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Enriched Categories and Quasi-uniform Spaces

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Abstract

The metric space representation in terms of enriched categories of [14] is extended to quasi-uniform spaces and uniformly continuous maps. The bicompletion of the quasi-uniform spaces [6] is related to the Cauchy-completion. On one hand *abstract quasi-uniformities* and uniformly continuous maps are defined. They generalize quasi-uniform spaces and their morphisms and form a locally preordered 2-category $AQUnif$. On the other hand a 2-category $Enr(H)$ of enrichments over different bases is defined. H stands here for a parameter 2-functor with domain a locally filtered 2-category and codomain the 2-category with objects monoidal partial orders and arrows the so-called *super* monoidal functors. Also the bases considered are partial orders (or "quantales"). In $Enr(H)$ the Cauchy-complete objects defines a fully reflective sub-2-category. Eventually we show a 2-equivalence $AQUnif \cong Enr(H)$ for a particular H. Thus one gets a completion for abstract quasi-uniformities, that reduces to the bicompletion for quasi-uniformities.

Keywords: enriched categories, quasi uniform spaces

1 Introduction

In [14] it is shown that enrichments over a particular monoidal closed category, say $[0, +\infty]$, may capture the notion of generalized metric spaces (also called pseudo quasi-metrics). The Cauchy-completion of $\mathcal{V}$-enrichments for a general monoidal closed $\mathcal{V}$ is defined. It reduces to the metric space completion when $\mathcal{V} = [0, +\infty]$. Also a general theory of (co)completions for enrichments is de-

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veloped in [9]. It is possible to parameterize the various completions according to classes of chosen colimits.

Since these fundamental works, the notion of completion of enrichments has been successfully applied to various areas of mathematics and computer science. Let us cite as a non-exhaustive lists of applications [23] for sheaves, [1] for fibrations, [22] for domain theory, [8], [2] for domain theory and metric spaces, also more recently [12] for metric spaces, [15] for ultra-metric spaces. Eventually we refer to [7] for many other mathematical examples of enrichments over quantales.

Nevertheless in some cases it seems necessary to develop further the theory of enrichments to render it more applicable. Let us mention for example the work on the change of base bicategories [10], that yielded the two-sided enrichments as bicategory morphisms. The original motivation for the change of base was to encode geometric morphisms using Walters' representation of sheaves [13].

Extending the original idea from [14], this paper treats quasi-uniformities and uniformly continuous maps. Its aim is to clarify the connection between quasi-uniform spaces together with uniformly continuous maps and enriched categories together with enriched functors. Again it will appear that the change of base techniques are required to capture the notion of uniformly continuous maps.

Quasi-uniformities arise from uniformities, as quasi-pseudo-metrics from pseudo metrics, by dropping the symmetry condition. These spaces were first studied in relation with topological groups (see [11] for a recent survey). They have also become a center of interest for computer science since the pioneering works of Smyth and Sunderhauf [16], [17], [18], [19], [20], [21]. From a technical point of view, a huge challenge is to define a correct theory of non symmetric space, that is, together with a well defined notion of non symmetric convergence. From this perspective, there is a hope that the analogy with enriched categories - which are essentially non-symmetric objects - should be fruitful. In this spirit, the connection established in the present paper enables the transposition of the categorical Cauchy-completion to the topological bicompletion.

Starting with the idea of [14] that enrichments are general metric spaces, and considering quasi-uniform spaces, a few remarks are in order. One may observe the following two points:

- The $[0, +\infty]$-functors correspond to non-increasing maps that are less general than the traditional uniformly continuous maps.
- Quasi-uniform spaces are more general than the metric ones and also admit a completion à la Cauchy, the so-called bicompletion [6]. After a little

thought, it seems reasonable that a quasi-uniform space (X,U) occurs as a $\mathcal{V}$-enrichment where the base category $\mathcal{V}$ is related to the quasi-uniformity U (indeed this is the case, as shown in this paper). But then it seems unrealistic to code the whole category of quasi-uniform spaces as a category of enrichments over a single a base category.

Therefore we investigated a theory of enrichments *over different bases* with in mind that:

($pb1$) Such a theory should capture the notion of quasi-uniform spaces with the right morphisms: the uniformly continuous maps;

($pb2$) There should exist a Cauchy-completion for enrichments over different bases that should capture the quasi-uniform space bicompletion.

It is fair to cite at this stage the recent work [3] that treated nicely ($pb1$). Nevertheless it is not obvious (at the moment for the author) that the general framework developed in the later paper may describe naturally the Cauchy completion. On that point our works may actually differ.

To present now an overview of the results obtained, it seems necessary to point out in an informal way the problems occurring with Lawvere's approach for quasi-uniform spaces. This should also provide some intuition for later technical developments.

Let us recall briefly some results of [14] regarding the (general) metric spaces.

$[0,+\infty]$ is the monoidal category with:

- objects: positive reals and $+\infty$;
- arrows: the reverse ordering, $x \to y$ if and only if $x \geq y$;
- tensor: the addition (with $+\infty + x = x + +\infty = +\infty$);
- unit: 0.

$[0,+\infty]$ is obviously symmetric. It is also closed as for any pair x, y of objects in $[0,+\infty]$, the exponential object $[x,y]$ is $max\{y-x, 0\}$.

A simple translation of the categorical definitions shows that $[0,+\infty]$-categories and $[0,+\infty]$-functors correspond respectively to the so-called general metric spaces and the non-increasing maps.

A *general metric space* or *gms* consists in a set of *objects* or *elements* denoted $Obj(A)$ or sometimes just A, together with a map $A(-,-) : Obj(A) \times Obj(A) \to [0,+\infty]$ that satisfies:

- for all $x, y, z \in A$, $A(y,z) + A(x,y) \geq A(x,z)$;
- for all $x \in A$, $A(x,x) = 0$.

For two gms a *map* $F : A \to B$ is non-increasing when

$$(*) \text{ for all } x, y \in A,\ A(x, y) \geq B(F(x), F(y)).$$

Considering $[0, +\infty]$-natural transformations one obtains then a preorder on non-increasing maps by $F \Rightarrow G : A \to B$ if and only if for all $x \in A$, $B(F(x), G(x)) = 0$. This preorder makes the category of general metric spaces and non-increasing maps, a locally preordered 2-category. Actually the sets of non-increasing maps between two general metric spaces inherits the structure of a general metric space.

$[0, +\infty]$-modules have a simple form. A $[0, +\infty]$-module $M : A \multimap B$ is a map $B \times A \to [0, +\infty]$ such that:

- for all $x, x' \in A$ and $y \in B$, $A(x, x') + M(y, x) \geq M(y, x')$;
- for all $x \in A$ and $y, y' \in B$, $M(y, x) + B(y', y) \geq M(y', x)$.

They compose as follows. Given $A \overset{M}{\multimap} B \overset{N}{\multimap} C$, the composite $N * M$ is defined for any $a \in A$ and $c \in C$ by $N * M(c, a) = \bigwedge_{b \in B} M(b, a) + N(c, b)$.

One has a 2-category $[0, +\infty]$-Mod of $[0, +\infty]$-modules and thus a formal notion of adjoints in this 2-category. It was observed in [14] that minimal Cauchy filters on a g.m.s. A are in one-to-one correspondence with left adjoint modules of the form $1 \overset{M}{\multimap} A$. This is crucial to establish that the full subcategory of $[A^{op}, [0, +\infty]]$ of left adjoint modules is, in term of general metric spaces, the Cauchy-completion of A.

Of course one would like to adopt this categorical framework to treat quasi-uniform spaces and their bicompletions. Nevertheless there are a number of technical problems that we shall enumerate now.

The first point is to describe quasi-uniform spaces as enrichments. According to the coding below for metric spaces, our intuition would tell us that a quasi-uniform space (X, U) defines an enrichment with objects the $x, y... \in X$ and homs given by:

$$(**) \quad X(x, y) = \{u \in U \mid (x, y) \in u\}.$$

Actually this is almost true but using this straightforward approach, one fails to define a monoidal category!

The second point is that there could not be one fixed base category for all quasi-uniform spaces. As one may suppose, there is description of the space (X, U) as a $\mathcal{V}$-category in the spirit of $(**)$ above. But here the base $\mathcal{V}$ would clearly depend from the quasi-uniformity U. Therefore one needs to consider different bases to code quasi-uniform spaces as enrichments.

A third point is to get a categorical characterization of the quasi-uniform maps. Until now one only knows such one for non-increasing maps. Let us consider quasi-uniform spaces (X, U) and (Y, T). Any map $f : X \to Y$ defines a relation $R \subseteq U \times T$ by $R(u, t)$ if and only if for all $x, y \in X$, $(x, y) \in u \Rightarrow (f(x), f(y)) \in t$. Then one has

$$(1)\ \ \exists_R(X(x, y)) \subseteq Y(fx, fy), \ \ \textit{for all } x, y \in X,$$

where $\exists_R$ denotes the direct image of R. Note that the f above is uniformly continuous exactly when for all $t \in T$, there exists $u \in U$ such that for all x, y, $(x, y) \in u \Rightarrow (f(x), f(y)) \in t$ that is just when

$$(2)\ \ \exists_R(U) = T.$$

The inequality (1) may be seen as a generalization of the condition $(*)$ defining non-increasing maps. The idea is then to characterize the uniformly continuous maps f between (X, U) and (Y, T) as the ones with an adequate change of base — i.e. some $R \subseteq U \times T$ satisfying (1) — and such that this change of base satisfies moreover (2).

So far we have only defined the changes of base in terms of uniformities. Actually we shall show that they may be defined from particular monoidal functors, the so-called *super* ones. Then finally, uniformly continuous maps $(X, U) \to (Y, T)$ are maps $f : X \to Y$ defining a $\mathcal{V}_T$-functor $F_{@}(X) \to Y$ where $F : \mathcal{V}_U \to \mathcal{V}_T$ is a suitable monoidal functor with induced change of base $F_{@}$.

An unpleasant feature of this coding is that the bases depend from the set of objects of the enrichments. To get rid of this, we generalize slightly the quasi-uniform spaces by considering pairs (X, U) where U is no more a filter of binary relations on X but more generally a semi group with a lattice structure. This with a bunch of axioms defines our *abstract* quasi-uniformities.

We introduce also the notion of *canonical* abstract quasi-uniformities. They are needed to make the eventual link between abstract quasi-uniformities and enrichments. Canonical quasi-uniformities are exactly enrichments: monoidal structures may be derived from them so that the above condition $(**)$ holds. It happens also that any abstract quasi-uniformity is equivalent to a canonical one (in a 2-categorical sense!).

The last difficulty was to find the good change of base functors, i.e. they should induce the morphisms of a 2-category of enrichments over different bases that admits a Cauchy-completion. Their definition was found by a careful inspection of the changes of base induced by quasi-uniformly continuous maps.

To sum up our results we have established the following chain of correspondences.

$$Enr(H) \rightharpoonup_{(1)} CAQUnif \rightharpoonup_{(2)} AQUnif \rightharpoonup_{(3)} QUnif.$$

$Enr(H)$ is a 2-category of enrichments over different bases. Bases considered are just partial orders or "quantales". $CAQunif$, $AQUinf$, $QUnif$ denote respectively the 2-categories of canonical abstract quasi-uniformities, abstract quasi-uniformities and quasi-uniformities. (1) is an isomorphism of 2-categories, (2) is an equivalence of 2-categories and (3) is a 2-coreflection. The 2-categorical Cauchy-completion in $Enr(H)$ migrates through (1) and (2) to $AQUnif$. We have also explicited this completion for abstract quasi-uniformities. From the latter and (3) one may retrieve the bicompletion for quasi-uniformities.

The rest of paper is technical and presents the above results. It is is organized as follows. There are two sides, one being enriched category theory, the other one topology, and the purpose is to reunify them. We start from enriched categories and do half of the way. Then start again from the quasi-uniformities and do the other half. The (canonical) abstract quasi-uniformities stand at the meeting point.

Section 2 recalls basic elements of quasi-uniformity theory. We introduce quasi-uniformities and their morphisms - that we call for short *uniformly continuous maps*. We point out that the set of uniformly continuous maps between two given quasi-uniformities may be preordered and hence quasi-uniformities and their morphisms form a locally preordered 2-category $QUnif$. Cauchy filters and the notions of separation and completeness for quasi-uniformities are recalled as well as the existence of a bicompletion for quasi-uniformities.

Chapter 3 recalls well-known very basic facts of the theory of $\mathcal{V}$-categories in the particular case where $\mathcal{V}$ is a monoidal partial order. The Cauchy-completion for $\mathcal{V}$-enrichments is detailed. The purpose of this chapter is to make the theory of the Cauchy completion accessible to non-specialists. The drawback is that fundamental notions of enriched category theory that are tight to the Cauchy-completion (functor categories, Yoneda, accessibility, free-cocompletions...) are just omitted. For any of these notions, we refer to Kelly's book [9]. The case of $\mathcal{V}$-enrichments with $\mathcal{V}$ non symmetric is treated in [4].

In chapter 4, we define a 2-category of enrichments over different bases. Here again, we only consider the case where the base monoidal categories are partial orders. We introduce the *super* monoidal functors between monoidal biclosed complete partial orders. They serve to define the 2-category $Enr(H)$, where H is a parameter 2-functor from a locally filtered 2-category to the 2-category $SMon$ of monoidal biclosed complete partial orders, super monoidal functors and monoidal transformations between them. For this 2-category the definition of morphisms between enrichments over different bases involves a change base monoidal functor that is super. We show that the Cauchy completion of enrichments is really a completion in $Enr(H)$. Precisely the Cauchy-completion yields a left 2-adjoint to the inclusion 2-functor $SC - Enr(H) \to Enr(H)$ where $SC - Enr(H)$ denotes the 2-category with skeletal and Cauchy-complete objects and with arrows and local preorders inherited respectively from $Enr(H)$. An important point is that $SC - Enr(H)$ is a full replete subcategory of $Enr(H)$.

Section 5 introduces the locally partially ordered 2-category QUT of quasi-uniform triples and their morphisms. Quasi-uniform triples correspond to an axiomatization of the lattice of quasi-uniformities and may be seen as quasi-uniformities "without points". We show how to build monoidal biclosed categories from quasi-uniform triples and super monoidal functors from morphisms of quasi-uniform triples. The notion of *canonical* morphisms of quasi-uniform triple is defined. It is needed to show the existence of a 2-functor $QUT \to SMon$. Indeed this 2-functor factorizes through $CQUT$ that is the locally partially ordered 2-category of quasi-uniform triples and canonical morphisms.

Section 6 introduces the locally preordered 2-category $AQUnif$ of *abstract quasi-uniformities* and their morphisms. As mentioned below abstract quasi-uniformities lie in between quasi-uniformities and enrichments - this statement will be made precise! - An abstract quasi-uniformity consists in a set of points with a "general distance" valued in a quasi-uniform triple. Any quasi-uniformity defines in a natural way an abstract one, (but the nature of this correspondence is postponed to section 8). We show that the 2-category $AQUnif$ is 2-equivalent to the 2-category $CAQUnif$ of *canonical* abstract quasi-uniformities, that is 2-isomorphic to $Enr(H)$ for the 2-functor $H : CQUT \to SMon$ defined in section 5. Let us just underline here that the notion of canonical abstract quasi-uniformities is linked with the crucial axiom

$$\forall u \in U, \exists v \in U, v^2 \subseteq u$$

occurring in the definition of a quasi-uniformity U.

Section 7 concerns the completion of abstract quasi-uniformities. We introduce the notions of Cauchy filters, neighborhood filters, separation and completeness for abstract quasi-uniformities. They correspond to the usual notions for quasi-uniformities. Via the 2-equivalence $AQUnif \cong Enr(H)$, sending the quasi-uniformity A to the corresponding enrichment $C(A)$, we see that:

- minimal Cauchy filters on A are in bijective correspondence with left adjoint modules of the form $1 \multimap C(A)$,

- the separation of A is equivalent to the skeletality of $C(A)$,
- the completeness of A is equivalent to the Cauchy-completeness of $C(A)$.

Since $AQUnif \cong Enr(H)$, one obtains as a direct application of the results of the section 4, a completion for abstract quasi-uniformities. Precisely, if $SC - AQUnif$ denotes the 2-category with objects the separated and complete abstract quasi-uniformities and arrows and local preorders inherited from $AQUnif$ then the inclusion 2-functor $SC - AQunif \to AQUnif$ has a left 2-adjoint that sends any abstract quasi-uniformity to its completion. Eventually we give an internal description of the completion of a quasi-uniformity.

Section 8 relates eventually usual quasi-uniformities to abstract ones. There is an obvious inclusion 2-functor $P : QUnif \to AQUnif$ that admits a right 2-adjoint Q such that $P \circ Q = 1$. Defining *concrete* abstract uniformities we show that $QUnif$ is 2-equivalent to the 2-category with objects the concrete abstract quasi-uniformities and arrows and local preorders inherited from $AQUnif$. Then we show how to retrieve the bicompletion of quasi-uniformities from the completion of abstract quasi-uniformities.

2 Quasi-uniformities

In this section we will recall briefly some elements of the theory of quasi-uniformities.

A quasi-uniformity U on a set X, also denoted (X, U), is a set of binary relations $U \subseteq \wp(X \times X)$ that satisfies:

(i) $\forall u \in U, \Delta \subseteq u$ (Δ is the diagonal relation),

(ii) U is a filter on $X \times X$,

(iii) $\forall u \in U, \exists v \in U, v^2 \subseteq u$.

A quasi-uniformity basis on X, is a filter basis for a quasi-uniformity (X, U).

Given quasi-uniformities (X, U) and (Y, V), a map $f : X \to Y$ is uniformly continuous when

$$\forall v \in V, \exists u \in U, \forall x, y \in X, (x, y) \in u \Rightarrow (fx, fy) \in v.$$

The set of uniformly continuous maps from (X, U) to (Y, V) is preordered by $\Rightarrow$ given for all $f, g : (X, U) \to (Y, V)$, by $f \Rightarrow g$ if and only if $\forall v \in V, \forall x \in X, (f(x), g(x)) \in v$. Quasi-uniformities, uniformly continuous maps and their preorders form a locally preordered 2-category $QUnif$.

Let (X, U) be a quasi-uniformity. A filter F on X is Cauchy (w.r.t U) when it satisfies:

$$\forall u \in U, \exists f \in F, f \times f \subseteq u.$$

The neighborhood filter of some $x \in X$ is the filter with basis the set of subsets of the form $\{y/(x,y) \in u \cap u^{-1}\}$, u ranging in U. Such a filter as above is Cauchy, it is also a minimal Cauchy one (This fact may be retrieved from results in subsections 7.2 and 8.2). (X, U) is said *complete* when any minimal Cauchy filter occurs as the neighborhood filter of some element of X. (X, U) is *separated* when the map sending the elements of X to their neighborhood filters is injective. Again we shall see in a slightly more general framework (7.10) that (X, U) is separated if and only if

$$\forall x, y \in X, x \neq y \Rightarrow \exists u \in U, (x, y) \notin u \cap u^{-1}.$$

We let $SC - QUnif$ denote the locally preordered 2-category with objects the separated and complete quasi-uniformities and with arrows and local preorders inherited from $QUnif$.

In section 8, we are going to show that the theorem below [6], may be obtained as a consequence of general results regarding enriched category theory 4.7 and 4.11.

Theorem 2.1 *The inclusion 2-functor $SC - QUnif \rightarrow QUnif$ has a left 2-adjoint.*

3 Enrichments over monoidal partial orders

This chapter recalls a few very basic elements of enriched category theory. All the results presented are well-known or belong to folklore. They are reproduced for completeness and to fix notations. This chapter introduces the Cauchy-completion of categories enriched over monoidal closed partial orders. Quite surprisingly this notion is accessible without referring to deep results in enriched category theory (Yoneda, indexed colimits, free cocompletions). We refer the reader to [9] for a general presentation of enriched category theory.

We call a monoidal category that is also a partial order, a *monoidal partial order*. In this section we will consider a monoidal biclosed and complete partial order $\mathcal{V}$, its tensor product is denoted $\otimes$, its identity I. An important fact is that $\mathcal{V}$ being complete is also cocomplete.

Definition 3.1 [$\mathcal{V}$-enrichments] An *enrichment* A over $\mathcal{V}$ or a *$\mathcal{V}$-category* is a set $Obj(A)$ — the *objects* of A — with a mapping $A(-,-) : Obj(A) \times Obj(A) \longrightarrow Obj(\mathcal{V})$ satisfying:

- for any object a of A, $I \leq A(a, a)$;
- for any objects a, b, c of A, $A(b, c) \otimes A(a, b) \leq A(a, c)$.

Given $\mathcal{V}$-categories A and B, a *$\mathcal{V}$-functor* f from A to B is a map $f : Obj(A) \longrightarrow Obj(B)$ such that for any objects a, a' of A, $A(a, a') \leq B(fa, fa')$.

Given two $\mathcal{V}$-functors $f, g : A \to B$ there is a *$\mathcal{V}$-natural transformation* from f to g, — which is denoted $f \Rightarrow g$, when for any object a of A, $I \leq B(fa, ga)$.

The usual definition of enrichment does not require $Obj(A)$ above to be a set. In this paper, we shall consider only “small” enrichments i.e. such that their sets of objects is small. The base categories will be also always small. This makes sense since the examples of enrichments that we will look at, are very “concrete”, and the fact that Cauchy-completion (3.10) preserves the smallness of objects is essential.

Proposition 3.2 *$\mathcal{V}$-categories, $\mathcal{V}$-functors and, $\mathcal{V}$-natural transformations constitute a locally preordered 2-category $\mathcal{V} - Cat$.*

In $\mathcal{V} - Cat$ the (horizontal) composition of $\mathcal{V}$-functors is given by the composition of underlying maps, and the identity functors correspond to identity maps. $\hat{1}$ denotes the enrichment on $\mathcal{V}$, with one object, say $*$, and $\hat{1}(*, *) = I$. If B is a $\mathcal{V}$-category, for any of its object b, $b : \hat{1} \to B$ denotes the $\mathcal{V}$-functor sending $*$ to b.

Definition 3.3 [$\mathcal{V}$-modules] Given two $\mathcal{V}$-categories A and B, a *$\mathcal{V}$-module* φ from A to B — denoted $\varphi : A \multimap B$ — is a map $\varphi : Obj(B) \times Obj(A) \to Obj(\mathcal{V})$ such that:

- for any objects a, a' of A, and b of B, $A(a, a') \otimes \varphi(b, a) \leq \varphi(b, a')$;
- for any objects a of A and, b, b' of B, $\varphi(b, a) \otimes B(b', b) \leq \varphi(b', a)$.

For any two $\mathcal{V}$-modules $\varphi : A \multimap B$ and $\psi : B \multimap C$, their composite $\psi \bullet \varphi : A \multimap C$ is defined by: $\forall a \in Obj(A), c \in Obj(C)$, $(\psi \bullet \varphi)(c, a) = \bigvee_{b \in Obj(B)} \varphi(b, a) \otimes \psi(c, b)$. The set $\mathcal{V} - Mod(A, B)$ of $\mathcal{V}$-modules from A to B is partially ordered by: $\varphi \leq \varphi' \Leftrightarrow \forall (b, a) \in Obj(B) \times Obj(A), \varphi(b, a) \leq \varphi'(b, a)$.

Proposition 3.4 *$\mathcal{V}$-categories and $\mathcal{V}$-modules with partial orders defined above, constitute a locally partially ordered 2-category denoted $\mathcal{V} - Mod$.*

In $\mathcal{V} - Mod$, the identity in A is the module with underlying map $A(-, -) : Obj(A) \times Obj(A) \to Obj(\mathcal{V})$ sending any (a, a') to $A(a, a')$. For any $\mathcal{V}$-categories A and B, a $\mathcal{V}$-module $\varphi : A \multimap B$ has right adjoint $\psi : B \multimap A$ if and only if $A(-, -) \leq \psi \bullet \varphi$ and $\varphi \bullet \psi \leq B(-, -)$. For any left adjoint module φ, $\tilde{\varphi}$ will denote its (unique!) right adjoint. Any $\mathcal{V}$-functor $f : A \to B$ corresponds to a pair of adjoint modules $f_\diamond \dashv f^\diamond$, $f_\diamond : A \multimap B$, $f^\diamond : B \multimap A$, as follows: for any objects a of A, and b of B, $f_\diamond(b,a) = B(b,fa)$ and $f^\diamond(a,b) = B(fa,b)$.

Further on $\mathcal{V} - AMod$ will denote the locally partially ordered 2-category with objects $\mathcal{V}$-categories, arrows: *left adjoint* $\mathcal{V}$-modules and local partial orders inherited from $\mathcal{V} - Mod$.

Proposition 3.5 *There is a 2-functor $J_\mathcal{V} : \mathcal{V} - Cat \to \mathcal{V} - Mod$ as follows. It is the identity on objects, and is the map $(-)_\diamond$ on arrows sending any $\mathcal{V}$-functor f to the $\mathcal{V}$-module $f_\diamond$. For any $\mathcal{V}$-functors $f, g : A \to B$, $f \Rightarrow g$ if and only if $f_\diamond \leq g_\diamond$.*

Definition 3.6 [Skeletality, Cauchy-completeness] A $\mathcal{V}$-category B is *skeletal*, respectively *Cauchy-complete* when for any $\mathcal{V}$-category A, the map $(-)_\diamond$: $\mathcal{V} - Cat(A,B) \to \mathcal{V} - AMod(A,B)$ is injective, respectively surjective.

Remark 3.7 For any $\mathcal{V}$-category B, the following assertions are equivalent:

- B is skeletal;
- The map $(-)_\diamond : \mathcal{V} - Cat(\hat{1}, B) \to \mathcal{V} - AMod(\hat{1}, B)$ is injective;
- For any of its objects a and b, if $I \leq B(a,b)$ and $I \leq B(b,a)$ then $a = b$.

Proposition 3.8 *A $\mathcal{V}$-category B is* Cauchy-complete *when the map $(-)_\diamond$: $\mathcal{V} - Cat(\hat{1}, B) \to \mathcal{V} - AMod(\hat{1}, B)$ is surjective. (i.e. for any left adjoint $\mathcal{V}$-module $\varphi : \hat{1} \multimap B$, $\varphi = b_\diamond$ for some object b of B).*

We define the following 2-categories.

- $\mathcal{V} - SkCcCat$: with objects skeletal and Cauchy-complete $\mathcal{V}$-categories, and arrows and local preorders inherited from $\mathcal{V} - Cat$.
- $\mathcal{V} - SkCcAMod$: with objects: skeletal and Cauchy-complete $\mathcal{V}$-categories, and arrows and local partial orders inherited from $\mathcal{V} - AMod$.

Proposition 3.9 *The restriction of $J_\mathcal{V}$ on $\mathcal{V} - SkCcCat$ is a 2-isomorphism onto $\mathcal{V} - SkCcAMod$.*

Definition 3.10 [Cauchy-completion] Let A be a $\mathcal{V}$-category. The *Cauchy-completion* of A is the $\mathcal{V}$-category $\bar{A}$ defined as follows. Its objects are the left-adjoint $\mathcal{V}$-modules of the form $\varphi : \hat{1} \multimap A$. For convenience we will consider modules in $Obj(\bar{A})$ as well as their adjoints, as maps with domains A. For any $\varphi, \psi \in Obj(\bar{A})$, $\bar{A}(\varphi, \psi) = (\tilde{\varphi} \bullet \psi)(*,*) = \bigvee_{a \in Obj(A)} \psi(a) \otimes \tilde{\varphi}(a)$. The map $Obj(A) \to Obj(\bar{A})$ sending any a to the $\mathcal{V}$-module $a_\diamond$, defines a $\mathcal{V}$-functor $j_A : A \to \bar{A}$.

Proposition 3.11 *For any $\mathcal{V}$-category A, $\bar{A}$ is skeletal and Cauchy-complete.*

Lemma 3.12 *For any $\mathcal{V}$-category A, $(j_A)_\diamond : A \multimap \bar{A}$ and $(j_A)^\diamond : \bar{A} \multimap A$ are inverse modules.*

With 3.9, 3.12 and 3.11, one may establish straightforwardly 3.13 and 3.14 below.

Proposition 3.13 *There is an equivalence of 2-categories $S : \mathcal{V} - AMod \cong \mathcal{V} - SkCcCat$, defined on objects by $S(A) = \bar{A}$ and on arrows by $S(\varphi) = f_\varphi$ where for any left adjoint $\mathcal{V}$-module $\varphi : A \multimap B$, $f_\varphi : \bar{A} \to \bar{B}$ is the unique $\mathcal{V}$-functor f satisfying $f_\diamond \bullet j_{A_\diamond} = j_{B_\diamond} \bullet \varphi$.*

Proposition 3.14 *The inclusion 2-functor $\mathcal{V} - SkCcCat \to \mathcal{V} - Cat$ has a left 2-adjoint.*

Here, the left 2-adjoint of the inclusion sends a $\mathcal{V}$-category A to $\bar{A}$ and the unit takes value $j_A : A \to \bar{A}$ in A.

According to 3.11 and 3.12, the following is coherent

Definition 3.15 [Morita-equivalence] Two $\mathcal{V}$-categories A and B are *Morita-equivalent* when one the following equivalent assertions is satisfied:

(i) They are isomorphic in $\mathcal{V} - Mod$;

(ii) Their Cauchy-completions are isomorphic in $\mathcal{V} - Cat$.

4 More on enrichments

In this chapter (4.1 and 4.2) we shall develop a 2-category of enrichments over different bases that will allow us to code quasi uniform spaces and their morphisms. As in the previous section we only treat enrichments over monoidal closed complete partial orders. We shall omit to recall the peculiar nature of the base monoidal categories.

4.1 About the change of base

The change of base for enrichments (over bicategories) has recently known some new developments [10]. Nevertheless we shall only use an old formulation of a classical result [5]. We introduce in this subsection the so-called *super* functors that were defined in [13]. These results will serve in the next section 4.2.

Proposition 4.1 *There is a 2-functor $(-)_@ : Mon \to 2 - CAT$ between the following 2-categories.*

Mon is the locally partially ordered 2-category with:

- *objects: small monoidal biclosed and complete partial orders,*
- *arrows: monoidal functors,*
- *2-cells: natural transformations (here, monoidal transformations are just natural transformations and there is a 2-cell $F \leq G : \mathcal{V} \to \mathcal{W}$ in Mon when for any object $v \in \mathcal{V}$, $Fv \leq Gv$ in $\mathcal{W}$).*

$2-CAT$ is the 2-category of large 2-categories, 2-functors and 2-natural transformations.

$(-)_@$ sends any category $\mathcal{V}$ to the 2-category $\mathcal{V}-Cat$. For any functor $F : \mathcal{V} \to \mathcal{W}$, $F_@ : \mathcal{V}-Cat \to \mathcal{W}-Cat$, is as follows. For any $\mathcal{V}$-category A, $F_@A$ is the $\mathcal{W}$-category with the same objects as A and such that for any objects a, b of A, $F_@A(a,b) = FA(a,b)$. For any $\mathcal{V}$-functor $f : A \to B$, $F_@f : F_@A \to F_@B$ is the $\mathcal{W}$-functor with the same underlying map as f. For any monoidal transformation $F \leq G$, there is a 2-natural transformation with value in A the $\mathcal{W}$-functor $F_@A \to G_@A$, with underlying map the identity of $Obj(A)$.

Proposition 4.2 *If $F : \mathcal{V} \to \mathcal{W}$ lies in Mon then there is a lax normal [3] functor $F_\sharp : \mathcal{V}-Mod \to \mathcal{W}-Mod$ that extends $F_@$, i.e. such that the diagram below commutes:*

$$\begin{tikzcd}
\mathcal{V}-Cat \arrow[r, "F_@"] \arrow[d, "J_\mathcal{V}"'] & \mathcal{W}-Cat \arrow[d, "J_\mathcal{W}"] \\
\mathcal{V}-Mod \arrow[r, "F_\sharp"'] & \mathcal{W}-Mod
\end{tikzcd}$$

For any $\mathcal{V}$-module $\varphi : A \multimap B$, $F_\sharp(\varphi) : F_@A \multimap F_@B$ is the map

$$\begin{cases} Obj(B) \times Obj(A) \to Obj(\mathcal{V}), \\ \forall a \in Obj(A), b \in Obj(B), F_\sharp(\varphi)(b,a) = F\varphi(b,a). \end{cases}$$

Proof. Let $\varphi : A \multimap B$. For any objects a, a' of A and b of B,

$$\begin{aligned} F_@A(a,a') \otimes F_\sharp(\varphi)(b,a) &= FA(a,a') \otimes F\varphi(b,a) \\ &\leq F(A(a,a') \otimes \varphi(b,a)) \\ &\leq F(\varphi(b,a')) \\ &= F_\sharp(\varphi)(b,a'). \end{aligned}$$

Similarly, for any object a of A and b, b' of B, $F\varphi(b,a) \otimes FB(b',b) \leq F\varphi(b',a)$. Thus $F_\sharp(\varphi)$ is a well defined module $F_@A \multimap F_@B$ in $\mathcal{W}-Mod$.

Trivially if $\varphi \leq \psi : A \multimap B$ then $F_\sharp\varphi \leq F_\sharp\psi : F_@A \multimap F_@B$.

Now given $\varphi : A \multimap B$ and $\psi : B \multimap C$, for any objects a of A and c

[3] A lax functor is *normal* when it preserves identities.

of C,

$$\begin{aligned} F_\sharp(\psi \bullet \varphi)(c,a) &= F(\psi \bullet \varphi)(c,a), \\ &= F(\bigvee_{b \in Obj(B)} \varphi(b,a) \otimes \psi(c,b)), \\ &\geq \bigvee_{b \in Obj(B)} F(\varphi(b,a) \otimes \psi(c,b)) \\ &\geq \bigvee_{b \in Obj(B)} F\varphi(b,a) \otimes F\psi(c,b) \\ &= (F_\sharp\psi \bullet F_\sharp\varphi)(c,a). \end{aligned}$$

By definition $F_\sharp(f_\diamond) = (F_@ f)_\diamond$ for any $\mathcal{V}$-functor f, thus for any $\mathcal{V}$-category A, $F_\sharp(1_{A\diamond}) = (F_@ 1_A)_\diamond = (1_{F_@ A})_\diamond$. □

Lemma 4.3 *Let $F : \mathcal{V} \to \mathcal{W}$ lie in Mon and $f : A \to B$ be a $\mathcal{V} - functor$. Then:*
(i) $F_\sharp(f_\diamond) \dashv F_\sharp(f^\diamond)$ in $\mathcal{W} - Mod$;
(ii) If $f_\diamond$ is an isomorphism in $\mathcal{V} - Mod$ then $F_\sharp(f^\diamond) \bullet F_\sharp(f_\diamond) = 1$.

Proof. (i): Results from the facts that for any $\mathcal{V}$-functor f, $F_\sharp(f_\diamond) = (F_@(f))_\diamond$ and also $F_\sharp(f^\diamond) = (F_@(f))^\diamond$.
(ii): Suppose $f_\diamond \bullet f^\diamond = 1$ and $f^\diamond \bullet f_\diamond = 1$. Composing the second inequality by $F_\sharp$ which is lax and normal, one gets $F_\sharp(f^\diamond) \bullet F_\sharp(f_\diamond) \leq F_\sharp(f^\diamond \bullet f_\diamond) = F_\sharp(1)$ $= 1$. Thus $F_\sharp(f^\diamond) \bullet F_\sharp(f_\diamond) = 1$ since $F_\sharp(f_\diamond) \dashv F_\sharp(f^\diamond)$. □

Definition 4.4 A monoidal functor $F : \mathcal{V} \to \mathcal{W}$ is *super* when for any family of pairs of objects $(v_i, w_i) \in Obj(\mathcal{V}) \times Obj(\mathcal{V})$, i ranging in I, if $I_\mathcal{V} \leq \bigvee_{i \in I} v_i \otimes w_i$ then $I_\mathcal{W} \leq \bigvee_{i \in I} F(v_i) \otimes F(w_i)$.

Note that the composition of super functors and the identity functors are super so we may define $SMon$ as the 2-category with objects the monoidal biclosed complete partial orders, with arrows the super monoidal functors and, with 2-cells inherited from Mon.

Lemma 4.5 *If F is super then $F_\sharp : \mathcal{V} - Mod \to \mathcal{W} - Mod$ preserves adjoint pairs: if $\varphi \dashv \tilde{\varphi}$ in $\mathcal{V} - Mod$ then $F_\sharp(\varphi) \dashv F_\sharp(\tilde{\varphi})$ in $\mathcal{W} - Mod$.*

Proof. Suppose that the monoidal $F : \mathcal{V} \to \mathcal{W}$ is super. Consider a left adjoint $\varphi : A \multimap B$ in $\mathcal{V} - Mod$.

Let us see that $1 \leq F_\sharp\tilde{\varphi} \bullet F_\sharp\varphi$. Since $1 \leq \tilde{\varphi} \bullet \varphi$, for any object a of A, $I_\mathcal{V} \leq A(a,a) \leq \bigvee_{b \in Obj(B)} (\varphi(b,a) \otimes \tilde{\varphi}(a,b))$ thus $I_\mathcal{W} \leq \bigvee_{b \in Obj(B)} F\varphi(b,a) \otimes F\tilde{\varphi}(a,b)$. Thus for any objects a, a' of A,

$$\begin{aligned} FA(a,a') &\leq FA(a,a') \otimes (\bigvee_{b \in Obj(B)} F\varphi(b,a) \otimes F\tilde{\varphi}(a,b)) \\ &= \bigvee_{b \in Obj(B)} ((FA(a,a') \otimes F\varphi(b,a)) \otimes F\tilde{\varphi}(a,b)) \\ &\leq \bigvee_{b \in Obj(B)} (F\varphi(b,a') \otimes F\tilde{\varphi}(a,b)) \\ &= (F_\sharp\tilde{\varphi} \bullet F_\sharp\varphi)(a,a'). \end{aligned}$$

That $F_\sharp\varphi \bullet F_\sharp\tilde{\varphi} \leq 1$ results from the facts that $F_\sharp$ is lax and normal and $\varphi \bullet \tilde{\varphi} \leq 1$:
$F_\sharp(\varphi) \bullet F_\sharp(\tilde{\varphi}) \leq F_\sharp(\varphi \bullet \tilde{\varphi}) \leq F_\sharp(1) = 1.$ □

Note that if $F : \mathcal{V} - Mod \to \mathcal{W}$ is super then according to the previous lemma $F_\sharp : \mathcal{V} - Mod \to \mathcal{W} - Mod$ will preserve in particular isomorphic modules, thus the Morita equivalence and the Cauchy-completion.

4.2 Enrichments over different bases

We investigate a kind of 2-categories with objects enrichments over different bases. These 2-categories are the $Enr(H)$ where $H : \mathcal{F} \to SMon$ is a parameter 2-functor with domain $\mathcal{F}$ locally filtered. The definition of the morphisms of $Enr(H)$ involves the super monoidal functors defined previously. It happens that for a particular H, $Enr(H)$ is 2-equivalent to the 2-category of abstract quasi-uniformities (see 6 further). Eventually we show that the Cauchy-completion yields again a left 2-adjoint to the inclusion $SC - Enr(H) \to Enr(H)$, where $SC - Enr(H)$ denotes the 2-category with objects the skeletal and Cauchy-complete enrichments and arrows and 2-cells inherited from $Enr(H)$. We introduce first the 2-category Enr^+ with objects enrichments over different bases. We establish a few technical results about it that will serve us further to reason conveniently about the 2-categories $Enr(H)$.

For convenience we write the "enrichment $(A, \mathcal{V})$" for the "$\mathcal{V}$-category A". Given a monoidal functor $F : \mathcal{V} \to \mathcal{W}$, enrichments $(A, \mathcal{V})$ and $(B, \mathcal{W})$, and a map $f : Obj(A) \to Obj(B)$, we say that F is *compatible* with the triple (f, A, B) when f defines a $\mathcal{W}$-functor $F_@A \to B$. Note

Remark 4.6 If $F \leq G : \mathcal{V} \to \mathcal{W}$ and G is compatible with (f, A, B) then so is F.

Let $\mathcal{E}$ be the locally preordered 2-category defined by the following data:

- objects: enrichments;
- arrows from $(A, \mathcal{V})$ to $(B, \mathcal{W})$: ordered pairs (f, F) where f is a map $Obj(A) \to Obj(B)$ and $F : \mathcal{V} \to \mathcal{W}$ is a monoidal functor compatible with (f, A, B);
- horizontal composition: the composite

$$(\,(B,\mathcal{W}) \xrightarrow{(g,G)} (C,\mathcal{X}\,)\,) \circ (\,(A,\mathcal{V}) \xrightarrow{(f,F)} (B,\mathcal{W})\,)$$

  is the pair $(g \circ f, GF : \mathcal{V} \to \mathcal{X})$;
- identities: $(1 : Obj(A) \to Obj(A), 1 : \mathcal{V} \to \mathcal{V})$ in $(A, \mathcal{V})$;

- 2-cells: the 2-cell $(f, F) \Rightarrow (g, G) : A \to B$ exists when there is a (monoidal) natural transformation $F \leq G$ and for any object a of A, there exists the arrow $I \to B(fa, ga)$ in $\mathcal{W}$.

Enr^+ is the locally preordered 2-category with objects enrichments, arrows those (f, F) of $\mathcal{E}$ with F *super*, and local preorders inherited from $\mathcal{E}$. It is straightforward to check that both $\mathcal{E}$ and Enr^+ are well-defined 2-categories. $SC - Enr^+$ will stand for the locally preordered 2-category with objects the skeletal and Cauchy-complete enrichments and with arrows and local preorders inherited from Enr^+.

Proposition 4.7 *The inclusion 2-functor $SC - Enr^+ \to Enr^+$ has a 2-left-adjoint. Precisely, for any enrichment $(A, \mathcal{V})$, the arrow $(A, \mathcal{V}) \xrightarrow{(j_A, 1)} (\bar{A}, \mathcal{V})$ (that lies in Enr^+) is such that:*

- *For any arrow $(A, \mathcal{V}) \xrightarrow{(f,F)} (B, \mathcal{W})$ in Enr^+ with B skeletal and Cauchy-complete, there is a unique arrow $(\bar{f}, \bar{F})$ that renders commutative the diagram in $\mathcal{E}$ below.*

$$\begin{tikzcd}
(A, \mathcal{V}) \arrow[r, "(j_A,1)"] \arrow[dr, "(f,F)"'] & (\bar{A}, \mathcal{V}) \arrow[d, "(\bar{f},\bar{F})"] \\
& (B, \mathcal{W})
\end{tikzcd}$$

This arrow $(\bar{f}, \bar{F})$ is in Enr^+. It is given by $\bar{F} = F$ and $\bar{f}_\diamond = f_\diamond \bullet F_\sharp({j_A}^\diamond)$.

- *There is a 2-cell $(f, F) \Rightarrow (g, G) : (A, \mathcal{V}) \to (B, \mathcal{W})$ in Enr^+ if and only if there is one $(\bar{f}, F) \Rightarrow (\bar{g}, G) : (\bar{A}, \mathcal{V}) \to (B, \mathcal{W})$ in Enr^+.*

Proof. Let $(f, F) : (A, \mathcal{V}) \to (B, \mathcal{W})$ be in Enr^+, B be skeletal and Cauchy-complete. We are looking for some arrow $(\bar{f}, \bar{F}) : \bar{A} \to B$ in Enr^+ such that $(\bar{f}, \bar{F}) \circ (j_A, 1_\mathcal{V}) = (f, F)$. Necessarily $\bar{F} = F$. Since F is super, $F_\sharp({j_A}_\diamond)$ is an isomorphism in $\mathcal{W} - Mod$ (4.5) and the composition on the right by $F_\sharp({j_A}_\diamond)$ (see below),

$$\begin{tikzcd}
F_@(A) \arrow[r, "F_\sharp({j_A}_\diamond)", "\circ" marking] \arrow[dr, "f"'] & F_@(\bar{A}) \arrow[d, "\bar{f}"] \\
& B
\end{tikzcd}$$

yields an isomorphism of categories $\mathcal{W} - Mod(F_@\bar{A}, B) \cong \mathcal{W} - Mod(F_@A, B)$. Thus by Cauchy-completeness and skeletality of B, the composition on the right by $F_@(j_A)$ is an isomorphism $\mathcal{W} - Cat(F_@\bar{A}, B) \cong \mathcal{W} - Cat(F_@A, B)$. Therefore the required $\bar{f}$ exists and is unique, it satisfies $\bar{f}_\diamond = f_\diamond \bullet F_\sharp({j_A}^\diamond)$. Given another functor $(g, G) : (A, \mathcal{V}) \to (B, \mathcal{W})$, we have to show that there is a natural transformation $(f, F) \Rightarrow (g, G)$ if and only if there is one $(\bar{f}, F) \Rightarrow (\bar{g}, G)$ where $\bar{g}$ is such that $\bar{g} \circ G_@(j_A) = g : G_@A \to B$ in $\mathcal{W} - Cat$. Let us write $\theta_A : F_@A \to G_@A$ and $\theta_{\bar{A}} : F_@\bar{A} \to G_@\bar{A}$ respectively for the components in A and $\bar{A}$ of the $\mathcal{V}$-natural transformation $\theta : F_@ \to G_@$ induced by $F \leq G$. θ_A and $\theta_{\bar{A}}$ are $\mathcal{W}$-functors with underlying maps the identities of $Obj(A)$ and $Obj(\bar{A})$. The diagram in $\mathcal{W} - Cat$ below commutes.

$$\begin{tikzcd}
F_@A \arrow[r, "F_@(j_A)"] \arrow[d, "\theta_A"'] & F_@\bar{A} \arrow[d, "\theta_{\bar{A}}"] \\
G_@A \arrow[r, "G_@(j_A)"'] & G_@\bar{A}
\end{tikzcd}$$

Then the situation in $\mathcal{W} - Mod$ is depicted below.

$$\begin{tikzcd}
 & F_@A \arrow[r, shift left, "F_\sharp(j_{A\diamond})"] \arrow[r, phantom, "\cong"] \arrow[dl, "(\theta_A)_\diamond"'] \arrow[dr, "f_\diamond"] & F_@\bar{A} \arrow[l, shift left, "F_\sharp(j_A{}^\diamond)"] \arrow[dl, "(\theta_{\bar{A}})_\diamond"'] \arrow[d, "\bar{f}_\diamond"] \\
G_@A \arrow[r, shift left, "G_\sharp(j_{A\diamond})"] \arrow[r, phantom, "\cong"] \arrow[dr, "g_\diamond"'] & G_@\bar{A} \arrow[l, shift left, "G_\sharp(j_A{}^\diamond)"] \arrow[d, "\bar{g}_\diamond"] \arrow[ur, phantom, "(1)"] & B \arrow[dl, "1"] \arrow[l, phantom, "(2)"] \\
 & B &
\end{tikzcd}$$

By 2-functoriality of $(-)_\diamond : \mathcal{W} - Cat \to \mathcal{W} - Mod$ and since $F_\sharp$ and $G_\sharp$ preserve isomorphisms, it is now straightforward that the 2-cell (1) exists if and only the 2-cell (2) exists. This is equivalent to the desired result according to 3.5 . $\square$

Consider 2-functors $H : \mathcal{C} \to \mathcal{A}$ and $K : \mathcal{B} \to \mathcal{A}$. Then one may construct the pullback $\mathcal{C} \times_{H,K} \mathcal{B}$ of H and K in the category $2 - Cat$, with projections π_H and π_K as below.

$$\begin{tikzcd}
\mathcal{C} \times_{H,K} \mathcal{B} \arrow[r, "\pi_K"] \arrow[d, "\pi_H"'] & \mathcal{B} \arrow[d, "K"] \\
\mathcal{C} \arrow[r, "H"'] & \mathcal{A}
\end{tikzcd}$$

If a 2-functor $H : \mathcal{C} \to SMon$ is such that $\mathcal{C}$ is locally partially ordered and U denotes the forgetful 2-functor $Enr^+ \to SMon$. $\mathcal{C} \times_{H,U} Enr^+$ is (2-isomorphic to) the locally preordered 2-category with:

- objects: (A, s) where s is an object of $\mathcal{C}$ and $(A, H(s))$ is an enrichment;
- arrows $(A, s) \to (B, t)$: the triples $(f, R : s \to t)$ where $R : s \to t$ is an arrow in $\mathcal{C}$ and $(f, H(R))$ is a morphism $(A, H(s)) \to (B, H(t))$ in Enr^+;
- 2-cells as follows: there is one 2-cell $(f, R : s \to t) \Rightarrow (g, P : s \to t) : (A, s) \to (B, t)$ if and only if $R \leq P$ in $\mathcal{C}$ and $I \leq B(fa, ga)$ in $H(t)$ for any object a of A.
- identity in (A, s): $(1 : Obj(A) \to Obj(A), 1 : s \to s)$;
- (horizontal) composition of

$$(A, s) \xrightarrow{(f,R)} (B, t) \xrightarrow{(g,P)} (C, u)$$

given by

$$(A, s) \xrightarrow{(gof, P \circ R)} (C, u)\,.$$

$\mathcal{C} \times_{H,U} SC - Enr^+$ is just (2-isomorphic) to the 2-category with objects the (A, s) of $\mathcal{C} \times_{H,U} Enr^+$ with $(A, H(s))$ skeletal and Cauchy-complete and arrows and 2-cells inherited from $\mathcal{C} \times_{H,U} Enr^+$.

From 4.7, it is straightforward that

Proposition 4.8 *Given a 2-functor $H : \mathcal{C} \to SMon$ with $\mathcal{C}$ locally partially ordered, the inclusion 2-functor $\mathcal{C} \times_{H,U} SC - Enr^+ \to \mathcal{C} \times_{H,U} Enr^+$ has a left 2 adjoint.*

Now we turn on to the 2-categories $Enr(H)$. For the rest of the section, we consider a 2-functor $H : \mathcal{F} \to SMon$ such that $\mathcal{F}$ is a locally partially ordered 2-category that is also *locally filtered* – i.e. the homsets of $\mathcal{F}$ are filtered partial orders.

The locally preordered 2-categories $Enr(H)$ and $SC - Enr(H)$ are as follows. $Enr(H)$ is the locally preordered 2-category with:

- objects: triples (A, s) where s is an object of $\mathcal{F}$ and $(A, H(s))$ is an enrichment;
- arrows $(A, s) \to (B, t)$: the maps $f : Obj(A) \to Obj(B)$ for which there exists an $R : s \to t$ in $\mathcal{F}$ such that $H(R)$ is compatible with $(f, (A, H(s)), (B, H(t)))$;
- 2-cells as follows: there is one 2-cell $f \Rightarrow g : (A, s) \to (B, t)$ if and only if for any object a, $I \leq B(fa, ga)$ in $H(t)$;
- identity in (A, s): the identity map on $Obj(A)$;

- horizontal composition of $(A,s) \xrightarrow{f} (B,t) \xrightarrow{g} (C,u)$ given by

$$(A,s) \xrightarrow{gof} (C,u)\,.$$

$SC - Enr(H)$ is the 2-category with objects the (A,s) of $Enr(H)$ where $(A,H(s))$ is skeletal and Cauchy-complete, and arrows and 2-cells inherited from $Enr(H)$.

It is important to note here that

Remark 4.9 If the enrichments $(A,\mathcal{V})$ and $(B,\mathcal{W})$ of are isomorphic in $Enr(H)$, then $(A,\mathcal{V})$ is Cauchy-complete if and only if $(B,\mathcal{W})$ is.

This results immediately from the fact that a change of base 2-functor $F_@ : \mathcal{V} - Cat \to \mathcal{W} - Cat$ is super and thus preserves the Cauchy-completeness.

According to 4.6, and since $\mathcal{F}$ is locally filtered,

Proposition 4.10 *There is a 2-cell $f \Rightarrow g : (A,s) \to (B,t)$ in $Enr(H)$ if and only if there exists an $R : s \to t$ in $\mathcal{F}$ such that there is a 2-cell $(f,R) \Rightarrow (g,R) : (A,s) \to (B,t)$ in $\mathcal{F} \times_{H,U} Enr^+$.*

Theorem 4.11 *The inclusion 2-functor $SC - Enr(H) \to Enr(H)$ has a left 2-adjoint.*

Proof. First, note that for any object (A,s) there exists the arrow $(j_A,1) : (A,s) \to (\bar{A},s)$ in $\mathcal{F} \times_{H,U} Enr^+$. Thus the map $j_A : Obj(A) \to Obj(\bar{A})$ defines a morphism of $Enr(H)$.

Now given a morphism $f : (A,s) \to (B,t)$ in $Enr(H)$ with B skeletal and Cauchy-complete, we know from 4.8 that the map $\bar{f} : Obj(\bar{A}) \to Obj(B)$ as in 4.7, defines a morphism of $Enr(H)$ such that the diagram

$$\begin{tikzcd} (A,s) \arrow[r, "j_A"] \arrow[dr, "f"'] & (\bar{A},s) \arrow[d, "\bar{f}"] \\ & (B,t) \end{tikzcd}$$

commutes in $Enr(H)$. We are going to show that $\bar{f}$ is the unique morphism in $Enr(H)$ that makes this diagram commute.

Let $(f, R : s \to t) : (A,s) \to (B,t)$ and $(g, P : s \to t) : (\bar{A},s) \to (B,t)$ in $\mathcal{F} \times_{H,U} Enr^+$ such that $g \circ j_A = f$.

Since $\mathcal{F}$ is locally filtered, there is a $Q : s \to t$ in $\mathcal{F}$ with $Q \leq R$, $Q \leq P$. Then $H(Q)$ $(\leq H(R), H(P))$, is compatible both with $(\bar{f}, \bar{A}, B)$ and $(g, \bar{A}, B)$. So the following diagram commutes in Enr^+:

$$\begin{tikzcd} (A, H(s)) \arrow[r, "{(j_A,1)}"] \arrow[d, "{(j_A,1)}"'] & (\bar{A}, H(s)) \arrow[d, "{(\bar{f},H(Q))}"] \\ (\bar{A}, H(s)) \arrow[r, "{(g,H(Q))}"'] & (B, H(t)) \end{tikzcd}$$

According to 4.7, one deduces that the maps $\bar{f}$ and g are equal.

Now let us see that for (A, s) and (B, t) with B skeletal and Cauchy-complete, the assignments $f \mapsto \bar{f}$ defines a preorder isomorphism:

$$Enr(H)[(A, s), (B, t)] \cong SC - Enr(H)[(\bar{A}, s), (B, t)].$$

For any morphisms $f, g : (A, s) \to (B, t)$ in $Enr(H)$, one has:

$f \Rightarrow g$

if and only if for some arrow $R : s \to t$ in $\mathcal{F}$,

$(f, H(R)) \Rightarrow (g, H(R)) : (A, H(r)) \to (B, H(t))$ in Enr^+ (according to 4.10)

if and only if for some arrow $R : s \to t$ in $\mathcal{F}$,

$(\bar{f}, H(R)) \Rightarrow (\bar{g}, H(R)) : (\bar{A}, H(s)) \to (B, H(t))$ in Enr^+ (according to 4.7)

if and only if $\bar{f} \Rightarrow \bar{g} : (\bar{A}, s) \to (B, t)$ in $Enr(H)$ (according to 4.10).

□

5 Quasi-uniform triples

This section introduces quasi-uniform triples and their morphisms. Quasi-uniform triples are just the algebraic version of the lattice of quasi-uniformities. They may be seen as quasi-uniformities without points. The purpose of this section is to establish the existence of a 2-functor $F_5 : QUT \to SMon$ from a locally partially ordered 2-category QUT of quasi-uniform triples and their morphisms to the locally partially ordered 2-category $SMon$ of monoidal partial orders and super monoidal functors between them. To get this result it appears necessary to introduce *canonical* morphisms of quasi-uniform triples. We obtain a locally partially ordered 2-category $CQUT$, with objects quasi-uniform triples and with arrows the canonical quasi-uniform triple morphisms. F_5 is obtained as a composition $QUT \to CQUT \to SMon$.

Definition 5.1 [Quasi-uniform triples] A *quasi-uniform triple* $(S, \cdot, \leq)$ consists of a semi-group $(S, \cdot)$ with base S, composition operation "$\cdot$", with S partially ordered by $\leq$. It satisfies moreover the following axioms:

(i) $\forall s, t \in S,\ (s \cdot t \geq s)\ \wedge\ (s \cdot t \geq t)$;
(ii) $\forall s, t, u \in S, t \leq u \Rightarrow (s \cdot t \leq s \cdot u) \wedge (t \cdot s \leq u \cdot s)$;
(iii) $\forall s \in S,\ \exists t \in S, t^2 \leq s$;
(iv) $\leq$ is filtered, i.e. $\forall s, t \in S, \exists u \in S, u \leq s \wedge u \leq t$.

5.1 From Quasi-uniform Triples to Complete Biclosed Categories

For the quasi-uniform triple $(S, \cdot, \leq)$, we define the following. The operation "$\cdot$" is extended to $\wp(S)$ by $V \cdot W = \{s \cdot t \mid s \in V, t \in W\}$. Further on, for some $V \subseteq S$, $[V]$ will denote the upper-closure of V with respect to $\leq$. $\mathcal{I}_S$ will stand for the set of upper-closed subsets of S. A binary operation $\otimes$ in $\mathcal{I}_S$ is defined by $V \otimes W = [V \cdot W]$. We define the operation ρ on $\mathcal{I}_S$ by $\rho(V) = S \otimes V \otimes S$. Note that according to 5.1-(iii), $S \otimes S = S$. From this, one checks that ρ is indeed a lower closure operation (it is monotonous and decreasing with respect to the inclusion and idempotent). $\mathcal{O}_S$ will stand for the sets of fixpoints for ρ. It is stable for $\otimes$ since for any $V, W \in \mathcal{O}_S$,

$$\begin{aligned} V \otimes W &= S \otimes V \otimes S \otimes S \otimes W \otimes S \\ &= S \otimes S \otimes V \otimes S \otimes S \otimes W \otimes S \otimes S \\ &= S \otimes V \otimes W \otimes S. \end{aligned}$$

Therefore $\mathcal{O}_S$ provided with the restriction of $\otimes$ – still denoted $\otimes$, is a monoid with S as neutral element.

Remark 5.2 Given a quasi-uniform triple $(S, \cdot, \leq)$, W and a family $(V_i)_{i \in I}$ both in $\mathcal{I}_S$, $W \otimes (\bigcup_{i \in I} V_i) = \bigcup_{i \in I} (W \otimes V_i)$ and $(\bigcup_{i \in I} V_i) \otimes W = \bigcup_{i \in I} (V_i \otimes W)$.

Proposition 5.3 *Given a quasi-uniform triple $(S, \cdot, \leq)$, a monoidal partial order $\mathcal{C}_S$ is defined by the following data:*
- *objects: elements of $\mathcal{O}_S$;*
- *order: inclusion;*
- *identity: S;*
- *tensor product: $\otimes$.*

$\mathcal{C}_S$ is cocomplete (and thus complete) and biclosed. The least upper bound a greatest lower bounds in $\mathcal{C}_S$ are as follows: for any family $(V_i)_{i \in I}$,

$$\bigvee_{i \in I} V_i = \bigcup_{i \in I} V_i,$$

$$\bigwedge_{i \in I} V_i = \rho(\bigcap_{i \in I} V_i).$$

Proof. Checking that $\mathcal{C}_S$ is a monoidal partial order is straightforward. Let us see that it is cocomplete. For any family $(V_i)_{i \in I}$ of elements $\mathcal{O}_S$, $\bigcup_{i \in I} V_i$ lies in $\mathcal{O}_S$, since it is upper closed and, according to 5.2, $S \otimes (\bigcup_{i \in I} V_i) \otimes S = \bigcup_{i \in I} S \otimes V_i \otimes S = \bigcup_{i \in I} V_i$. Therefore $\bigcup_{i \in I} V_i$ is the least upper bound of the V_i's in $\mathcal{O}_S$. Since $\rho : (\mathcal{I}_S, \subseteq) \to (\mathcal{O}_S, \subseteq)$ is a right adjoint, it preserves greatest lower bounds. Thus $\mathcal{C}_S$ has greatest lower bounds as above. $\mathcal{C}_S$ with inclusion ordering has a least upper bound S and greatest lower bound $\emptyset$. $\mathcal{C}_S$ is biclosed: for any $V, V', W \in \mathcal{O}_S$,

$V' \otimes V \subseteq W$ if and only if
$V' \subseteq \{s \in S \mid \forall t \in V, s \cdot t \in W\}$ if and only if
$V' \subseteq \rho(\{s \in S \mid \forall t \in V, s \cdot t \in W\})$.

Analogously, $V \otimes V' \subseteq W$ if and only if $V' \subseteq \rho(\{s \in S \mid \forall t \in V, t \cdot s \in W\})$.□

5.2 Morphisms of Quasi-uniform Triples

Recall that given a binary relation $R \subseteq S \times T$, the maps

$$\begin{cases} \exists_R : \wp(S) \to \wp(T), \\ V \mapsto \{u' \mid \exists u \in V, R(u, u')\} \end{cases}$$

and

$$\begin{cases} \forall_R : \wp(T) \to \wp(S), \\ V' \mapsto \{u \mid \forall u' \in T, R(u, u') \Rightarrow u' \in V'\} \end{cases}$$

defines an adjoint pair $\exists_R \dashv \forall_R : \wp(S) \to \wp(T)$, the powersets $\wp(S)$ and $\wp(T)$ being partially ordered with inclusion (For any $V \subseteq S$ and $V' \subseteq T$, $\exists_R(V) \subseteq V' \Leftrightarrow V \subseteq \forall_R(V')$).

The assignments $S \mapsto \wp(S)$ and $R \mapsto \exists_R$ defines a 2-functor from Rel, the locally partially ordered 2-category of relations with sets as objects, arrows as relations, and with relation inclusion for local partial orders, to Cat the 2-category of small categories. Moreover the local components of this 2-functor are full, i.e. for any $P, Q \subseteq S \times T$, $P \subseteq Q$ if and only if for the pointwise ordering $\exists_P \leq \exists_Q$. So $\exists$ as a functor is faithful. These local components also preserve least upper bounds: for any family of relations $(R_i \subseteq S \times T)_{i \in I}$, $\exists_{\bigcup_{i \in I} R_i} = \bigcup_{i \in I} \exists_{R_i}$.

Definition 5.4 Given two quasi-uniform triples S and T, a morphism from S to T is a relation $R \subseteq S \times T$ such that:

(i) $\exists_R(S) = T$;
(ii) $\forall s_1, s_2 \in S, \forall t \in T,\ R(s_2, t) \wedge (s_1 \leq s_2) \Rightarrow R(s_1, t)$;
(iii) $\forall s \in S, \forall t_1, t_2 \in T,\ R(s, t_1) \wedge (t_1 \leq t_2) \Rightarrow R(s, t_2)$;
(iv) $\forall s_1, s_2 \in S, \forall t_1, t_2 \in T,\ R(s_1, t_1) \wedge R(s_2, t_2) \Rightarrow R(s_1 s_2, t_1 t_2)$.

For any quasi-uniform triple S, its partial order $\leq \subseteq S \times S$ is a morphism of quasi-uniform triples. Quasi-uniform triples and their morphisms form a locally partially ordered bicategory QUT where:

- the horizontal composition of $S \xrightarrow{R} T \xrightarrow{P} U$ is $S \xrightarrow{P \circ R} U$;
- the identity at S is $\leq_S$ (thus QUT is NOT a subcategory of Rel);
- the local partial orders are the inclusion relations.

Proposition 5.5 *QUT is locally finitely complete. Precisely, given morphisms of quasi-uniform triples $P, Q : S \to T$, the relation $P \cap Q$ defines a morphism $S \to T$.*

Proof. For any $t \in T$, since P and Q satisfies (i), there are some $s_1, s_2 \in S$ such that $P(s_1, t)$ and $Q(s_2, t)$. Since S is filtered there is some $s \in S$ such that $s \leq s_1$ and $s \leq s_2$. Thus since P and Q satisfy (ii), $P(s, t)$ and $Q(s, t)$. That $P \cap Q$ satisfies axioms (ii), (iii) (iv) is trivial. $\square$

5.3 Canonical morphisms

Consider a quasi-uniform triple morphism $R : S \to T$. Since R satisfies 5.4-(iii) for any subsets $P \subseteq S$, $\exists_R(P)$ is upper closed.

Also,

Proposition 5.6 *For any subsets P, Q of S,*

$$\begin{aligned} \exists_R(P) \otimes \exists_R(Q) &= [\{v_1 \cdot v_2 \mid (\exists u_1 \in P, R(u_1, v_1)) \wedge (\exists u_2 \in Q, R(u_2, v_2))\}] \\ &\subseteq [\{v \mid \exists u_1 \in P, u_2 \in Q, R(u_1 \cdot u_2, v)\}], \textit{ by } 5.4\text{-}(iv) \\ &= \exists_R(P \otimes Q). \end{aligned}$$

Definition 5.7 A quasi-uniform triple morphism $R : S \to T$ is said *canonical* when one of the following equivalent assertions hold:

(1) For any $V \in \mathcal{I}_S$, $\exists_R(V)$ belongs to $\mathcal{O}_T$;
(2) For all $s \in S$, the upper set $\{t \in T \mid R(s, t)\}$ belongs to $\mathcal{O}_T$;
(3) $\forall s \in S, t \in T, R(s, t) \Rightarrow \exists t_1, t_2, t_3 \in T, t \geq t_1 t_2 t_3 \wedge R(s, t_2)$.

Let us just check the implication (2) $\Rightarrow$ (1), the other implications are trivial. Supposing (2), one has for any $V \in \mathcal{I}_V$,

$$\begin{aligned} \exists_R(V) &= \exists_R(\textstyle\bigcup_{s \in V} \uparrow s) \\ &= \textstyle\bigcup_{s \in V} \exists_R(\uparrow s) \\ &= \textstyle\bigcup_{s \in V} (S \otimes \exists_R(\uparrow s) \otimes S) \in \mathcal{O}_T \text{ according to } 5.3. \end{aligned}$$

Proposition 5.8 *Given a quasi-uniform triple morphism $R : U \to T$, there is a larger canonical morphism $\bar{R}$ amongst subrelations of R. It is defined for all $s \in S$ and $t \in T$ by:*

$$\bar{R}(s, t) \quad \Leftrightarrow \quad \exists t_1, t_2, t_3 \in T, t \geq t_1 t_2 t_3 \wedge R(s, t_2).$$

Proof. The $\bar{R}$ defined above is trivially a subrelation of R. Let us check that it is a morphism of quasi-uniform triples (5.4).

$\bar{R}$ satisfies (i). For any $t \in T$, there is some $t' \in T$ such that $t'^4 \leq t$. Since R satisfies axiom (i), there is an $s \in S$ such that $R(s, t')$, and thus $\bar{R}(s, t)$.

$\bar{R}$ trivially satisfies axioms (ii) and (iii). Let us see that $\bar{R}$ satisfies (iv). Suppose $s_1, s_2 \in S$, $t_1, t_2 \in T$, $\bar{R}(s_1, t_1)$ and $\bar{R}(s_2, t_2)$. Then there are $u_1, u_2, u_3, v_1, v_2, v_3 \in T$ such that $t_1 \geq u_1u_2u_3$, $R(s_1, u_2)$, $t_2 \geq v_1v_2v_3$ and $R(s_2, v_2)$. Then $t_1t_2 \geq u_1u_2u_3v_1v_2v_3 \geq u_1u_2v_2v_3$ and $R(s_1s_2, u_2v_2)$.

$\bar{R}$ is canonical. Suppose that $\bar{R}(s,t)$ then there are $t_1, t_2, t_3 \in T$ with $R(s, t_2)$ and $t \geq t_1t_2t_3$. There are t'_1, t'_3 such that $(t'_1)^2 \leq t_1$ and $(t'_3)^2 \leq t_3$. Thus for $t'_2 = t'_1t_2t'_3$, $\bar{R}(s, t'_2)$ and $t \geq t'_1t'_2t'_3$.

$\bar{R}$ is maximal amongst the canonical $P \subseteq R$. Given a canonical $P \subseteq R$, then for all $s \in S$, $t \in T$,

$$\begin{aligned} P(s,t) &\Rightarrow \exists t_1, t_2, t_3 \in T, P(s, t_2) \wedge t \geq t_1t_2t_3 \\ &\Rightarrow \exists t_1, t_2, t_3 \in T, R(s, t_2) \wedge t \geq t_1t_2t_3 \\ &\Leftrightarrow \bar{R}(s,t). \end{aligned}$$

□

Corollary 5.9 *For any quasi-uniform triple morphism R, $\bar{\bar{R}} = \bar{R}$.*

Proof. $\bar{\bar{R}} \subseteq \bar{R}$, and $\bar{\bar{R}}$ is the largest canonical morphism included in $\bar{R}$. Thus since $\bar{R}$ is canonical, $\bar{R} \subseteq \bar{\bar{R}}$. □

Consider the partial order $\leq_S \subseteq S \times S$, we will write $\sqsubseteq_S$ for the relation $\overline{\leq_S}$. $\sqsubseteq_S$ is the relation given for all $s, t \in S$ by: $s \sqsubseteq_S t \Leftrightarrow \exists t_1, t_3 \in S, t \geq t_1st_3$. Then for any quasi-uniform triple morphism $R : S \to T$, $\bar{R}$ is just the composite $\sqsubseteq_T \circ R$.

One has also

Proposition 5.10 *If the quasi-uniform triple morphism $R : S \to T$ is canonical then $R \circ \sqsubseteq_S = R$.*

Proof. Since $\sqsubseteq_S$ is a subrelation of $\leq_S$ then $(R \circ \sqsubseteq_S)$ is a subrelation of $(R \circ \leq_S) = R$. Conversely, suppose $R(s,t)$. Since R is canonical, there are $t_1, t_2, t_3 \in T$ such that $t \geq t_1t_2t_3$ and $R(s, t_2)$. Now there are some $s_1, s_3 \in S$ such that $R(s_1, t_1)$ and $R(s_3, t_3)$. Thus $R(s_1ss_3, t)$ showing $(R \circ \sqsubseteq_S)(s,t)$. □

Proposition 5.11 *The composition of canonical morphisms is canonical. Precisely, for any $\bullet \xrightarrow{P} \bullet \xrightarrow{Q} \bullet$ in QUT, $\overline{Q \circ P} = \bar{Q} \circ \bar{P}$.*

Proof.

$$\begin{aligned} \overline{Q \circ P} &= \sqsubseteq \circ Q \circ P \\ &= \bar{Q} \circ P \\ &= \bar{Q} \circ \sqsubseteq \circ P \text{ - according to 5.10} \\ &= \bar{Q} \circ \bar{P}. \end{aligned}$$

□

We define $CQUT$ as the partially ordered 2-category with quasi-uniform triples as objects, canonical morphisms as arrows and with composition of arrows and local partial orders inherited from QUT. The identity at S in $CQUT$ is $\sqsubseteq_S$, thus $CQUT$ is NOT a subcategory of QUT. According to the results of this section, one gets

Proposition 5.12 *There is a 2-functor* $\overline{(-)} : QUT \to CQUT$ *that is the identity on objects and sends a morphism* R *to* $\bar{R}$. *The components* $\overline{(-)}_{S,T} : QUT(S,T) \to CQUT(S,T)$ *are closure relations, i.e. they are right adjoints to the inclusions* $CQUT(S,T) \to QUT(S,T)$.

5.4 Canonical morphisms and super functors

Consider a quasi-uniform triple morphism $R : S \to T$. Then for any $V \in \mathcal{I}_S$, $\exists_R(V) \in \mathcal{I}_T$. Thus we define the two maps $\delta(R) : \mathcal{I}_S \to \mathcal{I}_T$ and $\bar{\delta}(R) : \mathcal{O}_S \to \mathcal{O}_T$ as follows. $\delta(R)$ is the restriction of $\exists_R$ on $\mathcal{I}_S$, and $\bar{\delta}(R) = \rho \circ \delta(R) \circ i$, where i denotes there the embedding $\mathcal{O}_S \to \mathcal{I}_S$ (see below)

$$\begin{array}{ccc} \mathcal{I}_S & \xrightarrow{\delta(R)} & \mathcal{I}_T \\ {\scriptstyle i}\uparrow & & \downarrow{\scriptstyle \rho} \\ \mathcal{O}_S & \xrightarrow[\bar{\delta}(R)]{} & \mathcal{O}_T \end{array}$$

By definition, for any morphism of quasi-uniform triple, $R : S \to T$, the following diagram commutes

$$\begin{array}{ccc} \wp(S) & \xrightarrow{\exists_R} & \wp(T) \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle i} \\ \mathcal{I}_S & \xrightarrow[\delta(R)]{} & \mathcal{I}_T \end{array}$$

From this one gets that for any $\bullet \xrightarrow{R} \bullet \xrightarrow{P} \bullet \in QUT$, $\delta(P \circ R) = \delta(P) \circ \delta(R)$. Also for any quasi-uniform triple S, $\delta(\leq_S)$ is the identity on $\mathcal{I}_S$, thus

Proposition 5.13 *The assignments* $S \mapsto (\mathcal{I}_S, \subseteq)$ *and* $R \mapsto \delta(R)$ *define a 2-functor* $F_1 : QUT \to Cat$.

Moreover the local components $F_{1S,T} : QUT(S,T) \to Cat(\mathcal{I}_S, \mathcal{I}_T)$ are full, thus F_1, as a functor, is faithful.

Given a quasi-uniform triple S, for any $V \in \mathcal{I}_S$, $\rho_S(V) = \exists_{\sqsubseteq_S}(V)$ i.e.

$$\delta(\sqsubseteq_S) = \mathcal{I}_S \xrightarrow{\rho} \mathcal{O}_S \xrightarrow{i} \mathcal{I}_S \,.$$

Lemma 5.14 *Any quasi-uniform triple morphism* $R : S \to T$ *is canonical if and only the diagram below commutes* $(*)$

$$\begin{tikzcd} \mathcal{I}_S \arrow[r, "\delta(R)"] \arrow[d, "\rho"'] & \mathcal{I}_T \\ \mathcal{O}_S \arrow[r, "\bar{\delta}(R)"'] & \mathcal{O}_T \arrow[u, "i"'] \end{tikzcd}$$

Proof. Suppose that R is canonical, then by 5.10, $R = \sqsubseteq \circ R \circ \sqsubseteq$ thus by functoriality of F_1, (I) below commutes.

$$\begin{tikzcd}[column sep=large, row sep=large] & \mathcal{O}_S \arrow[r, "\bar{\delta}(R)"] \arrow[d, "i"] \arrow[dl, phantom, "\mathrm{(II)}"] \arrow[dr, phantom, "\mathrm{(III)}"] & \mathcal{O}_T \arrow[dr, "i"] \arrow[dr, phantom, "\mathrm{(IV)}"', shift right=4] & \\ \mathcal{I}_S \arrow[ur, "\rho"] \arrow[r, "\delta(\sqsubseteq)"'] \arrow[rrr, bend right=40, "\delta(R)"'] \arrow[rrr, phantom, bend right=20, "\mathrm{(I)}"] & \mathcal{I}_S \arrow[r, "\delta(R)"'] & \mathcal{I}_T \arrow[u, "\rho"'] \arrow[r, "\delta(\sqsubseteq)"'] & \mathcal{I}_S \end{tikzcd}$$

(II), (IV), and (III) (by definition) also commute. Hence the diagram $(*)$ commutes. Conversely if $(*)$ commutes then $\delta(R)$ takes values in $\mathcal{O}_T$, i.e. R is canonical. $\square$

Composing on the right by $i : \mathcal{O}_S \to \mathcal{I}_S$ the diagram 5.14-$(*)$, one gets that for any canonical $R : S \to T$ the diagram below commutes

$$\begin{tikzcd} \mathcal{I}_S \arrow[r, "\delta(R)"] & \mathcal{I}_T \\ \mathcal{O}_S \arrow[u, "i"] \arrow[r, "\bar{\delta}(R)"'] & \mathcal{O}_T \arrow[u, "i"'] \end{tikzcd}$$

From this one deduces

Proposition 5.15 *The assignments $S \mapsto (\mathcal{O}_S, \subseteq)$ and $R \mapsto \bar{\delta}(R)$ define a 2-functor $F_2 : CQUT \to Cat$.*

Proposition 5.16 *Given $S \xrightarrow{R} T$ in QUT, $\bar{R}$ is the unique canonical morphism $S \xrightarrow{P} T$ such that $\bar{\delta}(P) = \bar{\delta}(R)$, i.e. F_2 as a functor is faithful. Precisely the local components of F_2 are full.*

Proof. If $R : S \to T$ is canonical then

$$\begin{aligned} \bar{\delta}(\bar{R}) &= \bar{\delta}(\sqsubseteq \circ R) \\ &= \rho \circ \delta(\sqsubseteq \circ R) \circ i \end{aligned}$$

$= \rho \circ \delta(\sqsubseteq) \circ \delta(R) \circ i$, by functoriality of F_1,
$= \rho \circ \delta(R) \circ i$
$= \bar{\delta}(R)$.

Now suppose that $R, P : S \to T$ are both canonical. Then according to 5.14, one has: $\bar{\delta}(R) \subseteq \bar{\delta}(P)$ if and only if $\delta(R) \subseteq \delta(P)$. The result follows then from the fact that the local components of F_1 are full. □

Eventually we define $F_3 = F_2 \circ \overline{(-)} : QUT \to Cat$. Note that according to 5.16, for any morphism R of quasi-uniform triple $F_3(R) = \bar{\delta}(R)$.

Proposition 5.17 *Given a morphism $R : S \to T$ of quasi-uniform triple,*
- $\delta(R) : \mathcal{I}_S \to \mathcal{I}_T$ preserves least upper bounds.
- $\bar{\delta}(R) : \mathcal{O}_S \to \mathcal{O}_T$ preserves least upper bounds.

Proof. Since $\exists_R : (\wp(S), \subseteq) \to (\wp(T), \subseteq)$ is a left adjoint and any union of upper-closed subsets is upper-closed, $\delta(R) : (\mathcal{I}(S), \subseteq) \to (\mathcal{I}(T), \subseteq)$ preserves unions, i.e. least upper bounds. Since $i : (\mathcal{O}(S), \subseteq) \to (\mathcal{I}(S), \subseteq)$ and $\rho : (\mathcal{I}(T), \subseteq) \to (\mathcal{O}(T), \subseteq)$ also preserves least upper bounds (see 5.2), $\bar{\delta}(R) = \rho \circ \delta(R) \circ i : (\mathcal{O}(S), \subseteq) \to (\mathcal{O}(T), \subseteq)$ preserves least upper bounds. □

Proposition 5.18 *For any morphism of quasi-uniform triples $R : S \to T$, the map $\bar{\delta}(R)$ defines a normal super monoidal functor $\mathcal{C}_S \to \mathcal{C}_T$.*

Proof. Since $\delta(R)(S) = T$, $\bar{\delta}(R)(S) = \rho \circ \delta(R)(S) = \rho(T) = T$.

For any $V, W \in \mathcal{O}_S$,

$$\begin{aligned}\bar{\delta}(R)(V \otimes W) &= \rho \circ \delta(R) \circ i(V \otimes W)\\ &\supseteq \rho(\delta(R) \circ i(V) \otimes \delta(R) \circ i(W)), \text{ (see 5.6)}\\ &\supseteq (\rho \circ \delta(R) \circ i)(V) \otimes (\rho \circ \delta(R) \circ i)(W)\\ &= \bar{\delta}(R)(V) \otimes \bar{\delta}(R)(W)\end{aligned}$$

Let us see that $\bar{\delta}(R)$ is super. Let $(V_i, W_i)_{i \in I}$ be a family of pairs of objects in $\mathcal{C}_S$ such that $S \subseteq \bigcup_{i \in I} V_i \otimes W_i$. Given $t \in T$ there is a $t' \in T$ such that $t'^2 \leq t$. Since $T = \bar{\delta}(R)(S) = \bar{\delta}(R)(\bigcup_{i \in I} V_i \otimes W_i) = \bigcup_{i \in I} \bar{\delta}(R)(V_i \otimes W_i)$, for this t' there is some $i \in I$, such that $t' \in \bar{\delta}(R)(V_i \otimes W_i)$. Thus $t' \in \bar{\delta}(R)(V_i) \cap \bar{\delta}(R)(W_i)$ and $t \in \bar{\delta}(R)(V_i) \otimes \bar{\delta}(R)(W_i)$. □

Eventually,

Proposition 5.19 *The assignments $S \mapsto \mathcal{O}_S$ and $R \mapsto \bar{\delta}(R)$ define a 2-functor $F_4 : QUT \to SMon$. Considering its restriction F_5 to $CQUT$, one has the factorization in $2 - CAT$*

$$\begin{tikzcd} QUT \arrow[rr, "\overline{(-)}"] \arrow[dr, "F_4"'] & & CQUT \arrow[dl, "F_5"] \\ & SMon & \end{tikzcd}$$

The local components of F_5 are full.

6 Abstract Quasi-uniformities

This section introduces abstract quasi-uniformities and their morphisms. They form together a locally preordered 2-category $AQUnif$ that "lies in between" the 2-category $QUnif$ of uniformities and uniformly continuous maps and the 2-category of enrichments $Enr(F_5)$. Any quasi-uniformity defines naturally an abstract one and the nature of the correspondence between $QUnif$ and $AQUnif$ will be detailed in section 8. In the other direction, we define *canonical* quasi-abstract uniformities that correspond to enrichments. They form together with abstract quasi-uniformity morphisms between them a locally preordered 2-category $CAQUnif$ that is 2-isomorphic to $Enr(F_5)$, ($F_5 : QUT \to SMon$ was defined in section 5) and 2-equivalent to $AQUnif$.

6.1 A few definitions

Definition 6.1 An *abstract quasi-uniformity A over a quasi-uniform triple S* denoted also (A, S), is a set $Obj(A)$ together with a map: $A(-,-) : Obj(A) \times Obj(A) \to \mathcal{I}_S$ that satisfies:
(i) $\forall x, y, z \in Obj(A)$, $A(y, z) \otimes A(x, y) \subseteq A(x, z)$,
(ii) $\forall x \in Obj(A)$, $S \subseteq A(x, x)$.

Let S, T, U be quasi-uniform triples, and A, B and, C be abstract quasi-uniformities respectively over S, T and U.

Definition 6.2 Let f be a map $Obj(A) \to Obj(B)$. A relation $R \subseteq S \times T$ is said *compatible* with the triple (f, A, B) — or simply "with f", when there is no ambiguity — when for all $s \in S$, $t \in T$, for all $x, y \in Obj(A)$, $s \in A(x, y) \wedge R(s, t) \Rightarrow t \in B(fx, fy)$.

Note that if $R : S \to T$ is a morphism then it is compatible with (f, A, B) if and only if for all objects x, y of A, $\delta(R)A(x, y) \subseteq B(fx, fy)$.

The set of compatible relations with some (f, A, B) is closed under arbitrary unions and subrelations. The largest relation compatible with (f, A, B) is denoted $R_{f,A,B}$ (or simply R_f), it is defined for all $s \in S$ and $t \in T$ by $R_f(s, t)$ if and only if $\forall x, y \in Obj(A), s \in A(x, y) \Rightarrow t \in B(fx, fy)$. R_f

satisfies the conditions (ii) and (iii) of 5.4. If the relation P is compatible with (f, A, B) and Q is compatible with (g, B, C) then $Q \circ P$ is compatible $(g \circ f, A, C)$.

We define the locally preordered 2-categories $AQUnif^+$ and $AQUnif$ as follows. Both have objects abstract quasi-uniformities. A morphism of $AQUnif^+ : (A, S) \to (B, T)$ consists in a pair (f, R) where f is a map $Obj(A) \to Obj(B)$ and R is a morphism $S \to T$ of quasi-uniform triples that is compatible with (f, A, B). The composite of arrows

$$(A, S) \xrightarrow{(f,P)} (B, T) \xrightarrow{(g,Q)} (C, U)$$

is $(A, S) \xrightarrow{(g \circ f, Q \circ P)} (C, U)$. The identity morphism in (A, S) is the pair $(1_A, \leq_S \subseteq S \times S)$. The local preorders $AQUnif^+((A, S), (B, T))$ are given by $(f, R) \Rightarrow (g, P)$ if and only if $R \subseteq P$ and for all object x of A, $T \subseteq B(fx, gx)$.

Morphisms $(A, S) \to (B, T)$ in $AQUnif$, that we shall call *morphisms of quasi-uniformities*, are the maps $f : Obj(A) \to Obj(B)$ for which there exists an $R : S \to T$ in QUT compatible with. The composition of arrows in $AQUnif$ is given by the underlying map composition and the identity morphisms correspond to identity maps. The local preorders $AQUnif((A, S), (B, T))$ are given by: $f \leq g$ if and only if for all object x of A, $T \subseteq B(fx, gx)$. There is an obvious forgetful 2-functor $AQUnif^+ \to AQUnif$.

Remark 6.3 Given a morphism of $AQUnif^+$, $(f, R) : (A, S) \to (B, T)$, since R satisfies 5.4-(i) and is compatible with f, then

$$\forall t \in T, \exists s \in S, \forall x, y \in Obj(A), s \in A(x, y) \Rightarrow t \in B(fx, fy)$$

6.2 Canonical abstract quasi-uniformities

Let S be some quasi-uniform triple. An enrichment of the form $(A, \mathcal{C}_S)$ corresponds exactly to an abstract quasi-uniformity (A, S) where the map $A(-, -)$ takes its values in $\mathcal{O}_S$. An abstract quasi-uniformity satisfying the latter property is called *canonical*.

First we define the locally preordered 2-category $CAQUnif^+$ that is 2-isomorphic to $CQUT \times_{F_5,U} Enr^+$. Its objects are the canonical abstract quasi-uniformities, its morphisms, the $(f, R) : (A, S) \to (B, T)$ of $AQUnif^+$ with R canonical. The unit at (A, S) is the pair $(1_A, \sqsubseteq_S)$ where 1_A is the identity map on $Obj(A)$ (Thus $CAQUnif^+$ is NOT a sub-2-category of $AQUnif^+$). The local preorders on $CAQUnif^+$ are inherited from $AQUnif^+$.

We define for any abstract quasi-uniformity A over S, the canonical abstract quasi-uniformity $C(A)$ over S, as follows. It has the same objects as

A and for any $x, y \in Obj(A)$, $C(A)(x, y) = \rho(A(x, y))$. For any morphism $(f, R) : A \to B$ in $AQUnif^+$, since for all $x, y \in Obj(A)$,

$$\begin{aligned}\delta(\bar{R}) \circ \rho(A(x,y)) &= \rho \circ \delta(R) \circ i \circ \rho(A(x,y)) \\ &\subseteq \rho \circ \delta(R)(A(x,y)) \\ &\subseteq \rho(B(fx, fy)),\end{aligned}$$

we let $C(f, R)$ denote the morphism $(f, \bar{R}) : C(A) \to C(B)$ in $CAQUnif^+$.

According to the 2-functoriality of $\overline{(-)} : QUT \to CQUT$ (5.12),

Proposition 6.4 *The assignments $A \mapsto C(A)$ and $(f, R) \mapsto C(f, R)$ define a 2-functor $C : AQUnif^+ \to CAQUnif^+$.*

Remark 6.5 Given a morphism $(A, S) \xrightarrow{(f,R)} (B, T)$ of abstract quasi-uniformities, since $\bar{R} \subseteq R$, the pair $(f, \bar{R})$ defines also a morphism $(A, S) \to (B, T)$.

According to this, the local components of $C : AQUnif^+ \to CAQUnif^+$ are coreflections.

Eventually we define the 2-category $CAQUnif$ of $AQUnif$. Its objects are the canonical abstract quasi-uniformities, its arrows and local preorders are inherited from $AQUnif$. According to 6.5, it is 2-isomorphic to $Enr(F_5)$ (from 5.5, QUT is locally filtered) and there is a forgetful 2-functor $CAQUnif^+ \to CAQUnif$ that is full as a functor.

Let (A, S) be an abstract quasi-uniformity. Since for any objects x, y of A, $\rho(A(x, y)) \subseteq A(x, y)$, there is a morphism $C(A) \to A$ of $AQUnif^+$ defined by the pair $(1_A, \leq_S)$. Now $\sqsubseteq_S$ is compatible with $(1_A, A, C(A))$. Thus the pair $(1_A, \sqsubseteq_S)$ defines a morphism $A \to C(A)$ of $AQUnif^+$. This shows that $A \cong C(A)$ in $AQUnif$, thus

Proposition 6.6 *$AQUnif$ and $CAQUnif$ are 2-equivalent. The following diagrams in $2-CAT$ involving that equivalence, the above C and the forgetful 2-functors commute.*

$$\begin{tikzcd} AQUnif^+ \arrow[r, "C"] \arrow[d] & CAQUnif^+ \arrow[d] \\ AQUnif \arrow[r, "\cong"'] & CAQUnif \end{tikzcd}$$

7 Completions

In this section we define notions of Cauchy filter, neighborhood filter, separation and completeness for abstract quasi-uniformities, that correspond to the classical notions for quasi-uniformities. We identify the equivalent notions in terms of enrichments. For an abstract quasi-uniformity A and the corresponding enrichment $C(A)$ via the equivalence $AQUnif \cong Enr(F_5)$: beginitemize

- Minimal Cauchy filters on A and left adjoint modules $1 \longrightarrow\!\!\circ\, C(A)$ are in bijective correspondence;
- The separation and completeness of A are equivalent respectively to the skeletality and the Cauchy-completeness of $C(A)$.

Via the above equivalence C, we infer from the Cauchy-completion in $Enr(F_5)$, a completion for abstract quasi-uniformities (the completion of the abstract quasi-uniformity A corresponds via C to the Cauchy-completion of $C(A)$). We will prove in chapter 8 how retrieve the classical bicompletion of quasi-uniformities from the completion of abstract ones.

7.1 Cauchy filters / adjoint modules

In this subsection, (A, S) is an abstract quasi-uniformity.

Definition 7.1 [Cauchy filters] A filter F on $Obj(A)$ is *Cauchy* when it satisfies

$$(*)\ \forall u \in U, \exists f \in F, \forall x, y \in f, u \in A(x, y)$$

A filter basis is Cauchy when the filter that it generates is Cauchy.

In the definition above we could have replaced the condition $(*)$ by:

$$(**) \quad \forall u \in U, \exists f \in F, \forall x, y \in f, u \in A(x, y) \otimes A(y, x).$$

Proposition 7.2 *Let $f : (A, S) \to (B, T)$ be a morphism of abstract quasi-uniformities. If F is a Cauchy filter on (A, S) then $f(F)$, the direct image of F by f is a Cauchy basis on (B, T).*

Proof. Since the direct image of any filter is a filter basis, it remains to check that $f(F)$ is Cauchy. Let $t \in T$. According to 6.3, we may find some $s \in S$ such that $\forall x, y \in Obj(A), s \in A(x, y) \Rightarrow t \in B(fx, fy)$. Since F is Cauchy, there is $p \in F$ such that $\forall x, y \in Obj(A), x, y \in p \Rightarrow s \in A(x, y)$ Then $\forall x, y \in Obj(A), x, y \in p \Rightarrow t \in B(fx, fy)$, i.e. $\forall x', y' \in Obj(B), x', y' \in f(p) \Rightarrow t \in B(x', y')$ □

According to this and since the identity map defines two morphisms $1_A : A \to C(A)$ and $C(A) \to A$, Cauchy filters on A and $C(A)$ are the same.

To simplify we shall suppose from now that A is canonical. We shall also denote by A the corresponding $\mathcal{C}_S$-category. We shall relate adjoint pairs

$\varphi \dashv \tilde{\varphi} : 1 \multimap A$ and Cauchy-filters on the abstract quasi-uniformity A (7.5). A Cauchy filter F is said *minimal* when it is minimal for the inclusion relation on $\wp(\wp(Obj(A)))$.

Proposition 7.3 *To any adjoint pair of $\mathcal{C}_S$-modules $\varphi \dashv \tilde{\varphi}$ with $\varphi : \hat{1} \multimap A$, corresponds a Cauchy filter $\bar{\Gamma}(\varphi)$ on $Obj(A)$ with basis the family $\Gamma(\varphi)$ of sets $\Gamma(\varphi)(u) = \{x \in Obj(A) \mid u \in \varphi(x) \otimes \tilde{\varphi}(x)\}$, u ranging in S. $\bar{\Gamma}(\varphi)$ is a minimal Cauchy filter. Precisely any Cauchy filter compatible with $\Gamma(\varphi)$ is finer than $\Gamma(\varphi)$.*

Proof. Consider an adjoint pair as above. By definition, the maps $\varphi : Obj(A) \to \mathcal{O}_S$, $\tilde{\varphi} : Obj(A) \to \mathcal{O}_S$ are such that

- $\forall x, y \in Obj(A), \varphi(x) \otimes A(y, x) \subseteq \varphi(y)$;
- $\forall x, y \in Obj(A), A(x, y) \otimes \tilde{\varphi}(x) \subseteq \tilde{\varphi}(y)$.

and

(1) $S \subseteq \bigcup_{x \in Obj(A)} \varphi(x) \otimes \tilde{\varphi}(x)$;
(2) $\forall x, y \in Obj(A), \tilde{\varphi}(y) \otimes \varphi(x) \subseteq A(x, y)$.

According to (1), for all $u \in S$, $\Gamma(\varphi)(u)$ is non empty. Since S is filtered, for all $u, v \in S$, there is some $w \in S$ with $w \leq u$, $w \leq v$ and $\Gamma(\varphi)(w) \subseteq \Gamma(\varphi)(u) \cap \Gamma(\varphi)(v)$. Therefore the family $\Gamma(\varphi)$ is a filter basis. Let us see now that this basis is Cauchy. Let $u \in S$. Consider $v \in S$ such that $v^2 \leq u$ and let $x, y \in \Gamma(\varphi)(v)$. Then $v^2 \in \varphi(y) \otimes \tilde{\varphi}(y) \otimes \varphi(x) \otimes \tilde{\varphi}(x) \subseteq \tilde{\varphi}(y) \otimes \varphi(x)$. From (2) we have $u \in A(x, y)$.

Suppose now that F is a Cauchy filter compatible with $\Gamma(\varphi)$. We show that F is finer than $\Gamma(\varphi)$. Consider $u \in S$, we have to find some $f \in F$ such that $f \subseteq \Gamma(\varphi)(u)$. Let $v \in S$, such that $v^4 \leq u$. Choose some $f \in F$ such that $\forall x, y \in f, v \in A(x, y)$. Since F is compatible with $\Gamma(\varphi)$, $\Gamma(\varphi)(v) \cap f \neq \emptyset$. Let $\bar{x} \in \Gamma(\varphi)(v) \cap f$. Then for all $y \in f$, $u \geq v^4 \in \varphi(\bar{x}) \otimes A(y, \bar{x}) \otimes A(\bar{x}, y) \otimes \tilde{\varphi}(\bar{x}) \subseteq \varphi(y) \otimes \tilde{\varphi}(y)$. □

Proposition 7.4 *If F is a filter basis on A then the maps*

$$\begin{cases} \varphi : X \to \mathcal{O}_S, \\ x \mapsto \bigcup_{f \in F} \bigwedge_{y \in f} A(x, y) \end{cases} \text{and,}$$

$$\begin{cases} \tilde{\varphi} : X \to \mathcal{O}_S, \\ x \mapsto \bigcup_{f \in F} \bigwedge_{y \in f} A(y, x) \end{cases}$$

define respectively modules $\varphi : \hat{1} \multimap A$ and, $\tilde{\varphi} : A \multimap \hat{1}$. If F is moreover Cauchy then $\varphi \dashv \tilde{\varphi}$ and the Cauchy filter $\bar{\Gamma}(\varphi)$ (see 7.3) is less fine than F.

Proof. φ is a module from $\hat{1}$ to A since for all objects x, y of A,

$$\begin{aligned}\varphi(x) \otimes A(y,x) &= (\bigcup_{f\in F} \bigwedge_{z\in f} A(x,z)) \otimes A(y,x)\\ &= \bigcup_{f\in F}(\ (\bigwedge_{z\in f} A(x,z)) \otimes A(y,x)\ )\\ &\subseteq \bigcup_{f\in F} \bigwedge_{z\in f}(A(x,z) \otimes A(y,x))\\ &\subseteq \varphi(y).\end{aligned}$$

Analogously $\tilde{\varphi}$ is a module $A \multimap 1$.

Now suppose that F is moreover a filter basis. Let us see that $\varphi \dashv \tilde{\varphi}$. Let us check first $S \subseteq \bigcup_{x\in Obj(A)} \varphi(x) \otimes \tilde{\varphi}(x)$.

$$\begin{aligned}\bigcup_{x\in Obj(A)} \varphi(x) \otimes \tilde{\varphi}(x) &= \bigcup_{x\in Obj(A)}((\bigcup_{f\in F} \bigwedge_{y\in f} A(x,y)) \otimes (\bigcup_{f\in F} \bigwedge_{y\in f} A(y,x)))\\ &= \bigcup_{x\in Obj(A)} \bigcup_{f_1,f_2\in F}(\bigwedge_{y\in f_1} A(x,y) \otimes \bigwedge_{y\in f_2} A(y,x))\\ &= \bigcup_{x\in Obj(A)} \bigcup_{f\in F}(\bigwedge_{y\in f} A(x,y) \otimes \bigwedge_{y\in f} A(y,x)),\end{aligned}$$

where the last equality follows from the fact that F is filtered. Let $u \in S$. There is some $v \in S$ such that $u \geq v^8$. Since F is Cauchy, there is some non-empty $f(v) \in F$ such that $x, y \in f(v) \Rightarrow v \in A(x,y)$. Then for some $x \in f(v)$, $v \in A(x,y) \cap A(y,x)$ for any $y \in f(v)$, $v^3 \in \bigwedge_{y\in f(v)} A(x,y) \cap \bigwedge_{y\in f(v)} A(y,x)$ and $u \in \bigwedge_{y\in f(v)} A(x,y) \otimes \bigwedge_{y\in f(v)} A(y,x)$.

Let us check that for any objects x, y of A, $\tilde{\varphi}(x) \otimes \varphi(y) \subseteq A(y,x)$. Let $x, y \in Obj(A)$,

$$\begin{aligned}\tilde{\varphi}(x) \otimes \varphi(y) &= (\bigcup_{f\in F} \bigwedge_{z\in f} A(z,x)) \otimes (\bigcup_{f\in F} \bigwedge_{z\in f} A(y,z))\\ &= \bigcup_{f_1,f_2\in F}(\bigwedge_{z\in f_1} A(z,x) \otimes \bigwedge_{z\in f_2} A(y,z))\\ &= \bigcup_{f\in F}(\bigwedge_{z\in f} A(z,x) \otimes \bigwedge_{z\in f} A(y,z)).\end{aligned}$$

Let $f \in F$. Since f is non empty then for some $z_0 \in f$, $\bigwedge_{z\in f} A(z,x) \subseteq A(z_0,x)$, $\bigwedge_{z\in f} A(y,z) \subseteq A(y,z_0)$ and, $\bigwedge_{z\in f} A(z,x) \otimes \bigwedge_{z\in f} A(y,z) \subseteq A(z_0,x) \otimes A(y,z_0)$. Thus $\tilde{\varphi}(x) \otimes \varphi(y) \subseteq A(y,x)$.

Now we prove that F is finer than $\Gamma(\varphi)$. Let $u \in S$. There is some $v \in S$ such that $v^2 \leq u$. Since $\Gamma(\varphi)(v) \neq \emptyset$, we choose some $\bar{x} \in \Gamma(\varphi)(v)$ i.e. $v \in \varphi(\bar{x}) \otimes \tilde{\varphi}(\bar{x})$, i.e. $v \geq v_1 \cdot v_2$ for some $v_1 \in \varphi(\bar{x})$ and $v_2 \in \tilde{\varphi}(\bar{x})$. Thus there are some $f_1, f_2 \in F$ such that

$$\begin{aligned}&v_1 \in \bigwedge_{y\in f_1} A(\bar{x},y) \subseteq \bigcap_{y\in f_1} A(\bar{x},y) \text{ and,}\\ &v_2 \in \bigwedge_{y\in f_2} A(y,\bar{x}) \subseteq \bigcap_{y\in f_2} A(y,\bar{x}).\end{aligned}$$

Thus given $y \in f_1 \cap f_2$, $v_1 \in A(\bar{x},y)$ and $v_2 \in A(y,\bar{x})$ and $u \geq v^2 \geq v_1v_2v_1v_2 \in \varphi(\bar{x}) \otimes A(y,\bar{x}) \otimes A(\bar{x},y) \otimes \tilde{\varphi}(\bar{x}) \subseteq \varphi(y) \otimes \tilde{\varphi}(y)$. Thus $f_1 \cap f_2 \subseteq \Gamma(\varphi)(u)$. □

Proposition 7.5 $\bar{\Gamma}$ *defines a bijection between adjoint pairs of modules of the form $\varphi \dashv \tilde{\varphi} :\ 1 \multimap A$ and minimal Cauchy-filters on A.*

This is a consequence of

Proposition 7.6 *Given an adjoint pair* $\varphi \dashv \tilde{\varphi} : 1 \multimap A$ *then it is the pair corresponding to the Cauchy filter* $\bar{\Gamma}(\varphi)$ *as in 7.4, — i.e. for any object* x *of* A, $\varphi(x) = \bigcup_{u \in S} \bigwedge_{y \in \Gamma(\varphi)(u)} A(x, y)$

Proof. It is enough to show that for any object x of A,

$$\varphi(x) \subseteq \bigcup_{u \in S} \bigwedge_{y \in \Gamma(\varphi)(u)} A(x, y)$$

since one should get analogously

$$\tilde{\varphi}(x) \subseteq \bigcup_{u \in S} \bigwedge_{y \in \Gamma(\varphi)(u)} A(y, x)$$

and we are considering two adjoint pairs. Let $x \in Obj(A)$ and $u \in \varphi(x)$. $u \geq v_1 v_2 u' v_3$ for some $v_1, v_2, v_3 \in S$ and $u' \in \varphi(x)$. For any $y \in \Gamma(\varphi)(v_2)$, by definition $v_2 \in \varphi(y) \otimes \tilde{\varphi}(y)$ thus $v_2 \in \tilde{\varphi}(y)$ and $v_2 u' \in \tilde{\varphi}(y) \otimes \varphi(x) \subseteq A(x, y)$. Therefore $u \geq v_1 v_2 u' v_3 \in \bigwedge_{y \in \Gamma(\varphi)(v_2)} A(x, y)$. $\square$

If F is a Cauchy-filter on A, we shall write φ_F for the corresponding left adjoint module $1 \multimap A$ defined in 7.4. 7.3 and 7.4 show that

Proposition 7.7 *For a given Cauchy filter* F, $\bar{\Gamma}(\varphi_F)$ *is the only minimal Cauchy filter less fine than* F.

7.2 Completion and separation / Cauchy-completion and skeletality

In this subsection, (A, S) will denote an abstract quasi-uniformity. and $C(A)$ will denote the canonical abstract quasi-uniformity as well as the corresponding enrichment.

Given some object x of A, let $B_1(x)$, $B_2(x)$, $B_3(x)$, $B_4(x)$ denote respectively the set of subsets of $Obj(A)$,

- $\{y \in Obj(A) \mid u \in A(y, x) \cap A(x, y)\}$, u ranging in S;
- $\{y \in Obj(A) \mid u \in A(y, x) \otimes A(x, y)\}$, u ranging in S;
- $\{y \in Obj(A) \mid u \in \rho \circ A(y, x) \cap \rho \circ A(x, y)\}$ u ranging in S;
- $\{y \in Obj(A) \mid u \in \rho \circ A(y, x) \otimes \rho \circ A(x, y)\}$, u ranging in S.

Then $B_1(x)$, $B_2(x)$, $B_3(x)$, $B_4(x)$ are Cauchy bases for (A, S) If (A, S) is canonical then $B_1(x) = B_3(x)$ and $B_2(x) = B_4(x)$. These bases are equivalent in the sense that they generate the same filter on $Obj(A)$. This filter is denoted $V(x)$, an element of $V(x)$ is called a *neighborhood of* x (in (A, S)). For any object x of A, $V(x)$ is just $\bar{\Gamma}(x_\diamond)$ for $x_\diamond : 1 \multimap C(A)$ Thus according to 7.3, $V(x)$ is a minimal Cauchy filter on (A, S).

This motivates the definition which is justified by 7.7

Definition 7.8 An abstract quasi-uniformity is said *complete* when one of the following equivalent assertion holds:

- any of its Cauchy filter is finer than a filter of neighborhoods for some element;
- any minimal Cauchy-filter occurs as a neighborhood filter.

Proposition 7.9 *The following assertions are equivalent:*

- *the abstract quasi-uniformity* (A, S) *is complete;*
- *the canonical abstract quasi-uniformity* $C(A, S)$ *is complete;*
- *the* $\mathcal{C}_S$*-enrichment* $C(A)$ *is Cauchy-complete.*

We define (dually)

Definition 7.10 (A, S) is said *separated* when one the following equivalent assertion holds:

- the map assigning any object x to the filter $V(x)$ is injective;
- For all objects x, y of A, if $x \neq y$ then there is $u \in S$ such that $u \notin A(x, y) \cap A(y, x)$.

The fact that the two assertions above are equivalent results from

Lemma 7.11 *For all objects* x*,* y *of* A*,* $V(x) \neq V(y)$ *if and and only if there is a* $u \in S$ *such that* $u \notin A(x, y) \cap A(y, x)$*.*

Proof. Clearly if there is a $u \in S$ such that $u \notin A(x, y) \cap A(y, x)$, then $y \notin \Gamma(x_\diamond)(u)$ or $x \notin \Gamma(y_\diamond)(u)$ and $V(x) \neq V(y)$.

To see the reverse implication, suppose $V(x) \neq V(y)$. Then by minimality of $V(y)$, $V(y)$ is not finer than $V(x)$. Thus there exists a $u \in S$ such that for any $v \in S$, $\Gamma(x_\diamond)(u) \not\supseteq \Gamma(y_\diamond)(v)$. For this u choose v such that $v^8 \leq u$. Then $v \notin A(x, y) \cap A(y, x)$ otherwise for any $z \in \Gamma(y_\diamond)(v)$, one would have $v^2 \in A(y, z) \otimes A(x, y) \subseteq A(x, z)$, $v^2 \in A(y, x) \otimes A(z, y) \subseteq A(z, x)$, thus $v^4 \in \rho(A(z, x)) \cap \rho(A(x, z))$ and $v^8 \in \rho(A(z, x)) \otimes \rho(A(x, z))$, and eventually $z \in \Gamma(x_\diamond)(u)$. □

Proposition 7.12 *The following assertions are equivalent:*

- *the abstract quasi-uniformity* (A, S) *is separated;*
- *the canonical abstract quasi-uniformity* $C(A, S)$ *is separated;*
- *the* $\mathcal{C}_S$*-enrichment,* $C(A)$ *is skeletal.*

7.3 A completion for abstract quasi-uniformities

Let us define the following 2-categories.

$SC - QAUnif$ with:
- objects: separated and complete abstract quasi-uniformities;
- arrows and local preorders inherited from $QAUnif$.

$SC - CQAUnif$ with:
- objects: separated and complete *canonical* abstract quasi-uniformities;
- arrows and local preorders inherited from $CQAUnif$.

Then

Theorem 7.13 *The inclusion functor $SC - QAUnif \to QAUnif$ has a left 2-adjoint.*

According to 6.6, 7.9 and 7.12, 7.13 is equivalent to

Theorem 7.14 *The inclusion functor $SC - CQAUnif \to CQAUnif$ has a left 2-adjoint.*

Which is just an instance of 4.11 for $H = F_5$, according again to 7.9 and 7.12.

Further on we shall write $\overline{\overline{(A,S)}}$ or simply $\overline{\overline{A}}$ for the "completion" of the abstract quasi-uniformity (A,S) that is its image by the left adjoint in 7.13. Let us describe $\overline{\overline{A}}$ up to isomorphism in $AQUnif$. Its elements are the Cauchy filters of A. We shall show that given Cauchy filters F_1 and F_2 and $u \in S$, $u \in \overline{\overline{A}}(F_1, F_2)$ if and only if there are some $f_1 \in F_1$ and $f_2 \in F_2$ such that $\forall x \in f_1, y \in f_2, u \in A(x,y)$.

Proof. First note that the map $\overline{\overline{A}}$ above defines an abstract quasi-uniformity. Then let us consider the enrichments $C(A)$ and $C(\overline{\overline{A}})$ corresponding respectively to the canonical abstract quasi-uniformities isomorphic to A and $\overline{\overline{A}}$. We show that $C(\overline{\overline{A}})$ is isomorphic to the Cauchy-completion $\overline{C(A)}$ of $C(A)$.

For two Cauchy filters F_1, F_2,

$$\overline{C(A)}(F_1, F_2) = \bigcup_{a \in A} ((\bigcup_{f_2 \in F_2} \bigwedge_{y \in f_2} A(a,y)) \otimes (\bigcup_{f_1 \in F_1} \bigwedge_{y \in f_1} A(y,a)))$$

since for any family $V_i \in \mathcal{I}_S$, $\rho(\cap_{i \in I} V_i) = \rho(\cap_{i \in I} \rho(V_i))$. Which means that $u \in \overline{C(A)}(F_1, F_2)$ if and only there is $a \in A$, $f_1 \in F$, $f_2 \in F_2$, $u_1, v_1, t_1, u_2, v_2, t_2 \in S$ such that $u \geq u_2 v_2 t_2 u_1 v_1 t_1$ and for all $y_1 \in f_1$, $y_2 \in f_2$, $u_1 \in A(y_1, a)$ and $u_2 \in A(a, y_2)$. Thus if $u \in \overline{C(A)}(F_1, F_2)$ considering the situation as above one has $u \geq u_2 v_2 v_1 t_1$ with $v_2 v_1 \in \overline{\overline{A}}(F_1, F_2)$, thus $u \in \rho(\overline{\overline{A}}(F_1, F_2))$.

Conversely, let $u \in \rho(\overline{\overline{A}}(F_1, F_2))$. Then there are $f_1 \in F_1$, $f_2 \in F_2$, $u_1, u_2, u_3 \in S$, such that $u \geq u_1 u_2 u_3$ and for all $y_1 \in f_1$, $y_2 \in f_2$, $u_2 \in A(y_1, y_2)$. Let t_1 such that ${t_1}^4 \leq u_3$. Choosing $g_1 \in F_1$ such that $\forall x, y \in g_1, t_1 \in A(x, y)$ and choosing $a \in f_1 \cap g_1$. Then $u \geq u_1 u_2 t_1^4$, for all $y_1 \in f_1 \cap g_1$, $t_1 \in A(y_1, a)$ and for all $y_2 \in f_2$, $u_2 \in A(a, y_2)$. This shows $u \in \overline{C(A)}(F_1, F_2)$. □

Eventually, the unit in A of 7.13 sends any object x to $V(x)$.

8 Quasi-uniformities

This section clarifies the connection between classical quasi-uniformities and abstract ones. The inclusion 2-functor $P : QUnif \to AQUnif$ has a right 2-adjoint Q. We define the *concrete* abstract uniformities. The 2-category with concrete abstract uniformities as objects and arrows and local preorders inherited from $AQUnif$ is 2-equivalent to $QUnif$. Both P, Q preserve the separation and completeness of objects. From this we can infer the classical bicompletion of quasi-uniformities from the completion of abstract ones.

8.1 Quasi-uniformities/abstract ones

Let (X, U) be a quasi-uniformity. One builds a quasi-uniform triple $U = (U, \circ, \subseteq)$, where $\circ$ denotes the composition of relations, and $\subseteq$ is the inclusion ordering. From (X, U), one can built $P(X, U)$, the abstract quasi-uniformity over U as follows. $P(X, U)$ has objects the elements of X, and for any $x, y \in X$, $P(X, U)(x, y) = (x, y)^{\in} = \{u \in U \mid (x, y) \in U\}$. Let (X, U), (Y, T) be quasi-uniformities, and let f be a map $X \to Y$. The relation $R_f = R_{f,P(X,U),P(Y,T)} \subseteq U \times T$ (see section 6.1) is for all $u \in U$ and $v \in T$

$$(u, v) \in R_f \Leftrightarrow (\ \forall x, y \in X, (x, y) \in u \Rightarrow (fx, fy) \in v\ ).$$

In this case, R_f also satisfies the condition 5.4-(iv). Now f is uniformly continuous exactly when $R_f(U) = T$, that is condition 5.4-(i). Thus f is uniformly continuous if and only if the map (f, R_f) is a morphism $P(X, U) \to P(Y, T)$ in $AQUnif^+$. So for an uniformly continuous map $f : (X, U) \to (Y, T)$, we let $P(f)$ denote the morphism $P(X, U) \to P(Y, T)$ in $AQUnif$ with underlying map $f : X \to Y$. Then

Proposition 8.1 *The assignments $(X, U) \mapsto P(X, U)$, $f : (X, U) \to (Y, T) \mapsto P(f)$, defines a 2-functor $P : QUnif \to AQUnif$. The local components of P are isomorphisms.*

Conversely, let (A, S) be an abstract quasi-uniformity, one may built a quasi-uniformity denoted $Q(A) = (A, \hat{S})$ with elements the objects of A, as follows. $\hat{S}$ has basis the set, say B, of sets $\hat{s} = \{(x,y) \mid s \in A(x,y)\}$, s ranging in S. Let us check that B is a basis of a quasi-uniformity. For any $s \in S$, $\Delta \subseteq \hat{s}$ since for any $s \in S$, $s \in A(x,x)$. The assignment $s \mapsto \hat{s}$ defines a monotonous map $S \to \wp(Obj(A) \times Obj(A))$, thus B is filtered since S is. Note that for any $s,t \in S$, $\hat{s} \circ \hat{t} \subseteq \widehat{s \circ t}$. So, for any $p \in B$, since $p = \hat{s}$ for some $s \in S$ there is some $t \in S$ such that $t^2 \leq s$ and $(\hat{t})^2 \subseteq \widehat{t^2} \subseteq p$.

Let $(f,R) : (A,S) \to (B,T)$ be a morphism of $AQUnif^+$. According to 6.3, f is a uniformly continuous map $Q(A) \to Q(B)$. We let $Q(f)$ denote this arrow in $QUnif$.

Proposition 8.2 *The assignments $A \mapsto Q(A)$ and $f : (A,S) \to (B,T) \mapsto Q(f)$ define a 2-functor Q $AQUnif \to QUnif$. Trivially the $Q \circ P \cong 1$.*

Definition 8.3 An abstract uniformity (A,S) is said *concrete* when it satisfies: $(\forall s,t \in S)(\forall x,y \in Obj(A))$

$$t \circ s \in A(x,y) \quad \Rightarrow \quad (\exists z \in Obj(A))\ s \in A(x,z) \wedge t \in A(z,y).$$

Observe that (A,S) is concrete exactly when for all $s,t \in S$, $\widehat{s \circ t} \subseteq \hat{s} \circ \hat{t}$. For any quasi-uniformity (X,U), the abstract quasi-uniformity $P(X,U)$ is concrete.

Now given two abstract uniformities (A,S) and (B,T) with A concrete and a map $f : Obj(A) \to Obj(B)$, the relation $R_{f,A,B}$ (see section 6.1) satisfies conditions of 5.4-(iv), as shown below. For all $s,s' \in S$ and $t,t' \in T$, such that $R_f(s,t)$ and $R_f(s',t')$, one has:

$$\begin{aligned}\forall x,y \in Obj(A), s' \circ s \in A(x,y) &\Rightarrow \exists z \in Obj(A),\ s \in A(x,z) \wedge s' \in A(z,y)\\ &\Rightarrow t \in B(fx,fz) \wedge t' \in B(fz,fy)\\ &\Rightarrow t' \circ t \in B(fx,fy).\end{aligned}$$

From this one gets that for any quasi-uniformity (X,U) and any abstract quasi-uniformity (A,S), a map $f : X \to Obj(A)$ is uniformly continuous $(X,U) \to Q(A,S)$ if and only $R_{f,P(X,U),(A,S)}$ is a morphism of quasi-uniform triples $U \to S$. Thus one gets a 2-natural isomorphism

$$QUnif((X,U),Q(A,S)) \cong_{(X,U),(A,S)} AQUnif(P(X,U),(A,S))$$

i.e.,

Proposition 8.4 *There is a 2-adjunction $P \dashv Q : QUnif \rightharpoonup AQUnif$.*

Let (A,S) be an abstract quasi-uniformity, the component in (A,S), $P \circ Q(A,S) \to (A,S)$, of the co-unit of the previous equivalence, has underlying map, the identity $Obj(A) \to Obj(A)$. Suppose now that the abstract quasi-uniformity (A,S) is moreover concrete. Then there is a morphism of quasi-uniform triples $R : S \to \hat{S}$ defined for all $s \in S$ and $p \in \hat{S}$ by $R(s,p) \Leftrightarrow \hat{s} \subseteq p$. It is straightforward that R satisfies conditions (i), (ii), (iii), of 5.4. It satisfies (iv) since for all $s,t \in S$, $p,q \in \hat{S}$ if $\hat{s} \subseteq p$ and $\hat{t} \subseteq q$ then $\widehat{s \circ t} = \hat{s} \circ \hat{t} \subseteq p \circ q$. This R is compatible with $(1_A, A, P \circ Q(A))$. This shows that the component in any concrete A of the co-unit of 8.4 is an isomorphism. Thus

Proposition 8.5 *$QUnif$ is 2-equivalent 2-category with objects concrete abstract uniformities, and morphisms and local preorders inherited from $AQUnif$.*

8.2 Bicompletion of quasi-uniformities

We show now how to retrieve 2.1 from 7.13 by proving that the inclusion $SC - QUnif \to QUnif$ has a left 2-adjoint with action on objects $(X,U) \mapsto Q\overline{\overline{P(X,U)}}$.

According to our definition of Cauchy filters and neighborhood filters on abstract quasi-uniformities (7.1 and section 7.2), P and Q preserve the separation and completeness of objects.

First we need to show

Lemma 8.6 *Given a quasi-uniformity (X,U), the component of the co-unit of the adjunction $P \dashv Q$ in $\overline{\overline{P(X,U)}}$ (8.4), $\epsilon_{\overline{\overline{P(X,U)}}} : PQ\overline{\overline{P(X,U)}} \to \overline{\overline{P(X,U)}}$ is an isomorphism.*

Proof. Since the components of unit of 8.4 are isomorphisms, the co-unit $\epsilon_{P(X,U)} : P \circ Q \circ P(X,U) \to P(X,U)$, in $P(X,U)$ is an isomorphism. Consider then the following commuting diagram in $QAUnif$.

$$\begin{tikzcd}
P \circ Q \circ P(X,U) \arrow[r, "\epsilon_{P(X,U)}", "\cong"'] \arrow[d, "P\circ Q(j_{P(X,U)})"'] & P(X,U) \arrow[d, "j_{P(X,U)}"] \\
P \circ Q \circ \overline{\overline{P(X,U)}} \arrow[r, "\epsilon_{\overline{\overline{P(X,U)}}}"'] & \overline{\overline{P(X,U)}}
\end{tikzcd}$$

where j denotes the unit of the 2-adjunction 7.13. Since $P \circ Q \circ \overline{\overline{P(X,U)}}$ is separated and complete, there is a unique morphism of quasi-uniformities r such that $r \circ j_{P(X,U)} = P \circ Q(j_{P(X,U)}) \circ (\epsilon_{P(X,U)})^{-1}$. Thus $j_{P(X,U)} = \epsilon_{\overline{\overline{P(X,U)}}} \circ r \circ j_{P(X,U)}$ showing $\epsilon_{\overline{\overline{P(X,U)}}} \circ r = 1$. Since $\epsilon_{\overline{\overline{P(X,U)}}}$ is also a monomorphism, it is an isomorphism. $\square$

Now one has the following sequence of 2-natural isomorphisms between 2-functors $QUnif \times SC - QUnif \to Cat$

$$QUnif((X,U),(Y,V)) \cong_{(X,U),(Y,V)}$$

$$
\begin{aligned}
&AQUnif(P(X,U),P(Y,V)) \cong_{(X,U),(Y,V)}\\
&SC-AQUnif(\overline{\overline{P(X,U)}},P(Y,V)) \cong_{(X,U),(Y,V)}\\
&SC-AQUnif(PQ\overline{\overline{P(X,U)}},P(Y,V)) \cong_{(X,U),(Y,V)}\\
&SC-QUnif(Q\overline{\overline{P(X,U)}},(Y,V)).
\end{aligned}
$$

References

[1] Betti R., Carboni A., Street R., Walters R., *Variation through enrichment*, J. Pure Appl. Algebra 29, 109–127, 1983.

[2] Bonsangue M. M., Van Breugel F., Rutten J.J.M.M., *Generalized metric spaces: completion, topology, and power domains via the Yoneda emdedding*, TCS 193, 1–51, 1998.

[3] Clementino M.M., Hofmann D., Tholen W., *One setting for all: metric, topology, uniformity, approach structure*, preprint. 2002.

[4] Day B.J., Street R., *Monoidal categories and Hopf algebroids*, Advances in math. 129, 99–157, 1997.

[5] Eilenberg S., Kelly G.M., *Closed categories*, Proceedings of the conference on Categorical Algebra at La Jolla Springer-Verlag, 421–562, 1966.

[6] Fletcher P., Lindgren W., *Quasi-uniform spaces*, Lecture Notes in Pure and Applied Maths - Marcel Dekker ed., 1982.

[7] Flagg R.C. *Quantales and Continuity Spaces*, Algebra Universalis 37, 257-276, 1997.

[8] , Flagg R.C., Kopperman R.D., *Continuity spaces: reconciling domains and metric spaces*, Theoretical Computer Science 177, 97, 111-138.

[9] Kelly G.M., *Basic Concepts of Enriched Category Theory*, London Mathematical Society Lecture Note Series 64 - Cambridge University Press, 1982.

[10] Kelly G.M., Labella A., Schmitt V., Street R., *categories enriched on two sides*, Journal of Pure and Applied Algebra 168, 53–98, 2002.

[11] Kunzi H.P.A., *Quasi-uniform spaces in the year 2001*, Chapter in: Recent Progress in General Topology II, eds. M. Husek and J.v. Mill, Elsevier, 313–344, 2002.

[12] Kunzi H.P.A., Schellekens M.P., *On the Yoneda completion of a quasi-metric space*, Theoretical Computer Science 278, 159–194, 2002.

[13] Labella A. and Schmitt V., *Change of base, Cauchy completeness and reversibility*, theory and Applications of Categories, vol.10, 187–219, 2002.

[14] Lawvere F.W., *Metric spaces, generalized logic, and closed categories*, Rend. Sem. Mat. Fis. di Milano 43, 135–166, 1973.

[15] Rutten J.J.M.M., *Elements of generalized ultrametric domain theory*, Theoretical Computer Science 170, 349–381, 1996.

[16] Smyth M.B., *Quasi-uniformities: Reconciling domains with metric spaces*, LNCS 298, Springer Verlag, 236–253, 1987.

[17] Smyth M.B., *Totally bounded spaces and compact ordered spaces as domains of computation*, G.M.Reed, A.W. Roscoe and R.F. Watcher, editors, "Topology and Category theory in Computer Science", Oxford University Press., 207–229, 1991.

[18] Smyth M.B., *Topology*, Handbook of Logic in Computer Science, ed. S.Abramsky, vol 1, Oxford: Clarendon Press, 641–761, 1992.

[19] Smyth M.B., *Completeness of quasi-uniform and syntopological spaces*, J. London Math. Soc. 49, 385–400, 1994.

[20] Sunderhauf P., *The Smyth completion of a quasi-uniform space*, M. Droste and Y. Gurevitch, editors, "Semantics of programming languages and model Theory", Algebra, Logic and Applications 5, Gordon and Breach Science Publ., 189–212, 1993.

[21] Sunderhauf P., *Quasi-uniform completeness in terms of Cauchy nets*, Acta Math. Hung. 69, 47–54, 1995.

[22] Wagner K.R., *Solving recursive domain equations with enriched categories*, PhD Thesis, Carnegie Mellon University, Pittsburg, July 94. Technical report CMU-CS-94-159.

[23] Walters R.F.C., *Sheaves and Cauchy-complete categories*, 3e Colloque sur les catégories, Amiens, juillet 80, Cah. Topo. Geom. Diff., vol XXII-3, 1981.