Available online at www.sciencedirect.com

ScienceDirect

Electronic Notes in Theoretical Computer Science 324 (2016) 135–150

Electronic Notes in Theoretical Computer Science

www.elsevier.com/locate/entcs

Interpretations on Quantum Fuzzy Computing: Intuitionistic Fuzzy Operations $\times$ Quantum Operators

Renata Reiser[1], Alexandre Lemke[1], Anderson Avila[1], Júlia Vieira[1], Maurício Pilla[1], André Du Bois[1]

Centro de Desenvolvimento Tecnológico, CDTEC
Universidade Federal de Pelotas
Pelotas, RS, Brazil

Abstract

Quantum processes provide a parallel model for fuzzy connectives. Calculations of quantum states may be simultaneously performed by the superposition of membership and non-membership degrees of each element regarding the intuitionistic fuzzy sets. This work aims to interpret Atanassov's intuitionistic fuzzy logic through quantum computing, where not only intuitionistic fuzzy sets, but also their basic operations and corresponding connectives (negation, conjuntion, disjuntion, difference, codifference, implication, and coimplication), are interpreted based on the traditional quantum circuit model.

Keywords: Quntum fuzzy computing, quantum computing, intuitionistic fuzzy logic, fuzzy implications, fuzzy difference.

1 Introduction

Intuitionistic fuzzy logic (IFL)[1] and quantum computing (CQ) are relevant research areas consolidating the analysis and the search for new solutions for difficult problems faster than the classical logical approach or conventional computing.

Similarities between these areas in the representation and modelling of uncertainty have been explored [2,3,4]. The uncertainty of human being's reasoning is modelled in fuzzy logic (FL) and its extensions as the fuzzy intuitionistic logic (IFL).

[1] Email: {reiser,alemke,abdavila,jkvieira,pilla, dubois}@inf.ufpel.edu.br

http://dx.doi.org/10.1016/j.entcs.2016.09.012
1571-0661/

By making use of the Fuzzy Set Theories (FSTs) as mathematical models inheriting the imprecision of natural language and uncertainty from membership and non-membership degrees, which are not necessarily complementary, intuitionistic fuzzy techniques help physicists and mathematicians to transform their imprecise ideas into new computational programs [5].

The uncertainty of the real world is concerned with fundamental concepts of quantum computing by making use of properties of quantum mechanics as the superposition, suggesting an improvement in the efficiency regarding complex tasks. In addition, simulations using classical computers allow the development and validation of basic quantum algorithms (QAs), anticipating the knowledge related to their behaviours when executed in a quantum computer. Quantum interpretations come from measurement operations performed on the corresponding quantum states.

In spite of quantum computers being restricted to a few research centers and laboratories, the studies from quantum information and quantum computation (QC) are a reality nowadays. In this context, new methods dealing with quantum fuzzy applications have been proposed [3]. In this work, the modelling and interpretation of intuitionistic fuzzy logic via quantum computing is considered, providing the description of representable fuzzy connectives by using pairs of quantum states and quantum registers from the traditional model of quantum circuits(qCs).

Some results of this research approach mainly related to interpretation of fuzzy connectives via quantum computing, as negation, conjuntion, disjuntion and implications can be found in [6,7,8,9] and more recently, by considering fuzzy exclusive or operators, in [10]. This work is the first step in order to extend this approach, towards an interpretation of intuitionistic fuzzy connectives from quantum computing.

In this regard, this paper considers the interpretation of uncertainty described by the membership and non-membership degrees of IFSs defined by intuitionistic fuzzy connectives by quantum states and quantum operators.

Moreover, it contributes to increase the interest in quantum algorithm applications representing IFSs operations, by exploring potentialities as quantum parallelism, entanglement and superposition of quantum states which can provide theoretical foundation for modelling humanoid behaviour based on intuitionistic fuzzy logic [11,12,13].

This paper is organized as follows:

Section 2 presents the fundamental concepts and properties of connectives of FL and IFL. Moreover, $IFSs$ can be obtained by intuitionistic fuzzy operators as presented in subsection 2.2. Section 3 brings the main concepts of QC which will be considered in the development of this work. In Section 4, a discussion about IFSs which can be obtained from quantum computing following the same methodology from previous work in order to model fuzzy operations from quantum registers. In Section 5, the approach for describing $IFSs$ using the QC is depicted. An interpretation of classical intuitionistic fuzzy sets from quantum states is also considered, presenting the operations on $IFSs$ modelled from quantum transformations, relating the intuitionistic fuzzy approach to the difference and implication operators, also including their dual constructions. Finally, conclusions and further studies are discussed in Section 6.

2 Preliminaries

This section reports main concepts of fuzzy logic (FL) and its corresponding extension, the Atanassov intuitionistic fuzzy logic (IFL).

2.1 Fuzzy connectives

FL connectives are studied from QC operators, overcoming the limitations related to quantum transitions as modelling smoothly their logical properties.

A **membership function** (MF) $f_A(x) : \mathcal{X} \to [0,1]$ determines the membership degree (MD) of the element $x \in \mathcal{X}$ to the fuzzy set A, such that $0 \leq f_A(x) \leq 1$. Thus, a **fuzzy set A** related to a set $\mathcal{X} \neq \emptyset$ is given as $A = \{(x, f_A(x)) : x \in \mathcal{X}\}$.

A function $N : [0,1] \to [0,1]$ is a **fuzzy negation** (FN) when the following holds:

[N1] $N(0) = 1$ and $N(1) = 0$; [N2] If $x \leq y$ then $N(x) \geq N(y)$, for all $x, y \in [0,1]$.

Fuzzy negations verifying the involutive property: [N3] $N(N(x)) = x$, for all $x \in [0,1]$, are called strong fuzzy negations. See, e.g., the standard negation $N_S(x) = 1 - x$.

Fuzzy connectives can be represented by aggregation functions. Herein, we consider triangular norms (t-norms) and triangular conorms (t-conorms).

A **triangular (co)norm** is an operation $(S)T : [0,1]^2 \to [0,1]$ such that, for all $x, y, z \in [0,1]$, the commutative and associative properties are verified along with the fact that it is an increasing function with neutral element (0) 1 .

Among different definitions of t-norms and t-conorms [14], in this work we consider the *Algebraic Product* and *Algebraic Sum*, respectively given as:

$$T_P(x,y) = x \cdot y; \quad \text{and} \quad S_P(x,y) = x + y - x \cdot y, \ \forall x, y \in [0,1]. \tag{1}$$

A binary function $I : U^2 \to U$ is an implication operator (implicator) if the following conditions are satisfied:

I0: $I(1,1) = I(0,1) = I(0,0) = 1$ and $I(1,0) = 0$.

Additional properties are considered to define a fuzzy implication from an implicator:

A **fuzzy implication** $I : U^2 \to U$ is an implicator verifying the antitonicity in the first argument, isotonicity in the second argument, falsity dominance in the antecedent and truth dominance in the consequent are verified.

In analogous manner, one can define a fuzzy implication $J : U^2 \to U$, which also satisfies the boundary conditions:

J0: $I(1,1) = I(1,0) = I(0,0) = 0$ and $I(0,1) = 1$.

A (S,N)**-(co)implication** is a fuzzy (co)implication $I_{(S,N)} : U^2 \to U$ defined by:

$$I_{(S,N)}(x,y) = S(N(x),y), \tag{2}$$
$$J_{(T,N)}(x,y) = T(N(x),y), \forall x,y \in U. \tag{3}$$

When N is involutive then $I_{(S,N)}$ is an S-implication. The Reichenbach (co)implication is given as:

$$I_{RB}(x,y) = 1 - x + xy, \tag{4}$$
$$I_{RB}(x,y) = x + y - xy, \forall x,y \in U, \tag{5}$$

is an S-implication (T-coimplication) obtained by a fuzzy negation $N_S(x) = 1 - x$ and a t-conorm $S_P(x,y) = x + y - x \cdot y$ (t-norm $T_P(x,y) = xy$) previously presented in Eqs. (1a) and (1b), respectively.

2.2 Intuitionistic fuzzy connectives

An intuitionistic fuzzy set (IFS) A_I in a non-empty and finite universe $\mathcal{X}$, expressed as

$$A_I = \{(\mathrm{x}, (\mu_{A_I}(\mathrm{x}), \nu_{A_I}(\mathrm{x}))) : \mathrm{x} \in \mathcal{X}, \mu_{A_I}(\mathrm{x}) + \nu_{A_I}(\mathrm{x})) \leq 1\},$$

extending a fuzzy set $A_I = \{(x, \mu_{A_I}(x), 1 - \mu_{A_I}(x)) : \mathrm{x} \in \mathcal{X}\}$, since the non-membership degree (NMD) $\nu_{A_I}(x)$ of an element $\mathrm{x} \in \mathcal{X}$ is less, at most equal to its complement, the membership degree (MD) $\mu_{A_I}(\mathrm{x})$. Hence, it does not necessarily equal to one.

Let $\tilde{U} \subset [0,1] \times [0,1]$, $\tilde{U} = \{\tilde{x} = (x_1, x_2) \in \tilde{U} : x_1 + x_2 \leq 1\}$ be the set of all pairs of MDs and NMDs and $l_{\tilde{U}}, r_{\tilde{U}} : \tilde{U} \to U$ be two projection functions on $\tilde{U}$, which are given by $l_{\tilde{U}}(\tilde{x}) = l_{\tilde{U}}(x_1, x_2) = x_1$ and $r_{\tilde{U}}(\tilde{x}) = r_{\tilde{U}}(x_1, x_2) = x_2$, respectively.

Thus, for all $\tilde{\mathbf{x}} = (\tilde{x}_1, \ldots, \tilde{x}_n) \in \tilde{U}^n$, such that $\tilde{x}_i = (x_{i1}, x_{i2})$ and $x_{i1} = N_S(x_{i2})$ when $1 \leq i \leq n$, let $l_{\tilde{U}^n}, r_{\tilde{U}^n} : \tilde{U}^n \times \tilde{U}^n \to \tilde{U}^n$ be the projections given by:

$$l_{\tilde{U}^n}(\tilde{\mathbf{x}}) = (x_{11}, x_{21}, \ldots x_{n1}) \text{ and } r_{\tilde{U}^n}(\tilde{\mathbf{x}}) = (x_{12}, x_{22}, \ldots x_{n2}). \tag{6}$$

Consider the order relation $\tilde{x} \leq_{\tilde{U}} \tilde{y}$ given by $x_1 \leq y_1$ and $x_2 \geq y_2$ such that $\tilde{0} = (0,1) \leq_{\tilde{U}} \tilde{x}$ and $\tilde{1} = (1,0) \geq_{\tilde{U}} \tilde{x}$, for all $\tilde{x}, \tilde{y} \in \tilde{U}$ [1].

The complement, intersection and union, implication and coimplication, difference and codifference operations are reported in the following.

2.2.1 Complement operation

An *intuitionistic fuzzy negation* (IFN) is a function $N_I : \tilde{U} \to \tilde{U}$ verifying:

[N1I]: If $N_I(\tilde{0}) = N_I(0,1) = \tilde{1}$ and $N_I(\tilde{1}) = N_I(1,0) = \tilde{0}$;

[N2I]: If $\tilde{x} \geq_{\tilde{U}} \tilde{y}$ then $N_I(\tilde{x}) \leq_{\tilde{U}} N_I(\tilde{y})$, for all $\tilde{x}, \tilde{y} \in \tilde{U}$.

And, N_I is a *strong intuitionistic fuzzy negation* (SIFN) if it is also an involutive function:

[N3I]: $N_I(N_I(\tilde{x})) = \tilde{x}, \forall \tilde{x} \in \tilde{U}$.

In this work, we consider the standard intuitionistic fuzzy negation expressed as
:

$$N_{I_S}(\tilde{x}) = N_{I_S}((x_1, x_2)) = (x_2, x_1), \text{for all } \tilde{x} = (x_1, x_2) \in \tilde{U}. \tag{7}$$

Definition 2.1 Let N_I be an IFN and A_I be IFS. The **complement of** A_I with respect to $\tilde{U}$ is a IFS indicated as A'_I and given by

$$A'_I = \{(\mathrm{x}, N_I(\mu_{A_I}(\mathrm{x}), \nu_{A_I}(\mathrm{x}))) : \mathrm{x} \in \mathcal{X}\}. \tag{8}$$

By the representation theorem [15], a SIFN N_I on $\tilde{U}$ can be given as

$$N_I(\mu_{A_I}(\mathrm{x}), \nu_{A_I}(\mathrm{x})) = (N(N_S(\nu_{A_I}(\mathrm{x})), N_S(N(\mu_{A_I}(\mathrm{x}))). \tag{9}$$

Additionally, when $N_S = N$ in Eq.(9), $N_I = N_{IS}$.

2.3 Union and intersection operations

A function $(S_I)T_I : \tilde{U}^2 \to \tilde{U}$ is a fuzzy triangular (co)norm (t-(co)norm shortly), if it is a commutative, associative and increasing function with neutral element $(\tilde{0})$ $\tilde{1}$, meaning that for all $\tilde{x}, \tilde{y}, \tilde{z} \in \tilde{U}$, the following properties hold:

[TI1]: $T_I(\tilde{x}, \tilde{1}) = \tilde{x}$;

[TI2]: $T_I(\tilde{x}, \tilde{y}) = T_I(\tilde{y}, \tilde{x})$;

[TI3]: $T_I(\tilde{x}, T_I(\tilde{y}, \tilde{y})) = T_I(T_I(\tilde{x}, \tilde{y}), \tilde{y})$;

[TI4]: if $\tilde{x} \leq \tilde{x}'$, $\tilde{y} \leq \tilde{y}'$, $T_I(\tilde{x}, \tilde{y}) \leq T_I(\tilde{x}', \tilde{y}')$.

[SI2]: $S_I(\tilde{x}, \tilde{0}) = \tilde{x}$;

[SI2]: $S_I(\tilde{x}, \tilde{y}) = S_I(\tilde{y}, \tilde{x}))$;

[SI3]: $S_I(\tilde{x}, S_I(\tilde{y}, \tilde{z})) = S_I(S_I(\tilde{x}, \tilde{y}), \tilde{z}))$;

[SI4]: if $\tilde{x} \leq \tilde{x}'$,$\tilde{y} \leq \tilde{y}'$, $S_I(\tilde{x}, \tilde{y}) \leq S_S(\tilde{x}', \tilde{y}')$.

Based on results of [16, Definition 3], an intuitionistic t-norm $(S_I)T_I : \tilde{U}^2 \to \tilde{U}$ is t-representable if there exist a t-norm $T : U^2 \to U$ and a t-conorm $S : U^2 \to U$ such that $T(x, y) \leq N_S(S(N_S(x), N_S(y)))$, it is given as:

$$T_I(\tilde{x}, \tilde{y}) = T_I((x_1, x_2), (y_1, y_2)) = (T(x_1, y_1), S(x_2, y_2)); \tag{10}$$

$$S_I(\tilde{x}, \tilde{y}) = S_I((x_1, x_2), (y_1, y_2)) = (S(x_1, y_1), T(x_2, y_2)). \tag{11}$$

We consider both intuitionistic aggregations: the Product t-norm T_{I_P} and Algebraic Sum S_{I_P}, respectively described by Eqs. (12) and (13), for all $\tilde{x} = (x_1, x_2), \tilde{y} = (y_1, y_2) \in \tilde{U}$ in the following:

$$T_{I_P}(\tilde{x}, \tilde{y}) = (T_P(x_1, y_1), S_P(x_2, y_2)); \tag{12}$$

$$S_{I_P}(\tilde{x}, \tilde{y}) = (S_P(x_1, y_1), T_P(x_2, y_2)). \tag{13}$$

Definition 2.2 Let $S_I, T_I : \tilde{U}^2 \to \tilde{U}$ be an intuitionistic fuzzy t-norm (IFT) and t-conorm (IFS). The **intersection** and **union** between the $IFSs$ A_I and B_I, both defined with respect to $x \in \mathcal{X}$ and such that $\tilde{x} = (\mu_{A_I}(\mathrm{x}), \nu_{A_I}(\mathrm{x})), \tilde{y} = (\mu_{B_I}(\mathrm{x}), \nu_{B_I}(\mathrm{x})) \in \tilde{U}$, resulting in the corresponding intuitionistic fuzzy sets:

$$A_I \cap B_I = \{(\mathrm{x}, (\mu_{A_I \cap B_I}(\mathrm{x}), \nu_{A_I \cap B_I)}(\mathrm{x}))) : \mathrm{x} \in \mathcal{X}\}; \tag{14}$$

$$A_I \cup B_I = \{(\mathrm{x}, (\mu_{A_I \cup B_I}(\mathrm{x}), \mu_{A_I \cap B_I}(\mathrm{x}))) : \mathrm{x} \in \mathcal{X}\}. \tag{15}$$

By the t-representability [15, Definiton 5] of an IFT and IFS in terms of a t-norm T and a t-conorm S, Eqs. (14a) and (14b) can be expressed as:

$$A_I \cap B_I = \{(\mathrm{x}, T(\mu_{A_I}(\mathrm{x}), \mu_{B_I}(x)), S(\nu_{A_I}(x), \nu_{B_I}(\mathrm{x}))) : \mathrm{x} \in \mathcal{X}\}; \tag{16}$$
$$A_I \cup B_I = \{(\mathrm{x}, S(\mu_{A_I}(\mathrm{x}), \mu_{B_I}(\mathrm{x})), T(\nu_{A_I}(\mathrm{x}), \nu_{B_I}(\mathrm{x}))) : \mathrm{x} \in \mathcal{X}\}. \tag{17}$$

In the following, by taking $\tilde{x} = (x_1, x_2)$ and $\tilde{y} = (y_1, y_2)$ such that $\mu_{A_I}(x) = x_1, \nu_{A_I}(x) = x_2, \mu_{B_I}(x) = y_1, \nu_{A_I}(x) = y_2 \in U$ the MFs and NMFs related to the intersection and union of IFSs A_I and B_I in Eqs.(16) and (17) are, respectively, given as:

$$\mu_{A_I \cap B_I}(\mathrm{x}) = x_1 \cdot y_1; \qquad \nu_{A_I \cap B_I}(\mathrm{x}) = x_2 + y_2 - x_2 \cdot y_2; \tag{18}$$
$$\mu_{A_I \cup B_I}(\mathrm{x}) = x_1 + y_1 - x_1 \cdot y_1; \qquad \nu_{A_I \cup B_I}(\mathrm{x}) = x_2 \cdot y_2. \tag{19}$$

2.3.1 Difference and codifference operations

According with [17], the difference between IFSs A and B is given by

$$A - B = \{(\mu_{A-B}(x), \nu_{A-B}(x)) | x \in \chi, 0 \leq \mu_{A-B}(x) + \nu_{A-B}(x) \leq 1\}$$

where $\mu_{A-B}, \nu_{A-B} : \chi \to U$ are the membership and non-membership function of $x \in \chi$ in the A-IFS $A - B$. By [17, Definition 3] an A-IFS $A - B$ can be expressed as:

$$D_I((\mu_A(x), \nu_A(x)), (\mu_B(x), \nu_B(x))) = (\mu_{A-B}(x), \nu_{A-B}(x)).$$

The next definition extends the results in [18]:

Definition 2.3 [17, Definition 4] $D_I(E_I) : \tilde{U}^2 \to \tilde{U}$ is an Atanassov intuitionistic fuzzy (co)difference (A-IFD (A-IFE)) if it satisfies the following axioms:

[DI1]: $D_I(\tilde{x}, \tilde{y}) \leq \tilde{x}$; [EI1]: $E_I(\tilde{x}, \tilde{y}) \geq \tilde{x}$;

[DI2]: $D_I(\tilde{x}, \tilde{0}) = \tilde{x}$; [EI2]: $E_I(\tilde{x}, \tilde{1}) = \tilde{x}$;

[DI3]: If $\tilde{y} \leq \tilde{z}$, $D_I(\tilde{x}, \tilde{y}) \geq D_I(\tilde{x}, \tilde{z})$; [EI3]: If $\tilde{y} \leq \tilde{z}$, $E_I(\tilde{x}, \tilde{y}) \geq E_I(\tilde{x}, \tilde{z})$;

[DI4]: If $\tilde{x} \leq \tilde{y}$, $D_I(\tilde{x}, \tilde{z}) \leq D_I(\tilde{y}, \tilde{z}))$; [EI4]: If $\tilde{x} \leq \tilde{y}$, $E_I(\tilde{x}, \tilde{z}) \leq E_I(\tilde{y}, \tilde{z}))$;

[DI5]: $D_I(\tilde{1}, \tilde{x}) = N_I(\tilde{x})$, N_I is a IFN; [EI5]: $E_I(\tilde{0}, \tilde{x}) = N_I(\tilde{x})$, N_I is a IFN.

Proposition 2.4 *For all $\tilde{x}, \tilde{y} \in U$, the operator $D_{I1}(E_{I1}) : U^2 \to U$ given by*

$$D_{I1}(\tilde{x}, \tilde{y}) = (T_I(\tilde{x}, N_I(\tilde{y})); \tag{20}$$
$$E_{I1}(\tilde{x}, \tilde{y}) = (S_I(\tilde{x}, N_I(\tilde{y})). \tag{21}$$

is an intuitionistic fuzzy (co)difference operator in the sense of Proposition2.3.

Proof Straightforward. □

Proposition 2.5 *Let $T_I(S_I)$ be a representable A-IFT (A-IFS). A function $D_I(E_I) : U^2 \to U$ in Eq.(20) and (21) can be expressed, for all $\tilde{x} = (x_1, x_2), \tilde{y} = (y_1, y_2) \in \tilde{U}$, as follows:*

$$D_I(\tilde{x}, \tilde{y}) = (T(x_1, y_2), S(x_2, y_1)); \tag{22}$$
$$E_I(\tilde{x}, \tilde{y}) = (S(x_1, y_2), T(x_2, y_1)). \tag{23}$$

Proof It follows from Eq.(10a) (Eq.(10b)). □

Example 2.6 By Eqs.(10) and (11) and results in Proposition 2.5, we have that:

$$D_{T_P,N_{S\,I}}(\tilde{x},\tilde{y}) = (x_1y_2, x_2 + y_1 - x_2y_1); \tag{24}$$
$$E_{S_P,N_{S\,I}}(\tilde{x},\tilde{y}) = (x_1 + y_2 - x_1y_2, x_2y_1), \tag{25}$$

is a representable A-IFT (A-IFS) obtained by the t-(co)norm T_P (S_P) and standard negation N_S.

2.3.2 Fuzzy intuitionistic (co)implications

The Atanassov's intuitionistic approach of fuzzy (co)-implications is considered in the following, discussing properties and projection functions in order to define representable fuzzy (co)-implications.

Definition 2.7 An **intuitionistic fuzzy (co)implicator** ((A-IFC) A-IFI) $(J_I)I_I : \tilde{U}^2 \to \tilde{U}$ is a binary function verifying the boundary conditions:

[II1]: $I_I(\tilde{0},\tilde{0}) = I_I(\tilde{0},\tilde{1}) = I_I(\tilde{1},\tilde{1}) = \tilde{1}$ and $I_I(\tilde{1},\tilde{0}) = \tilde{0}$;

[JI1]: $J_I(\tilde{0},\tilde{0}) = J_I(\tilde{1},\tilde{0}) = J_I(\tilde{1},\tilde{1}) = \tilde{0}$ and $J_I(\tilde{0},\tilde{1}) = \tilde{1}$;

Definition (2.7) can be reduced to a fuzzy (co)implication if $\tilde{x} = (x_1, x_2)$ and $\tilde{y} = (y_1, y_2) \in \tilde{U}$, such that $x_1 = N_S(x_2)$ e $y_1 = N_S(y_2)$ [1]. Intuitionistic fuzzy (co)-implications are defined in the sense of J. Fodor and M. Roubens [19,20].

Definition 2.8 An ((A-IFC) A-IFI) $(J_I)I_I : \widetilde{U}^2 \to \tilde{U}$ satisfies, for all $\tilde{x},\tilde{y},\tilde{z} \in \tilde{U}$, the conditions described as follows:

[II2]: $\tilde{x} \le \tilde{z} \Rightarrow I_I(\tilde{x},\tilde{y}) \ge I_I(\tilde{z},\tilde{y})$;	[JI2:] $\tilde{x} \le \tilde{z} \Rightarrow J_I(\tilde{x},\tilde{y}) \ge J_I(\tilde{z},\tilde{y})$;
[II3]: $\tilde{y} \le \tilde{z} \Rightarrow I_I(\tilde{x},\tilde{y}) \le I_I(\tilde{x},\tilde{z})$;	[JI3]: $\tilde{y} \le \tilde{z} \Rightarrow J_I(\tilde{x},\tilde{y}) \le J_I(\tilde{x},\tilde{z})$;
[II4]: $I_I(\tilde{0},\tilde{y}) = \tilde{1}$;	[JI4:] $J_I(\tilde{1},\tilde{y}) = \tilde{0}$;
[II5]: $I_I(\tilde{x},\tilde{1}) = \tilde{1}$;	[JI5]: $J_I(\tilde{x},\tilde{0}) = \tilde{0}$;

In this work, for all $\tilde{x} = (x_1, x_2), \tilde{y} = (y_1, y_2) \in \tilde{U}$, we consider the Reichenbach intuitionistic fuzzy S-(co)implication:

$$I_{RC\,I}(\tilde{x},\tilde{y}) = S_{P\,I}(N_{S\,I}(\tilde{x}),\tilde{y}) = (x_2 + y_1 - x_2y_1, x_1y_2) \tag{26}$$
$$J_{RC\,I}(\tilde{x},\tilde{y}) = T_{P\,I}(N_{S\,I}(\tilde{x}),\tilde{y}) = (x_2y_1, x_1 + y_2 - x_1y_2) \tag{27}$$

which can be defined based on Eqs.(4) and (5), respectively.

3 Obtaining intuitionistic fuzzy sets from quantum computing

Firstly, quantum computing concepts are reported.

3.1 Basic concepts of quantum computing

In *QC*, the *qubit* is the basic information unit, being the simplest quantum system, defined by a unitary and bi-dimensional state vector. Qubits are generally described, in Dirac's notation [21], by $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, and the coefficients α

and β are complex numbers for the amplitudes of the corresponding states in the computational basis (state space), respecting the condition $|\alpha|^2 + |\beta|^2 = 1$, which guarantees the unitary of the state vectors of the quantum system, represented by $(\alpha, \beta)^t$ [22].

The state space of a quantum system with multiple *qubits* is obtained by the tensor product of the space states of its subsystems. Considering a quantum system with two *qubits*, $|\psi\rangle = \alpha|0\rangle+\beta|1\rangle$ and $|\varphi\rangle = \gamma|0\rangle+\delta|1\rangle$, the state space comprehends the tensor product given by

$$|\Pi\rangle = |\psi\rangle \otimes |\varphi\rangle = \alpha \cdot \gamma|00\rangle + \alpha \cdot \delta|01\rangle + \beta \cdot \gamma|10\rangle + \beta \cdot \delta|11\rangle.$$

The state transition of a quantum systems is performed by controlled and unitary transformations associated with orthogonal matrices of order 2^N, with N being the number of *qubits* within the system, preserving norms, and thus, probability amplitudes [23].

For instance, the definition of the *Pauly X* transformation and its application over a one-dimensional and two-dimensional quantum systems are presented in Eq. (28).

$$X|\psi\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \beta \\ \alpha \end{pmatrix} ; X^{\otimes 2}|\Pi\rangle = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha \cdot \gamma \\ \alpha \cdot \delta \\ \beta \cdot \gamma \\ \beta \cdot \delta \end{pmatrix} = \begin{pmatrix} \alpha \cdot \gamma \\ \alpha \cdot \delta \\ \beta \cdot \delta \\ \beta \cdot \gamma \end{pmatrix} \tag{28}$$

Furthermore, a Toffoli transformation is also shown in Eq. (3.1), describing a controlled operation for athree dimensional quantum system. In this case, the *NOT* operator (*Pauly X*) is applied to the third *qubit* $|\sigma\rangle$ when the current states of the first two *qubits* $|\psi\rangle$ and $|\varphi\rangle$ are both $|1\rangle$.

Similarly to qTs of multiple *qubits* which were obtained by the tensor product performed over unitary transformations, Eq. (3.1) presents the matrix structure defining such transformation, when $|\mathcal{X}\rangle$ is the initial state:

In order to obtain information from a quantum system, it is necessary to apply measurement operators, defined by a set of linear operators M_m, called projections. The index m refers to the possible measurement results. If the state of a quantum system is $|\psi\rangle$ immediately before the measurement, the probability of an outcome occurrence is given by $p(|\psi\rangle) = \frac{M_m|\psi\rangle}{\sqrt{\langle\psi|M_m^\dagger M_m|\psi\rangle}}$. When measuring a *qubit* $|\psi\rangle$ with $\alpha, \beta \neq 0$, the probability of observing $|0\rangle$ and $|1\rangle$ are, respectively, given by the following expressions:

$$p(0)=\langle\phi|M_0^\dagger M_0|\phi\rangle=\langle\phi|M_0|\phi\rangle=|\alpha|^2 \text{ and } p(1)=\langle\phi|M_1^\dagger M_1|\phi\rangle=\langle\phi|M_1|\phi\rangle=|\beta|^2.$$

$$T|\mathcal{X}\rangle = \begin{pmatrix} 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&1&0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \\ \epsilon \\ \theta \\ \upsilon \\ \sigma \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \\ \epsilon \\ \theta \\ \sigma \\ \upsilon \end{pmatrix}$$

After the measuring process, the quantum state $|\psi\rangle$ has $|\alpha|^2$ as the probability to be in the state $|0\rangle$ and $|\beta|^2$ as the probability to be in the state $|1\rangle$.

3.2 Interpreting FS Operations from Quantum Transformations

According to [4], fuzzy sets can be interpreted by quantum superposition of classical fuzzy sets (CFSs) associated with quantum states. Additionally, interpretations for fuzzy operations such as complement, intersection and union are obtained from the NOT, AND and OR quantum transformations.

In order to simplify the paper notation, the MD defined by $\mu_A(\mathrm{x})$, which is related to an element $\mathrm{x} \in \mathcal{X}$ in the FS A, will be denoted by μ_A.

The modelling of fuzzy complement, intersection and union (NOT, AND and OR operators) are defined in [10] and summarized below. For that, let $\mu_A, \mu_B : \mathcal{X} \to [0,1]$ be membership functions (MFs) related to FSs A and B.

The corresponding pair $(|S_{\mu_A}\rangle, |S_{\mu_B}\rangle)$ of CFSs is given as:

$$|S_{\mu_A}\rangle = \sqrt{\mu_A(\mathrm{x})}|1\rangle + \sqrt{1-\mu_A(\mathrm{x})}|0\rangle, \tag{29}$$

$$|S_{\mu_B}\rangle = \sqrt{\mu_B(\mathrm{x})}|1\rangle + \sqrt{1-\mu_B(\mathrm{x})}|0\rangle. \tag{30}$$

respectively.

The **complement of a** FS is performed by the standard negation, which is obtained by the ***NOT* operator**, based on Eq.(9) and defined as

$$NOT(|S_{\mu_A}\rangle) = (\sqrt{\mu_A(\mathrm{x})}|0\rangle + \sqrt{1-\mu_A(\mathrm{x})}|1\rangle) \tag{31}$$

Moreover, when $|S_{\mu_A}\rangle = \otimes_{1\leq i\leq N}\left(\sqrt{1-\mu_A(\mathrm{x}_i)}|0\rangle + \sqrt{\mu_A(\mathrm{x}_i)}|1\rangle\right)$ the complement operator NOT^N preserves an N-dimensional quantum superposition of 1-*qubit* states described as $\mathcal{C}^{2^N}$ in the computational basis, and expressed by Eq. (32) below:

$$NOT^N(|S_{\mu_A}\rangle) = \otimes_{1\leq i\leq N}\left(\sqrt{\mu_A(\mathrm{x}_i)}|0\rangle + \sqrt{1-\mu_A(\mathrm{x}_i)}|1\rangle\right) \tag{32}$$

Let $|S_{\mu_A}\rangle$ and $|S_{\mu_B}\rangle$ be quantum states given by Eqs. (29) and (30).

(i) The **intersection** and **union** of FSs A and B are modelled by the ***AND*** and

OR operators which are respectively expressed through the *Toffoli* transformation as below:

$$AND(|S_{\mu_A}\rangle, |S_{\mu_B}\rangle) = T(|S_{\mu_A}\rangle, |S_{\mu_B}\rangle, |0\rangle) \quad (33)$$

$$OR(|S_{\mu_A}\rangle, |S_{\mu_B}\rangle) = NOT^3(T(NOT|S_{\mu_A}\rangle, NOT|S_{\mu_B}\rangle, |0\rangle)). \quad (34)$$

(ii) The fuzzy implication and coimplication of FSs A and B, are modelled by the ***IMP*** and ***COIMP*** operators which are respectively expressed through composition of NOT and *Toffoli* transformations as in the following:

$$IMP(|S_{\mu_A}\rangle, |S_{\mu_B}\rangle) = T(|S_{\mu_A}\rangle, NOT|S_{\mu_B}\rangle, |1\rangle). \quad (35)$$

$$COIMP(|S_{\mu_A}\rangle, |S_{\mu_B}\rangle) = T(NOT|S_{\mu_A}\rangle, |S_{\mu_B}\rangle, |0\rangle)). \quad (36)$$

(iii) The fuzzy difference and codifference of FSs A and B, are also modelled by the ***DIF*** and ***CODIF*** operators respectively expressed through composition of NOT and *Toffoli* transformations as follows:

$$DIF(|S_{\mu_A}\rangle, |S_{\mu_B}\rangle) = T(|S_{\mu_A}\rangle, NOT|S_{\mu_B}\rangle, |0\rangle)). \quad (37)$$

$$CODIF(|S_{\mu_A}\rangle, |S_{\mu_B}\rangle) = T(NOT|S_{\mu_A}\rangle, |S_{\mu_B}\rangle, |1\rangle)). \quad (38)$$

4 Interpreting CIFs from quantum states

The description of $IFSs$ from the QC viewpoint extends the work in [4] by modelling an element $\tilde{x} \in A_I$ which is given by $\tilde{x} = (\mu_{A_I}(\mathrm{x}), \nu_{A_I}(\mathrm{x}))$, such that $\mathrm{x} \in \chi \neq \emptyset$, by a pair of quantum register $(|S_{\mu_{A_I}}\rangle, |S_{\nu_{A_I}}\rangle)$ and fuzzy operators by quantum transformations.

Let $\mathcal{X}$ be a non-empty subset with cardinality N, meaning that $\mathcal{X} \neq \emptyset$, $|\mathcal{X}| = N$ and $i \in \mathcal{N}_N = \{1, 2, ..., N\}$. Let $\mu_{A_I}, \nu_{A_I} : \mathcal{X} \to [0,1] \times [0,1]$ be the MF and NMF related to an element $\mathrm{x}_i \in \mathcal{X}$ in the $IFSs$ A_I, respectively, and $\mu_{A_I}(\mathrm{x}_i), \nu_{A_I}(\mathrm{x}_i)$ be their corresponding MD and NMD.

Definition 4.1 A **classical intuitionistic fuzzy set** (CIFS) of N-*qubits* is a pair of N-dimensional quantum states, given by

$$(|S_{\mu_{A_I}}\rangle, |S_{\nu_{A_I}}\rangle) = \\ = \left(\bigotimes_{1\leq i\leq N}\left[\sqrt{1-\mu_{A_I}(\mathrm{x}_i)}|0\rangle + \sqrt{\mu_{A_I}(\mathrm{x}_i)}|1\rangle\right], \bigotimes_{1\leq i\leq N}\left[\sqrt{1-\nu_{A_I}(\mathrm{x}_i)}|0\rangle + \sqrt{\nu_{A_I}(\mathrm{x}_i)}|1\rangle\right]\right) \quad (39)$$

such that $\mu_{A_I}(\mathrm{x}_i) + \nu_{A_I}(\mathrm{x}_i) \leq 1$.

A pair of CIFSs of N-*qubits* is an N-dimensional quantum state given by Eq.(39). Taking $\mathcal{N}_1$, $\mu_{A_I}(\mathrm{x}_1) = x_1$, $\nu_{A_I}(\mathrm{x}_1) = x_2$ and $x_1, x_2 \in]0,1[$, the quantum states corresponding to $IFSs$ are obtained and expressed as

$$\left(|S_{\mu_{A_I}}\rangle, |S_{\nu_{A_I}}\rangle\right) = \left(\sqrt{x_1}|1\rangle + \sqrt{1-x_1}|0\rangle, \sqrt{x_2}|1\rangle + \sqrt{1-x_2}|0\rangle\right) \quad (40)$$

Moreover, by taking $\mathcal{N}_2$ and $\mu_{A_I}(\mathrm{x}_i) = x_{1i}, \nu_{A_I}(\mathrm{x}_i) = x_{2i} \in]0,1[$, the related superposition of bi-dimensional quantum states is given by $\left(|S_{\mu_{A_I}}\rangle, |S_{\nu_{A_I}}\rangle\right)$ and

$$|S_{\mu_{A_I}}\rangle = (\sqrt{x_{11}}|1\rangle + \sqrt{1-x_{11}}|0\rangle) \otimes (\sqrt{x_{12}}|1\rangle + \sqrt{1-x_{12}}|0\rangle) =$$
$$= \sqrt{(1-x_{11})(1-x_{12})}|00\rangle + \sqrt{(1-x_{11})x_{12}}|01\rangle + \sqrt{x_{11}(1-x_{12})}|10\rangle + \sqrt{x_{11}x_{12}}|11\rangle;$$
$$|S_{\nu_{A_I}}\rangle = (\sqrt{x_{21}}|1\rangle + \sqrt{1-x_{21}}|0\rangle) \otimes (\sqrt{x_{22}}|1\rangle + \sqrt{1-x_{22}}|0\rangle)$$
$$= \sqrt{(1-x_{21})(1-x_{22})}|00\rangle + \sqrt{(1-x_{21})x_{22}}|01\rangle + \sqrt{x_{21}(1-x_{22})}|10\rangle + \sqrt{x_{21}x_{22}}|11\rangle.$$

An application of an IMF f_I to each element in the image-set $f_I[\mathcal{X}]$ defines a quantum state. Thus, a canonical orthonormal basis in $\otimes^N\mathcal{C} \times \otimes^N\mathcal{C}$ denotes a pair of classical quantum registers of N-*qubits*, meaning that $\mu_{A_I}(\mathrm{x}_i), \nu_{A_I}(\mathrm{x}_i) \in \{0, 1\}$ in Eq.(39).

5 Interpreting IFS operations from quantum transformations

In the following, the composition of QTs applied to quantum registers also includes an interpretation based on results obtained by the quantum measurement operations. Moreover, fuzzy operators are considered to obtain the corresponding modelling of IFS operation as a composition of QTs applied to quantum registers

5.1 Modelling IFS complement operator by QTs

The **complement operator** $NOT_I{}^N$ applied to the state in an N-dimensional quantum superposition of 1-*qubit* states as described in Eq.(39), is given by

$$NOT_I{}^N(|S_{\mu_{A_I}}\rangle, |S_{\nu_{A_I}}\rangle) =$$
$$= \left(\bigotimes_{1\le i\le N} \left[\sqrt{1-\nu_{A_I}(\mathrm{x}_i)}|0\rangle + \sqrt{\nu_{A_I}(\mathrm{x}_i)}|1\rangle \right], \bigotimes_{1\le i\le N} \left[\sqrt{1-\mu_{A_I}(\mathrm{x}_i)}|0\rangle + \sqrt{\mu_{A_I}(\mathrm{x}_i)}|1\rangle \right] \right) \quad (41)$$

By restricting a CIFs as the pair of one-dimentional qubits as given in Eq.(40), the **complement operator** of an $IFS\ A_I$ is modelled by the NOT_I operator as:

$$NOT_I\left(|S_{\mu_{A_I}}\rangle, |S_{\nu_{A_I}}\rangle\right) = \left(|S_{\nu_{A_I}}\rangle, |S_{\mu_{A_I}}\rangle\right) \quad (42)$$

5.2 Modelling IFS intersection operation by QTs

Let A_I and B_I be IFSs both defined by $\mu_{A_I}(\mathrm{x}) = x_1$, $\nu_{A_I}(\mathrm{x}) = x_2$ and $\mu_{B_I}(\mathrm{x}) = y_1$, $\mu_{B_I}(\mathrm{x}) = y_1$ for $\mathrm{x} \in \mathcal{X}$. Consider the pairs of CIFSs of one-dimensional quantum registers given by Eqs. (43) and (44) in the following:

$$\left(|S_{\mu_{A_I}}\rangle, |S_{\nu_{A_I}}\rangle\right) = \left(\sqrt{x_1}|1\rangle + \sqrt{1-x_1}|0\rangle, \sqrt{x_2}|1\rangle + \sqrt{1-x_2}|0\rangle\right) \quad (43)$$
$$\left(|S_{\mu_{B_I}}\rangle, |S_{\nu_{B_I}}\rangle\right) = \left(\sqrt{y_1}|1\rangle + \sqrt{1-y_1}|0\rangle, \sqrt{y_2}|1\rangle + \sqrt{1-y_2}|0\rangle\right) \quad (44)$$

By making use of the Toffoli transformation, the **intersection** $A_I \cap B_I$ between the $IFSs\ A_I$ and B_I, is defined by the AND_I operator and described in the following:

$$AND_I \left(\left(|S_{\mu_{A_I}}\rangle, |S_{\nu_{A_I}}\rangle\right), \left(|S_{\mu_{B_I}}\rangle, |S_{\nu_{B_I}}\rangle\right)\right)$$
$$=\left(AND\left(|S_{\mu_{A_I}}\rangle, |S_{\mu_{B_I}}\rangle\right), OR\left(|S_{\nu_{A_I}}\rangle, |S_{\nu_{B_I}}\rangle\right)\right) \quad (45)$$

Expanding the components of Eq.(45) based on the fuzzy operators in Eqs.(33) and (34), we firstly consider the *AND* operator also expressed through the *Toffoli* quantum transformation as given in the next expressions:

$$AND\left(|S_{\mu_{A_I}}\rangle, |S_{\mu_{B_I}}\rangle\right) = T\left(|S_{\mu_{A_I}}\rangle, |S_{\mu_{B_I}}\rangle, |0\rangle\right) =$$
$$=T\left((\sqrt{x_1}|1\rangle + \sqrt{1-x_1}|0\rangle) \otimes (\sqrt{y_1}|1\rangle + \sqrt{1-y_1})|0\rangle) \otimes |0\rangle)\right)$$
$$=(\sqrt{1-x_1}\sqrt{1-y_1}|000\rangle + (\sqrt{1-x_1}\sqrt{y_1}|010\rangle + (\sqrt{x_1}\sqrt{1-y_1}|100\rangle + (\sqrt{x_1}\sqrt{y_1}|111\rangle) \quad (46)$$

Thus, a measurement performed over the third *qubit* ($|1\rangle$) in the first component of the final quantum state, which is expressed by Eq. (51), provides the output with probability $p_{AND}(1) = x_1 \cdot y_1$.

Analogously, a measurement of such third *qubit* ($|0\rangle$) in Eq. (51), returns an output state given with probability $p_{AND}(0) = 1 - x_1 \cdot y_1$.

Now, by applying the composition between NOT^3 and Toffoli operators we are able to express $OR(|S_{\nu_{A_I}}\rangle, |S_{\nu_{A_I}}\rangle)$ with similar calculations as the following:

$$OR(|S_{\nu_{A_I}}\rangle, |S_{\nu_{A_I}}\rangle) = NOT^3\left(T(NOT(|S_{\nu_{A_I}}\rangle), NOT(|S_{\nu_{A_I}}\rangle), |0\rangle)\right) =$$
$$=NOT^3(T((\sqrt{x_2}|0\rangle + \sqrt{1-x_2}|1\rangle) \otimes (\sqrt{y_2}|0\rangle + \sqrt{1-y_2}|1\rangle) \otimes |0\rangle))$$
$$=\sqrt{x_2y_2}|001\rangle + \sqrt{x_2(1-y_2)}|011\rangle + \sqrt{(1-x_2)y_2}|101\rangle + \sqrt{(1-x_2)(1-y_2)}|110\rangle \quad (47)$$

Observing a measure of the quantum state given in Eq. (50) performed on the third *qubit*:
(i) related to $|1\rangle$, returns $\frac{1}{\sqrt{x_2+y_2-x_2y_2}}\left(\sqrt{x_2y_2}|001\rangle + \sqrt{x_2(1-y_2)}|011\rangle + \sqrt{(1-x_2)y_2}|101\rangle\right)$ and the corresponding probability $p_{OR}(1) = x_2 + y_2 - x_2 \cdot y_2$.
(ii) related to $|0\rangle$ results on $|110\rangle$ with probability $p_{OR}(0) = (1 - x_2) \cdot (1 - y_2)$.

Based on Eqs. (51) and (50), it results in the state with corresponding components probabilities $(p_{AND}(1), p_{OR}(1))$ which is related to Eq. (18),

$$(\mu_{A_I \cap B_I}(\mathrm{x}), \nu_{A_I \cap B_I}(\mathrm{x})) = (x_1 \cdot y_1, x_2 + y_2 - x_2 \cdot y_2)$$

interpreting MD of $x \in \mathcal{X}$ belongs to the intersection $A_I \cap B_I$.

5.3 Modelling IFS union operation by QTs

The **union** $A_I \cup B_I$ between the $IFSs$ A_I and B_I both defined with respect to $\tilde{U}$, is analogously obtained. It is modelled by the OR_I operator described as:

$$OR_I\left(\left(|S_{\mu_{A_I}}\rangle, |S_{\nu_{A_I}}\rangle\right), \left(|S_{\mu_{B_I}}\rangle, |S_{\nu_{B_I}}\rangle\right)\right) =$$
$$\left(OR\left(|S_{\mu_{A_I}}\rangle, |S_{\mu_{B_I}}\rangle\right), AND\left(|S_{\nu_{A_I}}\rangle, |S_{\nu_{B_I}}\rangle\right)\right) \quad (48)$$

Similar to AND_I operator, the OR_I operator described in Eq.(48) can also be obtained based on Eqs. (51) and (50), resulting the state with corresponding components probabilities: $(p_{OR}(1), p_{AND}(1))$ which is related to Eq. (19) and given as follows:

$$(\mu_{A_I \cup B_I}(\mathrm{x}), \nu_{A_I \cup B_I}(\mathrm{x})) = (x_1 + y_1 - x_1 \cdot y_1, x_2 \cdot y_2)$$

interpreting the MD and NMD of an element $x \in \mathcal{X}$ in the union $A_I \cup B_I$.

5.4 Modelling IFS implication operation by QTs

By making use of the Toffoli transformation, the **implication** $A_I \to B_I$ between the $IFSs$ A_I and B_I, is defined by the IMP_I operator and described in the following:

$$\begin{aligned} IMP_I \quad & \left(\left(|S_{\mu_{A_I}}\rangle, |S_{\nu_{A_I}}\rangle\right), \left(|S_{\mu_{B_I}}\rangle, |S_{\nu_{B_I}}\rangle\right)\right) \\ = \quad & \left(OR\left(|S_{\nu_{A_I}}\rangle, |S_{\mu_{B_I}}\rangle\right), AND\left(|S_{\mu_{A_I}}\rangle, |S_{\nu_{B_I}}\rangle\right)\right) \end{aligned} \tag{49}$$

Expanding the components of Eq.(52), we firstly consider the OR operator also expressed through the *Toffoli* QT as given in next expression:

$$\begin{aligned} &OR(|S_{\nu_{A_I}}\rangle, |S_{\mu_{B_I}}\rangle) = T(NOT(|S_{\nu_{A_I}}\rangle), NOT(|S_{\mu_{B_I}}\rangle), |1\rangle) = \\ &= T((\sqrt{x_2}|0\rangle + \sqrt{1-x_2}|1\rangle) \otimes (\sqrt{y_1}|0\rangle + \sqrt{1-y_1}|1\rangle) \otimes |1\rangle)) \\ &= \sqrt{x_2 y_1}|001\rangle + \sqrt{x_2(1-y_1)}|011\rangle + \sqrt{(1-x_2)y_1}|101\rangle + \sqrt{(1-x_2)(1-y_1)}|110\rangle \end{aligned} \tag{50}$$

Observing a measure of the quantum state in Eq. (50) performed on the third *qubit* related to $|1\rangle$, returns $\frac{1}{\sqrt{x_2+y_1-x_2 y_1}}\left(\sqrt{x_2 y_1}|001\rangle + \sqrt{x_2(1-y_1)}|011\rangle + \sqrt{(1-x_2)y_1}|101\rangle\right)$ and the corresponding probability $p_{OR}(1) = x_2 + y_1 - x_2 \cdot y_1$. And, a measure operation related to $|0\rangle$ results on $|110\rangle$ with probability $p_{OR}(0) = (1 - x_2) \cdot (1 - y_1)$.

Now, by applying Toffoli QT we express $AND(|S_{\mu_{A_I}}\rangle, |S_{\nu_{B_I}}\rangle)$ in a similar way:

$$\begin{aligned} &AND\left(|S_{\mu_{A_I}}\rangle, |S_{\nu_{B_I}}\rangle\right) = T\left(|S_{\mu_{A_I}}\rangle, |S_{\nu_{B_I}}\rangle, |0\rangle\right) = \\ &= \sqrt{1-x_1}\sqrt{1-y_2}|000\rangle + \sqrt{1-x_1}\sqrt{y_2}|010\rangle + \sqrt{x_1}\sqrt{1-y_2}|100\rangle + (\sqrt{x_1}\sqrt{y_2}|111\rangle \end{aligned} \tag{51}$$

Thus, a measurement performed over the third *qubit* ($|1\rangle$) in the first component of the final quantum state, which is expressed by Eq. (51), provides the output with probability $p_{AND}(1) = x_1 \cdot y_2$. Analogously, a measurement of such third *qubit* ($|0\rangle$) in Eq. (51), returns an output state given with probability $p_{AND}(0) = 1 - x_1 \cdot y_2$.

Based on Eqs. (51) and (50), it results in the state with corresponding components probabilities $(p_{OR}(1), p_{AND}(1))$ such that $(\mu_{A_I \to B_I}(\mathrm{x}), \nu_{A_I \to B_I}(\mathrm{x})) = (x_2 + y_1 - x_2 \cdot y_1, x_1 \cdot y_2)$ interpreting MD of $x \in \mathcal{X}$ belongs to the **intuitionistic implication** $A_I \to B_I$.

5.5 Modelling IFS coimplication operation by QTs

By making use of the Toffoli transformation, the **co-implication** $A_I \leftarrow B_I$ between the $IFSs$ A_I and B_I, is defined by the $COIMP_I$ operator and described in the following:

$$COIMP_I\left(\left(|S_{\mu_{A_I}}\rangle,|S_{\nu_{A_I}}\rangle\right),\left(|S_{\mu_{B_I}}\rangle,|S_{\nu_{B_I}}\rangle\right)\right)=\left(AND\left(|S_{\nu_{A_I}}\rangle,|S_{\mu_{B_I}}\rangle\right),OR\left(|S_{\mu_{A_I}}\rangle,|S_{\nu_{B_I}}\rangle\right)\right)\tag{52}$$

Finally, analogous calculations result in the state with corresponding components probabilities $(p_{AND}(1), p_{OR}(1))$ such that

$$\begin{aligned}(\mu_{A_I\leftarrow B_I}(\mathrm{x}),\nu_{A_I\leftarrow B_I}(\mathrm{x})) &= (x_2\cdot y_1, x_1+y_2-x_1\cdot y_2)\\ &= (p_{AND}(0), p_{OR}(0))\end{aligned}$$

interprets MD of $x\in\mathcal{X}$ belongs to the **intuitionistic co-implication** $A_I \leftarrow B_I$.

5.6 Modelling IFS difference operation by QTs

The **difference** $A_I - B_I$ between $IFSs$ A_I and B_I both defined with respect to $\tilde{U}$, is analogously obtained. It is modelled by the DIF_I operator described as:

$$DIF_I\left(\left(|S_{\mu_{A_I}}\rangle,|S_{\nu_{A_I}}\rangle\right),\left(|S_{\mu_{B_I}}\rangle,|S_{\nu_{B_I}}\rangle\right)\right)=\left(AND\left(|S_{\mu_{A_I}}\rangle,|S_{\nu_{B_I}}\rangle\right),OR\left(|S_{\nu_{A_I}}\rangle,|S_{\mu_{B_I}}\rangle\right)\right)\tag{53}$$

Therefore, a measure performed over the operator DIF_I results on state with corresponding components probabilities $(p_{AND}(1), p_{OR}(1))$ which is related to Eq. (24) and given as follows,

$$\begin{aligned}(p_{AND}(0), p_{OR}(0)) &= (1-x_1\cdot y_1, 1-x_2-y_2+x_2\cdot y_2)\\ &= \left((\mu_{(A-B)}(x),\nu_{(A-B)}(x)\right))\end{aligned}\tag{54}$$

interpreting the MD and NMD of an element $x\in\mathcal{X}$ in the difference $A_I - B_I$.

5.6.1 Modelling IFS codifference operation by QTs

The **codifference** operation $A_I -_c B_I$ between $IFSs$ A_I and B_I both defined with respect to $\tilde{U}$, is analogously obtained. It is modelled by the $CODIF_I$ operator described as:

$$CODIF_I\left(\left(|S_{\mu_{A_I}}\rangle,|S_{\nu_{A_I}}\rangle\right),\left(|S_{\mu_{B_I}}\rangle,|S_{\nu_{B_I}}\rangle\right)\right)=\left(OR\left(|S_{\mu_{A_I}}\rangle,|S_{\nu_{B_I}}\rangle\right),AND\left(|S_{\nu_{A_I}}\rangle,|S_{\mu_{B_I}}\rangle\right)\right)\tag{55}$$

Analogously the above operators, the measure performed over the $CODIF_I$ operator results on state with corresponding components probabilities $(p_{OR}(1), p_{AND}(1))$ which is related to Eq. (25) as the follows:

$$\begin{aligned}(p_{AND}(0), p_{OR}(0)) &= (1-x_1\cdot y_1, 1-x_2-y_2+x_2\cdot y_2)\\ &= \left((\mu_{(A-_cB)}(x),\nu_{(A-_cB)}(x)\right))\end{aligned}\tag{56}$$

It provides interpretation to the MD and NMD of an element $x\in\mathcal{X}$ in the codifference $A_I -_c B_I$.

6 Conclusion and Final Remarks

This work is mainly focussed on the interpretation of Atanassov's intuitionistic fuzzy logic via QC, where not only the intuitionistic fuzzy sets but also complement, intersection, union, difference and codifference operations are interpreted based on the quantum circuit model, including IFSs obtained by representable (co)implications.

Further work aims to consolidate this specification including not only other fuzzy connectives but also constructors (e.i. automorphisms and reductions) and the corresponding extension of (de)fuzzyfication methodology from formal structures provided by QC.

Acknowledgment

This work is partially supported by the Brazilian funding agencies under the processes 309533/2013-9 (CNPq), 309533/2013-9 (FAPERGS) and 448766/2014-0 (MCTI/CNPQ).

References

[1] Atanassov, K., Gargov, G.: Elements of intuitionistic fuzzy logic. Fuzzy Sets and Systs. **9**(1) (1998) 39–52

[2] Pykacz, J.: Fuzzy quantum logic i. International Journal of Theoretical Physics **32** (1993) 1691–1708

[3] Kreinovich, V., Kohout, L., Kim, E.: Similarity between quantum logic and fuzzy logic (2009)

[4] Mannucci, M.: Quantum fuzzy sets: Blending fuzzy set theory and quantum computation. CoRR **abs/cs/0604064** (2006)

[5] Kosheleva, O., Reiser, R., Kreinovich, V.: Formalizing the informal, precisiating the imprecise: How fuzzy logic can help mathematicians and physicists by formalizing their intuitive ideas. In Trillas, E., Seising, R., Kacprycz, J., eds.: Lecture Notes in Computer Science. Volume 325 of Lecture Notes in Computer Science., Springer International Publishing, Netherlands, 2015 (2015) 301–321

[6] Maron, A.K., Visintin, L., Abeijon, A.M., REISER, R.H.S.: Interpreting fuzzy connectives from quantum computing – case study in reichenbach implication class. In: CBSF 2012 – Congresso Brasileiro de Sistemas Fuzzy, UFRN (2012) 1–12

[7] Visintin, L., Maron, A.K., Reiser, R.H.S., Abeijon, A.M.: Interpreting fuzzy set operations from quantum computing. In: WECIQ 2012 – IV Workshop-School in Quantum Computation and Information, UFF (2012) 142–151

[8] Visintin, L., Maron, A.K., Reiser, R.H.S., Abeijon, A.M., Kreinovich, V.: Relation between polling and likert-scale approaches to eliciting membership degrees clarified by quantum computing. In: (FUZZ–IEEE 2013), 2013 IEEE International Conference on Fuzzy Systems, IEEE Xplore Digital Library (2013) 1–6

[9] Maron, A.K., Visintin, L., Reiser, R.H.S., Abeijon, A.M., Kreinovich, V.: Aggregation operations from quantum computing. In: (FUZZ–IEEE 2013), 2013 IEEE International Conference on Fuzzy Systems, IEEE Xplore Digital Library (2013) 1–6

[10] Avila, A., Schmalfuss, M., Reiser, R., Kreinovich, V.: Fuzzy xor classes from quantum computing. In: Artificial Intelligence and Soft Computing - 14th International Conference, ICAISC 2015, Zakopane, Poland, June 14-28, 2015, Proceedings, Part II. (2015) 305–317

[11] Raghuvanshi, A., Perkowski, M.A.: Fuzzy quantum circuits to model emotional behaviors of humanoid robots. In: IEEE Congress on Evolutionary Computation, IEEE Xplore Digital Library (2010) 1–8

[12] Raghuvanshi, A., Fan, Y., Woyke, M., Perkowski, M.: Quantum robots for teenagers. In: Proceedings of the 37th International Symposium on Multiple-Valued Logic. (2007) 13–16

[13] Raghuvanshi, A., Perkowski, M.: Using fuzzy quantum logic to learn facial gestures of a schrdinger cat puppet for robot theatre. In: Proceedings of Read-Muller. Dallas Texas. (2008)

[14] Klir, G.J., Yuan, B.: Fuzzy Sets and Fuzzy Logic; Theory and Applications. Prentice Hall, Upper Saddle River, N. Y. (1995)

[15] G. Cornelis, G. Deschrijver, E.K.: Implications in intuitionistic fuzzy and interval-valued fuzzy set theory: construction, classification and application. International Journal of Approximate Reasoning (35) (2004) 55–95

[16] Baczyński, M.: On some properties of intuitionistic fuzzy implications. In Wagenknecht, M., Hampel, R., eds.: EUSFLAT Conf, University of Applied Sciences at Zittau/Görlitz, Germany (2003) 168–171

[17] Huawen, L.: Difference operation defined over the intuitionistic fuzzy sets. PhD thesis, School of Mathematics and System Sciences, Shandong University, Jinan, Shandong 250100, China

[18] Roberts, D.W.: An anticommutative difference operator for fuzzy sets and relations. Fuzzy Sets and Systems **21** (1987) 35–42

[19] Fodor, J.C.: On fuzzy implication operators. Fuzzy Sets and Systems **42**(3) (1991) 293–300

[20] Baczyński, M.: On some properties of intuitionistic fuzzy implications. Katowice, ul. Bankowa (14) (2007) 1–4

[21] Nielsen, M., Chuang, I.: Quantum Computation and Quantum Information. Cambridge University Publisher, Cambridge (2003)

[22] Kaye, P., Laflamme, R., Mosca, M.: An Introduction to Quantum Computing. Oxford University Press (2007)

[23] Imre, S., Balázs, F.: Quantum Computing and Communications An Engineering Approach. Johm Wiley & Sons, Ltd (2005)