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On the regular precession of an asymmetric rigid body acted upon by uniform gravity and magnetic fields

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ABSTRACT

In 1947 Grioli discovered that an asymmetric heavy rigid body moving about a fixed point can perform a regular precession, which is the rotation of the body about an axis fixed in it, while that axis precesses with the same uniform angular velocity about a non-vertical axis fixed in space.

In the present note, we show that a magnetized asymmetric rigid body moving about a fixed point while acted upon by uniform gravity and magnetic fields can perform a regular precession about a horizontal axis fixed in space orthogonal to the magnetic field. This motion does not contain Grioli's as a special case since the gravity and magnetic effects are coupled and can vanish only simultaneously.

1. Introduction

1.1. Historical

The subject of rigid body dynamics is two and half centuries old. It dates back to Euler, who introduced the basic notions and studied the motion of the torque-free body [1] (1758). Lagrange studied the case of an axi-symmetric body (top) in the uniform gravity field [2] (1788). It was found later that both cases of Euler and Lagrange have their general solutions as elliptic functions of the time variable t. The equations of motion are usually written in the form known as the Euler–Poisson equations and they admit three general integrals of motion: the total energy, the integral of areas and the geometric integral. The integrability of those equations requires the knowledge of a complementary (fourth) integral of motion, independent of those three (see e.g. [4]).

A whole century elapsed after Lagrange's work before Kowalevski found a third integrable case, now known after her name [3] (1889). She isolated this case by an interesting property: only in those three cases the general solution of the equations of motion of the heavy rigid body about a fixed point can be expressed for all initial conditions in terms of functions that have no singularities other than poles in the complex plane of the time variable t. She also found the complementary integral, which turned out to be, for the first time in

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dynamics, a polynomial of degree 4 in the angular velocity components and constructed the explicit solution in ultra-elliptic functions of time.

Goryachev and Chaplygin constructed the fourth integral for one more case. This case is conditionally integrable, i.e. integrable only on the zero level of the areas integral. This dynamical condition means that the motion is integrable only when the angular momentum of the body lies permanently in a horizontal plane. The conditional complementary integral for this case is a cubic polynomial in the angular velocities and explicit solution is expressed also in ultra-elliptic functions of time (e.g. [4]).

Kowalevski's results created a great interest all along the next century in exploring the deep relation between the branching of the solution of equations of motion in the complex $t-$ plane and algebraic integrability, i.e. existence of the fourth integral as a polynomial or algebraic function of the phase space. This research began with the works of Liouville, Husson and Burgatti [6–8] and was culminated with results of Kozlov and Ziglin [9–11]. In [10], it is shown that a meromorphic general integral of the equations of motion exists only in the cases of Euler, Lagrange and Kowalevski; and a conditional one only in the case of Goryachev and Chaplygin. For a review of those and related results, see e.g. [5].

After Kowalevski, the interest in the problem has shifted to the search for particular solutions of the equations of motion. Those are solutions under any conditions on the initial state of motion as well as on the distribution of mass in the body. The search for particular solutions produced 11 solutions, which, with the well-known motion of the body as a composite pendulum complete the list of 12 cases shown in the following Table 1:

For a detailed account of those cases see [25] or [26]. The last of them was found in 1970. Some of those solutions were generalized later through the addition of a gyrostatic moment and new solutions for a gyrostat were found by several authors: N.E.Zhoukovski, H.M.Yehia, L.N.Sretensky, D.N.Goryachev, A.I.Dokshevich, L.M.Kovaleva, G.V.Mozalevskaya, P.V.Kharlamov, E.I.Kharlamova. For the details see [25] and references therein.

Hess' case had a wide generalization including a gyrostatic moment and other potential and gyroscopic forces [27]. Grioli's case was also generalized to include an additional parameter, which transforms it into a solvable case of the dynamics of a rigid body by inertia in a fluid [28].

A direct, simple but very important generalization of the problem described above is that of motion of a rigid body under the action of a combination of two uniform fields. This problem is characterized by two vectors, constant in space which represent gravity and magnetic fields and two vectors, constant in the body, describing the centre of mass and the magnetic moment. The potential of this problem is a linear function in all the direction cosines of the two fields with respect to the body frame.

In spite of its practical importance, the problem of motion of rigid body under the action of more than one uniform fields has escaped attention for a long time. Despite the richness in its structure, integrable cases of this problem are still rare. Till now, none of the above results concerning properties of solutions in the complex plane of time or the existence of algebraic integrals could be generalized to cover this problem. The research in this problem was not carried out on a systematic basis, and only scattered results exist.

Although the integrals of motion were found so early as in 1893 in a much more complicated problem of motion of a rigid body influenced by the approximate Newtonian field of three attraction centres non-coplanar with the fixed point [29], the problem of motion of a rigid body influenced by constant gravity and magnetic fields was considered almost a century later, namely in 1984, by Bogoyavlensky [32]. He established that for Kowalevski's configuration $A = B = 2C$, this problem is Liouville integrable on a submanifold characterized by two invariant relations of the second degree. In our notation, this is equivalent to construction of a particular solution of the equations of motion. Shortly later, in our work [33] of 1986, we constructed a fourth-degree integral, which generalizes the famous integral of Kowalevski for the classical problem of one field by adding the second field and, simultaneously, attaching a gyrostatic moment. One more integral was still lacking to establish integrability in the new problem, since this problem admits no cyclic integral in general. In the same work [33], we isolated another integrable version of the problem with a cyclic integral corresponding to the sum (or difference) of the two angles of precession and proper rotation. This version does not stem out of Kowalevski's case, in the sense that it does not include that case of one field as a particular case,

Table 1 – Known particular solvable cases of the classical problem (in chronological order).

Case	1	2	3	4
Au.	Pendulum motion	Hess	Staude	Bobylev-Steklov
Year	–	1890	1894	1896
Ref.	–	[12]	[13]	[14,15]
Case	5	6	7	8
Au.	Goryachev	Steklov	Chaplygin	Kowalewski
Year	1899	1899	1904	1908
Ref.	[18]	[16]	[17]	[19]
Case	9	10	11	12
Au.	Grioli	Dokshevich	Konosevich-Pozdnyakovich	Dokshevich
Year	1947	1965	1968	1970
Ref.	[20]	[21]	[22]	[23]

since the intensity of the two fields are equal and can vanish only simultaneously. The next step in this problem was taken in [34] (1987), where the equations of motion were presented in the form of a Lax pair and the lacking general complementary integral was found, for arbitrary fields intensities, as a quadratic polynomial in the angular velocities. Together with our quartic integral, the last integral completes the requirements for integrability of the problem of motion of a body with Kowalevski's configuration in two uniform fields. Despite the fact that this integrable system has three degrees of freedom and is not generally reduced to quadratures, all its subsystems with two degrees of freedom were found and for two of them separation of variables was obtained (elliptic and hyperelliptic). For details see [35–38].

Two more general integrable cases were found in 1986. In both cases the body is of spherical dynamical symmetry (with three equal moments of inertia) and the potential is a linear function in the direction cosines. The first case is characterized by the presence of three quadratic integrals [31] and the second by three linear integrals [33]. Existence of other integrable cases is a matter of speculation and is in fact an open question. With no integrable cases in view, the search for particular solutions acquires great importance. However, not much is done in this respect. In [39] the equilibrium positions were classified and the stability of some positions was investigated. Plane motions of the body as a physical pendulum in two fields were also partially investigated in [40].

In the present note we consider the possibility of regular precessional motion of the rigid body about a fixed point in the presence of two fields. That is the rotation of the body with a constant angular velocity about an axis fixed in it, while this axis precesses with the same angular velocity about another axis fixed in space, keeping with it a fixed angle $\frac{\pi}{2}$. Motion of this type are usual for an axi-symmetric heavy body in a single uniform gravity field (Lagrange's top), with the axis of precession occupying the vertical position. But, as was firstly described by Grioli in 1947 [20], such motion is still possible for an asymmetric (triaxial) body in a single uniform gravity field. The axis of precession is inclined to the vertical in this case at an angle that depends on the moments of inertia [24].

The question whether the regular precessional motion is possible in the presence of two fields was not considered. We show here, on the example of uniform gravity and magnetic fields, that this is really the case. The conditions on the fields and on the parameters of the body are determined and explicit solution of the equations of motion is given.

1.2. Equations of motion

Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3), \boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3), \boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$ be the unit vectors along the axes of the inertial system $OXYZ$ and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ be the unit vectors along the axes of the system $Oxyz$, fixed in the body and $\boldsymbol{\omega} = (p, q, r)$ the angular velocity of the body, all being referred to the body system. The relative position of the two systems will be specified by the Eulerian angles: ψ-the angle of precession around the Z-axis, θ -the angle of nutation between z and Z, and φ the angle of rotation of the body around the z-axis.

The variables can be expressed in terms of Euler's angles: ψ, θ, φ. They have the form (e.g. the review book of Leimanis [4])

$$\begin{aligned}
\boldsymbol{\alpha} &= (\cos\psi\cos\varphi - \cos\theta\sin\psi\sin\varphi, -\cos\psi\sin\varphi - \cos\theta\sin\psi\cos\varphi, \sin\theta\sin\psi) \\
\boldsymbol{\beta} &= (\sin\psi\cos\varphi + \cos\theta\cos\psi\sin\varphi, -\sin\psi\sin\varphi + \cos\theta\cos\psi\cos\varphi, -\sin\theta\cos\psi) \\
\boldsymbol{\gamma} &= (\sin\theta\sin\varphi, \sin\theta\cos\varphi, \cos\theta)
\end{aligned} \tag{1}$$

and the angular velocity of the body

$$\boldsymbol{\omega} = (\dot\psi\sin\theta\sin\varphi + \dot\theta\cos\varphi, \dot\psi\sin\theta\cos\varphi - \dot\theta\sin\varphi, \dot\psi\cos\theta + \dot\varphi) \tag{2}$$

where dots denote derivative with respect to time.

Consider a heavy magnetized body of mass m and magnetic moment $\boldsymbol{\mu}$, in motion about the fixed point O, while acted upon by uniform gravity and magnetic fields $\boldsymbol{g} = g\boldsymbol{\alpha}$ and $H = h\boldsymbol{\beta}$, respectively. To suppress the number of parameters in the potential we normalize the fields so that $mg = 1, h = 1$. The potential of the problem can be written in the form

$$V = \mathbf{r}_0 \cdot \boldsymbol{\alpha} + \boldsymbol{\mu} \cdot \boldsymbol{\beta} \tag{3}$$

where $\mathbf{r}_0$ is the position vector of the centre of mass. In order that the gravity and magnetic effects cannot be reduced to one effect, we assume that the vectors $\mathbf{r}_0$ and $\boldsymbol{\mu}$ are not parallel and $|\mathbf{r}_0||\boldsymbol{\mu}| \neq 0$.

Let $\mathbf{I}$ be the inertia matrix of the body at the fixed point with respect to the system of axes Oxyz fixed in the body. The system of principal axes of inertia of the body is not the most suitable for describing the regular precessional motion about a non-vertical axis, so that we assume $\mathbf{I}$ in the form:

$$I = \begin{pmatrix} A & -F & -E \\ -F & B & -D \\ -E & -D & C \end{pmatrix} \tag{4}$$

The dynamical problem can be written in the form of Euler–Poisson equations

$$\begin{aligned}
&\dot{\boldsymbol{\omega}}\mathbf{I} + \boldsymbol{\omega} \times \boldsymbol{\omega}\mathbf{I} = \boldsymbol{\alpha} \times \mathbf{r}_0 + \boldsymbol{\beta} \times \boldsymbol{\mu} \\
&\dot{\boldsymbol{\alpha}} + \boldsymbol{\omega} \times \boldsymbol{\alpha} = \mathbf{0}, \dot{\boldsymbol{\beta}} + \boldsymbol{\omega} \times \boldsymbol{\beta} = \mathbf{0}
\end{aligned} \tag{5}$$

We shall proceed to construct a simple solution for those equations.

2. The solution

The regular precession is most simply described as the proper rotation of the body with a uniform angular velocity $\dot\varphi = \Omega$ about its $z-$ axis, which simultaneously precesses with the same angular velocity $\dot\psi = \Omega$ about the space axis Z keeping with it a fixed angle $\theta = \frac{\pi}{2}$. Substituting the values $\theta = \frac{\pi}{2}, \dot\theta = 0, \psi = \varphi = \Omega(t - t_0), \dot\psi = \dot\varphi = \Omega$, in (2) and (1), we obtain for the angular velocity and the three unit vectors $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ the following expressions:

$$\boldsymbol{\omega} = (\Omega\sin u, \Omega\cos u, \Omega) \tag{6}$$

$$\begin{aligned}\boldsymbol{\alpha} &= (\cos^2 u, -\sin u\cos u, \sin u),\\ \boldsymbol{\beta} &= (\sin u\cos u, -\sin^2 u, -\cos u),\\ \boldsymbol{\gamma} &= (\sin u, \cos u, 0)\end{aligned} \tag{7}$$

in which

$$u = \Omega(t - t_0) \tag{8}$$

and t_0 is some initial moment of time.

Substituting the last expressions in (5) we note that two Poisson equations are satisfied identically, while the dynamical equations give three equations involving powers of trigonometric functions of u. The conditions that each coefficient of the independent trigonometric terms must vanish lead to the following single set of values of parameters:

$$\Omega = \pm\sqrt{\frac{\mu_2}{C}} \tag{9}$$

$$\begin{aligned}&A = B, F = 0,\\ &\mathbf{r}_0 = \mu_2\left(-1, 0, \frac{E}{C}\right), \boldsymbol{\mu} = \mu_2\left(0, 1, -\frac{D}{C}\right)\end{aligned} \tag{10}$$

The angular velocity Ω is real only under the condition $\mu_2 > 0$. The case $\mu_2 = 0$ is excluded, since it gives only an equilibrium position $\psi = \varphi = 0, \theta = \frac{\pi}{2}$. Note also that

a) the condition $A = B$ means that the $x, y-$ axes lie in one of the two circular cross-sections of the ellipsoid of inertia of the body at the fixed point.

b) the conditions $A = B, F = 0$ guarantee that F (the the product of inertia of the body with respect to the $x, y-$ axes) vanishes for any other pair of axes in that plane. Thus, without loss of generality, we have the freedom to use this indeterminacy to rotate the $x, y-$ axes in their plane to satisfy the additional condition $D = 0$. This means that we choose the $y-$ axis to be the the principal axis of inertia lying in the circular cross-section of the inertia ellipsoid. That is the middle axis of inertia and it is the line of intersection of the two circular cross-sections at the fixed point with moment of inertia B.

The final choice of the parameters in (10) becomes

$$\begin{aligned}&A = B, D = F = 0,\\ &\mathbf{r}_0 = \mu_2\left(-1, 0, \frac{E}{C}\right), \boldsymbol{\mu} = (0, \mu_2, 0)\end{aligned} \tag{11}$$

and the potential of the problem may now be written as

$$V = \mu_2\left(-\alpha_1 + \frac{E}{C}\alpha_3\right) + \mu_2\beta_2 \tag{12}$$

3. The configuration

Although the uniform precessional motion is the same as the one described by Grioli for the case of a single gravity field, there are great differences in the configuration of the two fields and in the parameters of the body:

- As seen from (11), the magnetic moment of the body is directed along the middle principal axis of inertia (y-axis).
- The centre of mass lies in the principal plane orthogonal to it (xz− plane) in the direction inclined to the negative $x-$ axis at an angle

$$\delta = \tan^{-1}\frac{E}{C} \tag{13}$$

- The inertia matrix is now

$$I = \begin{pmatrix} B & 0 & -E \\ 0 & B & 0 \\ -E & 0 & C \end{pmatrix}$$

The middle principal moment (the radius of the circular

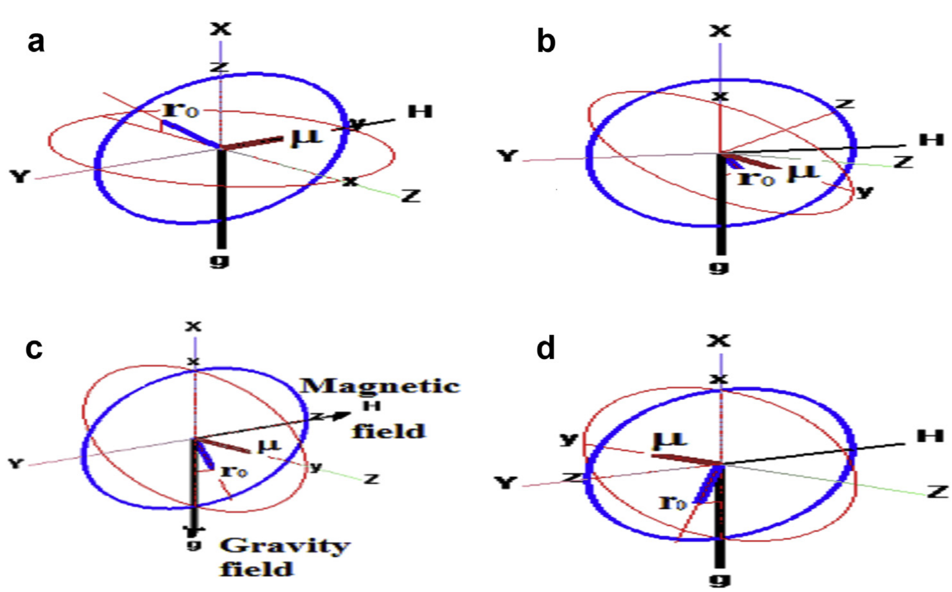

Fig. 1 – a) The configuration at t=0, b-at t=T/12, c-at t=T/4, d-at t=T/2.

cross-section of the inertia ellipsoid) is B. Denote the other two principal moments by A_0 and C_0. Those satisfy the relations

$$\begin{aligned} A_0 + C_0 &= B + C, \\ A_0 C_0 &= BC - E^2 \end{aligned} \tag{14}$$

Expressing the angle (13) in terms of the principal moments we get

$$\delta = \tan^{-1}\frac{\sqrt{(A_0 - B)(B - C_0)}}{A_0 - B + C_0} \tag{15}$$

This is the same angle as that, obtained by Gulyaev for the case of Grioli [24, 4], for the inclination of the space axis of precession to the vertical in the case of motion of a single gravity field.

Now we try to clarify the picture of motion of the body with respect to the space axes. The gravity field is in the negative direction of the X-axis. The positive half of the X-axis is vertical upwards. The Y-axis is horizontal and aligned in the direction of the magnetic field. The Z-axis is orthogonal to it in the horizontal plane.

For determinacy we choose the initial moment $t_0 = 0$. From (7) the $x-$ axis is initially aligned along the $X-$ axis vertically upwards, the $y-$ axis along the $Z-$ axis and the $z-$ axis (the body-axis of rotation) along the magnetic field (negative Y). The motion of the body is such that the body rotates with angular velocity Ω around its $z-$ axis while the last precesses in the vertical $XY-$ plane with the same angular velocity. The motion is periodic with a period

$$T = \frac{2\pi}{\Omega} = 2\pi\sqrt{\frac{C}{\mu_2}} \tag{16}$$

Fig. 1 depicts the configuration at four moments of time $t = 0, \frac{T}{12}, \frac{T}{4}, \frac{T}{2}$. The circular cross-section of the inertia ellipsoid is represented by the thin (red) circle in the $xy-$ plane, while the thick (blue) circle lying in the $XY-$ plane is the locus of the tip of the $z-$ axis. The centre of mass and the magnetic moment of the body are represented by thick blue and brown line segments, respectively.

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