Analyzing Reachability for Some Petri Nets With Fast Growing Markings

M. Praveen and Kamal Lodaya

The Institute of Mathematical Sciences, CIT Campus, Chennai 600113, India

Abstract

Using linear algebraic techniques, we analyse the computational complexity of testing reachability in Petri nets for which markings can grow very fast. This leads to two subclasses of Petri nets for which the reachability problem is PSPACE-complete. These subclasses are not contained in any other subclass for which complexity of the reachability problem was known, such as those given in Esparza and Nielsen's survey [5]. We give an example where further extension of our subclasses fails to maintain the upper bound.

Keywords: Petri net, reachability problem, T-invariant-less net, partially bounded net

1 Introduction

The reachability problem for Petri nets was first mentioned in [7] and remained open for a long time. Hopcroft and Pansiot [6] gave the example of a net whose reachability set is not semilinear. They also showed that when the reachability set is semilinear, the problem is decidable. Valk and Vidal-Naquet [23] gave an example of a family of bounded nets where the final marking grows according to Ackermann's function.

Mayr [16] and Kosaraju [8] gave algorithms for reachability over all nets. The algorithms in both proofs have non-primitive recursive complexity, and the exact complexity of the problem has been open since then. Reutenauer's book [21] has a detailed exposition of Kosaraju's proof. A simplification of this proof, still not primitive recursive, was given by Lambert [13]. Jančar points out a bug [18] in a paper claiming primitive recursive upper bound for the reachability problem. The best known lower bound remains exponential space, given by Lipton [15,4]. Many subclasses of Petri nets have been studied where a better complexity of the reachability problem has been established. Esparza and Nielsen's survey [5] contains a list of such results. We establish PSPACE complexity for two new subclasses.

Linear algebraic techniques for the analysis of Petri nets are well studied

[17,14,3,22,12], for example the concept of a T-invariant is used to establish recurrence of a marking. Recently, Kostin [9] gave an algorithm for reachability in T-invariant-less Petri nets and followed it up with an algorithm for a more general reachability analysis [10]. Hopcroft and Pansiot's example net [6] with non-semilinear reachability set falls into this subclass. As yet, there is no complexity analysis for these algorithms. We show that reachability for T-invariant-less Petri nets is PSPACE-complete. Our technique follows Rackoff [20], who used bounds on solutions of linear diophantine equations to give an upper bound on the complexity of the boundedness problem.

We then extend the technique to get a more general subclass of Petri nets while maintaining the PSPACE upper bound. Somewhat surprisingly, Valk and Vidal-Naquet's example family of nets [23] falls into this subclass, because the initial and the final marking are part of the input. But a small modification of this example shows that our technique cannot be extended further.

2   Notation and preliminary definitions

Let $\mathbb{Z}$ be the set of integers, $\mathbb{N}$ the set of natural numbers and $\mathbb{N}_+$ the set of positive integers. A **finite Petri net** is a 4-tuple $\mathcal{N} = (P, T, Pre, Post)$ where

- P is a set of m places,

- T is a set of n transitions,

- Pre and $Post$ are the incidence functions:

$$Pre : P \times T \to [0 \ldots D] \quad \text{(representing arcs going from places to transitions)},$$
$$Post : P \times T \to [0 \ldots D] \quad \text{(representing arcs going from transitions to places)},$$

where $D \in \mathbb{N}$. We assume a net is presented as two matrices, one each for Pre and $Post$. This has size $2mnd$ bits, where d is defined to be $\log D$. The $m \times n$ incidence matrix of the net $\boldsymbol{N} = [c_{ij}]$ ($1 \leq i \leq m$, $1 \leq j \leq n$) is given by $c_{ij} = Post(p_i, t_j) - Pre(p_i, t_j)$. In the rest of this paper, we will assume that a Petri net $\mathcal{N}$ has m places, n transitions, D is the maximum of range of Pre and $Post$, and that $\boldsymbol{N}$ is its incidence matrix.

A function $M : P \to \mathbb{N}$ is called a marking. With the implicit ordering $(p_0, p_1, \ldots, p_m)$ on the set of places, we also represent M as a column vector where i'th row contains $M(p_i)$. At a marking M, a place p is said to have $M(p)$ tokens. A net system $(\mathcal{N}, M_0, M_f)$ is a Petri net $\mathcal{N}$ with an initial marking M_0 and a final marking M_f. A transition $t \in T$ is enabled at marking M iff for all $p \in P$, $M(p) \geq Pre(p, t)$. If $t \in T$ is enabled at a marking M, then t may be fired yielding a new marking M' given by the equation $M'(p) = M(p) - Pre(p, t) + Post(p, t)$ for all $p \in P$, or $M' = M + \boldsymbol{N}t$, where $\boldsymbol{t}$ is the characteristic vector of the transition (i.e., a column vector having a 1 at the position corresponding to t and 0 everywhere else). $M \xrightarrow{t} M'$ denotes that M' is reached from M by firing t.

A finite sequence of transitions $\sigma = t_1 t_2 \dots t_r$ is a firing sequence of $(\mathcal{N}, M_0, M_f)$ iff there exist markings $M_1, M_2, \dots, M_r$ such that $\forall i : 1 \leq i \leq r$, we have $M_{i-1} \overset{t_i}{\to} M_i$. Its Parikh vector $\overline{\sigma} : T \to \mathbb{N}$ has as the i'th component the number of occurrences in σ of transition t_i. In the above case, we have $M_r = M_0 + \boldsymbol{N}\overline{\sigma}$. We will also use $\overline{\sigma}(U)$ to denote the number of occurrences in σ of transitions from a subset of transitions U.

We say that the marking M_r is reachable from M_0 by firing σ: $M_0 \overset{\sigma}{\to} M_r$. The reachability set $R(\mathcal{N}, M_0) = \{M : P \to \mathbb{N} \mid \exists$ finite firing sequence $\sigma : M_0 \overset{\sigma}{\to} M\}$ denotes the set of all markings reachable from M_0.

Definition 2.1 (Reachability problem) Given a net system $(\mathcal{N}, M_0, M_f)$, the reachability problem is to decide if $M_f \in R(\mathcal{N}, M_0)$. Let $r_0 = \max(\text{range}(M_0))$ and $r_f = \max(\text{range}(M_f))$. Let $m_0 = \log r_0$ and $m_f = \log r_f$. For the purpose of complexity analysis, we will take size of the input to be $N = 2mnd + m_0 + m_f$ bits, where $d = \log D$ as discussed above.

A net system $(\mathcal{N}, M_0, M_f)$ is said to be b-bounded for some $b \in \mathbb{N}$ if all markings in $R(\mathcal{N}, M_0)$ have at most b tokens in all places. A 1-bounded net system is commonly called a 1-Safe net. $(\mathcal{N}, M_0, M_f)$ is said to be bounded if it is b-bounded for some $b \in \mathbb{N}$. A Petri net $\mathcal{N}$ is said to be structurally bounded if for every initial marking M_i, there exists some $b_i \in \mathbb{N}$ such that all markings in $R(\mathcal{N}, M_i)$ has at most b_i tokens in all places. Given a net system $(\mathcal{N}, M_0, M_f)$, the boundedness problem is to decide whether it is bounded.

We will use $\boldsymbol{k}$ to denote a column vector of all k's: the dimension of the vector will be clear from the context. For example, $\boldsymbol{0}$ denotes a column vector of all 0's. If T is a finite set and $U \subseteq T$, then $\boldsymbol{e}[U]$ is the characteristic vector which has entry 1 in components corresponding to elements of U and 0 entries everywhere else. We use $\boldsymbol{I}$ for the identity matrix.

Definition 2.2 (T-invariant) Suppose $\mathcal{N} = (P, T, Pre, Post)$ is a Petri net. An integer vector (mapping) $\boldsymbol{J} : T \to \mathbb{Z}$ is a T-invariant iff for all $p \in P$,

$$\sum_{t \in T} \boldsymbol{J}(t) \left(Post(p, t) - Pre(p, t) \right) = 0 \ .$$

If all entries of a T-invariant $\boldsymbol{J}$ are non-negative (positive), then it is called a semi-positive (resp. positive) T-invariant. If σ is a finite firing sequence of $(\mathcal{N}, M_0, M_f)$ with Parikh vector $\overline{\sigma} = \boldsymbol{J}$, then, by definition of T-invariant, we get $M_0 \overset{\sigma}{\to} M_0$. Thus, semi-positive and positive T-invariants denote firing sequences whose net effect is zero on every place.

3   S-variants and T-invariant-less nets

All proofs of general Petri net reachability use Euler's theorem on a vector $\boldsymbol{v} \geq \boldsymbol{1}$ from which sufficiency of reachability is derived (for example in the conditions for

Kosaraju's sufficiency theorem, cf. [21]). An **S-variant** generalizes this kind of vector with weights.

Definition 3.1 (S-variant) Suppose $\mathcal{N} = (P, T, Pre, Post)$ is a Petri net. An integer vector (mapping) $\boldsymbol{V} : P \to \mathbb{Z}$ is an S-variant iff for all $t \in T$,

$$\sum_{p \in P} \boldsymbol{V}(p) \left(Post(p, t) - Pre(p, t) \right) \geq 1 \ .$$

If $\boldsymbol{V}$ is an S-variant and M is a marking, then by $\boldsymbol{V}(M)$, we denote $\sum_{p \in P} \boldsymbol{V}(p)M(p)$. Thus, if $\boldsymbol{V}$ is an S-variant and $M \xrightarrow{t} M'$, then $\boldsymbol{V}(M') > \boldsymbol{V}(M)$. We will represent S-variants in terms of expressions. If $\mathcal{N}$ is a net with 3 places p_1, p_2 and p_3, then the expression $2p_1 + 3p_2 - 4p_3$ denotes the S-variant $[2, 3, -4]^T$.

3.1 Characterizing nets with S-variants

For a Petri net $\mathcal{N}$, an S-variant $\boldsymbol{V}$ is an integral solution to the system of inequalities

$$\boldsymbol{N}^T \boldsymbol{v} \geq \boldsymbol{1} \ . \tag{1}$$

Also, a T-invariant $\boldsymbol{J}$ is an integral solution to the system of equations

$$\boldsymbol{N} \boldsymbol{j} = \boldsymbol{0} \ . \tag{2}$$

The following theorem (a proof is in [19, Section 3.1.1]) is an application of the well known Farkas lemma in linear algebra [17].

Theorem 3.2 *A Petri net has an S-variant iff it does not have any non-trivial semi-positive T-invariants.*

Theorem 3.3 *Given a Petri net $\mathcal{N}$, it can be checked in polynomial time whether $\mathcal{N}$ has S-variants or not.*

Proof. Checking for existence of S-variants is equivalent to checking existence of rational solutions to the system of inequalities $\boldsymbol{N}^T \boldsymbol{v} \geq \boldsymbol{1}$. This is the same as checking the feasibility of a linear programming instance with rational data. □

Example 3.4 Hopcroft and Pansiot's example to show that the reachability set of Petri nets need not be semilinear [6] is shown in Fig. 1. When the initial marking has one token each in places p_2 and p_3 and no tokens anywhere else, this net can reach markings where number of tokens in p_1 and p_2 is equal to or less than exponential of the number of tokens in p_5.

This net has S-variants, e.g. $\boldsymbol{V} = [3, 2, 0, 1, 2]^T$ is an S-variant for this net.

3.2 Reachability algorithm

In this section, we give a reachability algorithm for T-invariant-less nets and analyze its complexity.
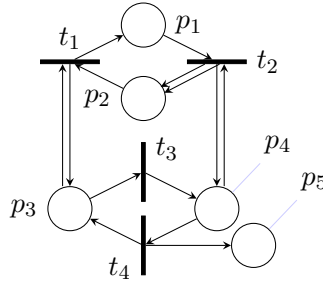
Fig. 1. Hopcroft and Pansiot's example net

Proposition 3.5 *If σ is a finite firing sequence of the net system $(\mathcal{N}, M_0, M_f)$ such that $M_0 \overset{\sigma}{\to} M$ and if $\boldsymbol{V}$ is an S-variant of $\mathcal{N}$, then the length of σ is at most $\boldsymbol{V}(M) - \boldsymbol{V}(M_0)$.*

Proof. Suppose $\sigma = t_1 t_2 \ldots t_r$ and $M_0 \overset{t_1}{\to} M_1 \overset{t_2}{\to} \ldots \overset{t_r}{\to} M_r = M$. By definition of S-variant, for $1 \le i \le r$, $\boldsymbol{V}(M_i) \ge \boldsymbol{V}(M_{i-1}) + 1$. Starting from $i = r$ and iteratively substituting $\boldsymbol{V}(M_{i-1})$ with its lower bound, we get $\boldsymbol{V}(M_r) \ge \boldsymbol{V}(M_0) + r$. Hence, $r \le \boldsymbol{V}(M) - \boldsymbol{V}(M_0)$. □

Our complexity analysis is based on the following result.

Theorem 3.6 *(Borosh and Treybig [1], Theorem 5) Let $\boldsymbol{A}$ be a $n \times r$ integer matrix and $\boldsymbol{B}$ be an $n \times 1$ integer matrix. Suppose $\boldsymbol{x}$ denotes a $r \times 1$ vector of variables and the system of equations $\boldsymbol{A}x = \boldsymbol{B}$ has a non-trivial positive integral solution. Also suppose that R is the maximum of absolute values of all minors of the augmented matrix $[\boldsymbol{A} \mid \boldsymbol{B}]$. If $\boldsymbol{A}$ is a full row rank matrix and $n \le r$, then the system of equations $\boldsymbol{A}x = \boldsymbol{B}$ has a positive integral solution where each entry of the solution is at most $Rn + nrR^2$.*

A result similar to the one below was used by Rackoff [20] to give exponential space upper bound for the boundedness problem of Petri nets. In the rest of this paper, some results will be provided only with a proof sketch, with the detailed proofs in Appendix A.

Lemma 3.7 *If a Petri net $\mathcal{N}$ has S-variants, it has one such that the absolute value of each entry is $O\left(mn^2(n!)^2 D^{2n}\right)$.*

Proof (sketch). An integral solution exists for $\boldsymbol{N}^T \boldsymbol{v} \ge \boldsymbol{1}$ iff a positive integral solution exists for the system of equations $\left[\boldsymbol{I} \mid \boldsymbol{N}^T \mid -\boldsymbol{N}^T\right] \boldsymbol{v'} = -\boldsymbol{1}$. The result follows by applying Theorem 3.6 to these equations. □

Proposition 3.8 *Suppose $(\mathcal{N}, M_0, M_f)$ is a net system with S-variants. If σ is a finite firing sequence of $(\mathcal{N}, M_0, M_f)$ such that $M_0 \overset{\sigma}{\to} M_f$, the length of σ is at most $O(m^2 n^2 (n!)^2 D^{2n} max(M_0, M_f))$.*

Proof. Since $\mathcal{N}$ has S-variants, Lemma 3.7 shows that there is an S-variant $\boldsymbol{V}$ such that the absolute value of each entry of $\boldsymbol{V}$ is $O(mn^2(n!)^2 D^{2n})$. By Proposition 3.5,

we have

$$\text{length of } \sigma \leq \boldsymbol{V}(M_f) - \boldsymbol{V}(M_0)$$
$$= \sum_{p \in P} \boldsymbol{V}(p)\,(M_f(p) - M_0(P))$$
$$\leq \sum_{p \in P} O(mn^2(n!)^2 D^{2n})\max(M_0, M_f)$$
$$\leq mO(mn^2(n!)^2 D^{2n})\max(M_0, M_f)$$
$$\text{Therefore, length of } \sigma \leq O(m^2 n^2 (n!)^2 D^{2n} \max(M_0, M_f)) \ .$$

$\square$

By Proposition 3.8, it is easy to see that the non-deterministic algorithm given in Algorithm 1 is correct. Kostin's algorithm in [9] is deterministic.

1: Let $c =$ bound on length of firing sequence given by Proposition 3.8.
2: Let $i = 1$, *currentMarking* $= M_0$.
3: **while** $i \leq c$ **do**
4:     $i \leftarrow i + 1$. Non deterministically guess a transition t.
5:     If t is not enabled in *currentMarking*, halt and reject.
6:     *currentMarking* $\leftarrow$ *currentMarking* $+ \boldsymbol{N} \cdot \boldsymbol{e}[t]$.
7:     If *currentMarking* $= M_f$, halt and accept.
8: **end while**
9: Halt and reject.

Algorithm 1: Reachability algorithm for Petri nets with S-variants

Now, we analyse the space complexity of Algorithm 1.

Lemma 3.9 *Algorithm 1 runs in polynomial space.*

Proof (sketch). Length of the firing sequence guessed by the algorithm is bounded. Calculating the amount of memory needed by all the variables based on this bound yields the desired result. $\square$

Theorem 3.10 *The reachability problem for T-invariant-less Petri nets (that is, those with S-variants) can be solved in polynomial space.*

Proof. Since by Lemma 3.9, there is a non-deterministic PSPACE algorithm for the problem being considered, we can apply the well known theorem of Savitch to conclude that there is a deterministic algorithm that solves the problem in PSPACE.$\square$

3.3   Lower bound

Now, we will prove that the reachability problem in Petri nets with S-variants is PSPACE-hard. Cheng, Esparza and Palsberg [2] gave a reduction from the problem of satisfiability of quantified boolean formulas (QBF-SAT) to the reachability problem in 1-Safe nets. We will use the same reduction and prove that the resulting net has S-variants. This will give us the necessary hardness proof.

Here, we give some prominent features of the Petri net to which QBF-SAT is reduced to in [2]. For a QBF formula G, a Petri net $\mathcal{N}_G$ is constructed. For our purposes here, we note the following important places of $\mathcal{N}_G$:

- The place G_in, standing for "initialize G".
- For every boolean variable x used in G, the places x_is_T and x_is_F.
- The places G_T and G_F

Intuitively, $\mathcal{N}_G$ starts with one token in G_in, indicating that testing of G has started. A token in x_is_T (resp. x_is_F) indicates that variable x is assigned to TRUE (resp. FALSE). When $\mathcal{N}_G$ finishes testing G (using many other places and transitions not mentioned here), it puts a token in G_T (resp. G_F) to indicate that G is TRUE (resp. FALSE). We refer the reader to the original paper [2] for further details. The following result shows that $\mathcal{N}_G$ has S-variants.

Theorem 3.11 *Suppose G is a Quantified Boolean Formula and $\mathcal{N}_G$ is the corresponding net as given in [2]. Then, for each $i \in \mathbb{N}_+$, $\mathcal{N}_G$ has a S-variant $\boldsymbol{V}(i)$ satisfying the following properties:*

(i) *$\boldsymbol{V}(i) + k \cdot G\_in$ is a S-variant of $\mathcal{N}_G$ for $0 \leq k \leq i - 1$.*

(ii) *The coefficient of G_in in $\boldsymbol{V}(i)$ is 0.*

(iii) *There exists a finite number $j_G(i)$ such that removing a token from G_T or G_F decreases $\boldsymbol{V}(i) + k \cdot G\_in$ by at most $j_G(i)$ for $0 \leq k \leq i - 1$.*

(iv) *The coefficient of X_is_T and x_is_F in $\boldsymbol{V}(i)$ is 0 for any variable x that is not bounded (by $\forall$ or $\exists$) in G.*

Proof (sketch). The S-variant can be constructed by induction on structure of $\mathcal{N}_G$, just like $\mathcal{N}_G$ itself is constructed by induction on structure of G.   □

4   Partial S-variants and partially bounded nets

Intuitively, Theorem 3.2 says that S-variants exist for a Petri net if the action of any transition cannot be "undone" by firing other transitions. Unlike transitions that are part of some semi–positive T-invariants, the action of a transition that is not part of a semi–positive T-invariant cannot be cancelled. With the firing of such a transition, the net makes some progress towards reaching the final marking. The following definition makes this formal.

Definition 4.1 Suppose $\mathcal{N}$ is a Petri net and t is a transition. t is said to be **progressive** if it is not part of any semi-positive T-invariant.

In a Petri net with S-variants, all transitions are progressive. Just like S-variants that measure the progress made by each transition, we introduce partial S-variants that measure progress made by progressive transitions.

Definition 4.2 Suppose $\mathcal{N} = (P, T, Pre, Post)$ is a Petri net and $\emptyset \neq U \subseteq T$ is a nonempty subset of progressive transitions. An integer vector (mapping) $\boldsymbol{V} : P \to \mathbb{Z}$ is a **partial S-variant** iff it satisfies the following properties:

(i) For all $t \in U$, $\sum_{p \in P} \boldsymbol{V}(p)(Post(p, t) - Pre(p, t)) \geq 1$.

(ii) For all $t \in T \setminus U$, $\sum_{p \in P} \boldsymbol{V}(p)(Post(p, t) - Pre(p, t)) \geq 0$.

As before, if M is a marking, $\boldsymbol{V}(M)$ denotes $\sum_{p \in P} \boldsymbol{V}(p)M(p)$. If $t \in U$ and $M \xrightarrow{t} M'$, then $\boldsymbol{V}(M') > \boldsymbol{V}(M)+1$. If $t \in T \setminus U$ and $M \xrightarrow{t} M'$, then $\boldsymbol{V}(M') \geq \boldsymbol{V}(M)$.

Example 4.3 Consider the Petri net shown in Fig. 2, which is same as the one in Fig. 1 with place p_5 removed. Due to this removal, transitions t_3 and t_4 that were progressive in the original net are now not progressive. However, transitions t_1 and t_2 are still progressive since they are not part of any semi-positive T-invariants. $3p_1 + 2p_2$ denotes a partial S-variant for this net.
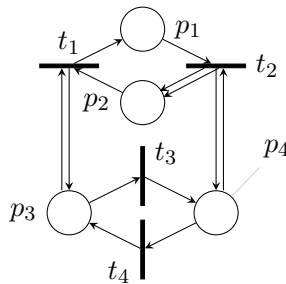

Fig. 2. Example of a Petri net with progressive transitions

If for a Petri net $\mathcal{N}$, $\boldsymbol{N}^T$ is represented as

$$\boldsymbol{N}^T = \left[\frac{\boldsymbol{N}_1}{\boldsymbol{N}_2} \right]$$

where $\boldsymbol{N}_1$ represents progressive transitions and $\boldsymbol{N}_2$ represents other transitions, then a partial S-variant is an integral solution to the system of inequalities

$$\left[\frac{\boldsymbol{N}_1}{\boldsymbol{N}_2} \right] \boldsymbol{v} \geq \left[\frac{\boldsymbol{1}}{\boldsymbol{0}} \right] . \tag{3}$$

The two following results extend to partial S-variants properties similar to those of S-variants in section 3. We refer to [19, Section 3.3.1] for proofs.

Theorem 4.4 *A Petri net has a partial S-variant iff it does not have any positive T-invariant.*

Lemma 4.5 *If a Petri net $\mathcal{N}$ has partial S-variants, it has one where the absolute value of each entry is $O(mn^2(n!)^2 D^{2n})$.*

Now, we extend the subclass of Petri nets with S-variants. Even if a Petri net doesn't have S-variants, we can use partial S-variants to bound the number of occurrences of progressive transitions in potential firing sequences. If other structural

and/or behavioural properties of the net system imply a bound on the number of occurrences of other transitions, we can bound the total length of the firing sequence. The following definition captures this intuition in a generic way.

Definition 4.6 Suppose $\mathcal{N} = (P, T, Pre, Post)$ is a Petri net. The net system $(\mathcal{N}, M_0, M_f)$ is called **partially bounded** if it satisfies the following properties:

(i) The set U of progressive transitions of $\mathcal{N}$ is not empty.

(ii) The non progressive transitions $V = T \setminus U$ are bounded by progressive ones, that is, there is a bound function f such that for every firing sequence σ with $M_0 \xrightarrow{\sigma} M$, there is another firing sequence τ with $M_0 \xrightarrow{\tau} M$ and $\overline{\tau}(V) \leq f(\overline{\tau}(U), M_0, M_f, D, n, m)$.

Different bound functions in the above definition lead to different complexities of the resulting reachability algorithm. We will now look at a sufficient condition for obtaining a PSPACE algorithm.

Lemma 4.7 *Consider the subclass of partially bounded net systems $(\mathcal{N}, M_0, M_f)$ with bound function f of the form $f = (p_1(\overline{\tau}(U), D, M_0, M_f))^{p_2(m,n)}$, for some polynomials p_1 and p_2. There is a PSPACE algorithm that solves the reachability problem in this subclass of Petri nets.*

Proof (sketch). If M_f is reachable, the number of occurrences of progressive transitions in the firing sequence can be bounded with the help of partial S-variants, with an argument similar to the one in Proposition 3.5. The bound function f then gives a bound on total length of the firing sequence. With this new bound, Algorithm 1 works in polynomial space.   □

Example 4.8 Fig. 3 shows an example family of net systems $\mathcal{N}_i, i \in \mathbb{N}$, given by Valk and Vidal-Naquet [23]. $\mathcal{N}_0$ is as shown in the top part and $\mathcal{N}_i$ is built on top of $\mathcal{N}_{i-1}$ as shown in the bottom part. This net system is bounded but can reach markings that are non-primitive recursive w.r.t. the initial marking and the size of the net. t_0, u_0, x_i, v_i and w_i are the only progressive transitions in $\mathcal{N}_i$. Transition t_0 plays an important role in enabling the net to reach non-primitive recursive markings. The analysis in Appendix B takes advantage of the fact that t_0 is progressive to get a bound function f that satisfies the properties required by Lemma 4.7.

It might seem surprising that a net with markings growing so fast can be analyzed in polynomial space. To see this simply observe that the final marking for which we analyze the reachability will have to be given as part of the input.

One disadvantage of partially bounded nets is their dependence on the bound function f. Given an arbitrary Petri net, there is no clear way of identifying whether a bound function exists and to compute it if it does exist. We fix this in the next section to identify a suitable subclass.
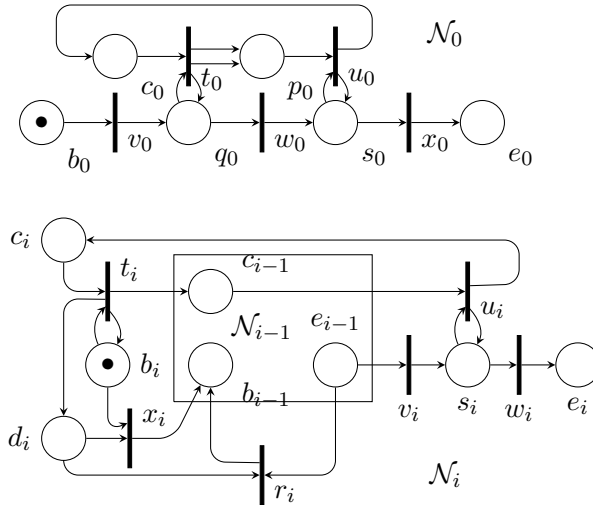
Fig. 3. Valk and Vidal-Naquet's example family of nets

4.1 Structurally partially bounded nets

One way of overcoming the disadvantage of partially bounded nets mentioned above is to look for simpler properties of Petri nets that automatically imply the existence of a bound function with properties required by Lemma 4.7. The subclass of Petri nets defined in Definition 4.9 below satisfy this requirement.

Definition 4.9 A Petri net $\mathcal{N} = (P, T, Pre, Post)$ is said to be **structurally partially bounded** if it satisfies the following properties:

(i) $\mathcal{N}$ has a non empty subset $U \subseteq T$ of progressive transitions.

(ii) $\mathcal{N}$, when restricted to $T \setminus U$ is structurally bounded.

It is known [17] that a Petri net $\mathcal{N}$ is structurally bounded iff the system of inequalities $\boldsymbol{N}^T \boldsymbol{y} \leq \boldsymbol{0}$ has a strictly positive integral solution. Using this, we will see that for structurally partially bounded nets, there will always be a bound function satisfying the requirements in Lemma 4.7.

Given an arbitrary Petri net, we can identify whether it is a structurally partially bounded net in polynomial time.

Lemma 4.10 *Let $\mathcal{N}$ be a Petri net. There is a polynomial time algorithm that checks whether $\mathcal{N}$ is a structurally partially bounded net.*

Proof. We first check if each transition is progressive. This can be done by formulating a linear programming problem that checks whether a transition is part of some semi-positive T-invariant and then testing its feasibility with rational data. With n transitions, n such tests need to be done. Then, we check if removing progressive transitions makes $\mathcal{N}$ structurally bounded. This can again be reduced to checking feasibility of a linear programming problem with rational data. Clearly, all the above operations can be done in polynomial time.   □

Our modification of Hopcroft and Pansiot's Petri net of Fig. 2 is structurally partially bounded, since removing the progressive transitions t_1 and t_2 results in a structurally bounded net. But Valk and Vidal-Naquet's family of nets (Fig. 3), although partially bounded, is not structurally partially bounded, since removing the progressive transitions t_0, u_0, x_i, v_i and w_i does not make the remaining net structurally bounded.

The proof of the following lemma is very similar to that of Lemma 4.5. [19] can be referred to for a proof.

Lemma 4.11 *If a Petri net $\mathcal{N}$ is structurally bounded, there exists a positive integral solution to the system of inequalities $\mathbf{N}^T\mathbf{y} \leq \mathbf{0}$ where each component is $O\left(mn^2(n!)^2D^{2n}\right)$.*

Theorem 4.12 *The reachability problem for the subclass of structurally partially bounded nets is PSPACE-complete.*

Proof. Since Petri nets with S-variants are structurally partially bounded, the PSPACE lower bound of Theorem 3.11 applies here also.

Suppose $(\mathcal{N}, M_0, M_f)$ is the net system for which we need to solve reachability, where $\mathcal{N}$ is structurally partially bounded. Since M_f is reachable, there is a firing sequence σ such that $M_0 \xrightarrow{\sigma} M_f$. Let σ consist of the components $\sigma_0 u_1 \sigma_1 u_2 \sigma_2 \cdots u_k \sigma_k$ for some k, where $u_1, u_2, \ldots, u_k$ are progressive transitions and $\sigma_0, \sigma_1, \ldots, \sigma_k$ consist of non-progressive transitions. Now, define firing sequence τ as $\tau = \tau_0 u_1 \tau_1 u_2 \tau_2 \cdots u_k \tau_k$, where τ_i is same as σ_i but with subsequences that start and end with the same marking removed. Thus, when τ_i is fired, all the intermediate markings that are reached are distinct from each other. It is easy to verify that $M_0 \xrightarrow{\tau} M_f$. It is also easy to see that length of τ_i is bounded by the number of distinct possible markings that can be reached during the firing of τ.

Suppose $V = T \setminus U$ is the set of non-progressive transitions and $\mathcal{N}_V$ is the Petri net $\mathcal{N}$ restricted to V. From Lemma 4.11, there exists a positive integral vector $\boldsymbol{\beta}$ where each component is greater than 0 and bounded by $O\left(mn^2(n!)^2D^{2n}\right)$, such that $\mathbf{N_V}^T\boldsymbol{\beta} \leq \mathbf{0}$. From Lemma A.1 (stated and proved in Appendix A), for any intermediate marking M reached during the firing of τ, $M^T\boldsymbol{\beta} \leq M_0^T\boldsymbol{\beta} + (\boldsymbol{k}\boldsymbol{D})^T\boldsymbol{\beta}$. Hence, for any place p, $M(p) \leq (M_0^T\boldsymbol{\beta} + (\boldsymbol{k}\boldsymbol{D})^T\boldsymbol{\beta})/\boldsymbol{\beta}(p)$. Therefore, during the firing of τ, each place will accumulate at most $M_0^T\boldsymbol{\beta} + (\boldsymbol{k}\boldsymbol{D})^T\boldsymbol{\beta}$ tokens. Since there are m places, total number of distinct markings possible is $(M_0^T\boldsymbol{\beta} + (\boldsymbol{k}\boldsymbol{D})^T\boldsymbol{\beta})^m$. As discussed above, this is a bound on the length of τ_i for each i, $0 \leq i \leq k$. Since there are $k+1$ sequences τ_i that make up all firings of non-progressive transitions in V, we get $\overline{\tau}(V) \leq (k+1)(M_0^T\boldsymbol{\beta} + (\boldsymbol{k}\boldsymbol{D})^T\boldsymbol{\beta})^m$. This is the function f required (where $k = \overline{\tau}(U)$) and it is easy to verify that this satisfies the requirements of Lemma 4.7. $\square$

5   Limitations and discussion

Using linear algebraic techniques, we established the computational complexity of testing reachability in some fast growing Petri nets. We identified Petri nets with

S-variants and partial S-variants as subclasses in which this approach works. Structurally partially bounded nets are an efficiently identifiable subclass of this kind.

Compared to boundedness, structural boundedness is a much stronger restriction. We could try weakening property (ii) in Definition 4.9 to say that the restricted net be bounded just for the particular initial marking M_0. But the essential limitation of our technique is that we consider the length of firing sequences in terms of markings and progressive transitions. To see an example of the kind of difficulty this raises, we replace the net $\mathcal{N}_0$ in Fig. 3 by the one below.
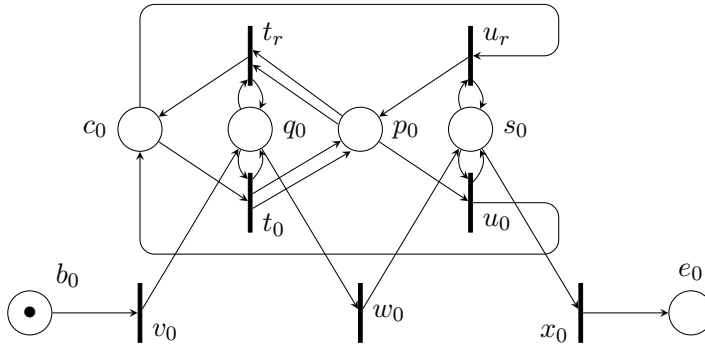

Fig. 4. Modification of $\mathcal{N}_0$ of Valk and Vidal-Naquet's example family of nets

The only modification from $\mathcal{N}_0$ given in Fig. 3 is the addition of "reverse" transitions t_r and u_r. Now, since t_0 and u_0 are not progressive transitions in the modified net, the bound function in terms of the progressive transitions x_i, v_i and w_i of the new net, as required in Definition 4.6, is no longer primitive recursive in its arguments.

Kostin [10,11] has a more general reachability algorithm based on further analysis of T-invariants (for example, see [12]). It would be interesting to analyze its complexity over subclasses larger than T-invariant-less Petri nets.

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A   Detailed proofs

Proof of Lemma 3.7. Since $\mathcal{N}$ has S-variants, an integral solution exists for the system $\boldsymbol{N}^T \boldsymbol{v} \geq \boldsymbol{1}$. If $\boldsymbol{I}$ is the identity matrix, then an integral solution exists for $\boldsymbol{N}^T \boldsymbol{v} \geq \boldsymbol{1}$ iff a positive integral solution exists for the system of equations $\left[\boldsymbol{I} \mid \boldsymbol{N}^T \mid -\boldsymbol{N}^T\right] \boldsymbol{v}' = -\boldsymbol{1}$.

Now we can use Theorem 3.6 on the system of equations $\left[\boldsymbol{I} \mid \boldsymbol{N}^T \mid -\boldsymbol{N}^T\right] \boldsymbol{v}' = -\boldsymbol{1}$. $\left[\boldsymbol{I} \mid \boldsymbol{N}^T \mid -\boldsymbol{N}^T\right]$ has full row rank due to the presence of $\boldsymbol{I}$. The number of columns of this coefficient matrix is $r = n + 2m \geq n$. The absolute value of minors of the augmented matrix is upper bounded by $R \leq n! D^n$. Hence, Theorem 3.6 is applicable and $Rn + nrR^2 \leq nn!D^n + n(n+2m)n!^2 D^{2n}$. Thus, we have a positive integral solution with each entry being $O\left(mn^2(n!)^2 D^{2n}\right)$.

It can be easily seen that a positive integral solution to $\left[\boldsymbol{I} \mid \boldsymbol{N}^T \mid -\boldsymbol{N}^T\right] \boldsymbol{v}' = -\boldsymbol{1}$ can be converted to an integral solution for $\boldsymbol{N}^T \boldsymbol{v} \geq \boldsymbol{1}$ without affecting the bounds. So we conclude that if $\mathcal{N}$ has an S-variant, it has one with the absolute value of each entry being $O\left(mn^2(n!)^2 D^{2n}\right)$.   □

Proof of Lemma 3.9. The algorithm first needs to calculate the constant c in line 1. Clearly, this can be done in space $O(\log m + n \log n + nd + m_0 + m_f)$, where m_0, m_f and d are as in Definition 2.1. The algorithm needs space to store the variables i and *currentMarking*. The maximum value in i will be $O(m^2 n^2 (n!)^2 D^{2n} \max(M_0, M_f))$ and requires $O(\log m + n \log n + nd + m_0 + m_f)$ bits. Since D is the maximum number of tokens that can be added to a place by one transition and we consider at most $O(m^2 n^2 (n!)^2 D^{2n} \max(M_0, M_f))$ transitions, the maximum value that will be stored for each place in the variable *currentMarking* is $O(m^2 n^2 (n!)^2 D^{2n+1} \max(M_0, M_f))$ and requires $O(m \log m + mn \log n + mnd + m(m_0 + m_f))$ bits. It is easy to see that space needed for guessing transitions and calculating resulting markings is dominated by the space required for the variables i and *currentMarking*. Thus, the whole algorithm runs in space $O(m \log m + mn \log n + mnd + m(m_0 + m_f))$ bits.

The input to the algorithm is $2mnd + m_0 + m_f$ bits. So the space needed by the algorithm is bounded by a polynomial in the size of the input.   □

Proof of Theorem 3.11. For a quantified boolean formula G, the net $\mathcal{N}_G$ given in [2] is shown in Fig. A.1 and Fig. A.2, for different structures of G. The nets shown in these figures are same as the ones shown in Fig. 2 of [2], drawn here in a slightly different format to make it easier to understand our construction of the S-variant.

Recall that an expression like $2p_1 + 3p_2 - 4p_3$ is an S-variant iff its value strictly increases whenever any transition in the net fires. We will construct the S-variant $\boldsymbol{V}(i)$ with the required properties by induction on structure of $\mathcal{N}_G$.

Base. $G = x$ for some variable x. See box **A** in Fig. A.1. $\boldsymbol{V}(i) = i \cdot x\_T + i \cdot x\_F$ is an S-variant that satisfies all the required properties with $j_x(i) = i$.

Step. $G = \neg P$. See box **B** in Fig. A.1. By induction hypothesis, we have a S-variant $\boldsymbol{V}_P(i)$ for the net corresponding to P that satisfies all the stated properties.
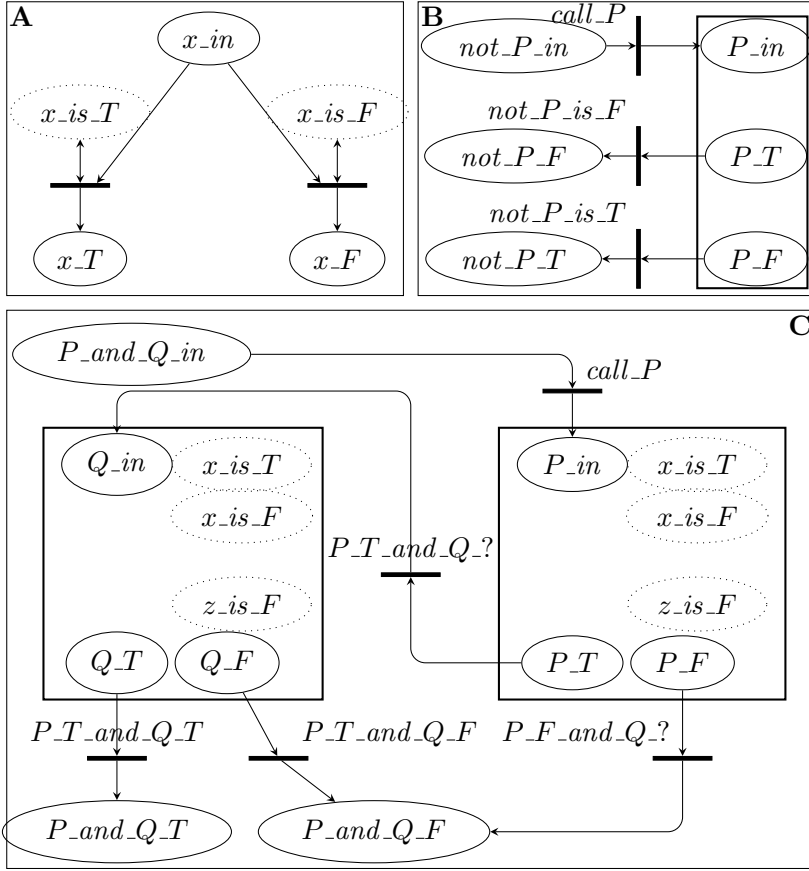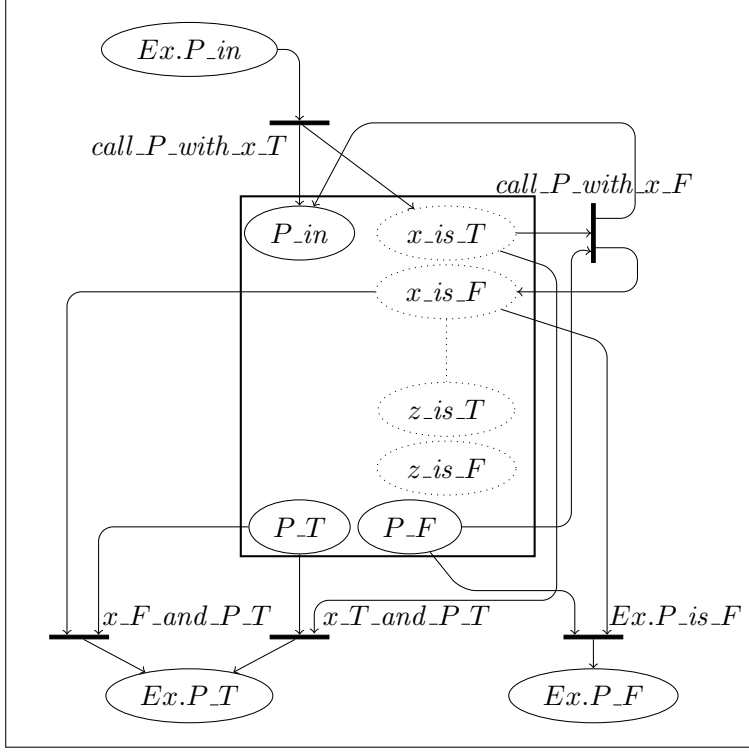
Fig. A.1. Reduction from $x,\neg P$ and $P \wedge Q$

The S-variant for the net corresponding to $\neg P$ is given by

$$\boldsymbol{V}(i) = \boldsymbol{V}_P(i+1) + i \cdot P\_in + (j_P(i+1)+1) \cdot not\_P\_T + (j_P(i+1)+1) \cdot not\_P\_F \ .$$

Since $\boldsymbol{V}_P(i+1) + i \cdot P\_in$ is a S-variant for $\mathcal{N}_P$ (the net corresponding to P) and transitions inside $\mathcal{N}_P$ do not affect the places not_P_T and not_P_F, $\boldsymbol{V}(i)$ strictly increases whenever any transition inside $\mathcal{N}_P$ fires. When the transition $call\_P$ fires, $\boldsymbol{V}(i)$ increases due to the presence of $i \cdot P\_in$. When the transition $not\_P\_is\_F$ (resp. $not\_P\_is\_T$) fires, $\boldsymbol{V}_i$ increases due to the presence of $\boldsymbol{V}_P(i+1) + (j_P(i+1)+1) \cdot not\_P\_F$ (resp. $\boldsymbol{V}_P(i+1) + (j_P(i+1)+1) \cdot not\_P\_T$). It is easy to see that $\boldsymbol{V}(i) + k \cdot not\_P\_in$ is a S-variant of $\mathcal{N}_{\neg P}$ for $0 \le k \le i-1$. In this case, $j_G(i) = j_P(i+1)+1$.

$G = P \wedge Q$. See box **C** in Fig. A.1. The S-variant for the net corresponding to $P \wedge Q$ is given by

$$\boldsymbol{V}(i) = \boldsymbol{V}_P(i+1) + i \cdot P\_in + \boldsymbol{V}_Q(j_P(i+1)+2) + (j_P(i+1)+1) \cdot Q\_in$$
$$+ (j_P(i+1) + j_Q(j_P(i+1)+2)+1) \cdot (P\_and\_Q\_T + P\_and\_Q\_F) \ .$$

Fig. A.2. Reduction from $\exists x.P$

Since $\boldsymbol{V}_P(i+1) + i \cdot P\_in$ is a S-variant for $\mathcal{N}_P$ and transitions inside $\mathcal{N}_P$ do not affect places in $\mathcal{N}_Q$ or $P\_and\_Q\_in$, $P\_and\_Q\_T$ or $P\_and\_Q\_F$, $\boldsymbol{V}(i)$ increases whenever any transition in $\mathcal{N}_P$ fires. Since $\boldsymbol{V}_Q(j_P(i+1)+2) + (j_P(i+1)+1) \cdot Q\_in$ is a S-variant for $\mathcal{N}_Q$ and transitions inside $\mathcal{N}_Q$ do not affect places in $\mathcal{N}_P$ or $P\_and\_Q\_in$, $P\_and\_Q\_T$ or $P\_and\_Q\_F$, $\boldsymbol{V}(i)$ increases whenever any transition in $\mathcal{N}_Q$ fires. When $call\_P$ fires, $\boldsymbol{V}(i)$ increases due to the presence of $i \cdot P\_in$. When transitions $P\_F\_and\_Q\_?$ or $P\_T\_and\_Q\_F$ fires, $\boldsymbol{V}(i)$ increases due to the presence of $(j_P(i+1) + j_Q(j_P(i+1)+2)+1) \cdot P\_and\_Q\_F$. When transition $P\_T\_and\_Q\_?$ fires, $\boldsymbol{V}(i)$ increases due to the presence of $(j_P(i+1)+1) \cdot Q\_in$. When transition $P\_T\_and\_Q\_T$ fires, $\boldsymbol{V}(i)$ increases due to the presence of $(j_P(i+1) + j_Q(j_P(i+1)+2)+1) \cdot P\_and\_Q\_T$. It is easy to see that $\boldsymbol{V}(i) + k \cdot P\_and\_Q\_in$ is a S-variant of $\mathcal{N}_{P \wedge Q}$ for $0 \le k \le i-1$. In this case, $j_G(i) = j_P(i+1) + j_Q(j_P(i+1)+2)+1$.

　　$G = \exists x.P$. See Fig. A.2. The S-variant for the net corresponding to $\exists x.P$ is given by

$$\boldsymbol{V}(i) = \boldsymbol{V}_P(i+1) + i \cdot P\_in + (2 + 2j_P(i+1)) \cdot Ex.P\_T$$
$$+ (1 + j_P(i+1)) \cdot x\_is\_F + (2 + 2j_P(i+1)) \cdot Ex.P\_F \ .$$

Since $\boldsymbol{V}_P(i+1) + i \cdot P\_in$ is a S-variant for $\mathcal{N}_P$ and co-efficient of x_is_F is 0 in $\boldsymbol{V}_P(i+1)$ (since x is not bound inside P), $\boldsymbol{V}(i)$ increases when any transition inside $\mathcal{N}_P$ fires (note that transitions inside $\mathcal{N}_P$ don't change token count of x_is_F,

$Ex.P\_T$ and $Ex.P\_F$). When transition $call\_P\_with\_x\_T$ fires, $\boldsymbol{V}(i)$ increases due to the presence of $i \cdot P\_in$. When transition $x\_T\_and\_P\_T$ fires, $\boldsymbol{V}(i)$ increases due to the presence of $(2+2j_P(i+1)) \cdot Ex.P\_T$. When transition $call\_P\_with\_x\_F$ fires, $\boldsymbol{V}(i)$ increases due to the presence of $(1+j_P(i+1)) \cdot x\_is\_F$. When transition $x\_F\_and\_P\_T$ fires, $\boldsymbol{V}(i)$ increases due to the presence of $(2+2j_P(i+1)) \cdot Ex.P\_T$. When transition $Ex.P\_is\_F$ fires, $\boldsymbol{V}(i)$ increases due to the presence of $(2 + 2j_P(i + 1)) \cdot Ex.P\_F$. It is easy to see that $\boldsymbol{V}(i) + k \cdot Ex.P\_in$ is a S-variant of $\mathcal{N}_{Ex.P}$ for $0 \leq k \leq i - 1$. In this case, $j_G(i) = 2 + 2j_P(i + 1)$. $\qquad\square$

Proof of Lemma 4.7. Suppose there is a firing sequence σ such that $M_0 \xrightarrow{\sigma} M_f$. Since $(\mathcal{N}, M_0, M_f)$ is partially bounded, there exists a firing sequence τ such that $M_0 \xrightarrow{\tau} M_f$ that satisfies the properties mentioned in Definition 4.6. We will first obtain a bound on $\overline{\tau}(U)$.

Suppose t is a transition in $\mathcal{N}$ and for some marking M, $M \xrightarrow{t} M'$. Since $(\mathcal{N}, M_0, M_f)$ is a partially bounded net, there is a partial S-variant $\boldsymbol{V}$ such that if $t \in U$, then $\boldsymbol{V}(M') \geq \boldsymbol{V}(M) + 1$ and if $t \in T \setminus U$, then $\boldsymbol{V}(M') \geq \boldsymbol{V}(M)$. Now, let $\tau = t_1 t_2 \dots t_r$ such that $M_0 \xrightarrow{t_1} M_1 \cdots \xrightarrow{t_r} M_r = M_f$. For $i \in \mathbb{N}_+ : 1 \leq i \leq r$, we have $\boldsymbol{V}(M_i) \geq \boldsymbol{V}(M_{i-1}) + 1$ if $t_i \in U$ and $\boldsymbol{V}(M_i) \geq \boldsymbol{V}(M_{i-1})$ if $t_i \in T \setminus U$. Starting with $i = r$ and iteratively substituting $\boldsymbol{V}(M_{i-1})$ with its lower bound, we get $\boldsymbol{V}(M_f) \geq \boldsymbol{V}(M_0) + \overline{\tau}(U)$. Therefore, $\overline{\tau}(U) \leq \boldsymbol{V}(M_f) - \boldsymbol{V}(M_0)$. Due to Lemma 4.5, we can assume w.l.o.g. that absolute value of each component of $\boldsymbol{V}$ is $O(mn^2(n!)^2 D^{2n})$. Thus, we get

$$\begin{aligned} \overline{\tau}(U) &\leq \boldsymbol{V}(M_f) - \boldsymbol{V}(M_0) \\ &= \sum_{p \in P} \boldsymbol{V}(p)(M_f(p) - M_0(p)) \\ &\leq \sum_{p \in P} O(mn^2(n!)^2 D^{2n}) \max(M_0, M_f) \ . \end{aligned}$$
$$\text{Therefore, } \overline{\tau}(U) \leq O(m^2 n^2 (n!)^2 D^{2n} \max(M_0, M_f)) \ .$$

Now, since $\overline{\tau}(T \setminus U) \leq (p_1(\overline{\tau}(U), D, M_0, M_f))^{p_2(m,n)}$, we get

$$\overline{\tau}(T) = O((p_1(\overline{\tau}(U), D, M_0, M_f))^{p_2(m,n)}).$$

Now, Algorithm 1 can be used here with the constant c in line 1 of the algorithm replaced with the bound given by the above equation. A space complexity analysis similar to the one done in Lemma 3.9 can be done for the modified algorithm, with the conclusion that it needs polynomial space. Again by Savitch's theorem, we conclude that there is a deterministic PSPACE algorithm that solves the reachability problem for the subclass of Petri nets mentioned in the statement of this lemma.$\square$

Lemma A.1 *Suppose the Petri net $\mathcal{N}$ is structurally partially bounded with set of progressive transitions U. Let $V = T \setminus U$ and $\mathcal{N}_V$ be the Petri net $\mathcal{N}$ restricted to V. Suppose $\boldsymbol{y}$ is a positive integral vector such that $\boldsymbol{N_V}^T \boldsymbol{y} \leq \boldsymbol{0}$. Suppose a firing sequence σ fires at M_0 such that $\overline{\sigma}(U) = k$. Then, for any intermediate marking M*

reached during firing of σ, $M^T \boldsymbol{y} \leq M_0^T \boldsymbol{y} + (\boldsymbol{kD})^T \boldsymbol{y}$.

Proof. Decompose the firing sequence σ as $\sigma = \sigma_0 u_1 \sigma_1 \cdots u_k \sigma_k$, where $u_1, \ldots, u_k$ are the progressive transitions and $\sigma_0, \ldots, \sigma_k$ are made up of the non-progressive transitions. Suppose the marking M is reached just after firing u_j or during firing of σ_j. We will prove by induction on j that $M^T \boldsymbol{y} \leq M_0^T \boldsymbol{y} + (\boldsymbol{jD})^T \boldsymbol{y}$. The result then follows since $j \leq k$.

For the base case $j = 0$, M is reached by firing transitions in $\mathcal{N}_V$ only. We have $M = M_0 + \boldsymbol{N_V} \overline{\sigma_0}$. Therefore, $M^T = M_0^T + \overline{\sigma_0}^T \boldsymbol{N_V}^T$ and hence $M^T \boldsymbol{y} = M_0^T \boldsymbol{y} + \overline{\sigma_0}^T \boldsymbol{N_V}^T \boldsymbol{y}$. Since $\boldsymbol{N_V}^T \boldsymbol{y} \leq \boldsymbol{0}$ and $\overline{\sigma_0} \geq \boldsymbol{0}$, we have $M^T \boldsymbol{y} \leq M_0^T \boldsymbol{y}$.

For the induction step, suppose $M_0 \overset{\sigma_0}{\to} \cdots \overset{\sigma_j}{\to} M_j \overset{u_{j+1}}{\to} M_{j+1} \overset{\sigma'_{j+1}}{\to} M$. By induction hypothesis, $M_j^T \boldsymbol{y} \leq M_0^T \boldsymbol{y} + (\boldsymbol{jD})^T \boldsymbol{y}$. Since D is the maximum entry in the incidence matrix $\boldsymbol{N}$ of $\mathcal{N}$, firing of u_{j+1} can add at most D tokens to any place. Hence, $M_{j+1} \leq M_j + \boldsymbol{D}$. Therefore, we have $M_{j+1}^T \boldsymbol{y} \leq M_0^T \boldsymbol{y} + ((\boldsymbol{j+1})\boldsymbol{D})^T \boldsymbol{y}$. Marking M was reached from M_{j+1} by firing only transitions in $\mathcal{N}_V$. Therefore, we have $M = M_{j+1} + \boldsymbol{N_V} \overline{\sigma'_{j+1}}$. Therefore, $M^T \boldsymbol{y} = M_{j+1}^T \boldsymbol{y} + \overline{\sigma'_{j+1}}^T \boldsymbol{N_V}^T \boldsymbol{y}$. Since $\boldsymbol{N_V}^T \boldsymbol{y} \leq \boldsymbol{0}$ and $\overline{\sigma'_{j+1}} \geq 0$, we get $M^T \boldsymbol{y} \leq M_{j+1}^T \boldsymbol{y} \leq M_0^T \boldsymbol{y} + ((\boldsymbol{j+1})\boldsymbol{D})^T \boldsymbol{y}$. This completes the induction and hence the proof. $\square$

B   Analysis of a partially bounded net

Valk and Vidal-Naquet [23] have given a family of Petri nets to demonstrate that the bound of a bounded Petri net can be non-primitive recursive. We will now show that all nets in this family are partially bounded, though not structurally partially bounded. In this family, there is one net $\mathcal{N}_i$ for each non-negative integer i. They are defined inductively as seen in Fig. 3.

The initial marking of $\mathcal{N}_i$ for any $i \geq 0$ is one token in b_i, n tokens in c_i and no tokens anywhere else. The working of this family of nets can be understood as follows. We start with $\mathcal{N}_0$ first. With one token in b_0 and n tokens in c_0, transition v_0 fires once to get a token in q_0. Transition t_0 can now fire n times to put $2n$ tokens in p_0. Transition w_0 can now fire once to put a token in s_0. Transition u_0 can now fire $2n$ times to put $2n$ tokens in c_0. Transition x_0 can now fire once to put a token in e_0. Thus, $\mathcal{N}_0$ began with one token in b_0 and ended with one token in e_0, and in the process, number of tokens in c_0 was doubled.

Working of $\mathcal{N}_i$ can now be understood in terms of $\mathcal{N}_{i-1}$ as follows. Suppose $f_{i-1}(n)$ is an upper bound on the number of tokens c_{i-1} can accumulate when $\mathcal{N}_{i-1}$ has finished with its' token in e_{i-1}, when it had n tokens in c_{i-1} at the beginning. With one token in b_i and n tokens in c_i, transition t_i can fire n times to put n tokens each in c_{i-1} and d_i. Firing x_i will now "initiate" $\mathcal{N}_{i-1}$ and it will "finish" with at most $f_{i-1}(n)$ tokens in c_{i-1}. $\mathcal{N}_{i-1}$ can be initiated again by firing r_i. This time, $\mathcal{N}_{i-1}$ finishes with at most $f_{i-1}^2(n)$ tokens in c_{i-1}. $\mathcal{N}_{i-1}$ can be initiated a maximum of n times like this to accumulate a maximum of $f_{i-1}^n(n)$ tokens in c_{i-1}. Now, transition v_i can be fired once to get a token into s_i. Transition u_i can be fired as many times as required to shift tokens from c_{i-1} to c_i. Finally, transition

w_i can be fired once to get a token in e_i. Thus, $\mathcal{N}_i$ began with one token in b_i and n tokens in c_i and finished with at most $f_i(n) = f_{i-1}^n(n)$ tokens in c_i. It is well known that the family of functions

$$f_0(n) = 2n$$
$$f_{i+1}(n) = f_i^n(n)$$

dominate any primitive recursive function. We will now show that any net in this family is partially bounded and that there is a PSPACE algorithm for solving reachability in any of these nets.

Proposition B.1 *Suppose $\mathcal{N}_i$ is a net from the above family, $i \geq 1$. Suppose σ_i is any firing sequence satisfying the following properties:*

  (i) $\overline{\sigma_i}(t_i) = \overline{\sigma_i}(r_i) = \overline{\sigma_i}(u_i) = 1$ *and* $\overline{\sigma_i}(x_i) = \overline{\sigma_i}(v_i) = \overline{\sigma_i}(w_i) = 0$.

  (ii) $\overline{\sigma_i}(r_k) = 1$ *for* $1 \leq k \leq i - 1$.

  (iii) $\overline{\sigma_i}(t_k) = \overline{\sigma_i}(u_k) = i - k + 1$ *for* $1 \leq k \leq i - 1$.

  (iv) $\overline{\sigma_i}(x_k) = \overline{\sigma_i}(v_k) = \overline{\sigma_i}(w_k) = i - k$ *for* $1 \leq k \leq i - 1$.

  (v) $\overline{\sigma_i}(t) = 0$ *for all other transitions t.*

Then the net effect of firing σ_i is to add i places to b_0 and remove i tokens from e_0.

Proof. We will prove by induction on j that firing transitions of σ_i in nets at level $i - j$ to i will result in $j + 1$ tokens being added to b_{i-j-1} and $j + 1$ tokens being removed from e_{i-j-1}. The result then follows by taking $j = i - 1$.

In what follows, we will represent the effect of firing a transition by expressions. $jw_i : -js_i + je_i$ means that firing transition w_i j times results in removal of j tokens from place s_i and addition of j tokens to place e_i.

When $i = 1$, t_1, r_1 and u_1 are the only transitions in $\overline{\sigma_i}$ and is not covered by the induction. In this case, effect of firing σ_i is given by the following expressions:

$$t_1 : -c_1 + c_0 + d_1$$
$$r_1 : -d_1 - e_0 + b_0$$
$$u_1 : -c_0 + c_1$$
$$total : b_0 - e_0$$

We will now begin with base case of the induction, $j = 1$. We want to prove that effect of firing transitions of σ_i that are at level $i - 1$ and i result in $2b_{i-2} - 2e_{i-2}$. This can be readily seen by observing that following expressions give the effect of each transition.

$$t_i : -c_i + c_{i-1} + d_i$$
$$r_i : -d_i - e_{i-1} + b_{i-1}$$
$$u_i : -c_{i-1} + c_i$$
$$r_{i-1} : -d_{i-1} - e_{i-2} + b_{i-2}$$
$$2t_{i-1} : -2c_{i-1} + 2c_{i-2} + 2d_{i-1}$$
$$2u_{i-1} : -2c_{i-2} + 2c_{i-1}$$
$$x_{i-1} : -b_{i-1} - d_{i-1} + b_{i-2}$$
$$v_{i-1} : -e_{i-2} + s_{i-1}$$
$$w_{i-1} : -s_{i-1} + e_{i-1}$$
$$total : 2b_{i-2} - 2e_{i-2}$$

For the induction step, assume that firing transitions in σ_i that are at levels $i - j$ through i results in $(j+1)b_{i-j-1} - (j+1)e_{i-j-1}$. We want to prove that firing transitions in σ_i that are at levels $i - j - 1$ through i results in $(j+2)b_{i-j-2} - (j+2)e_{i-j-2}$. This can again be verified by the following expressions.

$$\text{level } i - j \text{ and higher} : (j+1)b_{i-j-1} - (j+1)e_{i-j-1}$$
$$r_{i-j-1} : -d_{i-j-1} - e_{i-j-2} + b_{i-j-2}$$
$$(j+2)t_{i-j-1} : -(j+2)c_{i-j-1} + (j+2)c_{i-j-2} + (j+2)d_{i-j-1}$$
$$(j+2)u_{i-j-1} : -(j+2)c_{i-j-2} + (j+2)c_{i-j-1}$$
$$(j+1)x_{i-j-1} : -(j+1)b_{i-j-1} - (j+1)d_{i-j-1} + (j+1)b_{i-j-2}$$
$$(j+1)v_{i-j-1} : -(j+1)e_{i-j-2} + (j+1)s_{i-j-1}$$
$$(j+1)w_{i-j-1} : -(j+1)s_{i-j-1} + (j+1)e_{i-j-1}$$
$$total : (j+2)b_{i-j-2} - (j+2)e_{i-j-2}$$

$\square$

Proposition B.2 *In the net $\mathcal{N}_i$, except x_i, v_i, w_i, t_0 and u_0, all other transitions non-progressive transitions.*

Proof. It is sufficient to prove that all other transitions are part of some semi-positive T-invariant. For this, it is sufficient to take σ_i defined in proposition B.1 and add iv_0, iw_0 and ix_0 transitions to it. The resulting firing sequence τ_i when fired, doesn't change the number of tokens in any place. Hence, $\overline{\tau_i}$ is a semi-positive T-invariant whose support contains all transitions except x_i, v_i, w_i, t_0 and u_0. $\square$

Proposition B.3 *For $\mathcal{N}_i$, there exists a partial S-variant whose support consists of all the progressive transitions.*

Proof. The only progressive transitions are t_0, u_0, x_i, v_i and w_i. A partial S-variant is $\mathbf{V} = 2p_0 + (\Sigma_{1 \leq j \leq i} 3c_i) - b_i + s_i + 2e_i$. Table B.1 verifies this. $\square$

To establish that $\mathcal{N}_i$ is partially bounded, we need to prove that number of firings of non-progressive transitions is bounded by a function of number of firings of progressive transitions and input size. In what follows, the set of places $\{p_0, c_0, \ldots, c_i\}$ plays an important role. For convenience of notation, let $C_i = \{p_0, c_0, \ldots, c_i\}$ and

Transition	Places affected	$\sum_{p \in P} \boldsymbol{V}(\boldsymbol{p})(Post(p,t) - Pre(p,t))$
t_0	c_0, p_0	1
u_0	c_0, p_0	1
$t_j, 1 \leq j \leq i-1$	c_j, c_{j-1}, d_j	0
$x_j, 1 \leq j \leq i-1$	b_j, d_j, b_{j-1}	0
$r_j, 1 \leq j \leq i-1$	e_{j-1}, d_j, b_{j-1}	0
$v_j, 1 \leq j \leq i-1$	e_{j-1}, s_j	0
$u_j, 1 \leq j \leq i-1$	c_{j-1}, c_j	0
$w_j, 1 \leq j \leq i-1$	s_j, e_j	0
t_i	c_i, c_{i-1}, d_i	0
r_i	e_{i-1}, d_i, b_{i-1}	0
u_i	c_{i-1}, c_i	0
x_i	b_i, d_i, b_{i-1}	1
v_i	e_{i-1}, s_i	1
w_i	s_i, e_i	1

Table B.1
Effect of transitions on $\boldsymbol{V}$

$t(C_i)$ be the total number of tokens in all the places in C_i. For a marking M, let $M(C_i) = \sum_{p \in C_i} M(p)$.

Proposition B.4 *Suppose $\mathcal{N}_i$ has some initial marking M_i such that:*

(i) $M_i(b_i) = 1$.

(ii) $M_i(b_j) = M_i(s_j) = M_i(e_j) = 0, 0 \leq j \leq i-1$.

(iii) $M_i(q_0) = 0$.

(iv) $M_i(s_i) = M_i(e_i) = 0$.

Suppose $M_i \xrightarrow{\sigma} M$ and $\overline{\sigma}(t_0) = k$. Then we have $M(C_i) \leq M_i(C_i) + k$.

Proof. Among all transitions of $\mathcal{N}_i$, t_0 is the only one that can increase $t(C_i)$. We will now prove the result by induction on k.

For the base case $k = 0$, the result is a direct conclusion of the above observation. For the induction step, suppose $\overline{\sigma}(t_0) = k + 1$. Let us split σ as follows: $M_i \xrightarrow{\sigma_0} M_1 \xrightarrow{t_0} M_2 \xrightarrow{\sigma_1} M$ where $\overline{\sigma_1}(t_0) = 0$. By induction hypothesis, we get $M_1(C_i) \leq M_i(C_i) + k$. By inspecting the action of firing t_0, we can conclude that $M_2(C_i) \leq M_1(C_i) + 1$. Again by the observation made at the beginning of this proof, we can conclude that $M(C_i) \leq M_2(C_i)$. Therefore, we have $M(C_i) \leq M_i(C_i) + k + 1$.

This completes the induction step and hence the proof. □

Proposition B.5 *Suppose the net $\mathcal{N}_i$ (for some $i \geq 1$) has the initial marking with n_l tokens in c_i, 1 token in b_i and 0 tokens in all other places. In this initial marking, suppose a firing sequence σ is fired. If $\overline{\sigma}(t_0) = k$, then, for every $0 \leq j \leq i - 1$, $\overline{\sigma}(t_{i-j}) \leq (n_l + k)^{j+1}$.*

Proof. By induction on j. For the base case $j = 0$, we need to show that $\overline{\sigma}(t_i) \leq n_l + k$. This is true since in the given initial marking, t_i can fire at most n_l times.

By induction hypothesis, assume that $\overline{\sigma}(t_{i-j}) \leq (n_l + k)^{j+1}$. For the induction step, we need to show that $\overline{\sigma}(t_{i-j-1}) \leq (n_l + k)^{j+2}$. Note that for t_{i-j-1} to fire, there must be a token in b_{i-j-1}. Once there is a token in b_{i-j-1}, t_{i-j-1} can fire as many times as there are tokens in c_{i-j-1}. Once all tokens in c_{i-j-1} are exhausted, t_{i-j-1} can fire again only when more tokens are added to c_{i-j-1}. For adding more tokens to c_{i-j-1}, the token in b_{i-j-1} has to be removed. Therefore, for t_{i-j-1} to fire once more after exhausting all tokens in c_{i-j-1}, a token needs to be added to b_{i-j-1}. Let us call the period between adding a token to b_{i-j-1} and adding one token to b_{i-j-1} next time as one round. In one round, t_{i-j-1} can fire at most as many times as there are tokens in c_{i-j-1} at the beginning of the round (to add more tokens to c_{i-j-1}, the token in b_{i-j-1} has to be removed and this takes us to the next round). By proposition B.4, c_{i-j-1} has at most $n_l + k$ tokens at any time. Thus, $\overline{\sigma}(t_{i-j-1})$ is bounded by $n_l + k$ times number of times a token can be added to b_{i-j-1}.

Now, the only transitions that can add tokens to b_{i-j-1} are x_{i-j} and r_{i-j}. For every firing of x_{i-j} or r_{i-j}, a token is removed from d_{i-j}. Therefore, total number of times a token can be added to b_{i-j-1} is upper bounded by total number of tokens that can be added to d_{i-j}. The only transition that can add tokens to d_{i-j} is t_{i-j}. By induction hypothesis, $\overline{\sigma}(t_{i-j}) \leq (n_l + k)^{j+1}$. Therefore, $\overline{\sigma}(t_{i-j-1}) \leq (n_l + k)(n_l + k)^{j+1} = (n_l + k)^{j+2}$. This completes the induction and the proof. □

Proposition B.6 *Suppose the net $\mathcal{N}_i$, $i \geq 1$ has the initial marking with n_l tokens in c_i, 1 token in b_i and 0 tokens in all other places. In this initial marking, suppose a firing sequence σ is fired such that $\overline{\sigma}(t_0) = k$. Then the following are true.*

  (i) $\overline{\sigma}(x_{i-j}) \leq (n_l + k)^j$, $0 \leq j \leq i - 1$.

 (ii) $\overline{\sigma}(r_{i-j}) \leq (n_l + k)^{j+1}$, $0 \leq j \leq i - 1$.

(iii) $\overline{\sigma}(v_{i-j}) \leq (n_l + k)^j$, $0 \leq j \leq i - 1$.

(iv) $\overline{\sigma}(w_{i-j}) \leq (n_l + k)^j$, $0 \leq j \leq i - 1$.

 (v) $\overline{\sigma}(u_{i-j}) \leq (n_l + k)^{j+1}$, $0 \leq j \leq i - 1$.

(vi) $\overline{\sigma}(v_0), \overline{\sigma}(w_0), \overline{\sigma}(x_0) \leq (n_l + k)^i$.

Proof. (i) Every firing of x_{i-j} needs one token to be added to b_{i-j}. If $j = 0$, then x_{i-j} can fire only once. Otherwise, only transitions that can add tokens to b_{i-j} is x_{i-j+1} and r_{i-j+1}. Every firing of x_{i-j+1} or r_{i-j+1} needs one token to be added to d_{i-j+1}. The only transition that can add tokens to d_{i-j+1} is t_{i-j+1}. By proposition B.5, t_{i-j+1} can fire at most $(n_l + k)^j$ times. Therefore, x_{i-j} can fire at most $(n_l + k)^j$ times.

(ii) Every firing of r_{i-j} needs one token to be added to d_{i-j}. The only transition that can add tokens to d_{i-j} is t_{i-j}. Since by proposition B.5, t_{i-j} can fire at most $(n_l + k)^{j+1}$ times, r_{i-j} can fire at most $(n_l + k)^{j+1}$ times.

(iii) Between any two firings of x_{i-j}, v_{i-j} can fire at most once. Since x_{i-j} can fire at most $(n_l + k)^j$ times, v_{i-j} can also fire at most $(n_l + k)^j$ times.

(iv) Between any two firings of v_{i-j}, w_{i-j} can fire at most once. Since v_{i-j} can fire at most $(n_l + k)^j$ times, w_{i-j} can also fire at most $(n_l + k)^j$ times.

(v) Firing u_{i-j} needs a token to be present in s_{i-j}. Once a token is added to s_{i-j}, u_{i-j} can fire as many times as there are tokens in c_{i-j-1}. By proposition B.4, c_{i-j-1} will have at most $(n_l + k)$ tokens. Therefore, number of times u_{i-j} can be fired is bounded by $(n_l + k)$ times the number of times a token can be added to s_{i-j}. v_{i-j} is the only transition that can add tokens to s_{i-j} and it can fire at most $(n_l + k)^j$ times. Therefore, u_{i-j} can fire at most $(n_l + k)^{j+1}$ times.

(vi) Every firing of v_0, w_0 or x_0 needs one token to be added to b_0. Only transitions that can add tokens to b_0 are x_1 and r_1. Every firing of x_1 or r_1 needs one token to be added to d_1. t_1 is the only transition that can add tokens to d_1 and t_1 can fire at most $(n_l + k)^i$ times by proposition B.5. Hence, v_0, w_0 and x_0 can fire at most $(n_l + k)^i$ times. $\qquad\square$

In terms of Definition 4.6, $U = \{t_0, u_0, x_i, v_i, w_i\}$. The above proof gives a bound function that meets the requirements of Lemma 4.7. This leads to a polynomial space algorithm for reachability problem in any net $\mathcal{N}_i$.