq ab$ and $r(y_a) - \ell(y_1) > ab$;

From the last property, $r(x_b) - \ell(x_1) < r(y_a) - \ell(y_1)$, and therefore it does not hold that the intervals $y_1, \ldots, y_a$ lie between x_0 and x_{b+1}, which is not possible. Therefore, no such $\{a, b\}$-model of G exists.

For the case $\frac{b'}{a'} > \frac{b}{a}$, a similar reasoning can be used, by replacing all values a' by b', b' by a', a by b, and b by a in the scheme and the reasoning, and also conclude that $\textsc{Len}(a', b') \nsubseteq \textsc{Len}(a, b)$. Regarding the scheme, it is also necessary to change

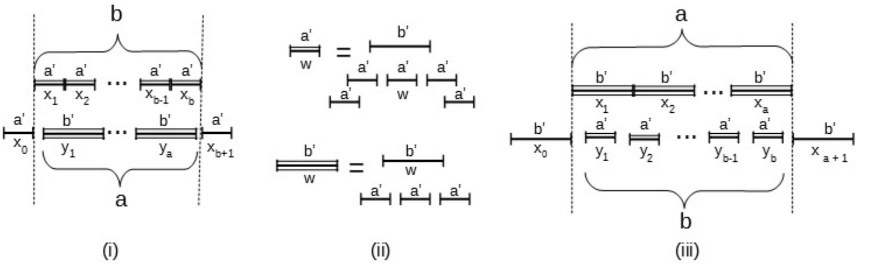

Fig. 2. Scheme of the model $\mathcal{R}$ used in Theorem 3.4.

the double bar intervals by triple bar intervals and vice versa. Figure 2 (iii) shows the modified scheme for this case.

Lastly, assume at least one of the values a', b', a, b is a rational non-natural number. Therefore, b'/a' and b/a are also rational numbers. Let $b'/a' = p_2/p_1$ and $b/a = q_2/q_1$ for natural numbers p_1, p_2, q_1, q_2. Let L be the least common multiple of p_1 and q_1. From Lemma 3.1, we have that $\textsc{Len}(a', b') = \textsc{Len}(1, b'/a') = \textsc{Len}(1, p_2/p_1) = \textsc{Len}(L, Lp_2/p_1)$. Similarly, $\textsc{Len}(a, b) = \textsc{Len}(L, Lq_2/q_1)$. By the first part of the argumentation, $\textsc{Len}(L, Lp_2/p_1) \not\subseteq \textsc{Len}(L, Lq_2/q_1)$. □

4 An extremal problem on the interval count

In this section, results on an extremal problem involving the interval count of trivially perfect and split orders and graphs will be presented. Fishburn [7] introduced the extremal problem of determining

$$\sigma(k) = \min\{|X| \mid P = (X, \prec) \text{ is an order and } IC(P) \geq k\}$$

and $\tilde{\sigma}(k)$ which is defined analogously for graphs. Fishburn [7] conjectured that $\sigma(k) = 3k - 2$, having proved the conjecture for all $k \leq 7$. He also established that $2k \leq \sigma(k) \leq 3k - 2$ for all $k \geq 2$ and that $\sigma(k) \leq \tilde{\sigma}(k)$. The orders which Fishburn [7] conjectured to have the smallest number of elements for given k are defined by their models, schematically depicted in Figure 3. In this figure, some of the extremes of intervals are not represented with a vertical dash, meaning that they can be increased or decreased as long as the interval is well-defined. Note that, in such orders, the number of elements is $3k - 2$, the conjectured value of $\sigma(k)$.

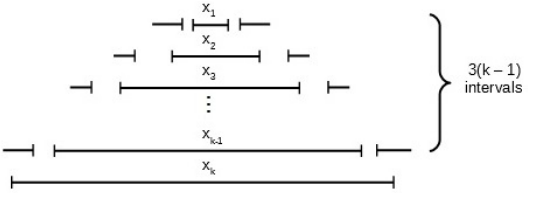

Fig. 3. The schematic model suggested by Fishburn [7].

In order to investigate further such a conjecture, we introduce the functions

$$\sigma_{\mathcal{C}}(k) = \min\{|X| \mid P = (X, \prec) \text{ is an order in class } \mathcal{C} \text{ and } IC(P) \geq k\}$$

and $\tilde{\sigma}_{\mathcal{C}}(k)$ which is defined analogously for graphs. Clearly, for $\mathcal{C}$ as the class of general interval orders (resp. graphs), we have that $\sigma_{\mathcal{C}}(k) = \sigma(k)$ and $\tilde{\sigma}_{\mathcal{C}}(k) = \tilde{\sigma}(k)$. For any $\mathcal{C}$, we have that $\sigma_{\mathcal{C}}(k) \geq \sigma(k)$ and $\tilde{\sigma}_{\mathcal{C}}(k) \geq \tilde{\sigma}(k)$. We will deal with the problem of determining $\sigma_{\mathrm{TP}}(k)$, $\tilde{\sigma}_{\mathrm{TP}}(k)$, $\sigma_{\mathrm{SPLIT}}(k)$, and $\tilde{\sigma}_{\mathrm{SPLIT}}(k)$.

Let G_k, for $k \geq 1$, a trivially perfect graph obtained from the following construction: if $k = 1$, G_1 consists of a single vertex u_1. If $k \geq 2$, G_k consists of three disjoint copies of G_{k-1} together with a universal vertex u_k. Figure 4 (i) illustrates a model of G_3.

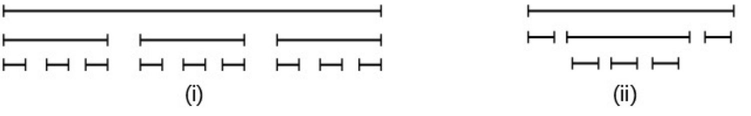

Fig. 4. (i) a model of G_3, (ii) a model of P_3.

Theorem 4.1 addresses the extremal problem restricted to the class of trivially perfect graphs.

Theorem 4.1 *Let $k \geq 1$. Let G be a smallest trivially perfect graph such that $IC(G) = k$. So, (i) $G \cong G_k$ and (ii) there exists a model $\mathcal{R}$ of G such that $IC(G) = k$ and the only interval having the largest length is the universal vertex of G.*

Proof. The proof is by induction on k. For $k = 1$, the smallest graph G such that $IC(G) = 1$ is the graph consisting of a single vertex, for which the result trivially holds. Let $k > 1$ and suppose the theorem holds for $G_{k'}$ for all $k' < k$. Let $\mathcal{R}$ be a model of G such that $IC(\mathcal{R}) = k$. Consider that G is obtained by the disjoint union of trivially perfect graphs $H_1, \ldots, H_\ell$, $\ell \geq 0$, plus the universal vertex u, where the intervals corresponding to the graphs $H_1, \ldots, H_\ell$ are found in this order from left to right in $\mathcal{R}$. Let P the order corresponding to $\mathcal{R}$ and P_i be the order corresponding to the submodel $\mathcal{R}_i$ of $\mathcal{R}$ induced by H_i. As P is an extended-bull free order [10], then $IC(P) = |\subset_A(P)|$. Note that, by construction of $\mathcal{R}$, the universal vertex of $\mathcal{R}$ nests every interval in each $\mathcal{R}_i$, $1 < i < \ell$, but not all intervals in $\mathcal{R}_1$ and $\mathcal{R}_\ell$, as for example, their respective universal vertices u_1 and u_ℓ. However, $\mathcal{R}$ nests every interval that u_1 and u_ℓ nest. Therefore, $IC(G) = \max\{IC(H_1), IC(H_2)+1, \ldots, IC(H_{\ell-1})+1, IC(H_\ell)\}$, since the left (resp. right) extreme of u_1 (resp. u_ℓ) can be decreased (resp. increased) until u_1 (resp. u_ℓ) has the same length of u. As G is the smallest trivially perfect graph having $IC(G) = k$, naturally $IC(H_i) \leq k-1$ for all $1 \leq i \leq \ell$. Therefore, there exists an H_i, for some $1 < i < \ell$, such that $IC(H_i) = k-1$. By the minimality of G, $\ell = 3$. Furthermore, note that $IC(H_1) = IC(H_3) = k-1$ since, for any of them, if (say) $IC(H_1) < k-1$, the model $\mathcal{R}'$ obtained from $\mathcal{R}$ by uniquely interchanging the positions of H_1 and H_2 would be such that $IC(\mathcal{R}') = \max\{IC(H_2), IC(H_1)+1, IC(H_3)\} = k-1$, contradicting $IC(G) = k$. Finally, as the number of vertices of G is minimum, the same must hold for each H_i. By applying the induction hypothesis for each H_i, from property (i), each $H_i \cong G_{k-1}$, for all $1 \leq i \leq 3$, and consequently, $G \cong G_k$, satisfying the property (i) for G. It is easy to realize that in $\mathcal{R}$ the largest length is that of u, satisfying also the property (ii). □

Theorem 4.2 *For all $k \geq 1$, $\tilde{\sigma}_{\mathrm{TP}}(k) = \frac{3^k-1}{2}$.*

Let P_k be the trivially perfect order for all $k \geq 1$ corresponding to the model $\mathcal{R}_k$ constructed as follows: for $k = 1$, $\mathcal{R}_1$ consists of a unique interval u_1. For $k \geq 2$, $\mathcal{R}_k$ is constructed as follows. Take a copy of the model $\mathcal{R}_{k-1}$, then add an interval which precedes all those in $\mathcal{R}_{k-1}$, another interval succeeding all those of $\mathcal{R}_{k-1}$, and a third interval which is universal (that is, corresponding to an element incomparable to all others). Figure 4 (ii) gives as an example, the order P_3.

Theorem 4.3 *Let $k \geq 1$. Let P a smallest trivially perfect order (in number of elements) such that $IC(P) = k$. Therefore, (i) $P \cong P_k$ and (ii) there exists a model $\mathcal{R}$ of P such that $IC(P) = k$ and the only interval having the largest length is the universal element of P.*

Theorem 4.4 *For all $k \geq 1$, $\sigma_{\mathrm{TP}}(k) = 3k - 2$.*

The extremal problem for the class of split graphs and orders is now addressed. Firstly, a family of split graphs will be presented in which, for all $k \geq 3$, there is a split graph G having $n = 3k - 1$ and $IC(G) = k$. Figure 5 presents a model of the split graph that, in particular, belongs to such a family, for $k = 3$ (and, therefore, for $n = 8$).

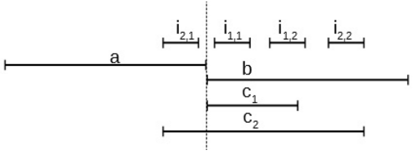

Fig. 5. A model of the split graph G having $IC(G) = 3$ and $n = 8$.

The construction of the family is by enumeration of each graph G_k member of the family, for all $k \geq 3$, by the definition of some model $\mathcal{R}_k$ of G_k. For $k = 3$, $\mathcal{R}_k$ is that depicted in Figure 5. For $k > 3$, $\mathcal{R}_k$ is obtained from $\mathcal{R}_{k-1}$ by the following transformation: three vertices $i_{k,1}$, $i_{k,2}$ and c_{k-1} are added such that $i_{k,1}$ (resp. $i_{k,2}$) is placed at left (resp. at right) of the first (resp. last) interval of $\mathcal{R}_k$ among those in $I(G_{k-1})$. Besides, $c_{k-1} = [\ell(i_{k,1}), r(i_{k,2})]$. Then, the extreme point $\ell(a)$ (resp. $r(b)$) is decreased (resp. increased) in such way that $|a|$ (resp. $|b|$) becomes equal to $|c_{k-1}|$. Theorem 4.5 establishes some properties of G_k. For an interval x, we say that $\ell(x)$ (resp. $r(x)$) is *free* if x belongs to the first (resp. last) maximal clique of the model.

Theorem 4.5 *The following statements hold with respect to G_k and $\mathcal{R}_k$:*

(i) *$\mathcal{R}_k$ is unique (with respect to the order of maximal cliques) up to reversal, and the reversal of the comparability relation between $i_{1,1}$ and $i_{1,2}$.*

(ii) *The intervals c_{k-1}, a, and b have the same length.*

(iii) *$IC(G_k) = IC(\mathcal{R}_k) = k$.*

(iv) *The interval a (resp. b) has free left (resp. right) extreme point.*

(v) $|V(G_k)| = 3k - 1$.

Fishburn established that $\sigma(k) \le \tilde{\sigma}(k)$. As it has been shown, there exists a family of split graphs G_k such that $|V(G_k)| = 3k - 1$ and $IC(G) = k$. We now prove that, for any split graph G with $|V(G)| = 3k - 2$, $IC(G) \le k - 1$ holds, and therefore $\tilde{\sigma}_{\text{SPLIT}}(k) = 3k - 1$.

Lemma 4.6 *Without loss of generality, in any model of a split order having the minimum number of lengths, the intervals of the independent set are intervals having the smallest length.*

Let $\mathcal{R}$ be a model of a split graph G and $v = (i, j) \in K_3(\mathcal{R})$ be the canonical representation of v. We will denote by $c(v)$ the number of intervals of $I(G)$ which intersect v, and call it the *covering* of v. That is, $c(v) = j - i - 1$.

Lemma 4.7 *Let P be a split order and G be the graph which agrees with P. Let $r(\mathcal{R})$ be the number of distinct lengths in $K_3(\mathcal{R}) \cup I(G)$, where $\mathcal{R}$ is a model of P. There exists a model $\mathcal{R}'$ of P such that $r(\mathcal{R}') \le \lfloor \frac{|I(G)|}{2} \rfloor$.*

Proof (Sketch) Let $\mathcal{R}$ be a model of P. Let p be a point belonging to all intervals of $K(G)$. Without loss of generality, p does not belong to any interval of $I(G)$. Build another model $\mathcal{R}'$ of G as follows. Firstly, define $|I(G)|$ intervals in $\mathcal{R}'$, numbered from 1 to $|I(G)|$, corresponding to the intervals of $I(G)$, all of them having unit length and equally distributed (same space between any two consecutive ones). Let t be such spacing between two consecutive intervals of $I(G)$. Suppose that p in $\mathcal{R}$ is found between, say, the intervals z and $z + 1$ of $I(G)$. Let $p' = \frac{(r(z)+\ell(z+1))}{2}$ be the point in $\mathcal{R}'$ that plays the same role as p in $\mathcal{R}$, that is, that corresponds to the point included in every interval of $K(G)$.

Now we will define an interval in $\mathcal{R}'$ for each interval of $K(G)$. Note that, whichever those intervals are defined, the length of each interval is bounded below and above, since the intervals of $I(G)$ are already defined in $\mathcal{R}'$. We define the intervals of $K_3(\mathcal{R}')$ in $C_0, C_1, \ldots, C_s$, where $s = \lfloor \frac{|I(G')|}{2} - 1 \rfloor$, such that $C_i = \{k \in K_3(\mathcal{R}') \mid \lfloor \frac{c(k)}{2} \rfloor = i\}$ for all $0 \le i \le s$. Such division may not consist of a partition since it is possible that some of such sets are empty. The proof consists in showing that it is possible to assign a same length to intervals of each set that respect those bounds and, besides, that the lengths of intervals in C_0 can be assigned the unitary length. Finally, it is easy to see that it is possible to insert in $\mathcal{R}'$ the intervals of $K_1(\mathcal{R})$, $K_2(\mathcal{R})$ and the universal one (if existing), all of them having a same length, since the intervals already positioned in $\mathcal{R}'$ keep the same comparability of those of $\mathcal{R}$, $\mathcal{R}'$ turning out to be a model of P in which $r(\mathcal{R}') \le \lfloor \frac{|I(G)|}{2} \rfloor$. □

Lemma 4.8 *Let P be a split order such that $|K_1(P)| > 1$. Let $\mathcal{R}$ be a model of P and $a \in K_1(\mathcal{R})$ be the interval having the smallest right extreme point. Then, $IC(P) = IC(P \setminus a)$.*

Lemma 4.9 *Let G be a connected split graph. There exists a model $\mathcal{R}$ of G such that $K_1(\mathcal{R}) \neq \emptyset$.*

Lemma 4.10 *Let $\mathcal{R}$ be a model of a split graph G such that $|K_3(\mathcal{R})| = 0$. Therefore, there exists a model $\mathcal{R}$ of G such that $IC(\mathcal{R}) = IC(G) \leq 2$.*

Theorem 4.11 *Let G be a connected split graph such that $n = 3k - 2$ for some natural k. Then, $IC(G) \leq k - 1$.*

Proof. Let $\mathcal{R}$ be a model of G. From Lemma 4.9, we consider without loss of generality that $K_1(\mathcal{R}) \neq \emptyset$. Thus, once $IC(\mathcal{R}) \leq k - 1$ is proved, then $IC(G) \leq IC(\mathcal{R}) \leq k - 1$, following the result. We also can assume that $|K_1(\mathcal{R})| = 1$ since, otherwise, we can decrease $|K_1(\mathcal{R})|$ to the unit, by successively applying the transformation described in the proof of Lemma 4.8, resulting into a model $\mathcal{R}'$ in which $|K_1(\mathcal{R}')| = 1$ without changing the interval count of the order P corresponding to $\mathcal{R}$. Therefore, once $IC(\mathcal{R}') \leq k - 1$ is proved, then $IC(G) \leq IC(P) = IC(\mathcal{R}') \leq k - 1$ follows. By analogous reasoning, and considering the operation of reversal, we can also assume that $K_2(\mathcal{R}) = \emptyset$ or $|K_2(\mathcal{R})| = 1$. Furthermore, as G is connected, $K_2(\mathcal{R}) \neq \emptyset$ or there exists a universal vertex u.

When $K_2(\mathcal{R}) \neq \emptyset$, it is possible to modify $\mathcal{R}$ so that all intervals of $K_1(\mathcal{R}) \cup K_2(\mathcal{R})$ have the same length (since all of them have a free extreme point, at left or at right). When $K_2(\mathcal{R}) = \emptyset$, then the lengths of the interval of $K_1(\mathcal{R})$ and u can assume the same value (since either the universal interval or the interval of $K_1(\mathcal{R})$ can be enlarged until their lengths become equal).

Also, it is assumed that in $\mathcal{R}$ all intervals of $I(G)$ have the smallest length of the model, applying the transformation of Lemma 4.6.

Let a be the interval of $K_1(\mathcal{R})$. If $K_2(\mathcal{R}) = \emptyset$, define $b = u$. Otherwise, let b be the interval of $K_2(\mathcal{R})$. To sum up, after all the implicit transformations mentioned so far, we suppose $\mathcal{R}$ a model in which there is a same length for all intervals of $I(G)$ and another length (possibly distinct of the former) for the intervals a and b. The proof continues by case analysis of the value of $|I(G)|$.

(i) If $|I(G)| \geq 2k - 1$, let $S = V(G) \setminus I(G) \setminus \{a, b\}$. Therefore, $|S| \leq (3k - 2) - (2k - 1) - 2 = k - 3$. From the previous assumptions on $\mathcal{R}$, we know all intervals of $I(G)$ have a same length, and the intervals a and b have also a same length. Therefore, $IC(G) \leq IC(\mathcal{R}) \leq k - 3 + 2 = k - 1$.

(ii) If $|I(G)| \leq 2k - 3$, applying the transformation of Lemma 4.7 on $\mathcal{R}$, we have that $r(\mathcal{R}) \leq \lfloor \frac{2k-3}{2} \rfloor = k - 2$. That is, there are at most $k - 2$ distinct lengths among the intervals of $K_3(\mathcal{R}) \cup I(G)$ and, considering one more length for both a and b, we have that $IC(G) \leq IC(\mathcal{R}) \leq k - 1$. Finally, note that if $K_2 \neq \emptyset$ and there exists a universal interval u, the lengths of u, a, b can be made equal, since all of them have a free extreme point.

(iii) If $|I(G)| = 2k - 2$, let $S = V(G) \setminus I(G) \setminus \{a, b\}$. Thus, $|S| = (3k - 2) - (2k - 2) - 2 = k - 2$. Consider the application of the transformation described in Lemma 4.7 in the model $\mathcal{R}$. Let $C_0, C_1, \ldots, C_s$ be the division of $K_3(\mathcal{R})$ obtained by such application with $s = \lfloor \frac{|I(G)|}{2} - 1 \rfloor = k - 2$. Since $|S| = k - 2$, some C_i must be empty, $0 \leq i \leq s$. If there exists $C_i = \emptyset$, with $0 < i \leq s$, then it follows from the transformation that $r(\mathcal{R}) \leq s = k-2$. Counting one more length for a, b and (if existing) u, $IC(\mathcal{R}) \leq k-1$ (the transformation of the length of u in that of a, b is done as in the case (ii)). Otherwise, as $|S| = k-2$, then $C_0 = \emptyset$, $|C_i| = 1$ for all $1 \leq i \leq s$ and the interval of C_i, produced by the transformation, has a distinct length of that of interval of C_j, $1 \leq i < j \leq s$. Let f and l be respectively the first and last intervals of $I(G)$. The proof continues accordingly to what b consists of.

(a) If $K_2(\mathcal{R}) = \{b\}$, let $c = (1, |I(G)|)$ be the canonical representation of the interval of C_s. Decrease $\ell(c)$ (if necessary) so that $r(f) < \ell(c) < \ell(b)$ and increase $r(c)$ (if necessary) so that $r(a) < r(c) < \ell(l)$. As a has free left extreme point and b has free right extreme point is possible to move those free extreme points so that the corresponding intervals have a same length and the largest of the model. The transformation of $\mathcal{R}$ continues as follows: move f (resp. l) to the left (resp. right) by x units and decrease $\ell(c)$ (resp. increase $r(c)$) also by x units. The goal is to make the intervals a, b and c have the same length after such transformation, which is possible by letting $x = |a| - |c|$ (since a and b will have length $|a| + x = 2|a| - |c|$ in the end of the transformation, whereas c will have length $|c| + 2x = 2|a| - |c|$). Thus, $IC(\mathcal{R}) \leq k-1$.

(b) If $u = b$ (and, therefore, $K_2(\mathcal{R}) = \emptyset$), then as a has free left extreme, we initially decrease $\ell(a)$ so that the length of a to become the largest of $\mathcal{R}$. Let c_s be the interval of C_s. Let $a = (-, j_a)$ and $c_i = (i_c, j_c)$ be the canonical representation of the interval of C_i, for some $1 \leq i \leq s$, having the smallest j_c such that $j_c \geq j_a$. Let z be the $(j_a - 1)$-th interval of $I(G)$ (from left to right) in $\mathcal{R}$. If necessary, increase $r(c_i)$ so that $r(c_i) > \max\{r(a), r(z)\}$. Now, modify the model $\mathcal{R}$ through increasing of $|a| - |c_i|$ every interval extreme point which is greater or equal to $r(c_i)$. By such modification, the lengths of a and c_i become the same. After that, move the interval l, the last interval of $I(G)$ in $\mathcal{R}$ so that l becomes the first of $I(G)$, with no intersection to a, and decrease $\ell(u)$ so that $\ell(u)$ becomes equal to $r(l)$. Due to this transformation, c_s now has free right extreme point and it is possible to increase $r(c_s)$ so that c_s and u have the same length. Finally, note that if $i = s$, then a also now has a free right extreme point and similarly it is possible to increase $r(a)$ so that $|a|$ becomes $|c_s| = |u|$. In the interval counting, if $i < s$, the intervals c_i and a have the same length, whereas c_s and u have also a same length. In case $i = s$, the length of each interval a, c_s, and u becomes the same. In any case, $IC(\mathcal{R}) \leq k-1$.

□

Theorem 4.12 *Let $k \geq 2$ be a natural number. There exists a split order $P = (X, \prec)$ such that $IC(P) = k$, for $|X| = 3k-2$.*

Proof. Let $\mathcal{R}$ be the model using k distinct lengths of P schematically defined in Figure 6. Note that $z \subset_A c_1 \subset_A c_2 \subset_A \cdots \subset_A c_{k-2} \subset_A u$. Therefore, $IC(P) = k$.

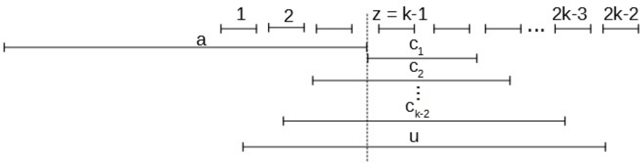

Fig. 6. Model $\mathcal{R}$ of a split order $P = (X, \prec)$ with $|X| = 3k - 2$ and $IC(P) = k$.

□

Theorem 4.13 *Let G be a connected split graph with $3k-3$ vertices and $P = (X, \prec)$ be a split order agreeing with G. Therefore, $IC(P) \leq k - 1$.*

5 Conclusion

In this work, we studied the inclusion hierarchy among the classes $\text{Len}(a, b)$, for all $0 \leq a < b$. In particular, we showed that $\text{Len}(a', b') \nsubseteq \text{Len}(a, b)$ if and only if $\frac{a'}{b'} \neq \frac{a}{b}$. In addition, motivated by the Fishburn's conjecture $\sigma(k) = 3k - 2$ [7], we have defined the parameters $\sigma_{\mathcal{C}}(k)$ and $\overline{\sigma}_{\mathcal{C}}(k)$, which specializes the parameters defined by Fishburn, and investigated them for $\mathcal{C} \in \{\text{TP}, \text{SPLIT}\}$. We proved that $\sigma_{\text{TP}}(k) = 3k - 2$, $\sigma_{\text{SPLIT}}(k) = 3k - 2$, confirming the conjecture if the orders are restricted to those classes. For graphs, for which there were neither a conjecture nor known exact values, we proved that $\tilde{\sigma}_{\text{TP}}(k) = (3^k - 1)/2$, $\tilde{\sigma}_{\text{SPLIT}}(k) = 3k - 1$.

References

[1] Cerioli, M. R., Oliveira, F. S., and Szwarcfiter, J. L., *The Interval Count of Interval Graphs and Orders: a short survey*, Journal of the Brazilian Computer Society, **18**, 103–112, 2012.

[2] Leibowitz, R., Assmann, S. F., and Peck, G. W., *Interval counts of interval graphs*. SIAM Journal on Algebraic and Discrete Methods, **3**, 485—494, 1982.

[3] Trotter, W. T, *Interval graphs, interval orders, and their generalizations*. SIAM Applications of Discrete Mathematics (eds. R. Ringeisen & F. Roberts), 45—58, 1988.

[4] Skrien, D., *Chronological orderings of interval graphs*, Discrete Applied Mathematics, **8**, 69—83, 1984.

[5] Rautenbach, D., Szwarcfiter, J.L.,*Unit and single point interval graphs*, Discrete Applied Mathematics, **160**, 1601-–1609, 2012.

[6] Boyadzjiska, S., Isaak, G., Trenk, A. N., *Interval orders with two interval lengths*, arXiv preprint arXiv, **1707.08093**, 2017.

[7] Fishburn, P. C., *Interval Orders and Interval Graphs*, John Wiley & Sons, 1985.

[8] Leibowitz, R., *Interval counts and threshold graphs*, PhD thesis, Rutgers University, 1978.

[9] Klavík, P., *Extension properties of graphs and structures*, PhD thesis, Charles University, 2017.

[10] Cerioli, M. R., Oliveira, F. S., and Szwarcfiter, J. L., *On counting interval lengths of interval graphs*, Discrete Applied Mathematics, **159**, 532–543, 2011.

[11] Cerioli, M. R., Oliveira, F. S., and Szwarcfiter, J. L., *Interval count of generalizations of threshold graphs*, Matemática Contemporânea, **42**, 9–16, 2014.

Appendix

Theorem 4.2 For all $k \geq 1$, $\tilde{\sigma}_{\mathrm{TP}}(k) = \frac{3^k-1}{2}$.

Proof. From Theorem 4.1, G_k is the smallest trivially perfect graph having $IC(G) = k$. Therefore, $\tilde{\sigma}_{\mathrm{TP}}(k) = |V(G_k)|$. We shall determine $|V(G_k)|$. By construction, note that such determination is reduced to the problem of solving a recurrence equation, described as follows:

$$\tilde{\sigma}_{\mathrm{TP}}(k) = \begin{cases} 1, & if\ k = 1, \\ 3\tilde{\sigma}_{\mathrm{TP}}(k-1) + 1, & if\ k > 1. \end{cases}$$

whose closed-form is given by $\tilde{\sigma}_{\mathrm{TP}}(k) = \frac{3^k-1}{2}$. □

Theorem 4.3 Let $k \geq 1$. Let P a smallest trivially perfect order (in number of elements) such that $IC(P) = k$. Therefore, (i) $P \cong P_k$ and (ii) there exists a model $\mathcal{R}$ of P such that $IC(P) = k$ and the only interval having the largest length is the universal element of P.

Proof. Let $\mathcal{R}$ be a model of P such that $IC(\mathcal{R}) = k$. Let G be the graph that agrees with P. Consider that G is obtained by the disjoint union of the trivially perfect graphs $H_1, \ldots, H_\ell$, $\ell \geq 0$, with the addition of a universal vertex u, where the intervals corresponding to the graphs $H_1, \ldots, H_\ell$ are found in such an order from left to right in $\mathcal{R}$. Note that it is possible to follow this proof by applying a similar induction as that in Theorem 4.1 and conclude that there exists an H_i, for some $1 < i < \ell$, such that $IC(H_i) = k - 1$. By the minimality of P, $\ell = 3$, $IC(H_1) = IC(H_3) = 1$ and $IC(H_2) = k - 1$. Also, by the induction hypothesis, the order corresponding to the intervals of H_2 in $\mathcal{R}$ is isomorphic to P_{k-1}. Consequently, $P \cong P_k$, satisfying the property (i). It is easy to realize that in $\mathcal{R}$ the largest length is that of the universal vertex of P, satisfying the property (ii). □

Theorem 4.4 For all $k \geq 1$, $\sigma_{\mathrm{TP}}(k) = 3k - 2$.

Proof. From Theorem 4.3, $P_k = (X, \prec)$ is the smallest order which is trivially perfect having $IC(P) = k$. Therefore, $\sigma_{\mathrm{TP}}(k) = |X|$. We have to determine $|X|$. By the construction, note that such a determination is reduced to the problem of solving a recurrence equation, defined as follows:

$$\sigma_{\mathrm{TP}}(k) = \begin{cases} 1, & if\ k = 1, \\ \sigma_{\mathrm{TP}}(k-1) + 3, & if\ k > 1. \end{cases}$$

It follows that $\sigma_{\mathrm{TP}}(k) = 3k - 2$, meeting the conjecture of Fishburn [7] even when the orders are restricted to the class of trivially perfect orders. □

Theorem 4.5 The following statements hold with respect to G_k and $\mathcal{R}_k$:

(i) $\mathcal{R}_k$ is unique (with respect to the order of maximal cliques) up to reversal, and the reversal of the comparability relation between $i_{1,1}$ and $i_{1,2}$.

(ii) The intervals c_{k-1}, a, and b have a same and the largest length of $\mathcal{R}_k$.

(iii) $IC(G_k) = IC(\mathcal{R}_k) = k$.

(iv) The interval a (resp. b) has free left (resp. right) extreme point.

(v) $|V(G_k)| = 3k - 1$.

Proof. We show the result by induction on k. For $k = 3$, it is easy to verify that $\mathcal{R}_3$ satisfies the properties $(i) - (v)$. In particular, the property (i) is realized by the fact that, when orienting arbitrarily an edge of $\overline{G_3}$, all other edges have their orientation forced in order to produce a transitive orientation, except for $(i_{1,1}, i_{1,2})$, following the result. Regarding the property (iii), note that $i_{1,1} \subset_A c_1 \subset_A c_2$ (or $i_{1,2} \subset_A c_1 \subset_A c_2$, if the reversal of comparability is considered) requiring therefore distinct lengths. Thus, $IC(G) = 3$. Let $k \geq 4$ and suppose the properties $(i) - (v)$ hold for G_{k-1} and $\mathcal{R}_{k-1}$. It will be shown that those properties also hold for G_k and $\mathcal{R}_k$.

(i) By considering the inclusion of $i_{k,1}$ and $i_{k,2}$ in $I(G_{k-1})$, note that $i_{k,1}$, a, b, $i_{k,2}$ induces a P_4, whose corresponding submodel is unique up to reversal. As, by the hypothesis induction, $\mathcal{R}_{k-1}$ is unique up to reversal, and the reversal of the comparability between $i_{1,1}$ and $i_{1,2}$, and $\mathcal{R}_k$ has thus a P_4 sharing their central vertices with vertices of $\mathcal{R}_{k-1}$, then $\mathcal{R}_k$ admits model which is unique up to reversal, and the reversal of the comparability between $i_{1,1}$ and $i_{1,2}$, since the interval c_{k-1} that is not taken into account yet is incomparable to all others.

(ii) As $c_{k-2} \subset_A c_{k-1}$ and, by hypothesis induction, c_{k-2} is the largest length in $\mathcal{R}_{k-1}$, then c_{k-1} requires a distinct length from the existing lengths in $\mathcal{R}_{k-1}$, which we can assume be the largest one. As a has left extreme point and b has free right extreme point, it is possible to enlarge them so their lengths are made equal to the largest length, which is that of c_{k-1}.

(iii) By the hypothesis induction, we have that $IC(\mathcal{R}_{k-1}) = k - 1$ and, from items (i) and (ii), it follows that the intervals c_{k-1}, a and b have a same and the largest length in $\mathcal{R}_k$ which is larger than those lengths in $\mathcal{R}_{k-1}$. Therefore, $IC(G_k) = IC(\mathcal{R}_k) = k$.

(iv) By the hypothesis induction, we have that a has a free left extreme point and b has a free right extreme point. As a and b intercept respectively $i_{k,1}$ and $i_{k,2}$ by construction and since the free extreme points of intervals a and b will be moved to the left and to right, respectively, in order to those intervals have the length of c_{k-1}, so a has free left extreme point and b has free right extreme point in $\mathcal{R}_k$.

(v) $|V(G_k)| = |V(G_{k-1})| + 3 = 3(k-1) - 1 + 3 = 3k - 1$.

□

Lemma 4.6 Without loss of generality, in any model of a split order having the minimum number of lengths, the intervals of the independent set are intervals having the smallest length.

Proof. Let $\mathcal{R}$ be a model of a split order P such that $IC(\mathcal{R}) = IC(P)$. Let G be the graph with which P agrees. Let m be the smallest length over the intervals of $\mathcal{R}$. Let $z \in I(G)$ be such that $|z| > m$. Note that it is possible, through the procedure of increasing $\ell(z)$ and decreasing $r(z)$, to make the length of z become m without affecting the intersections among intervals because, otherwise, there would exist two intervals a, b which intersect z having $\ell(b) - r(a) > m$. In this case, as a, b intersect z, they belong to $K(G)$, which is a contradiction since they would need to be mutually intersecting. The result follows by the iterative application of this transformation onto all intervals of $I(G)$ having length greater than m. □

Let $\mathcal{R}$ be a model of a split graph G and $v = (i, j) \in K_3(\mathcal{R})$ be the canonical representation of v. We will denote by $c(v)$ the number of intervals of $I(G)$ which intersect v, and call it *covering* of v. That is, $c(v) = j - i - 1$.

Lemma 4.7 Let P be a split order and G be the graph which agrees with P. Let $r(\mathcal{R})$ be the number of distinct lengths in $K_3(\mathcal{R}) \cup I(G)$, where $\mathcal{R}$ is a model of P. There exists a model $\mathcal{R}'$ of P such that $r(\mathcal{R}') \leq \lfloor \frac{|I(G)|}{2} \rfloor$.

Proof. Let $\mathcal{R}$ be a model of P. Let p be a point belonging to all intervals of $K(G)$. Without loss of generality, p does not belong to any interval of $I(G)$ (since it is possible to increase right extreme points and decrease left extreme points of intervals of K so that such a point p can be chosen). Build another model $\mathcal{R}'$ of G as follows. Firstly, define $|I(G)|$ intervals in $\mathcal{R}'$, numbered from 1 to $|I(G)|$, corresponding to the intervals of $I(G)$, all of them having unit length and distributed (same space between any two consecutive ones). Let t be such spacing between two consecutive intervals of $I(G)$. Suppose that p in $\mathcal{R}$ is found between, say, the intervals z and $z+1$ of $I(G)$. Let $p' = \frac{(r(z)+\ell(z+1))}{2}$ be the point in $\mathcal{R}'$ that plays the same role as p in $\mathcal{R}$, that is, that corresponds to the point included in every interval of $K(G)$.

Now we will define an interval in $\mathcal{R}'$ for each interval of $K(G)$. Note that, whichever those intervals are defined, the length of each interval is bounded below and above, since the intervals of $I(G)$ are already defined in $\mathcal{R}'$. More specifically, if $k \in K_3(\mathcal{R}')$, the upper bound is given by

(.1) $|k| < c(k) + (c(k) + 1)t$.

Regarding the lower bound, this depends on two types of intervals. For the intervals $(i, j) \in K_3(\mathcal{R}')$ such that $i \neq z$ and $j \neq z + 1$, the lower bound is given by

(.2) $|k| > c(k) - 2 + (c(k) - 1)t$.

For the intervals $(i, j) \in K_3(\mathcal{R}')$ having $i = z$ or $j = z + 1$, the lower bound is given by

(.3) $|k| > c(k) - 1 + (c(k) - 1)t + \dfrac{t}{2}$.

Figure .1 illustrates a model $\mathcal{R}$ such that the interval $k \in K_3(\mathcal{R})$ has covering $c(k) = 4$. Thus, in Figure .1 (i), the upper bound of $k \in K_3(\mathcal{R})$ is given by the inequation (.1) and in Figures .1 (ii),(iii), the lower bound of the interval $k \in K_3(\mathcal{R})$ are given by inequations (.2) and (.3), respectively.

The intervals of $K_3(\mathcal{R}')$ will be divided in $C_0, C_1, \ldots, C_s$, where $s = \lfloor \frac{|I(G')|}{2} - 1 \rfloor$. Such division may not be a partition since it will be possible that some of these sets are empty. The prove consists in showing that it is possible to assign a same length

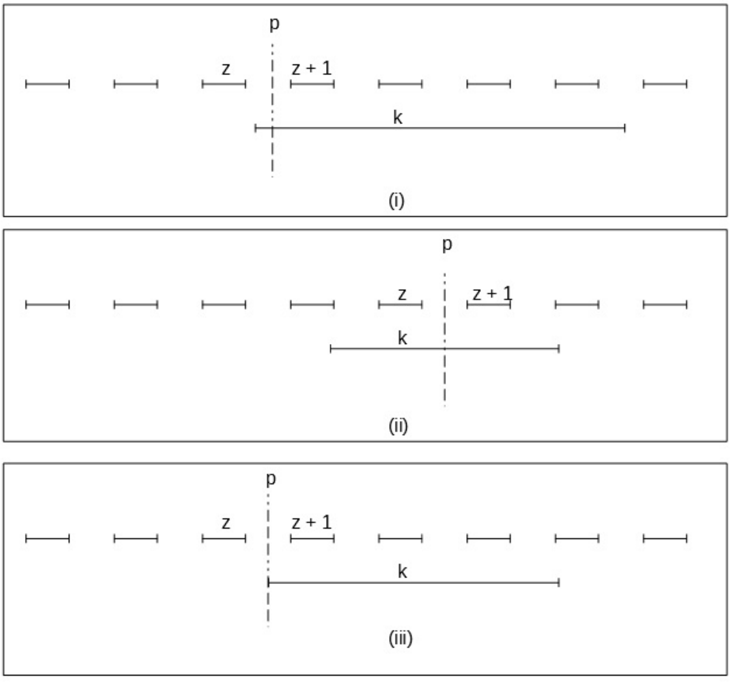

Fig. .1. Models of a split graph having $c(k) = 4$.

to all intervals in a same set and, besides that, that the lengths of the intervals in C_0 can be made unitary, following the result.

The division is done as follows: for all $0 \leq i \leq s$, $C_i = \{k \in K_3(\mathcal{R}') \mid \lfloor \frac{c(k)}{2} \rfloor = i\}$. In order to comply the goal of intervals in each C_i have a same length ℓ_i, it is necessary to show that it is possible to choose such ℓ_i satisfying the inequations (.1), (.2), and (.3) associated to all intervals of C_i. In other words,

$$\ell_i < c(k) + (c(k) + 1)t,$$

for all $1 \leq i \leq s$ and for all $k \in C_i$. By the construction of C_i, the above inequality can then be reduced to

$$\ell_i < 2i + (2i + 1)t,$$

for all $1 \leq i \leq s$. On the other hand, regarding the lower bound, we have that

$$\ell_i > c(k) - 2 + (c(k) - 1)t,$$

for all $1 \leq i \leq s$, and for all $k = (i', j') \in C_i$ with $i' \neq z$, $j' \neq z + 1$, and we have that

$$\ell_i > c(k) - 1 + (c(k) - 1)t + \frac{t}{2},$$

for all $1 \leq i \leq s$ and for all $k = (i', j') \in C_i$ with $i' = z$ or $j' = z + 1$. Note the constraints given by the second inequation are more restrictive than that given by the first (the second lower bound is the first lower bound added of $\frac{t}{2} + 1$). Therefore, we shall assume that the lower bound given by the second inequation must be satisfied for each C_i, since in this way both lower bounds are being satisfied.

Analogously to the rewriting of the upper bound, we can rewrite the lower bound as

$$\ell_i > (2i+1) - 1 + (2i+1-1)t + \frac{t}{2} = 2i + 2it + \frac{t}{2},$$

for all $1 \leq i \leq s$. Note that each each interval of possible values for ℓ_i is non-empty and a suitable value for ℓ_i can be chosen. For, as instance, $t = 2$, such interval of possible values for ℓ_i is given by

$$6i + 1 < \ell_i < 6i + 2.$$

Therefore, it is possible to adjust all intervals in C_i having the same length ℓ_i, for any value in the interval $(6i+1; 6i+2)$. Regarding the intervals $(i,j) \in C_0$, note that the coverings for each interval must be equal to 1 and, therefore, $i = z$ or $j = z+1$. We can define those intervals as unitary length and as either starting or finishing in p'.

Finally, it is easy to see that it is possible to define in $\mathcal{R}'$ the intervals of $K_1(\mathcal{R})$, $K_2(\mathcal{R})$ and the universal one (if existing), all of them having a same length, since the intervals already positioned in $\mathcal{R}'$ keep the same comparability of those of $\mathcal{R}$, $\mathcal{R}'$ turning out to be a model of P in which $r(\mathcal{R}') \leq \lfloor \frac{|I(G)|}{2} \rfloor$. □

Lemma 4.8 Let P be a split order such that $|K_1(P)| > 1$. Let $\mathcal{R}$ be a model of P and $a \in K_1(\mathcal{R})$ be the interval having the smallest right extreme point. Then, $IC(P) = IC(P \setminus a)$.

Proof. Let $\mathcal{R}$ be a model of P such that $IC(\mathcal{R}) = IC(P)$. Let $\mathcal{R}'$ the model obtained from $\mathcal{R}$ by the removal of the interval a. Naturally, $\mathcal{R}'$ is a model of $P \setminus a$ and, therefore, $IC(P \setminus a) \leq IC(\mathcal{R}') \leq IC(\mathcal{R}) = IC(P)$. On the other hand, let $\mathcal{R}'$ be a model of $P \setminus a$ such that $IC(\mathcal{R}') = IC(P \setminus a)$ and $\mathcal{R}$ be obtained from $\mathcal{R}'$ adding a interval a. As $a \in K_1(P)$, then the left extreme point of a is free. Thus, it is possible to assign a length for the interval a that is already used in $\mathcal{R}'$ (for instance, that of the interval of $K_1(\mathcal{R})$ which has the rightmost extreme point). Therefore, $IC(P \setminus a) = IC(\mathcal{R}') = IC(\mathcal{R}) \geq IC(P)$. Consequently, $IC(P) = IC(P \setminus a)$. □

Lemma 4.9 Let G be a connected split graph. There exists a model $\mathcal{R}$ of G such that $K_1(\mathcal{R}) \neq \emptyset$.

Proof. Let $\mathcal{R}$ be a model of G. If $K_1(\mathcal{R}) \neq \emptyset$, then we are done. If $K_2(\mathcal{R}) \neq \emptyset$, then the reversal of $\mathcal{R}$ is the model which proves the result. Assume therefore that $|K_1(\mathcal{R})| = |K_2(\mathcal{R})| = 0$. Since G is connected, G has a universal vertex u. Choose in $\mathcal{R}$ a interval $b = (i,j) \in K_3(\mathcal{R})$ such that i be minimum over all of them. The model can be transformed by transposing the intervals of $I(G)$ labeled by 1 up to i to the right of the last interval of $I(G)$, and then increasing $r(u)$ so that u keeps the intersection to those moved intervals, producing in whis way the model $\mathcal{R}'$. Note htat at least one interval of $K_3(\mathcal{R})$ has a free left extreme point in $\mathcal{R}'$, belonging to $K_1(\mathcal{R}')$. □

Lemma 4.10 Let $\mathcal{R}$ be a model of a split graph G such that $|K_3(\mathcal{R})| = 0$. Therefore, there exists a model $\mathcal{R}$ of G such that $IC(\mathcal{R}) = IC(G) \leq 2$.

Proof. As $|K_3(\mathcal{R})| = 0$, the proof follows from [11]. □

Theorem 4.13 Let G be a connected split graph with $3k-3$ vertices and $P = (X, \prec)$ be a split order agreeing with G. Therefore, $IC(P) \leq k-1$.

Proof. Let $\mathcal{R}$ be a model of P. Firstly, observe that the interval count of an order can not be different than that of the order obtained by reversing all comparability relations of P, since the operation of reversing a model $\mathcal{R}$ does not affect the lengths. Therefore, we shall assume only the cases in which both $K_1(\mathcal{R})$ and $K_2(\mathcal{R})$ are empty, both of them are non-empty, and $K_1(\mathcal{R}) \neq \emptyset$, $K_2(\mathcal{R}) = \emptyset$.

If $K_1(\mathcal{R}) \neq \emptyset$, then it can be assume without loss of generality that $|K_1(\mathcal{R})| = 1$ since, otherwise, it is possible to reduce $|K_1(\mathcal{R})|$ to 1 by successively applying the transformation described in Lemma 4.8, resulting in a model $\mathcal{R}'$ corresponding to the order P' in which $|K_1(\mathcal{R}')| = 1$ and $IC(P) = IC(P')$. Thus, if it is proved that $IC(\mathcal{R}') \leq k-1$, then $IC(P) \leq k-1$. By analogous reasoning, and considering again the operation of model reversal, it can also be assumed that either $K_2(\mathcal{R}) = \emptyset$, or $|K_2(\mathcal{R})| = 1$. Furthermore, as G is connected, $K_2(\mathcal{R}) \neq \emptyset$ or there exists a universal vertex u.

If $K_2(\mathcal{R}) \neq \emptyset$, we can modify $\mathcal{R}$ so that all intervals in $K_1(\mathcal{R}) \cup K_2(\mathcal{R})$ have the same length (since all of them have free extreme points). If $K_1(\mathcal{R}) \neq \emptyset$ and $K_2(\mathcal{R}) = \emptyset$, then the lengths of the interval of $K_1(\mathcal{R})$ and u can be made the same, since either u, or the interval of $K_1(\mathcal{R})$ can be enlarged so that the smallest between them have the length of the largest.

We can also consider that $\mathcal{R}$ have of intervals of $I(G)$ with the same and smallest length, applying the transformation of Lemma 4.6.

Let $a = u$ if $K_1(\mathcal{R}) = \emptyset$ or, otherwise, let a be interval of $K_1(\mathcal{R})$. Let $b = u$ if $K_2(\mathcal{R}) = \emptyset$ or, otherwise, let b the interval of $K_2(\mathcal{R})$. In summary, after all the previous transformations, we suppose $\mathcal{R}$ a model in which there is a same length for all intervals of $I(G)$ and a same length (possibly distinct of that of intervals of $I(G)$) for both intervals a and b. The proof continues by case analysis of the value of $|I(G)|$.

(i) If $|I(G)| \leq 2k-3$, then applying the transformation of Lemma 4.7 in $\mathcal{R}$, we have that $r(\mathcal{R}) \leq \lfloor \frac{2k-3}{2} \rfloor = k-2$. That is, there are at most $k-2$ distinct lengths over the intervals of $K_3(\mathcal{R}) \cup I(G)$. Counting one more length for both a and b, we have that $IC(G) \leq IC(\mathcal{R}) \leq k-1$. Note that if $K_2 \neq \emptyset$ and there exists a universal u, it is possible to assign a same length to u, a, b, since all of them have free extreme point.

(ii) If $|I(G)| \geq 2k-2$, then the proof is further divided in the following cases.
   (a) If $K_1(\mathcal{R}) \neq \emptyset$, then let $S = V(G) \setminus I(G) \setminus \{a, b\}$. Therefore, $|S| \leq (3k-3) - (2k-2) - 2 = k-3$. By the hypothesis on $\mathcal{R}$, all intervals of $I(G)$ have the same length, and both the intervals a and b have a same length. Therefore, $IC(G) \leq IC(\mathcal{R}) \leq k-3+2 = k-1$.
   (b) If $K_1(\mathcal{R}) = \emptyset$ and $|I(G)| \geq 2k-1$, let $S = V(G) \setminus I(G) \setminus \{u\}$. Therefore, $|S| \leq (3k-3)-(2k-1)-1 = k-3$. By the hypothesis on $\mathcal{R}$, all intervals of $I(G)$ have a same length. Therefore, $IC(G) \leq IC(\mathcal{R}) \leq k-3+2 = k-1$.

(c) If $K_1(\mathcal{R}) = \emptyset$ and $|I(G)| = 2k-2$, let $S = V(G) \setminus I(G) \setminus \{u\}$. Thus, $|S| = (3k-3)-(2k-2)-1 = k-2$. Consider the application of Lemma 4.7 in the model $\mathcal{R}$. Let $C_0, C_1, \ldots, C_s$ be the division of $K_3(\mathcal{R})$ obtained by such an application, where $s = \lfloor \frac{|I(G)|}{2} - 1 \rfloor = k-2$. As $|S| = k-2$, some C_i must be empty, with $0 \leq i \leq s$. If there exists $C_i = \emptyset$ for some $0 < i \leq s$, then it follows from the applied transformation that $r(\mathcal{R}) \leq s = k-2$. Counting one more length to u, we have that $IC(\mathcal{R}) \leq k-1$. Otherwise, as $|S| = k-2$, then $C_0 = \emptyset$, $|C_i| = 1$ for all $1 \leq i \leq s$ and the interval of C_i produced by the transformation has a length distinct of that of the interval in C_j, for all $1 \leq i < j \leq s$. In this case, by applying Lemma 4.7 using spacing $t = 1$, we can modify the interval $z = (i,j) \in C_1$ making its length assume that of the intervals of $I(G)$, by $\ell(z) = r(i+1)$ and $r(z) = \ell(i+2)$. Therefore, the number of distinct lengths in $K_3(\mathcal{R}) \cup I(G)$ is equal to $k-2$. Counting one more length to the universal vertex, $IC(G) \leq IC(\mathcal{R}) \leq k-1$.

□