

Towards a Directed Homotopy Type Theory

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Abstract

In this paper, we present a directed homotopy type theory for reasoning synthetically about (higher) categories and directed homotopy theory. We specify a new ‘homomorphism’ type former for Martin-Löf type theory which is roughly analogous to the identity type former originally introduced by Martin-Löf. The homomorphism type former is meant to capture the notions of morphism (from the theory of categories) and directed path (from directed homotopy theory) just as the identity type former is known to capture the notions of isomorphism (from the theory of groupoids) and path (from homotopy theory). Our main result is an interpretation of these homomorphism types into *Cat*, the category of small categories. There, the interpretation of each homomorphism type $\text{hom}_{\mathcal{C}}(a, b)$ is indeed the set of morphisms between the objects a and b of the category \mathcal{C} . We end the paper with an analysis of the interpretation in *Cat* with which we argue that our homomorphism types are indeed the directed version of Martin-Löf’s identity types.

Keywords: dependent type theory, directed homotopy theory, semantics of type theory

1 Introduction

Martin-Löf type theory, together with its identity type, is now often described as a synthetic theory of higher groupoids. This rich structure of the identity type was first discovered by Hofmann and Streicher in their disproof of the *uniqueness of identity proofs* [3]. There, the authors construct an interpretation of Martin-Löf’s identity type into the category of groupoids where for any terms a, b in a type \mathcal{G} , the identity type $\text{Id}_{\mathcal{G}}(a, b)$ is interpreted to be the set of morphisms $a \rightarrow b$ in the groupoid interpreting \mathcal{G} .

The full extent of this structure was later made explicit in Voevodsky’s interpretation of Martin-Löf type theory with the identity type (amongst other types) into the category of Kan complexes, the objects of which represent spaces or ∞ -groupoids

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[6]. In this interpretation, the identity type $\text{Id}_K(a, b)$ of any two points a, b in a Kan complex K is itself a Kan complex, or space.

In addition to these concrete interpretations, others (e.g. [13,2,9]) have compared Martin-Löf's specification of the identity types with the part of Quillen's axiomatization of homotopy theory that is today called the theory of weak factorization systems [11,1]. In [9], the author has shown that in any finitely complete category, models of Martin-Löf's identity type are in correspondence with weak factorization systems with two stability properties.

Such results present an illuminating perspective on the theory of higher groupoids. This theory is, to some extent, already very well understood by topologists who have studied concrete models of it for the better part of a century. However, if one thinks of the theory of higher groupoids as being a theory of (reversible) paths, then one might wonder whether there is also a type theory which describes a theory of irreversible, or *directed*, paths. One might hope that such a type theory would shed light on the less well understood theories of higher categories and directed spaces. Both of these theories have a notion of directed paths at their heart, and for both of these theories, there are many competing concrete models. Thus, having a synthetic theory of directed paths might shed light on this situation.

In this paper, we present the core of such a type theory. Our long-term goal is to use this type theory to prove and verify theorems about higher categories and directed homotopy theory.

1.1 Previous work

Several other type theories with semantics in category theories have been proposed. In [7], the authors construct a type theory that gives a synthetic theory of 1-categories. In [12], the authors construct a type theory that gives a theory of $(\infty, 1)$ -categories. Other works-in-progress [10,14] do hope to give a theory of ∞ -categories.

In this paper, we aim to give a syntax which not only describes ∞ -categories but also directed spaces. We have three design criteria: (1) that our notion of directed paths is introduced via a type former and not as a judgement, so that it can be iterated as Martin-Löf's identity type can be, (2) that our theory allows us to transport terms of dependent types along directed paths, and (3) that our theory does *not* always allow us to transport terms of the type of directed paths along identities, since such a transportation does not hold in categories of directed spaces.

The syntaxes of [7] and [10] do not meet our first criterion since their directed paths are introduced as a judgement and not as a type former. This seems to make the theory somewhat unwieldy: in [7], only the theory of 1-categories is introduced and [10] includes no semantics. The theory of [14] does not meet our second criterion, and the theory of [12] does not meet our third criterion.

1.2 Organization

In Section 2, we present the syntax for our type theory, which consists of one new type former hom within Martin-Löf type theory. In Section 3, we interpret the rules

of this new type former **hom** within a comprehension category derived from the category \mathcal{Cat} of small categories. Even though the objects of \mathcal{Cat} are simpler than those of our intended semantics (higher categories and directed spaces), we see this as an important test case since its theory is much better understood than those of our intended semantics.

2 The syntax

In this section, we define a type of homomorphisms within standard Martin-Löf type theory [8]. We only work in the bare type theory. That is, we assume that we have all the structural rules of [8], but none of the type-forming rules given there.

First, we require that for every type T , there are two new types

$$\frac{\Gamma \vdash T \text{ TYPE}}{\Gamma \vdash T^{\text{core}} \text{ TYPE}} \qquad \frac{\Gamma \vdash T \text{ TYPE}}{\Gamma \vdash T^{\text{op}} \text{ TYPE}}$$

together with the following functions.

$$\frac{\Gamma \vdash T \text{ TYPE} \quad \Gamma \vdash t : T^{\text{core}}}{\Gamma \vdash i(t) : T} \qquad \frac{\Gamma \vdash T \text{ TYPE} \quad \Gamma \vdash t : T^{\text{core}}}{\Gamma \vdash i^{\text{op}}(t) : T^{\text{op}}}$$

Now we can define our type of homomorphisms.

hom FORMATION :

$$\frac{\Gamma \vdash T \text{ TYPE} \quad \Gamma \vdash s : T^{\text{op}} \quad \Gamma \vdash t : T}{\Gamma \vdash \text{hom}_T(s, t) \text{ TYPE}}$$

hom INTRODUCTION :

$$\frac{\Gamma \vdash T \text{ TYPE} \quad \Gamma \vdash t : T^{\text{core}}}{\Gamma \vdash 1_t : \text{hom}_T(i^{\text{op}}t, it) \text{ TYPE}}$$

RIGHT hom ELIMINATION :

$$\frac{\begin{array}{c} \Gamma \vdash T \text{ TYPE} \\ \Gamma, s : T^{\text{core}} \vdash \Theta \text{ TYPE} \quad \Gamma, s : T^{\text{core}}, t : T, f : \text{hom}_T(i^{\text{op}}s, t), \theta : \Theta \vdash D \text{ TYPE} \\ \Gamma, s : T^{\text{core}}, \theta : \Theta \vdash d : D[is/t, 1_s/f] \\ \Gamma \vdash s' : T^{\text{core}} \quad \Gamma \vdash t' : T \quad \Gamma \vdash f' : \text{hom}_T(i^{\text{op}}s', t') \quad \Gamma \vdash \theta' : \Theta[s'/s] \end{array}}{\Gamma \vdash e_R(s.\Theta, s.t.f.\theta.D, s.\theta.d, s', t', f', \theta') : D[s'/s, t'/t, f'/f, \theta'/\theta]}$$

LEFT hom ELIMINATION :

$$\frac{\begin{array}{c} \Gamma \vdash T \text{ TYPE} \\ \Gamma, t : T^{\text{core}} \vdash \Theta \text{ TYPE} \quad \Gamma, s : T^{\text{op}}, t : T^{\text{core}}, f : \text{hom}_T(s, it), \theta : \Theta \vdash D \text{ TYPE} \\ \Gamma, t : T^{\text{core}}, \theta : \Theta \vdash d : D(i^{\text{op}}t/s, 1_t/f) \\ \Gamma \vdash s' : T^{\text{op}} \quad \Gamma \vdash t' : T^{\text{core}} \quad \Gamma \vdash f' : \text{hom}_T(s', it') \quad \Gamma \vdash \theta' : \Theta[t'/t] \end{array}}{\Gamma \vdash e_L(t.\Theta, s.t.f.\theta.D, t.\theta.d, s', t', f', \theta') : D[s'/s, t'/t, f'/f, \theta'/\theta]}$$

RIGHT hom COMPUTATION :

$$\frac{\begin{array}{c} \Gamma \vdash T \text{ TYPE} \\ \Gamma, s : T^{\text{core}} \vdash \Theta \text{ TYPE} \quad \Gamma, s : T^{\text{core}}, t : T, f : \text{hom}_T(\text{i}^{\text{op}}s, t), \theta : \Theta \vdash D \text{ TYPE} \\ \Gamma, s : T^{\text{core}}, \theta : \Theta \vdash d : D[\text{is}/t, 1_s/f] \quad \Gamma \vdash s' : T^{\text{core}} \quad \Gamma \vdash \theta' : \Theta[s'/s] \end{array}}{\Gamma \vdash \text{e}_R(s.\Theta, s.t.f.\theta.D, s.\theta.d, s', \text{is}', 1_{s'}, \theta') = d[s'/s, \theta'/\theta] : D[s'/s, \text{i}^{\text{op}}s'/t, 1_{s'}/f, \theta'/\theta]}$$

LEFT hom COMPUTATION :

$$\frac{\begin{array}{c} \Gamma \vdash T \text{ TYPE} \\ \Gamma, t : T^{\text{core}} \vdash \Theta \text{ TYPE} \quad \Gamma, s : T^{\text{op}}, t : T^{\text{core}}, f : \text{hom}_T(s, \text{it}), \theta : \Theta \vdash D \text{ TYPE} \\ \Gamma, t : T^{\text{core}}, \theta : \Theta \vdash d : D(\text{i}^{\text{op}}t/s, 1_t/f) \quad \Gamma \vdash t' : T^{\text{core}} \quad \Gamma \vdash \theta' : \Theta[t'/t] \end{array}}{\Gamma \vdash \text{e}_L(t.\Theta, s.t.f.\theta.D, t.\theta.d, \text{i}^{\text{op}}t', t', 1_{t'}, \theta') = d[t'/t, \theta'/\theta] : D[\text{i}^{\text{op}}t'/s, t'/t, 1_{t'}/f, \theta'/\theta]}$$

2.1 Transportation and composition

Before we develop semantics for these rules, we prove some illustrative results in the syntax.

First, we show that we can transport types along homomorphisms, which is analogous to the transportation along identities in homotopy type theory.

Proposition 2.1 *For any type $\Gamma, t : T \vdash E \text{ TYPE}$, any term $\Gamma, s : T^{\text{core}} \vdash e' : E[\text{is}/t]$, and any homomorphism $\Gamma \vdash s' : T^{\text{core}}, \Gamma \vdash t' : T, \Gamma \vdash f : \text{hom}_T(\text{i}^{\text{op}}s', t')$, we have a term $\Gamma \vdash \text{transport}_R(e', f) : E[t'/t]$.*

Proof. By plugging the data

- $\Gamma \vdash T \text{ TYPE}$
- $\Gamma, s : T^{\text{core}} \vdash E[\text{is}/t]$
- $\Gamma, s : T^{\text{core}}, t : T, f : \text{hom}_T(\text{i}^{\text{op}}s, t), e : E[\text{is}/t] \vdash E \text{ TYPE}$
- $\Gamma, s : T^{\text{core}}, e : E[\text{is}/t] \vdash e : E[\text{is}/t]$
- $\Gamma \vdash s' : T^{\text{core}}$
- $\Gamma \vdash t' : T$
- $\Gamma \vdash f : \text{hom}_T(\text{i}^{\text{op}}s', t')$
- $\Gamma \vdash e' : E[\text{is}'/t]$

into the right hom computation rule, we find a term

$$\Gamma \vdash \text{e}_R(s.E[\text{is}/t], s.t.f.e.E, s.e.e, s', t', f, e') : E[t'/t]$$

which we abbreviate by $\text{transport}_R(e', f)$. □

Now using this transportation, we can show that we can compose two homomorphisms.

Proposition 2.2 *For any type $\Gamma \vdash T \text{ TYPE}$, any terms $\Gamma \vdash r : T^{\text{op}}, \Gamma \vdash s : T^{\text{core}}, \Gamma \vdash t : T$, and any homomorphisms $\Gamma \vdash f : \text{hom}_T(r, \text{is})$ and $\Gamma \vdash g : \text{hom}_T(\text{i}^{\text{op}}s, t)$, we have a homomorphism $\Gamma \vdash \text{comp}_R(f, g) : \text{hom}_T(r, t)$.*

Proof. Using Proposition 2.1, we obtain a term $\Gamma \vdash \text{transport}_R(f, g) : \text{hom}_T(r, t)$ which we call $\text{comp}_R(f, g)$. \square

By the right hom computation rule, we find that for any $f : \text{hom}_T(r, is)$, the composition $\text{comp}_R(f, 1_s)$ is strictly equal to $f : \text{hom}_T(r, is)$. Using left hom elimination, we could analogously construct a left transportation transport_L and a left composition comp_L such that for any $g : \text{hom}_T(i^{\text{op}}s, t)$, the composition $\text{comp}_L(1_s, g)$ is strictly equal to g in the type $\text{hom}_T(i^{\text{op}}s, t)$.

3 The semantics in \mathcal{Cat}

In this section, we construct an interpretation in the category of categories.

We work in a set theory with two uncountable inaccessible cardinals $\kappa < \lambda$. Let \mathcal{Cat} denote the category of κ -small categories, and let \mathcal{CAT} denote the 2-category of λ -small categories. We have arranged this so that \mathcal{CAT} contains both \mathcal{Cat} and all the objects of \mathcal{Cat} . Note that there are no universes in our syntax, so this is as much of a tower of universes in the semantics as we need.

3.1 Grothendieck (op)fibrations

The theory of Grothendieck fibrations and opfibrations play a central role in this interpretation, so we briefly review it here. See [5] for more details.

Consider a functor $P : \mathcal{E} \rightarrow \mathcal{B}$ in \mathcal{CAT} . We say that a morphism $e : E' \rightarrow E$ in \mathcal{E} is *cartesian* if for any other morphism $e' : E'' \rightarrow E$ in \mathcal{E} , any factorization of $P(e')$ through $P(e)$ (that is, a morphism $b : P(E'') \rightarrow P(E')$ such that $P(e) \circ b = P(e')$) lifts uniquely to a factorization of e' through e (that is, there is a unique morphism $\ell : E'' \rightarrow E'$ such that $e \circ \ell = e'$ and $P(\ell) = b$).

$$\begin{array}{ccc}
 \mathcal{E} & & E'' \xrightarrow[\exists! \ell]{e'} E' \xrightarrow{e} E \\
 \downarrow P & & \\
 \mathcal{B} & & P(E'') \xrightarrow{b} P(E') \xrightarrow[P(e')]{P(e)} E
 \end{array}$$

The archetypical cartesian morphisms are pullback squares, in the following sense. First, let 2 denote the poset $0 \leq 1$. Then \mathcal{C}^2 is the category of morphisms of a category \mathcal{C} . Let $\text{COD} : \mathcal{C}^2 \rightarrow \mathcal{C}$ be the functor which sends each morphism to its codomain. Now consider any category \mathcal{C} in \mathcal{CAT} with all pullbacks. Then a morphism in \mathcal{C}^2 whose underlying diagram in \mathcal{C} is a pullback square is a cartesian morphism with respect to the functor COD .

A functor $P : \mathcal{E} \rightarrow \mathcal{B}$ in \mathcal{CAT} is a *Grothendieck fibration* if for any object E of \mathcal{E} , any morphism $b : B \rightarrow P(E)$ in \mathcal{B} can be lifted to a cartesian morphism $e : E' \rightarrow E$ (so that $P(e) = b$).

The archetypical Grothendieck fibration is a functor of the form $\text{COD} : \mathcal{C}^2 \rightarrow \mathcal{C}$ for any category \mathcal{C} with all pullbacks. This is because for any object f in \mathcal{C}^2 (which

is a morphism $f : X \rightarrow Y$ in \mathcal{C}) and morphism $g : Z \rightarrow \text{COD}(f)$ in \mathcal{C} , the pullback square $\pi : g^*f \rightarrow f$ is a cartesian morphism in \mathcal{C}^2 which is a lift of g .

Dually, we can define cocartesian morphisms and Grothendieck opfibrations. For a functor $P : \mathcal{E} \rightarrow \mathcal{B}$ in \mathcal{CAT} , a cocartesian morphism in \mathcal{E} is a cartesian morphism in \mathcal{E}^{op} with respect to the functor P^{op} . Then our archetypical cocartesian morphisms are pushout squares in any \mathcal{C}^2 with respect to the functor $\text{DOM} : \mathcal{C}^2 \rightarrow \mathcal{C}$. A *Grothendieck opfibration* is a functor $P : \mathcal{E} \rightarrow \mathcal{B}$ in \mathcal{CAT} such that P^{op} is a Grothendieck fibration. Our archetypical Grothendieck opfibration is $\text{DOM} : \mathcal{C}^2 \rightarrow \mathcal{C}$ for any category \mathcal{C} with all pushouts.

One reason that Grothendieck (op)fibrations have been of interest is that they are equivalent to pseudofunctors into \mathcal{Cat} . We will not make full use of that equivalence here, but we will make use of the *Grothendieck construction*. Consider any category \mathcal{C} in \mathcal{Cat} , and any functor $F : \mathcal{C} \rightarrow \mathcal{Cat}$. We can construct, following Grothendieck, a new category $\mathcal{C}.F$ as follows. The objects of $\mathcal{C}.F$ are pairs (X, Y) where X is an object of \mathcal{C} and Y is an object of $F(X)$. Morphisms $(X, Y) \rightarrow (X', Y')$ in $\mathcal{C}.F$ are pairs (f, g) where $f : X \rightarrow X'$ is a morphism in \mathcal{C} and $g : F(f)(Y) \rightarrow Y'$ is a morphism in $F(X')$. Then there is a projection functor $\pi_{\mathcal{C}} : \mathcal{C}.F \rightarrow \mathcal{C}$ which takes each pair (X, Y) to X . The functor $\pi_{\mathcal{C}}$ is a Grothendieck opfibration since given any (X, Y) in $\mathcal{C}.F$, we can lift any morphism $f : X \rightarrow X'$ in \mathcal{C} to the cocartesian morphism $(f, 1_{F(Y)}) : (X, Y) \rightarrow (X', F(Y))$ in $\mathcal{C}.F$.

Dually, given any functor of the form $F : \mathcal{C}^{op} \rightarrow \mathcal{Cat}$, one can construct a Grothendieck fibration $\pi_{\mathcal{C}} : \mathcal{C}.F \rightarrow \mathcal{C}$. The objects of $\mathcal{C}.F$ are still pairs (X, Y) such that X is an object of \mathcal{C} and Y is an object of $F(X)$, but the morphisms $(X, Y) \rightarrow (X', Y')$ of $\mathcal{C}.F$ are pairs (f, g) such that $f : X \rightarrow X'$ is a morphism in \mathcal{C} and $g : Y \rightarrow F(f)(Y')$ is a morphism in $F(X)$.

3.2 The comprehension category

A comprehension category over \mathcal{Cat} (following [4]) consists of a category \mathcal{E} , a functor $\chi : \mathcal{E} \rightarrow \mathcal{Cat}^2$, and a Grothendieck fibration $G : \mathcal{E} \rightarrow \mathcal{Cat}$ in \mathcal{CAT} making the following diagram commute and such that χ sends cartesian morphisms in \mathcal{E} to cartesian morphisms in \mathcal{Cat}^2 (that is, pullback squares).

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\chi} & \mathcal{Cat}^2 \\ & \searrow G & \swarrow \text{COD} \\ & \mathcal{Cat} & \end{array}$$

When interpreting a type theory into this framework, we take the objects of \mathcal{Cat} to be our collection of contexts, the morphisms of \mathcal{Cat} to be context morphisms, and each fiber $G^{-1}(\Gamma)$ to be the collection of types in the context Γ . The terminal category $*$ is the interpretation of the empty context. By applying the functor χ to an object T of $G^{-1}(\Gamma)$, we obtain a morphism $\chi(T)$ in \mathcal{Cat} with codomain Γ ; the domain of $\chi(T)$ is the context extension (or comprehension) of Γ by T . Thus, terms in a type T dependent on a context Γ are interpreted by sections of $\chi(T)$.

Problem 3.1 To construct a comprehension category over Cat .

Example 3.2 Before we construct the comprehension category, we indicate what $G^{-1}(*)$ (the types in the empty context) will be. This will be the category of functors $* \rightarrow Cat$, which is isomorphic to Cat itself. The comprehension functor χ will take a functor of the form $C : * \rightarrow Cat$ to a functor which is isomorphic to the unique functor $C(*) \rightarrow *$. Thus, terms of C are functors $* \rightarrow C(*)$, which are just objects of the category $C(*)$.

Construction 3.3 [for Problem 3.1] Consider the functor

$$[-, Cat] : Cat^{op} \rightarrow Cat$$

which takes each object Γ in Cat to the category $[\Gamma, Cat]$ of functors $\Gamma \rightarrow Cat$ in CAT and lax natural transformations between them (note that the notation $[\Gamma, Cat]$ is slightly nonstandard). Performing the Grothendieck construction on this functor, we get the Grothendieck fibration

$$\pi_{Cat} : Cat.[-, Cat] \rightarrow Cat.$$

The objects of $Cat.[-, Cat]$ are pairs (Γ, T) where Γ is a category in Cat and $T : \Gamma \rightarrow Cat$ is a functor. The morphisms $(\Gamma, T) \rightarrow (\Gamma', T')$ are pairs (F, τ) where $F : \Gamma \rightarrow \Gamma'$ is a functor and $\tau : T \Rightarrow T' \circ F$ is a lax natural transformation.

$$\begin{array}{ccc} \Gamma & \xrightarrow{T} & Cat \\ F \downarrow & \Downarrow \tau & \nearrow T' \\ \Gamma' & & \end{array}$$

Now we define the functor $\chi : Cat.[-, Cat] \rightarrow Cat^2$ to be the Grothendieck construction. That is, for any object (Γ, T) in $Cat.[-, Cat]$, we have

$$\chi(\Gamma, T) := \pi_{\Gamma} : \Gamma.T \rightarrow \Gamma.$$

For any morphism $(F, \tau) : (\Gamma, T) \rightarrow (\Gamma', T')$ we have that $\chi(F, \tau)$ is the morphism in Cat^2 with the following underlying commutative square in Cat

$$\begin{array}{ccc} \Gamma.T & \xrightarrow{F.\tau} & \Gamma'.T' \\ \downarrow \pi_{\Gamma} & & \downarrow \pi_{\Gamma'} \\ \Gamma & \xrightarrow{F} & \Gamma' \end{array}$$

where $F.\tau$ is the functor which sends each (X, Y) in $\Gamma.T$ to $(F(X), \tau_X(Y))$ in $\Gamma'.T'$.

By Lemma 3.4 below, the functor χ preserves cartesian morphisms.

Now it is straight-forward to check that the following diagram commutes.

$$\begin{array}{ccc} Cat.[-, Cat] & \xrightarrow{\chi} & Cat^2 \\ & \searrow \pi_{Cat} & \swarrow COD \\ & Cat & \end{array}$$

In particular, on objects, we have $\text{COD}\chi(\Gamma, T) = \text{COD}(\pi_\Gamma : \Gamma.T \rightarrow \Gamma) = \Gamma$ and $\pi_{\text{Cat}}(\Gamma, T) = \Gamma$. \square

Lemma 3.4 *The functor χ preserves cartesian morphisms.*

Proof. Consider a cartesian morphism $(F, \tau) : (\Gamma, T) \rightarrow (\Gamma', T')$ in $\text{Cat}[-, \text{Cat}]$.

$$\begin{array}{ccc} \Gamma & \xrightarrow{T} & \text{Cat} \\ F \downarrow & \Downarrow \tau & \nearrow T' \\ \Gamma' & & \end{array}$$

Applying χ to this morphism, we obtain the following morphism in Cat^2 , which is a commutative square in Cat .

$$\begin{array}{ccc} \Gamma.T & \xrightarrow{F.\tau} & \Gamma'.T' \\ \downarrow \pi_\Gamma & & \downarrow \pi_{\Gamma'} \\ \Gamma & \xrightarrow{F} & \Gamma' \end{array}$$

We want to show that this is cartesian with respect to $\text{COD} : \text{Cat}^2 \rightarrow \text{Cat}$.

So consider a morphism $\langle \alpha, \beta \rangle : Z \rightarrow \pi_{\Gamma'}$ in Cat^2 and a morphism $\gamma : Z_1 \rightarrow \Gamma$ in Cat such that $F\gamma = \beta$ as illustrated below.

$$\begin{array}{ccccc} & & \alpha & & \\ & \searrow \delta & & \nearrow & \\ Z_0 & \xrightarrow{\delta} & \Gamma.T & \xrightarrow{F.\tau} & \Gamma'.T' \\ \downarrow Z & & \downarrow \pi_\Gamma & & \downarrow \pi_{\Gamma'} \\ Z_1 & \xrightarrow{\gamma} & \Gamma & \xrightarrow{F} & \Gamma' \\ & \nearrow \beta & & \nwarrow & \end{array} \quad (*)$$

We need to show that there is a unique $\delta : Z_0 \rightarrow \Gamma.T$ which makes the diagram commute.

Consider the object $(Z_0, *_{Z_0})$ in $\text{Cat}[-, \text{Cat}]$ (where $*_{Z_0}$ is the functor $Z_0 \rightarrow \text{Cat}$ which sends everything to the terminal category). Then there is a morphism $(F\gamma Z, \bar{\alpha}) : (Z_0, *_{Z_0}) \rightarrow (\Gamma', T')$ where $\bar{\alpha}_z$ is the functor $* \rightarrow T'F\gamma Z(z)$ which picks out the object $\alpha(z)$ for any object $z \in Z_0$.

$$\begin{array}{ccc} Z_0 & \xrightarrow{*_{Z_0}} & \text{Cat} \\ \gamma Z \downarrow & \parallel \bar{\alpha} & \Downarrow \bar{\delta} \\ \Gamma & \xrightarrow{T} & \text{Cat} \\ F \downarrow & \parallel \tau & \Downarrow \tau \\ \Gamma' & \xrightarrow{T'} & \text{Cat} \end{array}$$

Since (F, τ) is cartesian, there is a unique lax natural transformation $\bar{\delta} : *_{Z_0} \Rightarrow$

$TF\gamma Z$ such that $\bar{\alpha} = \tau\bar{\delta}$. This gives rise to the functor $\delta : Z_0 \rightarrow \Gamma.T$ by setting $\delta(z) := (\gamma Z(z), \bar{\delta}_z(*))$. This makes the diagram (*) above commute.

If there were another such functor δ' making the diagram (*) commute, by making an analogous argument, we would find another lax natural transformation $\bar{\delta}'$. But then $\bar{\delta}' = \bar{\delta}$, and so $\delta' = \delta$. Thus, δ is the unique functor making the diagram (*) commute.

Thus, $\chi(F, \tau)$ is cartesian for every cartesian (F, τ) so χ is cartesian. \square

This comprehension category has a unit $\eta : Cat \rightarrow Cat.[-, Cat]$ which takes a category Γ to $(\Gamma, *_\Gamma : \Gamma \rightarrow Cat)$ where $*_\Gamma$ is the functor which is constant at the terminal category $*$. Then $\chi(\Gamma, *_\Gamma)$ is an isomorphism $\pi_\Gamma : \Gamma.*_\Gamma \rightarrow \Gamma$. Recall that for any (Γ, T) in $Cat.[-, Cat]$, we interpret the terms of T as sections of $\chi(\Gamma, T)$. These are in bijection with morphisms $(\Gamma, *_\Gamma) \rightarrow (\Gamma, T)$ in the fiber $\pi_{Cat}^{-1}\Gamma$ which are just morphisms $*_\Gamma \rightarrow T$ in the category $[\Gamma, Cat]$.

Thus, in the sections that follow, we will interpret contexts as categories Γ in Cat , we will interpret types in context Γ as objects of the category $[\Gamma, Cat]$ (which are functors $\Gamma \rightarrow Cat$), and we will interpret terms of such an object T in $[\Gamma, Cat]$ as lax natural transformations $*_\Gamma \Rightarrow T$.

3.3 core and op

In this section, we interpret the type formers **core** and **op** and the functions **i** and **i^{op}** into our comprehension category.

Problem 3.5 *To construct functorially, from each functor $\mathcal{T} : \Gamma \rightarrow Cat$, functors $\mathcal{T}^{core} : \Gamma \rightarrow Cat$ and $\mathcal{T}^{op} : \Gamma \rightarrow Cat$ and lax natural transformations $i : \mathcal{T}^{core} \Rightarrow \mathcal{T}$ and $i^{op} : \mathcal{T}^{core} \Rightarrow \mathcal{T}^{op}$.*

Example 3.6 Before we give the construction, we describe what it will be in the empty context.

Consider a type \mathcal{C} in the empty context, which is just a category. Then \mathcal{C}^{core} will be just the set³ of objects of \mathcal{C} and \mathcal{C}^{op} will be the usual opposite category of \mathcal{C} . Then $i : \mathcal{C}^{core} \rightarrow \mathcal{C}$ and $i^{op} : \mathcal{C}^{core} \rightarrow \mathcal{C}^{op}$ will be the injections which are the identity on objects.

Construction 3.7 [for Problem 3.5] Recall the two oft-encountered endofunctors $(-)^{core}$ and $(-)^{op}$ on Cat . For any category \mathcal{C} , the category \mathcal{C}^{core} is the set $\text{ob}(\mathcal{C})$. The objects of \mathcal{C}^{op} are the objects of \mathcal{C} , and for all objects X, Y of \mathcal{C}^{op} , the homomorphisms $X \rightarrow Y$ in \mathcal{C}^{op} are the homomorphisms $Y \rightarrow X$ in \mathcal{C} . There are moreover natural transformations $i : (-)^{core} \Rightarrow id_{Cat}$ and $i^{op} : (-)^{core} \Rightarrow (-)^{op}$ induced by the identity function on objects.

Postcomposition with the functors $(-)^{core}$ and $(-)^{op}$ gives two functors which take any $\mathcal{T} : \Gamma \rightarrow Cat$ to $\mathcal{T}^{core} : \Gamma \rightarrow Cat$ and $\mathcal{T}^{op} : \Gamma \rightarrow Cat$, respectively.

³ We will often talk about a set X in the category Cat . To be clear, we mean the category whose objects are the elements of X and whose morphisms are only identity morphisms.

Whiskering with i (resp. i^{op}) gives a functor which takes any $\mathcal{T} : \Gamma \rightarrow \mathcal{Cat}$ to the natural transformation $i : \mathcal{T}^{core} \Rightarrow \mathcal{T}$ (resp. $i^{op} : \mathcal{T}^{core} \Rightarrow \mathcal{T}^{op}$). \square

3.4 hom formation

In this section, we interpret the **hom** formation rule.

Problem 3.8 To functorially construct for each functor $\mathcal{T} : \Gamma \rightarrow \mathcal{Cat}$ and lax natural transformations $s : *_{\Gamma} \Rightarrow \mathcal{T}^{op}, t : *_{\Gamma} \Rightarrow \mathcal{T}$, a functor $hom_{\mathcal{T}}(s, t) : \Gamma \rightarrow \mathcal{Cat}$.

Example 3.9 In the empty context, our problem reduces to finding for each category \mathcal{C} and object $(x, y) \in \mathcal{C}^{op} \times \mathcal{C}$, a category $hom_{\mathcal{C}}(x, y)$. Our construction will produce the usual homset $hom_{\mathcal{C}}(x, y)$.

Construction 3.10 [for Problem 3.8] Let $(-)^{op} \times id_{\mathcal{Cat}} : \mathcal{Cat} \rightarrow \mathcal{CAT}$ denote the functor which takes any $\mathcal{C} \in \mathcal{Cat}$ to $\mathcal{C}^{op} \times \mathcal{C}$ in \mathcal{CAT} . Then let $c_{\mathcal{Set}} : \mathcal{Cat} \rightarrow \mathcal{CAT}$ be the constant functor at \mathcal{Set} , the category of κ -small sets.

There is a lax natural transformation $hom : (-)^{op} \times id_{\mathcal{Cat}} \Rightarrow c_{\mathcal{Set}}$ which at each $\mathcal{C} \in \mathcal{Cat}$ is the usual functor $hom_{\mathcal{C}} : \mathcal{C}^{op} \times \mathcal{C} \Rightarrow \mathcal{Set}$ (which maps an object $(X, Y) \in \mathcal{C}^{op} \times \mathcal{C}$ to the set $hom_{\mathcal{C}}(X, Y)$ of morphisms $X \rightarrow Y$ in \mathcal{C}).

$$\begin{array}{ccc} \mathcal{C}^{op} \times \mathcal{C} & \xrightarrow{F^{op} \times F} & \mathcal{D}^{op} \times \mathcal{D} \\ \downarrow hom_{\mathcal{C}} & \xRightarrow{hom_F} & \downarrow hom_{\mathcal{D}} \\ \mathcal{Set} & \xlongequal{\quad} & \mathcal{Set} \end{array}$$

Now for any functor $\mathcal{T} : \Gamma \rightarrow \mathcal{Cat}$ and lax natural transformations $s : *_{\Gamma} \Rightarrow \mathcal{T}^{op}, t : *_{\Gamma} \Rightarrow \mathcal{T}$, we have the following diagram of functors and lax natural transformations.

$$\begin{array}{ccccc} & & *_{\Gamma} & & \\ & \curvearrowright & & \curvearrowleft & \\ \Gamma & \xrightarrow{\mathcal{T}} & \mathcal{Cat} & \xrightarrow{(-)^{op} \times id} & \mathcal{CAT} \\ & & \downarrow (s, t) & \downarrow hom & \\ & & \mathcal{Set} & \xrightarrow{c_{\mathcal{Set}}} & \mathcal{CAT} \end{array} \quad (*)$$

By pasting together the diagram above, we obtain a lax natural transformation $hom_{\mathcal{T}}(s, t) : *_{\Gamma} \Rightarrow c_{\mathcal{Set}}\mathcal{T}$. At every object $\gamma \in \Gamma$, this gives a functor $hom_{\mathcal{T}}(s, t)\gamma : * \rightarrow \mathcal{Set}$ which is just a set $hom_{\mathcal{T}}(s, t)\gamma(*)$. For every morphism $\phi : \gamma \rightarrow \gamma'$ in Γ , this produces a lax natural transformation

$$\begin{array}{ccc} * & \xlongequal{\quad} & * \\ \downarrow hom_{\mathcal{T}}(s, t)\gamma & \xRightarrow{hom_{\mathcal{T}}(s, t)\phi} & \downarrow hom_{\mathcal{T}}(s, t)\gamma' \\ \mathcal{Set} & \xlongequal{\quad} & \mathcal{Set} \end{array}$$

which is just a function $hom_{\mathcal{T}}(s, t)\phi_* : hom_{\mathcal{T}}(s, t)\gamma(*) \rightarrow hom_{\mathcal{T}}(s, t)\gamma'(*)$. In other words, $hom_{\mathcal{T}}(s, t)$ is a functor $\Gamma \rightarrow \mathcal{Cat}$. \square

3.5 hom introduction

In this section, we interpret the hom introduction rule.

Problem 3.11 *To functorially construct for each functor $\mathcal{T} : \Gamma \rightarrow \mathcal{Cat}$ and lax natural transformation $t : *_{\Gamma} \Rightarrow \mathcal{T}^{core}$, a lax natural transformation $1_t : *_{\Gamma} \Rightarrow \text{hom}_{\mathcal{T}}(i^{op}t, it)$.*

Example 3.12 In the empty context, our problem reduces to finding for each category \mathcal{C} and object $t \in \mathcal{C}^{core}$, an object $1_t \in \text{hom}_{\mathcal{T}}(t, t)$. Our construction will produce the usual identity morphism $t \rightarrow t$.

Construction 3.13 [for Problem 3.11] For every category \mathcal{C} and for every $t \in \mathcal{C}^{core}$ (that is, an object t of \mathcal{C}), there is an element $1_t^{\mathcal{C}} \in \text{hom}_{\mathcal{C}}(t, t)$, the identity morphism $t \rightarrow t$. Since this is (vacuously) lax natural in t , it assembles into the lax natural transformation shown below

$$\begin{array}{ccc} \mathcal{C}^{core} & \xrightarrow{c_*} & \mathcal{Set} \\ & \Downarrow 1_{\bullet}^{\mathcal{C}} & \\ & \xrightarrow{\text{hom}_{\mathcal{C}}(i^{op} \times i)} & \end{array}$$

where c_* is the constant functor at the terminal set.

This is furthermore natural in \mathcal{C} , so it assembles into the following modification.

$$\begin{array}{ccc} \mathcal{Cat} & \xrightarrow{(-)^{core}} & \mathcal{CAT} \\ & \Downarrow c_{Set} & \\ & \xrightarrow{\text{hom}_{\mathcal{C}}(i^{op} \times i)} & \end{array}$$

c_* $\begin{array}{c} \curvearrowright \\ \equiv \\ \curvearrowleft \end{array}$ 1_{\bullet}

Now, consider a functor $\mathcal{T} : \Gamma \rightarrow \mathcal{Cat}$ and a lax natural transformation $t : *_{\Gamma} \Rightarrow \mathcal{T}^{core}$. We obtain the following diagram.

$$\begin{array}{ccc} \Gamma & \xrightarrow{\mathcal{T}} & \mathcal{Cat} \\ & \Downarrow t & \\ & \xrightarrow{(-)^{core}} & \mathcal{CAT} \end{array}$$

c_{Set} c_* $\begin{array}{c} \curvearrowright \\ \equiv \\ \curvearrowleft \end{array}$ 1_{\bullet} $\text{hom}_{\mathcal{C}}(i^{op} \times i)$

Then pasting together this diagram, we obtain a modification $c_* \Rightarrow \text{hom}_{\mathcal{C}}(i^{op}t, it)$ which unfolds into a lax natural transformation $1_t : *_{\Gamma} \Rightarrow \text{hom}(i^{op}t, it)$. This has at each $\gamma \in \Gamma$ the component $1_{t\gamma} : * \rightarrow \text{hom}(t\gamma, t\gamma)$ which picks out the identity morphism $t\gamma \rightarrow t\gamma$. \square

3.6 RIGHT hom elimination and computation

In this section, we interpret the right hom elimination rule so that it satisfies the strict equality required by the right hom computation rule.

Problem 3.14 To functorially construct for each functor $\mathcal{T} : \Gamma \rightarrow \text{Cat}$, each functor $\Theta : \Gamma.\mathcal{T}^{\text{core}} \rightarrow \text{Cat}$, each functor $\mathcal{D} : \Gamma.\mathcal{T}^{\text{core}}.\mathcal{T}.hom_{\mathcal{T}}(i^{op} \times 1).\Theta \rightarrow \text{Cat}$, and each lax natural transformation

$$d : *_{\Gamma.\mathcal{T}^{\text{core}}.\Theta} \Rightarrow \mathcal{D} \circ \Gamma.1_{\bullet}.\Theta : \Gamma.\mathcal{T}^{\text{core}}.\Theta \rightarrow \text{Cat},$$

a lax natural transformation

$$e_r(d) : *_{\Gamma.\mathcal{T}^{\text{core}}.\mathcal{T}.hom_{\mathcal{T}}(i^{op} \times 1).\Theta} \Rightarrow \mathcal{D} : \Gamma.\mathcal{T}^{\text{core}}.\mathcal{T}.hom_{\mathcal{T}}(i^{op} \times 1).\Theta \rightarrow \text{Cat},$$

such that $e_r(d) \circ \Gamma.1_{\bullet}.\Theta = d$.

Example 3.15 Where Γ and Θ are both empty contexts, our problem reduces to finding for each category \mathcal{C} , each functor $\mathcal{D} : \mathcal{C}^{\text{core}}.\mathcal{C}.hom_{\mathcal{C}}(i^{op} \times 1) \rightarrow \text{Cat}$, and each lax natural transformation $d : *_{\mathcal{C}^{\text{core}}} \Rightarrow \mathcal{D} \circ 1_{\bullet} : \mathcal{C}^{\text{core}} \rightarrow \text{Cat}$, a lax natural transformation

$$e_r(d) : *_{\mathcal{C}^{\text{core}}.\mathcal{C}.hom_{\mathcal{C}}(i^{op} \times 1)} \Rightarrow \mathcal{D} : \mathcal{C}^{\text{core}}.\mathcal{C}.hom_{\mathcal{C}}(i^{op} \times 1) \rightarrow \text{Cat},$$

such that $e_r(d)1_{\bullet} = d$.

The objects of $\mathcal{C}^{\text{core}}.\mathcal{C}.hom_{\mathcal{C}}(i^{op} \times 1)$ are triples (X, Y, f) such that $f : X \rightarrow Y$ is a morphism in \mathcal{C} . Its morphisms are of the form $g : (X, Y, f) \rightarrow (X, Y', g \circ f)$ where $g : Y \rightarrow Y'$ is a morphism of \mathcal{C} .

Since $e_r(d)1_{\bullet}$ must be d , we know that $e_r(d)_{(X, X, 1_X)}$ must be $d(X)$ for any object X of $\mathcal{C}^{\text{core}}$. Now for any other object (X, Y, f) of $\mathcal{C}^{\text{core}}.\mathcal{C}.hom_{\mathcal{C}}(i^{op} \times 1)$, there is a morphism $f : (X, X, 1_X) \rightarrow (X, Y, f)$. Then we get a morphism $\mathcal{D}(f) : \mathcal{D}(X, X, 1_X) \rightarrow \mathcal{D}(X, Y, f)$. We apply this functor to $d(X)$ to obtain $e_r(d)_f$. That is, we set

$$e_r(d)_f := \mathcal{D}(f)(dX).$$

Construction 3.16 [for Problem 3.14] Consider the data given in the Problem. The objects of the category $\Gamma.\mathcal{T}^{\text{core}}.\mathcal{T}.hom_{\mathcal{T}}(i^{op} \times 1).\Theta$ are quintuples $(\gamma, s, t, f, \theta)$ where γ is an object of Γ , $f : s \rightarrow t$ is a morphism in the category $\mathcal{T}(\gamma)$, and θ is an object of $\Theta(\gamma, s)$. Morphisms in $\Gamma.\mathcal{T}^{\text{core}}.\mathcal{T}.hom_{\mathcal{T}}(i^{op} \times 1).\Theta$ are of the form

$$(\phi, \alpha, \beta) : (\gamma, s, t, f, \theta) \rightarrow (\gamma', \mathcal{T}(\phi)(s), t', \alpha \circ \mathcal{T}(\phi)(f), \theta') \quad (*)$$

where $\phi : \gamma \rightarrow \gamma'$ is a morphism in Γ , $\alpha : \mathcal{T}(\phi)(t) \rightarrow t'$ is a morphism in $\mathcal{T}(\gamma')$, and $\beta : \Theta(\phi)(\theta) \rightarrow \theta'$ is a morphism in $\Theta(\gamma', \mathcal{T}(\phi)(s))$.

The objects of the category $\Gamma.\mathcal{T}^{\text{core}}.\Theta$ are triples (γ, t, θ) where $\gamma \in \Gamma$, $t \in \mathcal{T}^{\text{core}}(\gamma)$, and $\theta \in \Theta(\gamma, t)$. The morphisms are of the form $(\phi, \beta) : (\gamma, t, \theta) \rightarrow (\gamma', \mathcal{T}(\phi)(t), \theta')$ where $\phi : \gamma \rightarrow \gamma'$ is a morphism in Γ and $\beta : \Theta(\phi)\theta \rightarrow \theta'$ is a morphism in $\Theta(\gamma', \mathcal{T}(\phi)(t))$.

The functor $\Gamma.1_{\bullet}.\Theta : \Gamma.\mathcal{T}^{\text{core}}.\Theta \rightarrow \Gamma.\mathcal{T}^{\text{core}}.\mathcal{T}.hom_{\mathcal{T}}(i^{op} \times 1).\Theta$ takes a triple (γ, t, θ) to the quintuple $(\gamma, t, t, 1_t, \theta)$.

Now, to construct the lax natural transformation

$$e_r(d) : * \Rightarrow \mathcal{D} : \Gamma.\mathcal{T}^{\text{core}}.\mathcal{T}.hom_{\mathcal{T}}(i^{op} \times 1).\Theta \rightarrow \text{Cat},$$

we need to first specify all of its components $e_r(d)_{(\gamma,s,t,f,\theta)} : * \rightarrow \mathcal{D}(\gamma, s, t, f, \theta)$. So fix an object $(\gamma, s, t, f, \theta)$ of the category $\Gamma.\mathcal{T}^{core}.\mathcal{T}.hom_{\mathcal{T}}(i^{op} \times 1).\Theta$. To construct a functor $e_r(d)_{(\gamma,s,t,f,\theta)} : * \rightarrow \mathcal{D}(\gamma, s, t, f, \theta)$ is to find an object of the category $\mathcal{D}(\gamma, s, t, f, \theta)$. Consider the morphism $(1_\gamma, f, 1_\theta) : (\gamma, s, s, 1_s, \theta) \rightarrow (\gamma, s, t, f, \theta)$ in $\Gamma.\mathcal{T}^{core}.\mathcal{T}.hom_{\mathcal{T}}(i^{op} \times 1).\Theta$. By applying the functor \mathcal{D} , we get a functor

$$\mathcal{D}(1_\gamma, f, 1_\theta) : \mathcal{D} \circ \Gamma.1_\bullet(s).\Theta = \mathcal{D}(\gamma, s, s, 1_s, \theta) \rightarrow \mathcal{D}(\gamma, s, t, f, \theta).$$

Now we apply this functor to the object $d(\gamma, s, \theta)(*)$ of $\mathcal{D} \circ \Gamma.1_\bullet.\Theta(\gamma, s, \theta)$ to get $e_r(d)_{(\gamma,s,t,f,\theta)}$ as follows:

$$e_r(d)_{(\gamma,s,t,f,\theta)}(*) := \mathcal{D}(1_\gamma, f, 1_\theta)d(\gamma, s, \theta)(*).$$

We need to check that this choice is lax natural, which amounts to showing that for any morphism in $\Gamma.\mathcal{T}^{core}.\mathcal{T}.hom_{\mathcal{T}}(i^{op} \times 1).\Theta$ as in line (*) above, there is a natural transformation

$$\mathcal{D}(\phi, \alpha, \beta)e_r(d)_{(\gamma,s,t,f,\theta)} \implies e_r(d)_{(\gamma',T(\phi)(s),t',\alpha T(\phi)(f),\theta')}.$$

But we have

$$\begin{aligned} \mathcal{D}(\phi, \alpha, \beta)e(d)_{(\gamma,s,t,f,\theta)} &= \mathcal{D}(\phi, \alpha, \beta)\mathcal{D}(1_\gamma, f, 1_\theta)d(\gamma, s, \theta) \\ &= \mathcal{D}(\phi, \alpha T(\phi)(f), \beta)d(\gamma, s, \theta) \\ &= \mathcal{D}(1_{\gamma'}, \alpha T(\phi)(f), 1'_\theta)\mathcal{D}(\phi, 1_{T(\phi)s}, \beta)d(\gamma, s, \theta) \\ &\Rightarrow \mathcal{D}(1_{\gamma'}, \alpha T(\phi)(f), 1'_\theta)d(\gamma', T(\phi)(s), \theta') \\ &= e_r(d)_{(\gamma',T(\phi)(s),t',\alpha T(\phi)(f),\theta')} \end{aligned}$$

where the first equality and last equality come from our definition of $e_r(d)$, the second and third equalities come from composition in the category $\Gamma.\mathcal{T}^{core}.\mathcal{T}.hom_{\mathcal{T}}(i^{op} \times 1).\Theta$, and the natural transformation comes from the lax naturality of d .

Now we only need to check that $e_r(d) \circ \Gamma.1_\bullet.\Theta = d$. For any (γ, s, θ) of $\Gamma.\mathcal{T}^{core}.\Theta$, we have that

$$\begin{aligned} e_r(d)\Gamma.1_\bullet.\Theta(\gamma, s, \theta) &= e_r(d)(\gamma, s, s, 1_s, \theta) \\ &= \mathcal{D}(1_\gamma, 1_s, 1_\theta)d(\gamma, s, \theta) \\ &= d(\gamma, s, \theta) \end{aligned}$$

where the first equality is the definition of 1_\bullet , the second is the definition of $e_r(d)$, and the third is the functoriality of \mathcal{D} . For any $(\phi, \beta) : (\gamma, s, \theta) \rightarrow (\gamma', T(\phi)(s), \theta')$ of $\Gamma.\mathcal{T}^{core}.\Theta$, the natural transformation $\mathcal{D}(\phi, 1_{T(\phi)(s)}, \beta)e(d)_{(\gamma,s,s,1_s,\theta)} \implies e_r(d)_{(\gamma',T(\phi)(s),T(\phi)(s),1_{T(\phi)(s)},\theta')}$ defined above reduces to the natural transformation $\mathcal{D}(\phi, 1_{T(\phi)s}, \beta)d(\gamma, s, \theta) \implies d(\gamma', T(\phi)(s), \theta')$ given by the lax naturality of d . Therefore, we have that $e_r(d)1_\bullet = d$. \square

3.7 Left hom elimination and computation

In this section, we interpret the left hom elimination rule so that it satisfies the strict equality required by the left hom computation rule. However, it just takes an application of right hom elimination and computation.

Problem 3.17 *To functorially construct for each functor $\mathcal{T} : \Gamma \rightarrow \mathcal{Cat}$, each functor $\Theta : \Gamma.\mathcal{T}^{core} \rightarrow \mathcal{Cat}$, each functor $\mathcal{D} : \Gamma.\mathcal{T}^{op}.\mathcal{T}^{core}.\text{hom}_{\mathcal{T}}(1 \times i).\Theta \rightarrow \mathcal{Cat}$, and each lax natural transformation*

$$d : *_{\Gamma.\mathcal{T}^{core}.\Theta} \Rightarrow \mathcal{D} \circ \Gamma.1_{\bullet}.\Theta : \Gamma.\mathcal{T}^{core}.\Theta \rightarrow \mathcal{Cat},$$

a lax natural transformation

$$e_r(d) : *_{\Gamma.\mathcal{T}^{op}.\mathcal{T}^{core}.\text{hom}_{\mathcal{T}}(1 \times i).\Theta} \Rightarrow \mathcal{D} : \Gamma.\mathcal{T}^{op}.\mathcal{T}^{core}.\text{hom}_{\mathcal{T}}(1 \times i).\Theta \rightarrow \mathcal{Cat},$$

such that $e_r(d) \circ \Gamma.1_{\bullet}.\Theta = d$.

Construction 3.18 [for Problem 3.17] Note that $(\mathcal{T}^{op})^{core} = \mathcal{T}^{core}$, and

$$\Gamma.(\mathcal{T}^{op})^{core}.\mathcal{T}^{op}.\text{hom}_{\mathcal{T}^{op}}(i \times 1).\Theta \cong \Gamma.\mathcal{T}^{op}.\mathcal{T}^{core}.\text{hom}_{\mathcal{T}}(1 \times i).\Theta.$$

Then by substituting \mathcal{T}^{op} in for \mathcal{T} everywhere in the statement of the Problem 3.14, we find that our problem has already been solved. \square

3.8 The homotopy theory

In this section, we describe the (directed) homotopy theory that this interpretation inhabits in \mathcal{Cat} .

To understand this homotopy theory in \mathcal{Cat} , first recall that there are two weak factorization systems on \mathcal{Cat} [9]. We begin with the following two factorizations of the fold map

$$* + * \xrightarrow{0+1} 2 \xrightarrow{!} * \quad (\rightarrow)$$

$$* + * \xrightarrow{0+1} \mathbb{I} \xrightarrow{!} * \quad (\cong)$$

where 2 is the category generated by one morphism $0 \rightarrow 1$, \mathbb{I} is the category generated by one isomorphism $0 \cong 1$. This gives us, by exponentiation, two functorial factorizations of the diagonal $\mathcal{C} \rightarrow \mathcal{C} \times \mathcal{C}$ of any category \mathcal{C} .

$$\mathcal{C} \xrightarrow{\mathcal{C}^!} \mathcal{C}^2 \xrightarrow{\mathcal{C}^0 \times \mathcal{C}^1} \mathcal{C} \times \mathcal{C} \quad (\mathcal{C}^{\rightarrow})$$

$$\mathcal{C} \xrightarrow{\mathcal{C}^!} \mathcal{C}^{\mathbb{I}} \xrightarrow{\mathcal{C}^0 \times \mathcal{C}^1} \mathcal{C} \times \mathcal{C} \quad (\mathcal{C}^{\cong})$$

Then, we can functorially factor any functor $F : \mathcal{C} \rightarrow \mathcal{D}$ in two ways.

$$\mathcal{C} \xrightarrow{1 \times \mathcal{D}^! F} \mathcal{C} \times_{\mathcal{D}^0} \mathcal{D}^2 \xrightarrow{\mathcal{D}^1} \mathcal{D} \quad (\ell^{\rightarrow}, r^{\rightarrow})$$

$$\mathcal{C} \xrightarrow{1 \times \mathcal{D}^! F} \mathcal{C} \times_{\mathcal{D}^0} \mathcal{D}^{\mathbb{I}} \xrightarrow{\mathcal{D}^1} \mathcal{D} \quad (\ell^{\cong}, r^{\cong})$$

This gives the factorizations of two functorial weak factorization systems $(\mathcal{L}^\rightarrow, \mathcal{R}^\rightarrow)$ and $(\mathcal{L}^\cong, \mathcal{R}^\cong)$ on \mathcal{Cat} .

We can characterize the right maps of these weak factorization systems in another way. The maps in \mathcal{R}^\rightarrow are exactly those maps which have the enriched right lifting property against the inclusion of the domain into the generic morphism: $0 : * \rightarrow 2$. The maps in \mathcal{R}^\cong are exactly those maps which have the enriched right lifting property against the inclusion of the domain into the generic isomorphism: $0 : * \rightarrow \mathbb{I}$.

While the usual identity type of Martin-Löf type theory has an interpretation into the weak factorization system $(\mathcal{L}^\cong, \mathcal{R}^\cong)$ [9], we claim that we have constructed in the preceding sections an interpretation into the weak factorization system $(\mathcal{L}^\rightarrow, \mathcal{R}^\rightarrow)$. By this, we just mean that (1) given any interpretation of a dependent type $\mathcal{T} : \Gamma \rightarrow \mathcal{Cat}$, its comprehension $\pi_\Gamma : \Gamma.T \rightarrow \Gamma$ is in \mathcal{R}^\rightarrow , and (2) the elimination property can be seen as the existence of a lift between a map of \mathcal{L}^\rightarrow and a map of \mathcal{R}^\rightarrow .

To show that any comprehension $\pi_\Gamma : \Gamma.T \rightarrow \Gamma$ is in \mathcal{R}^\rightarrow , note first that it is a Grothendieck opfibration.

Proposition 3.19 *Any Grothendieck opfibration $p : E \rightarrow B$ is in \mathcal{R}^\rightarrow .*

Proof. We need to show that there is a lift ℓ in the following square.

$$\begin{array}{ccc} E & \xlongequal{\quad} & E \\ 1 \times B^! p \downarrow & \nearrow \ell & \downarrow p \\ E \times_{B^0} B^2 & \xrightarrow[B^1]{} & B \end{array}$$

An object of $E \times_{B^0} B^2$ is a pair (e, f) where e is an object of E and $f : pe \rightarrow b$ is a morphism in B . By the definition of Grothendieck opfibration, there is a lift of f , which we will denote $\widetilde{(e, f)} : e \rightarrow \ell(e, f)$ such that $p(\widetilde{(e, f)}) = p$. We will choose $\widetilde{(e, 1_{pe})}$ to always be 1_e .

A morphism $(e_0, f_0) \rightarrow (e_1, f_1)$ of $E \times_{B^0} B^2$ is a pair (ϵ, ϕ) where $\epsilon : e_0 \rightarrow e_1$ is morphism of E and $\phi : b_0 \rightarrow b_1$ is a morphism of B which makes the following square commute in B .

$$\begin{array}{ccc} pe_0 & \xrightarrow{p\epsilon} & pe_1 \\ \downarrow f_0 & & \downarrow f_1 \\ b_0 & \xrightarrow{\phi} & b_1 \end{array}$$

Now, in E we have the following diagram of solid arrows.

$$\begin{array}{ccc} e_0 & \xrightarrow{\epsilon} & e_1 \\ \downarrow \widetilde{(e_0, f_0)} & & \downarrow \widetilde{(e_1, f_1)} \\ \ell(e_0, f_0) & \dashrightarrow & \ell(e_1, f_1) \end{array}$$

By the definition of Grothendieck opfibration, we obtain a unique morphism $\ell(e_0, f_0) \rightarrow \ell(e_1, f_1)$, which we will denote $\ell(\epsilon, \phi)$, which makes the above diagram into a commutative square.

It is straightforward to check that ℓ is a functor which makes the first diagram of this proof commute. \square

Now the right elimination property (in the empty context) says that for any category \mathcal{C} , any functor $\mathcal{D} : \mathcal{C}^{core}.\mathcal{C}.hom_{\mathcal{C}} \rightarrow \mathcal{Cat}$, and any functor $d : \mathcal{C}^{core} \rightarrow \mathcal{C}^{core}.\mathcal{C}.hom_{\mathcal{C}}.\mathcal{D}$ making the following square commute, there is a lift ℓ making both triangles commute.

$$\begin{array}{ccc} \mathcal{C}^{core} & \xrightarrow{d} & \mathcal{C}^{core}.\mathcal{C}.hom_{\mathcal{C}}.\mathcal{D} \\ \downarrow 1_{\bullet} & \nearrow \ell & \downarrow \pi \\ \mathcal{C}^{core}.\mathcal{C}.hom_{\mathcal{C}} & \xlongequal{\quad} & \mathcal{C}^{core}.\mathcal{C}.hom_{\mathcal{C}} \end{array}$$

We want to see that this arises as the solution of a lifting problem between a left map and a right map of our weak factorization system. We just saw that π is a right map, so now we prove that 1_{\bullet} is a left map.

Proposition 3.20 *For any category \mathcal{C} , the functor*

$$1_{\bullet} : \mathcal{C}^{core} \rightarrow \mathcal{C}^{core}.\mathcal{C}.hom_{\mathcal{C}}$$

is in $\mathcal{L}^{\rightarrow}$.

Proof. Factor the functor $i : \mathcal{C}^{core} \rightarrow \mathcal{C}$.

$$\mathcal{C}^{core} \xrightarrow{1 \times C^!i} \mathcal{C}^{core} \times_{\mathcal{C}^0} \mathcal{C}^2 \xrightarrow{C^1} \mathcal{C}$$

We claim that the middle object, $\mathcal{C}^{core} \times_{\mathcal{C}^0} \mathcal{C}^2$, is isomorphic to $\mathcal{C}^{core}.\mathcal{C}.hom_{\mathcal{C}}$ and this makes 1_{\bullet} isomorphic to $1 \times C^!i$, the left factor of the factorization for the weak factorization system $(\mathcal{L}^{\rightarrow}, \mathcal{R}^{\rightarrow})$. Thus, 1_{\bullet} will be in $\mathcal{L}^{\rightarrow}$.

The objects of $\mathcal{C}^{core}.\mathcal{C}.hom_{\mathcal{C}}$ are pairs $((x, y), f)$ where (x, y) is an object of $\mathcal{C}^{core} \times \mathcal{C}$ and f is morphism $x \rightarrow y$.

The objects of $\mathcal{C}^{core} \times_{\mathcal{C}^0} \mathcal{C}^2$ are pairs (x, f) where x is an object of \mathcal{C}^{core} (that is, x is an object of \mathcal{C}) and f is morphism with domain x .

On objects, define the functor $\alpha : \mathcal{C}^{core}.\mathcal{C}.hom_{\mathcal{C}} \rightarrow \mathcal{C}^{core} \times_{\mathcal{C}^0} \mathcal{C}^2$ by $\alpha((x, y), f) := (x, f)$ and its inverse by $\alpha^{-1}(x, f) := ((x, \text{COD}f), f)$.

The morphisms $((x, y), f) \rightarrow ((x, y'), f')$ of $\mathcal{C}^{core}.\mathcal{C}.hom_{\mathcal{C}}$ are morphisms $(1_x, g) : (x, y) \rightarrow (x, y')$ of $\mathcal{C}^{core} \times \mathcal{C}$ such that $gf = f'$.

The morphisms $(x, f) \rightarrow (x, f')$ of $\mathcal{C}^{core} \times_{\mathcal{C}^0} \mathcal{C}^2$ are morphisms $(1_x, g) : f \rightarrow f'$ of \mathcal{C}^2 (so that $gf = f'$).

On morphisms, define the functor α and its inverse both by $(1_x, g) \mapsto (1_x, g)$.

This gives an isomorphism $\alpha : \mathcal{C}^{core}.\mathcal{C}.hom_{\mathcal{C}} \cong \mathcal{C}^{core} \times_{\mathcal{C}^0} \mathcal{C}^2$.

Now, we just check that $\alpha 1_{\bullet} = 1 \times C^!i$. For any object x of \mathcal{C}^{core} , we have that $\alpha 1_{\bullet}(x)$ and $1 \times C^!i(x)$ are both $(x, 1_x)$ (and \mathcal{C}^{core} has no nonidentity morphisms). \square

This concludes the interpretation of our syntax in \mathcal{Cat} .

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