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# Characterization of 1-d Periodic Boundary Reversible $CA$

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**Abstract**

This paper reports characterization of one dimensional 3-neighborhood periodic boundary cellular automata ($CA$). It targets characterization of CA rules for efficient synthesis of reversible $CA$. The concept of reachability tree, as it has been proposed in [5,6,7], is redefined to classify the $CA$ rules that can form a reversible $CA$. Such classification also enables synthesis of a reversible periodic boundary $CA$ in linear time.

*Keywords:* Non-uniform cellular automata, reversible $CA$, reachability tree, periodic boundary, rule class.

## 1 Introduction

Since its inception [15], the homogeneous structure of Cellular Automata ($CA$) has been employed for modeling physical systems. The $CA$ structure is significantly simplified during 1980s [16]. An 1-dimensional structure of $CA$, each cell having two states (0/1), has been proposed with the target to efficient analysis of $CA$ state space. The uniform 3-neighborhood (self, left neighbor and right neighbor) dependency of the $CA$ cells introduces the structural modularity. Though, it has been shown [17] that the 1-dimensional 3-neighborhood $CA$ exhibits excellent

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performance while modeling the physical systems, it is hard to view that the interacting objects in a dynamical system obey the same local rule (homogeneity) during its evolution. To model a wide variety of physical systems, non-homogeneous $CA$ structure (also called as *hybrid/non-uniform CA*) is evolved as an alternative.

A number of researchers have set their attention to hybrid $CA$ [3,4,8] since 1980s. Specially in $VLSI$ domain [4], the 1-dimensional hybrid $CA$ have gained a wide acceptance. A detail characterization of hybrid $CA$ and its applications in $VLSI$ domain [2,8] have been reported in [4].

In spite of its wide variations, the reversible $CA$ was the main focus of $CA$ research. The interesting properties of reversible $CA$ had attracted researchers for a long time to model a number of applications in hydrodynamics, dynamical systems, heat conduction, wave scattering, nucleation, dendritic growth, physical system modeling, etc. [14]. The dynamical properties of reversible cellular automata were investigated in [9,10]. For $VLSI$ applications, the reversible linear/additive $CA$ structure had also been developed [4].

The issue of reversibility in $CA$ states was addressed in [1,13]. In this work, we propose an alternative method to characterize reversible $CA$. The proposed characterization facilitates efficient analysis and synthesis of this class of $CA$. The set of 256 3-neighborhood $CA$ rules are classified based on its potential to form the reversible $CA$. This effectively enables synthesis of such a $CA$ in linear time.

The paper is organized as follows. The following section provides the preliminaries of cellular automata. Section 3 introduces the concept of reachability tree. Whereas Section 4 identifies the reversible $CA$, and Section 5 synthesizes reversible $CA$ utilizing the structure of reachability tree. The rules that take part in formation of reversible $CA$ are identified in Section 6. The classification of such rules for efficient synthesis of reversible $CA$ are reported in Section 7.

## 2 Preliminaries of Cellular Automata

A Cellular Automaton ($CA$) consists of a number of cells organized in the form of a lattice. It evolves in discrete space and time. Each cell of a $CA$ stores a discrete variable at time $t$ that refers to the present state of the cell. The next state of the cell at $(t+1)$ is affected by its state and the states of its *neighbors* at time $t$. In this work, we concentrate on such 3-neighborhood (self, left and right neighbors) $CA$, where a $CA$ cell is having two states - 0 or 1. Therefore, the next state $S_i^{t+1}$ of the $i^{th}$ $CA$ cell is specified by the *next state function* $f_i$ as

$$S_i^{t+1} = f_i(S_{i-1}^t, S_i^t, S_{i+1}^t) \tag{1}$$

$S_{i-1}^t$, $S_i^t$ and $S_{i+1}^t$ are the present states of left neighbor, self and right neighbor of the $i^{th}$ $CA$ cell at time $t$.

The collection of states of the cells $\mathcal{S}^t = (S_1^t, S_2^t, \cdots, S_n^t)$ at time $t$ is the present state of a $CA$. Therefore, the next state of an $n-$cell $CA$ is determined as

$$\mathcal{S}^{t+1} = (f_1(S_0^t, S_1^t, S_2^t), f_2(S_1^t, S_2^t, S_3^t), \cdots, f_n(S_{n-1}^t, S_n^t, S_{n+1}^t)) \tag{2}$$

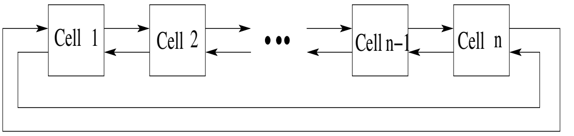



Fig. 1. Block diagram of an $n-$cell periodic boundary $CA$

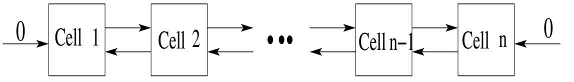



Fig. 2. Block diagram of an $n-$cell null boundary $CA$

Table 1
Truth table for rule 90, 150 and 75

| Present state : | 111 | 110 | 101 | 100 | 011 | 010 | 001 | 000 | *Rule* |
|---|---|---|---|---|---|---|---|---|---|
| $(RMT)$ | (7) | (6) | (5) | (4) | (3) | (2) | (1) | (0) | |
| (i) Next State : | 0 | 1 | 0 | 1 | 1 | 0 | 1 | 0 | 90 |
| (ii) Next State : | 1 | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 150 |
| (iii) Next State : | 0 | 1 | 0 | 0 | 1 | 0 | 1 | 1 | 75 |

Note: $RMT$ stands for *Rule Min Term*. The value 0/1 noted on $3^{rd}/4^{th}/5^{th}$ row shows the output of the three variable switching function.

If $S_0^t = S_n^t$ and $S_{n+1}^t = S_1^t$ (that is, left neighbor of the left most cell is the right most cell and vice versa), then the $CA$ is referred to as *periodic boundary* $CA$ (Fig. 1). On the other hand, if $S_0^t = 0$ (null) and $S_{n+1}^t = 0$ (null), the $CA$ is *null boundary* (Fig. 2). In this work, we concentrate on the characterization of periodic boundary $CA$.

If the next state function of the $i^{th}$ cell is expressed in the form of a truth table, then the decimal equivalent of its output is conventionally referred to as the 'Rule' $\mathcal{R}_i$ [16]. In a two-state 3-neighborhood $CA$, there can be a total of $2^8$ (256) rules. Three such rules 90, 150, and 75 are illustrated in Table 1. The first row of the table lists the possible $2^3$ (8) combinations of the present states of $(i-1)^{th}$, $i^{th}$ and $(i+1)^{th}$ cells at time $t$. The last three rows indicate the next states of the $i^{th}$ cell at $(t+1)$ for the rules, 90, 150 and 75 respectively.

**Definition 2.1** The set of rules $\mathcal{R} =< \mathcal{R}_1, \mathcal{R}_2, \cdots, \mathcal{R}_i, \cdots, \mathcal{R}_n >$ that configure the cells of a $CA$ is called the **rule vector**.

**Definition 2.2** A $CA$ is **uniform** if $\mathcal{R}_1 = \mathcal{R}_2 = \cdots = \mathcal{R}_n$; otherwise the $CA$ is **hybrid/non-uniform**.

**Rule Min Term ($RMT$):** From the view point of *Switching Theory*, a combination of the present states (as noted in the $1^{st}$ row of Table 1) can be viewed as the *Min Term* of a 3-variable $(S_{i-1}^t, S_i^t, S_{i+1}^t)$ switching function. Therefore, each column of the first row of Table 1 is referred to as **Rule Min Term ($RMT$)**. The column 011 in the truth table (Table 1) is the $3^{rd}$ $RMT$. The next states corresponding

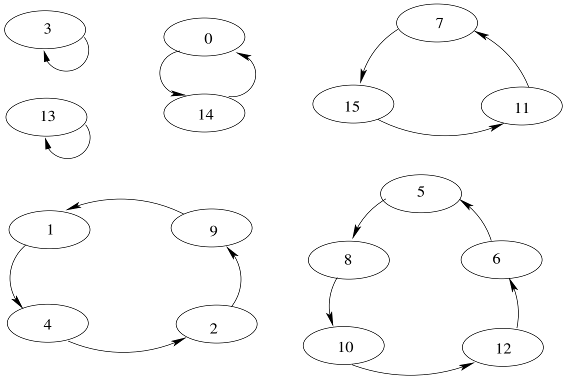



Fig. 3. State transitions of a reversible $CA < 202, 195, 105, 165 >$

to this $RMT$ are 1 for Rule 90 and 75, and 0 for Rule 150. The characterization reported in the following section is based on the analysis of $RMT$s of the $CA$ rules.

The next state functions $f_i$s for the rules 90 and 150 employ $XOR$ logic. These rules are called linear rules. On the other hand, rule 75 is a non-linear one. Out of total 256 rules, 14 employ only $XOR/XNOR$ logic. These are referred to as linear/additive rules. Other rules employ nonlinear logic functions ($AND$, $OR$, etc.).

**Definition 2.3** Whenever all the $\mathcal{R}_i$s $(i = 1, 2, \cdots, n)$ of a rule vector $\mathcal{R}$ are linear/additive, the $CA$ is referred to as **Linear/Additive** $CA$, otherwise the $CA$ is a **Nonlinear** one.

**Definition 2.4** A rule is **Balanced** if it contains equal number of 1s and 0s in its $8-$bit binary representation; otherwise it is an **Unbalanced** rule.

The rules shown in Table 1 are the balanced rules. On the other hand, rule 171 with five 1s in its 8-bit representation (10101011) is an unbalanced rule.

The sequence of states generated (state transitions) during its evolution with time directs the $CA$ behavior (Fig. 3 and Fig. 4). The state transition diagram of a $CA$ may contain *cyclic* and *non-cyclic* states (a state is called *cyclic* if it lies in a cycle) and based on this, the $CA$ can be categorized as either *reversible* or *irreversible* $CA$.

**Definition 2.5** A $CA$ is **reversible** if it contains only cyclic states in its state transition diagram; otherwise the $CA$ is **irreversible**.

In a reversible $CA$, the initial $CA$ state repeats after certain number of time steps (Fig. 3). Therefore, all the states of a reversible $CA$ are reachable from some other states and each state has exactly one predecessor. On the other hand, in an irreversible $CA$ (Fig. 4), there are some states that can not be reachable (*non-reachable* states) from any other state. Moreover, some states of such a $CA$ are having more than one predecessor [11,12]. For example, the states marked as 2, 4, 5, 10 and 15 of Fig. 4 are the non-reachable states, whereas 3, 6, 11 and 14 have more than one predecessor.

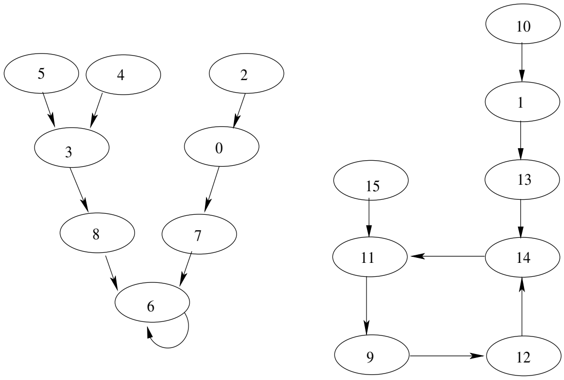



Fig. 4. State transitions of an irreversible $CA < 165, 171, 75, 202 >$

## 3 Reachability tree

Reachability Tree, we proposed in [5,6,7], is a binary tree that represents the reachable states of a $CA$. Each node of the tree is constructed with $RMT$(s) of a rule (Section 2). The left edge of a node is referred to as the 0-edge and the right edge is as 1-edge (Fig. 5). The number of levels in a reachability tree, for an $n-$cell $CA$, is $(n+1)$. The root node is at Level 0 and the leaf nodes are at Level $n$. The nodes at Level $i$ are constructed from the $RMT$s of $(i+1)^{th}$ $CA$ cell rule $\mathcal{R}_{i+1}$. The number of leaf nodes in the reachability tree denotes the number of reachable states of the $CA$. A sequence of edges from the root to a leaf node, representing an $n-$bit binary string, is the reachable state, the 0-edge represents 0 and 1-edge represents 1.

The $RMT$s of any two consecutive cell rules $\mathcal{R}_i$ and $\mathcal{R}_{i+1}$ influence the formation of next state of a $CA$. Since the $RMT$s are of 3-bit, a three bit window can be considered to get next state of the $CA$ [6]. For example, a window with value (101) at $i^{th}$ cell corresponds to $RMT$ 5 of $\mathcal{R}_i$. If the window for $i^{th}$ cell is $(b_{i-1}b_ib_{i+1})$, $b_i = 0/1$, then the window for $(i+1)^{th}$ cell is either $(b_ib_{i+1}0)$ or $(b_ib_{i+1}1)$. In other words, if the $i^{th}$ $CA$ cell changes its state following the $RMT$ $k$ (decimal equivalent of $b_{i-1}b_ib_{i+1}$) of rule $\mathcal{R}_i$, then the $(i+1)^{th}$ cell can generate the next state based on the $RMT$ $2k$ $mod$ 8 $(b_ib_{i+1}0)$ or $(2k+1)$ $mod$ 8 $(b_ib_{i+1}1)$ of rule $\mathcal{R}_{i+1}$. All such relationships between the $RMT$s of $\mathcal{R}_i$ and $\mathcal{R}_{i+1}$, while computing next state of a $CA$, is shown in Table 2. The relations, noted in the table, play an important role in characterizing the $CA$ behavior configured with different cell rules.

**Definition 3.1** Two $RMT$s are **sibling** at level $i+1$ if these are resulted in from the same $RMT$s at level $i$ of the Reachability Tree.

The $RMT$s 0 and 1 are the sibling $RMT$s as these two are resulted in either from $RMT$ 0 or from $RMT$ 4 (Table 2), and these sibling $RMT$s are associated with a single node. Therefore, if a node of Reachability Tree associates an $RMT$ $k$, it also associates the sibling of $k$.

The reachability tree for a $CA < 165, 171, 75, 202 >$ is shown in Fig. 5. The $RMT$s of $CA$ rules are noted in Table 3. The decimal numbers within a node of Fig. 5, at level $i$, represent the $RMT$s of the $CA$ cell rule $\mathcal{R}_{i+1}$ following which the

Table 2
Relationship between $RMT$s of cell $i$ and cell $(i+1)$ for next state computation

| $RMT$ at $i^{th}$ rule | $RMT$s at $(i+1)^{th}$ rule |
|---|---|
| 0 | 0, 1 |
| 1 | 2, 3 |
| 2 | 4, 5 |
| 3 | 6, 7 |
| 4 | 0, 1 |
| 5 | 2, 3 |
| 6 | 4, 5 |
| 7 | 6, 7 |

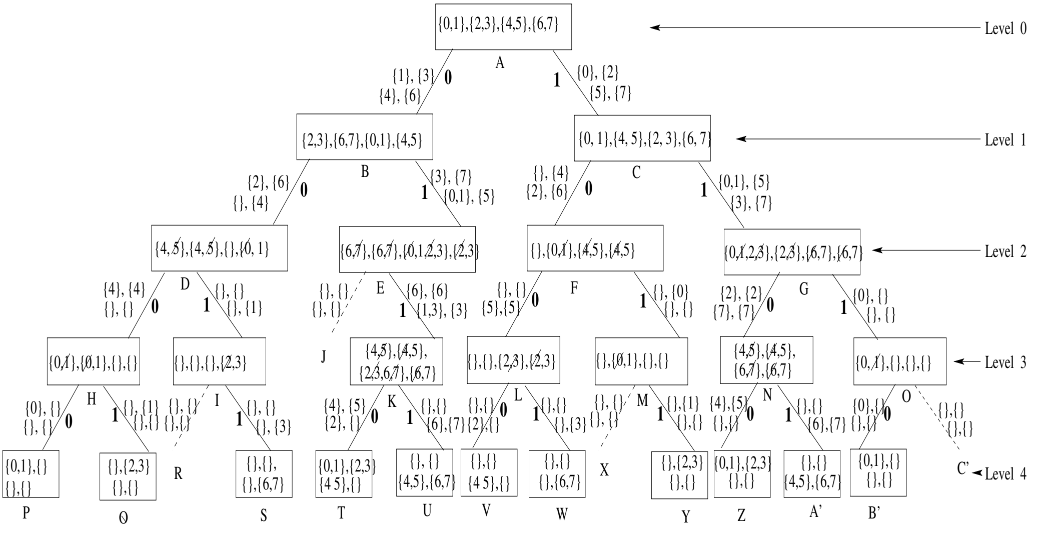


Fig. 5. Reachability Tree for the $CA < 165, 171, 75, 202 >$

cell $(i+1)$ can change its state. For example, the root node (level 0) is constructed with all the 8 $RMT$s – 0, 1, 2, 3, 4, 5, 6 and 7. The sibling $RMT$s (Definition 3.1) of root are grouped to form the sets – {0, 1}, {2, 3}, {4, 5} and {6, 7}. The $RMT$s (of a rule) for which we follow an edge (0-edge or 1-edge) are noted at the edge.

An $RMT$ of a rule is a member of $i^{th}$ set, implies that the $RMT$ is derived from set $i$ of the root ($0 \leq i \leq 3$ and set 0 is {0, 1}, set 1 is {2, 3}, set 2 is {4, 5} and {6, 7} is the set 3). Such grouping facilitates the characterization of periodic boundary $CA$.

For the $RMT$s 1 (set 0), 3 (set 1), 4 (set 2) and 6 (set 3) of 165 (Table 3), the

Table 3
Binary values of the $CA < 105, 128, 171, 65 >$ cell rules

| RMT | 111 (7) | 110 (6) | 101 (5) | 100 (4) | 011 (3) | 010 (2) | 001 (1) | 000 (0) | Rule |
|---|---|---|---|---|---|---|---|---|---|
| *First cell* | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 1 | 165 |
| *Second cell* | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 1 | 171 |
| *Third cell* | 0 | 1 | 0 | 0 | 1 | 0 | 1 | 1 | 75 |
| *Fourth cell* | 1 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 202 |

next states are 0 and it is 1 for the $RMT$s 0 (set 0), 2 (set 1), 5 (set 2) and 7 (set 3). Therefore, at level 1, the node following 0-edge of level 0 (node B of Fig. 5) contains the $RMT$s {2, 3}, {6, 7}, {0, 1} and {4, 5} (Table 2). Similarly, the node C following 1-edge of level 0, contains the $RMT$s {0, 1}, {4, 5}, {2, 3} and {6, 7}.

The dotted edge from a node (node E, node I, $\cdots$) indicates that the node does not have the corresponding edge (0-edge). The dotted 0-edge at node E signifies that any state started with 010 is non-reachable.

A number of $RMT$s are dropped from the nodes at level $(n-2)$ (level 2 of Fig. 5) and level $(n-1)$ – that is, level 3 of Fig. 5. The $RMT$s of the nodes at level $(n-2)$ correspond to the $CA$ cell rule $\mathcal{R}_{n-1}$. The $RMT$s of set 0 and set 1 assume that the cell $n$ is always 0 while we computing the next state, whereas the $RMT$s of set 2 and set 3 assume that the cell $n$ is always 1. Therefore, odd $RMT$s of set 0 and set 1, and even $RMT$s of set 2 and set 3 are invalid, and so striked out. Similarly, the $RMT$s of the nodes at level $(n-1)$ correspond to the $CA$ cell rule $\mathcal{R}_n$. Therefore, the $RMT$s of set 0 for $\mathcal{R}_n$ (at level $(n-1)$) should generate the $RMT$s of set 0 for $\mathcal{R}_1$, since next to the last cell is the first cell. However, few $RMT$s of set 0 at level $(n-1)$ may not generate set 0 for $\mathcal{R}_1$, these are marked as invalid, and striked out. Similar action is taken for other sets. In node H (*Figure 5*), $RMT$ 1 of set 0 is striked out as it can not generate set 0 for $\mathcal{R}_1$ ({0, 1}). Similarly, $RMT$ 0 of set 1 is striked out.

The reachability tree was defined originally to characterize null boundary $CA$ [5,6,7]. However, the structure shown in Fig. 5 can model a periodic boundary $CA$ considered in this work. This redefined structure of the reachability tree provides means to characterize the reversibility periodic boundary $CA$.

## 4 Identification of reversible CA

This section reports the theoretical background for identification of a reversible periodic boundary $CA$. The concept of reachability tree, introduced in earlier section, is utilized to develop the theoretical framework.

**Theorem 4.1** *The reachability tree of a reversible $CA$ is balanced.*

**Proof.** Since all states of a reversible $CA$ are reachable, the number of leaf nodes in the reachability tree, for the $n-$cell reversible $CA$, is $2^n$ (number of states). Therefore, the tree is balanced as it is a binary tree of $(n + 1)$ levels. □

The above theorem points to the fact that the identification of a reversible $CA$ can be done by constructing the reachability tree of the $CA$. If there is no non-reachable state in a tree, the $CA$ is reversible.

**Theorem 4.2** *The reachability tree of a 3-neighborhood periodic boundary $CA$ is balanced iff:*
*(i) each leaf edge is resulted from single $RMT$,*
*(ii) each immediate predecessor (edge) of the leaf edge is resulted from two $RMT$s, and*
*(iii) all other edges are resulted from 4 $RMT$s.*

**Proof.** Let consider, the number of $RMT$s, dictates an edge, is lesser than that is mentioned in $(i)$ to $(iii)$. That is,
$i$. there is no $RMT$ to dictate a leaf edge. It implies the tree is unbalanced.
$ii$. an edge predecessor of the leaf edge is resulted from single $RMT$. Therefore, the next node to that edge is constructed with 2 $RMT$s. Since the node is at level $(n - 1)$, one $RMT$ must be striked out since it is sibling of the other and both are the member of a single set. This single valid $RMT$ can generate single edge. Hence the other is empty. Therefore, the tree is unbalanced.
$iii$. Say, an intermediate edge is resulted from 3 $RMT$s. Therefore, the next node (N) to that edge is constructed with 6 $RMT$s. At the next level, the node N may have two edges that may be resulted from 1 and 5, 2 and 4, or 3 and 3 $RMT$s. In best case, the tree may remain balanced up to level $(n - 2)$. In level $(n - 2)$, one can find at least one node that comes from the edge which is resulted from 1 or 2 or 3 $RMT$s. Therefore, the node is constructed from 2, 4 or 6 $RMT$s. Since the node is at level $(n - 2)$, half of the $RMT$s are invalid. Hence, effectively the node is constructed with 1, 2 or 3 $RMT$s. If the number of $RMT$ is one, then obviously the tree is unbalanced. Otherwise, at least one edge of that node is resulted from single $RMT$, which results in an unbalanced tree $(ii)$.

If an intermediate edge, on the other hand, is resulted from more than 4 $RMT$s, then an edge can be found which is resulted from less than 4 $RMT$s. This implies, the tree is unbalanced $(iii)$. Now if a predecessor edge of the leaf edge is resulted from more than two $RMT$s, then the other edge is resulted from less than two $RMT$s, as the half of the $RMT$s that construct nodes at level $(n - 1)$ are striked out. Hence the tree is unbalanced (by $ii$). Similarly, if a leaf edge is resulted from more than 1 $RMT$, then the tree is also unbalanced. Therefore, if the number of $RMT$s, that dictate an edge, is not same as it is mentioned in $(i)$ to $(iii)$, the reachability tree is unbalanced. Hence the proof. □

**Corollary 4.3** *The nodes of a reachability tree corresponding to 3-neighborhood reversible periodic boundary $CA$ are constructed with*
*(i) 2 $RMT$s if the nodes are leaf nodes,*
*(ii) 4 $RMT$s if the nodes are immediate predecessors of the leaves, and*

Table 4
Binary values of the $CA < 202, 195, 105, 165 >$ cell rules

| RMT | 111 (7) | 110 (6) | 101 (5) | 100 (4) | 011 (3) | 010 (2) | 001 (1) | 000 (0) | Rule |
|---|---|---|---|---|---|---|---|---|---|
| *First cell* | 1 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 202 |
| *Second cell* | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 195 |
| *Third cell* | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 1 | 105 |
| *Fourth cell* | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 1 | 165 |

*(iii) 8 $RMT$s for all other nodes;*
*where the $RMT$s of a node may not be unique.*

**Proof.** According to Theorem 4.2, for a reversible $CA$
(i) leaf edges are resulted from single $RMT$,
(ii) the predecessors of the leaf edge are resulted from 2 $RMT$s, and
(iii) all other edges are resulted from 4 $RMT$s.

Hence, (i) the leaves are constructed with 2 $RMT$s, (ii) the immediate predecessors of the leaves are constructed with 4 $RMT$s, and (iii) all the other nodes are resulted from 8 $RMT$s. □

**Corollary 4.4** *The nodes of the reachability tree of a reversible $CA$ are balanced over their $RMT$s.*

**Proof.** Since the reachability tree for a reversible $CA$ is balanced, each node has a left child (0-edge) and a right child (1-edge). It is obvious from Theorem 4.2, if a child is resulted from $k$ number of $RMT$s, another child is also resulted from $k$ $RMT$s. Hence the node is balanced over its $RMT$s. □

**Example 4.5** Consider the 4-cell $CA$ $\mathcal{R} =< 202, 195, 105, 165 >$. Figure 6 is the reachability tree of the $\mathcal{R}$. The $CA$ is reversible. Figure 6 is balanced. Each node of the tree is also balanced over its $RMT$s.

Based on the above discussion, we next propose a method to identify a reversible $CA$. The following algorithm (*CheckReversible*) scans a $CA$ rule vector from left to right, and virtually constructs a reachability tree. It then verifies whether Theorem 4.2 is satisfied for the given $CA$ rule vector – that is, whether the edges are resulted from 4 $RMT$s (or in some special cases, 2 or 1 $RMT$) or not. If such an edge is found, it is an irreversible $CA$. The algorithm replaces the duplicate nodes having same $RMT$ sets, if any.

**Algorithm 1 CheckReversible**
**Input:** $< \mathcal{R}_1, \mathcal{R}_2, \cdots, \mathcal{R}_i, \cdots, \mathcal{R}_n >$ *(an n-cell $CA$)*
**Output:** *reversible or irreversible.*
Step 1: *Let $S_0 = \{s_0^0, s_0^1, s_0^2, s_0^3\}$ and $S_1 = \{s_1^0, s_1^1, s_1^2, s_1^3\}$ be the two sets of $RMT$ sets where (i) the $RMT$s of $S_0$ and $S_1$ are 0 and 1 respectively for $\mathcal{R}_1$, and (ii) the*

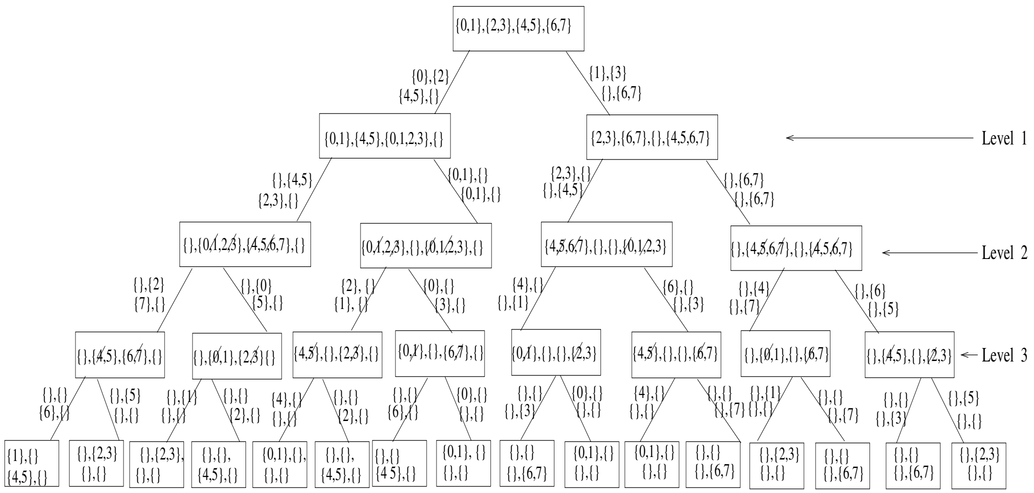



Fig. 6. Reachability tree of a reversible $CA < 202, 195, 105, 165 >$

*members of* $s_j^k$ *are either* $\phi$, *RMT* $2k$, *RMT* $(2k+1)$ *or RMTs* $2k$ & $(2k+1)$, $j = 0/1$ *and* $k = 0, 1, 2,$ *or 3.*

*If* $|s_0^0| + |s_0^1| + |s_0^2| + |s_0^3| \neq |s_1^0| + |s_1^1| + |s_1^2| + |s_1^3|$, *report the CA as irreversible and return.*

Step 2: *For* $i = 2$ *to* $n$ *repeat* Step 3 *to* Step 6

Step 3: *For each set of RMT sets,*

*find the RMTs that construct next level nodes of the reachability tree considering Table 2.*

Step 4: *(i) If* $i = n - 1$, *remove the odd RMTs from* $s_j^0$ *and* $s_j^1$, *and even RMTs from* $s_j^2$ *and* $s_j^3$.

*(ii) If* $i = n$, *remove those RMTs from* $s_j^k$ *that can generate RMTs* $2k$, $(2k+1)$ *for* $\mathcal{R}_1$.

Step 5: *Remove the duplicate nodes having same RMT sets, if any.*

Step 6: *Distribute the RMTs of each node into two sets* $S_0'$ *and* $S_1'$ *based on* $\mathcal{R}_i$, *where* $S_0' = \{s_0^{0'}, s_0^{1'}, s_0^{2'}, s_0^{3'}\}$ *and* $S_1' = \{s_1^{0'}, s_1^{1'}, s_1^{2'}, s_1^{3'}\}$, $s_j^{k'}$ *is derived from* $s_j^k$ *and the RMTs of* $S_0'$ *and* $S_1'$ *are 0 and 1 respectively.*

*If* $|s_0^{0'}| + |s_0^{1'}| + |s_0^{2'}| + |s_0^{3'}| \neq |s_1^{0'}| + |s_1^{1'}| + |s_1^{2'}| + |s_1^{3'}|$, *report the CA as irreversible and return.*

Step 7: *Report the CA is a reversible CA and return.*

**Complexity:** Since the maximum number of possible $RMT$ sets is fixed, the execution time of Algorithm 1 depends on $n$ only (step 2). Therefore, the complexity of Algorithm 1 is $O(n)$.

**Example 4.6** This example illustrates the execution steps of Algorithm 1. Consider the 4-cell $CA < 202, 195, 105, 165 >$ as input to Algorithm 1. For rule 202 ($\mathcal{R}_1$), $s_0^0 = \{0\}$, $s_0^1 = \{2\}$, $s_0^2 = \{4, 5\}$ & $s_0^3 = \{\}$, and $s_1^0 = \{1\}$, $s_1^1 = \{3\}$, $s_1^2 = \{\}$ & $s_1^3 = \{6, 7\}$ (*Table 4*). Here, $S_0 = \{\{0\}, \{2\}, \{4, 5\}, \{\}\}$ and $S_1 = \{\{1\}, \{3\}, \{\}, \{6, 7\}\}$. Hence, $|s_0^0| + |s_0^1| + |s_0^2| + |s_0^3| = |s_1^0| + |s_1^1| + |s_1^2| + |s_1^3|$.

As next step, for both the sets $S_0$ and $S_1$, the algorithm finds the $RMT$s for nodes at level 1 (step 3). The nodes are {{0, 1}, {4, 5}, {0, 1, 2, 3}, {}} and {{2, 3}, {6, 7}, {}, {4, 5, 6, 7}} (level 1 of Fig. 6). For each node, $S'_0$ and $S'_1$ are constructed (step 6). Since the number of $RMT$s in $S'_0$ & $S'_1$ are same (4) for both the nodes, no conclusion can be drawn. In the next step, we get 4 nodes as shown in level 2 of Fig. 6. The nodes correspond to $\mathcal{R}_{n-1} = 105$. So half of the $RMT$s of each node are striked out (step 4 ($i$)). Based on rule 105, we get the nodes of level 3. A number of $RMT$s are then removed (step 4 ($ii$)). However using rule 165, the leaf nodes are obtained. It is, therefore, declared in step 7 that the $CA$ is reversible.

## 5 Synthesis of reversible $CA$

Synthesis of reversible $CA$ is the reverse process of analysis/ identification reported in the earlier section. The algorithm *SynthesizeReversibleCA_1* proposes an efficient synthesis scheme. Input to the algorithm is $n$, the number of cells/ size of the $CA$ to be synthesized, and the output is an $n$-cell reversible $CA$. It determines the $(i+1)^{th}$ cell rule of the reversible $CA$ through inspection of $RMT$s of the rule $\mathcal{R}_i$, selected for the $CA$ cell. The $RMT$s are set in such a way that each edge of the reachability tree is resulted from four $RMT$s (in some special cases, two or one $RMT$) as guided by Theorem 4.2.

**Algorithm 2 SynthesizeReversibleCA_1**
**Input:** *$n$ (the number of $CA$ cells).*
**Output:** $< \mathcal{R}_1, \mathcal{R}_2, \cdots, \mathcal{R}_i, \cdots, \mathcal{R}_n >$ *(an $n-$cell reversible $CA$).*
Step 1: *Select a balanced rule as $\mathcal{R}_1$. Let $S_0 = \{s_0^0, s_0^1, s_0^2, s_0^3\}$ and $S_1 = \{s_1^0, s_1^1, s_1^2, s_1^3\}$ be the two sets of RMT sets, where*
*(i) the $RMT$s of $S_0$ and $S_1$ are 0 and 1 respectively for $\mathcal{R}_1$, and*
*(ii) the members of $s_j^k$ are either $\phi$, $RMT$ $2k$, $RMT$ $(2k+1)$, or $RMT$s $2k$ & $(2k+1)$, for $j = 0/1$ and $k = 0, 1, 2,$ or 3.*
Step 2: *For $i = 2$ to $n$ repeat* Step 3 *to* Step 6
Step 3: *For each set of $RMT$ sets*
*find the $RMT$s that construct next level nodes of the reachability tree, considering Table 2.*
Step 4: *(i) If $i = n - 1$, remove the odd $RMT$s from $s_j^0$ and $s_j^1$, and even $RMT$s from $s_j^2$ and $s_j^3$.*
*(ii) If $i = n$, remove those $RMT$s from $s_j^k$ that can generate $RMT$s $2k$, $(2k+1)$ for $\mathcal{R}_1$.*
Step 5: *Remove the duplicate nodes, if any.*
Step 6: *Find a rule $\mathcal{R}_i$ which partitions the $RMT$s of each node into two sets $S'_0$ and $S'_1$, where $S'_0 = \{s_0^{0'}, s_0^{1'}, s_0^{2'}, s_0^{3'}\}$ & $S'_1 = \{s_1^{0'}, s_1^{1'}, s_1^{2'}, s_1^{3'}\}$, and $|s_0^{0'}| + |s_0^{1'}| + |s_0^{2'}| + |s_0^{3'}| = |s_1^{0'}| + |s_1^{1'}| + |s_1^{2'}| + |s_1^{3'}|$; $s_j^{k'}$ is derived from $s_j^k$ and the $RMT$s of $S'_0$ and $S'_1$ are 0 and 1 respectively.*
Step 7: *Report $< \mathcal{R}_1, \mathcal{R}_2, \cdots, \mathcal{R}_i, \cdots, \mathcal{R}_n >$ as an $n-$cell reversible $CA$.*

**Complexity:** Since the maximum number of possible $RMT$ sets is fixed, the execution time of Algorithm 2 depends on $n$ only (step 2). Therefore the complexity of Algorithm 2 is $O(n)$.

From Theorem 4.2 and Algorithm 1, it can be observed that each rule of a $CA$ plays an important role in determining the reversible behavior of a $CA$.

**Definition 5.1** [6] A rule is a **irreversible rule** if its presence in a rule vector makes the $CA$ irreversible. Otherwise, it is a **reversible rule**.

**Example 5.2** The 4−cell $CA$ with rule vector $< 202, 195, 105, 165 >$ is a reversible $CA$. Therefore, all of the four rules are reversible rules. On the other hand, a $CA$ with rule vector $< 202, 196, 105, 165 >$ is an irreversible $CA$. The rule 196 makes the $CA$ irreversible. That is, 196 is an irreversible rule. 196 (11000100) is an unbalanced one. The number of 0s in the $RMT$s of 196 is 5.

Characterization of reversible rules also enables efficient synthesis of a reversible $CA$ of an arbitrary length. The following section reports such characterization. The reversible $CA$ synthesis scheme based on this characterization is introduced in Section 7.

## 6 Reversible rules

The reversible rules are considered to be the basic building blocks while synthesizing a reversible $CA$. This section explores the properties of reversible rules in 3-neighborhood dependency.

**Theorem 6.1** *An unbalanced rule is an irreversible rule.*

**Proof.** Let us consider a $CA$ $\mathcal{R} =< \mathcal{R}_1, \mathcal{R}_2, \cdots, \mathcal{R}_i, \cdots, \mathcal{R}_n >$, where $\mathcal{R}_i$ is an unbalanced rule and $\mathcal{R}'' =< \mathcal{R}_1, \mathcal{R}_2, \cdots, \mathcal{R}_i'', \cdots, \mathcal{R}_n >$ is a reversible $CA$. All the rules of $\mathcal{R}$ and $\mathcal{R}''$ are same except the $i^{th}$ rule.

The reachability tree of $\mathcal{R}$ is balanced up to $(i-1)^{th}$ level as $\mathcal{R}''$ is reversible $CA$ and having the same rules in $\mathcal{R}$ for 1 to $(i-1)^{th}$ cells.
Case 1: $i < n-1$: Since $\mathcal{R}_i$ is unbalanced, there exists at least one node at $(i-1)^{th}$ level with a child that is not resulted from exactly 4 $RMT$s.
Case 2: $i = n-1$: There exists at least one node at $(n-2)^{th}$ with a child which is resulted from 1 (or 3) $RMT$s.
Case 3: $i = n$: Since $\mathcal{R}_n$ is unbalanced, a few leaf edges are not resulted from exactly single $RMT$.
The above discussion implies that the tree is unbalanced (Theorem 4.2). Therefore, the $CA$ with rule vector $\mathcal{R}$ is irreversible. Hence the proof. □

There are ${}^8C_4 = 70$ balanced $CA$ rules in 3-neighborhood dependency. All the balanced rules are reversible rules in case of 3-neighborhood periodic boundary $CA$. These are listed in Table 5. The reversible rules can only form the reversible $CA$. However, any sequence of reversible rules in a $CA$ rule vector does not necessarily imply that the resulted $CA$ is a reversible $CA$.

**Theorem 6.2** *Only the specific sequence of reversible rules forms a reversible $CA$.*

Table 5
List of reversible rules in periodic boundary 3-neighborhood dependency

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 15 | 23 | 27 | 29 | 30 | 39 | 43 | 45 | 46 | 51 | 53 | 54 |
| 57 | 58 | 60 | 71 | 75 | 77 | 78 | 83 | 85 | 86 | 89 | 90 |
| 92 | 99 | 101 | 102 | 105 | 106 | 108 | 113 | 114 | 116 | 120 | 135 |
| 139 | 141 | 142 | 147 | 149 | 150 | 153 | 154 | 156 | 163 | 165 | 166 |
| 169 | 170 | 172 | 177 | 178 | 180 | 184 | 195 | 197 | 198 | 201 | 202 |
| 204 | 209 | 210 | 212 | 216 | 225 | 226 | 228 | 232 | 240 | | |

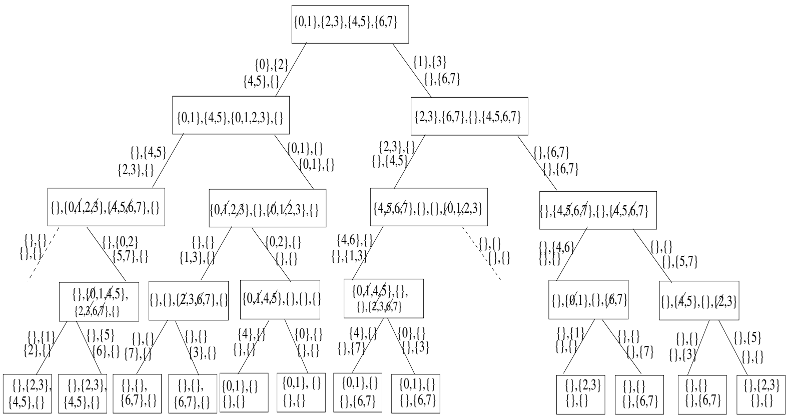


Fig. 7. Reachability tree for an irreversible $CA < 202, 195, 165, 105 >$ designed with reversible rules

**Proof.** Let us consider the reversible rules of an $n$-cell $CA$ are configured in such a way that the $CA$ loaded with any seed produces two types of states – $\{\cdots d_i d_{i+1} \cdots\}$ and $\{\cdots d'_i d'_{i+1} \cdots\}$, where $d_i (= 0/1)$ is the state of $i^{th}$ cell while $d'_i$ is its complement. Therefore, for $2^n$ number of current states, the next states are $S = \{\cdots d_i d_{i+1} \cdots, \cdots d'_i d'_{i+1} \cdots\}$. The maximum possible cardinality of $S$ is $2 \times 2^{n-2} = 2^{n-1}$. Since the number of next states is lesser than that of current states, there exists at least a state in $S$ with more than one predecessor. Therefore, the $CA$ is irreversible. Hence any sequence of reversible rules can't form reversible $CA$. □

**Example 6.3** The $CA < 202, 195, 105, 165 >$ is reversible (Example 4.6). However, the $CA$ $\mathcal{R} =< 202, 195, 165, 105 >$ is an irreversible $CA$ even though each of the rules in $\mathcal{R}$ is a reversible rule (Table 5). The reachability tree for $\mathcal{R}$ is shown in Fig. 7.

Theorem 6.2 directs that the reversible rules are interrelated. The sequence of reversible rules that form a reversible $CA$ follows a specific relation. The next section reports classification of 70 reversible rules based on the relation that must be followed to form a rule sequence for the reversible $CA$.

Table 6
Combinations of $RMT$s for nodes

| | | | | |
|---|---|---|---|---|
| [0, 0, 0, 0] | [0, 0, 0, 2] | [0, 0, 0, 4] | [0, 0, 0, 6] | [0, 0, 2, 2] |
| [0, 0, 2, 4] | [0, 0, 2, 6] | [0, 0, 4, 4] | [0, 0, 4, 6] | [0, 0, 6, 6] |
| [0, 2, 2, 2] | [0, 2, 2, 4] | [0, 2, 2, 6] | [0, 2, 4, 4] | [0, 2, 4, 6] |
| [0, 2, 6, 6] | [0, 4, 4, 4] | [0, 4, 4, 6] | [0, 4, 6, 6] | [0, 6, 6, 6] |
| [2, 2, 2, 2] | [2, 2, 2, 4] | [2, 2, 2, 6] | [2, 2, 4, 4] | [2, 2, 4, 6] |
| [2, 2, 6, 6] | [2, 4, 4, 4] | [2, 4, 4, 6] | [2, 4, 6, 6] | [2, 6, 6, 6] |
| [4, 4, 4, 4] | [4, 4, 4, 6] | [4, 4, 6, 6] | [4, 6, 6, 6] | [6, 6, 6, 6] |

# 7 Classification of reversible rules

It is reported in the earlier section that there are some specific relations among the reversible rules and that should be considered while synthesizing a reversible $CA$. This section identifies the relations and reports classification of reversible rules to find the desired sequence of rules for a reversible $CA$ rule vector. To facilitate the classification, the next subsection reports the characterization of reachability tree for reversible $CA$, whereas the classification of rules is presented in Section 7.2. Section 7.3 establishes the relationship among the rules of different classes.

## *7.1 Reachability tree for reversible CA*

Since the reachability tree for a reversible $CA$ is balanced (Theorem 4.1), there are $2^i$ nodes at level $i$ of the tree. However, all the nodes may not be unique. Two or more similar nodes at a level produce the same subtree. Moreover, since the nodes of level $i$ ($i < n-1$) are constructed with 8 $RMT$s (Corollary 4.3), the combinations of $RMT$s can be found same for a number of nodes at that level. The following lemma fixes the number of unique $RMT$ combinations that construct the nodes.

**Lemma 7.1** *Maximum possible unique RMT combinations at level i nodes of a reachability tree for n-cell reversible $CA$ is 35, where $i < n - 1$.*

**Proof.** Since the sibling $RMT$s (Definition 3.1) are associated with the same node in the reachability tree and there are 4 sets of sibling $RMT$s (0 & 1, 2 & 3, 4 & 5, and 6 & 7), $4^4$ combinations of $RMT$s are possible for the nodes. However, out of these a number of combinations are the same. For example, the $RMT$ combinations {2, 3, 4, 5, 0, 1, 6, 7} and {0, 1, 2, 3, 4, 5, 6, 7} are similar. Therefore, unique combinations of $RMT$s are lesser than $4^4$ combinations. The unique combinations of $RMT$s are noted in Table 6. Only even $RMT$s are recorded in the table and odd $RMT$ corresponding to each even $RMT$ is omitted for simplification. Number of combinations found in Table 6 is 35. □

The nodes of reachability tree for a reversible $CA$ are balanced over their $RMT$s (Corollary 4.4). Now if a node is constructed with a set of single sibling $RMT$s (for example, [0, 0, 0, 0], [2, 2, 2, 2], etc.), then that particular sibling $RMT$ is to be balanced (balanced over [0, 1], [2, 3], etc). Similarly, if a node, constructed with 4 sibling $RMT$s (that is, 8 $RMT$s), consists of 3 same sibling $RMT$s (for example, [2, 2, 2, 4], [4, 4, 4, 6], etc.), then the sibling $RMT$s are balanced (balanced over [2, 3] & [4, 5], [4, 5] & [6, 7], etc). Table 7 shows the $RMT$ groupings that are balanced over {0, 1} while a particular $RMT$ combination for a node is given. It is to be noted that in Column 2 only even $RMT$ of each sibling $RMT$ pair is reported for simplicity.

Table 7
Combinations of $RMT$s for nodes

| Node # | $RMT$ combinations | Grouping of balanced $RMT$s | Node # | $RMT$ combinations | Grouping of balanced $RMT$s |
|---|---|---|---|---|---|
| 1 | [0, 0, 0, 0] | [0, 1] | 19 | [0, 4, 6, 6] | [0, 1, 4, 5, 6, 6, 7, 7] |
| 2 | [0, 0, 0, 2] | [0, 1], [2, 3] | 20 | [0, 6, 6, 6] | [0, 1], [6, 7] |
| 3 | [0, 0, 0, 4] | [0, 1], [4, 5] | 21 | [2, 2, 2, 2] | [2, 3] |
| 4 | [0, 0, 0, 6] | [0, 1], [6, 7] | 22 | [2, 2, 2, 4] | [2, 3], [4, 5] |
| 5 | [0, 0, 2, 2] | [0, 1, 2, 3] | 23 | [2, 2, 2, 6] | [2, 3], [6, 7] |
| 6 | [0, 0, 2, 4] | [0, 0, 1, 1, 2, 3, 4, 5] | 24 | [2, 2, 4, 4] | [2, 3, 4, 5] |
| 7 | [0, 0, 2, 6] | [0, 0, 1, 1, 2, 3, 6, 7] | 25 | [2, 2, 4, 6] | [2, 2, 3, 3, 4, 5, 6, 7] |
| 8 | [0, 0, 4, 4] | [0, 1, 4, 5] | 26 | [2, 2, 6, 6] | [2, 3, 6, 7] |
| 9 | [0, 0, 4, 6] | [0, 0, 1, 1, 4, 5, 6, 7] | 27 | [2, 4, 4, 4] | [2, 3], [4, 5] |
| 10 | [0, 0, 6, 6] | [0, 1, 6, 7] | 28 | [2, 4, 4, 6] | [2, 3, 4, 4, 5, 5, 6, 7] |
| 11 | [0, 2, 2, 2] | [0, 1], [2, 3] | 29 | [2, 4, 6, 6] | [2, 3, 4, 5, 6, 6, 7, 7] |
| 12 | [0, 2, 2, 4] | [0, 1, 2, 2, 3, 3, 4, 5] | 30 | [2, 6, 6, 6] | [2, 3], [6, 7] |
| 13 | [0, 2, 2, 6] | [0, 1, 2, 2, 3, 3, 6, 7] | 31 | [4, 4, 4, 4] | [4, 5] |
| 14 | [0, 2, 4, 4] | [0, 1, 2, 3, 4, 4, 5, 5] | 32 | [4, 4, 4, 6] | [4, 5], [6, 7] |
| 15 | [0, 2, 4, 6] | [0, 1, 2, 3, 4, 5, 6, 7] | 33 | [4, 4, 6, 6] | [4, 5, 6, 7] |
| 16 | [0, 2, 6, 6] | [0, 1, 2, 3, 6, 6, 7, 7] | 34 | [4, 6, 6, 6] | [4, 5], [6, 7] |
| 17 | [0, 4, 4, 4] | [0, 1], [4, 5] | 35 | [6, 6, 6, 6] | [6, 7] |
| 18 | [0, 4, 4, 6] | [0, 1, 4, 4, 5, 5, 6, 7] | | | |

Table 7 points out that a few $RMT$ combinations, like $2^{nd}$ and $11^{th}$ combinations, result in the same type of $RMT$ groupings. Therefore, this kind of combinations can be supported by the same $CA$ rules. Hence, if we concentrate on groupings of $RMT$s instead of nodes, we can get lesser number of such groupings of $RMT$s. The following lemma directs us to reach unique $RMT$ combinations.

**Lemma 7.2** *There exists a node at any level of reachability tree that dictates the $RMT$ set {0, 1} is to be balanced iff there is another node for which $RMT$ set {2, 3} is balanced. Similarly, there exists a node at any level of reachability tree that dictates the $RMT$ set {4, 5} is to be balanced iff there is another node for which $RMT$ set {6, 7} is balanced.*

**Proof.** The $RMT$ set {0, 1} of a rule is to be balanced while the node is [0, 0, 0, 0, 1, 1, 1, 1] (Table 7). Therefore, the edge that connects the node with its parent is resulted from $RMT$ 0 or 4 (Table 2). Hence the other edge from the parent is to be resulted from 1 or 5. That is, the other child node of the parent is constructed with $RMT$s [2, 2, 2, 2, 3, 3, 3, 3]. Therefore, the $RMT$ set {2, 3} is to be balanced since the reachability tree is for a reversible $CA$. The reverse is also true. Hence the proof. Similarly, it can be proved that, a node dictates the $RMT$ set {4, 5} is to be balanced iff there is another node for which $RMT$ set {6, 7} is balanced. □

The lemma guides that if $RMT$ combinations for node 1 (Table 7) are supported by a reversible rule $\mathcal{R}_i$, then $\mathcal{R}_i$ also supports node 21. Similarly, if node 31 is supported by $\mathcal{R}_i$, then node 35 is also supported by $\mathcal{R}_i$.

**Lemma 7.3** *There exists a node at any level of reachability tree that dictates each of two sibling RMT sets is balanced iff there is another node for which each of another two sibling RMT sets is balanced.*

**Proof.** Let us consider two sibling $RMT$ sets – {0, 1} and {4, 5}. Now, each of these two sets is to be balanced while the node is either [0, 0, 0, 1, 1, 1, 4, 5] or [0, 1, 4, 4, 4, 5, 5, 5]. If the node is [0, 0, 0, 1, 1, 1, 4, 5], the edge that connects the node with its parent is resulted from $RMT$ 0 or 4, and 2 or 6 (Table 2). Hence the other edge from the parent is to be resulted from 1 or 5, and 3 or 7. That is, the other child node of the parent is constructed with $RMT$s [2, 2, 2, 3, 3, 3, 6, 7]. Hence, each of the sibling $RMT$ set {2, 3} and {6, 7} is to be balanced. On the other hand, if the node is [0, 1, 4, 4, 4, 5, 5, 5], the other child node of its parent is constructed with [2, 3, 6, 6, 6, 7, 7, 7]. Then also, each of the sibling $RMT$ set {2, 3} and {6, 7} is to be balanced. Therefore, the theorem holds for two sibling $RMT$ sets – {0, 1} and {4, 5}. In similar fashion it can be shown that the theorem holds for another two sibling sets {0, 1} and {6, 7}.

Now consider another two sibling $RMT$ sets {0, 1} and {2, 3} are balanced. In this case, the node is either $N_1$ =[0, 0, 0, 1, 1, 1, 2, 3] or $N_2$ =[0, 1, 2, 2, 2, 3, 3, 3]. If the node is of type $N_1$ =, then its one possible parent is $P_1$ =[0, 1, 2, 3, 4, 4, 5, 5]. $RMT$s 0, 1 and 4 of $P_1$ are grouped with the same $RMT$ value ($d$, $d = 0/1$) to form the $N_1$ =. That is, $RMT$s 2, 3 and 5 are also grouped with $d'$. It implies, there is another node of type $P_2$ =[0, 1, 2, 3, 6, 6, 7, 7] at the same level of $P_1$ (Lemma 7.2). To make $P_2$ balanced, $RMT$s 2, 3 & 6 or $RMT$s 2, 3 & 7 are to be grouped with $d'$. Therefore, one child of $P_2$ is either [4, 4, 4, 5, 5, 5, 6, 7] or [4, 5, 6, 6, 6, 7, 7, 7]. Hence, each of the sibling $RMT$ set {4, 5} and {6, 7} is balanced. The same is true for all possible parents of $N_1$. Similarly, if the node is $N_2$ instead of $N_1$, it can be shown that {4, 5} and {6, 7} are balanced. That is, the proposition – if {4, 5} and {6, 7} are balanced, then {0, 1} and {2, 3} are also balanced, can be proved following the similar logic. Hence the proof. □

The above lemma states that the rules, for which each sibling $RMT$ set is balanced, can support all the nodes where a single set of sibling $RMT$s appears thrice.

**Lemma 7.4** *If $RMT$s [i, i, j, k] construct a node (sibling of each RMT is omitted for simplicity) which is supported by a reversible rule $\mathcal{R}_i$, then the rule also supports another node with $RMT$s [l, l, j, k], where i, j, k and l are the representative $RMT$s for sibling $RMT$s.*

**Proof.** Since $\mathcal{R}_i$ is a reversible rule, so $RMT$s of $S_1$ =[i, i+1, j, j+1, k, k+1, l, l+1] are balanced over {0, 1}. Again, the $RMT$s of $S_2$ =[i, i+1, i, i+1, j, j+1, k, k+1] of a node are balanced (Corollary 4.4). Therefore, $S_1 + S_1 - S_2$ = [j, j+1, k, k+1, l, l+1, l, l+1] is also balanced. Hence, the rule can support the node with $RMT$s [l, l, j, k] (sibling $RMT$s are omitted for simplicity). □

Since our target is to classify reversible rules and to establish the relationship among the rules, we shall concentrate on rules, rather than individual nodes. It is evident from Lemma 7.2, Lemma 7.3 & Lemma 7.4 and Table 7 that to support all the unique nodes, lesser number of groupings of $RMT$s are sufficient. These groupings are noted in Table 8.

Table 8
Combinations of $RMT$s for nodes

| $RMT$ grouping# | Grouping of balanced $RMT$s | Combinations of Node# |
|---|---|---|
| 1 | $[0, 1], [2, 3], [4, 5, 6, 7]$ | $1, 21, 33$ |
| 2 | $[0, 1, 2, 3], [4, 5], [6, 7]$ | $5, 31, 35$ |
| 3 | $[0, 1], [2, 3], [4, 5], [6, 7]$ | $2, 3, 4, 11, 17, 20, 22, 23, 27, 30, 32, 34$ |
| 4 | $[0, 1, 2, 3], [4, 5, 6, 7]$ | $5, 33$ |
| 5 | $[0, 1, 4, 5], [2, 3, 6, 7]$ | $8, 26$ |
| 6 | $[0, 1, 6, 7], [2, 3, 4, 5]$ | $10, 24$ |
| 7 | $[0, 0, 1, 1, 2, 3, 4, 5], [2, 3, 4, 5, 6, 6, 7, 7]$ | $6, 29$ |
| 8 | $[0, 0, 1, 1, 2, 3, 6, 7], [2, 3, 4, 4, 5, 5, 6, 7]$ | $7, 28$ |
| 9 | $[0, 0, 1, 1, 4, 5, 6, 7], [2, 2, 3, 3, 4, 5, 6, 7]$ | $9, 25$ |
| 10 | $[0, 1, 2, 3, 4, 4, 5, 5], [0, 1, 2, 3, 6, 6, 7, 7]$ | $14, 16$ |
| 11 | $[0, 1, 2, 2, 3, 3, 6, 7], [0, 1, 4, 4, 5, 5, 6, 7]$ | $13, 18$ |
| 12 | $[0, 1, 4, 5, 6, 6, 7, 7], [0, 1, 2, 2, 3, 3, 4, 5]$ | $18, 12$ |

### 7.2 Formation of class

Let us consider, the rules $\mathcal{R}_1$, $\mathcal{R}_2$, $\cdots$, $\mathcal{R}_i$ are selected for the cell 1, cell 2, $\cdots$, cell $i$ respectively to form an $n-$cell reversible $CA$ satisfying Theorem 4.1 and Theorem 4.2. Further, consider $S$ is the set of all reversible rules ($|S| = 70$). Now, the $CA$ cell $(i + 1)$ can support a set of rules $S_j \in S$ so that any rule of $S_j$ can be selected as $\mathcal{R}_{i+1}$, satisfying the theorems 4.1 and 4.2. We refer the class of $(i+1)^{th}$ cell as $C$ that supports the rules of $S_j$. The term *class* for cell $(i + 1)$ as well as for the $S_j$ is used interchangeably.

**Lemma 7.5** *There are 15 possible classes of reversible $CA$ cells in 3-neighborhood dependency.*

**Proof.** There are $2^i$ nodes at the $i^{th}$ level of reachability tree of a group $CA$. However, the $RMT$ combinations for all the nodes may not be unique. Since there are 12 possible unique group of $RMT$s (Table 8), the rule $\mathcal{R}_{i+1}$, selected as $(i+1)^{th}$ $CA$ cell rule, can support any one of the 12 groups or the combination of two or more unique groups. Therefore, if only individual grouping is supported by $\mathcal{R}_{i+1}$, 12 classes are possible. The classes are marked as class 1, class 2, ..., class 12 in

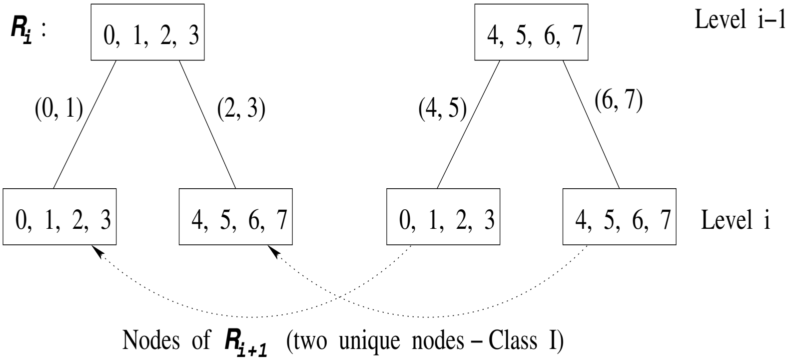
Table 9. If $\mathcal{R}_{i+1}$ is the combination of two unique groupings, another 3 classes are possible which support the combination of 4 & 5, 4 & 6 and 5 & 6 (these are as class 13, class 14 and class 15 of Table 9). Any other combination is the duplication of these 15 combinations. Therefore, there are 15 classes of reversible rules. □

Table 9
Class Table

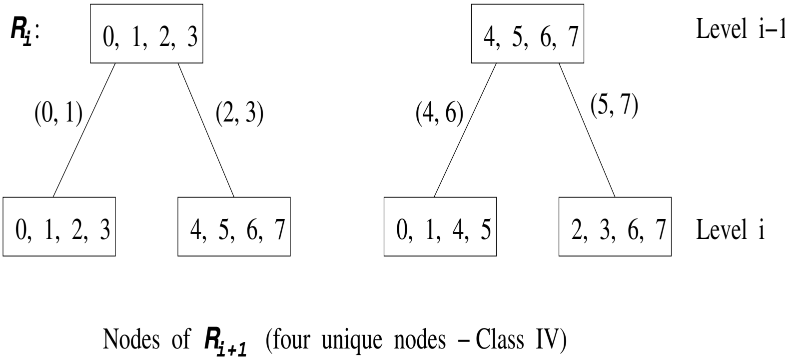
| Class | *RMT*s of nodes | Rules |
|---|---|---|
| 1 | {0, 1}, {2, 3}<br>{4, 5, 6, 7} | 53, 54, 57, 58, 85, 86, 89, 90, 101, 102, 105, 106,<br>149, 150, 153, 154, 165, 166, 169, 170, 197, 198, 201, 202 |
| 2 | {0, 1, 2, 3}<br>{4, 5}, {6, 7} | 83, 85, 86, 89, 90, 92, 99, 101, 102, 105, 106, 108,<br>147, 149, 150, 153, 154, 156, 163, 165, 166, 169, 170, 172 |
| 3 | {0, 1}, {2, 3}<br>{4, 5}, {6, 7} | 85, 86, 89, 90, 101, 102, 105, 106,<br>149, 150, 153, 154, 165, 166, 169, 170 |
| 4 | {0, 1, 2, 3}<br>{4, 5, 6, 7} | 51, 53, 54, 57, 58, 60, 83, 85, 86, 89, 90, 92,<br>99, 101, 102, 105, 106, 108, 147, 149, 150, 153, 154, 156,<br>163, 165, 166, 169, 170, 172, 195, 197, 198, 201, 202, 204 |
| 5 | {0, 1, 4, 5}<br>{2, 3, 6, 7} | 15, 29, 30, 45, 46, 60, 71, 75, 85, 86, 89, 90,<br>101, 102, 105, 106, 116, 120, 135, 139, 149, 150, 153, 154,<br>165, 166, 169, 170, 180, 184, 195, 209, 210, 225, 226, 240 |
| 6 | {0, 1, 6, 7}<br>{2, 3, 4, 5} | 15, 23, 27, 39, 43, 51, 77, 78, 85, 86, 89, 90,<br>101, 102, 105, 106, 113, 114, 141, 142, 149, 150, 153, 154,<br>165, 166, 169, 170, 177, 178, 204, 212, 216, 228, 232, 240 |
| 7 | {0, 0, 1, 1, 2, 3, 4, 5}<br>{2, 3, 4, 5, 6, 6, 7, 7} | 60, 77, 78, 85, 86, 89, 90, 101, 102, 105, 106, 149, 113,<br>114, 141, 142, 150, 153, 154, 165, 166, 169, 170, 177, 178, 195 |
| 8 | {0, 0, 1, 1, 2, 3, 6, 7}<br>{2, 3, 4, 4, 5, 5, 6, 7} | 29, 30, 45, 46, 51, 85, 86, 89, 90, 101, 102, 105, 106,<br>149, 150, 153, 154, 165, 166, 169, 170, 204, 209, 210, 225, 226 |
| 9 | {0, 0, 1, 1, 4, 5, 6, 7}<br>{2, 2, 3, 3, 4, 5, 6, 7} | 15, 53, 54, 57, 58, 85, 86, 89, 90, 101, 102, 105, 106,<br>149, 150, 153, 154, 165, 166, 169, 170, 197, 198, 201, 202, 240 |
| 10 | {0, 1, 2, 3, 4, 4, 5, 5}<br>{0, 1, 2, 3, 6, 6, 7, 7} | 15, 83, 85, 86, 89, 90, 92, 99, 101, 102, 105, 106, 149,<br>150, 153, 154, 156, 163, 165, 166,169, 170, 172, 240, 108, 147 |
| 11 | {0, 1, 2, 2, 3, 3, 6, 7}<br>{0, 1, 4, 4, 5, 5, 6, 7} | 23, 27, 29, 39, 43, 60, 85, 86, 90, 101, 102, 105, 106,<br>149, 150, 153, 154, 165, 169, 170, 195, 212, 216, 226, 228, 232 |
| 12 | {0, 1, 4, 5, 6, 6, 7, 7}<br>{0, 1, 2, 2, 3, 3, 4, 5} | 51, 71, 75, 85, 86, 89, 90, 101, 102, 105, 116, 120, 135,<br>139, 150, 153, 154, 165, 166, 169, 170, 180, 184, 204, 106, 149 |
| 13 | {0, 1, 2, 3}, {4, 5, 6, 7}<br>{0, 1, 4, 5}, {2, 3, 6, 7} | 60, 85, 86, 89, 90, 101, 105, 150, 165, 195<br>149, 153, 169, 154, 102, 166, 106, 170 |
| 14 | {0, 1, 2, 3}, {4, 5, 6, 7}<br>{0, 1, 6, 7}, {2, 3, 4, 5} | 51, 85, 86, 89, 90, 101, 102, 105, 106, 149,<br>150, 153, 154, 165, 166, 169, 170, 204 |
| 15 | {0, 1, 4, 5}, {2, 3, 6, 7}<br>{0, 1, 6, 7}, {2, 3, 4, 5} | 15, 85, 86, 89, 90, 102, 105, 150, 165, 240<br>149, 153, 101, 169, 154, 166, 106, 170 |

### *7.3 Relationship between $\mathcal{R}_i$ and $\mathcal{R}_{i+1}$*

The relationship between $\mathcal{R}_i$ and $\mathcal{R}_{i+1}$ signifies the identification of the class of $\mathcal{R}_{i+1}$ from the known $\mathcal{R}_i$ and its class. For example, let us consider the class of $\mathcal{R}_i$ is 4 (Fig. 8). Therefore, two unique nodes having $RMT$s {0, 1, 2, 3} and {4, 5, 6, 7} are available at the $(i-1)^{th}$ level of the reachability tree. Now consider the $RMT$s of $\mathcal{R}_i$ are clustered as {0, 1, 4, 5} and {2, 3, 6, 7}, where the $RMT$s of a set are the same, either 0 or 1. In Fig. 8(a), the $RMT$s {0, 1, 4, 5} are considered as 0, and the $RMT$s {2, 3, 6, 7} as 1. Therefore, the $RMT$s are grouped as (0, 1), (2, 3), (4, 5) and (6, 7). Each edge of the nodes is resulted from any one of these groups. Hence two edges connecting the node having $RMT$s {0, 1, 2, 3} with its children are resulted from (0, 1) and (2, 3). Therefore, the two children (for next level) of




$(a)$ Next rule class is 4




$(b)$ Next rule class is 13

Fig. 8. Determination of class relationship

that node are having $RMT$s {0, 1, 2, 3} and {4, 5, 6, 7} (Table 2) (Fig. 8(a)). Similarly, the children of another node having $RMT$s {4, 5, 6, 7} are constructed with $RMT$s {0, 1, 2, 3} and {4, 5, 6, 7} – that is, the nodes are same with the other two children. Therefore, the next level of the reachability tree contains two unique nodes having $RMT$s {0, 1, 2, 3} and {4, 5, 6, 7} (Fig. 8(a)). Hence the class of $\mathcal{R}_{i+1}$ is 4.

Further, if the $RMT$s of $\mathcal{R}_i$ are grouped as (0, 1), (2, 3), (4, 6), and (5, 7) (Fig. 8(b)), the nodes of level $i$, generated from the node of level $(i-1)$ with $RMT$s {0, 1, 2, 3}, are having $RMT$s {0, 1, 2, 3} and {4, 5, 6, 7}. The other two nodes at level $i$, generated from the node with $RMT$s {4, 5, 6, 7}, are having $RMT$s {0, 1, 4, 5} and {2, 3, 6, 7} (Fig. 8(b)). In this case, the next level of reachability tree contains four unique nodes having $RMT$s {0, 1, 2, 3}, {4, 5, 6, 7}, {0, 1, 4, 5}, and {2, 3, 6, 7} (Fig. 8(b)). Therefore, the organizations of $RMT$s support the property of both the Class 4 and Class 5. Therefore, the class of $\mathcal{R}_{i+1}$ is 13.

Table 10
Formation of class relationship between $\mathcal{R}_i$ and $\mathcal{R}_{i+1}$

| (1)<br>Class<br>of $\mathcal{R}_i$ | (2)<br>$RMT$s of nodes<br>at level $(i-1)$ | (3)<br>Groupings of $RMT$s<br>at level $(i-1)$ | (4)<br>$RMT$s of nodes<br>at level $i$ | (5)<br>Class<br>of $\mathcal{R}_{i+1}$ |
|---|---|---|---|---|
| 1 | {0, 1}, {2, 3}<br>{4, 5, 6, 7} | (0), (1), (2), (3)<br>* | {0, 1}, {2, 3},<br>{4, 5}, {6, 7} | 3 |
| 4 | {0, 1, 2, 3}<br>{4, 5, 6, 7} | (0, 1), (2, 3)<br>(4, 5), (6, 7) | {0, 1, 2, 3}<br>{4, 5, 6, 7} | 4 |
| | | (0, 2), (1, 3)<br>(4, 6), (5, 7) | {0, 1, 4, 5}<br>{2, 3, 6, 7} | 5 |
| | | (0, 3), (1, 2)<br>(4, 7), (5, 6) | {0, 1, 6, 7}<br>{2, 3, 4, 5} | 6 |
| | | {(0, 1), (2, 3)<br>(4, 6), (5, 7)}<br>or {(0, 2), (1, 3)<br>(4, 5), (6, 7)} | {0, 1, 2, 3}<br>{4, 5, 6, 7}<br>{0, 1, 4, 5}<br>{2, 3, 6, 7} | 13 |
| 5 | {0, 1, 4, 5}<br>{2, 3, 6, 7} | (0, 1), (4, 5)<br>(2, 3), (6, 7) | {0, 1, 2, 3}<br>{4, 5, 6, 7} | 4 |
| | | (0, 4), (1, 5)<br>(2, 6), (3, 7) | {0, 1}, {2, 3},<br>{4, 5}, {6, 7} | 3 |
| 8 | {0, 0, 1, 1, 2, 3, 6, 7}<br>{2, 3, 4, 4, 5, 5, 6, 7} | (0, 1), (2, 3, 6, 7)<br>(2, 3, 6, 7), (4, 5) | {0, 1, 2, 3}<br>{4, 5, 6, 7} | 4 |
| | | (0, 2, 3), (1, 6, 7)<br>(2, 3, 4), (5, 6, 7) | {0, 0, 1, 1, 4, 5, 6, 7}<br>{2, 2, 3, 3, 4, 5, 6, 7} | 9 |
| 13 | {0, 1, 2, 3}<br>{4, 5, 6, 7}<br>{0, 1, 4, 5}<br>{2, 3, 6, 7} | (0, 1), (2, 3)<br>(4, 5), (6, 7) | {0, 1, 2, 3}<br>{4, 5, 6, 7} | 4 |
| | | {(0, 1), (2, 3)<br>(4, 6), (5, 7)}<br>or {(0, 2), (1, 3)<br>(4, 5), (6, 7)} | {0, 1, 2, 3}<br>{4, 5, 6, 7}<br>{0, 1, 4, 5}<br>{2, 3, 6, 7} | 13 |

Table 10 partly displays the formation of relationship between reversible rules. Only 5 classes, 1, 4, 5, 8 and 13 are selected to illustrate the relationship. The first column shows the class of $\mathcal{R}_i$. Column 2 notes the $RMT$s of nodes at level $(i-1)$. The grouping of $RMT$s for $\mathcal{R}_i$ is shown in column 3. $RMT$s within the braces are balanced over {0, 1} and $RMT$s within each bracket are having the same value (0/1). The $RMT$s of nodes at level $i$ are shown in Column 4. Based on the nodes at level $i$, the class of $\mathcal{R}_{i+1}$ is decided and is reported in Column 5. The '*' in column 2 indicates any combination of remaining $RMT$s.

The details of relationship among the classes are reported in Table 11. The first and second columns of the table represent the class of $i^{th}$ cell and the rule $\mathcal{R}_i$ respectively, whereas the class of the $(i+1)^{th}$ cell corresponding to this pair (the class of $i^{th}$ cell and $\mathcal{R}_i$) is noted in the last column. It can be observed that a rule can be the member of more than one class. For example, rule 15, 240 are the members of class 5, 6, 9, 10, and 15. Moreover, there are a few rules which are the member of all classes. For example, the rules of class 3 are the member of all 15 classes. Such rules are referred to as the complete rules.

Table 11
Class relationship of $\mathcal{R}_i$ and $\mathcal{R}_{i+1}$

| Class of $\mathcal{R}_i$ | $\mathcal{R}_i$ | Class of $\mathcal{R}_{i+1}$ |
|---|---|---|
| 1 | 53, 54, 57, 58, 85, 86, 89, 90, 101, 102, 105, 106, 149, 150, 153, 154, 165, 166, 169, 170, 197, 198, 201, 202 | 3 |
| 2 | 83, 85, 86, 89, 90, 92, 99, 101, 102, 105, 106, 108, 147, 149, 150, 153, 154, 156, 163, 165, 166, 169, 170, 172 | 3 |
| 3 | 85, 86, 89, 90, 101, 102, 105, 106, 149, 150, 153, 154, 165, 166, 169, 170 | 3 |
| 4 | 51, 60, 195, 204 | 4 |
| | 85, 90, 165, 170 | 5 |
| | 102, 105, 150, 153 | 6 |
| | 53, 58, 83, 92, 163, 172, 197, 202 | 13 |
| | 54, 57, 99, 108, 147, 156, 198,201 | 14 |
| | 86, 89, 101, 106, 149, 154, 166, 169 | 15 |
| 5 | 29, 46, 89, 106, 149, 166, 209, 226 | 1 |
| | 71, 86, 101, 116, 139, 154, 169, 184 | 2 |
| | 85, 102, 153, 170 | 3 |
| | 15, 30, 45, 60, 75, 90, 105, 120, 135, 150, 165, 180, 195, 210, 225, 240 | 4 |
| 6 | 15, 51, 204, 240 | 4 |
| | 85, 105, 150, 170 | 5 |
| | 90, 102, 153, 165 | 6 |
| | 23, 43, 77, 113, 142, 178, 212, 232 | 13 |
| | 27, 39, 78, 114, 141, 177, 216, 228 | 14 |
| | 86, 89, 101, 106, 149, 154, 166, 169 | 15 |
| 7 | 77, 78, 85, 86, 89, 101, 102, 106, 113, 114, 141, 142, 149, 153, 154, 166, 169, 170, 177, 178 | 3 |
| | 60, 195 | 4 |
| | 90, 165 | 13 |
| | 105, 150 | 14 |
| 8 | 85, 170 | 5 |
| | 102, 153 | 6 |
| | 29, 30, 45, 46, 51, 89, 90, 105, 106, 149, 150, 165, 166, 204, 209, 210, 225, 226 | 9 |
| | 86, 101, 154, 169 | 15 |
| 9 | 53, 54, 57, 58, 85, 86, 89, 101, 102, 106, 149, 153, 154, 166, 169, 170, 197, 198, 201, 202 | 3 |
| | 15, 240 | 4 |
| | 105, 150 | 13 |
| | 90, 165 | 14 |
| 10 | 83, 85, 89, 92, 99, 101, 102, 106, 108, 147, 149, 153, 154, 156, 163, 166, 170, 172 | 3 |
| | 15, 240 | 4 |
| | 105, 150 | 13 |
| | 86, 90, 165, 169 | 14 |
| 11 | 23, 27, 39, 43, 85, 86, 89, 101, 102, 106, 149, 153, 154, 166, 169, 170, 212, 216, 228, 232 | 3 |
| | 60, 195 | 4 |
| | 90, 165 | 13 |
| | 105, 150 | 14 |

| Class of $\mathcal{R}_i$ | $\mathcal{R}_i$ | Class of $\mathcal{R}_{i+1}$ |
|---|---|---|
| 12 | 51, 204 | 4 |
| | 85, 170 | 5 |
| | 102, 153 | 6 |
| | 71, 75, 86, 90, 101, 105, 116, 120, 135, 139, 150, 154, 165, 169, 180, 184 | 10 |
| | 89, 106, 149, 166 | 15 |
| 13 | 85, 86, 89, 101, 102, 106, 149, 153, 154, 166, 169, 170 | 3 |
| | 60, 195 | 4 |
| | 90, 165 | 13 |
| | 105, 150 | 14 |
| 14 | 51, 204 | 4 |
| | 85, 170 | 5 |
| | 102, 153 | 6 |
| | 86, 89, 90, 101, 105, 106, 149, 150, 154, 165,166, 169 | 15 |
| 15 | 85, 86, 89, 101, 102, 106, 149, 153, 154, 166, 169, 170 | 3 |
| | 15, 240 | 4 |
| | 105, 150 | 13 |
| | 90, 165 | 14 |

**Definition 7.6** A rule is **complete** if it is the member of all classes.

### *7.4 Synthesis of $\mathcal{R}$*

The reversible $CA$ synthesis scheme is proposed in Algorithm 2 (Section 5) directly utilizing the structure of reachability tree. This subsection reports a relatively simpler method to synthesize an $n-$cell reversible $CA$ $\mathcal{R} =< \mathcal{R}_1, \mathcal{R}_2, \cdots, \mathcal{R}_n >$, based on the class relationship noted in Table 11. However, an additional table is required to select the first rule $\mathcal{R}_1$. A part of all the possible $\mathcal{R}_1$ is shown in Table 12. The details on formation of Table 12 are beyond the scope of this work.

The synthesis scheme selects $\mathcal{R}_1$ from Table 12 randomly and then class of $\mathcal{R}_2$. The $\mathcal{R}_{i+1}$s, for $i = 0$ to $n-1$, are selected from Table 11. Now based on the class of $\mathcal{R}_n$ and Table 11, the class of first rule is further determined. If this matches with the initially selected $\mathcal{R}_1$, then $\mathcal{R} = \mathcal{R}_1, \mathcal{R}_2, \cdots, \mathcal{R}_n$ is a reversible $CA$.

Table 12
First rule table

| First rule ($\mathcal{R}_1$) | Class of $\mathcal{R}_2$ |
|---|---|
| 51, 204 | 4 |
| 85, 170 | 5 |
| 102, 153 | 6 |
| 23, 53, 83, 113, 142, 172, 202, 232 | 7 |
| 27, 57, 78, 108, 147, 177, 198, 228 | 8 |
| 89, 106, 149, 166 | 9 |
| 86, 101, 154, 169 | 10 |
| 43, 58, 77, 92, 163, 178, 197, 212 | 11 |
| 39, 54, 99, 114, 141, 156, 201, 216 | 12 |

# 8 Conclusion

This work proposes the characterization of one dimensional 3-neighborhood periodic boundary reversible cellular automata ($CA$). The classified $CA$ rules that can form reversible $CA$ with the target to synthesize such a $CA$ in linear time.

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