Available online at www.sciencedirect.com
SCIENCE DIRECT®
Electronic Notes in Theoretical Computer Science
Electronic Notes in Theoretical Computer Science 120 (2005) 31–43
www.elsevier.com/locate/entcs

A General Constructive Proof Technique

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Abstract

In the constructive theory of uniform spaces there occurs a technique of proof in which the application of a weak form of the law of excluded middle is circumvented by purely analytic means. The essence of this proof–technique is extracted and then applied to three important problems in the theory of apartness and uniformity.

Keywords: Uniform structure, constructive mathematics.

[1] The authors thank the Marsden Fund of the Royal Society of New Zealand for generously supporting their research from April 1997 to October 2002; the Royal Society of New Zealand for the New Zealand Science & Technology Postdoctoral Research Fellowship held by Luminiţa Vîţă since November 2002; and the DAAD for the supporting Douglas Bridges as a Gastprofessor at Ludwig–Maximilians–Universität, München, during the revision of the paper. Finally, they thank Jeremy Clarke for suggesting several corrections and improvements to an earlier version of the paper.

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doi:10.1016/j.entcs.2004.06.032

1 Introduction

The theory of apartness spaces, a counterpart of the classical theory of proximity spaces, appears promising as a foundation for constructive [4] topology. In several of the papers dealing with metric and uniform apartness spaces [5,11,15], we have used *ad hoc* variants of what appears to be a general proof–technique, which we now outline.

We have $\mathbb{N} = P \cup Q$, where $\mathbb{N}$ is the set of positive integers, and the definition of the sets P, Q depends on certain additional hypotheses. We want to prove that $n \in P$ eventually. We first use additional information, such as the strong continuity (see below) of some mapping between uniform spaces, to establish that either $n \in P$ holds eventually [5] or else $n \in Q$ infinitely often. In order to rule out the second alternative, we then show that it implies a weak form of the law of excluded middle, and that if this weak form of excluded middle is added to intuitionistic logic, then it is contradictory that $n \in Q$ infinitely often. It follows from all this that $n \in P$ eventually.

In this note we expand these ideas into a surprisingly powerful proof–technique [6] that we then apply to three problems in the constructive theory of uniform spaces. We require only minimal knowledge of that theory, as found in [13]; but to assist the reader, we shall give some of the basic definitions at the start of Section 3.

2 The proof–technique

Our proof–technique comprises two lemmas and a proposition. The first of these results is a peculiarly constructive one that takes the sting out of a number of succeeding proofs and is made necessary by the constructive failure of what Bishop called the **limited principle of omniscience** (**LPO**):

For each binary sequence $(\lambda_n)_{n \geqslant 1}$ either $\lambda_n = 0$ for all n, or else there exists n such that $\lambda_n = 1$.

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[4] By *constructive mathematics* we mean mathematics with intuitionistic logic; see [1,2,4,14].

[5] Let A be a set of positive integers. We say that $n \in A$ **eventually**, or **for all sufficiently large** n, if there exists N such that $n \in A$ for all $n \geqslant N$, and that $n \in A$ **infinitely often** if we can construct a strictly increasing sequence $(n_k)_{k \geqslant 1}$ of positive integers such that $n_k \in A$ for each k.

[6] An earlier paper of ours [7] dealt with a proof technique in uniform space theory. That technique is strong enough to produce Theorem 3.3 below; but, contrary to the claim made in [7] and subsequently corrected in [8], it is not strong enough for the other two applications in that paper. Moreover, it does not have Ishihara's proof technique (see Proposition 4.2) as a consequence.

In its recursive interpretation (with classical logic), LPO entails the decidability of the halting problem ([4], Chapter 3).

Lemma 2.1 *Let S be a nonempty set, and H a set of sequences s in S such that if $s \in H$, then each subsequence of s belongs to H. Let T be a subset of S with the following property: if*

$$s \in H,\ \mathbb{N} = P \cup Q,\ and\, s_n \in T\, for\, each\, n \in Q, \tag{1}$$

then either $n \in P$ for all n or else there exists $n \in Q$. If (1) obtains, then either $n \in P$ eventually or else $n \in Q$ infinitely often.

Proof. In view of our hypotheses, we may assume without loss of generality that there exists $n_1 \in Q$. Set $\lambda_0 = \lambda_1 = 0$. Using dependent choice, we construct inductively an increasing binary sequence $(\lambda_k)_{k\geqslant 1}$ and a strictly increasing sequence $(n_k)_{k\geqslant 1}$ of positive integers such that for each $k > 1$,

- if $\lambda_k = 0$, then $n_k \in Q$, and
- if $\lambda_k = 1 - \lambda_{k-1}$, then $n \in P$ for all $n > n_{k-1}$.

To this end, suppose we have found n_{k-1} with the applicable properties. Define

$$P' = \{j \geqslant 1 : n_{k-1} + j \in P\},$$
$$Q' = \{j \geqslant 1 : n_{k-1} + j \in Q\},$$

and

$$s' = \left(s_{n_{k-1}+j}\right)_{j\geqslant 1}.$$

Then $s' \in H$, by our first hypothesis, and $\mathbb{N} = P' \cup Q'$. Moreover, if $j \in Q'$, then $n_{k-1} + j \in Q$ and so $s'_j \in T$. Applying our hypotheses with P, Q, s replaced by P', Q', s' respectively, we see that either $n \in P$ for all $n > n_{k-1}$, in which case we set $n_{k-1+j} = n_{k-1} + j$ for each $j \geqslant 1$, and $\lambda_k = 1$; or else there exists $n_k > n_{k-1}$ with $n_k \in Q$, and we set $\lambda_k = 0$. This completes our inductive construction.

Now let

$$P'' = \{k \in \mathbb{N} : \lambda_k = 0 \vee \lambda_{k-1} = 1\},$$
$$Q'' = \{k \in \mathbb{N} : \lambda_k = 1 - \lambda_{k-1}\},$$

and

$$s'' = \left(s_{n_{k-1}}\right)_{k\geqslant 2}.$$

We see that $s'' \in H$ and that $\mathbb{N} = P'' \cup Q''$. Moreover, if $k \in Q''$, then $n_{k-1} \in Q$, and $s''_k = s_{n_{k-1}} \in T$. Applying our hypotheses with P, Q, s_k replaced by P'', Q'', s''_k respectively, we see that either $k \in P''$ for all k or else there exists $k \in Q''$. In the first case, if $\lambda_j = 1 - \lambda_{j-1}$ for some j, then $j \notin P''$, a contradiction; whence $\lambda_k = 0$ for all k, and therefore $(n_k)_{k\geqslant 1}$ is a strictly increasing sequence of elements of Q. In the case there exists $k \in Q''$, we have $n \in P$ for all $n > n_{k-1}$. □

Our next lemma may seem bizarre, since it shows that under certain hypotheses the nonconstructive proposition LPO holds. However, it enables us to use LPO to rule out the unwanted second alternative in the conclusion of Lemma 2.1.

Lemma 2.2 *Under the hypotheses of* Lemma 2.1, *if $n \in Q$ infinitely often, then* LPO *holds.*

Proof. Choose a strictly increasing sequence $(n_k)_{k\geqslant 1}$ in Q, and note that $s_{n_k} \in T$ for all k. Consider any increasing binary sequence $(\lambda_k)_{k\geqslant 1}$. Applying Lemma 2.1 with P, Q, s_k replaced by

$$P' = \{k \in \mathbb{N} : \lambda_k = 0\},$$
$$Q' = \{k \in \mathbb{N} : \lambda_k = 1\},$$

and s_{n_k} respectively, we find that either $k \in P'$ for all k or else there exists $k \in Q'$. □

Proposition 2.3 *Under the hypotheses of* Lemma 2.1, *let $s \in H$, let $\mathbb{N} = P \cup Q$, and suppose that $s_n \in T$ for each $n \in Q$. Suppose also that*

$$\text{LPO} \Rightarrow \neg\left(\forall n \exists k > n \ \ (k \in Q)\right).$$

Then $n \in P$ eventually.

Proof. By Lemma 2.1, either $n \in P$ eventually or else $n \in Q$ infinitely often. In the second case, Lemma 2.2 shows that LPO holds, which, in view of our final hypothesis, is absurd. □

In order to apply the foregoing, we need to set up suitable S, H, T and show that if $s \in H$, $\mathbb{N} = P \cup Q$, and $s_n \in T$ for each $n \in Q$, then either $n \in P$ for all n or else there exists $n \in Q$. It will then follow from Proposition 2.3 that $n \in P$ eventually.

3 Continuity, convergence, and uniformity

Let X be a nonempty set, and let U, V be subsets of the Cartesian product $X \times X$. We define certain associated subsets as follows:

$$U \circ V = \{(x, y) : \exists z \in X \ \ ((x, z) \in U \wedge (z, y) \in V)\},$$
$$U^1 = U, \quad U^{n+1} = U \circ U^n \quad (n = 1, 2, \ldots),$$
$$U^{-1} = \{(x, y) : (y, x) \in U\}.$$

We say that U is **symmetric** if $U = U^{-1}$. The **diagonal** of $X \times X$ is the set

$$\Delta = \{(x, x) : x \in X\}.$$

A family $\mathcal{U}$ of subsets of $X \times X$ is called a **uniform structure**, or **uniformity**, on X if the following conditions hold.

U1 ***I*** Every finite intersection of sets in $\mathcal{U}$ belongs to $\mathcal{U}$.
II Every subset of $X \times X$ that contains a member of $\mathcal{U}$ is in $\mathcal{U}$.

U2 Every member of $\mathcal{U}$ contains both the diagonal Δ and a symmetric member of $\mathcal{U}$.

U3 For each $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $V^2 \subset U$.

U4 For each $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that

$$\forall \mathbf{x} \in X \times X \ (\mathbf{x} \in U \vee \mathbf{x} \notin V).$$

The pair $(X,\mathcal{U})$ —or, loosely, X alone—is called a **uniform space**, and the elements of $\mathcal{U}$ the **entourages** of X. For clarity, we sometimes denote the uniform structure on X by $\mathcal{U}_X$. The motivating example of a uniform space is a metric space (X, ρ), in which the unique uniform structure has a basis of sets of the form

$$\{(x, y) \in X \times X : \rho(x, y) \leqslant \varepsilon\} \tag{2}$$

with $\varepsilon > 0$.

Condition U1, and the fact that, by U2, each element of $\mathcal{U}$ is nonempty, show that $\mathcal{U}$ is a filter on $X \times X$. Classically axiom U4 is superfluous (we simply take $V = U$); but constructively it is essential for the development of the theory. A good reference for the classical theory of uniform spaces is [3].

The (uniform) topology on a uniform space $(X,\mathcal{U})$ is the one in which a base of neighbourhoods of a point $x \in X$ consists of the sets

$$U[x] = \{y \in X : (x, y) \in U\} \quad (U \in \mathcal{U}).$$

A topology τ on a set X is said to be **given by the uniform structure** $\mathcal{U}$ on X if it coincides with the uniform topology arising from $\mathcal{U}$.

We define a canonical inequality on a uniform space $(X,\mathcal{U})$ by

$$x \neq y \Leftrightarrow \exists U \in \mathcal{U} \ ((x, y) \notin U).$$

Note that, by axioms $\mathrm{U1}_{II}$ and U2, if $U \in \mathcal{U}$, then $U^{-1} \in \mathcal{U}$. It follows that if $x \neq y$, then $y \neq x$. Moreover, since $\mathcal{U}$ contains Δ, if $x \neq y$, then $\neg(x = y)$. Thus $\neq$ has the two properties that define an inequality relation in constructive mathematics. In turn, the inequality on X induces an associated inequality on $X \times X$ in the usual way. Note that an apparently stronger form of axiom U4 holds: for each $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that

$$\forall \mathbf{x} \in X \times X \ \left(\mathbf{x} \in U \vee \mathbf{x} \in \complement V\right),$$

where

$$\complement V = \{\mathbf{x} \in X \times X : \forall \mathbf{v} \in V \, (\mathbf{x} \neq \mathbf{v})\}$$

is the **complement** of V in $X \times X$.

For each positive integer n we define an n–**chain of entourages** of X to be an n–tuple $(U_1, \ldots, U_n)$ of entourages such that for each applicable k, U_k is symmetric, $U_k^2 \subset U_{k-1}$, and

$$\forall \mathbf{x} \in X \times X \left(\mathbf{x} \in U_{k-1} \vee \mathbf{x} \in \complement U_k\right).$$

Axiom U3 ensures that for each $U \in \mathcal{U}$ and each positive integer n there exists an n–chain $(U_1, \ldots, U_n)$ of entourages with $U_1 = U$.

We state two technical lemmas about uniform spaces for later use; these appear, with proofs, as Lemmas 7 and 8 of [7].

Lemma 3.1 *Let Y be a uniform space, let $(a_n)_{n\geqslant 1}, (b_n)_{n\geqslant 1}$ be sequences in Y, and let (V_1, V_2, V_3) be a 3–chain of entourages of Y such that $(a_n, b_n) \in \complement V_1$ for each n. Then it is impossible that for each n, $(a_n, b_k) \in V_3$ for all sufficiently large k.*

Lemma 3.2 *Assuming* LPO, *let $(a_n)_{n\geqslant 1}$ and $(b_n)_{n\geqslant 1}$ be sequences in a uniform space Y, and let $(V_1, \ldots, V_4)$ be a 4–chain of entourages of Y, such that $(a_n, b_n) \in \complement V_1$ for each n. Then there exists a strictly increasing sequence $(n_k)_{k\geqslant 1}$ of positive integers such that $(a_{n_j}, b_{n_k}) \in \complement V_4$ for all j and k.*

We say that two subsets A, B of a uniform space $(X,\mathcal{U})$ are **apart**, and we write $A \bowtie B$, if there exists an entourage U such that

$$A \times B \subset \complement U = \{\mathbf{x} \in X \times X : \forall \mathbf{y} \in U \, (\mathbf{x} \neq \mathbf{y})\}.$$

Two sequences $(x_n)_{n\geqslant 1}$ and $(x'_n)_{n\geqslant 1}$ in X are **eventually close** if for each entourage U of X we have $(x_n, x'_n) \in U$ eventually.

A mapping f from X into a uniform space Y is

- **strongly continuous** if for all subsets A, B of X,

$$f(A) \bowtie f(B) \Rightarrow A \bowtie B;$$

- **uniformly sequentially continuous** if for all sequences $(x_n)_{n\geqslant 1}$ and $(x'_n)_{n\geqslant 1}$ that are eventually close in X, the sequences $(f(x_n))_{n\geqslant 1}$ and $(f(x'_n))_{n\geqslant 1}$ are eventually close in Y.

These two types of continuity are weak forms of uniform continuity. We aim to prove the following result, whose metric space version is in [5].

Theorem 3.3 *A strongly continuous mapping between uniform spaces is uniformly sequentially continuous.*

Our proof will require yet more lemmas.

Lemma 3.4 *Let X, Y be uniform spaces, $f : X \to Y$ a strongly continuous function, and V an entourage of Y. Let $(\lambda_n)_{n\geqslant 1}$ be an increasing binary sequence, and $(A_n)_{n\geqslant 1}, (B_n)_{n\geqslant 1}$ sequences of subsets of X such that*

▷ *for each entourage U of X there exists N such that for each $n \geqslant N$, either $A_n \times B_n = \emptyset$ or else $A_n \times B_n$ intersects U;*

▷ *if $\lambda_n = 0$, then $A_n = \emptyset$; and*

▷ *if $\lambda_n = 1 - \lambda_{n-1}$, then $A_n \neq \emptyset, B_n \neq \emptyset$, $f(A_n) \times f(B_n) \subset \complement V$, and $A_k = \emptyset$ for all $k > n$.*

Then there exists N such that $\lambda_n = \lambda_N$ for all $n \geqslant N$.

Proof. Note that if $\lambda_1 = 1$, then there is nothing to prove. Writing

$$A = \bigcup_{n\geqslant 1} A_n,$$
$$B = \bigcup_{n\geqslant 1} \{B_n : \lambda_n = 1 - \lambda_{n-1}\},$$

we see that $f(A) \times f(B) \subset \complement V$: for if $x \in A$, then there exists n such that $A = A_n, B = B_n$, and $f(A_n) \times f(B_n) \subset \complement V$. Hence $f(A) \bowtie f(B)$ and therefore, by the strong continuity of f, there exists an entourage U of X such that $A \times B \subset \complement U$. Choose N such that for each $n \geqslant N$, either $A_n \times B_n = \emptyset$ or else $A_n \times B_n$ intersects U. Either $\lambda_N = 1$ and therefore $\lambda_n = 1$ for all $n \geqslant N$, or else $\lambda_N = 0$. In the latter case, if $\lambda_m = 1 - \lambda_{m-1}$ for some $m > N$, then $A = A_m \neq \emptyset, B = B_m \neq \emptyset$, and $A \times B$ intersects U. This contradicts our choice of U; whence $\lambda_n = 0$ for all $n \geqslant N$. □

Lemma 3.5 *Let X, Y be uniform spaces, $f : X \to Y$ a strongly continuous function, and V an entourage of Y. Let S be the space $X \times X$,*

$$H = \left\{ s \in S^{\mathbb{N}} : \forall U \in \mathcal{U}_X \exists N \forall n \geqslant N \, (s_n \in U) \right\},$$

and

$$T=\left\{(x,x')\in S:(f(x),f(x'))\in \complement V\right\}.$$

If $s\in H$, $\mathbb{N}=P\cup Q$, and $s_n\in T$ for each $n\in Q$, then either $n\in P$ for all n or else there exists $n\in Q$.

Proof. For each n write $s_n=(x_n,x'_n)$. Construct an increasing binary sequence $(\lambda_n)_{n\geqslant 1}$ such that

- if $\lambda_n=0$, then $k\in P$ for all $k\leqslant n$;
- if $\lambda_n=1-\lambda_{n-1}$, then $n\in Q$.

We may assume that $\lambda_1=0$. If $\lambda_n=0$, set $A_n=B_n=\emptyset$. If $\lambda_n=1-\lambda_{n-1}$, set $A_n=\{x_n\}$ and $B_n=\{x'_n\}$, and note that, as $n\in Q$, we have $(x_n,x'_n)\in T$ and therefore $f(A_n)\times f(B_n)\subset \complement V$; also set $A_k=\emptyset=B_k$ for each $k\geqslant n$. Now consider any entourage U of X. Since $s\in H$, there exists ν such that $(x_n,x'_n)\in U$ for all $n\geqslant\nu$. For each such n, if $\lambda_n=0$ or $\lambda_{n-1}=1$, then $A_n\times B_n=\emptyset$. On the other hand, if $\lambda_n=1$, then $A_n\times B_n=\{(x_n,x'_n)\}\subset U$. Thus the hypotheses of Lemma 3.4 are satisfied. Applying that lemma, we produce N such that $\lambda_n=\lambda_N$ for all $n\geqslant N$. If $\lambda_N=0$, then $n\in P$ for all n; whereas if $\lambda_N=1$, there exists $k\leqslant n$ such that $k\in Q$. □

To prove Theorem 3.3, let $S=X\times X$, and define H as in the preceding lemma. Given two sequences $(x_n)_{n\geqslant 1},(x'_n)_{n\geqslant 1}$ in X that are eventually close, let $s=((x_n,x'_n))_{n\geqslant 1}$. Let V be any entourage of Y, and construct a 5–chain $(V_1,\dots,V_5)$ of entourages of Y with $V_1=V$. Define

$$P=\{n:(f(x_n),f(x'_n))\in V\},$$
$$Q=\{n:(f(x_n),f(x'_n))\in\complement V_2\},$$

and

$$T=\left\{(x,x')\in S:(f(x),f(x'))\in\complement V_2\right\}.$$

Then $\mathbb{N}=P\cup Q$, $s\in H$, and $s_n\in T$ for each $n\in Q$. It follows from Lemmas 3.5 and 2.1 that either $n\in P$ eventually or else $n\in Q$ infinitely often. Supposing that $n\in Q$ infinitely often, we see from Lemma 2.2 that LPO holds. Thus, by Lemma 3.2, there exists a strictly increasing sequence $(n_k)_{k\geqslant 1}$ of positive integers such that $\left(f(x_{n_j}),f(x'_{n_k})\right)\in\complement V_5$ for all j,k. Writing

$$A=\left\{x_{n_j}:j\geqslant 1\right\},\quad B=\left\{x'_{n_k}:k\geqslant 1\right\},$$

we see that $f(A)\times f(B)\subset\complement V_5$, so $f(A)\bowtie f(B)$ in Y. Since f is strongly continuous, $A\bowtie B$ in X, and therefore there exists an entourage U of X such that $A\times B\subset\complement U$. But this is absurd, since $\left(x_{n_k},x'_{n_k}\right)\in U$ eventually. Referring to Proposition 2.3, we conclude that $n \in P$ —that is, $(f(x_n), f(x'_n)) \in V$ — eventually. This completes the proof.

It is shown in [11] that Theorem 3.3 is the best we can produce constructively without introducing a principle that, while valid in the standard models (intuitionistic, recursive, classical) of constructive mathematics, appears not to be derivable using only intuitionistic logic and dependent choice. Under certain additional hypotheses, Theorem 3.3 can be strengthened to produce the uniform continuity of the strongly continuous function f; see [5].

The uniform continuity theorem for continuous mappings from compact metric spaces into metric spaces follows easily from Theorem 3.3 with classical logic. In the same way, a standard classical result about uniform convergence can be obtained from our next application of Proposition 2.3. This needs more definitions.

Let X be a nonempty set, Y a uniform space, $(\phi_n)_{n\geqslant 1}$ a sequence in Y^X, and $\phi \in Y^X$. We say that (ϕ_n) is **proximally convergent** to ϕ if

$$\phi(A) \bowtie B \Rightarrow \exists N \forall n \geqslant N \left(\phi_n(A) \bowtie B\right);$$

and that (ϕ_n) is **uniformly sequentially convergent** to ϕ if for each sequence $(x_n)_{n\geqslant 1}$ in X, the sequences $(\phi(x_n))_{n\geqslant 1}$ and $(\phi_n(x_n))_{n\geqslant 1}$ are eventually close. We want to prove the following theorem which first appeared in [15].

Theorem 3.6 *Let X be a nonempty set, Y a uniform space, and $(\phi_n)_{n\geqslant 1}$ a sequence in Y^X that converges proximally to $\phi \in Y^X$. Then (ϕ_n) is uniformly sequentially convergent to ϕ.*

To do this, we need counterparts of Lemmas 3.4 and 3.5.

Lemma 3.7 *Let X be a nonempty set, and Y a uniform space. Let $(\phi_n)_{n\geqslant 1}$ be a sequence in Y^X that converges proximally to $\phi \in Y^X$. Let $(x_n)_{n\geqslant 1}$ be a sequence in X, and V an entourage of Y. Let $(\lambda_n)_{n\geqslant 1}$ be an increasing binary sequence, and $(A_n)_{n\geqslant 1}, (B_n)_{n\geqslant 1}$ sequences of subsets of X, Y respectively, such that*

- ▷ *if $\lambda_n = 0$, then $A_n = B_n = \emptyset$, and*
- ▷ *if $\lambda_n = 1 - \lambda_{n-1}$, then $A_n = \{x_n\}, B_n = \{\phi_n(x_n)\}, \phi(A_n) \times B_n \subset \complement V$, and $A_k = B_k = \emptyset$ for all $k > n$.*

Then there exists N such that $\lambda_n = \lambda_N$ for all $n \geqslant N$.

Proof. Write

$$A = \bigcup_{n\geqslant 1} A_n,$$
$$B = \bigcup_{n\geqslant 1} \{B_n : \lambda_n = 1 - \lambda_{n-1}\}.$$

If $x \in A$, then there exists n such that $A = A_n = \{x_n\}$, $B = B_n = \{\phi_n(x_n)\}$, and $\phi(A_n) \times B_n \subset \complement V$. Hence $\phi(A) \times B \subset \complement V$ and therefore $\phi(A) \bowtie B$. By proximal convergence, there exists N such that $\phi_n(A) \bowtie B$ for all $n \geqslant N$. Suppose that $\lambda_m = 1 - \lambda_{m-1}$ for some $m > N$. Then $A = \{x_m\}, B = \{\phi_m(x_m)\}$, and $\phi_m(x_m) \bowtie \phi_m(x_m)$, which is absurd. Hence $\lambda_n = \lambda_N$ for all $n \geqslant N$. □

Lemma 3.8 *Let X be a set, and Y a uniform space. Let $(\phi_n)_{n\geqslant 1}$ be a sequence in Y^X that converges proximally to $\phi \in Y^X$. Let $(x_n)_{n\geqslant 1}$ be a sequence in X, and V an entourage of Y. Let P, Q be sets of positive integers such that $\mathbb{N} = P \cup Q$, and suppose that $(\phi(x_n), \phi_n(x_n)) \in \complement V$ for all $n \in Q$. Then either $n \in P$ for all n or else there exists $n \in Q$.*

Proof. Construct an increasing binary sequence $(\lambda_n)_{n\geqslant 1}$ such that

- if $\lambda_n = 0$, then $k \in P$ for all $k \leqslant n$;
- if $\lambda_n = 1 - \lambda_{n-1}$, then $n \in Q$.

We may assume that $\lambda_1 = 0$. If $\lambda_n = 0$, set $A_n = B_n = \emptyset$. If $\lambda_n = 1 - \lambda_{n-1}$, set $A_n = \{x_n\}$ and $B_n = \{\phi_n(x_n)\}$, and note that, as $n \in Q$, we have $\phi(A_n) \times B_n \subset \complement V$; also set $A_k = \emptyset = B_k$ for each $k \geqslant n$. By Lemma 3.7, there exists N such that $\lambda_n = \lambda_N$ for all $n \geqslant N$. If $\lambda_N = 0$, then $n \in P$ for all n; if $\lambda_N = 1$, then $n \in Q$ for some $n \leqslant N$. □

To prove Theorem 3.6, given a set X and a uniform space Y, let $S = X \times Y^X$ and $H = S^{\mathbb{N}}$. Let $(\phi_n)_{n\geqslant 1}$ be a sequence in Y^X that converges proximally to $\phi \in Y^X$, let U be an entourage of Y, and construct a 5–chain (U, V_1, V_2, V_3, V_4) of entourages of Y. Define

$$T = \left\{(x, f) \in S : (\phi(x), f(x)) \in \complement V_1\right\}.$$

Let $(x_n)_{n\geqslant 1}$ be a sequence in X, and let $s = ((x_n, \phi_n))_{n\geqslant 1} \in H$. Define

$$P = \{n : (\phi(x_n), \phi_n(x_n)) \in U\},$$
$$Q = \left\{n : (\phi(x_n), \phi_n(x_n)) \in \complement V_1\right\}.$$

Applying Lemmas 3.8 and 2.1, we see that either $n \in P$ eventually or else $n \in Q$ infinitely often. Suppose that there exists a strictly increasing sequence $(n_k)_{k\geqslant 1}$ of positive integers in Q. Then LPO holds, by Lemma 2.2, so we can apply Lemma 3.2 with

$$a_k = \phi\left(x_{n_k}\right), \ b_k = \phi_{n_k}\left(x_{n_k}\right), \ \text{and } r = \frac{\varepsilon}{2},$$

to construct a strictly increasing sequence $(k_i)_{i=1}^{\infty}$ of positive integers such that

$$\left(\phi\left(x_{n_{k_j}}\right), \phi_{n_{k_i}}\left(x_{n_{k_i}}\right)\right) \in \complement V_4 \quad (i, j \geqslant 1)\,. \tag{3}$$

Let

$$A = \left\{x_{n_{k_i}} : i \geqslant 1\right\},$$
$$B = \left\{\phi_{n_{k_j}}\left(x_{n_{k_j}}\right) : j \geqslant 1\right\}.$$

We see from (3) that $\phi(A) \bowtie B$. Hence there exists N such that $\phi_n(A) \bowtie B$ for all $n \geqslant N$. In particular, for all $i \geqslant N$ we obtain the contradiction

$$\phi_{n_{k_i}}\left(x_{n_{k_i}}\right) \bowtie \phi_{n_{k_i}}\left(x_{n_{k_i}}\right).$$

Thus

$$\neg\left(\forall n \exists k > n\,(k \in Q)\right),$$

and so, by Proposition 2.3, $n \in P$ eventually. That is, $(\phi(x_n), \phi_n(x_n)) \in U$ for all sufficiently large n. This completes the proof.

4 Ishihara's tricks

In the seminal paper [12], Hajime Ishihara introduced two lemmas, subsequently dubbed *Ishihara's tricks*, that have had many significant applications in constructive analysis. These lemmas were examined in a more general setting in [9]. We end this paper by showing that they are also related to our foregoing proof–technique. We first prove a slight generalisation of Ishihara's first trick ([12], Lemma 1) that sets up our application of Proposition 2.1.

Lemma 4.1 *Let X be a complete metric space, $x \in X$, and f a strongly extensional mapping of X into a metric space Y. Let H be the set of all sequences in X that converge to x, let $\alpha > 0$, and define*

$$T = \{y \in X : \rho(f(x), f(y)) > \alpha\}.$$

If $(x_n)_{n \geqslant 1} \in H$, $\mathbb{N} = P \cup Q$, and $x_n \in T$ for all $n \in Q$, then either $n \in P$ for all n, or else there exists $n \in Q$.

Proof. Suppose that $(x_n)_{n \geqslant 1} \in H$, $\mathbf{N}^+ = P \cup Q$, and $x_n \in T$ for all $n \in Q$. Construct an increasing binary sequence $(\lambda_n)_{n \geqslant 1}$ such that

- if $\lambda_n = 0$, then $k \in P$ for all $k \leqslant n$;
- if $\lambda_n = 1 - \lambda_{n-1}$, then $n \in Q$.

We may assume that $\lambda_1 = 0$. If $\lambda_n = 0$, set $y_n = x$. If $\lambda_n = 1-\lambda_{n-1}$, set $y_k = x_n$ for all $k \geqslant n$. Then $(y_n)_{n\geqslant 1}$ is a Cauchy sequence in X and so converges to a limit $y \in X$. Either $\rho(f(x), f(y)) > 0$ or $\rho(f(x), f(y)) < \alpha$. In the first case, since f is strongly extensional, we have $x \neq y$. So there exists N such that $x \neq y_N$, from which it follows that $\lambda_N = 1$ and hence $n \in Q$ for some $n \leqslant N$. In the case $\rho(f(x), f(y)) < \alpha$ we must have $\lambda_n = 0$, and therefore $n \in P$, for all n. □

We now easily deduce Ishihara's second trick ([12], Lemma 2).

Proposition 4.2 *Let X be a complete metric space, f a strongly extensional mapping of X into a metric space Y, and $(x_n)_{n\geqslant 1}$ a sequence converging to a limit $x \in X$. Then for all positive α, β with $\alpha < \beta$, either $\rho(f(x_n), f(x)) < \beta$ eventually or else $\rho(f(x_n), f(x)) > \alpha$ infinitely often.*

Proof. Take $S = X$, and let H and T be defined as in the statement of Lemma 4.1. That lemma shows that the hypotheses of Lemma 2.1 hold. Applying that lemma with

$$P = \{n : \rho(f(x_n), f(x)) < \beta\},$$
$$Q = \{n : \rho(f(x_n), f(x)) > \alpha\},$$

we immediately obtain the desired conclusion. □

Thus our proof–technique can be viewed as a strong extension of Ishihara's ideas in [12], lifting them from the restrictive context of a complete metric space to a general setting that permits applications such as the ones we have given in the theory of uniform spaces.

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