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A Fuzzy Modal Logic for Fuzzy Transition Systems

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Abstract

This paper intends to contribute with a new fuzzy modal logic to model and reason about transition systems involving uncertainty in behaviours. Our formalism supports fuzziness at transitions and on the proposition symbols assignment levels.
Against of other approaches in the literature, our bisimulation and bisimilarity notions generalise the analogous standard notions of classic modal logic and of process algebras. Moreover, the outcome of our logic is also fuzzy, with the semantic interpretation of connectives supported by the Gödel algebra.

Keywords: bisimulation; simulation; fuzzy transition systems;modal logic;Gödel algebra

1 Introduction

For 50 years, fuzzy sets and fuzzy logic have been an area of active research (cf. [13]). Fuzzy automata [17], fuzzy Markov processes [1], fuzzy petri nets [25,16], fuzzy reactive frames [24] and fuzzy discrete event systems [22] are some of the formalisms that have been considering to model computational systems that deal with uncertainty and fuzzy sets. In this work, we will focus on fuzzy transition systems or fuzzy labelled transition systems, which are a generalisation of transition

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systems or labelled transition systems (widely used in computer science) with $[0,1]$-weights on the transitions [26,27,15].

It is well known that bisimulations and simulations are a worth way of comparing two transition systems. They have been considered in several frameworks, such as fuzzy automata [7,8] fuzzy Markov process [9], fuzzy discrete systems [22], weighted labelled transition system [26,27,15]. All of them have special motivations and, consequently, different formulations. For example, in [26,6,5] bisimulations are defined as equivalence relations. There are other approaches that focus on horizontal and vertical bisimulations [15] and some other which define bisimulations as fuzzy relations [7].

In the literature, there are several logics to reason about fuzzy transition systems (or better about state transition systems with assertions on the states). As in the classical case, such logics are variants of modal logic. Among these approaches, both the accessibility and the proposition symbols used to represent assertions on the states can be crisp or fuzzy. For example in [10,11] the propositions are considered crisp and many-valued accessibility relations evaluated in finite Heyting algebras. Bou et al. in [4] adopted the truth support of finite integral commutative residuated lattices. In some research works the truth lattice is a chain ([3,23]), where any multi-valued relation can be expressed as a decreasing family of crispy modal relations which is indexed by the support of the respective lattice. The reference [21], presents a multi-valued logic over the Gödel algebra assuming crisp frames. There are many other ways to define weighted accessibility relation like the approaches used in [20,19]. Addressing many-valued dynamic logic in [14] where J. Hughes *et al.* introduced a propositional dynamic logic over the continuum truth $[0,1]$-lattice with the standard fuzzy residues. On the other hand, C. Liau [18] introduced a many-valued dynamic logic over the specific continuum truth $[0,1]$-lattice. This approach is quite different from [14] as it has parametrized the implication.

In this work, we consider the uncertain and the boolean reasoning based on the Gödel algebra, considering the fuzziness both in the accessibility relation and in the logic. As far as we know there is no research work that focuses on bisimulation and simulation for (full) fuzzy modal logic. We define bisimulation and simulation for fuzzy models using the ideas from the work already done in bisimulation and simulation for fuzzy transition systems [26,27,15] and fuzzy modal logic [12].

Our work is close to the work in [27]. However there are two important aspects that distinguish both approaches: (i) against of what is done there, our logic is a "full" fuzzy modal logic, in the sense that the value of a formula is not bivalent, it is a value in $[0,1]$, and (ii) our bisimulation notion generalises the analogous concept for standard labelled transition systems (cf. Theorem 4.9). Although in [27], closed sets are used to define bisimulation and simulation, which is similar but not exactly the same as the sets that we have defined (cf. corollary 4.14) as $U = E^{-1}[E[U]]$ and $U' = E[E^{-1}[U']]$.

The work in [12] also closely resembles our paper in the sense of defining the fuzzy model but they have not defined bisimulations and simulations for fuzzy modal logic. They have defined fuzzy propositions the same way as we do but unlike our definition of fuzzy accessibility relation, they have defined relations to be reflexive, symmetric and t-norm transitive. Moreover, they define logic using Lukasiewicz's logic as the underlying algebra while we use the Gödel algebra.

Outline of the paper. The remaining of the paper is organised as follows: Section 2 recalls the standard modal logic, and some properties relevant for this work. Then, in Section 3 we define a fuzzy modal logic based in a Gödel algebra and in Section 4 are introduced notions of bisimulation and of simulation for this logic. Then, in Section 5 we proved the existence of a bisimilarity which is the largest bisimulation relation (also an equivalence relation) defined on a fuzzy model. Finally, the modal invariance property for this framework is proved in Section 6. We conclude the paper in Section 7 with some consideration about the future work.

2 Classic Modal Logic

The long tradition in the study of logic is reasoning the scenarios that involve change; since the age of Aristotle. This family of logics which is known as Modal logics represents a classic topic in Logic and Philosophy. The developments of Kripke semantics in the 60's, based on relational structures, constituted a very important tool to reason about state-based systems. This section briefly reviews the basic definition of propositional modal logic and some of its main results.

Given a set of (atomic) propositional symbols Prop, the Prop-formulas are defined by the grammar

$$\varphi ::= p \mid \Diamond\varphi \mid \Box\varphi \mid \sim\varphi \mid \varphi \vee \varphi \mid \varphi \wedge \varphi$$

where $p \in$ Prop. The Kripke models are state transition structures, with propositions assigned to set of states. Formally, a Prop-model is a tuple $M = \big(W\ , V\ , R\big)$ where

- W is a non-empty set.
- $V : \text{Prop} \to \mathcal{P}(W)$ is a function.
- $R \subseteq W \times W$ is binary relation.

The modal satisfaction of a Prop-formula φ at a state w in a model M is recursively defined as follows:

- $M, w \models p$ iff $w \in V(p)$
- $M, w \models \Diamond\varphi$ iff there is a $w' \in W$ such that $(w, w') \in R$ and $M, w' \models \varphi$
- $M, w \models \Box\varphi$ iff for any $w' \in W$ such that $(w, w') \in R$ we have $M, w' \models \varphi$
- $M, w \models \sim\varphi$ iff it is false that $M, w \models \varphi$
- $M, w \models \varphi \wedge \varphi'$ iff $M, w \models \varphi$ and $M, w \models \varphi'$
- $M, w \models \varphi \vee \varphi'$ iff $M, w \models \varphi$ or $M, w \models \varphi'$

Modal logic has been used to reason about statements such as "it is possible", "it is known" etc. Bisimulation is an important notion in modal logic that relates the states in (Kripke) models with the same behaviour.

Definition 2.1 [Simulation and Bisimulation] A *simulation* between two models $M = (S, R, V)$ and $M' = (S', R', V')$ is a non empty relation $E \subseteq S \times S'$ such that whenever $s\ E\ s'$:

- Atoms: for any $p \in \text{Prop}$, $V(p) \subseteq V'(p)$
- Zig: If $s\ R\ v$ then, there exists a $v' \in M'$ such that $s'\ R'\ v'$ and $v\ E\ v'$.

A *bisimulation* between two models $M = (S, R, V)$ and $M' = (S', R', V')$ is a non empty simulation relation $E \subseteq S \times S'$ such that whenever $s\ E\ s'$:

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- Zag: If $s'\ R\ v'$ then there is a $v \in M$ such that $s\ R\ v$ and $v\ E\ v'$.

It is well known that if two states are bisimilar then they satisfy the same formulas (**Modal Invariance**). Moreover, the converse also holds for the models with finite image (**Hennessy-Milner Theorem**). These results can be found in [2].

3 Fuzzy Modal Logic

Kripke semantics for modal logic consist of graphs labelled with propositional symbols on each edge. Hence, they can be used to model many situations, such as network science, graph theory, epistemic logic and also for reasoning about time, beliefs, computational systems, etc. However, there are situations where we cannot say that a transition exists (edges in graphs) or not; the best we can do is to assign a degree of certainty to its existence. This leads to fuzzy state transitions.

Definition 3.1 [Fuzzy Frame and Fuzzy Model] A *fuzzy frame* is a pair $F = (S, R)$ where

- S is a finite non empty set of states;
- $R : S \times S \to [0, 1]$ is the fuzzy accessibility function.

A *fuzzy model* is a tuple $M = (S, R, V)$ where

- (S, R) is a fuzzy frame;
- $V : S \times \text{Prop} \to [0, 1]$ is a (fuzzy valuation) function.

An example of a fuzzy frame is presented in Figure 1. We are going to define a fuzzy modal logic based on the Gödel algebra. The definition of Gödel algebra is as follows:

Definition 3.2 [Gödel algebra] The Gödel algebra is the structure $G = ([0, 1], \max, \min, I, 0, 1, N)$, where max , min are the usual maximum and minimum operations, respectively and

- $I(x, y) = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$
- $N(x) = \begin{cases} 1 & \text{if } \ x = 0 \\ 0 & \text{if } x > 0 \end{cases}$

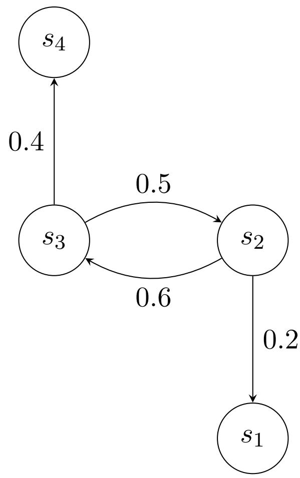


Fig. 1. Fuzzy Frame example

Definition 3.3 [Formulas] For a given set of propositional symbols Prop, we defined the set of Prop-formulas Fm(Prop), by the following grammar:

$$\varphi := \top \mid \bot \mid p \mid\ \sim \varphi \mid \varphi \wedge \phi \mid \varphi \vee \phi \mid \varphi \to \phi \mid \Diamond\varphi \mid \Box\varphi$$

The satisfaction relation is defined as a function on $[0,1]$ i.e. it is considered as a fuzzy relation.

Definition 3.4 [Satisfaction] The satisfaction relation in a fuzzy model M consists of a function

$$\models : S \times \mathrm{Fm(Prop)} \to [0,1]$$

recursively defined as follows:

- $(M, s \models \top) = 1$
- $(M, s \models \bot) = 0$
- $(M, s \models p) = V(s, p)$, for $p \in$ Prop and $s \in S$
- $(M, s \models \varphi \wedge \varphi') = \min\big\{(M, s \models \varphi), (M, s \models \varphi')\big\}$
- $(M, s \models \varphi \vee \varphi') = \max\big\{(M, s \models \varphi), (M, s \models \varphi')\big\}$
- $(M, s \models \varphi \to \varphi') = I((M, s \models \varphi), (M, s \models \varphi'))$
- $(M, s \models \sim \varphi) = N(M, s \models \varphi)$
- $(M, s \models \Diamond\varphi) = \max\big\{\min\big(R(s,u), (M, u \models \varphi)\big) | u \in S\big\}$
- $(M, s \models \Box\varphi) = \min\big\{I(R(s,u), (M, u \models \varphi)) | u \in S\big\}$

Example 3.5 Consider the fuzzy frame $(S\ , R)$ in Figure 1. The labels on the transitions mean the value of the relation between such pairs; for example $R(s_2, s_1) = 0.2$. Moreover, when there is no transition that means that the value is 0; for example $R(s_2, s_4) = 0$. Let us assume $M = (S\ , R\ , V)$, Prop $= \{p\}$, $V(s_3, p) = 0.8$ and $V(s_1, p) = 0.7$. Then,

$$
\begin{aligned}
(M, s_2 \models \Diamond p) &= \max\Big\{ \min\big(R(s_2, s_3), (M, s_3 \models p)\big); \min\big(R(s_2, s_1), (M, s_1 \models p)\big)\Big\} \\
&= \max\Big\{ \min(0.6, 0.8); \min\big(0.2, 0.7\big)\Big\} \\
&= \max\{0.6; 0.2\} = 0.6 \\
(M, s_2 \models \Box p) &= \min\Big\{I\big(R(s_2, s_3), (M, s_3 \models p)\big); I\big(R(s_2, s_1), (M, s_1 \models p)\big)\Big\} \\
&= \min\big\{I(0.6, 0.8); I(0.2, 0.7)\big\} \\
&= \min\{1; 1\} = 1
\end{aligned}
$$

4 Simulation and bisimulation

In this section, we propose definitions of simulation and bisimulation for fuzzy models which are a generalisation of the classical ones. This is altogether a new approach in which we compare two models which have different fuzzy accessibility relations and fuzzy valuation functions.

Definition 4.1 Let $E \subseteq S \times S'$, $U \subseteq S$ and $U' \subseteq S'$. Then,

$E[U] := \{s' \in S' : u\, E\, s' \text{ for some } u \in U\}$

$E^{-1}[U] := \{s \in S : s\, E\, u' \text{ for some } u' \in U'\}$

Definition 4.2 [Simulation] Let $M = (S, R, V)$ and $M' = (S', R', V')$ be two fuzzy models and $E \subseteq S \times S'$. We say that E is a *simulation from M to M'* if for every $w \in S$ and $w' \in S'$ such that $w\ E\ w'$ we have

Atoms for any $p \in \text{Prop}$, $V(w, p) \le V'(w', p)$

Fzig For any $u \in S, R(w, u) \le \max\limits_{u' \in\ E[\{u\}]} R'(w', u')$

Moreover, we say that E is a *simulation from the fuzzy frame (S, R) to (S', R')* if the Fzig condition holds.

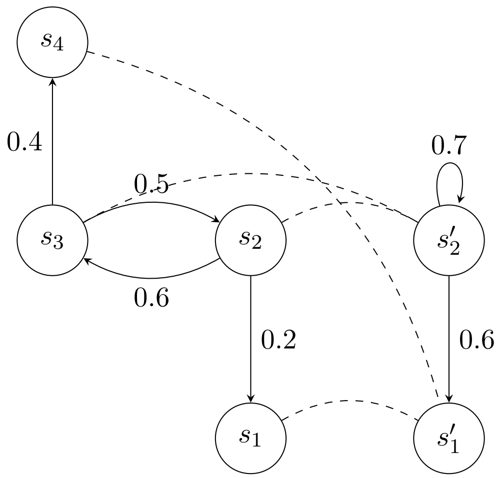
Example 4.3 Consider the fuzzy models $M = (S, R, V)$ and $M' = (S', R', V')$ in Figure 2(a) where

$$S = \{s_1, s_2, s_3, s_4\}, S' = \{s'_1, s'_2\}, E = \{(s_1, s'_1), (s_2, s'_2), (s_3, s'_2), (s_4, s'_1)\}, \text{Prop} = \{p\}.$$

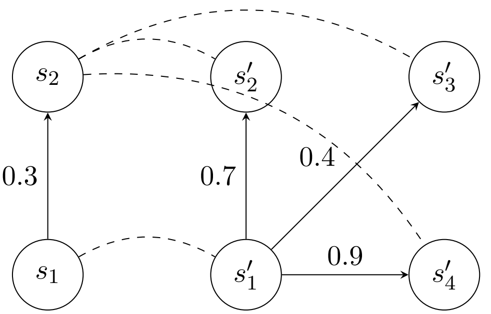
and $\forall\ p \in \text{Prop}$,

$$V(s_2, p) \le V'(s'_2, p); V(s_1, p) \le V'(s'_1, p); V(s_3, p) \le V'(s'_2, p); V(s_4, p) \le V'(s'_1, p).$$

The labels on transitions mean the value of relations between pairs, for example $R(s_3.s_4) = 0.4; R'(s'_2, s'_1) = 0.6$ etc.
It is easy to see that the relation $E \subseteq S \times S'$ represented by the dashed lines is a simulation relation from M to M'.
In fact, the atomic conditions hold by assumption. To check the Fzig condition we have to check for each pair in E.



(a) Simulation 1



(b) Simulation 2.

Fig. 2. Simulation Examples

Let us show for $s_2\ E\ s'_2$. For that, we have to show that

$$\forall u \in S, R(s_2, u) \leq \max_{u' \in\ E[\{u\}]} R'(s'_2, u').$$

Since, the transitions from s_2 that are different from 0 are only (s_2, s_3) and (s_2, s_1). So, we show the Fzig condition for only $u = s_1$ and $u = s_3$.

$(u = s_1)$; $\quad R(s_2, s_1) = 0.2 \leq \max_{u' \in\ E[\{u\}]} R'(s'_2, u') = R'(s'_2, s'_1) = 0.6$

$(u = s_3)$; $\quad R(s_2, s_3) = 0.6 \leq \max_{u' \in\ E[\{u\}]} R'(s'_2, u') = R'(s'_2, s'_2) = 0.7$

The remaining cases can be checked in a similar way.

Example 4.4 Consider the fuzzy frames $F = (S, R)$ and $F' = (S', R')$ in Figure 2(b) where

$$S = \{s_1, s_2\}, S' = \{s'_1, s'_2, s'_3, s'_4\}, E = \{(s_1, s'_1), (s_2\ , s'_4), (s_2\ , s'_2), (s_2\ , s'_3)\}.$$

It is easy to see that the relation $E \subseteq S \times S'$ represented by the dashed lines is a

simulation relation from frame F to F'.
To check the Fzig condition we have to check for each pair in E. Let us check for the case (s_1, s'_1).
Clearly,

$$R(s_1, s_2) = 0.3 \leq \max\{R'(s'_1, s'_2), R'(s'_1, s'_3), R'(s'_1, s'_4)\} = \max\{0.7, 0.4, 0.9\} = 0.9. \tag{1}$$

The Fzig condition can also be checked for the pairs $(s_2\ , s'_4), (s_2\ , s'_2)$ and $(s_2\ , s'_3)$.

Definition 4.5 [Bisimulation] Let $M = (S, R, V)$, $M' = (S', R', V')$ be two fuzzy models and $E \subseteq S \times S'$. We say that E is a *bisimulation from M to M'* if for every $w \in S$ and $w' \in S'$ such that $w\ E\ w'$ we have

Atoms for any $p \in \text{Prop}$, $V(w, p) = V'(w', p)$

Fzig for any $u \in S, R(w, u) \leq \max\limits_{u' \in\ E[\{u\}]} R'(w', u')$

Fzag for any $u' \in S', R'(w', u') \leq \max\limits_{u \in E^{-1}[\{u'\}]} R(w, u)$

Moreover, we say that E is a *bisimulation from the fuzzy frame* (S, R) to (S', R') if the Fzig and Fzag conditions hold.

As in the standard case, a bisimulation is a binary relation such that itself and its inverse are simulations[5] .

Lemma 4.6 *Let $M = (S, R, V)$ and $M' = (S', R', V')$ be two fuzzy models and $E \subseteq S \times S'$. Then,*
E is a bisimulation from M to M' iff E and E^{-1} are simulations from M to M' and M' to M, respectively.

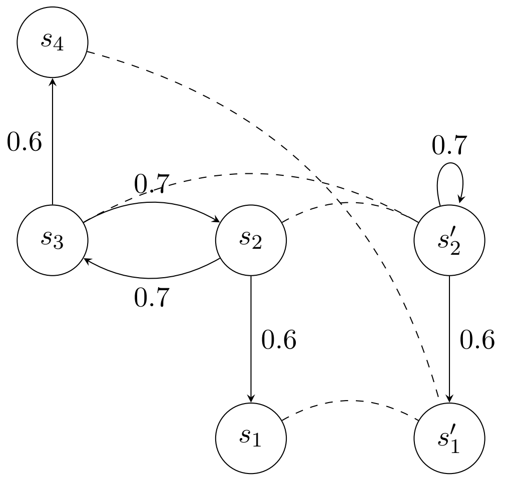
Proof. ($\Rightarrow$) Suppose that E is a bisimulation relation. Then clearly, E is a simulation relation.
To show that E^{-1} is a simulation relation we need to note that the Fzag condition for E gives the Fzig condition for E^{-1} and the atomic condition holds trivially. Thus, E and E^{-1} are simulations from M to M' and M' to M, respectively.
($\Leftarrow$) Suppose that E and E^{-1} are simulation relations from M to M' and from M' to M, respectively. Similar to the above, both of these conditions imply the Fzig and Fzag condition for E. Also, the atomic conditions together give the atomic condition for E. Hence, E is a bisimulation relation from M to M'. $\square$

Example 4.7 Consider the fuzzy frames $F = (S, R)$ and $F' = (S', R')$ given in Figure 3(a) where

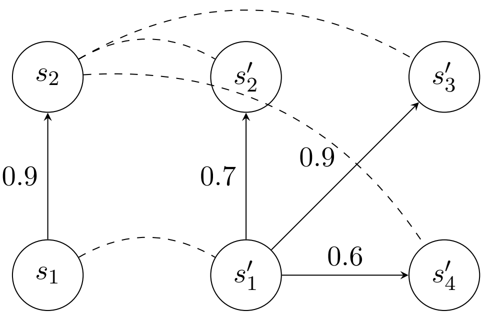
$$S = \{s_1, s_2, s_3, s_4\}, S' = \{s'_1, s'_2\}, E = \{(s_1, s'_1), (s_2, s'_2), (s_3, s'_2), (s_4, s'_1)\}$$

We will check the Fzig and the Fzag conditions for the specific pair (s_2, s'_2). So,

[5] The inverse of a relation E, is the relation $E^{-1} := \{(a, b) : (b, a) \in E\}$.



(a) Bisimulation 1



(b) Bisimulation 2.

Fig. 3. Bisimulation Examples

we have to check

$$\forall\, u \in S\ , R(s_2, u) \leq \max_{u' \in E[\{u\}]} R'(s'_2, u') \ \text{ and } \ \forall\, u' \in S'\ , R'(s'_2, u') \leq \max_{u \in E^{-1}[\{u\}]} R(s_2, u)$$

[Fzig] The transitions from s_2 which are different from 0 are only (s_2, s_1) and (s_2, s_3). So we check for $u = s_1, s_3$.

$$(u = s_1)\ :\ R(s_2, s_1) = 0.6 \leq \max_{u' \in\ E[\{s_1\}]} R'(s'_2, u') = R'(s'_2, s'_1) = 0.6$$

$$(u = s_3)\ :\ R(s_2, s_3) = 0.7 \leq \max_{u' \in\ E[\{s_3\}]} R'(s'_2, u') = R'(s'_2, s'_2) = 0.7$$

[Fzag] The transitions from s'_2 which are different from 0 are only (s'_2, s'_2) and (s'_2, s'_1). So we check for $u' = s'_1, s'_2$.

$$(u' = s'_1)\ :\ R'(s'_2, s'_1) = 0.6 \leq \max_{u \in\ E^{-1}[\{s'_1\}]} R(_2, u) = R(s_2, s_1) = 0.6$$

$$(u' = s'_2)\ :\ R'(s'_2, s'_2) = 0.7 \leq \max_{u \in\ E^{-1}[\{s'_2\}]} R(s_2, u) = R(s_2, s_3) = 0.7$$

For the remaining pairs in E, the Fzig and Fzag conditions can be proved in a similar way.

$\diamond$

Example 4.8 Consider the fuzzy models $M = (S, R, V)$ and $M' = (S', R', V')$ represented by Figure 3(b) where

$$S = \{s_1, s_2\}, S' = \{s'_1, s'_2, s'_3, s'_4\}, E := \{(s_1, s'_1), (s_2, s'_2), (s_2, s'_3), (s_2, s'_4)\}, Prop = \{p\}$$

.

and $\forall\ p \in \text{Prop}, V(s_1, p) = V'(s'_1, p); V(s_2, p) = V'(s'_2, p); V(s_2, p) = V'(s'_3, p); V(s_2, p) = V'(s'_4, p)$
It is easay to see that E is a bisimulation relation from M to M'. We just check Fzig and Fzag for the pair (s_1, s'_1).So,
$\forall\, u \in S\ , R(s_1, u) \leq \max_{u' \in E[\{u\}]} R'(s'_1, u')$ and $\forall\, u' \in S'\ , R'(s'_1, u') \leq \max_{u \in E^{-1}[\{u\}]} R(s_1, u)$

[Fzig] The transitions from s_1 which are different from 0 are only (s_1, s_2). So, we check for $u = s_2$.

$(u = s_2)\ ; R(s_1, s_2) = 0.9 \leq \max_{u' \in E[\{s_2\}]} R'(s'_1, u') = \max\{R'(s'_1, s'_2), R'(s'_1, s'_3), R'(s'_1, s'_4)\} = 0.9$

[Fzag] We need to check for $u' = s'_2; u' = s'_3; u' = s'_4$.

$(u' = s'_2)\ ;\ R'(s'_1, s'_2) = 0.7 \leq \max_{u \in\ E^{-1}[\{s'_2\}]} R(s_1, u) = R(s_1, s_2) = 0.9$

$(u' = s'_3)\ ;\ R(s'_1, s'_3) = 0.9 \leq \max_{u \in\ E^{-1}[\{s'_3\}]} R(s_1, u) = R(s_1, s_2) = 0.9$

$(u' = s'_4)\ ;\ R(s'_1, s'_4) = 0.6 \leq \max_{u \in\ E^{-1}[\{s'_4\}]} R(s_1, u) = R(s_1, s_2) = 0.9$

The remaining cases can be checked in a similar way.

The following theorem shows that our notion is a generalization of the classical (crisp) case. This is one of the most important differences from our notion of bisimulation and the one proposed in [5,6,26,27].

Theorem 4.9 *Let $M = (S, R, V)$ and $M' = (S', R', V')$ be two classical models and $E \subseteq S \times S'$. Let $M^f = (S\ , R^f\ , V^f\ )$ and $M'^f = (S', R'^f, V'^f)$ be the natural corresponding fuzzy models, i.e. the functions R^f and R'^f are defined as*

$$R^f(s,t) = \begin{cases} 1 & s\,R\,t \\ 0 & \textit{otherwise} \end{cases} \quad \textit{and } R'^f(s',t') = \begin{cases} 1 & s'R't' \\ 0 & \textit{otherwise} \end{cases}$$

and the function $V'^f : S' \times \text{Prop} \to [0,1]$ and $V^f : S \times \text{Prop} \to [0,1]$ are defined as

$$V'^f(s',p) = \begin{cases} 1 & s' \in V'(p) \\ 0 & \textit{otherwise} \end{cases} \quad \textit{and} \quad V^f(s,p) = \begin{cases} 1 & s \in V(p) \\ 0 & \textit{otherwise} \end{cases}$$

Then the following are equivalent

(i) *E is a bismulation from M to M'.*

(ii) *E is a bisimulation from fuzzy models M^f to M'^f.*

Proof. (i) $\Rightarrow$ (ii) Suppose that E is a bisimulation from M to M'. Let $w \in S$ and $w' \in S'$ such that $w \; E \; w'$.
The atomic condition is trivial as the fuzzy valuations are defined. To prove the Fzig condition, let $u \in S$.

- If $R^f(w,u) = 0$ then Fzig condition automatically holds.
- If $R^f(w,u) = 1$, then $w \; R \; u$. Then by (classical) zig condition there exist a $u' \in S'$ such that $w' \; R' \; u'$ and $u \; E \; u'$.
  Whenever $R^f(w,u) = 1$ then there exist a $u' \in S'$ such that $R'^f(w',u') = 1$ and $u' \in \; E[\{u\}]$.
  Thus, $\max\limits_{u' \in E[\{u\}]} R'^f(w',u') = 1$.

Therefore, $R^f(w,u) \leq \max\limits_{u' \in \; E[\{u\}]} R'^f(w',u')$. Similarly we can prove the Fzag condition.
(ii) $\Rightarrow$ (i) Suppose that E is a bisimulation from M^f to M'^f. Let $w \in S$ and $w' \in S'$ such that $w \; E \; w'$.
The atomic condition is trivial as the fuzzy valuations are defined.
To prove the (classical) zig condition, let $u \in S$ such that $w \; R \; u$. Hence $R^f(w,u) = 1$.
By the Fzig condition $\max\limits_{u' \in \; E[\{u\}]} R'^f(w',u') = 1$.
Since we are assuming S' to be finite, there is a $u' \in E[\{u\}]$ such that $R'^f(w',u') = 1$. This means that there is
$u' \in S'$ such that $w' \; R' \; u'$ and $u \; E \; u'$. Therefore, zig condition holds. Similarly for zag condition. □

The following theorem states that given two bisimilar states if the value of the transition from one is non-empty then the value of the transition from the other is also non-empty.

Theorem 4.10 *Let $M = (S, R, V)$ and $M' = (S', R', V')$ be two fuzzy models and $E \subseteq S \times S'$ a bisimulation from M to M'. Let $w \in S$ and $w' \in S'$ such that $w \; E \; w'$. Then,*

(i) *for any $u \in S$, if $R(w,u) \neq 0$ then there exists $u' \in E[\{u\}]$ such that $R'(w',u') \neq 0$.*

(ii) *for any $u' \in S'$, if $R'(w',u') \neq 0$ then there exists $u \in E^{-1}[\{u'\}]$ such that $R(w,u) \neq 0$.*

Proof. (i) Let $u \in S$ such that $R(w,u) \neq 0$. As, $w \; E \; w'$, $\max\limits_{u' \in \; E[\{u\}]} R'(w',u') > 0$ (by Fzig condition). This implies that there is a $u' \in E[\{u\}]$ such that $R'(w',u') \neq 0$.
(ii) The proof is similar (using Fzag condition). □

The following two lemmas state alternative (set-based) conditions for a relation to be a bisimulation (cf. Theorem 4.13).

Lemma 4.11 *Let $M = (S, R, V)$ and $M' = (S', R', V')$ be two fuzzy models and $E \subseteq S \times S'$. Then for any $w \in S$ and $w' \in S'$ the following conditions are equivalent:*

(i) *for any $u \in S, R(w, u) \leq \max_{u' \in E[\{u\}]} R'(w', u')$*

(ii) *for any $U \subseteq S$, $\max_{u \in U} R(w, u) \leq \max_{u' \in E[U]} R'(w', u')$*

Proof. (ii) $\Rightarrow$ (i) Just take $U = \{u\} \subseteq S$ in condition (ii).

(i) $\Rightarrow$ (ii) Let $U \subseteq S$. Since for any $u \in U$ $E[\{u\}] \subseteq E[U]$; we have $\max_{u' \in E[\{u\}]} R'(w', u') \leq \max_{u' \in E[U]} R'(w', u')$.

Hence, by the above inequation and (i), we have that for any $u \in U, R(w, u) \leq \max_{u' \in E[U]} R'(w', u')$.

As U is arbitrary, for any $U \subseteq S$, $\max_{u \in U} R(w, u) \leq \max_{u' \in E[U]} R'(w', u')$. □

Lemma 4.12 *Let $M = (S, R, V)$ and $M' = (S', R', V')$ be two fuzzy models and $E \subseteq S \times S'$. Then for any $w \in S$ and $w' \in S'$ the following conditions are equivalent:*

(i) *for any $u' \in S', R'(w', u') \leq \max_{u \in E^{-1}[\{u'\}]} R(w, u)$*

(ii) *for any $U' \subseteq S'$, $\max_{u' \in U'} R'(w', u') \leq \max_{u \in E^{-1}[U']} R(w, u)$*

Proof. The proof is similar to the proof of the previous lemma. □

As a consequence of the previous lemmas we have the following theorem.

Theorem 4.13 *Let $M = (S, R, V)$ and $M' = (S', R', V')$ be two fuzzy modals and $E \subseteq S \times S'$. Then the following are equivalent*

(i) *E is a bisimulation from M to M'*

(ii) *for any $(w, w') \in E$ the following conditions hold,*
- *for any $p \in \mathrm{Prop}$, $V(w, p) = V'(w', p)$*
- *for any $U \subseteq S$, $\max_{u \in U} R(w, u) \leq \max_{u' \in E[U]} R'(w', u')$*
- *for any $U' \subseteq S'$, $\max_{u' \in U'} R'(w', u') \leq \max_{u \in E^{-1}[U']} R(w, u)$.*

Proof. The result is directly entailed by Lemma 4.10 and Lemma 4.11. □

Corollary 4.14 *Let $M = (S, R, V)$ and $M' = (S', R', V')$ be two fuzzy models and $E \subseteq S \times S'$. Then the following are equivalent*

(i) *E is a bisimulation from M to M'*

(ii) *for any $(w, w') \in E$ the following conditions hold,*
- *for any $p \in \mathrm{Prop}, V(w, p) = V(w', p)$*

- *for any* $U \subseteq S$ *such that* $U = E^{-1}[E[U]]$, $\max\limits_{u' \in E[U]} R'(w', u') = \max\limits_{u \in U} R(w, u)$
- *for any* $U' \subseteq S'$ *such that* $U' = E[E^{-1}[U']]$, $\max\limits_{u' \in U'} R'(w', u') = \max\limits_{u \in E^{-1}[U']} R(w, u)$

Proof. (i) $\Rightarrow$ (ii) Let $U \subseteq S$ such that $U = E^{-1}[E[U]]$.

On one hand, by the equivalent condition of Fzig in Lemma 4.11, we have

$$\max_{u \in U} R(w, u) \leq \max_{u' \in E[U]} R'(w', u') \tag{2}$$

On the other hand, by the equivalent condition of Fzag in Lemma 4.12,we have

$$\forall\, U' \subseteq S', \max_{u' \in U'} R'(w', u') \leq \max_{u \in E^{-1}[U']} R(w, u) \tag{3}$$

In particular, for $U' = E[U]$ in (2), we get

$$\max_{u' \in E[U]} R'(w', u') \leq \max_{u \in E^{-1}[E[U]]} R(w, u) \tag{4}$$

Combining equation (1) and (3) we get: $\max\limits_{u' \in E[U]} R'(w', u') = \max\limits_{u \in U} R(w, u)$.

Similarly, we can prove the other conditions of (ii) by using Lemma 4.10 and also clearly, the atoms condition hold.

(ii) $\Rightarrow$ (i) Let $(w, w') \in E$ and $U \subseteq S$. Consider $U_0 = E^{-1}[E[U]]$.

It is easy to see that $U \subseteq U_0$, $E[U] = E[U_0]$ and $U_0 = E^{-1}[E[U_0]]$. Then,

$$\begin{aligned}
\max_{u \in U} R(w, u) &\leq \max_{u \in U_0} R(w, u) \\
&= \max_{u' \in E[U_0]} R'(w', u') \text{ (by hypothesis)} \\
&= \max_{u' \in E[U]} R'(w', u')
\end{aligned}$$

The proof of the other condition in Theorem 4.13 is similar. Hence, E is a bisimulation from M to M'. □

5 Bisimilarity

This section establishes some properties of bisimulations on fuzzy models, making the analogy with the well known results of standard bisimulation. As happens in the classic settings, these results figure out the usual ingredients to introduce a notion of fuzzy bisimilarity — the largest bisimulation.

With this aim, we start observing that the diagonal relation is a bisimulation and that bisimulations are closed under unions and compositions.

Theorem 5.1 *Let* $M = (S, R, V)$, $M' = (S', R', V')$ *and* $M'' = (S'', R'', V'')$ *be fuzzy models. Then, the following properties hold*

(i) *The diagonal relation* $\Delta_s \subseteq S \times S := \{(s,s) | s \in S\}$ *is a bisimulation (from* M *to itself);*

(ii) *If* E *and* F *are bisimulations from model* M *to* M'*, then* $E \cup F$ *is also a bisimulation from* M *to* M'*;*

(iii) *If* E *is a bisimulation from* M *to* M' *and* E' *is a bisimulation from* M' *to* M''*, then* $E \circ E'$ *is a bisimulation from* M *to* M''*.*

(iv) *If* E *is a bisimulation from* M *to* M'*, then* E^{-1} *is a bisimulation from* M' *to* M*.*

Proof. (i) The proof is direct since the (Fzig), (Fzag) and (atom) trivially holds at equal points.

(ii) Let us suppose $s\,(E \cup F)\,s'$. Then $s\ E\ s'$ or $s\ F\ s'$. Suppose that $s\ E\ s'$.
Then, the (atom) property for $(E \cup F)$ is assured by the (atom) property of E. By (Fzig) condition of E (As, E is a bisimulation) and $E \subseteq (E \cup F)$; $\forall\ u \in S$, $R(s,u) \ \leq \max\limits_{u' \in\ E[\{u\}]} R'(s',u') \ \leq \max\limits_{u' \in\ E\cup F[\{u\}]} R'(s',u')$.
Therefore (Fzig) condition holds for $(E \cup F)$. The condition (Fzag) can be proved in a similar way.

(iii) Let us assume $w\,(E \circ E')\,w''$. Then by the definition there exists ; $w' \in S'$ such that $w\ E\ w'$and $w'\ E'\ w''$.
Atom condition for $(E \circ E')$ is assured by the (Atom) conditions of E and E'; we have

$$\forall\ p \in \text{Prop}, V(w,p) = V'(w',p) = V''(w'',p).$$

Now we prove the (Fzig) condition for $(E \circ E')$. As, E is a bisimulation relation by (Fzig) condition of E, we have

$$\forall\ u \in S,\ R(w,u) \leq \max_{u' \in\ E[\{u\}]} R'(w',u') \tag{5}$$

and by (Fzig) condition of E',

$$\forall\ v' \in S',\ R'(w',v') \leq \max_{v'' \in\ E'[\{v'\}]} R''(w'',v'') \tag{6}$$

in particular,

$$\forall\ u' \in E[\{u\}],\ R'(w',u') \leq \max_{v'' \in\ E'[\{E\{u\}\}]} R''(w'',v''). \tag{7}$$

Hence,

$$\max_{u' \in\ E[\{u\}]} R'(w',u') \leq \max_{v'' \in\ E'[\{E\{u\}\}]} R''(w'',v''). \tag{8}$$

Using (5) and (8) we get the (Fzig) condition for $(E \circ E')$:

$$\forall\ u \in S, R(w,u) \leq \max_{v'' \in\ E'[\{E\{u\}\}]} R''(w'',v'') \tag{9}$$

By similar way we can prove the (Fzag) condition for $(E \circ E')$.

(iv) Suppose that E is bisimulation relation. We are going to prove that $E^{-1} := \{(s', s) : (s, s') \in E\} \subseteq S' \times S$ is a bisimulation relation. Let $(s, s') \in E$, then the Fzig condition of E is given by

$$\forall\ u \in S, \quad R(s, u) \leq \max_{u' \in\ E[\{u\}]} R'(s', u') \tag{10}$$

Note that the Fzig condition of E (for the pair (s, s')) is exactly same as the Fzag condition for E^{-1} (for the pair (s', s)). Similarly, the Fzag condition of E is given by,

$$\forall\ v' \in S', \quad R'(s', v') \leq \max_{v \in\ E^{-1}[\{v'\}]} R(s, v) \tag{11}$$

Note that the Fzag condition of E (for the pair (s, s')) coincides with the Fzig condition for E^{-1} (for the pair (s', s)) and the atom condition for E^{-1} is assured by the atom condition of E.

□

We have now all the ingredients to characterise a bisimilarity notion for fuzzy models.

Definition 5.2 [Bisimilarity] Let $M = (S, R, V)$ be a fuzzy model. The *bisimilarity* on M is the relation:

$$\sim_S := \bigcup \{E \subseteq S \times S \mid E \text{ is a bisimulation relation from } M \text{ to itself}\}$$

Theorem 5.3 *Let $\sim_S$ the bisimilarity relation defined on a fuzzy model M. Then*

(i) *$\sim_S$ is a bisimulation from M to itself;*

(ii) *$\sim_S$ is an equivalence relation on S.*

Proof. Property (i) is a direct consequence of (ii) of Theorem 5.1. In order to prove (ii) we just have to observe that *reflexivity*, *transitivity* and *symmetry* of $\sim_S$ are consequence of (i) and (iii) and (iv) of Theorem 5.1, respectively. □

We observe that these results are crucial for the further development of this framework. In particular, the equivalence relation structure opens the door for new theoretical developments, including minimization and conductive proof methods.

6 Modal Invariance

The importance of Invariance in Modal logic is well known. Here we show that invariance by bisimulation also holds in Fuzzy modal logic.

Theorem 6.1 *Let $M = (S, R, V)$ and $M' = (S', R', V')$ be two fuzzy models, and $E \subseteq S \times S'$ a bisimulation from M to M'. Then, for any formula $\phi \in Fm(\text{Prop})$ and for any two states $w \in S$, $w' \in S$, such that $w\ E\ w'$,*

$$(M, w \models \phi) = (M', w' \models \phi)$$

Proof. We prove this result by induction on the structure of formulas.
For the invariance of the formula $\top$, $(M, w \models \top) = 1 = (M', w' \models \top)$ and similarly we can prove for the formula $\bot$.
Invariance of atomic propositional symbols $p \in$ Prop, is a direct consequence of (Atoms) property,

$$(M, w \models p) = V(w, p) = V'(w', p) = (M', w' \models p).$$

For the invariance of formulas $\sim \phi$, we observe that

$$(M, w \models \sim \phi) = \begin{cases} 1 & (M, w \models \phi) = 0 \\ 0 & (M, w \models \phi) > 0 \end{cases} =^{I.H.} \begin{cases} 1 & (M', w' \models \phi) = 0 \\ 0 & (M', w' \models \phi) > 0 \end{cases} = (M', w' \models \sim \phi).$$

For the invariance of formulas $\phi \wedge \varphi$, we observe that

$$\begin{aligned} (M, w \models \phi \wedge \varphi) &= \min\{(M, w \models \phi), (M, w \models \varphi)\} \\ &=^{I.H.} \min\{(M', w' \models \phi), (M', w' \models \varphi)\} = (M', w' \models \phi \wedge \varphi) \end{aligned}$$

and the proof for the invariance of formulas $\phi \vee \varphi$ can be proved similarly. For the invariance of formulas $\phi \to \varphi$;

$$(M, w \models \phi \to \varphi) = \begin{cases} 1 & \text{if } (M, w \models \phi) \leq (M, w \models \varphi) \\ (M, w \models \varphi) & \text{otherwise} \end{cases}$$

and by I.H. this is equivalent to

$$(M, w \models \phi \to \varphi) = \begin{cases} 1 & \text{if } (M', w' \models \phi) \leq (M', w' \models \varphi) \\ (M', w' \models \varphi) & \text{otherwise} \end{cases}$$

Hence, $(M, w \models \phi \to \varphi) = (M', w' \models \phi \to \varphi)$.

For the invariance of formulas $\Diamond\phi$, we observe that By (Fzig) condition we have

$$\forall u \in S, R(w, u) \leq \max_{u' \in\ E[\{u\}]} R'(w', u') = R'(w', u'_u) \text{ for some } u'_u \in S' \tag{12}$$

Since for every $u \in S; u'_u \in E[\{u\}]$ iff $u\ E\ u'_u$. By I. H., we have $(M, u \models \phi) = (M', u'_u \models \phi)$ and, by (12),

$$\forall u \in S, \min\{R(w, u), (u \models \phi)\} \leq \min\{R'(w', u'_u), (u'_u \models \phi)\} \tag{13}$$

and, in particular,

$$\max_{u \in S}(\min\{R(w, u), (u \models \phi)\}) \leq \max_{u'_u : u \in S}(\min\{R'(w', u'_u), (u'_u \models \phi)\}) \tag{14}$$

Since $\{u'_u : u \in S\} \subseteq \{u' : u' \in S'\}$ we have $\max\{u'_u : u \in S\} \leq \max\{u' : u' \in S'\}$ and by (14)

$$\max_{u \in S}(\min\{R(w, u), (u \models \phi)\}) \leq \max_{u' \in S'}(\min\{R'(w', u'), (u' \models \phi)\}) \tag{15}$$

i.e. $(M, w \models \Diamond\phi) \leq (M', w' \models \Diamond\phi)$. Similarly we can prove $(M, w \models \Diamond\phi) \geq (M', w' \models \Diamond\phi)$ by using Fzag condition.

For the invariance of formulas $\Box\phi$, since $w\ E\ w'$ we have by (Fzig) condition

$$\forall u \in S, R(w, u) \leq \max_{u' \in\ E[\{u\}]} R'(w', u') = R'(w', u'_u) \text{ for some } u'_u \in S' \tag{16}$$

Since for every $u \in S, u'_u \in E[\{u\}]$ iff $u \in S, u\ E\ u'_u$. Hence, by I.H.

$$(M, u \models \phi) = (M', u'_u \models \phi). \tag{17}$$

It follows from the definition of I that $x_0 \leq x_1$ implies $I(x_0, y) \geq I(x_1, y)$. Then, from (16) and (17) we have

$$\forall u \in S, I\big(R(w, u), (u \models \phi)\big) \geq I\big(R'(w', u'_u), (u'_u \models \phi)\big)$$

and in particular

$$\min_{u \in S}(I\big(R(w, u), (u \models \phi)\big)) \geq \min_{u'_u : u \in S}(I\big(R'(w', u'_u), (u'_u \models \phi)\big)). \tag{18}$$

Since $\{u'_u : u \in S\} \subseteq \{u' : u' \in S'\}$, we have $\min\{u'_u : u \in S\} \geq \min\{u' : u' \in S'\}$ and hence

$$\min_{u \in S}(I\big(R(w, u), (u \models \phi)\big)) \geq \min_{u' \in S'}(I\big(R'(w', u'), (u' \models \phi)\big)). \tag{19}$$

Therefore $(M, w \models \Box\phi) \geq (M', w' \models \Box\phi)$. The proof for $(M, w \models \Box\phi) \leq (M', w' \models \Box\phi)$ is analogous. $\Box$

As a straightforward consequence we have the following result.

Corollary 6.2 *Let $M = (S, R, V)$ and $M' = (S', R', V')$ be two fuzzy models, and $E \subseteq S \times S'$ a bisimulation relation from M to M'. For any two states $w \in S$ and $w' \in S$ such that $w\ E\ w'$ we have,*

$$\max_{u \in S} R(w, u) = \max_{u' \in S'} R'(w', u')$$

Proof. Since $w\ E\ w'$, we have by Theorem 6.1 that

$$(M, w \models \Diamond\top) = (M', w' \models \Diamond\top)$$

i.e.

$$\max_{u \in S}(\min\{R(w, u), (u \models \top)\}) = \max_{u' \in S'}(\min\{R(w', u'), (u' \models \top)\})$$

By satisfaction definition, this is the same as

$$\max_{u \in S}(\min\{R(w, u), 1\}) = \max_{u' \in S'}(\min\{R(w', u'), 1\})$$

and since $R(w,u) \leq 1, R'(w',u') \leq 1$, we have:

$$\max_{u \in S} R(w,u) = \max_{u' \in S'} R'(w',u') \qquad \square$$

7 Conclusion

This paper proposed a new bisimulation notion for fuzzy models, i.e. fuzzy transition systems with a fuzzy valuation. Moreover, based on these models we introduced a fuzzy modal logic, supporting the interpretation of connectives by a Gödel algebra. Then we pursued an analogy with the standard treatment of bisimulation relation, by establishing some standard results for the proposed bisimulation notion. This includes the existence of the bisimilarity equivalence in Theorem 5.3. Finally, we were able to prove the modal invariance of fuzzy modal logic with respect to the proposed bisimulation as in Theorem 6.1. A core motivation for this research was to set up a theory that actually generalises the standard modal logic. Against of other bisimulations proposed in the literature for fuzzy transition systems (eg. [5,6,26,27], our bisimulation relation, in fact, generalises the bisimulation notion proposed for modal logic and process algebras as in Theorem 4.9.

There are a number of lines of open research that emerges from this starting paper. First, there is a natural follow up to be done in this work, including the study of the Hennessy-Milner Theorem (the converse implication of the invariance theorem) and the establishment of a standard translation to first-order logic. Next, it was our intention to generalise these results to other variants of our logic based on fuzzy algebras other than the Gödel. This would follow the parametric strategy adopted by the authors in [19].

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