Electronic Notes in Theoretical Computer Science 40 (2001)
URL: http://www.elsevier.nl/locate/entcs/volume40.html 16 pages

A graphical approach to monad compositions

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Abstract

In this paper we show how composite expressions involving natural transformations can be pictorially represented in order to provide graphical proof support for providing monad compositions. Examples are drawn using powerset monads composed with the term monad.

1 Introduction

Monads have shown to be useful in different fields related to computer science. In functional programming monad compositions are applied to structuring of functional programs [11]. In particular, in functional programs like parsers or type checkers the monad needed is often a composed monad [13]. In logic programming, unification has been identified as the provision of co-equalisers in Kleisli categories of term monads [12].

The foundational understanding of monads has been well-known for decades, but proof techniques, especially related to monad compositions have not been developed. As monad compositions are basically built upon operations of corresponding natural transformations, proof techniques require an adequate handling of the basic combinatorial properties of functors and natural transformations (Godement rules). In [4,7] it was discovered that these combinatorial properties can be represented more visually, in that the basic observation relates to distributivity of the star product of natural transformations with respect to composition of natural transformations.

This improves readability of expressions involving compositions of natural transformations and supports proofs involving more complex properties. This visual technique is not widely known and has been used mainly in purely algebraic contexts [1].

[1] Supported by the Swedish Research Council for Engineering Sciences.
[2] Partially supported by Spanish DGI project BFM2000-1054-C02-02

The aim of this paper is to further develop these ideas about graphical representation, and to demonstrate the use of this technique on a case study for providing some concrete examples on generalised terms where various set functors are composed with the conventional term functor [6,5].

The structure of the paper is the following: in Section 2, the basic definitions and notation of the graphical approach are given. In Section 3, some conditions for extending the composition of monads to a monad are presented and proved using the graphical approach. As an example, in Section 4 we introduce the problem of generalising sets of terms as the composition of suitable powerset monads and the term monads, which raise our interest in the problem of composing monads and in a visual interface to make the corresponding calculations. Some conclusions are presented in Section 5. Finally, an appendix is introduced containing a standard proof of Proposition 3.2, just to see how the properties of natural transformations are more naturally handled in the graphical approach, so that one can abstract them from the main line of reasoning.

2 Notations and pictorial representations

The notational conventions followed in this paper are those presented in [2]. Let $\mathtt{C}$ be a category and consider (covariant) endofunctors, denoted with capital letters $F, G, H, \ldots : \mathtt{C} \to \mathtt{C}$, together with natural transformations, denoted with greek letters $\tau, \sigma, \ldots$, between such endofunctors. For $\tau: F \to G$ and $\sigma: G \to H$, let $\sigma \circ \tau: F \to H$ be the usual vertical composition of natural transformations, and for $\tau': F' \to G'$, let $\tau' \star \tau: F' \circ F \to G' \circ G$ be the star product given by

(1) $\tau' \star \tau = \tau' G \circ F' \tau = G' \tau \circ \tau' F.$

The star product, like composition, is associative.

For the identity transformation $id_F: F \to F$, also written as 1_F or 1, note that

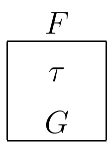
(2) $1_F \star 1_G = 1_{F \circ G}.$

For a natural transformation $\tau: F \to G$, and a functor H, $(H\tau)_X = H\tau_X$ and $(\tau H)_X = \tau_{HX}$, or equivalently, $H\tau = 1_H \star \tau$ and $\tau H = \tau \star 1_H$. The following distributivity laws hold:

(3) $1 \star (\sigma \circ \tau) = (1 \star \sigma) \circ (1 \star \tau),$

(4) $(\sigma \circ \tau) \star 1 = (\sigma \star 1) \circ (\tau \star 1).$

A natural transformation $\tau: F \to G$ as a basic building block is depicted as



.

Blocks $\tau: F \to G$ and $\sigma: G \to H$ are built, or composed, vertically as

$$\begin{array}{|c|} \multicolumn{1}{c}{F} \\ \hline \tau \\ G \\ \hline \sigma \\ H \\ \hline \end{array} \quad = \quad \begin{array}{|c|} \multicolumn{1}{c}{F} \\ \hline \\ \sigma \circ \tau \\ \\ H \\ \hline \end{array} \, .$$

For $\tau\colon F \to G$ and $\tau'\colon F' \to G'$, the horizontal composition of τ' followed by τ, denoted by the star product $\tau' \star \tau$ is visually denoted by the juxtaposition of two building blocks.

$$\begin{array}{|c|c|} \multicolumn{1}{c}{F'} & \multicolumn{1}{c}{F} \\ \hline \tau' & \tau \\ G' & G \\ \hline \end{array} \quad = \quad \begin{array}{|c|} \multicolumn{1}{c}{F' \quad F} \\ \hline \tau' \star \tau \\ G' \quad G \\ \hline \end{array} \, .$$

Note in particular that the juxtaposition order reflects the syntactic order of $\tau' \star \tau$. As an application of the previous construction, note that equation (1) can be pictorially represented by

$$\begin{array}{|c|} \multicolumn{1}{c}{F' \quad F} \\ \hline \tau' \star \tau \\ G' \quad G \\ \hline \end{array} \quad = \quad \begin{array}{|c|} \multicolumn{1}{c}{F' \quad F} \\ \hline 1_{F'} \star \tau \\ F' \quad G \\ \hline \tau' \star 1_G \\ G' \quad G \\ \hline \end{array} \quad = \quad \begin{array}{|c|} \multicolumn{1}{c}{F' \quad F} \\ \hline \tau' \star 1_F \\ G' \quad F \\ \hline 1_{G'} \star \tau \\ G' \quad G \\ \hline \end{array} \, .$$

Equation (3) can be written as

$$\begin{array}{|c|c|} \multicolumn{1}{c}{K} & \multicolumn{1}{c}{F} \\ \hline & \\ 1_K & \sigma \circ \tau \\ & \\ K & H \\ \hline \end{array} \quad = \quad \begin{array}{|c|} \multicolumn{1}{c}{K \quad F} \\ \hline 1_K \star \tau \\ K \quad G \\ \hline 1_K \star \sigma \\ K \quad H \\ \hline \end{array} \, ,$$

i.e., in this case building blocks can be applied in any order. The same holds for equation (4).

For natural transformations $F \xrightarrow{\tau} G \xrightarrow{\sigma} H$ and $F' \xrightarrow{\tau'} G' \xrightarrow{\sigma'} H'$ we then have

$$
\begin{array}{|c|}
\multicolumn{1}{c}{F' \quad F} \\ \hline
\tau' \star \tau \\
G' \quad G \\ \hline
\sigma' \star \sigma \\
H' \quad H \\ \hline
\end{array}
\overset{(1)}{=}
\begin{array}{|c|}
\multicolumn{1}{c}{F' \quad F} \\ \hline
\tau' \star 1_F \\
G' \quad F \\ \hline
1_{G'} \star \tau \\
G' \quad G \\ \hline
1_{G'} \star \sigma \\
G' \quad H \\ \hline
\sigma' \star 1_H \\
H' \quad H \\ \hline
\end{array}
\overset{(3)}{=}
\begin{array}{|c|}
\multicolumn{1}{c}{F' \quad F} \\ \hline
\tau' \star 1_F \\
G' \quad F \\ \hline
1_{G'} \star (\sigma \circ \tau) \\
G' \quad H \\ \hline
\sigma' \star 1_H \\
H' \quad H \\ \hline
\end{array}
$$

$$
\overset{(1)}{=}
\begin{array}{|c|}
\multicolumn{1}{c}{F' \quad F} \\ \hline
\tau' \star 1_F \\
G' \quad F \\ \hline
\sigma' \star 1_F \\
H' \quad F \\ \hline
1_{H'} \star (\sigma \circ \tau) \\
H' \quad H \\ \hline
\end{array}
\overset{(4)}{=}
\begin{array}{|c|}
\multicolumn{1}{c}{F' \quad F} \\ \hline
(\sigma' \circ \tau') \star 1_F \\
H' \quad F \\ \hline
1_{H'} \star (\sigma \circ \tau) \\
H' \quad H \\ \hline
\end{array}
\overset{(1)}{=}
\begin{array}{|c|c|}
\multicolumn{1}{c}{F'} & \multicolumn{1}{c}{F} \\ \hline
\sigma' \circ \tau' & \sigma \circ \tau \\
H' & H \\ \hline
\end{array}\ ,
$$

i.e., we have (re)proved the Interchange Law

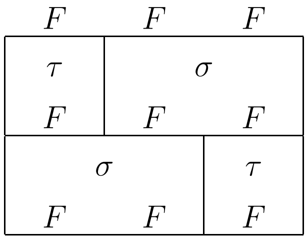
$$(\sigma' \circ \tau') \star (\sigma \circ \tau) = (\sigma' \star \sigma) \circ (\tau' \star \tau) \tag{5}$$

which can be summarized as

$$
\begin{array}{|c|c|}
\multicolumn{1}{c}{F'} & \multicolumn{1}{c}{F} \\ \hline
\tau' & \tau \\
G' & G \\ \hline
\sigma' & \sigma \\
H' & H \\ \hline
\end{array}
=
\begin{array}{|c|c|}
\multicolumn{1}{c}{F'} & \multicolumn{1}{c}{F} \\ \hline
 & \\
\sigma' \circ \tau' & \sigma \circ \tau \\
 & \\
H' & H \\ \hline
\end{array}
=
\begin{array}{|c|}
\multicolumn{1}{c}{F' \quad F} \\ \hline
\tau' \star \tau \\
G' \quad G \\ \hline
\sigma' \star \sigma \\
H' \quad H \\ \hline
\end{array}
$$

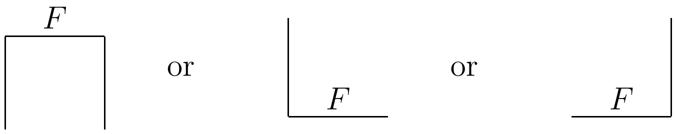
showing how blocks with particular positions generally can be attached vertically and horizontally in any order without changing the resulting transformation.

Note in the transformation



that the composition $(\sigma \star \tau) \circ (\tau \star \sigma)$ indeed exists, but neither $\tau \circ \sigma$ nor $\sigma \circ \tau$ do. This indicates how the applicability of the Interchange Law is more easily seen in the pictorial representation of the transformation.

In order to further improve readability of transformation expressions, identity transformations $1_F\colon F \longrightarrow F$ as blocks within transformation expressions are depicted as



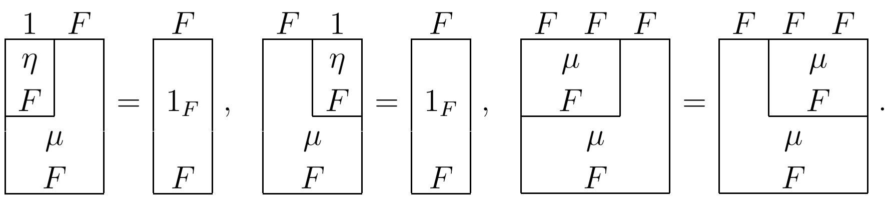
This choice for the representation of identity transformations will allow the use of asymmetric stacking of boxes.

3 Monad compositions

A *monad* (or *triple*, or *algebraic theory*) over $\mathtt{C}$ is written as $\mathbf{F} = (F, \eta, \mu)$, where $F\colon \mathtt{C} \to \mathtt{C}$ is a (covariant) functor, and $\eta\colon id_{\mathtt{C}} \to F$ and $\mu\colon F \circ F \to F$ are natural transformations such that

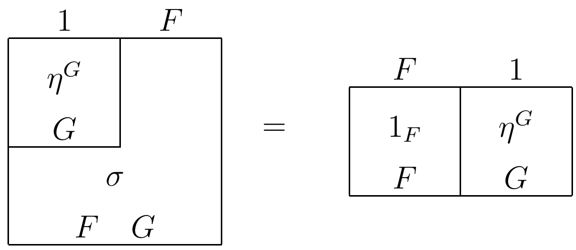
(6) $\mu \circ (\eta \star 1_F) = 1_F$,
(7) $\mu \circ (1_F \star \eta) = 1_F$,
(8) $\mu \circ (1_F \star \mu) = \mu \circ (\mu \star 1_F)$.

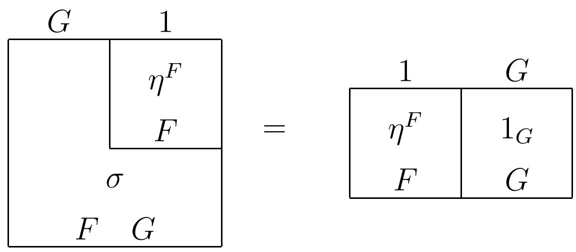
We say that η is respectively a left and right unit, and that the multiplication μ is associative. These monad conditions, with the identity functor $id_{\mathtt{C}}\colon \mathtt{C} \to \mathtt{C}$ written as 1, can be depicted as

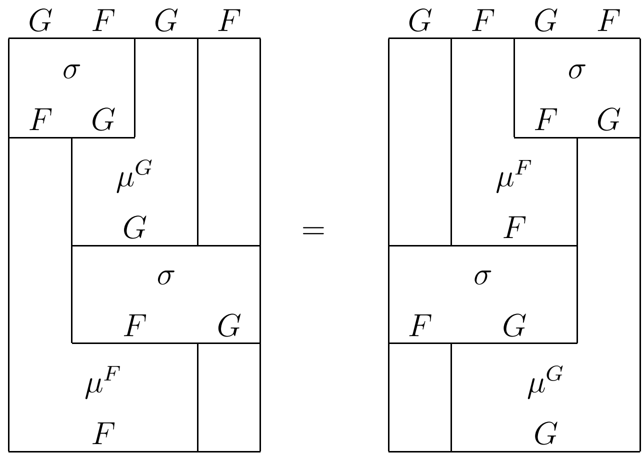


The following proposition appears in [5]. Similar results concerning composability of monads appeared in [4,6,9], and originally also in [3].

Proposition 3.1 *Let* $\mathbf{F} = (F, \eta^F, \mu^F)$ *and* $\mathbf{G} = (G, \eta^G, \mu^G)$ *be monads. Let* $\sigma\colon G \circ F \to F \circ G$*, called a 'swapper', be a natural transformation such that the following properties hold:*

(9) 

(10) 

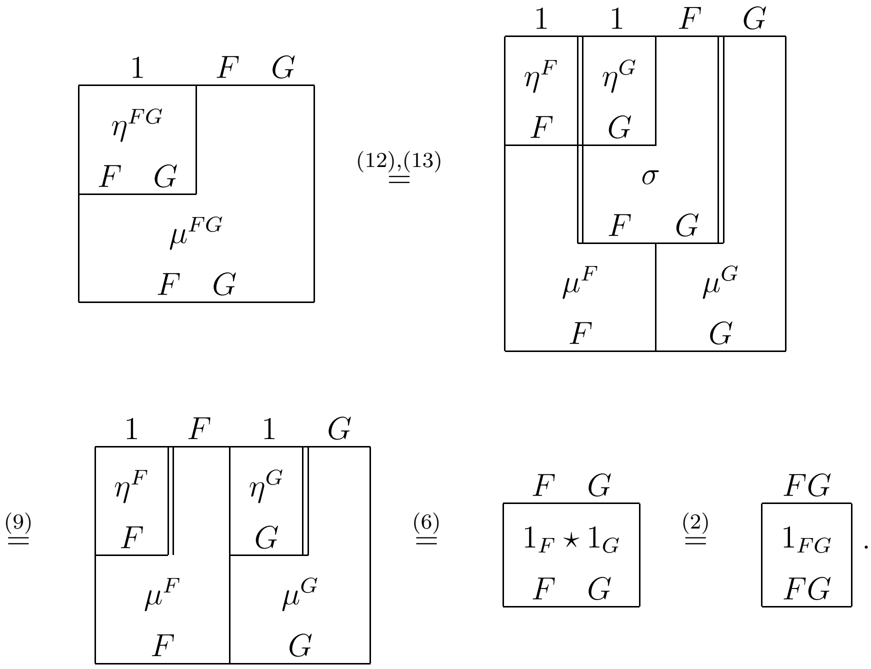
(11) 

Then $\mathbf{F} \bullet \mathbf{G} = (F \circ G, \eta^{FG}, \mu^{FG})$ *is a monad, where*

(12) $\eta^{FG} = \eta^F \star \eta^G$,
(13) $\mu^{FG} = (\mu^F \star \mu^G) \circ (1_F \star \sigma \star 1_G)$.

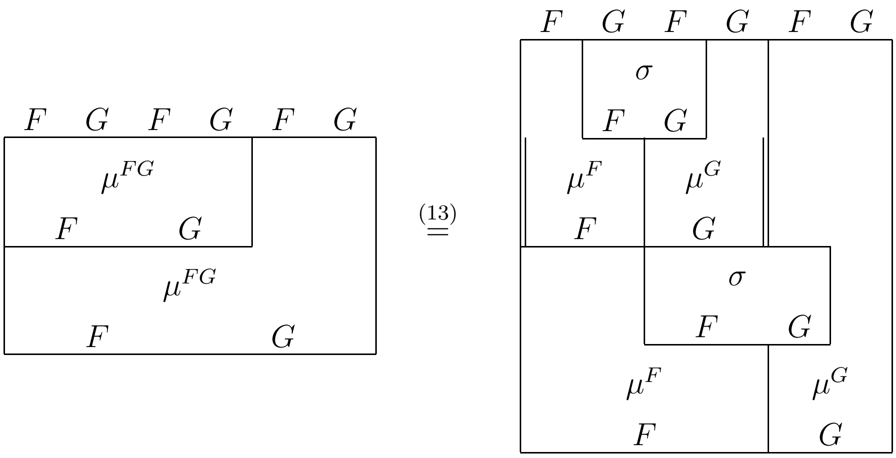
Proof. The following proof demonstrates the use of our pictorial representations.

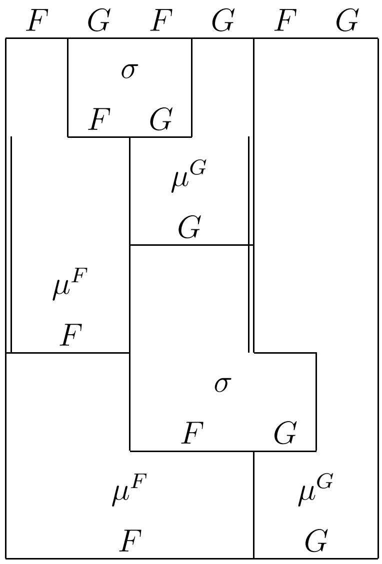
Firstly, we show that η^{FG} is a left unit.

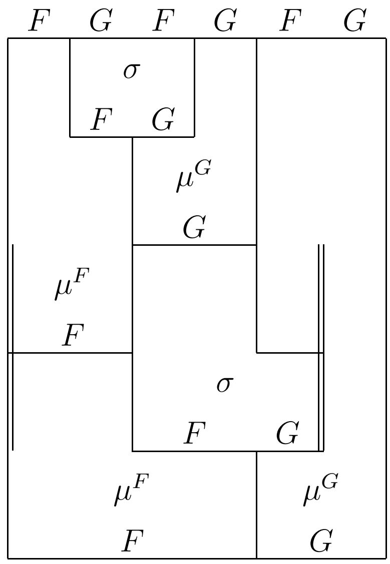


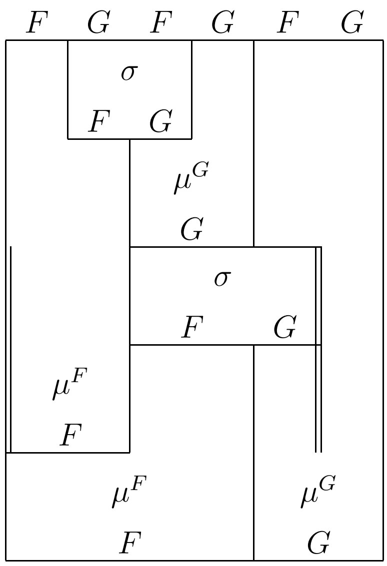
Note how the ‘highlighting’ of subexpressions is due to the Interchange Law. The right unit property is shown similarly.

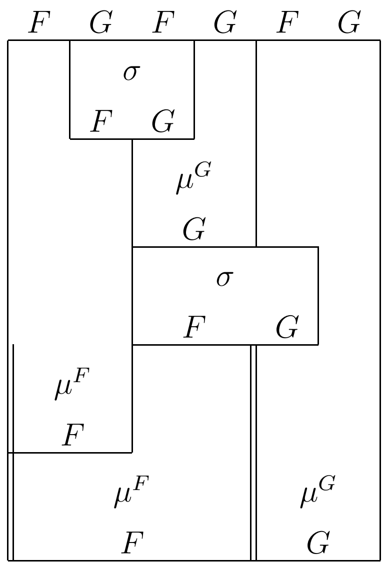
Secondly, we show that μ^{FG} is associative.

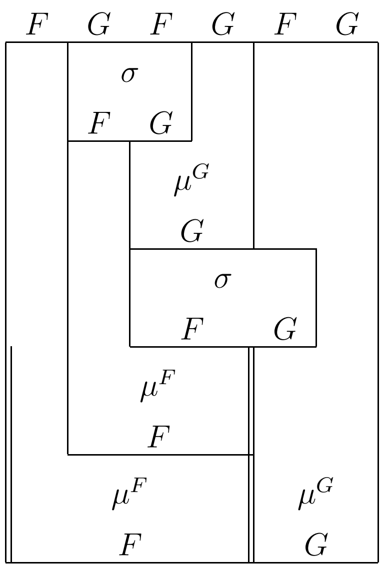


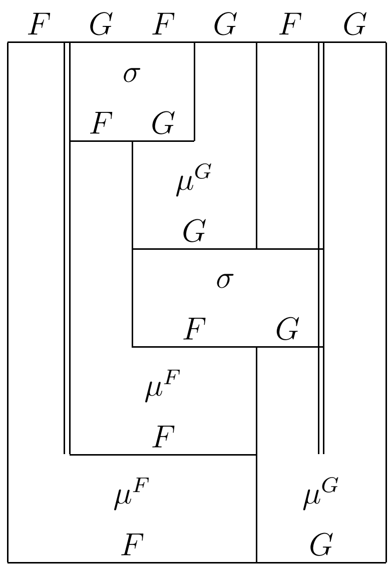
$\overset{(1)}{=}$ 

$\overset{(5)}{=}$ 

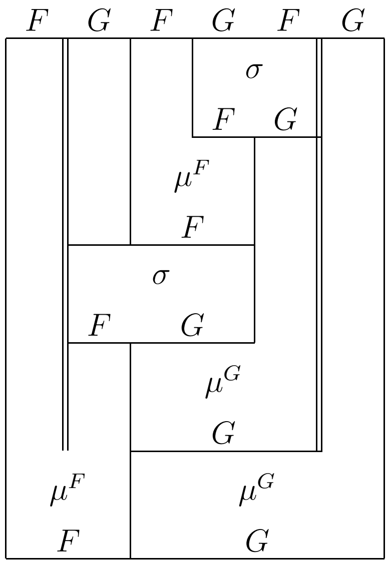
$\overset{(1)}{=}$ 

$\overset{(5)}{=}$ 

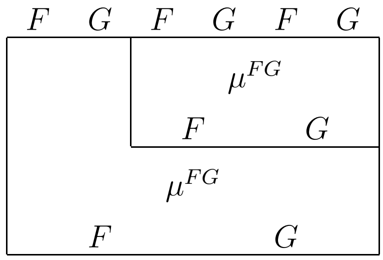
$\overset{(8)}{=}$ 

$\overset{(5)}{=}$ 

$\overset{(11)}{=}$



$\overset{revert}{=}$[3]



$\square$

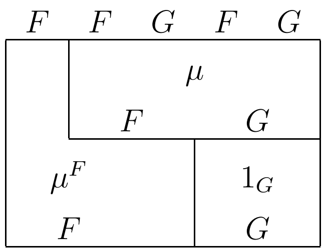
A converse result can be partially achieved under some additional assumptions on the behaviour of the multiplication of the composite monad w.r.t. either the multiplications or the units of the base monads.

Proposition 3.2 *If* $\mathbf{F} \bullet \mathbf{G} = (F \circ G, \eta^F \star \eta^G, \mu)$ *is a monad, then a natural transformation* $\sigma_\mu : G \circ F \to F \circ G$ *can be defined by*

(14) $\sigma_\mu = \mu \circ (\eta^F \star 1_{GF} \star \eta^G)$

such that conditions (9) and (10) are satisfied. In addition, condition (11) holds and $\mu = \mu^{FG}$*, with* μ^{FG} *related to* σ_μ *given by (13), under the assumption that at least one of the conditions*

(15)



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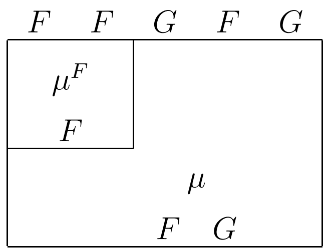
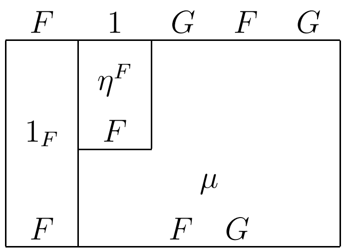


(16)



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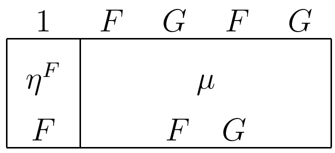
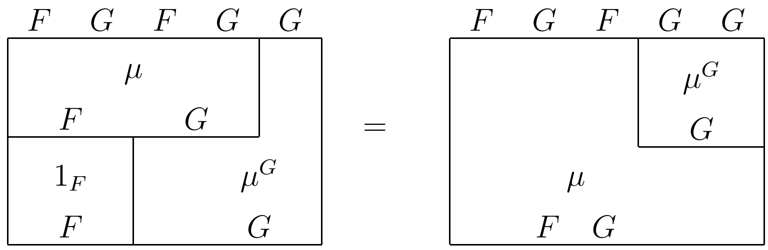


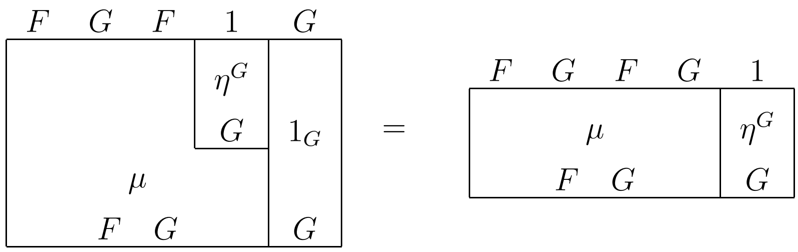
hold together with at least one of the conditions

[3] Simply apply the same steps in reverse ordering.

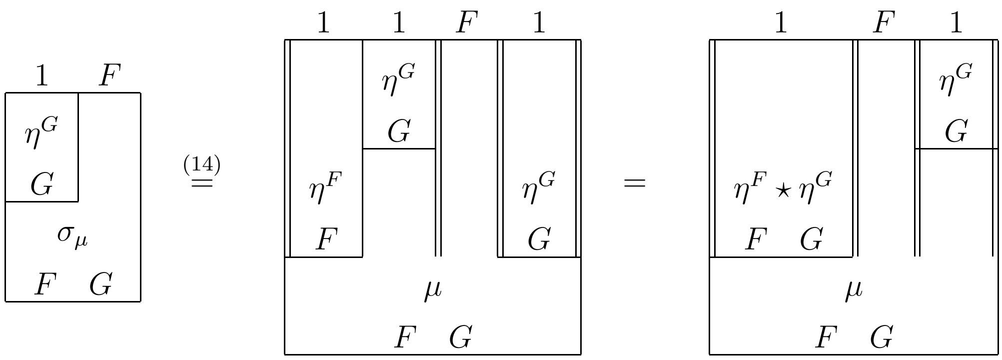
(17)

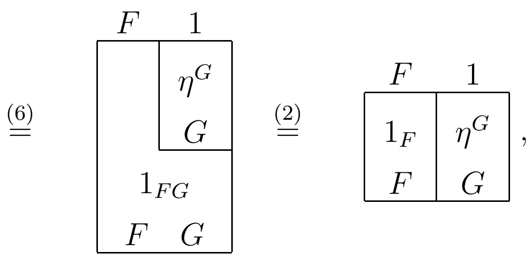


(18)



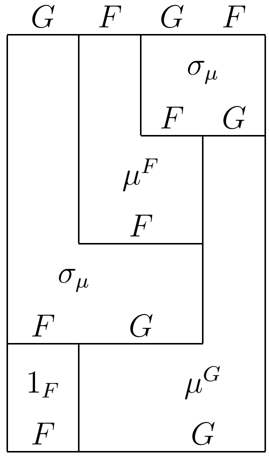
Proof. Condition (9) follows from



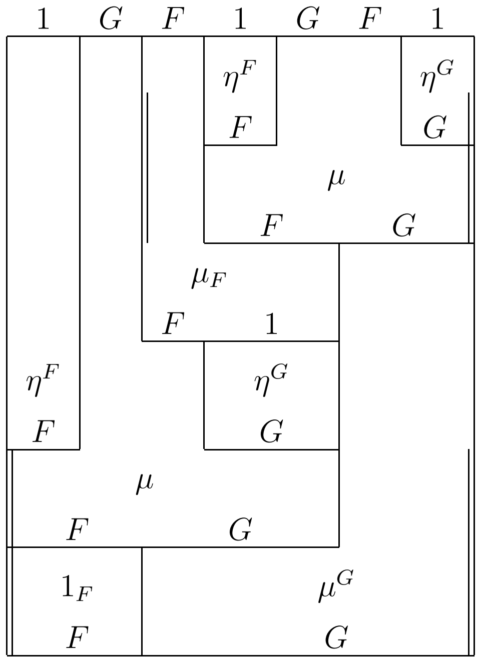


and condition (10) can be shown similarly.

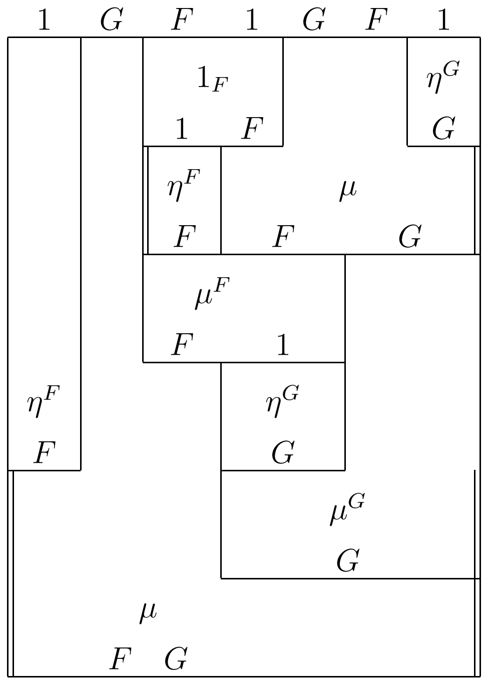
Now, required combinations of conditions (15)-(18) imply condition (11), as shown by



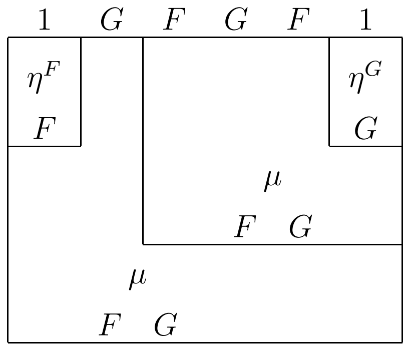
$\overset{(14)}{=}$

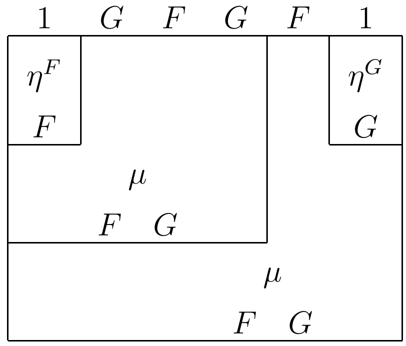
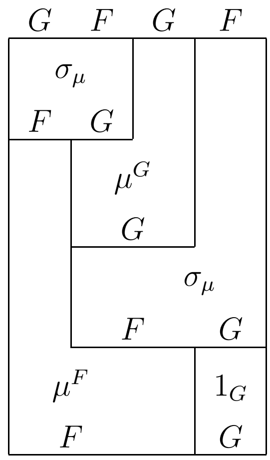


$\overset{(16),(17)}{=}$



$\overset{(6)}{=}$



$\overset{(8)}{=}$  $\overset{revert}{=}$  .

This provides a proof sketch. Other combinations of conditions can easily be applied in a similar way, also likewise in order to prove $\mu = \mu^{FG}$. □

4 Examples of monad compositions

In this section we refer to some examples of monads and swappers that provide monad compositions. This example is interesting in that sufficient conditions for a composition of monads to be a monad were obtained using the graphical approach introduced above, and it illustrates how the complexity of checking monad conditions can be reduced to checking only a few sufficient conditions.

Let L be a completely distributive lattice. For $L = \{0,1\}$, write $L = 2$. The covariant powerset functor L_{id} is obtained by $L_{id}X = L^X$, i.e., the set of mappings $A\colon X \to L$, and following [8], for a morphism $f\colon X \to Y$ in $\mathtt{Set}$, by defining

$$L_{id}f(A)(y) = \bigvee_{f(x)=y} A(x).$$

Further, define $\eta\colon 1 \to L_{id}$ by

$$\eta_X(x)(x') = \begin{cases} 1 & \text{if } x = x' \\ 0 & \text{otherwise} \end{cases}$$

and $\mu\colon L_{id} \circ L_{id} \to L_{id}$ by

$$\mu_X(\mathsf{M})(x) = \bigvee_{A \in L_{id}X} A(x) \wedge \mathsf{M}(A).$$

Then, $\mathbf{L}_{id} = (L_{id}, \eta, \mu)$ is a monad [10], and $\mathbf{2}_{id}$ is the usual covariant powerset monad $\mathbf{P} = (P, \eta, \mu)$, where PX is the set of subsets of X, $\eta_X(x) = \{x\}$ and $\mu_X(\mathsf{B}) = \bigcup \mathsf{B}$.

These powerset monads are suitably composed with the term monad $\mathbf{T}_\Omega = (T_\Omega, \eta^{T_\Omega}, \mu^{T_\Omega})$ [10], where $T_\Omega X$ is the usual set of terms over an operator domain Ω and variables in X, i.e., $T_\Omega X = \bigcup_{k=0}^\infty T_\Omega^k(X)$, where $T_\Omega^0(X) = X$ and $T_\Omega^{k+1}(X) = \{(n, \omega, (m_i)_{i \le n}) \mid \omega \in \Omega_n, n \in N, m_i \in T_\Omega^k(X)\}$. In [6], a swapper $\sigma\colon T_\Omega \circ L_{id} \to L_{id} \circ T_\Omega$ was given by $\sigma_{X|T^0LX} = (1_{L_{id}})_X$ and for

$l = (n, \omega, (l_i)_{i \leq n}) \in T^\alpha LX$, $\alpha > 0$, $l_i \in T^{\beta_i} LX$, $\beta_i < \alpha$, by

$$\sigma_X(l)((n', \omega', (m_i)_{i \leq n})) = \begin{cases} \bigwedge_{i \leq n} \sigma_X(l_i)(m_i) & \text{if } n = n' \text{ and } \omega = \omega', \\ \\ 0 & \text{otherwise,} \end{cases}$$

and it was shown that $\mathbf{L}_{id}$ and $\mathbf{T}_\Omega$ together with σ satisfy conditions in Propositions 3.1 and 3.2.

5 Conclusions

The impact of the paper is two-fold. On one hand, we contribute to methods and tools for generating monad compositions. In particular, we focus on composing various powerset monads with the term monad in order to provide generalised terms for extended many-valued logic programming. On the other hand, we provide a categorical instrumentation for unification in the framework of using generalised terms.

We have shown how compositions and star products of natural transformations can be pictorially represented in order to provide proof support. Handling conditions for monad compositions involve manipulations of rather complicated expressions involving natural transformations, and it is important to continue investigations on how to construct new monads from given ones.

Proving composability conditions is complicated as the complexity of the functors increase. The graphical support is beneficial in that composability proofs are expected to reveal further examples of monad compositions that provide useful scenarios for generalised terms. Not only is the graphical approach a theoretical tool for a better understanding of the composition of natural transformations, but computing with natural transformations could be, to some extent, automatised and managed with such a graphical interface.

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6 Appendix: A standard proof of Proposition 3.2

For comparison purposes, we present here a standard proof of the result stated in Proposition 3.2. Firstly, the statement in rephrased in more conventional terms; then, the proof itself is given.

Proposition 3.2 *If* $\mathbf{F} \bullet \mathbf{G} = (F \circ G, \eta^F G \circ \eta^G, \mu)$ *is a monad, then a natural transformation* $\sigma_\mu : G \circ F \to F \circ G$ *can be defined by* $\sigma_\mu = \mu \circ FGF\eta^G \circ \eta^F GF$ *such that conditions (9) and (10) are satisfied. In addition, condition (11) holds and* $\mu = \mu(\sigma_\mu)$ *under the assumption of any pair of properties* (A_i, B_j) *with* $i, j \in \{1, 2\}$*, where*

(A_1*)* $\mu^F G \circ F\mu = \mu \circ \mu^F GFG$

(A_2*)* $F\mu \circ F\eta^F GFG = F\mu \circ \eta^F FGFG$

(B_1*)* $F\mu^G \circ \mu G = \mu \circ FGF\mu^G$

(B_2*)* $\mu G \circ FGF\eta^G G = \mu G \circ FGFG\eta^G$

Proof.

Condition (9) follows from the fact that the unit transformation of the composed monad is the composition of the units of F and G.

$$\begin{aligned}
\sigma_\mu \circ \eta^G F &= \mu \circ FGF\eta^G \circ \eta^F GF \circ \eta^G F \\
&= \mu \circ FGF\eta^G \circ \eta^{FG} F \\
&= \mu \circ \eta^{FG} FG \circ F\eta^G \\
&= id_{FG} \circ F\eta^G \\
&= F\eta^G.
\end{aligned}$$

and condition (10) can be shown similarly.

Unfolding the definition of σ_μ, the left hand side (LHS) of equation (11) can be written as

$$\mu^F G \circ F\mu \circ F\eta^F GFG \circ FGF\eta^G \circ F\mu^G F \circ \mu GF \circ FGF\eta^G GF \circ \eta^F GFGF$$

and the right hand side (RHS) as

$$F\mu^G \circ \mu G \circ FGF\eta^G G \circ \eta^F GFG \circ G\mu^F G \circ GF\mu \circ GF\eta^F GFG \circ GFGF\eta^G$$

Assume, for instance, the properties (A_2, B_1).

For the LHS of equation (11) we have

$$\begin{aligned}
&\mu^F G \circ F\mu \circ F\eta^F GFG \circ FGF\eta^G \circ F\mu^G F \circ \mu GF \circ FGF\eta^G GF \circ \eta^F GFGF \overset{A_2,B_1}{=} \\
&\mu^F G \circ F\mu \circ \eta^F FGFG \circ FGF\eta^G \circ \mu F \circ FGF\mu^G F \circ FGF\eta^G GF \circ \eta^F GFGF = \\
&\qquad = \mu \circ \mu^F GFG \circ \eta^F FGFG \circ FGF\eta^G \circ \mu F \circ \eta^F GFGF \\
&\qquad = \mu \circ FGF\eta^G \circ \mu F \circ \eta^F GFGF.
\end{aligned}$$

For the RHS we have

$$\begin{aligned}
&F\mu^G \circ \mu G \circ FGF\eta^G G \circ \eta^F GFG \circ G\mu^F G \circ GF\mu \circ GF\eta^F GFG \circ GFGF\eta^G \overset{B_1,A_2}{=} \\
&\mu \circ FGF\mu^G \circ FGF\eta^G G \circ \eta^F GFG \circ G\mu^F G \circ GF\mu \circ G\eta^F FGFG \circ GFGF\eta^G \\
&\qquad = \mu \circ \eta^F GFG \circ G\mu \circ G\mu^F GFG \circ G\eta^F FGFG \circ GFGF\eta^G \\
&\qquad = \mu \circ \eta^F GFG \circ G\mu \circ GFGF\eta^G.
\end{aligned}$$

Note that, in fact, assuming any of the combinations (A_i, B_j), it can be proved that the LHS of equation (11) is equivalent to

$$\mu \circ FGF\eta^G \circ \mu F \circ \eta^F GFGF$$

whereas the RHS is equivalent to

$$\mu \circ \eta^F GFG \circ G\mu \circ GFGF\eta^G$$

The equivalence of the two previous equations follows directly from the graphical representation, although at the level of equational representation it cannot be easily observed. The proof follows as a consequence of the properties of natural transformations, as shown below:

$$\begin{aligned}\mu \circ FGF\eta^G \circ \mu F \circ \eta^F GFGF &= \mu \circ \mu FG \circ FGFGF\eta^G \circ \eta^F GFGF \\ &= \mu \circ FG\mu \circ \eta^F GFGFG \circ GFGF\eta^G \\ &= \mu \circ \eta^F GFG \circ G\mu \circ GFGF\eta^G.\end{aligned}$$

□