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# An Effective Tietze-Urysohn Theorem for QCB-Spaces (Extended Abstract)

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**Abstract**

The Tietze-Urysohn Theorem states that every continuous real-valued function defined on a closed subspace of a normal space can be extended to a continuous function on the whole space. We prove an effective version of this theorem in the Type Two Model of Effectivity (TTE). Moreover, for qcb-spaces we introduce a slightly weaker notion of normality than the classical one and show that this property still admits an Extension Theorem for continuous functions.

*Keywords:* Computable Analysis, Qcb-spaces, Topological spaces

## 1 Introduction

Theorems about extendability of continuous functions belong to the most important theorems in the field of topological spaces. Extendability of a continuous function $f$ onto a larger space $\mathsf{Y}$ means the existence of a continuous function $F$ on $\mathsf{Y}$ which coincides with $f$ on the domain of $f$. A famous example of an extension theorem is the Tietze-Urysohn Theorem for normal topological spaces (cf. [2]). Of similar interest are theorems about extendability of *computable* functions. A computable version of the Tietze-Urysohn Theorem for computable metric spaces has been proved by Weihrauch in [9].

In this paper we prove a continuous and a computable Extension Theorem for a subclass of qcb-spaces that contains all computable metric spaces. Qcb-spaces [7] are known to form exactly the class of topological spaces which can be handled by the representation based approach to Computable Analysis, the Type Two Model of Effectivity (TTE). The category $\mathsf{QCB}$ of qcb-spaces has excellent closure properties, for example it is cartesian closed [1].

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Unfortunately, many interesting Hausdorff qcb-spaces fail to be normal. For example, it was recently proved that the space $\mathbb{N}^{(\mathbb{N}^{\mathbb{N}})}$ of Kleene-Kreisel continuous functionals of order 2 is not regular [6]. Moreover the space of real-valued continuous functions on a computable metric space need not necessarily be normal in $\mathsf{QCB}$ (cf. [6]). Hence the classical Tietze-Urysohn Theorem, which requires normality, can not be applied to these kinds of spaces.

In this paper we introduce a weaker notion of normality called *quasi-normality*. This notion may be considered as a substitute for normality in the class of qcb-spaces (cf. Section 3). We show that quasi-normal qcb-spaces admits extendability of continuous functions defined on functionally closed subspaces (cf. Section 4). The category of quasi-normal qcb-spaces forms a cartesian closed subcategory of $\mathsf{QCB}$ and contains all separable metrisable spaces.

In Section 5 we establish a computable version of the Tietze-Urysohn Theorem. It is formulated for qcb-spaces that satisfy the property of *effective quasi-normality*.

Since this is an extended abstract, most proofs are omitted.

## 2 Preliminaries

After fixing some notations, we repeat some notions and basic facts of topological spaces, of the used computational model, of qcb-spaces and of pseudobases.

### 2.1 Notations

We write $\mathbb{N}$ for the set of natural numbers (including 0) and also for the discrete topological space with carrier set $\mathbb{N}$. The set of infinite sequences over $\mathbb{N}$ is denoted by $\mathbb{N}^{\mathbb{N}}$, the set of finite words over $\mathbb{N}$ by $\mathbb{N}^*$, and, for a word $w \in \mathbb{N}^*$, the set of sequences with prefix $w$ by $w\mathbb{N}^{\mathbb{N}}$. We write $p^{<k}$ for the prefix of $p \in \mathbb{N}^{\mathbb{N}}$ of length $k$ and $\sqsubseteq$ for the prefix relation on $\mathbb{N}^* \cup \mathbb{N}^{\mathbb{N}}$.

Depending on the context, $\langle . \rangle$ stands for a computable bijection either from $(\mathbb{N}^{\mathbb{N}})^k$ to $\mathbb{N}^{\mathbb{N}}$ or from $(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$ to $\mathbb{N}^{\mathbb{N}}$ or from $\mathbb{N}^k$ to $\mathbb{N}$, as defined in [5]. Moreover, we denote by $\mathsf{w}\colon \mathbb{N} \to \mathbb{N}^*$ an effective bijection between $\mathbb{N}$ and $\mathbb{N}^*$. For a subset $M \subseteq \mathbb{R}$, $\varrho_M$ stands for the binary signed-digit representation corestricted to $M$.

### 2.2 Computability theory

As the underlying computational model we use the representation-based approach to Computable Analysis, the Type-2 Theory of Effectivity (TTE). We assume that the reader is familiar with basic concepts of TTE, see [8,10].

We repeat here the less known notion of a computable multi-function. A *multi-function* $H$ from $X$ to $Y$ is a relation between $X$ and $Y$. The domain of $H$ is the set $\mathrm{dom}(H) := \{x \in X \mid \exists y.\, (x,y) \in H\}$. Given two representations $\delta :\subseteq \mathbb{N}^{\mathbb{N}} \to X$ of $X$ and $\gamma :\subseteq \mathbb{N}^{\mathbb{N}} \to Y$ of $Y$, a multi-function $H\colon X \rightrightarrows Y$ is called *computable*, if there is partial computable function $g :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ which maps any name $p$ of an element $x \in \mathrm{dom}(H)$ to a name of *one* possible result for $x$, i.e. for all $p \in \mathrm{dom}(\delta)$

with $\delta(p) \in \mathrm{dom}(H)$ we have $\gamma(g(p)) \in \{y \in Y \mid (x,y) \in H\}$. Computability of ordinary functions is defined correspondingly.

### 2.3 Topological spaces and sequential spaces

To denote topological spaces, we use sans-serif letters like $\mathsf{X}, \mathsf{Y}$ etc. We write $\mathcal{O}(\mathsf{X})$ for the topology of a space $\mathsf{X}$, $\mathcal{A}(\mathsf{X})$ for the family of closed sets of $\mathsf{X}$ and $\mathcal{G}(\mathsf{X})$ for the family of $\mathcal{G}_\delta$-sets of $\mathsf{X}$, which are countable intersections of open sets. We will often denote the carrier set of a space $\mathsf{X}$ by the symbol $\mathsf{X}$.

A subset $A$ of a topological space $\mathsf{X}$ is called *sequentially closed*, if $A$ contains any limit of any convergent sequence of elements in $A$. Complements of sequentially closed sets are called *sequentially open*. For a given topology $\tau$, we denote the topology of sequentially open sets by $seq(\tau)$. Spaces such that every sequentially open set is open are called *sequential*. The sequentialisation (or sequential coreflection) $seq(\mathsf{X})$ of $\mathsf{X}$ is the topological space that carries the topology $seq(\mathcal{O}(\mathsf{X}))$ of sequentially open sets of $\mathsf{X}$. The operator $seq$ is idempotent.

A subset $A$ of a topological space $\mathsf{X}$ is called *functionally closed*, if there is a continuous function $f$ from $\mathsf{X}$ to the unit interval $\mathbb{I} = [0,1]$ (endowed with the usual Euclidean topology) such that $f^{-1}\{0\} = A$. Complements of functionally closed sets are called *functionally open*. A common term for “functionally closed set” is *zero-set*, and for “functionally open set” is *cozero-set*. Two disjoint functionally closed sets $A, B$ can be strongly separated in the sense that there is a continuous function $h\colon \mathsf{X} \to [0,1]$ satisfying $h^{-1}\{0\} = A$ and $h^{-1}\{1\} = B$.

We denote the family of functionally open sets of $\mathsf{X}$ by $\mathcal{FO}(\mathsf{X})$ and the family of functionally closed sets by $\mathcal{FA}(\mathsf{X})$. $T_0$-spaces such that all open sets are functionally open are called *perfectly normal*. If $\mathsf{X}$ is a hereditarily Lindelöf space (i.e. any open cover of any subset has a countable subcover) then $\mathcal{FO}(\mathsf{X})$ forms a topology. It has the property that every real-valued function $f$ on $\mathsf{X}$ is continuous w.r.t. the original topology $\mathcal{O}(\mathsf{X})$ if, and only if, $f$ is continuous w.r.t. $\mathcal{FO}(\mathsf{X})$. Regularity, normality [2] and perfect normality are equivalent for hereditarily Lindelöf spaces (and thus for qcb-spaces, see below).

For more details about the theory of topological spaces we refer to [2,11].

### 2.4 Qcb-spaces and admissible representations

A qcb-space [7] is a topological *q*uotient of a *c*ountably-*b*ased topological space. Qcb$_0$-spaces, i.e. qcb-spaces that satisfy the $T_0$-property, are exactly the class of sequential spaces which have an *admissible* representation and which therefore can be handled by the Type Two Model of Effectivity. Admissibility is a property guaranteeing topological well-behavedness of representations (cf. [4]). The final topology of an admissible representation of a sequential space is equal to the topology of that space.

[2] A *normal* space is a $T_0$-space such that for a pair of disjoint closed sets $(A, B)$ there exists a pair of disjoint open sets $(U, V)$ such that $A \subseteq U$ and $B \subseteq V$. Note that some authors omit the $T_0$-condition.

Qcb-spaces are hereditarily Lindelöf and sequential. The category $\mathsf{QCB}$ of qcb-spaces as objects and of continuous functions as morphisms is cartesian closed. Moreover $\mathsf{QCB}$ has all countable limits and all countable colimits. For two admissible representations $\delta_{\mathsf{X}}$ and $\delta_{\mathsf{Y}}$ of qcb$_0$-spaces $\mathsf{X}$ and $\mathsf{Y}$ we denote by $[\delta_{\mathsf{X}} \to \delta_{\mathsf{Y}}]$ the usual function space representation of $\mathsf{Y}^{\mathsf{X}}$ as defined in [5] or [8].

More information can be found in [1,4,5,7].

### 2.5 Pseudobases and pseudo-open decompositions

Given a topological space $\mathsf{X}$, we say that a family $\mathcal{A}$ of subsets of $\mathsf{X}$ is a *pseudo-open decomposition* of a subset $M$, if $M = \bigcup \mathcal{A}$ holds and for every sequence $(x_n)_n$ that converges to some element $x_\infty \in M$ there is some set $B \in \mathcal{A}$ and some $n_0 \in \mathbb{N}$ such that $\{x_n, x_\infty \,|\, n \geq n_0\} \subseteq B \subseteq M$ holds. Clearly, a set has a pseudo-open decomposition if, and only if, it is sequentially open.

A *pseudobase* for $\mathsf{X}$ is a family $\mathcal{B}$ of subsets such that every open set has a pseudo-open decomposition into sets in $\mathcal{B}$. Any base of topological space is a pseudobase, but not vice versa. Pseudobases are of interest, when they are countable. Every admissible representation $\delta$ of a topological space $\mathsf{X}$ induces a countable pseudobase for $\mathsf{X}$, namely the family $\mathcal{B}_\delta := \big\{\emptyset, \delta(w\mathbb{N}^\omega) \,\big|\, w \in \mathbb{N}^*\big\}$. Using the bijection $\mathsf{w}\colon \mathbb{N} \to \mathbb{N}^*$ from Section 2.1, we equip $\mathcal{B}_\delta$ with a numbering $B_\delta$ defined by $B_\delta(0) := \emptyset$ and $B_\delta(i+1) := \delta(\mathsf{w}(i)\mathbb{N}^{\mathbb{N}})$. Conversely, if $\mathcal{A}$ is a pseudobase of a sequential $T_0$-space, then the space has an admissible representation such that the induced pseudobase is equal to the closure of $\mathcal{A}$ under finite intersection. Hence a sequential $T_0$-space is a qcb-space if, and only if, it has a countable pseudobase.

## 3 Quasi-normal Qcb-Spaces

In this section we introduce and investigate the notion of a quasi-normal qcb-space.

The classical Tietze-Urysohn Theorem is formulated for normal spaces. However, many interesting Hausdorff qcb-spaces fail to be normal. For example, a recent result states that the function space $\mathbb{N}^{(\mathbb{N}^\mathbb{N})}$ formed in the category $\mathsf{QCB}$ is not normal [6]. Hence the final topology of the natural representation on $\mathbb{N}^{(\mathbb{N}^\mathbb{N})}$ is not normal, because it is equal to the topology of the qcb-space $\mathbb{N}^{(\mathbb{N}^\mathbb{N})}$. Moreover, the space $\mathbb{R}^{(\mathbb{R}^\mathbb{R})}$ of continuous real-valued function from $\mathbb{R}^\mathbb{R}$ to $\mathbb{R}$ is not normal either, despite the fact that the compact-open topology on $\mathbb{R}^{(\mathbb{R}^\mathbb{R})}$ is normal and the sequentialisation of the latter yields the topology of $\mathbb{R}^{(\mathbb{R}^\mathbb{R})}$.

Therefore we need an appropriate substitute for the property of normality. The idea behind the following definition is the fact that finite products and function spaces in the category $\mathsf{QCB}$ are constructed as the sequentialisation of their counterparts in classical topology, which enjoy the property of preserving regularity and even normality in the case of countably-pseudobased spaces.

**Definition 3.1** A qcb-space $\mathsf{X}$ is called *quasi-normal*, if $\mathsf{X}$ is the sequentialisation of a normal space.

In other words, a qcb-space is quasi-normal if, and only if, its convergence relation is induced by a normal topology. Simple examples of quasi-normal spaces are countably based normal spaces, because countably based spaces are equal to their sequentialisation. In [3, Example 1.2] Michael gave an example of a regular space such that its sequentialisation is not regular. This sequentialisation turns out to be a qcb-space, thus it is an example of a quasi-normal, but not normal qcb-space.

We will give now two characterisations of quasi-normality.

**Proposition 3.2** *A qcb-space $\mathsf{X}$ is quasi-normal if, and only if, its convergence relation is induced by the topology of functionally open sets.*

Note that $\mathcal{FO}(\mathsf{X})$ is indeed a topology, if $\mathsf{X}$ is a qcb-space, because qcb-spaces are hereditarily Lindelöf spaces.

**Proof.** Omitted. □

We characterise quasi-normal qcb-spaces in terms of properties of pseudobases. Recall that qcb-spaces are known to be those sequential spaces that have a countable pseudobase (cf. [5,7]).

**Proposition 3.3** *A qcb-space is quasi-normal if, and only if, it is a $T_0$-space and has a countable pseudobase consisting of functionally closed sets.*

We omit the proof which is based on the following surprising lemma. By a *functional $\mathcal{G}_\delta$-set* we mean a set that is a countable intersection of functionally open sets.

**Lemma 3.4** *Let $\mathsf{X}$ be a qcb-space equipped with a countable pseudobase consisting of functionally closed sets. Then every open functional $\mathcal{G}_\delta$-set is functionally open.*

**Proof.** Let $G_0, G_1, \dots$ be a sequence of functionally open sets such that $V := \bigcap_{j=0}^{\infty} G_j$ is open. Let $(\beta_i)_i$ be a pseudo-open decomposition of $V$ (see Section 2.5) into pseudobase sets. Since the functionally closed set $\bigcup_{i=0}^{n} \beta_i$ is contained in $G_n$, there exists a continuous function $h_n \colon \mathsf{X} \to [0,1]$ with $h_n^{-1}\{0\} = \mathsf{X} \backslash G_n$ and $h_n^{-1}\{1\} = \bigcup_{i=0}^{n} \beta_i$ by [2, Theorem 1.5.14]. We define a function $f \colon \mathsf{X} \to [0,1]$ by $f(x) := \inf_{n \in \mathbb{N}} h_n(x)$ and show that $f$ is sequentially continuous with $f^{-1}\{0\} = \mathsf{X} \setminus V$. Let $(x_n)_n$ be a sequence converging in $\mathsf{X}$ to some $x_\infty$.

(1) Let $x_\infty \in V$. Then there is some $i_0, n_0 \in \mathbb{N}$ such that $\{x_n \,|\, n \geq n_0\} \subseteq \beta_{i_0}$. Thus for all $j \geq i_0$ and $n \geq n_0$ (including $n = \infty$) we have $h_j(x_n) = 1$ and $f(x_n) = \min\{h_0(x_n), \dots, h_{i_0}(x_n)\}$. This implies that $(f(x_n))_n$ converges to $f(x_\infty)$. Moreover, since $h_j(x_\infty) \neq 0$ for all $j \leq i_0$, $f(x_\infty) \neq 0$.

(2) Let $x_\infty \notin V$. Then there is some $j \in \mathbb{N}$ with $x_\infty \notin G_j$, hence $f(x_\infty) = h_j(x_\infty) = 0$. As $(h_j(x_n))_n$ converges to 0, $(f(x_n))_n$ converges to 0 as well.

Hence $f$ is sequentially continuous and therefore (topologically) continuous, because $\mathsf{X}$ is sequential. So $f$ is a witness for $V$ being functionally open. □

One can show that forming (i) countable products, (ii) subspaces, (iii) countable coproducts, (iv) function spaces in the category of qcb-spaces preserves quasi-normality. Hence:

**Theorem 3.5** *The category of quasi-normal qcb-spaces is cartesian closed. Moreover it has all countable limits and all countable coproducts.*

## 4 An Extension Theorem for Quasi-Normal Qcb-Spaces

In this section we prove an Extension Theorem for quasi-normal qcb-spaces. It states that every continuous function from a functionally closed subset into the unit interval can be extended to a continuous function on the whole space.

### 4.1 A transitivity property for zero-sets

It is well-known that the subspace operator on topological spaces has the following transitivity property: Any functionally open subset of a functionally open subspace is functionally open in the original space, whereas the analogous statement for functionally closed sets is false in general (cf. [2, 2.1.B]).

Validity of the transitivity property for zero-sets (= functionally closed sets) is related to extendability of continuous functions. Let $\mathsf{X}$ be a functionally closed subspace of a topological space $\mathsf{Y}$. If any continuous $[0,1]$-valued function on $\mathsf{X}$ is extendable onto $\mathsf{Y}$, then any functionally closed subset $M$ of $\mathsf{X}$ is functionally closed in $\mathsf{Y}$: Take continuous functions $f\colon \mathsf{X} \to [0,1]$ and $g\colon \mathsf{Y} \to [0,1]$ with $f^{-1}\{0\} = M$ and $g^{-1}\{0\} = \mathsf{X}$ and extend $f$ to a continuous function $F\colon \mathsf{Y} \to [0,1]$. Then $\lambda_{y\in\mathsf{Y}}.\max\{F(y), g(y)\}$ is a continuous function witnessing that $M$ is functionally closed in $\mathsf{Y}$.

It follows from [2, 2.1.J] that the reverse implication is true as well. So we will prove at first that quasi-normal qcb-spaces have the property that any zero-set of any functionally closed subspace is also a zero-set of the original space.

### 4.2 Proof of the transitivity property for zero-sets

Let $\mathsf{Y}$ be a quasi-normal qcb-space and $\mathsf{X}$ be a functionally closed subspace of $\mathsf{Y}$. By Proposition 3.3, $\mathsf{Y}$ has a countable pseudobase $\mathcal{B}$ consisting of functionally closed sets. We define $\tau$ to be the topology on $\mathsf{Y}$ given by

$$\tau := \big\{U \in \mathcal{O}(\mathsf{Y}) \,\big|\, U \cap \mathsf{X} \in \mathcal{FO}(\mathsf{X}) \text{ and } U \setminus \mathsf{X} \in \mathcal{FO}(\mathsf{Y})\big\} \tag{1}$$

and show that $\tau$ is equal to the topology $\mathcal{FO}(\mathsf{Y})$ of functionally open sets of $\mathsf{Y}$. Note that $\tau$ is indeed closed under arbitrary union, because $\mathcal{O}(\mathsf{Y})$, $\mathcal{FO}(\mathsf{X})$ and $\mathcal{FO}(\mathsf{Y})$ are all hereditarily Lindelöf topologies by having a countable pseudobase.

Clearly we have $\mathcal{FO}(\mathsf{Y}) \subseteq \tau \subseteq \mathcal{O}(\mathsf{Y})$. The proof of the reverse inclusion $\mathcal{FO}(\mathsf{Y}) \supseteq \tau$ is based on three lemmas about $\mathcal{G}_\delta$-sets, namely Lemma 3.4 and the following two lemmas. They are direct consequences of the existence of a countable functionally closed pseudobase for $\mathsf{Y}$.

**Lemma 4.1** *Let $V$ be open and let $\{\beta_i \,|\, i \in \mathbb{N}\}$ be a pseudo-open decomposition of $V$ into pseudobase elements in $\mathcal{B}$. Moreover, let $(U_j)_j$ be a sequence of open sets such that $i \leq j$ implies $\beta_i \subseteq U_j$. Then the $\mathcal{G}_\delta$-set $V \cap \bigcap_{j=0}^{\infty} U_j$ is open.*

**Proof.** Omitted. □

The complement of any closed subset of $\mathsf{Y}$ has a decompostion into sets of the countable and functionally closed pseudobase $\mathcal{B}$. Hence:

**Lemma 4.2** *Every closed subset of* $\mathsf{Y}$ *is a functional* $\mathcal{G}_\delta$*-set of* $\mathsf{Y}$.

The key step of the proof of the transitivity property for zero-sets (Proposition 4.6) is to show that the topology $\tau$ satisfies the following 'normality' property.

**Lemma 4.3** *For every functionally closed set* $A \in \mathcal{FA}(\mathsf{Y})$ *and every set* $U \in \tau$ *containing* $A$ *there is a set* $U' \in \tau$ *and a functionally closed set* $A' \in \mathcal{FA}(\mathsf{Y})$ *satisfying* $A \subseteq U' \subseteq A' \subseteq U$.

**Proof.** We omit the non-trivial proof. □

We employ Lemma 4.3 to show the following separation lemma. It resembles Urysohn's Separation Lemma which states that two disjoint closed sets in a normal space can be separated by a continuous real-valued function (cf. [2, Theorem 1.5.11]).

**Lemma 4.4** *For every functionally closed set* $A \in \mathcal{FA}(\mathsf{Y})$ *and every set* $U \in \tau$ *with* $A \subseteq U$ *there is a continuous function* $h\colon \mathsf{Y} \to [0,1]$ *with* $A \subseteq h^{-1}\{0\}$ *and* $\mathsf{Y} \setminus U \subseteq h^{-1}\{1\}$.

**Proof idea:** The proof is a variation of the standard proof of Urysohn's Separation Lemma using the weaker normality property stated in Lemma 4.3 in place of standard normality. Details are omitted. □

From Lemma 4.4 we can deduce:

**Lemma 4.5** *The topology* $\tau$ *is equal to the family of functionally open sets of* $\mathsf{Y}$.

**Proof.** Omitted. □

Finally we obtain our transitivity result for functionally closed sets.

**Proposition 4.6** *Let* $\mathsf{X}$ *be a functionally closed subspace of a quasi-normal qcb-space* $\mathsf{Y}$. *Then every set that is functionally closed in* $\mathsf{X}$ *is functionally closed in* $\mathsf{Y}$.

**Proof.** Let $A \in \mathcal{FA}(\mathsf{X})$. Then the set $U := \mathsf{Y} \setminus A = (\mathsf{X} \setminus A) \cup (\mathsf{Y} \setminus \mathsf{X})$ is an element of $\tau$ and thus functionally open in $\mathsf{Y}$ by Lemma 4.5. Hence $A \in \mathcal{FA}(\mathsf{Y})$. □

### *4.3 The Extension Theorem for Continuous Functions*

In this section we formulate and prove the Extension Theorem for quasi-normal qcb-spaces. The proof follows the lines of the proof of the original Tietze-Urysohn Theorem (cf. [2, Theorem 2.1.8]), using Proposition 4.6 in place of Urysohn's Separation Lemma.

**Theorem 4.7** *Let* $\mathsf{X}$ *be a functionally closed subspace of a quasi-normal qcb-space* $\mathsf{Y}$. *Then every continuous function* $f\colon \mathsf{X} \to [0,1]$ *can be extended to a continuous function* $F\colon \mathsf{Y} \to [0,1]$ *satisfying* $F(x) = f(x)$ *for all* $x \in \mathsf{X}$.

**Proof.** Omitted. □

In general closed subspaces of a quasi-normal qcb-space which are not functionally closed do not admit extendability of continuous real-valued functions.

# 5 An Effective Version of the Extension Theorem

In this section we establish an effective version of the Tietze-Urysohn Extension Theorem. This theorem is formulated for qcb-spaces that satisfy a computable notion of quasi-normality, which we call *effective quasi-normality.*

## 5.1 Representations for families of subsets

Given an admissible representation $\delta$ of a qcb-space $\mathsf{Y}$, we introduce representations (derived from $\delta$) for the following families of subsets of $\mathsf{Y}$: the open sets, the closed sets, the functionally opens sets, the functionally closed sets, and the functional $G_\delta$-sets (= countable intersections of functionally open sets).

To define the representations of $\mathcal{O}(\mathsf{Y})$ and $\mathcal{A}(\mathsf{Y})$, we use the fact that every open set has a pseudo-open decomposition into elements of the pseudobase $\mathcal{B}_\delta$ induced by $\delta$. Using the effective bijective numbering $\mathsf{w}\colon \mathbb{N} \to \mathbb{N}^*$ of $\mathbb{N}^*$ from Section 2.1, we define the representations $\delta^{\mathcal{O}}$ of $\mathcal{O}(\mathsf{Y})$ and $\delta^{\mathcal{A}}$ of $\mathcal{A}(\mathsf{Y})$ by

$$\delta^{\mathcal{O}}(q) = V \;:\Longleftrightarrow\; \delta^{-1}(V) = \big\{p \in \mathrm{dom}(\delta) \,\big|\, \exists i.\, q(i) > 0 \wedge \mathsf{w}(q(i)-1) \sqsubseteq p\big\}$$

and $\delta^{\mathcal{A}}(q) := \mathsf{Y} \setminus \delta^{\mathcal{O}}(q)$. One can show that $\delta^{\mathcal{O}}$ is computably equivalent to the Sierpiński representation of $\mathcal{O}(\mathsf{Y})$, which encodes an open set $V$ via its characteristic function $cf_V$ from $\mathsf{Y}$ into the Sierpiński space. The Sierpiński space has $\{\bot, \top\}$ as its underlying set and $\{\bot\}$ is its only closed singleton.

By using the standard function representation $[\delta \to \varrho_{[0,1]}]$ of the space the continuous functions from $\mathsf{Y}$ to $[0,1]$, we define representations $\delta^{\mathcal{FO}}$ of $\mathcal{FO}(\mathsf{Y})$ and $\delta^{\mathcal{FA}}$ of $\mathcal{FA}(\mathsf{Y})$ by

$$\delta^{\mathcal{FA}}(q) := \big\{y \in \mathsf{Y} \,\big|\, [\delta \to \varrho_{[0,1]}](q)(y) = 0\big\} \;\text{ and }\; \delta^{\mathcal{FO}}(q) := \mathsf{Y} \setminus \delta^{\mathcal{FA}}(q)\,.$$

Finally, we define the representation $\delta^{\mathcal{FG}}$ of the family of functional $\mathcal{G}_\delta$-sets by

$$\delta^{\mathcal{FG}}(\langle q_0, q_1, \dots\rangle) := \textstyle\bigcap_{j=0}^{\infty} \delta^{\mathcal{FO}}(q_j)\,,$$

where $\langle\,.\,\rangle$ denotes a standard computable bijection from $(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$ to $\mathbb{N}^{\mathbb{N}}$.

With standard methods of TTE, one can prove the following lemma. It presents effective versions of known theorems in the theory of topological spaces.

**Lemma 5.1**

(i) *Finite union and finite intersection (on the respective family of subsets) are computable w.r.t. each of the representations* $\delta^{\mathcal{O}}, \delta^{\mathcal{A}}, \delta^{\mathcal{FO}}, \delta^{\mathcal{FA}}, \delta^{\mathcal{FG}}$.

(ii) *The multi-function that maps two disjoint functionally closed sets $A, B$ to all continuous functions $h\colon \mathsf{Y} \to [0,1]$ satisfying $h^{-1}\{0\} = A$ and $h^{-1}\{1\} = B$ is computable w.r.t.* $\delta^{\mathcal{FA}}$ *and* $[\delta \to \varrho_{[0,1]}]$.

(iii) *The function that maps a continuous function* $h\colon X \to \mathbb{R}$ *and two real numbers* $r, s \in \mathbb{R}$ *to the functionally closed set* $h^{-1}[r, s]$ *is computable w.r.t.* $[\delta \to \varrho_{\mathbb{R}}]$, $\varrho_{\mathbb{R}}$ *and* $\delta^{\mathcal{FA}}$.

(iv) *The representation* $\delta^{\mathcal{FO}}$ *is computably reducible to* $\delta^{\mathcal{O}}$, *and* $\delta^{\mathcal{FA}}$ *is computably reducible to* $\delta^{\mathcal{A}}$.

(v) *Let* $\cdot^{\mathsf{op}}$ *be any operator in* $\{\cdot^{\mathcal{O}}, \cdot^{\mathcal{A}}, \cdot^{\mathcal{FO}}, \cdot^{\mathcal{FA}}\}$. *For any* $\delta^{\mathsf{op}}$*-computable subset* $M$, *the function* $A \mapsto A \cap M$ *is computable w.r.t.* $\delta^{\mathsf{op}}$ *and* $(\delta|^M)^{\mathsf{op}}$.

(vi) *Let* $\cdot^{\mathsf{op}}$ *be any operator in* $\{\cdot^{\mathcal{O}}, \cdot^{\mathcal{A}}, \cdot^{\mathcal{FO}}\}$. *For any* $\delta^{\mathsf{op}}$*-computable subset* $M$, $(\delta|^M)^{\mathsf{op}}$ *is computably reducible to* $\delta^{\mathsf{op}}$.

Here $\delta|^M$ denotes the corestriction of $\delta$ to the subset $M$.

### 5.2 Effectively quasi-normal spaces

We introduce an effectivised version of the notion of a quasi-normal space.

**Definition 5.2** *Let* $\mathsf{Y}$ *be a qcb-space.*

(i) An admissible representation $\delta$ of $\mathsf{Y}$ is called *effectively functionally closed*, if the pseudobase $\mathcal{B}_\delta = \{B_\delta(n) \mid n \in \mathbb{N}\}$ induced by $\delta$ consists of functionally closed sets and the sequence $(B_\delta(n))_n$ is computable w.r.t. the representation $\delta^{\mathcal{FA}}$.

(ii) The space $\mathsf{Y}$ is called *effectively quasi-normal*, if $\mathsf{Y}$ has an admissible and effectively functionally closed representation.

By having a functionally closed pseudobase, an effectively quasi-normal space is indeed quasi-normal (cf. Proposition 3.3). The standard construction in [4] of an admissible representation built from a functionally closed pseudobase yields a representation that induces a functionally closed pseudobase. An example of an effectively functionally closed representation is the signed-digit representation $\varrho_{\mathbb{R}}$, because the function $(a, b) \mapsto [a, b]$ is computable w.r.t. $\varrho_{\mathbb{R}}$ and $\varrho_{\mathbb{R}}^{\mathcal{FA}}$. Computable equivalence of representations do not preserve this effectivity property, simply because there are effective representations of the Euclidean space that induce pseudobases containing non-closed sets.

### 5.3 The effective Tietze-Urysohn Extension Theorem

Now we are ready to formulate the effective Tietze-Urysohn Extension Theorem for quasi-normal qcb-spaces. We state a uniform and a non-uniform version.

**Theorem 5.3** *Let* $\mathsf{Y}$ *be a quasi-normal qcb-space equipped with an admissible effectively functionally closed representation* $\delta$. *Moreover let* $X$ *be a* $\delta^{\mathcal{FA}}$*-computable subset of* $\mathsf{Y}$. *Then the multi-function that maps any continuous function* $f\colon X \to [0, 1]$ *to all its continuous extensions* $F\colon \mathsf{Y} \to [0, 1]$ *is computable w.r.t.* $[\delta|^X \to \varrho_{[0,1]}]$ *and* $[\delta \to \varrho_{[0,1]}]$.

The non-uniform version reads as follows:

**Theorem 5.4** *Let* $\mathsf{Y}$ *be a quasi-normal qcb-space equipped with an admissible effec-*

*tively functionally closed representation $\delta$. Moreover let $X$ be a $\delta^{\mathcal{FA}}$-computable subset of $\mathsf{Y}$. Then every $(\delta|^X, \varrho_{[0,1]})$–computable function $f\colon X \to [0,1]$ has a $(\delta, \varrho_{[0,1]})$–computable extension $F\colon \mathsf{Y} \to [0,1]$.*

### 5.4 Sketch of Proof of the effective Extension Theorem

The effective Tietze-Urysohn Theorem can be deduced from the following proposition along with Lemma 5.1 by carefully effectivising the proof of Theorem 4.7.

**Proposition 5.5** *Let $\mathsf{Y}$ be a quasi-normal qcb-space equipped with an admissible effectively functionally closed representation $\delta$. For every $\delta^{\mathcal{FA}}$-computable subset $X \subseteq \mathsf{Y}$, the representation $(\delta|^X)^{\mathcal{FA}}$ is computably reducible to $\delta^{\mathcal{FA}}$.*

This effectivisation of Proposition 4.6 can be proved by showing effective versions of the lemmas in Section 4 on which Proposition 4.6 is based. Their proofs can be obtained by effectivising the proofs of their topological counterparts using Lemma 5.1.

In the sequel, we assume that $\delta$ is an admissible effectively functionally closed representation of $\mathsf{Y}$ and that $X$ is a $\delta^{\mathcal{FA}}$-computable subset of $\mathsf{Y}$. By Lemma 5.1, $\delta|^X$ is an effectively functionally closed representation of $X$ endowed with the sequential subspace topology inherited from $\mathsf{Y}$. As a pseudobase for $\mathsf{Y}$ we use the functionally closed pseudobase induced by $\delta$.

At first we introduce a representation $\Omega$ of the topology $\tau$ from Equation (1). We define it by

$$\Omega\langle q, r, s\rangle = U \;:\Longleftrightarrow\; \big(\delta^{\mathcal{O}}(q) = U,\, (\delta|^X)^{\mathcal{FO}}(r) = U \cap X,\, \delta^{\mathcal{FO}}(s) = U \setminus X\big)\,.$$

Here $\langle \cdot,\cdot,\cdot\rangle$ denotes a computable bijection between $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ and $\mathbb{N}^{\mathbb{N}}$.

The effective version of Lemma 3.4 states that any $\delta^{\mathcal{O}}$-name of a functionally open set $V$ can be converted into a $\delta^{\mathcal{FO}}$-name, when additionally the information about $V$ as a functional $\mathcal{G}_\delta$-set is given by means of a $\delta^{\mathcal{FG}}$-name. To formulate Lemma 5.6 precisely, we represent the family of all open functional $\mathcal{G}_\delta$-sets by the conjunction [3] of the representations $\delta^{\mathcal{O}}$ and $\delta^{\mathcal{FG}}$.

**Lemma 5.6** *The representation $\delta^{\mathcal{O}} \wedge \delta^{\mathcal{FG}}$ is computably equivalent to $\delta^{\mathcal{FO}}$, and $(\delta|^X)^{\mathcal{O}} \wedge (\delta|^X)^{\mathcal{FG}}$ is computably equivalent to $(\delta|^X)^{\mathcal{FO}}$.*

**Proof.** Omitted. □

The following technical lemma effectivises Lemma 4.1.

**Lemma 5.7** *There is a computable function $g\colon \mathbb{N}^{\mathbb{N}} \times (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that*

$$\delta^{\mathcal{O}}\big(g(q, s_0, s_1, \dots)\big) = \delta^{\mathcal{O}}(q) \cap \textstyle\bigcap_{j=0}^{\infty} \delta^{\mathcal{O}}(s_j)$$

*holds for all $q, s_0, s_1, \dots \in \mathrm{dom}(\delta^{\mathcal{O}})$ satisfying $i \le j \implies B_\delta(q(i)) \subseteq \delta^{\mathcal{O}}(s_j)$.*

**Proof.** Omitted. □

An effective version of Lemma 4.2 reads as follows:

[3] The conjunction $\delta^{\mathcal{O}} \wedge \delta^{\mathcal{FG}}$ is defined by $(\delta^{\mathcal{O}} \wedge \delta^{\mathcal{FG}})(\langle q, s\rangle) = V :\Longleftrightarrow \delta^{\mathcal{O}}(q) = \delta^{\mathcal{FG}}(s) = V$, cf. [5,8].

**Lemma 5.8** *The representation $\delta^{\mathcal{A}}$ is computably reducible to $\delta^{\mathcal{FG}}$.*

**Proof.** Any $\delta^{\mathcal{A}}$-name $q$ of a closed set $A$ provides a sequence $(\beta_i)_i$ of pseudobase elements in $\mathcal{B}_\delta$ such that their union is the complement of $A$. By the effectivity condition on $\delta$ and by Lemma 5.1, we can convert $q$ into a $\delta^{\mathcal{FG}}$-name of the set $A = \bigcap_{i=0}^{\infty}(\mathsf{Y} \setminus \beta_i)$. $\square$

Lemmas 4.3 and 4.4 can be effectivised by stating computability of appropriate multi-functions.

**Lemma 5.9** *The multi-function which maps a functionally closed set $A \in \mathcal{FA}(\mathsf{Y})$ and a set $U \in \tau$ with $A \subseteq U$ to all pairs $(U', A') \in \tau \times \mathcal{FA}(\mathsf{Y})$ satisfying $A \subseteq U' \subseteq A' \subseteq U$ is computable w.r.t. the representations $\delta^{\mathcal{FA}}$ and $\Omega$.*

**Lemma 5.10** *The multi-function which maps a functionally closed set $A \in \mathcal{FA}(\mathsf{Y})$ and a set $U \in \tau$ with $A \subseteq U$ to all continuous functions $h \colon \mathsf{Y} \to [0,1]$ satisfying $A \subseteq h^{-1}\{0\}$ and $\mathsf{Y} \setminus U \subseteq h^{-1}\{1\}$ is computable w.r.t. the representations $\delta^{\mathcal{FA}}$, $\tau$ and $[\delta \to \varrho_{[0,1]}]$.*

By Lemma 4.5, the topology is $\tau$ is equal to $\mathcal{FO}(\mathsf{Y})$. We express this property in terms of computable equivalence of representations.

**Lemma 5.11** *The representations $\Omega$ and $\delta^{\mathcal{FO}}$ are computably equivalent.*

Lemmas 5.9, 5.10 and 5.11 can be proven by effectivisations of the proofs of their topological counterparts using Lemmas 5.1, 5.6, 5.7 and 5.8. We omit the details.

## 6 Discussion

We have shown that quasi-normality yields a reasonable substitute for the property of normality in the category of qcb-spaces. It admits a continuous and, in its effective version, a computable Extension Theorem for functions defined on functionally closed subspaces. The category of quasi-normal qcb-spaces contains all countably based normal spaces and enjoys excellent closure properties, for example it is cartesian closed. By contrast, the category of normal qcb-spaces is not cartesian closed: one can use the Extension Theorem 4.7 and a non-regularity result from [6] to prove that for any separable metric space $\mathsf{M}$ that is not locally compact the function space $\mathbb{R}^{\mathsf{M}}$ formed in $\mathsf{QCB}$ is not normal. An open question is whether or not the category of qcb-spaces endowed with an admissible effectively functionally closed representation is cartesian closed.

## References


[1] Battenfeld, I., Schröder, M., and Simpson, A.: *A Convenient Category of Domains*, Electronic Notes in Theoretical Computer Science **172** (2007), pp. 69–99.

[2] Engelking, R.: *General Topology*, Heldermann, Berlin (1989).

[3] Michael, E.: *On k-spaces, $k_R$-spaces and $k(X)$*; Pacific Journal of Mathematics **47(2)** (1973), 487–498.

[4] Schröder, M.: *Extended Admissibility*, Theoretical Computer Science **284** (2002), pp. 519–538.

[5] Schröder, M.: *Admissible Representations for Continuous Computations*, Ph.D. Thesis, Informatik–Berichte 299, FernUniversität Hagen (2003).

[6] Schröder, M.: *The sequential topology on* $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ *is not regular*, submitted (2008).

[7] Simpson, A.: *Towards a Convenient Category of Topological Spaces*, in: Proceedings of Thirteenth ALGI Workshop (2003).

[8] Weihrauch, K.: *Computable Analysis*, Springer, Berlin (2000).

[9] Weihrauch, K.: *On Computable Metric Spaces Tietze-Urysohn Extension is Computable*, Lecture Notes in Computer Science **2064** (2001), pp. 357-368.

[10] Weihrauch, K.: *The Computable Multi-Functions on Multi-Represented Sets are Closed under Programming*, Journal of Universal Computer Science **14(6)** (2008), pp. 801–844.

[11] Willard, S.: *General Topology*, Addison–Wesley Publishing Company, Reading Massachusetts (1970).