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Equitable Total Chromatic Number of $K_{r\times p}$ for p Even[1]

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Abstract

A total coloring is equitable if the number of elements colored by any two distinct colors differs by at most one. The equitable total chromatic number of a graph (χ''_e) is the smallest integer for which the graph has an equitable total coloring. Wang (2002) conjectured that $\Delta+1 \leq \chi''_e \leq \Delta+2$. In 1994, Fu proved that there exist equitable $(\Delta+2)$-total colorings for all complete r-partite p-balanced graphs of odd order. For the even case, he determined that $\chi''_e \leq \Delta+3$. Silva, Dantas and Sasaki (2018) verified Wang's conjecture when G is a complete r-partite p-balanced graph, showing that $\chi''_e = \Delta+1$ if G has odd order, and $\chi''_e \leq \Delta+2$ if G has even order. In this work we improve this bound by showing that $\chi''_e = \Delta+1$ when G is a complete r-partite p-balanced graph with $r \geq 4$ even and p even, and for r odd and p even.

Keywords: Equitable total coloring, complete r-partite p-balanced graphs, graph coloring.

1 Introduction

Throughout this paper all graphs analyzed are finite, undirected and simple. Let $G = (V, E)$ be a graph. A *k-total coloring* of G is an assignment of k colors to the vertices and edges of G so that adjacent or incident elements have different colors. The *total chromatic number of* G, denoted by χ'', is the smallest k for which G has a

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k-total coloring. From the definition of total coloring, we have that $\chi'' \geq \Delta+1$ and the Total Coloring Conjecture (TCC) (Behzad [2], Vizing[13]) states that the total chromatic number of any graph is at most $\Delta+2$, where Δ is the maximum degree of the graph. In 1989, Sánchez-Arroyo [10] proved that the problem of determining the total chromatic number of an arbitrary graph is NP-hard, and it remains NP-hard even for cubic bipartite graphs.

An *equitable total coloring* is a total coloring that satisfies the additional property that the difference between the cardinalities of any two color classes is at most 1. The *equitable total chromatic number* of a graph G, denoted by χ''_e, is the least integer for which G has an equitable total coloring. In 2002, Wang [14] conjectured that the equitable total chromatic number of any graph is at most $\Delta+2$ (Equitable Total Coloring Conjecture (ETCC)). Ever since, many papers have been published in this subject [5,7,8]. In 2016, Dantas, de Figueiredo, Mazzuoccolo, Preissmann, dos Santos and Sasaki [5] proved that the problem of determining the equitable total chromatic number of a cubic bipartite graph is NP-complete.

In 1974, the total chromatic number of all complete r-partite p-balanced graphs was determined by Bermond [3]. A *complete r-partite p-balanced graph*, denoted by $K_{r\times p}$, is a graph where the vertex set can be partitioned into r independent sets $X_1, \cdots, X_r$, such that $|X_i| = p$, $i = 1, \ldots, r$, and there is an edge between any two vertices of different parts. In 1994, Fu [6] determined that the equitable total coloring of complete bipartite graphs is $\Delta+2$ and proved that there exist equitable $(\Delta+2)$-total colorings for all complete r-partite graphs of odd order. Silva, Dantas and Sasaki [11] determined the equitable total chromatic number for two classes of complete r-partite p-balanced graphs: $r \geq 4$ even and p odd ($\chi''_e = \Delta+2$); and r and p odd ($\chi''_e = \Delta+1$).

In this paper, we improve the existing previous bounds by proving that $\chi''_e = \Delta+1$ when the graph is a complete r-partite p-balanced with $r \geq 4$ even and p even, or r odd and p even, concluding all cases of this class.

2 Preliminaries

We adopt the following convention regarding the complete r-partite p-balanced graphs. The display of the vertices of $K_{r\times p}$ is similar to a matrix with r columns and p rows, where each column represents a part X_i of the partition of the vertex set. The vertex x_{ij} is the j-th vertex of the part X_i and it is assigned to the j-th row and i-th column. We define a *horizontal edge* as an edge $x_{ij}x_{i'j}$ (see Figure 1c). Also, a *matching of distance ℓ* between rows j and j' ($1 \leq j < j' \leq p$) is defined as the matching $\{x_{ij}x_{i+\ell,j'} | 1 \leq i \leq r\}$, where the index $i+\ell$ is taken modulo r. It is easy to see that there are $r-1$ matchings of distance linking the vertices of any two rows, say j and j', because there are $r-1$ edges linking a given vertex in the j-th row and the vertices of the j'-th row and each one of these edges belong to a different matching of distance.

In a graph total coloring, we say that a color is *represented* in a vertex if it is either the color of the vertex itself or if it is the color of an incident edge to the vertex in question.

Throughout the paper, we use matchings of the complete graph. The graph K_r is the complete graph having r vertices, in which r represents the number of parts of $K_{r\times p}$. In this case, we denote its matchings by R_t. Similarly, the graph K_p is the complete graph having p vertices, in which p represents the number of vertices in each part of $K_{r\times p}$. In this case, we denote its matchings by P_t. The following two results about matchings of complete graphs are due to Soifer [12]. Let K_n be the complete graph on n vertices. If $n \geq 4$ is even, then this graph has $n-1$ disjoint perfect matchings; and if $n \geq 3$ is odd, then this graph has n disjoint matchings.

A *Latin square* of order r is an $r \times r$ matrix whose entries are the elements of the set $\{1, 2, \cdots, r\}$ such that each symbol occurs precisely once per row and per column. Given a Latin square of order r, a *transversal* is a set of r different entries of different rows and columns. McKay, McLeod and Wanless [9] proved the following theorem:

Theorem 2.1 *(McKay, McLeod and Wanless, 2006) Let $T(r)$ be the maximum number of transversals over all Latin squares of order r, then $b^k \leq T(k)$ for $k \geq 5$, where $b \approx 1,719$.*

It is possible to use this result to prove that there exists a Latin square of even order $r \geq 4$ whose elements in the main diagonal are pairwise different. In fact, for our purposes, we will need Latin squares which are more restricted, as follows.

Lemma 2.2 *There exists a Latin square of even order $r \geq 4$ whose main diagonal is $1, 2, 3, \ldots, r$.*

Proof. Let A be the Latin square we are building and let $a_{ii'}$ be the entry of the i-th row and i'-th column of A. We observe that indices must be taken modulo r and if the index is congruent 0 modulo r, then such index is r, instead of 0. Let c be one of the elements of the set $\{1, \cdots, r\}$, which will be the entries of the Latin square A.

- The entries which receive color $c = 1$ are: $a_{1,1}$; $a_{i,i+1}$ for $2 \leq i \leq r-1$; and $a_{r,2}$.
- The entries which receive color c, for $2 \leq c \leq r-2$ are: $a_{i,(i+c+1)}$ for $1 \leq i \leq c-1$; a_{cc}; $a_{i,(i+c)}$ for $c+1 \leq i \leq r-1$; and $a_{n,(c+1)}$.
- The entries which receive color $c = r-1$ are: $a_{i,2i}$ for $1 \leq i \leq r/2$; and $a_{i,(2i-r+1)}$ for $(r/2)+1 \leq i \leq r$.
- The entries which receive color $c = r$ are: $a_{i,(2i+1)}$ for $1 \leq i \leq r/2$; and $a_{i,(2i-r)}$ for $(r/2)+1 \leq i \leq r$.

By construction, it is easy to see that the diagonal entries are $1, 2, \cdots, r$, as claimed. Supposing that two entries in the same row or column receive the same element, it is easy to reach a contradiction. $\square$

For classical results in Latin squares, we also refer to [4]. An example of the output of the algorithm of Lemma 2.2 for the case $r = 4$ can be seen in the following matrix.

$$\begin{bmatrix} 1 & 3 & 4 & 2 \\ 4 & 2 & 1 & 3 \\ 2 & 4 & 3 & 1 \\ 3 & 1 & 2 & 4 \end{bmatrix}$$

3 $K_{r\times p}$, with $r \geq 4$ even and p even

Next, we prove that $K_{r\times p}$ has $\chi''_e = \Delta + 1$ for $r \geq 4$ even and p even by showing three different algorithms according to the value of p. The general idea of these algorithms is to color the vertices first and to represent such colors on the other vertices. Then, the uncolored remaining elements are edges that can be grouped into matchings of distance. For the case $p = 4$, it was necessary to introduce an algorithm different than the one for $p = 2$ because the pattern for the coloring of the vertices of the part X_4 needed to be different, as we will explain later. The case $p = 4$ could not be extended for $p \geq 6$ because of the coloring of the horizontal edges and so we develop a third algorithm for that case.

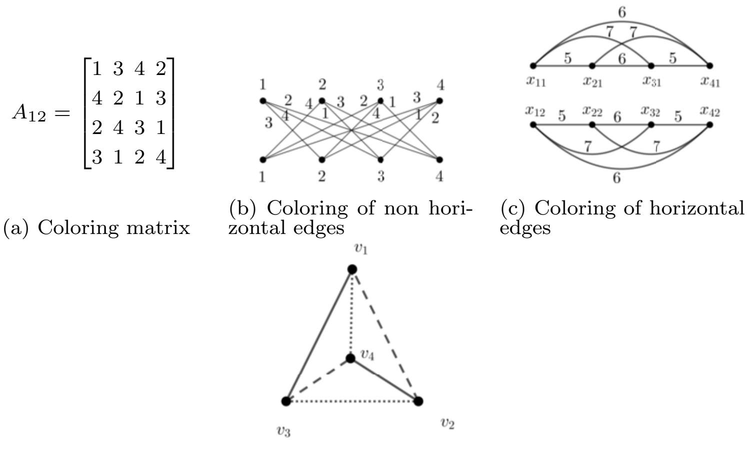
Algorithm for the case $p = 2$

We show that $\chi''_e(K_{r\times 2}) = \Delta + 1 = 2r - 1, r \geq 4$ even. The idea of the coloring is: we first obtain a Latin square of order r to determine the colors of vertices and non horizontal edges. Then, we use the matchings of K_r to determine the colors of horizontal edges.

We construct a coloring matrix A_{12} that will give the colors of edges of distance between rows 1 and 2 and the colors of the vertices of $K_{r\times 2}$ of order r, in which the entry $a_{ii'}$ represents the color of the edge $x_{i1}x_{i'2}$, $1 \leq i < i' \leq r$; and the entry a_{ii} represents the color of each vertex in part X_i.

Since $\Delta + 1 = 2r - 1$ in this case, $r - 1$ colors still need to be used. They are attributed to the horizontal edges as follows. We obtain the $r - 1$ matchings R_t of K_r. For each $t = 1, 2, \cdots, r - 1$, if $R_t = \{v_iv_{i'}, \cdots, v_{i''}v_{i'''}\}$, then the edges $x_{i1}x_{i'1}, \cdots, x_{i''1}x_{i'''1}$ and $x_{i2}x_{i'2}, \cdots, x_{i''2}x_{i'''2}$ of $K_{r\times 2}$ receive the same color. For example, for the case $K_{4\times 2}$, we consider the following matchings of K_4: $R_1 = \{v_1v_2, v_3v_4\}$, $R_2 = \{v_2v_3, v_1v_4\}$, $R_3 = \{v_1v_3, v_2v_4\}$ (see the example of $K_{4\times 2}$ in Figures 1c and 1d).

The matrix A_{12} is a Latin square whose elements of the main diagonal are all distinct. Lemma 2.2 gives a construction of such a matrix (see an example in Figures 1a and 1b). Since the vertices of different parts are adjacent, the fact that the elements of the main diagonal are all distinct implies that vertices of different parts do not receive the same color. The fact that elements do not occur more than once per rows or columns implies that non horizontal edges and vertices that are adjacent or incident do not receive the same color. Since the other colors are attributed to horizontal edges determined by the matchings R_t, then adjacent edges do not receive the same color. The first set of colors appears r times in the coloring matrix A_{12}, being $r-1$ times in edges and twice in vertices, totalizing $r+1$ times. Each one of the $r-1$ remaining colors is used in a perfect matching of $K_{r\times 2}$, that is, in r edges. Therefore, the difference between the cardinalities of any two color classes is at most 1, as desired.



(a) Coloring matrix (b) Coloring of non horizontal edges (c) Coloring of horizontal edges

(d) Matchings of K_4

Fig. 1. Example of coloring $K_{4\times 2}$

Algorithm for the case $p = 4$

We show that $\chi''_e(K_{r\times 4}) = \Delta + 1 = 4r - 3$, $r \geq 4$ even. The idea of this coloring is the following. First we color the vertices using colors $1, 2, \ldots, 2r$. Then, we use such colors on the horizontal edges of $K_{r\times 4}$. Next, we apply the $r-1$ colors $2r+1$ to $3r-1$ on matchings of distance linking vertices of rows 1 and 3; 2 and 4. The next step is to take the $2r$ colors of the vertices and color the edges that do not have these colors represented in their extreme vertices or adjacent edges. After this step, the edges that were not colored form a 2-regular graph, which can be colored with 2 colors.

Let $c(x)$ be the color of vertex x. We color the vertices of $K_{r\times 4}$ in the following way: for $i = 1, \ldots, r-1$, we have $c(x_{i1}) = c(x_{i2}) = i$ and $c(x_{i3}) = c(x_{i4}) = i+(r-1)$; $c(x_{r1}) = c(x_{r4}) = 2r-1$, $c(x_{r2}) = c(x_{r3}) = 2r$. We depict an example of this coloring for $K_{4\times 4}$ in Figure 2a.

The $r-1$ colors used in vertices of rows 1 and 2 of parts $X_1, X_2, \cdots, X_{r-1}$ are applied in horizontal edges of rows 3 and 4 according to the $r-1$ matchings of K_r, whereas the $r-1$ colors used in vertices of rows 3 and 4 of parts $X_1, X_2, \cdots, X_{r-1}$ are applied in horizontal edges of rows 1 and 2 according to the $r-1$ matchings of K_r. That is, for each $t = 1, 2, \cdots, r-1$, if $R_t = \{v_i v_{i'}, \cdots, v_{i''} v_{i'''}\}$, then use the color of vertices x_{t1} and x_{t2} on edges $x_{ij}x_{i'j}, \cdots, x_{i''j}x_{i'''j}$, $j = 3, 4$ of $K_{r\times 4}$; and use the color of the vertices x_{t3} and x_{t4} on edges $x_{ij}x_{i'j}, \cdots, x_{i''j}x_{i'''j}$, $j = 1, 2$ of $K_{r\times 4}$. See an example in Figure 2b.

Like in the previous case, we define $\binom{p}{2}$ matrices of order r, namely A_{13}, A_{24}, A_{12}, A_{34}, A_{14} and A_{23}, which store the colors of their respective matchings of distance. The entry $a_{ii'}$ of the matrix $A_{jj'}$, $1 \leq j < j' \leq p$ represents the color of the edge $x_{ij}x_{i'j'}$, $1 \leq i < i' \leq r$; and the entry a_{ii} stays empty. We refer to Figure 2c for an example where $r = 4$. In this figure, the entries of matrices A_{12}, A_{14}, A_{34} and A_{23} with asterisk $(*)$ receive colors later (we reinforce that they are distinct from the empty entries of the main diagonal).

Matrices A_{13} and A_{24}: we use $r-1$ colors (different than the ones used in vertices) in matchings of distance linking vertices of rows 1 and 3, or of rows 2 and 4. This means that matrices A_{13} and A_{24} are filled as follows: the entries of the main diagonal stay empty; the first row receive the numbers in ascending order from $2r+1$ to $3r-1$ and each row below is filled with the elements of the row above shifted one unity to the right.

Matrices A_{12} and A_{34}: these matrices are Latin squares of order r with a transversal in the main diagonal constructed from Lemma 2.2 with some changes. The entries of the matrix A_{12} are the $r-1$ colors used in the vertices of rows 1 and 2 of parts $X_1, X_2, \cdots, X_{r-1}$, whereas the entries of the matrix A_{34} are the colors used in the vertices of rows 3 and 4 of parts $X_1, X_2, \cdots, X_{r-1}$. We place an asterisk in the entries with color equal to the color of a_{rr} and remove the main diagonal entries. The entries marked with an asterisk will receive colors later. Since these matrices come from Latin squares of order r, after this process they are filled with $r-1$ colors.

Since we are presenting an equitable $(\Delta+1)$-total coloring of a regular graph, all colors must be represented in every vertex. After coloring horizontal edges and filling matrices A_{12} and A_{34}, we finish representing colors used in the vertices of parts $X_1, X_2, \cdots, X_{r-1}$ in every vertex.

We recall that $\Delta + 1 = 4r - 3$ and, up to this point, we used $2r + (r-1)$ colors, i.e., $2r$ colors in vertices or horizontal edges or A_{12} or A_{34}, and $r-1$ colors in the matrices A_{13} and A_{24}, and hence there are $r-2$ available unused colors. Also, the colors of the vertices of X_r need to be represented in every vertex. Among these $r-2$ available colors, 2 will be used in special matchings and the other $r-4$ in matchings of distance linking vertices of rows 1 and 4; and of the rows 2 and 3; that is, they are applied in the matrices A_{14} and A_{23}.

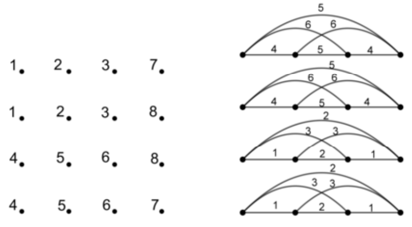
Matrices A_{14} and A_{23}: colors $2r-1$ and $2r$ have been used respectively in the vertices x_{r1} and x_{r4}; and in the vertices x_{r2} and x_{r3}. Thus, color $2r-1$ is used in entries $a_{1,r-1}, a_{21}, a_{32}, \cdots, a_{r-1,r-2}$ of the matrix A_{14}, whereas color $2r$ occupies entries $a_{12}, a_{23}, \cdots, a_{r-1,r}, a_{r1}$ of the matrix A_{14}. Color $2r-1$, by occupying the entries of the matrix A_{14} cited above, is applied to the edges $x_{11}x_{r-1,4}$, $x_{21}x_{14}$, $x_{31}x_{24}$, $\cdots$, $x_{r-1,1}x_{r-2,4}$. The first one of these edges belongs to the matching of distance $r-2$, whereas the others belong to the matching of distance $r-1$ (e.g. by the definition of matching of distance $r-1$ above, the edge $x_{21}x_{14}$ can be written as $x_{21}x_{(2+(4-1)mod4),4}$). Color $2r$, by occupying the entries of matrix A_{14} cited above, is used in a matching of distance 1 linking vertices of rows 1 and 4. So, colors $2r-1$

and $2r$ are used in edges that belong to matchings of distance $r-2, r-1$ and 1 linking vertices of rows 1 and 4. Since there are $r-1$ matchings of distance, this means that $r-4$ matchings of distance remain. Therefore, we can fit $r-4$ colors in matchings of distance linking vertices of rows 1 and 4.

Note that color $2r$, by being applied in entries of the matrix A_{14}, is not applied in some edges that are part of the matchings of distance $r-1$ and $r-2$, that are represented by the following entries of the matrix A_{14}: $a_{2r}, a_{31}, a_{42}, \cdots, a_{r,r-2}, a_{1r}$ and $a_{r,r-1}$. Matrix A_{23} is constructed from matrix A_{14} by replacing the entries with $2r-1$ by $2r$ and vice versa.

So far there are entries in the matrices A_{12}, A_{34}, A_{14} and A_{23} with $(*)$. When we look to the i-th row of matrix $A_{jj'}$, all entries of this row represent edges that have x_{ij} as one of its ends. Analogously, if we look to the i-th column of the same matrix, the entries of that column represent edges that have x_{ij} as one of its ends. By the process of filling the matrices A_{12}, A_{34}, A_{23} and A_{14} that we explained above, it can be verified that the asterisk entries form a 2-regular subgraph H of $K_{r\times 4}$. It is known that a graph is 2-regular if and only if its connected components are cycles. We claim that none of the connected components of H is a cycle of odd size. Indeed, the edges that were not assigned to any color yet are the asterisk entries of the matrices A_{12}, A_{34}, A_{23} and A_{14}, that is, they link vertices of rows 1 and 2, 3 and 4, 2 and 3, 1 and 4. Suppose, by contradiction, that the subgraph H contains a cycle C_k of odd size. Assume, without loss of generality, that the first vertex of C_k, here denoted by v_1 is a vertex of the first row. Consequently, the vertex v_2 is a vertex of row 2 or 4. Regardless of the possible options for the row where the vertex v_2 is, we have that vertex v_3 is a vertex of row 1 or 3. Proceeding with this reasoning, we have that the k-th vertex of C_k is either in row 1 or 3, since we are assuming that C_k has odd size. However, since the edge v_1v_k is of C_k, it follows that v_k cannot be in row 1 nor 3. Thus, we get a contradiction. It follows that none of the connected components of H is a cycle of odd size. Thus, the components of H are cycles of even size, whose edges can be colored with 2 colors, as desired. One of these colors is color $4r-4=\Delta$ and the other one is color $4r-3=\Delta+1$.

Colors $1, 2, \cdots, \frac{r}{2}$ were used in two vertices and in $2r-1$ edges, so they were used in a total of $2r+1$ elements each. Colors $\frac{r}{2}+1, \cdots, 4r-3$ were used in perfect matchings of $K_{r\times 4}$, totalizing $2r$ edges each. Therefore, the coloring is equitable, as desired.



(a) Coloring of vertices (b) Coloring of horizontal edges

$$A_{13} = \begin{bmatrix} & 9 & 10 & 11 \\ 11 & & 9 & 10 \\ 10 & 11 & & 9 \\ 9 & 10 & 11 & \end{bmatrix} \quad A_{12} = \begin{bmatrix} & 3 & * & 2 \\ * & & 1 & 3 \\ 2 & * & & 1 \\ 3 & 1 & 2 & \end{bmatrix} \quad A_{14} = \begin{bmatrix} & 8 & 7 & * \\ 7 & & 8 & * \\ * & 7 & & 8 \\ 8 & * & * & \end{bmatrix}$$

$$A_{24} = \begin{bmatrix} & 9 & 10 & 11 \\ 11 & & 9 & 10 \\ 10 & 11 & & 9 \\ 9 & 10 & 11 & \end{bmatrix} \quad A_{34} = \begin{bmatrix} & 6 & * & 5 \\ * & & 4 & 6 \\ 5 & * & & 4 \\ 6 & 4 & 5 & \end{bmatrix} \quad A_{23} = \begin{bmatrix} & 7 & 8 & * \\ 8 & & 7 & * \\ * & 8 & & 7 \\ 7 & * & * & \end{bmatrix}$$

(c) Coloring matrices

Fig. 2. Example of coloring $K_{4\times 4}$

Algorithm for the case $p \geq 6$

We show that $\chi''_e(K_{r\times p}) = \Delta + 1 = rp - p + 1$, $r \geq 4$ and $p \geq 6$ even. The general idea of the algorithm is the following. We use a perfect matching P_1 of K_p to determine the vertices's colors. By this step, each color is represented in two vertices of the same independent set, and thus they are obviously represented in two different rows in this set. Suppose that a color c is assigned to vertices of rows j and j'. With the next step we represent color c in every vertex of rows j and j', and we do so using Latin squares, similar to the ones defined for the case $p = 2$. Since we are constructing an equitable $(\Delta + 1)$-total coloring of a regular graph, all the colors must be represented in every vertex. The colors used so far were represented in some vertices, but not all. So, we use a result of Alspach and Gavlas [1] to finish representing these colors in all vertices of the graph. The remaining colors are used in perfect matchings of $K_{r\times p}$ determined by the matchings of distance of K_p (for non horizontal edges) and by the matchings of K_r (for horizontal edges).

We color the vertices of $K_{r\times p}$, $p \geq 6$ in the following way. Let $c(x)$ be the color of vertex x and let P_1 be a perfect matching of the graph K_p (with $\frac{p}{2}$ edges). Thus, for each edge $e_k = v_j v_{j'} \in P_1$, $k = 1, \cdots, (\frac{p}{2})$, we have $c(x_{ij}) = c(x_{ij'}) = i + (k-1)r$, for $i = 1, \cdots, r$. We depict an example of this coloring for $K_{4\times 6}$ in Figure 3a.

Consider the $k = |P_1|$ matrices $A_{jj'}, \cdots, A_{j''j'''}$ as described in the beginning of this section for the case $p = 2$. We apply Theorem 2.1 and Lemma 2.2 to obtain Latin squares with a transversal in the main diagonal , i.e., each edge $e_k = v_j v_{j'} \in P_1$, $k = 1, \cdots, (\frac{p}{2})$ defines a Latin square $A_{jj'}$ with colors from $(k-1)r + 1$ to kr.

By the end of this step, colors $1, 2, \cdots, \frac{rp}{2}$ were used in every vertex of two rows. However, those colors still need to be represented in the vertices of the other rows. To do so, we use the following result of Alspach and Gavlas [1]:

Claim 3.1 *(Alspach and Gavlas, 2001) For positive even integers m and n with $4 \leq m \leq n$, the graph $K_n - I$ can be decomposed in cycles of size m if and only if the number of edges in $K_n - I$ is a multiple of m, where I is a 1-factor.*

For the next step of the algorithm we need to obtain $\frac{p}{2}$ cycles of size $p-2$ of the graph K_p minus a 1-factor. We observe that $K_p - I$ has $\binom{p}{2} - \frac{p}{2} = \frac{p(p-2)}{2}$ edges. Making $m = p - 2$ and $n = p$ in Claim 3.1, we conclude that $K_p - I$ can be decomposed into $\frac{p}{2}$ cycles of size $p - 2$, as desired.

Suppose, without loss of generality, that $K_p - I = K_p - P_1$, with P_1 being a perfect matching of K_p. It is known that edges of every cycle of even size can be partitioned into two perfect matchings. So we divide each cycle in two perfect matchings and we associate them with the edges of P_1, so that each edge $v_j v_{j'}$ of P_1 is associated to the matchings of the cycle of $K_p - P_1$ that does not contain the vertices v_j and $v_{j'}$.

With the process of decomposition of $K_p - P_1$, we obtain $\frac{p}{2}$ cycles. Let M_k and M'_k be the matchings obtained from the k-th cycle of the decomposition of $K_p - P_1$, that does not contain the edge $v_j v_{j'}$. Then, the colors used in the vertices of rows j and j' of the parts $X_1, X_2, \cdots, X_{\frac{r}{2}}$ must be used in matchings of distance linking vertices of rows determined by M_k, whereas the colors used in the vertices of rows

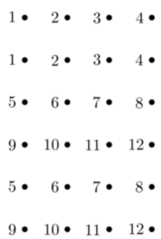
j and j' of parts $X_{\frac{r}{2}+1}, \cdots, X_r$ are used in matchings of distance linking vertices of rows determined by M'_k. Since there are $r-1$ matchings of distance linking vertices of any two rows and since we used only $\frac{r}{2}(< r-1)$ matchings of this kind, we conclude that this is a valid action.

The objective of decomposing $K_p - P_1$ into cycles to apply the colors that were used in vertices in matchings of distance is to ensure that, at the end of this process, in the matchings $P_2, P_3, \cdots, P_{p-1}$, each pair of rows of the graph $K_{r\times p}$ was used the same amount of times. With the step described above, we make sure that each pair of rows and, consequently, each matching from P_2 to P_{p-1} was used $\frac{r}{2}$ times in matchings of distance, from a total of $r-1$ matchings of this type. This means that there are still $r - 1 - \frac{r}{2} = \frac{r}{2} - 1$ matchings P_t $(2 \leq t \leq r)$. In other words, $\left(\frac{r}{2} - 1\right)(p-2)$ colors can be applied in those available matchings of distance. Note that each one of these colors is applied in a perfect matching of $K_{r\times p}$, that is, the colors are represented in all the vertices, as desired, since this is an equitable total coloring with $\Delta + 1$ colors of a regular graph (see an example in Figure 3c).

Finally, we apply $r-1$ new colors in horizontal edges determined by the matchings of K_r as follows. If $R_t = \{v_i v_{i'}, \cdots, v_{i''} v_{i'''}\}$, then we apply one of the new colors in the edges $x_{ij}x_{i'j}, \cdots, x_{i''j}x_{i'''j}$ for all $j = 1, 2, \cdots, p$. Using a different color to each matching R_t of K_r, we conclude that $r-1$ colors are used in this step, as claimed.

We use $\frac{rp}{2} + \left(\frac{r}{2} - 1\right)(p-2) + (r-1) = rp - p + 1 = \Delta + 1$ colors and, by construction, they were not applied in incident or adjacent elements of the graph.

Consider the graph $K_{4\times 6}$. We show its coloring by the algorithm provided above. The decomposition of $K_6 - P_1$ into cycles that we used was $\{v_3v_4v_5v_6, v_2v_4v_1v_6, v_1v_3v_2v_5\}$.



(a) Coloring of vertices of $K_{4\times 6}$ determined by using $P_1 = \{v_1v_2, v_3v_5, v_4v_6\}$

$$A_{12} = \begin{bmatrix} 1 & 3 & 4 & 2 \\ 4 & 2 & 1 & 3 \\ 2 & 4 & 3 & 1 \\ 3 & 1 & 2 & 4 \end{bmatrix} \quad A_{46} = \begin{bmatrix} 9 & 11 & 12 & 10 \\ 12 & 10 & 9 & 11 \\ 10 & 12 & 11 & 9 \\ 11 & 9 & 10 & 12 \end{bmatrix}$$

$$A_{35} = \begin{bmatrix} 5 & 7 & 8 & 6 \\ 8 & 6 & 5 & 7 \\ 6 & 8 & 7 & 5 \\ 8 & 5 & 6 & 8 \end{bmatrix}$$

(b) Matrices A_{12}, A_{35} and A_{46} determined by using P_1

Color	Distance	Rows
1	1	3 and 4; 5 and 6
2	2	3 and 4; 5 and 6
3	1	4 and 5; 3 and 6
4	2	4 and 5; 3 and 6
5	1	2 and 4; 1 and 6
6	2	2 and 4; 1 and 6
7	1	4 and 1; 2 and 6
8	2	4 and 1; 2 and 6
9	1	1 and 3; 2 and 5
10	2	1 and 3; 2 and 5
11	1	3 and 2; 1 and 5
12	2	3 and 2; 1 and 5

(c) Colors 1 to 12 in matchings of distance using P_1

Fig. 3. Coloring of $K_{4\times 6}$

There is still a matching of distance 3 for P_2, P_3, P_4 and P_5. Thus, we use colors 13, 14, 15 e 16 to each one of the matchings of distance 3 linking vertices of rows determined by P_2, P_3, P_4 e P_5.

Since $r = 4$, we need to obtain the matchings of K_4, which are: $R_1 = \{v_1v_2, v_3v_4\}$, $R_2 = \{v_2v_3, v_1v_4\}$ and $R_3 = \{v_1v_3, v_2v_4\}$. Thus, we use color 17 on edges $x_{1j}x_{2j}$ and $x_{3j}x_{4j}$ for all $j = 1, 2, \cdots, 6$. Analogously, we use color 18 on edges $x_{2j}x_{3j}$ and $x_{1j}x_{4j}$ and color 19 on edges $x_{1j}x_{3j}$ and $x_{2j}x_{4j}$ for all $j = 1, 2, \cdots, 6$.

4 $K_{r\times p}$, with r odd and p even

Algorithm for the case $p = 2$

Briefly, the algorithm assign one color to the two vertices of each independent set and use such colors in horizontal edges oriented by the matchings of K_r. Then, we use the remaining colors on matchings of distance linking vertices of rows 1 and 2.

The vertices of part X_i, $i = 1, 2, \cdots, r$, receive color i. These colors are also used in horizontal edges as follows.

Let $R_t = \{v_iv_{i'}, \cdots, v_{i''}v_{i'''}\}$ be one of the r matchings of K_r, and let v_t be remaining vertex, i.e., the vertex that is not extreme of any edge of R_t. We use color t to color edges $x_{i1}x_{i'1}, \cdots, x_{i''1}x_{i'''1}$ and also to color edges $x_{i2}x_{i'2}, \cdots, x_{i''2}x_{i'''2}$.

Since v_t is the remaining vertex of matching R_t and the edges that receive color t are associated with the referred matching, the vertices and edges that received color t are not incident nor adjacent, by construction. Since each matching of K_r has cardinality $\frac{r-1}{2}$, we conclude that each one of the r colors is used $2 + \frac{r-1}{2}2 = r + 1$ times.

Since this is a coloring with $\Delta + 1 = 2r - 1$ colors and r colors were used in the first step, $r-1$ colors remained to be used in non horizontal edges. We use each one of the $r - 1$ remaining colors in a matching of distance, which finishes the coloring.

Algorithm for the case $p \geq 4$

We show that $\chi''_e(K_{r\times p}) = \Delta + 1 = rp - p + 1$, r odd and $p \geq 4$ even. The first step to obtain an equitable $(\Delta + 1)$-total coloring of this class of graphs consists in obtaining $p - 1$ matchings of K_p. Thus, we make a table where each matching is written $r-1$ times. Each copy is associated to matching of distance l ($\ell = 1, \cdots r-1$) which receive color c ($c = 1, \cdots, (p-1)(r-1)$). In other words, suppose that in the c-th row of the table, color c is associated with the matching of distance l determined by the matching $P_t = \{v_jv_{j'}, \cdots, v_{j''}v_{j'''}\}$. This means that color c is applied in a matching of distance l between the rows determined by P_t, that is, between rows j and j', $\cdots$, j'' and j'''. In this process we used $(r - 1)(p - 1) = rp - r - p + 1$ colors and $\Delta + 1 = rp - p + 1$, so it remains r colors to be used. For this, in the second step we change the colors of some edges in order to use this r remaining colors.

The second step of the coloring consists in changing part of what was done in the first step and $\left(\frac{p}{2} - 1\right) r$ rows of the table will be changed. Considering the table, we remove the first edge from each one of the copies of the matchings P_t, $t = 1, 3, 5, \cdots, p - 3$, and the first edge from the first row associated with each matching P_t, $t = 2, 4, 6, \cdots, p - 2$. These removed edges will receive new colors in the following way. We assign $r - 1$ colors of the r remaining colors to the matchings of distance l determined by the new matching obtained from the edges removed from P_t, $t = 1, 3, 5, \cdots, p-3$; and we assign the last color r to the matching of distance 1 determined by the new matching obtained from the edges removed from P_t, $t = 2, 4, 6, \cdots, p-2$.

The $\left(\frac{p}{2}-1\right) r$ colors of each row of the table we have changed must be represented in vertices of rows j and j' associated to the first edge $v_j v_{j'}$ of the matching P_t that was removed in the previous step. To represent those colors in these vertices, we will use them in the coloring of vertices and horizontal edges.

If a given color c had been applied in a matching of distance l, but transferred the element $v_j v_{j'}$ of the matching P_t (t odd), then color c has to be used in the coloring of vertices x_{lj} and $x_{lj'}$. In addition, if a given color c had been applied in a matching of distance l, but transferred the element $v_j v_{j'}$ of the matching P_t (t even), then color c has to be used in vertices x_{rj} and $x_{rj'}$. The $r-1$ colors that were inserted in the second step must color vertices $x_{l,p-1}$ and x_{lp} if the corresponding distance in the table is l. The last color r is used in vertices x_{r1} and x_{rp}. Horizontal edges are colored as in the graph $K_{r\times p}$ with r and p even presented before, using the matchings of K_r.

Some colors were used only in edges. Such colors were used in matchings of distance between rows that were determined by the matchings of K_p. Since the matchings are pairwise disjoint, the pairs of rows determined by the matchings of K_p are distinct and, therefore, there are no colors being applied to adjacent edges. By construction it is clear that incident and adjacent elements were not assigned to the same color. The colors used only in edges were used in perfect matchings of the graph, totalizing $\frac{rp}{2}$ times. The colors used in vertices and edges were used in 2 vertices, in horizontal edges of two rows, totalizing $2\frac{r-1}{2}$ times and also in $\frac{p-2}{2}r$ non horizontal edges. Therefore, each one of these colors was used $2+2\frac{r-1}{2}+\frac{p-2}{2}r = \frac{rp+2}{2}$ times. We conclude that the difference between the cardinalities of any two color classes is at most 1, as desired.

Table 1
Initial distribution of colors on edges of $K_{3\times 8}$

Color	Distance	Matching
1	1	$\{v_1v_2, v_3v_7, v_4v_6, v_5v_8\}$
2	2	$\{v_1v_2, v_3v_7, v_4v_6, v_5v_8\}$
3	1	$\{v_2v_3, v_1v_4, v_5v_7, v_6v_8\}$
4	2	$\{v_2v_3, v_1v_4, v_5v_7, v_6v_8\}$
5	1	$\{v_3v_4, v_2v_5, v_1v_6, v_7v_8\}$
6	2	$\{v_3v_4, v_2v_5, v_1v_6, v_7v_8\}$
7	1	$\{v_4v_5, v_3v_6, v_2v_7, v_1v_8\}$
8	2	$\{v_4v_5, v_3v_6, v_2v_7, v_1v_8\}$
9	1	$\{v_5v_6, v_4v_7, v_1v_3, v_2v_8\}$
10	2	$\{v_5v_6, v_4v_7, v_1v_3, v_2v_8\}$
11	1	$\{v_6v_7, v_1v_5, v_2v_4, v_3v_8\}$
12	2	$\{v_6v_7, v_1v_5, v_2v_4, v_3v_8\}$
13	1	$\{v_1v_7, v_2v_6, v_3v_5, v_4v_8\}$
14	2	$\{v_1v_7, v_2v_6, v_3v_5, v_4v_8\}$

Table 2
Final distribution of colors on edges of $K_{3\times 8}$

Color	Distance	Matching
1	1	$\{v_3v_7, v_4v_6, v_5v_8\}$
2	2	$\{v_3v_7, v_4v_6, v_5v_8\}$
3	1	$\{v_1v_4, v_5v_7, v_6v_8\}$
4	2	$\{v_2v_3, v_1v_4, v_5v_7, v_6v_8\}$
5	1	$\{v_2v_5, v_1v_6, v_7v_8\}$
6	2	$\{v_2v_5, v_1v_6, v_7v_8\}$
7	1	$\{v_3v_6, v_2v_7, v_1v_8\}$
8	2	$\{v_4v_5, v_3v_6, v_2v_7, v_1v_8\}$
9	1	$\{v_4v_7, v_1v_3, v_2v_8\}$
10	2	$\{v_4v_7, v_1v_3, v_2v_8\}$
11	1	$\{v_1v_5, v_2v_4, v_3v_8\}$
12	2	$\{v_6v_7, v_1v_5, v_2v_4, v_3v_8\}$
13	1	$\{v_1v_7, v_2v_6, v_3v_5, v_4v_8\}$
14	2	$\{v_1v_7, v_2v_6, v_3v_5, v_4v_8\}$
15	1	$\{v_1v_2, v_3v_4, v_5v_6\}$
16	2	$\{v_1v_2, v_3v_4, v_5v_6\}$
17	1	$\{v_2v_3, v_4v_5, v_6v_7\}$

For an example, consider the graph $K_{3\times 8}$. The tables of the first and the second steps are Tables 1 and 2. The coloring of the vertices of $K_{3\times 8}$ is done as follows: x_{11} and x_{12} receive color 1, x_{13} and x_{14} receive color 5, x_{15} and x_{16} receive color 9, x_{17} and x_{18} receive color 15, x_{21} and x_{22} receive color 2, x_{23} and x_{24} receive color 6, x_{25} and x_{26} receive color 10, x_{27} and x_{28} receive color 16, x_{32} and x_{33} receive color 3, x_{34} and x_{35} receive color 7, x_{36} and x_{37} receive color 11 and x_{31} and x_{38} receive color 17.

5 Conclusion and perspectives

In this paper we prove that the equitable total chromatic number of $K_{r\times p}$ is $\Delta+1$ for r and p even ($r \geq 4$); and for r odd and p even. This paper, alongside with [6,11] concludes the work of determining the equitable total chromatic number for all complete r-partite p-balanced graphs, verifying the ETCC for this class of graphs. We summarize these results in Table 3. Future work includes, but is not limited to, determining the equitable total chromatic number of complete r-partite non-balanced graphs.

Table 3
The equitable total chromatic number of the complete r-partite p-balanced graphs

r	p	χ''_e	Ref.
$r = 2$	–	$\Delta + 2$	[6]
$r \geq 4$ even	odd	$\Delta + 2$	[11]
odd	odd	$\Delta + 1$	[11]
$r \geq 4$ even	even	$\Delta + 1$	this work
odd	even	$\Delta + 1$	this work

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