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Almost Every Domain is Universal

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Abstract

We endow the collection of ω-bifinite domains with the structure of a probability space, and we will show that in this space the collection of all universal domains has measure 1. For this, we present a probabilistic way to extend a finite partial order by one element. Applying this procedure iteratively, we obtain an infinite partial order. We show that, with probability 1, the cpo-completion of this infinite partial order is the universal homogeneous ω-bifinite domain. By alternating the probabilistic one-point extension with completion procedures we obtain almost surely the universal and homogeneous ω-algebraic lattice, ω-Scott domain, and ω-bifinite L-domain, respectively.
We also show that in the projective topology, the set of universal and homogeneous ω-bifinite domains is residual (i.e., comeagre), and we present an explicit number-theoretic construction of such a domain.

Keywords: domain theory, universal homogeneous domains, probabilistic systems, constructive mathematics, topological models

1 Introduction

In the theory of denotational semantics of programming languages, several authors established the existence of particular kinds of 'universal' domains. Scott [23] provided a universal domain for the class of all ω-algebraic lattices and showed that in this domain calculations can be handled by a calculus of retracts. Universal domains for the classes of all coherent, respectively bounded-complete, ω-algebraic domains were given by Plotkin [21] and Scott [24]. Gunter and Jung [15] and Droste and Göbel [7] described a systematic way of constructing universal - even saturated, or universal homogeneous - domains. Let us recall that a domain U of a class C of domains is called universal, if each other domain of C can be embedded (via an embedding-projection pair) into U, and U is homogeneous, if each isomorphism between two finite subdomains of U extends to an automorphism of U; intuitively,

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homogeneity means that U has the highest possible degree of structural symmetry. Such universal homogeneous domains are unique up to isomorphism.

For further results on various universal domains, see [16], [17], [15], [6], [9] and [20].

In this paper, we will present a probabilistic construction of domains, and we will show that with probability 1 it will produce a universal homogeneous domain. More specifically, we will achieve this for four classes of domains: ω-algebraic lattices, ω-Scott domains (= bounded-complete ω-algebraic domains), ω-bifinite L-domains, and ω-bifinite domains.

Our construction of the universal and homogeneous ω-bifinite domain proceeds as follows. Note that any domain is uniquely determined by the structure of its subposet of compact elements. Since this set is countable, we can assume it to be the set $\mathbb{N}$ of natural numbers (with 1 below any element). We will then determine a partial order on $\mathbb{N}$. For this, we proceed inductively. Assume we have obtained a partial order on the elements of $\{1, ..., n\}$, for some $n \in \mathbb{N}$. In order to extend it to the set $\{1, ..., n+1\}$, we only have to specify the order-relations between $n+1$ and i, for each element $i \in \{1, ..., n\}$. We will do this in a random way, subject only to few natural side-conditions that we stay inside our given class of domains and that $\{1, ..., n\}$ becomes a subdomain of $\{1, ..., n+1\}$. By induction, we thus obtain an order on $\mathbb{N}$. Since each $\{1, \ldots, n\}$ will be a subdomain of the order on $\mathbb{N}$, we can view our procedure as a construction of finite approximations of the order on $\mathbb{N}$. Our main result states that its completion to a domain is, with probability 1, a universal and homogeneous bifinite domain.

To construct universal homogeneous domains in the classes of all ω-algebraic lattices, of all ω-Scott domains resp. of all ω-bifinite L-domains, we proceed in exactly the same way for extending the order structure from $\{1, ..., n\}$ probabilistically to an order on $\{1, ..., n+1\}$. However, in the next step we construct a completion of this poset to a lattice, Scott domain, resp. L-domain, in a deterministic way. This completion is finite, and now we apply again the probabilistic procedure to it to extend the order by one more element. By induction, we get our partial order on $\mathbb{N}$, and we show that its cpo-completion is, with probability 1, a universal homogeneous algebraic lattice, Scott domain, resp. L-domain.

In fact, our probabilistic constructions will depend on a given sequence of parameters. Our result shows that regardless of the choice of these parameters, with probability 1 we get the same universal homogeneous domain.

We will describe how to phrase our intuitively given construction more precisely, i.e. how to construct the corresponding probability spaces. Our probabilistic extension of the order $\leq_n$ of $\{1, ..., n\}$ to an order $\leq_{n+1}$ on $\{1, ..., n+1\}$ essentially describes the conditional probability of obtaining $\leq_{n+1}$ given $\leq_n$. Hence we obtain in a natural way a projective system of finite discrete probability spaces, and its limit is the probability space of the constructed partial orders on $\mathbb{N}$.

Scott [23] showed how to construct domains which embed, or which are even isomorphic to, their own function space; such domains give rise to weakly extensional (resp. extensional) models of the untyped λ-calculus, and the method is crucial for constructing solutions of domain equations. As an immediate consequence of our results, it follows for each of the four classes of domains we consider that the collection of all those domains which embed their own function space has measure 1, whereas the collection of all domains which are isomorphic to their function space has measure 0.

In our second main result, we show that each of the four classes of ω-domains can be endowed with the structure of a complete metric space. There is a topological notion for a subset of such a space to be "large" or to contain "almost all" of the space. This is the notion of being residual, i.e. the complement of a meagre set. After describing the background, we show that in each of the four corresponding metric spaces, the subset of all universal homogeneous domains is residual, i.e. large in this topological sense.

Finally, we will present a simple inductive number-theoretic construction of the universal homogeneous bifinite domain. It will follow that our construction is effective and that the order on each initial segment $\{1, ..., n\}$ of $\mathbb{N}$ can be determined in time polynomial in n. In fact, this is the simplest construction of the universal homogeneous ω-bifinite domain known to us.

The investigation of universal structures dates back at least to Cantor [5] who showed that in the class of countable linear orders, the chain $(\mathbb{Q}, \leq)$ is universal. Clearly, $(\mathbb{Q}, \leq)$ is also homogeneous. Fraïssé [13] showed that the class of all countable undirected graphs contains a universal homogeneous object (which is unique up to isomorphism). Erdős and Rényi [11] gave a probabilistic construction of countable graphs which, with probability 1, produces this universal homogeneous graph, therefore also called the random graph. This amazing result spurred much further research on such graphs and on probabilistic laws in finite model theory. In [8], the present authors gave a probabilistic construction of the universal homogeneous countable partial order. For further results, see e.g. [2]. Recently, in [10] a probabilistic construction of universal homogeneous causal sets was given (these are partially ordered sets which have been proposed as a basic model for discrete space-time in quantum gravity). The present results show that also in the area of domain theory a probabilistic, hence intuitively "chaotic", construction leads to domains which are universal and carry maximal degree of symmetry.

2 Background

We first summarize our notation (which is mostly standard) for the convenience of the reader. Let $(D, \leq)$ be a partially ordered set (a *poset*). A non-empty subset $A \subseteq D$ is called *directed*, if for any $a, b \in A$ there exists $c \in A$ with $a \leq c$ and $b \leq c$. We say that $(D, \leq)$ is a *cpo*, if it has a smallest element, denoted $\perp$, and each directed subset of D has a supremum in D. An element $x \in D$ is *compact*, if for any directed subset A of D for which $\sup A$ exists and $x \leq \sup A$ there is $a \in A$ with $x \leq a$. The set of all compact elements of D is denoted by D^0. Then $(D, \leq)$ is *algebraic*, if for each $d \in D$ the set $\{x \in D^0 : x \leq d\}$ is directed and has supremum d. An algebraic cpo D will be called a *domain*. It is well-known that any domain $(D, \leq)$ is determined uniquely (up to isomorphism) by the structure of its subset of compact elements $(D^0, \leq)$. A domain $(D, \leq)$ is called a *Scott domain* (*L-domain*), if each non-empty subset A of D which is bounded above in D has a supremum (an infimum) in D. Thus an *algebraic lattice* can be considered as a Scott domain containing a greatest element. We use the prefix $\omega-$ for a domain to denote that its set of compact elements is countable.

Let $(D, \leq)$ again be a poset. For $A \subseteq D$ and $x \in D$, we write $A \leq x$ to denote that $a \leq x$ for all $a \in A$. Let $\mathrm{mub}(A)$ denote the set of minimal upper bounds of A in D. A subset S of D is called *bounded-complete* in D, denoted $S \lhd D$, if whenever $A \subseteq S$ is a finite subset, $x \in D$ and $A \leq x$, then there exists $s \in S$ such that $A \leq s \leq x$. Then $(D, \leq)$ is called *bifinite*, if each finite subset $A \subseteq D^0$ is contained in some finite bounded-complete subset $S \subseteq D^0$.

Let $(P, \leq), (Q, \leq)$ be two posets. A function $f : P \to Q$ is *continuous*, if it preserves suprema of directed subsets of P. Furthermore, f is a *mub-embedding*, if for any finite subset A of P, we have $f(\mathrm{mub}(A)) = \mathrm{mub}(f(A))$. Note that then, in particular, f is an *order-embedding* (i.e. $a \leq b$ iff $f(a) \leq f(b)$ for any $a, b \in P$); moreover, provided that P, Q have smallest elements $\perp_P, \perp_Q$, respectively, we have $f(\perp_P) = \perp_Q$ (let $A = \emptyset$ and observe that $\mathrm{mub}(\emptyset) = \{\perp_P\}$). As usual, an order-embedding which is onto is called *isomorphism*.

Now, let $f : P \to Q$, $g : Q \to P$ be continuous. Then (f, g) is called an *embedding-projection pair (EPP)* from $(P, \leq)$ into $(Q, \leq)$, if $g \circ f = \mathrm{id}_P$ and $f \circ g \leq \mathrm{id}_Q$. If f is an isomorphism, then $g = f^{-1}$, and the EPP is also called an *isomorphism*. We denote an EPP (f, g) by φ. The composition of two EPPs $(f, g) : (P \leq) \to (Q, \leq)$, $(h, k) : (Q, \leq) \to (R, \leq)$ is defined to be $(h \circ f, g \circ k)$ and is again an EPP. We let $\omega\mathrm{B}^{epp}$, $\omega\mathrm{BL}^{epp}$, $\omega\mathrm{S}^{epp}$, $\omega\mathrm{Lat}^{epp}$ denote the categories of all ω-bifinite domains, ω-bifinite L-domains, ω-Scott domains, and ω-algebraic lattices, respectively, in each case with EPPs as morphisms. Embedding-projection pairs, mub-embeddings and bounded-complete subsets are closely related, as is well-known (cf., e.g., [9]).

Proposition 2.1 *Let $(D_1, \leq)$, $(D_2, \leq)$ be two bifinite domains and $f : D_1^0 \to D_2$ be a mapping. Then the following are equivalent:*

(i) *There exists an EPP $(\bar{f}, g)$ from $(D_1, \leq)$ into $(D_2, \leq)$ such that $\bar{f}\restriction_{D_1^0} = f$.*

(ii) *f is a mub-embedding and $f(D_1^0) \subseteq D_2^0$.*

(iii) *f is an order-embedding and $f(D_1^0) \lhd D_2^0$.*

Let $\mathfrak{C}$ be one of the categories $\omega\mathrm{B}^{epp}$, $\omega\mathrm{BL}^{epp}$, $\omega\mathrm{S}^{epp}$, or $\omega\mathrm{Lat}^{epp}$, and $(U, \leq) \in \mathfrak{C}$. Then $(U, \leq)$ is called

- *universal in* $\mathfrak{C}$, if for each domain $(D, \leq) \in \mathfrak{C}$ there exists an EPP $\varphi : (D, \leq) \to (U, \leq)$;
- *homogeneous*, if for any finite domain $(D, \leq) \in \mathfrak{C}$ and EPPs $\varphi_i : (D, \leq) \to (U, \leq)$ $(i = 1, 2)$ there exists an isomorphism $\psi : (U, \leq) \to (U, \leq)$ such that $\psi \circ \varphi_1 = \varphi_2$.

Moreover, we say that $(U^0, \leq)$ *realizes all one-point extensions of finite subdo-*

mains, if for any two finite domains $(A, \leq_A)$ and $(B, \leq_B)$ in $\mathfrak{C}$ with $(A, \leq_A) \lhd (U^0, \leq)$, $(A, \leq_A) \lhd (B, \leq_B)$ and $B = A \uplus \{y\}$ there exists $z \in U^0$ such that $\mathrm{id}_A \cup \{(y, z)\} : (B, \leq_B) \to (U^0, \leq)$ is a mub-embedding; equivalently, $A \cup \{z\} \lhd (U^0, \leq)$ and $\mathrm{id}_A \cup \{(y, z)\} : (B, \leq_B) \to (A \cup \{z\}, \leq)$ is an isomorphism.

As is well-known in model theory (cf. [14,4,18]), the properties of universality and homogeneity are intimately related with realizations of one-point extensions; in our setting such realizations of one-point extensions were already utilized in [15,7,9], and we have the following useful and essential characterization.

Proposition 2.2 ([9, Prop. 2.2]) *Let $(U, \leq) \in \mathfrak{C}$ where $\mathfrak{C}$ is one of the categories $\omega\mathrm{B}^{epp}$, $\omega\mathrm{BL}^{epp}$, $\omega\mathrm{S}^{epp}$, or $\omega\mathrm{Lat}^{epp}$. Then the following are equivalent:*

(i) $(U, \leq)$ *is universal and homogeneous in* $\mathfrak{C}$.

(ii) $(U^0, \leq)$ *realizes all one-point extensions of finite subdomains.*

The main result of [7] is the following

Theorem 2.3 ([7]) *Each of the categories $\omega\mathrm{B}^{epp}$, $\omega\mathrm{BL}^{epp}$, $\omega\mathrm{S}^{epp}$, or $\omega\mathrm{Lat}^{epp}$ contains a universal and homogeneous domain. This universal and homogeneous domain is unique up to isomorphism.*

To prove this result, [7] demonstrate that the class of finite domains in $\mathfrak{C}$ has the amalgamation property. Then the result follows from a category-theoretic generalization of Fraïssé's theorem from model theory, cf. [7, Thm. 1.1].

3 The universal bifinite domain

For an introduction into our construction, we briefly recall the random construction of the universal homogeneous graph. In the following, a structure (V, R) is a *graph*, if V is a non-empty set and $R \subseteq V \times V$ is irreflexive and symmetric, i.e., graphs are loopless and undirected. *Embeddings* of graphs are one-to-one functions which both preserve and reflect the edge relation. A countable graph U is *universal*, if any countable graph can be embedded into it, and U is *homogeneous*, if each isomorphism between two finite subgraphs of U extends to an automorphism. Fraïssé [13] showed that there exists a countable universal homogeneous graph U; moreover, U is unique up to isomorphism with these properties.

Now, let us describe a probabilistic construction of this graph U. As underlying set, we take $\mathbb{N}$, the natural numbers. Choose an enumeration of all 2-subsets $S_i = \{a_i, b_i\}$ $(i \in \mathbb{N})$ of $\mathbb{N}$. Then toss a fair coin to decide whether a_i and b_i become connected by an edge or not. Since the choices are completely independent of each other, they can be made by following the enumeration, any other order, or even concurrently. Then, in any case, with probability 1 we obtain the universal homogeneous graph.

To phrase this heuristic construction more precisely, we describe the underlying probability space (cf. Erdős and Spencer [12], Cameron [4]).

We assume that our graphs have $\mathbb{N}$, the natural numbers, as underlying set. Let $\Omega = \{R \subseteq \mathbb{N} \times \mathbb{N} \mid (\mathbb{N}, R) \text{ is a graph}\}$ and $\Omega_n = \{R \subseteq A_n^2 \mid (A_n, R) \text{ is a graph}\}$

where $A_n = \{1, 2, \ldots, n\}$ ($n \in \mathbb{N}$). For $R \in \Omega_n$, let $(R, n)^\sim = \{S \in \Omega \mid S \cap A_n^2 = R\}$. Let $\mathfrak{A}$ denote the σ-algebra on Ω generated by the basic sets $(R, n)^\sim$ where $R \in \Omega_n$. Then there exists a unique probability measure μ on Ω satisfying $\mu((R, n)^\sim) = 2^{-\binom{n}{2}}$ for any $R \in \Omega_n$ as above. The main statement above says that the set $\{R \in \Omega \mid (\mathbb{N}, R)$ is a universal and homogeneous graph$\}$ has measure 1 in $(\Omega, \mathfrak{A}, \mu)$.

Next, we wish to develop our probabilistic construction of bifinite domains. We will again first proceed heuristically and afterwards indicate the precise definition of the underlying probability space.

Note that any domain $(D \leq)$ is determined uniquely up to isomorphism by the structure of its subposet $(D^0, \leq)$ of compact elements. Indeed, if $(D, \leq)$ and $(E, \leq)$ are domains and $f : (D^0, \leq) \to (E^0, \leq)$ is an isomorphism, then f extends *uniquely* to an isomorphism $\bar{f} : (D, \leq) \to (E, \leq)$ with $\bar{f}\restriction_{D^0} = f$. Hence we may describe properties of the class of all domains $(D, \leq)$ by corresponding properties for the sets $(D^0, \leq)$. We will assume that our infinite domains $(D, \leq)$ satisfy $D^0 = \mathbb{N}$, and that they are defined in a unique and uniform way (e.g., via the cpo-completion) over their poset $(\mathbb{N}, \leq)$ of compact elements.

Construction 3.1 We fix some discrete probability distribution $\nu : \mathbb{N} \to [0, 1]$ and, for each $i \in \mathbb{N}$, some probability $p_i \in [0, 1]$. We describe how to extend a finite partial order $\leq_n$ on $A_n = \{1, 2, \ldots, n\}$ probabilistically to a finite partial order $\leq_{n+1}$ on $A_{n+1} = \{1, 2, \ldots, n + 1\}$ such that $(A_n, \leq_n) \lhd (A_{n+1}, \leq_{n+1})$. For this, we only have to determine the order relations between $n + 1$ and i for each $i \in A_n$. In order to achieve that $(A_n, \leq_n) \lhd (A_{n+1}, \leq_{n+1})$, there will have to be a greatest element x in $(A_n, \leq_n)$ with $x <_{n+1} n + 1$.

We proceed as follows. We first choose some $x \in A_n$ with probability $\nu(x)$ for $x < n$ and with probability $\sum_{y \geq n} \nu(y)$ for $x = n$. Next we define a binary relation R in $\{n+1\} \times \{a \in A_n \mid x <_n a\}$ as follows. Decide independently for each $a \in A_n$ with $x <_n a$ with probability p_a that $(n + 1, a) \in R$, and with probability $1 - p_a$ that $(n + 1, a) \notin R$. Then let $\leq_{n+1}$ be the reflexive and transitive closure of $\leq_n \cup \{(x, n + 1)\} \cup R$ on A_{n+1}. It is clear that $\leq_{n+1}$ is a partial order on A_{n+1}. Since the new element $n + 1$ has a unique lower neighbor x in $(A_{n+1}, \leq_{n+1})$, we obtain $(A_n, \leq_n) \lhd (A_{n+1}, \leq_{n+1})$.

Construction 3.2 We apply the above construction iteratively yielding a sequence of finite partial orders $(A_n, \leq_n)$ with $A_n = \{1, 2, \ldots, n\}$ and

$$(A_1, \leq_1) \lhd (A_2, \leq_2) \lhd (A_3, \leq_3) \lhd \ldots$$

Then $\preceq\; = \bigcup_{n \geq 1} \leq_n$ is a partial order on $\mathbb{N}$ with minimal element 1. Clearly $(A_n, \leq_n) \lhd (\mathbb{N}, \preceq)$ for each $n \in \mathbb{N}$ and the cpo-completion of $(\mathbb{N}, \preceq)$ is an ω-bifinite domain.

We now show that, under mild assumptions on the discrete probability distribution ν and the sequence of probabilities $(p_i)_{i \in \mathbb{N}}$, the cpo-completion of this partial order $(\mathbb{N}, \preceq)$ is almost surely a universal and homogeneous ω-bifinite domain.

Theorem 3.3 *Let $\nu : \mathbb{N} \to [0, 1]$ be a discrete probability distribution such that*

$\nu(i) > 0$ for all $i \in \mathbb{N}$. Furthermore, for each $i \in \mathbb{N}$, let $p_i \in (0,1)$ be some probability such that $\sum_{i\in\mathbb{N}} p_i < \infty$. Then, with probability 1, Construction 3.2 produces a partial order $\preceq$ on $\mathbb{N}$ whose cpo-completion is a universal and homogeneous bifinite domain.

Proof. By Prop. 2.2, it suffices to show that with probability 1, the resulting partial order $(\mathbb{N}, \preceq)$ realizes all one-point extensions of finite subdomains. Note that there are, up to isomorphism, only countably many such one-point extensions $(A \leq_A) \lhd (B, \leq_B)$ with $A \subseteq \mathbb{N}$. Since the intersection of countably many events of probability 1 again has probability 1, it suffices to consider an arbitrary fixed one-point extension $(A, \leq_A) \lhd (B, \leq_B)$ with $A \uplus \{y\} = B$ and $(A, \leq_A) \lhd (A_m, \leq_m)$ where $m \in \mathbb{N}$. We claim that then with probability 1, there exists $n \in \mathbb{N}$ such that $f_{n+1} = \mathrm{id}_A \cup \{(y, n+1)\}) : (B, \leq_B) \to (A_{n+1}, \leq_{n+1})$ is a mub-embedding.

To this aim, consider any integer $n \in \{m, m+1, \dots\}$ such that $(A_n, \leq_n)$ is constructed. We wish to compute a lower bound for the probability of constructing $\leq_{n+1}$ on A_{n+1} such that f_{n+1} is a mub-embedding.

Let $x \in A$ be the greatest element of $(A, \leq)$ such that $x <_B y$. The probability that in Construction 3.1 we choose this element x from A_n is at least $\nu(x) > 0$. Now consider the relation $R = \{(n+1, a) \mid a \in A, y <_B a\} \subseteq \{n+1\} \times \{a \in A_n \mid x \leq_n a\}$. To estimate its probability, note that

$$\prod\{1 - p_a \mid a \in A_n, x <_n a, y \not<_B a\} \geq \prod\{1 - p_i \mid i \in \mathbb{N}\} > 0$$

by a standard result of analysis on infinite products, since $\sum_{i\in\mathbb{N}} p_i < \infty$. Hence the probability of choosing in our construction of $\leq_{n+1}$ the above relation R is at least

$$\begin{aligned}\prod\{p_a \mid a \in A, y <_B a\} \cdot \prod\{1 - p_a \mid a \in A_n, x <_n a, y \not<_B a\} \\ \geq \prod\{p_a \mid a \in A, y <_B a\} \cdot \prod\{1 - p_i \mid i \in \mathbb{N}\}.\end{aligned}$$

Evidently, there is a (small but) fixed $r > 0$ such that at least with probability r all these choices of x and R are done as described, and r depends only on $(A, \leq_A)$ and $(B, \leq_B)$ (but not on n).

Assume that R is constructed in this way and let $\leq_{n+1}$ be the reflexive and transitive closure of $\leq_n \cup \{(x, n+1)\} \cup R$. We demonstrate that then the mapping $f_{n+1} = \mathrm{id}_A \cup \{(y, n+1)\} : (B, \leq_B) \to (A \cup \{n+1\}, \leq_{n+1})$ is an order-isomorphism. Let $a \in A$. By the choice of x, the fact that $(A, \leq_A) \subseteq (A_n, \leq_n)$, and the definition of $\leq_{n+1}$, we obtain $a <_B y$ iff $a \leq_B x$ iff $a \leq_A x$ iff $a \leq_n x$ iff $a <_{n+1} n+1$. If $y <_B a$, we have $(n+1, a) \in R$ by construction, hence $n+1 <_{n+1} a$. Conversely, let $n+1 <_{n+1} a$. Then there is $a' \in A$ with $(n+1, a') \in R$ and $a' \leq_n a$. Hence, by the definition of R, we have $y <_B a'$ implying $y <_B a$. Hence f_{n+1} is an order-isomorphism as claimed.

To show that f_{n+1} is a mub-embedding into $(A_{n+1}, \leq_{n+1})$, it remains to see that $A \uplus \{n+1\} \lhd (A_{n+1}, \leq_{n+1})$. So let $S \subseteq A \cup \{n+1\}$ and $c \in A_{n+1} \setminus S$ with $S \leq_{n+1} c$. If $n+1 \notin S$, then $A \lhd (A_m, \leq_m) \lhd (A_n, \leq_n) \lhd (A_{n+1}, \leq_{n+1})$ implies the existence of $b \in A$ with $S \leq_{n+1} b \leq_{n+1} c$. Now suppose $n+1 \in S$. Since $n+1 <_{n+1} c$, there exists $a' \in A$ with $(n+1, a') \in R$ and $a' \leq_n c$ (and therefore $a' \leq_{n+1} c$). As above, we find $b \in A$ with $(S \setminus \{n+1\}) \cup \{a'\} \leq_{n+1} b \leq_{n+1} c$ implying $S \leq_{n+1} b \leq_{n+1} c$ since $n+1 \leq_{n+1} a'$.

Thus, the probability that f_{n+1} is a mub-embedding of $(B, \leq_B)$ into $(A_{n+1}, \leq_{n+1})$ is at least r. Hence the probability that it is not such a mub-embedding is at most $1-r$. Consequently, the probability that for *all* $n \in \{m, m+1, \dots\}$, the mapping f_{n+1} is not a mub-embedding is $(1-r)^\omega = 0$. Hence, almost surely the one-point extension is realized after some number of steps and therefore the result follows. □

Our procedure in Construction 3.1 of extending a given finite partial order by one more element depends on the underlying probability distribution $\nu : \mathbb{N} \to [0,1]$ and the sequence of probabilities $(p_i)_{i\in\mathbb{N}}$. As seen, Theorem 3.3 holds quite generally for *any* choice of these parameters $\nu, (p_i)_{i\in\mathbb{N}}$ satisfying the requirements stated in Theorem 3.3. However, we now show that a further slight generalization of this construction leads to totally different domains.

Clearly, we could formulate (and thereby generalize) Construction 3.1 by replacing ν by a sequence of discrete probability distributions $(\nu_n)_{n\in\mathbb{N}}$. Then when extending the order from A_n to A_{n+1}, we should proceed exactly as before, replacing ν by ν_n.

Now assume that the sequence $(\nu_n)_{n\in\mathbb{N}}$ satisfies $\sum_{n\in\mathbb{N}} \nu_n(1) < \infty$. That is, we just prescribe that $(\nu_n(1))_{n\in\mathbb{N}}$ converges sufficiently quickly to 0. Then, when extending the order from A_n to A_{n+1}, the probability that we choose $x = 1$ to be the greatest element below $n+1$ equals 1 if $n = 1$ (i.e., $1 <_2 2$), and $\nu_n(1)$ if $n \geq 2$. That is, if $n \geq 2$, with probability $1 - \nu_n(1)$ we obtain $1 <_{n+1} i <_{n+1} n+1$ for some $2 \leq i \leq n$. Consequently, with probability $p = \prod_{n\geq 2}(1 - \nu_n(1))$, for any $n \geq 2$, there is some $i \leq n$ with $1 <_{n+1} i <_{n+1} n+1$. In particular, with probability p, we have $2 \preceq n+1$ for all $n \geq 2$, for otherwise we could choose the least natural number $n \geq 2$ such that $2 \not\preceq n+1$, and then $n+1$ would be minimal above 1 in $(A_{n+1}, \leq_{n+1})$, a contradiction. Again we have $p > 0$ by the assumption that $\sum_{n\in\mathbb{N}} \nu_n(1) < \infty$.

This sharply contrasts the situation for any universal and homogeneous bifinite domain $(U, \leq)$. There, for any $a \in U^0$ with $\perp < a$ there is some $z \in U^0$ satisfying $\perp < z < a$. For, consider the one-point extension $(A, \leq_A) \lhd (B, \leq_B)$ with $A = \{\perp, a\}$, $B = \{\perp, y, a\}$ and $\perp <_B y <_B a$ and apply Prop. 2.2 to obtain our claim. Hence the above construction produces with positive probability a domain which is certainly not universal and homogeneous. A further analysis of the structure of the domains obtained in these more general situations seems to require intricate combinatorial arguments.

We now describe the probability space underlying Theorem 3.3 more formally. A *bifinite partial order* is a partial order $\preceq$ on some initial segment A of the natural numbers such that, for any $1 \leq n < |A|$, we have $\{1, 2, \dots, n\} \lhd (A, \preceq)$. For $n \in \mathbb{N}$, let $\mathfrak{B}_n$ denote the set of bifinite partial orders $\preceq$ on $\{1, 2, \dots, n\}$ and let $\mathfrak{B}$ denote the set of bifinite partial orders $\preceq$ on $\mathbb{N}$.

We wish to define discrete probability distributions μ_n on each $\mathfrak{B}_n$ $(n \in \mathbb{N})$.

Let $n \in \mathbb{N}$ and assume that we have defined μ_n on B_n. Let $\leq_n \in \mathfrak{B}_n$. In Construction 3.1, we extend $\leq_n$ to an order $\leq_{n+1} \in \mathfrak{B}_{n+1}$. Our probabilistic procedure thus determines the intuitive conditional probability $P(\leq_{n+1}|\leq_n)$ of constructing $\leq_{n+1}$ given its restriction $\leq_n$ to A_n. Putting $\mu_1(\leq_1) = 1$ and, inductively, $\mu_{n+1}(\leq_{n+1}) = P(\leq_{n+1}|\leq_n) \cdot \mu_n(\leq_n)$ for any $\leq_{n+1} \in \mathfrak{B}_{n+1}$ and $\leq_n = \leq_{n+1} \cap A_n^2$, we obtain a discrete distribution μ_{n+1} on $\mathfrak{B}_{n+1}$ such that now $P(\leq_{n+1}|\leq_n)$ becomes the exact conditional probability of $\leq_{n+1}$ given $\leq_{n+1} \cap A_n^2 = \leq_n$. Inductively, for any $\leq_{n+1} \in \mathfrak{B}_{n+1}$ we obtain $\mu_{n+1}(\leq_{n+1}) = \prod_{1 \leq m \leq n} P(\leq_{m+1}|\leq_m)$ where $\leq_m = \leq_{n+1} \cap A_m^2$ for $1 \leq m \leq n+1$.

Also note that there is a natural mapping $g_n : \mathfrak{B}_{n+1} \to \mathfrak{B}_n$ which maps $\leq_{n+1}$ to its (uniquely determined) restriction $\leq_{n+1} \cap A_n^2$ on A_n. As is easily verified, for any $\leq_n \in \mathfrak{B}_n$ we have $\sum\{P(\leq_{n+1}|\leq_n) \mid \leq_{n+1} \in \mathfrak{B}_{n+1}, \leq_{n+1} \cap A_n^2 = \leq_n\} = 1$, and thus $\mu_n(\leq_n) = \sum\{\mu_{n+1}(\leq_{n+1}) \mid \leq_{n+1} \in \mathfrak{B}_{n+1}, g_n(\leq_{n+1}) = \leq_n\}$. Hence we obtain a projective system (in the natural way) whose projective limit has the set $\mathfrak{B}$ of all bifinite partial orders on $\mathbb{N}$ as underlying set.

For each bifinite partial order $\preceq$ on $\mathbb{N}$ and $n \in \mathbb{N}$, let $\pi_n(\preceq)$ denote its restriction to the set A_n. Then the Borel σ-algebra $\mathfrak{A}$ on the projective limit is generated by the collection $\{\pi_n^{-1}(\leq_n) \mid n \in \mathbb{N}, \leq_n \in \mathfrak{B}_n\}$. A weak version of the Prokhorov extension theorem (cf. [1, Section 3.1] for a lucid short introduction into the background) asserts that there exists a unique probability measure μ on $\mathfrak{A}$ satisfying $\mu(\pi_n^{-1}(\leq_n)) = \mu_n(\leq_n)$ for all $n \in \mathbb{N}$ and $\leq_n \in \mathfrak{B}_n$.

Note that, given $n \in \mathbb{N}$ and $\leq_n \in \mathfrak{B}_n$, $\pi_n^{-1}(\leq_n)$ is the set of those bifinite orders on $\mathbb{N}$ to which $\leq_n$ can be extended; thus the probability of this generating set in $\mathfrak{A}$ equals $\mu_n(\leq_n)$, the probability of having constructed $\leq_n$ on A_n (in finitely many steps), as is intuitively expected. Therefore the probability space $(\mathfrak{B}, \mathfrak{A}, \mu)$ is the space of all possible orders that we consider.

Now we assume that for each $\preceq \in \mathfrak{B}$ we have a uniquely defined cpo-completion $(D, \leq)$ of $(\mathbb{N}, \preceq)$; then $(D, \leq)$ is an ω-bifinite domain. Let $\overline{\mathfrak{B}}$ comprise all these cpo-completions of bifinite partial orders $(\mathbb{N}, \preceq)$. Clearly, we can define a probability measure $\overline{\mu}$ on $\overline{\mathfrak{B}}$ making $(\overline{\mathfrak{B}}, \overline{\mathfrak{A}}, \overline{\mu})$ naturally isomorphic to $(\mathfrak{B}, \mathfrak{A}, \mu)$. We regard $(\overline{\mathfrak{B}}, \overline{\mathfrak{A}}, \overline{\mu})$ as our probability space of ω-bifinite domains. Now the result of Theorem 3.3 means that in $(\overline{\mathfrak{B}}, \overline{\mathfrak{A}}, \overline{\mu})$, the set $\mathcal{U} = \{(D, \leq) \in \overline{\mathfrak{B}} \mid (D, \leq)$ is universal and homogeneous$\}$ has measure 1. We just note here that it can be shown (e.g. by Theorem 5.3 below) that $\mathcal{U}$ is measurable in $\overline{\mathfrak{B}}$. Now we show:

Corollary 3.4 *In the probability space $(\overline{\mathfrak{B}}, \overline{\mathfrak{A}}, \overline{\mu})$, almost any domain $(D, \leq)$ embeds its own function space $[D \to D]$, but almost no domain $(D, \leq)$ is isomorphic to $[D \to D]$.*

Proof. Let $\mathcal{U}$ be the collection of universal homogeneous ω-bifinite domains in $\overline{\mathfrak{B}}$. If $(D, \leq)$ is any universal ω-bifinite domain, then $[D \to D]$ is again ω-bifinite, hence embeds into $(D, \leq)$. Hence the set of domains in $\overline{B}$ which embed their own function space contains $\mathcal{U}$ which has probability 1 by Theorem 3.3. On the other hand, the set of domains in $\overline{\mathfrak{B}}$ isomorphic to their own function space is disjoint from $\mathcal{U}$ [9] and thus has measure 0. $\square$

We remark that the proof of Corollary 3.4 shows the following. Let P be a property of domains which is isomorphism closed, i.e. if $(D, \leq)$, $(D', \leq')$ are two isomorphic domains and $(D, \leq)$ satisfies P, then so does $(D', \leq')$. Then either almost any domain in $\overline{\mathfrak{B}}$ has property P, or else almost no domain in $\overline{\mathfrak{B}}$ has property P; the first alternative holds iff the universal homogeneous ω-bifinite domains satisfy P.

4 Further universal domains

In this section, we wish to construct universal homogeneous domains in the categories $\omega\mathrm{Lat}^{epp}$, $\omega\mathrm{S}^{epp}$ and $\omega\mathrm{BL}^{epp}$. First, we need a few order-theoretic preparations.

A poset $(Q, \leq)$ is *complete* if any of its subsets has a supremum. It is *Dedekind-complete* if any of its upper bounded non-empty subsets has a supremum; equivalently each non-empty lower bounded subset of Q has an infimum in Q. Finally, $(Q, \leq)$ is *L-complete* if any of its upper bounded non-empty subsets has an infimum.

Furthermore, let $(C, \leq)$ be a complete poset and $Q \subseteq C$. Then Q is *Dedekind-closed in* $(C, \leq)$ if, for any $X \subseteq Q$ and $x \in Q$ with $X \leq x$, we have $\sup_{(C,\leq)} X \in Q$. Further, Q is *L-closed in* $(C, \leq)$ if, for any $X \subseteq Q$ and $x \in Q$ with $X \leq x$, we have $\inf_{(C,\leq)} X \in Q$. Recall that the MacNeille-completion $(C, \leq)$ of $(P, \leq)$ is a complete lattice containing P such that the identity mapping $\mathrm{id}_P : P \to C$ preserves all suprema and infima existing in $(P, \leq)$, cf. [3].

Definition 4.1 Let $(P, \leq)$ be a poset and $(C, \leq)$ its MacNeille-completion. Its *Dedekind-completion* (*L-completion*, resp.) is the smallest Dedekind-closed (L-closed) subset of $(C, \leq)$ that contains P.

Note that the intersection of Dedekind-closed subsets of $(C, \leq)$ exists and is Dedekind-closed, again. The same holds for L-closed subsets. Hence the Dedekind- and L-completions of a poset exist. It is easy to see that the Dedekind-completion of $(P, \leq)$ consists of all elements $\sup_{(C,\leq)} X \in C$ where $X \subseteq P$ has an upper bound in P. A similar characterization of the L-completion requires an iterative procedure of adding necessary infima (see proof below). Since the MacNeille-completion of a finite poset is finite, again, so are the Dedekind- and L-completions.

Lemma 4.2 *Let $(P, \leq)$ be a poset with smallest element, and let $A \lhd (P, \leq)$.*

(a) If $(A, \leq)$ is complete, then $(A, \leq)$ is bounded-complete in the MacNeille-completion of $(P, \leq)$.

(b) If $(A, \leq)$ is Dedekind-complete, then it is bounded-complete in the Dedekind-completion of $(P, \leq)$.

(c) If $(A, \leq)$ is L-complete, then it is bounded-complete in the L-completion of $(P, \leq)$.

Proof. The proof is similar but more general than the argument used for [7, Prop. 3.6]. Let $(C, \leq)$ be the MacNeille-completion of $(P, \leq)$.

(a) Assume that $(A, \leq)$ is complete. Let $X \subseteq A$ be finite and $c \in C$ with $X \leq c$.

By assumption, there exists $s = \sup_{(A,\leq)} X \in A$. Then $s = \sup_{(P,\leq)} X$ by $A \lhd (P, \leq)$, and thus also $s = \sup_{(C,\leq)} X$ since $(C, \leq)$ is the MacNeille-completion of $(P, \leq)$. Hence $X \leq s \leq c$ proving $A \lhd (C, \leq)$.

(b) Let $(A, \leq)$ be Dedekind-complete and let $(Q, \leq)$ be the Dedekind-completion of $(P, \leq)$. Let $X \subseteq A$ be finite and $q \in Q$ with $X \leq q$. The above characterization of the Dedekind-completion $(Q, \leq)$ of $(P, \leq)$ implies that there exists $p \in P$ with $q \leq p$. From $A \lhd (P, \leq)$ we now obtain some $a \in A$ with $X \leq a \leq p$. Since $(A, \leq)$ is Dedekind-complete, there exists $s = \sup_{(A,\leq)} X \in A$ and, by $A \lhd (P, \leq)$, we have $s = \sup_{(P,\leq)} X$. Since $(C, \leq)$ is the MacNeille-completion of $(P, \leq)$, this implies $s = \sup_{(C,\leq)} X$ and therefore $X \leq s \leq q$, hence $A \lhd (Q, \leq)$.

(c) Let $(A, \leq)$ be L-complete and let $(R, \leq)$ be the L-completion of $(P, \leq)$. First, we assume that P is finite. Let $n = |R|$. We define an ascending chain of subsets $R_1 \subseteq R_2 \subseteq \cdots \subseteq R_n \subseteq R$ as follows. We put $R_1 = P$. Now assume that $1 \leq j < n$ and that R_j is already defined. Then let

$$R_{j+1} = \{y \in R \mid \exists S \subseteq R_j, z \in R_j : S \leq z \text{ and } y = \inf_{(C,\leq)} S\}.$$

Since $R = |n|$, it follows that $R_{n-1} = R_n$ is an L-closed subset of $(C, \leq)$ that contains P. Because of the minimality of $(R, \leq)$, it follows that $R_n = R$.

By assumption, we have $A \lhd R_1$. Now let $1 \leq j < n$ and assume that $A \lhd R_j$. We claim that $A \lhd R_{j+1}$. For this, let $X \subseteq A$ and $y \in R_{j+1}$ with $X \leq y$. Choose $S \subseteq R_j$ and $z \in R_j$ such that $S \leq z$ and $y = \inf_{(C,\leq)} S$. For each $s \in S$, we have $X \leq y \leq s$, hence by $A \lhd R_j$, there exists $a_s \in A$ with $X \leq a_s \leq s \leq z$. Again by $A \lhd R_j$, we obtain $\{a_s \mid s \in S\} \leq a' \leq z$ for some $a' \in A$. Since $(A, \leq)$ is L-complete, there exists $x = \inf_{(A,\leq)}\{a_s \mid s \in S\} \in A$. Then $X \leq x \leq s$ for each $s \in S$, thus, $X \leq x \leq y$ and our claim $A \lhd R_{j+1}$ follows. By induction, we obtain $A \lhd R_n = P$ as needed.

The case where P is infinite can be treated similarly as above by transfinite induction (cf. [7, proof of Prop. 3.6(d)]). But since only the finite case will be needed here, we leave the infinite case to the reader.

□

Now we wish to derive analogues of Constructions 3.1 and 3.2 for lattices, Scott-domains, and L-domains, respectively. This consists of Construction 3.1 followed by a suitable completion process, and then we iterate this procedure.

Construction 4.3 Let $\nu : \mathbb{N} \to [0, 1]$ be some discrete probability distribution and, for $i \in \mathbb{N}$, let $p_i \in [0, 1]$. We describe three ways to probabilistically extend a finite partial order $(A, \leq)$ with $A = \{1, 2, \ldots, k\}$ into a finite partial order $(C, \leq)$ that will eventually lead to the universal homogeneous (a) ω-algebraic lattice, (b) ω-Scott domain, and (c) ω-bifinite L-domain.

The first step is to apply Construction 3.1 to obtain probabilistically a partial order $(B, \leq)$ with $B = \{1, 2, \ldots, k + 1\}$. Secondly, let $(C, \leq)$ be the

(a) MacNeille-completion

(b) Dedekind-completion

(c) L-completion

of $(B, \leq)$, respectively. Recall that $(A, \leq) \lhd (B, \leq)$ as explained in Construction 3.1. If $(A, \leq)$ is (a) a lattice, (b) a Scott domain, or (c) an L-domain, then Lemma 4.2 implies $(A, \leq) \lhd (C, \leq)$ for the respective completion $(C, \leq)$ of $(B, \leq)$. Moreover, since C is finite, we may assume (by some systematic renaming of the elements of $C \setminus B$) that $C = \{1, \dots, |C|\}$.

Construction 4.4 We apply this construction iteratively yielding a sequence of finite partial orders $(A_n, \leq_n)$ with $A_n = \{1, 2, \dots, k_n\}$ for some $k_n \in \mathbb{N}$ with $k_n < k_{n+1}$ for all n and

$$(A_1, \leq_1) \lhd (A_2, \leq_2) \lhd (A_3, \leq_3) \lhd \dots$$

Then $\preceq = \bigcup_{n \geq 1} \leq_n$ is a partial order on $\mathbb{N}$ with minimal element 1 and $(A_n, \leq_n) \lhd (\mathbb{N}, \preceq)$ for each $n \in \mathbb{N}$. Furthermore, the cpo-completion of $(\mathbb{N}, \preceq)$ is (a) a lattice, (b) a Scott domain, or (c) an L-domain, respectively.

We now show that, under mild assumptions on the discrete probability distribution ν and the sequence of probabilities $(p_i)_{i \in \mathbb{N}}$, the cpo-completion of this partial order $(\mathbb{N}, \preceq)$ is almost surely a universal and homogeneous ω-algebraic lattice, ω-Scott domain, ω-bifinite L-domain, respectively.

Theorem 4.5 *Let $\nu : \mathbb{N} \to [0, 1]$ be a discrete probability distribution such that $\nu(i) > 0$ for all $i \in \mathbb{N}$. Furthermore, for each $i \in \mathbb{N}$, let $p_i \in (0, 1)$ be some probability such that $\sum_{i \in \mathbb{N}} p_i < \infty$.*

Then, with probability 1, Construction 4.4 produces a partial order $\preceq$ on $\mathbb{N}$ whose cpo-completion is a universal and homogeneous

(a) ω-algebraic lattice,

(b) ω-Scott domain,

(c) ω-bifinite L-domain, respectively.

Proof. The proof is analogous for all these three statements and actually follows the line of the proof of Theorem 3.3. We only sketch the necessary changes.

Let $(\mathbb{N}, \preceq)$ be any outcome of Construction 4.4. Then there are $1 = k_1 < k_2 < k_3 < \dots$ such that if $A_n = \{1, 2, \dots, k_n\}$ then one application of Construction 4.3 yields $(A_{n+1}, \preceq)$ from $(A_n, \preceq)$, and $A_n \lhd A_{n+1} \lhd (\mathbb{N}, \preceq)$.

Let $(A, \leq_A) \lhd (B, \leq_B)$ be a one-point extension in $\omega\mathrm{Lat}^{epp}$ $(\omega\mathrm{S}^{epp}, \omega\mathrm{BL}^{epp}$, respectively) with $B = A \uplus \{y\}$ and $(A, \leq_A) \lhd (A_m, \leq_m)$ where $m \in \mathbb{N}$. As in the proof of Theorem 3.3, it suffices to show that this one-point extension is realized with probability 1 in $(\mathbb{N}, \preceq)$. As in the proof of Theorem 3.3, let $n \in \{m, m+1, \dots\}$ and consider $(A_n, \leq_n)$. Then there is some fixed $r > 0$ such that, with probability at least r, the first step of Construction 4.3 (i.e., Construction 3.1) results in a poset $(A_n \cup \{k_n + 1\}, \leq'_n)$ that embeds $(B, \leq_B)$ via the mub-embedding $f_{n+1} = \mathrm{id}_A \cup \{(y, k_n + 1)\}$. Then $(A_{n+1}, \leq_{n+1})$ is the MacNeille-completion (Dedekind-completion, L-completion, respectively) of $(A_n \cup \{k_n + 1\}, \leq'_n)$ and $(A_n, \leq_n) \lhd (A_{n+1}, \leq_{n+1})$. By Lemma 4.2, the mapping f_{n+1} is also a mub-embedding of $(B, \leq_B)$ into $(A_{n+1}, \leq_n)$. Hence, also with probability at least r, Construction 4.3 yields a poset $(A_{n+1}, \leq_{n+1})$ that realizes the one-point extension $(A, \leq_A) \lhd (B, \leq_B)$. The remaining arguments can be taken from the proof of Theorem 3.3. $\square$

To indicate the precise probability spaces underlying the above theorem, we can proceed similarly to the case of Theorem 3.3. We sketch the case of the universal homogeneous ω-Scott domain.

A *Scott partial order* is a partial order $\preceq$ on some initial segment A of $\mathbb{N}$ such that there exist $1 = k_1 < k_2 < \cdots < k_n = |A|$ if A is finite or $1 = k_1 < k_2 < \ldots$ if $A = \mathbb{N}$ such that for any $1 \leq i (< n)$

$$\{1, 2, \ldots, k_i\} \lhd (\{1, 2, \ldots, k_i + 1\}, \preceq) \text{ and } (\{1, 2, \ldots, k_{i+1}\}, \preceq) \text{ is the Dedekind-completion of } (\{1, 2, \ldots, k_i + 1\}, \preceq). \quad (*)$$

If A is finite and $k_n = |A|$, we call n the *Dedekind-length of* $(A, \preceq)$. For $n \in \mathbb{N}$, let $\mathfrak{S}_n$ denote the set of all finite Scott partial orders of Dedekind-length n, and let $\mathfrak{S}$ be the set of all infinite Scott partial orders. Now the definition of the probability distributions on $\mathfrak{S}_n$ and $\mathfrak{S}$ follow the same line as those in Section 3 for $\mathfrak{B}_n$ and $\mathfrak{B}$, and we define the probability space $(\overline{\mathfrak{S}}, \overline{\mathfrak{A}}, \overline{\mu})$ of ω-Scott domains again as isomorphic to the one on $\mathfrak{S}$.

The following corollary can be shown in the same way as we proved Cor. 3.4.

Corollary 4.6 *In the probability space $(\overline{\mathfrak{S}}, \overline{\mathfrak{A}}, \overline{\mu})$, almost any ω-Scott domain $(D, \leq)$ embeds its own function space $[D \to D]$, but almost no ω-Scott domain $(D, \leq)$ is isomorphic to $[D \to D]$.*

For the ω-algebraic lattices and ω-bifinite L-domains, we can proceed very similarly, replacing in $(*)$ above the Dedekind-completion by the MacNeille-completion and the L-completion, respectively.

5 Topological interpretation of "almost"

In this section, we wish to endow the collection of all bifinite domains $(D, \leq)$ with $D^0 = \mathbb{N}$ with a metric, and we will show that in this metric space, the collection of all universal homogeneous bifinite domains forms a "large" subset in a topological sense. We first recall some basic notions from topology, cf. e.g. [19]. Let (X, d) be a complete metric space. A subset S of X is *open* if for each $s \in S$ there is $\varepsilon > 0$ such that the ε-ball $B_\varepsilon(s) = \{x \in X \mid d(s, x) < \varepsilon\}$ of s is contained in S. The set S is *dense* if its closure equals X, i.e., S meets every non-empty open set. A subset R is *residual* if R contains the intersection of countably many open dense sets (in equivalent terminology, R is the complement of a meagre set).

Proposition 5.1 (Baire category theorem) *A residual set in a complete metric space is dense.*

Hence, if we can show that the collection of all universal homogeneous domains forms a residual subset of some complete metric space, it follows that there *exists* such a domain. In fact, a residual set R is considered as "large" containing "almost all" of the space; also the intersection of countably many residual sets is again residual. We refer the reader to [4] for applications of this in algebra.

Now we construct a metric space (X, d) as follows. For each $n \geq 1$, let again $A_n = \{1, 2, \dots, n\}$. Let

$$X = \{R \mid R \subseteq \mathbb{N}^2\}$$

and set $d(R_1, R_2) = 2^{-n}$ where $n = \sup\{m \in \mathbb{N} \mid R_1 \cap A_m^2 = R_2 \cap A_m^2\}$ with $2^{-\infty} = 0$. The topology on X is determined by the basic open sets $\{R \in X \mid R \cap A_n^2 = F\}$ for finite sets $F \subseteq A_n$ and $n \in \mathbb{N}$. Then it is well-known that (X, d) is a complete metric space, and the topology of X is compact.

As before, we will call a partial order $\preceq$ on $\mathbb{N}$ *bifinite* if $A_n \lhd (\mathbb{N}, \preceq)$ for each $n \geq 1$. Thus we have $A_1 \lhd A_2 \lhd \cdots \lhd (\mathbb{N}, \preceq)$. We note that this will not be an essential restriction since for any ω-bifinite domain $(D, \leq)$, we can enumerate $D^0 = \{d_i \mid i \geq 1\}$ with $d_1 = \min(D^0, \leq)$ such that $\{d_i \mid 1 \leq i \leq n\} \lhd D^0$ for each $n \geq 1$. Now let

$$P_{\text{bif}} = \{\preceq \in X \mid \preceq \text{ is a bifinite partial order on } \mathbb{N}\}.$$

For any $R \in X$, we have $R \notin P_{\text{bif}}$ iff for some $n \geq 1$, the restriction $R \cap A_n^2$ is not a bifinite partial order on the finite set A_n. Hence $X \setminus P_{\text{bif}}$ is a union of basic open sets and thus open, showing that P_{bif} is a closed subset of X. Hence (P_{bif}, d) is again a complete metric space. If follows from general results that (P_{bif}, d) is homeomorphic to the Cantor set.

If $\preceq$ is a bifinite partial order on $\mathbb{N}$, we let $D(\preceq)$ denote the (up to isomorphism uniquely determined) bifinite domain with $(D(\preceq)^0, \leq) = (\mathbb{N}, \preceq)$. Now, for each one-point extension $((A, \leq_A), (B, \leq_B))$ in $\omega\mathrm{B}^{epp}$ with $A \subseteq \mathbb{N}$ finite, we introduce a set $S_{A,B} \subseteq P_{\text{bif}}$:

$$\begin{aligned} S_{A,B} = \{\preceq \in P_{\text{bif}} \mid \text{ if } A \lhd (\mathbb{N}, \preceq), \text{ then there exists a mub-embedding} \\ g : (B, \leq_B) \to (\mathbb{N}, \preceq) \text{ such that } g \restriction_A = \mathrm{id}_A\}. \end{aligned}$$

Lemma 5.2 *Let $((A, \leq_A), (B, \leq_B))$ be a one-point extension in $\omega\mathrm{B}^{epp}$ with $A \subseteq \mathbb{N}$ finite. Then $S_{A,B}$ is an open and dense subset of (P_{bif}, d).*

Proof. First we show that $S_{A,B}$ is open. It is easily seen that $S_{A,B}$ is the union of the following open sets

- $\{\preceq \in P_{\text{bif}} \mid$ there is a mub-embedding $g : (B, \leq_B) \to (A_n, \preceq \cap A_n^2)$ with $g\restriction_A = \mathrm{id}_A\}$ where $n \in \mathbb{N}$,
- $\{\preceq \in P_{\text{bif}} \mid$ not $(A, \leq_A) \lhd (A_m, \preceq \cap A_m^2)\}$ where $m \in \mathbb{N}$ is minimal with $A \subseteq A_m$.

Hence $S_{A,B}$ is a union of open sets and therefore open.

Second, we show that $S_{A,B}$ is dense, i.e., intersects any basic open set. So let $(C, \leq_C)$ be some finite partial order with $C \subseteq \mathbb{N}$. Let V be the basic open set determined by $\leq_C$. We claim that $S_{A,B} \cap V \neq \emptyset$. If there is no bifinite partial order $\preceq$ such that $(A, \leq_A), (C, \leq_C) \lhd (\mathbb{N}, \preceq)$, then $V \subseteq S_{A,B}$. Otherwise, let $\preceq \in P_{\text{bif}}$ be a bifinite partial order such that $(A, \leq_A), (C, \leq_C) \lhd (\mathbb{N}, \preceq)$.

Since A and C are finite, there exists $n \in \mathbb{N}$ with $(A, \leq_A), (C, \leq_C) \lhd (A_n, \leq_n)$ where $\leq_n = \preceq \cap A_n^2$. It suffices to show that $S_{A,B}$ intersects the basic open set determined by $\leq_n$, i.e., that $S_{A,B}$ contains some bifinite partial order $\preceq'$ with $(A_n, \leq_n) \lhd (\mathbb{N}, \preceq')$. For this, let $B = A \cup \{y\}$ and let x be the greatest element of $(A, \leq_A)$ with $x <_B y$. Then define $\leq_{n+1} \subseteq A_{n+1}^2$ as the reflexive and transitive closure of

$$\leq_n \cup \{(x, n+1)\} \cup \{(n+1, a) \mid a \in A, y <_B a\}.$$

As in the proof of Theorem 3.3, $(A_n, \leq_n) \lhd (A_{n+1}, \leq_{n+1})$ and the mapping $g = \mathrm{id}_A \cup \{(y, n+1)\} : (B, \leq_B) \to (A_{n+1}, \leq_{n+1})$ is a mub-embedding. Now extend $\leq_{n+1}$ to a bifinite partial order, denoted $\preceq'$, on $\mathbb{N}$. Then g is a mub-embedding of $(B, \leq_B)$ into $(\mathbb{N}, \preceq')$, i.e., $\preceq' \in S_{A,B}$ and $(A, \leq_A) \lhd (A_{n+1}, \leq_{n+1}) \lhd (\mathbb{N}, \preceq')$ as needed. □

Theorem 5.3 *In the metric space (P_{bif}, d), the subset*

$$S = \{\preceq \in P_{\mathrm{bif}} \mid D(\preceq) \textit{ is universal and homogeneous}\}$$

is residual.

Proof. We claim that S contains the intersection of all sets $S_{A,B}$ where $(A, \leq_A) \lhd (B, \leq_B)$ is a one-point extension in $\omega\mathrm{B}^{epp}$. Indeed, if $\preceq$ belongs to this intersection, $(\mathbb{N}, \preceq)$ realizes all one-point extensions of finite subdomains, hence $D(\preceq)$ is universal and homogeneous by Prop. 2.2. Clearly, there are just countably many such sets $S_{A,B}$, proving that S is residual. □

To derive a similar result for ω-algebraic lattices, ω-Scott domains, and ω-bifinite L-domains, one can proceed very similarly, combining it with arguments of Section 4. We only sketch the case of ω-Scott domains. Recall the definition of Scott partial orders from section 4. Let

$$P_{\mathrm{Scott}} = \{\preceq \in X \mid \preceq \text{ is a Scott partial order on } \mathbb{N}\}.$$

This is a closed subset of X. Now follow the previous argument, replacing P_{bif} by P_{Scott}; for the corresponding density claim of Lemma 5.2 apply the construction given in the proof of Theorem 4.5. It follows that in the metric space (P_{Scott}, d), the subset

$$S = \{\preceq \in P_{\mathrm{Scott}} \mid \text{ the cpo-completion of } (\mathbb{N}, \preceq) \text{ is universal and homogeneous}\}$$

is residual.

6 A number-theoretic representation

As is well-known [22], there is also an explicit number-theoretic representation of the universal homogeneous graph: as underlying set, take the natural numbers $\mathbb{N}$. For $i, j \in \mathbb{N}$, let i and j be connected by an edge iff 2^i occurs in the unique expansion of j as a sum of distinct powers of 2. Then the graph obtained realizes all one-point extensions, hence is universal and homogeneous.

Now we wish to derive a similar representation for the compact elements of the universal and homogeneous ω-bifinite domain. Again, the situation is more complicated than for graphs since we have to stay inside the class of bifinite posets. We proceed inductively and similarly to the probabilistic construction.

For this, let again $A_n = \{1, \ldots, n\}$ ($n \in \mathbb{N}$). Let $n \in \mathbb{N}$ and assume that $(A_n, \leq_n)$ is a domain with least element 1. We wish to define the order $\leq_{n+1}$ on A_{n+1} such that $(A_n, \leq_n) \lhd (A_{n+1}, \leq_{n+1})$. We consider the unique ternary expansion of $n+1$ as a sum of distinct powers of 3 with coefficients in $\{0, 1, 2\}$. Choose the smallest number $x \in A_n$ (in the natural order on $\mathbb{N}$) such that 3^x occurs in this expansion of $n+1$ with coefficient 0. Also let R denote the set of pairs $(n+1, a)$ with $a \in A_n$, $x <_n a$, and such that 3^a occurs in the expansion of $n+1$ with coefficient 2. Then let $\leq_{n+1}$ be the reflexive and transitive closure of $\leq_n \cup \{(x, n+1)\} \cup R$. Since x is the unique lower neighbor of $n+1$, we have $(A_n, \leq_n) \lhd (A_{n+1}, \leq_{n+1})$. Then we put $\preceq = \bigcup_{n \in \mathbb{N}} \leq_n$. Then the cpo-completion $(U, \leq)$ of $(\mathbb{N}, \preceq)$ is a bifinite domain.

Theorem 6.1 *The bifinite domain $(U, \leq)$ is universal and homogeneous.*

Proof. By Prop. 2.2, it suffices to show that $(\mathbb{N}, \preceq)$ realizes all one-point extensions of finite subdomains. So let $((A, \leq_A), (B, \leq_B))$ be a one-point extension with $(A, \leq_A) \lhd (\mathbb{N}, \preceq)$ finite and $B = A \uplus \{y\}$. Let $x \in A$ be the greatest element of $(A, \leq_A)$ with $x <_B y$. Choose $m, n \in \mathbb{N}$ with $A \subseteq A_{m-1}$ and

$$n + 1 = \sum_{i=1}^{x-1} 3^i + \sum_{a \in A, y <_B a} 2 \cdot 3^a + 3^m.$$

Then $A \lhd A_{m-1} \lhd A_n$. Observe that if $a \in A$ and $y <_B a$, then also $x <_B a$ and thus $x <_A a$ showing $x <_n a$. Hence $\leq_{n+1}$ is the reflexive and transitive closure of $\leq_n \cup \{(x, n+1)\} \cup R$ with $R = \{(n+1, a) \mid a \in A, y <_B a\}$. The proof of Theorem 3.3 shows that the mapping $f_{n+1} = \mathrm{id}_A \cup \{(y, n+1)\}$ is a mub-embedding of $(B, \leq_B)$ into $(A_{n+1}, \leq_{n+1})$, and the result follows. □

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