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Towards a Classification of Behavioural Equivalences in Continuous-time Markov Processes

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Abstract

Bisimulation is a concept that captures behavioural equivalence of states in a transition system. In [6], we proposed two equivalent definitions of bisimulation on continuous-time stochastic processes where the evolution is a *flow* through time. In the present paper, we develop the theory further: we introduce different concepts that correspond to different behavioural equivalences and compare them to bisimulation. In particular, we study the relation between bisimulation and symmetry groups of the dynamics. We also provide a game interpretation for two of the behavioural equivalences. We then compare those notions to their discrete-time analogues.

Keywords: Stochastic Processes, Markov Processes, continuous time, bisimulation, diffusion.

1 Introduction

Bisimulation [16,18,20] is a fundamental concept in the theory of transition systems capturing a strong notion of behavioural equivalence. In particular, it is a notion stronger than that of trace equivalence. Bisimulation has been widely studied for discrete time systems where transitions happen as steps, both on discrete [15] and continuous state spaces [4,9,17]. In all these types of systems a crucial ingredient of the definition of bisimulation is the ability to talk about *the next step*. Thus, the general format of the definition of bisimulation is that one has some property that

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must hold "now" (in the states being compared) and then one says that the relation is preserved in the *next* step.

Outside of computer science, there is a vast range of systems that involve continuous-time evolution: deterministic systems governed by differential equations and stochastic systems governed by "noisy" differential equations called stochastic differential equations. These have been extensively studied for over a century since the pioneering work of Einstein [12] on Brownian motion.

In [6], we introduced a notion of bisimulation for stochastic systems with true continuous-time evolution. Some attempts had previously been made to talk about continuous-time [10], but even in what are called continuous-time Markov chains there is a discrete notion of time *step*; it is only that there is a real-valued duration associated with each state that makes such systems continuous time. They are often called "jump processes" in the mathematical literature, see, for example, [19,21], a phrase that better captures the true nature of such processes.

We focused on a class of systems called Feller-Dynkin processes for which a good mathematical theory exists. These systems are Markov processes defined on continuous state spaces and with continuous time evolution. Such systems encompass Brownian motion and its many variants.

In this paper, we explore four other notions of behavioural equivalence for such continuous-time processes. The strongest notion is that of a group of symmetries. It is stronger than the notion of bisimulation introduced in [6] and it captures the symmetries of the system. We do not know yet if all bisimulations can be accounted for by groups of symmetries.

Temporal equivalence is a notion that is weaker than bisimulation. It looks closer to the definition of bisimulation in discrete time than the definition we provided in [6], however it also strongly relies on trajectories. Temporal equivalence can be summed up as trace equivalence with some additional step-like constraints. It is still an open question if there is a class of processes for which temporal equivalence and bisimulation are equivalent notions or not.

The third notion is that of trace equivalence. It is the weakest of all these behavioural equivalences and an example in [6] shows that it is a strictly weaker notion.

Finally, we give two game interpretations, one for bisimulation and one for temporal equivalence. They closely mirror the one provided in [13,7]. The game for bisimulation also emphasizes the importance of trajectories for the study of behavioural equivalences in continuous time.

The relations between these different behavioural equivalences can be displayed as follows.

$$\begin{tikzcd}
 & \text{trace equivalence} & \\
\text{discrete time bisimulation} \arrow[r, <->, dotted] & \text{temporal equivalence} \arrow[u] \arrow[r, <->] & \text{game interpretation} \\
 & \text{bisimulation} \arrow[u] \arrow[r, <->] & \text{game interpretation} \\
 & \text{group of symmetries} \arrow[u] &
\end{tikzcd}$$

We end this paper by studying discrete-time systems and by revisiting the examples provided in our previous study. This seems to indicate that the correct notion that extends bisimulation to continuous-time systems is that of temporal equivalence and not the initial definition provided in [6].

Some proofs that were either too long or not crucial have been moved to the appendix.

2 Feller-Dynkin Processes

2.1 Definition of Feller-Dynkin Processes

We assume that basic concepts like topology, measure theory and basic concepts of probability on continuous spaces are well known; see, for example [3,11,17].

The basic arena for the action is a probability space.

Definition 2.1 A *probability space* is a triple (X, Σ, P) where X is a space (usually a topological space), Σ is a σ-algebra (usually its Borel algebra) and P is a probability measure on S.

Given a measurable space (X, Σ) a *(sub)-Markov kernel* is a map $\tau : X \times \Sigma \to [0, 1]$ which is measurable in its first argument, i.e. $\tau(\cdot, A \in \Sigma) : X \to \mathbb{R}$ is measurable for any fixed A in Σ and for any fixed $x \in E$, $\tau(x, \cdot)$ is a (sub)probability measure. These kernels describe transition probability functions.

Definition 2.2 A *filtration* on a measurable space (X, Σ) is a nondecreasing family $(\mathcal{F}_t)_{t \geq 0}$ of sub-σ-algebras of Σ, *i.e.* $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \Sigma$ for $0 \leq s < t < \infty$.

This concept is used to capture the idea that at time t what is "known" or "observed" about the process is encoded in the sub-σ-algebra $\mathcal{F}_t$.

Definition 2.3 A *stochastic process* is a collection of random variables $(Y_t)_{0 \leq t < \infty}$ on a measurable space (Ω, Σ_1) that take values in a second measurable space (X, Σ_2) called the *state space*. We say that a stochastic process is *adapted* to a filtration $(\mathcal{F}_t)_{t \geq 0}$ if for each $t \geq 0$ we have X_t is $\mathcal{F}_t$-measurable.

Note that a stochastic process is always adapted to the filtration $(\mathcal{G}_t)_{t \geq 0}$, where for each $t \geq 0$, $\mathcal{G}_t$ is defined as the σ-algebra generated by all the random variables $\{Y_s | s \leq t\}$. The filtration $(\mathcal{G}_t)_{t \geq 0}$ is also referred to as the natural filtration associated to $(X_t)_{t \geq 0}$.

Before stating the definition of the continuous-time processes we will be interested in, let us first start by recalling the definition of their discrete-time counterparts.

Definition 2.4 A *labelled Markov process (LMP)* is a triple (X, Σ, τ) where (X, Σ) is a measurable space and τ is a Markov kernel.

We will quickly review the theory of continuous-time processes on continuous state space; much of this material is adapted from "Diffusions, Markov Processes and Martingales, Volume I" by Rogers and Williams [19] and we use their notations. Another useful source is "Functional analysis for probability and stochastic processes" by A. Bobrowski [5]. Let E be a locally compact, Hausdorff space with countable base which is also σ-compact and Polish, and let E be equipped with the Borel σ-algebra $\mathcal{E} = \mathcal{B}(E)$. E_∂ is the one-point compactification of E: $E_\partial = E \uplus \{\partial\}$. The physical picture is that the added state, ∂, represents a point at infinity; we will view it as an absorbing state. Denoting $\mathcal{O}$ the topology on E, the space E_∂ is equipped with the topology $\mathcal{O}_\partial = \mathcal{O} \cup \{\{\partial\} \cup K^c \mid K \text{ compact in } (E, \mathcal{O})\}$.

Definition 2.5 A *semigroup* of operators on any Banach space is a family of linear continuous (bounded) operators T_t indexed by $t \in \mathbf{R}^{\geq 0}$ such that

$$\forall s, t \geq 0, T_s \circ T_t = T_{s+t}$$

and

$$T_0 = I \qquad \text{(the identity)}.$$

The first equation above is called the semigroup property. The operators in a semigroup are continuous however there is a useful continuity property of the semigroup as a whole.

Definition 2.6 For X a Banach space, we say that a semigroup $T_t : X \to X$ is *strongly continuous* if

$$\forall x \in X, \lim_{t \downarrow 0} T_t x = x$$

which is equivalent to saying

$$\forall x \in X, \lim_{t \downarrow 0} \|T_t x - x\| \to 0.$$

We say that a continuous real-valued function f on E "vanishes at infinity" if for every $\varepsilon > 0$ there is a compact subset $K \subset E$ such that $\forall x \in E \setminus K$ we have $|f(x)| \leq \varepsilon$. The space $C_0(E)$ of continuous real-valued functions that vanish at infinity is a Banach space with the sup norm.

Definition 2.7 A *Feller-Dynkin (FD) semigroup* is a strongly continuous semigroup $(\hat{P}_t)_{t \geq 0}$ of linear operators on $C_0(E)$ satisfying the additional condition:

$$\forall t \geq 0 \quad \forall f \in C_0(E), \text{ if } \quad 0 \leq f \leq 1, \text{ then } \quad 0 \leq \hat{P}_t f \leq 1$$

The following important proposition relates these FD semigroups with Markov processes which allows one to see the connection with more familiar probabilistic transition systems.

Proposition 2.8 *Given such an FD semigroup, it is possible to define a unique family of sub-Markov kernels* $(P_t)_{t\geq 0} : E \times \mathcal{E} \to [0,1]$ *such that for all* $t \geq 0$ *and* $f \in C_0(E)$,

$$\hat{P}_t f(x) = \int f(y) P_t(x, dy).$$

A very important ingredient in the theory is the space of trajectories of a FD process (FD semigroup) as a probability space. This space does not appear explicitly in the study of labelled Markov processes but one does see it in the study of continuous-time Markov chains and jump processes.

Definition 2.9 We define a *trajectory* ω on E_∂ to be a *cadlag* [5] function $[0,\infty) \to E_\partial$ such that if either $\omega(t-) := \lim_{s<t, s\to t} \omega(s) = \partial$ or $\omega(t) = \partial$ then $\forall u \geq t, \omega(u) = \partial$. We can extend ω to a map from $[0,\infty]$ to E_∂ by setting $\omega(\infty) = \partial$.

As a first intuition, the reader can think of these trajectories as continuous paths. Cadlagness allows for some jumping in a reasonable fashion. The intuition behind the additional condition is that once a trajectory has reached the terminal state ∂, it stays in that state and similarly if a trajectory "should" have reached the terminal state ∂, then there cannot be a jump to avoid ∂ and once it is in ∂, it stays there.

It is possible to associate to such an FD semigroup a *canonical FD process.* Let Ω be the set of trajectories $\omega : [0,\infty) \to E_\partial$.

Definition 2.10 The *canonical FD process* associated to the FD semigroup $(\hat{P}_t)$ is

$$(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0}, (X_t)_{0\leq t\leq\infty}, (\mathbb{P}^x)_{x\in E_\partial})$$

where

- $X_t(\omega) = \omega(t)$
- $\mathcal{G} = \sigma(X_s \mid 0 \leq s < \infty)$ [6], $\mathcal{G}_t = \sigma(X_s \mid 0 \leq s \leq t)$
- given any probability measure μ on E_∂, by the Daniell-Kolmogorov theorem, there exists a unique probability measure $\mathbb{P}^\mu$ on $(\Omega, \mathcal{G})$ such that for all $n \in \mathbb{N}, 0 \leq t_1 \leq t_2 \leq ... \leq t_n$ and $x_0, x_1, ..., x_n$ in E_∂,

$$\mathbb{P}^\mu(X_0 \in dx_0, X_{t_1} \in dx_1, ..., X_{t_n} \in dx_n) = \mu(dx_0) P^{+\partial}_{t_1}(x_0, dx_1) ... P^{+\partial}_{t_n - t_{n-1}}(x_{n-1}, dx_n)$$ [7]

  where $P^{+\partial}_t$ is the Markov kernel extending the Markov kernel P_t to E_∂ by $P^{+\partial}_t(x, \{\partial\}) = 1 - P_t(x, E)$ and $P^{+\partial}_t(\partial, \{\partial\}) = 1$. We set $\mathbb{P}^x = \mathbb{P}^{\delta_x}$.

---

[5] By *cadlag* we mean right-continuous with left limits.

[6] The σ-algebra $\mathcal{G}$ is the same as the one induced by the Skorohod metric, see theorem 16.6 of [2]

[7] The dx_i in this equation should be understood as infinitesimal volumes. This notation is standard in probabilities and should be understood by integrating it over measurable state sets C_i.

This is the version of the system that will be most useful for us. In order to bring it more in line with the kind of transition systems that have hitherto been studied in the computer science literature we introduce a countable set of atomic propositions AP and such a FD process is equipped with a function $obs : E \to 2^{AP}$. This function is extended to a function $obs : E_\partial \to 2^{AP} \uplus \{\partial\}$ by setting $obs(\partial) = \partial$.

Instead of following the dynamics of the system step by step as one does in a discrete system we have to study the behaviour of sets of trajectories. The crucial ingredient is the distribution $\mathbb{P}^x$ which gives a measure on the space of trajectories for a system started at the point x.

2.2 Brownian motion as a FD process

Brownian motion is a stochastic process describing the irregular motion of a particle being buffeted by invisible molecules. Now its range of applicability extends far beyond its initial application [14]. The following definition is from [14].

Definition 2.11 A standard one-dimensional Brownian motion is a Markov process adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$,

$$B = (W_t, \mathcal{F}_t), 0 \leq t < \infty$$

defined on a probability space $(\Omega, \mathcal{F}, P)$ with the properties

(i) $W_0 = 0$ almost surely,

(ii) for $0 \leq s < t$, $W_t - W_s$ is independent of $\mathcal{F}_s$ and is normally distributed with mean 0 and variance $t - s$.

In this very special process, one can start at any place, there is an overall translation symmetry which makes calculations more tractable. In order to do any calculations we use the following fundamental formula: If the process is at x at time 0 then at time t the probability that it is in the (measurable) set D is given by

$$P_t(x, D) = \int_{y\in D} \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(x-y)^2}{2t}\right) \mathrm{d}y.$$

The associated FD semigroup is the following: for $f \in C_0(\mathbb{R})$ and $x \in \mathbb{R}$,

$$\hat{P}_t(f)(x) = \int_y \frac{f(y)}{\sqrt{2\pi t}} \exp\left(-\frac{(x-y)^2}{2t}\right) \mathrm{d}y.$$

3 Bisimulation

We introduced a notion of bisimulation in [6] that we will recall in this section. We will also define two weaker notions: trace equivalence and temporal equivalence. The latter one seems to be a better generalization of bisimulation in discrete time systems than our original definition of bisimulation. For the remaining of this paper, we consider a FD-process as in section 2.

3.1 Naive approach

Let us start by illustrating why the naive definition does not work and why we need a more complex definition.

The key idea of bisimulation is that "what can be observed now is the same" and bisimulation is preserved by the evolution. In order to capture this we need two conditions: the first captures what is immediately observable and the second captures the idea that the evolution preserves bisimulation.

Consider the naive extension of bisimulation in discrete time (see definition 6.1): let R be an equivalence relation on the state space E such that whenever $x \; R \; y$ $(x, y \in E)$:

- **(initiation)** $obs(x) = obs(y)$, and
- **(induction)** for all R-closed (see definition 3.1) sets C in $\mathcal{E}$, for all times t, $P_t(x, C) = P_t(y, C)$

Let us illustrate on an example why this definition is not enough.

We consider the case of Brownian motion on the reals where there is a single atomic proposition marking 0: $obs(0) = 1$ and $obs(x) = 0$ for $x \neq 0$. Intuitively, we would like that two states x and y are bisimilar if and only if $|x| = |y|$ as the only symmetry that this system has is point reflection with respect to 0.

However, the two conditions (initiation) and (induction) are not strong enough to enforce that this equivalence relation is the greatest bisimulation.

Let us define the equivalence

$$R = (\mathbb{R}^* \times \mathbb{R}^*) \cup \{(0, 0)\} \text{ where } \mathbb{R}^* = \mathbb{R} \setminus \{0\}.$$

This equivalence satisfies both conditions (induction) and (initiation). The last one follows directly from the definitions of R and *obs*.

For the induction condition, the only R-closed sets are $\emptyset, \{0\}, \mathbb{R}^*$ and $\mathbb{R}$, and for any state $z \neq 0$ and time $t \geq 0$, $P_t(z, \emptyset) = P_t(z, \{0\}) = 0$ and $P_t(z, \mathbb{R}^*) = P_t(z, \mathbb{R}) = 1$.

This shows that with this naive definition of bisimulation, any two non-zero states are bisimilar, which does not correspond to our intuition for this example.

3.2 Definition

As we have just shown, unlike in the discrete-time case we cannot just say that the "next step" preserves the relation. This is why the definition of bisimulation that we introduced in [6] dealt with trajectories instead of steps. There are two conditions (initiation and induction conditions) that can be modified to account for trajectories. Depending on which one is adapted, we get either temporal equivalence or bisimulation.

Definition 3.1 Given an equivalence R on E, a subset C of the state space E is *R-closed* if for every states x and y such that $x \; R \; y$, $x \in C$ if and only if $y \in C$.

Definition 3.2 Given an equivalence R on E extended to E_∂ by setting $\partial\ R\ \partial$, a set B of trajectories is *time-R-closed* if for every trajectories ω and ω' such that for every time $t \geq 0$, $\omega(t)\ R\ \omega'(t)$ (where R is extended to E_∂ by setting $\partial\ R\ \partial$), $\omega \in B$ if and only if $\omega' \in B$.

A set B is called *time-obs-closed* if it is time-R-closed where R is the equivalence defined by $x\ R\ y$ if and only if $obs(x) = obs(y)$.

The following is the definition of bisimulation introduced in [6]. In discrete time, bisimulation was only about the next step. Here, we take the full trajectories into consideration.

Definition 3.3 A *bisimulation* is an equivalence relation R on E such that for all $x, y \in E$, if $x\ R\ y$, then

(initiation) $obs(x) = obs(y)$, and

(induction) for all measurable time-R-closed sets B, $\mathbb{P}^x(B) = \mathbb{P}^y(B)$.

Another well-known concept is that of trace equivalence.

Definition 3.4 Two states are *trace equivalent* if and only if for all measurable time-*obs*-closed sets B, $\mathbb{P}^x(B) = \mathbb{P}^y(B)$.

Temporal equivalence can be viewed as trace equivalence which additionally accounts for step-like branching. As such, it is weaker than bisimulation but stronger than trace equivalence. As shown in section 6.1, it seems to be the notion that best generalizes discrete-time bisimulation since the induction requirement of bisimulation is actually very strong.

Definition 3.5 A *temporal equivalence* is an equivalence relation R on E such that for all $x, y \in E$, if $x\ R\ y$, then

(initiation) for all measurable time-*obs*-closed sets B, $\mathbb{P}^x(B) = \mathbb{P}^y(B)$, and

(induction) for all measurable R-closed sets C, for all times t, $P_t(x, C) = P_t(y, C)$.

Remark 3.6 Two states that are related by a bisimulation are called *bisimilar*. There is a greatest bisimulation that corresponds to this equivalence.

Similarly, two states that are related by a temporal equivalence are called *temporally equivalent*. There is a greatest temporal equivalence that corresponds to this equivalence.

Example 3.7 Consider for example Brownian motion. In the first case, we will distinguish zero from the other states, i.e. $obs(x) = 1$ if and only if $x = 0$. In the second case, we will distinguish the integers from the other states, i.e. $obs(x) = 1$ if and only if $x \in \mathbb{Z}$.

In the first case, the equivalence R defined by $x\ R\ y$ if and only if $x = y$ or $-y$ is both the greatest bisimulation, temporal equivalence and the trace equivalence.

This equivalence is also a bisimulation and a temporal equivalence for the second case. However, there are other bisimulations and temporal equivalences, the greater ones being that two states x and y are related if and only if they have same distance to the closest integer, i.e. $x - \lfloor x \rfloor = y - \lfloor y \rfloor$ or $\lceil y \rceil - y$. This equivalence is also trace equivalence.

Theorem 3.8 *A bisimulation is also a temporal equivalence. If two states are temporally equivalent, then they are trace equivalent.*

We provide in section 6.2.2 an example where the greatest temporal equivalence is strictly greater than trace equivalence. It is still an open question as to whether bisimulation and temporal equivalence are the same notions or if they are only for a class of processes.

4 Symmetry groups of the process

Given a function $h : E_\partial \to E_\partial$, we define $h_* : \Omega \to \Omega$ by $h_*(\omega) = h \circ \omega$.
A group of symmetries is a set of bijections on the state space that respect the dynamics of the FD-process. Intuitively speaking, once a symmetry is applied, the system is transformed but the new system behaves exactly like the old one.

Definition 4.1 A *group of symmetries* is a group (closed under inverse and composition) $\mathcal{H}$ of homeomorphisms on the state space E extended to E_∂ by setting $h(\partial) = \partial$ for every $h \in \mathcal{H}$ such that

- for all $h \in \mathcal{H}$, $obs \circ h = obs$, and
- for all $x \in E_\partial$, for all $f \in \mathcal{H}$ and for all measurable sets B such that for all $h \in \mathcal{H}$, $h_*(B) = B$,
$$\mathbb{P}^x(B) = \mathbb{P}^{f(x)}(B).$$

Example 4.2 A rather trivial example of a group of symmetries is the group with a single element $\{id\}$.

Consider again the example of Brownian motion with zero distinguished from the rest, the set $\{s, id\}$ where $s(x) = -x$ for $x \in \mathbb{R}$ is a group of symmetries.

This set is also a group of symmetries for Brownian motion with integers distinguished from the rest. However, there are several other possible groups of symmetries: $\{id, t_k \mid k \in \mathbb{Z}\}$ and $\{id, t_k, s_k \mid k \in \mathbb{Z}\}$ where $t_k(x) = x + k$ and $s_k(x) = k - x$ for every $k \in \mathbb{Z}$ and $x \in \mathbb{R}$.

Definition 4.3 Given a group of symmetries $\mathcal{H}$ on E, we denote $R_\mathcal{H}$ the equivalence defined on E as follows: $x \; R_\mathcal{H} \; y$ if and only if there exists $h \in \mathcal{H}$ such that $h(x) = y$. The fact that $\mathcal{H}$ is a group guarantees that $R_\mathcal{H}$ is an equivalence.

One of the requirements for being a group of symmetries is to be closed under inverse and composition. This condition is useful for getting an equivalence on the state space, however, it is usually easier (if possible) to view a group of symmetries as generated by a set of homeomorphisms.

Lemma 4.4 *Consider a set of homeomorphisms $\mathcal{H}_{gen}$ on the state space E and define $\mathcal{H}$ is the closure under inverse and composition of the set $\mathcal{H}_{gen}$. Assume that* *the set $\mathcal{H}_{gen}$ satisfies the following conditions:*

- *for all $f \in \mathcal{H}_{gen}$, $obs \circ f = obs$, and*
- *for all measurable sets B such that for all $f \in \mathcal{H}_{gen}$, $f_*(B) = B$, for all $x \in E_\partial$ and for all $g \in \mathcal{H}_{gen}$, $\mathbb{P}^x(B) = \mathbb{P}^{g(x)}(B)$.*

Then the set $\mathcal{H}$ is a group of symmetries.

Example 4.5 The two groups of symmetries $\{id, s_k \mid k \in \mathbb{Z}\}$ and $\{id, s_k, t_k \mid k \in \mathbb{Z}\}$ are generated by $\{id, t_1\}$ and $\{id, t_1, s\}$ respectively. If one carefully reads the proofs of the examples provided in [6], these two homeomorphisms clearly appear.

Theorem 4.6 *Given a group of symmetries $\mathcal{H}$, the equivalence $R_\mathcal{H}$ is a bisimulation.*

Example 4.7 The groups of symmetries $\{id, s\}$, $\{id, s_k \mid k \in \mathbb{Z}\}$ and $\{id, s_k, t_k \mid k \in \mathbb{Z}\}$ respectively correspond to the equivalences

$$
\begin{aligned}
&x\ R_1\ y \text{ if and only if } \ x = y \text{ or } -y\\
&x\ R_2\ y \text{ if and only if } \ x - y \in \mathbb{Z}, \text{ i.e. } x - \lfloor x \rfloor = y - \lfloor y \rfloor\\
&x\ R_3\ y \text{ if and only if } \ x - y \text{ or } x + y \in \mathbb{Z}, \text{ i.e. } x - \lfloor x \rfloor = y - \lfloor y \rfloor \text{ or } \lceil y \rceil - y
\end{aligned}
$$

The equivalence R_1 is the greatest bisimulation for Brownian motion with zero distinguished. All three equivalences are bisimulations for Brownian motion with the integers distinguished and R_3 is the greatest possible.

Remark 4.8 Given a group of symmetries $\mathcal{H}$, a set C of states is $R_\mathcal{H}$-closed if and only if for every $h \in \mathcal{H}$, $h(C) = C$. It is tempting to find a nice characterization of time-$R_\mathcal{H}$-closed sets too, however in the case of trajectories, this is much more complicated. It is true that if a set B is time-$R_\mathcal{H}$-closed, then for every $h \in \mathcal{H}$, $h_*(B) = B$ (which we used in the proofs of the examples considered in [6]) but this is no longer an equivalence.

To illustrate this, consider Brownian motion with zero distinguished and the group of symmetries $\{s, id\}$. Define the following trajectories:

$$
\omega_1(t) = \begin{cases} 1 - t & \text{for } t < 2 \\ t - 1 & \text{for } t \geq 2 \end{cases} \qquad \omega_1(t) = \begin{cases} 1 - t & \text{for } t < 1 \\ t - 1 & \text{for } t \geq 1 \end{cases} \qquad \omega_3(t) = t - 1
$$

For every time t, $|\omega_1(t)| = |\omega_2(t)| = |\omega_3(t)|$, which means that any time-$R_\mathcal{H}$-closed set that contains one of these trajectories should contain all of them. Define $B_i = \{\omega_i, -\omega_i\}$ for $i = 1, 2$ or 3. It is clear that $s_*(B_i) = B_i$ but $\omega_j \notin B_i$ for $i \neq j$.

To account for this, we can make the condition more complex by allowing to "use" different functions from $\mathcal{H}$ as time goes by. More formally, define the set $\mathcal{H}_{traj}$ as the set of functions F obtained in the following way.
Given a set $\mathcal{I}$ such that $\mathcal{I} = \mathbb{N}$ or $[\![0, m]\!]$ (for $m \in \mathbb{N}$), an $\mathcal{I}$-indexed family of times t_i such that $t_0 = 0 < t_1 < t_2 < ...$ such that $\bigcup_{i \in \mathcal{I}} [t_i, t_{i+1}) = \mathbb{R}_{\geq 0}$ (where t_{m+1} is understood as ∞) and an $\mathcal{I}$-indexed family of homeomorphisms $f_i \in \mathcal{H}$, we can

define $F : \Omega \to \Omega$ such that for $\omega \in \Omega$,

$$\begin{aligned} F(\omega) : \mathbb{R}_{\geq 0} &\to E_\partial \\ t &\mapsto f_n(\omega(t)) \qquad \text{where } n \text{ is such that } t_n \leq t < t_{n+1} \end{aligned}$$

If a set B is time-$R_\mathcal{H}$-closed, then for all $F \in \mathcal{H}_{traj}$, $F(B) = B$.

However, this is not an equivalence: consider Brownian motion as previously and the trajectories $\omega(t) = t \times \sin(1/t)$ $(\omega(0) = 0)$ and $\omega'(t) = |\omega(t)|$. These two functions are $R_\mathcal{H}$-related at all times but they cannot be accounted for by a "fixed grid" as is done by defining the set $\mathcal{H}_{traj}$. In order to fully understand time-$R_\mathcal{H}$-closed sets, we would need to have some sort of "dynamic grid" that we can refine. This will be tackled in future work.

Remark 4.9 In [6], we also introduced the notion of FD-homomorphism in order to extend the discrete-time notion of zigzags [8].

Given two FD processes, a continuous function $f : E_1 \to E_2$ is called a FD-homomorphism if it satisfies

- $obs_1 = obs_2 \circ f$
- for every $x \in E_1$ and every measurable $B_2 \subset \Omega_2$, $\mathbb{P}_2^{f(x)}(B_2) = \mathbb{P}_1^x(B_1)$ where $B_1 = \{\omega \in \Omega_1 \mid f \circ \omega \in B_2\}$.

We showed that cospans of FD-homomorphisms and bisimulations correspond to one another. In particular, if R is a bisimulation, the quotient by R yields a FD-homomorphism (which is not a homeomorphism if R is non trivial). This is quite different from group of symmetries that identify symmetries of the system.

To illustrate another difference with groups of symmetries, consider Brownian motion with zero distinguished and its group of symmetries $\{id, s\}$. Now look at the set $\{\omega \mid \omega(3) \in [1, 2]\}$. This set is accounted for in the definition of FD-homomorphism, but not in the definition of group of symmetries which, in our case, only allows to consider $\{\omega \mid \omega(3) \in [-2, -1] \cup [1, 2]\}$. Section 6.4 provides an example of a set of FD-homomorphisms which is not a group of symmetries.

5 Game Interpretation

The following games are adaptations from [13,7] to our setting of continuous-time processes. The proofs in this section closely mirror those in [13], they can also be found in the appendix. However, it is especially interesting to note that the game interpretation of bisimulation emphasizes once again the role of trajectories in that concept whereas the game interpretation of temporal equivalence resembles that in discrete time very closely.

5.1 Game interpretation of bisimulation

Definition 5.1 Two trajectories ω and ω' are time-bisimilar if for all times $t \geq 0$, $\omega(t)$ and $\omega'(t)$ are bisimilar.

Lemma 5.2 *Two states x and y are bisimilar if and only if the trajectories ω_x and ω_y are time-bisimilar where ω_z is the trajectory defined by $\omega_z(t) = z$ for all times $t \geq 0$ for a given state z.*

We define the following game. Duplicator's plays are pairs of trajectories that he claims are time-bisimilar. Spoiler is trying to prove him wrong.

- Given two trajectories ω and ω', Spoiler chooses $t \geq 0$ and $B \neq \emptyset \in \mathcal{G}$ such that $\mathbb{P}^{\omega(t)}(B) \neq \mathbb{P}^{\omega'(t)}(B)$
- Duplicator answers by choosing $\omega_0 \in B$ and $\omega_1 \notin B$ such that $obs \circ \omega_0 = obs \circ \omega_1$ and the game continues from (ω_0, ω_1)

A player who cannot make a move at any point loses. Duplicator wins if the game goes on forever. The only way for Spoiler to win is to choose a time-*obs*-closed set.

Theorem 5.3 *Two trajectories ω and ω' are time-bisimilar if and only if Duplicator has a winning strategy from (ω, ω').*

Corollary 5.4 *Two states x and y are bisimilar if and only if Duplicator has a winning strategy from (ω_x, ω_y).*

5.2 Game interpretation of temporal equivalence

We define the following game. Duplicator's plays are pairs of states that he claims are bisimilar. Spoiler is trying to prove him wrong.

- Given two states x and y, Spoiler chooses $t \geq 0$ and $C \neq \emptyset \in \mathcal{E}$ such that $P_t(x, C) \neq P_t(y, C)$.
- Duplicator answers by choosing $x_1 \in C$ and $y_1 \notin C$ that are trace-equivalent and the game continues from (x_1, y_1)

A player who cannot make a move at any point loses. Duplicator wins if the game goes on forever. The only way for Spoiler to win is to choose a set that is closed under trace equivalence. Duplicator's only valid moves are pairs of trace equivalent states.

Theorem 5.5 *Two states x and y are temporally equivalent if and only if Duplicator has a winning strategy from (x, y).*

6 Justification for these behavioural equivalences

The goal of this work is to extend the notion of bisimulation that exists in discrete time to a continuous-time setting. Therefore two important questions are the following: do we get back the definition of bisimulation that existed in discrete time when we restrict Feller-Dynkin processes to (some kind of) discrete-time processes? How well do these notions behave on examples?

6.1 Discrete-time Case

It is common in discrete time to consider several actions. Everything that was exposed in this paper can easily be adapted to accommodate several actions. However, we will not mention actions in this section either for the sake of readability.

Given an LMP $(X, \Sigma, \tau, (\chi_A)_{A \in AP})$ where $\Sigma = \sigma(\mathcal{T})$ where $\mathcal{T}$ is a topology on X, we can always view it as a FD process where transitions happen at every time unit. Since the process has to remain Markovian, it cannot keep track of how long it has spent in a state of the LMP. This is why a state of the FD process is a pair of a state in X and a time explaining how long it has been since the last transition. For trajectories to be cadlag, that time is in $[0, 1)$.

We will write $obs(x) = (\chi_A(x))_{A \in AP}$ to mimick what we have in continuous time.

Formally, the state space of the FD process is $(E, \mathcal{E})$ where the space is defined as $E = X \times [0, 1)$ and is equipped with a topology $\mathcal{O} = \mathcal{T} \times \mathcal{O}([0, 1))$ (where $\mathcal{O}([0, 1))$ denotes the usual topology on the interval $[0, 1)$) and the following σ-algebra $\mathcal{E} = \Sigma \otimes \mathcal{B}([0, 1))$ is generated by this topology. The corresponding kernel is defined for all $x \in X$ and $C \in \mathcal{E}$, $t \geq 0$ and $s \in [0, 1)$ as $P_t((x, s), C) = \tau_{\lfloor t+s \rfloor}(x, C')$ where $C' = \{z \mid (z, t + s - \lfloor t + s \rfloor) \in C\}$ and for $k \geq 1$,

$$\tau_0(x, C') = \mathbb{1}_{C'}(x), \quad \tau_1(x, C') = \tau(x, C') \quad \text{and} \quad \tau_{k+1}(x, C') = \int_{y \in X} \tau(x, dy) \tau_k(y, C')$$

We also define $obs(x, s) = obs(x)$ (i.e. $(obs(x, s))_i = \chi_{A_i}(x)$ where $AP = \{A_1, A_2, ...\}$).

Recall the definition of bisimulation in discrete time.

Definition 6.1 Given an LMP $(X, \Sigma, \tau, (\chi_A)_{A \in AP})$, a *discrete time bisimulation (DT-bisimulation)* R is an equivalence relation on X such that if $x \; R \; y$, then

- $obs(x) = obs(y)$ (i.e. for all $A \in AP$, $\chi_A(x) = \chi_A(y)$)
- for all R-closed sets $C \in \Sigma$, $\tau(x, C) = \tau(y, C)$.

The following lemma is in [6]

Lemma 6.2 *Consider a DT-bisimulation R. If $x \; R \; y$, then for all $n \geq 1$, for all R-closed sets $A_1, ..., A_n$,*

$$\begin{aligned} &\int_{x_1 \in A_1} \cdots \int_{x_n \in A_n} \tau(x, dx_1) \tau(x_1, dx_2) ... \tau(x_{n-1}, dx_n) = \\ &\qquad \int_{x_1 \in A_1} \cdots \int_{x_n \in A_n} \tau(y, dx_1) \tau(x_1, dx_2) ... \tau(x_{n-1}, dx_n) \end{aligned}$$

It is possible to define the notion of trajectories in the LMP and that of trace equivalence just as we did in the case of FD-processes. A *trajectory* is a function $\omega : \mathbb{N} \to X \uplus \{\partial\}$ such that if $\omega(n) = \partial$, then for every $k \geq n$, $\omega(k) = \partial$. Similarly to what was done in section 2, for a state x we can define the probability distribution $\mathbb{P}^x$ on the set of trajectories that extends the finite-time distributions using the

Daniell-Kolmogorov theorem. Two states x and y are *trace equivalent* if for every set B of trajectories that is measurable and time-*obs*-closed, $\mathbb{P}^x(B) = \mathbb{P}^y(B)$.

We have the following result that will be later useful to us.

Lemma 6.3 *In an LMP with countably many atomic propositions, any two states x and y that are DT-bisimilar are trace equivalent.*

Proof. Consider a non-empty, measurable set B that is time-*obs*-closed. We will denote A_0 the atomic proposition corresponding to ∂.

Define the following sets for $k \in \mathbb{N}$:

$$\begin{aligned}
D_{k,l} &= \{\gamma : [\![0,k]\!] \times [\![0,l]\!] \to \{0,1\} \mid \exists \omega \in B\ \forall i \in [\![0,k]\!]\ \forall j \in [\![0,l]\!]\ \chi_{A_j}(\omega(i)) = \gamma(i,j)\} \\
B_{k,l} &= \{\omega : \mathbb{N} \to X \uplus \{\partial\} \mid \exists \gamma \in D_{k,l}\ \forall i \in [\![0,k]\!]\ \forall j \in [\![0,l]\!]\ \chi_{A_j}(\omega(i)) = \gamma(i,j)\} \\
&= \{\omega : \mathbb{N} \to X \uplus \{\partial\} \mid \exists \omega' \in B\ \forall i \in [\![0,k]\!]\ \forall j \in [\![0,l]\!]\ \chi_{A_j}(\omega(i)) = \chi_{A_j}(\omega'(i)) \\
B_{k,l}(\gamma) &= \{\omega : \mathbb{N} \to X \uplus \{\partial\} \mid \forall i \in [\![0,k]\!]\ \forall j \in [\![0,l]\!]\ \chi_{A_j}(\omega(i)) = \gamma(i,j)\}
\end{aligned}$$

with $\gamma \in D_{k,l}$. The set $D_{k,l}$ is finite for every $k, l \in \mathbb{N}$.

First, note that $B = \bigcap_{k,l\in\mathbb{N}} B_{k,l}$. Indeed, by definition $B \subset B_{k,l}$ for every $k, l \in \mathbb{N}$ which proves the direct inclusion. For the reverse inclusion, note that

$$\begin{aligned}
B &= \textstyle\bigcup_{\omega'\in B} \bigcap_{i\in\mathbb{N}} \bigcap_{j\in\mathbb{N}} \{\omega \mid \chi_{A_j}(\omega(i)) = \chi_{A_j}(\omega'(i))\} \\
B_{k,l} &= \textstyle\bigcup_{\omega'\in B} \bigcap_{i=0}^{k} \bigcap_{j=0}^{l} \{\omega \mid \chi_{A_j}(\omega(i)) = \chi_{A_j}(\omega'(i))\}
\end{aligned}$$

The first equality holds since B is time-R-closed. Using these expressions and infinite distributivity of set union and intersections, the equality $B = \bigcap_{k,l\in\mathbb{N}} B_{k,l}$ follows.

Second, for $\gamma \in D_{k,l}$,

$$B_{k,l}(\gamma) = \bigcap_{i=0}^{k} \{\omega \mid \omega(i) \in \bigcap_{j=0}^{l} (\chi_{A_j})^{-1}(\gamma(i,j))\}$$

This proves that $B_{k,l}(\gamma)$ is measurable. Furthermore, $\mathbb{P}^x(B_{k,l}(\gamma)) = \mathbb{P}^y(B_{k,l}(\gamma))$ using lemma 6.2.

Now, $B_{k,l} = \bigcup_{\gamma\in D_{k,l}} B_{k,l}(\gamma)$ is measurable since $D_{k,l}$ is finite. Since for $\gamma \neq \gamma'$, $B_{k,l}(\gamma) \cap B_{k,l}(\gamma') = \emptyset$, we also have that $\mathbb{P}^x(B_{k,l}) = \sum_{\gamma\in D_{k,l}} \mathbb{P}^x(B_{k,l}(\gamma))$ and similarly for $\mathbb{P}^y(B_{k,l})$, which shows that for every $k, l \in \mathbb{N}$, $\mathbb{P}^x(B_{k,l}) = \mathbb{P}^y(B_{k,l})$.

Define $B_k = \bigcap_{l',k'\leq k} B_{k',l'}$. If $l \leq l'$, $B_{k,l'} \subset B_{k,l}$ and similarly if $k \leq k'$, $B_{k',l} \subset B_{k,l}$. This means that $B_k = B_{k,k}$ and $B = \bigcap_{k\in\mathbb{N}} B_k$. Finally, $B_0 \supset B_1 \supset \ldots \supset B_n \supset B_{n+1} \supset \ldots \supset \bigcap_{k\in\mathbb{N}} B_k = B$ and therefore $\mathbb{P}^x(B) = \mathbb{P}^y(B)$ by down-continuity of measures $\mathbb{P}^x$ and $\mathbb{P}^y$. □

Proposition 6.4 *If the equivalence R is a DT-bisimulation, then the relation R' defined as*

$$R' = \{((x,s),(y,s)) \mid s \in [0,1), x\ R\ y\}$$

is a temporal equivalence.

Proof. Consider $(x, s)\ R'\ (y, s)$, $t \geq 0$ and a measurable and R'-closed set C. By definition of P_t, $P_t((x, s), C) = \tau_{\lfloor t+s \rfloor}(x, C')$ where $C' = \{z \mid (z, s') \in C\}$ with $s' = t + s - \lfloor t + s \rfloor$ (and similarly for y).

The set C' is R-closed. Indeed, consider two states $z \in C'$ and $z' \in X$ such that $z\ R\ z'$. These conditions imply that $(z, s') \in C$ and $(z, s')\ R'\ (z', s')$. Since the set C is R'-closed, $(z', s') \in C$ and hence by definition of the set C', $z' \in C'$.

Since $(x, s)\ R'\ (y, s)$, we also have that $x\ R\ y$. By lemma 6.2, we have that $\tau_{\lfloor t+s \rfloor}(x, C') = \tau_{\lfloor t+s \rfloor}(y, C')$. This allows us to conclude that $P_t((x, s), C) = P_t(y, s), C)$.

The initiation condition (trace equivalence) is a direct consequence of lemma 6.3. □

Remark 6.5 In remark 4.8, we hinted at how complex it is to characterize measurable time-R-closed sets when R is defined from a group of symmetries.

A very tempting way to do the proof of Lemma 6.3 would be to state that measurable time-*obs*-closed sets are generated by the sets $X_s^{-1}(obs^{-1}(A))$ where $s \in \mathbb{R}_{\geq 0}$ and $A \subset \{0, 1\}$. Along those lines, in [6], we compared bisimulation and DT-bisimulation and to do so, we used the fact that time-R-closed measurable sets were generated by $X_s^{-1}(C)$ where C is a measurable and R-closed set.

However some further studies led us to realize that understanding time-R-closed sets is much harder than expected when R is an equivalence, even in the seemingly simple context of discrete-time systems. In particular, it is still an open question whether the σ-algebra of measurable time-R-closed (resp time-*obs*-closed) sets is generated by the $X_t^{-1}(C)$ where $t \in \mathbb{R}$ and C is a measurable and R-closed subset of the state space (resp $C = obs^{-1}(A)$ with $A \subset \{0, 1\}$).

In Proposition 6.4, we have corrected the results from [6] but this has required us to use quite different machinery, instead we used down-continuity.

Proposition 6.6 *If the equivalence R is a temporal equivalence, then the relation R' defined as the transitive closure of the relation*

$$\{(x, y) \mid \exists t, t' \in [0, 1) \text{ such that } (x, t)\ R\ (y, t')\}$$

is a DT-bisimulation.

Proof. First note that R' is indeed an equivalence.

Let us consider $x\ R'\ y$, i.e. there exists $(x_i)_{i=1,\ldots,n}$, $(t_i)_{i=1,\ldots,n-1}$ and $(t'_i)_{i=1,\ldots,n-1}$ such that $x = x_1$, $y = x_n$ and for every $1 \leq i \leq n - 1$, $(x_i, t_i)\ R\ (x_{i+1}, t'_i)$.

The fact that $obs(x) = obs(y)$ is a direct consequence of the initiation condition of a temporal equivalence.

Now, consider a Σ-measurable and R'-closed set C'. Define the set $C = \{(z, s) \mid z \in C', s \in [0, 1)\}$. It is $\mathcal{E}$-measurable and R-closed. Since R is a temporal equivalence, for every i, $P_1((x_i, t_i), C) = P_1((x_{i+1}, t'_i), C)$ for every $i \leq n$. Additionally, note that for every $z \in E$ and every $s \in [0, 1)$, $P_1((z, s), C) = \tau(z, C')$. This proves that $\tau(x, C') = \tau(y, C')$. □

Remark 6.7 This result is stronger than the one in [6] since we only ask for R to be a temporal equivalence instead of a bisimulation and additionally, we do not impose further restrictions on the equivalence R such as time-coherence in [6].

These results can be summed up in the following theorem relating temporal equivalence and DT-bisimulation.

Theorem 6.8 *Two states x and y (in the LMP) are DT-bisimilar if and only if for all $t \in [0, 1)$, the states (x, t) and (y, t) (in the Feller-Dynkin process) are temporally equivalent.*

Remark 6.9 The result presented in this paper seems to indicate that temporal equivalence is actually the notion that extends DT-bisimulation to continuous time and that the definition of bisimulation in [6] may be too strong in some contexts. There remains to understand which contexts.

6.2 Basic examples

We now revisit some examples from [6] and clarify them in our new framework.

6.2.1 Deterministic Drift

Consider a deterministic drift on the real line $\mathbb{R}$ with constant speed $a \in \mathbb{R}_{>0}$ and a single atomic proposition. We studied two cases in [6] with 0 as the only distinguished point and with all the integers distinguished from the other points. In both cases, trace equivalence and greatest bisimulation are the same.

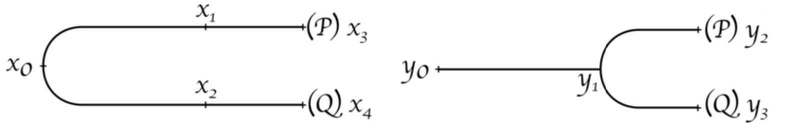
atomic proposition distinguishes...	trace equivalence/ bisimulation: $x\ R\ y$ if and only if	group of symmetries
zero	$x = y$ or $x, y > 0$	generated by $\{f_{x,y} \mid x, y > 0\}$
integers	$x - \lfloor x \rfloor = y - \lfloor y \rfloor$	$\{t_k \mid k \in \mathbb{Z}\}$ generated by $\{t_1\}$

where we define for every $x \in \mathbb{R}$ and every $k \in \mathbb{Z}$, $t_k(x) = x+k$ and for $x, y \in \mathbb{R}_{>0}$ the function $f_{x,y} : \mathbb{R} \to \mathbb{R}$ by

$$f_{x,y}(z) = \begin{cases} z & \text{if } z \leq 0 \\ \frac{y}{x} z & \text{if } 0 \leq z \leq x \\ z - x + y & \text{if } z > x \end{cases}$$

6.2.2 Fork

This example can be found in [6] in order to show how important the induction condition is in the definition of bisimulation. It is an extension of the standard "vending machine" example in discrete time to our continuous-time setting. The process is a deterministic drift at constant speed with a single probabilistic fork (the process then goes up or down with probability 1/2). We compare the case where the fork is at the start with the case where the fork happens later. There are two atomic propositions P and Q which enable the process to tell the difference between the two ends of each fork.



We write $[x_1, x_3]$ for the branch between the states x_1 and x_3 (and similarly for other branches) and we denote $|y - x|$ the time necessary to go from x to y (if applicable).

Two different states x and y are bisimilar if and only if either $x \in [x_1, x_3]$, $y \in [y_1, y_2]$ and $|x_3 - x| = |y_2 - y|$, or $x \in [x_2, x_4]$, $y \in [y_1, y_3]$ and $|x_4 - x| = |y_3 - y|$ (and similar cases by exchanging x and y).

This bisimulation is generated by the group of symmetries $\{f, g, id\}$ where

$$f(z) = \begin{cases} y \in [y_1, y_2] \text{ such that } |y_2 - y| = |x_3 - z| \text{ if } z \in [x_1, x_3] \\ x \in [x_1, x_3] \text{ such that } |y_2 - z| = |x_3 - x| \text{ if } z \in [y_1, y_2] \\ z \text{ otherwise} \end{cases}$$

and g similarly permutes the two branches $[x_2, x_4]$ and $[y_1, y_3]$.

This example is interesting because trace equivalence is strictly greater than bisimulation or temporal equivalence: the states x_0 and y_0 are trace equivalent even though they are neither bisimilar, nor temporally equivalent.

6.3 Examples based on Brownian motion

It is especially interesting to read the proofs that were done for these examples with this new framework in mind. All the proofs follow the same steps. First we define an equivalence. We then state that it is a bisimulation, by actually displaying a set of functions which is a group of symmetries. Second, we show that it corresponds to trace equivalence and hence that it is the greatest bisimulation (and also temporal equivalence/group of symmetries). We will only restate the equivalence on the state space and the group of symmetries. Details and proofs can be found in [6]. All these examples are some variants of Brownian motion with either a single or no (first two cases of absorbing wall) atomic proposition.

6.3.1 Standard Brownian Motion

atomic proposition distinguishes...	trace equivalence/ bisimulation: $x\ R\ y$ if and only if	group of symmetries
zero	$\|x\| = \|y\|$	$\{s, id\}$
integers	$x - \lfloor x \rfloor = y - \lfloor y \rfloor$ or $\lceil y \rceil - y$	$\{id, t_k, s_k \mid k \in \mathbb{Z}\}$ generated by $\{s, t_1\}$
interval $[-1, 1]$	$\|x\| = \|y\|$	$\{s, id\}$

where for every $x \in \mathbb{R}$, $s(x) = -x$, $s_k(x) = k - x$ and $t_k(x) = x + k$ for every $k \in \mathbb{Z}$.

6.3.2 Brownian motion with drift

Let us consider a Brownian process with an additional drift: $W'_t = W_t + at$ (where W_t is the standard Brownian motion and $a > 0$).

atomic proposition distinguishes...	trace equivalence/ bisimulation: $x\ R\ y$ if and only if	group of symmetries
zero	$x = y$	$\{id\}$
integers	$x - \lfloor x \rfloor = y - \lfloor y \rfloor$	$\{id, t_k \mid k \in \mathbb{Z}\}$ generated by $\{t_1\}$
interval $[-1, 1]$	$x = y$	$\{id\}$

where for every $x \in \mathbb{R}$ and every $k \in \mathbb{Z}$, $t_k(x) = x + k$.

6.3.3 Brownian motion with absorbing wall

Another usual variation on Brownian motion is to add boundaries and to consider that the process does not move anymore or dies once it has hit a boundary. The boundary is absorbing/killing the process.

absorbing boundary at	state space	atomic proposition distinguishes	trace equivalence/ bisimulation: $x\ R\ y$ if and only if	group of symmetries
0	$\mathbb{R}_{>0}$	-	$x = y$	$\{id\}$
0 and $b > 0$	$(0, b)$	-	$x = y$ or $x = b - y$	$\{id, s_b\}$
0 and $2b > 0$	$(0, 2b)$	b	$x = y$ or $x = 2b - y$	$\{id, s_{2b}\}$
0 and $4b > 0$	$(0, 4b)$	b	$x = y$	$\{id\}$

where $s_k(x) = k - x$ for every $x \in \mathbb{R}$ and $k = b$ or $2b$.

6.4 Poisson process

This is an example that we did not consider in [6]. Poisson process models the number of customer arriving at a taxi stop for instance. It is a continuous-time process $(N_t)_{t\geq 0}$ on the set of natural numbers $\mathbb{N}$, a discrete space. We define the set Ω of trajectories as usual on the state space. The probability distribution on the set Ω is defined as

$$\mathbb{P}^k(N_t = n) = \frac{(\lambda t)^{n-k}}{(n-k)!}e^{-\lambda t} \qquad \text{for } n \geq k$$

We are going to study two cases. In the first case, we are able to test if there is an even or odd number of customers that have arrived. In the second case, we are able to test if there are more customers than a critical value.

Testing evenness of number of customers:

There is a single atomic proposition on the state space: $obs(k) = 1$ if and only if k is even.

Proposition 6.10 *Two states x and y are bisimilar (resp. temporally equivalent, trace equivalent) if and only if $x \equiv y \mod 2$.*

Proof. Let us write R for the corresponding equivalence.

First, it is indeed a bisimulation. Consider $y = x + 2n$ where $n \in \mathbb{N}$ (note that $obs(x) = obs(y)$) and B a measurable, time-R-closed set.

For a measurable set B', $\mathbb{P}^x(B') = \mathbb{P}^{x+2n}(B' + 2n)$, where $B' + 2n = \{t \mapsto \omega(t) + 2n\} \mid \omega \in B\}$. In particular $\mathbb{P}^x(B) = \mathbb{P}^y(B + 2n)$. Since B is time-R-closed, $B + 2n \subset B$, so $\mathbb{P}^x(B) \leq \mathbb{P}^y(B)$.

For the reverse direction, let M be the set of non-decreasing trajectories, and write $B_0 = B \cap M \cap \{\omega \mid \omega(0) = y\}$. Note that the process can only realize non-decreasing trajectories, therefore $\mathbb{P}^y(B) = \mathbb{P}^y(B_0)$. Define $B_1 = \{t \mapsto \omega(t) - 2n \mid \omega \in B_0\}$. Note that for every $\omega' \in B_1$ and $s \geq 0$, $\omega'(s) \in \mathbb{N}$ since for every $\omega \in B_0$ and $t \geq 0$, $\omega(t) \geq \omega(0) = x + 2n$. Since $B_1 + 2n = B_0$, we have that $\mathbb{P}^y(B_0) = \mathbb{P}^x(B_1)$. Furthermore, $B_1 \subset B$ since B is time-R-closed. Putting all this together: $\mathbb{P}^y(B) = \mathbb{P}^y(B_0) = \mathbb{P}^x(B_1) \leq \mathbb{P}^x(B)$.

This concludes the proof that R is a bisimulation. Now, notice that $x \, R \, y$ if and only if $obs(x) = obs(y)$. Since this is weaker than trace equivalence, we have that R is trace equivalence and the greatest bisimulation and temporal equivalence. $\square$

Remark 6.11 This situation may look a lot like the deterministic or Brownian drift with parity as the atomic proposition. However, there is one key difference here: we are preventing the set of translations by an even number to be a group of symmetries by only allowing positive numbers. These translations are however FD-homomorphisms. Proving that there is no greater group of symmetries than $\{id\}$ is not as trivial as it may look.

Testing for a critical value:

Fix $m \in \mathbb{N}_{\geq 0}$, we define the function *obs* by $obs(x) = 1$ if and only if $x \geq m$.

Proposition 6.12 *Two states $x \neq y$ are bisimilar if and only if $x, y \geq m$.*

Proof. Denote

$$R = \{(x, x) \mid x < m\} \cup \{(x, y) \mid x, y \geq m\}$$

Let us show that it is a bisimulation. Consider $x\ R\ y$ and assume $x \neq y$. This means that $x, y \geq m$ and hence $obs(x) = obs(y)$.

Now consider a measurable time-R-closed set B. Define $B' = B \cap M \cap \{\omega \mid \omega(0) \geq m\}$ where M is the set of non-decreasing functions. Similarly to previous example, M is measurable. Note that the process can only realize non-decreasing trajectories, therefore $\mathbb{P}^y(B) = \mathbb{P}^y(B')$ (and similarly for x).

There are now two cases to consider:

- If B' is empty, $\mathbb{P}^y(B) = \mathbb{P}^x(B) = 0$.
- Otherwise, there exists $\omega' \in B'$. Note that for every time $t \geq 0$, $\omega'(t) \geq \omega'(0) \geq m$ since ω' is non-decreasing.

  We claim that $B' = \{\omega \mid \omega(0) \geq m\} \cap M$. The direct inclusion is by definition. For the reverse implication, consider a non-decreasing trajectory ω such that for every time $\omega(0) \geq m$. This implies that for every time $t \geq 0$, $\omega(t) \geq m$ and since we also have that $\omega'(t) \geq m$, in particular $\omega(t)\ R\ \omega'(t)$. Since $\omega' \in B' \subset B$ and B is time-R-closed, $\omega \in B$ and hence $\omega \in B'$.

  So this means that for every $z \geq m$, $\mathbb{P}^z(B') = \mathbb{P}^z(\{\omega \mid \omega(0) \geq m\} \cap M) = 1$. And in particular, this shows that $\mathbb{P}^x(B) = \mathbb{P}^y(B) = 1$.

To prove that it is the greatest bisimulation, we show that it corresponds to trace equivalence. The proof above can be easily adapted to show that $x, y \geq m$ are trace equivalent.

Clearly if $x < m$ and $y \geq m$, then x and y cannot be trace equivalent since $\mathbb{P}^x(\{\omega \mid obs(\omega(0)) = 0\}) = 1$ and $\mathbb{P}^y(\{\omega \mid obs(\omega(0)) = 0\}) = 0$.

Consider the case when $x \neq y$ are both less than m. Consider a time $t > 0$ and define $B_t = \{\omega \mid \omega(t) \geq m\}$. This set is time-*obs*-closed and for $k < m$,

$$\mathbb{P}^k(B_t) = \sum_{n \geq m} \mathbb{P}^k(N_t = n) = e^{-\lambda t} \sum_{n \geq m-k} \frac{(\lambda t)^n}{n!}$$

This allows us to conclude that if $x \neq y$, $\mathbb{P}^x(B_t) \neq \mathbb{P}^y(B_t)$. □

7 Conclusion

The main lesson we have learned is that the continuous-time setting is far more complex and richer than the discrete-time setting. There are entirely new phenomena at work, for example, the concept of *local time* or the fact that exit and entry times are not always easily definable when the state space is also a continuum. Not surprisingly, there are different possible extensions of the discrete-time equivalences to the continuous-time setting. We have uncovered a few different behavioural equivalences; we expect some of them to be equivalent with some reasonable restrictions on the systems studied. The question of when they are really different is open and tends to get mired in measurability issues.

One of the interesting prospects is the pursuit of the symmetry group point of view. There are Nöther-like theorems for such systems [1] and it would be very interesting to explore such theorems in our setting.

There is still ongoing work to extend logical characterization and event bisimulation to the continuous-time setting. We are also exploring behavioural metrics in this setting.

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A Bisimulation

Lemma A.1 *There is a greatest temporal equivalence.*

Proof. Define $\mathcal{M}$ the set of temporal equivalences and R the transitive closure of $\bigcup_{R' \in \mathcal{R}} R'$.

First note that the relation R is an equivalence. The equivalence $\{(x, x) \mid x \in E\}$ is a bisimulation and hence $x \ R \ x$. Furthermore, if $x \ R \ y$, it means there exists $(x_i)_{i=0,...,n}$ in E and $(R_j)_{j=0,...,n-1}$ in $\mathcal{R}$ such that $x_0 = x$, $x_n = y$ and for every $i \in \{0, ..., n-1\}$, $x_i \ R_i \ x_{i+1}$. Since R_i is an equivalence, $x_{i+1} \ R_i \ x_i$, and hence $y \ R \ x$. Finally, by definition the relation R is transitive.

Now, we can prove that R is a temporal equivalence. Consider $x \ R \ y$, i.e. there exists $(x_i)_{i=0,...,n}$ in E and $(R_j)_{j=0,...,n-1}$ in $\mathcal{R}$ such that $x_0 = x$, $x_n = y$ and for every $i \in \{0, ..., n-1\}$, $x_i \ R_i \ x_{i+1}$.

For the initiation condition, consider a measurable time-*obs*-closed set B. For every $i \in \{0, ..., n-1\}$, $\mathbb{P}^{x_i}(B) = \mathbb{P}^{x_{i+1}}(B)$, since R_i is a temporal equivalence. This proves that

$$\mathbb{P}^x(B) = \mathbb{P}^{x_0}(B) = ... = \mathbb{P}^{x_n}(B) = \mathbb{P}^y(B)$$

For the induction condition, consider $t \geq 0$ and a measurable R-closed set C. Then the set C is R_i-closed for every $i \in \{0, ..., n-1\}$: consider $z \ R_i \ z'$ and $z \in C$, then by definition of R, $z \ R \ z'$ and since C is R-closed, $z' \in C$. Since R_i is a temporal equivalence, $P_t(x_i, C) = P_t(x_{i+1}, C)$ and hence

$$P_t(x, C) = P_t(x_0, C) = ... = P_t(x_n, C) = P_t(y, C)$$

which concludes the proof. □

Proof. [Proof of Theorem 3.8] Let R' be a bisimulation and consider two states x and y such that $x \ R' \ y$.

Consider a time-*obs*-closed set B. Then it is also time-R'-closed: consider two trajectories ω and ω' such that $\omega \in B$ and for every time t, $\omega(t) \ R' \ \omega'(t)$. Then for every time t, $obs(\omega(t)) = obs(\omega'(t))$ (By the initiation condition of bisimulation). Since B is time-*obs*-closed and $\omega \in B$, ω' is also in B. Using the induction condition of bisimulation, we get that $\mathbb{P}^x(B) = \mathbb{P}^y(B)$.

Consider a measurable R-closed set C and a time t. Define the set $B = \{\omega \mid \omega(t) \in C\} = X_t^{-1}(C)$. It is measurable and time-R-closed. We can then apply the induction condition and we get

$$P_t(x, C) = \mathbb{P}^x(B) = \mathbb{P}^y(B) = P_t(y, C)$$

This concludes the proof that R' is a temporal equivalence.

The second part of the lemma follows directly from the initiation condition of a temporal equivalence: this is precisely trace equivalence. □

B Symmetry groups of the process

Proof. [Proof of lemma 4.4] Consider $h = f_1 \circ ... \circ f_n$ such that f_i or f_i^{-1} is in $\mathcal{H}_{gen}$ for every i. First note that since for every $f \in \mathcal{H}_{gen}$, $obs \circ f = obs$ implies that $obs = obs \circ f^{-1}$ and hence for every $i \in [\![1, n]\!]$, $obs \circ f_i = obs$. Finally, we get that

$$obs \circ f_1 \circ ... \circ f_n = obs \circ f_2 \circ ... \circ f_n = ... = obs.$$

Now, consider a set B such that for all $h' \in \mathcal{H}$, $h'_*(B) = B$. In particular, for every $f \in \mathcal{H}_{gen}$, $f_*(B) = B$. This implies that for every y in E_∂ and for every f_i, $\mathbb{P}^y(B) = \mathbb{P}^{f_i(y)}(B)$, which means that for every $x \in E_\partial$,

$$\mathbb{P}^x(B) = \mathbb{P}^{f_n(x)}(B) = \mathbb{P}^{f_{n-1} \circ f_n(x)}(B) = ... = \mathbb{P}^{f_1 \circ ... \circ f_n(x)}(B) = \mathbb{P}^{h(x)}(B).$$

□

The proof of Theorem 4.6 requires two additional lemmas.

Lemma B.1 *Consider a time-$R_\mathcal{H}$-closed set B. Then for every $f \in \mathcal{H}$, $f_*(B) \subset B$.*

Proof. Consider $\omega \in B$ and any time $t \geq 0$. Then $f^{-1}(f_*(\omega)(t)) = f^{-1} \circ f \circ \omega(t) = \omega(t)$. Since f^{-1} is in $\mathcal{H}$, we know that $f(\omega(t))\ R_\mathcal{H}\ \omega(t)$. This is true for any time t and since B is time-$R_\mathcal{H}$-closed, we have that $f_*(\omega) \in B$. □

Lemma B.2 *Given a group of symmetries $\mathcal{H}$, if a set B is time-$R_\mathcal{H}$-closed, then for every $h \in \mathcal{H}$, $h_*(B) = B$.*

Proof. First, using lemma B.1, $h_*(B) \subset B$.

To prove the converse implication, consider $\omega \in B$. We define ω' as $\omega' = (h^{-1})_*(\omega)$. Note that for all times t, $h(\omega'(t)) = \omega(t)$, i.e. $\omega(t)\ R_\mathcal{H}\ \omega'(t)$.

Since B is time-$R_\mathcal{H}$-closed, $\omega' \in B$. We have defined $\omega' = (h^{-1})_*(\omega) = h^{-1} \circ \omega$. A direct consequence is that $\omega = h \circ \omega' = h_*(\omega')$, and therefore $\omega \in h_*(B)$. □

Proof. [Proof of Theorem 4.6] Consider two equivalent states $x\ R_\mathcal{H}\ y$, i.e. there exists $h \in \mathcal{H}$ such that $h(x) = y$.

First, $obs(x) = obs \circ h(x) = obs(y)$.

Second, let us consider a measurable, time-$R_\mathcal{H}$-closed B. Using lemma B.2, we know that for every $f \in \mathcal{H}$, $f_*(B) = B$, and hence $\mathbb{P}^x(B) = \mathbb{P}^{h(x)}(B) = \mathbb{P}^y(B)$, which concludes the proof. □

C Game Interpretation

C.1 Game interpretation of bisimulation

Proof. [Proof of theorem 5.3] Denote R the greatest bisimulation.

For the first implication, if two trajectories ω and ω' are time-bisimilar, we know that for all $t \geq 0$, for all time-R-closed sets B', $\mathbb{P}^{\omega(t)}(B') = \mathbb{P}^{\omega'(t)}(B')$. Spoiler chooses a time $t \geq 0$ and a measurable set B such that $\mathbb{P}^{\omega(t)}(B) \neq \mathbb{P}^{\omega'(t)}(B)$. This means that the set B that Spoiler chose cannot be time-R-closed. That is why Duplicator can find two trajectories $\omega_0 \in B$ and $\omega_1 \notin B$ that are time-bisimilar. This strategy is winning for Duplicator, since it is allowing him to respond to every move from Spoiler and Duplicator wins all infinite plays.

For the reverse implication, define the following relation R' on trajectories: $\omega\ R'\ \omega'$ if and only if duplicator has a winning strategy from (ω, ω').

Note that R' is an equivalence:

- reflexivity: Spoiler has no valid move from (ω, ω), hence duplicator wins.
- symmetry: Assume $\omega\ R'\ \omega'$. Whatever move (B, t) Spoiler does when Duplicator says (ω', ω) is also a valid move from (ω, ω'). Duplicator can then play as he would have from (ω, ω') and if he had a winning strategy then, it is also a winning strategy now. This means that $\omega'\ R'\ \omega$.
- transitivity: Assume $\omega\ R'\ \omega'$ and $\omega'\ R'\ \omega''$. Now consider the game when duplicator starts by saying (ω, ω''). Spoiler then says (B, t) such that $\mathbb{P}^{\omega(t)}(B) \neq \mathbb{P}^{\omega''(t)}(B)$. In this case, note that we have $\mathbb{P}^{\omega(t)}(B) \neq \mathbb{P}^{\omega'(t)}(B)$ or $\mathbb{P}^{\omega'(t)}(B) \neq \mathbb{P}^{\omega''(t)}(B)$ (or both). Duplicator then picks one of those situation (or if only one of them is true, he picks this one) and replies what he would have replied in the game starting with the corresponding start: (ω, ω') or (ω', ω''). Since Duplicator had a winning strategy in both game, he has one here. Hence $\omega\ R'\ \omega''$.

Define the following relation R_1 on states: $z\ R_1\ z'$ if and only if $\omega_z\ R'\ \omega_{z'}$. This relation is an equivalence (this is a direct consequence of the fact that R' is itself an equivalence). Furthermore, this relation is a bisimulation. To prove this, assume it is not a bisimulation. I.e. there exists $x\ R_1\ y$ such that either $obs(x) \neq obs(y)$ or there exists a measurable time-R_1-closed set B of trajectories such that $\mathbb{P}^x(B) \neq \mathbb{P}^y(B)$. We can start by excluding the first case. Indeed we know that $\omega_x\ R'\ \omega_y$, which means that $obs \circ \omega_x = obs \circ \omega_y$, i.e. $obs(x) = obs(y)$.
We can now show that there is a contradiction. Consider the game starting from (ω_x, ω_y). Spoiler says $(B, 0)$. Now, whatever move (ω, ω') ($\omega \in B$ and $\omega' \notin B$) duplicator picks, there exists $t \geq 0$ such that $\omega(t)$ and $\omega'(t)$ are not R_1-related (since B is time-R_1-closed). This means that Spoiler has a winning strategy from $(\omega_{\omega(t)}, \omega_{\omega'(t)})$ that he can play. This contradicts the fact that duplicator has a winning strategy from (ω_x, ω_y). Which proves that R_1 is a bisimulation. $\square$

Remark C.1 We can also define the relation R_2 on states: $x\ R_2\ y$ if and only if there exists ω, ω', t such that $\omega\ R'\ \omega'$, $\omega(t) = x$ and $\omega'(t) = y$.

Trivially, if $x \ R_1 \ y$, then $x \ R_2 \ y$.

Now assume that $x \ R_2 \ y$, and consider ω, ω', t according to the definition of R_2. Duplicator has a winning strategy from (ω_x, ω_y). Indeed, either Spoiler is stuck from the start, in which case duplicator wins, or spoiler says (B, t). This means that $\mathbb{P}^x(B) \neq \mathbb{P}^y(B)$. Duplicator then replies what he would have said in the game starting from (ω, ω') if Spoiler had said (B, t). This proves that $R_1 = R_2$.

C.2 Game interpretation of temporal equivalence

Proof. [Proof of Theorem 5.5] Denote R the greatest temporal equivalence.

For the first implication, if two states x and y are temporally equivalent, we know that for all $t \geq 0$, for all R-closed sets C', $P_t(x, C') = P_t(y, C')$. Spoiler chooses a time $t \geq 0$ and a measurable set C such that $P_t(x, C) \neq P_t(y, C)$. This means that the set C that Spoiler chose cannot be R-closed. That is why Duplicator can find two states $x_1 \in C$ and $y_1 \notin C$ that are temporally equivalent. This strategy is winning for Duplicator, since it is allowing him to respond to every move from Spoiler and Duplicator wins all infinite plays.

For the reverse implication, define the following relation R' on the state space: $x \ R' \ y$ if and only if Duplicator has a winning strategy from (x, y).

Note that R' is an equivalence:

- reflexivity: Spoiler has no valid move from (x, x), hence duplicator wins.
- symmetry: Assume $x \ R' \ y$. Whatever move (C, t) Spoiler does when Duplicator says (x, y) is also a valid move from (y, x). Duplicator can then play as he would have from (x, y) and if he had a winning strategy then, it is also a winning strategy now. This means that $y \ R' \ x$.
- transitivity: Assume $x \ R' \ y$ and $y \ R' \ z$. Now consider the game when duplicator starts by saying (x, z). Spoiler then says (C, t) such that $P_t(x, C) \neq P_t(z, C)$. In this case, $P_t(x, C) \neq P_t(y, C)$ or $P_t(y, C) \neq P_t(z, C)$ (or both). Duplicator then picks one of those situation (or if only one of them is true, he picks this one) and replies what he would have replied in the game starting with the corresponding start: (x, y) or (y, z). Since Duplicator had a winning strategy in both game, he has one here. Hence $x \ R' \ z$.

Furthermore, this relation is a temporal equivalence. To prove this, assume it is not a temporal equivalence, i.e. there exists $x \ R' \ y$ such that either x and y are not trace equivalent, or there exists a measurable R'-closed set C and a time t such that $P_t(x, C) \neq P_t(y, C)$.

Duplicator's only valid moves are pairs of trace equivalent states, so only the second case is possible. Now consider (C, t) to be Spoiler's move from (x, y). Whatever move (x_1, y_1) Duplicator chooses, it is not possible to have $x_1 \ R' \ y_1$ since C is R'-closed. Since the game is determined, Spoiler has a winning strategy from (x', y') which contradicts the fact that Duplicator has a winning strategy from (x, y). Which proves that R' is a temporal equivalence. □