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Quasi-continuous Yoneda Complete Quasi-Metric Space

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Abstract

We introduce the notion of quasi-continuous Yoneda complete quasi-metric spaces and the hyperspace of finitely-generated maps. We show that the former can be completely characterized via two ways: (i) A certain class of quasi-continuous dcpos of formal balls and (ii) their hyperspace of finitely-generated maps.

Keywords: quasi-metric, Yoneda complete, quasi-continuous dcpo, quasi-continuous domain, continuous quasi-metric space

1 Introduction

One motivation of domain theory is to study the properties of spaces and to provide them with suitable computational models (e.g., [6,1]). One example of this in action comes in the form of formal balls. The posets of formal balls were first introduced by Weichrauch and Screiber in 1981 to provide an environment into which metric spaces can be embedded ([19]). In 1998, Edalat and Heckmann demonstrated the versatility of formal balls in several aspects; for instance, they showed that for any metric space with the open ball topology, its poset of formal balls endowed with the Scott topology provides a continuous model for it ([5]). Furthermore, it was shown that a metric space is complete if and only if its poset of formal balls is a domain; a direct corollary of this equivalence is that the poset of formal balls is a domain model for any complete metric space. The aforementioned results proven by Edalat

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and Heckmann establish a strong link between Domain Theory and Metric Space Theory.

Quasi-metric spaces are a generalization of metric spaces, dropping off the requirement for symmetry; and these structures have been extensively studied by several people, such as F. W. Lawvere in [14], and M. M. Bonsangue, F. van Breugel and J. J. M. M. Rutten in [3]. It is thus natural to ask if links between Domain Theory and Quasi-metric Space Theory can be established in the style of Edalat and Heckmann. In particular, one asks for characterization of properties of a quasi-metric space in terms of some order-theoretic properties of its poset of formal balls, that typically reads as follows:

> "(X, d) is a 'such-and-such' quasi-metric space if and only if $(B(X), \sqsubseteq)$ is a 'such-and-such' poset."

Four important such characterizations stand out amongst others:

Theorem 1.1 *(i)* (X, d) *is Yoneda-complete if and only if* $(B(X), \sqsubseteq)$ *is a dcpo ([13])*[5] *;*

(ii) (X, d) *is continuous Yoneda complete if and only if* $(B(X), \sqsubseteq)$ *is a domain ([9])*[6] *;*

(iii) (X, d) *is Smyth-complete if and only if* $(B(X), \sqsubseteq)$ *is a domain with* $\prec^{d^+}$ *as the way-below relation ([17]);*

(iv) (X, d) *is d-algebraic Yoneda-complete if and only if* $(B(X), \sqsubseteq)$ *is a domain with basis* $\{(x, r) \mid x$ *is d-finite*$\}$ *([2]).*

A closer scrutiny of these four statements yield a common trait: the various notions of completeness and approximation of quasi-metric spaces are reflected by some corresponding notions of completeness and approximation of their posets of formal balls as exemplified by the preceding theorem. A natural next step is to investigate if such correspondence between the realms of quasi-metric spaces and posets of formal balls hold for other meaningful notions of completeness and approximation in Domain Theory.

This brings us to the notion of quasi-continuity in Domain Theory. Quasi-continuous dcpos are often considered by domain theorists (e.g. [7,6,15,11,10]) as they enjoy many properties that the smaller class of continuous dcpos possesses. Thus a natural first question is to ask for some of the necessary and sufficient conditions of Yoneda complete quasi-metric spaces for which their posets of formal balls are quasi-continuous dcpos.

At the same time, quasi-metric spaces can in fact be seen as a generalization of posets, as we can define a quasi-metric space out of every poset. So can we define a notion of quasi-continuous Yoneda complete quasi-metric spaces? Naturally, this definition needs to encompass that of quasi-continuous dcpos, since every dcpo can

[5] In fact, Kostanek and Waszkiewicz considered Q-categories, which generalizes quasi-metric spaces as the latter are essentially $[0, \infty]$-categories. However, in this paper, we will apply their results to the specific case of quasi-metric spaces.

[6] This results strengthens an existing result given in [13, Theorem 9.1].

be seen as a Yoneda complete quasi-metric space. Following the fact that the notion of quasi-continuous dcpos is a generalization of that of continuous dcpos in posets, it is also natural to expect the proposed notion of quasi-continuous Yoneda complete quasi-metric space to be a generalization of that of the analogous continuous Yoneda complete quasi-metric space studied by Kostanek and Waszkiewicz in [13].

In this paper, we propose a definition of quasi-continuous Yoneda complete quasi-metric spaces. We give a positive answer to the above questions by showing that this notion generalizes that of the continuous Yoneda complete quasi-metric spaces. Furthermore, we have that the poset of formal balls of a quasi-continuous Yoneda complete quasi-metric space is a quasi-continuous dcpo. We also obtain the converse, albeit in the presence of some other condition.

We outline the paper as follows. In Section 2, we provide the relevant definitions and also some basic results related to the hyperspaces of quasi-metric spaces. We then study some properties of the formal balls of quasi-continuous Yoneda complete quasi-metric spaces in Section 3 and obtain our first main result, which characterizes quasi-continuous Yoneda complete quasi-metric spaces using certain quasi-continuous dcpos of formal balls. We then proceed to consider the notion of the hyperspace of finitely-generated subsets of a quasi-metric space in Section 4 and show that it can characterize quasi-continuous Yoneda complete quasi-metric space, which is the second of our main results. This new characterization extends [11, Proposition 4.5]. Finally, we conclude in Section 5.

2 Preliminaries and Basic Results

In this section, we gather some notions of hemi-metric and quasi-metric spaces which are essential for the discussion in the paper. We point the reader to [16] for basic topology, [4] for order theory and [1,6,8] for domain theory. For a more in-depth discussion on hemi- and quasi-metric spaces, the reader may also refer to [8].

In this paper, we write x for $\{x\}$ whenever there is no confusion. Let $\mathcal{P}_f(X)$ denote the set of all nonempty finite subsets of any given set X. For any preordered set $(P, \leq)$, $x \in P$ and a nonempty finite subset $F \subseteq P$, we denote $\uparrow x := \{y \in P \mid x \leq y\}$ and $\uparrow F := \bigcup_{x \in F} \uparrow x$. Also, we say that x is a supremum of a subset $D \subseteq P$ if $\uparrow x = \bigcap_{d \in D} \uparrow d$. In the case where the existing supremum of a subset D is unique (e.g. when $\leq$ is a partial order), we denote the supremum by $\bigvee D$.

Definition 2.1 [Hemi-metric, quasi-metric] Given a map $d : X \times X \longrightarrow [0, \infty]$, we say that d is a *hemi-metric* on X if for each $x, y, z \in X$,

(i) $d(x, x) = 0$,

(ii) $d(x, y) \leq d(x, z) + d(z, y)$,

Furthermore, if for each $x, y \in X$, $d(x, y) = d(y, x) = 0$ always implies $x = y$, then we say that d is a quasi-metric on X.

If for each $x, y \in X$, $d(x, y) = 0$ always implies $x = y$, then we say that d is a T_1 quasi-metric on X.

If d is a hemi-metric (resp., quasi-metric), we call the pair (X, d) a *hemi-metric* (resp., *quasi-metric space*).

Example 2.2 (i) Every metric space is a quasi-metric space, and hence a hemi-metric space.

(ii) On $[0,\infty]$, consider $d_{\mathbb{R}}$ where for each $x, y \in [0,\infty]$, $d_{\mathbb{R}}(x,y) := \max\{x-y, 0\} := x \dot{-} y$ (we adopt the convention that $\infty - \infty = 0$). Then $([0,\infty], d_{\mathbb{R}})$ is a quasi-metric space.

(iii) For any hemi-metric (resp., quasi-metric) space (X,d), define $d^{op} : X \times X \longrightarrow [0,\infty]$ by $d^{op}(x,y) := d(y,x)$. Then (X, d^{op}) is also a hemi-metric (resp., quasi-metric) space.

(iv) Let $(X,\le)$ be a preordered set (resp., poset). Define $d_{\le}$ on X by

$$d_{\le}(x,y) = \begin{cases} 0 & \text{if } x \le y; \\ \infty & \text{otherwise.} \end{cases}$$

Then $(X, d_{\le})$ is a hemi-metric (resp., quasi-metric) space.

Remark 2.3 We simply say that (X,d) is a preordered set (resp., poset), instead of $(X,d_{\le})$, if it is a hemi-metric space (resp., quasi-metric space) defined from a preordered set (resp., poset) $(X,\le)$.

Definition 2.4 [Cauchy nets, d-limits, Yoneda complete] Let (X,d) be a hemi-metric space.

(i) A net $(x_i)_{i\in I}$ of (X,d) is *Cauchy* if for each $\epsilon > 0$, there exists $i_\epsilon \in I$ such that for each $i, i' \in I$ where $i_\epsilon \le i \le i'$, it holds that $d(x_i, x_{i'}) < \epsilon$.

(ii) We say that x is a *d-limit* of a Cauchy net $(x_i)_{i\in I}$ if for each $y \in X$, $d(x,y) = \limsup_{i\in I} d(x_i, y)$, where for any net $(r_i)_{i\in I}$ in $[0,\infty]$, $\limsup_{i\in I} r_i := \inf_{i\in I} \sup_{i'\in I, i\le i'} r_i$.

(iii) We say that (X,d) is *Yoneda complete* if every Cauchy net has a d-limit.

Remark 2.5 (i) If (X,d) is a quasi-metric space, every Cauchy net has at most one d-limit. However, this is not true if (X,d) is only a hemi-metric space.

(ii) Although d-limits can also be defined for nets which are not Cauchy, we will only consider d-limits for Cauchy nets in this paper. Also, whenever we say that a net $(h_i)_{i\in I}$ has d-limit h, we mean that $(h_i)_{i\in I}$ is a Cauchy net.

Let (X,d) be a quasi-metric space in the sequel unless otherwise stated. We are now ready to propose the notions of the way-below map W^d on the hyperspace of nonempty finite subsets of (X,d) and quasi-continuous Yoneda complete quasi-metric space.

Definition 2.6 Let (X,d) is a quasi-metric space.

(i) Let $S^d : \mathcal{P}_f(X) \times \mathcal{P}_f(X) \longrightarrow [0,\infty], S^d(F,G) := \max_{y\in G} \min_{x\in F} d(x,y)$.
It can be easily verified that $(\mathcal{P}_f(X), S^d)$ is a hemi-metric space.

(ii) (Continuous Yoneda complete quasi-metric space)[18, Definitions 3.2 and 3.4]

(a) Define the way-below mapping between any two elements by $w^d : X \times X \longrightarrow [0, \infty]$,

$$w^d(x,y) := \sup_{(h_i)_{i\in I} \text{ has } d\text{-limit } h} \limsup_{i\in I} d(x,h_i) \dot{-} d(y,h).$$

(b) A Yoneda complete quasi-metric space (X,d) is *continuous* if for each $x \in X$, there exists some Cauchy net $(y_i)_{i\in I}$ in (X,d) where

(i) $w^d(-,x) = \limsup_{i\in I} d(-,y_i)$, and

(ii) x is the d-limit of $(y_i)_{i\in I}$.

(iii) (Quasi-continuous Yoneda complete quasi-metric space)

(a) Define the way-below mapping between any two finite subsets by $W^d : \mathcal{P}_f(X) \times \mathcal{P}_f(X) \longrightarrow [0,\infty]$,

$$W^d(F,G) := \sup_{(h_i)_{i\in I} \text{ has } d\text{-limit } h} \limsup_{i\in I} S^d(F,h_i) \dot{-} S^d(G,h),$$

(b) A Yoneda complete quasi-metric space (X,d) is *quasi-continuous* if for each $x \in X$, there exists some Cauchy net $(G_i)_{i\in I}$ in $(\mathcal{P}_f(X), S^d)$ where

(i) $W^d(-,x) = \limsup_{i\in I} S^d(-,G_i)$ for some Cauchy net $(G_i)_{i\in I}$, and

(ii) x is an S^d-limit of $(G_i)_{i\in I}$.

Remark 2.7 (i) As $(\mathcal{P}_f(X), S^d)$ is a hemi-metric space, the S^d-limit of Cauchy nets in $(\mathcal{P}_f(X), S^d)$ may not be unique.

(ii) If (X,d) is a poset, then $S^d(F,G) = 0$ if and only if $\uparrow F \supseteq \uparrow G$ if and only if $F \leq G$ by the definition of the Smyth preorder.

(iii) Also, $W^d(F,G) = 0$ if and only if $F \ll G$, where $\ll$ is the usual way-below relation defined for finite subsets: $F \ll G$ if for each directed subset $D \subseteq X$ with supremum $\bigvee D$, $G \leq \bigvee D$ implies that $F \leq d$ for some $d \in D$.

(iv) An alert reader may have realised that one can also propose the way-below mapping between finite subsets as follows.

$$W'^d(F,G) := \sup_{(h_i)_{i\in I} \text{ has } d\text{-limit } h} \min_{x\in F} \limsup_{i\in I} d(x,h_i) \dot{-} \min_{y\in G} d(y,h).$$

This is because such a map will still give the usual way-below relation when (X,d) is a poset. We shall show that such a definition is equivalent to our proposed definition for any quasi-metric space by Lemma 2.12.

(v) When we 'restrict' the domain of W^d to $X \times X$, i.e, by equating $\{x\}$ with x, d and S^d coincide, and W^d and w^d coincide.

(vi) In [18, Section 4], Waszkiewicz considered several mappings between finite subsets of X. In particular, the S map considered is exactly our S^d, which generalizes the Smyth preorder. However, we point out that the corresponding map $s : \mathcal{P}_f(X) \times \mathcal{P}_f(X) \longrightarrow [0,\infty]$ considered in the section is different from our W^d. Waszkiewicz defined s to be

$$s(F,G) := \max_{y\in G}\min_{x\in F} w^d(x,y) = \max_{y\in G}\min_{x\in F} \sup_{(h_i)_{i\in I} \text{ has } d\text{-limit } h} \limsup_{i\in I} d(x,h_i)\dot{-}d(y,h).$$

Note that $s(F,G)$ does not 'reduce' to the usual way-below relation defined between finite subsets in the event when (X,d) is a poset, whereas our definition of W^d does (see (iii)).

Proposition 2.8 *For any $F,G \in \mathcal{P}_f(X)$, $W^d(F,G) \geq S^d(F,G)$.*

Proof. We pick any $y_0 \in G$, and we simply observe that y_0 is a d-limit of the constant net (y_0).

So $W^d(F,G) \geq \limsup \min_{x\in F} d(x,y_0)\dot{-}\min_{y\in G} d(y,y_0) = \min_{x\in F} d(x,y_0) = S^d(F,y_0)$. Hence $W^d(F,G) \geq \max_{y_0\in G} S^d(F,y_0) = S^d(F,G)$. □

Remark 2.9 In the case where (X,d) is a poset, the preceding implies that whenever $F \ll G$, then $F \leq G$. Here, we write $\leq$ for the preorder on $\mathcal{P}_f(X)$ where $\leq$ is inherited from the underlying order on the set X.

Proposition 2.10 *x is a d-limit of the Cauchy net $(x_i)_{i\in I}$ if and only if x is an S^d-limit of the Cauchy net $(x_i)_{i\in I}$.*

Proof. We simply show the ($\Longrightarrow$) direction as the ($\Longleftarrow$) direction is clear.

Let $G \in \mathcal{P}_f(X)$ be given. By definition, $\limsup_{i\in I} S^d(x_i,G) = \limsup_{i\in I} \max_{y\in G} d(x_i,y)$ which is equivalent to $\max_{y\in G} \limsup_{i\in I} d(x_i,y)$ by [8, Exercise 7.1.12]. By our supposition, $d(x,y) = \limsup_{i\in I} d(x_i,y)$. Therefore,

$$\begin{aligned}\limsup_{i\in I}\max_{y\in G} d(x_i,y) &= \max_{y\in G}\limsup_{i\in I} d(x_i,y)\\ &= \max_{y\in G} d(x,y)\\ &= S^d(x,G).\end{aligned}$$

□

Remark 2.11 The preceding proposition extends the observation that a directed family $\{x_i\}_{i\in I}$ of a poset (X,d) has the supremum x if and only if $\{\{x_i\}\}_{i\in I}$ has the supremum $\{x\}$ in $(\mathcal{P}_f(X),\leq)$, i.e., $\bigcap_{i\in I} \uparrow x_i = \uparrow x$.

Lemma 2.12 *Let $(x_i)_{i\in I}$ be a Cauchy net and $G \in \mathcal{P}_f(X)$. $\limsup_{i\in I} S^d(G,x_i) = \min_{y\in G} \limsup_{i\in I} d(y,x_i)$.*

In particular, $W^d = W'^d$.

Proof. We simply prove the first claim as the second claim follows immediately.

Recall that $\limsup_{i\in I} S^d(G,x_i) = \limsup_{i\in I} \min_{y\in G} d(y,x_i)$. Clearly, $\min_{y\in G} \limsup_{i\in I} d(y,x_i) \geq \limsup_{i\in I} \min_{y\in G} d(y,x_i)$. Let $\epsilon > 0$ be given. There exists $i_0 \in I$ such that for all $i_0 \leq i \leq i'$, $d(x_i,x_{i'}) < \epsilon$ as $(x_i)_{i\in I}$ is Cauchy. Consider such $x_i, x_{i'}$, and choose $y_1 \in G$ such that $d(y_1,x_i) = \min_{y\in G} d(y,x_i)$. So

$d(y_1, x_{i'}) \leq d(y_1, x_i) + d(x_i, x_{i'}) < d(y_1, x_i) + \epsilon$. Hence by taking lim sup over i' on both sides,

$$\begin{aligned}\limsup_{i' \in I, i \leq i'} d(y_1, x_{i'}) &\leq d(y_1, x_i) + \epsilon \\ &= \min_{y \in G} d(y, x_i) + \epsilon,\end{aligned}$$

and

$$\begin{aligned}\limsup_{i' \in I} d(y_1, x_{i'}) &= \limsup_{i' \in I, i \leq i'} d(y_1, x_{i'}) \\ &\leq \min_{y \in G} d(y, x_i) + \epsilon\end{aligned}$$

by [8, Exercise 7.1.14].

So $\min_{y \in G} \limsup_{i' \in I} d(y, x_{i'}) \leq \min_{y \in G} d(y, x_i) + \epsilon$. Taking lim sup over i on both sides, we have $\min_{y \in G} \limsup_{i' \in I} d(y, x_{i'}) \leq \limsup_{i \in I} \min_{y \in G} d(y, x_i) + \epsilon$. As ϵ is arbitrary, it follows that $\min_{y \in G} \limsup_{i \in I} d(y, x_i) \leq \limsup_{i \in I} \min_{y \in G} d(y, x_i)$, and $\min_{y \in G} \limsup_{i \in I} d(y, x_i) = \limsup_{i \in I} \min_{y \in G} d(y, x_i)$ as claimed. □

The notion of continuous quasi-metric spaces was considered by Waszkiewicz in [18, Section 3]. We have the following.

Proposition 2.13 *Every continuous Yoneda complete quasi-metric space (X, d) is quasi-continuous.*

Proof. This is immediate from Theorem 3.25. However, we show the proof for completeness and for us to discuss some properties which will be useful in the later part of the paper.

Since (X, d) is continuous, $w^d(-, x) = \limsup_{i \in I} d(-, x_i)$, where $(x_i)_{i \in I}$ a Cauchy net in (X, d) and has d-limit x. Clearly, $(\{x_i\})_{i \in I}$ is a Cauchy net in $(\mathcal{P}_f(X), S^d)$. We claim that $W^d(-, x) = \limsup_{i \in I} S^d(-, x_i)$, i.e., for each $G \in \mathcal{P}_f(X)$, $W^d(G, x) = \limsup_{i \in I} S^d(G, x_i) = \limsup_{i \in I} \min_{y \in G} d(y, x_i)$.

Firstly,

$$\begin{aligned}W^d(G, x) &= \sup_{(h_i)_{i \in I} has\ d\text{-limit } h} \min_{y \in G} \limsup_{i \in I} d(y, h_i) \dot{-} d(x, h) \qquad \text{(by Lemma 2.12)} \\ &\leq \min_{y \in G} \sup_{(h_i)_{i \in I} has\ d\text{-limit } h} \limsup_{i \in I} d(y, h_i) \dot{-} d(x, h) \\ &= \min_{y \in G} w^d(y, x) \\ &= \min_{y \in G} \limsup_{i \in I} d(y, x_i).\end{aligned}$$

Suppose for the sake of contradiction that the inequality is strict. So for each Cauchy net $(h_i)_{i \in I}$ with d-limit h, $\min_{y \in G} \limsup_{i \in I} d(y, h_i) \dot{-} d(x, h) < \min_{y \in G} \limsup_{i \in I} d(y, x_i)$.

In particular, consider the Cauchy net $(x_i)_{i\in I}$ which has the d-limit x. Pick any $y_0 \in G$ such that $\limsup_{i\in I} d(y_0, x_i) = \min_{y\in G} \limsup_{i\in I} d(y, x_i)$. Hence

$$\begin{aligned} \limsup_{i\in I} d(y_0, x_i) \dot{-} d(x,x) &= \limsup_{i\in I} d(y_0, x_i) \\ &< \min_{y\in G} \limsup_{i\in I} d(y, x_i) \\ &= \limsup_{i\in I} d(y_0, x_i), \end{aligned}$$

which is a contradiction. It follows that

$$W^d(G,x) = \min_{y\in G} \limsup_{i\in I} d(y, x_i) = \min_{y\in G} w^d(y,x). \qquad (*)$$

Hence $W^d(G,x) = \limsup_{i\in I} \min_{y\in G} d(y,x_i) = \limsup_{i\in I} S^d(G,x_i)$ by Lemma 2.12.

By Proposition 2.10, x is the S^d-limit of $(x_i)_{i\in I}$. Hence the proof is complete.□

Corollary 2.14 (i) *Proposition 2.13 generalizes the result that every continuous dcpo is quasi-continuous.*

(ii) *By its proof, we also have that whenever $w^d(-,x) = \limsup_{i\in I} d(-,x_i)$ for some Cauchy net $(x_i)_{i\in I}$ that has d-limit x, then $W^d(-,x) = \limsup_{i\in I} S^d(-,x_i)$ and $(x_i)_{i\in I}$ has d-limit x.*

(iii) *By (*) in the proof of Proposition 2.13, we obtain a generalization for the result that in a continuous dcpo, a finite subset G is such that $G \ll x$ if and only if there exists $y \in G$, $y \ll x$.*

Following Proposition 2.13, any complete metric space and the Sorgenfrey line are quasi-continuous as quasi-metric spaces as they are already continuous. For an example of quasi-continuous Yoneda complete quasi-metric space which is not continuous, we can simply retrieve it from the quasi-continuous posets.

Example 2.15 We shall see in Proposition 3.27 that a dcpo (X,d) is quasi-continuous as a quasi-metric space if and only if it is quasi-continuous as a poset. This is a direct consequence of Theorem 3.25.

So far we have given examples of quasi-continuous Yoneda complete quasi-metric spaces using continuous Yoneda complete quasi-metric spaces and quasi-continuous dcpos. We shall see in Example 2.16 that the class of quasi-continuous Yoneda complete quasi-metric spaces which is not continuous is strictly larger than that of quasi-continuous dcpos which are not continuous.

Example 2.16 Let $X := \{(0,0),\omega\} \cup (\{1\}\times\mathbb{N})$. There is a natural order on X, where for each $x \in X, x \le \omega$ and $(1,m) \le (1,n)$ if $m \le n$. We shall use this order to aid our description of the proposed quasi-metric d.

Define d on X as follows: If $x \nleq y$, $d(x,y) := \infty$; otherwise:

(i) $d((1,m),(1,n)) := 1/2^m - 1/2^n$;

(ii) $d((0,0),\omega) := 0$;

(iii) $d((1,n),\omega) := 1/2^n$.

Figure 1 shows a pictorial representation of the space (X,d); an arrow is drawn from one node x to another node y if and only if $d(x,y) \neq \infty$, with the value of $d(x,y)$ annotating the arrow.

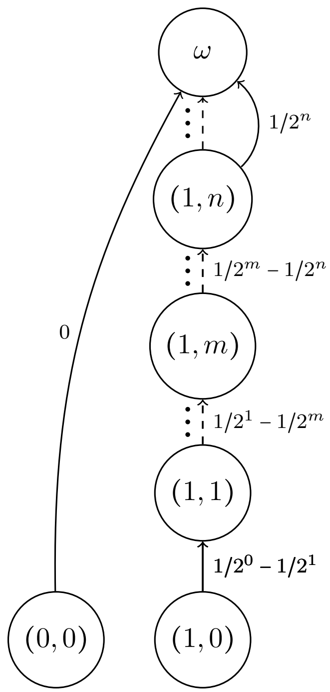

Fig. 1. Example of a Quasi-continuous Yoneda Complete Quasi-metric Space

Then (X,d) is a quasi-continuous Yoneda complete quasi-metric space which is not continuous.

Remark 2.17 (i) We refer the reader to the Appendix for the verification of this example.

(ii) One can check that the poset $(X,\leq)$, where $\leq$ is the order described in the example is a quasi-continuous dcpo which is not continuous. In particular, no element is way-below $(0,0)$, each $\{(0,0),(1,n)\}$ is way-below $(0,0)$ and $\{(0,0),(1,n)\}_{n\in\mathbb{N}}$ is a directed family with supremum $(0,0)$.

In the remaining section, we discuss some basic properties of S^d-limits of Cauchy nets of the hemi-metric space $(\mathcal{P}_f(X),S^d)$ and the mapping $W^d : \mathcal{P}_f(X)\times\mathcal{P}_f(X) \longrightarrow [0,\infty]$.

Since the Cauchy nets of the hemi-metric space $(\mathcal{P}_f(X),S^d)$ can have more than one S^d-limit, it is natural to ask if there are any relationships between the S^d-limits of a given Cauchy net.

Proposition 2.18 *If $\{x\}$ and F are S^d-limits of a Cauchy net $(F_i)_{i\in I}$ in*

$(\mathcal{P}_f(X), S^d)$, *then* $x \in F$.

In particular, it follows that if $F = \{y\}$, *then* $x = y$.

Proof. We note that $S^d(x, F) = \limsup_{i \in I} S^d(F_i, F) = S^d(F, F) = 0$ and $S^d(F, x) = \limsup_{i \in I} S^d(F_i, x) = S^d(x, x) = 0$. So $\max_{y \in F} d(x, y) = 0$ and $\min_{y \in F} d(y, x) = 0$. The first equality gives that for each $y \in F$, $d(x, y) = 0$ while the second equality gives that there exists $y_0 \in F$ such that $d(y_0, x) = 0$. Since (X, d) is a quasi-metric space, combining the two equalities we have that $x = y_0$, and $x \in F$ as claimed. □

We shall look at some properties of the hyperspace of $([0, \infty], d_{\mathbb{R}})$.

Example 2.19 Let (X, d) be the Yoneda quasi-metric space $([0, \infty], d_{\mathbb{R}})$. We have the following.

(i) For any $F, G \in \mathcal{P}_f(X)$, $S^d(F, G) = \min(F) \dot{-} \min(G)$.

(ii) For any Cauchy net $(F_i)_{i \in I}$ of $(\mathcal{P}_f(X), S^d)$, its S^d-limit is any $F \in \mathcal{P}_f(X)$ with $\min(F) = \limsup_{i \in I} \min(F_i)$.

(iii) $(\mathcal{P}_f(X), S^d)$ is Yoneda complete.

Proof. For (i), clearly, we have

$$\begin{aligned} S^d(F, G) &= \max_{y \in G} \min_{x \in F} d(x, y) \\ &= \max_{y \in G} \min_{x \in F} (x \dot{-} y) \\ &= \min_{x \in F} x \dot{-} \min_{y \in G} y \\ &= \min(F) \dot{-} \min(G). \end{aligned}$$

For (ii), let $(F_i)_{i \in I}$ be a Cauchy net of $(\mathcal{P}_f(X), S^d)$. So for any $\epsilon > 0$, there exists $i_0 \in I$ such that for each $i, i' \in I$, $i_0 \leq i \leq i'$, $S^d(F_i, F_{i'}) = \min(F_i) \dot{-} \min(F_{i'}) < \epsilon$, so $(\min(F_i))_{i \in I}$ forms a Cauchy net.

The d-limit of $(\min(F_i))_{i \in I}$ is given by $\limsup_{i \in I} \min(F_i)$ (see [8, Exercise 7.1.16]). We now prove our claim by showing that F is the S^d-limit of $(F_i)_{i \in I}$. Let $G \in \mathcal{P}_f(X)$. $S^d(F, G) = \min(F) \dot{-} \min(G) = \limsup_{i \in I} \min(F_i) \dot{-} \min(G) = \limsup_{i \in I} S^d(F_i, G)$.

Following (ii), (iii) is immediate as for any net $(r_i)_{i \in I}$ in (X, d), $\limsup_{i \in I} r_i$ exists. □

While $(\mathcal{P}_f(X), S^d)$ is generally not a quasi-metric space, it is if (X, d) is T_1.

Example 2.20 Let (X, d) be a T_1 quasi-metric space (e.g., the quasi-metric space $(\mathbb{R}, d_l)$ representing the Sorgenfrey line, where $d_l(x, y) = y - x$ if $x \leq y$ and ∞ otherwise).

We now show that $(\mathcal{P}_f(X), S^d)$ is a quasi-metric space.

Proof. Let $F, G \in \mathcal{P}_f(X)$ such that $S^d(F, G) = S^d(G, F) = 0$. So $\max_{y \in G} \min_{x \in F} d(x, y) = 0$, i.e., for every $y \in G$, there exists $x \in F$ such that

$d(x,y) = 0$. Hence $\uparrow F \supseteq \uparrow G$. Since (X,d) is T_1, $F = \uparrow F \supseteq \uparrow G = G$ and by symmetry, $G \supseteq F$. So $F = G$. □

There are many pleasing properties of the way below relation $\ll$ defined on the usual posets. The following is one such example.

Proposition 2.21 *[11, Corollary 4.3] Let $(P,\leq)$ be a poset and $F,G \in \mathcal{P}_f(P)$. $F \ll G$ if and only if $F \ll y$ for each $y \in G$.*

We shall show that some of these hold for the case of W^d defined on quasi-metric spaces.

Proposition 2.22 *For any $F,G \in \mathcal{P}_f(X)$, $W^d(F,G) = \max_{y\in G} W^d(F,y)$.*

Proof. We fix a Cauchy net $(h_i)_{i\in I}$ which has a d-limit h.

Then

$$\begin{aligned} W^d(F,G) &= \sup_{(h_i)_{i\in I} \text{ has } d\text{-limit } h} \min_{x\in F} \limsup_{i\in I} d(x,h_i) \dot{-} \min_{y\in G} d(y,h) \\ &= \sup_{(h_i)_{i\in I} \text{ has } d\text{-limit } h} \max_{y\in G} \min_{x\in F} (\limsup_{i\in I} d(x,h_i) \dot{-} d(y,h)) \\ &= \max_{y\in G} \sup_{(h_i)_{i\in I} \text{ has } d\text{-limit } h} \min_{x\in F} (\limsup_{i\in I} d(x,h_i) \dot{-} d(y,h)) \\ &= \max_{y\in G} W^d(F,y). \end{aligned}$$

This completes the proof. □

This is analogous to Proposition 2.21, as can be seen in the following.

Corollary 2.23 *If (X,d) is a poset, then $F \ll G$ if and only if $F \ll y$ for all $y \in G$.*

Proof. We note that $F \ll G$ if and only if $W^d(F,G) = 0$ if and only if $\max_{y\in G} W^d(F,y) = 0$ if and only if for all $y \in G$, $F \ll y$. □

Recall that in order theory, we have the notion of an auxiliary binary relation on any preordered set. Similarly, we can have the following.

Definition 2.24 [See [18, Defintion 3.1]] Let (X,d) be a hemi-metric space. A mapping $v : X \times X \longrightarrow [0,\infty]$ is *auxiliary* if

(i) $d(x,y) \leq v(x,y)$,

(ii) $v(x,t) \leq d(x,y) + v(y,z) + d(z,t)$.

Proposition 2.25 *The mapping $W^d : (\mathcal{P}_f(X), S^d) \times (\mathcal{P}_f(X), S^d) \longrightarrow [0,\infty]$ is auxiliary.*

Proof. Condition (i) for W^d to be auxiliary is clear following Proposition 2.8. We shall now show that condition (ii) holds for W^d.

Let $F,G,H,T \in \mathcal{P}_f(X)$ be given and fix a Cauchy net $(s_i)_{i\in I}$ with d-limit s.

We have that $S^d(F,s_i) \leq S^d(F,G)+S^d(G,s_i)$, so $\limsup_{i\in I} S^d(F,s_i) \leq S^d(F,G)+\limsup_{i\in I} S^d(G,s_i)$.

Also, $S^d(H,s) \le S^d(H,T) + S^d(T,s)$, so $-S^d(T,s) \le S^d(H,T) - S^d(H,s)$.

Combining the two inequalities, we have $\limsup_{i\in I} S^d(F,s_i) - S^d(T,s) \le S^d(F,G) + \limsup_{i\in I} S^d(G,s_i) - S^d(H,s) + S^d(H,T)$, and hence $\limsup_{i\in I} S^d(F,s_i) \dot{-} S^d(T,s) \le S^d(F,G) + \limsup_{i\in I} S^d(G,s_i) \dot{-} S^d(H,s) + S^d(H,T)$.

Since $(s_i)_{i\in I}$ is arbitrary, $W^d(F,T) \le S^d(F,G) + W^d(G,H) + S^d(H,T)$ as claimed. □

Remark 2.26 Compare this with the case where (X,d) is a poset: this result implies that for any $F,G,H,T \in \mathcal{P}_f(X)$, $F \le G \ll H \le T$ implies that $S^d(F,G) = W^d(G,H) = S^d(H,T) = 0$, which implies that $W^d(F,T) = 0$, i.e., $F \ll T$.

We then have the following:

Corollary 2.27 *The mapping $W : (\mathcal{P}_f(X), S^d) \times (\mathcal{P}_f(X), S^d) \longrightarrow [0,\infty]$ satisfies the triangle inequality axiom, i.e., for any $F,G,H \in \mathcal{P}_f(X)$,*

$$W^d(F,G) \le W^d(F,H) + W^d(H,G).$$

Proof. This is straightforward because W^d is auxiliary and hence $W^d(F,G) \le S^d(F,F) + W^d(F,H) + S^d(H,G) \le W^d(F,H) + W^d(H,G)$. □

Proposition 2.28 *If (X,d) is a metric space, then*

(i) *w^d and d coincide.*

(ii) *W^d and S^d coincide.*

Proof. It suffices to show (ii) as (i) can be obtained by 'restricting' the domain of W^d and S^d to $X \times X$, i.e., by equating $\{x\}$ with x.

Let $F,G \in \mathcal{P}_f(X)$ be given, and $(h_i)_{i\in I}$ be a Cauchy net with d-limit h. Then

$$\begin{aligned}
\limsup_{i\in I} S^d(F,h_i) \dot{-} S^d(G,h) &= \min_{x\in F} \limsup_{i\in I} d(x,h_i) \dot{-} \min_{y\in G} d(y,h) \quad \text{(by Lemma 2.12)}\\
&= \min_{x\in F} \limsup_{i\in I} d(h_i,x) \dot{-} \min_{y\in G} d(y,h)\\
&= \min_{x\in F} d(h,x) \dot{-} \min_{y\in G} d(y,h)\\
&= \max_{y\in G} \min_{x\in F} (d(h,x) \dot{-} d(y,h)).
\end{aligned}$$

Note that for each $x_0 \in F, y_0 \in G$ and $h \in X$, $d(x_0,h) \le d(x_0,y_0) + d(y_0,h)$, so $\min_{x\in F} d(x,h) \dot{-} d(y_0,h) \le d(x_0,h) \dot{-} d(y_0,h) \le d(x_0,y_0)$. Since $x_0 \in F$ is chosen arbitrarily, $\min_{x\in F} d(x,h) \dot{-} d(y_0,h) \le \min_{x\in F} d(x_0,y_0) \le \max_{y\in G} \min_{x\in F} d(x,y) = S^d(F,G)$. Thus $\limsup_{i\in I} S^d(F,h_i) \dot{-} S^d(G,h) \le S^d(F,G)$ and since $(h_i)_{i\in I}$ are chosen arbitrarily, $W^d(F,G) \le S^d(F,G)$. □

3 Formal Balls of Quasi-continuous Yoneda Complete Quasi-metric Space

We shall begin this section by introducing the notion of formal balls and investigating some properties of the set of formal balls induced by spaces like (X,d) and

$(\mathcal{P}_f(X), S^d)$.

Definition 3.1 [Formal balls][19] Let (X, d) be a hemi-metric space. We denote $B(X) := X \times \mathbb{R}^+$, and define $\leq^{d^+}$ on $B(X)$ by $(x, r) \leq^{d^+} (y, s)$ if $d(x, y) \leq r - s$.

We call the elements of $B(X)$ *formal balls*.

Remark 3.2 (i) We simply write $\sqsubseteq$ to mean the preorder or partial order on any set of formal balls.

(ii) If (X, d) is a hemi-metric space (resp., quasi-metric space), $(B(X), \sqsubseteq)$ is a preordered set (poset).

(iii) Since $(\mathcal{P}_f(X), S^d)$ may not be a quasi-metric space, $(B(\mathcal{P}_f(X)), \sqsubseteq)$ may not be a poset.

(iv) Consequently, for $(B(\mathcal{P}_f(X)), \leq)$, a subset may have more than one supremum.

We list some quick and useful observations about $(B(X), \leq)$ which we will use in this paper.

Fact 3.3 *Let $x \in X$ and $r, s \in \mathbb{R}^+$.*

(i) *If $(x, r) \sqsubseteq (y, s)$, $r \geq s$.*

(ii) *If $(x, r) \ll (y, s)$, $r > s$.*

(iii) *If $r \geq s$, $(x, r) \sqsubseteq (x, s)$.*

(iv) *For any $a \in \mathbb{R}^+$, $(x, r) \sqsubseteq (y, s) \iff (x + a) \sqsubseteq (y, s + a)$.*

We say that a map $f : (P, \leq) \longrightarrow (Q, \preceq)$ between preordered sets $(P, \leq)$ and $(Q, \preceq)$ is a *preorder-embedding* if for each $x, y \in P$, $x \leq y$ if and only if $f(x) \preceq f(y)$.

Hence we have the following.

Lemma 3.4 *The map*

$$f : ((B(\mathcal{P}_f(X))), \sqsubseteq) \longrightarrow (\mathcal{P}_f(B(X)), \sqsubseteq)$$
$$(F, r) \mapsto \{(x, r) \mid x \in F\},$$

where we write $\sqsubseteq$ on $\mathcal{P}_f(B(X))$ for the Smyth preorder inherited from $(B(X), \sqsubseteq)$, is a preorder-embedding.

Proof. We simply observe the following.

$$\begin{aligned}
&(F, r) \sqsubseteq (G, s) \\
\iff &S^d(F, G) = \max_{y \in G} \min_{x \in F} d(x, y) \leq r - s \\
\iff &(\forall y \in G)(\exists x \in F) d(x, y) \leq r - s \\
\iff &(\forall y \in G)(\exists x \in F)(x, r) \sqsubseteq (y, s) \\
\iff &\{(x, r) \mid x \in F\} \sqsubseteq \{(y, s) \mid y \in G\}.
\end{aligned}$$

□

Remark 3.5 In view of the above, we shall write $[F, r] := \{(x, r) \mid x \in F\}$. We point out that the $[F, r]$'s represent these special finite subsets of $B(X)$ where each formal ball in $[F,r]$ has the same second coordinate r while the (F,r)'s are elements of $B(\mathcal{P}(X))$.

In Lemma 3.4, we showed that $(B(\mathcal{P}_f(X)),\sqsubseteq)$ is embedded as a sub-preordered set of $(\mathcal{P}_f(B(X)),\sqsubseteq)$.

We can in fact say more for certain directed suprema in the two structures.

Lemma 3.6 *Let $(F_i,r_i)_{i\in I}$ be a directed family of $(B(\mathcal{P}_f(X)),\sqsubseteq)$.*

Then, (F,r) is a supremum of $(F_i,r_i)_{i\in I}$ in $(B(\mathcal{P}_f(X)),\sqsubseteq)$ if and only if $[F,r]$ is a supremum of $[F_i,r_i]_{i\in I}$ in $(\mathcal{P}_f(B(X)),\sqsubseteq)$.

Proof. ($\Longrightarrow$): It suffices to show that for any upper bound (y,s) of $[F_i,r_i]_{i\in I}$ in $(B(X),\sqsubseteq)$, $[F,r]\sqsubseteq(y,s)$, as it follows immediately that $[F,r]$ is below any upper bound $G\in\mathcal{P}_f(B(X))$ of $[F_i,r_i]_{i\in I}$, where $G:=\{(y_1,s_1),(y_2,s_2),\ldots,(y_k,s_k)\}$. First observe that (y,s) is an upper bound of $(F_i,r_i)_{i\in I}$ in $(B(P_f(X)),\sqsubseteq)$ by Lemma 3.4. By supposition, $(F,r)\sqsubseteq(y,s)$ in $(B(P_f(X)),\sqsubseteq)$, hence $[F,r]\sqsubseteq(y,s)$ in $(B(X),\sqsubseteq)$.

($\Longleftarrow$): Let (G,s) be an upper bound of $(F_i,r_i)_{i\in I}$ in $(B(\mathcal{P}_f(X)),\sqsubseteq)$. So for each $y\in G$, (y,s) is an upper bound of $(F_i,r_i)_{i\in I}$ in $(B(\mathcal{P}_f(X)),\sqsubseteq)$, hence (y,s) is an upper bound of $[F_i,r_i]_{i\in I}$ in $(B(X),\sqsubseteq)$ again by Lemma 3.4. By supposition, $[F,r]\sqsubseteq(y,s)$ for each $y\in G$. So $(F,r)\sqsubseteq(G,s)$ in $(B(\mathcal{P}_f(X)),\sqsubseteq)$ and this completes the proof. □

We take a short detour by discussing a result due to Heckmann and Keimel ([11, Lemma 4.1]) which we will make use of in Lemma 3.19.

Lemma 3.7 *Let $(P,\leq)$ be a dcpo. The following are equivalent.*

1. *(1) $F\ll G$ in $(P,\leq)$, i.e., whenever $G\leq\bigvee D$ for some directed family $D\subseteq X$, $F\leq d$ for some $d\in D$.*
2. *(2) $\uparrow F\ll\uparrow G$ in the hyperspace of finitely-generated subsets ordered by reverse inclusion, i.e., whenever $\uparrow G\supseteq\uparrow H$, where $\uparrow H$ is the filtered intersection of some $\{\uparrow H_i\}_{i\in I}$, then $\uparrow F\supseteq\uparrow H_i$ for some $i\in I$.*
3. *(3) $F\ll G$ in the hyperspace of nonempty finite subsets with the usual Smyth preorder, i.e., whenever $G\leq H$, where H is the supremum of some directed family $\{H_i\}_{i\in I}$, then $F\leq H_i$ for some $i\in I$.*

We recall [8, Trick 5.1.20]. To show that a poset is continuous, we can simply show that for each x, there exists a directed family $\{y_i\}_{i\in I}$ such that each $y_i\ll x$ and $\{y_i\}_{i\in I}$ has supremum x. This is in contrast to the method where we show $\twoheaddownarrow x$ is directed and has supremum x. [11, Lemma 4.1] allows us to consider an analogous trick.

Trick 1 *To show that a dcpo $(P,\leq)$ is quasi-continuous, we can simply show that for each x, there exists a directed family $\{F_i\}_{i\in I}$ such that each $F_i\ll x$ and $\{F_i\}_{i\in I}$ has supremum $\{x\}$.*

Proof. Let $G_1,G_2\ll x$ in $(X,\leq)$. Therefore $G_1,G_2\ll\{x\}$ in the hyperspace of nonempty finite subsets with the usual Smyth preorder. Since $\{F_i\}_{i\in I}$ is directed and has supremum x in $(X, \leq)$, $\{F_i\}_{i \in I}$ is directed and has supremum x in the hyperspace of nonempty finite subsets with the usual Smyth preorder. So $G_1 \leq F_1$ and $G_2 \leq F_2$ for some $F_1, F_2 \in \{F_i\}_{i \in I}$. Thus there exists $F_3 \in \{F_i\}_{i \in I}$, $G_j \leq F_j \leq F_3$ for $j = 1, 2$. So $\Downarrow x$ is directed and contains $\{F_i\}_{i \in I}$ as a cofinal subset. The proof is then complete. □

Definition 3.8 A fin-basis of a dcpo $(P, \leq)$ is a subcollection $\mathcal{B} \subseteq \mathcal{P}_f(P)$ such that for each $x \in P$, the set $\Downarrow x = \{F \in \mathcal{P}_f(P) \mid F \ll x\} \cap \mathcal{B}$ is directed and has supremum $\{x\}$.

Following Trick 1, we immediately have:

Proposition 3.9 *The following are equivalent for a dcpo* $(P, \leq)$.

(1) $(P, \leq)$ *is quasi-continuous.*

(2) $(P, \leq)$ *has a fin-basis* $\mathcal{B}$.

(3) For each $x \in P$*, there exists a directed subset* $\mathcal{F} \subseteq \mathcal{B}$ *such that for each* $F \in \mathcal{F}$, $F \ll x$ *and* $\mathcal{F}$ *has supremum* $\{x\}$.

We now give a general overview of the structures that we shall use in the paper. The reader may refer to Figure 2 for a chart of the relationships between these structures.

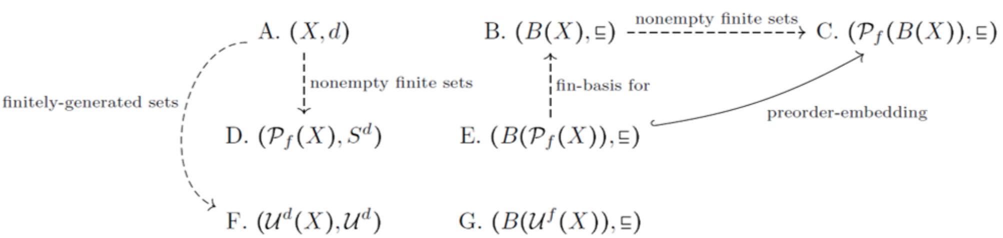

Fig. 2. Overview of the Structures

By the classical definition of quasi-continuity for any poset, we are required to show that it has a fin-basis consisting of its finite subsets following Trick 1. Applying this to the case where the poset is the poset of formal balls (B), we need to obtain this using its finite subsets from the preordered set (C). We note that by Lemma 3.4, (E) can be seen as a subset of (C), where (E) is the preordered set of formal balls arising from the hemi-metric space (D), which consists of elements in the form of finite subsets of (A). We shall show in Theorem 3.25 (E) serves as a fin-basis for (B).

In Section 4 of this paper, we consider a notion of the finitely-generated subsets of a quasi-metric space, (F). By passing through the passage of formal balls (B) and (G), we then obtain an extension of the result in [11, Proposition 4.5] where a dcpo is quasi-continuous if and only if its hyperspace of nonempty finitely-generated sets ordered by reverse inclusion is continuous. This result is presented in Corollary 4.14.

There are many interesting connections between the Cauchy nets in (X, d) and the directed families of $(B(X), \sqsubseteq)$ as can be observed from [13, Lemmas 7.7 and 7.8] [7]. For instance, we have the following.

Lemma 3.10 *Let (X, d) be a hemi-metric space and $(x_i, r_i)_{i \in I}$ be a directed family in $(B(X), \sqsubseteq)$. the following hold:*

(i) *$(x_i)_{i \in I}$ is a Cauchy net in (X, d); and*

(ii) *if x is a d-limit of $(x_i)_{i \in I}$, then $(x_i, r_i)_{i \in I}$ has supremum (x, r), where $r = \inf_{i \in I} r_i$.*

The converse to Lemma 3.10(ii) is not true in general, as one can verify directly using the example given in [9, Remark 2.3]. However, it is true if $(B(X), \sqsubseteq)$ is a dcpo or as we shall see more generally, when (X, d) is standard.

Definition 3.11 [9, Definition 2.1] (X, d) is *standard* if for every directed family of formal balls $(x_i, r_i)_{i \in I}$, for every $s \in \mathbb{R}^+$, $(x_i, r_i)_{i \in I}$ has a supremum if and only $(x_i, r_i + s)_{i \in I}$ has a supremum in $(B(X), \sqsubseteq)$.

Lemma 3.12 *[9, Proposition 2.4 and Lemma 5.15] In a standard quasi-metric space (X, d), if a directed family of formal balls $(x_i, r_i)_{i \in I}$ has a supremum (x, r), the following hold:*

(i) *$r = \inf_{i \in I} r_i$; and*

(ii) *for every $s \in [-r, \infty)$, $(x, r + s)$ is a supremum of $(x_i, r_i + s)_{i \in I}$;*

(iii) *$(x_i)_{i \in I}$ has d-limit x.*

Example 3.13 [9, Proposition 2.2] The following quasi-metric spaces are standard.

(i) Metric spaces

(ii) Yoneda complete quasi-metric spaces

(iii) Posets

To establish the connection between Cauchy nets in (X, d) and directed families in $(B(X), \sqsubseteq)$, we can consider the notion of Cauchy-weighted nets and Cauchy-weightable nets in (X, d), as discussed in [8, Definition 7.2.6]. We introduce the notion of a Cauchy-weighted net by making just one modification to the definition of the former.

Definition 3.14 Let (X, d) be a hemi-metric space. A *Cauchy-weighted net* $(x_i, r_i)_{i \in I}$ is a net in $X \times [0, \infty)$ such that

(i) for each $i, j \in I$, $i \leq j$ implies $d(x_i, x_j) \leq r_i - r_j$, i.e., $(x_i, r_i) \sqsubseteq (x_j, r_j)$,

(ii) $\inf_{i \in I} r_i = 0$.

[7] The reader may also refer to [8, Lemmas 7.4.25 and 7.4.26] which deal directly with hemi-metric spaces instead of Q-categories.

We say that a Cauchy net $(x_i)_{i\in I}$ is *Cauchy-weightable* if there exists a net $(r_i)_{i\in I}$ such that $(x_i, r_i)_{i\in I}$ is a Cauchy-weighted net.

Remark 3.15 It can be observed directly that if $(x_i, r_i)_{i\in I}$ is a Cauchy-weighted net in (X, d), it is a directed family in $(B(X), \sqsubseteq)$.

Lemma 3.16 *Let (X, d) be a hemi-metric space.*

(i) *For every Cauchy net $(x_i)_{i\in I}$, there exists a Cauchy-weighted net $(x_j, r_j)_{j\in J}$ where $(x_j)_{j\in J}$ is a subnet of $(x_i)_{i\in I}$.*

(ii) *If $(x_j)_{j\in J}$ is a subnet of some Cauchy net $(x_i)_{i\in I}$, then $(x_j)_{j\in J}$ is a Cauchy net and x is a d-limit of $(x_i)_{i\in I}$ if and only if x is a d-limit of $(x_j)_{j\in J}$.*

(iii) *(X, d) is Yoneda complete if and only if every Cauchy-weightable net has a d-limit.*

Proof. (i): See the proof of [8, Lemma 7.2.8].

(ii): The first part can be verified directly. For the second part, see [8, Lemma 7.4.6 and Exercise 7.4.7].

Following (i) and (ii), (iii) is immediate. □

The mappings W^d can be used to characterize the way-below relation between finite subsets in $(B(X), \leq)$ as can be seen in Lemmas 3.17 and 3.21.

Lemma 3.17 *Let (X, d) be a Yoneda complete quasi-metric space. If $W^d(F, G) < r - s$, then $[F, r] \ll [G, s]$ in $(B(X), \sqsubseteq)$.*

Proof. Suppose $[G, s] \sqsubseteq (h, t)$, where (h, t) is the supremum of $(h_i, t_i)_{i\in I}$. Then $S^d(G, h) \leq s - t$, i.e., $\min_{y\in G} d(y, h) \leq s - t$.

By supposition, $\min_{x\in F} \limsup_{i\in I} d(x, h_i) < \min_{y\in G} d(y, h) + r - s$. So there exists $\epsilon > 0$ such that

$$\begin{aligned}\min_{x\in F} \limsup_{i\in I} d(x, h_i) &< \min_{y\in G} d(y, h) + r - s - \epsilon \\ &\leq s - t + r - s - \epsilon \\ &= r - t - \epsilon.\end{aligned}$$

This implies that there exists $x \in F$, there exists $i_0 \in I$ such that for all $i \in I$ where $i_0 \leq i$, $d(x, h_i) < r - t - \epsilon$. Since (X, d) is Yoneda complete and hence standard, $\inf_{i\in I} t_i = t$ by Lemma 3.12. So there exists $i_1 \in I$ above i_0 such that for all $i \in I$ where $i_1 \leq i$, it holds that $t_i \leq t + \epsilon$, which gives $-t - \epsilon < -t_i$. In particular, $d(x, h_{i_1}) < r - t - \epsilon < r - t_{i_1}$. This implies that $[F, r] \sqsubseteq (h_{i_0}, t_{i_0})$ and hence $[F, r] \ll [G, s]$. □

To investigate whether there is a form of converse to Lemma 3.17, we need to first consider a few lemmas, where one of which uses the idea of the inspiring result of [11, Lemma 4.1].

Lemma 3.18 *Recall from the beginning of Section 2 that we denote*

$$\uparrow [H, t] := \bigcup_{z\in H} \{(x, r) \in B(X) \mid (z, t) \sqsubseteq (x, r)\}.$$

The map $\psi : (B(\mathcal{P}(X)), \sqsubseteq) \longrightarrow (\{\uparrow [H,t] \mid H \in \mathcal{P}_f(X)\}, \supseteq)$ *defined by* $(F,r) \mapsto \uparrow [F,r]$ *is a surjective preorder-embedding. (Caution:* ψ *may not be an order-isomorphism as* $(B(\mathcal{P}(X)), \sqsubseteq)$ *may not be a poset.)*

Proof. This is clear as $(F,r) \sqsubseteq (G,s)$ if and only if $[F,r] \sqsubseteq [G,s]$ by Lemma 3.4, and $[F,r] \sqsubseteq [G,s]$ if and only if $\uparrow [F,r] \supseteq \uparrow [G,s]$ by the definition of $\sqsubseteq$ on $\mathcal{P}_f(B(X))$. □

Lemma 3.19 *Let* (X,d) *be a Yoneda complete quasi-metric space.*

The following are equivalent.

(1) $[F,r] \ll [G,s]$ *in* $(B(X), \sqsubseteq)$.

(2) $\uparrow [F,r] \ll \uparrow [G,s]$ *in* $(\{\uparrow [H,t] \mid H \in \mathcal{P}_f(B(X))\}, \supseteq)$.

(3) $(F,r) \ll (G,s)$ *in* $(B(\mathcal{P}_f(X)), \sqsubseteq)$.

Proof. (2) and (3) are equivalent by the preceding lemma.

For (1) and (2), we first note that (X,d) is Yoneda complete, $(B(X), \sqsubseteq)$ is a dcpo by [13, Theorem 7.1]. Although [11, Lemma 4.1] deals with all nonempty finite subset of a dcpo while we only deal with a certain collection of nonempty finite subsets, it is straightforward to verify that we can show that (1) and (2) are equivalent using the same idea. However, we present the proof for completeness by showing (1) and (3) are equivalent.

(3) $\implies$ (1): Suppose $[G,s] \sqsubseteq (h,t)$, where (h,t) is the supremum of $(h_i,t_i)_{i\in I}$ in $(B(X), \sqsubseteq)$. So $(G,s) \sqsubseteq (h,t)$ in $(B(\mathcal{P}_f(X)), \sqsubseteq)$, and $(F,r) \sqsubseteq (h_i,t_i)$ for some $i \in I$. It follows that $[F,r] \sqsubseteq (h_i,t_i)$ for some $i \in I$ and $[F,r] \ll [G,s]$.

(1) $\implies$ (3): Suppose $(G,s) \sqsubseteq (H,t)$, where (H,t) is a supremum of $(H_i,t_i)_{i\in I}$, and suppose for the sake of contradiction that $(F,r) \not\sqsubseteq (H_i,t_i)$ for all $i \in I$, which equivalently gives $[F,r] \not\sqsubseteq [H_i,t_i]$ for all $i \in I$. For each $i \in I$, define $H_i' := H_i - H_i''$, where $H_i'' := \{z \in H_i \mid (F,r) \sqsubseteq (z,t_i)\}$. So each H_i' is nonempty by supposition.

We show that $(H_i',t_i)_{i\in I}$ is directed. Let $(H_1',t_1),(H_2',t_2)$ be given. There exists (H_3,t_3) such that $(H_1,t_1),(H_2,t_2) \sqsubseteq (H_3,t_3)$. We claim that $(H_1',t_1),(H_2',t_2) \sqsubseteq (H_3',t_3)$. By symmetry, it suffices to show that $S^d(H_1',H_3') \le t_1 - t_3$, i.e., for each $z_3 \in H_3'$, there exists $z_1 \in H_1'$ such that $d(z_1,z_3) \le t_1 - t_3$. Clearly, we know that there exists $z_1 \in H_1$ such that $d(z_1,z_3) \le t_1 - t_3$, i.e., $(z_1,t_1) \sqsubseteq (z_3,t_3)$. Suppose $z_1 \notin H_1'$, that means $(F,r) \sqsubseteq (z_1,t_1)$ and hence $(F,r) \sqsubseteq (z_3,t_3)$ by transitivity, a contradiction. So $z_1 \in H_1'$. Similarly, $S^d(H_2',H_3') \le t_2 - t_3$, and the claim is true.

We can now rewrite $[H_i',t_i]_{i\in I}$ as some directed subset $\mathcal{D} := \{(z,t_i) \mid z \in H_i'\}_{i\in I} \subseteq \mathcal{P}(B(X))$ using Lemma 3.4. By Jung's version of Rudin's lemma [12, Theorem 4.11], we can find a directed set $D \subseteq \bigcup \mathcal{D} \subseteq B(X)$ such that $D \cap H_i' \neq \emptyset$ for each $i \in I$. Since (X,d) is Yoneda complete, $(B(X), \sqsubseteq)$ is a dcpo by [13, Theorem 7.1] and hence $\bigvee D$ exists. In particular, we note that (X,d) is Yoneda complete and thus standard, so $\inf_{i\in I} t_i = t$ by Lemma 3.12. We write $\bigvee D = (a,t)$ for some $a \in X$.

Since $H_i' \subseteq H_i$, it is immediate that $S^d(H_i,H_i') = 0$ and $[H_i,t_i] \sqsubseteq [H_{i'},t_i]$. So for each $i \in I$, $(H_i',t_i) \sqsubseteq (a,t)$, and $[H_i',t_i] \sqsubseteq (a,t)$. Hence $[H_i,t_i] \sqsubseteq [H_i',t_i] \sqsubseteq (a,t)$. This implies that $[H,t] \sqsubseteq (a,t)$, as $[H,t]$ is the supremum of $[H_i,t_i]$, and in particular, $[G,s] \sqsubseteq [a,t]$ by transitivity. Thus there exists some $(z_0,t_0) \in D$ such that $[F,r] \sqsubseteq (z_0,t_0) \in D$ in $(\mathcal{P}(B(X)),\sqsubseteq)$. However, each z_0 is in H_i' if and only if $[F,r] \notin (z_0,t_i)$ for some t_i, which is again a contradiction. Thus $(F,r) \sqsubseteq (H_i,t_i)$ for some $i \in I$. □

Lemma 3.20 *If $(B(X),\sqsubseteq)$ is a quasi-continuous dcpo with fin-basis of the form $\{[F,r] \mid F \in \mathcal{P}_f(X)\}$, then for any $a \in \mathbb{R}^+$, the following hold:*

(i) *$[F,r] \ll (y,s)$ in $(B(X),\sqsubseteq)$ implies $[F,r+a] \ll (y,s+a)$ in $(B(X),\sqsubseteq)$; and*

(ii) *$[F,r] \ll [G,s]$ in $(B(X),\sqsubseteq)$ implies $[F,r+a] \ll [G,s+a]$ in $(B(X),\sqsubseteq)$ and $(F,r+a) \ll (G,s+a)$ in $(B(\mathcal{P}_f(X)),\sqsubseteq)$.*

Proof. Since $(B(X),\sqsubseteq)$ has a fin-basis $\{[F,r] \mid F \in \mathcal{P}_f(X)\}$, there exists a directed family $[F_i,r_i]_{i\in I}$, where each $[F_i,r_i] \ll (y,s+a)$ and $[F_i,r_i]_{i\in I}$ has $(y,s+a)$ as supremum. Furthermore, for each $i \in I$, $r_i \geq s+a$. Hence it can be verified directly that $[F_i,r_i-a]_{i\in I}$ is a directed family. We show that $[F_i,r_i-a]_{i\in I}$ has (y,s) as supremum. Indeed, if (z,t) is an upper bound of $[F_i,r_i-a]_{i\in I}$, then $(z,t+a)$ is an upper bound of $[F_i,r_i]_{i\in I}$, therefore $(y,s+a) \sqsubseteq (z,t+a)$. So $(y,s) \sqsubseteq (z,t)$ and this proves the claim.

Now, since $[F,r] \ll (y,s)$ in $(B(X),\sqsubseteq)$, $(F,r) \ll (y,s)$ in $(B(\mathcal{P}(X)),\sqsubseteq)$ by Lemma 3.19, and hence $(F,r) \sqsubseteq (F_i,r_i-a)$ for some $i \in I$ by Lemma 3.6. Thus $(F,r+a) \sqsubseteq (F_i,r_i)$, so $[F,r+a] \sqsubseteq [F_i,r_i]$. So $[F,r+a] \ll (y,s+a)$ as claimed.

For (ii), we realise that $[F,r] \ll [G,s]$ if and only if $[F,r] \ll (y,s)$ for each $y \in G$ by Proposition 2.21. By (1), $[F,r+a] \ll (y,s+a)$ for each $y \in G$ and thus again by Proposition 2.21, $[F,r+a] \ll [G,s+a]$. Following Lemma 3.19, it is then immediate that $(F,r+a) \ll (G,s+a)$. □

We can now obtain a form of converse to Lemma 3.17.

Lemma 3.21 *If $(B(X),\sqsubseteq)$ is a quasi-continuous dcpo with fin-basis of the form $\{[F,r] \mid F \in \mathcal{P}_f(X)\}$, then $[F,r] \ll [G,s]$ in $(B(X),\sqsubseteq)$ implies $W^d(F,G) \leq r-s$.*

Proof. Fix any Cauchy-weighted net $(h_i,s_i)_{i\in I}$, where $(h_i)_{i\in I}$ has d-limit h. By Lemma 3.10, $\bigvee(h_i,s_i)_{i\in I} = (h,0)$, as $\inf_{i\in I} s_i = 0$. Since (X,d) is Yoneda complete, $\bigvee(h_i,s_i+s)_{i\in I} = (h,s)$ by [9, Proposition 2.2].

Case 1: $S^d(G,h) = \infty$.

By definition of S^d, $\min_{y\in G} d(y,h) = \infty$. Clearly, $\limsup_{i\in I} d(x,h_i) \dot{-} \min_{y\in G} d(y,h) = 0 \leq r-s$.

Case 2: $S^d(G,h) < \infty$.

We observe that $[G,S^d(G,h)+s] \sqsubseteq (h,s)$, so $[F,S^d(G,h)+r] \sqsubseteq (h_i,s_i+s)$ for $i \in I$ large enough as $[F,S^d(G,h)+r] \ll [G,S^d(G,h)+s]$ by Lemma 3.20. So there exists $x_0 \in F$ such that for all $i \in I$ large enough, $(x_0,S^d(G,h)+r) \sqsubseteq (h_i,s_i+s)$, therefore $d(x_0,h_i) \leq \min_{y\in G} d(y,h) + r - s_i - s$. Thus $\min_{x\in F} \limsup_{i\in I} d(x,h_j) \leq \limsup_{i\in I} d(x_0,h_j) \leq S^d(G,h) + r - s$.

Since $(h_i)_{i\in I}$ is chosen arbitrarily, $W^d(F,G) = W'^d(F,G) \leq r-s$ as claimed. □

The following uses the construction of the Cauchy-weightable subnet of any given Cauchy net given in [8, Lemma 7.2.8]. We will study more properties of this construction in Fact 4.8, but for this section, the following suffices.

Lemma 3.22 *Let (X,d) be a quasi-continuous Yoneda complete quasi-metric space. For each $x \in X$, there exists a Cauchy-weighted net $(F_{\alpha(E)}, 1/2^{|E|})_{E \in \mathcal{P}_f(I)}$ such that for any $r \in \mathbb{R}^+$,*

(i) *$(F_{\alpha(E)})_{E \in \mathcal{P}_f(I)}$ has S^d-limit x;*

(ii) *$(F_{\alpha(E)}, 1/2^{|E|} + r) \ll (x,r)$ in $(B(\mathcal{P}_f(X)), \sqsubseteq)$; and*

(iii) *$(F_{\alpha(E)}, 1/2^{|E|} + r)_{E \in \mathcal{P}_f(I)}$ has supremum (x,r).*

Proof. By supposition, $W(-,x) = \limsup_{i \in I} S^d(-,G_j)$ for some Cauchy net $(G_j)_{j \in J}$ with S^d-limit x in $(\mathcal{P}_f(X), S^d)$. We consider the construction of the Cauchy-weightable subnet of $(G_j)_{j \in J}$ given in [8, Lemma 7.2.8].

For each finite subset E of I, $\alpha(E)$ is defined as follows: First find $i_0 \in I$ such that for all $i,j \in I$, $i_0 \le i \le j$, $S^d(F_i, F_j) < 1/2^{|E|+1}$, and then find i_1 which is above i_0 and $\alpha(E')$ for every proper subset E' of E. Then define $\alpha(E) = i_1$. So $\limsup_{j \in I} S^d(F_{\alpha(E)}, F_j) \le 1/2^{|E|+1} < 1/2^{|E|}$. Thus $W^d(F_{\alpha(E)}, x) < 1/2^{|E|} + r - r$ and by Lemmas 3.17 and 3.19, $(F_{\alpha(E)}, 1/2^{|E|} + r) \ll (x,r)$. Hence (ii) is shown. Also, (i) is clear by Lemma 3.16, since $(F_{\alpha(E)})_{E \in \mathcal{P}_f(I)}$ is a subnet of $(G_j)_{j \in J}$ and $(G_j)_{j \in J}$ has S^d-limit x.

Finally, using Lemma 3.10, we have that (x,r) is a supremum of $(F_{\alpha(E)}, 1/2^{|E|} + r)$. □

The following is analogous to [9, Lemma 3.2].

Lemma 3.23 *Let (X,d) be a Yoneda complete quasi-metric space. For every $a \in \mathbb{R}^+$, if $[F, r+a] \ll [G, s+a]$, then $[F,r] \ll [G,s]$.*

Proof. Suppose $[G,s] \le (h,t)$, where (h,t) is the supremum of the directed family $(h_i,t_i)_{i \in I}$. Then $(h_i, t_i + a)_{i \in I}$ is a directed family with supremum $(h, t+a)$ as (X,d) is Yoneda complete and hence standard. In particular, $(h, t+a)$ is above $[G, s+a]$. By supposition, there exists $i \in I$ such that $[F, r+a] \sqsubseteq (h_i, t_i + a)$, and hence $[F,r] \sqsubseteq (h_i, t_i)$. □

We are now ready to consider an alternative version to Lemma 3.12, where we replace the condition of standardness by some other conditions. This will turn out to be very useful for us in the proof for (3) $\implies$ (1) of Theorem 3.25, as at the point of writing, we are unable to show that $(\mathcal{P}_f(X), S^d)$ is standard in general even when (X,d) is Yoneda complete.

Lemma 3.24 *Let $(B(X), \sqsubseteq)$ be a quasi-continuous dcpo with a fin-basis $\{[G,s] \mid G \in \mathcal{P}_f(X)\}$. If $[G_i, s_i]_{i \in I}$ is a directed family with supremum $(x,0)$ and for each $i \in I$, $[G_i, s_i] \ll (x,0)$, then*

(i) *for any $t \in \mathbb{R}^+$, $[G_i, s_i + t]_{i \in I}$ has supremum (x,t),*

(ii) *$\inf_{i \in I} s_i = 0$ and $(G_i)_{i \in I}$ is a Cauchy net in $(\mathcal{P}_f(X), S^d)$ with S^d-limit x.*

Proof. We use Jung's version of Rudin's lemma [12, Theorem 4.11] to obtain a directed family $(y_i, s_i)_{i\in I}$ such that $y_i \in G_i$ for each $i \in I$. Since $(B(X), \sqsubseteq)$ is a dcpo, the supremum of $(y_i, s_i)_{i\in I}$ exists, let it be (y, s). We use Lemma 3.12 to obtain that $s = \inf_{i\in I} s_i$. Also, note that (y, s) is an upper bound of $[G_i, s_i]_{i\in I}$, so $(x, 0) \le (y, s)$, so $d(x, y) \le 0 - s$, i.e., $s = 0$.

We now show (i), i.e., for every $t \in \mathbb{R}^+$, the directed family $[G_i, s_i + t]_{i\in I}$ has the supremum (x, t). First, observe that each $[G_i, s_i + t] \ll (x, t)$ by Lemma 3.20. Also, for each $[G, s] \in \downarrow\!\!\!\downarrow (x, t)$, $[G, s - t] \in \downarrow\!\!\!\downarrow (x, 0)$ by Lemma 3.23. Since $[G_i, s_i]_{i\in I}$ is a directed family with the supremum $(x, 0)$, $(G_i, s_i)_{i\in I}$ is a directed family with the supremum $(x, 0)$ by Lemma 3.6. Hence $(G, s-t) \sqsubseteq (G_i, s_i)$ for some $i \in I$. It follows that $(G, s) \sqsubseteq (G_i, s_i + t)$ for some $i \in I$, so $[G_i, s_i + t]_{i\in I}$ is a cofinal directed subset of $\downarrow\!\!\!\downarrow (x, t)$. Thus $[G_i, s_i + t]_{i\in I}$ has the supremum (x, t).

To show (ii), we follow the idea of the last part of the proof of [8, Lemma 7.4.26]. For each $i \in I$, $S^d(G_i, x) \le s_i$. So for any $H \in \mathcal{P}_f(X)$, $S^d(G_i, H) \le S^d(G_i, x) + S^d(x, H) \le s_i + S^d(x, H)$. Taking sups over $i \in I$, we have $\sup_{i\in I}(S^d(G_i, H) - s_i) \le S^d(x, H)$.

Suppose for the sake of contradiction that the inequality is strict. So $s := \sup_{i\in I}(S^d(G_i, H) - s_i) < S^d(x, H)$. Thus we deduce that $s < \infty$. For each $i \in I$, $S^d(G_i, H) - r_i \le s$, so $(G_i, r_i + s) \sqsubseteq (H, 0)$. Since $[G_i, s_i + t]_{i\in I}$ has the supremum (x, t) for all $t \in \mathbb{R}^+$, $(x, s) \sqsubseteq (H, 0)$. Thus $S^d(x, H) \le s$, which is a contradiction.

Hence we have that $S^d(x, H) = \sup_{i\in I}(S^d(G_i, H) - s_i)$. We also observe that $(G_i, s_i)_{i\in I}$ is Cauchy-weighted, so by [8, Lemma 7.4.9], x is the S^d-limit of $(G_i)_{i\in I}$. □

Let us gather all the essential ingredients and present our first main result.

Theorem 3.25 *Let (X, d) be a Yoneda complete quasi-metric space. The following are equivalent.*

(1) (X, d) is quasi-continuous.

(2) $(B(X), \sqsubseteq)$ is a quasi-continuous dcpo with a fin-basis of $\{[G, s] \mid G \in \mathcal{P}_f(X)\}$.

(3) For each $(x, r) \in B(X)$, the set of elements with $(G, s) \ll (x, r)$ in $(B(\mathcal{P}_f(X)), \sqsubseteq)$ is directed and has (x, r) as supremum.

Proof. We first note that by [13, Theorem 7.1], (X, d) is Yoneda complete if and only if $(B(X), \sqsubseteq)$ is a dcpo.

Let $(x, r) \in B(X)$ be given.

(2) $\Longrightarrow$ (3): By supposition, there exists a directed subset $[G_i, s_i]_{i\in I}$ such that each $[G_i, s_i] \ll (x, r)$ in $(B(X), \sqsubseteq)$ and $[G_i, s_i]_{i\in I}$ has supremum (x, r). So $(G_i, s_i) \ll (x, r)$ in $(B(\mathcal{P}(X)), \sqsubseteq)$ by Lemma 3.19 and $\{(G_i, s_i)\}_{i\in I}$ has supremum (x, r) by Lemma 3.6.

(3) $\Longrightarrow$ (2): Since there exists a directed set of elements with $(G, s) \ll (x, r)$ in $(B(\mathcal{P}(X)), \sqsubseteq)$, $[G, s] \ll (x, r)$ in $(B(X), \sqsubseteq)$ again by Lemma 3.19. Since $\{(G, s) \mid (G, s) \ll (x, r)\}$ has supremum (x, r) in $(B(\mathcal{P}_f(X)), \sqsubseteq)$, $\{[G, s] \mid [G, s] \ll (x, r)\}$ has supremum (x, r) in $(B(X), \sqsubseteq)$ by Lemma 3.6, (2) is shown.

(1) $\Longrightarrow$ (3) is true by Lemma 3.22 and Trick 1.

Now we show (3) $\implies$ (1). Let $x \in X$ be given. We have that $\Downarrow (x,0) = (G_i, s_i)_{i\in I}$ for some directed family $(G_i, s_i)_{i\in I}$ and $(x,0)$ is a supremum of $(G_i, s_i)_{i\in I}$ in $(B(X), \sqsubseteq)$.

Since (2) and (3) are equivalent, $(B(X), \sqsubseteq)$ is a quasi-continuous dcpo with a fin-basis of $\{[G,s] \mid G \in \mathcal{P}_f(X)\}$. Hence by Lemma 3.24(ii), $(G_i)_{i\in I}$ has S^d-limit x. It remains to show that $W^d(-,x) = \limsup_{i\in I} S^d(-,G_i)$.

Let $F \in \mathcal{P}_f(X)$, $\epsilon > 0$ be given, and define $r := W^d(F,x) + \epsilon$. It is trivial that $\limsup_{i\in I} S^d(F,G_i) \leq W^d(F,x)$ when $r = \infty$. Now suppose $r < \infty$. So $W^d(F,x) < r$, and $(F,r) \ll (x,0)$ by Lemma 3.17. This implies that $(F,r) \sqsubseteq (G_i,s_i)$ for some $i \in I$ large enough and hence $S^d(F,G_i) \leq r - s_i \leq W^d(F,x) + \epsilon - s_i$, so $\limsup_{i\in I} S^d(F,G_i) \leq W^d(F,x) + \epsilon$ as $\limsup_{i\in I}(-s_i) = 0$. Since ϵ is arbitrary, $\limsup_{i\in I} S^d(F,G_i) \leq W^d(F,x)$.

Now let $r' := \limsup_{i\in I} S^d(F,G_i) + 2\epsilon$. Again, it is trivial when $r' = \infty$ that $W^d(F,G_i) \leq \limsup_{i\in I} S^d(F,G_i)$. So suppose $r' < \infty$. Thus $\limsup_{i\in I} S^d(F,G_i) < r' - \epsilon$. There exists $i_0 \in I$ such that for all $i_0 \leq i$, $S^d(F,G_i) < r' - \epsilon$. Therefore $(F,r') \sqsubseteq (G_i,\epsilon)$. Choose i_1 above i_0 such that for all $i \in I$ where $i_1 \leq i$, $s_i < \epsilon$ since $\inf_{i\in I} s_i = 0$. So $S^d(F,G_i) < r' - \epsilon < r' - s_i$. Using Lemmas 3.17 and 3.19, it follows that $(F,r') \sqsubseteq (G_i,s_i) \ll (x,0)$ and hence $(F,r') \ll (x,0)$. Now by Lemma 3.21, $W^d(F,x) \leq r' = \limsup_{i\in I} S^d(F,G_i) + 2\epsilon$. So $W^d(F,x) \leq \limsup_{i\in I} S^d(F,G_i)$.

We have thus shown that $W^d(F,x) = \limsup_{i\in I} S^d(F,G_i)$. This completes the proof. $\square$

At this point, we highlight the importance of the imposed condition that $\{(G_i,s_i)_{i\in I} \mid G_i \in \mathcal{P}_f(X)\}$ is a fin-basis of $(B(X),\sqsubseteq)$ in showing (3) $\implies$ (1) of Theorem 3.25.

To show that (X,d) is quasi-continuous, we require a Cauchy net $(G_i)_{i\in I}$ in $(\mathcal{P}_f(X), S^d)$ which is such that $W^d(-,x) = \limsup_{i\in I} S^d(-,G_i)$ and has x as an S^d-limit for every $x \in X$. By imposing the abovementioned condition, we then have a very suitable candidate for this Cauchy net as we have that $\{(G_i,s_i)_{i\in I}\} \cap \Downarrow (x,0)$ is a cofinal directed subset of $\Downarrow (x,0)$.

If we do not have the condition, then we only have that $\Downarrow (x,0) := \{F_i \in \mathcal{P}_f(B(X)) \mid F_i \ll (x,0)\}_{i\in I}$ is directed and has $(x,0)$ as supremum. We point out that at the point of writing, we are unable to determine whether $(\pi_1(F_i))_{i\in I}$, where $\pi_1(F) := \{y \in X \mid (y,s) \in F\}$, is a Cauchy net in $(\mathcal{P}_f(X), S^d)$ in general.

Remark 3.26 We mentioned earlier that Proposition 2.13 is immediate following Theorem 3.25. Indeed, if (X,d) is continuous Yoneda complete, $(B(X),\sqsubseteq)$ is a continuous dcpo by [9, Theorem 3.7]. In particular, $(B(X),\sqsubseteq)$ is a quasi-continuous dcpo which has a fin-basis $\{(x,r) \mid x \in X\} \subseteq \{[G,s] \mid F \in \mathcal{P}_f(X)\}$ and hence (X,d) is quasi-continuous Yoneda complete.

Proposition 3.27 *If (X,d) is a dcpo, X is quasi-continuous as a quasi-metric space if and only if it is quasi-continuous as a poset.*

Proof. We use the fact that $B(X)$ and $X \times (-\infty,0]$ are order isomorphic by [9, Example 1.8]. So $B(X)$ is quasi-continuous if and only if X is quasi-continuous as a poset. The result then follows immediately. □

In [18, Lemma and Definition 3.2(4)], Waszkiewicz considered the notion of interpolative mappings and he obtained as a corollary to Lemma 3.2 that w^d is interpolative when (X,d) is a continuous Yoneda complete quasi-metric space. Using the close relationship between the quasi-continuity of Yoneda complete quasi-metric spaces and the quasi-continuity of their formal balls as shown in Theorem 3.25, we obtain the following analogous result for quasi-continuous Yoneda complete quasi-metric space (X,d) where $(\mathcal{P}_f(X), S^d)$ is standard and (X,d) has a particular property.

Proposition 3.28 *Let (X,d) be a quasi-continuous Yoneda complete quasi-metric space where $(\mathcal{P}_f(X), S^d)$ is standard. $W : \mathcal{P}_f(X) \times \mathcal{P}_f(X) \longrightarrow [0,\infty]$ is interpolative, i.e.,*

$$W^d(F,y) = \inf_{H \in \mathcal{P}_f(X)} (W^d(F,H) + W^d(H,y))$$

for all $F \in \mathcal{P}_f(X)$, $y \in X$.

Proof. We first note that W^d satisfies the triangle inequality, and hence for any $H \in \mathcal{P}_f(X)$, $W^d(F,y) \leq W^d(F,H) + W^d(H,y)$, so $W^d(F,y) \leq \inf_{H \in \mathcal{P}_f(X)}(W^d(F,H) + W^d(H,y))$. We now show the reverse inequality. It is trivially true when $W^d(F,y) = \infty$.

Suppose $W^d(F,y) < \infty$. Define $r := W^d(F,y) + \epsilon$. So $W^d(F,y) < r$. Hence $[F,r] \ll (y,0)$ by Lemma 3.17. By the ($\Longrightarrow$) direction of Theorem 3.25, $(B(X), \sqsubseteq)$ is a quasi-continuous dcpo with a fin-basis $\{[H,t] \mid H \in \mathcal{P}_f(X)\}$. So there exists $[H_0,t_0]$ such that $[F,r] \ll [H_0,t_0] \ll (y,0)$ by [6, Proposition III-3.5]. By Lemma 3.21, this implies that $W^d(F,H_0) + W^d(H_0,y) \leq r - t_0 + t_0 = r$. So $\inf_{H \in \mathcal{P}_f(X)}(W^d(F,H) + W^d(H,y)) \leq r = W^d(F,y) + \epsilon$. Since ϵ is arbitrary, $\inf_{H \in \mathcal{P}_f(X)}(W^d(F,H) + W^d(H,y)) \leq W^d(F,y)$. This completes the proof. □

4 Finitely-Generated HyperSpace of Quasi-continuous Yoneda Complete Quasi-metric Space

In [18, Section 2.3], the author considered the notion of a lower map. A map $l : (X,d) \longrightarrow ([0,\infty], d_{\mathbb{R}})$ is called a *lower map* if l is non-expansive from (X, d^{op}) to $([0,\infty], d_{\mathbb{R}}^{op})$, i.e., for any $x,y \in X$, $d_{\mathbb{R}}(l(x), l(y)) \leq d(x,y)$. For example, for any $x \in X$, $d(-,x) : (X,d) \longrightarrow ([0,\infty], d_{\mathbb{R}})$ is a lower map since for any $y,z \in X$, $d(y,x) \leq d(y,z) + d(z,x)$, i.e., $d(y,z) \geq d(y,x) \dot{-} d(z,x) = d_{\mathbb{R}}(d(y,x), d(z,x))$.

Naturally, we can also consider the notion of an upper map $u : (X,d) \longrightarrow ([0,\infty], d_{\mathbb{R}})$ for a hemi-metric space (X,d). We call u an *upper map* if u is non-expansive from (X,d) to $([0,\infty], d_{\mathbb{R}}^{op})$. For example, for any $x \in X$, $d(x,-)$ is an upper map, since for any $y,z \in X$, $d(x,z) \leq d(x,y) + d(y,z)$, i.e., $d(y,z) \geq d(x,z) \dot{-} d(x,y) = d_{\mathbb{R}}^{op}(d(x,y), d(x,z))$.

We now consider the following subset of upper maps of $[(X,d) \longrightarrow ([0,\infty], d_{\mathbb{R}})]$.

Definition 4.1 Let

$$\mathcal{U}^d(X) := \{S^d(F,-) : (X,d) \longrightarrow ([0,\infty], d_{\mathbb{R}}) \mid F \in \mathcal{P}_f(X)\}$$

and for any upper maps $f, g : (X,d) \longrightarrow ([0,\infty], d_{\mathbb{R}})$,

$$\mathcal{U}^d(f,g) := \sup_{H \in \mathcal{P}_f(X)} (f(H) \dot{-} g(H)).$$

In particular,

$$\mathcal{U}^d(S^d(F,-), S^d(G,-)) := \sup_{H \in \mathcal{P}_f(X)} (S^d(F,H) \dot{-} S^d(G,H)).$$

Remark 4.2 (i) By considering the lower maps in [18], Waszkiewicz was able to discuss notions like ideals, Scott-closed sets and supremum, which are traditionally notions from the posetal aspect, on Q-categories.

For our case, as we wish to study the result obtained by Heckmann and Keimel on the hyperspace of nonempty finitely-generated subsets via the lens of quasi-metric [11, Proposition 4.5], it is natural to use the dual notions of lower maps, which are the upper maps, to serve as our link.

(ii) Note that we considered $S^d(F,-)$ with the domain restricted to (X,d), instead of $(\mathcal{P}_f(X), S^d)$. This is to ensure that the definition is compatible with the notion of $\uparrow F$, which denotes the set of all $y \in X$ above some $x \in F$.

By doing so, if (X,d) is a poset, we can recover $\uparrow F$ by $S^d(F,-)^{-1}(\{0\})$. Also, $(\mathcal{U}^d(X,d), \mathcal{U}^d)$ can be identified as the collection of all nonempty finitely generated upper sets ordered by reverse inclusion considered in [11, Section 4.1].

Lemma 4.3 (i) *For each $F \in \mathcal{P}_f(X)$, $S^d(F,-) : (X,d) \longrightarrow ([0,\infty], d_{\mathbb{R}})$ is an upper map.*

(ii) *$(\mathcal{U}^d(X), \mathcal{U}^d)$ is a quasi-metric space.*

(iii) *(Yoneda lemma) For all $F, G \in \mathcal{P}_f(X)$, $\mathcal{U}^d(S^d(F,-), S^d(G,-)) = S^d(F,G)$.*

Proof. (i) is immediate from the discussion at the start of this section.

For (ii), suppose $\mathcal{U}^d(S^d(F,-), S^d(G,-)) = \mathcal{U}^d(S^d(G,-), S^d(F,-)) = 0$, so $S^d(F,G) = S^d(G,F) = 0$. Given any $H \in \mathcal{P}_f(X)$, $S^d(F,H) \leq S^d(F,G) + S^d(G,H) = S^d(G,H)$ and $S^d(G,H) \leq S^d(G,F) + S^d(F,H) = S^d(F,H)$. So $S^d(F,H) = S^d(G,H)$, and hence $S^d(F,-) = S^d(G,-)$.

For (iii), we observe that $\mathcal{U}^d(S^d(F,-), S^d(G,-)) = \sup_{H \in \mathcal{P}_f(X)} (S^d(F,H) \dot{-} S^d(G,H)) \geq S^d(F,G) \dot{-} S^d(G,G) = S^d(F,G)$; and for any $H \in \mathcal{P}_f(X)$, $S^d(F,H) \leq S^d(F,G) + S^d(G,H)$, so $S^d(F,H) \dot{-} S^d(G,H) \leq S^d(F,G)$, i.e., $S^d(F,G) \geq \sup_{H \in \mathcal{P}_f(X)} (S^d(F,H) \dot{-} S^d(G,H)) = \mathcal{U}^d(S^d(F,-), S^d(G,-))$. This gives that $\mathcal{U}^d(S^d(F,-), S^d(G,-)) = S^d(F,G)$. □

Lemma 4.4 *Consider $\uparrow(F,r) := \{(G,s) \in B(\mathcal{P}(X)) \mid (F,r) \sqsubseteq (G,s)\} \subseteq B(\mathcal{P}(X))$. The following statements are true.*

(i) *The map* $\rho : (\{\uparrow [H,t] \mid H \in \mathcal{P}_f(X)\} \supseteq) \longrightarrow (B(\mathcal{U}^d(X)), \sqsubseteq)$ *defined by* $\uparrow [F,r] \mapsto (S^d(F,-), r)$ *is an order-isomorhpism.*

(ii) *The map* $\tau : (B(\mathcal{U}^d(X)), \sqsubseteq) \longrightarrow (B(\mathcal{P}_f(X)), \sqsubseteq)$ *defined by* $(S^d(F,-), r) \mapsto (F, r)$ *is a surjective preorder-embedding. (Caution: Again,* τ *may not an order-isomorphism as* $(B(\mathcal{P}_f(X)), \sqsubseteq)$ *may not be a poset.)*

Proof. For (i), we simply observe that $(S^d(F,-), r) \sqsubseteq (S^d(G,-), s)$ if and only if $\mathcal{U}^d(S^d(F,-), S^d(G,-)) = S^d(F,G) \le r - s$ if and only if $\max_{y\in G} \min_{x\in F} d(x,y) \le r - s$ if and only if for each $y \in G$, there exists $x \in F$ such that $d(x,y) \le r - s$ if and only if for each $y \in G$, there exists $x \in F$ such that $(x,r) \sqsubseteq (y,s)$.

(ii) is clear by (i) and Lemma 3.18. □

Lemma 4.5 *A directed family* $(F_i, r_i)_{i\in I}$ *has supremum* (F,r) *in* $(B(\mathcal{P}_f(X)), \sqsubseteq)$ *if and only if* $(S^d(F_i,-), r_i)_{i\in I}$ *has supremum* $(S^d(F,-), r)$ *in* $(B(\mathcal{U}^d(X)), \sqsubseteq)$.

Proof. This is clear by Lemma 4.4. □

We can now make an addition to Lemma 3.19.

Lemma 4.6 *Let* (X,d) *be a Yoneda complete quasi-metric space and let* $F, G \in \mathcal{P}_f(X)$*. The following are equivalent.*

(1) $[F,r] \ll [G,s]$ *in* $(B(X), \sqsubseteq)$.

(2) $\uparrow [F,r] \ll \uparrow [G,s]$ *in* $(\{\uparrow [H,t] \mid H \in \mathcal{P}_f(X)\}, \supseteq)$.

(3) $(F,r) \ll (G,s)$ *in* $(B(\mathcal{P}_f(X)), \sqsubseteq)$.

(4) $(S^d(F,-), r) \ll (S^d(G,-), s)$ *in* $(B(\mathcal{U}^d(X)), \sqsubseteq)$.

Proof. This is clear by Lemmas 3.19 and 4.4. □

We shall now turn our attention towards the characterization of quasi-continuous Yoneda complete quasi-metric space using its hyperspace of finitely-generated set.

At first glance, [8, Lemma 7.2.8] informs that there exists a Cauchy-weightable subnet to every Cauchy net in a hemi-metric space, and this is very useful in establishing a link between quasi-metric spaces and their posets of formal balls. By Lemma 3.22, we realise we can even say more about the Cauchy-weightable net constructed when (X,d) is quasi-continuous. It turns out that there is an even greater use of this construction, which we will come back to it later in Fact 4.8.

In [11, Proposition 4.5], Heckmann and Keimel showed that for a dcpo $(P, \le)$, its hyperspace of finitely-generated subsets ordered by reverse inclusion is continuous if and only if the dcpo is quasi-continuous. Readers who are familiar with this result will be able to identify that the ($\Longleftarrow$) is the harder direction. We shall present this proof so as to facilitate our discussion.

Proof. ($\Longleftarrow$): Let $F \in \mathcal{P}_f(P)$ be given. By hypothesis, for each $x \in F$, $\twoheaddownarrow \{x\} = \{F_i\}_{i\in I_x}$ is an ideal and has $\{x\}$ as supremum.

Step 1: For any $F_1, F_2 \in \mathcal{P}_f(P)$, $F_1 \cup F_2$ is an infimum of F_1, F_2. It can be verified directly that $\bigcap_{x\in F} \{F_i\}_{i\in I_x}$ is an ideal, and hence directed.

Step 2: We now show that $\bigcap_{x\in F}\{F_i\}_{i\in I_x}$ has a supremum F. Let $z \in X$ such that $F \nleq \{z\}$. So for each $x \in F$, $x \nleq z$. In particular, for each $x \in F$, there exists $i_x \in I_x$ such that $F_{i_x} \nleq \{z\}$.

Consider $F' := \bigcup_{x\in F} F_{i_x}$. Then $F' \nleq \{z\}$. So $\bigcap_{x\in F}\{F_i\}_{i\in I_x}$ has a supremum F. $\square$

We point out some problems we face when we apply this to the case of formal balls.

Let (X, d) be a quasi-continuous Yoneda complete quasi-metric space and $F \in \mathcal{P}_f(X)$ be given. For each $x \in F$, $W^d(-, x) = \limsup_{i\in I_x} S^d(-, F_i)$ for some Cauchy net $(F_i)_{i\in I_x}$ and $(F_i)_{i\in I_x}$ has S^d-limit x. For each $x \in X$, there exists a Cauchy-weighted net $(F_j, r_j)_{j\in J_x}$ where $(F_j)_{j\in J_x}$ is a subnet of $(F_i)_{i\in I_x}$ by [8, Lemma 7.2.8]. Furthermore, it can be shown each $(F_{j_x}, r_{j_x}) \ll (x, 0)$ in $(B(\mathcal{P}_f(X)), \sqsubseteq)$, and $(F_j, r_j)_{j\in J_x}$ has supremum $(x, 0)$. So for each $x \in F$, $(F_j, r_j)_{j\in J_x}$ is a cofinal subset of $\Downarrow (x, 0)$.

We now illustrate the problems we faced in applying the technique by playing out each step of the above proof to our case.

Step 1*: $(B(\mathcal{P}_f(X)), \sqsubseteq)$ need not be a semilattice: For $(F_1, r_1), (F_2, r_2)$ given, one possible candidate for their infimum is $(F_1 \cup F_2, \max\{r_1, r_2\})$. However, we are unable to show that this is indeed their infimum and hence are unable to conclude that $\bigcap_{x\in F}(F_j, r_j)_{j\in J_x}$ is directed using this strategy.

Step 2*: To show that $\bigcap_{x\in F}(F_j, r_j)_{i\in I_x}$ has supremum $(F, 0)$, we can again consider any (h, t) such that $(F, 0) \not\sqsubseteq (h, t)$. We have that for each $x \in F$, $(x, 0) \not\sqsubseteq (h, t)$, so there exists (F_{j_x}, r_{j_x}) such that $(F_{j_x}, r_{j_x}) \not\sqsubseteq (h, t)$. Following the idea in Step 2 of the proof above, we then need to produce some (F', r') using $(F_{j,x}, r_{j_x})$'s. In particular, we have to decide how to obtain r'.

Naively, we can keep the first coordinate of each (F_{j_x}, r_{j_x}) constant and set each of the second coordinate to $\min_{x\in F} r_{j_x}$ and then consider $(\bigcup_{x\in F} F_{j_x}, \min_{x\in F} r_{j_x})$. Note that this will imply $(F_{j_x}, \min_{x\in F} r_{j_x}) \not\sqsubseteq (h, t)$ as desired, as each $(F_{j,x}, r_{j_x}) \sqsubseteq (F_{j_x}, \min_{x\in F} r_{j_x})$. However, we note that this may not guarantee that $(F_{j_x}, \min_{x\in F} r_{j_x}) \ll (x, 0)$. So $(\bigcup_{x\in F} F_{j_x}, \min_{x\in F} r_{j_x})$ may not be way-below $(F, 0)$.

Suppose instead that we set each of the second coordinate to $\max_{x\in F} r_{j_x}$ and consider $(\bigcup_{x\in F} F_{j_x}, \max_{x\in F} r_{j_x})$ so as to repair the problem above. This will guarantee that each $(F_{j_x}, \max_{x\in F} r_{j_x}) \ll (x, 0)$, since $(F_{j_x}, \max_{x\in F} r_{j_x}) \sqsubseteq (F_{j_x}, \min_{x\in F} r_{j_x}) \ll (x, 0)$. However, we note that it may not ensure that each of the $(F_{j_x}, \max_{x\in F} r_{j_x})$'s is not below below (h, t). In other words, such $(\bigcup_{x\in F} F_{j_x}, \max_{x\in F} r_{j_x})$ may be below (h, t).

Remark 4.7 From the above, we observe that this method of fixing the first coordinate of each of these formal balls while varying the second coordinate does not appear to work. Without this method of constructing a directed family for which each element is way-below $(F, 0)$, we are unable to construct a desired Cauchy net $(S^d(F_i, -))_{i\in I}$ in $(\mathcal{U}^d(X), \mathcal{U}^d)$ that has $\mathcal{U}^d$-limit $S^d(F, -)$, and in turn show that $(\mathcal{U}^d(X), \mathcal{U}^d)$ is a continuous quasi-metric space.

We now make the following observations following Lemma 3.22.

Fact 4.8 *For each $n \geq 1$,*

(i) *there exists some $(F_{\alpha(E)}, 1/2^n)$ from the Cauchy-weighted net $(F_{\alpha(E)}, 1/2^{|E|})_{E \in Fin(I)}$.*

(ii) *there exists some $(F_{\alpha(E')}, 1/2^{n+1})$ such that $(F_{\alpha(E)}, 1/2^n) \sqsubseteq (F_{\alpha(E')}, 1/2^{n+1})$ in $(B(\mathcal{P}_f(X), \sqsubseteq))$.*

Proof. (i) is clear as we can choose any E with $|E| = n$. To show (ii), consider any $i \in I - E$. Then by setting $E' := E \cup \{i\}$ will give the desired result, as whenever $E \subseteq E'$, $(F_{\alpha(E)}, 1/2^n) \sqsubseteq (F_{\alpha(E')}, 1/2^{n+1})$. □

Proposition 4.9 *F is an S^d-limit of a Cauchy net $(F_i)_{i \in I}$ if and only if $S^d(F, -)$ is a $\mathcal{U}^d$-limit of $(S^d(F_i, -))_{i \in I}$ in $(\mathcal{U}^d(X), \mathcal{U}^d)$.*

Proof. This is clear as for any $G \in \mathcal{P}_f(X)$ given, $S^d(F, G) = \limsup_{i \in I} S^d(F_i, G)$ if and only if $\mathcal{U}^d(S^d(F, -), S^d(G, -)) = \limsup_{i \in I} \mathcal{U}^d(S^d(F_i, -), S^d(G, -))$. □

We are now ready to present our second main result.

Theorem 4.10 *Let (X, d) be a Yoneda complete quasi-metric space. The following are equivalent.*

(1) (X, d) is quasi-continuous.

(2) $(B(X), \sqsubseteq)$ is a quasi-continuous dcpo with a fin-basis of $\{[G, s] \mid G \in \mathcal{P}_f(X)\}$.

(3) For each $(x, r) \in B(X)$, the set of elements with $(G, s) \ll (x, r)$ in $(B(\mathcal{P}_f(X)), \sqsubseteq)$ is directed and has (x, r) as supremum.

(4) $(B(\mathcal{U}^d(X)), \sqsubseteq)$ is a continuous poset.

Proof. (1), (2) and (3) are equivalent by Theorem 3.25.

We first show that (1) implies (4). Let $(S^d(F, -), r) \in B(\mathcal{U}^d(X))$ be given and let $F := \{x_1, x_2, ..., x_k\}$. By supposition, for each $1 \leq k \leq m$, $W^d(-, x_k) = \limsup_{i \in I_k} S^d(-, F_{k,i})$, where $(F_{k,i})_{i \in I_k}$ is a Cauchy net and has S^d-limit x_k.

For each k, we construct the Cauchy-weighted net $(F_{k,\alpha(E_{n,k})}, 1/2^{|E_{n,k}|})_{E_{n,k} \in Fin(I_k)}$, where $(F_{k,\alpha(E_k)})_{E_k \in Fin(I_k)}$ is a subnet of $(F_{k,i})_{i \in I_k}$ following Lemma 3.22. Hence for each $n \geq 1$, we have the following table.

$$\begin{array}{c|ccccc}
x_1 & (F_{1,\alpha(E_1)}, 1/2^n) & (F_{1,\alpha(E_1')}, 1/2^n) & (F_{1,\alpha(E_1'')}, 1/2^n) & \dots \\
x_2 & (F_{2,\alpha(E_2)}, 1/2^n) & (F_{2,\alpha(E_2')}, 1/2^n) & (F_{2,\alpha(E_2'')}, 1/2^n) & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
x_m & (F_{m,\alpha(E_m)}, 1/2^n) & (F_{m,\alpha(E_m')}, 1/2^n) & (F_{m,\alpha(E_m'')}, 1/2^n) & \dots
\end{array}$$

Essentially, for each $n \geq 1$ and for each k, we consider the $\alpha(E_k^{(-)})$'s where the finite subsets $E_k^{(-)}$'s of I_k have the same size n. Note that for each k, there need not be a countable number of formal balls created with second projection $1/2^n$; we simply list a few as examples.

Define each $F_{n,j}$ by taking the union of exactly one $F_{k,\alpha(E_k^{(-)})}$ from every row, and we call the collection of all such j's over all $n \geq 1$ as J. Since $(F_{n,j}, 1/2^{n,j}) \ll (x_k, 0)$ for each k, by Proposition 2.21, it holds that $(F_{n,j}, 1/2^{n,j}) \ll (F, 0)$ for each $(n,j) \in J$. Furthermore, by Theorem 3.25, $(B(X), \sqsubseteq)$ is a quasi-continuous dcpo with a fin-basis $\{[G,s] \mid G \in \mathcal{P}_f(X)\}$. So we can use Lemma 3.20 to obtain that $(F_{n,j}, 1/2^{n,j} + r) \ll (F, r)$ for each $(n,j) \in J$.

We now show that $(F_{n,j}, 1/2^{n,j})_{(n,j)\in J}$ is a directed set. Let $(F_{n,j}, 1/2^{n,j}), (F_{n',j'}, 1/2^{n',j'})$ be given. Without loss of generality, suppose $F_{n,j} = \bigcup_k F_{k,\alpha(E_k)}$ and $F_{n',j'} = \bigcup_k F_{k,\alpha(E'_k)}$. So for each k, there exists E''_k such that $(F_{k,\alpha(E_k)}, 1/2^n), (F_{k,\alpha(E'_k)}, 1/2^{n'}) \sqsubseteq (F_{k,\alpha(E''_k)}, 1/2^{|E''_k|})$. Furthermore, without loss of generality, we can assume that the $|E''_k|$'s are equal, otherwise we can simply increase each $|E''_k|$ and find $(F_{k,\alpha(E'''_k)}, 1/2^{|E'''_k|})$ which is above $(F_{k,\alpha(E''_k)}, 1/2^{|E''_k|})$ and $|E'''_k| = \max_{1\leq k\leq m} |E''_k|$ by Fact 4.8. So $(\bigcup_k F_{k,\alpha(E''_k)}, 1/2^{|E''_k|})$ is an upper bound of $(F_{n,j}, 1/2^{n,j})$ and $(F_{n',j'}, 1/2^{n',j'})$, and $(F_{n,j}, 1/2^{n,j})_{(n,j)\in J}$ is directed as claimed. In particular, $(F_{n,j}, 1/2^{n,j} + r)_{(n,j)\in J}$ is directed.

We proceed to show that $(F_{n,j}, 1/2^n + r)_{(n,j)\in J}$ has a supremum (F, r) using the same idea from [11, Proposition 4.5]. It is obvious that (F, r) is an upper bound of $(F_{n,j}, 1/2^n + r)_{(n,j)\in J}$. We note that for each k, $(F_{k,\alpha(E_{n,k})}, 1/2^{|E_{n,k}|})_{E_{n,k}\in Fin(I_k)}$ has supremum $(x_k, 0)$ by Lemma 3.10. So $(F_{k,\alpha(E_{n,k})}, 1/2^{|E_{n,k}|} + r)_{E_{n,k}\in Fin(I_k)}$ has supremum (x_k, r) by Lemma 3.24(i).

Now suppose $(F, r) \not\sqsubseteq (G, s)$. It follows that $S^d(F, G) \not\leq r - s$, so there is a $y \in G$ such that for each k, $(x_k, r) \not\sqsubseteq (y, s)$. Hence for each k, there exists some $(F_{k,\alpha(E_k)}, 1/2^{|E_k|} + r) \not\sqsubseteq (G, s)$. We can again assume without loss of generality that the $|E_k|$'s are equal by Fact 4.8. This means that $(\bigcup_k F_{k,\alpha(E_k)}, 1/2^{|E_k|}) \not\sqsubseteq (G, s)$. So $(F_{n,j}, 1/2^{n,j} + r)_{(n,j)\in J}$ has supremum (F, r). In particular $(S^d(F_{n,j}, -), 1/2^{n,j} + r)_{(n,j)\in J}$ is directed in $(B(\mathcal{U}^d(X)), \sqsubseteq)$, where each $(S^d(F_{n,j}, -), 1/2^{n,j} + r) \ll (S^d(F, -), r)$, and $(S^d(F_{n,j}, -), 1/2^{n,j} + r)_{(n,j)\in J}$ has supremum $(S^d(F, -), r)$ by Lemmas 4.6 and 4.5. By [8, Trick 5.1.20], we have shown that $(B(\mathcal{U}^d(X)), \sqsubseteq)$ is a continuous poset.

Finally, we show (4) implies (3).

Let $(x, r) \in B(X)$ be given. By supposition, $\Downarrow (S^d(x, -), r) \coloneqq (S^d(G_i, -), s_i)_{i\in I}$ is directed and has supremum $(S^d(x, -), r)$ in $(B(\mathcal{U}^d(X)), \sqsubseteq)$. We make some observations. $(G_i, s_i)_{i\in I}$ is directed and has supremum (x, r) in $(B(\mathcal{P}_f(X)), \sqsubseteq)$ by Lemma 4.5. Also, each $(G_i, s_i) \ll (x, r)$ by Lemma 4.6. Since $(G_i, s_i)_{i\in I}$ is directed where and has supremum (x, r) in $(B(\mathcal{P}_f(X)), \sqsubseteq)$, the proof is complete. □

Remark 4.11 From the proof of Theorem 4.10 (1) $\implies$ (4), we observe the importance of the construction in Lemma 3.22. In particular, the construction allows us to control the second coordinate of each of the elements of the desired directed family and overcome the problem stated in Remark 4.7.

Waskiewicz considered the notion of continuous Yoneda complete quasi-metric spaces [18]. However, just like the case of continuous dcpos, it is meaningful to also consider the notion of continuity on quasi-metric spaces when the completeness condition is dropped.

Definition 4.12 A quasi-metric space (X, d) is *continuous* if $(B(X), \leq)$ is a continuous poset.

Remark 4.13 Compare this definition of continuous quasi-metric space with [9, Definition 3.10], where continuity for quasi-metric spaces is considered only when they are standard. However, we drop this condition following the fact that $(\mathcal{U}^d(X), \mathcal{U}^d)$ may not be standard even if (X, d) is Yoneda complete and by Theorem 4.10. (At the point of writing, we do not have a counterexample for this.)

As a result, we have the following.

Corollary 4.14 *A Yoneda complete quasi-metric space* (X, d) *is quasi-continuous if and only if* $(\mathcal{U}^d(X), \mathcal{U}^d)$ *is continuous.*

Remark 4.15 In the case where (X, d) is a poset, Corollary 4.14 is exactly the following:

A dcpo P is quasi-continuous if and only if its hyperspace of finitely-generated subsets ordered by reverse inclusion is continuous.

5 Conclusion

In this paper, we introduced the notion of quasi-continuous Yoneda complete quasi-metric spaces. We also study a subclass of quasi-continuous dcpos of formal balls, namely those with a particular fin-basis. It then turns out that such a dcpos of formal balls can fully characterize the corresponding quasi-continuous Yoneda complete quasi-metric spaces. Furthermore, we obtain a full characterization of such quasi-metric spaces using the continuous quasi-metric spaces of the finitely-generated spaces. Following [11] and [9], these results appear to suggest that the notion of quasi-continuous Yoneda complete quasi-metric spaces introduced is an appropriate one, as it allows us successfully extend the results mentioned in the abovementioned papers to quasi-metric spaces. The reader may have observed a great use of the corresponding dcpos of formal balls in our proof of this investigation. This further supports the slogan that '*formal balls are the essence of quasi-metric spaces*' [9].

Finally, we list them some of the questions which can be our possible future work.

Question 5.1 (i) *To show Theorem 3.25 (2) $\implies$ (1), we require a fin-basis of* $\{[G, s] \mid G \in \mathcal{P}_f(X)\}$ *in the supposition. Can we remove this condition? i.e., is the following true:*

If $(B(X), \sqsubseteq)$ *is a quasi-continuous dcpo, then* (X, d) *is a quasi-continuous.*

(ii) *In the lead-up to Lemma 3.24, we mentioned that we are unable to show that* $(\mathcal{P}_f(X), S^d)$ *is standard even when* (X, d) *is Yoneda complete. One can verify that* $(\mathcal{P}_f(X), S^d)$ *is standard if* (X, d) *is a poset or* $([0, \infty], d_{\mathbb{R}})$*. Can this be shown to be generally true?*

(iii) *If* (X, d) *is a quasi-metric space,* $(\mathcal{U}^d(X), \mathcal{U}^d)$ *is a quasi-metric space, and hence there is the way-below mapping* $w^{\mathcal{U}^d}$*. Is it true that* $w^{\mathcal{U}^d}(S^d(F, -), S^d(G, -)) = W^d(F, G)$ *for all* $F, G \in \mathcal{P}_f(X)$*?*

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Appendix

Here, we give the proof that the quasi-metric space (X, d) given in Example 2.16 is quasi-continuous Yoneda complete and is not continuous.

Proof. We make the following observations. It is easy to verify that (X, d) is a quasi-metric space. Also, it can be verified directly that a net $(x_i)_{i\in I}$ in (X, d) is Cauchy if $(x_i)_{i\in I}$ is eventually constant or it is eventually strictly increasing along the $(1, n)$'s column, $n \in \mathbb{N}$. For the latter, it has d-limit ω, as for any $y \neq \omega$, $\limsup_{i\in I} d(x_i, y) = \infty = d(\omega, y)$ and $y = \omega$, $\limsup_{i\in I} d(x_i, y) = 0 = d(\omega, \omega)$. Thus (X, d) is Yoneda complete.

We show that (X, d) is not continuous by showing that there is no Cauchy net $(x_j)_{j\in J}$ such that

$$w^d(-, (0, 0)) = \limsup_{j\in J} d(-, x_j), \qquad (\diamond)$$

and

$$(x_j)_{j\in J} \text{ has } d\text{-limit } (0, 0). \qquad (\diamond\diamond)$$

In order to satisfy $(\diamond\diamond)$, $(x_j)_{j\in J}$ must be the constant net $((0, 0))$. We show that this net will not satisfy $(\diamond)$ by showing that $w^d((0, 0), (0, 0)) \neq d((0, 0), (0, 0))$. Clearly, the right-hand side of the equation is 0. Consider the Cauchy net $(1, n)_{n\in\mathbb{N}}$ which has d-limit ω. So $w^d((0, 0), (0, 0)) \geq \limsup_{n\in\mathbb{N}} d((0, 0), (1, n)) \dot{-} d((0, 0), \omega) = \infty$. Thus (X, d) is not continuous.

We now show that (X, d) is quasi-continuous. We state an observation which we will use in the sequel: If a Cauchy net $(x_i)_{i\in I}$ has d-limit ω and it is not eventually constant, then $(x_i)_{i\in I}$ is a subnet of $(1, k)_{k\in\mathbb{N}}$ (this is without loss of generality, more accurately, there exists $i_0 \in I$, $(x_i)_{i\in\uparrow i_0}$ is a subnet of $(1, k)_{k\in\mathbb{N}}$).

There are three types of elements we need to consider: $(1, n)$, $n \in \mathbb{N}$, ω, and $(0, 0)$.

(1) Claim: $W^d(-, (1, n)) = S^d(-, (1, n))$.

It suffices to show that for any $x \in X$, $w^d(x, (1, n)) = d(x, (1, n))$ by Corollary 2.14(ii).

Case 1.1: $d(x, (1, n)) = \infty$. So $w^d(x, (1, n)) \leq d(x, (1, n))$. But $w^d(x, (1, n)) \geq d(x, (1, n))$ in general. So $w^d(x, (1, n)) = d(x, (1, n))$.

Case 1.2: $d(x, (1, n)) = r < \infty$. We deduce that $x = (1, m) \leq (1, n)$, and $d((1, m), (1, n)) = 1/2^m - 1/2^n$.

Now let $(h_i)_{i\in I}$ be a Cauchy net with d-limit h. It is clear that when $d((1, n), h) = \infty$, $\limsup_{i\in I} d(x, h_i) \dot{-} d((1, n), h) = 0 \leq r$. So suppose $s := d((1, n), h) < \infty$.

Case 1.2(a): If $h = \omega$, then $d((1, n), h) = 1/2^n$. Also, $(h_i)_{i\in I}$ is a subnet of $(1, n)_{n\in\mathbb{N}}$. Hence $\limsup_{i\in I} d(x, h_i) \dot{-} d((1, n), h) = \limsup_{k\in\mathbb{N}} (1/2^m - 1/2^k) \dot{-} 1/2^n = 1/2^m - 1/2^n = d(x, (1, n))$.

Case 1.2(b): If $h \neq \omega$, then $(h_i)_{i\in I}$ is eventually constant and equal to some $(1, p) \neq \omega$. Hence $\limsup_{i\in I} d(x, h_i) \dot{-} d((1, n), h) = d((1, m), (1, p)) \dot{-} d((1, n), (1, p)) = (1/2^m - 1/2^n) \dot{-} (1/2^n - 1/2^p) = 1/2^m - 1/2^n = d(x, (1, n))$.

Either way, we have that $w^d(x,(1,n)) \leq d(x,(1,n))$, hence $w^d(x,(1,n)) = d(x,(1,n))$.

(2) Claim: $W^d(-,\omega) = \limsup_{n\in\mathbb{N}} S^d(-,(1,n))$.

We again just need to show that for any $x \in X$, $w^d(x,\omega) = \limsup_{n\in\mathbb{N}} d(x,(1,n))$ by Corollary 2.14(ii).

Case 2.1: $x = \omega$. Thus $\limsup_{n\in\mathbb{N}} d(x,(1,n)) = \infty$. Consider the Cauchy net $(1,n)_{n\in\mathbb{N}}$ has d-limit ω. So $w^d(x,(1,n)) \geq \limsup_{n\in\mathbb{N}} d(x,(1,n)) \dot{-} d((1,n),\omega) = \infty$. Hence $w^d(x,(1,n)) = \limsup_{i\in I} d(x,(1,n))$ for this case.

Case 2.2: $x = (0,0)$. So $\limsup_{n\in\mathbb{N}} d(x,(1,n)) = \infty$. Also, $w^d(x,\omega) \geq \limsup_{n\in\mathbb{N}} d(x,(1,n)) \dot{-} d(\omega,\omega) = \infty$. Hence $w^d(x,\omega) = \limsup_{n\in\mathbb{N}} d(x,(1,n))$.

Case 2.3: $x = (1,m)$ for some $m \in \mathbb{N}$. So $\limsup_{n\in\mathbb{N}} d((1,m),(1,n)) = 1/2^m$. Let $(h_i)_{i\in I}$ be a Cauchy net with d-limit h.

Case 2.3(a): $h \neq \omega$. So $d(\omega,h) = \infty$, and hence $\limsup_{i\in I} d(x,h_i) \dot{-} d(\omega,h) = 0$. Thus $w^d(x,\omega) = \limsup_{n\in\mathbb{N}} d(x,(1,n))$.

Case 2.3(b): $h = \omega$. So $(h_i)_{i\in I}$ is a subnet of $(1,k)_{k\in\mathbb{N}}$. It follows that $\limsup_{i\in I} d(x,h_i) \dot{-} d(\omega,h) = \limsup_{i\in I} d(x,h_i) = 1/2^m$. Thus $w^d(x,\omega) = \limsup_{n\in\mathbb{N}} d(x,(1,n))$ for either case.

(3) Claim: $W^d(-,(0,0)) = \limsup_{n\in\mathbb{N}} S^d(-,F_n)$, where $F_n := \{(0,0) \cup (1,n)\}$, $n \in \mathbb{N}$.

Let $G \in \mathcal{P}_f(X)$ be given.

Case 3.1: $G \cap (\{1\} \times \mathbb{N}) = \varnothing$. Then $\limsup_{n\in\mathbb{N}} S^d(G,F_n) = \infty$. Again consider the Cauchy net $(1,k)_{k\in\mathbb{N}}$ which has d-limit ω. So $W^d(G,(0,0)) \geq \limsup_{k\in\mathbb{N}} S^d(G,(1,k)) \dot{-} S^d(\omega,\omega) = \infty$, i.e., $W^d(G,(0,0)) = \infty$.

Case 3.2: $G \cap (\{1\} \times \mathbb{N}) \neq \varnothing$, and $(0,0) \notin G$. It follows that $\limsup_{n\in\mathbb{N}} S^d(G,F_n) = \infty$. Also, using the constant net $(0,0)$ which trivially has d-limit $(0,0)$, $W^d(G,(0,0)) \geq S^d(G,(0,0)) \dot{-} S^d((0,0),(0,0)) = \infty$.

Case 3.3: $G \cap (\{1\} \times \mathbb{N}) \neq \varnothing$, and $(0,0) \in G$. Without loss of generality, select $(1,m)$ where $m = \max_{(1,k)\in G} k$. So $\limsup_{n\in\mathbb{N}} S^d(G,F_n) = 1/2^m$. Let $(h_i)_{i\in I}$ be a Cauchy net with d-limit h.

Case 3.3(a): $h = \omega$. Hence $(h_i)_{i\in I}$ is a subnet of $(1,k)_{k\in\mathbb{N}}$. So $\limsup_{i\in I} S^d(G,h_i) \dot{-} S^d((0,0),h) = 1/2^m \dot{-} 0 = 1/2^m$.

Case 3.3(b): $h = (0,0)$. So $(h_i)_{i\in I}$ is the constant net $(0,0)$. Hence $\limsup_{i\in I} S^d(G,h_i) \dot{-} S^d((0,0),h) = 0$

Case 3.3(c): $h = (1,k)$ for some $k \in \mathbb{N}$. Then $\limsup_{i\in I} S^d(G,h_i) \dot{-} S^d((0,0),h) = 0$, since $S^d((0,0),h) = \infty$.

Since for all $G \in \mathcal{P}_f(X)$, $W^d(G,(0,0)) = \limsup_{n\in\mathbb{N}} S^d(G,F_n)$, we have shown that $W^d(-,(0,0)) = \limsup_{n\in\mathbb{N}} S^d(-,F_n)$.

We shall now consider whether the nets $(1,n)$, $(1,n)_{n\in\mathbb{N}}$ and $(F_n)_{n\in\mathbb{N}}$ are Cauchy, and have S^d-limits $(1,n)$, ω and $(0,0)$ respectively. Clearly, the constant net $(1,n)$ and the net $(1,n)_{n\in\mathbb{N}}$ are Cauchy. It is clear that $(F_n)_{n\in\mathbb{N}}$ is Cauchy as we see that for any $m,n \in \mathbb{N}$, $m \leq n$, $S^d(F_m,F_n) = 1/2^m - 1/2^n$.

It is trivial that the constant net $(1,n)$ has d-limit $(1,n)$, and hence $(1,n)$ has S^d-limit $(1,n)$ by Proposition 2.10.

Similarly, it is clear that the net $(1,n)_{n\in\mathbb{N}}$ has d-limit ω, and hence S^d-limit ω.

We now show that the Cauchy net $(F_n)_{n\in\mathbb{N}}$ has S^d-limit $(0,0)$, i.e., for any $G \in \mathcal{P}_f(X)$, $S^d((0,0),G) = \limsup_{n\in\mathbb{N}} S^d(F_n, G)$ and hence complete the proof.

If $G \cap (\{1\} \times \mathbb{N}) = \varnothing$, $S^d((0,0),G) = 0 = \limsup_{n\in\mathbb{N}} S^d(F_n, G)$.

If $G \cap (\{1\} \times \mathbb{N}) \neq \varnothing$, then $S^d((0,0),G) = \infty = \limsup_{n\in\mathbb{N}} S^d(F_n, G)$.

This completes the proof. □