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Open-independent, Open-locating-dominating Sets in Complementary Prism Graphs

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Abstract

For a finite, simple, undirected graph $G = (V(G), E(G))$, an open-dominating set $S \subseteq V(G)$ is such that every vertex in G has at least one neighbor in S. An open-independent, open-locating-dominating set $S \subseteq V(G)$ (OLD_{OIND}-set for short) is such that no two vertices in G have the same set of neighbors in S and each vertex in S is open-dominated exactly once by S. The problem of deciding whether or not a given graph has an OLD_{OIND}-set is known to be $\mathcal{NP}$-complete. The complementary prism of G is the graph $G\bar{G}$, formed from the disjoint union of G and its complement $\bar{G}$ by adding the edges of a perfect matching between the corresponding vertices of G and $\bar{G}$. We provided a logarithmic lower bound on the size of an OLD_{OIND}-set in any graph. Various properties of and bounds on OLD_{OIND}-sets in complementary prisms were presented and the cases of cliques, paths and cycles have been completely solved. It has been shown that for any graph with girth at least five, it can be decided in polynomial time whether or not its complementary prism has an OLD-OIND-set (and also the set can be found in polynomial time if it exists).

Keywords: open-independent sets, open-locating-dominating sets, complementary prisms of graphs, complexity.

1 Introduction

Consider the situation where a graph G models a facility or a multiprocessor network, with limited-range detection devices placed at chosen vertices of G. The purpose of these devices is to detect and precisely identify the location of an intruder such as a thief, saboteur, fire or faulty processor that may be present at any vertex. Sometimes such a device can determine if an intruder is in its neighborhood but cannot detect if the intruder is at its own location. In this case, it is required to find a so-called, *open-locating-dominating* vertex subset S, which is a dominating

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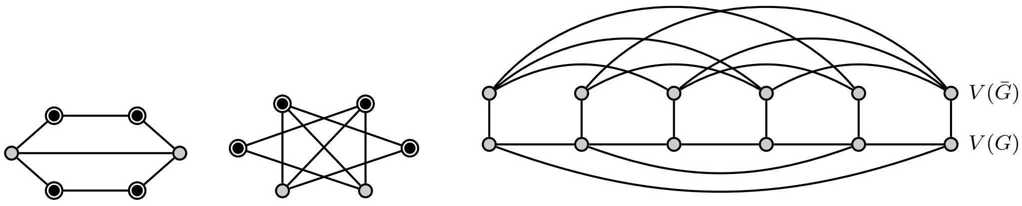
set of G, such that every vertex in G has at least one neighbor in S, and no two vertices in G have the same set of neighbors in S. When a device may be prevented from detecting an intruder at its own location, it is necessary to install another device in its neighborhood. A natural way to analyze such situations is to make use of open neighborhood sets which may have useful additional properties, such as being open-independent, dominating, open-dominating or open-locating-dominating.

Throughout this paper, $\lg(x)$ denotes $\log_2(x)$, and $G = (V(G), E(G))$ is a finite, simple, undirected graph with vertex set $V(G)$ and edge set $E(G)$. Let $S \subseteq V(G)$. The subgraph of G induced by S is denoted by $G[S]$.

The *open neighborhood* of a vertex $v \in V(G)$ is $N_G(v) = \{u \in V(G) \mid vu \in E(G)\}$, and its *closed neighborhood* is $N_G[v] = N_G(v) \cup \{v\}$. The open neighborhood of S is $N_G(S) = \bigcup_{v \in S} N_G(v)$ and its closed neighborhood is $N_G[S] = N_G(S) \cup S$. Such a set S is termed *dominating* if $N_G[S] = V(G)$ and *open-dominating* (or *total dominating) if* $N_G(S) = V(G)$. A vertex $v \in V(G)$ is *dominated* by S if $|N[v] \cap S| \geqslant 1$ and is *open-dominated* by S if $|N(v) \cap S| \geqslant 1$. The set S is a *locating-dominating set* if it is a dominating set and for every pair of distinct vertices $u, v \in V(G) \setminus S$, $N(u) \cap S \neq N(v) \cap S$. S is an *open-locating-dominating set* (OLD-set for short) if it is an open-dominating set and no two distinct vertices in G have the same set of neighbors in S, i.e., for all $u, v \in V(G)$, with $u \neq v$, $N(u) \cap S \neq N(v) \cap S$. For this last condition, u and v are said to be *distinguished* by S. This is an analogue of the well-studied identifying code problem in the literature [10]. The parameter $OLD(G)$ denotes the minimum size of an OLD set $S \subseteq V(G)$. The concept of an open-locating-dominating set was first considered by Seo and Slater [12]. The authors showed that to decide if a graph G has such a set is an $\mathcal{NP}$-complete decision problem and provided some useful results on OLD-sets in trees and grid graphs. S is *independent* if no two vertices in S are adjacent, i.e., $\forall\, v \in S,\ |N[v] \cap S| = 1$. S is an *open-independent set* (OIND-set for short) if every vertex in S is open-dominated by S at most once, i.e., $\forall\, v \in S,\ |N(v) \cap S| \leqslant 1$.

If an open-independent, open-locating-dominating set (OLD_{OIND}-set for short) exists in a given graph G, it is often of interest to establish the minimum size among such sets in G, which is denoted by $OLD_{OIND}(G)$. If S is an OLD_{OIND}-set for G, each component of $G[S]$ is isomorphic to K_2 (a complete graph on two vertices). See, for example, the graphs in Figures 1(a) and 1(b), where an OLD_{OIND}-set of each graph is represented by the black vertices. Seo and Slater [13] demonstrated that the problem of deciding whether or not a given graph has an OLD_{OIND}-set is $\mathcal{NP}$-complete. The authors also presented some results on OLD_{OIND}-sets in paths, trees and infinite grid graphs, and characterized OLD_{OIND}-sets in graphs with girth at least five.

Haynes et al. [6] introduced the so-called *complementary product* as a generalization of the well-known *Cartesian product* of graphs. As a particular case of complementary products, the authors define the *complementary prism* of a graph G, denoted by $G\bar{G}$, as the graph formed from the disjoint union of G and its complement $\bar{G}$ by adding the edges of the perfect matching between the corresponding vertices of G and $\bar{G}$. Note that $V(G\bar{G}) = V(G) \cup V(\bar{G})$. For the purposes of illustra-



(a) G.

(b) $\bar{G}$.

(c) $G\bar{G}$.

Fig. 1. Example of a graph, its complement and the resulting complementary prism.

tion, a graph G, its complement $\bar{G}$ and the complementary prism $G\bar{G}$ are depicted in Figure 1. To simplify matters, we use G and $\bar{G}$ to refer to the subgraph copies of G and $\bar{G}$, respectively, in $G\bar{G}$. We term as *primary* the vertices and edges in G, as *complementary* the vertices and edges in $\bar{G}$ and as *matching* any edge $v\bar{v}$ that directly connects a vertex $v \in V(G)$ and a vertex $\bar{v} \in V(\bar{G}$. For a set $X \subseteq V(G)$, let $\bar{X}$ denote the corresponding vertices of X in $V(\bar{G})$.

Haynes et al. [6] investigated several graph theoretic properties of complementary prisms, such as independence, distance and domination. For further study on domination parameters in complementary prisms, see [1,3,4,7,8,9,11]. Cappelle et al. [2] described a polynomial time recognition algorithm for complementary prisms and Duarte et al. [5] studied algorithmic/complexity properties of complementary prisms with respect to cliques, independent sets, domination, and convexity. Here we study open-independent, open-locating-dominating sets in the complementary prisms of graphs. Various properties of and bounds on OLD_{OIND}-sets in complementary prisms were presented and the cases of cliques, paths and cycles have been completely solved. Our main result is that if the girth of G is at least five, then the OLD_{OIND}-set of $G\bar{G}$, if it exists, can be found in polynomial time.

2 Some preliminary results for OLD_{OIND}-sets

The following theorem provides necessary and sufficient conditions for the existence of an OLD_{OIND}-set in a graph G that has girth $g(G) \geqslant 5$. For general graphs (with arbitrary girth), the conditions stated in Theorem 2.1 are necessary but not sufficient, as is stated in Lemma 2.2.

Theorem 2.1 ([13]) *If a graph G has girth $g(G) \geqslant 5$ and $S \subseteq V(G)$, then S is an OLD_{OIND}-set if and only if (i) each $v \in S$ is open-dominated exactly once, and (ii) each $v \notin S$ is open-dominated at least twice.*

Lemma 2.2 *If $S \subseteq V(G)$ is an OLD_{OIND}-set of a graph G, then (i) each $v \in S$ is open-dominated exactly once, and (ii) each $v \in V(G) \setminus S$ is open-dominated at least twice.*

A corollary to Lemma 2.2 is that if S exists, the components of $G[S]$ are of order two. The complexity of deciding whether or not a given graph has an OLD_{OIND}-set is established in the following theorem.

Theorem 2.3 ([13]) *Deciding, for a given graph G, whether or not G has an OLD_{OIND}-set is an $\mathcal{NP}$-complete decision problem.*

The next proposition provides a sharp logarithmic lower bound on the size of an OLD_{OIND}-set of any graph G, if G has such a set.

Proposition 2.4 *For a given graph G of order n, if G has an OLD_{OIND}-set, then*

$$OLD_{OIND}(G) \geqslant \lg(n+1). \tag{1}$$

Proof. Let S be an OLD_{OIND}-set of G with $|S| = k$ and $A = V(G) \setminus S$. By Lemma 2.2, for every $v \in A$, we have $|N(v) \cap S| \geqslant 2$. Since for any two vertices $u, v \in A$, $N(u) \cap S \neq N(v) \cap S$, the size of A is bounded by all combinations of at least two vertices taken from S. So, $|A| \leqslant 2^k - k - 1$. Hence $n \leqslant 2^k - 1$ and thus $k \geqslant \lg(n+1)$. □

It is possible to construct an infinite class of graphs that attains the bound in (1). Let $k = 2p$, where p is a positive integer and $n = 2^k - 1$. For every $i = 1, \dots, n$, let v_i be a vertex of G. For every $j = 1, \dots, p$, let $v_{2j-1}v_{2j}$ be an edge of G. Let $X = \{X_{k+1}, \dots, X_n\}$ be the set of sets of all combinations of at least two vertices taken from $\{v_1, \dots, v_k\}$. Now, for $i = k+1, \dots, n$, let v_i be adjacent to each vertex of X_i and $S = \{v_1, \dots, v_k\}$. Note that every vertex of S is open-dominated exactly once by the vertices of S and every vertex of $V(G) \setminus S$ is open-dominated at least twice. Furthermore, for any two vertices $u, v \in V(G) \setminus S$, $N(u) \cap S \neq N(v) \cap S$. Therefore, $n = 2k - 1$ and S is an OLD_{OIND}-set of G of size $k = \lg(n+1)$.

3 Some results for complementary prisms

Next we consider OLD_{OIND}-sets in the complementary prisms of graphs. In any OLD_{OIND}-set of a complementary prism there is at most one pair of matched vertices, as proved in Lemma 3.1. Moreover, the OLD_{OIND}-set has at least one vertex of G and at least one vertex of $\bar{G}$, as demonstrated in Lemma 3.2.

Lemma 3.1 *For a given graph G, if $G\bar{G}$ has an OLD_{OIND}-set $S = S_0 \cup \bar{S}_1$ with $S_0 \subseteq V(G)$ and $\bar{S}_1 \subseteq V(\bar{G})$, then there is at most one matching edge $u\bar{u} \in G\bar{G}[S]$ with $u \in S_0$ and $\bar{u} \in \bar{S}_1$.*

Proof. Suppose S is an OLD_{OIND}-set in $G\bar{G}$ as described and assume there are two distinct matching edges $u\bar{u}$ and $v\bar{v}$ say, in $G\bar{G}[S]$, such that $\{u, v\} \subseteq S_0$ and $\{\bar{u}, \bar{v}\} \subseteq \bar{S}_1$. Since $G\bar{G}$ is a complementary prism, either (i) edge $uv \in G$ (and vertices u and v open-dominate each other) or else (ii) edge $\bar{u}\bar{v} \in \bar{G}$ (and vertices $\bar{u}$ and $\bar{v}$ open-dominate each other). But vertices u and $\bar{u}$ open-dominate each other, and vertices v and $\bar{v}$ open-dominate each other. Thus, condition (i) of Lemma 2.2 does not hold, which is a contradiction. □

Lemma 3.2 *For a given graph G, if $G\bar{G}$ has an OLD_{OIND}-set $S = S_0 \cup \bar{S}_1$ with $S_0 \subseteq V(G)$ and $\bar{S}_1 \subseteq V(\bar{G})$, then $S_0 \neq \emptyset$ and $\bar{S}_1 \neq \emptyset$.*

Proof. For a proof by contradiction, suppose $S_0 = \emptyset$. Let $v \in V(G)$. By Lemma 2.2, vertex v has to be open-dominated at least twice. Since v has at most one neighbor in $\bar{S}_1$, we can conclude that S is not an OLD_{OIND}-set in $G\bar{G}$. The proof for $\bar{S}_1$ follows analogously. □

In the following Lemma we present some bounds on the sizes of some subsets of vertices of $G\bar{G}$, when $G\bar{G}$ has an OLD_{OIND}-set.

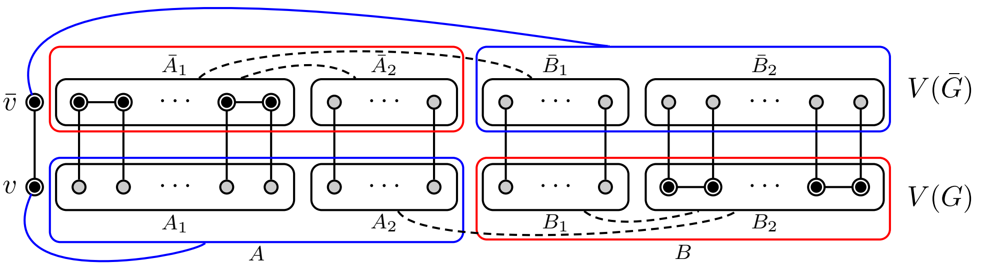
Lemma 3.3 *Let G be a graph of order n such that $G\bar{G}$ has an OLD_{OIND}-set $S = S_0 \cup \bar{S}_1$ with $S_0 \subseteq V(G)$, $\bar{S}_1 \subseteq V(\bar{G})$, then*

(i) if S contains the endpoints of a matching edge $v\bar{v}$, consider the following sets: $A = N_G(v)$, $B = V(G) \setminus (A \cup \{v\})$, $A_1 = (V(G) \cap N(\bar{S}_1)) \setminus \{v\}$, $A_2 = A \setminus A_1$, and $B_1 = B \setminus S_0$. Then, $|A_2| \leqslant \min\{2^{|S_0|-1} - 1, 2^{|\bar{S}_1|-1} - |\bar{S}_1|\}$ and $|B_1| \leqslant \min\{2^{|\bar{S}_1|-1} - 1, 2^{|S_0|-1} - |S_0|\}$.

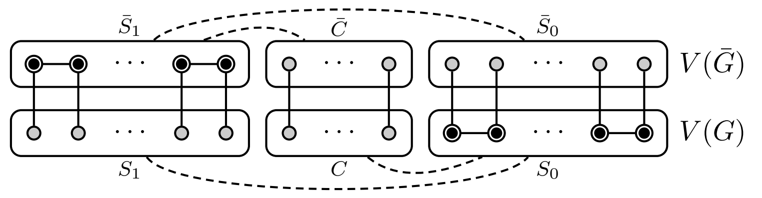
(ii) if S does not contain the endpoints of a matching edge and $r = \min\{|S_0|, |\bar{S}_1|\}$, then $n - |S_0| - |\bar{S}_1| \leqslant 2^r - r - 1$.

Proof. Assume that $S = S_0 \cup \bar{S}_1$ is an OLD_{OIND}-set in $G\bar{G}$ with $S_0 \subseteq V(G)$, $\bar{S}_1 \subseteq V(\bar{G})$. The reader is referred to Figure 2. To prove (i), let A, B, A_1, A_2, and B_1 be the sets as described above and $B_2 = B \setminus B_1 = S_0 \setminus \{v\}$. Observe that $S_0 = \{v\} \cup B_2$ and $\bar{S}_1 = \{\bar{v}\} \cup \bar{A}_1$. For every $X \in \{A_1, A_2, B_1, B_2\}$ let $\bar{X}$ be the set of corresponding vertices of X in $\bar{G}$. Note that the sets A, B and $\{v\}$ together partition $V(G)$ and $\bar{A}$, $\bar{B}$ and $\{\bar{v}\}$ together partition $V(\bar{G})$. The vertices in A_1 are open-dominated by v and distinguished from each other by the vertices in $\bar{S}_1$. Note that $N_{G\bar{G}}(A_2) \cap \bar{S}_1 = \emptyset$. Since $A_2 \subseteq N_{G\bar{G}}(v)$, every $u \in A_2$ is open-dominated by v and distinguished by the vertices in B_2. So, for every distinct pair $u, x \in A_2$, the sets $(N_G(u) \cap B_2)$ and $(N_G(x) \cap B_2)$ are distinct and non empty. Thus $|A_2| \leqslant 2^{|B_2|} - 1 = 2^{|S_0|-1} - 1$. Now consider the set $\bar{A}_2$. Every $\bar{w} \in \bar{A}_2$ must be open-dominated and distinguished by the vertices in $\bar{A}_1$. So, for every distinct pair $\bar{w}, \bar{y} \in A_2$, the sets $(N_{\bar{G}}(\bar{w}) \cap B_2)$ and $(N_{\bar{G}}(\bar{y}) \cap B_2)$ are distinct and non empty. Thus $|\bar{A}_2| \leqslant 2^{|\bar{A}_1|} - |\bar{A}_1| - 1$, and since $|A_2| = |\bar{A}_2|$ and $|\bar{A}_1| = |\bar{S}_1| - 1$, we can conclude that $|A_2| \leqslant \min\{2^{|S_0|-1} - 1, 2^{|\bar{S}_1|-1} - |\bar{S}_1|\}$. By symmetry, $|B_1| \leqslant \min\{2^{|A_1|} - 1, 2^{|B_2|} - |B_2|\}$, i.e. $|B_1| \leqslant \min\{2^{|\bar{S}_1|-1} - 1, 2^{|S_0|-1} - |S_0|\}$.

To prove (ii), assume $r = \min\{|S_0|, |\bar{S}_1|\}$ and S does not contain the endpoints of a matching edge. Following Figure 2(b), let S_1 be the set of corresponding vertices of $\bar{S}_1$ in $V(G)$, $\bar{S}_0$ be the set of corresponding vertices of S_0 in $V(\bar{G})$, C be the set $V(G) \setminus (S_1 \cup S_0)$ and $\bar{C}$ be the corresponding vertices of C in $V(\bar{G})$. Note that the sets S_1, C and S_0 together partition $V(G)$, and $\bar{S}_0$, $\bar{C}$ and $\bar{S}_1$ together partition $V(\bar{G})$. The vertices in S_1 are distinguished from each other by $\bar{S}_1$ and every vertex in S_1 is also dominated at least once by the vertices of S_0. By construction, $N(C) \cap \bar{S}_1 = \emptyset$. For every distinct pair $u, x \in C$, we have that $(N(u) \cap S_0) \neq (N(x) \cap S_0)$. So, using the same argument as in the proof of Theorem 2.4, we can conclude that $|C| \leqslant 2^{|S_0|} - |S_0| - 1$. Analogously, $|\bar{C}| \leqslant 2^{|\bar{S}_1|} - |\bar{S}_1| - 1$. Since $|C| = |\bar{C}|$ and $r = \min\{|S_0|, |\bar{S}_1|\}$, $|C| = n - |S_0| - |\bar{S}_1|$ is bounded by $2^r - r - 1$, where n is the order of G. □



(a) Lemma 3.3(i).



(b) Lemma 3.3(ii).

Fig. 2. An illustration of the sets described in Lemma 3.3.

Next we sharpen the bound presented in Proposition 2.4 for complementary prisms.

Theorem 3.4 *Let G be a graph of order n such that $G\bar{G}$ has an OLD_{OIND}-set. Then,*

$$2\lg(n+1) - 2 \leqslant OLD_{OIND}(G\bar{G}) \leqslant n+1. \tag{2}$$

Furthermore, these bounds are sharp.

Proof. Let G be a graph of order n such that $G\bar{G}$ has an OLD_{OIND}-set $S = S_0 \cup \bar{S}_1$ with $S_0 \subseteq V(G), \bar{S}_1 \subseteq V(\bar{G})$. To prove the upper bound in (2), assume that $OLD_{OIND}(G\bar{G}) \geqslant n+2$. Then $G\bar{G}[S]$ has at least two matching edges. But, by Lemma 3.1, every OLD_{OIND}-set has at most one matching edge, a contradiction, which completes the proof. Note that the upper bound in (2) is tight, for instance, when $G\bar{G} = P_3\bar{P}_3$ or $C_5\bar{C}_5$.

To begin the proof of the lower bound in (2), let $|S_0| = k_0$ and $|\bar{S}_1| = k_1$ with $k_0, k_1 \geqslant 1$. We consider the cases where (i) S contains the endpoints of a matching edge, and (ii) S does not contain the endpoints of a matching edge. By symmetry, for each of (i) and (ii), we can assume that $k_0 \geqslant k_1$, and need consider only the subcases (a) $2^{k_1} \geqslant k_0$ and (b) $k_0 > 2^{k_1}$.

Case (i). By the definition of an OLD_{OIND}-set, k_0 and k_1 are positive, odd integers. Consider the sets A_2 and B_1 defined in Lemma 3.3(i). Since $|A_2| \leqslant \min\{2^{k_0-1}-1, 2^{k_1-1}-k_1\}$, $|B_1| \leqslant \min\{2^{k_1-1}-1, 2^{k_0-1}-k_0\}$ and $n = k_0+k_1-1+|A_2|+|B_1|$ it follows that $n \leqslant k_0+k_1-1+2^{k_1-1}-k_1+2^{k_1-1}-1$ and thus

$$n + 2 \leqslant k_0 + 2^{k_1}. \tag{3}$$

Subcase $i(a)$. From (3) we have that

$$n + 2 \leqslant 2^{k_1+1} \tag{4}$$

and thus

$$\lg(n + 2) - 1 \leqslant k_1. \tag{5}$$

Subcase $i(b)$. From (3) we have that

$$n + 2 \leqslant 2k_0. \tag{6}$$

and thus

$$\lg(n + 2) - 1 \leqslant \lg(k_0). \tag{7}$$

Case (ii). Here k_0 and k_1 are positive, even integers. Consider the set C defined in Lemma 3.3(ii). Since $|C| \leqslant \min\{2^{k_0} - k_0 - 1, 2^{k_1} - k_1 - 1\}$ and $n = k_0 + k_1 + |C|$ it follows that $n \leqslant k_0 + k_1 + 2^{k_1} - k_1 - 1$ and thus

$$n + 1 \leqslant k_0 + 2^{k_1}. \tag{8}$$

Subcase $ii(a)$. From (8) we have that

$$n + 1 \leqslant 2^{k_1+1} \tag{9}$$

and thus

$$\lg(n + 1) - 1 \leqslant k_1. \tag{10}$$

Subcase $ii(b)$. From (8) we have that

$$n + 1 \leqslant 2k_0. \tag{11}$$

and thus

$$\lg(n + 1) - 1 \leqslant \lg(k_0). \tag{12}$$

To prove the lower bound in (2) we use the obvious fact that $|S| = k_0 + k_1$. Using (5) and (10), this implies that for Cases $i(a)$ and $ii(a)$, $2\lg(n + 1) - 2 \leqslant 2k_1 \leqslant |S|$. Similarly, using (7) and (12) implies that for Cases $i(b)$ and $ii(b)$, $2\lg(n + 1) - 2 \leqslant 2\lg(k_0) \leqslant |S|$, and the result follows.

To show that the lower bound in (2) is tight, let $G\bar{G}$ be the graph constructed as in Figure 2(b) with $2^{k_0} = n+1$ and $2^{k_1+2} = n+1$. Note that $2^{k_1} = n+1 > \lg(n+1) = k_0 > k_1$ and thus Subcase $ii(a)$ applies. $|C| = 2^{k_1} - k_1 - 1 = (n+1)/4 - \lg(n+1) + 1$. Finally, $S_0 \cup \bar{S}_1$ is an OLD_{OIND}-set in $G\bar{G}$ with size $k_0 + k_1 = 2\log(n + 1) - 2$, which complete the proof. □

Lemma 3.5 *For a given, nontrivial graph G, if $G\bar{G}$ has an OLD_{OIND}-set $S = S_0 \cup \bar{S}_1$ with $S_0 \subseteq V(G), \bar{S}_1 \subseteq V(\bar{G})$ and $|\bar{S}_1| = 1$, then G is a disconnected graph.*

Furthermore, G is the disjoint union of an isolated vertex and a collection of ℓ K_2 subgraphs, where $\ell \geqslant 1$.

Proof. Let G and S be as described above. If $|\bar{S}_1| = 1$, S contains the endpoints of a matching edge, say $v\bar{v}$, of $G\bar{G}$. Thus G consists of an isolated vertex v and a collection of $\ell \geqslant 1$ disjoint K_2 subgraphs and $V(G) \cup \{\bar{v}\}$ is an OLD_{OIND}-set for $G\bar{G}$. □

3.1 Complementary prisms of graphs with girth at least five

Seo and Slater [13] presented some results about open-independent, open-locating-dominating sets (OLD_{OIND}-sets) in trees. The authors showed that every leaf and its neighbor are contained in any OLD_{OIND}-set of any tree T, if T has such a set. Furthermore, they recursively defined the collection of trees such that each has a unique OLD_{OIND}-set. In this section we study OLD_{OIND}-sets in the complementary prisms of graphs that have girth at least five, which includes all trees. In particular, for a given graph G with girth $g(G) \geqslant 5$, the following result bounds the number of vertices of $V(\bar{G})$ that can be members of an OLD_{OIND}-set of the complementary prism $G\bar{G}$, of G, if $G\bar{G}$ has such a set.

Proposition 3.6 *For a given connected graph G whose girth satisfies $g(G) \geqslant 5$, if $G\bar{G}$ has an OLD_{OIND}-set $S = S_0 \cup \bar{S}_1$ with $S_0 \subseteq V(G)$ and $\bar{S}_1 \subseteq V(\bar{G})$, then $2 \leqslant |\bar{S}_1| \leqslant 3$. Furthermore,*

- *(i) if $|\bar{S}_1| = 2$, then $\bar{S}_1$ consists of the endpoints of a complementary edge in $\bar{G}$.*
- *(ii) if $|\bar{S}_1| = 3$, then $\bar{S}_1$ consists of three distinct vertices in $\bar{G}$, two of which are the endpoints of a complementary edge in $\bar{G}$, and the three corresponding vertices in G induce a 3-path in G.*

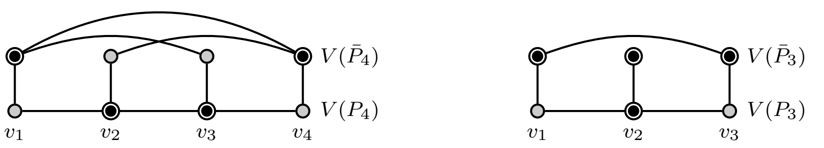
Proof. Suppose a connected graph G whose girth satisfies $g(G) \geqslant 5$ is given and $G\bar{G}$ has an OLD_{OIND}-set $S = S_0 \cup \bar{S}_1$ with $S_0 \subseteq V(G)$ and $\bar{S}_1 \subseteq V(\bar{G})$.

By Lemma 3.2, $\bar{S}_1 \neq \emptyset$ and, since G is connected, by Lemma 3.5, $|\bar{S}_1| > 1$. Suppose that $|\bar{S}_1| \geqslant 4$. By Proposition 3.1, there is at most one matching edge induced by the vertices in S. Hence if $|\bar{S}_1| \geqslant 4$, there are two distinct complementary edges $\bar{u}_1\bar{u}_2$ and $\bar{v}_1\bar{v}_2 \in G\bar{G}$, with $\{\bar{u}_1, \bar{u}_2, \bar{v}_1, \bar{v}_2\} \subseteq \bar{S}_1$. By Lemma 2.2, the set $\{\bar{u}_1, \bar{u}_2, \bar{v}_1, \bar{v}_2\}$ induces a graph isomorphic to $2K_2$ in $\overline{G}$, hence $\{u_1, u_2, v_1, v_2\}$ induces a cycle of order 4 in G. This contradicts the assumption that $g(G) \geqslant 5$ which completes this part of the proof.

To prove (i), let $\bar{S}_1 = \{\bar{u}, \bar{v}\}$. Since, by Lemma 3.1, there is at most one matching edge in $G[S]$ between the vertices in S_0 and in $\bar{S}_1$, we can conclude that $\bar{S}_1$ consists of the endpoints of the complementary edge $\bar{u}\bar{v}$ in $G\bar{G}$. In order to prove (ii), let $\bar{S}_1 = \{\bar{u}, \bar{v}, \bar{x}\}$. By Lemma 3.1 and the definition of an OLD_{OIND}-set, $G[S]$ must contain at least two components isomorphic to K_2, one of them containing a matching edge and the other a complementary edge. So $\bar{S}_1$ consists of a vertex that is an endpoint of a matching edge in $G\bar{G}$ and the endpoints of a complementary edge in $\bar{G}$. Suppose that edges in $G\bar{G}$ match $\bar{u}, \bar{v}$ and $\bar{x}$ with u, v and x in G, respectively. Without loss of generality, suppose there is an edge $\bar{u}\bar{v} \in E(G\bar{G})$.

Since S in an OLD_{OIND}-set, there is no edge between $\bar{u}$ and $\bar{x}$, nor between $\bar{v}$ and $\bar{x}$. So u, v and x induce a P_3 in G. □

We present some examples of complementary prisms to illustrate Proposition 3.6. As an illustration of (i), consider the path graph $P_4 = \langle v_1, v_2, v_3, v_4\rangle$. Then $\{\bar{v}_1, v_2, v_3, \bar{v}_4\}$ is an OLD_{OIND}-set for $P_4\bar{P}_4$ with $\bar{S}_1 = \{\bar{v}_1, \bar{v}_4\}$ and the complementary edge $\bar{v}_1\bar{v}_4 \in E(\bar{P}_4)$ (see Figure 3(a)). As an illustration of (ii), consider $P_3 = \langle v_1, v_2, v_3\rangle$. Then $\{\bar{v}_1, v_2, \bar{v}_2, \bar{v}_3\}$ is an OLD_{OIND}-set for $P_3\bar{P}_3$ with $\bar{S}_1 = \{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$, the matching edge $v_2, \bar{v}_2 \in E(P_3\bar{P}_3)$ and the complementary edge $\bar{v}_1\bar{v}_3 \in E(\bar{P}_3)$ (Figure 3(b)). Furthermore, there are instances where neither the graph nor its complement has an OLD_{OIND}-set but the complementary prism of the graph does (cf. P_3 in (ii) above). Conversely, there are instances where both a graph and its complement have OLD_{OIND}-sets but the complementary prism of the graph does not. An example is the graph G of Figure 1(a).



(a) $P_4\bar{P}_4$. (b) $P_3\bar{P}_3$.

Fig. 3. OLD_{OIND}-sets in complementary prisms of path graphs.

In view of Theorem 2.3, another question that arises is, for a given graph G, if the problem of deciding whether or not $G\bar{G}$ has an OLD_{OIND}-set, remains $\mathcal{NP}$-complete. However, the following result establishes that when we have girth $g(G) \geqslant 5$, the decision problem can be solved in polynomial time.

Theorem 3.7 *If G is a nontrivial, connected graph with $g(G) \geqslant 5$, we can decide in polynomial time whether or not $G\bar{G}$ has an OLD_{OIND}-set S. Furthermore, if S exists, it can be determined in polynomial time.*

Proof. Let G be a nontrivial connected graph whose girth satisfies $g(G) \geqslant 5$. We begin by assuming that $G\bar{G}$ has an OLD_{OIND}-set $S = S_0 \cup \bar{S}_1$ with $S_0 \subseteq V(G)$ and $\bar{S}_1 \subseteq V(\bar{G})$. By Proposition 3.6, $|\bar{S}_1| \in \{2, 3\}$. If $|\bar{S}_1| = 2$, S does not contain the endpoints of a matching edge. Consider the set C defined in Lemma 3.3 (ii). Then, $n = |S_0| + |\bar{S}_1| + |C|$. By Lemma 3.3 (ii), $|C| \leqslant 1$ and it can be concluded that $|V(G) \setminus S_0| \leqslant 3$. If $|\bar{S}_1| = 3$, S contains the endpoints of exactly one matching edge. Consider the sets A_2 and B_1 defined in the proof of item (i) of Lemma 3.3. Then $n = |S_0| + |\bar{S}_1| - 1 + |A_2| + |B_1|$. Since by Lemma 3.3 (i), $|A_2| \leqslant 1$ and $|B_1| \leqslant 3$, it follows that $|V(G) \setminus S| \leqslant 7$. In both cases, by using a brute force approach, it is possible to verify in polynomial time which vertices of $G\bar{G}$ must correspond to S_0 and $\bar{S}_1$.

Since $\bar{S}_1$ is bounded by a constant, if S exists, it can be found in polynomial time and S_0 can be deduced from it in polynomial time. It must be considered all possible guesses for $\bar{S}_1$ and either successfully complete the corresponding solution S or decide that such a set is impossible. Assume that every checking step succeeds.

If something within any step goes wrong, then the corresponding combination of guesses cannot lead to a solution.

Note that the algorithm is based on the fact that if graph $G\bar{G}$ has an OLD_{OIND}-set S with $S_0 \subseteq V(G), \bar{S}_1 \subseteq V(\bar{G})$, then $|\bar{S}_1| \in \{2, 3\}$. Since it is possible that more than one case can occur, it is necessary to compare the sizes of S_0 induced by each of the cases, assume the best case and proceed with the lower size. □

For some graphs G it is not possible to bound the size of the set $\bar{S}_1$ in $G\bar{G}$ and the size of S can be arbitrarily large. Consequently, we present Conjecture 3.8 which states that the problem of deciding if a graph has an OLD_{OIND}-set remains $\mathcal{NP}$-complete even for complementary prisms.

Conjecture 3.8 *Deciding, for a given graph G, whether or not $G\bar{G}$ has an open-independent, open-locating-dominating set is an $\mathcal{NP}$-complete decision problem.*

3.2 Complementary prisms of some particular graph classes

We study OLD_{OIND}-sets in the complementary prisms of some particular graph classes. It is easy to see that if G is a complete graph K_n, with $n \geqslant 1$, $G\bar{G}$ has an OLD_{OIND}-set if and only if $n = 1$. Next we consider the complementary prism of cycles and paths.

The complementary prisms of the n-cycle C_n ($n \geqslant 3$) and the n-path P_n ($n \geqslant 1$) are denoted by $C_n\bar{C}_n$ and $P_n\bar{P}_n$, respectively. When $n = 5$, $C_n\bar{C}_n$ is the Petersen graph which has domination number 3 and independence number 4. Henceforth, the vertex set of C_n or P_n is represented as the set $\{v_1, v_2, \ldots, v_n\}$ and the vertex set of $\bar{C}_n$ or $\bar{P}_n$ is represented as $\{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n\}$. We identify indexes of vertices of G modulo n.

As will be seen, only a few cycle graphs have complementary prisms with an OLD_{OIND}-set. It can easily be checked that for $n \in \{3, 4, 7\}$ the graph $C_n\bar{C}_n$ has no OLD_{OIND}-set. On the other hand, $OLD_{OIND}(C_5\bar{C}_5) = OLD_{OIND}(C_6\bar{C}_6) = 6$ and $OLD_{OIND}(C_8\bar{C}_8) = OLD_{OIND}(C_9\bar{C}_9) = 8$. For $n \geqslant 10$, the proof that $C_n\bar{C}_n$ does not have an OLD_{OIND}-set is given in the next theorem.

Theorem 3.9 *For $n \geqslant 10$, $C_n\bar{C}_n$ does not have an OLD_{OIND}-set.*

Proof. For a proof by contradiction, assume $n \geqslant 10$ and that $C_n\bar{C}_n$ has an OLD_{OIND}-set $S = S_0 \cup \bar{S}_1$ such that $S_0 \subseteq V(C_n)$ and $\bar{S}_1 \subseteq V(\bar{C}_n)$.

First we claim that no two vertices in $V(C_n) \setminus S_0$ are adjacent in C_n, that is, if $v \in V(C_n) \setminus S_0$, then $|N(v) \cap S_0| = 2$. Suppose $|N(v) \cap S_0| \leqslant 1$. So, v has a neighbor in C_n, say $u \in V(C_n) \setminus S_0$. By Lemma 2.2, u and v must be open-dominated at least twice, which implies $\{\bar{u}, \bar{v}\} \subset \bar{S}_1$. Since $\bar{u}$ and $\bar{v}$ must be open-dominated and they are not neighbors, $\bar{u}$ has a neighbor in $\bar{C}_n$, say $\bar{u}'$, that belongs to $\bar{S}_1$, also $\bar{v}$ has a neighbor in $\bar{C}_n$, say $\bar{v}'$, that belongs to $\bar{S}_1$. So $|S_1| \geqslant 4$ and this fact contradicts Proposition 3.6. Thus, we may conclude that $|N(v) \cap S_0| = 2$. This implies that S_0 consists of either (i) $n/3$ components of order two (which is possible only if n mod $3 = 0$), or (ii) $\lfloor n/3 \rfloor$ components of order two and exactly one isolated vertex (which is possible only if n mod $3 = 2$.) From now on, we assume n mod $3 \in \{0, 2\}$.

Reference to Figure 4 may aid the understanding of the following reasoning, where for $\bar{C}_n$, instead of indicating the edges, we indicate the non-edges by dashed lines. By Proposition 3.6, $|\bar{S}_1| \in \{2,3\}$. If $|\bar{S}_1| = 2$, assume $\bar{S}_1 = \{\bar{v}_i, \bar{v}_j\}$ for some integers $i, j, 1 \leqslant i < j \leqslant n$. In order to dominate all vertices in $V(\bar{C}_n)$, $i+3 \leqslant j$. If $|\bar{S}_1| = 3$, then $\bar{S}_1 = \{\bar{v}_i, \bar{v}_{i+1}, \bar{v}_{i+2}\}$, for some integer $i, 1 \leqslant i \leqslant n-2$.

Now let $W = V(C_n) \setminus S_0$, let $\bar{W}$ denote the corresponding vertices of W in $V(\bar{C}_n)$, and let $\bar{W}' = \bar{W} \setminus \bar{S}_1$. If $n \bmod 3 = 0$, we have that $|S_0| = 2n/3$ and $|\bar{S}_1| = 2$, and if $n \bmod 3 = 2$, then $|S_0| = 1 + (2n-4)/3$ and $|\bar{S}_1| = 3$. In both cases, since $n \geqslant 10$, $|\bar{W}'| \geqslant 2$. Let $\bar{v}_k, \bar{v}_\ell \in \bar{W}'$ for some integers $k, \ell, 1 \leqslant k < \ell \leqslant n$. Then, $(N(\bar{v}_k) \cap S) = (N(\bar{v}_\ell) \cap S) = \bar{S}_1$, implying that S is not an OLD_{OIND}-set for $C_n\bar{C}_n$. □

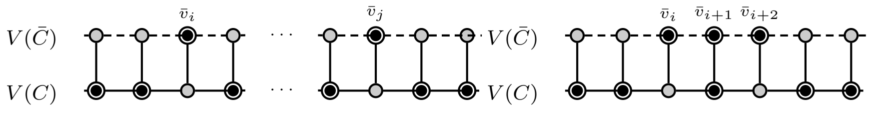


Fig. 4. The only two possible configurations for $S_0 \cup \bar{S}_1$ to be an OLD_{OIND}-set in $C_n\bar{C}_n$.

Next we consider the path graph $P_n, n \geqslant 2$. We can easily check that the graph $P_2\bar{P}_2$ does not have an OLD_{OIND}-set and that $OLD_{OIND}(P_3\bar{P}_3) = OLD_{OIND}(P_4\bar{P}_4) = 4$, $OLD_{OIND}(P_5\bar{P}_5) = OLD_{OIND}(P_6\bar{P}_6) = 6$, $OLD_{OIND}(P_7\bar{P}_7) = OLD_{OIND}(P_8\bar{P}_8) = OLD_{OIND}(P_9\bar{P}_9) = 8$, and $OLD_{OIND}(P_{10}\bar{P}_{10}) = OLD_{OIND}(P_{11}\bar{P}_{11}) = 10$. We prove in the next theorem that $P_n\bar{P}_n$ does not have an OLD_{OIND}-set when $n \geqslant 12$.

Theorem 3.10 *For $n \geqslant 12$, $P_n\bar{P}_n$ does not have an OLD_{OIND}-set. (cf Proposition 3.2 of [13] concerning the existence of OLD_{OIND}-sets in path graphs.)*

Proof. As $E(P_n) = E(C_n) \setminus \{v_1v_n\}$ and $E(\bar{P}_n) = E(\bar{C}_n) \cup \{v_1v_n\}$, the proof closely follows that of Theorem 3.9. The claim that no two vertices in $V(P_n) \setminus S_0$ are adjacent in P_n mirrors that in Theorem 3.9 except when $v = v_1$ or v_n. Clearly, in these cases v must be open-dominated by its solitary neighbor in P_n and $|N(v) \cap S_0| = 1$, but the claim still holds. Following the notation and reasoning in Theorem 3.9, when $n \geqslant 12$, $|W| = |\bar{W}| \geqslant 4$. Since $|\bar{W}'| \geqslant 2$ in all cases, the result follows. □

4 Concluding remarks

A logarithmic lower bound on the size of an OLD_{OIND}-set in any graph has been provided. Various properties of and bounds on OLD_{OIND}-sets in complementary prisms were presented and the cases of cliques, paths and cycles have been completely solved. It has been shown that for any graph with girth at least five, it can be decided in polynomial time whether or not its complementary prism has an OLD-OIND-set (and also the set can be found in polynomial time if it exists). Furthermore, we conjecture that the problem of deciding if a graph has an OLD_{OIND}-set remains $\mathcal{NP}$-complete even for complementary prisms and pose the following open questions:

(i) Which families of graphs attain the bounds of Theorem 3.4?

(ii) What are the additional conditions necessary to extend Theorem 2.1 for general graphs? and for complementary prisms?

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