

Correlated Information: A Logic for Multi-Partite Quantum Systems

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Abstract

In this paper we analyze classical and quantum correlations using the tools of epistemic logic. Our main contribution consists of two new logical systems. The first one is called General Epistemic Logic (GEL), it extends traditional epistemic logic with operators that allow us to reason about the information carried by a complex system composed of several parts. The second system is called the Logic of Correlated Knowledge (LCK), which extends GEL with sentences that describe the observational capabilities of an agent. On the semantic side we introduce *correlation models*, as a generalization of the “interpreted systems” semantics. We use this setting to investigate several types of informational correlations (e.g. distributed information, quantum correlated information) that complex systems can exhibit. We also provide an informational-logical characterization of the notion of “quantum entanglement”.

Keywords: quantum correlations, correlation models, general epistemic logic, quantum information, entanglement

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1 Introduction

Our topic in this paper is the logical analysis of informational correlations. These are taken to be correlations between the information carried by each of the parts of a complex system. Examples of complex systems include multi-partite physical systems, groups of “agents” with observational capabilities, logical inferences etc. Essential is that such complex systems are composed of elementary parts that can either be independent of each other or may exhibit informational correlations. For instance, in a group of agents, the agent’s observations can be correlated due to previous communication. While in physical systems, correlations may be due to the effect of quantum entanglement. Hence, different types of informational correlations are manifested in different types of complex systems⁴. Our study of these informational dependencies will allow us to mark the difference between classical and quantum correlations.

Our approach in this paper uses the formalism of modal logic. In particular, we use a version of *epistemic logic* to analyze the mentioned notions. Note however that we identify the role played by an *epistemic “agent”* in traditional epistemic logic, with the role of *the elementary parts* (“subsystems”, or “locations”) *in a complex physical system*. Hence, our notion of an “agent’s knowledge” differs from the accounts linked to traditional epistemic logic. Indeed, we adopt an “external view” of knowledge. This view is standard in Computer Science and has been advocated for in [11]. In this sense, every complex system can be regarded as a “group of agents”. Moreover, any localized part of a complex system can be seen as an “agent”. So in our setting, an agent’s “implicit knowledge” refers to the *information that is potentially available at the corresponding location*. Something is “implicitly known” if it is a *consequence of features that can in principle be observed at that location*. So, implicit knowledge gives us the information (about the overall system) that is carried by a part of the system.⁵ We highlight in our approach the *spatial features of epistemic logic*, as these will turn out to be essential in our study of the informational correlations between the different parts of a complex system. In contrast to other spatial logics that are based on topologies or metric spaces, our logic is aimed to capture only “*local*” *features of space*, indicating *which information is carried by which part of the system*.

We are interested in the ways in which the (information carried by the) individual parts of a complex system can be combined. In the context of logic, this relates to the question regarding what is the “implicit” knowledge that can be assigned

⁴ Our main concern is with the “qualitative” (“logical”, or “semantic”) aspects of the information carried by the parts of a complex system. We will not focus on syntactic aspects or quantitative measures of information (be it in terms of Shannon entropy or von Neumann entropy).

⁵ The distinction between “explicit” and “implicit” knowledge is standard in Computer Science. The first notion refers to the actual information possessed by a real agent, stored in its “database” or subjective internal state. While implicit knowledge can be thought of as what a *virtual agent could* in principle *come to “know”* by performing observations on that system and deriving logical consequences. Hence implicit knowledge is what “we” externally assign to an agent (or to a particle or even just a spatial location), based on what is in principle observable at that location. Note that we use the notion of “information carried by a system” interchangeably with the notion of “implicit knowledge” (in line also with [5,7]). In contrast to explicit knowledge, implicit knowledge can be assumed to be always truthful and closed under logical inference and under (positive and negative) introspection and as such it satisfies the axioms of the modal system known as S5.

to a *group* of agents. In our setting, such implicit “group knowledge” captures the information carried by a (whole) complex physical system. In the literature on logic for Computer Science, the standard formalization of implicit group knowledge is in terms of *distributed knowledge*. This refers to the information obtainable by pooling together (and closing under logical inference) the “knowledge” of each of the “parts” (agents) of a complex system. The standard view in epistemic logic is that the implicit knowledge of a group would be the same as distributed knowledge: the information carried by a complex system is then nothing more than the “sum” of the information carried by its parts.

But in the context of quantum mechanics, it is almost obvious that this standard view cannot be correct. The information carried by an entangled quantum system is more than the sum of the information carried by its parts. Hence we claim that, while the standard answer is adequate for classical physics, it does *fail for quantum systems*. As a solution we propose the notion of “*correlated knowledge*”. As the name suggests, correlated knowledge takes into account the correlations between the pieces of information carried by the parts of the system. We will argue that correlated knowledge is a better model for the information carried by an arbitrary complex system (or group of agents).

In the next section we introduce the logical system called “General Epistemic Logic”(GEL). GEL is based on an extension of traditional epistemic logic with operators for group knowledge. In section 3 we show how GEL can be regarded as a generalization of the “interpreted systems” semantics of [11] and we discuss the notion of distributed knowledge. In section 4 we show how GEL can be used to reason about implicit quantum knowledge. The formalism of GEL allows us to give an *informational-logical characterization of the properties of “separability” and “entanglement”*. In the last section we extend GEL with operators that explicitly capture the agents’ observational capabilities. The obtained system is called the “Logic of Correlated Knowledge” (LCK) and is interpreted on so-called “correlation models”. A complete axiomatization of LCK has been presented in an extended version of this paper in [6].

2 General Epistemic Logic

In this section we introduce a *generalized “epistemic logic”* with epistemic operators for both *individuals* (or “agents”) and for *groups* of agents. In the following, we use the notion of component of a complex system and the notion of an agent interchangeably. We think about this metaphorically, associating to *each subsystem a virtual agent that can “observe” only the state of that subsystem*. When considering a physical system, this sums up all the information that is obtainable by performing *local measurements* on that subsystem (and closing under logical inference).

To introduce our notation, consider a complex system composed of n basic *components* (or “locations”). Each basic component is denoted with a label from a given (finite) set $N = \{1, \dots, n\}$. Note that the information carried by this complex system may be concentrated or *distributed* over specific spatial “locations” or

“components” of the system. So it is important to note that some information is potentially available only at some locations but not at others. We also consider the information carried by *subsystems* that are composed of several (but not necessarily all) components. *Sets* of labels $I \subseteq N$ are used to denote complex subsystems or groups of “agents”. N is the largest group (the “whole world”), while the smallest groups are singletons $\{i\}$.

For each subsystem I , we have an “information” (or “implicit knowledge”) operator K_I in our logic. Again, the notion of “knowledge” is used in this paper only in the *implicit, external* sense. Hence it refers to the “information that is in principle available” at a given location. Intuitively, we read the proposition $K_I P$ as saying that the subsystem, or group, I (potentially) carries the information that P is the case. For singleton groups $\{i\}$ of one agent, we use the simplified notation K_i instead of $K_{\{i\}}$. Our use of these K_I -operators in order to capture the qualitative spatial features of complex systems does extend our previous approach in [5], which in its turn is based on [4,2] and inspired by [9,1].

Observational Equivalence. For s and s' , two possible states of the world, if the implicit information carried by system I is *the same* in these states then we call these states “*observationally equivalent*”, or “*indistinguishable*” for system I , and denote this as $s \stackrel{I}{\sim} s'$. We simply write $s \stackrel{i}{\sim} s'$ in case $I = \{i\}$. Observational equivalence for I means in agent’s terms that the virtual agent or group of agents (associated to) I *can make exactly the same observations* (at location I) in two states of the world. The relation of observational equivalence is next used to give an interpretation to the K_I modal operators:

General Epistemic Frame (GEF). We define a *general epistemic frame* to be a Kripke frame (or multi-modal frame) $(\Sigma, \{\stackrel{I}{\sim}\}_{I \subseteq N})$ consisting of a given set N of basic components, a set of states (or “possible worlds”) Σ , a family of binary relations $\{\stackrel{I}{\sim}\}_{I \subseteq N} \subseteq \Sigma \times \Sigma$ for every subsystem $I \subseteq N$, and subject to the following conditions:

- (i) all $\stackrel{I}{\sim}$ are (labeled) *equivalence relations*;
- (ii) *Information is monotonic w.r.t. groups*: if $I \subseteq J$ then $\stackrel{J}{\sim} \subseteq \stackrel{I}{\sim}$;
- (iii) *Observability Principle*: if $s \stackrel{N}{\sim} s'$ then $s = s'$;
- (iv) *Vacuous Information*: $s \stackrel{\emptyset}{\sim} s'$, for all $s, s' \in \Sigma$.

These conditions seem natural for the interpretation of $\stackrel{I}{\sim}$ as “observational equivalence”. The first condition is self-explanatory. The second condition captures the fact that information behaves monotonic with respect to group inclusion. It assumes that every observation that can be performed by an agent of a group is in principle available to the whole group. Or in other words, this condition captures the fact that the members of a group *have the capacity to share information* among themselves. The third condition is called the “*observability*” principle: it identifies states that differ in ways that are not observable even by the whole world N . The last condition captures the fact that the empty group carries no non-trivial information, it cannot make observations and hence cannot tell any two states apart.

Local Information State. Given a GEF $(\Sigma, \{\sim^I\}_{I \subseteq N})$, we define the notion of an I -local state, for each subsystem $I \subseteq N$: $s_I := \{s' \in \Sigma : s \sim^I s'\}$.

Σ -Propositions and Models. Take a GEF Σ , we call a Σ -proposition any subset $P \subseteq \Sigma$. Intuitively, we say that a state s satisfies proposition P if $s \in P$. We define a general epistemic model to be a structure $\Sigma = (\Sigma, \{\sim^I\}_{I \subseteq N}, \|\cdot\|)$, consisting of a GEF and a valuation map $\|\cdot\|: \Omega \rightarrow P(\Sigma)$ (where Ω is a given set of atomic sentences). We adopt the standard notation for satisfaction of atomic sentences in a given state of model Σ denoted by $s \models p$ or $s \in \|p\|$. For every model Σ , we have the usual Boolean operators on Σ -propositions: $P \wedge Q := P \cap Q$, $P \vee Q := P \cup Q$, $\neg P := \Sigma \setminus P$, $P \Rightarrow Q := \neg P \vee Q$. We have the constants $\top_\Sigma := \Sigma$ and $\perp_\Sigma := \emptyset$. The “knowledge” operator is defined on Σ -propositions as $K_I P := \{s \in \Sigma : t \in P \text{ for every } t \sim^I s\}$.

General Epistemic Logic. The language of the logic GEL has the following syntax, where $p \in \Omega$ denote the atomic sentences:

$$\varphi := p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K_I \varphi$$

The semantics is given by an *interpretation map* associating to each sentence φ of GEL a proposition $\|\varphi\|$. The definition is by induction in terms of the obvious compositional clauses. Let us focus only on the K_I modality in particular:

$s \models K_I \varphi$ iff $t \models \varphi$ for all states $t \sim^I s$. Hence $K_I \varphi$ is true in a state s , (or a system carries the information that φ is the case in state s) if and only if φ holds in all states of the world that are observationally equivalent for I to s .

Proof System. The proof system of GEL includes the rules and axioms of propositional logic in addition to the following list:

- (i) *K_I -Necessitation.* From $\vdash \varphi$, infer $\vdash K_I \varphi$
- (ii) *Kripke’s Axiom.* $\vdash K_I(\varphi \Rightarrow \psi) \Rightarrow (K_I \varphi \Rightarrow K_I \psi)$
- (iii) *Truthfulness.* $\vdash K_I \varphi \Rightarrow \varphi$
- (iv) *Positive Introspection.* $\vdash K_I \varphi \Rightarrow K_I K_I \varphi$
- (v) *Negative Introspection.* $\vdash \neg K_I \varphi \Rightarrow K_I \neg K_I \varphi$
- (vi) *Monotonicity of Group “Knowledge”.* For $I \subseteq J$, we have $\vdash K_I \varphi \Rightarrow K_J \varphi$
- (vii) *Observability.* $\vdash \varphi \Rightarrow K_N \varphi$

Using standard results in Modal Correspondence theory (see e.g. [8]), a proof for the following result has been sketched in an extended version of this paper [6].

Proposition 1 *The above proof system is sound and complete with respect to general epistemic frames.*

3 Distributed Knowledge

The notion of distributed knowledge traces back to [12]. In [11] we read that a group has distributed knowledge of a proposition φ if roughly speaking the agents’

combined knowledge implies φ . Intuitively this means that the agents of a group can “combine” their knowledge by sharing all they know, in addition to making individual observations. It boils down to the fact that *group members can announce to the group the knowledge obtained on the basis of their separate observations*.

To capture this notion formally, we introduce a modality DK_I for the distributed knowledge of a group I . DK_I is defined via a “distributed observational equivalence” relation, which is given by the intersection $\bigcap_{i \in I} \overset{i}{\sim}$ of all the individual observational equivalence relations. It captures the idea that two states are indistinguishable for the group iff they are indistinguishable for *all* the members of the group. The operator DK_I is the Kripke modality for $\bigcap_{i \in I} \overset{i}{\sim}$, given explicitly as: $s \models DK_I \varphi$ iff for every state $t \in \Sigma$, if $\forall i \in I \ s \overset{i}{\sim} t$ then $t \models \varphi$.

The authors in [11] do identify implicit knowledge of a group I with the distributed knowledge DK_I of that group. As mentioned in the introduction, we cast doubt on the fact that this is really the correct definition to adopt in all cases. Recall that implicit knowledge was explained as what the agent/group could come to know based on potential observations. Hence the question amounts to what are the observational capabilities of a group in general? Clearly, the use of the intersection of individual observational equivalencies (as adopted in the definition of distributed knowledge) makes sense only if one assumes that a group’s observations are nothing but observations done by either of the members of the group. We claim however that this is in general a highly unreasonable assumption that precludes the possibility of any joint observations by the group. To obtain distributed knowledge, each agent only shares with the group the end-result of all her separate observations, the agents are not allowed to correlate (the results of) their observations. The natural notion of group knowledge according to us is not distributed knowledge but rather something that could be called “correlated knowledge”, that what a group could come to know by performing joint observations and sharing the results.

Separability. For system $J \subseteq I$, we say that system I is *J-separable* in state s if $s_J \cap s_{I \setminus J} = s_I$. System I is said to be *fully separable* in state s if $\bigcap_{i \in I} s_i = s_I$. *Full separability means that group I ’s knowledge in state s is the same as its distributed knowledge*, i.e. $s \models K_I P$ iff $s \models DK_I P$, for all sets $P \subseteq \Sigma$. If I is fully separable then it is *J-separable* for all $J \subseteq I$, but the converse is false in general. We call a state *I-entangled* if it is not *I-separable*.

Classical Epistemic Frame. A GEF $(\Sigma, \{\overset{I}{\sim}\}_{I \subseteq N})$ is called **classical** if *all its states are fully separable*, i.e. if it satisfies $\overset{I}{\sim} = \bigcap_{i \in I} \overset{i}{\sim}$ for all systems I . A frame is classical iff *any group’s “knowledge” in any state (coincides with its distributed knowledge, and hence) can be obtained by pooling together the information of each of its components*: $K_I = DK_I$.

Not only classical or macroscopic systems will satisfy the conditions of a classical epistemic frame. We may encounter classical epistemic frames in the *quantum world* for instance when the subsystems are *separated*.

Interpreted Systems. Classical epistemic frames are close connected to S5 Kripke

models. To show this, we take for each i , $\Sigma_i := \{s_i : s \in \Sigma\}$ the set of all i -local states. *Every classical epistemic frame Σ can be canonically embedded via some embedding e into the Cartesian product $\Sigma_1 \times \Sigma_2 \times \cdots \Sigma_n$ in such a way that $s \stackrel{I}{\sim} s'$ iff $e(s)_i = e(s')_i$ for all $i \in I$.* This is essentially the well-known “interpreted systems” representation of epistemic (S5) Kripke models, in the style of [11], in which global states are simply taken to be tuples of local states, with identity of the i -th components as the indistinguishability relation $\stackrel{i}{\sim}$. Note that such a representation is not possible in the general (non-classical) case.

4 Quantum “Knowledge”

We represent a single quantum system by a state space Σ consisting of *rays* in a Hilbert space H . A quantum system composed of N subsystems $\Sigma_1, \dots, \Sigma_n$ is represented by the state space $\Sigma_1 \otimes \cdots \otimes \Sigma_n$ corresponding to the *tensor product* $H_1 \otimes \cdots \otimes H_n$. The tensor product is richer than the Cartesian product, so we *cannot* view a composed quantum system as a *classical epistemic frame*. Nevertheless, as we will show further in this section, they *can be thought of as (non-classical) general epistemic frames*.

First, consider the question: what is the “state” of an entangled subsystem I ? Or, what is the state of component 1 in the binary system $|00\rangle + |11\rangle$? For *I-separable* states $s = s_I \otimes s_{N \setminus I}$ the answer is simply given by the local state s_I . But we saw that one cannot talk in any meaningful way about the I -local state of an i -entangled system. In the following, we first give the standard definition used in Quantum Mechanics, which is given in terms of density operators. After that we justify its usefulness for our purposes by looking at the results of local observations.

The State of a Subsystem. If a global system is in state s (having an associated density operator ρ_s), then QM describes the state $s_{(I)}$ of any of its subsystems I (possibly entangled with its environment $N \setminus I$ in the state s) by the density operator $s_{(I)} := \text{tr}_{N \setminus I}(\rho_s)$. The “state” of subsystem I is obtained by taking the partial trace $\text{tr}_{N \setminus I}$ (with respect to the subsystem’s environment $N \setminus I$) of (the density operator associated to) the global state s .

In case the subsystem I is entangled with its environment $N \setminus I$, the above description does not really give us a “state” in the sense of this paper (i.e. a *pure state*), but a “mixed state”. But as an abstract description, it can still give us an *indistinguishability relation* $\stackrel{I}{\sim}$ on global states. Namely, the specific definition of the “state” of a subsystem is not relevant for us in itself, but only the resulting notion of “identity of states” of the given subsystem. This leads to the following definition:

Observational Equivalence in Quantum Systems. Two quantum states s, s' of a global quantum system N are *observationally equivalent* (“indistinguishable”) for a subsystem $I \subseteq N$ if the mixed states of subsystem I are *the same* in s and s' . Formally: $s \stackrel{I}{\sim} s'$ iff $\text{tr}_{N \setminus I}(\rho_s) = \text{tr}_{N \setminus I}(\rho_{s'})$.

The definition using density operators may look unnatural, although it can be

justified via what an observer can learn about an entangled subsystem I *by observing only that subsystem* (so by performing local measurements on I). While it is known that a mixed state corresponds to a probability measure over pure states, what is not always well-appreciated is the *meaning* of a mixed state $s_{(I)}$ describing a (possibly entangled) subsystem:

Quantum I -equivalence via local observations. Assuming that we have *an unlimited supply* of identical I -entangled systems, all prepared in the *same* (entangled) global state s . And imagine that a virtual agent associated to I can perform *all possible local measurements* (in various bases) on (various copies of) subsystem I . Suppose that the agent can also repeat the same tests on different copies and observe the *frequency of each result*. After many tests, this agent can approximate the *probability of every given result, for each possible local measurement*. The list of all these probabilities gives us the “information carried by subsystem I ”, or the “information obtainable by local observations at location I ”. Hence two global states s, s' are I -indistinguishable if all these probabilities are the same in s and s' , i.e. if the two states behave the same way under I -local measurements.

Quantum I -equivalence via remote evolutions. Observational equivalence can alternatively also be defined via the *invariance* under changes that do *not* affect the information carried by subsystem I (see also [5]). An evolution (or unitary map) U is said to be I -remote (or “remote from I ”) if it corresponds to applying only a local unitary map on the subsystem $N \setminus I$ (the “non- I ” part of the system, also known as I ’s “environment”): i.e., if U is of the form $Id_I \otimes U_{N \setminus I}$, where Id_I is the identity map on subsystem I and $U_{N \setminus I}$ is a unitary map on the subsystem $N \setminus I$. Hence, U is I -remote if it is $N \setminus I$ -local. Intuitively, I -remote evolutions should not affect the “state” of subsystem I ; hence, we could *define* the “state” of I as what is left *invariant* by all I -remote evolutions. As a consequence, two states will be I -indistinguishable if they differ only by some I -remote evolution.

These three ways of defining observational equivalence are equivalent:

Proposition 2 *For $I \subseteq N$, $s, s' \in \Sigma$, the following are equivalent:*

- (1). $tr_{N \setminus I}(\rho_s) = tr_{N \setminus I}(\rho_{s'})$;
- (2). *for every I -local measurement, the probability of obtaining any given result is the same in state s as in state s' ;*
- (3). $s' = U(s)$ for some I -remote unitary map;

We can define the quantum equivalence relation $s \stackrel{I}{\sim} s'$, and hence the *implicit knowledge* K_I , by any of the clauses given above. If we adopt the third clause, we obtain the following: $K_I P$ holds at s iff, for all I -remote evolutions U , P holds at $U(s)$. If P is implicitly known by I in state s , i.e. if $s \in K_I P$, then we say that *subsystem I carries the information that P* .

Call a *quantum epistemic frame* a (state space Σ associated to a) Hilbert space endowed with the quantum I -equivalence relations $\stackrel{I}{\sim}$ (as defined above) for every subsystem I .

Proposition 3 *Quantum epistemic frames satisfy all the postulates of general epistemic frames.*

Properties. In addition to the properties of the K_I operator in GEL, we add:

- If s is I -separable, then $s \stackrel{I}{\sim} s'$ iff $s_I = s'_I$
- If I is fully separable then we have: $s \stackrel{I}{\sim} s'$ iff $s \stackrel{i}{\sim} s'$ for all $i \in I$. As a consequence, the quantum “group” knowledge K_I of a fully separated system I is the same as the “distributed knowledge” DK_I of the “group” I .
- In general (for non-fully separated systems I), the previous statement is false: the information K_I carried by a quantum (sub)system I is *not* the “sum” $DK_{i \in I}$ of the information carried by its i -component systems. *Quantum epistemic frames are “non-classical”.*

Example. In a Bell state when the information stored in two subsystems is correlated according to the identity rule, the agents associated to these subsystems will *never recover fully the information possessed by the global system if they cannot correlate the results of their individual observations.*

Informational Characterizations of Separability and Entanglement

We already gave an “epistemic” characterization of entanglement and separability in a GEF: *A state s is I -separable iff I ’s knowledge in state s is the same as its distributed knowledge.* In the special case of quantum systems, this gives us the standard QM notion of separability. However, this characterization cannot be expressed in the language of epistemic logic as it involves a second-order quantifier over all subsets P of the state space. It basically requires that, for every such subset, s satisfies $K_I P$ iff it satisfies $DK_I P$. But in line with our presentation in [5], we *can* use epistemic logic to give “informational characterizations” of separability and entanglement in a quantum system, *provided we are given only one (logical constant denoting a) fully separable state.* Take $w = w_0 \otimes \dots \otimes w_n$ to be some (fixed) fully separable state; for example, we may take $w = \bar{0} = |0\rangle^{\otimes N} = |0\rangle \otimes |0\rangle \dots \otimes |0\rangle$. Then: two subsystems I and J are *entangled* in a (global) state s iff s satisfies $K_J K_I \neg w$ or, equivalently, $K_I K_J \neg w$. The state s is *I -entangled* iff the subsystem I and $N \setminus I$ are entangled in s , i.e. if s satisfies $K_I K_{N \setminus I} \neg w$. The system is separable if it is not entangled: $\neg K_I K_{N \setminus I} \neg w$. To summarize: *two physical systems are entangled if and only if they potentially carry (non-trivial) information about each other* (assuming no prior communication).

Example. For $n = 2$, take the set $\Sigma_{(1)}$ of all 1-separable (=2-separable=fully separable) global states, as our model. Given that the system is in state $|00\rangle$, subsystem 1 is in state $|0\rangle$ and “implicitly knows” his own state. 1 implicitly knows that it is not possible that the whole system is in state $|10\rangle$. Hence, $|00\rangle \models K_1 \neg |10\rangle$. Moreover, $|00\rangle \models \neg K_1 K_2 \neg |11\rangle$.

5 Correlated Knowledge

The complex systems we consider in this paper can be modeled more accurately if we add some extra structure to our GEF. This can be done for instance by enriching the language of the logic to capture the observational capabilities of the individual agents and of the groups explicitly. Before we deal with the language in detail, we will first generalize the concrete semantics given by “interpreted systems” to a type of GEF that we call *correlation models*.

We generalize interpreted systems in three stages: the first type of models that we consider are *relation-based models*. Here the *states are relations between the agents’ possible observations*. Given sets O_1, \dots, O_n of possible observations for each agent, a *joint observation* will be a tuple of observations in $O_1 \times \dots \times O_n$. A state of the world can be characterized by the joint observations that can be performed on it, so a state is a set of such tuples namely a relation. A model will have as its state space any set $\Sigma \subseteq \mathcal{P}(O_1 \times \dots \times O_n)$. The state s_I of subsystem I of a global system in state s will be naturally given by the projection: $s_i = \{(o_i)_{i \in I} : \vec{o} \in s\}$. So the observational equivalence is given by $s \stackrel{I}{\sim} t$ iff $\{(o_i)_{i \in I} : \vec{o} \in s\} = \{(o_i)_{i \in I} : \vec{o} \in t\}$.

The second stage of generalization is given by *multi-set models*. We now consider multi-sets instead of sets of tuples of observations as in the relational models. Using multisets has the advantage that we can model the case when agents record the frequencies of their observations. Now states are multi-sets of joint observations, i.e. functions s from tuples of observations from $O_1 \times \dots \times O_n$ into natural numbers. The state s_I of subsystem I in global state s is given by

$$s_I((x_i)_{i \in I}) := \sum \{s(\vec{o}) : \vec{o} \in O_1 \times \dots \times O_n \text{ such that } o_i = x_i \text{ for all } i \in I\}$$

The third type of models that we consider are *correlation models*. In this stage, we generalize natural numbers to an abstract set R of possible *observational results*, together with some abstract operation $\sum : \mathcal{P}(R) \rightarrow R$ of *composing results*. This operation may be *partial* (i.e. defined only for some subsets $A \subseteq R$), but it is required to satisfy the condition: $\sum\{A_k : k \in K\} = \sum(\bigcup_{k \in K} A_k)$ whenever $\{A_k : k \in K\}$ are pairwise disjoint. Working with this type, (R, \sum) is called a *result structure*.

Correlation Models Given a result structure R and a tuple $\vec{O} = (O_i)_{i \in N}$ of sets of possible observations, a *correlation model over (R, \vec{O})* is given by a set $\Sigma \subseteq \{s : s \text{ is a function } : O_1 \times \dots \times O_n \rightarrow R\}$ of maps assigning results to joint observations $\vec{o} = (o_i)_{i \in N}$. So global states will then be functions from $O_1 \times \dots \times O_n$ into R . We put $O_I := \times_{i \in I} O_i = \{(o_i)_{i \in I} : o_i \in O_i \text{ for every } i \in I\}$. As before, in global state s , the state s_I of subsystem I will be a map $s_I : O_I \rightarrow R$, given by $s_I((x_i)_{i \in I}) := \sum\{s(\vec{o}) : \vec{o} \in O_1 \times \dots \times O_n \text{ such that } o_i = x_i \text{ for all } i \in I\}$.

Correlated Knowledge *Correlation models are general epistemic models*, if we take our *observational equivalence to be identity of the corresponding local states*: $s \stackrel{I}{\sim} t$ iff $s_I = t_I$. The “group knowledge” K_I in a correlation model will be called *correlated knowledge*.

In general, *correlation models are not necessarily classical (as epistemic frames)*. Hence, *correlated knowledge is in general different from distributed knowledge*.

Examples The *relation-based models* can be recovered as special cases of correlation models, if we take $R = \{0, 1\}$ and logical *disjunction* as the composition operation. *Interpreted systems* are special cases of relation-based models (in which every state is a singleton consisting of only one joint observation), and hence they also are correlation models. The *multi-sets models* are also correlation models, with R being the set of natural numbers, and *addition* as the composition operation. *Quantum epistemic systems* $\Sigma_1 \otimes \Sigma_2 \otimes \cdots \otimes \Sigma_n$ are also correlation models with the sets of observations O_i given by the (state spaces associated to) Hilbert spaces Σ_i . Joint observations $(o_i)_{i \in I}$ are interpreted as projectors onto the corresponding state in $\bigotimes_{i \in I} \Sigma_i$, i.e. local measurements yielding the given outcome $(o_i)_{i \in I}$. The result structure is the interval $R = [0, 1]$ with renormalized addition. The “result” of a joint observation $(o_i)_{i \in I}$ made on a state s is interpreted as the *probability* that the outcome of a local measurement (in any basis that includes $o = \bigotimes_{i \in I} o_i$) of the I -subsystem of a (global system in) state s will be o . Indeed, it is known that any quantum state $s \in \bigotimes_{i \in N} \Sigma_i$ is uniquely characterized (up to multiplication by a non-zero scalar) by the function mapping every fully separable state $o = o_1 \otimes \cdots \otimes o_n \in \Sigma_1 \times \cdots \times \Sigma_n$ to the probability $| \langle s, o \rangle |^2$ of s collapsing to o (after a measurement in any basis that includes o).⁶

The Logic of Correlated Knowledge. The logic of correlated knowledge LCK extends the general logic GEL with atomic sentences describing the results of possible joint observations by groups of agents:

$$\varphi ::= p \mid o_I^r \mid \neg\varphi \mid \varphi \wedge \psi \mid K_I\varphi$$

with $r \in R$ and $o_I = (o_i)_{i \in O_I} \in O_I$ is a I -tuple of observations. (Recall that $O_I := \times_{i \in I} O_i$.) The *semantics* of o_I^r is naturally given by: $s \models o_I^r$ iff $s_I(o_I) = r$.

Proposition 4 *For every finite set $N = \{1, \dots, n\}$ of agents, every finite result structure (R, \sum) and every tuple of finite observation sets $\vec{O} = (O_1, \dots, O_n)$, there exists a complete axiomatization of the above logic LCK with respect to correlation models over (R, \vec{O}) .⁷*

Examples of interesting axioms:

Observations have unique results; i.e. for $r \neq p$, we have $o_I^r \Rightarrow \neg o_I^p$.

Groups know the results of their (joint) observations: $o_I^r \Rightarrow K_I o_I^r$.

Group knowledge is correlated knowledge (based on joint observations): for every tuple $(r_o)_{o \in O_I}$ of results, one for each possible joint observation $o = (o_i)_{i \in I} \in O_I$ by group I , we have $K_I\varphi \wedge \bigwedge \{o^{r_o} : o \in O_I\} \Rightarrow K_\emptyset (\bigwedge \{o^{r_o} : o \in O_I\} \Rightarrow \varphi)$.

⁶ A different type of relational models for a generalized version of QM is proposed in [10]. Note that our models are “relational” in the sense that *quantum “states” correspond in our settings to relations* (in relation-based models) or *functions* (in correlation models). In contrast, in the categorical approach of [10], *relations* (between finite sets) *play the role of morphisms*, i.e. they are the *analogue of linear maps* (between Hilbert spaces) in QM.

⁷ The details are presented in an extended version of this paper in [6].

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