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Streams, d-Spaces and Their Fundamental Categories

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Abstract

We describe an abstract framework in which the notion of fundamental category can be defined. The structures matching this framework are categories endowed with some additional structure. Provided we have a suitable adjunction between two of them, the fundamental categories defined in both cases can be easily compared. Each of these structures has a "natural" functor to the category of d-spaces [10] and provide a Van Kampen like theorem. As an application we compare the fundamental categories of streams [17,18] and d-spaces, actually proving that streams and d-spaces are almost the same notion.

Keywords: stream, d-space, fundamental category, directed algebraic topology, directed geometric realisation

1 Introduction and Basics

Directed topology is a recent area of mathematics, studying the intrinsic structure of spaces equipped with some notion of direction. The motivating examples and primary focus of the field continue to be the state spaces of concurrent processes. A program made of n concurrent processes is modelled by an n-dimensional such structure and the instruction pointer of the program is thought as a point moving on the model according to the direction carried by the model.

We formally compare three point-set models of "directed spaces" in the literature: *d-spaces*, *streams*, and *pospaces*. This comparison is formally achieved

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by constructing suitable adjunctions between appropriate categories and proving they commute with the fundamental category, an oft-used algebraic invariant in the field. Provided we slightly strengthen the axioms for d-spaces and the axioms for streams given in [10] and [18] we actually prove their categories are isomorphic. In addition the directed geometric realisation construction makes sense in both of them since they are complete and cocomplete. From the practitioner of concurrency point of view, these theoretic features imply that modelling concurrent processes by means of d-spaces or streams is just a matter of taste.

We denote the category of topological space by Top, the category of small categories by Cat and the category of sets by Set. We identify a reflexive binary relation ρ over a set X with its **graph** that is to say

$$\mathsf{graph}(\rho) := \big\{(x, x') \in X \times X \;\big|\; x \;\rho\; x'\big\}$$

the underlying set X is also refered to as $|\rho|$. Given a set X we denote the **diagonal** of X by

$$\Delta_X := \big\{(x, x) \;\big|\; x \in X\big\}$$

The **underlying set** of a topological space X is denoted by UX. By a slight abuse of language we define the diagonal of a topological space as the diagonal of its underlying set. We recall that a topological space X is Hausdorff iff (the graph of) its diagonal is closed in $X \times X$. In particular, a topological product of Hausdorff spaces is a Hausdorff space. Given any set map f from X to Y, the sets X and Y are called the **domain** and **codomain** of f, they are denoted by $\mathsf{dom}(f)$ and $\mathsf{cod}(f)$. We also define the **(direct) image** of f as the following subset of Y

$$\mathsf{im}(f) := \{f(x) \mid x \in X\}$$

Given two relations ρ_1 and ρ_2, one says that ρ_1 is **coarser** than ρ_2 when

$$\mathsf{graph}(\rho_1) \subseteq \mathsf{graph}(\rho_2)$$

A morphism of relations from ρ_1 to ρ_2 is a map f from $|\rho_1|$ to $|\rho_2|$ such that

$$\forall x, x' \in |\rho_1|,\; x \;\rho_1\; x' \Rightarrow f(x) \;\rho_2\; f(x')$$

The (binary reflexive) relations[3] and their morphisms forms a category denoted by Rel. A relation $\preccurlyeq$ is said to be **transitive (reflexive)** when

$$\forall x, x', x'' \in |\preccurlyeq| \; (x \preccurlyeq x' \text{ and } x' \preccurlyeq x'') \;\Rightarrow\; x \preccurlyeq x''$$

A transitive relation $\preccurlyeq$ is called a **preorder**. The category of preorders Pre is defined as the full subcategory of Rel whose objects are the preorders. A relation ρ

[3] From now on we write "relation" to mean "binary reflexive relation".

is said to be **antisymetric** when

$$\forall x, x' \in X\ (x\rho x' \text{ and } x'\rho x)\ \Rightarrow\ x = x'$$

By definition, an antisymetric preorder is a **partially ordered set** or **poset**. So we have the category of preordered sets Pre, the category of partially ordered sets Ps and two obvious forgetful functors respectively from Pre and Ps to Set. The forgetful functor from Pre to Set has a left adjoint provided by the diagonal relation

$$X \in \mathsf{Set} \quad \mapsto \quad \big(X, \{(x,x) | x \in X\}\big) \in \mathsf{Pre}$$

and a right adjoint provided by the (chaotic) relation

$$X \in \mathsf{Set} \quad \mapsto \quad \big(X, X \times X\big) \in \mathsf{Pre}$$

Let $(X_i, \rho_i)_{i\in I}$ be a family with ρ_i being a binary relation over X_i, we put

$$X = \bigcup_{i\in I} X_i$$

and given two elements x and x' of X, we write $x \preccurlyeq_X x'$ if $x = x'$ or when there is a finite subset $\{x_0, \ldots, x_n\}$ of X, $n \in \mathbb{N}\backslash\{0\}$, such that $x_0 = x$, $x_n = x'$ and for all $k \in \{0, \ldots, n-1\}$ there exists some $i_k \in I$ such that $x_k \rho_{i_k} x_{k+1}$. The relation $\preccurlyeq_X$ is actually the least (with respect to inclusion over the subsets of $X \times X$) preorder on X which contains ρ_i (seen as a subset of $X \times X$) for i running through I. Following [18] we write

$$(X, \preccurlyeq_X) = \bigvee_{i\in I} (X_i, \rho_i)$$

and we say that the preordered set $(X, \preccurlyeq_X)$ is **generated** by the family $(X_i, \rho_i)_{i\in I}$. An open covering of a topological space X is a family $(O_i)_{i\in I}$ of open subsets of X such that

$$\bigcup_{i\in I} O_i = X$$

Given $J \subseteq I$ the family $(O_i)_{i\in J}$ is called a sub-covering of $(O_i)_{i\in I}$ when $(O_i)_{i\in J}$ is still a covering of X, the sub-covering is said to be finite when J is so. Moreover X is said to be compact when every open covering of X has a finite sub-covering.

2 Framework for Fundamental Category

The notion of fundamental category arises from an adaptation of the usual notion of fundamental groupoid [3,15,23]. Actually we can define the fundamental category of a (locally) partially ordered space [8,20], d-space [10] or stream [18]. In each case the construction follows the same pattern and leads to a Van Kampen like theorem [10,8]. We briefly recall their construction following a general setting inspired from [14]. From this framework we give a common proof for the Van Kampen theorem and provide a result comparing the various fundamental category functors.

We recall the construction of the fundamental groupoid [3,15]. Given a topological space X, a **Moore path** on X is a continuous map δ from $[0,r]$ to X with $r \in \mathbb{R}_+$. The parameter r is called the **length** of the path while its source and target are defined as $\mathsf{s}(\delta) := \delta(0)$ and $\mathsf{t}(\delta) := \delta(r)$. Given two Moore paths (on X) δ and γ of lengths r and s such that $\mathsf{s}(\gamma) = \mathsf{t}(\delta)$, we define the **concatenation** $\gamma * \delta$ as follows

$$\begin{aligned} \gamma * \delta : \ [0, r+s] &\longrightarrow X \\ t &\longmapsto \begin{cases} \delta(t) & \text{if } t \in [0,r] \\ \gamma(t-r) & \text{if } t \in [r, r+s] \end{cases} \end{aligned}$$

Defined this way, concatenation is obviously associative thus we obtain $P(X)$, the **(Moore) path category** of X, the objects are the points of X while its identities are the paths of null length. Actually this construction leads to a functor P from Top to Cat.

Given two paths γ and δ on a topological space X sharing the same length r, a **homotopy** from γ to δ is a morphism $h \in \mathsf{Top}[[0,r] \times [0,\rho], X]$ such that

1) the following mappings are constant

$$x \in [0,\rho] \mapsto h(0,x) \in X$$

$$x \in [0,\rho] \mapsto h(r,x) \in X$$

2) and for all $t \in [0,r]$, $h(t,0) = \gamma(t)$ and $h(t,\rho) = \delta(t)$

Then we write $\gamma \sim \delta$ to assert that there exist two constant paths c_γ, c_δ and a homotopy from $c_\gamma * \gamma$ to $c_\delta * \delta$. The relation $\sim$ is actually a congruence (in the sense of [19]) on $P(X)$ and the **fundamental groupoid** [4] of X, denoted by $\Pi_1(X)$, is then defined as the quotient $P(X)/\sim$ and we denote by q_X the quotient functor from $P(X)$ to $\Pi_1(X)$.

Lemma 2.1 *The functors q_X for X running through the collection of topological spaces form a natural transformation q from P to Π_1.*

Proof. Let f be a continous map from X to Y and γ be a morphism of $P(X)$ that is to say a continuous map from $[0,r]$ to X. Then $\big(q_Y \circ P(f)\big)(\gamma)$ and $\big(\Pi_1(f) \circ q_X\big)(\gamma)$ are two equivalence classes (up to homotopy of paths in Y) that share the element $f \circ \gamma$ hence they are equal. $\square$

[4] The inverse of the $\sim$-class of a path δ of length r is the $\sim$-class of the path $t \mapsto \delta(r-t)$.

$$\begin{array}{ccc} P(X) & \xrightarrow{P(f)} & P(Y) \\ \downarrow{\scriptstyle q_X} & & \downarrow{\scriptstyle q_Y} \\ \Pi_1(X) & \xrightarrow[\Pi_1(f)]{} & \Pi_1(Y) \end{array}$$

We will apply these ideas to concrete categories over Top enjoying some additional properties.

Let C be a category together with a faithful functor U from C to Top admitting a left adjoint F. Also assume the category C has a family of objects $(\mathbb{I}_\iota)_{\iota\in\mathcal{I}}$ indexed by the set $\mathcal{I}$ of all sub-intervals of $\mathbb{R}$ (including $\emptyset$ and the singletons). Yet, for each real $r \geq 0$ the notation $\mathbb{I}_r$ stands for $\mathbb{I}_{[0,r]}$.
Axiom 1) for all n-uple $(\iota_1, \ldots, \iota_n)$ the n-fold product $\mathbb{I}_{\iota_1} \times \cdots \times \mathbb{I}_{\iota_n}$ exists and we suppose that $F(\{0\}) = \mathbb{I}_0$. By convention the 0-fold product is the terminal object of C.
Axiom 2) for all continuous order[5] preserving maps β from $\iota_1 \times \cdots \times \iota_n$ to $\iota'_1 \times \cdots \times \iota'_{n'}$ there exists a morphism α of C from $\mathbb{I}_{\iota_1} \times \cdots \times \mathbb{I}_{\iota_n}$ to $\mathbb{I}_{\iota'_1} \times \cdots \times \mathbb{I}_{\iota'_{n'}}$ such that $U(\alpha) = \beta$. As a consequence, for all $\iota \in \mathcal{I}$ we have $U(\mathbb{I}_\iota) = \iota$.

Recall that the homeomorphisms between two intervals of $\mathbb{R}$ are the increasing or decreasing bijections between them (such a map is necessarily continuous). Hence, as a first consequence of the Axiom 2, note that for all intervals ι and ι', if there exists an increasing bijection between ι and ι' (each of which being equiped with the standard order over real numbers), then the objects $\mathbb{I}_\iota$ and $\mathbb{I}_{\iota'}$ are isomorphic.

From now on we choose the singleton $\{0\}$ as a representative of the terminal object of Top. Formally a point of a topological space T is a morphism from $\{0\}$ to T. Then remark that we have a bijection (since $F \dashv U$, $F(\{0\}) = \mathbb{I}_0$ and then $\eta_{\{0\}} = \mathsf{id}_{\{0\}}$)

$$\begin{array}{ccc} \mathsf{C}[\mathbb{I}_0, X] & \longrightarrow & \mathsf{Top}[\{0\}, U(X)] \\ p & \longmapsto & U(p) \end{array}$$

As a consequence, when f belongs to the homset $\mathsf{C}[X, Y]$ and x is a point of $U(X)$, we write $f(x)$ to denote the unique element of $\mathsf{C}[\mathbb{I}_0, Y]$ whose image under U is $(U(f))(x)$. Given an object X of C the **directed paths**[6] on X are the elements of

$$\bigcup_{r\in\mathbb{R}_+} \mathsf{C}[\mathbb{I}_r, X]$$

thus a path on X induces a (continuous) map from $[0, r]$ to $U(X)$ i.e. a path on $U(X)$ in the usual sense. Since there is a bijection between the elements of $\mathsf{C}[\mathbb{I}_0, \mathbb{I}_r]$

[5] Here we mean product order.
[6] We sometimes just write "paths" for short.

and the points of $[0, r]$ it is sound to define the source and the target of a path δ as $\mathsf{s}(\delta) := \delta(0)$ and $\mathsf{t}(\delta) := \delta(r)$. We still call r the **length** of the path though the parameter r still has no geometric meaning. Given a path α of length r on X and a continuous increasing map θ from $[0, s]$ to $[0, r]$, the composite $\alpha \circ \theta$ is a path of length s on X since by the Axiom 2 the map θ is morphism of C. For $x \in [0, r]$ the notation $\mathbb{I}_0 \xrightarrow{x} \mathbb{I}_r$ represents the unique element of $\mathsf{C}[\mathbb{I}_0, \mathbb{I}_r]$ whose image under U is x. Given $x, r, s \in \mathbb{R}_+$ such that $x + r \leqslant s$, the Axiom 2 allows us to define $i^s_{x,r}$ as the unique morphism of C from $\mathbb{I}_r$ to $\mathbb{I}_s$ so that $U(i^s_{x,r})$ be the following map.

$$\begin{array}{ccc} [0, r] & \longrightarrow & [0, s] \\ t & \longmapsto & x + t \end{array}$$

With the notation previously introduced, the following square is commutative since its image under the faithful functor U is actually a pushout square in Top. Furthermore we suppose

Axiom 3) the following diagram is a pushout square in C

$$\begin{tikzcd} & \mathbb{I}_{r+s} & \\ \mathbb{I}_r \arrow[ur, "i^{r+s}_{0,r}"] & & \mathbb{I}_s \arrow[ul, "i^{r+s}_{r,s}"'] \\ & \mathbb{I}_0 \arrow[ul, "r"] \arrow[ur, "0"'] & \end{tikzcd}$$

and for all finite sequences $(\mathbb{I}_{r_1}, \ldots, \mathbb{I}_{r_n})$ and all $i \in \{1, \ldots, n\}$, it is preserved by the endofunctor of C which sends each X to the Cartesian product

$$\mathbb{I}_{r_1} \times \cdots \times \mathbb{I}_{r_{i-1}} \times X \times \mathbb{I}_{r_{i+1}} \times \cdots \times \mathbb{I}_{r_n}$$

which is the case when $\mathbb{I}_{[0,1]}$ is **exponentiable** that is to say when for all X the product $\mathbb{I}_{[0,1]} \times X$ exists and the induced functor $\mathbb{I}_{[0,1]} \times -$ admits a right adjoint. Any category C provided with a functor U and a family $(\mathbb{I}_\iota)_{\iota \in \mathcal{I}}$ satisfying the Axioms 1, 2 and 3 is called a **framework for fundamental category** or just **fffc** for short.

The category Top with the identity functor as U provides a trivial example of fffc.

A **partially ordered (topological) space** (or **pospace**) [20] is a topological space X equiped with a partial order (over its underlying set) whose graph is closed in $X \times X$. Together with the order preserving continous maps between them, the pospaces form a category denoted by P. The underlying space of a pospace is separated (in the sense of Hausdorff) and the forgetful functor from P to the category of Hausdorff spaces H is faithful and admits a left adjoint (by providing a Hausdorff space with the diagonal odrer). Composing with the inclusion of H into Top (which also admits a left adjoint) we obtain the functor U. Then, the standard order over the set of real numbers provides $\mathbb{R}$ and any of its intervals with a structure of pospace thus turning P into a framework for fundamental category.

We describe the construction of the fundamental category functor over a given fffc. Given $\delta \in \mathsf{C}[\mathbb{I}_r, X]$ and $\gamma \in \mathsf{C}[\mathbb{I}_s, X]$ such that $\gamma(0) = \delta(r)$ the outter shape of the left hand diagram commutes. By axiom 3 there is a unique path $\gamma * \delta$ of length $r+s$ on X that makes the left hand diagram commute. By definition $\gamma * \delta$ is called the **concatenation** of δ followed by γ. Moreover $U(\gamma * \delta)$ makes the right hand diagram commute which implies that $U(\gamma * \delta) = U(\gamma) * U(\delta)$.

$$
\begin{tikzcd}
 & X & \\
 & \mathbb{I}_{r+s} \arrow[u, dotted, "\gamma*\delta"] & \\
\mathbb{I}_r \arrow[uur, bend left, "\delta"] \arrow[ur, "i^{r+s}_{0,r}"] & & \mathbb{I}_s \arrow[uul, bend right, "\gamma"'] \arrow[ul, "i^{r+s}_{r,s}"'] \\
 & \mathbb{I}_0 \arrow[ul, "r"] \arrow[ur, "0"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & U(X) & \\
 & {[0,r+s]} \arrow[u, "U(\gamma*\delta)=U(\gamma)*U(\delta)"] & \\
{[0,r]} \arrow[uur, bend left, "U(\delta)"] \arrow[ur, "t\mapsto t"] & & {[0,s]} \arrow[uul, bend right, "U(\gamma)"'] \arrow[ul, "t\mapsto s+t"'] \\
 & \{0\} \arrow[ul, "r"] \arrow[ur, "0"'] &
\end{tikzcd}
$$

The concatenation we have defined is actually associative hence we have a small category $\overrightarrow{P}(X)$ whose objects are the points of X and whose identities are the paths of null length. The construction actually induces a functor from C to Cat in the natural way and we have an obvious embedding of $\overrightarrow{P}(X)$ into $P\big(U(X)\big)$ which leads to a natural transformation from $\overrightarrow{P}$ to $P \circ U$.

We then come to the notion of (directed) homotopy between (directed) paths. A path is said to be constant if it factors, as a morpshim of C, through the terminal object of C. Given two paths γ and δ, we write $\gamma \preccurlyeq \delta$ when there exists two constant paths c_γ, c_δ and some $h \in \mathsf{C}[\mathbb{I}_r \times \mathbb{I}_\rho, X]$ such that $U(h)$ is a homotopy from $U(c_\gamma * \gamma)$ to $U(c_\delta * \delta)$. The constant paths are needed so we can relate two directed paths whose lengths differ. Such a morphism h is a **(directed) homotopy** [7]. Considering $\rho = 0$ we check that the relation $\preccurlyeq$ is reflexive. The axiom 3 allows one to paste homotopies the same way as in classical theory [3,15] thus proving the relation $\preccurlyeq$ is transitive and compatible with concatenation that is to say if $\delta \preccurlyeq \delta'$, $\gamma \preccurlyeq \gamma'$ and the source of γ is the target of δ, then the source of γ' is the target of δ' and $\gamma * \delta \preccurlyeq \gamma' * \delta'$. Denoting by $\sim$ the least equivalence relation containing $\preccurlyeq$ we have a congruence over $\overrightarrow{P}(X)$ (in the sense of [19]). It is worth to notice that we have $\gamma \sim \delta$ if and only if there exists a zigzag of directed homotopies between γ and δ that is to say a finite sequence $(\xi_0, \dots, \xi_n)$ of paths on X such that $\xi_0 = \gamma$, $\xi_n = \delta$ and for all $k \in \{0, \dots, n-1\}$ $\xi_k \preccurlyeq \xi_{k+1}$ or $\xi_{k+1} \preccurlyeq \xi_k$. So we define the fundamental category of X as the quotient

$$\overrightarrow{P}(X)/\sim$$

We denote it by $\overrightarrow{\Pi_1}(X)$ without any reference to the category C when the context leaves no ambiguity [8] and we denote by $\overrightarrow{q_X}$ the quotient functor from $\overrightarrow{P}(X)$ to $\overrightarrow{\Pi_1}(X)$. The relations $\preccurlyeq$ and $\sim$ are called the **homotopy preorder** and the

[7] Note h preserves endpoints because so does $U(h)$.

[8] We write $\overrightarrow{P}^{(\mathsf{C})}(X)$ and $\overrightarrow{\Pi_1}^{(\mathsf{C})}(X)$ if we need to insist on the fact that the underlying fffc is C.

homotopy congruence over X.

If we consider the framework Top, then $\overrightarrow{\Pi_1}$ is the classical fundamental groupoid functor.

Lemma 2.2 *The functors $\overrightarrow{q_X}$ for X running through the collection of objects of C is a natural transformation $\overrightarrow{q}$ from P to $\overrightarrow{\Pi_1}$.*

Proof. Adapting the proof of the Lemma 2.1. □

As a first application, we can compare fundamental categories through functors between frameworks for fundamental category. Let (C, U) and (C', U') be two fffc's and D be a functor from C to C' such that $U' \circ D = U$ and $D(\mathbb{I}_r) = \mathbb{I}'_r$. Let X be an object of C. Given $x, r, s \in \mathbb{R}^+$ such that $x + r \leqslant s$ one has

$$U'\big(D(i^s_{x,r})\big) = U(i^s_{x,r}) = U'(i'^s_{x,r}) = \left\{ \begin{array}{ccc} [0,r] & \longrightarrow & [0,s] \\ t & \longmapsto & x+t \end{array} \right.$$

hence for U' is faithful we have $D(i^s_{x,r}) = i'^s_{x,r}$ and it follows that

$$D\left(\begin{tikzcd} & X & \\ & \mathbb{I}_{r+s} \arrow[u, "\gamma*\delta"] & \\ \mathbb{I}_r \arrow[ur, "i^{r+s}_{0,r}"] \arrow[uur, bend left, "\delta"] & & \mathbb{I}_s \arrow[ul, "i^{r+s}_{r,s}"'] \arrow[uul, bend right, "\gamma"'] \\ & \mathbb{I}_0 \arrow[ul, "r"] \arrow[ur, "0"'] & \end{tikzcd}\right) = \begin{tikzcd} & D(X) & \\ & \mathbb{I}'_{r+s} \arrow[u, "D(\gamma*\delta)"'] & \\ \mathbb{I}'_r \arrow[ur, "i'^{r+s}_{0,r}"] \arrow[uur, bend left, "D(\delta)"] & & \mathbb{I}'_s \arrow[ul, "i'^{r+s}_{r,s}"'] \arrow[uul, bend right, "D(\gamma)"'] \\ & \mathbb{I}'_0 \arrow[ul, "r"] \arrow[ur, "0"'] & \end{tikzcd}$$

from which we deduce that $D(\gamma * \delta) = D(\gamma) * D(\delta)$. Hence we have defined a functor α_X from $\overrightarrow{P}^{(\mathsf{C})}\big(X\big)$ to $\overrightarrow{P}^{(\mathsf{C}')}\big(DX\big)$ whose object part is the identity of the underlying set of X and which sends a path δ on X to the path $D(\delta)$ on DX.

Lemma 2.3 *The collection of functors α_X for X running through the collection of objects of C forms a natural transformation from $\overrightarrow{P}^{(\mathsf{C})}$ to $\overrightarrow{P}^{(\mathsf{C}')}$. Moreover the functor α_X is an identity if and only if for all $r \in \mathbb{R}_+$ we have $\mathsf{C}[\mathbb{I}_r, X] = \mathsf{C}'[\mathbb{I}'_r, DX]$.*

Proof. An immediate consequence of the definition of the category of paths. □

Now we add the condition that $D(\mathbb{I}_{r_1} \times \mathbb{I}_{r_2}) = \mathbb{I}'_{r_1} \times \mathbb{I}'_{r_2}$. The notations $\preccurlyeq_X$, $\preccurlyeq_{DX}$, $\sim_X$ and $\sim_{DX}$ refer to the homotopy preorders and homotopy congruences over X and DX. If h is a directed homotopy from γ to δ then $D(h)$ is a directed homotopy from $D(\gamma)$ to $D(\delta)$. It follows that $\gamma \sim_X \delta$ implies $D(\gamma) \sim_{DX} D(\delta)$. So we have a functor β_X from $\overrightarrow{\Pi_1}^{(\mathsf{C})}(X)$ to $\overrightarrow{\Pi_1}^{(\mathsf{C}')}(DX)$ whose object part is the identity of the underlying set of X. Precisely, the morphisms of $\overrightarrow{\Pi_1}^{(\mathsf{C})}(X)$ are the $\sim_X$-equivalence classes of morphisms of $\overrightarrow{P}^{(\mathsf{C})}\big(X\big)$ (i.e. paths on X). Given such a class E, the morphism part of the functor is defined as the $\sim_{DX}$-equivalence classe of $D(\delta)$ where δ is any element of E. In other words for all path δ on X we have

$$\beta_X\big([\delta]_X\big) = \big[D(\delta)\big]_{DX}$$

where $[\delta]_X$ and $[D(\delta)]_{DX}$ are the equivalence classes of the paths δ and $D(\delta)$.

Lemma 2.4 *The collection of functors β_X for X running through the collection of objects of C forms a natural transformation from $\overrightarrow{\Pi}_1^{(\mathsf{C})}$ to $\overrightarrow{\Pi}_1^{(\mathsf{C}')}$. Moreover the functor β_X is an iso if and only if for all directed paths δ'_1 and δ'_2 on DX such that $\delta'_1 \sim_{DX} \delta'_2$ there exist two directed paths δ_1 and δ_2 on X such that $D(\delta_1) \sim_{DX} \delta'_1$, $D(\delta_2) \sim_{DX} \delta'_2$ and $\delta_1 \sim_X \delta_2$.*

Proof. From the description of the morphism part of the functor β_X. □

Corollary 2.5
Given an object X of C, if for r and ρ running through $\mathbb{R}^+$ the maps A_r are bijective and the maps $B_{r,\rho}$ are surjective

$$\begin{array}{ccc} \mathsf{C}[\mathbb{I}_r, X] & \xrightarrow{A_r} & \mathsf{C}'[\mathbb{I}'_r, D(X)] \\ \delta & \longmapsto & D(\delta) \end{array} \qquad \begin{array}{ccc} \mathsf{C}[\mathbb{I}_r \times \mathbb{I}_\rho, X] & \xrightarrow{B_{r,\rho}} & \mathsf{C}'[\mathbb{I}'_r \times \mathbb{I}'_\rho, D(X)] \\ h & \longmapsto & D(h) \end{array}$$

then we have $\overrightarrow{P}(X) \cong \overrightarrow{P}\big(D(X)\big)$ and $\overrightarrow{\Pi}_1(X) \cong \overrightarrow{\Pi}_1\big(D(X)\big)$.

Proof. As an immediate consequence of the Lemmas 2.3 and 2.4. □

Suppose D admits a left adjoint S, denoting by η the unit of the adjunction $S \dashv D$ we also make the assumption that for all $r \in \mathbb{R}_+$ we have $\eta_{\mathbb{I}'_r} = \mathsf{id}_{\mathbb{I}'_r}$. In addition, assume we have for all $r_1, r_2 \in \mathbb{R}_+$

$$S(\mathbb{I}'_{r_1} \times \mathbb{I}'_{r_2}) = \mathbb{I}_{r_1} \times \mathbb{I}_{r_2}$$

In practice, the technical extra hypotheses about the adjunction $S \dashv D$ are often trivially satisfied.

Corollary 2.6
Under the assuptions made above, the natural transformations α from $\overrightarrow{P}^{(\mathsf{C})}\big(X\big)$ to $\overrightarrow{P}^{(\mathsf{C}')}\big(DX\big)$ and β from $\overrightarrow{\Pi}_1^{(\mathsf{C})}(X)$ to $\overrightarrow{\Pi}_1^{(\mathsf{C}')}(DX)$ are isomorphisms.

Proof. The mappings below are bijections because S is left adjoint to D.

$$\begin{array}{ccc} \mathsf{C}[S(\mathbb{I}'_r), X] & \longrightarrow & \mathsf{C}'[\mathbb{I}'_r, D(X)] \\ \delta & \longmapsto & D(\delta) \circ \eta_{\mathbb{I}'_r} \end{array} \qquad \begin{array}{ccc} \mathsf{C}[S(\mathbb{I}'_r \times \mathbb{I}'_\rho), X] & \longrightarrow & \mathsf{C}'[\mathbb{I}'_r \times \mathbb{I}'_\rho, D(X)] \\ h & \longmapsto & D(h) \circ \eta_{\mathbb{I}'_r \times \mathbb{I}'_\rho} \end{array}$$

In addition we have

$$\mathsf{C}'[\mathbb{I}'_r, D(X)] = \mathsf{C}[S(\mathbb{I}'_r), X] = \mathsf{C}[\mathbb{I}_r, X]$$

and

$$\mathsf{C}'[\mathbb{I}'_r \times \mathbb{I}'_\rho, D(X)] = \mathsf{C}[S(\mathbb{I}'_r \times \mathbb{I}'_\rho), X] = \mathsf{C}[\mathbb{I}_r \times \mathbb{I}_\rho, X]$$

then we conclude applying the Corollary 2.5 because $\eta_{\mathbb{I}'_r} = \mathsf{id}_{\mathbb{I}'_r}$ and according to the next basic [9] lemma we have $\eta_{\mathbb{I}'_r \times \mathbb{I}'_\rho} = \eta_{\mathbb{I}'_r} \times \eta_{\mathbb{I}'_\rho} = \mathsf{id}_{\mathbb{I}'_r \times \mathbb{I}'_\rho}$.

Lemma 2.7 *Let $F \dashv U : \mathcal{A} \underset{F}{\overset{U}{\rightleftarrows}} \mathcal{B}$ be an adjunction and let B, B' be two objects of $\mathcal{B}$ such that $F(B \times B') = FB \times FB'$, then $\eta_{B \times B'} = \eta_B \times \eta_{B'}$.*

We now prove the Lemma 2.7 using basic results about adjunction that can be found in [1]. We denote by η and ε the unit and the counit of the adjunction. In order to conclude, it suffices to check that

$$\begin{cases} \eta_B \circ \Pi_B = \Pi_{UFB} \circ \eta_{B \times B'} \\ \eta_{B'} \circ \Pi_B = \Pi_{UFB'} \circ \eta_{B \times B'} \end{cases}$$

Taking into account the fact that $F(B \times B') = FB \times FB'$, it is a general fact about adjunction that the (set theoretic) maps are bijections

$$\begin{array}{rcl} \mathcal{A}[FB \times FB', FB] & \rightleftarrows & \mathcal{B}[B \times B', UFB] \\ \gamma & \longmapsto & U(\gamma) \circ \eta_{B \times B'} \\ \varepsilon_{FB} \circ F(\delta) & \longleftarrow\!\dashv & \delta \end{array}$$

It is another general fact that $\varepsilon_{FB} \circ F(\eta_B) = \mathsf{id}_{FB}$ hence we have $\varepsilon_{FB} \circ F(\eta_B \circ \Pi_B) = \big(\varepsilon_{FB} \circ F(\eta_B)\big) \circ F(\Pi_B) = F(\Pi_B)$. Now for F preserves the product $B \times B'$ we have $F(\Pi_B) = \Pi_{FB}$. Thus, according to the preceding bijection we have $U(\Pi_{FB}) \circ \eta_{B \times B'} = \eta_B \circ \Pi_B$. Then for U has a left adjoint it preserves products hence $U(\Pi_{FB}) = \Pi_{UFB}$ and we have $\eta_B \circ \Pi_B = \Pi_{UFB} \circ \eta_{B \times B'}$ □

However as we shall see, there might be an object X' of C' such that $\overrightarrow{\Pi_1}(X') \ncong \overrightarrow{\Pi_1}\big(S(X')\big)$. Yet, if X' is an object of C' such that $D \circ S(X') = X'$ then $\overrightarrow{\Pi_1}\big(S(X')\big) = \overrightarrow{\Pi_1}\big(D \circ S(X')\big) = \overrightarrow{\Pi_1}(X')$.

We now provide a tool for actual calculations of the fundamental category. An element α of $\mathsf{C}[X, Y]$ is called an **inclusion** when $U(X)$ is a subspace of $U(Y)$ and $U(\alpha)$ is the corresponding inclusion. In this case the notation $U(\overset{\circ}{X})$ stands for the topological interior of $U(X)$ seen as a subset of $U(Y)$. Then we have a generic form of the Van Kampen theorem.

Theorem 2.8 (Seifert - Van Kampen)

A square of inclusions of C such that $U(\overset{\circ}{X_1})$ and $U(\overset{\circ}{X_2})$ cover $U(X)$ and $U(X_0) =$

[9] I explicitly provide the statement and its proof for I could not find them anywhere in the "classics" of Category Theory.

$U(X_1) \cap U(X_2)$ is sent to pushout squares of Cat *by the functors* $\overrightarrow{P}$ *and* $\overrightarrow{\Pi_1}$.

$$
\begin{array}{ccc}
\begin{array}{ccc} X_0 & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ X_2 & \longrightarrow & X \end{array}
&
\begin{array}{ccc} \overrightarrow{P}(X_0) & \longrightarrow & \overrightarrow{P}(X_1) \\ \downarrow & & \downarrow \\ \overrightarrow{P}(X_2) & \longrightarrow & \overrightarrow{P}(X) \end{array}
&
\begin{array}{ccc} \overrightarrow{\Pi_1}(X_0) & \longrightarrow & \overrightarrow{\Pi_1}(X_1) \\ \downarrow & & \downarrow \\ \overrightarrow{\Pi_1}(X_2) & \longrightarrow & \overrightarrow{\Pi_1}(X) \end{array}
\end{array}
$$

Proof. Provided we pay some attention to the details pointed out below, it suffices to mimic the proof of the classical Van Kampen theorem for groupoids given in [15]. Compactness actually remains the cornerstone of the argumentation.

First we need that for all $\gamma \in \mathsf{C}[\mathbb{I}_r, X]$, all $h \in \mathsf{C}[\mathbb{I}_\rho \times \mathbb{I}_s, X]$, all closed interval $\iota \subseteq [0, r]$ and all closed rectangles $\iota_1 \times \iota_2 \subseteq [0, \rho] \times [0, s]$ the restriction of α to ι and the restriction of h to $\iota_1 \times \iota_2$ induce morphisms of C. Writting these restrictions as $\iota \hookrightarrow [0, r] \xrightarrow{\alpha} X$ and $\iota_1 \times \iota_2 \hookrightarrow [0, \rho] \times [0, s] \xrightarrow{h} X$ it is an immediate consequence of the Axiom 2.

The classical proof also uses the fact that any two paths on a rectangle sharing the same extremities are homotopic. Our context requires a similar result involving increasing paths and zigzag of directed homotopies. Given α and β two continuous increasing maps from $[0, r]$ to some rectangle $R = [0, a] \times [0, b]$ such that $\alpha(0) = \beta(0)$ and $\alpha(r) = \beta(r)$, we remark that the map γ from $[0, r]$ to R defined by $\gamma(t) = \max(\alpha(t), \beta(t))$ is still continuous and increasing. It follows that the map h from $[0, r] \times [0, 1]$ to R defined by

$$h(t, s) := (1 - s) \cdot \alpha(t) + s \cdot \gamma(t) = \alpha(t) + s \cdot (\gamma(t) - \alpha(t))$$

is continuous and increasing with respect to the product order on R. Therefore h induces a morphism of C in virtue of Axiom 2 and h is then a directed homotopy from α to γ. The same way, we obtain a directed homotopy from β to γ thus providing the expected zigzag. $\square$

In the framework Top, the Theorem 2.8 is reduced to the classical Van Kampen Theorem for fundamental groupoids. The Theorem 2.8 can be summarized by the following commutative cube whose upper and lower faces are pushout squares.

$$
\begin{tikzcd}
 & & \overrightarrow{P}(X_1) \arrow[drr] \arrow[ddd] & & \\
\overrightarrow{P}(X_0) \arrow[urr] \arrow[dr] \arrow[ddd] & & & & \overrightarrow{P}(X) \arrow[ddd] \\
 & & \overrightarrow{P}(X_2) \arrow[urr] \arrow[ddd] & & \\
 & & \overrightarrow{\Pi}_1(X_1) \arrow[drr] & & \\
\overrightarrow{\Pi}_1(X_0) \arrow[urr] \arrow[drr] & & & & \overrightarrow{\Pi}_1(X) \\
 & & \overrightarrow{\Pi}_1(X_2) \arrow[urr] & &
\end{tikzcd}
$$

3 Examples of frameworks for fundamental categories

3.1 d-Spaces

The notion of d-space is due to *Marco Grandis* [10,11]. A **subpath** of length s of a path δ of length r is a path of the form $\delta \circ \theta$ where θ is an increasing continous map from $[0, s]$ to $[0, r]$. The map θ needs not to be one-to-one nor onto. We slightly differ from the definition of [10] in that we allow paths to be defined on any non empty compact interval (including singletons) instead of just $[0, 1]$. As one can imagine, it does not make any significant difference but better fits with the structures described in the preceding section.

Definition 3.1 A **d-space** X is a topological space $U(X)$ together with a collection dX of paths on $U(X)$ which is stable under subpaths, stable under concatenation and contains all constant paths.

The standard example of d-space is the compact segment $[0, r]$ equipped with the collection of increasing continuous maps from $[0, s]$ to $[0, r]$ with s running through $\mathbb{R}_+$, we denote it by $\uparrow\mathbb{I}_r$.

Definition 3.2 A **d-space morphism** from X to Y is a continuous map f from $U(X)$ to $U(Y)$ such that $\forall \delta \in dX$, $f \circ \delta \in dY$.

The d-spaces and their morphisms form a category denoted by dT. There is an obvious forgetful functor U from dT to Top which has both a left adjoint (providing a topological space with its constant paths) and a right adjoint (providing a topological space with all its paths). Moreover any interval $\iota \subseteq \mathbb{R}$ can be provided with a structure of d-space by considering all its increasing continuous paths. Hence the category dT can be endowed with a framework for fundamental category structure and in the sequel, we refer to it as the **directed framework of** dT. However there are other alternatives, we can provide every interval with the set of all its constant paths (**chaotic framework of** dT) or provide every interval with the set of all its paths (**discrete framework of** dT).

Actually the directed framework of dT plays a special role. Recall that given an object X of C the collection of directed paths on X consists on the union of the homsets $\mathsf{C}[\mathbb{I}_r, X]$ for r running through the non negative numbers. This collection is stable under subpaths (by Axiom 2), contains all the constant paths (by Axiom 2) and is stable under concatenation (by Axiom 3) : thus we have a d-space $D(X)$. Moreover, a morphism $f \in \mathsf{C}[X,Y]$ clearly induces a morphism $D(f) \in \mathsf{dT}[D(X), D(Y)]$ by setting

$$\big(D(f)\big)(\gamma) := f \circ \gamma$$

Then we have defined a faithful functor D from C to dT which clearly depends on the structure of framework for fundamental category carried by C. Furthermore, in [10] the elements of dX are called the "directed paths" of X while in the directed framework of dT, a "directed path" on X is a morphism of $\mathsf{dT}[\uparrow\mathbb{I}_r, X]$ for some $r \geq 0$. The next result asserts that both terminology coincide.

Lemma 3.3 *If* dT *is equiped with its directed framework, then the directed paths on a d-space* X *are the elements of* dX *that is to say*

$$\bigcup_{r \in \mathbb{R}_+} \mathsf{dT}[\uparrow\mathbb{I}_r, X] = dX$$

Proof. Let $\alpha \in \mathsf{dT}[\uparrow\mathbb{I}_r, X]$ and θ denote the identity map on $[0,r]$, then θ is continuous and increasing hence $\alpha = \alpha \circ \mathsf{id}_{[0,r]}$ belongs to dX for α is a morphism of dT and the d-space structure of $\uparrow\mathbb{I}_r$ consists on all the increasing continuous paths on $[0,r]$. Conversely, any $\delta \in dX$ is in particular a continous map from $[0,r]$ to X for some $r \geq 0$. Since dX is closed under subpaths one has $\alpha \circ \theta \in dX$ for all $s \geq 0$ and all continuous increasing maps θ from $[0,s]$ to $[0,r]$. Yet, according to the d-space structure of $\uparrow\mathbb{I}_r$, it precisely means that δ belongs to $\mathsf{dT}[\uparrow\mathbb{I}_r, X]$. $\square$

Note that if we had chosen any of the two other "pathological" framework over dT the preceding lemma would have not hold. On one hand the choatic framework does not bring anything new. More precisely

Lemma 3.4 *If* dT *is equiped with its chaotic framework, then the directed paths on a d-space* X *are the paths on* $U(X)$ *and the fundamental category of* X *is the fundamental groupoid of* $U(X)$.

Proof. A path α on $U(X)$ defined on $[0,r]$ is a directed path on X if and only if for all constant path c on $[0,r]$ the path $\alpha \circ c$ belongs to dX, which is the case since dX contains all constant paths. $\square$

On the other hand, the discrete framework is also irrelevant since in this case, the fundamental category of the directed unit interval (that is to say $[0,1]$ with the set of continuous increasing paths on it) is just the discrete category whose set of objects is $[0,1]$.

Lemma 3.5 *If* dT *is equiped with its discrete framework, then one has*

$$\bigcup_{r\in\mathbb{R}_+} \mathsf{dT}[\mathbb{I}_r, X] \subseteq dX$$

and the equality occurs iff for all $\delta \in dX$ *defined on* $[0,r]$ *and all continuous map* θ *from* $[0,s]$ *to* $[0,r]$ *the map* $\delta \circ \theta$ *belongs to* dX*.*

Proof. Let $\alpha \in \mathsf{dT}[\mathbb{I}_r, X]$ and θ denote the identity map on $[0,r]$, then θ is continuous hence $\alpha = \alpha \circ \theta$ belongs to dX for α is a morphism of dT and the d-space structure of $\mathbb{I}_r$ consists on all the continuous paths on $[0,r]$. The case of equality immediately comes from the fact that each interval is equiped with the set of all continuous paths on it. $\square$

Any set of paths on a topological space induces a structure of d-spaces. Indeed, given a topological space X and a collection P of paths over X, this structure can be described as the collection dX_P of paths on X which are constant or can be written as

$$(\delta_n * \ldots * \delta_1) \circ \theta$$

where $n \geq 1$, $\delta_k \in P$ for all $k \in \{1,\ldots,n\}$ and $\theta : [0,r] \to [0,s]$ is continuous and increasing (where s is the sum of the lengths of the paths δ_k and r ranges in $\mathbb{R}_+$). Actually dX_P is the least (with respect to inclusion) collection of paths on X which contains P_X and provides X with a structure of d-space. The structure is then said to be **generated** by P.

3.2 Streams, Circulations and Precirculations

These notions are due to *Sanjeevi Krishnan* [17,18].

Definition 3.6 A **circulation** on a topological space T is a mapping which sends any open subset $W \subseteq T$ to a preorder relation $\preccurlyeq_W$ on W and satisfies the following property : given any open subset $W \subseteq T$ and any open covering $(O_i)_{i\in I}$ of W, the relation $\preccurlyeq_W$ is the preorder on W generated by the family of relations $(\preccurlyeq_{O_i})_{i\in I}$ i.e.

$$(W, \preccurlyeq_W) = \bigvee_{i\in I} (O_i, \preccurlyeq_{O_i})$$

A **stream** X is a topological space $U(X)$ together with a circulation on $U(X)$.

Definition 3.7 A **morphism of stream** from X to Y is a continuous map from $U(X)$ to $U(Y)$ such that for every open subset $W \subseteq X$ and every open subset $V \subseteq Y$ such that $f(W) \subseteq V$, one has

$$\forall w, w' \in W \quad w \preccurlyeq^X_W w' \;\Rightarrow\; f(w) \preccurlyeq^Y_V f(w')$$

In particular given some open subset $V \subseteq Y$ we have

$$\forall w, w' \in f^{-1}(V) \quad w \preccurlyeq^X_{f^{-1}(V)} w' \;\Rightarrow\; f(w) \preccurlyeq^Y_V f(w')$$

Conversely, assume the previous assertion is satisfied and consider some open subset $W \subseteq X$ such that $f(W) \subseteq V$ and let $w, w' \in W$ be such that $w \preccurlyeq^X_W w'$. Since $W \subseteq f^{-1}(V)$ and $\preccurlyeq$ is a circulation we have $w \preccurlyeq^X_{f^{-1}(V)} w'$ and therefore $f(w) \preccurlyeq^X_V f(w')$. It clearly follows that the morphisms of stream compose and thus form a category that we denote by St. There is an obvious forgetful functor U from St to Top which admits a left adjoint provided by the diagonal relations on the open subsets of any topological space. It also admits a right adjoint by associating any topological space with its "greatest" stream structure [18], yet, this structure cannot be easily described.

The category of streams enjoys many nice properties, in particular it is complete. However the products in St are not easily described and requires to generalize the notion of stream. We still follow the terminology of [18] calling **precirculation** on a topological space T any mapping which sends any open subset $W \subseteq T$ to a preorder relation $\preccurlyeq_W$ on W such that if $W_1 \subseteq W_2$ are two open subsets of T, then for all $w, w' \in W_1, w \preccurlyeq_{W_1} w' \Rightarrow w \preccurlyeq_{W_2} w'$. As suggested by the terminology, any circulation is a precirculation. Then we call **prestream** a topological space together with a precirculation on it. Defining the morphisms of prestream the same way as the morphisms of streams we have a category pSt and a full inclusion of St into pSt. We also have a forgetful functor from pSt to Top which admits both a left adjoint (defined as for the streams) and a right adjoint which consists on assigning to each open subset W its chaotic preorder that is to say the one whose graph is $W \times W$. Still we write $\preccurlyeq^X$ to denote the precirculation of a prestream X. Given two precirculations $\preccurlyeq$ and $\preccurlyeq'$ on the same toplogical space, one says that $\preccurlyeq$ is **contained** in $\preccurlyeq'$ (or that $\preccurlyeq$ is less that $\preccurlyeq'$) when for all open subset W and all $w, w' \in W$ one has $w \preccurlyeq_W w' \Rightarrow w \preccurlyeq'_W w'$ i.e. one has the graph inclusion $\mathsf{graph}(\preccurlyeq) \subseteq \mathsf{graph}(\preccurlyeq')$. Then, the **cosheafification** of a prestream X is defined, for all open subset W of $U(X)$, by the preorder on W generated by all the preorders $\preccurlyeq_W$, where $\preccurlyeq$ ranges in the collection of all circulations on $U(X)$ contained in $\preccurlyeq^X$. By a slight abuse of language, one may say that a prestream X is less that a prestream X' when so are their corresponding precirculations. Then, in an abstract way, the cosheafification of a prestream X is the greatest stream contained in X. Following the notation of [18], the cosheafification of a prestream X is denoted by $X^!$. As one can imagine, the cosheafification construction induces a right adjoint to the inclusion $\mathsf{pSt} \hookrightarrow \mathsf{St}$. In particular, the following commutative diagram lead us to write the right adjoint to the forgetful functor from St to Top as a composite of right adjoints.

$$\begin{tikzcd}
 & \mathsf{Top} \\
\mathsf{St} \arrow[ur, "U"] \arrow[r, hook] & \mathsf{pSt} \arrow[u, "U"']
\end{tikzcd}$$

Furthermore, the product of a family of streams is given by the cosheafification of its product in pSt [18]. In other words the inclusion functor from St to pSt preserves products. Yet, it is noticed in [18] that the product of a family of streams in pSt almost always differ from its product in St.

Any interval of $\mathbb{R}$ can be provided with a structure of stream assigning to each of its open subsets W the following preorder : for w and w' in W write $w \preccurlyeq_W w'$ when w is less than w' (as real numbers) and the segment $[w, w']$ is contained in W. In particular for all $r \in \mathbb{R}_+$ we denote by $\overrightarrow{\mathbb{I}}_r$ the stream on the compact interval $[0, r]$. From this example, we immediately obtain a framework for fundamental categories on St, the Axiom 3 being given by the fact that $\overrightarrow{\mathbb{I}}_r$ is exponentiable.

Corollary 3.8 *A prestream X and its cosheafification $X^!$ have isomorphic fundamental categories i.e.*

$$\overrightarrow{\Pi}_1(X) \cong \overrightarrow{\Pi}_1(X^!)$$

Proof. Applying the Corollary 2.5. □

3.3 Partially Ordered Spaces

Mathematicians have studied the pospaces since the early forties [6]. Later motivated by functional analysis, *Leopoldo Nachbin* generalized many basic facts about topological spaces to pospaces in [20]. Later on, in theoretical computer science, *Edsger Wybe Dijkstra* introduced the notion of **progress graph** in order to modelize the phenomena encountered in **concurrency** [5]. Then *Scott D. Carson*, *Paul F. Reynolds Jr* and *Vaughan Pratt* formalized the geometric idea that holes in the structure modelizing a concurrent program actually represent the lack of ressource that may occur during its execution [4,22]. It was therefore natural to consider algebraic topology as a way of adressing issues met in concurrency, thus pospaces provided the first playground for directed algebraic topologists [7]. By the way, it is worth to note the forgetful functor from P to H does not preserve colimits. Indeed, identifying the endpoints of the directed compact unit segment $[0, 1]$ in P result in identifying all the points of $[0, 1]$ thus producing a singleton. On the other hand, identifying the endpoints of the compact unit segment $[0, 1]$ in H produces a circle. Since directed loops cannot be represented as pospaces several generalizations, like local pospaces [8], d-spaces [10] or streams [17,18], have been introduced.

The colimits of Top (or any of its "nice" subcategory) occur everywhere in algebraic topology. Then one of the first step towards the algebraic topology of pospaces is to prove that they form a cocomplete category. Doing so, we establish an adjunction that will be used in the next section to compare the fundamental categories of pospaces to the fundamental categories of their corresponding streams.

A **weakly ordered (topological) space** or **wospace** is a Hausdorff space X equiped with a closed reflexive binary relation ρ. In the sequel we write X to denote both the wospace (X, ρ) and its underlying topological space X, while ρ_X is put for the relation that comes with X. The morphisms of spaces with relation from X to Y are the continuous maps $f : X \to Y$ such that for all $x, x' \in X$,

$x \,\rho_X\, x' \;\Rightarrow\; f(x) \,\rho_Y\, f(y)$. The spaces with relation together with their morphisms form the category W. As we shall see, the cocompleteness of P actually arises from the directed version of the following standard result from basic general topology.

Lemma 3.9 *Let X be a topological space and $\sim$ be an equivalence relation over UX, there exists a unique, up to isomorphism, Hausdorff space Y as well as a unique continuous map $q : X \to Y$ such that for any Hausdorff space Z and any continuous map $f : X \to Z$ satisfying $x \sim x' \;\Rightarrow\; f(x) = f(x')$ there is a unique continuous map $g : Y \to Z$ such that $f = g \circ q$. Moreover, if $\sim$ is the relation $\{(x, x') \mid f(x) = f(x')\}$ the the mapping g is one-to-one.*

Proof. Let $\sim^\star$ be the least closed equivalence relation on UX containing $\sim$. The topological space Y is the set theoretic quotient of UX by $\sim^\star$, that we also denote by UY, equiped with the largest topology that makes the set theoretic quotient map $q : UX \to UY$ continuous. The closed subsets of this topology are precisely the subsets V of UY such that $q^{-1}(V)$ is a closed subset of X. By definition of q, the set $q^{-1}(\Delta_{UY})$ is exactly the graph of $\sim^\star$ which is closed by construction. Thus by definition of the topology on Y, the set Δ_{UY} is closed in $Y \times Y$ which exactly means that Y is Hausdorff.
Now consider a continuous map f from X to a Hausdorff space Z such that for all $x, x' \in X$, $x \sim^\star x'$ implies $f(x) = f(x')$. The set $(f \times f)^{-1}(\Delta_{UZ})$ is clearly (the graph of) an equivalence relation which is closed since f is continuous, hence it contains (the graph of) $\sim^\star$. So there exists a unique set theoretic map $g : UY \to UZ$ such that $f = g \circ q$. Let C be a closed subset of Z, then $f^{-1}(C) = q^{-1}(g^{-1}(C))$ is a closed subset of X since f is continuous. It follows that $g^{-1}(C)$ is closed in Y by definition of the topology of Y and thus g is continuous.
Now suppose the relation $\sim$ is actually the following

$$(f \times f)^{-1}(\Delta_Z) = \{(x, x') \mid f(x) = f(x')\}$$

then we have $\sim = \sim^\star$. Hence given $x, x' \in X$ we have $g(q(x)) = g(q(x'))$ iff $f(x) = f(x')$ by definition of f, then $f(x) = f(x')$ iff $x \sim x'$ by definition of $\sim$, and $x \sim x'$ iff $q(x) = q(x')$ by definition of q which is therefore one-to-one. □

The category of topological spaces is denoted by Top, its full subcategory of Hausdorff spaces is denoted by H. The inclusion functor of H in Top has a left adjoint whose counit is the identity while its unit is given by the Lemma 3.9 by taking the equivalence $\sim$ to be Δ_{UX} for each topological space X. The next result is the adaptation of the Lemma 3.9 to the spaces with relation, the proof of the former actually relies on the later.

Lemma 3.10 *Let X be a space with relation and $\sim$ be an equivalence relation over UX, there exists a unique, up to isomorphism, space with relation Y as well as a unique morphism $q : X \to Y$ such that for any space with relation Z and any morphism $f : X \to Z$ satisfying $x \sim x' \;\Rightarrow\; f(x) = f(x')$ there is a unique morphism $g : Y \to Z$ such that $f = g \circ q$. Moreover*

1) the underlying topological space of Y is the quotient of X by $\sim$ in H. *See the Lemma 3.9*

2) if $\sim = (f \times f)^{-1}(\Delta_Z)$ then g is one-to-one

3) if Z is a pospace and $\sim = (f \times f)^{-1}(\Delta_Z)$ then g is actually a morphism of pospaces.

Proof. The underlying Hausdorff space of Y as well as the continuous map q are given by the Lemma 3.9 and the relation on Y is the least closed reflexive relation on Y that turns q into a morphism of W.

Now let $f : X \to Z$ be a morphism of W, from the Lemma 3.9 we have a unique continous map $g : Y \to Z$ such that $f = g \circ q$, then it suffices to check that g is actually a morphism of W. Since f is a morphism of W we have $\rho_X \subseteq (f \times f)^{-1}(\rho_Z)$ i.e. $\rho_X \subseteq (q \times q)^{-1}\big((g \times g)^{-1}(\rho_Z)\big)$ therefore $(g \times g)^{-1}(\rho_Z)$ is a reflexive binary relation that turns q into a morphism of W. Moreover (the graph of) the relation $(g \times g)^{-1}(\rho_Z)$ is closed since g is continuous, thus $\rho_Y \subseteq (g \times g)^{-1}(\rho_Z)$ i.e. g is a morphism of W. By construction of Y we have proven 1) and 2) which are direct consequences of the Lemma 3.9.

Suppose $\sqsubseteq_Z$ is a closed partial order, we want to check that $(g \times g)^{-1}(\sqsubseteq_Z)$ is so. The only point which remains to check is the antisymetry. Let $y, y' \in Y$ such that $g(y) \sqsubseteq_Z g(y')$ and $g(y') \sqsubseteq_Z g(y)$, then we have $g(y) = g(y')$ since $\sqsubseteq_Z$ is antisymetric and from 2) we know that g is one-to-one, so we have $y = y'$. □

The object Y is called the **quotient** of X by $\sim$. The obvious forgetful functor from W to H has both a left adjoint and a right adjoint which are respectively given by (X, Δ_X) and $(X, X \times X)$ i.e. the relation whose graph is $X \times X$. As a consequence, the forgetful functor preserves limits and colimits that exist in W. Furthermore the forgetful functor from W to H has a left adjoint, however, as we shall see, it has no right adjoint.

Lemma 3.11 *The categores* W *and* P *are complete and the inclusion functor* P $\hookrightarrow$ W *preserves limits.*

Proof. By a standard result of basic category theory [1] it suffices to prove that W and P admit products and equalizers and that they are preserved by the inclusion functor. From the preceding remark, we have no choice about the underlying topological spaces. The product of a family $(X_i)_{i \in \mathcal{I}}$ of spaces with relation is the product of their underlying spaces equiped with the product relation i.e. $(x_i)_{i \in \mathcal{I}}$ and $(x'_i)_{i \in \mathcal{I}}$ are related when $x_i \ \rho_{X_i} \ x'_i$ for all $i \in \mathcal{I}$. By the way if each relation ρ_{X_i} is a partial order then so is the product, hence the inclusion functor preserves the products. The equalizer of two parallel morphisms $f_1, f_2 \in \mathsf{W}[X, Y]$ is the following topological subspace of X

$$\big\{(x, x') \;\big|\; f(x) = f(x')\big\}$$

equiped with the relation induced by the product relation $\rho_X \times \rho_X$. In particular it is a pospace when so is X, hence the inclusion functor also preserves equalizers and therefore all limits. □

Lemma 3.12 *The category* W *is cocomplete, the category* P *admits coproducts and they are preserved by the inclusion functor* $\mathsf{P} \hookrightarrow \mathsf{W}$ *.*

Proof. Once again it suffices to prove that W admits coproducts and coequalizers and for the inclusion functor admits a right adjoint we have no choice about the underlying topological spaces.

The coproduct of a family $(X_i)_{i\in\mathcal{I}}$ of spaces with relation is the coproduct of their underlying spaces equiped with the coproduct relation i.e. x and x' are related when x and x' belong to the same component X_i and $x\ \rho_{X_i}\ x'$. The coproduct relation is a partial order when each ρ_{X_i} is so therefore P admits coproducts and they are preserved by the inclusion functor.

The coequalizer in W of two parallel morphisms $f_1, f_2 \in \mathsf{W}[X', X]$ is provided by the Lemma 3.10 applied with the equivalence relation generated by $\big\{(f_1(x), f_2(x)) \bigm| x \in X\big\}$. □

Lemma 3.13 *The category* P *is an epireflective subcategory of* W*.*

Proof. From the Lemma 3.11 we know that P is complete and the inclusion functor $\mathsf{P} \hookrightarrow \mathsf{W}$ preserves limits. Let $\mathcal{S}'$ be the set of equivalence relation on X such that the quotient of X by $\sim$ is a pospace and let $\mathcal{S}$ the set of quotient of X by $\sim$ with $\sim$ running through $\mathcal{S}'$. From the Lemma 3.10 we know that any morphism of W from X to some pospace Z can be factorized through an element of $\mathcal{S}$ i.e. $\mathcal{S}$ is a solution set [1] for X. Then we conclude applying the Freyd's Adjoint Functor Theorem [1]. □

The following corollary is a direct consequence of a result from general category theory which claims that any reflective subcategory of a cocomplete (resp. complete) category is also cocomplete (resp. complete) [1]. Since we already have checked that P has all coproducts (Lemma 3.12) it only remains to check that P has all coequalizers, yet, they are immediately provided by the third point of the Lemma 3.10.

Corollary 3.14 *The category of partially ordered spaces is cocomplete.*

Before leaving pospaces we give some remarkable facts about them.

The image of a path δ on a pospace X is understood as the pospace structure induced over the set $\mathsf{im}(\delta)$ by X. Moreover, we recall that $\mathbb{I}_0$ is the singleton $\{0\}$ (with its unique pospace structure) and $\mathbb{I}_1$ is the compact segment $[0,1]$ equiped with the standard order on real numbers. Then we have

Theorem 3.15
The image of a path on a pospace is isomorphic to $\mathbb{I}_1$ *or isomorphic to* $\mathbb{I}_0$*.*

Proof. The demonstration of the Theorem 3.15 heavily relies on the tight relation between the standard topology of $[0,1]$ and its total order, and more precisely on the two following facts.

Recall that a topological space is said to be **separable** when it is not reduced to a singleton and admits a countable subset which intersects any of its (non empty) open subsets. A **continuum** is a compact connected Hausdorff space, a point x of a connected topological space X is said to be **non-separating** when $X\backslash\{x\}$ is still connected, and an **arc** is a continuum with exactly two non-separating points. The first fact claims that any separable arc is homeomorphic to $[0,1]$. Moreover, a pospace X is said to be **linear** when its underlying order is so, that is to say when for all x and x' in X, one has $x \sqsubseteq x'$ or $x' \sqsubseteq x$. The second fact is well-known in point-set topology, it claims that given an arc A there exists exactly two linear pospaces whose underlying space is A and actually, each of them is isomorphic to the opposite of the other.

Admitting these assertions the proof becomes easy. Let δ be a morphism of pospace from $\mathbb{I}_r$ to X, its image $\mathsf{im}(\delta)$ inhertis a pospace structure from X. The underlying space of $\mathsf{im}(\delta)$ is a continuum as the direct image of the continuum $[0,r]$ by the continuous map δ whose codomain is Hausdorff. It is clearly separable considering $\{\delta(x) \mid x \in \mathbb{Q} \cap [0,r]\}$. According to the first claim, the underlying space of $\mathsf{im}(\delta)$ is homeomorphic to $[0,1]$. Furthermore, the order inherited by $\mathsf{im}(\delta)$ from X is linear. Indeed, if $x, y \in \mathsf{im}(\delta)$ there exist $t, t' \in [0,1]$ such that $\delta(t) = x$ and $\delta(t') = y$, since one has $t \leqslant t'$ or $t' \leqslant t$ we have $x \leqslant y$ or $y \leqslant x$. Hence according to the second claim, the pospace structure of $\mathsf{im}(\delta)$ is isomorphic to $\mathbb{I}_1$. □

The Theorem 3.15 has no obvious counterpart in general topology : for example the *Peano* curves are known to provide continous maps from $[0,1]$ onto $[0,1] \times [0,1]$. In fact the study of continuous images of the compact unit segment is a mathematical research subject on its own [21]. Beyond its interest as a characterization result, the Theorem 3.15 as a striking consequence upon dihomotopy classes of paths on a pospace.

Corollary 3.16 *Two dipaths on the same pospace sharing the same image are dihomotopic.*

Proof. Let γ and δ be two paths on a pospace X such that $\mathsf{im}(\delta) = \mathsf{im}(\gamma)$ and denote by K the pospace structure induced on $\mathsf{im}(\delta)$ by X. Without loss of generality we can suppose $\mathsf{dom}(\delta) = \mathsf{dom}(\gamma) = \mathbb{I}_1$ and define $\alpha \in \mathsf{P}[\mathbb{I}_1, X]$ by $\alpha(t) := \max(\gamma(t), \delta(t))$. Then α is a path on X such that $\mathsf{im}(\alpha) = \mathsf{im}(\delta)$. Now supposing that $\mathbb{K} \ncong \mathbb{I}_0$ and applying the Theorem 3.15, let $\phi \in \mathsf{P}[\mathbb{I}_1, K]$ be an isomorphism. Then define

$$H_1(s,t) := \phi\Big(s * \phi^{-1}\big(\alpha(t)\big) + (1-s) * \phi^{-1}\big(\gamma(t)\big)\Big)$$
$$H_2(s,t) := \phi\Big(s * \phi^{-1}\big(\alpha(t)\big) + (1-s) * \phi^{-1}\big(\delta(t)\big)\Big)$$

thus obtaining two dihomotopies H_1 and H_2 respectively from γ and δ to α. □

It is worth to notice that due to the antisymetry of a partial order, the forgetful functor from P to H does not preserve coequalizers indeed, in one hand one has the following coequalizing diagram in Top

$$\{0,1\} \overset{t\mapsto t}{\underset{t\mapsto 1-t}{\rightrightarrows}} [0,1] \xrightarrow{t\mapsto e^{(2i\Pi t)}} S^1$$

while in the other hand one has the following coequalizing diagram in P

$$\{0,1\} \overset{t\mapsto t}{\underset{t\mapsto 1-t}{\rightrightarrows}} \overrightarrow{[0,1]} \xrightarrow{t\mapsto 0} \{0\}$$

Imagine one has pinned a finite sequence of "instructions" along the segment $[0,1]$. Consider this sequence forms a "program" whose "execution" consists on performing each of its instruction following the order provided by the standard ordering of real numbers. Then identifying 0 and 1 should produce another program whose instructions are pinned on a "directed circle" thus giving rise to infinite repetition of the same sequence of instructions. Clearly, the behaviour of coequalizers in P does not fit with this point of view. Saying to be **loop-free** [10] any category $\mathcal{C}$ such that

$$\forall x,y \in \mathsf{Ob}(\mathcal{C})\ \mathcal{C}[x,y] \neq \emptyset \text{ and } \mathcal{C}[y,x] \neq \emptyset \implies x = y \text{ and } \mathcal{C}[x,x] = \{\mathsf{id}_x\}$$

it is easy to check that the fundamental category of any pospace is loop-free.

We conclude the section showing the notion pospace is rather rigid in regard of the fundamental category invariant. Indeed there is a unique (up to isomorphism) pospace whose fundamental category is the standard poset [11] $[0,1]$. In opposition, there is at least two non isomorphic d-spaces whose fundamental category is the standard poset $[0,1]$, the topology and the collection of directed paths of a d-space being too loosely bound.

Recall that $\uparrow\mathbb{I}_1$ is the d-space over the compact unit segment $[0,1]$ whose directed paths are the continuous increasing maps from $[0,r]$ to $[0,1]$ with $r \in \mathbb{R}^+$. Denote by X the d-space on the coarse topology of $[0,1]$ whose directed paths are those of $\uparrow\mathbb{I}_1$. As we have remarked, if δ_1 and δ_2 are two continuous increasing maps from $[0,r]$ to $[0,1]$ the map δ_3 defined by $\delta_3(t) := \max(\delta_1(t), \delta_2(t))$ is still so and the maps h_1 and h_2 defined by $h_i(t) := (1-s)\delta_i(t) + s\delta_3(t)$ for $i \in \{1,2\}$ prove that δ_1 and δ_2 are dihomotopic. Yet $\uparrow\mathbb{I}_1$ and X are clearly not isomorphic since their underlying topological spaces are not. In comparison we have

Proposition 3.17 *Any pospace whose fundamental category is isomorphic to the standard poset* $[0,1]$ *is isomorphic to the directed compact unit segment.*

Proof. Let X be a pospace such that $\overrightarrow{\Pi_1}(X)$ is isomorphic to the standard poset $[0,1]$. We can suppose the underlying set of X is $[0,1]$. We denote by $\leqslant$ the

[10] The loop-free categories have been introduced by *André Haefliger* as **small categories without loops** or **scwols** [2,12,13].

[11] The poset $[0,1]$ is equiped with the standard order on real numbers and seen as a small category.

standard order over real numbers restricted to $[0,1]$ and $\sqsubseteq_X$ the partial order of X. Let $x, y \in [0,1]$ such that $x \leqslant y$, by construction of the fundamental category there exists a morphism of pospaces δ from $\mathbb{I}_1$ to X such that $\mathsf{s}(\delta) = x$ and $\mathsf{t}(\delta) = y$. As a morphism of pospaces δ is increasing hence $x \sqsubseteq_X y$. Furthermore the order on real numbers is total hence $\sqsubseteq_X$ is actually $\leqslant$. Moreover if we take x and y to be 0 and 1 then we have a continuous map from the compact unit segment onto $U(X)$ which is Hausdorff as the underlying topological space of a pospace [20]. Hence $U(X)$ is compact Hausdorff. In order to conclude, we just need to prove that the topology of X is finer than the standard topology. Then remark that for X is a pospace the subsets $\{x \in X \mid a \sqsubseteq_X x\}$ and $\{x \in X \mid x \sqsubseteq_X a\}$ are closed [20], thus $\{x \in X \mid a > x\}$ and $\{x \in X \mid x > a\}$ are open, and therefore the topology on X is finer than the standard topology on $[0,1]$. $\square$

4 The Compactly Generated Weakly Hausdorff feature

It's a well-known fact that Top is not Cartesian closed, this issue is classically adressed by considering the full subcategory of compactly generated spaces CG. Obviously, the same problem arises with frameworks for fundamental categories and it is adressed substituting CG to Top.

Given a topological space X we denote by $\mathcal{K}(X)$ the poset of compact Hausdorff sub-spaces of X, ordered by inclusion. The poset $\mathcal{K}(X)$ can be seen as a subcategory of Top, then X is said to be **compactly generated** when

$$\mathsf{colim}\Big(\mathcal{K}(X) \hookrightarrow \mathsf{Top}\Big) \cong X$$

One easily checks that a topological space X is compactly generated if and only if for all $C \subseteq X$, if $K \cap C$ is closed in K for K running through $\mathcal{K}(X)$ then C is closed in X. This is actually the definition given in [16]. As pointed out by the referee, one usually considers the weakly Hausdorff compactly generated (WHGC) spaces because the quotient of a WHGC space by some k-closed equivalence relation remains weakly Hausdorff and the category of weakly Hausdorff compactly generated spaces is still complete, cocomplete and Cartesian closed. Formally, a topological space X is said to be **weakly Hausdorff** when for all compact Hausdorff spaces K and all continuous map $u : K \to X$, the set $u(K)$ is a closed subset of X. Every weakly Hausdorff (respectively Hausdorff) space is T_1 (respectively weakly Hausdorff) but the converse is false.

5 Comparing some examples of framework

The section is devoted to the description of adjunctions between the different frameworks described above, and the fact that they behave well with respect to fundamental categories.

$$\mathsf{St} \underset{S}{\overset{D}{\rightleftarrows}} \mathsf{dT} \qquad \mathsf{pSt} \underset{S}{\overset{D}{\rightleftarrows}} \mathsf{dT}$$

Lemma 5.1 *Given a d-space X the mapping that sends each open subset $W \subseteq X$ to the relation*

$$\left\{(w, w') \in W \times W \middle| \ \exists \delta \in dX,\ \mathsf{im}(\delta) \subseteq W,\ \mathsf{s}(\delta) = w,\ \mathsf{t}(\delta) = w'\right\}$$

forms a circulation on $U(X)$, hence a structure of stream denoted by $S(X)$.

Proof. Suppose $(O_i)_{i \in I}$ is an open covering of W. Since dX is stable under concatenation, the relation $\preccurlyeq_W$ contains the preorder generated by the relations $\preccurlyeq_{O_i}$ for i running through I. Conversely, consider an element δ of dX such that $\mathsf{im}(\delta) \subseteq W$. Each $\delta^{-1}(O_i)$ is an open subset of $[0, r]$ hence a disjoint union of open sub-intervals of $[0, r]$. For $[0, r]$ is compact we have a finite family of open sub-intervals of $[0, r]$ each of which being contained in some $\delta^{-1}(O_i)$. From this finite covering we deduce a finite family $\gamma_n, \ldots, \gamma_0$ of subpaths of δ such that $\mathsf{s}(\gamma_{j+1}) = \mathsf{t}(\gamma_j)$, $\mathsf{s}(\gamma_0) = \mathsf{s}(\delta)$ and $\mathsf{t}(\gamma_n) = \mathsf{t}(\delta)$. It follows that for all j in $\{0, \ldots, n\}$ we have some $i_j \in I$ such that $\mathsf{im}(\gamma_j) \subseteq O_{i_j}$ and $\mathsf{s}(\gamma_j) \preccurlyeq^X_{O_{i_j}} \mathsf{t}(\gamma_j)$, and thus $\preccurlyeq^X_W$ is the preorder on W generated by the relations $\preccurlyeq^X_{O_i}$. □

For example we have $S(\uparrow \mathbb{I}) = \overrightarrow{\mathbb{I}}$.

Lemma 5.2 *Given the d-spaces X and Y one has*

$$\mathsf{dT}[X, Y] \subseteq \mathsf{St}[S(X), S(Y)]$$

Proof. By definition, if f is a morphism of d-space and $\delta \in dX$, then $f \circ \delta \in dY$.□

As a consequence of Lemma 5.2, we have a functor S from dT to St such that any morphism of d-space f one has $S(f) = f$.

Lemma 5.3 *The functor S preserves finite products and the terminal object.*

Proof. Let A and B be two d-spaces while $\preccurlyeq^{S(A)}$ and $\preccurlyeq^{S(B)}$ are the circulations of $S(A)$ and $S(B)$. As we have noticed, the circulation of $S(A \times B)$ is contained in the precirculation of the product of $S(A)$ and $S(B)$ in pSt. It remains to see that it is the biggest one. Let $\preccurlyeq$ be a circulation on $U(A \times B) = U(A) \times U(B)$ contained in the precirculation of the product of $S(A)$ and $S(B)$ in pSt. Denote by Π_A and Π_B the projections from $U(A) \times U(B)$ to respectively $U(A)$ and $U(B)$. Let W be an open subset of $U(A) \times U(B)$.
First we treat the case where $W = W_A \times W_B$ with W_A and W_B two open subsets of $U(A)$ and $U(B)$. Suppose $w \preccurlyeq w'$, then by hypothesis one has some $\delta_A \in dA$ and some $\delta_B \in dB$ such that $\mathsf{s}(\delta_A) = \Pi_A(w)$, $\mathsf{t}(\delta_A) = \Pi_A(w')$ and $\mathsf{im}(\delta_A) \subseteq U(A)$, and $\mathsf{s}(\delta_B) = \Pi_B(w)$, $\mathsf{t}(\delta_B) = \Pi_B(w')$ and $\mathsf{im}(\delta_B) \subseteq U(B)$. Then $\delta := \delta_A \times \delta_B$ belongs to $d(A \times B) = dA \times dB$ and satisfies $\mathsf{s}(\delta) = w$, $\mathsf{t}(\delta) = w'$ and $\mathsf{im}(\delta) \subseteq W$.
General case : let $(C_j)_{j \in J}$ be an open covering of W where each C_j is a rectangle i.e. the product of an open subset of $U(A)$ and an open subset of $U(B)$. Since $\preccurlyeq$ is a circulation one has a finite sequence $w = x_0 \preccurlyeq_{C_{j_1}} x_1 \preccurlyeq_{C_{j_2}} x_2 \cdots \preccurlyeq_{C_{j_n}} x_n = w'$. Applying the preceding case for each C_{j_k} with $k \in \{1, \ldots, n\}$ one has a finite

sequence $\delta_1, \ldots, \delta_n$ of elements of $d(A \times B)$ such that for all $k \in \{1, \ldots, n\}$ one has $\mathsf{s}(\delta_k) = x_{k-1}$, $\mathsf{t}(\delta_k) = x_k$ and $\mathsf{im}(\delta_k) \subseteq C_{j_k} \subseteq W$. Considering the concatenation $\delta := \delta_n * \ldots * \delta_1$ we have $\mathsf{s}(\delta) = w$, $\mathsf{t}(\delta) = w'$ and $\mathsf{im}(\delta) \subseteq W$, in other words $w \preccurlyeq^{S(A \times B)} w'$. □

Lemma 5.4 *The functor S preserves products and the terminal object.*

Proof. Let $(A_j)_{j \in J}$ be a non empty family of d-spaces while $\preccurlyeq^{S(A_j)}$ is the circulation of $S(A_j)$ for j running through the indexing set J. As we have noticed, the circulation of

$$S\Big(\prod_{j \in J} A_j\Big)$$

is contained in the precirculation of the product of $\big(S(A_j)\big)_{j \in J}$ in pSt. It remains to see that it is the biggest one. Let $\preccurlyeq$ be a circulation on

$$P := U\Big(\prod_{j \in J} A_j\Big) = \prod_{j \in J} U(A_j)$$

contained in the precirculation of the product of $\big(S(A_j)\big)_{j \in J}$ in pSt. For each $j \in J$ denote by Π_j the projection

$$\Pi_j : \prod_{j' \in J} U(A_{j'}) \longrightarrow U(A_j)$$

Then let W be an open subset of the product P. First we treat the case where W belongs to the canonical base of open sets of P in other words we suppose that

$$W = \prod_{j \in J} W_j \text{ with each } W_j \text{ being an open subset of } U(A_j)$$

and only finitely many of them differing from $U(A_j)$. Suppose $w \preccurlyeq w'$, then by hypothesis one has for all $j \in J$ some $\delta_j \in dA_j$ such that $\mathsf{s}(\delta_j) = \Pi_j(w)$, $\mathsf{t}(\delta_j) = \Pi_j(w')$ and $\mathsf{im}(\delta_j) \subseteq U(A_j)$. Then putting

$$\delta := \prod_{j \in J} \delta_j$$

one has an element of the following set theoretic product

$$d\Big(\prod_{j \in J} A_j\Big) = \prod_{j \in J} dA_j$$

which satisfies $\mathsf{s}(\delta) = w$, $\mathsf{t}(\delta) = w'$ and $\mathsf{im}(\delta) \subseteq W$.
General case : let $(C_\xi)_{\xi \in X}$ be an open covering of W where each C_ξ belongs to the canonical base of P. Since $\preccurlyeq$ is a circulation one has a finite sequence $w = x_0 \preccurlyeq_{C_{\xi_1}} x_1 \preccurlyeq_{C_{\xi_2}} x_2 \cdots \preccurlyeq_{C_{\xi_n}} x_n = w'$. Applying the preceding case for each C_{ξ_k} with $k \in \{1, \ldots, n\}$ one has a finite sequence $\delta_1, \ldots, \delta_n$ of elements of $d(A \times B)$ such

that for all $k \in \{1, \dots, n\}$ one has $\mathsf{s}(\delta_k) = x_{k-1}$, $\mathsf{t}(\delta_k) = x_k$ and $\mathsf{im}(\delta_k) \subseteq C_{j_k} \subseteq W$. Considering the concatenation $\delta := \delta_n * \ldots * \delta_1$ we have $\mathsf{s}(\delta) = w$, $\mathsf{t}(\delta) = w'$ and $\mathsf{im}(\delta) \subseteq W$, in other words $w \preccurlyeq^{S(\prod A_j)} w'$. □

Given a stream X, the elements of the following set are called the directed paths over X.

$$\bigcup_{r \in \mathbb{R}^+} \mathsf{St}[\overrightarrow{\mathbb{I}}_r, X]$$

Lemma 5.5 *Given a stream X, the set of directed paths on X provides the underlying topological space of X with a structure of d-space denoted by $D(X)$.*

Proof. Let $\gamma \in \mathsf{St}[\overrightarrow{\mathbb{I}}_s, X]$ and $\delta \in \mathsf{St}[\overrightarrow{\mathbb{I}}_r, X]$ such that $\gamma(0) = \delta(r)$. Given an open subset $W \subseteq X$, let t and t' be two elements of $(\gamma * \delta)^{-1}(W)$ such that $t \leqslant t'$, if $t \leqslant r \leqslant t'$ then we have

$$(\gamma * \delta)(t) = \delta(t) \preccurlyeq^X_W \delta(r) = (\gamma * \delta)(r) = \gamma(0) \preccurlyeq^X_W \gamma(t' - r) = (\gamma * \delta)(t')$$

if $t \leqslant t' \leqslant r$ then we have

$$(\gamma * \delta)(t) = \delta(t) \preccurlyeq^X_W \delta(t') = (\gamma * \delta)(t')$$

if $r \leqslant t \leqslant t'$ then we have

$$(\gamma * \delta)(t) = \gamma(t - r) \preccurlyeq^X_W \gamma(t' - r) = (\gamma * \delta)(t')$$

which proves that $\gamma * \delta$ is a directed path. The increasing continuous map from $[0, r]$ to $[0, r]$ are precisely the endomorphisms of the stream $\overrightarrow{\mathbb{I}}_r$: any increasing continuous map induces an endomorphism since St admits a fffc structure (and thus satisfy the Axiom 2), conversely given f a stream endomorphism of $\overrightarrow{\mathbb{I}}_r$ and $w, w' \in [0, r]$ such that $w \leqslant w'$, we have $f(w) \leqslant f(w')$ by definition of the circulation of $\overrightarrow{\mathbb{I}}_r$. Therefore $\mathsf{St}[\overrightarrow{\mathbb{I}}_r, X]$ is stable under subpaths and obviously contains all the constant paths. □

As an example, for all $r \in \mathbb{R}_+$ we have $D(\overrightarrow{\mathbb{I}}_r) = \uparrow\!\mathbb{I}_r$.

Lemma 5.6 *Given the streams X and Y one has*

$$\mathsf{St}[X, Y] \subseteq \mathsf{dT}[D(X), D(Y)]$$

Proof. The morphisms of stream compose. □

As a consequence of Lemma 5.6, we have a functor D from St to dT such that any morphism of stream f one has $D(f) = f$.

Lemma 5.7 *Given a d-space X, any element of dX induces a directed path on $S(X)$ i.e.*

$$dX \subseteq \bigcup_{r \in \mathbb{R}_+} \mathsf{St}[\overrightarrow{\mathbb{I}}_r, S(X)]$$

Proof. Let δ belong to dX, U be an open subset of X and t, t' be two elements of $\delta^{-1}(U)$ such that $t \leqslant t'$ (as real numbers) and $[t, t'] \subseteq \delta^{-1}(U)$. Let θ be a continuous increasing map from $[0, 1]$ to $\mathsf{dom}(\delta)$ such that $\theta(0) = t$ and $\theta(1) = t'$, then $\delta \circ \theta$ belongs to dX and satisfy $\mathsf{im}(\delta \circ \theta) \subseteq U$, hence $\delta(t) = \delta \circ \theta(0) \preccurlyeq^{S(X)} \delta \circ \theta(1) = \delta(t')$. $\square$

The Lemma 5.7 can be restated saying that for any d-space X, the identity of the underlying topological space of X induces a morphism η_X of d-space from X to $D(S(X))$. As we shall see, the collection of morphisms η_X, for X running through the collection of d-spaces, is the unit η of the adjunction. First, the commutativity of the following diagram is obvious since the underlying set theoretic maps of the morphisms of d-space f and $D(S(f))$ are equal while the underlying set theoretic map of η_X and η_Y are the identities of their underyling sets. Thus we have a natural transformation η from the identity functor of dT to $D \circ S$.

$$\begin{tikzcd} dX \arrow[r, "f"] \arrow[d, "dX"'] & dY \arrow[d, "dY"] \\ D(S(dX)) \arrow[r, "D(S(f))"'] & D(S(dY)) \end{tikzcd}$$

Lemma 5.8 *Given a stream X, the identity of the underlying topological space of X induces a morphism of streams ε_X from $S(D(X))$ to X.*

Proof. Let W be an open subset of X. Suppose $w \preccurlyeq^{S(D(X))} w'$ i.e. there exists a directed path δ on X such that $\mathsf{im}(\delta) \subseteq W$, $\mathsf{s}(\delta) = w$ and $\mathsf{t}(\delta) = w'$. For δ is in particular a morphism of stream we have $w = \mathsf{s}(\delta) \preccurlyeq^X \mathsf{t}(\delta) = w'$. $\square$

As we shall see, the collection of morphisms ε_X, for X running through the collection of stream, is the co-unit ε of the adjunction. First, the commutativity of the following diagram is obvious since the underlying continous maps of the morphisms of stream f and $S(D(f))$ are equal while the underlying continuous map of ε_X and ε_Y are the identities of their underyling sets. Thus we have a natural transformation ε from $S \circ D$ to the identity functor of St.

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ S(D(X)) \arrow[r, "S(D(f))"'] \arrow[u, "\varepsilon_X"] & S(D(Y)) \arrow[u, "\varepsilon_Y"'] \end{tikzcd}$$

Lemma 5.9 *The following set theoretic maps are identities*

$$\begin{array}{ccc} \mathsf{St}[S(X), Y] & \longrightarrow & \mathsf{dT}[X, D(Y)] \\ g & \longmapsto & D(g) \circ \eta_X \end{array} \qquad \begin{array}{ccc} \mathsf{dT}[X, D(Y)] & \longrightarrow & \mathsf{St}[S(X), Y] \\ f & \longmapsto & \varepsilon_Y \circ S(f) \end{array}$$

Proof. We have seen that $S(f) = f$ and $D(g) = g$ (by Lemmas 5.2 and 5.6), moreover the underlying continuous maps of η_X and ε_Y are the identities of X and Y. $\square$

As a straightforward consequence we have the adjunction $S \dashv D$ whose unit and co-unit are respectively η and ε.

Proposition 5.10 *$D \circ S \circ D = D$ and $S \circ D \circ S = S$*

Proof. Let X be a stream, the underlying map of the morphism of d-spaces $\big(D{*}\varepsilon_X\big)$ from $DSD(X)$ to $D(X)$ is id_X so any directed path on $DSD(X)$ is a directed path on $D(X)$. Conversely, let δ be a directed path on $D(X)$ and W be an open subset of X. Suppose $t \leqslant t'$ with $t, t' \in \mathsf{dom}(\delta)$ and $[t, t'] \subseteq \delta^{-1}(W)$. Consider a subpath $\delta \circ \theta$ of δ with θ continous and increasing from $[0,1]$ onto $[t,t']$. Then $\delta \circ \theta$ is a directed path of $D(X)$ satisfying $\mathsf{im}(\delta \circ \theta) \subseteq W$, $\delta \circ \theta(0) = \delta(t)$ and $\delta \circ \theta(1) = \delta(t')$, hence by definition $\delta(t) \preccurlyeq_W^{SD(X)} \delta(t')$. Then δ is a directed path on $DSD(X)$. Let W be an open subset of X, the underlying map of the morphism of stream $\big(S * \eta_X\big)$ from $S(X)$ to $SDS(X)$ is id_X so $w \preccurlyeq_W^{S(X)} w'$ implies $w \preccurlyeq_W^{SDS(X)} w'$ for all $w, w' \in W$. Conversely, if $w \preccurlyeq_W^{SDS(X)} w'$ then we have a directed path γ of $DS(X)$ such that $\mathsf{im}(\gamma) \subseteq W$, $\mathsf{s}(\gamma) = w$ and $\mathsf{t}(\gamma) = w'$. In particular γ is a morphism of stream so we have $w = \mathsf{s}(\gamma) \preccurlyeq_W^{S(X)} \mathsf{t}(\gamma) = w'$. □

We apply the preceding results to compare fundamental categories of streams and d-spaces.

Corollary 5.11 *For all streams X we have $\overrightarrow{\Pi_1}(X) = \overrightarrow{\Pi_1}\big(D(X)\big)$*

Proof. We have seen that D is right adjoint to S. From the Lemma 5.19 we know that S preserves products. Furthermore, it is obvious that $S(\uparrow\!\mathbb{I}_r) = \overrightarrow{\mathbb{I}}_r$ for all non negative real r. Then we conclude applying the Corollary 2.5. □

Corollary 5.12 *For all d-space X, if there exists a stream X' such that $D(X') = X$ then $\overrightarrow{\Pi_1}\big(S(X)\big) = \overrightarrow{\Pi_1}(X)$*

Proof. By the Proposition 5.10 we have $DSDX' = DX'$ and by the Corollary 5.11 we have $\overrightarrow{\Pi_1}\big(D \circ S \circ D(X')\big) = \overrightarrow{\Pi_1}\big(S \circ D(X')\big)$. □

From [18] we know that different d-spaces may have the same image under S : it suffices to consider the paths on the plane $\mathbb{R}^2$ which are increasing in both coordinates and the more pathological example of the directed paths generated by the paths of the following form with $x \in [0,1]$ and θ increasing.

$$\begin{array}{rcl} v_x : [0,1] & \longrightarrow & [0,1]^2 \\ t & \longmapsto & (x, \theta(t)) \end{array} \quad \text{and} \quad \begin{array}{rcl} h_x : [0,1] & \longrightarrow & [0,1]^2 \\ t & \longmapsto & (\theta(t), x) \end{array}$$

Those paths will be called the **staircases** in reference to the shape of their image. The two d-spaces we have described clearly yield to the same stream.

It is worth to notice that neither D nor S are full. First we treat the case of D by considering the set of rational numbers equipped with the stream structure inherited from the real line $\mathbb{R}$, we denote it by $\mathbb{Q}$. Then the elements of $\mathsf{St}[\mathbb{Q}, \mathbb{Q}]$ consist on the continuous increasing maps from $\mathbb{Q}$ to $\mathbb{Q}$ while $\mathsf{dT}[D(\mathbb{Q}), D(\mathbb{Q})]$ contains all the continuous maps from $\mathbb{Q}$ to $\mathbb{Q}$ since the only directed paths on $\mathbb{Q}$ are the constant ones. Then we come to the case of S by considering the staircases d-space X and the following map f

$$\begin{array}{rcl} \mathbb{R}^2 & \xrightarrow{f} & \mathbb{R}^2 \\ (x,y) & \longmapsto & (x+2y, 2x+y) \end{array}$$

which belongs to $\mathsf{St}[SX, SX]$ since SX is the plane $\mathbb{R}^2$ with the stream structure induced by the standard order while it obviously not belongs to $\mathsf{dT}[X, X]$ since it does not preserve the staircases.

Nevertheless, the functors S and D are almost isomorphisms. First remark that from the Proposition 5.10 we know that for all streams (respectively d-spaces) X, there exists a d-space (respectively stream) Y such that $X = SY$ (respectively $X = DY$) if and only if $X = SDX$ (respectively $X = DSX$). Then write dT^* and St^* for the full subcategories of dT and St whose collections of objects are respectively

$$\Big\{D(X) \;\Big|\; X \text{ object of } \mathsf{St}\Big\} \quad \text{and} \quad \Big\{S(X) \;\Big|\; X \text{ object of } \mathsf{dT}\Big\}$$

and denote by $\overline{S}$ and $\overline{D}$ for the restriction of S and D to dT^* and St^*. Then

Theorem 5.13 *The functors $\overline{S}$ and $\overline{D}$ are inverse of each other.*

$$\begin{tikzcd} \mathsf{St}^* \arrow[r, shift left, "\overline{D}=\overline{S}^{-1}"] & \mathsf{dT}^* \arrow[l, shift left, "\overline{S}=\overline{D}^{-1}"] \end{tikzcd}$$

Proof. From Proposition 5.10 we deduce that the object parts of the functors $\overline{S}$ and $\overline{D}$ are inverse of each other. Now let X and X' be two d-spaces and let Y and Y' be two streams such that $X = DY$ and $X' = DY'$ (or equivalently $SX = Y$ and $SX' = Y'$). Then in particular we have $\mathsf{dT}[X, X'] = \mathsf{dT}[X, DSX']$, $\mathsf{St}[Y, Y'] = \mathsf{St}[Y, SDY']$ and by Lemma 5.9 the following maps are identities (we have already seen that the underlying maps of the morphisms η_X and ε_Y are identities).

$$\begin{array}{rcl} \mathsf{St}[Y,Y'] & \longrightarrow & \mathsf{dT}[DY, DY'] \\ g & \longmapsto & D(g) \end{array} \qquad \begin{array}{rcl} \mathsf{dT}[X,X'] & \longrightarrow & \mathsf{St}[SX, SX'] \\ f & \longmapsto & S(f) \end{array}$$

□

Corollary 5.14 *The categories St^* and dT^* are complete and cocomplete.*

Proof. Denote by I the full inclusion of dT^* in dT and let F be a functor from some small category $\mathcal{C}$ to dT^*. Then by the Proposition 5.10 one has $I \circ F = D \circ S \circ F$ and for D is a right adjoint, it preserves limits hence we have

$$\lim_{\mathsf{dT}} I \circ F = \lim_{\mathsf{dT}} D \circ S \circ F = D\big(\lim_{\mathsf{St}} S \circ F\big)$$

In particular the morphisms forming the limiting cone of $I \circ F$ in dT belong to dT^* which is a full subcategory of dT, so the limiting cone of F in dT^* exists and we have

$$\lim_{\mathsf{dT}^*} F = \lim_{\mathsf{dT}} I \circ F$$

Now exchange the roles of D and S to prove that St^* is cocomplete. It follows by the Theorem 5.13 that both dT^* and St^* are complete and cocomplete. □

By definition St^* and dT^* are the codomains of the functors S and D thus we have two functors $S' : \mathsf{dT} \to \mathsf{St}^*$ and $D' : \mathsf{St} \to \mathsf{dT}^*$ characterized by the following factorizations $D = (\mathsf{dT}^* \hookrightarrow \mathsf{dT}) \circ D'$ and $S = (\mathsf{St}^* \hookrightarrow \mathsf{St}) \circ S'$.

Proposition 5.15
The functor $\overline{D} \circ S'$ if the left adjoint to the inclusion $I : \mathsf{dT}^* \hookrightarrow \mathsf{dT}$ and dT^* is a mono and epi reflective subcategory[12] *of dT.*

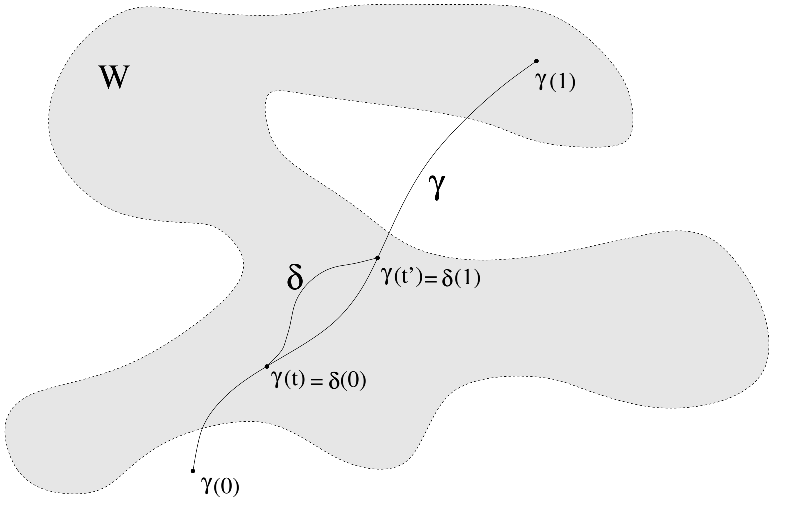
Proof. First note that $\overline{D} \circ S' \circ I = \mathsf{id}_{\mathsf{dT}^*}$ (cf. Proposition 5.10). Given a d-space X, the identity mapping of the underlying space of X induces an element η_X of $\mathsf{dT}[X, I \circ \overline{D} \circ S'(X)]$, thus providing a natural transformation η from $\mathsf{id}_{\mathsf{dT}}$ to $I \circ \overline{D} \circ S'(X)$. Then given an object Y of dT^* and some morphism $f \in \mathsf{dT}[X, IY]$ we have $\overline{D} \circ S'(f) \in \mathsf{dT}^*[\overline{D} \circ S'(X), Y]$ and $\overline{D} \circ S'(f) \circ \eta_X = f$ by construction of $\overline{D}$, S' and η_X. The natural transformation η is thus the unit of the adjunction and for all d-space X, the morphism η_X is both mono and epi since its underlying continuous map is an identity. In addition, if $X \cong X'$ and X belongs to the collection of objects of dT^*, then so does X'. □

Proposition 5.16
The functor $\overline{S} \circ D'$ if the right adjoint to the inclusion $I : \mathsf{St}^* \hookrightarrow \mathsf{St}$ and St^* is a mono and epi coreflective subcategory[7] *of St.*

Proof. Carbon copy the proof of Proposition 5.15 exchanging the roles of S' and D'. In particular, given a stream X, the identity of the underlying space of X induces a morphism ε_X from $\overline{S} \circ D'(X)$ to X thus providing a natural transformation from $I \circ \overline{S} \circ D'$ to $\mathsf{id}_{\mathsf{St}}$ which is the counit of the adjunction. □

A path γ on the underlying space UX of a d-space X is said to be **pseudo directed** when for all open subset $W \subseteq UX$, for all $t, t' \in \mathbb{R}_+$ such that $t \leq t'$ and $[t, t'] \subseteq \gamma^{-1}(W)$, there exists some $\delta \in dX$ such that $\mathsf{s}(\delta) = \gamma(t)$, $\mathsf{t}(\delta) = \gamma(t')$ and $\mathsf{im}(\delta) \subseteq W$. Any directed path is pseudo directed because the collection of directed paths is stable under subpaths. The object part of the image of $D \circ S$ is thus rather easy to determine. Indeed given a d-space X, the elements of $d(D \circ S(X))$ are the pseudo directed paths of X.

[12] See [1].



Actually all the results of this section remains valid for the category pSt and the related proofs just consist on carbon copies of the preceding ones, some of them are even easier.

Lemma 5.17 *Given a d-space X the mapping that sends each open subset $W \subseteq X$ to the relation*

$$\Big\{(w, w') \in W \times W \Big| \; \exists \delta \in dX,\ \mathsf{s}(\delta) = w,\ \mathsf{t}(\delta) = w'\Big\}$$

forms a precirculation on $U(X)$, hence a structure of prestream denoted by $S(X)$.

Proof. It is a consequence of the fact that dX is stable under concatenation. □

Lemma 5.18 *Given the d-spaces X and Y one has*

$$\mathsf{dT}[X, Y] \subseteq \mathsf{pSt}[S(X), S(Y)]$$

Proof. By definition, if f is a morphism of d-space and $\delta \in dX$, then $f \circ \delta \in dY$. □

As a consequence of Lemma 5.18, we have a functor S from dT to pSt such that any morphism of d-space f one has $S(f) = f$.

Lemma 5.19 *The functor S preserves products and the terminal object.*

Proof. By construction of S and the fact that if $(A_j)_{j \in J}$ is a non empty family of d-spaces, then we have the set theoretic equality

$$d\Big(\prod_{j \in J} A_j\Big) = \prod_{j \in J} dA_j$$

□

Given a prestream X, the elements of the following set are still called the directed paths over X.

$$\bigcup_{r\in\mathbb{R}^+} \mathsf{pSt}[\overrightarrow{\mathbb{I}}_r, X]$$

Lemma 5.20 *Given a stream X, the set of directed paths on X provides the underlying topological space of X with a structure of d-space denoted by $D(X)$.*

Proof. Carbon copy of the proof of Lemma 5.5. □

Lemma 5.21 *Given the prestreams X and Y one has*

$$\mathsf{pSt}[X,Y] \subseteq \mathsf{dT}[D(X), D(Y)]$$

Proof. The morphisms of prestream compose. □

As a consequence of Lemma 5.21, we have a functor D from pSt to dT such that any morphism of stream f one has $D(f) = f$.

Lemma 5.22 *Given a d-space X, any element of dX induces a directed path on $S(X)$ i.e.*

$$dX \subseteq \bigcup_{r\in\mathbb{R}_+} \mathsf{pSt}[\overrightarrow{\mathbb{I}}_r, S(X)]$$

Proof. Carbon copy of the proof of Lemma 5.7. □

The Lemma 5.22 can be restated saying that for any d-space X, the identity of the underlying topological space of X induces a morphism η_X of d-space from X to $D(S(X))$. As we shall see, the collection of morphisms η_X, for X running through the collection of d-spaces, is the unit η of the adjunction. First, the commutativity of the following diagram is obvious since the underlying set theoretic maps of the morphisms of d-space f and $D(S(f))$ are equal while the underlying set theoretic map of η_X and η_Y are the identities of their underyling sets. Thus we have a natural transformation η from the identity functor of dT to $D \circ S$.

$$\begin{tikzcd}
dX \arrow[r, "f"] \arrow[d, "dX"'] & dY \arrow[d, "dY"] \\
D(S(dX)) \arrow[r, "D(S(f))"'] & D(S(dY))
\end{tikzcd}$$

Lemma 5.23 *Given a stream X, the identity of the underlying topological space of X induces a morphism of prestreams ε_X from $S(D(X))$ to X.*

Proof. Carbon copy of the proof of Lemma 5.8. □

As we shall see, the collection of morphisms ε_X, for X running through the collection of prestreams, is the co-unit ε of the adjunction. First, the commutativity of the following diagram is obvious since the underlying continous maps of the morphisms of prestream f and $S(D(f))$ are equal while the underlying continuous map of

ε_X and ε_Y are the identities of their underyling sets. Thus we have a natural transformation ε from $S \circ D$ to the identity functor of pSt.

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ S(D(X)) \arrow[u, "\varepsilon_X"] \arrow[r, "S(D(f))"'] & S(D(Y)) \arrow[u, "\varepsilon_Y"'] \end{tikzcd}$$

Lemma 5.24 *The following set theoretic maps are identities*

$$\begin{array}{ccc} \mathsf{pSt}[S(X),Y] & \longrightarrow & \mathsf{dT}[X,D(Y)] \\ g & \longmapsto & D(g)\circ\eta_X \end{array} \qquad \begin{array}{ccc} \mathsf{dT}[X,D(Y)] & \longrightarrow & \mathsf{pSt}[S(X),Y] \\ f & \longmapsto & \varepsilon_Y\circ S(f) \end{array}$$

Proof. Carbon copy of the proof of the Lemma 5.9.

We have seen that $S(f) = f$ and $D(g) = g$ (by Lemmas 5.2 and 5.6), moreover the underlying continuous maps of η_X and ε_Y are the identities of X and Y. □

As a straightforward consequence we have the adjunction $S \dashv D$ whose unit and co-unit are respectively η and ε.

Proposition 5.25 $D \circ S \circ D = D$ *and* $S \circ D \circ S = S$

Proof. Carbon copy of the proof of the Lemma 5.10. □

We apply the preceding results to compare fundamental categories of prestreams and d-spaces.

Corollary 5.26 *For all prestreams X we have* $\overrightarrow{\Pi_1}(X) = \overrightarrow{\Pi_1}\big(D(X)\big)$

Proof. We have seen that D is right adjoint to S. From the Lemma 5.19 we know that S preserves products. Furthermore, it is obvious that $S(\uparrow\mathbb{I}_r) = \overrightarrow{\mathbb{I}}_r$ for all non negative real r. Then we conclude applying the Corollary 2.5. □

In the next statement, I denotes the inclusion functor from St to pSt.

Corollary 5.27 *For all stream X we have* $\overrightarrow{\Pi_1}(X) = \overrightarrow{\Pi_1}\big(I(X)\big)$

Proof. As a consequence of the fact that I is full and faithful. □

6 Directed Geometric Realisation of Cubical Sets

6.1 Description and Examples

We focus on a construction which provides an easy way to describe the objects commonly met in directed (and classical) algebraic topology. Given $n \in \mathbb{N}$ we denote by $[n]$ the initial segment $\{0,\ldots,n-1\}$ and $[2]^{[n]}$ the set of maps from $[n]$

to $[2]$ i.e. the finite sequences of elements of $\{0,1\}$ whose length is n. The cube category (also called the box category) is usually denoted by $\square$ and defined as the subcategory of Set whose objects are the sets $[2]^{[n]}$ for $n \in \mathbb{N}$ while its morphisms are generated by the following maps:

$$\delta^n_{i,\varepsilon} : [2]^{[n]} \to [2]^{[n+1]} \quad \text{for } n \in \mathbb{N} \text{ and } i \in [n+1]$$

$$\sigma^n_i : [2]^{[n+1]} \to [2]^{[n]} \quad \text{for } n \in \mathbb{N} \text{ and } i \in [n]$$

where

$$\begin{array}{rcl} \delta^n_{i,\varepsilon}(s) : [n+1] & \longrightarrow & [2] \\ k & \longmapsto & \begin{cases} s_k & \text{if } k < i \\ \varepsilon & \text{if } k = i \\ s_{k-1} & \text{if } k > i \end{cases} \end{array} \qquad \begin{array}{rcl} \sigma^n_i(s) : [n] & \longrightarrow & [2] \\ k & \longmapsto & \begin{cases} s_k & \text{if } k < i \\ \\ s_{k+1} & \text{if } k > i \end{cases} \end{array}$$

The box category can be described in a "computer scientist" way. Indeed it is isomorphic to the category whose set of objects is $\mathbb{N}$ and whose homset from n to m is the (finite) set of ordered pairs (n, w) where w is a word of length m on the alphabet $\{0,1\} \cup \{x_0, \ldots, x_{n-1}\}$ such that for all $i, j \in [n]$ if $w(i) = x_{i'}$, $w(j) = x_{j'}$ and $i < j$, then $i' < j'$. The composition being defined by

$$w' \circ w(k) = \begin{cases} w'(k) \text{ if } w'(k) \in \{0,1\} \\ w(k') \text{ if } w'(k) = x_{k'} \end{cases}$$

while the identity of n is represented by the word $(n, x_0 \cdots x_{n-1})$. For example one has

$$(5, 01x_0 0x_4 111) \circ (7, x_1 01 x_3 0) = (7, 01x_1 00111)$$

In particular the morphisms $\delta^n_{i,\varepsilon}$ and σ^n_i are represented by $(n, x_0 \cdots x_{i-1} \varepsilon x_i \cdots x_{n-1})$ and $(n+1, x_0 \cdots x_{i-1} x_{i+1} \cdots x_n)$. This approach is actually better fitted to the description of the geometric realisation of a cubical space. First note that any morphism (n, w) of $\square$ is canonically associated with a map ϕ from $\mathbb{R}^n$ to $\mathbb{R}^m$ (with m being the length of w), indeed the k^{th} component of the image of $(t_0, \ldots, t_{n-1})$ by ϕ is $w(k)$ if $w(k) \in \{0,1\}$ and $t_{k'}$ if $w(k) = x_{k'}$. The category of **cubical sets** is the presheaf $\mathsf{Set}^{\square^{\mathsf{op}}}$ and we denote is by CSet.

We define the geometric realisation of cubical sets adapting the approach of [9]. The ***n*-standard cube** $\square_n$ is the functor $\square[-, n]$ from $\square^{\mathsf{op}}$ to Set taking each $k \in \mathbb{N}$ to $\square[k, n]$ and each morphism f of $\square^{\mathsf{op}}[k, k']$ to the set theoretic application $\square[f, n]$ which sends any element $c \in \square[k, n]$ to $c \circ f \in \square[k', n]$. The right hand side diagram summarizes the definition.

$$\begin{array}{ccc} \square^{\mathsf{op}} & \longrightarrow & \mathsf{Set} \\ \\ k & & \square[k,n] \\ {\scriptstyle f}\downarrow & \longmapsto & \downarrow{\scriptstyle(-\circ f)} \\ k' & & \square[k',n] \end{array}$$

Then given some cocomplete category $\mathcal{C}$
The objects of the category $\square \downarrow K$ are the morphisms of cubical sets from $\square_n$ to K, the morphisms from $X : \square_n \to K$ to $Y : \square_m \to K$ are the elements $\phi \in \mathsf{CSet}[\square_n, \square_m]$ such that $X = Y \circ \phi$.

$$\begin{tikzcd} \square_n \arrow[rr, "\phi"] \arrow[dr, "X"'] & & \square_m \arrow[dl, "Y"] \\ & K & \end{tikzcd}$$

Then we have the obvious functor F_K from $\square \downarrow K$ taking an object $X : \square_n \to K$ to $\mathbb{I}^n$ and an morphism ϕ to its associated canonical map. The **geometric realisation** in C is then defined as the colimit of F_K. This construction is functorial. If the endomophisms of $\mathbb{I}$ are exactly the *nondecreasing* continuous maps from $\mathbb{I}$ to $\mathbb{I}$ then the geometric realisation is said to be **directed**, it is the case when $\mathbb{I}$ is the stream $\overrightarrow{\mathbb{I}}_1$ or the d-space $\uparrow\mathbb{I}_1$. Thus we already have four directed geometric realisation functors $\rceil - \lfloor_s$, $\rceil - \lfloor_{s^*}$, $\rceil - \lfloor_d$ and $\rceil - \lfloor_{d^*}$ according to the framework (St, St^*, dT and dT^*) we are working in. We would like to compare them as well as their fundamental categories.

Proposition 6.1 *For all cubical sets K we have $S(\rceil K\lfloor_d) = \rceil K\lfloor_s$, and we also have $D(\rceil K\lfloor_s) = \rceil K\lfloor_d$ if and only if $\rceil K\lfloor_d \in \mathsf{dT}^*$*

Proof. The first equality immediately comes from the fact that S preserves colimits (as a left adjoint) and $S(\uparrow\mathbb{I}_1^n) \cong \overrightarrow{\mathbb{I}}_1^n$. Now suppose that $\rceil K\lfloor_d \in \mathsf{dT}^*$, then applying the Theorem 5.13 we have $D(\rceil K\lfloor_s) \cong \rceil K\lfloor_d$. □

Proposition 6.2 *For all cubical sets K we have $\rceil K\lfloor_{s^*} = \rceil K\lfloor_s$, and we also have $\rceil K\lfloor_{d^*} = \rceil K\lfloor_d$ if and only if $\rceil K\lfloor_d \in \mathsf{dT}^*$*

Proof. By the Proposition 5.16 the inclusion functor $I : \mathsf{St}^* \hookrightarrow \mathsf{St}$ has a right adjoint, hence it preserves colimits and we have $I(\rceil K\lfloor_{s^*}) = \rceil K\lfloor_s$. Then we conclude because $I(\rceil K\lfloor_{s^*}) = \rceil K\lfloor_{s^*}$. By definition $\rceil K\lfloor_d$ is the colimit in dT of a functor F_K whose image is contained in dT^* while $\rceil K\lfloor_{d^*}$ is its colimit in dT^*. Now suppose $\rceil K\lfloor_d \in \mathsf{dT}^*$, then both colimits match since dT^* is a full subcategory of dT. □

The next proposition comes from the fact that the functors $\overline{S}$ and $\overline{D}$ are inverse of each other.

Proposition 6.3 *For all cubical sets K we have $\overline{S}\big(\rceil K\lfloor_{d^*}\big) = \rceil K\lfloor_{s^*}$ and $\overline{D}\big(\rceil K\lfloor_{s^*}\big) = \rceil K\lfloor_{d^*}$*

Then given some cubical set K we have

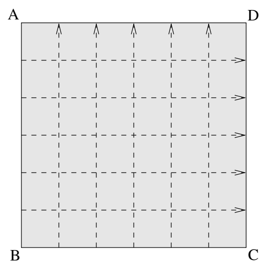
$$\begin{array}{rcll} \overrightarrow{\Pi_1}\big(D(\rceil K\lfloor_s)\big) & = & \overrightarrow{\Pi_1}\big(\rceil K\lfloor_s\big) & \text{by the Corollary 5.11} \\ \overrightarrow{\Pi_1}\big(\rceil K\lfloor_s\big) & = & \overrightarrow{\Pi_1}\big(\rceil K\lfloor_{s^*}\big) & \text{by the Proposition 6.2} \\ \overrightarrow{\Pi_1}\big(\rceil K\lfloor_{s^*}\big) & = & \overrightarrow{\Pi_1}\big(S(\rceil K\lfloor_{d^*})\big) & \text{by the Proposition 6.3} \\ \overrightarrow{\Pi_1}\big(S(\rceil K\lfloor_{d^*})\big) & = & \overrightarrow{\Pi_1}\big(\rceil K\lfloor_{d^*}\big) & \text{by the Corollary 5.12} \end{array}$$

The next result summarizes the preceding remarks.

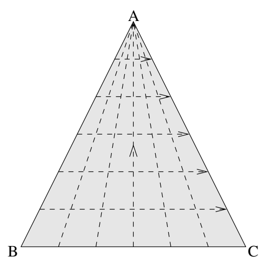
Corollary 6.4 *Given a cubical set K the fundamental categories of the following are isomorphic $D(\upharpoonleft K\downharpoonright_s)$, $\upharpoonleft K\downharpoonright_s$, $\upharpoonleft K\downharpoonright_{s^*}$, $S(\upharpoonleft K\downharpoonright_{d^*})$ and $\upharpoonleft K\downharpoonright_{d^*}$.*

Actually it might happen that the d-spaces $D(\upharpoonleft K\downharpoonright_s)$ and $\upharpoonleft K\downharpoonright_d$ are not isomorphic. Indeed, suppose $(\phi_A : A \to X)_{A\in|\mathcal{C}|}$ is the colimiting cone for some functor F from $\mathcal{C}$ to St. Then $(D(\phi_A) : D(A) \to D(X))_{A\in|\mathcal{C}|}$ is a colimiting cone of $D \circ F$ if and only if all $\gamma \in \mathsf{St}[\overrightarrow{\mathbb{I}}_1, X]$ can be written as a concatenation $\gamma = (D(\xi_n) * \cdots * D(\xi_0)) \circ \theta$ for some finite sequence $(A_0, \ldots, A_n)$ of objects of $\mathcal{C}$ such that $\xi_k \in \mathsf{St}[\overrightarrow{\mathbb{I}}_1, A_k]$ for $k \in \{0, \ldots, n\}$ and $\theta \in \mathsf{St}[\overrightarrow{\mathbb{I}}_1, \overrightarrow{\mathbb{I}}_1]$ surjective. We describe a cubical set K which provides a counter-example. It is worth to notice this situation is not pathological since it is actually the kind of model obtained when a linear process (i.e. containing neither loop nor fork) spawns a new process which is reduced to a loop. In this case we should also have a directed segment whose end is glue to the vortex. The vortex thus corresponds to the state where the new process is created.

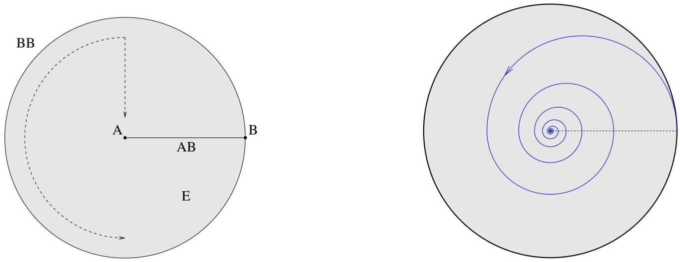
Start with $\overrightarrow{\mathbb{I}}_1 \times \overrightarrow{\mathbb{I}}_1$, the directed square of St pictured below and call it E. In other words we have a 2-dimensional element E which thus belongs to K_2.



Then identify all the points of the upper segment AD with a single one and call this point A. In the cubical set way of speaking $A \in K_0 \cap K_1$, $K(\sigma_0^1)(A) = A$ and $K(\delta_{1,1}^1)(E) = A$. We obtain the following triangle.



The vortex is then created by identifing the segments AB and AC which is formally interpreted by the relation $AB = K(\delta_{0,1}^1)(E) = K(\delta_{0,0}^1)(E) = AC$. Consequently the points B and C are identified and $K(\delta_{1,0}^1)(E) = BB$, the border of the following disk.



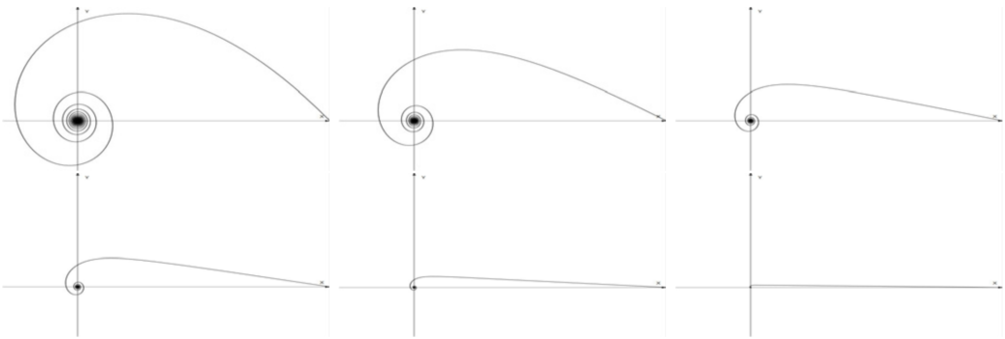
Then the following path, that we can see as a "downward spiral", provides a directed path of $D(\upharpoonleft K\downharpoonleft_s)$ which is not a directed path of $\mathsf{colim}(D \circ K)$.

$$t \in [0,1] \mapsto (1-t)\exp\left(\frac{it}{1-t}\right)$$

However, there is a homotopy from $t \in [0,1] \mapsto (1-t,0)$ to the downward spiral

$$(t,s) \in [0,1]\times[0,1] \mapsto (1-t)\exp\left(\frac{ist}{1-t}\right)$$

Some intermediate paths for the values 1, 0.5, 0.2, 0.15, 0.05 and 0.01 of the parameter s.



The fundamental category of $\overrightarrow{\mathbb{C}}$ can be described as follows: its set of objects is $\mathbb{C}$ and its set of morphisms is

$$\big\{(x,n,y) \;\big|\; x \neq 0 \;;\; |x| \leq |y| \;;\; n \in \mathbb{N}\big\} \cup \big\{(x,-\infty,0) \;\big|\; x \in \mathbb{C}\big\}$$

By definition the source and the target of (x,n,y) and (x,n,y) are x and y. Given $x \in \mathbb{C}\backslash\{0\}$ we define $\mu(x) := \frac{x}{|x|}$ and $\overset{\frown}{xy}$ as the anticlockwise arc from $\mu(x)$ to $\mu(y)$ for any $x, y \in \mathbb{C}\backslash\{0\}$. The composition is defined by

$$(y,m,z)\circ(x,n,y) = \begin{cases} (x, n+m, z) & \text{if } \overset{\frown}{xy} \cup \overset{\frown}{yz} \neq S^1 \\ (x, n+m+1, z) & \text{if } \overset{\frown}{xy} \cup \overset{\frown}{yz} = S^1 \end{cases}$$

Note that neither $\mu(y)$ nor $\overset{\frown}{xy}$ are well-defined if $y = 0$, however if $n = -\infty$ or $m = -\infty$ then $n + m = n + m + 1 = -\infty$.

6.2 The directed graphs case

The directed geometric realisation of a cubical set in dT may differ from the one in dT^*. Yet, in this section, we will prove that if the cubical set is 1-dimensional, which means it is a directed graph, then both realisations coincide. Given a directed graph $A \underset{\mathsf{t}}{\overset{\mathsf{s}}{\rightrightarrows}} V$ we give an extensive description of its directed geometric realisation. We suppose that $A \cap V = \emptyset$ then consider the topological space $X := A \times [0,1] \sqcup V$ where A and V are the discrete space on the sets A and V respectively. The underlying space is the quotient $Q := A \times [0,1] \sqcup V/\sim$ where $\sim$ is the least equivalence relation over X such that for all $a \in A$ we have $(a,0) \sim v$ if $\mathsf{s}(a) = v$ and $(a,1) \sim v$ if $\mathsf{t}(a) = v$.
By definition, this quotient topology is the finest one making the quotient map q continuous i.e. any $U \subseteq X$ is open iff so is $q^{-1}(U)$. The elements of $V_Q := \{q(v) \mid v \in V\}$ are called the **vertices** of X. A point of $x \in X$ is **isolated** (i.e. $\{x\}$ is both open and closed) iff it is a vertex such that for all $a \in A$, $q(\mathsf{s}(a)) \neq x$ and $q(\mathsf{t}(a)) \neq x$. The space Q is Hausdorff. Moreover it is compact iff the graph is finite (i.e. both A and V are finite). Indeed V_Q is closed and the mapping q induces an homeomorphism from $A \times]0,1[$ onto $Q \backslash V_Q$ whose inverse is denoted by g. Then remark that the connected components of $q(Q \backslash A \times [\frac{1}{3}, \frac{2}{3}])$ provide a family U_v of open subsets of Q such that $v \in U_v$ and $U_v \cap U_{v'} = \emptyset$ for all vertices v and v'. Hence if A or V is infinite then Q is not compact. On the contrary, if both A and V are finite, then X is compact hence so is $Q = q(X)$. One can also check that Q is locally compact iff for all vertices $v \in V$ the following set is finite

$$\{a \in A \mid \mathsf{s}(a) = v \text{ or } \mathsf{t}(a) = v\}$$

The set $A \times [0,1] \sqcup V$ admits a natural order setting $(a,t) \sqsubseteq (a',t')$ when $a = a'$ and $t \leq t'$. The topology and the order over $A \times [0,1] \sqcup V$ thus give rise to a pospace still denoted by X. Then we define the directed paths on X as the nondecreasing ones and the collection of directed paths on Q as the least turning the quotient map q into a morphism of d-spaces.

In particular, for all $a \in A$, the mapping q induces a homeomorphism from $\{a\} \times]0,1[$ onto its image. In addition if $q(a,0) \neq q(a,1)$ then $q(\{a\} \times [0,1])$ is homeomorphic to $[0,1]$ otherwise it is homeomorphic to the circle. In fact we have a global result

Lemma 6.5 *The morphism of d-spaces q induces an isomorphism of d-spaces from from $A \times]0,1[$ onto $Q \backslash V_Q$ whose inverse is induced by g*

Lemma 6.6 *Given δ a path on Q such that $\delta^{-1}(Q \backslash V_Q)$ is connected there is a path γ on X such that $\delta = q \circ \gamma$. If δ is pseudo directed then both γ and δ are directed. If $\delta^{-1}(Q \backslash V_Q) \neq \emptyset$ then γ is unique and there is a unique $\alpha \in A$ such that $\mathsf{im}(\delta) \subseteq q(\{\alpha\} \times [0,1])$. If $\mathsf{s}(\delta), \mathsf{t}(\delta) \in V_Q$ and (δ is pseudo directed or $\mathsf{s}(\delta) \neq \mathsf{t}(\delta)$) then $\mathsf{im}(\delta) = q(\{\alpha\} \times [0,1])$*

Proof. If $\delta^{-1}(Q \backslash V_Q) = \emptyset$ then $\mathsf{im}(\delta) \subseteq V_Q$ which is discrete, hence δ is constant.

Since q is onto we have the expected γ. Note however that it may not be unique. Now suppose $\delta^{-1}(Q\backslash V_Q) \neq \emptyset$ and define

$$\begin{array}{rcl} \gamma \ : \ \delta^{-1}(Q\backslash V_Q) & \longrightarrow & X \\ t & \longmapsto & g \circ \delta(t) \end{array}$$

Let $a := \inf\left(\delta^{-1}(Q\backslash V_Q)\right)$ and $b := \sup\left(\delta^{-1}(Q\backslash V_Q)\right)$. We would like to prove that γ can be extended to a continous map on $[a,b]$. If $\delta(a) \notin V_Q$ then γ is already defined at point a hence suppose $\delta(a) \in V_Q$. Suppose γ cannot be extended to a continous map at point a that is to say for all $x \in X$ there is some W_x neighbourhood of x such that for all $\varepsilon > 0$ we have $\gamma(]a, a+\varepsilon[) \nsubseteq W_x$. For all $\alpha \in A$ choose some ε_α such that $\{\alpha\} \times [0, \varepsilon_\alpha[$ and $\{\alpha\} \times]1-\varepsilon_\alpha, 1]$ are neighbourhoods of $(\alpha, 0)$ and $(\alpha, 1)$ as above. Then

$$U := \bigcup_{\alpha \in A} \{\alpha\} \times \left([0, \varepsilon_\alpha[\cup]1-\varepsilon_\alpha, 1]\right)$$

is an open subset of X whose direct image by q is open in Q. Since δ is continuous at point a we have some $\varepsilon' > 0$ such that $\delta(]a-\varepsilon', a+\varepsilon'[) \subseteq q(U)$. In particular we have $\delta(]a, a+\varepsilon'[) \subseteq q(U)$ which is equivalent to $\gamma(]a, a+\varepsilon'[) \subseteq U$ by the Lemma 6.5 and the fact that $\delta(]a, a+\varepsilon'[) \subseteq Q\backslash V_Q$. The subspace $\gamma(]a, a+\varepsilon'[)$ meets exactly one connected component of U since it is connected (as the continuous direct image of a connected space). In other words we have $\gamma(]a, a+\varepsilon'[) \subseteq \{\alpha\} \times [0, \varepsilon_\alpha[$ or $\gamma(]a, a+\varepsilon'[) \subseteq \{\alpha\} \times]1-\varepsilon_\alpha, 1]$ for some $\alpha \in A$ which is a contradiction. Thus γ can be extended to a continuous map at a and the same way we check it can be extended to a continuous map at b. Since $\delta^{-1}(Q\backslash V_Q)$ is open in $\mathsf{dom}(\delta) := [0, r]$ we have $a \in \delta^{-1}(Q\backslash V_Q) \Rightarrow a = 0$ and $b \in \delta^{-1}(Q\backslash V_Q) \Rightarrow b = r$. Suppose $a \notin \delta^{-1}(Q\backslash V_Q)$, it follows that for all $t \in [0, a]$ we have $\delta(t) \in V_Q$ which is discrete. Hence the restriction of δ to $[0, a]$ is constant as well as its restriction to $[b, r]$. Thus γ is uniquely defined as expected. In particular we have a unique $\alpha \in A$ such that $\mathsf{im}(\gamma) \subseteq \{\alpha\} \times [0,1]$ i.e. $\mathsf{im}(\delta) \subseteq q(\{\alpha\} \times [0,1])$.

Suppose δ is pseudo directed (and nonconstant otherwise the result is obvious). Given $t, t' \in \delta^{-1}(Q\backslash V_Q)$ such that $t < t'$, there is a unique $\alpha \in A$ such that $\delta([t, t']) \subseteq q(\{\alpha\} \times]0,1[)$ which is an open subset of Q. Since δ is pseudo directed we have a directed path ξ from $\delta(t)$ to $\delta(t')$ with $\mathsf{im}(\xi) \subseteq q(\{\alpha\} \times]0,1[)$. By the Lemma 6.5 $g \circ \xi$ is a directed path on X hence $g \circ \xi(t) \sqsubseteq g \circ \xi(t')$ i.e. $\gamma(t) \sqsubseteq \gamma(t')$. Hence the restriction of γ to $\delta^{-1}(Q\backslash V_Q)$ is directed and for X comes from a pospace, we have $\gamma(a) \sqsubseteq \gamma(t) \sqsubseteq \gamma(b)$ for all $t \in \delta^{-1}(Q\backslash V_Q)$ see [20]. Hence both γ and δ are directed.

Suppose $\delta^{-1}(Q\backslash V_Q) \neq \emptyset$ and $\mathsf{s}(\delta), \mathsf{t}(\delta) \in V_Q$. If $\mathsf{s}(\delta) \neq \mathsf{t}(\delta)$ or γ is directed then $\{\mathsf{s}(\gamma), \mathsf{t}(\gamma)\} = \{(\alpha, 0), (\alpha, 1)\}$. Therefore $\mathsf{im}(\gamma) = \{\alpha\} \times [0,1]$ since $\mathsf{im}(\gamma)$ is connected. □

Corollary 6.7 *A path δ on Q such that $\delta^{-1}(V_Q)$ has finitely many connected components can be written as a concatenation $(q \circ \gamma_n) * \cdots * (q \circ \gamma_0)$ with γ_i path on X for all $i \in \{0, \ldots, n\}$. Moreover t.f.a.e*

- *δ is directed*

- δ is pseudo directed

- each γ_i is directed

Proof. Suppose $\delta^{-1}(V_Q)$ has finitely many connected components $I_0, \dots, I_n$, all of them are compact intervals of $\mathbb{R}$. We can suppose that for all $k \in \{0, \dots, n-1\}$ we have $\forall t \in I_k\ \forall t' \in I_{k+1}\ t < t'$. Then pick one element t_k from every I_k and consider the restriction of δ to $[t_k, t_{k+1}]$ for all $k \in \{0, \dots, n-1\}$. By the Lemma 6.6 each of these restrictions can be written as $q \circ \gamma$ for some path γ on X. The first point clearly implies the second one. If the second point is satisfied, then from the Lemma 6.6 we know that each γ_i is directed. If the third point is satisfied, the each $q \circ \gamma_i$ is directed hence so is the composite δ. □

Corollary 6.8 *For any δ nonconstant pseudo directed path on Q such that $\mathsf{s}(\delta), \mathsf{t}(\delta) \in V_Q$ there is some $\alpha \in A$ such that $q(\{\alpha\} \times [0,1]) \subseteq \mathsf{im}(\delta)$*

Proof. Let C be a connected component of $\delta^{-1}(Q \backslash V_Q)$ which is nonempty since δ is nonconstant. For $\mathsf{s}(\delta), \mathsf{t}(\delta) \in V_Q$ we can suppose $\inf C \notin C$ and $\sup C \notin C$ in other words $C =]a, b[$ for some $a, b \in \mathbb{R}$ such that $a < b$. Then the restriction of δ to $[a, b]$ induces a pseudo directed path $\tilde{\delta}$ on Q such that $\mathsf{s}(\tilde{\delta}), \mathsf{t}(\tilde{\delta}) \in V_Q$ and $\tilde{\delta}^{-1}(Q \backslash V_Q) =]a, b[$ is connected. Applying the Lemma 6.6 we know there is some $\alpha \in A$ such that $\{\alpha\} \times [0,1] \subseteq \mathsf{im}(\tilde{\delta})$. □

Proposition 6.9 *Given a directed graph K we have $\rceil K \lfloor_d = \rceil K \lfloor_{d^*}$*

Proof. By the Corollary 6.7 the demonstration amounts to proving that for any *pseudo* directed path δ on $\rceil K \lfloor_d$ the subspace $\delta^{-1}(V_Q)$ has finitely many connected components. Suppose it is not the case. We have a sequence of points of $\delta^{-1}(V_Q)$ whose intersection with any connected components of $\delta^{-1}(V_Q)$ contains at most one element. The set V_Q of vertices of Q is closed hence so is $\delta^{-1}(V_Q)$ in $\mathsf{dom}(\delta)$ which is compact. Hence $\delta^{-1}(V_Q)$ is compact and we can extract a subsequence $(t_n)_{n \in \mathbb{N}}$ converging to $\tau \in \delta^{-1}(V_Q)$. Since two different terms of the sequence are picked from two different connected component of $\delta^{-1}(V_Q)$ we can suppose no term of $(t_n)_{n \in \mathbb{N}}$ belongs to the connected component of τ. Then we define the following neighbourhood of $\delta(\tau)$

$$W := q\Big(\bigcup_{a \in A} \{a\} \times \big([0;1] \backslash \{\frac{1}{2}\}\big)\Big)$$

hence $\delta^{-1}(W)$ is a neighbourhood of τ. Since δ is continuous and $(t_n)_{n \in \mathbb{N}}$ converges to τ we have some $n \in \mathbb{N}$ such that $\delta(I) \subseteq W$ where I denotes the segment $[t_n, \tau]$ if $t_n < \tau$ and $[\tau, t_n]$ otherwise. Since t_n and τ do not belong to the same connected component of $\delta^{-1}(V_Q)$ we have some $t' \in I$ such that $\delta(t') \notin V_Q$. Hence the restriction of δ to I is nonconstant and pseudo directed, therefore applying the Corollary 6.8 we know its image is not included in W which is a contradiction. □

The following lemma is a basic result about d-spaces that can be easily checked.

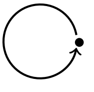
Lemma 6.10 *Let X and Y be Hausdorff spaces and dX be a collection of paths on X providing it with a d-space structure. Given a continuous map $f : X \to Y$ the least d-space structure on Y such that f becomes a morphism of d-spaces is the collection of paths $\gamma_n * \cdots * \gamma_0$ with (($\gamma_k = f \circ \delta_k$ and $\delta_k \in dX$) or γ_k constant) for all $k \in \{0, \ldots, n\}$*

As an immediate consequence of the Lemma 6.10 we have the following result.

Corollary 6.11 *Given a path δ on Q the following are equivalent*

- *δ is directed*
- *δ is pseudo directed*
- *$\delta = (q \circ \gamma_n) * \cdots * (q \circ \gamma_0)$ with γ_i directed path on X for all $i \in \{0, \ldots, n\}$*

Proof. The equivalence between the two first points is another way state the Proposition 6.9. The equivalence between the first and the last points comes from the Lemma 6.10 and the fact that the directed structure on Q is the least one turning the quotient map q into a morphism of d-spaces. □



The directed circle is actually the directed geometric realisation of the directed graph with one vertex and one arrow.

The pathology described at the end of the preceding section directly derives from the presence of the vortex. Intuitively, it seems impossible to have a vertex in the directed geometric realisation of some cubical set K without using a degeneracy. Thus we conjecture the Proposition 6.9 actually holds for any *precubical* set. Yet, by the Propositions 6.2 and 6.3 we have

$$\overline{D}\big(\rceil K \lfloor_{s^*}\big) = \rceil K \lfloor_{d^*} \quad \text{and} \quad \rceil K \lfloor_{s^*} = \rceil K \lfloor_{s}$$

so when dealing with directed geometric realisation one may as well work in St, St^* or even dT^* to avoid pathologies.

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