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Unique Fixed Points in Domain Theory

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Abstract

We unveil new results based on measurement that guarantee the existence of unique fixed points which *need not* be maximal. In addition, we establish that least fixed points are *always* attractors in the μ topology, and then explore the consequences of these findings in analysis. In particular, an extension of the Banach fixed point theorem on compact metric spaces [3] is obtained.

1 Introduction

The standard fixed point theorem in domain theory states that a Scott continuous map $f : D \to D$ on a dcpo D with least element $\perp$ has a least fixed point given by

$$\mathrm{fix}(f) := \bigsqcup_{n \geq 0} f^n(\perp).$$

This is perhaps the single most important result in domain theory, given its effectiveness in handling the semantics of recursion, and the fact that its reasoning extends naturally to the categorical level to explain why it is that equations like $D \simeq [D \to D]$ may be solved.

It could be argued that one of its faults is that it only applies to *continuous* mappings, since there are now more general fixed point theorems available [5]. However, within the context of continuous mappings, the only criticism that seems plausible is that its canonical fixed points are not as canonical as they could be. There is, after all, one thing more satisfying than a least fixed point: A unique fixed point.

Using ideas all originally introduced in [6], we establish that there are natural fixed point theorems in domain theory which guarantee the existence of unique, attractive fixed points. In the next three sections, we discuss domains, content and invariance. These are preliminary ideas needed later on. We then introduce contractions on domains and prove a fixed point theorem about them very reminiscent of the Banach theorem in analysis. In fact, this new result has the Banach theorem as one of its consequences.

Finally, in the case of a domain with a least element, we learn that least fixed points are *always* attractive in the μ topology and that the results on contractions also hold for a larger and more natural class of *nonexpansive* mappings. Because of the latter, an improvement of the Banach fixed point theorem on compact metric spaces [3] can be obtained.

2 Background

A *poset* is a partially ordered set [1].

Definition 2.1 Let $(P, \sqsubseteq)$ be a partially ordered set. The *least element* $\perp$ of P satisfies $\perp \sqsubseteq x$ for all x, when it exists. A nonempty subset $S \subseteq P$ is *directed* if $(\forall x, y \in S)(\exists z \in S)\, x, y \sqsubseteq z$. The *supremum* of a subset $S \subseteq P$ is the least of all its upper bounds provided it exists. This is written $\bigsqcup S$. A *dcpo* is a poset in which every directed subset has a supremum.

Definition 2.2 For a subset X of a dcpo D, set

$$\uparrow\! X := \{y \in D : (\exists x \in X)\, x \sqsubseteq y\} \;\;\&\;\; \downarrow\! X := \{y \in D : (\exists x \in X)\, y \sqsubseteq x\}.$$

We write $\uparrow x = \uparrow\{x\}$ and $\downarrow x = \downarrow\{x\}$ for elements $x \in X$. The set of *maximal elements* in a dcpo D is $\max D = \{x \in D : \uparrow x = \{x\}\}$.

By Hausdorff maximality, every dcpo has at least one maximal element.

Definition 2.3 A subset U of a dcpo D is *Scott open* if

(i) U is an upper set: $x \in U \;\&\; x \sqsubseteq y \Rightarrow y \in U$, and

(ii) U is inaccessible by directed suprema: For every directed $S \subseteq D$,

$$\bigsqcup S \in U \Rightarrow S \cap U \neq \emptyset.$$

The collection σ_D of all Scott open sets on D is called the Scott topology.

Proposition 2.4 *A map $f : D \to E$ between dcpo's is Scott continuous iff*

(i) *f is monotone: $x \sqsubseteq y \Rightarrow f(x) \sqsubseteq f(y)$.*

(ii) *f preserves directed suprema: For every directed $S \subseteq D$,*

$$f(\bigsqcup S) = \bigsqcup f(S).$$

Definition 2.5 In a dcpo $(D, \sqsubseteq)$, $a \ll x$ iff for all directed subsets $S \subseteq D$, $x \sqsubseteq \bigsqcup S \Rightarrow (\exists s \in S)\, a \sqsubseteq s$. We set $\mathord{\downarrow\!\!\downarrow} x = \{a \in D : a \ll x\}$. A dcpo D is *continuous* if $\mathord{\downarrow\!\!\downarrow} x$ is directed with supremum x for all $x \in D$.

The sets $\mathord{\uparrow\!\!\uparrow} x = \{y \in D : x \ll y\}$ for $x \in D$ form a basis for the Scott topology on a continuous dcpo D. Finally, we adopt the following definition of 'domain' in this paper.

Definition 2.6 A *domain* is a continuous dcpo D such that for all $x, y \in D$, there is $z \in D$ with $z \sqsubseteq x, y$.

For example, a continuous dcpo with *least element* $\perp$ is a domain in the present sense.

3 Content

The ideas in this and the next section are covered in embarassing detail in [6]. Let $[0,\infty)^*$ be the domain of nonnegative reals ordered as $x \sqsubseteq y \Leftrightarrow y \le x$.

Definition 3.1 A Scott continuous map $\mu : D \to [0,\infty)^*$ on a continuous dcpo D *induces the Scott topology near* $X \subseteq D$ if for all $x \in X$ and all sequences $x_n \ll x$,

$$\lim_{n\to\infty} \mu x_n = \mu x \Rightarrow \bigsqcup x_n = x,$$

and this supremum is directed. We write this as $\mu \to \sigma_X$.

That is, if we *observe* that a sequence (x_n) of approximations calculates x, it *actually does* calculate x. The map μ measures the information content of the objects in X. For this reason, we sometimes say that μ *measures* X.

Definition 3.2 A *measurement* is a Scott continuous map $\mu : D \to [0,\infty)^*$ that measures the set $\ker \mu = \{x \in D : \mu x = 0\}$.

The nature of the idea is that information imparted to us by a measurement about the environment may be taken as true. Here is an illustration of this principle.

Proposition 3.3 *Let $\mu : D \to [0,\infty)$ be a measurement with $\mu \to \sigma_D$. Then*

(i) *For all $x \in D$, $\mu x = 0 \Rightarrow x \in \max D$.*

(ii) *For all $x, y \in D, x \sqsubseteq y$ & $\mu x = \mu y \Rightarrow x = y$.*

(iii) *A monotone map $f : D \to D$ is Scott continuous iff $\mu f : D \to [0,\infty)^*$ is Scott continuous.*

One of the first motivations for measurement was the desire to prove useful fixed point theorems. Generally speaking, 'useful' means theorems which are easy to apply to nonmonotonic mappings, or results which say more about monotonic maps than the Scott fixed point theorem. Here is the first example ever found of the latter type [6].

Theorem 3.4 *Let $f : D \to D$ be a monotone map on a domain D with a measurement μ for which there is a constant $c < 1$ such that*

$$(\forall x)\, \mu f(x) \le c \cdot \mu x.$$

If there is a point $x \in D$ with $x \sqsubseteq f(x)$, then

$$x^\star = \bigsqcup_{n\ge 0} f^n(x) \in \max D$$

is the unique fixed point of f on D. Furthermore, $x^\star$ is an attractor in two different senses:

(i) *For all $x \in \ker\mu$, $f^n(x) \to x^\star$ in the Scott topology on $\ker\mu$, and*

(ii) *For all $x \sqsubseteq x^\star$, $\bigsqcup_{n\geq 0} f^n(x) = x^\star$, and this supremum is a limit in the Scott topology on D.*

The only problem with this theorem is that it requires fixed points maximal. Very shortly we will uncover some new results that overcome this difficulty in what appears to be a more elegant approach.

4 Invariance

All ways of measuring a domain appeal to a common objective.

Definition 4.1 The μ *topology* on a continuous dcpo D has as a basis all sets of the form $\Uparrow\! x \cap \downarrow\! y$, for $x, y \in D$. It is denoted μ_D.

One unsatisfying aspect of the Scott topology is its weak notion of limit. Ideally, one would hope that any sequence with a limit in the Scott topology had a supremum. But this is far from true. For instance, on a continuous dcpo with least element $\perp$, all sequences converge to $\perp$.

Lemma 4.2 (Martin [6]) *Let D be a continuous dcpo. Then*

(i) *A sequence (x_n) converges to a point x in the μ topology iff it converges to x in the Scott topology and $(\exists n)\, x_k \sqsubseteq x$, for all $k \geq n$.*

(ii) *If $x_n \to x$ in the μ topology, then there is a least integer n such that*

$$\bigsqcup_{k\geq n} x_k = x.$$

(iii) *If (x_n) is a sequence with $x_n \sqsubseteq x$, then $x_n \to x$ in the μ topology iff $x_n \to x$ in the Scott topology.*

In a phrase, μ limits are the Scott limits with computational significance.

Proposition 4.3 *A monotone map $f : D \to E$ between continuous dcpo's is μ continuous iff it is Scott continuous.*

So what does all this have to do with information content? Given a measurement $\mu \to \sigma_D$, consider the elements ε-close to $x \in D$, for $\varepsilon > 0$, given by

$$\mu_\varepsilon(x) := \{y \in D : y \sqsubseteq x \;\&\; |\mu x - \mu y| < \varepsilon\}.$$

Regardless of the measurement we use, these sets are *always* a basis for the μ topology. In fact, it is this property which *defines* content on a domain.

Theorem 4.4 (Martin [6]) *For a Scott continuous map $\mu : D \to [0,\infty)^*$, $\mu \to \sigma_D$ iff $\{\mu_\varepsilon(x) : x \in D \;\&\; \varepsilon > 0\}$ is a basis for the μ topology on D.*

This realization not only improves our understanding of the μ topology, it also allows us to make more effective use of measurement.

Lemma 4.5 *Let (D,μ) be a continuous dcpo with a measurement $\mu \to \sigma_D$.*

(i) *If (x_n) is a sequence with $x_n \sqsubseteq x$, then*

$$x_n \to x \text{ in the } \mu \text{ topology iff } \lim_{n\to\infty} \mu x_n = \mu x.$$

(ii) *A monotone map $f : D \to D$ is Scott continuous iff $\mu f : D \to [0,\infty)^*$ is μ continuous.*

Notice that the Scott topology can always be recovered from the μ topology as $\sigma_D = \{\uparrow U : U \in \mu_D\}$.

5 Fixed points of contractions

In this section, (D,μ) is a *domain* with a measurement $\mu \to \sigma_D$.

Definition 5.1 Let f be a monotone selfmap on (D,μ). If there exists a constant c such that

$$x \sqsubseteq y \Rightarrow \mu f(x) - \mu f(y) \leq c \cdot (\mu x - \mu y),$$

for all $x, y \in D$, then f is a *contraction* if $c < 1$ and *nonexpansive* if $c \leq 1$.

Proposition 5.2 *A contraction is Scott continuous.*

Proof. First, μf is μ continuous. By Theorem 4.4, the μ topology on D is first countable, and so we can work with sequences in verifying this assertion.

Let $x_n \to x$ in the μ topology on D. Then we can assume $x_n \sqsubseteq x$. Hence

$$0 \leq \mu f(x_n) - \mu f(x) \leq c \cdot (\mu x_n - \mu x)$$

which means

$$\lim_{n\to\infty} \mu f(x_n) = \mu f(x)$$

since $\mu x_n \to \mu x$. Then μf is μ continuous. By Lemma 4.5(ii), f is Scott continuous. □

The last proposition does not require a contraction: The same proof works for any value of $c \geq 0$. It is our next result that requires $c < 1$.

Theorem 5.3 *Let f be a contraction on (D,μ). If there is a point $x \sqsubseteq f(x)$, then*

$$\text{fix}(f) := \bigsqcup_{n\geq 0} f^n(x)$$

is the unique fixed point of f on D.

Proof. By Prop. 5.2, f is Scott continuous, so it is clear that fix(f) is a fixed point of f.

Let $x = f(x)$ and $y = f(y)$ be two fixed points of f. By our assumption on D, there is an element $z \in D$ with $z \sqsubseteq x, y$. Then

$$f^n(z) \sqsubseteq x = f^n(x),$$

for all $n \geq 1$. By induction, we have

$$\mu f^n(z) - \mu f^n(x) = \mu f^n(z) - \mu x \leq c^n \cdot (\mu z - \mu x),$$

for all $n \geq 1$. Then $\mu f^n(z) \to \mu x$ and so

$$\bigsqcup_{n \geq 0} f^n(z) = x$$

since $\mu \to \sigma_D$. But the same argument applies to y. □

In fact, careful inspection of the proof of the last theorem shows that the unique fixed point is an attractor in the μ topology.

Definition 5.4 A fixed point $p = f(p)$ of a continuous map $f : X \to X$ on a space X is called an *attractor* if there is an open set U around p such that

$$f^n(x) \to p$$

for all $x \in U$. We also refer to p as *attractive.*

Examples of attractive fixed points are easy to find: In the analysis of hyperbolic iterated function systems, where they are sometimes called fractals, or in the study of iterative methods in numerical analysis like Newton's method, where they arise as the solutions to nonlinear equations.

Corollary 5.5 *Let f be a contraction on (D, μ). If $a \sqsubseteq \mathrm{fix}(f)$, then*

$$\bigsqcup_{n \geq 0} f^n(a) = \mathrm{fix}(f),$$

and this supremum is a limit in the μ topology. That is, $\mathrm{fix}(f)$ *is an attractor in the μ topology.*

Proof. The claim is implicitly established in the previous theorem. For the attractor bit, let $U = \downarrow\!\mathrm{fix}(f)$. This is a lower set and hence μ open. □

Thus, beginning with *any* approximation a of $\mathrm{fix}(f)$, the iterates $f^n(a)$ converge to $\mathrm{fix}(f)$, even if $a \not\sqsubseteq f(a)$. In addition, we can obtain a good estimate of how many iterations are required to achieve an ε-approximation of $\mathrm{fix}(f)$.

Proposition 5.6 *Let f be a contraction on (D, μ) with unique fixed point* $\mathrm{fix}(f)$*. Then for any $x \sqsubseteq \mathrm{fix}(f)$ and $\varepsilon > 0$,*

$$n > \frac{\log(\mu x - \mu\,\mathrm{fix}(f)) - \log \varepsilon}{\log(1/c)} \;\Rightarrow\; |\mu f^n(x) - \mu\,\mathrm{fix}(f)| < \varepsilon,$$

for any integer $n \geq 0$, provided $x \neq \mathrm{fix}(f)$.

Proof. If $c = 0$, then f is constant, and the statement holds trivially, adopting the convention that $\log(1/0) = \log \infty = \infty$. Let $0 < c < 1$.

For an integer $n \geq 0$, we have

$$|\mu f^n(x) - \mu\,\mathrm{fix}(f)| = \mu f^n(x) - \mu\,\mathrm{fix}(f) \leq c^n \cdot (\mu x - \mu\,\mathrm{fix}(f)).$$

Thus,

$$n > \frac{\log(\mu x - \mu\,\mathrm{fix}(f)) - \log \varepsilon}{\log(1/c)} \Rightarrow c^n \cdot (\mu x - \mu\,\mathrm{fix}(f)) < \varepsilon,$$

which proves the claim. □

Of course, for the estimate to be useful we must know the measure of $\mathrm{fix}(f)$. One case when this is easy to calculate is if $\mu f(x) \leq c \cdot \mu x$. Then $\mu\,\mathrm{fix}(f) = 0$. Surprisingly, this condition amounts to saying that f is the extension to D of a continuous map on $\ker \mu$.

Proposition 5.7 *For a contraction f on (D, μ) with $\ker \mu = \max D$, the following are equivalent:*

(i) *The map f preserves maximal elements.*

(ii) *For all $x \in D$, $\mu f(x) \leq c \cdot \mu x$.*

In either case, $\mathrm{fix}(f) \in \max D$.

Proof. Let f have contraction constant c.

(i) $\Rightarrow$ (ii): Let $x \in D$. By the directed completeness of D, there is an element $y \in \uparrow\!x \cap \max D$. Then

$$\mu f(x) - \mu f(y) = \mu f(x) \leq c \cdot (\mu x - \mu y) = c \cdot \mu x$$

which holds since $\mu f(y) = 0$ by (i).

(ii) $\Rightarrow$ (i): Let $x \in \max D$. Then $\mu x = 0$ so $0 \leq \mu f(x) \leq c \cdot \mu x = 0$. Thus, $f(x) \in \ker \mu = \max D$. □

In fact, every contraction on a complete metric space can be represented as a contraction on a domain of the type above.

Example 5.8 Let $f : X \to X$ be a contraction on a complete metric space X with Lipschitz constant $c < 1$. The mapping $f : X \to X$ extends to a monotone map $\bar{f} : \mathbf{B}X \to \mathbf{B}X$ on the formal ball model $\mathbf{B}X$ [2] given by

$$\bar{f}(x, r) = (fx, c \cdot r),$$

which satisfies

$$\pi \bar{f}(x, r) - \pi \bar{f}(y, s) = c \cdot \pi(x, r) - c \cdot \pi(y, s) = c \cdot (\pi(x, r) - \pi(y, s)),$$

where $\pi : \mathbf{B}X \to [0, \infty)^*$, $\pi(x, r) = r$, is the standard measurement on $\mathbf{B}X$. Now choose r so that $(x, r) \sqsubseteq \bar{f}(x, r)$. By Theorem 5.3, $\bar{f}$ has a unique fixed point which implies that f does too.

Thus, Theorem 5.3 has the Banach fixed point theorem as a consequence. A constant mapping taking any value off the top is a contraction with a unique fixed point that is not maximal, and hence not of the sort mentioned in Prop. 5.7. We will see a more substantial example later on.

6 Fixed points of nonexpansive maps

We begin with a fundamental result on a well-known theme.

Theorem 6.1 *Let $f : D \to D$ be a Scott continuous map on a continuous dcpo D with least element $\perp$. Then its least fixed point is an attractor in the μ topology: For all $x \sqsubseteq \mathrm{fix}(f)$,*

$$\bigsqcup_{n\geq 0} f^n(x) = \mathrm{fix}(f),$$

and this supremum is a limit in the μ topology.

Proof. By monotonicity, we have

$$f^n(\perp) \sqsubseteq f^n(x) \sqsubseteq \mathrm{fix}(f),$$

for all $n \geq 0$. Then since $f^n(\perp) \to \mathrm{fix}(f)$ in the μ topology, $f^n(x) \to \mathrm{fix}(f)$ in the μ topology. ☐

Proposition 6.2 *Let f be a monotone map on (D, μ) with least element $\perp$ and measurement $\mu \to \sigma_D$ such that*

$$x \sqsubseteq y \Rightarrow \mu f(x) - \mu f(y) < \mu x - \mu y,$$

for all distinct pairs $x, y \in D$. Then

$$\mathrm{fix}(f) := \bigsqcup_{n\geq 0} f^n(\perp)$$

is the unique fixed point of f on D.

Proof. The map f is nonexpansive and hence Scott continuous by the remark following Prop. 5.2. Thus, $\mathrm{fix}(f)$ is its least fixed point.

If x is any fixed point of f, then $\mathrm{fix}(f) \sqsubseteq x$. If these two are different, then $\mu\,\mathrm{fix}(f) - \mu x = \mu f(\mathrm{fix}(f)) - \mu f(x) < \mu\,\mathrm{fix}(f) - \mu x$. Then they are the same. This proves that $\mathrm{fix}(f)$ is the only fixed point of f. ☐

If the map in the last result preserved $\max D = \ker \mu$, it would satisfy $\mu f(x) < \mu x$, for $\mu x > 0$. Happily, for maps like these, we can prove the last result assuming only a measurement.

Theorem 6.3 *Let (D, μ) be a continuous dcpo with measurement μ and least element $\perp$. If $f : D \to D$ is a Scott continuous map with $\mu f(x) < \mu x$ for $\mu x > 0$, then*

$$\mathrm{fix}(f) := \bigsqcup_{n\geq 0} f^n(\perp) \in \max D$$

is the unique fixed point of f on D. In addition, if $f(\ker \mu) \subseteq \ker \mu$, then

$$(\forall x \in \ker \mu)\; f^n(x) \to \mathrm{fix}(f),$$

in the relative Scott topology on $\ker \mu$.

Proof. If $\mu\,\mathrm{fix}(f) > 0$, then $\mu\,\mathrm{fix}(f) = \mu f(\mathrm{fix}(f)) < \mu\,\mathrm{fix}(f)$. Hence $\mu\,\mathrm{fix}(f) = 0$ which means $\mathrm{fix}(f) \in \ker \mu \subseteq \max D$. But if a least fixed point is maximal, it must be unique.

To see that $\mathrm{fix}(f)$ is an attractor in the relative Scott topology on $\ker\mu$, let $U \subseteq D$ be a Scott open set around $\mathrm{fix}(f)$. Then there is K such that

$$n \geq K \Rightarrow f^n(\perp) \in U$$

which means

$$n \geq K \Rightarrow f^n(x) \in U \cap \ker\mu$$

since $f^n(\perp) \sqsubseteq f^n(x)$ and $f(\ker\mu) \subseteq \ker\mu$. Hence, $f^n(x) \to \mathrm{fix}(f)$ in the relative Scott topology on $\ker\mu$, for any initial guess $x \in \ker\mu$. $\square$

This is the same result as Theorem 3.4 extended to a larger class of mappings on domains with least elements. In addition, in each of the last two results, Theorem 6.1 implies $\mathrm{fix}(f)$ is an attractor in the μ topology on D. Now for why all this matters.

7 Applications

Time to be sixteen again. Let $f : [0, \pi/2] \to [0, \pi/2]$ be $f(x) = \sin x$. As is well-known, beginning with any point $x \in [0, \pi/2]$ and successively applying f yields a sequence of iterates $(f^n(x))$ that magically tends to zero. Why?

At first glance, one thinks of the Banach theorem, which explains that contractions behave this way. Upon closer inspection, however, we see that things are more interesting in the case of the sine wave. Because $f'(0) = 1$, $f(x) = \sin x$ is not a contraction on $[0, \pi/2]$, and so the Banach theorem is not applicable. But domain theory is.

Example 7.1 Let $D = [0, \pi/2]^*$ be the domain with

$$x \sqsubseteq y \Leftrightarrow y \leq x$$

and natural measurement $\mu x = x$.

The function $f(x) = \sin x$ is a monotone selfmap on D. By the mean value theorem, if $x \sqsubseteq y$ and $x \neq y$, there is $c \in (y, x)$ such that

$$\mu f(x) - \mu f(y) = f(x) - f(y) = f'(c)(x - y) = (\cos c)(\mu x - \mu y).$$

Hence, $\mu f(x) - \mu f(y) < \mu x - \mu y$, since $0 < \cos c < 1$.

Then Theorem 6.2 implies that f has a unique fixed point, given by

$$\mathrm{fix}(f) = \bigsqcup_{n \geq 0} f^n(\pi/2).$$

However, $f(0) = 0$, so we must have $\mathrm{fix}(f) = 0$, by uniqueness.

Now the interesting part. By Theorem 6.1, $\mathrm{fix}(f)$ is an attractor in the μ topology. Thus, for *any* $x \sqsubseteq \mathrm{fix}(f)$, $f^n(x) \to \mathrm{fix}(f)$ in the μ topology. But convergence in the μ topology on D implies convergence in the euclidean topology. Thus, for all $x \in [0, \pi/2]$, $f^n(x) \to 0$.

In fact, the reasoning in the last example extends to *any* compact metric space, since they can all be modeled [4] as the kernel of a measurement.

Proposition 7.2 ([3]) *Let $f : X \to X$ be a function on a compact metric space (X, d) such that*

$$d(fx, fy) < d(x, y)$$

for all $x, y \in X$ with $x \neq y$. Then f has a unique fixed point x^* such that for all $x \in X$, $f^n(x) \to x^*$.

Proof. Let $\mathbf{U}X$ be the domain of nonempty compact subsets of X ordered under reverse inclusion. The map f has a Scott continuous extension to $\mathbf{U}X$ given by $\bar{f} : \mathbf{U}X \to \mathbf{U}X :: K \mapsto f(K)$.

The domain $\mathbf{U}X$ has a natural measurement, $\mu x = \operatorname{diam} x$, the diameter mapping derived from the metric d. In addition, the space X can be recovered as $\ker \mu = \{\{x\} : x \in X\} = \max \mathbf{U}X \simeq X$ in the relative Scott topology.

For $\mu x > 0$, either $\mu \bar{f}(x) = 0 < \mu x$, or the compactness of x yields distinct points $a, b \in x$ such that

$$\mu \bar{f}(x) = d(fa, fb) < d(a, b) \leq \mu x.$$

The result now follows from Theorem 6.3. □

That is, the Banach fixed point theorem holds on compact metric spaces under weaker assumptions. The impressive aspect of the last result is *not* the uniqueness of x^*: It is that x^* is a global attractor.

Corollary 7.3 *Let $f : [a, b] \to [a, b]$ be a continuous map on a nonempty compact interval. If $|f'(x)| < 1$ for all $x \in (a, b)$, then f has a unique fixed point $x^* \in [a, b]$ such that $f^n(x) \to x^*$, for all $x \in [a, b]$.*

Proof. By the mean value theorem,

$$|f(b) - f(a)| = |f'(c)||b - a| < |b - a|$$

for some $c \in (a, b)$. Now Prop. 7.2 applies. □

The map $f(x) = \sin x$ satisifes $0 < f'(x) < 1$ on $(0, \pi/2)$, but as we have already seen, it is not a contraction. Thus, Theorem 6.3 yields another insight into why the sine wave behaves the way it does. But the reader should not be misled into thinking that these ideas are only applicable to domain theoretic fragments of classical mathematics.

Example 7.4 Let $D = [\mathbb{N} \to \mathbb{N}_\perp]$ be the domain of partial functions on the naturals. The elements of D are measured as

$$\mu f = \sum_{f(n) = \perp} \frac{1}{2^{n+1}}.$$

Then $\ker \mu = \{f \in D : \operatorname{dom}(f) = \mathbb{N}\}$ is the set of total functions. Now consider the operator $\phi : D \to D$ given by

$$\phi(f)(n) = \begin{cases} n & \text{if } n = 0 \text{ or } n = 1; \\ n + f(n-2)/\cos(n\pi/2) & \text{otherwise.} \end{cases}$$

Because $\phi(f)(n) = \perp \Rightarrow f(n-2) = \perp$ or ($n > 1$ and odd), we are able to write for elements $f \sqsubseteq g$ that

$$\begin{aligned}\mu\phi(f) - \mu\phi(g) &= \sum_{f(n-2)=\perp} \frac{1}{2^{n+1}} - \sum_{g(n-2)=\perp} \frac{1}{2^{n+1}} \\ &= \sum_{f(n)=\perp} \frac{1}{2^{n+3}} - \sum_{g(n)=\perp} \frac{1}{2^{n+3}} \\ &= \frac{1}{4}(\mu f - \mu g).\end{aligned}$$

By Theorem 5.3, ϕ has a unique fixed point. Clearly this fixed point is not maximal: It is undefined on the set $\{2k+1 : k > 1\}$.

8 Ideas

It would be nice to see a metric based approach to semantics replaced with one based on results like Theorem 5.3. Especially on a model of CSP. Trying to obtain estimates in the spirit of Prop. 5.6 for *nonexpansive* maps also seems like a fun question. It would be neat to find out if the informatic derivative [6] (derivative of a map on a domain with respect to a measurement) can be useful in this regard.

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