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Sigma Coloring on Powers of Paths and Some Families of Snarks

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Abstract

Consider a vertex coloring of a graph where each color is represented by a natural number. The color sum of a vertex is the sum of the colors of its adjacent vertices. The Sigma Coloring Problem concerns determining the sigma chromatic number of a graph G, $\sigma(G)$, which is the least number of colors for a coloring of G such that the color sum of any two adjacent vertices are different. In this article, we proved that $\sigma(P_n^k) \leq 3$ when $2 \leq k \leq \frac{n}{3} - 1$, and we determined the sigma chromatic number for P_n^k in the remaining cases. This article also presents the sigma chromatic number for Blanuša 1st and 2nd families of snarks, Flower snarks, Goldberg and Twisted Goldberg snarks.

Keywords: Sigma Coloring, Power of Path, Snark, Vertex Coloring

1 Introduction

A coloring of a graph G is an assignment of colors for the vertices of G. In this work, the colors are represented by natural numbers, $c : V(G) \to \mathbb{N}$ is a vertex coloring of a graph G, and $c(v)$ denotes the color of a vertex v. If any two adjacent vertices u and v have $c(u) \neq c(v)$, then c is a proper vertex coloring of G.

Consider a non-proper vertex coloring of G. The color sum of a vertex v, denoted $\sigma(v)$, is the sum of the colors of all vertices adjacent to v. If for every two adjacent vertices u and v, $\sigma(u) \neq \sigma(v)$, then the coloring is a sigma coloring of G. The

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minimum number of colors for a sigma coloring of a graph G is called the sigma chromatic number of G and it is denoted $\sigma(G)$. The Sigma Coloring Problem is, given a graph G, to determine the sigma chromatic number of G.

The Sigma Coloring Problem was introduced by Chartrand and Zhang [1] in 2008 as a study project. In 2010 Chartrand et al. presented the first paper with results to this problem [2], determining the sigma chromatic number for complete graphs, cycles, and complete r-partite graphs with $r \geq 2$. In the same work, [2] proved that for any graph G, $\sigma(G) \leq \chi(G)$, where $\chi(G)$ is the least number of colors to a proper vertex coloring of G. To the best of our knowledge, there are few other works on the Sigma Coloring Problem. Dehghan et al. [3] showed that to decide if a graph G has $\sigma(G) = s$, for any fixed $s \geq 3$, is an NP-Complete Problem. Dehghan et al. [3] also proved that to decide if a cubic graph G has $\sigma(G) = 2$ is an NP-complete Problem. For circulant graphs, Luzon et al. [7] determined the sigma chromatic number for $C_n(1,2)$, $C_n(1,3)$, $C_{2n}(1,n)$.

Given a graph G, the degree of a vertex v, $deg(v)$, is the number of vertices that are adjacent to v in G and the maximum degree of G, $\Delta(G)$, is the greatest degree of a vertex of G. Note that $\sigma(G) = 1$ if and only if $deg(u) \neq deg(v)$ for any pair of adjacent vertices u and v in G. So, any path graph with n vertices, P_n, has $\sigma(P_n) \geq 2$ when $n \geq 2$ and $n \neq 3$. Since Chartrand et al. [2] proved that for any graph G, $\sigma(G) \leq \chi(G)$ and it is a well known result that $\chi(P_n) = 2$ for $n \geq 2$, then $\sigma(P_n) = 2$ when $n \geq 2$ and $n \neq 3$. By inspection, one can check that $\sigma(P_3) = 1$.

A power of a path, P_n^k, with $n > 0$ and $0 < k < n$, is a simple graph with n vertices $v_0, v_1, \ldots, v_{n-1}$ such that $(v_i, v_j) \in E(G)$ if, and only if, $0 < |i-j| \leq k$. Note that the path graph with n vertices is the graph P_n^1, for which the sigma chromatic number is known. Moreover, the complete graphs are the powers of paths P_n^{n-1}, for which Chartrand et al. [2] showed $\sigma(P_n^{n-1}) = n$. Considering these results, this work expands the knowledge on the sigma chromatic number of powers of paths. We determine $\sigma(P_n^k)$ when $k = 2$ and when $k > \frac{n}{3} - 1$. The proofs presented in this work provide polynomial time algorithms for an optimum sigma coloring of these graphs. Note that many of theses graphs have $\sigma(P_n^k) \geq 3$ and according to Dehghan et al. [3] to decide if a graph G has $\sigma(G) = s$, for any fixed $s \geq 3$, is an NP-Complete Problem. We also prove that $\sigma(P_n^k) \leq 3$ for the remaining open cases.

Since Dehghan et al. [3] proved that to decide if a cubic graph G has $\sigma(G) = 2$ is an NP-complete Problem, it is interesting to consider the Sigma Coloring Problem for subclasses of cubic graphs. So, this article also presents the sigma chromatic number for Blanuša 1st and 2nd families of snarks, Flower snarks, Goldberg and Twisted Goldberg snarks. For all these classes of cubic graphs the proofs results in polynomial time algorithms for an optimum sigma coloring. So the decision version of the Sigma Coloring Problem can be solved in polynomial time for these classes of snarks.

2 Theoretical Framework

This section presents the definitions and previous results that are essential to the development of this work.

The neighborhood of v, $N(v)$, is the set of all vertices adjacent to v. the closed neighborhood of v, $N[v]$, is the set $N(v) \cup \{v\}$. If for two vertices v and u have $N[u] = N[v]$ they are called strong twins.

If a graph G has every vertex v with $deg(v) = 3$, then G is a cubic graph. Snarks are cubic graphs with chromatic index 4 and without any bridge[3].

A canonical ordering of a graph P_n^k is a linear ordering of the vertices $(v_0, v_1, ...v_{n-1})$ such that vertices are adjacent if, and only if, $0 < |i - j| \leq k$.

Remark 2.1 provides a lower bound for the sigma chromatic number of any graph that has adjacent vertices with same degree.

Remark 2.1 [2] Let G be a nontrivial connected graph. Then $\sigma(G) = 1$ if, and only if, every two adjacent vertices of G have different degrees.

Remark 2.2 increases the lower bound previously established by the number of strong twins on any graph.

Remark 2.2 [2] If u and v are two adjacent vertices in a graph G such that $N[u] = N[v]$, then $c(u) \neq c(v)$ for every sigma coloring c of G.

Remark 2.3 is a direct result of Remark 2.2 and it is a lower bound for the sigma chromatic number of every complete graph. In fact, this lower bound is tight due to Theorem 2.4.

Remark 2.3 [2] If H is a complete subgraph of order k in a graph G such that $N[u] = N[v]$ for every two vertices u and v of H , then $\sigma(G) \geq k$.

The Theorem 2.4 sets an upper bound for the sigma chromatic number of any graph.

Theorem 2.4 *[2] For every graph G, $\sigma(G) \leq \chi(G)$.*

Dehghan et al. [3] define a sigma partition of a graph G as a partition of the set of vertices of G, $[P_0, P_1, \ldots, P_{\sigma(G)-1}]$, such that for every edge uv there is an index i where u and v have different numbers of neighbors in P_i. Let $s \geq \Delta(G) + 1$ be a constant. If the vertices of each set P_i, $0 \leq i \leq \sigma(G) - 1$, are colored with the color s^i, we obtain an optimum sigma coloring of G [3]. The Remark 2.5 is a direct consequence of the observations presented in the work of Dehghan et al. [3].

Remark 2.5 Let G be a nontrivial connected graph with a sigma coloring using the set of colors $\{x_1, x_2, x_3, \ldots, x_n\}$ such that the relation $x_{i+1} > \Delta(G)x_i$ is satisfied for $1 \leq i \leq n$. Then no two vertices u and v in G with $deg(v) \neq deg(u)$ will have $\sigma(v) = \sigma(u)$.

[3] A bridge is an edge e in a connected graph G such that $G - e$ is disconnected.

3 Results on powers of paths

The results presented in this section are divided based on the relation between k and n for a P_n^k as follows: Theorem 3.1 defines $\sigma(P_n^k)$ when $k \geq \lceil \frac{n}{2} \rceil$; Theorem 3.2 determines $\sigma(P_n^k)$ when $\frac{n}{3} - 1 < k < \frac{n}{2}$; Theorem 3.3 presents an upper bound for the sigma chromatic number when $2 \leq k \leq \frac{n}{3} - 1$; and Theorem 3.4 determines $\sigma(P_n^k)$ when $k = 2$ and $n > 3$.

The following theorem defines the sigma chromatic number of a graph when $k \geq \lceil \frac{n}{2} \rceil$, notice that when $k = n - 1$ the graph P_n^k is a K_n and the theorem still holds true.

Theorem 3.1 *In a graph P_n^k, where $k \geq \lceil \frac{n}{2} \rceil$, $\sigma(P_n^k) = n_\Delta$, where n_Δ is the number of vertices with degree $\Delta(P_n^k)$.*

Proof. Since every pair of vertices u and v in a P_n^k, with $deg(v) = deg(u) = \Delta(P_n^k)$ are strong twins, then they need to be colored with a unique color so their color sums are pairwise distinct 2.2. For this reason it is possible to present a sigma coloring with n_Δ colors for P_n^k.

To present a sigma coloring of a P_n^k with n_Δ colors, consider that the set of vertices of this graph, ordered according to the canonical order will be partitioned in three subsets A, B and D such that $A = \{v_i : 0 \leq i \leq n - k - 2\}$, $B = \{v_i : k + 1 \leq i < n\}$ and D contains all the vertices with degree $\Delta(P_n^k)$. Let a_i be the vertex with degree $\Delta(P_n^k) - i$ in A, $1 \leq i < n - k$. Similarly, let b_i be the vertex with degree $\Delta(P_n^k) - i$ in B, $1 \leq i < n - k$. Consider that the pairwise distinct colors $c_1, c_2, ..., c_{n_\Delta - 2}, x, y$ are positive natural numbers such that $c_{i+1} \geq (\Delta(P_n^k) + 1)c_i$, $1 \leq i \leq n_\Delta - 3$, and $c_1 \geq (\Delta(P_n^k) + 1)y \geq (\Delta(P_n^k) + 1)^2 x$. Color the vertices from $A \cup \{b_1\}$ with the color x, the vertices from $B \setminus \{b_1\}$ with the color y and the vertices from D with distinct colors from $\{c_1, c_2, ..., c_{n_\Delta - 2}, x, y\}$.

We will demonstrate that this is a valid sigma coloring. When two vertices have different degrees, by the Remark 2.5, their color sum are different, by the choice of the colors. Then, consider that the vertices v and u have the same degree. If they belong to D, then they are twins as previously stated and we are done. So consider that $u \in A$ and $v \in B$, without loss of generality. So there is an index i, $1 \leq i < n - k$, such that u is a_i and v is b_i. Let d_j, $1 \leq j \leq n_\Delta$, be the vertices with degree $\Delta(P_n^k)$ and, for any vertex w, let $\lambda(w)$ be the color of w. The color sum of a_i and b_i are: $\sigma(a_i) = (\sum_{i=1}^{n_\Delta} \lambda(d_i)) - \lambda(a_i) + \sum_{j=1}^{\frac{n - n_\Delta}{2}} \lambda(a_j) + \sum_{j=1}^{k - n_\Delta - i + 1} \lambda(b_j)$ and $\sigma(b_i) = (\sum_{i=1}^{n_\Delta} \lambda(d_i)) - \lambda(b_i) + \sum_{j=1}^{\frac{n - n_\Delta}{2}} \lambda(b_j) + \sum_{j=1}^{k - n_\Delta - i + 1} \lambda(a_j)$.

Now suppose by contradiction that the color sum of a_i and b_i are equal, we have: $(\sum_{i=1}^{n_\Delta} \lambda(d_i)) - \lambda(a_i) + \sum_{j=1}^{\frac{n - n_\Delta}{2}} \lambda(a_j) + \sum_{j=1}^{k - n_\Delta - i + 1} \lambda(b_j) = (\sum_{i=1}^{n_\Delta} \lambda(d_i)) - \lambda(b_i) + \sum_{j=1}^{\frac{n - n_\Delta}{2}} \lambda(b_j) + \sum_{j=1}^{k - n_\Delta - i + 1} \lambda(a_j)$, $\left(\sum_{j=k-n_\Delta-i+2}^{\frac{n - n_\Delta}{2}} \lambda(a_j) \right) - \lambda(a_i) = \left(\sum_{j=k-n_\Delta-i+2}^{\frac{n - n_\Delta}{2}} \lambda(b_j) \right) - \lambda(b_i)$, $\left(\sum_{j=k-n_\Delta-i+2}^{\frac{n - n_\Delta}{2}} \lambda(a_j) \right) + \lambda(b_i) =$

$\left(\sum_{j=k-n_\Delta-i+2}^{\frac{n-n_\Delta}{2}} \lambda(b_j)\right) + \lambda(a_i)$. Since $n = 2k - n_\Delta + 2$, $\frac{n-n_\Delta}{2} = k - n_\Delta + 1$, then: $\left(\sum_{j=k-n_\Delta-i+2}^{k-n_\Delta+1} \lambda(a_j)\right) + \lambda(b_i) = \left(\sum_{j=k-n_\Delta-i+2}^{k-n_\Delta+1} \lambda(b_j)\right) + \lambda(a_i)$.

Consider the case where $1 < i < n - n_\Delta$. Since $\lambda(a_i) = x$ and $\lambda(b_i) = y$, this equation has the form $cx + y = dy + x$, with $c, d \in \mathbb{N}$. Since $deg(a_i) = deg(b_i)$, $c = d$, therefore $(d-1)x = (d-1)y$ and $x = y$, which is a contradiction since $x \neq y$.

When $i = 1$, we have: $\left(\sum_{j=k-n_\Delta-1+2}^{k-n_\Delta+1} \lambda(a_j)\right) + \lambda(b_1) = \left(\sum_{j=k-n_\Delta-1+2}^{k-n_\Delta+1} \lambda(b_j)\right) + \lambda(a_1)$. Then, $\lambda(a_{k-n_\Delta+1}) + \lambda(b_1) = \lambda(b_{k-n_\Delta+1}) + \lambda(a_1)$. Since $\lambda(a_1) = \lambda(b_1)$, $\lambda(a_{k-n_\Delta+1}) = \lambda(b_{k-n_\Delta+1})$ which is a contradiction seeing that by the hypothesis $\lambda(a_{k-n_\Delta+1}) \neq \lambda(b_{k-n_\Delta+1})$. So the proposed coloring of the graph is a valid sigma coloring and $\sigma(P_n^k)$ on those cases is n_Δ. □

When k is sufficiently large it is possible to find a sigma coloring using only two colors, as it is demonstrated by the Theorem 3.2.

Theorem 3.2 *In a graph P_n^k with $\frac{n}{3} - 1 < k < \frac{n}{2}$, $\sigma(P_n^k) = 2$.*

Proof. By Remark 2.1, $\sigma(P_n^k) > 1$ in this case, since there are adjacent vertices with the same degree.

First, consider the case where $k > 3$ and $k = \frac{n-2}{3}$. Suppose that the vertices of the graph P_n^k are in the canonical order. Partition the vertices of P_n^k in four sets X^0, X^1, X^2 and X^3, such that the first $k+1$ vertices of P_n^k are in X^0, $X^1 = \{v_{k+1}\}$, $X^2 = \{v_{k+2}, v_{k+3}, \ldots, v_{2k+1}\}$ and $X^3 = \{v_{2k+2}, v_{2k+3}, \ldots, v_{3k+1}\}$. Denote the ith vertex of each set X^j by x_i^j, $0 \leq i < |X^j|$ and $0 \leq j \leq 3$.

Consider two colors a and b such that $b > \Delta(P_n^k)a$. Color every vertex of the sets X^0 and X^2 with the color a and every vertex of the sets X^1 and X^3 with the color b. This way, vertices with different degrees have different color sums by the Remark 2.5. Therefore, the k first vertices of the set X^0 and the vertices of the set X^3 have different color sum from their neighbors by Remark 2.5.

The color sum of the vertices of this coloring are: $\sigma(x_0^0) = ka$; $\sigma(x_i^0) = (k+i-1)a + b$, for $1 \leq i \leq k$; $\sigma(x_0^1) = 2ka$; $\sigma(x_i^2) = (2k-2-i)a + (2+i)b$, for $0 \leq i < k$; and $\sigma(x_i^3) = (k-i)a + (k-1)b$, for $0 \leq i < k$. Thus, using the proposed coloring, all the vertices of the graph have different color sums and $\sigma(P_n^k) = 2$.

When $k > \frac{n}{3} - 1$, use the same technique to color a power of path P_{3k+2}^k. It is possible to remove the last vertex of the graph P_{3k+2}^k (in the canonical order), successively, at most $k-1$ times in such a way that the color sum will still be different in every vertex. After that it is possible to remove the first vertex of the resulting graph for as long as $k < \frac{n}{2}$, the color sum of every vertex will continue to be different to all others. That way every graph P_n^k, with $\frac{n}{2} > k > \frac{n}{3} - 1$ can be obtained, and $\sigma(P_n^k) = 2$. □

When k is not large enough to fit the description of Theorem 3.2, then an upper bound can be found, as proved on Theorem 3.3. Notice that the upper bound, if not tight, is at most one greater then the sigma chromatic number.

Theorem 3.3 *In a graph P_n^k with $2 \leq k \leq \frac{n}{3} - 1$, $\sigma(P_n^k) \leq 3$.*

Proof. Consider a graph P_n^k, with $2 \leq k \leq \frac{n}{3} - 1$ and $n \equiv 0 \pmod{2k+1}$, in such a way that its vertices are in the canonical order. Let the vertex set of P_n^k be partitioned in blocks $G_0, G_1, \ldots, G_{\frac{n}{2k+1}-1}$, such that the first $2k+1$ vertices in the canonical order belong to G_0, the next $2k+1$ vertices belong to G_1 and so on. Label the vertices in G_i as $v_{i,0}, v_{i,1}, \ldots, v_{i,2k}$, $0 \leq i < \frac{n}{2k+1}$. Consider the colors a, b and c, where $b > \Delta(P_n^k)a$ and $c > \Delta(P_n^k)b$. This way, it is possible to use the following function $\lambda : V(G) \to \{a, b, c\}$ to color the vertices of P_n^k, as follows: a, if $j \neq k$, $0 \leq j \leq 2k$, and $i \equiv 0 \pmod 3$; b, if $j = k$ and $i \equiv 0 \pmod 3$; b, if $j \neq k$, $0 \leq j \leq 2k$, and $i \equiv 1 \pmod 3$; c, if $j = k$ and $i \equiv 1 \pmod 3$; c, if $j \neq k$, $0 \leq j \leq 2k$, and $i \equiv 2 \pmod 3$; and a, if $j = k$ and $i \equiv 2 \pmod 3$.

By the Remark 2.5, vertices with different degrees have different color sum. Thus, each vertex from the first and last k vertices of the graph (in the canonical order) have different color sum from their neighbors, since vertices with the same degree are not adjacent if the degree is less then $2k$. It remains to prove that vertices with same degree have different color sums.

First consider blocks G_i with $i \equiv 1 \pmod 3$. In those cases the color sum are as follows: $\sigma(v_{i,j}) = (k-j)a + (k-1+j)b + c$, when $0 \leq j < k$ (case 1); $\sigma(v_{i,j}) = 2kb$, when $j = k$ (case 2); and $\sigma(v_{i,j}) = (3k-1-j)b + (1+j-k)c$, when $k+1 \leq j \leq 2k$ (case 3).

The vertices of the case 1 are different from each other because $\sigma(v_{i,j}) < \sigma(v_{i,j+1})$, since $\sigma(v_{i,j+1}) = \sigma(v_{i,j}) - a + b$. There is only one vertex in the case 2. The vertices from case 3 are different because $\sigma(v_{i,j}) < \sigma(v_{i,j+1})$, since $\sigma(v_{i,j+1}) = \sigma(v_{i,j}) - b + c$.

Now consider blocks G_i with $i \equiv 0 \pmod 3$. In those cases the color sum are as follows: $\sigma(v_{i,j}) = (k-j)c + (k-1+j)a + b$, if $0 \leq j < k$ (case 1); $\sigma(v_{i,j}) = 2ka$, if $j = k$ (case 2); and $\sigma(v_{i,j}) = (3k-1-j)a + (1+j-k)b$, if $k+1 \leq j \leq 2k$ (case 3).

At last consider blocks G_i with $i \equiv 2 \pmod 3$. In those cases the color sum are as follows: $\sigma(v_{i,j}) = (k-j)b + (k-1+j)c + b$ when $0 \leq j < k$ (case 1); $\sigma(v_{i,j}) = 2kc$ when $j = k$ (case 2); and $\sigma(v_{i,j}) = (3k-1-j)c + (1+j-k)a$ when $k+1 \leq j \leq 2k$ (case 3).

By analogue arguments to those used on blocks G_i with $i \equiv 1 \pmod 3$, vertices have different color sum on the same block when $i \equiv 0 \pmod 3$ and $i \equiv 2 \pmod 3$. Then it remains to prove that two vertices with the same degree in different blocks have different color sums.

Let $v_{i,p}$ and $v_{i+1,q}$ be two adjacent vertices with the same degree, $0 \leq i < \frac{n}{2k+1} - 2$. Then, $\sigma(v_{i,p}) \neq \sigma(v_{i+1,q})$ because no vertex of the block G_i have neighbors with the color of $v_{i+1,q}$ in its color sum, which is by turn adjacent to all the vertices in the block G_{i+1}.

Now consider the graph P_n^k in which n is not a multiple of $2k+1$. It is possible to obtain a sigma coloring for the graph P_n^k by coloring a graph $H = P^k_{n+2k+1-(n \bmod 2k+1)}$ and removing the last vertices of this graph until P_n^k is obtained.

Consider the power of path H with a sigma coloring like the one defined previously. Remove the last vertex of the graph H and observe that the new graph has a sigma coloring, since the removing of the vertex will only affect the color sum of the vertices v_{p-k-1} to v_{p-1}, where $p = n + 2k + 1 - (n \bmod 2k + 1)$. Such vertices are the new last k vertices of the graph, in the canonical order, and therefore have different degrees which are less then $2k$. By the Remark 2.5 the color sum of those vertices are different between themselves and they are never adjacent to the first k vertices with same degrees. This way the sigma coloring is still valid.

Using the same proceeding it is possible to remove the vertices of H until P_n^k is obtained with a valid sigma coloring when $k \leq \frac{n}{3} - 1$.

Therefore $\sigma(P_n^k) \leq 3$ when $k \leq \frac{n}{3} - 1$. □

If $k = 2$ the sigma chromatic number is defined to any value of $n > 3$. If $n = 3$ then P_3^2 is a K_3, and so $\sigma(P_3^2) = 3$. If $n = 2$, then no simple graph has $k = 2$.

Theorem 3.4 *In a graph P_n^k with $k = 2$ and $n > 3$, then $\sigma(P_n^k) = 2$.*

Proof. Consider the graph P_n^k with $k = 2$, and vertices in the canonical order. Color the first six vertices (if existent) with the colors b, a, b, b, a, a, on that order, and $b \geq \Delta(P_n^k)a$. The next six vertices are painted with the same sequence of colors and so on. We can define this coloring by the function $\lambda : V(P_n^2) \to \{a, b\}$ such that $\lambda(v_{6i}) = b, \lambda(v_{6i+1}) = a, \lambda(v_{6i+2}) = b, \lambda(v_{6i+3}) = b, \lambda(v_{6i+4}) = a, \lambda(v_{6i+5}) = a$, with $i \in \mathbb{N}$.

So $\sigma(v_0)$, $\sigma(v_1)$, $\sigma(v_{n-2})$, $\sigma(v_{n-1})$ have different color sums from the rest for having different degrees, by the Rermark 2.5. To the remaining vertices, we have: $\sigma(v_{3j+2}) = 2a + 2b$; $\sigma(v_{3j+3}) = 3a + b$; and $\sigma(v_{3j+4}) = a + 3b$, with $j \in \mathbb{N}$.

Note that v_0 and v_{n-1} are never adjacent. If v_1 and v_{n-2} are adjacent they have color sum $\sigma(v_1) = 3b$ and $\sigma(v_{n-2}) = a + 2b$ or $\sigma(v_1) = 3b$ and $\sigma(v_{n-2}) = 2a + b$.

Therefore, $\forall i$, $\sigma(v_i) \neq \sigma(v_i \pm 1)$ and $\sigma(v_i) \neq \sigma(v_i \pm 2)$. Since $k = 2$ the graph has a valid sigma coloring. □

Based on computational tests, done using a brute force algorithm, after observing the results from theorems 3.2 and 3.3. A minimum sigma coloring for all P_n^k with $n \leq 27$ and $k \leq 8$ was obtained. The following conjecture was then enunciated.

Conjecture 3.5 *Let P_n^k be a power of a path. If $2 < k \leq \frac{n}{3} - 1$, then $\sigma(P_n^k) = 3$.*

4 Results on snarks

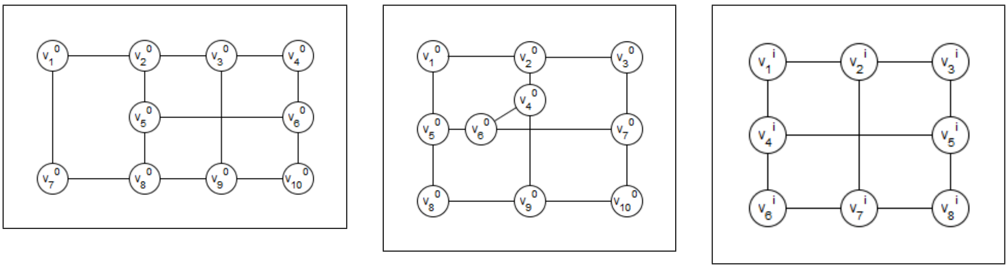
This section presents the results about the sigma chromatic number of snarks, in particular for the families of Blanuša first and second families, the flower snark, the Goldberg snarks and the Twisted Goldberg snarks.

Watkins [8] presented two infinite families of snarks, the Blanuša First family and the Blanuša second family. Each snark in the Blanuša First family is denoted by B_n^1 and it uses a copy of the block H_0^1 illustrated in Figure 1a and n copies of the block B, presented in Figure 1c. The ith copy of the block B is denoted B_i, $1 \leq i \leq n$. Similarly, each snark in the Blanuša Second family is denoted by B_n^2 and it uses a copy of the block H_0^2 illustrated in Figure 1b and n copies of the block B.

Each block H_0^1 is connected to others by creating the edges (v_1^0, v_8^n), (v_7^0, v_3^n), (v_4^0, v_6^1) and (v_{10}^0, v_1^1). Each block H_0^2 is connected to others by creating the edges (v_1^0, v_8^n), (v_8^0, v_3^n), (v_3^0, v_6^1) and (v_{10}^0, v_1^1). Each block B_i will connect to another block B_{i+1} by creating the edges (v_3^i, v_6^{i+1}) and (v_8^i, v_1^{i+1}), if $1 \leq i < n$.

Fig. 1. Blanuša blocks

(a) Block H_0^1 (b) Block H_0^2 (c) Block B_i



Theorem 4.1 *If a graph is a B_n^1, so $\sigma(B_n^1) = 2$.*

Proof. Let G be a snark of the Blanuša first family. The proof is direct and is divided into two cases, according to the parity of the number of blocks B in G.

First consider that the number of blocks B in G is odd. Color each block B_i, i even, such that $c(v_j^i) = 1$ when $j \in \{1, 3, 4, 5, 7, 8\}$ and $c(v_j^i) = 2$ otherwise. Color each block B_i, i odd, such that $c(v_j^i) = 1$ when $j \in \{3, 7\}$ and $c(v_j^i) = 2$ otherwise. The coloring of the vertices of the block H_0^1 are as follows: $c(v_j^0) = 2$ when $j \in \{6, 7, 9\}$ and $c(v_j^0) = 1$ otherwise.

The color sum of each vertex in the block B_i, i even, is $\sigma(v_j^i) = 5$ when $j \in \{1, 3, 7\}$, $\sigma(v_j^i) = 4$ when $j \in \{4, 8\}$ and $\sigma(v_j^i) = 3$ otherwise. The color sum of each vertex in the block B_i, i odd, is $\sigma(v_j^i) = 4$ when $j \in \{2, 6, 8\}$, $\sigma(v_j^i) = 5$ when $j \in \{1, 5\}$ and $\sigma(v_j^i) = 6$ otherwise. The color sum of each vertex v_j^0 is $\sigma(v_j^0) = 6$ when $j \in \{10\}$, $\sigma(v_j^0) = 5$ when $j \in \{1, 4, 8\}$, $\sigma(v_j^0) = 4$ when $j \in \{3, 5\}$ and $\sigma(v_j^0) = 3$ otherwise.

Now consider that the number of blocks B in G is even. Color each block B_i, i even, such that $c(v_j^i) = 2$ when $j \in \{4, 8\}$ and $c(v_j^i) = 1$ otherwise. Color each block B_i, i odd and $i \neq 1$, such that $c(v_j^i) = 1$ when $j \in \{1, 5\}$ and $c(v_j^i) = 2$ otherwise. The coloring of each vertex v_j^0 in H_0^1 are as follows: $c(v_j^0) = 2$ when $j \in \{6, 7, 9\}$ and $c(v_j^0) = 1$ otherwise. The coloring of the vertices v_j^1 in B_1 are as follows: $c(v_j^1) = 2$ when $j \in \{3, 6, 8\}$ and $c(v_j^1) = 1$ otherwise.

The color sum of each vertex in the block B_i, i odd and $i \neq 1$, is $\sigma(v_j^i) = 6$ when $j \in \{1, 5, 7\}$, $\sigma(v_j^i) = 5$ when $j \in \{2, 6\}$ and $\sigma(v_j^i) = 4$ otherwise. The color sum of each vertex in the block B_i, i even, is $\sigma(v_j^i) = 5$ when $j \in \{1, 5, 6\}$, $\sigma(v_j^i) = 4$ when $j \in \{3, 7\}$ and $\sigma(v_j^i) = 3$ otherwise. The color sum of each vertex in the block H_0^1 is $\sigma(v_j^0) = 5$ when $j \in \{1, 4, 8, 10\}$, $\sigma(v_j^0) = 4$ when $j \in \{3, 5\}$ and $\sigma(v_j^0) = 3$ otherwise. The color sum of each vertex on the block B_1 is $\sigma(v_j^1) = 5$ when $j \in \{5, 7\}$, $\sigma(v_j^1) = 4$ when $j \in \{2, 4\}$ and $\sigma(v_j^1) = 3$ otherwise.

Therefore, since no two adjacent vertices have the same color sum on the sigma coloring described in this proof, $\sigma(B_n^1) \leq 2$. By Remark 2.1, $\sigma(B_n^1) = 2$. □

Theorem 4.2 *If a graph is a B_n^2, so $\sigma(B_n^2) = 2$.*

Proof. Let G be a snark of the Blanuša second family. The proof is direct and is divided into two cases, according to the parity of the number of blocks B in G.

First consider that the number of blocks B in G is odd. Color each block B_i, i even, with the following coloring of vertices: $c(v_j^i) = 1$ when $j \in \{3, 6, 8\}$ and $c(v_j^i) = 2$ otherwise. Color each block B_i, i odd, with the following coloring of vertices: $c(v_j^i) = 2$ when $j \in \{3, 6, 8\}$ and $c(v_j^i) = 1$ otherwise. The coloring of the vertices of the block H_0^2 are as follows: $c(v_j^0) = 2$ when $j \in \{5, 7, 9\}$ and $c(v_j^0) = 1$ otherwise.

The color sum of each vertex in B_i, i even, is $\sigma(v_j^i) = 6$ when $j \in \{1, 3, 6\}$, $\sigma(v_j^i) = 5$ when $j \in \{2, 4, 8\}$ and $\sigma(v_j^i) = 4$ otherwise. The color sum of each vertex in B_i, i odd and $j \neq 10$, is $\sigma(v_j^i) = 5$ when $j \in \{5, 7\}$, $\sigma(v_j^i) = 4$ when $j \in \{2, 4, \}$ and $\sigma(v_j^i) = 3$ otherwise. When i is odd and $j = 10$, $\sigma(v_{10}^i) = 3$ if $i = n$ and $\sigma(v_{10}^i) = 4$ otherwise. Notice that $\sigma(v_{10}^i)$ are fine for all cases. The color sum of each vertex v_j^0 is $\sigma(v_j^0) = 6$ when $j \in \{6, 8\}$, $\sigma(v_j^0) = 5$ when $j \in \{1, 3, 10\}$, $\sigma(v_j^0) = 4$ when $j \in \{4\}$ and $\sigma(v_j^0) = 3$ otherwise.

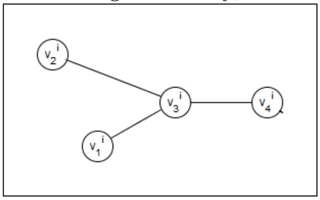
Now consider that the number of blocks B in G is even. Color each block B_i, i even, such that $c(v_j^i) = 2$ when $j \in \{1, 3, 6\}$ and $c(v_j^i) = 1$ otherwise. Color each block B_i, i odd and $i \neq 1$, such that $c(v_j^i) = 1$ when $j \in \{1, 3, 6\}$ and $c(v_j^i) = 2$ otherwise. The coloring of each vertex of the block H_0^2 are as follows: $c(v_j^0) = 2$ when $j \in \{5, 7, 9\}$ and $c(v_j^0) = 1$ otherwise. The coloring of the vertices of the block B_1 are as follows: $c(v_j^1) = 2$ when $j \in \{5, 6, 7, 8\}$ and $c(v_j^1) = 1$ otherwise.

The color sum of each vertex in B_i, i even, is $\sigma(v_j^i) = 5$ when $j \in \{2, 4\}$, $\sigma(v_j^i) = 4$ when $j \in \{1, 5, 7\}$ and $\sigma(v_j^i) = 3$ otherwise. The color sum of each vertex in B_i, i odd and $i \neq 1$, is $\sigma(v_j^i) = 6$ when $j \in \{3, 6, 8\}$, $\sigma(v_j^i) = 5$ when $j \in \{1, 5, 7\}$ and $\sigma(v_j^i) = 4$ otherwise. The color sum of each vertex v_j^0 is $\sigma(v_j^0) = 6$ when $j \in \{6, 8\}$, $\sigma(v_j^0) = 5$ when $j \in \{10\}$, $\sigma(v_j^0) = 4$ when $j \in \{1, 4\}$, and $\sigma(v_j^0) = 3$ otherwise. The color sum of each vertex in B_1 is $\sigma(v_j^1) = 6$ when $j \in \{8\}$, $\sigma(v_j^1) = 5$ when $j \in \{3, 4, 7\}$, $\sigma(v_j^1) = 4$ when $j \in \{2, 5, 6, \}$, and $\sigma(v_j^1) = 3$ otherwise.

Since no two adjacent vertices have the same color sum, $\sigma(B_n^2) \leq 2$. By Remark 2.1, $\sigma(B_n^2) = 2$. □

Other infinite family of snarks, the Flower snarks, were found in 1975 [6]. Each Flower snark, denoted F_n, is formed by a set of n blocks f with 4 vertices each, n odd and $n \geq 3$. The ith block in a flower snark is denoted f_i, $1 \leq i \leq n$. A block f_i is illustrated on Figure 2.

Fig. 2. Block f_i



Each flower snark F_n is made by creating the edges $(v_1^i, v_1^{(i \bmod n)+1})$, $(v_2^i, v_2^{(i \bmod n)+1})$ and $(v_4^i, v_4^{i+1 \pmod n})$ for each block f_i and $f_{(i \bmod n)+1}$, $1 \leq i \leq n$. This way connecting all blocks on the graph.

Theorem 4.3 *If a graph is a flower snark F_n, then $\sigma(F_n) = 2$.*

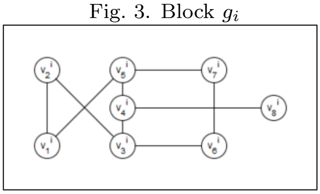
Proof. The proof is direct. Color each block f_i, with odd i and $i < n$ as follows: $c(v_j^i) = 1$ when $j \in \{3\}$ and $c(v_j^i) = 2$ otherwise. Color each block f_i, with even i, as follows: $c(v_j^i) = 1$ when $j \in \{1, 2, 4\}$ and $c(v_j^i) = 2$ otherwise. The coloring of the vertices in block f_n is $c(v_j^n) = 2$ for every vertex.

The color sum of each vertex v_j^i with odd i and $i \neq 1$ is $\sigma(v_j^i) = 3$ when $j \in \{1, 2, 4\}$ and $\sigma(v_3^i) = 6$. If $i = 1$ then the color sum of each vertex is $\sigma(v_j^i) = 4$ when $j \in \{1, 2, 4\}$ and $\sigma(v_3^i) = 6$, both values of color sums for odd i are valid in this coloring. The color sum of each vertex in f_i with even i is $\sigma(v_j^i) = 6$ if $j \in \{1, 2, 4\}$ and $\sigma(v_3^i) = 3$. The color sum of each vertex on the block f_n is $\sigma(v_j^n) = 5$ when $j \in \{1, 2, 4\}$ and $\sigma(v_3^n) = 6$.

Therefore, since no two adjacent vertices have the same color sum on the sigma coloring described in this proof, $\sigma(F_n) \leq 2$. By Remark 2.1, $\sigma(F_n) = 2$. □

The infinite family of Goldberg snarks were found in 1981 [5]. Each Goldberg snark is denoted G_n, and is formed by a set of n blocks g with 8 vertices, n odd and $n \geq 3$. The block g is illustrated on Figure 3 and the ith copy of the block g is denoted by g_i.

Fig. 3. Block g_i



The vertex v_k of a block g_i is denoted g_k^i, as illustrated in Figure 3. Each Goldberg snark G_n is constructed by adding the edges $(v_2^i, v_1^{(i \bmod n)+1})$, $(v_7^i, v_6^{(i \bmod n)+1})$ and $(v_8^i, v_8^{(i \bmod n)+1})$ to each block g_i and $g_{(i \bmod n)+1}$. The Twisted Goldberg snark is constructed by "twisting" two edges on a G_n and it is denoted TG_n. Twisting two edges on the graph consists in removing edges $(v_2^i, v_1^{(i \bmod n)+1})$ and $(v_7^i, v_6^{(i \bmod n)+1})$, and creating edges $(v_2^i, v_6^{(i \bmod n)+1})$ and $(v_7^i, v_1^{(i \bmod n)+1})$, for a pair of blocks g_i and $g_{(i \bmod n)+1}$. Twisting more then two pairs of blocks will not result in a new graph [4].

Theorem 4.4 *If a graph is a Goldberg snark G_n, then $\sigma(G) = 2$.*

Proof. The proof is direct. Color each block g_i, with i odd and $i > 1$ as follows: $c(v^i_j) = 1$ when $j \in \{1, 6, 7\}$ and $c(v^i_j) = 2$ otherwise. The coloring of the vertices in the block g_1 are as follows: $c(v^1_j) = 1$ when $j \in \{1, 2, 3, 4, 7\}$ and $c(v^1_j) = 2$ otherwise. Color each block g_i, i even, as follows: $c(v^i_j) = 1$ when $j \in \{1, 7, 8\}$ and $c(v^i_j) = 2$ otherwise.

The color sum of each vertex in g_i, with i odd is $\sigma(v^i_j) = 6$ when $j \in \{1, 4\}$, $\sigma(v^i_j) = 5$ when $j \in \{3, 7\}$ and $\sigma(v^i_j) = 4$ when $j \in \{2, 5, 6\}$. For the vertex v^i_8, $\sigma(v^i_8) = 4$ when $i \neq n$ and $\sigma(v^i_8) = 5$ when $i = n$. The color sum of each vertex in the block g_1 is $\sigma(v^1_j) = 5$ when $j \in \{1, 4\}$, $\sigma(v^1_j) = 4$ when $j \in \{3, 8\}$, $\sigma(v^1_j) = 3$ when $j \in \{2, 5, 6\}$ and $\sigma(v^1_7) = 6$. The color sum of each vertex in g_i, i even, is $\sigma(v^i_j) = 4$ when $j \in \{2, 5, 6\}$, $\sigma(v^i_j) = 5$ when $j \in \{4, 7\}$, $\sigma(v^i_j) = 6$ when $j \in \{3, 8\}$. When $j \in \{1\}$, $\sigma(v^2_1) = 5$ if $i = 2$ and $\sigma(v^2_1) = 6$ otherwise.

Therefore, since no two adjacent vertices have the same color sum on the sigma coloring described in this proof, $\sigma(G_n) \leq 2$. By Remark 2.1, $\sigma(G_n) = 2$. □

Theorem 4.5 *The twisted Goldberg TG_n has $\sigma(G) = 2$ if $n > 3$, and $\sigma(G) = 3$ if $n = 3$.*

Proof. The proof is direct and there are two cases. If $n > 3$, color each block g_i, with i odd and $i > 1$, as follows: $c(v^i_j) = 1$ when $j \in \{1, 4, 7\}$ and $c(v^i_j) = 2$ otherwise. Color each block g_i with i even and $i > 2$ as follows: $c(v^i_j) = 1$ when $j \in \{1, 2, 3, 6, 7\}$ and $c(v^i_j) = 2$ otherwise. Remains to color the vertices in the blocks g_1 and g_2. Assign the colors to vertices in g_1 as follows: $c(v^1_j) = 1$ when $j \in \{1, 3, 8\}$ and $c(v^1_j) = 2$ otherwise. Color the block g_2 as follows: $c(v^2_j) = 1$ when $j \in \{2, 3, 4, 6, 7\}$ and $c(v^2_j) = 2$ otherwise.

The color sum of each vertex in g_i, with i odd is $\sigma(v^i_j) = 6$ when $j \in \{4\}$, $\sigma(v^i_j) = 5$ when $j \in \{1, 3\}$, $\sigma(v^i_j) = 4$ when $j \in \{2, 6\}$, and $\sigma(v^i_j) = 3$ when $j \in \{5\}$. When $i = n$ then $\sigma(v^n_7) = 6$ and $\sigma(v^n_8) = 4$, if $i \neq n$ then $\sigma(v^i_7) = \sigma(v^i_8) = 5$. The color sum of each vertex in g_i with i even is $\sigma(v^i_j) = 6$ when $j \in \{8\}$, $\sigma(v^i_j) = 5$ when $j \in \{1, 4, 7\}$, $\sigma(v^i_j) = 4$ when $j \in \{3, 5\}$, and $\sigma(v^i_j) = 3$ otherwise. The color sum of each vertex on the block g_1 is $\sigma(v^1_j) = 6$ when $j \in \{1, 3, 7, 8\}$, $\sigma(v^1_j) = 5$ when $j \in \{5\}$, $\sigma(v^1_j) = 4$ when $j \in \{4, 6\}$, and $\sigma(v^1_2) = 3$. The color sum of each vertex in g_2 is $\sigma(v^2_j) = 5$ when $j \in \{1, 4, 7\}$, $\sigma(v^2_j) = 4$ when $j \in \{2, 5, 6, 8\}$, and $\sigma(v^2_3) = 3$. Therefore, since no two adjacent vertices have the same color sum, this is a sigma coloring and $\sigma(TG_n) \leq 2$. By Remark 2.1, $\sigma(TG_n) = 2$ when $n > 3$.

If $n = 3$, the graph TG_3 has no valid sigma coloring with two colors. This fact was established after computational inspection of all combinations of coloring with two colors[4]. However, it is possible to find a coloring with 3 colors. Follow the previous instructions and color v^2_8 with 3. The color sum of the vertices v^1_8, v^2_4 and v^3_8 will change to 7, 6 and 5 respectively, and the sigma coloring will be valid. Therefore $\sigma(TG_3) = 3$. □

[4] The code for test is available in sheilaalmeida.pro.br/codigos/ProofForTG3.txt.

5 Conclusion

For any power of a path P_n^k, the sigma chromatic number was determined except when $2 < k \leq \frac{n}{3} - 1$. In this last case, we present a sigma coloring with at most $\sigma(P_n^k) + 1$ colors. Considering the snarks, the sigma chromatic number was determined for the Blanuša First and Second families, Flower snarks, Goldberg and twisted Goldberg snarks. Those results are interesting since for cubic graphs the complexity the decision version of the Sigma Coloring Problem is NP-Complete, and for those families the results can be found in polynomial time using algorithms that can be obtained from the constructive proofs presented in this work.

It is interesting to emphasize that the sigma chromatic number of a graph does not have a simple behavior when analyzing similar structures in different graphs. As an example, the result found for the graph TG_3 is different from the result on the graph G_3, even those graphs being different only by two edges.

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