Reflection in Membership Equational Logic, Many-Sorted Equational Logic, Horn Logic with Equality, and Rewriting Logic

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Abstract

We show that the generalized variant of rewriting logic where the underlying equational specifications are membership equational theories, and where the rules are conditional and can have equations, memberships and rewrites in the conditions is reflective. We also show that membership equational logic, many-sorted equational logic, and Horn logic with equality are likewise reflective. These results provide logical foundations for reflective languages and tools based on these logics, and in particular for the Maude language itself.

Key words: Reflection, reflective logics, universal theory, reflective rewriting logic, reflective membership equational logic, reflective Horn logic with equality

1 Introduction

Reflection is a very powerful and useful feature of rewriting logic, motivating work on metalogical reflection theorems. Clavel and Meseguer have formerly given detailed proofs for increasingly general fragments of rewriting logic, namely: (1) unsorted and unconditional [3], (2) unsorted conditional [7]; and (3) many-sorted conditional [7]. This paper generalizes these previous results to the case of conditional rewrite theories whose underlying equational specifications are theories in *membership equational logic* [11]. Conditional rules in this latter case are very general, since they can involve not only other rewrites,

but also equations and memberships as conjuncts. The work presented here is also related to Palomino's own research on rewriting logic reflection [12].

But what about other logics? What about membership equational logic itself? What about many-sorted equational logic? What about Horn logic with equality? We have for long conjectured that these logics are also reflective, and that the same methods developed for rewriting logic can be used to obtain reflection theorems for these new logics. The present paper confirms the truth of these conjectures. Furthermore, our constructions shed light on the question of how the universal theories of related logics are themselves related. For example, membership equational logic is itself a sublogic of rewriting logic, and this sublogic relation is expressed at the reflective level by the fact that the universal theory of membership equational logic is itself a *subtheory* of the universal theory for the more general version of rewriting logic where the underlying equational specifications are membership equational theories.

Therefore, our results make clear that reflection is available as a very powerful feature not only for this more general variant of rewriting logic, namely the one supported by Maude [4,5], but also for other computational logics of great importance in formal specification and declarative programming, such as membership equational logic, many-sorted equational logic, and Horn logic with equality. This can then serve as a basis for the theoretically-grounded design of *declarative reflective programming languages* in those logics.

Having an explicit specification of the corresponding universal theories is of great practical importance for metalogical reasoning. In joint work with David Basin [1,2] we have investigated the good properties of membership equational logic and of rewriting logic as *reflective metalogical frameworks* that combine induction, parameterization, and reflection to support metalogical reasoning about logics represented in them. In such metareasoning, as well as whenever *proof objects* are required to justify reflective proofs, it is essential to make an explicit use of the corresponding universal theories. In particular, the results in this paper have several important consequences for the Maude language, in that they both serve as a foundation for its `META-LEVEL` module, and they provide a general method for combining efficient reflective computation using the built-in functionality of the `META-LEVEL` module with the ability to generate proof objects by means of the universal theories when this is required.

The paper is organized as follows. First, in Section 2, we summarize the axioms characterizing the notion of a reflective logic. Then, in Sections 3, 4, 5, and 6, we prove, respectively, that membership equational logic, many-sorted equational logic, many-sorted Horn logic with equality, and rewriting logic are reflective in our axiomatic sense. Finally, in Section 7 we compare these results with previous work, and in Section 8 we draw conclusions.

2 Reflection in General Logics

We present below in summarized form the axiom characterizing the notion of a *reflective logic*. We introduce first the notions of *syntax* and of *entailment system*, used in our axiomatization. These notions are defined using the language of category theory, but do not require any acquaintance with categories beyond the basic notions of category and functor.

Syntax.

Syntax can typically be given by a *signature* Σ providing a grammar on which to build *sentences*. For first-order logic, a typical signature consists of a set of function symbols and a set of predicate symbols, each with a prescribed number of arguments, which are used to build up the usual sentences. We assume that for each logic there is a category **Sign** of possible signatures for it, and a functor *sen* assigning to each signature Σ the set $sen(\Sigma)$ of all its sentences. We call the pair (**Sign**, *sen*) a *syntax*.

Entailment systems.

For a given signature Σ in **Sign**, *entailment* (also called *provability*) of a sentence $\varphi \in sen(\Sigma)$ from a set of axioms $\Gamma \subseteq sen(\Sigma)$ is a relation $\Gamma \vdash_\Sigma \varphi$ which holds if and only if we can prove φ from the axioms Γ using the rules of the logic. We make this relation relative to a signature.

In what follows, $|\mathcal{C}|$ denotes the collection of objects of a category $\mathcal{C}$.

Definition 2.1 [9] An *entailment system* is a triple $\mathcal{E} = (\mathbf{Sign}, sen, \vdash)$ such that

- (**Sign**, *sen*) is a syntax,
- $\vdash$ is a function associating to each $\Sigma \in |\mathbf{Sign}|$ a binary relation $\vdash_\Sigma \subseteq \mathcal{P}(sen(\Sigma)) \times sen(\Sigma)$, called Σ-*entailment*, that satisfies the following properties:
 (i) *reflexivity*: for any $\varphi \in sen(\Sigma)$, $\{\varphi\} \vdash_\Sigma \varphi$,
 (ii) *monotonicity*: if $\Gamma \vdash_\Sigma \varphi$ and $\Gamma' \supseteq \Gamma$ then $\Gamma' \vdash_\Sigma \varphi$,
 (iii) *transitivity*: if $\Gamma \vdash_\Sigma \varphi$, for all $\varphi \in \Delta$, and $\Gamma \cup \Delta \vdash_\Sigma \psi$, then $\Gamma \vdash_\Sigma \psi$,
 (iv) $\vdash$-*translation*: if $\Gamma \vdash_\Sigma \varphi$, then for any $H : \Sigma \to \Sigma'$ in **Sign** we have $sen(H)(\Gamma) \vdash_{\Sigma'} sen(H)(\varphi)$.

Definition 2.2 [9] Given an entailment system $\mathcal{E}$, its category **Th** of *theories* has as objects pairs $T = (\Sigma, \Gamma)$ with Σ a signature and $\Gamma \subseteq sen(\Sigma)$. A *theory morphism* $H : (\Sigma, \Gamma) \to (\Sigma', \Gamma')$ is a signature morphism $H : \Sigma \to \Sigma'$ such that if $\varphi \in \Gamma$, then $\Gamma' \vdash_{\Sigma'} sen(H)(\varphi)$.

Note that we can extend the functor *sen* to a functor on theories by taking $sen(\Sigma, \Gamma) = sen(\Sigma)$.

2.1 Reflective Logics

A reflective logic is a logic in which important aspects of its metatheory can be represented at the object level in a consistent way, so that the object-level representation correctly simulates the relevant metatheoretic aspects. Two obvious metatheoretic notions that can be so reflected are theories and the entailment relation $\vdash$. This leads us to the notion of a universal theory. However, universality may not be absolute, but only relative to a class $\mathcal{C}$ of *representable* theories.

Definition 2.3 Given an entailment system $\mathcal{E}$ and a set of theories $\mathcal{C} \subseteq |\mathbf{Th}|$, a theory U is $\mathcal{C}$-*universal* if there is a function, called a *representation function*,

$$\overline{(\_ \vdash \_)} : \bigcup_{T \in \mathcal{C}} \{T\} \times sen(T) \longrightarrow sen(U)$$

such that for each $T \in \mathcal{C}$, $\varphi \in sen(T)$,

$$T \vdash \varphi \iff U \vdash \overline{T \vdash \varphi}.$$

If, in addition, $U \in \mathcal{C}$, then the entailment system $\mathcal{E}$ is called $\mathcal{C}$-*reflective*.

To take into account computability considerations, we should further require that the representation function $\overline{(\_ \vdash \_)}$ is *recursive*. Finally, to rule out unfaithful representations, we should require that the function $\overline{(\_ \vdash \_)}$ is *injective*.

3   Reflection in Membership Equational Logic

3.1 Membership Equational Logic

A *signature* in membership equational logic—in short, MEqtl—is a triple $\Omega = (K, \Sigma, S)$, with K a set of *kinds*, Σ a K-kinded signature $\Sigma = \{\Sigma_{w,k}\}_{(w,k) \in K^* \times K}$, and $S = \{S_k\}_{k \in K}$ a pairwise disjoint K-kinded family of sets. We call S_k the set of *sorts* of kind k. The pair (K, Σ) is what is usually called a many-sorted signature of function symbols; however we call the elements of K kinds because each kind k now has a set S_k of associated *sorts*, which in the models will be interpreted as subsets of the carrier for the kind. As usual, we denote by T_Σ the K-kinded algebra of ground Σ-terms, and by $T_\Sigma(X)$ the K-kinded algebra of Σ-terms on the K-kinded set of variables X.

The atomic formulae of MEqtl are either *equations* $t = t'$, where t and t' are Σ-terms of the same kind, or *membership assertions* of the form $t : s$, where the term t has kind k and $s \in S_k$. Sentences are Horn clauses on these atomic formulae, i.e., sentences of the form

$$\forall (x_1, \ldots, x_m).\, A_1 \wedge \ldots \wedge A_n \Longrightarrow A_0 \,,$$

where each A_i is either an equation or a membership assertion, and each x_j is a K-kinded variable. A theory in membership equational logic is a pair

(Ω, E), where E is a set of sentences—(conditional) equations or (conditional) membership axioms—in MEqtl over the signature Ω. To simplify the definition of the universal theory for MEqtl in Section 3.2, we will work with theories with *nonempty kinds*, that is, for each kind, the elements of that kind in the initial algebra form a nonempty set. This is a relatively minor restriction that avoids the well-known complications with quantification in many-sorted equational deduction [8].[4] Thus, from now on, we will omit the quantifiers in all sentences.

Finally, we introduce our notions and notations for contexts and substitutions. Given a signature $\Omega = (K, \Sigma, S)$, a K-kinded set of variables X, and a K-kinded set of new constants $\{\imath_k\}_{k \in K}$, a *context* is a term C^k which contains exactly one subterm $t' = \imath_k$, called its "hole," for some $k \in K$. We define $C^\imath_\Sigma(X)$ to be the set of contexts. Given a context C^k and a term $t \in T_\Sigma(X)$ of kind k, $C^k[t] \in T_\Sigma(X)$ is the term that results from replacing the "hole" $\imath_k$ in C^k by t. When not needed, we omit mentioning the kind of the "hole" in the context. Given a signature $\Omega = (K, \Sigma, S)$, and a K-kinded set of variables X, we define $\mathcal{S}_{(\Sigma, X)}$ to be the set of *substitutions*[5]

$$\mathcal{S}_{(\Sigma, X)} = \{(x_1 \mapsto w_1, \dots, x_n \mapsto w_n) \mid \forall i, j \leq n, i \neq j \Rightarrow x_i \neq x_j,$$
$$\text{and } \forall i \leq n, x_i \text{ and } w_i \text{ have the same kind}\}.$$

Given a term t and a substitution $\sigma = (x_1 \mapsto w_1, \dots, x_n \mapsto w_n)$, we denote by $t\sigma$ the term $t(w_1/x_1, \dots, w_n/x_n)$ obtained from t by simultaneously substituting w_i for x_i, $i = 1, \dots, n$.

3.1.1 The Rules of MEqtl

We now introduce the rules of deduction of MEqtl. Our formulation is slightly different from, but equivalent to and simpler for our purposes than, that in [11], in that the congruence rule is removed and is taken into account as part of the "modus ponens" rule.

Given a membership equational theory $T = (\Omega, E)$, we say that T *entails* a sentence ϕ and write $T \vdash \phi$ if and only if ϕ can be obtained by finite application of the following *rules of deduction* :

(i) **Reflexivity**. For each $t \in T_\Sigma(X)$, $\dfrac{}{T \vdash t = t}$.

(ii) **Modus ponens**. For each equation $(C_{mb} \wedge C_{eq} \Longrightarrow t = t')$ in E, with t, t' of kind k, context $C^k \in C^\imath_\Sigma(X)$, and substitution σ, where $C_{mb} \triangleq (u_1 :$

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[4] The specification and proof of correctness of a universal theory in the more general case in which some kinds can be empty should follow very similar lines. The main difference is that the universal quantifiers need to be metarepresented in sentences, and the inference rules must keep track of such quantifiers.

[5] Conceptually, a substitution is a function from variables to terms. For technical convenience, we choose to define substitutions as a special case of lists of pairs formed by variables and terms.

$s_1 \wedge \ldots \wedge u_j : s_j)$ and $C_{eq} \triangleq (v_1 = v_1' \wedge \ldots \wedge v_k = v_k')$,

$$\frac{T \vdash u_1\sigma : s_1 \quad \cdots \quad T \vdash u_j\sigma : s_j \quad T \vdash v_1\sigma = v_1'\sigma \quad \cdots \quad T \vdash v_k\sigma = v_k'\sigma}{T \vdash C[t\sigma] = C[t'\sigma]}.$$

Similarly, for each membership axiom $(C_{mb} \wedge C_{eq} \implies t : s)$ in E, and substitution σ, where C_{mb} and C_{eq} are as before,

$$\frac{T \vdash u_1\sigma : s_1 \quad \cdots \quad T \vdash u_j\sigma : s_j \quad T \vdash v_1\sigma = v_1'\sigma \quad \cdots \quad T \vdash v_k\sigma = v_k'\sigma}{T \vdash t\sigma : s}.$$

(iii) **Symmetry**. $\dfrac{T \vdash t = t'}{T \vdash t' = t}$.

(iv) **Transitivity**. $\dfrac{T \vdash t = t'' \quad T \vdash t'' = t'}{T \vdash t = t'}$.

(v) **Membership**. $\dfrac{T \vdash t = u \quad T \vdash u : s}{T \vdash t : s}$.

(vi) **Implication introduction**. For each sentence $A_1 \wedge \ldots \wedge A_n \implies A_0$ over the signature of T, where each A_i is either an equation or a membership assertion,

$$\frac{(\Omega(X), E \cup \{A_1, \ldots, A_n\}) \vdash A_0}{(\Omega, E) \vdash A_1 \wedge \ldots \wedge A_n \implies A_0},$$

where $\Omega(X)$ is the signature Ω extended with the elements of X as additional new constants.

3.2   A Universal Theory for MEqtl

In this section, we introduce the universal theory U_{MEqtl} and a representation function $\overline{(\_\vdash\_)}$ that encodes pairs consisting of a theory T and a sentence over its signature, as a sentence in U_{MEqtl}. We also sketch the proof of the universality of U_{MEqtl}, as formalized in Definition 2.3. In what follows, we will be dealing with finitely presentable theories in MEqtl.

3.2.1   The Signature of U_{MEqtl}

The signature of the theory U_{MEqtl} contains constructors to represent terms, contexts, substitutions, kinds, sorts, signatures, axioms, and theories. For brevity we only declare the subsignature of U_{MEqtl} explicitly used in the paper. We use Maude notation and write a kind enclosed in square brackets.

To represent the decomposition of a term $t = C[t']$ into a context C and a potential redex t', the signature of U_{MEqtl} includes the constructor

```
op _[_] : [Context] [Term] -> [Term] .
```

and to represent the decomposition of a term $t = t'\sigma$ into a term t' and a substitution σ, it includes the constructor [6]

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[6]   Notice that we indeed use these two operations as constructors, and not as defined

```
op __ : [Term] [Substitution] -> [Term] .
```

To represent the context consisting only of a "hole," and to represent the empty substitution, the signature of U_MEqtl includes the constructors

```
op * : -> [Context] . op - : -> [Substitution] .
```

Also, to represent (possibly conditional) equations and membership axioms, the signature of U_MEqtl includes the constructors

```
op _=_ : [Term] [Term] -> [Atom] [comm] .
op _:_ : [Term] [Sort] -> [Atom] .
op none : -> [Condition] .
op _/\_ : [Atom] [Condition] -> [Condition] .
op _if_ : [Atom] [Condition] -> [Axiom] .
```

where the constant `none` is used to represent the empty set of conditions, and the attribute `comm` indicates that matching a term built with this constructor is done *modulo* commutativity.

In addition, the signature of U_MEqtl contains a Boolean function `parse` to decide whether a term is well-formed with respect to a many-kinded signature, a function `extVar` to extract the variables occurring in a sentence, a function `addVarAsCnst` to extend a signature by adding some variables as new constants, and a function `addEq` to extend a set of axioms by adding the atomic formulae in a condition as new axioms.

```
op parse : [Term] [Signature] -> [Bool] .
op extVar : [Term] [Term] [Condition] -> [VarSet] .
op extVar : [Term] [Condition] -> [VarSet] .
op addVarAsCnst : [Signature] [VarSet] -> [Signature] .
op addEq : [AxiomSet] [Condition] -> [AxiomSet] .
```

Finally, the signature of U_MEqtl contains two Boolean operations

```
op _:_if_in_ : [Term] [Sort] [Condition] [Theory] -> [Bool] .
op _=_if_in_ : [Term] [Term] [Condition] [Theory] -> [Bool] .
```

to represent entailment of sentences in a given membership equational theory; the main axioms of U_MEqtl, including those in Figure 1, define these two operations.

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operations; that is, application of a substitution to a term is not really defined. Instead, in addition to equations dealing with the rules of deduction of MEqtl, the universal theory U_MEqtl has equations for composing/decomposing both terms appearing in an equation into different contexts and substitutions (these equations are not included here due to space limitations), so that the "modus ponens" rule can be applied. In particular, a term t can for example be represented as $*[\bar{t}-]$, where $\bar{t}$ is the metarepresentation of t itself, and $*$ and $-$ represent, respectively, the "hole" and the empty substitution as explained below. The reason for this choice is that it makes our proofs simpler.

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3.2.2 The Representation Function

We next define the representation function $\overline{(\_\vdash\_)}$. For all membership equational theories T, and sentences ϕ over the signature of T,

- if $\phi = (A_1 \wedge \ldots \wedge A_n \Longrightarrow t : s)$, where each A_i is an atomic formula, then $\overline{T \vdash \phi} \triangleq (\ast[\overline{t}\,\text{-}]:\overline{s}$ if $\overline{A_1 \wedge \ldots \wedge A_n}$ in $\overline{T}) = \texttt{true}$, and

- if $\phi = (A_1 \wedge \ldots \wedge A_n \Longrightarrow t = t')$, where each A_i is an atomic formula, then $\overline{T \vdash \phi} \triangleq (\ast[\overline{t}\,\text{-}] = \ast[\overline{t'}\,\text{-}]$ if $\overline{A_1 \wedge \ldots \wedge A_n}$ in $\overline{T}) = \texttt{true}$,

where $\overline{(\_)}$ is a representation function defined recursively over theories, signatures, sets of axioms, individual equational and membership axioms, conjunctions of atomic formulae, terms and sorts, and returns, respectively, ground terms of kind [Theory], [Signature], [AxiomSet], [Axiom], [Condition], [Term], and [Sort]. We also define a representation function $\overline{(\_)}$ over contexts and substitutions that returns, respectively, ground terms of kind [Context] and [Substitution].

3.2.3 The Axioms of U_{MEqtl}

Finally, we define the axioms of U_{MEqtl}, which include equations that correspond to the inference rules of MEqtl, along with equations to decide whether a term is well-formed with respect to a many-kinded signature, to compose/decompose a term (see footnote 6), to extract the variables occurring in a sentence, to extend a signature by adding some variables as new constants, and to extend a set of axioms by adding the atomic formulae in a condition as new axioms. Due to space limitations we only show here, in Figure 1, the equations in U_{MEqtl} that correspond to the inference rules of MEqtl,[7] and state below some essential properties satisfied by its axioms.

Proposition 3.1 For $T = (\Omega, E)$ a finitely presentable membership equational theory with nonempty kinds, with $\Omega = (K, \Sigma, S)$, for all terms $t \in T_\Sigma(X)$, it holds that

$$\text{U}_{\text{MEqtl}} \vdash \texttt{parse}(\overline{t}, \overline{\Omega}) = \texttt{true}.$$

Proposition 3.2 For $T = (\Omega, E)$ a finitely presentable membership equational theory with nonempty kinds, with $\Omega = (K, \Sigma, S)$, for all terms w in U_{MEqtl} of kind [Term], it holds that if

$$\text{U}_{\text{MEqtl}} \vdash \texttt{parse}(w, \overline{\Omega}) = \texttt{true},$$

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[7] To ease the understanding of these equations, we replace the usual variable notation by the corresponding representations of the entities to be placed in such variable positions. For example, $\overline{\Omega}$ is a normal variable, but the notation suggests that the terms that the variable will match will typically be representations of signatures. Also, in the actual theory U_{MEqtl}, the conditions of the "modus ponens" equations are formalized using a Boolean function that checks at the metalevel whether an instantiated conjunction of conditions holds in a given membership equational theory. To ease readability, in Figure 1 we have "expanded out" this Boolean function as a conjunction of checks for each of the atoms in the condition.

then there is a term $t \in T_\Sigma(X)$ such that $w = \bar{t}$.

Proposition 3.3 *For $T = (\Omega, E)$ a finitely presentable membership equational theory with nonempty kinds, with $\Omega = (K, \Sigma, S)$, for all terms $t, t', u, u' \in T_\Sigma(X)$, contexts $C \in \mathrm{C}^i_\Sigma(X)$, and substitutions $\sigma \in \mathcal{S}_{(\Sigma, X)}$, it holds that, if $t = C[u\sigma]$ and $t' = C[u'\sigma]$, then*

$$\mathrm{U}_{\mathrm{MEqtl}} \vdash (\ast[\bar{t}\text{-}] = \ast[\bar{t'}\text{-}] \text{ if none in } \overline{T}) = (\overline{C}[\overline{u}\,\overline{\sigma}] = \overline{C}[\overline{u'}\overline{\sigma}] \text{ if none in } \overline{T}).$$

Similarly, for all terms $t, u \in T_\Sigma(X)$ and substitutions $\sigma \in \mathcal{S}_{(\Sigma, X)}$, it holds that, if $t = u\sigma$ then

$$\mathrm{U}_{\mathrm{MEqtl}} \vdash (\ast[\bar{t}\text{-}]:\overline{s} \text{ if none in } \overline{T}) = (\ast[\overline{u}\,\overline{\sigma}]:\overline{s} \text{ if none in } \overline{T}).$$

3.3 The Correctness of the Universal Theory $\mathrm{U}_{\mathrm{MEqtl}}$

We sketch now the proof of the correctness of the universal theory $\mathrm{U}_{\mathrm{MEqtl}}$.

Theorem 3.4 *For all finitely presentable membership equational theories with nonempty kinds $T = (\Omega, E)$, and sentences $A_1 \wedge \ldots \wedge A_n \Longrightarrow t{:}s$ over Ω, where each A_i is an atomic formula,*

$$T \vdash A_1 \wedge \ldots \wedge A_n \Longrightarrow t{:}s$$
$$\Longleftrightarrow \mathrm{U}_{\mathrm{MEqtl}} \vdash (\ast[\bar{t}\text{-}]:\overline{s} \text{ if } \overline{A_1 \wedge \ldots \wedge A_n} \text{ in } \overline{T}) = \texttt{true}.$$

Similarly, for all sentences $A_1 \wedge \ldots \wedge A_n \Longrightarrow t = t'$ over Ω, where each A_i is an atomic formula,

$$T \vdash A_1 \wedge \ldots \wedge A_n \Longrightarrow t = t'$$
$$\Longleftrightarrow \mathrm{U}_{\mathrm{MEqtl}} \vdash (\ast[\bar{t}\text{-}] = \ast[\bar{t'}\text{-}] \text{ if } \overline{A_1 \wedge \ldots \wedge A_n} \text{ in } \overline{T}) = \texttt{true}.$$

The $(\Rightarrow)$-direction of this theorem is proved by structural induction on MEqtl proofs, using Propositions 3.1 and 3.3. Examples of similar proofs are given in detail in [3,7]. The $(\Leftarrow)$-direction of the theorem is proved as a corollary of the following

Lemma 3.5 *For all finitely presentable membership equational theories with nonempty kinds $T = (\Omega, E)$, with $\Omega = (K, \Sigma, S)$, terms t in $T_\Sigma(X)$, and sorts s in some S_k,*

$$T \vdash t{:}s \Longleftarrow \mathrm{U}_{\mathrm{MEqtl}} \vdash (\ast[\bar{t}\text{-}]:\overline{s} \text{ if none in } \overline{T}) = \texttt{true}.$$

Similarly, for all terms t, t' in $T_\Sigma(X)$,

$$T \vdash t = t' \Longleftarrow \mathrm{U}_{\mathrm{MEqtl}} \vdash (\ast[\bar{t}\text{-}] = \ast[\bar{t'}\text{-}] \text{ if none in } \overline{T}) = \texttt{true}.$$

Proof. Notice, first, that a proof in $\mathrm{U}_{\mathrm{MEqtl}}$ of an equality of the form

$$(\ast[\bar{t}\text{-}]:\overline{s} \text{ if none in } \overline{T}) = \texttt{true} \text{ or } (\ast[\bar{t}\text{-}] = \ast[\bar{t'}\text{-}] \text{ if none in } \overline{T}) = \texttt{true}$$

$$\tag{1}$$

```
*** reflexivity
eq (*[t̄ -] = *[t̄ -] if none in (Ω̄, Ē)) = true.
*** modus ponens
```
*** where $E = \{t = t' \ if \ C_{mb} \wedge C_{eq}\} \cup E'$, with
*** $C_{mb} = (u_1 : s_1 \wedge \ldots \wedge u_j : s_j)$ and $C_{eq} = (v_1 = v'_1 \wedge \ldots \wedge v_k = v'_k)$.
```
ceq (C̄[t̄σ̄] = C̄[t̄'σ̄] if none in (Ω̄, Ē)) = true
    if (*[ū₁σ̄]:s̄₁ if none in (Ω̄, Ē)) = true
    ⋮

    /\ (*[ūⱼσ̄]:s̄ⱼ if none in (Ω̄, Ē)) = true
    /\ (*[v̄₁σ̄] = *[v̄'₁σ̄] if none in (Ω̄, Ē)) = true
    ⋮

    /\ (*[v̄ₖσ̄] = *[v̄'ₖσ̄] if none in (Ω̄, Ē)) = true .
*** modus ponens
```
*** where $E = \{t : s \ if \ C_{mb} \wedge C_{eq}\} \cup E'$, with
*** $C_{mb} = (u_1 : s_1 \wedge \ldots \wedge u_j : s_j)$ and $C_{eq} = (v_1 = v'_1 \wedge \ldots \wedge v_k = v'_k)$.
```
ceq (*[t̄σ̄]:s̄ if none in (Ω̄, Ē)) = true
    if (*[ū₁σ̄]:s̄₁ if none in (Ω̄, Ē)) = true
    ⋮

    /\ (*[ūⱼσ̄]:s̄ⱼ if none in (Ω̄, Ē)) = true
    /\ (*[v̄₁σ̄] = *[v̄'₁σ̄] if none in (Ω̄, Ē)) = true
    ⋮

    /\ (*[v̄ₖσ̄] = *[v̄'ₖσ̄] if none in (Ω̄, Ē)) = true .
*** symmetry
ceq (*[t̄ -] = *[t̄' -] if none in (Ω̄, Ē)) = true
    if (*[t̄' -] = *[t̄ -] if none in (Ω̄, Ē)) = true .
*** transitivity
ceq (*[t̄ -] = *[t̄' -] if none in (Ω̄, Ē)) = true
    if parse(t̄'', Ω̄) = true
    /\ (*[t̄ -] = *[t̄'' -] if none in (Ω̄, Ē)) = true
    /\ (*[t̄'' -] = *[t̄' -] if none in (Ω̄, Ē)) = true .
*** membership
ceq (*[t̄ -]:s̄ if none in (Ω̄, Ē)) = true
    if parse(ū, Ω̄) = true
    /\ (*[t̄ -] = *[ū -] if none in (Ω̄, Ē)) = true
    /\ (*[ū -]:s̄ if none in (Ω̄, Ē)) = true .
*** implication introduction
```
*** where $Cond = A_1 \wedge \ldots \wedge A_n$, with each A_i an atomic formula.
```
ceq (*[t̄ -] = *[t̄' -] if C̄ōn̄d̄ in (Ω̄, Ē)) = true
    if (*[t̄ -] = *[t̄' -] if none in
       (addVarAsCnst(Ω̄, extVar(t̄, t̄', C̄ōn̄d̄)), addEq(Ē, C̄ōn̄d̄))) = true .
ceq (*[t̄ -]:s̄ if C̄ōn̄d̄ in (Ω̄, Ē)) = true
    if (*[t̄ -]:s̄ if none in
       (addVarAsCnst(Ω̄, extVar(t̄, C̄ōn̄d̄)), addEq(Ē, C̄ōn̄d̄))) = true .
```

Fig. 1. The universal theory U_{MEqtl} (fragment).

must consist of an application (possibly after a number of applications of the equations for decomposing/composing) of the "modus ponens" rule of inference using the equations in Figure 1. Notice also that: i) all equations but one in this figure are conditional; ii) their conditions include equalities that are either of the form (1), or can be reduced to that form by Proposition 3.3; and iii) these equalities must be proved before the conditional equations can be used in an inference. Thus, each proof in U_{MEqtl} of an equality of the form (1) will have a certain depth, which we denote as the *conditional depth* of the proof [7]. The lemma is proved by induction on this conditional depth. Examples of similar proofs are given in detail in [7]. □

As a corollary of the reflective result proved in this section, we will show in the next two sections the reflective nature of two other related logics: many-sorted equational logic and Horn logic with equality.

4 Reflection in Many-Sorted Equational Logic

Many-sorted equational logic—in short, MSEqtl—is a sublogic of MEqtl, namely the sublogic obtained by making the set of sorts empty [11]; in particular, for all theories T in MSEqtl, and sentences ϕ over the signature of T, it holds that $T \vdash_{\mathrm{MSEqtl}} \phi \iff T \vdash_{\mathrm{MEqtl}} \phi$. But then, since we have only used kinds and conditional equations not involving any memberships in the definition of U_{MEqtl}, we have that U_{MEqtl} is a theory in MSEqtl, and therefore the following

Theorem 4.1 U_{MEqtl} *is a universal theory in* MSEqtl *for the class of finitely presentable theories having nonempty sorts.*

Proof. For all finitely presentable theories T in MSEqtl having nonempty sorts, and sentences ϕ over the signature of T, since, by definition, $\overline{T \vdash \phi}$ is a sentence in MSEqtl,

$$T \vdash_{\mathrm{MSEqtl}} \phi \iff T \vdash_{\mathrm{MEqtl}} \phi$$
$$\iff U_{\mathrm{MEqtl}} \vdash_{\mathrm{MEqtl}} \overline{T \vdash \phi} \iff U_{\mathrm{MEqtl}} \vdash_{\mathrm{MSEqtl}} \overline{T \vdash \phi}.$$

□

5 Reflection in Horn Logic with Equality

In [11] it is shown that MEqtl is equivalent to many-sorted Horn logic with equality—in short, $\mathrm{MSHorn}^{\overline{=}}$. It is not surprising then that the reflective results about MEqtl can be translated straightforwardly to $\mathrm{MSHorn}^{\overline{=}}$.

A signature in $\mathrm{MSHorn}^{\overline{=}}$ is a triple (L, Σ, Π), with L a set of sorts, $\Sigma = \{\Sigma_{w,l}\}_{(w,l) \in L^* \times L}$ a family of sets of function symbols, and $\Pi = \{\Pi_w\}_{w \in L^*}$ a family of sets of predicate symbols. A signature $\Omega = (K, \Sigma, S)$ in MEqtl can then be mapped to a signature $\Omega^\star = (K, \Sigma, S^\star)$ in $\mathrm{MSHorn}^{\overline{=}}$ by taking $S_k^\star = S_k$ for $k \in K$, and $S_w^\star = \emptyset$ for any $w \in K^* \setminus K$. Thus, if we adopt

a postfix notation $\_:s$ for each predicate in $\Omega^\star$, corresponding to a sort s in Ω, each sentence over Ω in MEqtl can be seen as a sentence over $\Omega^\star$ in $\mathrm{MSHorn}^{\bar{\bar{n}}}$. We will write $(\Omega, E)^\star$ for $(\Omega^\star, E)$. Then, for all sentences ϕ over Ω it holds that $(\Omega, E) \vdash_{\mathrm{MEqtl}} \phi \iff (\Omega, E)^\star \vdash_{\mathrm{MSHorn}^{\bar{\bar{n}}}} \phi$. Moreover, [11] defines a translation α mapping theories and sentences in $\mathrm{MSHorn}^{\bar{\bar{n}}}$ into MEqtl, satisfying $T \vdash_{\mathrm{MSHorn}^{\bar{\bar{n}}}} \phi \iff \alpha(T) \vdash_{\mathrm{MEqtl}} \alpha(\phi)$. But then, we have the following

Theorem 5.1 $\mathrm{U}^\star_{\mathrm{MEqtl}}$ *is a universal theory in* $\mathrm{MSHorn}^{\bar{\bar{n}}}$ *for the class of all finitely presentable theories with nonempty sorts.*

Proof. For all finitely presentable theories with nonempty sorts T in $\mathrm{MSHorn}^{\bar{\bar{n}}}$, and sentences ϕ over T,

$$
\begin{aligned}
T \vdash_{\mathrm{MSHorn}^{\bar{\bar{n}}}} \phi &\iff \alpha(T) \vdash_{\mathrm{MEqtl}} \alpha(\phi) \\
&\iff \mathrm{U}_{\mathrm{MEqtl}} \vdash_{\mathrm{MEqtl}} \overline{\alpha(T) \vdash_{\mathrm{MEqtl}} \alpha(\phi)} \\
&\iff \mathrm{U}^\star_{\mathrm{MEqtl}} \vdash_{\mathrm{MSHorn}^{\bar{\bar{n}}}} \overline{\alpha(T) \vdash_{\mathrm{MEqtl}} \alpha(\phi)}
\end{aligned}
$$

$\square$

6 Reflection in Rewriting Logic

6.1 Rewriting Logic

Rewriting logic—in short, Rwl—is parameterized with respect to an underlying equational logic; here we use MEqtl as this underlying logic. Given a MEqtl signature $\Omega = (K, \Sigma, S)$, the *sentences* of rewriting logic are "sequents" of the form $t \longrightarrow t'$, where t and t' are Ω-terms of the same kind possibly involving some variables from a K-kinded set of variables X.

A *rewrite theory* T is a 3-tuple $T = (\Omega, E, R)$ where (Ω, E) is a MEqtl theory, and R is a set of *(conditional) rewrite rules* of the form

$$
t \longrightarrow t' \ if \ \bigwedge_j (u_j : s_j) \wedge \bigwedge_k (v_k = v'_k) \wedge \bigwedge_h (w_h \longrightarrow w'_h),
$$

where t, t' are Σ-terms of the same kind, each u_j is a Σ-term of the kind of the sort s_j, v_k, v'_k are Σ-terms of the same kind, and w_h, w'_h are also terms of the same kind. As before, we assume that the underlying MEqtl theory has nonempty kinds.

6.1.1 The Rules of Rwl
We now introduce the rules of deduction of Rwl. Our formulation is slightly different from, but generalizes, that in [10], in that we give an explicit rule of E-equality for rewrite sequents $t \longrightarrow t'$, instead of absorbing such a rule in sequents $[t] \longrightarrow [t']$ between E-equivalence classes. In addition, our underlying equational logic is now MEqtl.

Given a rewrite theory $T = (\Omega, E, R)$, we say that T *entails* a sequent $t \longrightarrow t'$ and write $T \ \vdash \ t \longrightarrow t'$ if and only if $t \longrightarrow t'$ can be obtained by finite application of the following *rules of deduction* :

(i) **Reflexivity**. For each $t \in T_\Sigma(X)$, $\dfrac{}{T \vdash t \longrightarrow t}$.

(ii) **Replacement**. For each rewrite rule $(t \longrightarrow t' \ if \ C_{mb} \wedge C_{eq} \wedge C_{rl})$ in R, with t, t' of kind k, context $C^k \in C^k_\Sigma(X)$, and substitution σ, where $C_{mb} \triangleq (u_1 : s_1 \wedge \ldots \wedge u_j : s_j)$, $C_{eq} \triangleq (v_1 = v'_1 \wedge \ldots \wedge v_k = v'_k)$, and $C_{rl} \triangleq (w_1 \longrightarrow w'_1 \wedge \ldots \wedge w_h \longrightarrow w'_h)$,

$$\frac{\begin{array}{ccc} (\Omega, E) \vdash u_1\sigma : s_1 & \cdots & (\Omega, E) \vdash u_j\sigma : s_j \\ (\Omega, E) \vdash v_1\sigma = v'_1\sigma & \cdots & (\Omega, E) \vdash v_k\sigma = v'_k\sigma \\ T \vdash w_1\sigma \longrightarrow w'_1\sigma & \cdots & T \vdash w_h\sigma \longrightarrow w'_h\sigma \end{array}}{T \vdash C[t\sigma] \longrightarrow C[t'\sigma]}.$$

(iii) **Transitivity**. $\dfrac{T \vdash t \longrightarrow t' \qquad T \vdash t' \longrightarrow t''}{T \vdash t \longrightarrow t''}$.

(iv) **Equality**. $\dfrac{(\Omega, E) \vdash t = u \qquad (\Omega, E) \vdash t' = u' \qquad T \vdash t \longrightarrow t'}{T \vdash u \longrightarrow u'}$.

6.2 A Universal Theory for Rwl

Here, we introduce the universal theory U_{Rwl} and a representation function $\overline{(\_ \vdash \_)}$ that encodes pairs consisting of a theory T and a sentence over its signature, as a sentence in U_{Rwl}. We also sketch the proof of the universality of U_{Rwl}. The key observation is that U_{Rwl} is an extension of U_{MEqtl}, so that we can use the universality of U_{MEqtl} in the proof of the universality of U_{Rwl}. In what follows, we will be dealing with finitely presentable theories in Rwl.

6.2.1 The Signature of U_{Rwl}

The signature of the theory U_{Rwl} is an extension of the signature of U_{MEqtl}. To represent (possibly conditional) rules, the signature of U_{Rwl} includes the constructor:

```
op _=>_if_ : [Term] [Term] [RuleCondition] -> [Rule] .
```

In addition, the signature of U_{Rwl} contains a Boolean function

```
op _=>_in_ : [Term] [Term] [Theory] -> [Bool] .
```

to represent provability of sentences in a given rewrite theory; the main axioms of U_{Rwl} define this operation.

6.2.2 The Representation Function

We next define the representation function $\overline{(\_ \vdash \_)}$. For all finitely presentable rewrite theories with nonempty kinds T, and sentences $t \longrightarrow t'$ over the sig-

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nature of T,

$$\overline{T \vdash t \longrightarrow t'} \triangleq (\ast[\overline{t}\,\text{-}] \Rightarrow \ast[\overline{t}'\,\text{-}] \text{ in } \overline{T}) \longrightarrow \texttt{true}.$$

where $\overline{(\_)}$ is an extension of the representation function defined for U_{MEqtl}.

6.2.3 The Axioms of U_{Rwl}

Finally, we define the axioms of U_{Rwl}, which include rules that correspond to the inference rules of Rwl, along with equations to compose/decompose a term and all the equations in U_{MEqtl}. Due to space limitations, we only show here, in Figure 2, the rules in U_{Rwl} that correspond to the inference rules of Rwl; the same remarks as in footnote 7 apply to our notation for these rules. As with U_{MEqtl}, we have the following

Proposition 6.1 *For $T = (\Omega, E, R)$ a finitely presentable rewrite theory with nonempty kinds, with $\Omega = (K, \Sigma, S)$, for all terms $t, t', u, u' \in T_\Sigma(X)$, contexts $C \in \text{C}^{\iota}_\Sigma(X)$, and substitutions $\sigma \in \mathcal{S}_{(\Sigma,X)}$, it holds that, if $t = C[u\sigma]$ and $t' = C[u'\sigma]$, then*

$$\text{U}_{\text{Rwl}} \vdash \ (\ast[\overline{t}\,\text{-}] \Rightarrow \ast[\overline{t}'\,\text{-}] \text{ in } \overline{T}) \longrightarrow (\overline{C}[\overline{u}\,\overline{\sigma}] \Rightarrow \overline{C}[\overline{u}'\overline{\sigma}] \text{ in } \overline{T}).$$

6.3 The Correctness of the Universal Theory U_{MEqtl}

We sketch now the proof of the correctness of the universal theory U_{Rwl}.

Theorem 6.2 *For all finitely presentable rewrite theories with nonempty kinds $T = (\Omega, E, R)$, with $\Omega = (K, \Sigma, S)$, and terms t, t' in $T_\Sigma(X)$*

$$T \vdash t \longrightarrow t' \iff \text{U}_{\text{Rwl}} \vdash (\ast[\overline{t}\,\text{-}] \Rightarrow \ast[\overline{t}'\,\text{-}] \text{ in } \overline{T}) \longrightarrow \texttt{true}.$$

Proof. The proof is analogous to that of Theorem 3.4, using in key steps of the proof the fact that U_{Rwl} is an extension of U_{MEqtl}. $\qquad\square$

7 Comparison with Previous Results

The work discussed in this paper generalizes and extends our previous work on reflection in rewriting logic [3,7,12]. The results presented here *generalize* our previous results on reflection in rewriting logic to its more general variant, namely, to the case of conditional rewrite theories whose underlying equational specifications are theories in membership equational logic. To simplify the correctness proof of the universal theory, we have, however, adopted a different approach for defining the universal theory. Essentially, in [3,7], an entailment of the form $T \vdash t \longrightarrow t'$ was reflected as $U \vdash \langle \overline{T}, \overline{t} \rangle \longrightarrow \langle \overline{T}, \overline{t}' \rangle$. Accordingly, the "transitivity" rule of deduction did not have to be explicitly reified in the universal theory, since, if $T \vdash t_1 \longrightarrow t_3$ was proved by transitivity from $T \vdash t_1 \longrightarrow t_2$ and $T \vdash t_2 \longrightarrow t_3$, then, of course, $U \vdash \langle \overline{T}, \overline{t}_1 \rangle \longrightarrow \langle \overline{T}, \overline{t}_3 \rangle$,

```
including U_MEqtl .
*** reflexivity
rl (*[t̄-] => *[t̄-] in (Ω̄, Ē, R̄)) => true.
*** replacement
```

*** where $R = \{t \longrightarrow t'\ if\ C_{mb} \wedge C_{eq} \wedge C_{rl}\} \cup R'$, with

*** $\quad\quad C_{mb} = (u_1 : s_1 \wedge \ldots \wedge u_j : s_j)$,

*** $\quad\quad C_{eq} = (v_1 = v'_1 \wedge \ldots \wedge v_k = v'_k)$, and

*** $\quad\quad C_{rl} = (w_1 \longrightarrow w'_1 \wedge \ldots \wedge w_h \longrightarrow w'_h)$

```
crl (C̄[t̄σ̄] => C̄[t̄'σ̄] in (Ω̄, Ē, R̄)) => true
    if (*[ū₁σ̄]:s̄₁ in (Ω̄, Ē)) = true
    ⋮
    /\ (*[ūⱼσ̄]:s̄ⱼ in (Ω̄, Ē)) = true
    /\ (*[v̄₁σ̄] = *[v̄'₁σ̄] in (Ω̄, Ē)) = true
    ⋮
    /\ (*[v̄ₖσ̄] = *[v̄'ₖσ̄] in (Ω̄, Ē)) = true
    /\ (*[w̄₁σ̄] => *[w̄'₁σ̄] in (Ω̄, Ē, R̄)) => true
    ⋮
    /\ (*[w̄ₕσ̄] => *[w̄'ₕσ̄] in (Ω̄, Ē, R̄)) => true .
*** transitivity
crl (*[t̄-] => *[t̄'-] in (Ω̄, Ē, R̄)) => true
    if parse(t̄'', Ω̄) = true
    /\ (*[t̄-] => *[t̄''-] in (Ω̄, Ē, R̄)) => true
    /\ (*[t̄''-] => *[t̄'-] in (Ω̄, Ē, R̄)) => true .
*** equality
crl (*[ū-] => *[ū'-] in (Ω̄, Ē, R̄)) => true
    if parse(t̄, Ω̄) = true
    /\ parse(t̄', Ω̄) = true
    /\ (*[t̄-] = *[ū-] in (Ω̄, Ē)) = true
    /\ (*[t̄'-] = *[ū'-] in (Ω̄, Ē)) = true
    /\ (*[t̄-] => *[t̄'-] in (Ω̄, Ē, R̄)) => true .
```

Fig. 2. The universal theory U_{Rwl} (fragment).

would also be proved by transitivity from $U \vdash \langle \overline{T}, \bar{t}_1 \rangle \longrightarrow \langle \overline{T}, \bar{t}_2 \rangle$ and $U \vdash \langle \overline{T}, \bar{t}_2 \rangle \longrightarrow \langle \overline{T}, \bar{t}_3 \rangle$. In our current approach, however, an entailment of the form $T \vdash t \longrightarrow t'$ is reflected as $U \vdash (\bar{t} \Rightarrow \bar{t}'\ in\ \overline{T}) \longrightarrow true$, and the "transitivity" rule (and also the "symmetry" rule in the case of membership equational logic) has to be explicitly reflected in the universal theory. [8]

In addition, the results presented here *extend* in a natural way our previous results on reflection in rewriting logic to other related logics, namely, membership equational logic, many-sorted equational logic, and Horn logic with equality. The extensions are very natural, in the sense that the proposed

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[8] Since we have only given here sketches of the proofs of correctness for the different universal theories, we leave for another occasion a comparison between both approaches, showing the advantages of the current approach.

universal theories are themselves related.

8   Conclusion

We have shown that the generalized variant of rewriting logic where the underlying equational specifications are membership equational theories, and where the rules are conditional and can have equations, memberships and rewrites in the conditions is reflective. We have also shown that membership equational logic, many sorted equational logic, and Horn logic with equality are likewise reflective. These results provide logical foundations for reflective languages and tools based on these logics, and in particular for the Maude language itself. The results presented here can be further developed and generalized in several directions, including:

- giving proofs of reflection for other more restrictive but frequently used logics, such as Horn logic without equality;

- further extending the rewriting logic results to theories where some of the operators are *frozen*, so that no rewriting is allowed under them, and to theories where some kinds can be empty;

- developing adequate strategies to execute the universal theories of rewriting logic and of membership equational logic in Maude, so that proof objects can be associated to reflective proofs when desired.

This work is one step further within a broader effort, whose first results appeared in [6], to develop a general theory of reflection for logics and declarative languages. In this regard, the results presented in this paper raise the issue of how the universal theories of related logics are themselves related. We expect that the metalogical foundations provided by the theory of general logics [9], which are at the base of our axiomatic approach to the study of reflection, will provide the concepts needed to address in a precise and formalism-independent way this issue.

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References

[1] D. Basin, M. Clavel, and J. Meseguer. Rewriting logic as a metalogical framework. In S. Kapoor and S. Prasad, editors, *Twentieth Conference on the Foundations of Software Technology and Theoretical Computer Science, New Delhi, India, December 13–15, 2000, Proceedings*, volume 1974 of *Lecture Notes in Computer Science*, pages 55–80. Springer-Verlag, 2000.

[2] D. Basin, M. Clavel, and J. Meseguer. Reflective metalogical frameworks. Submitted for publication, 2002.

[3] M. Clavel. *Reflection in Rewriting Logic: Metalogical Foundations and Metaprogramming Applications*. CSLI Publications, 2000.

[4] M. Clavel, F. Durán, S. Eker, P. Lincoln, N. Martí-Oliet, J. Meseguer, and J. F. Quesada. Maude: Specification and programming in rewriting logic. Manual distributed as documentation of the Maude system, Computer Science Laboratory, SRI International. `http://maude.csl.sri.com/manual`, Jan. 1999.

[5] M. Clavel, F. Durán, S. Eker, P. Lincoln, N. Martí-Oliet, J. Meseguer, and J. F. Quesada. Towards Maude 2.0. In K. Futatsugi, editor, *Proceedings Third International Workshop on Rewriting Logic and its Applications, WRLA 2000, Kanazawa, Japan, September 18–20, 2000*, volume 36 of *Electronic Notes in Theoretical Computer Science*, pages 297–318. Elsevier, 2000. `http://www.elsevier.nl/locate/entcs/volume36.html`.

[6] M. Clavel and J. Meseguer. Axiomatizing reflective logics and languages. In G. Kiczales, editor, *Proceedings of Reflection'96, San Francisco, California, April 1996*, pages 263–288, 1996.

[7] M. Clavel and J. Meseguer. Reflection in conditional rewriting logic. *Theoretical Computer Science*, 285(2), Aug. 2002.

[8] J. Goguen and J. Meseguer. Completeness of many-sorted equational logic. *Houston Journal of Mathematics*, 11(3):307–334, 1985.

[9] J. Meseguer. General logics. In H.-D. Ebbinghaus et al., editor, *Logic Colloquium'87*, pages 275–329. North-Holland, 1989.

[10] J. Meseguer. Conditional rewriting logic as a unified model of concurrency. *Theoretical Computer Science*, 96(1):73–155, 1992.

[11] J. Meseguer. Membership algebra as a logical framework for equational specification. In F. Parisi-Presicce, editor, *Recent Trends in Algebraic Development Techniques, 12th International Workshop, WADT'97, Tarquinia, Italy, June 3–7, 1997, Selected Papers*, volume 1376 of *Lecture Notes in Computer Science*, pages 18–61. Springer-Verlag, 1998.

[12] M. Palomino Tarjuelo. Comparing Meseguer's rewriting logic with the logic CRWL. In M. Hanus, editor, *Proceedings WFLP 2001, International Workshop on Functional and (Constraint) Logic Programming*, volume 64 of *Electronic Notes in Theoretical Computer Science*. Elsevier, 2001.