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Generalizing Topological Set Operators

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Abstract

It is well-known that topological spaces can be axiomatically defined by the topological closure operator, i.e., Kuratowski Closure Axioms. Equivalently, they can be axiomatized by other set operators reflecting primitive notions of topology, such as interior operator, derived set operator (or dually, exterior operators, co-derived set operators), or boundary operator. It is also known that topological closure operators (and dually, topological interior operators) can be weakened as in generalized closure (interior) systems. What about boundary operator, exterior operator, and derived set (and co-derived set) operator in the weakened systems? Our paper completely answers this question by showing that these six operators can all be weakened in an appropriate way such that their relationships remain essentially the same as in topological spaces. Our results indicate that topological semantics can be fully relaxed to the weakened systems.

Keywords: closure, interior, derived set, co-derived set, exterior, boundary, operator

1 Introduction

Let us recall the notion of a topological space [5]. A topology on a set X is a collection $\mathcal{T}$ of subsets of X including $\emptyset$ and X which is closed under arbitrary union and finite intersection, and $(X, \mathcal{T})$ is called a topological space. Those subsets of X, which are members of $\mathcal{T}$, are called *open* (sub)set in the space X. A subset $F \subseteq X$ is called *closed* in $(X, \mathcal{T})$ if its complement $X \setminus F$ is an open set. From De Morgan's Law, we infer that the collection of closed sets includes $\emptyset$ and X, and that

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it is closed under finite union and arbitrary intersection. So specifying a collection of open sets amounts to specifying the collection of closed sets.

Associated with any topology $\mathcal{T}$ is the *topological closure operator*, denoted $\mathbf{Cl}$, which gives, for any subset $A \subseteq X$, the smallest closed set containing A. Obviously, a set A is closed if and only if $\mathbf{Cl}(A) = A$.

Denote $\mathcal{P}(X)$ as the powerset of X. Then $\mathbf{Cl}$ as defined above can be viewed as an operator $\mathbf{Cl} : \mathcal{P}(X) \to \mathcal{P}(X)$ that satisfies the following properties (for arbitrary $A, B \subseteq X$):
(CO1) $\mathbf{Cl}(\emptyset) = \emptyset$;
(CO2) $A \subseteq \mathbf{Cl}(A)$;
(CO3) $\mathbf{Cl}(\mathbf{Cl}(A)) = \mathbf{Cl}(A)$;
(CO4) $\mathbf{Cl}(A \cup B) = \mathbf{Cl}(A) \cup \mathbf{Cl}(B)$.

Indeed, any operator $\mathbf{Cl}$ on $\mathcal{P}(X)$ that satisfies the above four axioms (called Kuratowski Closure Axioms) defines a *topological closure operator*. Its fixed points $\{A\colon \mathbf{Cl}(A) = A\}$ form a set system that can be properly identified as a system of closed sets; taking complement of each of these closed sets gives rise to a set system that will properly be a topology. In this sense, we can say that an operator satisfying the Kuratowski Closure Axioms (CO1)-(CO4) defines a topological space $(X, \mathcal{T})$.

Dual to the topological closure operator is the *topological interior operator* $\mathbf{Int}$, which satisfies the following four axioms (for any set $A, B \subseteq X$):
(IO1) $\mathbf{Int}(X) = X$;
(IO2) $\mathbf{Int}(A) \subseteq A$;
(IO3) $\mathbf{Int}(\mathbf{Int}(A)) = \mathbf{Int}(A)$;
(IO4) $\mathbf{Int}(A \cap B) = \mathbf{Int}(A) \cap \mathbf{Int}(B)$.
The fixed points of $\mathbf{Int}$, $\{A : \mathbf{Int}(A) = A\}$ form a set system that can be properly identified as open sets, hence defining the topological space $(X, \mathcal{T})$.

The equivalence of the above two axiomatically defined operators on $\mathcal{P}(X)$ in specifying any topology $\mathcal{T}$ is well-known. Furthermore, topological semantics of $\mathbf{Cl}(A)$ and $\mathbf{Int}(A)$ as they operate on an arbitrary subset $A \subseteq X$ are compatible with the corresponding set of axioms defining each operator.

In addition to the closure or interior operators defining a topological space, there are other four set operators widely used as primitive notions in topology. They are the *exterior operator*, the *boundary operator*, the *derived set operator*, and the dually-defined *co-derived set operator*. All these operators have been shown to be able to specify an identical topology $\mathcal{T}$ – they are equivalent to one another, as with $\mathbf{Cl}$ and $\mathbf{Int}$ operators. Therefore, all *six* operators $\mathcal{P}(X) \to \mathcal{P}(X)$ provide equivalent characterizations of a topological space $(X, \mathcal{T})$. Taken together, they provide comprehensive topological semantics to ground first-order modal logic.

In parallel with these various axiomatizations of what can be called Topological System, it is also widely established that the topological closure operator can be relaxed to the more general setting of a Closure System in which the closure operator satisfies, instead of (CO1)-(CO4), three similar axioms (see below), without enforcing axiom (CO4). The fixed points associated to this *generalized closure operator* are called *(generalized) closed sets*. Viewed in this way, the closed-set system of a Topological System is just a special case of this generalized Closure System. Other applications of the Closure System include Formal Concept Analysis [8], Matroid and Anti-Matroid/Learning Space [4,6], in which the generalized closure operator is enhanced with an additional exchange axiom (for matroid) or anti-exchange (for anti-matroid) axiom. Closure Systems also play an important role in Category Theory [1,2,14] and also in Domain Theory, e.g. [9].

With these theoretical backdrop, one immediate question is whether there exist meaningful generalizations of the other five operators in a Topological System to a corresponding Closure System. Implied in this "meaningfulness" is the requirement that the behavior of these generalized operators would mirror their roles of their topological counterparts, such that the relations interlocking one operator to another are preserved. If a meaningful generalization can be achieved, then we can claim that the Closure System is a strict weakening of the Topological System while the semantics of the operations are preserved.

In the present work, we provide a complete answer to the above question. We provide an axiomatic system for the suite of generalized operators. Some of the generalizations are straightforward, for instance, the generalized interior and generalized exterior operators can be linked to the generalized closure operator in a direct, immediate fashion (which involves only set complement). Others are more involved. After carefully analyzing the axiomatization schemes for the topological boundary operator and for the topological derived set operator [7,10,12,13], we obtain a generalization of boundary, derived set, co-derived set operators from the setting of Topological System to that of Closure System, such that the relationships between themselves and the closure/interior/exterior operators mimic those in the topological context. In doing so, we obtained a full axiomatic characterization of relevant operators in a Closure System.

The remaining part of the paper is organized as follows. In Section 2, we review the various axiomatization of topological set operators, while at the same time highlighting some important properties of the boundary operator and derived set operator. In Section 3, starting from the generalized closure operator, generalized interior operator, and generalized exterior operator, we provide a generalization of the boundary operator (Section 3.2), and a generalization of derived set operators (Section 3.3) and co-derived set operators (Section 3.4). We close our paper with a short summary and discussion (Section 4).

2 Equivalent Characterizations of a Topological System

Topological Systems are specified by the collection of open sets, or equivalently, the collection of closed sets as set systems. In addition to the topological closure and topological interior operators to characterize a topology, there are four other operators commonly used in topology, namely, exterior operator, boundary operator, derived set operator, and co-derived set operator. They can also be used to completely characterize a Topological System, as shown by [7,10,12,13]. We review below.

2.1 Exterior and Boundary Operator

Let us first discuss the exterior and the boundary operator in a topological space. [7] provided axiomatizations for both.

In addition to axiomatically defining **Cl** and **Int** operators, one can make use of the so-called *topological exterior operator* **Ext** related to **Int** by $\mathbf{Ext}(A) = \mathbf{Int}(A') = \mathbf{Int}(X \setminus A)$, where $'$ denotes set-wise complement. Just as the set $\mathbf{Int}(A)$ gives the interior of A, the set $\mathbf{Ext}(A)$ gives the exterior of A in the space $(X, \mathcal{T})$.

Definition 2.1 (Topological Exterior Operator).
A mapping $\mathbf{Ext}\colon \mathcal{P}(X) \to \mathcal{P}(X)$ is called an exterior operator if for any $A, B \subseteq X$, **Ext** satisfies the following four axioms:
(EO1) $\mathbf{Ext}(\emptyset) = X$;
(EO2) $A \cap \mathbf{Ext}(A) = \emptyset$;
(EO3) $\mathbf{Ext}(X \setminus \mathbf{Ext}(A)) = \mathbf{Ext}(A)$;
(EO4) $\mathbf{Ext}(A \cup B) = \mathbf{Ext}(A) \cap \mathbf{Ext}(B)$.

Given an operator **Ext** satisfying the above four axioms, then we can obtain $\mathcal{T} = \{U \in \mathcal{P}(X) \mid \mathbf{Ext}(X \setminus U) = U\}$, which is a topology. Moreover, $\mathcal{T}$ is the only topology that is compatible with this "exterior" meaning for this operator, i.e., dual to the interior operator whose fixed points forms the system of open sets of $\mathcal{T}$.

Definition 2.2 (Topological Boundary Operator).
A mapping $\mathbf{Fr}\colon \mathcal{P}(X) \to \mathcal{P}(X)$ is called a boundary operator (sometimes also called *frontier*) if for any set $A, B \subseteq X$, **Fr** satisfies the following five axioms:
(FO1) $\mathbf{Fr}(\emptyset) = \emptyset$;
(FO2) $\mathbf{Fr}(A) = \mathbf{Fr}(X \setminus A)$;
(FO3) $A \subseteq B \Rightarrow \mathbf{Fr}(A) \subseteq B \cup \mathbf{Fr}(B)$;
(FO4) $\mathbf{Fr}(\mathbf{Fr}(A)) \subseteq \mathbf{Fr}(A)$;
(FO5) $\mathbf{Fr}(A \cup B) \subseteq \mathbf{Fr}(A) \cup \mathbf{Fr}(B)$.

For a boundary operator **Fr**, we can define $\mathcal{T} = \{U \in \mathcal{P}(X) \mid \mathbf{Fr}(X \setminus U) \subseteq X \setminus U\}$. $\mathcal{T}$ is a topology. For $A \subseteq X$, $\mathbf{Fr}(A)$ is the boundary of A in the space $(X, \mathcal{T})$. Moreover, $\mathcal{T}$ is the only topology satisfying this condition.

We now investigate the role of (FO5), which is to be removed when relaxing to generalized Closure System.

Proposition 2.3
(FO4) *and* (FO5) *imply*
(FO4)* $\mathbf{Fr}(A \cup \mathbf{Fr}(A)) \subseteq \mathbf{Fr}(A)$,
which then implies $\mathbf{Fr}(A \cup \mathbf{Fr}(A)) \subseteq A \cup \mathbf{Fr}(A)$.

Proof. By (FO5), $\mathbf{Fr}(A \cup \mathbf{Fr}(A)) \subseteq \mathbf{Fr}(A) \cup \mathbf{Fr}(\mathbf{Fr}(A))$. Because of (FO4), $\mathbf{Fr}(A) \cup \mathbf{Fr}(\mathbf{Fr}(A)) = \mathbf{Fr}(A)$. Then $\mathbf{Fr}(A \cup \mathbf{Fr}(A)) \subseteq \mathbf{Fr}(A)$ holds. Obviously, $\mathbf{Fr}(A \cup \mathbf{Fr}(A)) \subseteq A \cup \mathbf{Fr}(A)$ also holds. $\square$

If we drop axiom (FO5) in the definition of $\mathbf{Fr}$, we do not have (FO4)*. On the other hand, we have the following result.

Proposition 2.4
(FO2), (FO3) *and* (FO4)* *implies* (FO4).

Proof. Suppose that set operator $\mathbf{Fr}$ only satisfies (FO2) and (FO3) in Definition 2.2. For any $A \subseteq X$, $\mathbf{Fr}(A) \subseteq A \cup \mathbf{Fr}(A)$. An application of (FO3) gives $\mathbf{Fr}(\mathbf{Fr}(A)) \subseteq A \cup \mathbf{Fr}(A) \cup \mathbf{Fr}(A \cup \mathbf{Fr}(A)) = A \cup \mathbf{Fr}(A)$, where the last step invokes (FO4)*. Then $\mathbf{Fr}(\mathbf{Fr}(A)) \subseteq A \cup \mathbf{Fr}(A)$ holds. Likewise, for the complement $X \setminus A$, $\mathbf{Fr}(\mathbf{Fr}(X \setminus A)) \subseteq (X \setminus A) \cup \mathbf{Fr}(X \setminus A)$ holds. By (FO2), we have $\mathbf{Fr}(\mathbf{Fr}(A)) \subseteq A \cup \mathbf{Fr}(A)$ and $\mathbf{Fr}(\mathbf{Fr}(A)) \subseteq (X \setminus A) \cup \mathbf{Fr}(A)$. Therefore, $\mathbf{Fr}(\mathbf{Fr}(A)) \subseteq (A \cup \mathbf{Fr}(A)) \cap ((X \setminus A) \cup \mathbf{Fr}(A)) = (A \cap (X \setminus A)) \cup \mathbf{Fr}(A)) = \emptyset \cup \mathbf{Fr}(A)) = \mathbf{Fr}(A)$, i.e., $\mathbf{Fr}(\mathbf{Fr}(A)) \subseteq \mathbf{Fr}(A)$.□

From the above two Propositions, it follows that (FO4) in the axiomatical definition of boundary operator can be equivalently replaced by (FO4)*. Then we have another axiomatization of topological boundary operator $\mathbf{Fr}$.

Definition 2.5 (Topological Boundary Operator, Alternative Definition).
A mapping $\mathbf{Fr}$: $\mathcal{P}(X) \to \mathcal{P}(X)$ is called a boundary operator if for any set $A, B \subseteq X$, $\mathbf{Fr}$ satisfies the following five axioms:
(FO1) $\mathbf{Fr}(\emptyset) = \emptyset$;
(FO2) $\mathbf{Fr}(A) = \mathbf{Fr}(X \setminus A)$;
(FO3) $A \subseteq B \Rightarrow \mathbf{Fr}(A) \subseteq B \cup \mathbf{Fr}(B)$;
(FO4)* $\mathbf{Fr}(A \cup \mathbf{Fr}(A)) \subseteq \mathbf{Fr}(A)$;
(FO5) $\mathbf{Fr}(A \cup B) \subseteq \mathbf{Fr}(A) \cup \mathbf{Fr}(B)$.

2.2 Derived Set and Co-Derived Set Operator

We will turn to the derived set operator and the co-derived set operator. For axioms of derived set operator in topological spaces, two versions were suggested by Harvey [10] and Spira [12] , respectively. We first consider the scheme by Harvey.

Definition 2.6 (Topological Derived Set Operator).
A mapping $\mathbf{Der}$: $\mathcal{P}(X) \to \mathcal{P}(X)$ is called a derived set operator if for any set $A, B \subseteq X$, $\mathbf{Der}$ satisfies the following four axioms:
(DO1) $\mathbf{Der}(\emptyset) = \emptyset$;
(DO2) $x \in \mathbf{Der}(A) \Leftrightarrow x \in \mathbf{Der}(A \setminus \{x\})$;
(DO3) $\mathbf{Der}(A \cup \mathbf{Der}(A)) \subseteq A \cup \mathbf{Der}(A)$;
(DO4) $\mathbf{Der}(A \cup B) = \mathbf{Der}(A) \cup \mathbf{Der}(B)$.

Proposition 2.7
A derived set operator $\mathbf{Der}$ *has the following property:*
(DO3)* $\mathbf{Der}(\mathbf{Der}(A)) \subseteq A \cup \mathbf{Der}(A)$ *for any* $A \subseteq X$.
Moreover, (DO3)* *is equivalent to* (DO3) *under* (DO4).

Proof. First, we show that the derived set operator $\mathbf{Der}$ is monotone: for any $A, B \subseteq X$, $A \subseteq B$ implies $\mathbf{Der}(A) \subseteq \mathbf{Der}(B)$. By (DO4) and $A \subseteq B$, $\mathbf{Der}(A \cup B) = \mathbf{Der}(B) = \mathbf{Der}(A) \cup \mathbf{Der}(B)$, which implies $\mathbf{Der}(A) \subseteq \mathbf{Der}(B)$.

For any $A \subseteq X$, $\mathbf{Der}(A) \subseteq A \cup \mathbf{Der}(A)$ holds. By the monotone property of $\mathbf{Der}$ and (DO3), we have $\mathbf{Der}(\mathbf{Der}(A)) \subseteq \mathbf{Der}(A \cup \mathbf{Der}(A)) \subseteq A \cup \mathbf{Der}(A)$. Then $\mathbf{Der}(\mathbf{Der}(A)) \subseteq A \cup \mathbf{Der}(A)$, so (DO3)* holds.

On the other hand, suppose that $\mathbf{Der}$ only has the property (DO4): $\mathbf{Der}(A \cup B) = \mathbf{Der}(A) \cup \mathbf{Der}(B)$. Then $\mathbf{Der}(A \cup \mathbf{Der}(A)) = \mathbf{Der}(A) \cup \mathbf{Der}(\mathbf{Der}(A))$. By (DO3)*, $\mathbf{Der}(A \cup \mathbf{Der}(A)) \subseteq A \cup \mathbf{Der}(A)$, i.e., (DO3) holds. Therefore, (DO3)* is equivalent to (DO3) under (DO4). □

From the above proposition, it follows that we can equivalently substitute (DO3)* for (DO3) in the definition of axiomatical derived set operator. Denote (DO2)*: $x \notin \mathbf{Der}(\{x\})$ for any $x \in X$.
Spira [12] showed that axiom (DO2) is equivalent to (DO2)* under axioms (DO1) and (DO4). Therefore, we have an alternative, simpler version of axiomatical derived set operator.

Definition 2.8 (Topological Derived Set Operator, Alternative Definition).
A mapping $\mathbf{Der}\colon \mathcal{P}(X) \to \mathcal{P}(X)$ is called a derived set operator if for any set $A, B \subseteq X$, $\mathbf{Der}$ satisfies the following four axioms:
(DO1) $\mathbf{Der}(\emptyset) = \emptyset$;
(DO2)* For any $x \in X$, $x \notin \mathbf{Der}(\{x\})$;
(DO3)* $\mathbf{Der}(\mathbf{Der}(A)) \subseteq A \cup \mathbf{Der}(A)$;
(DO4) $\mathbf{Der}(A \cup B) = \mathbf{Der}(A) \cup \mathbf{Der}(B)$.

From a derived set operator $\mathbf{Der}$, we can dually define an operator through complementation, $\mathbf{Cod}$: for any $A \subseteq X$, $\mathbf{Cod}(A) = (\mathbf{Der}(A'))'$ (by the definition, $\mathbf{Der}(A) = (\mathbf{Cod}(A'))'$). Steinsvold [13] used the co-derived set operator as the semantics for belief in his PhD thesis.

Definition 2.9 (Topological Co-Derived Set Operator).
A mapping $\mathbf{Cod}\colon \mathcal{P}(X) \to \mathcal{P}(X)$ is called a co-derived set operator if for any set $A, B \subseteq X$, $\mathbf{Cod}$ satisfies the following four axioms:
(i) $\mathbf{Cod}\ (X) = X$;
(ii) $x \in \mathbf{Cod}(A) \Leftrightarrow x \in \mathbf{Cod}(A \cup \{x\})$;
(iii) $\mathbf{Cod}(A \cap \mathbf{Cod}(A)) \supseteq A \cap \mathbf{Cod}(A)$;
(iv) $\mathbf{Cod}(A \cap B) = \mathbf{Cod}(A) \cap \mathbf{Cod}(B)$.

Both derived set and co-derived set can be used to define a topology. Any subset $A \subseteq X$ is called *closed* when $\mathbf{Der}(A) \subseteq A$. Then $\mathcal{T} = \{U \in \mathcal{P}(X) \mid X \setminus U$ is closed$\} = \{U \in \mathcal{P}(X) \mid \mathbf{Der}(X \setminus U) \subseteq X \setminus U\}$ is a topology on X and the derived set operator induced by $\mathcal{T}$ is just $\mathbf{Der}$. Moreover, $\mathcal{T}$ is the only topology satisfying this condition. Dually, $\mathcal{T}' = \{U \in \mathcal{P}(X) \mid U \subseteq \mathbf{Cod}(U)\}$ is also a topology on X. A derived set operator and its dual co-derived set operator generate the same topology. That is, the above two topologies are the same, i.e., $\mathcal{T} = \mathcal{T}'$.

3 Equivalent Characterizations of a Closure System

In this Section, we first review the relaxation from topological closure to a generalized closure operator, also denoted **Cl** here. The three axioms for **Cl** can turn equivalently to axiom system for generalized interior operator **Int** and generalized exterior operator **Ext**. Note that all operators treated in this Section refers to the "generalized" version, despite of using the same bold-face symbols.

3.1 Generalized Closure, Interior, and Exterior Operators

We first recall the generalization of closure operator.

Definition 3.1 (Closure Operator).
A mapping **Cl**: $\mathcal{P}(X) \to \mathcal{P}(X)$ is called a *generalized closure operator* (or simply, *closure operator*) if for any $A, B \subseteq X$, **Cl** satisfies the following three axioms:
(C1) $A \subseteq \mathbf{Cl}(A)$;
(C2) $A \subseteq B \Rightarrow \mathbf{Cl}(A) \subseteq \mathbf{Cl}(B)$;
(C3) $\mathbf{Cl}(\mathbf{Cl}(A)) = \mathbf{Cl}(A)$.

Dually, we can define a generalized interior operator.

Definition 3.2 (Interior Operator).
A mapping **Int**: $\mathcal{P}(X) \to \mathcal{P}(X)$ is called *the generalized interior operator* (or simply, *interior operator*) if for any set $A, B \subseteq X$, **Int** satisfies the following three axioms:
(I1) $\mathbf{Int}(A) \subseteq A$;
(I2) $A \subseteq B \Rightarrow \mathbf{Int}(A) \subseteq \mathbf{Int}(B)$;
(I3) $\mathbf{Int}(\mathbf{Int}(A)) = \mathbf{Int}(A)$.

The interior operator **Int** is dual to the closure operator **Cl**, in the sense that for any $A \subset X$, $\mathbf{Int}(A) = (\mathbf{Cl}(A'))'$ and $\mathbf{Cl}(A) = (\mathbf{Int}(A'))'$.

In light of the relation between exterior operators and interior operators: $\mathbf{Int}(A) =: \mathbf{Ext}(A')$ for any subset A of X, we immediately obtain a generalization of topological exterior operator as follows.

Definition 3.3 (Exterior Operator).
A mapping **Ext**: $\mathcal{P}(X) \to \mathcal{P}(X)$ is called a *generalized exterior operator* (or simply, *exterior operator*) if for any set $A, B \subseteq X$, **Ext** satisfies the following three axioms:
(E1) $A \cap \mathbf{Ext}(A) = \emptyset$;
(E2) $A \subseteq B \Rightarrow \mathbf{Ext}(A) \supseteq \mathbf{Ext}(B)$;
(E3) $\mathbf{Ext}(X \setminus \mathbf{Ext}(A)) = \mathbf{Ext}(A)$.

3.2 Generalized Boundary Operator

Compared with how topological closure operator becomes the generalized closure operator, (FO1) and (FO5) in the definition of topological boundary operator can be dropped to obtain a generalized boundary operator. (FO2) shows the essence of boundary of a set. (FO3) shows **Fr** is "monotone" in some sense. (FO4)* corresponds to "idempotency" of the closure operator. Therefore, we only keep (FO2), (FO3) and (FO4)* to obtain the definition of generalized derived set operator.

Definition 3.4 (Boundary Operator).
A mapping $\mathbf{Fr}$: $\mathcal{P}(X) \to \mathcal{P}(X)$ is called a *generalized boundary operator* (or simply, *boundary operator*) if for any set $A, B \subseteq X$, $\mathbf{Br}$ satisfies the following three axioms:
(F1) $\mathbf{Fr}(A) = \mathbf{Fr}(X \setminus A)$;
(F2) $A \subseteq B \Rightarrow \mathbf{Fr}(A) \subseteq B \cup \mathbf{Fr}(B)$;
(F3) $\mathbf{Fr}(A \cup \mathbf{Fr}(A)) \subseteq \mathbf{Fr}(A)$.

Proposition 3.5
The boundary operator $\mathbf{Fr}$ *has the following property:*
(F3)* $\mathbf{Fr}(\mathbf{Fr}(A)) \subseteq \mathbf{Fr}(A)$ *for any* $A \subseteq X$.

Proof. See the proof of Proposition 2.4. □

As we recall, (F3)* as stated above is axiom (FO4) in the topological boundary operator. However, (F3)* cannot be an alternative axiom in the definition of generalized boundary operator — in fact, (F3)* is strictly weaker than (F3) under (F1) and (F2). It can be seen from the following example.
Example Let $X = \{1, 2, 3\}$. Define an operator $\mathbf{Fr}$:

$$\mathbf{Fr}(A) = \begin{cases} \emptyset & A = \emptyset \text{ or } X, \\ \{2\} & A = \{1\} \text{ or } \{2,3\} \text{ or } \{2\} \text{ or } \{1,3\}, \\ \{3\} & A = \{3\} \text{ or } \{1,2\}. \end{cases}$$

$\mathbf{Fr}$ satisfies (F1), (F2), and (F3)*. But $\mathbf{Fr}$ does not satisfy axiom (F3): for $A = \{1\}$, $\mathbf{Fr}(A) = \{2\}$, $\mathbf{Fr}(A \cup \mathbf{Fr}(A)) = \mathbf{Fr}(\{1, 2\}) = \{3\}$. Obviously, $\mathbf{Fr}(A \cup \mathbf{Fr}(A)) \nsubseteq \mathbf{Fr}(A)$.

Theorem 3.6 (*From* $\mathbf{Fr}$ *to* $\mathbf{Cl}$).
Let $\mathbf{Fr}$*:* $\mathcal{P}(X) \to \mathcal{P}(X)$ *be a boundary operator. Define the operator* $\mathbf{Cl}$ *as* $\mathbf{Cl}(A) =:$ $A \cup \mathbf{Fr}(A)$ *for any subset* A *of* X*. Then* $\mathbf{Cl}$ *is a generalized closure operator.*

Proof. For axiom (C1), for any subset A of X, we have $A \subseteq A \cup \mathbf{Fr}(A) = \mathbf{Cl}(A)$, where the last step is by the definition of the operator $\mathbf{Cl}$. Therefore, $A \subseteq \mathbf{Cl}(A)$.

For axiom (C2), given $A \subseteq B \subseteq X$, axiom (F2) gives $\mathbf{Fr}(A) \subseteq B \cup \mathbf{Fr}(B)$. Therefore, we have $A \cup \mathbf{Fr}(A) \subseteq B \cup \mathbf{Fr}(B)$. This, by the definition of $\mathbf{Cl}$, is $\mathbf{Cl}(A) \subseteq \mathbf{Cl}(B)$.

For axiom (C3), using the definition of $\mathbf{Cl}$ twice, we have $\mathbf{Cl}(\mathbf{Cl}(A)) = \mathbf{Cl}(A \cup \mathbf{Fr}(A)) = A \cup \mathbf{Fr}(A) \cup \mathbf{Fr}(A \cup \mathbf{Fr}(A))$. By axiom (F3), then $\mathbf{Cl}(A \cup \mathbf{Fr}(A)) = A \cup \mathbf{Fr}(A) = \mathbf{Cl}(A)$, that is, $\mathbf{Cl}(\mathbf{Cl}(A)) = \mathbf{Cl}(A)$ holds. □

Theorem 3.7 (*From* $\mathbf{Cl}$ *to* $\mathbf{Fr}$).
Let $\mathbf{Cl}$*:* $\mathcal{P}(X) \to \mathcal{P}(X)$ *be a closure operator. Define* $\mathbf{Fr}(A) = \mathbf{Cl}(A) \cap \mathbf{Cl}(A')$ *for any subset* A *of* X*. Then* $\mathbf{Fr}$ *is a generalized boundary operator.*

Proof. For (F1), from the definition of $\mathbf{Fr}$, $\mathbf{Fr}(A') = \mathbf{Cl}(A') \cap \mathbf{Cl}((A')') = \mathbf{Cl}(A') \cap \mathbf{Cl}(A) = \mathbf{Fr}(A)$.

For (F2), First, by axiom (C1), $B \subseteq \mathbf{Cl}(B)$. Again, by axiom (C1), $B \cup \mathbf{Cl}(B') \supseteq B \cup B' = X$, so $B \cup \mathbf{Cl}(B') = X$. Therefore, apply the definition of $\mathbf{Fr}$, $B \cup \mathbf{Fr}(B) = B \cup (\mathbf{Cl}(B) \cap \mathbf{Cl}(B')) = (B \cup \mathbf{Cl}(B)) \cap (B \cup \mathbf{Cl}(B')) = \mathbf{Cl}(B) \cap X = \mathbf{Cl}(B)$. By axiom (C2), for any subsets $A, B \subseteq X$ with $A \subseteq B$, $\mathbf{Cl}(A) \subseteq \mathbf{Cl}(B)$, i.e., $\mathbf{Fr}$ satisfies (F2).

For (F3), from the above proof of (F2), it follows that $A \cup \mathbf{Fr}(A) = \mathbf{Cl}(A)$ for any $A \subseteq X$. We only need to check $\mathbf{Fr}(\mathbf{Cl}(A)) \subseteq \mathbf{Fr}(A)$. Again by the definition of $\mathbf{Fr}$, $\mathbf{Fr}(\mathbf{Cl}(A)) = \mathbf{Cl}(\mathbf{Cl}(A)) \cap \mathbf{Cl}((\mathbf{Cl}(A))')$. The first term on the right-hand side, by axiom (C3), becomes $\mathbf{Cl}(\mathbf{Cl}(A)) = \mathbf{Cl}(A)$. As for the second term, $\mathbf{Cl}((\mathbf{Cl}(A))')$, by axioms (C1), (C2), and (C3), we have $A \subseteq \mathbf{Cl}(A)$ which implies $(\mathbf{Cl}(A))' \subseteq A'$, then $\mathbf{Cl}((\mathbf{Cl}(A))') \subseteq \mathbf{Cl}(A')$. Therefore, $\mathbf{Fr}(\mathbf{Cl}(A)) = \mathbf{Cl}(\mathbf{Cl}(A)) \cap \mathbf{Cl}((\mathbf{Cl}(A))') \subseteq \mathbf{Cl}(A) \cap \mathbf{Cl}(A') = \mathbf{Fr}(A)$. □

In the proof of Theorem 3.6, we do not use (F1) in the definition of generalized boundary operator. So we can further weaken the notion of generalized boundary set operator as follows:

Definition 3.8 (Pre-Boundary Operator $\mathbf{Pb}$).
An operator on $\mathcal{P}(X)$ is called a *pre-boundary operator*, denoted $\mathbf{Pb}$, if $\mathbf{Pb}$ satisfies the following two conditions:
(i). $A \subseteq B \Rightarrow \mathbf{Pb}(A) \subseteq B \cup \mathbf{Pb}(B)$;
(ii). $\mathbf{Pb}(A \cup \mathbf{Pb}(A)) \subseteq \mathbf{Pb}(A)$.

Theorem 3.9
Let $\mathbf{Pb}$ *be a pre-boundary operator.*

(i) *Define* $\mathbf{Cl}$ *as* $\mathbf{Cl}(A) =: A \cup \mathbf{Pb}(A)$ *for any subset* A *of* X*. Then* $\mathbf{Cl}$ *is a closure operator.*

(ii) *Define* $\mathbf{Fr}(A) =: \mathbf{Cl}(A) \cap \mathbf{Cl}(A')$ *as the boundary operator associated to* $\mathbf{Cl}$*. Then the following two statements are equivalent:*
(i). *For any subset* $A \subseteq X$, $\mathbf{Pb}(A) = \mathbf{Pb}(A')$;
(ii). *For any subset* $A \subseteq X$, $\mathbf{Pb}(A) = \mathbf{Fr}(A)$.

Proof. From (i) to (ii). By the construction of $\mathbf{Fr}$, For any subset $A \subseteq X$, $\mathbf{Fr}(A) = \mathbf{Cl}(A) \cap \mathbf{Cl}(A') = (A \cup \mathbf{Pb}(A)) \cap (A' \cup \mathbf{Pb}(A'))= (A \cap A') \cup (A \cap \mathbf{Pb}\ (A')) \cup (A' \cap \mathbf{Pb}(A)) \cup (\mathbf{Pb}(A) \cap \mathbf{Pb}(A')) = (A \cap \mathbf{Pb}\ (A')) \cup (A' \cap \mathbf{Pb}(A)) \cup (\mathbf{Pb}(A) \cap \mathbf{Pb}(A'))$. By (i), for any subset $A \subseteq X$, $\mathbf{Pb}(A) = \mathbf{Pb}(A')$, then $\mathbf{Fr}(A) = (A \cap \mathbf{Pb}\ (A)) \cup (A' \cap \mathbf{Pb}(A)) \cup (\mathbf{Pb}(A) \cap \mathbf{Pb}(A)) = (A \cap \mathbf{Pb}\ (A)) \cup (A' \cap \mathbf{Pb}(A)) \cup \mathbf{Pb}(A) = \mathbf{Pb}(A)$, i.e., (i) implies (ii).

From (ii) to (i). This is through the definition of $\mathbf{Fr}$, with axiom (F1) stating that $\mathbf{Fr}(A) = \mathbf{Fr}(X \setminus A)$. □

From the above theorem, we can see that in the axiomatical definition of a boundary operator, axiom (F1), $\mathbf{Fr}(A) = \mathbf{Fr}(X \setminus A)$, is indispensable, which guarantees the one-to-one correspondence between boundary operators and closure operators.

3.3 Generalized Derived Set Operator

In this section, we consider the generalization of derived set operator. Compared to the generalized closure operator, (DO1) should be omitted and (DO5) in the definition of derived set operator should be weakened to be monotone. From Proposition 2.4 and its proof, (DO3) should be kept. (DO2) shows the essence of the notion of derived set. Then we have the following definition of generalized derived set operator.

Definition 3.10 (Derived Set Operator **Der**).
A mapping **Der**: $\mathcal{P}(X) \to \mathcal{P}(X)$ is called a *generalized derived set operator* (or simply, *derived set operator*) if for any $A, B \subseteq X$, **Der** satisfies the following three axioms:
(D1) $x \in \mathbf{Der}(A) \Leftrightarrow x \in \mathbf{Der}(A \setminus \{x\})$;
(D2) $A \subseteq B \Rightarrow \mathbf{Der}(A) \subseteq \mathbf{Der}(B)$;
(D3) $\mathbf{Der}(A \cup \mathbf{Der}(A)) \subseteq A \cup \mathbf{Der}(A)$.

Proposition 3.11
A derived set operator **Der** *satisfies*
(D3)* $\mathbf{Der}(\mathbf{Der}(A)) \subseteq A \cup \mathbf{Der}(A)$, *for any* $A \subseteq X$.

Proof. By the monotonicity of **Der**. □

In the case of topological derived set operator, property (DO3) and (DO3)* are substitutable. However, their equivalence does not hold in the situation of a generalized derived set operator. In fact, (D3)* is strictly weaker than (D3) in the case of generalized derived set operator. The following example can show this result.
Example. Let $X = \{1, 2, 3\}$. Define an operator **Der**:

$$\mathbf{Der}(A) = \begin{cases} \emptyset & A = \emptyset \text{ or } \{2\} \text{ or } \{3\} \text{ or } \{2,3\}, \\ \{2\} & A = \{1\} \text{ or } \{1,3\}, \\ \{2,3\} & A = \{1,2\} \text{ or } X. \end{cases}$$

Der satisfies (D1), (D2), and (D3)*. But **Der** does not satisfy axiom (D3): for $A = \{1\}$, $A \cup \mathbf{Der}(A) = \{1, 2\}$ $\mathbf{Der}(A \cup \mathbf{Der}(A)) = \mathbf{Der}(\{1, 2\}) = \{2, 3\}$. We can see that $\mathbf{Der}(A \cup \mathbf{Der}(A)) \nsubseteq A \cup \mathbf{Der}(A)$.

In topology, derived set operators and closure operators have a one-to-one correspondence. Such correspondence still holds in their respective generalizations.

Theorem 3.12 (*From* **Der** *to* **Cl**).
Let **Der**: $\mathcal{P}(X) \to \mathcal{P}(X)$ *be a derived set operator. Define* **Cl** *as* $\mathbf{Cl}(A) =: A \cup \mathbf{Der}(A)$ *for any subset* A *of* X. *Then* **Cl** *is a closure operator.*

Proof. For axiom (C1), since $A \subseteq A \cup \mathbf{Der}(A)$ for any subset A of X, use the definition of the operator **Cl**, $\mathbf{Cl}(A) = A \cup \mathbf{Der}(A)$, we obtain $A \subseteq \mathbf{Cl}(A)$.

For axiom (C2), suppose $A \subseteq B \subseteq X$. By (D2), $\mathbf{Der}(A) \subseteq \mathbf{Der}(B)$. Therefore $A \cup \mathbf{Der}(A) \subseteq B \cup \mathbf{Der}(B)$, namely, $\mathbf{Cl}(A) \subseteq \mathbf{Cl}(B)$.

For axiom (C3), using the definition of **Cl** twice, we have $\mathbf{Cl}(\mathbf{Cl}(A))$=$\mathbf{Cl}(A \cup$

$\mathbf{Der}(A))= A \cup \mathbf{Der}(A) \cup \mathbf{Der}(A \cup \mathbf{Der}(A))$. By (D3), then $\mathbf{Cl}(A \cup \mathbf{Der}(A)) = A \cup \mathbf{Der}(A) = \mathbf{Cl}(A)$, that is, the idempotent law $\mathbf{Cl}(\mathbf{Cl}(A)) = \mathbf{Cl}(A)$ holds. □

Conversely, we also can get a derived set operator from a closure operator.

Theorem 3.13 (*From* **Cl** *to* **Der**).
Let $\mathbf{Cl}$*:* $\mathcal{P}(X) \to \mathcal{P}(X)$ *be a closure operator. Define* $\mathbf{Der}(A) = \{x \in X \mid x \in \mathbf{Cl}(A \setminus \{x\})\}$ *for any subset* A *of* X*. Then* **Der** *is a derived set operator.*

Proof. For (D1), assume that $x \in \mathbf{Der}(A)$, by the definition of **Der** here, $x \in \mathbf{Cl}(A \setminus \{x\})$. Since $A \setminus \{x\} = (A \setminus \{x\}) \setminus \{x\}$, we have $x \in \mathbf{Cl}(A \setminus \{x\}) = \mathbf{Cl}((A \setminus \{x\}) \setminus \{x\})$. Again, by the definition of **Der**, $x \in \mathbf{Der}(A \setminus \{x\})$ holds. Every step above can be reversed. Therefore, $x \in \mathbf{Der}(A) \Leftrightarrow x \in \mathbf{Der}(A \setminus \{x\})$ holds.

For (D2), given that any subsets, $A, B \subseteq X$, $A \subseteq B$, so $A \setminus \{x\} \subseteq B \setminus \{x\}$. For any $x \in \mathbf{Der}(A)$, by the definition of **Der**, we have $x \in \mathbf{Cl}(A \setminus \{x\})$. By (C2), we have $x \in \mathbf{Cl}(A \setminus \{x\}) \subseteq \mathbf{Cl}(B \setminus \{x\})$. Again by the definition of **Der**, $x \in \mathbf{Der}(B)$. Therefore, $\mathbf{Der}(A) \subseteq \mathbf{Der}(B)$ holds.

For (D3), let us first show $A \cup \mathbf{Der}(A) = \mathbf{Cl}(A)$ for any $A \subseteq X$. For any $x \in \mathbf{Der}(A)$, we have $x \in \mathbf{Cl}(A \setminus \{x\})$ by the definition of **Der**. Since $\mathbf{Cl}(A \setminus \{x\}) \subseteq \mathbf{Cl}(A)$ by (C2), then $x \in \mathbf{Cl}(A)$. So $\mathbf{Der}(A) \subseteq \mathbf{Cl}(A)$. Together with (C1), $A \cup \mathbf{Der}(A) \subseteq \mathbf{Cl}(A)$ holds. On the other hand, for every $x \in \mathbf{Cl}(A)$, assume that $x \notin A$, then $A = A \setminus \{x\}$. So $x \in \mathbf{Cl}(A) = \mathbf{Cl}(A \setminus \{x\})$, namely, again by the definition of **Der**, $x \in \mathbf{Der}(A)$, so $\mathbf{Cl}(A) \subseteq A \cup \mathbf{Der}(A)$. Therefore, $A \cup \mathbf{Der}(A) = \mathbf{Cl}(A)$. Because of this $\mathbf{Der}(A) \subseteq \mathbf{Cl}(A)$. So $\mathbf{Der}(A \cup \mathbf{Der}(A)) =\mathbf{Der}(\mathbf{Cl}(A)) \subseteq \mathbf{Cl}(\mathbf{Cl}(A)) = \mathbf{Cl}(A)$, by (C3). Therefore, $\mathbf{Der}(A \cup \mathbf{Der}(A)) \subseteq= \mathbf{Cl}(A) = A \cup \mathbf{Der}(A)$, which is (D3), namely, $\mathbf{Der}(A \cup \mathbf{Der}(A)) \subseteq A \cup \mathbf{Der}(A)$. □

As with Theorem 3.12, we do not use (D1) in the definition of generalized derived set operator **Der**. A further weakening of generalized derived set operator can be obtained:

Definition 3.14 (Pre-Derived Set Operator **Pd**).
A mapping on $\mathcal{P}(X)$ is called a *pre-derived set operator*, denoted **Pd**, if **Pd** satisfies the following conditions:
(i). $A \subseteq B \Rightarrow \mathbf{Pd}(A) \subseteq \mathbf{Pd}(B)$;
(ii). $\mathbf{Pd}(A \cup \mathbf{Pd}(A)) \subseteq A \cup \mathbf{Pd}(A)$.

So the other version of Theorem 3.12 can be given.

Theorem 3.15
Let **Pd** *be a pre-derived set operator.*

(i) *Define the operator* **Cl** *by* $\mathbf{Cl}(A) =: A \cup \mathbf{Pd}(A)$ *for any subset* A *of* X*. Then* **Cl** *is a closure operator.*

(ii) *Define the operator* **Der** *by* $\mathbf{Der}(A) = \{x \in X \mid x \in \mathbf{Cl}(A \setminus \{x\})\}$*. Then the following two statements are equivalent:*
(i)*. For any subset* $A \subseteq X$*,* $x \in \mathbf{Pd}(A) \Leftrightarrow x \in \mathbf{Pd}(A \setminus \{x\})$*;*
(ii)*. For any subset* $A \subseteq X$*,* $\mathbf{Pd}(A) =\mathbf{Der}(A)$*.*

Proof. From (i) to (ii). Assume that (i) holds, namely, for any subset $E \subseteq X$, $x \in \mathbf{Pd}(E) \Leftrightarrow x \in \mathbf{Pd}(E \setminus \{x\})$, By the definitions of **Cl** and **Der**, for any $x \in \mathbf{Der}(E)$, then $x \in \mathbf{Cl}(E \setminus \{x\}) = (E \setminus \{x\}) \cup \mathbf{Pd}(E \setminus \{x\})$, which implies $x \in \mathbf{Pd}(E \setminus \{x\})$. By the given condition (i), we have $x \in \mathbf{Pd}(E)$, then $\mathbf{Der}(E) \subseteq \mathbf{Pd}(E)$. Similarly, for the other direction, for any $x \in \mathbf{Pd}(E)$, by the given condition (i), $x \in \mathbf{Pd}(E \setminus \{x\})$, so $x \in \mathbf{Cl}(E \setminus \{x\})$ by the definition of **Cl**. Again by the definition of **Der**, $x \in \mathbf{Der}(E)$. Therefore, $\mathbf{Pd}(E) \subseteq \mathbf{Der}(E)$. That is to say, (ii) holds.

From (ii) to (i). If (ii) holds, **Pd** is a generalized derived set operator. Then **Pd** satisfies (i). So the proof is completed. □

3.4 Generalized Co-Derived Set Operator

Dual to a generalized derived set operator, we can define a generalized co-derived set operator.

Definition 3.16 (Co-Derived Set Operator **Cod**).
A *generalized co-derived set operator* (or simply, *co-derived set operator*), denoted **Cod**, is defined as a mapping on $\mathcal{P}(X)$ which satisfies:
(i). $x \in \mathbf{Cod}(A) \Leftrightarrow x \in \mathbf{Cod}(A \cup \{x\})$;
(ii). $A \subseteq B \Rightarrow \mathbf{Cod}(A) \subseteq \mathbf{Cod}(B)$;
(iii). $\mathbf{Cod}(A \cap \mathbf{Cod}(A)) \supseteq A \cap \mathbf{Cod}(A)$.

That the derived set operator **Der** is dual to the coderived set operator **Cod** is reflected in $\mathbf{Cod}(A) = (\mathbf{Der}(A'))'$ and $\mathbf{Der}(A) = (\mathbf{Cod}(A'))'$.

Combining the duality between the derived set operator and the co-derived set operator and the duality between the closure operator and the interior operator, we have the following results dual to previous theorems for derived set operators.

Theorem 3.17 (*From* **Cod** *to* **Int**).
Let **Cod**: $\mathcal{P}(X) \to \mathcal{P}(X)$ *be a co-derived set operator. Define* $\mathbf{Int}(A) =: A \cap \mathbf{Cod}(A)$ *for any subset* A *of* X*. Then* **Int** *is an interior operator.*

Theorem 3.18 (*From* **Int** *to* **Cod**).
Let **Int**: $\mathcal{P}(X) \to \mathcal{P}(X)$ *be an interior operator. Define* $\mathbf{Cod}(A) := \{x \in X \mid x \in \mathbf{Int}(A \cup \{x\})\}$ *for any subset* A *of* X*. Then* **Cod** *is a co-derived set operator.*

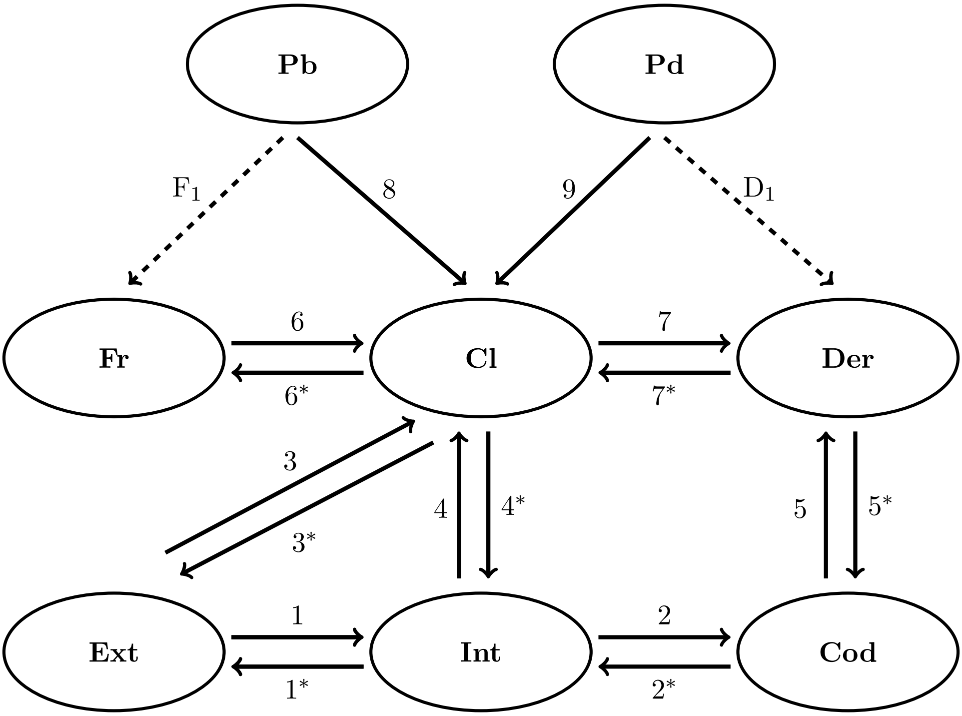
Definition 3.19 (Pre-Co-Derived Set Operator **Pcd**.)
An operator on $\mathcal{P}(X)$ is called a pre-co-derived set operator, denoted **Pcd**, if it satisfies the following conditions:
(i). $A \subseteq B \Rightarrow \mathbf{Pcd}(A) \subseteq \mathbf{Pcd}(B)$;
(ii). $\mathbf{Pcd}(A \cup \mathbf{Pcd}(A)) \supseteq A \cup \mathbf{Pcd}(A)$.

Theorem 3.20
Let **Pcd** *be a pre-co-derived set operator. Denote* **Int** *as the interior operator generated by* **Pcd**: $\mathbf{Int}(A) =: A \cap \mathbf{Pcd}(A)$*, and* **Cod** *as the resulting co-derived set operator:* $\mathbf{Cod}(A) = \{x \in X \mid x \in \mathbf{Int}(A \cup \{x\})\}$*, for any subset* $A \subseteq X$*. Then the following two statements are equivalent:*

(i) For any subset $E \subseteq X$, $x \in \mathbf{Pcd}(E) \Leftrightarrow, x \in \mathbf{Pcd}(E \cup \{x\})$;
(ii) For any subset $E \subseteq X$, $\mathbf{Pcd}(E) = \mathbf{Cod}(E)$.

4 Summary and Discussions

In the paper, we consider set operators in Closure System as extension of those in Topological System. Four generalized axiomatical set operators are obtained: generalized boundary operator, generalized exterior operator, generalized derived set operator, and generalized co-derived set operator. Those, in addition to the generalized interior and generalized closure operator, provide a complete generalization of the set of six axiomatic operators encountered in topology. Both generalized derived set operator and generalized boundary operator are in one-to-one correspondence with the generalized closure operator. Likewise, there is a one-to-one correspondence between generalized co-derived set operator, generalized exterior operator, and generalized interior operator, respectively. The results are summarized in the following figure:



In the figure, a solid arrow means one set operator induces the other one and a dashed arrow means the operator together with an additional condition (F1 in Definition 3.4 or D1 in Definition 3.10) becomes the directed one. The numbers index the corresponding transforming formulas:

$$1.\ \mathbf{Int}(A) = \mathbf{Ext}(A'),\ 1^*.\ \mathbf{Ext}(A) = \mathbf{Int}(A')$$
$$2.\ \mathbf{Cod}(A) = \{x \in X \mid x \in \mathbf{Int}(A \cup \{x\})\},\ 2^*.\ \mathbf{Int}(A) = A \cap \mathbf{Cod}(A)$$
$$3.\ \mathbf{Cl}(A) = (\mathbf{Ext}(A))'\ 3^*.\ \mathbf{Ext}(A) = (\mathbf{Cl}(A))'$$

$$4.\ \mathbf{Cl}(A) = (\mathbf{Int}(A'))',\ 4^*.\ \mathbf{Int}(A) = (\mathbf{Cl}(A'))'$$
$$5.\ \mathbf{Der}(A) = (\mathbf{Cod}(A'))',\ 5^*.\ \mathbf{Cod}(A) = (\mathbf{Der}(A'))'$$
$$6.\ \mathbf{Cl}(A) = A \cup \mathbf{Fr}(A),\ 6^*.\ \mathbf{Fr}(A) = \mathbf{Cl}(A) \cap \mathbf{Cl}(A')$$
$$7.\ \mathbf{Der}(A) := \{x \in X \mid x \in \mathbf{Cl}(A \setminus \{x\})\},\ 7^*.\ \mathbf{Cl}(A) = A \cup \mathbf{Der}(A).$$
$$8.\ \mathbf{Cl}(A) = A \cup \mathbf{Pb}(A)$$
$$9.\ \mathbf{Cl}(A) = A \cup \mathbf{Pd}(A)$$

Having the generalization of the complete suite of topological operators allows us to extend the topological semantics to those in Closure Systems in general. This could be useful because there are other non-topological closure systems, such as matroid/independent system and anti-matroid/accessible system. Our results are also significant for axiomatic operators on lattice (posets which are closed with respect to meet and join operations), because closure system correspond to complete lattices. Our axiomatization of set operators will shed new lights to the interplay of topology, lattice, and logic [2,3].

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