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Anti-Ramsey Threshold of Cycles for Sparse Graphs[1,2]

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Abstract

For graphs G and H, let $G \xrightarrow[\text{p}]{\text{rb}} H$ denote the property that for every *proper* edge colouring of G there is a *rainbow* copy of H in G. Extending a result of Nenadov, Person, Škorić and Steger (2017), we prove that $n^{-1/m_2(C_\ell)}$ is the threshold for $G(n,p) \xrightarrow[\text{p}]{\text{rb}} C_\ell$ when $\ell \geq 5$. Thus our result together with a result of the third author which says that the threshold for $G \xrightarrow[\text{p}]{\text{rb}} C_4$ is $n^{-3/4}$ settles the problem of determining the threshold for $G \xrightarrow[\text{p}]{\text{rb}} C_\ell$ for all values of ℓ.

Keywords: anti-Ramsey, proper colouring, random graphs, threshold.

1 Introduction

For a positive integer r and graphs G and H, let $G \to (H)_r$ denote the following Ramsey property: for every edge colouring of G with at most r colours, there is a

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monochromatic copy of H in G. In random graph theory, we are interested in a function $\hat{p}\colon \mathbb{N} \to [0,1]$ such that with $p \gg \hat{p}$ the probability that $G(n,p)$ satisfies some graph property tends to 1 as n tends to infinity. We then say that $G(n,p)$ satisfies this property asymptotically almost surely (a.a.s.) and if additionally we have that for $p \ll \hat{p}$ we a.a.s. do not have this property, then we call $\hat{p}$ the *threshold* for this property. Note that often there are more precise statements known, but for simplicity we will only discuss this kind of threshold.

A classical result in extremal combinatorics by Rödl and Ruciński [9] determines the threshold for $G(n,p) \to (H)_r$, where $G(n,p)$ is the binomial random graph. In particular, when H contains a cycle, the threshold is expressed in terms of the *maximum 2-density* $m_2(H) = \max\{(e(J)-1)/(v(J)-2) : J \subseteq H, v(J) \geq 3\}$.

Theorem 1.1 ([9]) *Let H be a graph which contains a cycle. Then the threshold function $p_H = p_H(n)$ for the property $G(n,p) \to (H)_r$ is given by $p_H(n) = n^{-1/m_2(H)}$.*

In this paper we investigate an *anti-Ramsey* property of sparse random graphs. Given graphs G and H, we denote by $G \xrightarrow[\text{p}]{\text{rb}} H$ the following anti-Ramsey property: every *proper* edge colouring of G contains a *rainbow* copy of H in G, i.e. a subgraph of G isomorphic to H in which all edges have distinct colours. Since $G \xrightarrow[\text{p}]{\text{rb}} H$ is an increasing property, there exists a threshold $p_H^{\text{rb}} = p_H^{\text{rb}}(n)$ for any fixed graph H (see [1]).

Rödl and Tuza [10] already studied anti-Ramsey properties of random graphs in the early 90's, obtaining an upper bound for the threshold p_C^{rb} for $G(n,p) \xrightarrow[\text{p}]{\text{rb}} C$, where C is a fixed cycle. In [4], an upper bound for the threshold p_H^{rb} for any fixed graph H was obtained.

Theorem 1.2 ([4]) *Let H be a fixed graph. Then there exists a constant $C > 0$ such that a.a.s. $G(n,p) \xrightarrow[\text{p}]{\text{rb}} H$ whenever $p = p(n) \geq Cn^{-1/m_2(H)}$. In particular, $p_H^{rb} \leq n^{-1/m_2(H)}$.*

The threshold for a graph H is not always given by $n^{-1/m_2(H)}$, as in [5] it was proved that there are infinitely many graphs H for which the threshold p_H^{rb} is asymptotically smaller than $n^{-1/m_2(H)}$. However, Nenadov, Person, Škorić and Steger [8] proved that at least for sufficiently large cycles and complete graphs H the lower bound for p_H^{rb} matches the upper bound $n^{-1/m_2(H)}$ of Theorem 1.2.

Theorem 1.3 ([8]) *If H is a cycle on at least 7 vertices or a complete graph on at least 19 vertices, then $p_H^{rb} = n^{-1/m_2(H)}$.*

In [6], extending Theorem 1.3, it was proved that for complete graphs K_k with $k \geq 5$ the threshold $p_{K^k}^{\text{rb}}$ is in fact $n^{-1/m_2(K_k)}$ and that for K_4 we have $p_{K_4}^{\text{rb}} = n^{-7/15} \ll n^{-1/m_2(K_4)}$. Our result determines the threshold $p_{C_\ell}^{\text{rb}}$ for every cycle C_ℓ on $\ell \geq 5$ vertices.

Theorem 1.4 *If H is a cycle on at least 5 vertices, then $p_H^{rb} = n^{-1/m_2(H)}$.*

In Section 2 we prove Theorem 1.4. Similarly to what happens for complete graphs, the situation for C_4 is different. In a short abstract of the third author [7], it is proved that $p_{C_4}^{\mathrm{rb}} = n^{-3/4} \ll n^{-1/m_2(C_4)}$. For completeness, we exhibit in Section 3 the proof of threshold $p_{C_4}^{\mathrm{rb}}$ obtained in [7]. We use standard notation and terminology (see e.g. [2] and [3]). In particular, C_ℓ denotes a cycle on ℓ vertices, $v(G)$ and $e(G)$ denote the number of vertices and edges of G, respectively, and we write $G - H$ for $G - V(H)$.

2 Cycles on at least five vertices

The *maximum density* of a graph G is denoted by $m(G) = \max\{e(J)/v(J) : J \subseteq G, v(J) \geq 1\}$. The results of Nenadov, Person, Škorić, and Steger [8] provide a general framework that allows for a transference of random Ramsey problems into questions for deterministic graphs with bounded density. The proof of Theorem 1.3 for cycles relies on the following lemma.

Lemma 2.1 ([8]) *Let $\ell \geq 7$ be an integer and G be a graph such that $m(G) < m_2(C_\ell)$. Then $G \xrightarrow[\mathrm{p}]{\mathrm{rb}} C_\ell$.*

Note that in fact they prove a slightly stronger statement for which they need a non-strict inequality for the density [8, Corollary 13]. The condition $\ell \geq 7$ above is simply a consequence of the proof of the lemma, as observed by the authors [8]. We extend Lemma 2.1, proving the following result, which we state in words for clarity. It is easy to see that $m_2(C_\ell) = (\ell - 1)/(\ell - 2)$.

Lemma 2.2 *Let $\ell \geq 5$ be an integer and G be a graph such that $m(G) < (\ell - 1)/(\ell - 2)$. Then there exists a proper edge colouring of G such that every ℓ-cycle has two non-adjacent edges with the same colour.*

Theorem 1.4 thus follows by replacing Lemma 2.1 with our Lemma 2.2 in the proof of Nenadov et al. [8]. We remark that the proof of Lemma 2.2 considers all the cycle lengths in the range $\ell \geq 5$; it is not a proof only for the cases $\ell = 5$ and $\ell = 6$.

Throughout this section let $\ell \geq 5$ be an integer and G be a graph with $m(G) < (\ell - 1)/(\ell - 2)$. For the proof of our lemma, we shall define a *partial* proper edge colouring of G such that every ℓ-cycle has two non-adjacent edges with the same colour. Clearly, having defined such a partial edge colouring, we can extend it to a proper edge colouring (for instance, the uncoloured edges may be assigned distinct colours).

Let $\mathcal{C}_\ell(G)$ be the set of all ℓ-cycles of G. The *edge intersection graph* of $\mathcal{C}_\ell(G)$ is the graph whose vertex set is $\mathcal{C}_\ell(G)$ and whose edges correspond to pairs $\{C, C'\}$, $C \neq C'$, such that $E(C) \cap E(C') \neq \emptyset$. A subgraph $H \subseteq G$ is a C_ℓ-*component* of G if it is the union of all the ℓ-cycles corresponding to the vertices of some component of the edge intersection graph of $\mathcal{C}_\ell(G)$. Clearly H can be constructed from an ℓ-cycle H_1 in G as follows. Suppose we have defined $H_1 \subseteq \cdots \subseteq H_i$, $i \geq 1$. If there is an ℓ-cycle C in G such that $C \nsubseteq H_i$ and $E(C) \cap E(C') \neq \emptyset$ for some ℓ-cycle $C' \subseteq H_i$ then we put $H_{i+1} = H_i \cup C$; otherwise we terminate the construction and set $H = H_i$. Note that H can be reconstructed by this procedure starting from any ℓ-cycle $H_1 \subseteq H$. Also note that two ℓ-cycles belonging to distinct C_ℓ-components may share vertices (obviously they do not share edges).

We start the colouring procedure in some C_ℓ-component H of G. Once we have the partial colouring of H, we proceed to assign colours different from those used in H to edges of a C_ℓ-component of $G - E(H)$, according to the same steps as those taken for the colouring of H. We continue in this manner (taking an uncoloured C_ℓ-component, colouring it and removing its edges), until we have considered all the ℓ-cycles of G. So our aim is to describe the colouring procedure of an arbitrary C_ℓ-component H of G.

Let t be such that $H = H_t$, i.e. t is the number of steps taken for the construction of H starting from some ℓ-cycle H_1 in G. We may assume $t \geq 2$, since the case $t = 1$ is trivial. The following proposition describes the possibilities for a step of construction of H. Some configurations are impossible because they imply that $e(H)/v(H) \geq (\ell-1)/(\ell-2)$, which contradicts $m(G) < (\ell-1)/(\ell-2)$.

Proposition 2.3 *Let $1 \leq i \leq t-1$. Let C be an ℓ-cycle which is added to H_i to form H_{i+1}. There exists a labelling $C = u_1u_2\cdots u_\ell u_1$ such that exactly one of the following configurations of H_{i+1} (with respect to C) occurs, where $2 \leq k \leq \ell$ and $3 \leq j \leq \ell$:*

- (A_k) *$u_1u_2\cdots u_k$ is a k-path in H_i and $u_{k+1}, \ldots, u_\ell \notin V(H_i)$;*
- (B_j) *$u_1u_2 \in E(H_i)$, $u_2u_3 \notin E(H_i)$, $\{u_3, \ldots, u_\ell\} \setminus \{u_j\} \subseteq V(H_{i+1}) \setminus V(H_i)$, $u_j \in V(H_i)$.*

With this proposition at hand the proof of Lemma 2.2 proceeds as follows: For configurations (A_k) with $2 \leq k \leq \ell-2$ we assign a new colour to two non-adjacent new edges. All other configurations appear at most twice and in these cases we will colour all previous configurations more carefully so that we are able to proceed.

To prove Proposition 2.3 and that bad configurations are rare, we heavily use $m(G) < (\ell-1)/(\ell-2)$. For each $1 \leq j \leq i$, let r_j be the number of edges in $E(H_{j+1}) \setminus E(H_j)$, s_j be the number of vertices in $V(H_{j+1}) \setminus V(H_j)$, and c_j be the number of components of $H_{j+1} - H_j$. We have that $r_j \geq s_j + 1$. In fact, if $s_j = 0$ then $r_j \geq 1$, and if $s_j \geq 1$ then the components of $H_{j+1} - H_j$ are paths and the number of edges of H_{j+1} with at least one end in $H_{j+1} - H_j$ is $r_j = s_j + c_j \geq s_j + 1$.

We have that $s_j \leq \ell - 2$, because an ℓ-cycle is added to H_j with at least one edge in $E(H_j)$. By an adequate number of applications of the inequality $a/b > (a+1)/(b+1)$ (which is equivalent to $a > b$), we obtain

$$\frac{\ell-1}{\ell-2} > \frac{e(H_{i+1})}{v(H_{i+1})} = \frac{e(H_i)+r_i}{v(H_i)+s_i} \geq \frac{\ell + r_i + \sum_{j=1}^{i-1}(s_j+1)}{\ell + s_i + \sum_{j=1}^{i-1} s_j} \geq \frac{\ell + r_i + (i-1)(\ell-1)}{\ell + s_i + (i-1)(\ell-2)}, \tag{1}$$

which is equivalent to $r_i < ((\ell-1)s_i + \ell)/(\ell-2)$.

Proof of Proposition 2.3 Using this last inequality, one can easily check that $s_i =$

$\ell-3$ implies $r_i \leq \ell-1$. We get that for $s_i = \ell-3$ exactly one of the following three alternatives holds: $H_{i+1} - H_i$ has one component and no edge with both ends in $V(H_i)$ is added to H_i, or $H_{i+1} - H_i$ has one component and one edge with both ends in $V(H_i)$ is added to H_i, or $H_{i+1} - H_i$ has two components and no edge with both ends in $V(H_i)$ is added to H_i. These alternatives correspond to Configurations (A_3) and (B_j) with $3 \leq j \leq \ell$, respectively. The inequality also gives us that $0 \leq s_i \leq \ell-4$ implies $r_i \leq s_i + 1$; we have thus Configurations (A_k) with $4 \leq k \leq \ell$. Configuration (A_2) is clearly the only possibility for $s_i = \ell - 2$. □

For arguments involving similar calculations to (1) we will refer to this as the *density argument*. For example, when H_{i+1} has Configuration (A_ℓ), then $s_i = 0$ and $r_i = 1$, which with (1) immediately implies that there cannot be another occurrence of (A_ℓ). Similarly, we can show that Configuration $(A_{\ell-1})$, where $s_i = 1$ and $r_i = 2$, appears at most twice and any (B_j), where $s_i = \ell-3$ and $r_i = \ell-1$, at most once. Furthermore, when one of the configurations appears, the occurrence of the other (A_k) with $3 \leq k \leq \ell-2$ is restricted, while only (A_2) can appear arbitrarily often.

Proof of Lemma 2.2 Choose an arbitrary ℓ-cycle H_1 and assign a colour c_1 to some pair of non-adjacent edges of H_1. Let $H = H_t$, $t \geq 2$, be the C_ℓ-component of G constructed from H_1.

Next we consider the cases according to which configurations given by Proposition 2.3 occur during the construction of H. For each $1 \leq i \leq t-1$ and each ℓ-cycle which is added to H_i to get H_{i+1}, we shall choose two non-adjacent edges of the cycle and a colour c, and assign c to the chosen edges.

The ℓ-cycles added to H_i are the ones of H_{i+1} which pass through edges of components of $H_{i+1} - H_i$ (recall that these components are paths) or through edges in $E(H_{i+1}) \setminus E(H_i)$ with both ends in $V(H_i)$. Thus, if all the edges in $E(H_{i+1}) \setminus E(H_i)$ belong to components of $H_{i+1} - H_i$ and each of these components has at least two vertices, then the task is easy: for each component we choose two non-adjacent edges of it and assign a new colour to them, and this guarantees that H_{i+1} contains no rainbow copy of C_ℓ. If nothing else is explicitly stated we will always do this when H_{i+1} has Configuration (A_k) with $2 \leq k \leq \ell-2$. Hence the effort in the proof consists in dealing with the other configurations. These will receive colours first. By the density argument Configuration (A_ℓ) appears at most once, $(A_{\ell-1})$ at most twice, and any (B_j) at most once.

Case 1 *For all* $1 \leq i \leq t-1$, H_{i+1} *has Configuration* (A_k) *with* $2 \leq k \leq \ell-2$.

For $1 \leq i \leq t-1$, assign a new colour c_{i+1} to two non-adjacent edges in $E(H_{i+1}) \setminus E(H_i)$.

Case 2 *There is* $1 \leq i_1 \leq t-1$ *such that* H_{i_1+1} *has Configuration* (A_ℓ) *with respect to some* C.

In this case, for all $i \neq i_1$, H_{i+1} has Configuration (A_k) with $2 \leq k \leq \ell-2$, by the density argument. Moreover, for at most one $1 \leq i_2 \leq t-1$, H_{i_2+1} has Configuration (A_3) with respect to some C'.

Let $C = u_1u_2\cdots u_\ell u_1$, where $P = u_1u_2\cdots u_\ell$ is an ℓ-path in H_{i_1} and $u_\ell u_1 \notin E(H_{i_1})$. The number of ℓ-cycles in H_{i_1+1} which are not in H_{i_1} equals that of ℓ-paths in H_{i_1} with ends u_1 and u_ℓ. First, we consider the case in which the number of such paths is greater than one, and, for this case, we define the colouring of H in such a way that each of these paths contains two non-adjacent edges with the same colour.

Suppose that H_{i_1} contains some $P' = u_1x_2\cdots x_{\ell-1}u_\ell$, $P' \neq P$. We have that $P \cup P'$ contains an even cycle with length at most $2\ell-2$, and in H_{i_1} the only such cycles are ℓ-cycles (when ℓ is even), or $(2\ell-4)$-cycles, or $(2\ell-2)$-cycles. A $(2\ell-2)$-cycle appears during the construction only if, for some i, H_i has two ℓ-cycles C and C' such that $C \cap C'$ is a 2-path. Similarly a $(2\ell-4)$-cycle appears during the construction only if, for some i, H_i has two ℓ-cycles C and C' such that $C \cap C'$ is a 3-path.

First consider that $P \cup P'$ is a $(2\ell-2)$-cycle in H_{i_1} (P' and P are internally disjoint). So we may assume without loss of generality that $i_1 = 2$, H_2 has Configuration (A_2) and $P \cup P' \subseteq H_2$. Assign a colour c_1 to the edges of $P \cup P'$ alternately. Note that, if ℓ is even then there may be a third ℓ-path in H_2 between u_1 and u_ℓ, and one can easily check that such a path will have two edges with the same colour.

Consider that $P \cup P'$ contains a $(2\ell-4)$-cycle. We may assume that there is no ℓ-path P'' in H_{i_1} between u_1 and u_ℓ such that $P'' \cup P$ or $P'' \cup P'$ is a $(2\ell-2)$-cycle. Without loss of generality $x_2 = u_2$, H_2 has Configuration (A_3) and $(P \cup P') - u_1 \subseteq H_2$. Alternately colour the edges of $(P \cup P') - u_1$ with two colours c_1 and c_2. If ℓ is even then there may be a third $(\ell-1)$-path in H_2 between u_2 and u_5, and one can easily check that such a path will have two edges with the same colour.

Now consider that $P \cup P'$ contains an ℓ-cycle, ℓ even. We may assume that there is no ℓ-path P'' in H_{i_1} between u_1 and u_ℓ such that $P'' \cup P$ or $P'' \cup P'$ contains a cycle with length at least $2\ell-4$. Without loss of generality H_1 is an ℓ-cycle contained in $P \cup P'$. and we alternately colour the edges of H_1 with two colours c_1 and c_2.

Finally, assume that H_{i_1} contains no ℓ-path linking u_1 and u_ℓ other than P. Suppose that P has two consecutive edges in some ℓ-cycle C'. Obviously, C' cannot contain both u_1u_2 and $u_{\ell-1}u_\ell$. Assign a colour c_1 to two non-adjacent edges in $E(C')$ in such a way that, if $\{u_1u_2, u_2u_3\} \subseteq E(C')$, then c_1 is not assigned to u_2u_3, and, if $\{u_{\ell-2}u_{\ell-1}, u_{\ell-1}u_\ell\} \subseteq E(C')$, then c_1 is not assigned to $u_{\ell-2}u_{\ell-1}$. Without loss of generality $H_1 = C'$. For $1 \leq i \leq t-1$, $i \neq i_1$, assign a new colour c_{i+1} to two non-adjacent edges in $E(H_{i+1}) \setminus E(H_i)$. We have that some edge in $E(P) \setminus \{u_2u_3, u_{\ell-2}u_{\ell-1}\}$ is uncoloured. Assign a new colour c_{i_1+1} to $u_\ell u_1$ and to an uncoloured edge in $E(P) \setminus \{u_2u_3, u_{\ell-2}u_{\ell-1}\}$.

Suppose now that no two consecutive edges of P lie in the same ℓ-cycle in H_{i_1}. We have that P cannot have two non-consecutive edges in same ℓ-cycle, otherwise P would have length greater than $\ell-1$. So any ℓ-cycle in H_{i_1} has at most one edge of P. For $1 \leq i \leq t-1$, $i \neq i_1, i_2$, assign a new colour c_{i+1} to two non-adjacent edges in $E(H_{i+1}) \setminus (E(H_i) \cup E(P))$. Colour two non-adjacent edges in $E(H_{i_2+1}) \setminus E(H_{i_2})$ (when $\ell = 5$ there are exactly two such edges) with a new colour c_{i_2+1}. Again some edge in $E(P) \setminus \{u_2u_3, u_{\ell-2}u_{\ell-1}\}$ is uncoloured. Assign a new colour c_{i_1+1} to $u_\ell u_1$

and to an uncoloured edge in $E(P) \setminus \{u_2u_3, u_{\ell-2}u_{\ell-1}\}$.

Case 3 *There are $1 \leq i_1 < i_2 \leq t-1$ such that H_{i_1+1} and H_{i_2+1} have Configuration $(A_{\ell-1})$ with respect to some C and to some C', respectively.*

By the density argument, this case occurs only if $\ell = 5$ and we have that H_{i+1} has Configuration (A_2) for all $i \neq i_1, i_2$. Let $C = u_1u_2u_3u_4u_5u_1$, and let $C' = v_1v_2v_3v_4v_5v_1$, where $P = u_1u_2u_3u_4$ and $P' = v_1v_2v_3v_4$ are 4-paths in H_{i_1}, $u_5 \notin V(H_{i_1})$ and $v_5 \notin V(H_{i_2})$. We see that P is the only 4-path between u_1 and u_4 in H_{i_1} and thus C is the only 5-cycle added to H_{i_1} to form H_{i_1+1}. As for P', there may be a 4-path P'' in H_{i_2} between v_1 and v_4 other than P'. If such is the case then we have that $P' \cup P''$ contains a 4-cycle or a 6-cycle.

One of the following three alternatives holds for P in H_{i_1}: the three edges of P lie in the same 5-cycle, or two consecutive edges of P lie in the same 5-cycle but the third one does not, or any 5-cycle in H_{i_1} contains at most one edge of P.

Consider that the first alternative holds for P. Without loss of generality all the edges of P lie in H_1 and $i_1 = 1$. Hence H_1 is of the form $H_1 = u_1u_2u_3u_4x_5u_1$ for some x_5. Note that $C'' = u_1x_5u_4u_5u_1$ is a 4-cycle in H_2. Suppose that all the edges of P' lie in H_2. Therefore, without loss of generality $i_2 = 2$. If the ends of P' are two adjacent vertices in $V(C'')$ then we colour u_1u_2 and u_3u_4 with c_1 and we colour two non-adjacent edges of C' with a new colour c_2. If the ends of P' are u_1 and u_4 then we colour u_1u_2 and u_3u_4 with c_1. If the ends of P' are x_5 and a vertex in $\{u_2, u_3\}$ then we assign c_1 to u_1u_2 and u_3u_4 and we assign a new colour c_2 to v_5x_5 and u_2u_3. The case in which the ends of P' are u_5 and a vertex in $\{u_2, u_3\}$ is symmetric. Suppose that the ends of P' are u_1 and u_3. Note that there are two 4-paths between u_1 and u_3 (and thus two possibilities for P'): $u_1u_5u_4u_3$ or $u_1x_5u_4u_3$. We assign c_1 to u_4x_5, u_1u_2 and u_3v_5, and assign c_2 to u_2u_3, u_4u_5 and v_5u_1. The case in which the ends of P' are u_4 and u_2 is symmetric.

Now suppose that P' has at most two edges in H_2. If P' has two edges in H_2 then these must be consecutive. Hence we may assume without loss of generality that $v_3v_4 \notin E(H_2)$. Since there is no 6-cycle in H_{i_2} and the unique 4-cycle in H_{i_2} has its edges in H_2, a 4-path $P'' \neq P'$ between v_1 and v_4 must pass through v_3v_4. Colour u_1u_2 and u_3u_4 with c_1, and v_3v_4 and v_5v_1 with a new colour c_{i_2}.

Let us consider the second alternative for P. Without loss of generality H_1 contains the edges u_1u_2 and u_2u_3 but does not contain u_3u_4. Thus H_1 is of the form $H_1 = u_1u_2u_3x_4x_5u_1$ for some x_4 and x_5. Note that $C'' = u_1x_5x_4u_3u_4u_5u_1$ is a 6-cycle in H_{i_1+1}. We see that C'' is the unique 6-cycle in H_{i_2}, and that H_{i_2} contains no 4-cycle. Hence, the number of 4-paths linking v_1 and v_4 is at most two. If there are two such paths, these correspond to two internally disjoint paths along C''. Suppose that $E(P') \subseteq E(C'')$. Alternately colour the edges of C'' with two colours c_1 and c_2 and, for $1 \leq i \leq t-1$, $i \neq i_1, i_2$, assign a new colour c_{i+2} to two non-adjacent edges in $E(H_{i+1}) \setminus (E(H_i) \cup \{u_3u_4\})$. Assume that $E(P') \nsubseteq E(C'')$. Thus P' is the unique 4-path between v_1 and v_4. If $E(P') \subseteq E(H_1)$ then $E(P') \cap \{u_1u_2, u_2u_3\} \neq \emptyset$ (by assumption P' cannot be $u_1x_5x_4u_3$), and we colour u_4u_5 and the two non-adjacent edges in $E(P')$ with c_1. Assign a new colour c_{i+1} to two non-adjacent edges in $E(H_{i+1}) \setminus E(H_i)$, for $1 \leq i \leq t-1$, $i \neq i_1, i_2$. Thus assume that $E(P') \not\subseteq E(H_1)$ (possibly $P' = P$). Therefore P' has an edge v_jv_{j+1} which does not belong to $E(H_1)$. Colour u_2u_3, x_4x_5 and an edge in $\{u_5u_1, u_5u_4\} \setminus \{v_jv_{j+1}\}$ with c_1, and give a new colour c_{i_2+1} to v_jv_{j+1} and to some edge in $\{v_5v_1, v_5v_4\}$ not incident with v_j nor with v_{j+1}.

Finally, consider the third alternative for P. In H_{i_2} there are neither 4-cycles nor 6-cycles, and therefore P' is the unique 4-path between v_1 and v_4. We may assume without loss of generality that H_1 contains u_2u_3 and that u_2u_3 is uncoloured. Suppose that $P' = P$. Colour u_2u_3, u_5u_1 and v_5u_4 with a new colour c_{i_1+1}, and assign a new colour c_{i+1} to two non-adjacent edges in $E(H_{i+1}) \setminus E(H_i)$, for $1 \leq i \leq t-1$, $i \neq i_1, i_2$. Now suppose that $P' \neq P$. Since P' is the unique 4-path in H_{i_2} linking v_1 and v_4, P' and P cannot have both ends in common. Without loss of generality $v_1 \neq u_1$. Colour u_2u_3 and u_5u_1 with a new colour c_{i_1+1}. If $v_2v_3 = u_2u_3$ then colour v_5v_1 with c_{i_1+1}; otherwise colour v_2v_3 and v_5v_1 with a new colour c_{i_2+1}.

Case 4 *There is exactly one $1 \leq i_1 \leq t-1$ such that H_{i_1+1} has Configuration ($A_{\ell-1}$) with respect to some C.*

By the density argument, H_{i+1} has Configuration (A_k) with $2 \leq k \leq 4$ for all $i \neq i_1$. Let $C = u_1u_2 \cdots u_\ell u_1$, where $P = u_1 \cdots u_{\ell-1}$ is an $(\ell-1)$-path in H_{i_1} and $u_\ell \notin V(H_{i_1})$. The number of ℓ-cycles in H_{i_1+1} which are not in H_{i_1} equals that of $(\ell-1)$-paths with ends u_1 and $u_{\ell-1}$. Next we consider the case in which the number of such paths is greater than one, and, for this case, we define the colouring of H in such a way that each of these paths contains two non-adjacent edges with the same colour.

Suppose that H_{i_1} contains some $P' = u_1x_2 \cdots x_{\ell-2}u_{\ell-1}$, $P' \neq P$. We see that $P \cup P'$ contains an even cycle with length at most $2\ell-4$, and in H_{i_1} the only such cycles are ℓ-cycles (when ℓ is even), or $(2\ell-6)$-cycles, or $(2\ell-4)$-cycles. A $(2\ell-4)$-cycle appears during the construction only if, for some i, H_i has two ℓ-cycles C and C' such that $C \cap C'$ is a 3-path. Similarly a $(2\ell-6)$-cycle appears during the construction only if, for some i, H_i has two ℓ-cycles C and C' such that $C \cap C'$ is a 4-path.

First, consider that $P \cup P'$ is a $(2\ell-4)$-cycle in H_{i_1} (P' and P are internally disjoint). So we may assume without loss of generality that $i_1 = 2$, H_2 has Configuration (A_3) and $P \cup P' \subseteq H_2$. Alternately colour the edges of $P \cup P'$ with two colours c_1 and c_2. Note that if ℓ is even then there may be a third $(\ell-1)$-path in H_2 between u_1 and $u_{\ell-1}$, and one can easily check that such a path will have two edges with the same colour.

Consider that $P \cup P'$ contains a $(2\ell-6)$-cycle. We may assume that there is no $(\ell-1)$-path P'' in H_{i_1} between u_1 and $u_{\ell-1}$ such that $P'' \cup P$ or $P'' \cup P'$ is a $(2\ell-4)$-cycle. Without loss of generality $x_2 = u_2$, H_2 has Configuration (A_4) and $(P \cup P') - u_1 \subseteq H_2$. Alternately colour the edges of $(P \cup P') - u_1$ with two colours c_1 and c_2, and colour the two non-adjacent edges of $C' \cap H_1$ with a new colour c_3. If ℓ is even then there may be a third $(\ell-2)$-path in H_2 between u_2 and $u_{\ell-1}$. Such a path passes through the edges of $C' \cap H_1$, and therefore will have two edges with the same colour.

Now consider that $P \cup P'$ contains an ℓ-cycle, ℓ even. We may assume that there is no $(\ell-1)$-path P'' in H_{i_1} between u_1 and u_ℓ such that $P'' \cup P$ or $P'' \cup P'$ contains a cycle with length at least $2\ell - 6$. Without loss of generality H_1 is an ℓ-cycle contained in $P \cup P'$. Alternately colour the edges of H_1 with two colours c_1 and c_2, and assign a new colour c_{i+2} to two non-adjacent edges in $E(H_{i+1}) \setminus E(H_i)$ for $1 \leq i \leq t-1$, $i \neq i_1$.

Finally, assume that P is the only $(\ell-1)$-path in H_{i_1} between u_1 and $u_{\ell-1}$. Let C' be an ℓ-cycle containing the edge u_2u_3. Assign a colour c_1 to two non-adjacent edges in $E(C') \setminus \{u_2u_3\}$. Without loss of generality $H_1 = C'$. For $1 \leq i \leq i_1 - 1$, assign a new colour c_{i+1} to two non-adjacent edges in $E(H_{i+1}) \setminus E(H_i)$. We have that u_2u_3 is uncoloured. Assign a new colour c_{i_1+1} to $u_\ell u_1$ and u_2u_3.

Case 5 *There is* $1 \leq i_1 \leq t-1$ *such that* H_{i_1+1} *has Configuration* (B_j) *with* $3 \leq j \leq \ell$ *with respect to some* C.

By the density argument, H_{i+1} has Configuration (A_2) for all $i \neq i_1$. Let $C = u_1u_2\cdots u_\ell u_1$, where $u_j \in V(H_i)$ for some $3 \leq j \leq \ell$, $\{u_3, \ldots, u_\ell\} \setminus \{u_j\} \subseteq V(H_{i+1}) \setminus V(H_i)$. If there is a path P in H_{i_1} between u_1 and u_j such that $V(P) \cup \{u_{j+1}, \ldots, u_\ell\}$ induces an ℓ-cycle in H_{i_1+1} or there is a path P' in H_{i_1} between u_2 and u_j such that $V(P') \cup \{u_3, \ldots, u_{j-1}\}$ induces an ℓ-cycle in H_{i_1+1}, then H_{i_1+1} can be constructed in such a way that each of the last two steps has Configuration $(A_{\ell-j+3})$ and (A_j), respectively, and therefore Case 1, 2, 3, or 4 happens. So we may suppose that H_{i_1} contains none of these paths and colour u_2u_3 and $u_\ell u_1$ with colour c_{i_1+1}.

□

3 Cycle on four vertices

In this section we give a sketch of the proof that $p^{\mathrm{rb}}_{C^4} = n^{-3/4}$. By a classical result of Bollobás (see [3]), we know that if $p \gg n^{-3/4}$, then a.a.s. $G(n,p)$ contains a copy of $K_{2,4}$. It is not hard to see that in any proper colouring of the edges of $K_{2,4}$ there is a rainbow copy of C_4, which implies that $p^{\mathrm{rb}}_{C^4} \leq n^{-3/4}$.

For the lower bound, define a sequence $F = C_4^1, \ldots, C_4^\ell$ of copies of C_4 in G as a C_4-*chain* if for any $2 \leq i \leq \ell$ we have $E(C_4^i) \cap \left(\bigcup_{j=1}^{i-1} E(C_4^i)\right) \neq \emptyset$. Let $p \ll n^{-3/4}$ and $G = G(n,p)$. We want to show that a.a.s. there exists a proper colouring of G that contains no rainbow copy of C_4. It is enough to consider C_4-chains that are maximal with respect to the number of C_4's. The first step is to properly colour some edges in all maximal C_4-chains so that all C_4's in G will be non-rainbow. Then, since all C_4's are coloured we can just give a new colour for each one of the remaining uncoloured edges. For the first step, we use Markov's inequality and a union bound to obtain that a.a.s. G does not contain any graph H with $m(H) \geq 4/3$ and $|V(H)| \leq 12$.

Let $F = C_4^1, \ldots, C_4^\ell$ be an arbitrary C_4-chain in G with $m(F) \geq 4/3$. Let $2 \leq i \leq \ell$ be the smallest index such that $F' = C_4^1, \ldots, C_4^i$ has density $m(F') \geq 4/3$. Then, since $F'' = C_4^1, \ldots, C_4^{i-1}$ has density $m(F'') < 4/3$, we can explore the structure of G to conclude that $|V(F'')| \leq 10$, which implies $|V(F')| \leq 12$, a contradiction.

Therefore, a.a.s. G contains no copies of C_4-chains F with $m(F) \geq 4/3$. Since F is not too dense, a careful analysis of such chains shows that it is possible to obtain the desired colouring, which proves that $p_{C^4}^{\mathrm{rb}} \geq n^{-3/4}$.

References

[1] Bollobás, B. and A. Thomason, *Threshold functions*, Combinatorica **7** (1987), pp. 35–38.

[2] Diestel, R., "Graph theory," Graduate Texts in Mathematics **173**, Springer, Heidelberg, 2010, fourth edition, xviii+437 pp.
URL http://dx.doi.org/10.1007/978-3-642-14279-6

[3] Janson, S., T. Łuczak and A. Ruciński, "Random graphs," Wiley-Interscience, New York, 2000, xii+333 pp.

[4] Kohayakawa, Y., P. B. Konstadinidis and G. O. Mota, *On an anti-Ramsey threshold for random graphs*, European Journal of Combinatorics **40** (2014), pp. 26–41.

[5] Kohayakawa, Y., P. B. Konstadinidis and G. O. Mota, *On an anti-Ramsey threshold for sparse graphs with one triangle*, Journal of Graph Theory **87** (2017), pp. 176–187.

[6] Kohayakawa, Y., G. O. Mota, O. Parczyk and J. Schnitzer, *Anti-Ramsey threshold of complete graphs*, arXiv:1902.00306, 2019 .
URL http://arxiv.org/abs/1902.00306

[7] Mota, G. O., *Advances in anti-Ramsey theory for random graphs*, Encontro de Teoria da Computação (ETC–CSBC) **2** (2017).

[8] Nenadov, R., Y. Person, N. Škorić and A. Steger, *An algorithmic framework for obtaining lower bounds for random Ramsey problems*, J. Combin. Theory Ser. B (2017), http://dx.doi.org/10.1016/j.jctb.2016.12.007.

[9] Rödl, V. and A. Ruciński, *Threshold functions for Ramsey properties*, J. Amer. Math. Soc. **8** (1995), pp. 917–942.
URL http://dx.doi.org/10.2307/2152833

[10] Rödl, V. and Z. Tuza, *Rainbow subgraphs in properly edge-colored graphs*, Random Structures Algorithms **3** (1992), pp. 175–182.