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Electronic Notes in Theoretical Computer Science 329 (2016) 27–38

Electronic Notes in Theoretical Computer Science

www.elsevier.com/locate/entcs

Multiobjective Optimization in a Quantum Adiabatic Computer

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Abstract

In this work we propose what we consider the first quantum algorithm for multiobjective combinatorial optimization, at least to the best of our knowledge. The proposed algorithm is based on the adiabatic algorithm of Farhi et al. and it is constructed by mapping a multiobjective combinatorial optimization problem into a Hamiltonian using a convex combination among objectives. We present mathematical properties of the eigenspectrum of the associated Hamiltonian and prove that the quantum adiabatic algorithm can find Pareto-optimal solutions provided certain convex combinations of objectives are used and the underlying multiobjective problem meets certain restrictions.

Keywords: multiobjective optimization, quantum adiabatic computing, combinatorial optimization

1 Introduction

Optimization problems are pervasive in everyday applications like logistics, communication networks, artificial intelligence and many other areas. Consequently, there is a high demand of efficient algorithms for these problems. Many algorithmic and engineering techniques applied to optimization problems are being developed to make an efficient use of computational resources in optimization problems. In fact, several engineering applications are multiobjective optimization problems, where several objectives must be optimized at the same time. For a survey on multiobjective optimization see for example [8,20]. In this work, we present what we consider the first algorithm for multiobjective optimization using a quantum adiabatic computer.

Quantum computation is a promising paradigm for the design of highly efficient algorithms based on the principles of quantum mechanics. Researchers have studied the computational power of quantum computers by showing the advantages it

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http://dx.doi.org/10.1016/j.entcs.2016.12.003
1571-0661/

presents over classical computers in many applications. Two of the most well-know applications are in unstructured search and the factoring of composite numbers. In structured search, Grover's algorithm can find a single marked element among n elements in time $O(\sqrt{n})$, whereas any other classical algorithm requires time at least n [10]. Shor's algorithm can factor composite numbers in polynomial time—any other known classical algorithm factors composite numbers in subexponential time (it is open whether a classical algorithm can factor numbers in polynomial time) [18].

Initially, before the year 2000, optimization problems were not easy to construct using quantum computers. This was because most studied models of quantum computers were based on quantum circuits which presented difficulties for the design of optimization algorithms. The first paper reporting on solving an optimization problem was in the work of Dürr and Høyer [7]. Their algorithm finds a minimum inside an array of n numbers in time $O(\sqrt{n})$. More recently, Baritompa, Bulger and Wood [3] presented an improved algorithm based on [7]; this latter algorithm, however, does not have a proof of convergence in finite time. The algorithms of [7] and [3] are based on Grover's search, an hence, in the quantum circuit model.

Farhi et al. [9] presented a new quantum algorithm and computation paradigm more friendly to optimization problems known as *Quantum Adiabatic Computing*. This new paradigm is based on a natural phenomenon of quantum annealing [6]; thus, analogously to classical annealing, optimization problems are mapped onto a natural optimization phenomenon, and hence, optimal solutions are found by just letting this phenomenon to take place.

The algorithms of [7] and [3] are difficult to extend to multiobjective optimization and at the same time prove convergence in finite time. Hence, quantum adiabatic computing presents itself as a more suitable model to achieve the follwing two goals: (i) to propose a quantum algorithm for multiobjective optimization, and (ii) prove convergence in finite time of the algorithm.

In this work, as our main contribution, we show that the quantum adiabatic algorithm of Farhi et al. [9] can be used to find Pareto-optimal solutions in finite time provided certain restrictions are met. In Theorem 4.1, we identify two structural properties that any multiobjective optimization problem must fulfill in order to use the abovementioned adiabatic algorithm.

The outline of this paper is the following. In Section 2 we present a brief overview of multiobjective combinatorial optimization and introduce the notation used throughout this work; in particular, several new properties of multiobjective combinatorial optimization are also presented that are of independent interest. In Section 3 we explain the quantum adiabatic theorem, which is the basis of the adiabatic algorithm. In Section 4 we explain the adiabatic algorithm and its application to combinatorial multiobjective optimization. In Section 5 we sketch a proof of our main result of Theorem 4.1. In Section 6 we show how to use the adiabatic algorithm in a concrete problem. Finally, in Section 7 we present a list of challenging open problems. Full proofs of all theorems and lemmas appear in [2].

2 Multiobjective Combinatorial Optimization

In this section we introduce the notation used throughout this paper and the main concepts of multiobjective optimization. The set of natural numbers (including 0) is denoted $\mathbb{N}$, the set of integers is $\mathbb{Z}$, the set of real number is denoted $\mathbb{R}$ and the set of positive real numbers is $\mathbb{R}^+$. For any $i, j \in \mathbb{N}$, with $i < j$, we let $[i, j]_\mathbb{Z}$ denote the discrete interval $\{i, i+1, \ldots, j-1, j\}$. The set of binary words of length n is denoted $\{0,1\}^n$.

2.1 Definition

A *multiobjective combinatorial optimization problem* (or MCO) is an optimization problem involving multiple objectives over a finite set of feasible solutions. These objectives typically present trade-offs among solutions and in general there is no single optimal solution. In this work, we follow the definition of Kung, Luccio and Preparata [12].

Let $S_1, \ldots, S_d$ be totally ordered sets and let $\leq_i$ be an order on set S_i for each $i \in [1, d]_\mathbb{Z}$. We also let n_i be the cardinality of S_i. Define the natural partial order relation $\prec$ over the cartesian product $S_1 \times \cdots \times S_d$ in the following way. For any $u = (u_1, \ldots, u_d)$ and $v = (v_1, \ldots, v_d)$ in $S_1 \times \cdots \times S_d$, we write $u \prec v$ if and only if for any $i \in [1, d]_\mathbb{Z}$ it holds that $u_i \leq_i v_i$. An element $u \in S$ is a *minimal element* if there is no $v \in S$ such that $v \prec u$ and $v \neq u$. Moreover, we say that u is *non-comparable* with v if $u \not\prec v$ and $v \not\prec u$ and succinctly write $u \sim v$. In the context of multiobjective optimization, the relation $\prec$ as defined here is often referred to as the *Pareto-order* relation [12].

Definition 2.1 A *multiobjective combinatorial optimization problem* (or shortly, MCO) is defined as a tuple $\Pi = (D, R, d, \mathcal{F}, \prec)$ where D is a finite set called domain, $R \subseteq \mathbb{R}^+$ is a set of values, d is a positive integer, $\mathcal{F}$ is a finite collection of functions $\{f_i\}_{i \in [1,d]_\mathbb{Z}}$ where each f_i maps from D to R, and $\prec$ is the Pareto-order relation on R^d (here R^d is the d-fold cartesian product on R). Define a function f that maps D to R^d as $f(x) = (f_1(x), \ldots, f_d(x))$ referred as the *objective vector* of Π. If $f(x)$ is a minimal element of R^d we say that x is a *Pareto-optimal solution* of Π. For any two elements $x, y \in D$, if $f(x) \prec f(y)$ we write $x \prec y$; similarly, if $f(x) \sim f(y)$ we write $x \sim y$. For any $x, y \in D$, if $x \prec y$ and $y \prec x$ we say that x and y are *equivalent* and write $x \equiv y$. The set of all Pareto-optimal solutions of Π is denoted $P(\Pi)$.

A canonical example of a multiobjective optimization problem is the Two-Parabolas problem. In this problem we have two objective functions defined by two parabolas that intersect in a single point, see Fig.1. In this work, we will only be concerned with a combinatorial version of the Two-Parabolas problem where each objective function only takes values on a finite set of numbers.

Considering that the set of Pareto-optimal solutions can be very large, we are mostly concerned on finding a subset of the Pareto-optimal solutions. Kung, Luccio and Preparata [12] give optimal query algorithms to find all Pareto-optimal

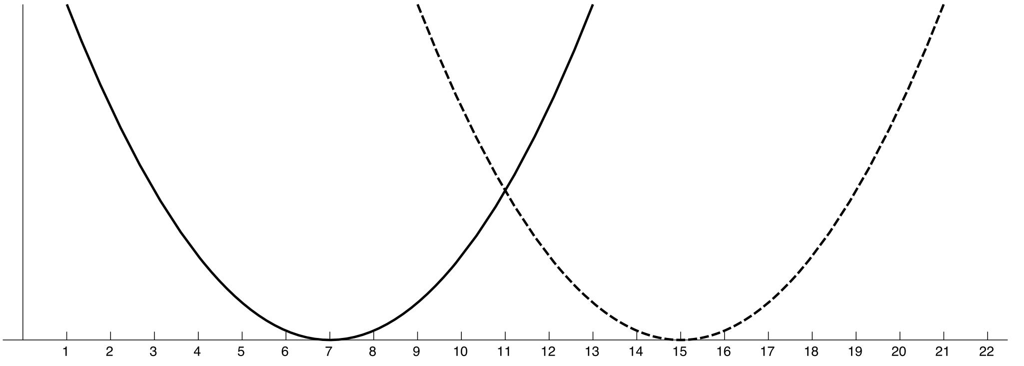

Fig. 1. The Two-Parabolas Problem. The first objective function f_1 is represented by the bold line and the second objective function f_2 by the dashed line. For MCOs, each objective function takes values only on the natural numbers. Note that there are no equivalent elements in the domain. In this particular example, all the solutions between 7 and 15 are Pareto-optimal.

solutions for $d = 2, 3$ and almost tight upper and lower bounds for any $d \geq 4$ up to polylogarithmic factors. Papadimitriou and Yannakakis [15] showed that an approximation to all Pareto-optimal solutions can be found in polynomial time.

For the remaining of this work, $\prec$ will always be the Pareto-order relation and will be omitted from the definition of any MCO. Furthermore, for convenience, we will often write $\Pi_d = (D, R, \mathcal{F})$ as a short-hand for $\Pi = (D, R, d, \mathcal{F})$. As a final remark, we will assume for this work that each function $f_i \in \mathcal{F}$ is computable in polynomial time and each $f_i(x)$ is bounded by a polynomial in the number of bits of x.

2.2 Some Foundational Properties

In this section we study properties of MCOs that will be necessary later in our work.

Definition 2.2 An MCO $\Pi_d = (D, S, \mathcal{F})$ is *normal* if for each $f_i \in \mathcal{F}$ there is a unique $x \in D$ such that $f_i(x) = 0$ and if $f_i(x) = 0$ and $f_j(y) = 0$, for $i \neq j$, then $x \neq y$.

In a normal MCO, the value of an optimal solution in each f_i is 0, and all optimal solutions are different. In Fig.1, solutions 7 and 15 are optimal solutions of f_1 and f_2 with value 0, respectively; hence, the Two-Parabolas problem of Fig.1 is normal.

Definition 2.3 An MCO Π_d is *collision-free* if given $\lambda = (\lambda_1, \ldots, \lambda_d)$, with each $\lambda_i \in \mathbb{R}^+$, for any $i \in [1, d]_{\mathbb{Z}}$ and any pair $x, y \in D$ it holds that $|f_i(x) - f_i(y)| > \lambda_i$. If Π_d is collision-free we write succinctly as Π_d^λ.

The Two-Parabolas problem of Fig.1 is not collision-free; for example, for solutions 5 and 9 we have that $f_1(5) = f_1(9)$. In Section 6 we show how to turn the Two-Parabolas problem into a collision-free MCO.

Definition 2.4 A Pareto-optimal solution x is *trivial* if x is an optimal solution of some $f_i \in \mathcal{F}$.

In Fig.1, solutions 7 and 15 are trivial Pareto-optimal solutions, whereas any x between 7 and 15 is non-trivial.

Lemma 2.5 *For any normal MCO Π_d, if x and y are trivial Pareto-optimal solutions of Π_d, then x and y are not equivalent.*

Let W_d be a set of of normalized vectors in $[0,1]^d$ defined as

$$W_d = \left\{ w = (w_1, \ldots, w_d) \in [0,1)^d \,\middle|\, \sum_{i=1}^{d} w_i = 1 \right\}. \tag{1}$$

For any $w \in W_d$, define $\langle f(x), w\rangle = \langle w, f(x)\rangle = w_1 f_1(x) + \cdots + (w_d) f_d(x)$.

Lemma 2.6 *Given $\Pi_d = (D, R, \mathcal{F})$, any two elements $x, y \in D$ are equivalent if and only if for all $w \in W_d$ it holds that $\langle f(x), w\rangle = \langle f(y), w\rangle$.*

Lemma 2.7 *Let $\Pi_d = (D, S, \mathcal{F})$ be any MCO. For any $w \in W_d$ there exists $x \in D$ such that if $\langle f(x), w\rangle = \min_{y \in D}\{\langle f(y), w\rangle\}$, then x is a Pareto-optimal solution of Π_d.*

In this work, we will concentrate on finding non-trivial Pareto-optimal solutions. Finding trivial elements can be done by letting $w_i = 1$ for some $i \in [1, d]_{\mathbb{Z}}$ and then running and optimization algorithm for f_i; hence, in Eq.(1) we do not allow for any w_i to be 1. The process of mapping several objectives to an single-objective optimization problem is sometimes referred as a *linearization* of the MCO [8].

From Lemma 2.7, we know that there are Pareto-optimal solutions that are not optimal for any $w \in W_d$. We define the set of *non-supported Pareto-optimal solutions* as the set $N(\Pi)$ of all Pareto-optimal solutions x such that $\langle f(x), w\rangle$ is not optimal for any linearization $w \in W_d$. We also define the set $S(\Pi)$ of *supported Pareto-optimal solutions* as the set $S(\Pi) = P(\Pi) \setminus N(\Pi)$ [8].

Note that there may be non-dominated Pareto-optimal solutions x and y that are non-comparable and $\langle f(x), w\rangle = \langle f(y), w\rangle$ for some $w \in W_d$. That is equivalent to say that the objective function obtained from the linearization of an MCO is not an injective function.

Definition 2.8 Any two Pareto-optimal solutions $x, y \in D$ are *weakly-equivalent* if there exists $w \in W_d$ such that $\langle f(x), w\rangle = \langle f(y), w\rangle$.

Any two equivalent solutions x, y are weakly-equivalent, by Lemma 2.6; the other way, however, does not hold in general. For example, consider two objective vectors $f(x) = (1, 2, 3)$ and $f(y) = (1, 3, 2)$. Clearly, x and y are not equivalent; however, if $w = (1/3, 1/3, 1/3)$ we can see that x and y are indeed weakly-equivalent. In Fig.1, points 10 and 12 are weakly-equivalent.

3 Quantum Adiabatic Computation

Starting from this section we assume basic knowledge of quantum computation. For a thorough treatment of quantum information science we refer the reader to the book by Nielsen and Chuang [14].

Let $\mathcal{H}$ be a Hilbert space with a finite basis $\{|u_i\rangle\}_i$. For any vector $|v\rangle = \sum_i \alpha_i|u_i\rangle$, the ℓ_2-norm of $|v\rangle$ is defined as $\|v\| = \sqrt{\sum_i |\alpha_i|^2}$. For any matrix A acting on $\mathcal{H}$, we define the operator norm of A induced by the ℓ_2-norm as $\|A\| = \max_{\|v\|=1} \|A|v\rangle\|$.

The Hamiltonian of a quantum system gives a complete description of its time evolution, which is governed by the well-known Schrödinger's equation

$$i\hbar\frac{d}{dt}|\Psi(t)\rangle = H(t)|\Psi(t)\rangle, \tag{2}$$

where H is a Hamiltonian, $|\Psi(t)\rangle$ is the state of the system at time t, $\hbar$ is Planck's constant and $i = \sqrt{-1}$. For simplicity, we will omit $\hbar$ and i from now on. If H is time-independent, it is easy to see that a solution to Eq.(2) is simply $|\Psi(t)\rangle = U(t)|\Psi(0)\rangle$ where $U(t) = e^{-itH}$ using $|\Psi(0)\rangle$ as a given initial condition. However, when the Hamiltonian depends on time, Eq.(2) is not in general easy to solve and much research is devoted to it; nevertheless, there are a few known special cases.

Say that a closed quantum system is described by a time-dependent Hamiltonian $H(t)$. If $|\Psi(t)\rangle$ is the minimum energy eigenstate of $H(t)$, adiabatic time evolution keeps the system in its lower energy eigenstate as long as the change rate of the Hamiltonian is "slow enough." This natural phenomenon is formalized in the *Adiabatic Theorem*, first proved by Born and Fock [4]. Different proofs where given along the years, see for example [11,13,17,16,1]. In this work we make use of a version of the theorem presented in [1].

Consider a time-dependent Hamiltonian $H(s)$ for $0 \le s \le 1$, where $s = t/T$ so that T controls the rate of change of H for $t \in [0, T]$.

Theorem 3.1 (Adiabatic Theorem [4,11,1]) *Let $H(s)$ be a nondegenerate Hamiltonian, let $|\psi(s)\rangle$ be one of its eigenvectors and $\gamma(s)$ the corresponding eigenvalue. For any $\lambda \in \mathbb{R}^+$ and $s \in [0,1]$, assume that for any other eigenvalue $\hat{\gamma}(s)$ it holds that $|\gamma(s) - \hat{\gamma}(s)| > \lambda$. Consider the evolution given by H on initial condition $|\psi(0)\rangle$ for time T and let $|\phi\rangle$ be the state of the system at T. For any $\delta \in \mathbb{R}^+$, if $T \ge \frac{10^5}{\delta^2}.\max\{\frac{\|H'\|^3}{\lambda^4}, \frac{\|H'\|\cdot\|H''\|}{\lambda^3}\}$ then $\|\phi - \psi(1)\| < \delta$.*

4 The Quantum Adiabatic Algorithm

The adiabatic theorem can be used to construct quantum algorithms for optimization problems. Consider a function $f : \{0,1\}^n \to \mathbb{R}^+$ whose optimal solution $\bar{x}$ gives $f(\bar{x}) = 0$. Let H_1 be a Hamiltonian defined as

$$H_1 = \sum_x f(x)|x\rangle\langle x|. \tag{3}$$

Notice that $H_1|\bar{x}\rangle = 0$, and hence, $|\bar{x}\rangle$ is an eigenvector. Thus, an optimization problem reduces to finding the eigenstate with minimum eigenvalue [9]. For any $s \in [0,1]$, let $H(s) = (1-s)H_0 + sH_1$, where H_0 is an initial Hamiltonian chosen accordingly. If we initialize the system in the lowest energy eigenstate $|\psi(0)\rangle$, the adiabatic theorem guarantees that T at least $1/\mathrm{poly}(\lambda)$ suffices to obtain a quantum state close to $|\psi(1)\rangle$, and hence, to our desired optimal solution. We call H_1 and H_0 the final and initial Hamiltonians, respectively. The only requirement in order to make use of the Adiabatic Theorem is that H_0 and H_1 must not commute.

In this section we show how to construct the initial and final Hamiltonians for MCOs. Given any normal and collision-free MCO $\Pi_d^\lambda = (D, R, \mathcal{F})$ we will assume with no loss of generality that $D = \{0,1\}^n$, that is, D is a set of binary words of length n.

For each $i \in [1, d]_{\mathbb{Z}}$ define a Hamiltonian $H_{f_i} = \sum_{x\in\{0,1\}^n} f_i(x)|x\rangle\langle x|$. The minimum eigenvalue of each H_{f_i} is nondegenerate and 0 because Π_d^λ is normal and collision-free. For any $w \in W_d$, the final Hamiltonian H_w is defined as

$$\begin{aligned} H_w &= w_1 H_{f_1} + \cdots + w_d H_{f_d} \\ &= \sum_{x\in\{0,1\}^n} \big(w_1 f_1(x) + \cdots + w_d f_d(x)\big)|x\rangle\langle x| \\ &= \sum_{x\in\{0,1\}^n} \langle f(x), w\rangle |x\rangle\langle x|. \end{aligned} \tag{4}$$

Following the works of [9], we choose as initial Hamiltonian one that does not diagonalizes in the computational basis. Let $|\hat{0}\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ and $|\hat{1}\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$. A state $|\hat{x}\rangle$ for any $x \in \{0,1\}^n$ is obtained by applying the n-fold Walsh-Hadamard operation $F^{\otimes n}$ on $|x\rangle$. The set $\{|\hat{x}\rangle\}$ is known as the Hadamard basis. The initial Hamiltonian is thus defined over the Hadamard basis as

$$H_0 = \sum_{x\in\{0,1\}^n} h(x)|\hat{x}\rangle\langle\hat{x}|, \tag{5}$$

where $h(0^n) = 0$ and $h(x) \geq 1$ for all $x \neq 0^n$. It is easy to see that the minimum eigenvalue is nondegenerate with corresponding eigenstate $|\hat{0}^n\rangle = \frac{1}{\sqrt{2}} \sum_{x\in\{0,1\}^n} |x\rangle$.

After defining the initial and final Hamiltonians, the Adiabatic Theorem guarantees that we can find a Pareto-optimal solution in finite time.

Theorem 4.1 *Given any normal and collision-free MCO Π_d^λ, if there are no equivalent Pareto-optimal solutions, then there exists $w \in W_d$ such that the quantum adiabatic algorithm can find the Pareto-optimal solution x corresponding to w in finite time. Moreover, if each w is chosen appropriately, then the quantum adiabatic algorithm can find all supported solutions.*

By Lemma 2.7, all supported solutions can be found by choosing any $w \in W_d$. Thus, to prove Theorem 4.1 we show in the following section that there always exists an appropriate w that makes H_w nondegenerate in its minimum eigenvalue.

5 Eigenspectrum of the Final Hamiltonian

In this section we sketch a proof of Theorem 4.1. Note that if the initial Hamiltonian does not commute with the final Hamiltonian, it suffices to prove that the final

Hamiltonian is nondegenerate in its minimum eigenvalue [9]. For the remaining of this work, we let σ_w and α_w be the smallest and second smallest eigenvalues of H_w.

Lemma 5.1 *For any $w \in W_d$, it holds that $\sigma_w \geq \sum_{i\in N} w_i\lambda_i$. In particular, for any $w \in (0,1)^d$, it holds that $\sigma_w \geq \langle w, \lambda\rangle$.*

Lemma 5.2 *For any $w \in W_d$, let H_w be a Hamiltonian with a nondegenerate minimum eigenvalue. The eigenvalue gap between the smallest and second smallest eigenvalues of H_w is at least $\langle \lambda, w\rangle$.*

Lemma 5.3 *If there are no weakly-equivalent Pareto-optimal solutions, then the Hamiltonian H_w is non-degenerate in its minimum eigenvalue.*

We further show that even if Π_d has weakly-equivalent Pareto-optimal solutions, we can have a nondegenerate Hamiltonian.

Lemma 5.4 *Let $x_1, \ldots, x_\ell$ be Pareto-optimal solutions of Π_d^λ that are not pairwise equivalent. If there exists $w \in W_d$ such that $\langle f(x_1), w\rangle = \cdots = \langle f(x_\ell), w\rangle = \sigma_w$ for some $\sigma_w \in \mathbb{R}^+$, then there exists $w' \in W_d$ and $i \in [1,\ell]_\mathbb{Z}$ such that for all $j \in [1,\ell]_\mathbb{Z}$, with $j \neq i$, it holds $\langle f(x_i), w'\rangle < \langle f(x_j), w'\rangle$. In particular, if σ_w is minimum among all $y \in D$, then w' can be chosen such that $\langle f(x_i), w'\rangle$ is unique and minimum among all $y \in D$.*

Lemma 5.5 *Let Π_d^λ be a MCO with no equivalent Pareto-optimal solutions and let H_w be a degenerate Hamiltonian in its minimum eigenvalue with corresponding minimum eigenstates $|x_1\rangle, \ldots, |x_\ell\rangle$. There exists $w' \in W_d$ and $i \in [1,\ell]_\mathbb{Z}$ such that H'_w is nondegenerate in its smallest eigenvalue with corresponding eigenvector $|x_i\rangle$.*

From Lemma 5.5 Theorem 4.1 immediately follows.

6 Application of the Adiabatic Algorithm to the Two-Parabolas Problem

To make use of the Adiabatic algorithm of Section 4 in the Two-Parabolas problem we need to consider a collision-free version of the problem. Let $TP_2^\lambda = (D, R, \mathcal{F})$ be a normal and collision-free MCO where $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^+ \times \mathbb{R}^+$, $D = \{0,1\}^n$, $R \subseteq \mathbb{R}^+$ and $\mathcal{F} = \{f_1, f_2\}$. Let x_0 and x'_0 be the optimal solutions of f_1 and f_2, respectively. We will use x_i to indicate the ith solution of f_1 and x'_i for f_2. Moreover, we assume that $|x_0 - x'_0| > 1$. This latter assumption will ensure that there is at least one non-trivial Pareto-optimal solution. Note that if $|x_0 - x'_0| \leq 1$, the problem only has trivial solutions.

To make TP_2^λ a Two-Parabolas problem, we impose the following conditions.

(i) For each $x \in [0, x_0]$, the functions f_1 and f_2 are decreasing;

(ii) for each $x \in [x'_0, 2^n - 1]$, the functions f_1 and f_2 are increasing;

(iii) for each $x \in [x_0 + 1, x'_0 - 1]$, the function f_1 is increasing and the function f_2 is decreasing.

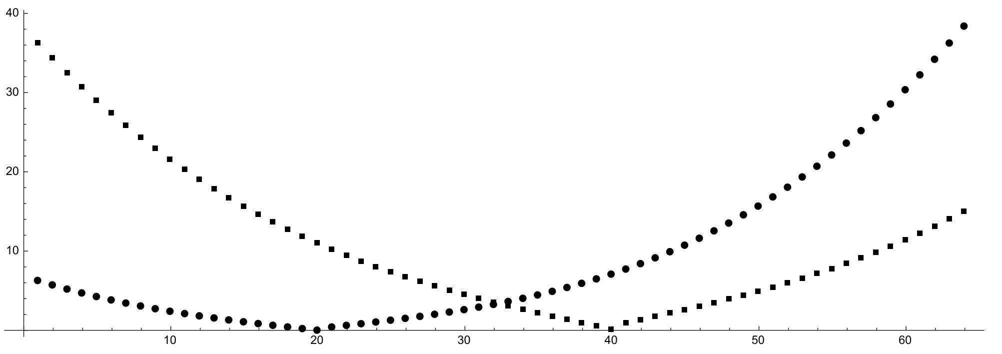

Fig. 2. A discrete Two-Parabolas problem on six qubits. Each objective function f_1 and f_2 is represented by the rounded points and the squared points, respectively. The gap vector $\lambda = (0.2, 0.4)$. The trivial Pareto-optimal points are 20 and 40.

The final and initial Hamiltonians are as in Eq.(4) and Eq.(5), respectively. In particular, in Eq.(5), we define the initial Hamiltonian as

$$\hat{H}_0 = \sum_{x \in \{0,1\}^n \setminus \{0^n\}} |\hat{x}\rangle\langle\hat{x}|. \tag{6}$$

Thus, the Hamiltonian of the entire system for TP_2^λ is

$$H_w(s) = (1-s)H_0 + sH_w. \tag{7}$$

Let $\Delta_{max} = \max_s \|\frac{d}{ds}H_w(s)\|_2$ and $g_{min} = \min_s g(s)$, where $g(s)$ is the eigenvalue gap of $H_w(s)$. It can be proved that $T = O(\frac{\Delta_{max}}{g_{min}^2})$ suffices to find a supported solution corresponding to w [19]. The solution is therefore found in finite time.

The quantity Δ_{max} is usually easy to estimate. The eigenvalue gap g_{min} is, however, very difficult to compute; indeed, determining for any Hamiltonian if $g_{min} > 0$ is undecidable [5].

We present a concrete example of the Two-Parabolas problem on six qubits and estimate the eigenvalue gap. In Fig.2 we show a discretized instance as explained above whereas Table 1 presents a complete specification of all points.

For this particular example we use as initial Hamiltonian $3H_0$, that is, Eq.(6) multiplied by 3. Thus, the minimum eigenvalue of $3H_0$ is 0, whereas any other eigenvalue is 3.

In Fig.3 we present the eigenvalue gap of TP_2^λ for $w = 0.54$ where we let $w_1 = w$ and $w_2 = 1 - w_1$; for this particular value of w the Hamiltonian $H_{F,w}$ has a unique minimum eigenstate which corresponds to Pareto-optimal solution 32. The two smallest eigenvalues never touch, and exactly at $s = 1$ the gap is $|\langle w, f(x_0)\rangle - \langle w, f(x_1)\rangle|$, where $x_0 = 32$ and $x_1 = 31$ are the smallest and second smallest solutions with respect to w, which agrees with lemmas 5.1 and 5.2.

Similar results can be observed for different values of w and a different number of qubits. Therefore, the experimental evidence lead us to conjecture that in the Two-Parabolas problem $g_{min} \leq |\langle w, f(x)\rangle - \langle w, f(y)\rangle|$, where x and y are the smallest and second smallest solutions with respect to w.

Table 1
Complete definition of the Two-Parabolas example of Fig.2 for six qubits.

x	$f_1(x)$	$f_2(x)$	x	$f_1(x)$	$f_2(x)$	x	$f_1(x)$	$f_2(x)$	x	$f_1(x)$	$f_2(x)$
1	6.27	36.139	2	5.709	34.218	3	5.185	32.374	4	4.696	30.605
5	4.24	28.909	6	3.815	27.284	7	3.419	25.728	8	3.05	24.239
9	2.706	22.815	10	2.385	21.454	11	2.085	20.154	12	1.804	18.913
13	1.54	17.729	14	1.291	16.6	15	1.055	15.524	16	0.83	14.499
17	0.614	13.523	18	0.405	12.594	19	0.201	11.71	20	0	10.869
21	0.401	10.069	22	0.605	9.308	23	0.814	8.584	24	1.03	7.895
25	1.255	7.239	26	1.491	6.614	27	1.74	6.018	28	2.004	5.449
29	2.28	4.905	30	2.585	4.384	31	2.906	3.884	32	3.25	3.403
33	3.619	2.939	34	4.015	2.49	35	4.44	2.054	36	4.896	1.629
37	5.385	1.213	38	5.909	0.804	39	6.47	0.4	40	7.07	0
41	7.711	0.8	42	8.395	1.204	43	9.124	1.613	44	9.9	2.029
45	10.725	2.454	46	11.601	2.89	47	12.53	3.339	48	13.514	3.803
49	14.555	4.284	50	15.655	4.784	51	16.816	5.305	52	18.04	5.849
53	19.329	6.418	54	20.685	7.014	55	22.11	7.639	56	23.606	8.295
57	25.175	8.984	58	26.819	9.708	59	28.54	10.469	60	30.34	11.269
61	32.221	12.11	62	34.185	12.994	63	36.234	13.923	64	38.37	14.899

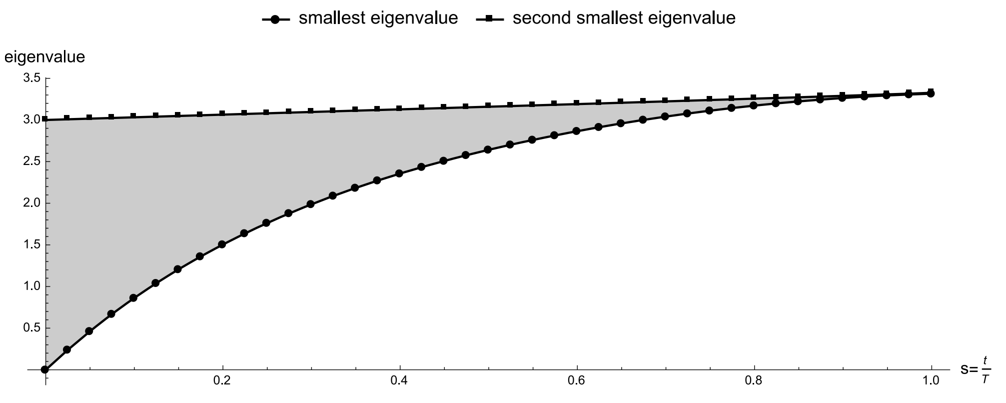

Fig. 3. Eigenvalue gap (in gray) of the Two-Parabolas problem of Fig.2 for $w = 0.54$. The eigenvalue gap at $s = 1$ is exactly $|\langle w, f(x)\rangle - \langle w, f(y)\rangle|$, where $x = 32$ and $y = 31$ are the smallest and second smallest solutions with respect to w.

7 Concluding Remarks and Open Problems

In this work we showed that the quantum adiabatic algorithm of Farhi et al. [9] can be used for multiobjective combinatorial optimization problems. In particular, a simple linearization of the objective functions suffices to guarantee convergence to a Pareto-optimal solution provided the linearized single-objective problem has an unique optimal solution. However, even if a linearization of objectives does not gives an unique optimal solution, then it is always possible to choose an appropriate linearization that does.

We end this paper by listing a few promising and challenging open problems.

(i) To make any practical use of Theorem 4.1 we need to chose $w \in W_d$ in such a way that the optimal solution of the linearization of an MCO has an unique solution. It is very difficult, however, to know a priori which w to chose in order to use the adiabatic algorithm. Therefore, more research is necessary to learn how to select these linearizations. One way could be to constraint the domain of an MCO in order to minimize the number of weak-equivalent solutions.

(ii) Another related open issue is how to solve multiobjective problems in the presence of equivalent solutions. A technique of mapping an MCO with equivalent solutions to Hamiltonians seems very difficult; that is because the smallest eigenvalue must be unique in order to apply the adiabatic theorem.

(iii) According to Theorem 4.1, we can only find all supported solutions. Other works showed that the number of non-supported solutions can be much larger than the number of supported solutions [8]. Hence, it is interesting to construct a quantum algorithm that could find an approximation to all Pareto-optimal solutions.

(iv) Prove our conjecture of Section 6 that the eigenvalue gap of the Hamiltonian of Eq.(7), corresponding to the Two-Parabolas problem, is at most the difference between the smallest solution and second smallest solution for any given linearization of the objective functions.

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