Available online at www.sciencedirect.com

ScienceDirect

Electronic Notes in Theoretical Computer Science

Electronic Notes in Theoretical Computer Science 346 (2019) 221–228

www.elsevier.com/locate/entcs

Large Immersions in Graphs with Independence Number 3 and 4

S. Bustamante[1,2,5]

Departamento de Ingeniería Matemática
Universidad de Chile
Santiago, Chile

D.A. Quiroz[1,5] M. Stein[1,3,5]

Departamento de Ingeniería Matemática y Centro de Modelamiento Matemático
Universidad de Chile
Santiago, Chile

J. Zamora[4,6]

Departamento de Matemáticas
Universidad Andres Bello
Santiago, Chile

Abstract

The analogue of Hadwiger's conjecture for the immersion order, a conjecture independently posed by Lescure and Meyniel, and by Abu-Khzam and Langston, states that every graph G which does not contain the complete graph K_{t+1} as an immersion satisfies $\chi(G) \leq t$. If true, this conjecture would imply that every graph with n vertices and independence number α contains $K_{\lceil \frac{n}{\alpha} \rceil}$ as an immersion (and if $\alpha = 2$, the two statements are known to be equivalent).

The immersion conjecture has been tackled with more success than its graph minors counterpart: not only is a linear upper bound known for the chromatic number of K_{t+1}-immersion-free graphs, but the best bound currently known is very close to optimal. Namely, the currently best bound in this respect is due to Gauthier, Le and Wollan, who recently proved that every graph not containing K_{t+1} as an immersion satisfies $\chi(G) \leq 3.54t + 7$. Their result implies that any graph with n vertices and independence number α contains $K_{\lceil \frac{n}{3.54\alpha} - c \rceil}$ as an immersion, where $c < 1.98$. Moreover, the same authors prove that every graph of independence number 2 contains $K_{2\lfloor \frac{n}{5} \rfloor}$ as an immersion.

We show that any graph with n vertices and independence number 3 contains a clique immersion on at least $\lfloor \frac{2n}{9} \rfloor - 1$ vertices, and any graph with n vertices and independence number 4 contains a clique immersion on at least $\lfloor \frac{4n}{27} \rfloor - 1$ vertices. Thus, comparing to the bound from above, in both cases we roughly double the size of the immersion obtained.

Keywords: Graph immersion, independence number, chromatic number, Hadwiger's conjecture

https://doi.org/10.1016/j.entcs.2019.08.020

1571-0661/

1 Introduction

For a graph G, let $\chi(G)$ denote its chromatic number. A famous conjecture of Hadwiger [13] states that every graph G which does not contain the complete graph K_{t+1} as a minor satisfies $\chi(G) \leq t$. The conjecture is only known to be true for $1 \leq t \leq 5$; the proofs for cases $t = 4$ and $t = 5$ depend on the Four Color Theorem. For general t, the currently best upper bound on the chromatic number springs from (independent) results of Kostochka [16] and Thomason [20] and is $\chi(G) \in \mathcal{O}(t\sqrt{\log t})$.

Letting $\alpha(G)$ denote the independence number of a graph G, it is well known and easy to see that $\chi(G) \geq \lceil \frac{n}{\alpha(G)} \rceil$. Therefore, the Four Color Theorem implies that every planar graph on n vertices has an independent set of size $\lceil \frac{n}{4} \rceil$. In 1968, Erdős asked whether this corollary could be proved without using the Four Color Theorem. This question is still open (see [5]).

Similarly, Hadwiger's conjecture, if true, implies that a graph G on n vertices excluding K_{t+1} as a minor contains an independent set of size $\lceil \frac{n}{t} \rceil$, which would be best possible as evidenced by the disjoint union of copies of K_t. Alternatively, we could say Hadwiger's conjecture implies that every n-vertex graph contains a clique minor of size at least $\lceil \frac{n}{\alpha(G)} \rceil$.

As mentioned above, it is not known whether every graph G excluding the complete graph K_{t+1} as a minor satisfies $\chi(G) \leq 2t$ (and indeed, any linear upper bound on Hadwiger's conjecture would be a major breakthrough). However, in 1982, Duchet and Meyniel [9] proved that any graph G on n vertices contains a clique minor of size $\lceil \frac{n}{2\alpha(G)-1} \rceil$. There have been several improvements on this bound [3,11,14,15,22], the best bound, due to Balogh and Kostochka [2], being $\lceil \frac{n}{c\alpha(G)} \rceil$, where c is a constant with $c < 1.95$.

In this extended abstract, we shall focus on the immersion relation on graphs, which has received extensive attention in recent years. The immersion relation can be seen as a variant of the topological minor relation. A graph G is said to contain another graph H as an *immersion* if there exists an injective function $\phi\colon V(H) \to V(G)$ such that:

(i) For every $uv \in E(H)$, there is a path in G, denoted P_{uv}, with endpoints $\phi(u)$ and $\phi(v)$.

(ii) The paths in $\{P_{uv} \mid uv \in E(H)\}$ are pairwise edge disjoint.

We call the vertices in $\phi(V(H))$, the *branch vertices* of the immersion. By *splitting off the path* P_{uv} we mean we delete the edges of the path and add an edge uv to G (unless it is already present). So, H is an immersion of G, if it can be obtained from

[1] Supported by CONICYT, PIA/Apoyo a Centros Científicos y Tecnológicos de Excelencia con Financiamiento Basal AFB170001.

[2] Supported by CONICYT Doctoral Fellowship grant 21141116.

[3] Supported by FONDECYT Regular grant 1183080.

[4] Supported by FONDECYT Regular grant 1180994.

[5] Email: {dquiroz, mstein, sbustamante}@dim.uchile.cl

[6] Email: josezamora@unab.cl

G through operations of vertex deletion, edge deletion and splitting off paths P_{uv} (in any order).

The minor order and the immersion order are not comparable. The class of planar graphs, while excluding K_5 as a minor, contains all cliques as immersions. On the other hand, the class of graphs with maximum degree at most 3, while excluding K_4 as an immersion, contains all cliques as minors.

However, the two relations do share some important similarities. Both of them are well-quasi-orders, as proved by Robertson and Seymour [18,19]. Additionally, if a graph G contains a graph H as a topological minor then G contains H both as a minor and as an immersion. Possibly inspired by such similarities, Lescure and Meyniel [17] and, independently, Abu-Khzam and Langston [1] proposed the following conjecture.

Conjecture 1.1 ([1,17]) *Every graph G excluding the complete graph K_{t+1} as an immersion satisfies $\chi(G) \leq t$.*

This conjecture has received much attention recently, and has been tackled with more success than its minor order counterpart. The cases $1 \leq t \leq 3$ follow from the fact that Hajos' Conjecture (replace 'immersion' by 'topological minor' in Conjecture 1.1) is true for these cases [8]. The cases $4 \leq t \leq 6$ were solved independently by Lescure and Meyniel [17] and DeVos, Kawarabayashi, Mohar, and Okamura [7].

For general t, DeVos, Dvořák, Fox, McDonald, Mohar, and Scheide [6] were the first to give a linear upper bound for Conjecture 1.1. They proved that every graph G excluding K_{t+1} as an immersion satisfies $\chi(G) \leq 200t$. Dvořák and Yepremyan [10] improved that upper bound to $11t + 6$. The best upper bound currently known is due to Gauthier, Le and Wollan [12] who showed that if G excludes K_{t+1} as an immersion, then G satisfies $\chi(G) \leq 3.54t + 7$. This implies the following.

Theorem 1.2 (Gauthier, Le and Wollan [12]) *Let G be a graph on n vertices and with independence number α. Then G contains $K_{\lceil \frac{n}{3.54\alpha} - c \rceil}$ as an immersion, where $c < 1.98$ is a constant.*

It seems natural to ask whether this can be improved without necessarily improving Gauthier, Le and Wollan's bound on the chromatic number of K_{t+1}-immersion-free graphs, as Duchet and Meyniel did in the context of graph minors.

The first attempt in this direction (actually earlier than [12]) has been carried out by Vergara [21]. She conjectured that every graph on n vertices and independence number 2 contains an immersion of $K_{\lceil \frac{n}{2} \rceil}$, and showed that this conjecture is equivalent to Conjecture 1.1 for graphs of independence number 2.

In support of her conjecture, Vergara proved that every graph on n vertices and independence number 2 contains a $K_{\lceil \frac{n}{3} \rceil}$-immersion. This result was improved by Gauthier, Le and Wollan [12] as follows.

Theorem 1.3 (Gauthier, Le and Wollan [12]) *Let G be a graph on n vertices with independence number 2. Then G contains $K_{2\lfloor \frac{n}{5} \rfloor}$ as an immersion.*

For general independence number α we make the following conjecture, extending the conjecture by Vergara. Our conjecture is best possible, since, as already mentioned earlier, the disjoint union of complete graphs of order t does not contain an immersion of K_{t+1}.

Conjecture 1.4 [4] *Let G be a graph on n vertices with independence number α. Then G contains $K_{\lceil \frac{n}{\alpha} \rceil}$ as an immersion.*

Our main results, which give more evidence in support of Conjecture 1.4, are the following.

Theorem 1.5 [4] *Let G be a graph on n vertices with independence number at most 3. Then G contains a clique immersion on at least $\lfloor \frac{2n}{9} \rfloor - 1$ vertices.*

Theorem 1.6 [4] *Let G be a graph on n vertices with independence number at most 4. Then G contains a clique immersion on at least $\lfloor \frac{4n}{27} \rfloor - 1$ vertices.*

Both results greatly improve on the size of the immersion obtained from Theorem 1.2. It is our hope that further development of the techniques we implement here may help bring further evidence of Conjecture 1.4 and, consequently, of Conjecture 1.1. In fact, while this extended abstract was being reviewed we managed to prove the following result, which improves on Theorem 1.2 for every value of α.

Theorem 1.7 [4] *Let G be a graph on n vertices with independence number at most α. Then G contains $K_{\lceil \frac{n}{3\alpha-2} \rceil}$ as an immersion.*

The remainder of this extended abstract is dedicated to the proof of Theorem 1.5. Theorem 1.6 relies on similar ideas, but is more complicated. We refer the interested reader to [4] for the proof of Theorem 1.6, and the proof of Theorem 1.7.

2 Graphs with independence number 3

In this section we prove Theorem 1.5. Before getting to the actual proof, we shall give a quick outline of the basic ideas.

2.1 An overview of the proof

We will apply induction on the number of vertices of the graph G. In the inductive step, we separate three vertices, a_1, a_2 and a_3, and use the induction hypothesis to find an immersion H of a complete graph in $G - \{a_1, a_2, a_3\}$.

This immersion H might be one short of having the size we need. In that case we will try to connect one of a_1, a_2, a_3 to the branch vertices of H, making the chosen vertex a_i a new branch vertex of the larger immersion. The paths used for connecting a_i to the other branch vertices will belong entirely to the bipartite graph between $\{a_1, a_2, a_3\}$ and $V(G) - \{a_1, a_2, a_3\}$, and therefore, they will not interfere with any of the paths from the original immersion H.

Which of the three vertices we choose as the new branch vertex a_i will depend on the sizes of their respective neighborhoods and their intersections in the different parts of $G-\{a_1,a_2,a_3\}$ (the part containing the branch vertices and the remainder). A more precise description of this choice is given by the first of the two main conditions of Lemma 2.1 below. Lemma 2.1 will be responsible for finding the connecting paths for the new branch vertex.

It might happen, however, that none of the three vertices is suitable, because there are simply not enough paths to make the necessary connections (that is, if the second of the two main conditions of Lemma 2.1 is not met). In this case, we will be able to show that we are in one of the following two scenarios. Either, for one the vertices a_i, its non-neighborhood in $G-\{a_1,a_2,a_3\}$ (which spans a graph of independence number at most 2) is large enough to apply Theorem 1.3 and find the desired immersion there. Or, there are two vertices, a_i and a_j, such that their joint non-neighborhood (which spans a complete graph) is of the size we need for our immersion. In either case, this finishes the proof.

2.2 The proof of Theorem 1.5

Let us now make this idea more precise. We need the following lemma for making the connections to the new branch vertex. (Later on, in the application of the lemma, the set M will be the set of 'old' branch vertices.)

For a graph G and $v\in V(G)$, let $\bar{N}(v):=V(G)\setminus(N(v)\cup v)$. A set of vertices A *dominates* a set of vertices B, if every vertex in B has a neighbor in A.

Lemma 2.1 *Let G be a bipartite graph with bipartition $(M\cup Q,\{a_1,a_2,a_3\})$, where M and Q are disjoint, and where the set $\{a_1,a_2,a_3\}$ dominates M. Suppose that $k=3$ maximizes*

$$\left|\Big(N(a_i)\cap N(a_j)\cap Q\Big)\setminus N(a_k)\right|$$

over all choices of distinct $i,j,k\in\{1,2,3\}$. Further, assume that

$$|\bar{N}(a_3)\cap M|\leq\left|\Big(N(a_1)\cup N(a_2)\Big)\cap N(a_3)\cap Q\right|.$$

Then, G contains an immersion of a star whose centre is a_3 and whose leaves are in M.

Proof. Given G as in the statement of the lemma, first observe that every vertex from $\bar{N}(a_3)\cap(M\cup Q)$ has a neighbor in $\{a_1,a_2\}$. We partition $\bar{N}(a_3)\cap M$ into two sets N_1^* and N_2^* such that for $i=1,2$, every vertex in N_i^* is adjacent to a_i. (This partition need not be unique.)

For each distinct $i,j,k\in\{1,2,3\}$, let

$$Y_{ij}:=(N(a_i)\cap N(a_j)\cap Q)\setminus N(a_k).$$

By the hypotheses of the lemma, we know that

$$|Y_{12}|\geq\max\{|Y_{13}|,|Y_{23}|\},\tag{1}$$

and

$$|\bar{N}(a_3)\cap M|\leq|Y_{13}|+|Y_{23}|+\big|X\big|,\tag{2}$$

donde $X := Q \cap \bigcap_{j=1,2,3} N(a_j)$.

First, suppose it is possible to partition X into two sets, X_1 and X_2, such that $|Y_{i3} \cup X_i| \geq |N_i^*|$ for both $i \in \{1,2\}$. In this case, we can pair, for $i = 1,2$, each vertex $x \in N_i^*$ with a vertex $y_x \in Y_{i3} \cup X_i$ in such a way that $y_x \neq y_{x'}$ if $x \neq x'$. This means that for every $x \in N_1^* \cup N_2^*$ there is a path of the form $xa_iy_xa_3$, and all these paths are edge disjoint. Splitting off all these paths, and also using the edges between a_3 and M, we obtain a star having a_3 as its center and M as its set of leaves, which is as desired.

We will therefore assume from now on that it is not possible to partition X into sets X_1 and X_2 such that $|Y_{i3} \cup X_i| \geq |N_i^*|$ for both $i \in \{1,2\}$. This implies that, modulo swapping indices 1 and 2, we have

$$|Y_{23} \cup X| < |N_2^*|.$$

Now, in a similar way as before, we pair each vertex $x \in N_1^*$ with a vertex $y_x \in Y_{13}$, and we pair each vertex $x \in N_2^*$ with a vertex $y_x \in Y_{13} \cup Y_{23} \cup X$, in a way that all the vertices y_x are distinct. This is possible by (2). For all vertices $x \in N_1^*$, and for each vertex $x \in N_2^*$ which has not been paired to a vertex from Y_{13}, we split off the paths of the form $xa_iy_xa_3$.

For vertices $x \in N_2^*$ which have been paired to a vertex from Y_{13}, we do the following. Pair each such vertex with a second vertex $v_x \in Y_{12}$, such that all vertices v_x are disjoint. This is possible because of (1). Now, split off the path $xa_2v_xa_1y_xa_3$. After doing this for all remaining vertices $x \in N_2^*$, we obtain the desired star immersion. □

We are now ready to prove Theorem 1.5.

Proof. [Proof of Theorem 1.5] We use induction on n to prove the statement of Theorem 1.5 for all graphs. The statement is trivially true for all $n < 13$. Thus, we can assume the given graph G has order $n \geq 14$.

If G has independence number at most 2, we can use Theorem 1.3. Hence, we can assume the independence number of G is equal to 3. Therefore, we can remove an independent set $I = \{a_1, a_2, a_3\}$ from G. By the induction hypothesis, we know that $G - I$ contains a immersion of $H := K_{\lfloor \frac{2(n-3)}{9} \rfloor}$, with branch vertices M.

Set $Q := V(G) \setminus (I \cup M)$. Then clearly,

$$|Q| \;\geq\; n - 3 - \left\lfloor \frac{2(n-3)}{9} \right\rfloor \;>\; \frac{7}{9}n - 3. \tag{3}$$

Our aim is to either add a new branch vertex to the immersion of H, or to find a different immersion of the correct size.

Set $N_i := N(a_i)$ and $\bar{N}_i := \bar{N}(a_i)$ for $i = 1,2,3$. Since G has independence number 3, we know that $G[\bar{N}_i]$ has independence number at most 2, for every $i = 1,2,3$. Hence, we may assume that

$$|\bar{N}_i| < \frac{5}{9}n + 1, \text{ for every } i = 1,2,3, \tag{4}$$

as otherwise (by Theorem 1.3) we would have that some $\bar{N}_i$ contains an immersion of the complete graph on $2\lfloor \frac{n}{9}+1\rfloor \geq \lfloor \frac{2n}{9}\rfloor$ vertices.

We may assume, without loss of generality, that a_1, a_2 are such that $\{i,j\} = \{1,2\}$ maximizes $|(N_i \cap N_j \cap Q) \setminus N_k|$ over all choices of distinct i,j,k in $\{1,2,3\}$.

Since I is an independent set of maximum size, I dominates $G-I$. If we could, by means of splitting off paths which are edge disjoint from $G-I$, make a_3 adjacent to $\bar{N}_3 \cap M$, then we would obtain the desired immersion of $K_{\lfloor \frac{2n}{9}\rfloor}$. So, by Lemma 2.1, we can assume that

$$|\bar{N}_3 \cap M| \geq |Z|+1, \tag{5}$$

where

$$Z := (N_1 \cup N_2) \cap N_3 \cap Q.$$

Since Q is dominated by $\{a_1, a_2, a_3\}$ we have

$$Q = Z \cup (\bar{N}_1 \cap \bar{N}_2 \cap Q) \cup (\bar{N}_3 \cap Q). \tag{6}$$

Now, since G has independence number 3, the set $\left(\bar{N}_1 \cap \bar{N}_2\right) \cup \{a_3\}$ induces a clique. We can assume this clique to have size at most $\lfloor \frac{2n}{9}\rfloor - 1$, as otherwise we have found the desired immersion.

Therefore, and by (6), we can deduce that

$$|Q| \ \leq \ |Z| + \left\lfloor \frac{2n}{9} \right\rfloor - 2 + |\bar{N}_3 \cap Q| \ \leq \ \left\lfloor \frac{2n}{9} \right\rfloor + |\bar{N}_3| - 3 \ \leq \ \left\lfloor \frac{2n}{9} \right\rfloor + \frac{5}{9}n - 3 \ < \ \frac{7}{9}n - 3,$$

where the second inequality comes from (5), and the third inequality comes from (4). This contradicts (3). □

References

[1] F. Abu-Khzam, and M. Langston, *Graph coloring and the immersion order*. In: T. Warnow, and B. Zhu, ed., *Computing and Combinatorics*. Lecture Notes in Comput. Sci., vol. 2697. Springer, Berlin, Heidelberg, 2003.

[2] J. Balogh, and A.V. Kostochka, *Large minors in graphs with given independence number*. Discrete Math., 311(20):2203–2215, 2011.

[3] J. Balogh, J. Lenz, and H. Wu, *Complete minors, independent sets, and chordal graphs*. Discuss. Math. Graph Theory, 31(4):639–674, 2011.

[4] S. Bustamante, D.A. Quiroz, M. Stein, J. Zamora, *Clique immersions and independence number*. In preparation.

[5] D.W. Cranston, and L. Rabern, *Planar graphs have independence ratio at least 3/13*. Electron. J. Combin., 23(3), 2016.

[6] M. DeVos, Z. Dvořák, J. Fox, J. McDonald, B. Mohar, and D. Scheide, *A minimum degree condition forcing complete graph immersion*. Combinatorica, 34(3):279–298, 2014.

[7] M. DeVos, K.-i. Kawarabayashi, B. Mohar, and H. Okamura, *Immersing small complete graphs*. Ars Math. Contemp., 3(2):139–146, 2010.

[8] G.A. Dirac, *A property of 4-chromatic graphs and some remarks on critical graphs*. J. London Math. Soc., 27:85–92, 1952.

[9] P. Duchet, and H. Meyniel, *On Hadwiger's number and the stability number*. Ann. Discrete Math., 13:71–74, 1982.

[10] Z. Dvořák, and L. Yepremyan, *Complete graph immersions and minimum degree*. J. Graph Theory, 88(1):211–221, 2017.

[11] J. Fox, *Complete minors and independence number*. SIAM J. Discrete Math., 24(4):1313–1321, 2010.

[12] G. Gauthier, T.-N. Le, and P. Wollan, *Forcing clique immersions through chromatic number*. arXiv:1703.08235 [math.CO], 2017.

[13] H. Hadwiger, *Über eine Klassifikation der Streckenkomplexe*. Vierteljschr. Naturforsch. Ges. Zürich, 88:133–143, 1943.

[14] K.-i. Kawarabayashi, M. Plummer, and B. Toft, *Improvements of the theorem of Duchet and Meyniel on Hadwiger's conjecture*. J. Combin. Theory Ser. B, 95(1):152–167, 2005.

[15] K.-i. Kawarabayashi, and Z.-X. Song, *Independence number and clique minors*. J. Graph Theory, 56(3):219–226, 2007.

[16] A.V. Kostochka, *Lower bound on the Hadwiger number of graphs by their average degree*. Combinatorica, 4(4):307–316, 1984.

[17] F. Lescure, and H. Meyniel, *On a problem upon configurations contained in graphs with given chromatic number*. Ann. Discrete Math., 41:325–331, 1989.

[18] N. Robertson, and P. Seymour, *Graph minors. XX. Wagner's conjecture*. J. Combin. Theory Ser. B, 92(2):325–357, 2004.

[19] N. Robertson, and P. Seymour, *Graph minors XXIII. Nash-Williams' immersion conjecture*. J. Combin. Theory Ser. B, 100(2):181–205, 2010.

[20] A. Thomason, *An extremal function for contractions of graphs*. Math. Proc. Cambridge Philos. Soc., 95(2):261265, 1984

[21] S. Vergara, *Complete graph immersions in dense graphs*. Discrete Math., 340(5):1019–1027, 2017.

[22] D.R. Wood, *Independent sets in graphs with an excluded clique minor*. Discrete Math. Theor. Comput. Sci., 9(1):171–175, 2007.