

A Characterisation of Expressivity for Coalgebraic Bisimulation and Simulation

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Abstract

In a previous paper we introduced internal models for coalgebraic modal logics and showed how they characterise expressivity for bismulation. Here we extend this work by enriching over preordered sets, and in so doing derive a characterisation that subsumes expressivity for both bisimulation and simulation.

Keywords: Enriched logical connection; Preordered set; Coalgebra; Bisimulation; Simulation.

1 Introduction

In [15] an abstract, category theoretic, characterisation of expressivity for coalgebraic bisimulation was given. This characterisation is in terms of internal models for a modal logic, and these were also introduced in [15]. In this paper we extend that work to include both bisimulation and simulation.

Our new characterisation is achieved by enriching over the preordered sets of [13], and thus the carrier of each coalgebra carries a preorder relation. It is this relation that gives our notion of simulation. This idea was first proposed in [6,7], where the authors enriched over **Pos** (partial orders), but we generalise this to incorporate bisimulation by noting (as in [13]) that sets can be viewed as preordered sets with the discrete preorder - i.e. **Set** \cong **DiscSetoid** (discrete setoids).

In fact we go further than this, and observe that by enriching over **Preord** (preorders), **Pos**, **Setoid** (setoids), and **DiscSetoid**, that we can characterise simulation, simulation where mutual simulation is bisimulation, mutual simulation, and bisimulation respectively.

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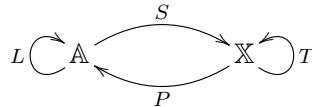
The notion of expressivity for a coalgebraic modal logic with respect to bisimulation states that, two states are logically equivalent if and only if they are behaviourally equivalent. Here logically equivalent means “have the same theory”, and behaviourally equivalent means “can be identified in a model”, where the identification is by means of coalgebra homomorphisms. Thus there is an implicit reliance on the equality relation associated with the set of all theories, and the equality relation associated with the carrier of each coalgebra. It is this that we generalise, and instead use the preorder relations that these objects now carry.

For brevity we summarise the important ideas from [15] that we need, and it should be consulted for further details.

A general outline of this paper is as follows. In Section 2 we give an overview of the framework in which we work. Then in Section 3 we define the preordered sets. In Section 4 we recall from [15] the definitions and results we need, and in Section 5 define our generalised notion of expressivity. Section 6 contains our main characterisation result, and finally in Section 7 we work through the well known expressivity result for simulation of finite branching labelled transition systems as an example.

2 Dual-Adjunction Framework

Increasingly, the standard approach to coalgebraic modal logic is to formulate it in a dual-adjunction framework [10,8,5]. In [12] this is extended to an enriched setting, where the enrichment is over a symmetric monoidal closed category that is complete and cocomplete. It is this enriched version that we use, and in the remainder of this paper all categories, functors, etc. should be assumed to be enriched.



The framework consists of two categories \mathbb{A} and \mathbb{X} , and two contravariant functors P and S that form a dual adjunction i.e. there exists a natural isomorphism

$$\Phi: \mathbb{A}(-_1, P(-_2)) \Rightarrow \mathbb{X}(-_2, S(-_1))$$

Such a dual-adjunction is often referred to as a **logical connection** [14], and we denote the unit and counit by

$$\begin{aligned} \rho: id_{\mathbb{A}} &\Rightarrow PS \\ \sigma: id_{\mathbb{X}} &\Rightarrow SP \end{aligned}$$

The category \mathbb{X} represents a collection of state spaces, and a collection of generalised transition systems is defined on these state spaces as coalgebras for an endofunctor T . Similarly, the category \mathbb{A} represents a collection of base logics to which modal operators are to be added. These are introduced via an endofunctor L , and the corresponding modal logics are the L -algebras. The semantics of

these modal logics is given in two stages. First the dual adjunction gives a semantics for the base logics in terms of the state spaces, and then secondly, a natural transformation

$$\delta: LP \Rightarrow PT$$

gives the semantics of the modal operators in terms of the transition structures introduced by T [9,11].

The literature contains many examples for the unenriched case, for example see [5], and in Example 3.4 we will develop an enriched example.

3 Enrichment over Preordered Sets

Recall the category **Preord** of preordered sets and monotone functions, the objects of which are pairs consisting of a set, and a preorder relation on that set. Similarly, the categories **Pos** (partially ordered sets), **Setoid** (setoids), and **DiscSetoid** (discrete setoids) have for objects pairs consisting of a set and respectively, a partial order, equivalence relation, or the equality relation, on that set. In [13] these examples are collectively known as the preordered sets, and they have significance for the coalgebraic analysis of simulation, which we shall come back to later.

We can consider these examples together by means of the following definition, where by a relation of “type R ” we mean either a *preorder*, *partial order*, *equivalence relation*, or *equality*. The type is fixed, and every object in the category **Set_R** must have a relation of that type.

Definition 3.1 The category **Set_R** has for objects pairs (X, R_X) consisting of a set X , and R_X a binary relation of type R on X . The morphisms are the R -preserving functions i.e. $f: (X, R_X) \rightarrow (Y, R_Y)$ is a morphism if and only if for all $x, x' \in X$

$$xR_Xx' \Rightarrow f(x)R_Yf(x')$$

To be explicit we have the following four cases:

- (i) If R is the type preorder, then **Set_R** is **Preord**.
- (ii) If R is the type partial order, then **Set_R** is **Pos**.
- (iii) If R is the type equivalence relation, then **Set_R** is **Setoid**.
- (iv) If R is the type equality, then **Set_R** is **DiscSetoid** (which is obviously isomorphic to **Set**).

It is easy to verify that the forgetful from **Set_R** to **Set** creates limits and colimits - the product of (X, R_X) and (Y, R_Y) is given by $(X \times Y, R_{X \times Y})$, where $(x, y)R_{X \times Y}(x', y') \Leftrightarrow xR_Xx'$ and yR_Yy' , and the final object is $(\mathbf{1}, R_{\mathbf{1}})$, where $\mathbf{1}$ is the singleton set, and $R_{\mathbf{1}} = \mathbf{1} \times \mathbf{1}$.

It is also easy to verify that binary product and the final object form the tensor and unit of a symmetric monoidal category. To make **Set_R** also closed we need internal hom objects $[(X, R_X), (Y, R_Y)]$ such that $[(Y, R_Y), -]$ is right adjoint to $- \times (Y, R_Y)$. The obvious definition for $[(X, R_X), (Y, R_Y)]$ is the set of all R -preserving

functions from X to Y carrying the relation

$$fR_{[(X,R_X),(Y,R_Y)]}g \Leftrightarrow \forall x \in X \ f(x)R_Yg(x)$$

Further, we can define the unit of the adjunction

$$\eta_{(X,R_X)}: (X, R_X) \rightarrow [(Y, R_Y), (X, R_X) \times (Y, R_Y)]$$

by $\eta_{(X,R_X)}(x) = f_x: (Y, R_Y) \rightarrow (X, R_X) \times (Y, R_Y)$, where $f_x(y) = (x, y)$, and the counit of the adjunction

$$\varepsilon_{(Z,R_Z)}: [(Y, R_Y), (Z, R_Z)] \times (Y, R_Y) \rightarrow (Z, R_Z)$$

by $\varepsilon_{(Z,R_Z)}(g, y) = g(y)$. Thus we have the following proposition.

Proposition 3.2 *The category \mathbf{Set}_R is cartesian closed.*

For the rest of this paper we make the following assumptions.

Assumption 1 *The categories \mathbb{A} and \mathbb{X} are enriched over \mathbf{Set}_R for some fixed type R of relations. Further, the categories \mathbb{A} and \mathbb{X} are concrete categories i.e. the objects are sets with some additional structure and carry a relation of type R , and the morphisms have underlying R -preserving functions.*

We do this for two reasons. Firstly enriching over \mathbf{Set}_R is an obvious generalisation of the approach of [6,7], and secondly to investigate expressivity we need to access individual states of objects in \mathbb{X} .

Example 3.3 In the case where R is the type equality, then enrichment over \mathbf{Set}_R is just ordinary category theory, and so we have all the examples of logical connections from the literature, see for example [5].

The following example is our leading example, and will feature prominently when we discuss expressivity.

Example 3.4 There is a logical connection enriched over \mathbf{Set}_R between **MSL** (meet semilattices with top) and \mathbf{Set}_R itself. To see this we need to observe that the objects of **MSL** come with two built-in preorders. The first is the well known partial order defined by $x \leq y \Leftrightarrow x = x \wedge y$, and the second is equality. In what follows, if type R represents preorders or partial orders, then objects of **MSL** should be considered as having the standard partial order, and if R represents equivalence relations or equality, then they should be considered as carrying the equality relation. The functor $P: \mathbf{Set}_R \rightarrow \mathbf{MSL}$ sends an object (X, R_X) to the meet semilattice of its right R -closed subsets. A subset $U \subseteq X$ is right R -closed if $x \in U$ and xR_Xy implies $y \in U$. $P(X, R_X)$ is either ordered by inclusion or equality, depending upon the type R . The functor $S: \mathbf{MSL} \rightarrow \mathbf{Set}_R$ sends a meet semilattice A to the set of its filters, again either ordered by inclusion or equality depending upon the type R .

4 Models, Internal Models, and R -Models

In [15] we introduced the concepts of models and internal models for an L -algebra. We summarise these definitions below, and extend them to the case of enrichment over \mathbf{Set}_R .

Definition 4.1 [Models] For an L -algebra (A, α) we define $\mathbf{Mod}(A, \alpha)$, the category of **models** for (A, α) , with objects given by pairs

$$((X, \gamma), f: X \rightarrow S(A))$$

where (X, γ) is a T -coalgebra, and f is a morphism such that

$$\begin{array}{ccc} X & \xrightarrow{f} & S(A) \\ \gamma \downarrow & & \downarrow S(\alpha) \\ T(X) & \xrightarrow{T(f)} & TS(A) \\ & & \uparrow \delta_A^* \\ & & SL(A) \end{array}$$

commutes. Such an f is called a **theory map**. Here $\delta^*: TS \Rightarrow SL$ is defined as

$$\delta^* = SL\rho \circ \delta^\flat S$$

where ρ is the unit of the logical connection, and δ^\flat is the adjunct of δ under the logical connection. The morphisms of $\mathbf{Mod}(A, \alpha)$

$$g: ((X_1, \gamma_1), f_1) \rightarrow ((X_2, \gamma_2), f_2)$$

are given by T -coalgebra morphisms $g: (X_1, \gamma_1) \rightarrow (X_2, \gamma_2)$ such that $f_1 = f_2 \circ g$.

Like the categories $\mathbf{Alg}(L)$ and $\mathbf{CoAlg}(T)$, the category $\mathbf{Mod}(A, \alpha)$ is enriched over \mathbf{Set}_R , and the relation (of type R) on each homobject is given by the corresponding relation on the underlying homobject in \mathbf{Set}_R .

Definition 4.2 [Internal Models] Given a class M of monomorphisms in \mathbb{X} , we define the category $\mathbf{IntMod}_M(A, \alpha)$ to be the full subcategory of $\mathbf{Mod}(A, \alpha)$ where the theory maps are in M , and write

$$G: \mathbf{IntMod}_M(A, \alpha) \rightarrow \mathbf{Mod}(A, \alpha)$$

for the corresponding inclusion functor.

We parameterise by the class M , as sometimes we require the morphisms of M to have additional properties, for example, that the members of M are preserved by T . In [15] this was exploited for the Giry functor which does not preserve all monomorphisms, but does preserve a particular subclass of them.

In [15] internal models were used to characterise expressivity for bisimulation. As we shall see in Section 6, to characterise simulation we shall need the following generalised definition.

Definition 4.3 [*R*-Models] The category $\mathbf{R}\text{-Mod}(A, \alpha)$ is the full subcategory of $\mathbf{Mod}(A, \alpha)$ where the theory maps are *R*-reflecting. A function $f: X \rightarrow Y$ is *R*-reflecting if for all $x, y \in X$ if $f(x)R_Y f(y)$ then $xR_X y$. We write

$$H: \mathbf{R}\text{-Mod}(A, \alpha) \rightarrow \mathbf{Mod}(A, \alpha)$$

for the corresponding inclusion functor.

It should be noted that if the monomorphisms of the class M are also *R*-reflecting, then the morphisms of M correspond to embeddings of the type *R* relations, and $\mathbf{IntMod}_M(A, \alpha)$ is a full subcategory of $\mathbf{R}\text{-Mod}(A, \alpha)$.

The following definition is very important for our characterisation of expressivity, and we shall make extensive use of it.

Definition 4.4 We say a model X in $\mathbf{Mod}(A, \alpha)$ **factors** via the internal model I in $\mathbf{IntMod}_M(A, \alpha)$ if there exists a morphism $g: X \rightarrow G(I)$ in $\mathbf{Mod}(A, \alpha)$. Similarly, X factors via the *R*-model J in $\mathbf{R}\text{-Mod}(A, \alpha)$ if there exists a morphism $h: X \rightarrow H(J)$ in $\mathbf{Mod}(A, \alpha)$.

In future sections we will also need colimits of models. Because we are working in an enriched setting the general case is that of weighted colimits, but we shall only need conical colimits i.e. those that correspond to colimits in the ordinary categorical sense. However, we must not forget that homobjects of mediating morphisms and cocones (for a fixed diagram and object) must be isomorphic as objects in \mathbf{Set}_R , not just as sets.

The following two theorems follow immediately from the corresponding results of [15]. This is because created colimits are preserved, and the relation of type *R* on the set of cocones from a diagram to an object, corresponds to the underlying one in \mathbf{Set}_R .

Theorem 4.5 *The forgetful functor $U: \mathbf{CoAlg}(T) \rightarrow \mathbb{X}$ creates small conical colimits.*

Theorem 4.6 *The forgetful functor $U: \mathbf{Mod}(A, \alpha) \rightarrow \mathbb{X}$ creates small conical colimits.*

5 Expressivity

To develop a notion of expressivity that covers both bisimulation and simulation, we first need to extend the notions of logical equivalence and behavioural equivalence.

Definition 5.1 Given two models X_1, X_2 in $\mathbf{Mod}(A, \alpha)$, and states $x_1 \in X_1, x_2 \in X_2$, we say x_1 and x_2 are **logically R-related** if

$$f_1(x_1)R_{S(A)}f_2(x_2)$$

where f_1 and f_2 are the theory maps of X_1 and X_2 respectively.

Definition 5.2 Given two models X_1, X_2 in $\mathbf{Mod}(A, \alpha)$, and states $x_1 \in X_1, x_2 \in X_2$, we say x_1 and x_2 are **behaviourally R-related** if there exists in $\mathbf{Mod}(A, \alpha)$ a cospan

$$X_1 \xrightarrow{f_1} X_3 \xleftarrow{f_2} X_2$$

such that $f_1(x_1) R_{X_3} f_2(x_2)$.

To see that these are the correct definitions we first consider the case where the type R is equality. In this case logically R -related simply becomes equality of theories, as expected. For the definition of behaviourally R -related we see that the forgetful functor from $\mathbf{Mod}(A, \alpha)$ to $\mathbf{CoAlg}(T)$ yields the usual definition of behavioural equivalence as a cospan in $\mathbf{CoAlg}(T)$ [9], but in addition, the forgetful functor to \mathbb{X} yields a condition that the theory maps are compatible. This is because we are working with arbitrary L -algebras³, and not just the initial L -algebra, and is similar to the definition of bisimulation in [2]. Thus when the type R represents equality we have, as in [15], the case of bisimulation.

In the case where the type R represents preorders, we want to interpret the definition that x is behaviourally R -related to y , as saying that x is simulated by y . How is this so?

The usual way of approaching simulation in coalgebras is via something called a relator [4,3,13]. In the case of a functor $F: \mathbf{Set} \rightarrow \mathbf{Set}$, an F -relator is a functor $\Gamma: \mathbf{Rel} \rightarrow \mathbf{Rel}$ that satisfies certain additional properties. Using this a general notion of Γ -simulation for F -coalgebras can be defined. Further, associated with F and Γ is a functor $T: \mathbf{Preord} \rightarrow \mathbf{Preord}$ [4, Lemma 5.5] and [13, Definition 11] given by

$$T(X, R_X) = (F(X), \Gamma(R_X))$$

and under certain conditions [4, Theorem 9.4] the final T -coalgebra is the final F -coalgebra with the preorder given by the Γ -similarity relation. This final T -coalgebra characterises Γ -similarity of F -coalgebras as every set carries a discrete preorder (equality). Thus for every F -coalgebra there is a corresponding T -coalgebra, and given two F -coalgebras, the Γ -similarity relation on those two F -coalgebras is given by the preorder on the images of states under the corresponding unique cospan of morphisms to the final T -coalgebra [3, Remark 21].

Now in our general framework, for the initial L -algebra, every T -coalgebra has a unique theory map making it a model. Therefore if there exists a final T -coalgebra, it is a model, and moreover every other model factors uniquely via it. It is thus the final model Z . So for any cospan of models

$$X_1 \xrightarrow{f_1} X_3 \xleftarrow{f_2} X_2$$

such that $f_1(x_1) R_{X_3} f_2(x_2)$, there exists a unique model morphism $g: X_3 \rightarrow Z$, and this gives $g \circ f_1(x_1) R_Z g \circ f_2(x_2)$. So if T is given by an F -relator as above, our

³ The use of arbitrary L -algebras allows propositional variables to be treated directly, rather than as nullary modalities.

notion of similarity coincides with Γ -similarity.

Our notion of simulation can thus be seen as taking the F -relator notion of simulation and extending it to arbitrary cospans in $\mathbf{Mod}(A, \alpha)$, not just those with the final T -coalgebra as the target, and also to an arbitrary functor T , rather than one arising from a functor F on \mathbf{Set} and an F -relator Γ .

The idea of approaching coalgebraic simulation via enrichment first appeared in [7], though there the authors enrich over \mathbf{Pos} , not \mathbf{Preord} . It is also in [7] that the link between enrichment over preordered sets and the relator approach to simulation was first made.

Example 5.3 Consider the logical connection of Example 3.4, and where we fix the type R to be preorders. Then logically R -related corresponds to inclusion of theories. To explain behaviourally R -related we can consider a generalisation of the finite powerset functor on \mathbf{Set}_R . We define

$$\mathcal{P}_{\text{fin}}(X, R_X) = (\mathcal{P}_{\text{fin}}(X), R_{\mathcal{P}_{\text{fin}}(X)})$$

where

$$UR_{\mathcal{P}_{\text{fin}}(X)}V \Leftrightarrow \forall x \in U \exists y \in V. xR_X y$$

(this can be seen to be an example of the *Sim* relator of [13]). Then using this we can define

$$T(X, R_X) = \mathcal{P}_{\text{fin}}((\Sigma, R_\Sigma) \times (X, R_X))$$

making the T -coalgebras finite branching labelled transition systems with labels from Σ . Further, we assume that R_Σ is the equality relation. Then we see that for a T -coalgebra $\gamma: (X, R_X) \rightarrow T(X, R_X)$, the fact that γ must be R -preserving, means that $xR_X y$ implies for all $(l, z) \in \gamma(x)$ there exists $(l, z') \in \gamma(y)$ such that $zR_X z'$, which will be seen to be the usual notion that x is simulated by y .

The other two case for the type R are variations on simulation, and are summarised as follows:

- (i) If R is the type preorder, then we have simulation.
- (ii) If R is the type partial order, then we have simulation where mutual simulation is bisimulation.
- (iii) If R is the type equivalence relation, then we have mutual simulation.
- (iv) If R is the type equality, then we have bisimulation.

The type R determines which type of behavioural relation we have e.g. simulation or bisimulation, but it is the definition of the functor T , specifically the way it defines an R relation on the codomain (in practice probably via a relator), that determines what is actually meant by simulation for the T -coalgebras.

The following result is a simple consequence of the fact that morphisms in \mathbb{X} are R -preserving.

Proposition 5.4 *Given two models X_1, X_2 in $\mathbf{Mod}(A, \alpha)$, and states $x_1 \in X_1$, $x_2 \in X_2$, if x_1 and x_2 are behaviourally R -related then x_1 and x_2 are logically R -related.*

Finally, our expressivity definition is as follows.

Definition 5.5 An L -algebra (A, α) is **R-expressive** for $\mathbf{Mod}(A, \alpha)$ if for all models in $\mathbf{Mod}(A, \alpha)$, states are logically R -related if and only if they are behaviourally R -related.

6 Characterisation of Expressivity

In [15] we gave an abstract, category theoretic, characterisation of expressivity for bisimulation. Here we extend this to include simulation. In actual fact the changes are minimal, and amount to little more than replacing $\mathbf{IntMod}_M(A, \alpha)$ with $\mathbf{R-Mod}(A, \alpha)$, and injective functions with R -reflecting ones.

The following theorem gives sufficient conditions for R -expressivity.

Theorem 6.1 *Given an L -algebra (A, α) , if the following hold:*

- (i) *every model in $\mathbf{Mod}(A, \alpha)$ factors via some model in $\mathbf{R-Mod}(A, \alpha)$,*
- (ii) *for every pair I_1, I_2 in $\mathbf{R-Mod}(A, \alpha)$ there is a cospan $I_1 \rightarrow I_3 \leftarrow I_2$ in $\mathbf{R-Mod}(A, \alpha)$,*

then (A, α) is R -expressive for $\mathbf{Mod}(A, \alpha)$.

Proof. Take any pair of models X_1 and X_2 in $\mathbf{Mod}(A, \alpha)$. Then these factor via the R -models I_1 and I_2 respectively, and by assumption there exists an R -model I_3 such that there exists a cospan $I_1 \rightarrow I_3 \leftarrow I_2$ in $\mathbf{R-Mod}(A, \alpha)$. Thus both X_1 and X_2 factor via I_3 .

Spelling this out, the models $((X_1, \gamma_1), f_1)$ and $((X_2, \gamma_2), f_2)$ factor via the R -model $((I_3, \zeta_3), m_3)$ via T -coalgebra morphisms $g_1: (X_1, \gamma_1) \rightarrow (I_3, \zeta_3)$ and $g_2: (X_2, \gamma_2) \rightarrow (I_3, \zeta_3)$ such that $f_1 = m_3 \circ g_1$ and $f_2 = m_3 \circ g_2$.

Now suppose two states $x_1 \in X_1$ and $x_2 \in X_2$ are logically R -related for (A, α) . Then $f_1(x_1) R_{S(A)} f_2(x_2)$, which means $m_3 \circ g_1(x_1) R_{S(A)} m_3 \circ g_2(x_2)$, and since m_3 is R -reflecting, $g_1(x_1) R_{I_3} g_2(x_2)$, and x_1 and x_2 are behaviourally R -related.

The converse direction is given by Proposition 5.4. □

In addition to Assumption 1, for several of the results that follow we will also need to make assumptions about the category $\mathbf{Mod}(A, \alpha)$. These assumptions contain the precise category theoretic properties that we require to prove our results, however, in Corollary 6.5 we shall see that these assumptions actually follow from more basic assumptions about the category \mathbb{X} and the functor T . Whenever we require these additional assumptions this will be indicated in the premises of the relevant proposition, lemma, or theorem.

Assumption 2 *Given an L -algebra (A, α) the category $\mathbf{Mod}(A, \alpha)$ has small pushouts, a factorisation system $(E_{\mathbf{Mod}(A, \alpha)}, M_{\mathbf{Mod}(A, \alpha)})$, and is $E_{\mathbf{Mod}(A, \alpha)}$ -cowellpowered, where $M_{\mathbf{Mod}(A, \alpha)}$ is a subclass of those morphisms in $\mathbf{Mod}(A, \alpha)$ with R -reflecting underlying functions, and $E_{\mathbf{Mod}(A, \alpha)}$ is a subclass of those morphisms in $\mathbf{Mod}(A, \alpha)$ with surjective underlying functions.*

Using Assumption 2 we can prove a converse to Theorem 6.1. The most difficult part is the proof that expressivity of (A, α) for $\mathbf{Mod}(A, \alpha)$ implies that all models factor via R -models.

Theorem 6.2 *Given an L -algebra (A, α) , and assuming the conditions of Assumption 2 hold, if (A, α) is R -expressive for $\mathbf{Mod}(A, \alpha)$ then every model in $\mathbf{Mod}(A, \alpha)$ factors via an R -model in $\mathbf{R-Mod}(A, \alpha)$.*

Proof. The proof is quite long and technical, but essentially the same as the bisimulation version of [15]. Thus we shall only sketch the outline:

- (i) Observe that all model morphisms have an $(E_{\mathbf{Mod}(A, \alpha)}, M_{\mathbf{Mod}(A, \alpha)})$ -factorisation.
- (ii) For a model $((X, \gamma), f)$ use that $\mathbf{Mod}(A, \alpha)$ is $E_{\mathbf{Mod}(A, \alpha)}$ -cowellpowered to take the pushout $((\coprod_{<e_j>} I_j, \zeta), f^\dagger)$ of the $E_{\mathbf{Mod}(A, \alpha)}$ -quotient objects of $((X, \gamma), f)$.
- (iii) Construct a model epimorphism $h: ((X, \gamma), f) \rightarrow ((\coprod_{<e_j>} I_j, \zeta), f^\dagger)$.
- (iv) Use the diagonalisation property of the factorisation system to show $h, p_j \in E_{\mathbf{Mod}(A, \alpha)}$ for all $j \in J$.
- (v) Use that morphisms in $M_{\mathbf{Mod}(A, \alpha)}$ have underlying functions that are R -reflecting to show that f^\dagger has an underlying function that is R -reflecting.
- (vi) Observe that this makes $((\coprod_{<e_j>} I_j, \zeta), f^\dagger)$ an R -model.

□

Corollary 6.3 *Given an L -algebra (A, α) , and with the following assumptions:*

- (i) *the conditions of Assumption 2 hold,*
- (ii) $\mathbf{Mod}(A, \alpha)$ *has binary coproducts,*

if (A, α) is R -expressive for $\mathbf{Mod}(A, \alpha)$ then for every pair I_1, I_2 in $\mathbf{R-Mod}(A, \alpha)$ there is a cospan $I_1 \rightarrow I_3 \leftarrow I_2$ in $\mathbf{R-Mod}(A, \alpha)$.

Proof. Given two R -models I_1 and I_2 , since they are also models their coproduct exists, and by Theorem 6.2 the coproduct factors via an R -model, say I_3 , and this induces an obvious cospan between I_1 and I_2 . □

From Theorems 6.1, 6.2, and Corollary 6.3 we obtain our main expressivity result - an abstract, category theoretic, characterisation of R -expressivity.

Theorem 6.4 *Given an L -algebra (A, α) , and with the following assumptions:*

- (i) *the conditions of Assumption 2 hold,*
- (ii) $\mathbf{Mod}(A, \alpha)$ *has binary coproducts,*

(A, α) is R -expressive for $\mathbf{Mod}(A, \alpha)$ if and only if

- (i) *every model in $\mathbf{Mod}(A, \alpha)$ factors via an R -model in $\mathbf{R-Mod}(A, \alpha)$,*
- (ii) *for every pair I_1, I_2 in $\mathbf{R-Mod}(A, \alpha)$ there is a cospan $I_1 \rightarrow I_3 \leftarrow I_2$ in $\mathbf{R-Mod}(A, \alpha)$.*

The conditions of Assumption 2 follow from appropriate conditions on the category \mathbb{X} and the functor T . Essentially what is required is that \mathbb{X} have enough colimits, and that \mathbb{X} has a proper factorisation system, the monomorphisms of which are preserved by T .

Corollary 6.5 *Given an L -algebra (A, α) , and with the following assumptions:*

- (i) \mathbb{X} is a concrete category (over \mathbf{Set}_R) that has small conical colimits, has a factorisation system $(E_{\mathbb{X}}, M_{\mathbb{X}})$, and is $E_{\mathbb{X}}$ -cowellpowered, where $M_{\mathbb{X}}$ is chosen to be a subclass of those morphisms in \mathbb{X} that have underlying functions that are injective and R -reflecting, and $E_{\mathbb{X}}$ is chosen to be a subclass of those morphisms in \mathbb{X} with underlying functions that are surjective,
 - (ii) T preserves $M_{\mathbb{X}}$ i.e. $m \in M_{\mathbb{X}} \Rightarrow T(m) \in M_{\mathbb{X}}$,
- (A, α) is R -expressive for $\mathbf{Mod}(A, \alpha)$ if and only if
- (i) every model in $\mathbf{Mod}(A, \alpha)$ factors via an R -model in $\mathbf{R-Mod}(A, \alpha)$,
 - (ii) for every pair I_1, I_2 in $\mathbf{R-Mod}(A, \alpha)$ there is a cospan $I_1 \rightarrow I_3 \leftarrow I_2$ in $\mathbf{R-Mod}(A, \alpha)$.

Proof. We have to show that the premises of Theorem 6.4 hold. Firstly we observe that by Theorem 4.6 $\mathbf{Mod}(A, \alpha)$ has small conical colimits.

To show that the factorisation system of \mathbb{X} lifts to $\mathbf{Mod}(A, \alpha)$ we note that in [5] it is observed that if T preserves $M_{\mathbb{X}}$, and the members of $M_{\mathbb{X}}$ are monomorphisms, then the factorisation system of \mathbb{X} lifts to $\mathbf{CoAlg}(T)$, and it is easy to see that this extends to $\mathbf{Mod}(A, \alpha)$.

Finally, $\mathbf{Mod}(A, \alpha)$ is $E_{\mathbb{X}}$ -cowellpowered since the morphisms in $E_{\mathbb{X}}$ are epimorphisms, and this ensures that given a span in $\mathbf{Mod}(A, \alpha)$ where the underlying morphisms are in $E_{\mathbb{X}}$, there is an isomorphism between the two so defined $E_{\mathbb{X}}$ -quotient objects in $\mathbf{Mod}(A, \alpha)$, if and only if, there is an isomorphism between the underlying $E_{\mathbb{X}}$ -quotient objects in \mathbb{X} . \square

7 A Simulation Example

In order to better illustrate what actually is going on with R -models, we shall cast into our framework the well known result that the logic given by the syntax

$$\mathcal{L} \ni \phi ::= \text{tt} \mid \phi \wedge \phi \mid \langle l \rangle \phi \quad \text{where } l \in \Sigma$$

is expressive for simulation of finite branching labelled transition systems. This continues Example 5.3, and again the type R is fixed to be preorders.

What we aim to do is define an L and δ such that \mathcal{L} is the initial L -algebra, then any T -coalgebra has a unique theory map that makes it a model for \mathcal{L} . For

any such model $((X, R_X), \gamma, f)$ we then factor it as follows

$$\begin{array}{ccccc}
 & & f & & \\
 & \searrow & \xrightarrow{\quad} & \searrow & \\
 (X, R_X) & \xrightarrow{e} & (I, R_I) & \xrightarrow{m} & S(\mathcal{L}) \\
 \downarrow \gamma & & \downarrow \zeta & & \downarrow S(\cong) \\
 T(X, R_X) & \xrightarrow{T(e)} & T(I, R_I) & \xrightarrow{T(m)} & TS(\mathcal{L}) \\
 & \swarrow & \xleftarrow{\quad} & \swarrow & \\
 & & T(f) & &
 \end{array}
 \quad \begin{array}{c}
 SL(\mathcal{L}) \\
 \uparrow \delta_{\mathcal{L}}^*
 \end{array}$$

where $((I, R_I), \zeta, m)$ is an R -model for \mathcal{L} .

We start by defining the modalities using the forgetful functor $U: \mathbf{MSL} \rightarrow \mathbf{Set}_R$ and its left adjoint $F: \mathbf{Set}_R \rightarrow \mathbf{MSL}$ that creates free meet semilattices with a top element. Specifically, $F(X, R_X)$ is the usual free meet semilattice $F(X)$, over the set of variables X , with the relation $R_{F(X)}$ given by the equality relation extended by $[x]R_{F(X)}[y] \Leftrightarrow xR_X y$ for all $x, y \in X$. The functor L is then defined by $L(A) = F \coprod_{l \in \Sigma} U(A)$, and we choose δ as follows

$$\begin{aligned}
 \delta_{(X, R_X)}: LP(X, R_X) &\rightarrow PT(X, R_X) \\
 \top_{LP(X, R_X)} &\mapsto \mathcal{P}_{\text{fin}}((\Sigma, R_\Sigma) \times (X, R_X)) \\
 [V_l]_{LP(X, R_X)} &\mapsto \{W \in \mathcal{P}_{\text{fin}}((\Sigma, R_\Sigma) \times (X, R_X)) \mid \exists (l, x) \in W. x \in V_l\} \\
 [V_{l_1} \wedge V_{l_2}]_{LP(X, R_X)} &\mapsto \delta_{(X, R_X)}([V_{l_1}]_{LP(X, R_X)}) \cap \delta_{(X, R_X)}([V_{l_2}]_{LP(X, R_X)})
 \end{aligned}$$

where the notation V_l indicates that V is from the copy of $UP(X, R_X)$ indexed by l . This corresponds to a modal operator $\langle l \rangle$ for each $l \in \Sigma$, where $\langle l \rangle a$ is satisfied at a state if there is an l transition from that state to one where a is satisfied. From this we get

$$\begin{aligned}
 \delta_A^*: TS(A) &\rightarrow SL(A) \\
 V &\mapsto \{[W]_{L(A)} \in L(A) \mid V \in \delta_{S(A)} \circ L(\rho_A)([W]_{L(A)})\}
 \end{aligned}$$

where $\rho_A(a) = \{s \in S(A) \mid a \in s\}$ is the unit of the logical connection, and so

$$\begin{aligned}
 \delta_{S(A)} \circ L(\rho_A): L(A) &\rightarrow PTS(A) \\
 \top_{L(A)} &\mapsto \mathcal{P}_{\text{fin}}((\Sigma, R_\Sigma) \times (S(A), R_{S(A)})) \\
 [a_l]_{L(A)} &\mapsto \{V \in \mathcal{P}_{\text{fin}}((\Sigma, R_\Sigma) \times (S(A), R_{S(A)})) \mid \exists (l, s) \in V. a_l \in s\} \\
 [a_{l_1} \wedge a_{l_2}]_{L(A)} &\mapsto \delta_{S(A)} \circ L(\rho_A)([a_{l_1}]_{L(A)}) \cap \delta_{S(A)} \circ L(\rho_A)([a_{l_2}]_{L(A)})
 \end{aligned}$$

where again the notation a_l indicates that a is from the copy of $U(A)$ indexed by l .

To go back to our plan, what we need to do next is factor a theory map f via an R -reflecting morphism m . We do this using the well known result that Hennessy-

Milner logic (\mathcal{L} with negation) is expressive for bisimulation of finite branching labelled transition systems.

Specifically, using a functor $U: \mathbb{X} \rightarrow \mathbb{X}$ that assigns to every object the discrete preorder (in other words, forgets the current preorder), any model $((X, R_X), \gamma), f)$ can be quotiented via a surjective T -coalgebra morphism $e: (X, \gamma) \rightarrow (I, \zeta)$, where I is a subset of the ultrafilters (maximally consistent sets) of Hennessy-Milner logic. There is then an obvious function $m: I \rightarrow US(A)$ that maps an ultrafilter in Hennessy-Milner logic to the corresponding filter in \mathcal{L} by throwing out all the formulae that contain negation, and moreover, $U(f) = m \circ e$. The way to think of this, is that a filter in \mathcal{L} lists all the possible future things a state in a transition system can do, and an ultrafilter in Hennessy-Milner logic explicitly adds all the things it cannot do. Note: it is well known that mutual simulation is not the same as bisimulation, therefore m is clearly not injective.

Now $S(A)$ is ordered by inclusion, and it is easy to see that I can be given a preorder R_I such that e is R -preserving, and m is both R -preserving and R -reflecting. Specifically, we can order the ultrafilters of I by the inclusion order on their negation free subsets.

Further, since e is surjective, $((I, U(\zeta)), U(m))$ is a model for \mathcal{L} . What remains to be shown is that ζ preserves the preorder R_I , for if that is the case, then $((I, R_I), \zeta), m)$ is an R -model for \mathcal{L} . It is easily seen that this is the case if T preserves R -reflecting morphisms, and $\delta_{\mathcal{L}}^*$ is R -reflecting. The former is not very hard to show, so what remains is to show that $\delta_{\mathcal{L}}^*$ is R -reflecting. In fact we shall show this for an arbitrary L -algebra (A, α) .

To do this suppose $V\mathcal{R}_{TS(A)}V'$, then

$$V\mathcal{R}_{TS(A)}V' \Leftrightarrow \exists(l, s) \in V. \forall(l', s') \in V' \text{ either } l \neq l' \text{ or } s\mathcal{R}_{S(A)}s'$$

Now, our plan is to find $[a_l]_{L(A)} \in L(A)$ such that $a_l \in s$, and for all $(l', s') \in V'$ either $l \neq l'$ or $a_l \notin s'$. If there is no $(l', s') \in V'$ such that $l = l'$ then we can take $a_l = (\top_A)_l$. If that is not the case, then there is a finite set of pairs $(l, s') \in V'$ such that $s\mathcal{R}_{S(A)}s'$. Now $s\mathcal{R}_{S(A)}s'$ means $s \not\subseteq s'$, so it is possible to find an element of s that is not in any of the s' (do it pairwise and then take the meet - we can do this as V' is finite). Therefore $\delta_A^*(V) \not\subseteq \delta_A^*(V')$, which means $\delta_A^*(V)\mathcal{R}_{SL(A)}\delta_A^*(V')$, and thus δ_A^* is R -reflecting.

We have thus shown that every model for \mathcal{L} factors via an R -model. Further, since \mathbf{Set}_R has coproducts, by Theorem 4.6 the coproduct of any pair of R -models, as models, exists, and since any model factors via an R -model, this yields a cospan of R -models. Therefore we have satisfied the premises of Theorem 6.1 and so we can conclude, as expected, that \mathcal{L} is expressive for simulation of finite branching labelled transition systems.

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