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Invariants of monadic coalgebras

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Abstract

In this paper we consider invariants of computations described by monadic coalgebras, that is, coalgebras for a functor endowed with the structure of a monad. Following the idea of Pöschel and Rößiger [8], we propose another concept of invariants of such coalgebras, namely, the one based on co-relations. We introduce the clone-theoretic apparatus for monadic coalgebras and show that co-relations can be taken for a general representation of their invariants. We then demonstrate that not only subuniverses, but arbitrary λ-simulations can be thought of as invariants of monadic coalgebras, and that the approach to invariants via λ-simulations is inferior in comparison to the one via co-relations. In some cases invariant co-relations uniquely determine the monadic coalgebra. Since the same does not hold in general, to every monadic coalgebra we associate a coalgebra for the same monad which emulates the original one, and has the pleasant property of being uniquely determined by its invariant co-relations.

Key Words and Phrases: coalgebras, monads, invariants, co-relations, clone theory

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1 Introduction

Coalgebras provide an elegant and unified apparatus for investigation of various models of both computation and data structures. A particularly intriguing approach to modeling computer programs formally was offered by E. Moggi in [5,6]. The idea is to represent programs by coalgebras $X \to T(X)$, where T is a functor endowed with the structure of a monad, with the intuition that programs are to be thought of as mappings from data (elements of X) to computations (elements of $T(X)$). The requirement that T be equipped with the

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structure of a monad is a way of coping with the need to compose simpler programs into larger ones. Note that the carrier of a coalgebra is usually thought of as a set of internal states of the computation model under investigation. Following the idea of E. Moggi, in this paper we accept the other approach and think of it as the set of data.

Treating programs as monads has been widely accepted by the functional programming community, where it has led to elegant solutions to many practical problems. The use of monads in functional programs makes easier both implementation and formal description of some key concepts such as purely functional input/output, destructive arrays, lazy evaluation and modularity [10,11].

The formal investigation of invariant properties of computations has started long time ago and has played an important role in the development of computer science ever since. In this paper, we consider invariants of computations described by monadic coalgebras. Invariants of arbitrary coalgebras were introduced in [4,3] as subsets of the carrier closed with respect to the coalgebraic structure. Inspired by the research presented in [8], we propose another concept: invariant co-relations.

We first show that co-relations and monadic coalgebras can be bound together by means of a pair of standard clone-theoretic operators. The Galois connection which emerges from this construction shows that co-relations can be taken for a general representation of invariants of monadic coalgebras. We then demonstrate that not only subuniverses, but arbitrary λ-simulations can be thought of as invariants of such coalgebras, and that the approach to invariants via λ-simulations is inferior in comparison to the one via co-relations.

As its main tool, this paper introduces clone-theoretic apparatus for monadic coalgebras. The "classical" clone theory [7] can be understood as a general theory of invariants of sets of operations. We say that an operation f preserves a relation ϱ if

$$\begin{bmatrix} a_1 \\ b_1 \\ \vdots \end{bmatrix}, \begin{bmatrix} a_2 \\ b_2 \\ \vdots \end{bmatrix}, \ldots, \begin{bmatrix} a_n \\ b_n \\ \vdots \end{bmatrix} \in \varrho \quad \text{implies} \quad \begin{bmatrix} f(a_1, \ldots, a_n) \\ f(b_1, \ldots, b_n) \\ \vdots \end{bmatrix} \in \varrho.$$

By the fact that f preserves ϱ we actually mean that f has the property "encoded" by ϱ. We also say that ϱ is an invariant of f. For example, "f preserves $\varrho_0 := \{0\}$" means that $f(0, \ldots, 0) = 0$, while "f preserves $\leq$", where $\leq$ is a partial order, means that f is monotone.

To every set of operations F we can associate the set $\operatorname{Inv} F$ of all invariants common to all the elements of F. Dually, to every set of relations Q we can associate the set $\operatorname{Pol} Q$ of all the operations having each of the properties encoded by relations from Q. The pair $\langle \operatorname{Pol}, \operatorname{Inv} \rangle$ forms a Galois connection between sets of operations and sets of relations. Galois closed sets of operations are called *clones* of operations. Clones of operations can be thought of as maximal sets of operations with the given set of properties. It turns out that there is another, equivalent, characterization: clones of operations are composition closed sets of operations containing all trivial operations. Let us note that the emphasis in clone-theoretic investigations is not on properties of *one* operation, but on *sets* of operations having certain *sets* of properties.

This paper is motivated by [8] and relies on notions and results presented there. In [8], a coalgebra is taken to be a pair $\langle X, F\rangle$ where X is a set and F a set of *co-operations* on X. In that context (first proposed by K. Drbohlav in 1971, [2]) an n-ary co-operation is a mapping $f : X \to X + \ldots + X$ (n times). Clones of co-operations understood in this way were introduced in [1], while the notion of co-relation (which is crucial for our purposes) and the standard clone-theoretic apparatus were introduced in [8]. The authors of [8] had a strong intuition that their results should somehow carry over to T-coalgebras. They expressed this opinion in the form of a problem posed in Remark 6.10. In this paper, we solve the problem for the class of monadic coalgebras.

2 Preliminaries

Co-operations and coalgebras.

Let X be a set and $T : \mathcal{S}et \to \mathcal{S}et$ a functor. A *T-co-operation* is any mapping $\alpha : X \to T(X)$. A *T-coalgebra on X* is any pair $\langle X, \alpha\rangle$ where α is a T-co-operation. To keep the terminology simple, we shall omit the prefix “T-” and we shall not distinguish between coalgebras and co-operations – in the sequel, the term *coalgebra* will refer to both.

If T is the identity functor, then T-coalgebras are just mappings $X \to X$. For a set X let T_X denote the set of all mappings $X \to X$. For $f, g \in \mathsf{T}_X$ and $x \in X$ let $(f \cdot g)(x) := (g \circ f)(x) := g(f(x))$.

Let $\langle X, \alpha\rangle$ and $\langle Y, \beta\rangle$ be coalgebras. A mapping $h : X \to Y$ is a *homomorphism* between $\langle X, \alpha\rangle$ and $\langle Y, \beta\rangle$, in symbols $h : \langle X, \alpha\rangle \to \langle Y, \beta\rangle$, if the adjacent diagram commutes. A bijective homomorphism is called *isomorphism*. We write $\langle X, \alpha\rangle \cong \langle Y, \beta\rangle$ to denote that the two coalgebras are isomorphic.

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "\alpha"'] & Y \arrow[d, "\beta"] \\ T(X) \arrow[r, "T(h)"] & T(Y). \end{tikzcd}$$

We say that $\langle X, \alpha\rangle$ is a *subcoalgebra* of $\langle Y, \beta\rangle$ if $X \subseteq Y$ and the inclusion mapping $i_X : X \to Y : x \mapsto x$ is a homomorphism. In that case we say that X is a *subuniverse* of $\langle Y, \beta\rangle$.

Let $\langle X, \alpha\rangle$ be a coalgebra, $\lambda > 0$ an ordinal and $\varrho \subseteq X^\lambda$ a relation on X. We say that ϱ is a *λ-simulation* of $\langle X, \alpha\rangle$ if there exists a coalgebra $\gamma : \varrho \to T(\varrho)$ such that π^λ_ν is a homomorphism between $\langle \varrho, \gamma\rangle$ and $\langle X, \alpha\rangle$ for every $\nu < \lambda$. Here, π^λ_ν is the projection mapping $\pi^\lambda_\nu(\langle x_\xi : \xi < \lambda\rangle) = x_\nu$. In case of $\lambda = 2$, we write π_1 and π_2 instead of (more correct) π^2_0 and π^2_1, respectively.

Note that 1-simulations are precisely the subuniverses of the coalgebra, while 2-simulations are generally referred to as *bisimulations on* $\langle X, \alpha \rangle$.

A relation $\varrho \subseteq X \times Y$ is called a *bisimulation between coalgebras* $\langle X, \alpha \rangle$ *and* $\langle Y, \beta \rangle$ if there exists a coalgebra $\gamma : \varrho \to T(\varrho)$ such that $\pi_1 : \varrho \to X$ and $\pi_2 : \varrho \to Y$ are homomorphisms.

$\mathcal{S}et_T$ denotes the category whose objects are T-coalgebras, and whose morphisms are homomorphisms between T-coalgebras.

Monads.

Let $\mathcal{C}$ be a category and let $T : \mathcal{C} \to \mathcal{C}$ an endofunctor. A $\mathcal{C}$-*monad* is a triple $\langle T, \mu, \eta \rangle$ where $\mu : T^2 \to T$ and $\eta : \mathrm{id} \to T$ are natural transformations such that:

$$\begin{tikzcd} T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"'] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"] & T \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} T \arrow[r, "\eta T"] \arrow[dr, "\mathrm{id}"'] & T^2 \arrow[d, "\mu"] & T \arrow[l, "T\eta"'] \arrow[dl, "\mathrm{id}"] \\ & T & \end{tikzcd}$$

Recall that for every coalgebra $\alpha : X \to T(X)$ we have $\mu_X \circ T(\alpha) \circ \eta_X = \alpha$.

Monadic coalgebras.

Let $\mathcal{M} := \langle T, \mu, \eta \rangle$ be a $\mathcal{S}et$-monad. To emphasise that T carries the structure of a monad, coalgebras for functor T will be referred to as $\mathcal{M}$-coalgebras. Also, the category $\mathcal{S}et_T$ will be denoted by $\mathcal{S}et_{\mathcal{M}}$. For a set X, the pair $\langle \mathcal{M}, X \rangle$ will be abbreviated to $\mathcal{M}X$.

For a set X, let $\mathrm{cA}_{\mathcal{M}X}$ denote the set of all $\mathcal{M}$-coalgebras on X. Note that for every $f \in \mathsf{T}_X$, the monounary algebra $\langle X, f \rangle$ is a coalgebra for the identity monad $\langle \mathrm{id}, \mathrm{id}, \mathrm{id} \rangle$.

For $\alpha \in \mathrm{cA}_{\mathcal{M}X}$, let $\alpha^* := \mu_X \circ T(\alpha) \in \mathsf{T}_{T(X)}$ (the *Kleisli star*). Let $\mathsf{T}_{\mathcal{M}X} := \{ f \in \mathsf{T}_{T(X)} \mid (f \circ \eta_X)^* = f \}$. The *superposition* of $\mathcal{M}$-coalgebras α and β, in symbols $\alpha \cdot \beta$, is defined in the usual way: $\alpha \cdot \beta := \mu_X \circ T(\beta) \circ \alpha = \beta^* \circ \alpha$. It is clear that $\langle \mathrm{cA}_{\mathcal{M}X}, \cdot, \eta_X \rangle$ is a monoid isomorphic to $\langle \mathsf{T}_{\mathcal{M}X}, \cdot, \mathrm{id}_{T(X)} \rangle$ under the isomorphism $\alpha \mapsto \alpha^*$.

For $\alpha \in \mathrm{cA}_{\mathcal{M}X}$ and a nonnegative integer k, we define α^k by: $\alpha^0 := \eta_X$ and $\alpha^k := \alpha \cdot \ldots \cdot \alpha$ (k times). A coalgebra α is called *idempotent* if $\alpha^2 = \alpha$. Let $\mathcal{S}et^{\mathrm{idp}}_{\mathcal{M}}$ denote the full subcategory of $\mathcal{S}et_{\mathcal{M}}$ whose objects are idempotent coalgebras.

Submonoids of $\langle \mathrm{cA}_{\mathcal{M}X}, \cdot, \eta_X \rangle$ will be referred to as *monoids of coalgebras*. For $F \subseteq \mathrm{cA}_{\mathcal{M}X}$ let $\mathrm{Mon}_{\mathcal{M}X}\, F$ denote the least monoid of coalgebras containing F.

Co-relations.

For a set Y let $\lceil Y \rceil$ denote the least infinite cardinal exceeding $|Y|$:

$$\lceil Y \rceil := \begin{cases} \aleph_0, & |Y| < \aleph_0 \\ \aleph_{\xi+1}, & |Y| = \aleph_\xi, \xi \geq 0. \end{cases}$$

Let Y be a set and let $\lambda > 0$ be an ordinal. A *λ-ary co-vector* on Y is any mapping $\mathbf{r} : Y \to \lambda$. A *$\lambda$-ary co-relation* on Y [8] is any set of λ-ary co-vectors. If σ is a λ-ary co-relation, we write $\mathrm{ar}(\sigma) = \lambda$. Let $\mathrm{cR}_Y^{(\lambda)}$ denote the set of all λ-ary co-relations on Y and let $\mathrm{cR}_Y = \bigcup_{0<\lambda<\lceil Y \rceil} \mathrm{cR}_Y^{(\lambda)}$.

We say that $f \in \mathsf{T}_Y$ *preserves* $\varrho \in \mathrm{cR}_Y$ [8] if $\mathbf{r} \circ f \in \varrho$ for all $\mathbf{r} \in \varrho$. For $F \subseteq \mathsf{T}_Y$, let $\mathrm{cInv}_Y F := \{\varrho \in \mathrm{cR}_Y \mid \text{every } f \in F \text{ preserves } \varrho\}$ (see [8]). For $Q \subseteq \mathrm{cR}_Y$ let $\mathrm{cEnd}_Y Q := \{f \in \mathsf{T}_X \mid f \text{ preserves every } \varrho \in Q\}$.

Clones of co-relations.

Let ξ, λ and λ_i ($i \in I$) be ordinals such that $\xi > 0$ and $0 < \lambda, \lambda_i < \lceil Y \rceil$, $i \in I$. Further, let $\varrho_i \in \mathrm{cR}_Y^{(\lambda_i)}$, $i \in I$, and let $\varphi : \xi \to \lambda$ and $\varphi_i : \xi \to \lambda_i$, $i \in I$, be arbitrary mappings. The *generalised superposition of* $\langle \varrho_i : i \in I \rangle$ [8] is the λ-ary co-relation defined by $\bigwedge_{i \in I}{}^{\varphi}_{\varphi_i} \varrho_i := \{\varphi \circ \mathbf{r} \mid \mathbf{r} : Y \to \xi \text{ and } (\forall i \in I)\ \varphi_i \circ \mathbf{r} \in \varrho_i\}$.

For a nonempty subset $B \subseteq \lambda$ we define the *λ-ary B-co-diagonal* [8] by $\delta_B^\lambda := \{\mathbf{r} \mid \mathbf{r} : Y \to \lambda \text{ and } \mathbf{r}[Y] \subseteq B\}$. We say that $Q \subseteq \mathrm{cR}_Y$ is a *clone of co-relations* [8] if

- Q contains all δ_B^λ, where $0 < \lambda < \lceil Y \rceil$ and $\varnothing \neq B \subseteq \lambda$, and
- Q is closed with respect to all generalised superpositions.

We say that a clone Q of co-relations is *complete* if $\bigcup S \in Q$ for every $S \subseteq Q \cap \mathrm{cR}_{\mathcal{M}X}^{(\lambda)}$ and every ordinal λ such that $0 < \lambda < \lceil Y \rceil$. (It is easy to see that every clone of co-relations is closed with respect to arbitrary intersections; hence the name. What we here call complete clones of co-relations are referred to as *1-locally closed* clones of co-relations in [8].)

The next proposition is a slight modification of [8, Theorems 5.1 and 5.2]. Although the paper considers finitary co-relations only, the proofs of the two theorems easily generalise to cardinals less than $\lceil Y \rceil$. Thus, we have:

Proposition 2.1 *Let Y be a nonempty set.*

(i) *$M \subseteq \mathsf{T}_Y$ is a transformation monoid if and only if $M = \mathrm{cEnd}_Y \mathrm{cInv}_Y M$.*

(ii) *$Q \subseteq \mathrm{cR}_Y$ is a complete clone of co-relations if and only if $Q = \mathrm{cInv}_Y \mathrm{cEnd}_Y Q$.*

3 Monoids of coalgebras and clones of co-relations

In this section we extend the notions of co-relation, preservation, cEnd and cInv to coalgebras for a given monad. We characterise closed sets for the Galois connection $\langle \mathrm{cEnd}, \mathrm{cInv} \rangle$, and give a clone-theoretic description of λ-simulations. We pay additional attention to 1-simulations.

The clone-theoretic toolbox.

For a set X and a $\mathcal{S}et$-monad $\mathcal{M} := \langle T, \mu, \eta \rangle$ let $|\mathcal{M}X| := |T(X)|$ and

$$\lceil \mathcal{M}X \rceil := \begin{cases} \aleph_0, & |\mathcal{M}X| < \aleph_0 \\ \aleph_{\alpha+1}, & |\mathcal{M}X| = \aleph_\alpha, \alpha \geq 0. \end{cases}$$

A λ-*ary co-relation* on $\mathcal{M}X$ is any λ-ary co-relation on $T(X)$. Let $\mathrm{cR}^{(\lambda)}_{\mathcal{M}X}$ denote the set of all λ-ary co-relations on $\mathcal{M}X$ and let $\mathrm{cR}_{\mathcal{M}X} = \bigcup_{0<\lambda<\lceil \mathcal{M}X \rceil} \mathrm{cR}^{(\lambda)}_{\mathcal{M}X}$.

Let $\alpha \in \mathrm{cA}_{\mathcal{M}X}$ be a coalgebra and let $\mathbf{r} : T(X) \to \lambda$ be a co-vector. The *action of* α *on* $\boldsymbol{r}$ is the λ-ary co-vector $\alpha \cdot \mathbf{r}$ defined by $\alpha \cdot \mathbf{r} := \mathbf{r} \circ \mu_X \circ T(\alpha) = \mathbf{r} \circ \alpha^*$. We say that a coalgebra $\alpha \in \mathrm{cA}_{\mathcal{M}X}$ *preserves* a co-relation σ and that the co-relation σ is an *invariant* of α if $\alpha \cdot \mathbf{r} \in \sigma$ whenever $\mathbf{r} \in \sigma$.

For $F \subseteq \mathrm{cA}_{\mathcal{M}X}$, let $\mathrm{cInv}_{\mathcal{M}X} F := \{\varrho \in \mathrm{cR}_{\mathcal{M}X} \mid \text{every } \alpha \in F \text{ preserves } \varrho\}$. For $Q \subseteq \mathrm{cR}_{\mathcal{M}X}$ let $\mathrm{cEnd}_{\mathcal{M}X} Q := \{\alpha \in \mathrm{cA}_{\mathcal{M}X} \mid \alpha \text{ preserves every } \varrho \in Q\}$.

Characterisations.

The pair $\langle \mathrm{cEnd}, \mathrm{cInv} \rangle$ forms a Galois connection between the lattices $\mathcal{P}(\mathrm{cA}_{\mathcal{M}X})$ and $\mathcal{P}(\mathrm{cR}_{\mathcal{M}X})$. Our intention is to characterize the corresponding Galois closed sets of coalgebras and co-relations.

Lemma 3.1 *Let $\alpha, \beta \in \mathrm{cA}_{\mathcal{M}X}$ and $F \subseteq \mathrm{cA}_{\mathcal{M}X}$. Further, let $\boldsymbol{r}$ be a co-vector, $\varrho \in \mathrm{cR}_{\mathcal{M}X}$ and $Q \subseteq \mathrm{cR}_{\mathcal{M}X}$. Then*

(i) $(\alpha \cdot \beta) \cdot \boldsymbol{r} = \alpha \cdot (\beta \cdot \boldsymbol{r})$.

(ii) *α preserves ϱ if and only if α^* preserves ϱ.*

(iii) $\mathrm{cInv}_{T(X)}(F^*) = \mathrm{cInv}_{\mathcal{M}X} F$.

(iv) *If $F^* = \mathrm{cEnd}_{T(X)} Q$ then $F = \mathrm{cEnd}_{\mathcal{M}X} Q$.*

Proof. (i) follows by an easy calculation: $(\alpha \cdot \beta) \cdot \mathbf{r} = \mathbf{r} \circ \mu_X \circ T(\alpha \cdot \beta) = \mathbf{r} \circ \mu_X \circ T(\mu_X \circ T(\beta) \circ \alpha) = \mathbf{r} \circ \mu_X \circ T(\mu_X) \circ T^2(\beta) \circ T(\alpha) = \mathbf{r} \circ \mu_X \circ T(\beta) \circ \mu_X \circ T(\alpha) = \alpha \cdot (\mathbf{r} \circ \mu_X \circ T(\beta)) = \alpha \cdot (\beta \cdot \mathbf{r})$, while (ii), (iii) and (iv) are immediate. □

Proposition 3.2 *Let $F \subseteq \mathrm{cA}_{\mathcal{M}X}$. The following statements are equivalent:*

(i) *F is a monoid of coalgebras, i.e. $F = \mathrm{Mon}_{\mathcal{M}X} F$,*

(ii) *$F = \mathrm{cEnd}_{\mathcal{M}X} \mathrm{cInv}_{\mathcal{M}X} F$, and*

(iii) *$F = \mathrm{cEnd}_{\mathcal{M}X} Q$ for some $Q \subseteq \mathrm{cR}_{\mathcal{M}X}$.*

Proof. (ii) $\Rightarrow$ (iii) is trivial. (iii) $\Rightarrow$ (i) is an easy consequence of the Lemma 3.1 (i). As for (i) $\Rightarrow$ (ii), note that F^* is a transformation monoid, so $F^* = \mathrm{cEnd}_{T(X)} \mathrm{cInv}_{T(X)}(F^*)$ (Proposition 2.1). According to Lemma 3.1 (iii) and (iv), we have $F = \mathrm{cEnd}_{\mathcal{M}X} \mathrm{cInv}_{\mathcal{M}X} F$, as required. $\square$

The description of Galois closed sets of co-relations requires several technical prerequisits. Let $\lambda := |T(X)|$. For arbitrary bijection $\varphi : T(X) \to \lambda$, define a λ-ary co-relation $\delta^{\varphi}_{\mathcal{M}X}$ by $\delta^{\varphi}_{\mathcal{M}X} := \{\alpha \cdot \varphi \mid \alpha \in \mathrm{cA}_{\mathcal{M}X}\}$, and let $\mathrm{c}\Delta_{\mathcal{M}X} := \{\delta^{\varphi}_{\mathcal{M}X} \mid \varphi \text{ is a bijection from } T(X) \text{ to } \lambda\}$.

Lemma 3.3 (i) $\mathrm{cEnd}_{T(X)}\, c\Delta_{\mathcal{M}X} = \mathsf{T}_{\mathcal{M}X}$.

(ii) $c\Delta_{\mathcal{M}X} \subseteq \mathrm{cInv}_{T(X)} \mathsf{T}_{\mathcal{M}}$.

Proof. It is clear that (ii) follows immediately from (i). So let us show (i).

$\subseteq$: Take any $f \in \mathrm{cEnd}_{T(X)} \mathrm{c}\Delta_{\mathcal{M}X}$ and let $\delta^{\varphi}_{\mathcal{M}X} \in \mathrm{c}\Delta_{\mathcal{M}X}$ be arbitrary. Then f preserves $\delta^{\varphi}_{\mathcal{M}X}$. Since $\varphi = \eta_X \cdot \varphi \in \delta^{\varphi}_{\mathcal{M}X}$, we have that $\varphi \circ f \in \delta^{\varphi}_{\mathcal{M}X}$. So, $\varphi \circ f = \alpha \cdot \varphi = \varphi \circ \alpha^*$ for some $\alpha \in \mathrm{cA}_{\mathcal{M}X}$. But then $f = \alpha^* \in \mathsf{T}_{\mathcal{M}X}$ (since φ is bijective, and $(\mathrm{cA}_{\mathcal{M}X})^* = \mathsf{T}_{\mathcal{M}X}$).

$\supseteq$: Take any $\alpha^* \in \mathsf{T}_{\mathcal{M}X}$ and any $\delta^{\varphi}_{\mathcal{M}X} \in \mathrm{c}\Delta_{\mathcal{M}X}$. Let $\beta \cdot \varphi \in \delta^{\varphi}_{\mathcal{M}X}$ be arbitrary. Then $\alpha \cdot (\beta \cdot \varphi) = (\alpha \cdot \beta) \cdot \varphi \in \delta^{\varphi}_{\mathcal{M}X}$. $\square$

Proposition 3.4 *Let $Q \subseteq \mathrm{cR}_{\mathcal{M}X}$. The following are equivalent:*

(i) *Q is a complete clone of co-relations containing $\mathrm{c}\Delta_{\mathcal{M}X}$,*

(ii) *$Q = \mathrm{cInv}_{\mathcal{M}X} \mathrm{cEnd}_{\mathcal{M}X} Q$, and*

(iii) *$Q = \mathrm{cInv}_{\mathcal{M}X} F$ for some $F \subseteq \mathrm{cA}_{\mathcal{M}X}$.*

Proof. (ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (i): Let $Q = \mathrm{cInv}_{\mathcal{M}X} F$ for some $F \subseteq \mathrm{cA}_{\mathcal{M}X}$. According to Lemma 3.1 (iii), $Q = \mathrm{cInv}_{T(X)}(F^*)$. From $F^* \subseteq \mathsf{T}_{\mathcal{M}X}$ it follows that $Q = \mathrm{cInv}_{T(X)}(F^*) \supseteq \mathrm{cInv}_{T(X)} \mathsf{T}_{\mathcal{M}X} \supseteq \mathrm{c}\Delta_{\mathcal{M}X}$. Therefore, Q contains $\mathrm{c}\Delta_{\mathcal{M}X}$.

Let us now show that Q is a complete clone of co-relations. In view of Proposition 2.1, it suffices to show that $Q = \mathrm{cInv}_{T(X)} \mathrm{cEnd}_{T(X)} Q$. As we have already seen, $Q = \mathrm{cInv}_{T(X)}(F^*)$. So, $Q = \mathrm{cInv}_{T(X)}(F^*) = \mathrm{cInv}_{T(X)} \mathrm{cEnd}_{T(X)} \mathrm{cInv}_{T(X)}(F^*) = \mathrm{cInv}_{T(X)} \mathrm{cEnd}_{T(X)} Q$. Here, we used the fact that $\langle \mathrm{cInv}, \mathrm{cEnd} \rangle$ is a Galois connection, whence $\mathrm{cInv}\, \mathrm{cEnd}\, \mathrm{cInv} = \mathrm{cInv}$.

(i) $\Rightarrow$ (ii): Since Q is a complete clone of co-relations, according to Proposition 2.1, we conclude that $Q = \mathrm{cInv}_{T(X)} \mathrm{cEnd}_{T(X)} Q$. Let $G := \mathrm{cEnd}_{T(X)} Q$. Since $Q \supseteq \mathrm{c}\Delta_{\mathcal{M}X}$, we have $\mathrm{cEnd}_{T(X)} Q \subseteq \mathrm{cEnd}_{T(X)} \mathrm{c}\Delta_{\mathcal{M}X} = \mathsf{T}_{\mathcal{M}X}$ (Lemma 3.3). Now, let $F := \{f \circ \eta_X \mid f \in G\}$. From $G \subseteq \mathsf{T}_{\mathcal{M}X}$ it follows that $G = F^*$. So, $Q = \mathrm{cInv}_{T(X)}(F^*)$, whence $Q = \mathrm{cInv}_{\mathcal{M}X} F$ (Lemma 3.1). On the other hand $F^* = G = \mathrm{cEnd}_{T(X)} Q$, whence $F = \mathrm{cEnd}_{\mathcal{M}X} Q$ (the same lemma). Therefore, $Q = \mathrm{cInv}_{\mathcal{M}X} \mathrm{cEnd}_{\mathcal{M}X} Q$. $\square$

Thus, Galois closed sets of coalgebras and co-relations are, respectively, the monoids of coalgebras and certain complete clones of co-relations. We

$$
\begin{tikzcd}
\varrho \arrow[r, "\eta_\varrho"] \arrow[d, "\pi^\lambda_\nu"'] & T(\varrho) \arrow[d, "T(\pi^\lambda_\nu)"] \\
X \arrow[r, "\eta_X"] & T(X)
\end{tikzcd}
\qquad
\begin{tikzcd}
\varrho \arrow[r, "\gamma"] \arrow[d, "\pi^\lambda_\nu"'] & T(\varrho) \arrow[d, "T(\pi^\lambda_\nu)"] \\
X \arrow[r, "\alpha"] & T(X)
\end{tikzcd}
\qquad
\begin{tikzcd}
\varrho \arrow[r, "\delta"] \arrow[d, "\pi^\lambda_\nu"'] & T(\varrho) \arrow[d, "T(\pi^\lambda_\nu)"] \\
X \arrow[r, "\beta"] & T(X).
\end{tikzcd}
$$

(a) (b) (c)

$$
\begin{tikzcd}
\varrho \arrow[r, "\gamma"] \arrow[d, "\pi^\lambda_\nu"'] & T(\varrho) \arrow[r, "T(\delta)"] \arrow[d, "T(\pi^\lambda_\nu)"] & T^2(\varrho) \arrow[r, "\mu_\varrho"] \arrow[d, "T^2(\pi^\lambda_\nu)"] & T(\varrho) \arrow[d, "T(\pi^\lambda_\nu)"] \\
X \arrow[r, "\alpha"] & T(X) \arrow[r, "T(\beta)"] & T^2(X) \arrow[r, "\mu_X"] & T(X)
\end{tikzcd}
$$

(d)

Fig. 1. The four diagrams from the proof of Proposition 3.5

see here a nice parallel with invariants in the sense of [4]. It is a well known fact that the set of subuniverses of a coalgebra for a sufficiently nice functor is closed with respect to arbitrary unions and intersections. The same holds for co-relations: the set of all invariant co-relations of a monadic coalgebra is a complete clone of co-relations, and hence, closed with respect to arbitrary unions and intersections of co-relations of the same arity. As an immediate consequence of this observation we have that, given $\alpha \in \mathrm{cA}_{\mathcal{M}X}$, for every $\varrho \in \mathrm{cR}^{(\lambda)}_{\mathcal{M}X}$ there exists the least λ-ary invariant of α containing ϱ, and the greatest λ-ary invariant of α contained in ϱ. If we denote the former by $\langle \varrho \rangle_\alpha$ and the latter by $[\varrho]_\alpha$, then $\langle \varrho \rangle_\alpha = \{\beta \cdot \mathbf{r} \mid \beta \in \mathrm{Mon}_{\mathcal{M}X}\{\alpha\}, \mathbf{r} \in \varrho\}$ and $[\varrho]_\alpha = \{\mathbf{r} \in \varrho \mid \langle \mathbf{r} \rangle_\alpha \subseteq \varrho\}$.

Note also that the closure operator $\langle \cdot \rangle_\alpha$ resembles the operator $\Gamma_F(\cdot)$ which plays the analogous role in [8] (see Definition 2.1 and Proposition 3.8 in case $n = 1$).

Co-relations and λ-simulations.

¿From the clone-theoretic point of view, certain property is considered to be invariant if the set of all the objects having that property is composition closed and contains the trivial objects. The reason for this is simple. Composition closed sets of objects can be described by means of relations – the standardised invariants. Thus, the property under consideration can also be represented by the standardised invariants, and, therefore, is an invariant itself. In view of that, we shall now show that arbitrary λ-simulations can also be considered as invariants of monadic coalgebras.

Proposition 3.5 *Let $\lambda > 0$ be an ordinal and $\varrho \subseteq X^\lambda$ a relation on X. The set of all coalgebras $\alpha \in \mathrm{cA}_{\mathcal{M}X}$ with the property that ϱ is a λ-simulation of $\langle X, \alpha \rangle$ forms a monoid of coalgebras.*

Proof. Let $F = \{\alpha \in \mathrm{cA}_{\mathcal{M}X} \mid \varrho \text{ is a } \lambda\text{-simulation of } \langle X, \alpha\rangle\}$. Since $\eta : \mathrm{id} \to T$ is natural, the diagram (a) in Fig. 1 commutes i.e., $\eta_X \in F$. Now, let $\alpha, \beta \in F$. There exist $\gamma, \delta : \varrho \to T(\varrho)$ such that diagrams (b) and (c) in Fig. 1 commute. Then diagram (d) in Fig. 1 commutes whence $\alpha \cdot \beta \in F$, $\mu_\varrho \circ T(\delta) \circ \gamma$ being the corresponding coalgebra structure $\varrho \to T(\varrho)$. $\square$

Corollary 3.6 *Let $\lambda > 0$ be an ordinal and $\varrho \subseteq X^\lambda$ a relation on X. There exists a set of co-relations $Q_\varrho \subseteq \mathrm{cR}_{\mathcal{M}X}$ such that ϱ is a λ-simulation of $\langle X, \alpha\rangle$ if and only if $Q_\varrho \subseteq \mathrm{cInv}_{\mathcal{M}X}\{\alpha\}$.*

Proof. Let $F := \{\alpha \in \mathrm{cA}_{\mathcal{M}X} \mid \varrho \text{ is a } \lambda\text{-simulation of } \langle X, \alpha\rangle\}$. Then F is a monoid of coalgebras, whence $F = \mathrm{cEnd}_{\mathcal{M}X}\,\mathrm{cInv}_{\mathcal{M}X}\,F$. Now, put $Q_\varrho := \mathrm{cInv}_{\mathcal{M}X}\,F$. $\square$

The corollary says that, at least in principle, one can describe λ-simulations by co-relations. Thus, when speaking of properties of coalgebras, co-relations suffice and there is no need to consider λ-simulations. However, λ-simulations are much easier to work with, and we shall use them whenever appropriate. As an illustration, we now present a constructive description of 1-simulations (that is, invariants in the sense of [4]) by co-relations.

For $S \subseteq X$ we define a binary co-relation Θ_S as follows: $\mathbf{r} \in \Theta_S$ if and only if $\mathbf{r}$ is a mapping $T(X) \to \{0,1\}$ which makes the adjacent diagram commute. Here, c_0 is the constant mapping $c_0(x) = 0$, and $i_S : S \to X$ is the inclusion mapping.

$$\begin{tikzcd} T(S) \arrow[r, "T(i_S)"] \arrow[d, "c_0"'] & T(X) \arrow[d, "\mathbf{r}"] \\ \{0\} \arrow[r, "i_{\{0\}}"] & \{0,1\} \end{tikzcd} \qquad (*)$$

Proposition 3.7 *Let $\langle X, \alpha\rangle$ be a coalgebra and $S \subseteq X$. S is a subuniverse of $\langle X, \alpha\rangle$ if and only if $\Theta_S \in \mathrm{cInv}_{\mathcal{M}X}\{\alpha\}$.*

Proof. $\Rightarrow$: Suppose S is a subuniverse of $\langle X, \alpha\rangle$. There exists a mapping $\alpha' : S \to T(S)$ such that $\langle S, \alpha'\rangle$ is a subcoalgebra of $\langle X, \alpha\rangle$. By applying T to the corresponding diagram we obtain

$$\begin{tikzcd} T(S) \arrow[r, "T(i_S)"] \arrow[d, "T(\alpha')"'] & T(X) \arrow[d, "T(\alpha)"] \\ T^2(S) \arrow[r, "T^2(i_S)"] & T^2(X) \end{tikzcd} \qquad (1)$$

The naturality of μ yields

$$\begin{tikzcd} T^2(S) \arrow[r, "T^2(i_S)"] \arrow[d, "\mu_S"'] & T^2(X) \arrow[d, "\mu_X"] \\ T(S) \arrow[r, "T(i_S)"] & T(X) \end{tikzcd} \tag{2}$$

Take any $\mathbf{r} \in \Theta_S$. Then $\mathbf{r}$ makes the diagram $(*)$ commute. By pasting (1), (2) and $(*)$ we obtain

$$\begin{tikzcd} T(S) \arrow[r, "T(i_S)"] \arrow[d, "c_0 \circ \mu_S \circ T(\alpha')"'] & T(X) \arrow[d, "\mathbf{r} \circ \mu_X \circ T(\alpha)"] \\ \{0\} \arrow[r, "i_{\{0\}}"] & \{0,1\} \end{tikzcd}$$

Since $c_0 \circ \mu_S \circ T(\alpha') = c_0$ and $\mathbf{r} \circ \mu_X \circ T(\alpha) = \alpha \cdot \mathbf{r}$, we have that $\alpha \cdot \mathbf{r} \in \Theta_S$. This proves that α preserves Θ_S.

$\Leftarrow$: Now suppose that $\Theta_S \in \mathrm{cInv}_{\mathcal{M}X}\{\alpha\}$. We have to find $\beta : S \to T(S)$ such that $\alpha \circ i_S = T(i_S) \circ \beta$. To do so, it suffices to show that $\alpha[S] \subseteq j_S[T(S)]$, where $j_S := T(i_S)$. (Note that if this inclusion holds, then $\beta : S \to T(S) : s \mapsto j_S^{-1}(\alpha(s))$ will do.)

Suppose to the contrary that there exists an $s \in S$ such that $\alpha(s) \notin j_S[T(S)]$. Define $\mathbf{r} : T(X) \to \{0,1\}$ by

$$\mathbf{r}(x) = \begin{cases} 1, & x = \alpha(s), \\ 0, & \text{otherwise.} \end{cases}$$

Obviously, $\mathbf{r} \in \Theta_S$. Let us show that $\alpha \cdot \mathbf{r} \notin \Theta_S$. Since $\eta : \mathrm{id} \to T$ is natural and since $\langle T, \mu, \eta \rangle$ is a monad, we have

$$\begin{tikzcd} S \arrow[r, "i_S"] \arrow[d, "\eta_S"'] & X \arrow[r, "\alpha"] \arrow[d, "\eta_X"'] & T(X) \arrow[d, "\eta_{T(X)}"'] \arrow[dr, "\mathrm{id}_{T(X)}"] & \\ T(S) \arrow[r, "T(i_S)"] & T(X) \arrow[r, "T(\alpha)"] & T^2(X) \arrow[r, "\mu_X"] & T(X) \end{tikzcd}$$

i.e. $\mu_X \circ T(\alpha) \circ T(i_S) \circ \eta_S = \alpha \circ i_S$. Now $((\alpha \cdot \mathbf{r}) \circ T(i_S) \circ \eta_S)(s) = (\mathbf{r} \circ \mu_X \circ T(\alpha) \circ T(i_S) \circ \eta_S)(s) = (\mathbf{r} \circ \alpha \circ i_S)(s) = \mathbf{r}(\alpha(s)) = 1$.

On the other hand, for all $\mathbf{r} \in \Theta_S$ and all $s \in S$ we have $(\mathbf{r} \circ T(i_S) \circ \eta_S)(s) = 0$. Indeed, since $\mathbf{r} \in \Theta_S$ and $\eta_S : S \to T(S)$, the adjacent diagram commutes, whence $(\mathbf{r} \circ T(i_S) \circ \eta_S)(s) = (i_{\{0\}} \circ c_0 \circ \eta_S)(s) = 0$. Therefore, $\alpha \cdot \mathbf{r} \notin \Theta_S$, which contradicts the assumption $\Theta_S \in \mathrm{cInv}_{\mathcal{M}X}\{\alpha\}$.

$$\begin{tikzcd} S \arrow[r, "\eta_S"] & T(S) \arrow[r, "T(i_S)"] \arrow[d, "c_0"'] & T(X) \arrow[d, "\mathbf{r}"] \\ & \{0\} \arrow[r, "i_{\{0\}}"] & \{0,1\} \end{tikzcd}$$

This concludes the proof of $\alpha[S] \subseteq j_S[T(S)]$. $\square$

Example 3.8 Let us take a look at an example concerning Turing machines. Let Σ be an alphabet and let Tapes(Σ) be the set of all mapings $t : \mathbf{Z} \to \{\square\} + \Sigma$ with the property that the set $\{m \in \mathbf{Z} \mid t(m) \neq \square\}$ is finite. Further, let $X := \mathbf{Z} \times \mathrm{Tapes}(\Sigma)$ be the set of configurations, where $\langle k, t\rangle$ represents the tape t with the head scanning the k-th cell. Then a Turing machine can be modelled by a coalgebra $\alpha : X \to T(X)$, where $T(X) = X^+ + X^\omega$ (recall that X^+ is the set of all nonempty finite sequences of elements of X). So, $\alpha(x)$ is the sequence of configurations representing the behaviour of the machine on input x. This sequence is either finite or infinite, depending on whether the machine stops on the input or not.

Let us now describe the monad for T where $\alpha \cdot \beta$ will mean "start β on the outcome of α". Let $\eta_X(x) = \langle x\rangle$, a 1-element sequence. As for μ, let us first note that it operates on sequences of sequences which we shall depict as column vectors. For finite sequences of (finite or infinite) sequences we set

$$\mu_X \begin{bmatrix} \langle \boxed{a_{11}}, a_{12}, \ldots\rangle \\ \langle \boxed{a_{21}}, a_{22}, \ldots\rangle \\ \vdots \\ \langle \boxed{a_{n1}, a_{n2}, \ldots} \rangle \end{bmatrix} = \langle a_{11}, a_{21}, \ldots, a_{n1}, a_{n2}, \ldots\rangle,$$

while for infinite sequences we just collect the first coordinates:

$$\mu_X \begin{bmatrix} \langle \boxed{a_{11}}, a_{12}, \ldots\rangle \\ \langle \boxed{a_{21}}, a_{22}, \ldots\rangle \\ \vdots \\ \langle \boxed{a_{n1}}, a_{n2}, \ldots\rangle \\ \vdots \end{bmatrix} = \langle a_{11}, a_{21}, \ldots, a_{n1}, \ldots\rangle.$$

Then $\langle T, \mu, \eta\rangle$ is a monad. For T-coalgebras α and β, $\alpha \cdot \beta(x)$ has the following meaning: if $\alpha(x)$ terminates with the outcome y, apply β on y; if not, β is never applied (since α goes on for ever).

Let σ be the set of all co-vectors $\mathbf{r} : T(X) \to \{0, 1\}$ with the property that $\mathbf{r}[X^+] = \{0\}$. Then it is not hard to see that σ is an invariant of a T-coalgebra α if and only if α terminates on all inputs.

4 Characterising coalgebras by invariant co-relations

Invariants are often used to show that two systems are not equal or not isomorphic. In this section we are going to show that the concept of co-relation as a tool for encoding properties of monadic coalgebras is finer than that of a λ-simulation. Namely, it may happen that two coalgebras cannot be distinguished by means of λ-simulations, but can easily be distinguished by means of invariant co-relations.

We start with two examples. In the first example, we present a pair of isomorphic coalgebras which have the same λ-simulations and distinct invariant co-relations. The second example shows that the same can happen with coalgebras that are not even bisimilar. After the examples, we investigate to what extent the knowledge of invariants of a monadic coalgebra determines the coalgebra. In some cases invariant co-relations uniquely determine the coalgebra, but this does not hold in general.

Example 4.1 Let $X = \{a, b, c, d, e, f\}$ and consider the monad $\langle T, \mu, \eta\rangle$ given by: $T(X) = X + X$ (which we understand as $\{0, 1\} \times X$), $\eta_X(x) = \langle 0, x\rangle$ and

$$\mu_X(\langle p, \langle q, x\rangle\rangle) = \begin{cases} \langle 0, x\rangle, & \langle p, q\rangle \in \{\langle 0, 0\rangle, \langle 1, 1\rangle\} \\ \langle 1, x\rangle, & \langle p, q\rangle \in \{\langle 0, 1\rangle, \langle 1, 0\rangle\}. \end{cases}$$

Consider coalgebras $\langle X, \alpha\rangle$ and $\langle X, \beta\rangle$ given by

$$\alpha := \begin{pmatrix} a & b & c & d & e & f \\ 0b & 0c & 0a & 1e & 1f & 1d \end{pmatrix} \quad \text{and} \quad \beta := \begin{pmatrix} a & b & c & d & e & f \\ 0b & 0c & 0a & 1f & 1d & 1e \end{pmatrix},$$

where $0a$ abbreviates $\langle 0, a\rangle$, etc. The mapping $\varphi = \begin{pmatrix} a\ b\ c\ d\ e\ f \\ a\ b\ c\ f\ e\ d \end{pmatrix}$ is an isomorphism between the two coalgebras.

It is easy to check that for every ordinal $\lambda > 0$ and every $\varrho \subseteq X^\lambda$, ϱ is a λ-simulation of $\langle X, \alpha\rangle$ if and only if ϱ is a λ-simulation of $\langle X, \beta\rangle$. Therefore, these two coalgebras cannot be distinguished by means of λ-simulations.

Now, consider the 6-ary co-relation $\varrho := \{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$ where

$$\mathbf{r}_1 := \begin{pmatrix} 0a & 0b & 0c & 0d & 0e & 0f & 1a & 1b & 1c & 1d & 1e & 1f \\ 0 & 1 & 2 & 3 & 4 & 5 & 0 & 1 & 2 & 3 & 4 & 5 \end{pmatrix},$$

$$\mathbf{r}_2 := \begin{pmatrix} 0a & 0b & 0c & 0d & 0e & 0f & 1a & 1b & 1c & 1d & 1e & 1f \\ 1 & 2 & 0 & 4 & 5 & 3 & 1 & 2 & 0 & 4 & 5 & 3 \end{pmatrix}, \text{ and}$$

$$\mathbf{r}_3 := \begin{pmatrix} 0a & 0b & 0c & 0d & 0e & 0f & 1a & 1b & 1c & 1d & 1e & 1f \\ 2 & 0 & 1 & 5 & 3 & 4 & 2 & 0 & 1 & 5 & 3 & 4 \end{pmatrix}.$$

An easy computation shows that $\alpha \cdot \mathbf{r}_1 = \mathbf{r}_2$, $\alpha \cdot \mathbf{r}_2 = \mathbf{r}_3$ and $\alpha \cdot \mathbf{r}_3 = \mathbf{r}_1$, whence $\varrho \in \mathrm{cInv}_{\mathcal{M}X}\{\alpha\}$. However $\varrho \notin \mathrm{cInv}_{\mathcal{M}X}\{\beta\}$ since

$$\beta \cdot \mathbf{r}_1 := \begin{pmatrix} 0a & 0b & 0c & 0d & 0e & 0f & 1a & 1b & 1c & 1d & 1e & 1f \\ 1 & 2 & 0 & 5 & 3 & 4 & 1 & 2 & 0 & 5 & 3 & 4 \end{pmatrix} \notin \varrho$$

Example 4.2 Let $X = \{a, b\}$ and consider the monad $\langle T, \mu, \eta \rangle$ from Example 4.1. Let α and β be the following coalgebras:

$$\alpha := \eta_X = \begin{pmatrix} a & b \\ 0a & 0b \end{pmatrix} \qquad \text{and} \qquad \beta := \begin{pmatrix} a & b \\ 1a & 1b \end{pmatrix}.$$

Since $\varnothing$ is the only bisimulation between $\langle X, \alpha \rangle$ and $\langle X, \beta \rangle$, the two coalgebras are "coalgebraically unrelated". However, it is easy to see that $\varrho \subseteq X^\lambda$ is a λ-simulation of $\langle X, \alpha \rangle$ if and only if it is a λ-simulation of $\langle X, \beta \rangle$.

Now consider the binary co-relation $\sigma := \{\mathbf{r}\}$ where $\mathbf{r}$ is the following co-vector:

$$\mathbf{r} := \begin{pmatrix} 0a & 0b & 1a & 1b \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

It is easy to see that

$$\beta \cdot \mathbf{r} = \begin{pmatrix} 0a & 0b & 1a & 1b \\ 1 & 1 & 0 & 0 \end{pmatrix} \notin \sigma,$$

whence $\sigma \notin \mathrm{cInv}_{\mathcal{M}X}\{\beta\}$. On the other hand, η_X preserves any co-relation, so $\sigma \in \mathrm{cInv}_{\mathcal{M}X}\{\alpha\}$.

We say that $\alpha \in \mathrm{cA}_{\mathcal{M}X}$ is *singular* if $\alpha \neq \eta_X$ and $\alpha^k = \alpha$ for some integer $k > 1$. Otherwise, α is called *regular*. For a singular $\alpha \in \mathrm{cA}_{\mathcal{M}X}$, the least integer $k > 1$ for which $\alpha^k = \alpha$ will be called the *singularity index* of α. If s is the singularity index of a singular coalgebra α, then $s-1$ is called the *period* of α and will be denoted by $\omega(\alpha)$.

Let $f \in \mathsf{T}_Y$ be a mapping. A nonempty subset $C \subseteq Y$ is called a *cycle of f of length l* if $C = \{y_0, \dots, y_{l-1}\}$ where $y_i \neq y_j$ for $i \neq j$, and $f(y_i) = y_{i+1}$ (addition modulo l). Note that if f is singular and C is a cycle of f, then $|C|$ divides the period of f.

Proposition 4.3 *Let α and β be coalgebras such that* $\mathrm{cInv}_{\mathcal{M}X}\{\alpha\} = \mathrm{cInv}_{\mathcal{M}X}\{\beta\}$.

(i) *If at least one of α, β is regular, then $\alpha = \beta$.*

(ii) *If at least one of α, β is idempotent, then $\alpha = \beta$.*

Proof. (i) Suppose that α is a regular coalgebra. From $\mathrm{cInv}_{\mathcal{M}X}\{\alpha\} = \mathrm{cInv}_{\mathcal{M}X}\{\beta\}$ it follows that $\mathrm{cEnd}_{\mathcal{M}X}\,\mathrm{cInv}_{\mathcal{M}X}\{\alpha\} = \mathrm{cEnd}_{\mathcal{M}X}\,\mathrm{cInv}_{\mathcal{M}X}\{\beta\}$, and hence, by Proposition 3.2, $\mathrm{Mon}_{\mathcal{M}X}\{\alpha\} = \mathrm{Mon}_{\mathcal{M}X}\{\beta\}$. If $\alpha = \eta_X$, then $\mathrm{Mon}_{\mathcal{M}X}\{\alpha\} = \{\eta_X\}$ so $\beta = \eta_X$, too. Now, suppose $\alpha \neq \eta_X$ and $\beta \neq \eta_X$. Then $\alpha = \beta^k$ and $\beta = \alpha^l$ for some positive integers k and l, whence $\alpha = \alpha^{kl}$. Since α is regular, $kl = 1$. Therefore, $k = 1$ and $\alpha = \beta$.

(ii) Suppose α is idempotent, and $\alpha \neq \eta_X$ and $\beta \neq \eta_X$. As in (i), $\mathrm{Mon}_{\mathcal{M}X}\{\alpha\} = \mathrm{Mon}_{\mathcal{M}X}\{\beta\}$. But $\mathrm{Mon}_{\mathcal{M}X}\{\alpha\} = \{\eta_X, \alpha\}$ and thus $\beta = \alpha$. □

The situation concerning non-idempotent singular coalgebras is not so clear. In only a few cases we can show that invariant co-relations determine the coalgebra uniquely. We present one such case.

Lemma 4.4 *Let $f, g \in \mathsf{T}_Y$ be singular mappings such that* $\mathrm{cInv}_Y\{f\} = \mathrm{cInv}_Y\{g\}$.

(i) *Let $\varrho \subseteq X^\lambda$ for some $\lambda > 0$. Then: ϱ is a λ-simulation of $\langle Y, f\rangle$ if and only if ϱ is a λ-simulation of $\langle Y, g\rangle$.*

(ii) *For $\varnothing \neq C \subseteq X$ we have: C is a cycle of f if and only if C is a cycle of g.*

(iii) $\omega(f) = \omega(g)$.

(iv) $\ker f = \ker g$.

(v) *If $\omega(f) = 2$ or $\omega(g) = 2$, then $f = g$.*

Proof. (i) Let $\varrho \subseteq X^\lambda$ be a λ-simulation of $\langle Y, f\rangle$. According to Corollary 3.6, there exists a set Q_ϱ of co-relations such that for every $h \in \mathsf{T}_Y$ we have: ϱ is a λ-simulation of h if and only if $Q_\varrho \subseteq \mathrm{cInv}_Y\{h\}$. Therefore, $Q_\varrho \subseteq \mathrm{cInv}_Y\{f\} = \mathrm{cInv}_Y\{g\}$, and so ϱ is a λ-simulation of g.

(ii) Follows from (i) and the following simple observation: Let f be a singular mapping. Then $C \neq \varnothing$ is a cycle of f if and only if C is a subuniverse of $\langle Y, f\rangle$ and no proper subset of C has this property.

(iii) Note that $\omega(f) = \max\{|C| \mid C \text{ is a cycle of } f\}$. The statement now follows from (ii).

(iv) Let $x \neq y$. It is easy to see that $\langle x, y\rangle \in \ker f$ if and only if $\varrho_{xy} := \{\langle x, y\rangle\} \cup \{\langle z, z\rangle \mid z \in Y\}$ is a bisimulation of $\langle Y, f\rangle$ and at most one of $\{x\}$,

$\{y\}$ is a cycle of f. The statement now follows from (i) and (ii).

(v) The statement follows from (ii) and (iv). $\square$

Proposition 4.5 *Let α and β be singular coalgebras such that* $\mathrm{cInv}_{\mathcal{M}X}\{\alpha\} = \mathrm{cInv}_{\mathcal{M}X}\{\beta\}$ *and $\alpha = \alpha^3$ or $\beta = \beta^3$. Then $\alpha = \beta$.*

Proof. Suppose $\alpha = \alpha^3$. Then $\alpha^* = (\alpha^*)^3$. From Lemma 3.1 (iii) it follows that $\mathrm{cInv}_{T(X)}\{\alpha^*\} = \mathrm{cInv}_{T(X)}\{\beta^*\}$. So, $\alpha^* = \beta^*$ by Lemma 4.4 (v), whence $\alpha = \beta$. $\square$

Example 4.6 We can easily provide an example of two distinct coalgebras having the same invariant co-relations. Let $X = \{a, b, c\}$, let $f : X \to X$ be the mapping $a \mapsto b \mapsto c \mapsto a$ and $g = f^{-1}$. Then id-coalgebras $\langle X, f\rangle$ and $\langle X, g\rangle$ have the same invariant co-relations.

5 Emulating coalgebras by idempotent coalgebras

We have seen in Section 4 that invariant co-relations need not characterise the coalgebra uniquely, and that they fail to do so in some rather irregular cases. We would, therefore, like to single out certain well-behaved representatives and show that every coalgebra can be somehow reduced to one of them. In our case, the well-behaved representatives are idempotent coalgebras.

In this section we first introduce the notion of emulation, and then show that for every monadic coalgebra there exists an idempotent coalgebra for the same monad which emulates the former one. Intuitively, to a coalgebra $\alpha : X \to T(X)$ we shall associate a coalgebra $\overline{\alpha} : Y \to T(Y)$ in such a way that $\overline{\alpha}$(some suitable representation of x) = some suitable representation of $\langle x, \alpha(x)\rangle$. For example, if α were a model of a Turing machine, say M, then $\overline{\alpha}$ would correspond to a Turing machine with two tapes which first copies the input data from tape 0 to tape 1, and then proceeds as machine M would have, operating on tape 1 only.

Coalgebras of the form $\langle Y, \overline{\alpha}\rangle$ can be thought of as models of computations that follow a simple discipline of not destroying the input data. It is intuitively acceptable that, at a fairly low cost, every computation can be emulated by such a computation. Coalgebras corresponding to such computations are obviously idempotent and, therefore, have the pleasant property of being uniquely determined by their invariant co-relations.

Formally, we shall show that the mapping $\langle X, \alpha\rangle \to \langle Y, \overline{\alpha}\rangle$ is functorial, and, more over, that this is an embedding of the category $\mathcal{S}et_{\mathcal{M}}$ into its full subcategory spanned by idempotent coalgebras.

Definition 5.1 Let X and Y be arbitrary sets.

We say that a coalgebra $\langle Y, \beta \rangle$ *emulates* a coalgebra $\langle X, \alpha \rangle$ if there exist two mappings, an *encoding* mapping $e : X \to Y$ and a *decoding* mapping $d : T(Y) \to T(X)$, such that $d \circ T(e) = \mathrm{id}_{T(X)}$ and $\alpha = d \circ \beta \circ e$. We also say that $\langle Y, \beta \rangle$ is an $\langle e, d \rangle$*-emulation* of $\langle X, \alpha \rangle$.

$$\begin{tikzcd} Y \arrow[r, "\beta"] & T(Y) \arrow[d, "d"] \\ X \arrow[u, "e"] \arrow[r, "\alpha"] & T(X). \end{tikzcd}$$

For a set X, let $X^{\otimes T} := X \times T(X)$, let $\partial_X : X \to X^{\otimes T} : x \mapsto \langle x, \eta_X(x) \rangle$ and define $p_X : T(X^{\otimes T}) \to T(X)$ by $p_X = \mu_X \circ T(\pi_2)$.

Finally, for a coalgebra $\alpha : X \to T(X)$ let $\overline{\alpha} : X^{\otimes T} \to T(X^{\otimes T})$ be the coalgebra defined by $\overline{\alpha} := \eta_{X^{\otimes T}} \circ \langle \pi_1, \alpha \circ \pi_1 \rangle$. Note that $\overline{\alpha}(\langle x, t \rangle) = \eta_{X^{\otimes T}}(\langle x, \alpha(x) \rangle)$, that is, the first component always contains the input data, while the second component contains the outcome of the computation.

Proposition 5.2 *For every coalgebra $\langle X, \alpha \rangle$, $\overline{\alpha}$ is a $\langle \partial_X, p_X \rangle$-emulaton of α. Moreover, $\overline{\alpha}$ is idempotent.*

Proof. Let us first show that $\langle \partial_X, p_X \rangle$ is indeed a pair of encoding-decoding mappings, i.e. that $p_X \circ T(\partial_X) = \mathrm{id}_{T(X)}$: $p_X \circ T(\partial_X) = \mu_X \circ T(\pi_2) \circ T(\partial_X) = \mu_X \circ T(\pi_2 \circ \partial_X) = \mu_X \circ T(\eta_X) = \mathrm{id}_{T(X)}$.

Next, let us show that $\overline{\alpha}$ is a $\langle \partial_X, p_X \rangle$-emulaton of α: $p_X \circ \overline{\alpha} \circ \partial_X(x) = p_X \circ \overline{\alpha}(\langle x, \eta_X(x) \rangle) = p_X \circ \eta_{X^{\otimes T}}(\langle x, \alpha(x) \rangle) = \mu_{X^{\otimes T}} \circ T(\pi_2) \circ \eta_{X^{\otimes T}}(\langle x, \alpha(x) \rangle) = \pi_2(\langle x, \alpha(x) \rangle) = \alpha(x)$.

Finally, let us show that $\overline{\alpha} \cdot \overline{\alpha} = \overline{\alpha}$: $\overline{\alpha} \cdot \overline{\alpha}(\langle x, t \rangle) = \mu_{X^{\otimes T}} \circ T(\overline{\alpha}) \circ \overline{\alpha}(\langle x, t \rangle) = \mu_{X^{\otimes T}} \circ T(\overline{\alpha}) \circ \eta_{X^{\otimes T}}(\langle x, \alpha(x) \rangle) = \overline{\alpha}(\langle x, \alpha(x) \rangle) = \eta_{X^{\otimes T}}(\langle x, \alpha(x) \rangle) = \overline{\alpha}(\langle x, t \rangle)$.□

Proposition 5.3 *Functor $G : \mathcal{S}et_{\mathcal{M}} \to \mathcal{S}et^{\mathrm{idp}}_{\mathcal{M}}$ given by $G(\langle X, \alpha \rangle) = \langle X^{\otimes T}, \overline{\alpha} \rangle$ and $G(f) = f \times T(f)$ is an embedding of $\mathcal{S}et_{\mathcal{M}}$ into $\mathcal{S}et^{\mathrm{idp}}_{\mathcal{M}}$ (i.e. G is one-to-one and faithful).*

Proof. Let us show that for every homomorphism $f : \langle X, \alpha \rangle \to \langle Y, \beta \rangle$, $G(f)$ is a homomorphism between $\langle X^{\otimes T}, \overline{\alpha} \rangle$ and $\langle Y^{\otimes T}, \overline{\beta} \rangle$, i.e. that $\overline{\beta} \circ G(f) = T(G(f)) \circ \overline{\alpha}$. Take any $\langle x, t \rangle \in X^{\otimes T}$. Then

$$\begin{aligned}
\overline{\beta} \circ (f \times T(f))(\langle x, t \rangle) &= \eta_{Y^{\otimes T}} \circ \langle \pi_1, \beta \circ \pi_1 \rangle (\langle f(x), T(f)(t) \rangle) \\
&= \eta_{Y^{\otimes T}}(\langle f(x), \beta \circ f(x) \rangle) \\
&= \eta_{Y^{\otimes T}}(\langle f(x), T(f) \circ \alpha(x) \rangle) \\
&\qquad \text{(since } f \text{ is a homomorphism)} \\
&= \eta_{Y^{\otimes T}} \circ (f \times T(f))(\langle x, \alpha(x) \rangle) \\
&= T(f \times T(f)) \circ \eta_{X^{\otimes T}}(\langle x, \alpha(x) \rangle) \\
&\qquad \text{(since } \eta \text{ is natural)} \\
&= T(f \times T(f)) \circ \eta_{X^{\otimes T}} \circ \langle \pi_1, \alpha \circ \pi_1 \rangle (\langle x, t \rangle) \\
&= T(f \times T(f)) \circ \overline{\alpha}(\langle x, t \rangle).
\end{aligned}$$

Now, it is easy to see that G is indeed a functor, that it is one-to-one and faithful. It is well-defined since coalgebras of the form $\overline{\alpha}$ are idempotent. This concludes the proof that G is an embedding of $\mathcal{S}et_{\mathcal{M}}$ into $\mathcal{S}et_{\mathcal{M}}^{\mathrm{idp}}$. $\square$

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