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On Orthomodular Posets Generated by Transition Systems

Luca Bernardinello[1] Lucia Pomello[1] Stefania Rombolà[1]

Università degli studi di Milano–Bicocca
DISCo–viale Sarca 336, U14–20126 Milano (Italy)

Abstract

We study orthomodular structures formed by some sets of states of finite transition systems. These sets, called regions, can be interpreted as local states of a distributed, concurrent system, that can be modelled by a Petri net. The main result shows that such orthomodular structures have enough elements to represent meets of certain subsets of elements.

Keywords: concurrency, Petri nets, orthomodular posets, quantum logic

1 Introduction

We study a class of orthomodular structures arising from finite automata. The elements of such structures are defined as certain subsets of the set of states of an automaton. These subsets can be interpreted as *local states*, in contrast to the given global states of an automaton, in a sense which will be explained in the following.

This line of research comes from our interest in the theory of Petri nets, a formal model of concurrent and distributed systems, introduced by Carl Adam Petri with the aim of building a theory of systems embodying principles derived from modern physics, specifically from the special theory of relativity and from quantum mechanics ([4]).

A Petri net models a system in terms of local states (conditions) and local changes of state, or events ([5], [6]). Events are fully characterized by the changes that their occurrences produce in the local states of a system: the conditions which were true before the occurrence of an event e, and become false afterwards, are the *preconditions* of e, while the conditions which were false and become true are the *postconditions*.

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In this way, the model explicitly captures the relations of causal dependence, replacing temporal order by causal order, and simultaneity by *concurrency*.

Building on these simple elements, Petri defines an essentially asynchronous computational model, endowed with a set of universal, reversible logic gates.

The theory of Petri nets has later been developed also in other directions, based on a specific operational semantics: certain aggregations of local states form potential *global states* of the system; the behaviour of the system can then be described by a finite state automaton (or *transition system*), built on the global states, where state transitions are labelled by the names of local events.

In this paper, we present some results towards the characterization of the class of orthomodular structures generated from the finite automata which describe the behaviour of nets of conditions and events, which form a basic class of Petri nets.

2 Preliminary definitions

In this section we introduce the basic definitions needed in the following.

2.1 Condition Event Net Systems

Condition Event net systems (shortly, CE net systems) were introduced by Carl Adam Petri ([5]) as a basic model of general systems.

Definition 2.1 A *net* is a triple $N = (B, E, F)$, where B and E are finite sets, $F \subseteq (B \times E) \cup (E \times B)$, and

(i) $B \cap E = \emptyset$;

(ii) $\mathrm{dom}(F) \cup \mathrm{ran}(F) = B \cup E$.

The elements of B are called *local states* or *conditions*, the elements of E *local changes of state* or *events*, and F is called the *flow relation*.

For each $x \in B \cup E$, define ${}^\bullet x = \{y \in B \cup E | (y, x) \in F\}$, $x^\bullet = \{y \in B \cup E | (x, y) \in F\}$. For $e \in E$, an element $b \in B$ is a *precondition* of e if $b \in {}^\bullet e$; it is a *postcondition* of e if $b \in e^\bullet$.

A net $N = (B, E, F)$ is *simple* iff for each $x, y \in B \cup E$ (${}^\bullet x = {}^\bullet y$ and $x^\bullet = y^\bullet$) $\Rightarrow$ $x = y$; $N = (B, E, F)$ is *pure* iff for each $e \in E$ ${}^\bullet e \cap e^\bullet = \emptyset$.

Given a net $N = (B, E, F)$, define $K = \mathbb{P}(B)$, where $\mathbb{P}(X)$ denotes the powerset of X. The elements of K are called *cases* of N. A case is a potential global state of the system that N models. Occurrences of transitions, which are subject to the firing rule, defined below, change the current case.

Let $m_1, m_2 \subseteq B$ be two cases of N, and $e \in E$ be an event. We will say that the occurrence of e at m_1 brings the net system from m_1 to m_2, denoted by $m_1 [e\rangle m_2$, if

$${}^\bullet e \subseteq m_1, e^\bullet \cap m_1 = \emptyset, m_2 = (m_1 \setminus {}^\bullet e) \cup e^\bullet.$$

When $m_1 [e\rangle m_2$, for some $e \in E$, we say that m_2 is *forward reachable in one step* from m_1. We call ρ the relation of forward reachability in one step. Then $(\rho \cup \rho^{-1})^*$ is the *full reachability* relation of N.

Definition 2.2 A *Condition Event net system* (CE net system) is a quadruple $N = (B, E, F, C)$, where (B, E, F) is a simple and pure net, and C is an equivalence class of the full reachability relation. The elements of C are called *reachable cases*

The semantics of CE net systems can be defined in several ways. One kind of operational semantics is given by sequential case graphs.

Definition 2.3 The *sequential case graph* of a CE net system N is the triple $SCG(N) = (C, E, T)$, where $T = \{(c_1, e, c_2) \mid c_1, c_2 \in C, e \in E, c_1 [\, e\rangle\, c_2\}$.

Different net systems can be equivalent with respect to this semantics, in the sense that they generate isomorphic case graphs.

For any CE net system there is an equivalent CE net system which is *saturated* of local states (conditions), as shown by Ehrenfeucht and Rozenberg ([3]).

Definition 2.4 Let $N = (B, E, F, C)$. Then it exists $N' = (B', E, F', C')$ such that:

(i) $SCG(N)$ is isomorphic to $SCG(N')$

(ii) $B \subseteq B'$ and $F \subseteq F'$;

(iii) N' is *saturated*, namely it is not possible to add a new condition to B' without generating a non-simple net or changing the behaviour of N'.

The set of local states of a saturated net system can be endowed with a partial ordering, corresponding to a form of logical implication.

Definition 2.5 Let $N = (B, E, F, C)$ a CE net system.

$$\forall b_1, b_2 \in B : b_1 \ \leq_N \ b_2 \Leftrightarrow \forall c \in C : b_1 \in c \Rightarrow b_2 \in c.$$

The algebraic structure generated by $\leq_N$ on the conditions of N' will be discussed in the following.

2.2 Transition systems

We start by defining the kind of finite state automata of interest.

Definition 2.6 A *transition system* is a structure $A = (S, E, T)$, where S is a set of *states*, E a set of *events*, $T \subseteq S \times E \times S$ is a set of *transitions*.

A transition system is *finite* if S and E are finite. In the rest of the paper we will only consider finite transition systems satisfying the following axioms:

(i) the underlying graph of the transition system is simply connected;

(ii) $\forall (s_1, e, s_2) \in T : \ s_1 \neq s_2$;

(iii) $\forall (s_1, e_1, s_2), (s_1, e_2, s_3) \in T : s_2 = s_3 \Rightarrow e_1 = e_2$;

(iv) $\forall e \in E : \ \exists (s_1, e, s_2) \in T$.

A region is a set of states such that all occurrences of a given event have the same crossing relation (entering, leaving or non-crossing) with respect to the region itself, and this property holds for all events [3]. A region can be interpreted as a local state of a system whose behaviour is described by the transition system. Such local states correspond to *conditions*, which are either true or false in a global state. A region r is the set of global states in which the corresponding condition is true.

Definition 2.7 Let $A = (S, E, T)$ be a transition system. A set of states $r \subseteq S$ is said to be a *region* iff $\forall e \in E$, $\forall (s_1, e, s_2), (s_3, e, s_4) \in T$ we have $(s_1 \in r \wedge s_2 \notin r) \Rightarrow (s_3 \in r \wedge s_4 \notin r) \wedge (s_1 \notin r \wedge s_2 \in r) \Rightarrow (s_3 \notin r \wedge s_4 \in r)$.

The set of all regions of A will be denoted by R_A. From the definition it follows that: $\emptyset, S \in R_A$ and $\forall r \in R_A : S \setminus r \in R_A$. For each $s \in S$, R_s will denote the set of regions containing s.

The conditions defining regions allow us to formalize the crossing relation between events and regions. This is captured by the notions of pre- and post-sets of regions and pre- and post-sets of events.

Definition 2.8 Let $A = (S, E, T)$ be a transition system. Let $r \in R_A$. Then the *pre-set* of r, denoted by ${}^\bullet r$, and the *post-set* of r, denoted by $r^\bullet$, are defined by: ${}^\bullet r = \{e \in E | \; \exists (s_1, e, s_2) \in T : s_1 \notin r \text{ and } s_2 \in r\}$, $r^\bullet = \{e \in E | \; \exists (s_1, e, s_2) \in T : s_1 \in r \text{ and } s_2 \notin r\}$. Let $e \in E$. Then the *pre-set* and the *post-set* of e, denoted by, respectively, ${}^\bullet e$ and $e^\bullet$, are defined by: ${}^\bullet e = \{r | \; r \in R_A \text{ and } e \in r^\bullet\}$, $e^\bullet = \{r | \; r \in R_A \text{ and } e \in {}^\bullet r\}$.

Now we are ready to give the definition of condition event transition systems (CE transition systems), which form the class of transition systems isomorphic to the sequential case graphs of CE net systems.

Definition 2.9 A finite transition system $A = (S, E, T)$ is a *condition event transition system* (CE transition system) iff it satisfies the following axioms:

A1. $\forall s_1, s_2 \in S : R_{s_1} = R_{s_2} \Rightarrow s_1 = s_2$;

A2. $\forall s \in S \; \forall e \in E : {}^\bullet e \subseteq R_s \Rightarrow \exists s_1 \in S \; (s, e, s_1) \in T$;

A3. $\forall s \in S \; \forall e \in E : e^\bullet \subseteq R_s \Rightarrow \exists s_1 \in S \; (s_1, e, s) \in T$.

The sequential case graph of a CE net system is a CE transition system. Vice versa, given an abstract CE transition system $A = (S, E, T)$, it is always possible to build a CE net system $N(A)$ such that its sequential case graph is isomorphic to A. The conditions of $N(A)$ are the regions of A, and $N(A)$ is saturated of conditions.

2.3 Orthomodular posets

Orthomodular posets can be considered as a generalization of Boolean algebras, where meet and join are partial operations, while each element has a complement.

Definition 2.10 An *orthomodular poset* $P = \langle P, \leq, 0, 1, (.)' \rangle$ is a partially ordered set $\langle P, \leq \rangle$, equipped with a minimum and a maximum element, respectively denoted by 0 and 1, and with a map $(.)' : P \to P$, such that the following conditions are verified (where $\vee$ and $\wedge$ denote, respectively, the least upper bound and the greatest lower bound with respect to $\leq$, when they exist): $\forall x, y \in P$

$$
\begin{aligned}
&(x')' = x;\\
&x \leq y \Rightarrow y' \leq x';\\
&x \leq y \Rightarrow y = x \vee (y \wedge x');\\
&x \leq y' \Rightarrow x \vee y \in P;\\
&x \wedge x' = 0.
\end{aligned}
$$

The third condition above is known as *orthomodular law*.

Two elements, x, y, of an orthomodular poset P are *orthogonal* (denoted by $x \perp y$) if and only if $x \leq y'$; x and y are *compatible* (denoted by $x \$ y$) if there are three elements, $x_0, y_0, z \in P$, pairwise orthogonal, such that $x = x_0 \vee z$ and $y = y_0 \vee z$. An orthomodular poset P is *coherent* if for all $x, y, z \in P$, pairwise compatible, it holds $(x \vee y) \$ z$.

In the following, we will use OMP as a shorthand for orthomodular poset.

We are interested in certain subsets of an OMP, here called *prime filters*, which can be seen as a generalization of ultrafilters in Boolean algebras. They correspond to *two-valued states* as defined in the literature on quantum logics (see, for example, [7]); for our purposes, it is convenient to see them as subsets rather than as maps.

Definition 2.11 Let $P = \langle P, \leq, 0, 1, (.)' \rangle$ be an OMP. A non empty subset $f \subseteq P$ is a *prime filter* iff for all $x, y \in P$:

(i) $(x \in f \text{ and } x \leq y) \Rightarrow y \in f$

(ii) $(x, y \in f \text{ and } x \$ y) \Rightarrow x \wedge y \in f$

(iii) $0 \notin f$

(iv) $\forall x, y \in P$: $(x \, \$ \, y \text{ and } x \vee y \in f) \Rightarrow (x \in f \text{ or } y \in f)$

The set of all prime filters of P will be denoted by $FP(P)$. For $x \in P$, $[x]$ will denote the set of all prime filters of P which contain x.

We will call an OMP P *prime* if the set of prime filters has enough elements to "separate" distinct elements of P, that is, if, for all $x, y \in P$, there exists $f \in FP(P)$ such that $x \in f \Leftrightarrow y \notin f$.

3 Regional orthomodular posets

Let $A = (S, E, T)$ be a transition system. The set of regions of A is partially ordered by set inclusion. The resulting partially ordered set has a minimum, the empty set, and a maximum, the set S of all states. We can actually prove that it is a coherent and prime orthomodular poset (see [1]). We will denote this poset by $H(A) = (R_A, \subseteq, (.)', \emptyset, S)$. In general, $H(A)$ is not a lattice. If A is a CE transition system, $H(A)$ is a Boolean algebra only if A can be realized as the case graph of a *state machine* net system, namely a CE net system such that any event has exactly one precondition and one postcondition, and each reachable case consists in exactly one condition.

When A is a CE transition system, there is a strong relation between its states and the prime filters of $H(A)$. Let $s \in S$; the set of regions containing s is a prime filter of R_A; this filter will be denoted by f_s. On the other hand, in general not all prime filters of $H(A)$ correspond to states of A. Nonetheless, we can interpret all filters as states of another transition system, in which A is embedded. More generally, given a coherent and prime OMP, we can always define a CE transition system (hence, also a CE net system), as follows.

Definition 3.1 Let P be a coherent and prime OMP. Define

$$J(P) = (FP(P), E_P, T_P)$$

where $E_P = \{\langle f\backslash g, g\backslash f\rangle | f, g \in FP(P), f \neq g\}$, and $(f, e, g) \in T_P$ iff $e = \langle f\backslash g, g\backslash f\rangle$.

Notice that different pairs of prime filters can produce the same event; this means that the event can occur in different global states, with the same local effect.

We now have two maps: H associates a coherent, prime orthomodular poset to a transition system; J goes the other way round. In [1], it was shown that $J(P)$ belongs to the class of transition systems generated by nets of condition and events (namely, CE transition systems). In the same paper, it was shown that A is embedded in $JH(A)$ for all CE transition system A, and that P is embedded in $HJ(P)$. In both cases, the embedding is given by a natural notion of morphism.

We conjecture that $H(A)$ and $HJH(A)$ are isomorphic OMPs, and that $J(P)$ and $JHJ(P)$ are isomorphic transition systems, but these remain open problems. If the two conjectures were true, we could interpret HJ and JH as a kind of saturation in the respective domains.

A related open problem is the characterization of orthomodular posets generated by transition systems via regions. In the following, we describe a step towards that end.

Definition 3.2 An orthomodular poset P is *regional* if it is isomorphic to $H(A)$ for some transition system A.

Every regional OMP is coherent and prime; however, there are coherent and prime OMPs which are not regional; a simple case is described in the following example.

Example 3.3 Let $B = \{1, \ldots, 6\}$ and $\Omega = \{X \subseteq \{1, \ldots, 6\} || X| \text{ is an even number}\}$. It is easy to verify that $P = \langle \Omega, \subseteq, \emptyset, B, (.)' \rangle$, where the orthocomplementation is the complement relative to B, is a coherent and prime orthomodular poset. P has six prime filters; a typical prime filter is given by the set of all $X \in \Omega$ containing a given $i \in B$. The corresponding transition system, $J(P)$, has six states, and is such that all subsets of states are regions. Hence, $HJ(P)$ is a Boolean algebra with six atoms, and P is embedded into $HJ(P)$.

As this example shows, the class of coherent and prime OMPs is strictly larger than the class of regional OMPs.

In our search for a characterization of regional orthomodular posets, we look for peculiar properties of concrete regions which can be translated in the abstract setting of OMPs.

Lemma 3.4 *Let A be a CE transition system. Let x, y, z be regions of A and $x \cap y = y \cap z = z \cap x$; then $x \cap y \cap z$ is a region of A.*

The proof of this lemma consists in a routine verification. An analogous property can be stated for OMPs.

Definition 3.5 Let P be a coherent prime OMP. P is said to be *rich* if the following property holds:

$$[x] \cap [y] = [y] \cap [z] = [z] \cap [x] \Rightarrow \exists w \in P : [w] = [x] \cap [y] \cap [z]$$

Notice that the OMP described in Example 3.3 is coherent and prime, but not rich, and not regional.

Theorem 3.6 *Every regional OMP is rich.*

Proof. Let P be a regional OMP. Then there exists a CE transition system $A = (S, E, T)$ such that P is isomorphic to $H(A)$. In this proof, we work on $H(A)$.

Take $x, y, z \in R_A$ such that $[x] \cap [y] = [y] \cap [z] = [z] \cap [x]$.

We have already remarked that the correspondence between the prime filters of $H(A)$ and the states of A is not bijective; more exactly, there can be prime filters of $H(A)$ that do not correspond to states of A. We can then decompose the set of prime filters $[x]$ into two disjoint subsets: $[x] = X_1 \cup X_2$, where $X_1 = \{f \in [x] | f = f_s$ for some $s \in S\}$ and $X_2 = [x] \setminus X_1$. In a similar way, we decompose $[y] = Y_1 \cup Y_2$, and $[z] = Z_1 \cup Z_2$.

Then $X_1 \cap Y_1 = Y_1 \cap Z_1 = Z_1 \cap X_1$. To prove this statement, note that $Y_1 \cap X_2 = \emptyset$, and $Y_2 \cap X_1 = \emptyset$, so that $[x] \cap [y] = (X_1 \cap Y_1) \cup (X_2 \cap Y_2)$, and similarly for $[y] \cap [z]$ and $[x] \cap [z]$.

The set X_1 contains all filters corresponding to states of A, and containing x; we can identify it with x as a region of A, because the correspondence between states and filters is injective. Then, we deduce that $x \cap y = y \cap z = z \cap x$, where we consider x, y, z as regions. From Lemma 3.4, it follows that $w = x \cap y \cap z$ is a region of A, and an element of $H(A)$.

It is easy to show that, in any orthomodular poset P, for all $p, q \in P$ such that $p \,\$\, q$ it holds $[p] \cap [q] = [p \wedge q]$. In $H(A)$, we have $w = x \cap y$, hence $w = x \wedge y$, so $[w] = [x] \cap [y] = [x] \cap [y] \cap [z]$. □

It is still an open problem whether all coherent, prime, and rich OMPs are regional.

While working on the characterization of regional OMPs, we plan to study mutual relations among various properties of OMPs. In particular, we know that an OMP can be rich and non prime (take for instance an OMP without prime filters); we will investigate whether richness and the existence of enough prime filters imply coherence.

4 Orthomodular lattices generated by occurrence nets

We have shown that a natural order relation defined on the local states of a CE net system gives rise to an orthomodular poset, when the net system is saturated with respect to local states. A different way of generating orthomodular posets from Petri nets builds on occurrence nets. These are acyclic nets of conditions and events, originally introduced as a model for concurrent processes; they can naturally be used to define a semantics of general CE net systems, expressed within net theory.

Petri has defined an occurrence net with a regular structure as a kind of discrete representation of a relativistic spacetime. It is then possible to define a closure system on subsets of elements of the net, and a corresponding orthocomplement. The closed subsets form an orthomodular lattice; this result resembles results by several authors (see, for example, [2]) on the usual Minkowski spacetime, and suggests a different line of research.

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