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Generalized Scott Topology on Sets with Families of Pre-orders

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Abstract

[5, Lei Fan] proposed a class of sets with families of pre-orders ($\mathcal{R}$-posets for short). They are not only a non-symmetric generalization of sfe [8, L.Monteiro] but also a special case of quasi-metric spaces (qms, [10, M. B. Smyth]) and generalized ultrametric spaces (gums, [9, J. J. M. M. Rutten]). In this paper, we define a kind of generalized Scott topology on $\mathcal{R}$-posets and discuss some basic properties of the topology. Some relevant interesting examples are offered. It is worth pointing out that an $\mathcal{R}$-monotone functions is $\mathcal{R}$-continuous if and only if (iff for short) it's continuous with respect to (w.r.t for short) the generalized Scott topology.

Keywords: $\mathcal{R}$-posets, Scott Topology, Generalized Scott Topology, $\mathcal{R}$-continuous.
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1 Introduction and Preliminaries

Domain theory is an important branch of computer science, motivated by providing a mathematical foundation of computer functional languages. Information orderings is proposed and applied to interpret quantity or extent of approximating to

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some object. Since information need not to be always comparable, the information ordering is a partial order.

Definition 1.1 A partial order is a binary relation $\sqsubseteq$ over a set P which is antisymmetric, transitive, and reflexive. In other words, a partial order is an antisymmetric preorder. A set with a partial order is called a partially ordered set (also called a poset).

As a supplement and enrichment of partial order, the second author of this work proposed a structure of sets with families of pre-orders in [5]. Inside this mathematical structure, there will be a possibility to interpret or compare information yielded from complex computation process.

Definition 1.2 [[5], L. Fan] Let $(P, \sqsubseteq)$ be a poset and let $(\omega, \leq)$ be the set of natural numbers. If $\mathcal{R} = (\sqsubseteq_n)_{n\in\omega}$ is a family of pre-orders on P, where $\sqsubseteq_0 = P \times P$, such that (i)$\forall n, m \in \omega, m \leq n$ implies $\sqsubseteq_n \subseteq \sqsubseteq_m$, and (ii)$\bigcap_{n\in\omega} \sqsubseteq_n = \sqsubseteq$, then we call $(P, \sqsubseteq)$ a poset with the pre-order family $\mathcal{R}$. We call it $\mathcal{R}$-poset or rpos for short and denote it briefly by $(P, \sqsubseteq; (\sqsubseteq_n)_{n\in\omega})$ or $(P, \sqsubseteq; \mathcal{R})$. $(P, \sqsubseteq)$ is said to be a trivial rpos when $\mathcal{R} = (\sqsubseteq)_{n\in\omega}$.

J.P Gao simulated state transition systems by the notion of rpos in [6] and gave a general method to obtain rpos from sets.

Example 1.3 [[6,8,9]] Consider a transition system $\langle S, A, \rightarrow\rangle$. For $a \in A$ and $U \subseteq S$, let $p_a U = \{s | (\exists t \in U) s \xrightarrow{a} t\}$ be the set of a-predecessors of U. Extend this to traces by $p_\varepsilon U = U$ and $p_{av} = p_a p_v U$. Then $s \in p_v S$ if and only if s has trace v. Let $\mathcal{U}_n = \{p_v S | v \text{ has length } n\}$, so that $\mathcal{U}_0 \cup \cdots \cup \mathcal{U}_n = \{p_v S | v \text{ has length } \leq n\}$. We define $s \sqsubseteq_n t$ if and only if, for every trace v of length $\leq n$, $s \in p_v S$ implies $t \in p_v S$, that is, if s has traces of length $\leq n$ then t has the same traces of length $\leq n$. Then $(\sqsubseteq_n)_{n\in\omega}$ is a pre-order family on S.

Example 1.4 [[6,8]]Let S be a set and $(\mathcal{U}_n)_{n\in\omega}$ a family of sets $\mathcal{U}_n$ of subsets of S where $\mathcal{U}_0 = \{S\}$. Define $s \sqsubseteq_n t$ by requiring that $s \in U$ implies $t \in U$ for every $U \in \mathcal{U}_0 \cup \cdots \cup \mathcal{U}_n$. This gives an rpos.

Example 1.5 We give several interesting examples of rpos in Figures A.1–A.6 in Appendix A. In each figure, the orders in $(\sqsubseteq_n)_{n\in\omega}$ and $\sqsubseteq = \bigcap_{n\in\omega} \sqsubseteq_n$ are represented as Hasse diagrams. Throughout the paper we always assume that the pre-orders in each figure are in turn $\sqsubseteq_1, \sqsubseteq_2, \sqsubseteq_3, \cdots$, and $\sqsubseteq = \bigcap_{n\in\omega} \sqsubseteq_n$ unless otherwise noted.

For a poset $(P, \sqsubseteq)$ with a pre-order family $\mathcal{R} = (\sqsubseteq_n)_{n\in\omega}$, it is easy to see that $\sqsubseteq \subseteq \sqsubseteq_n$ for all $n \in \omega$, thus pre-order family $(\sqsubseteq_n)_{n\in\omega}$ can be seen as a simplicity sequence of $\sqsubseteq$ on P. We interpret $x \sqsubseteq_n y$ as indicating the extent $(1/n)_{n\in\omega}$ (the smaller the better) to which the transitions of x can be simulated by y. Thus, it is not surprising that we assume that any two elements are in relation $\sqsubseteq_0$ and that $\sqsubseteq_n \subseteq \sqsubseteq_m$ for $m \leq n$ in ω. A typical application of this notion is to objects that can be structured or evaluated in stepwise manner, where it makes sense to state that an object can be simulated by another object up to level n.

If we replace pre-orders '$(\sqsubseteq_n)_{n\in\omega}$' in rpos with equivalences '$(\equiv_n)_{n\in\omega}$' then we obtain the notion of set with a family of equivalences (sfe) [8, L.Monteiro]. According to this idea, rpos is an order-version of sfe. The generalization from sfe to $\mathcal{R}$-poset is motivated by the desire to have a better world of reconciling metric spaces with domains. We define a distance $d(x, y) = 2^{-n}$ when x can be simulated by y up to the greatest (level) n (if x can be simulated by y up to every level n then $d(x, y) = 0$). Thus rpos's are also a particular case of quasi-metric spaces (qms) [10] and generalized ultrametric spaces (gums) [9]. The quantification mechanism in rpos's partially agrees with one of the basic ideas of constructive analysis, that is, replacing the arbitrary $\epsilon \geq 0$ with a particular rational sequence such as $(1/n)_{n\in\omega}$ or $(2^{-n})_{n\in\omega}$. rpos is also a special type of L-fuzzy domains([3,4]). The advantage of restriction to the rpos is that it simplifies definitions and allows a larger set of constructions. Furthermore, in contrast to quantitative framework of metrics, the presentation of conclusions and proofs on rpos's is uncomplicated and much closer to standard domain theory since there are only orders and natural numbers in rpos.

Scott topology on posets is one of the fundamental structures in theoretical computer science. In this work, we propose a kind of generalized Scott topology on rpos as a generalization of Scott topology on posets. This paper continues the work of [15] and partially rewrites the work of [2,17]. Section 2 give definition and some basic facts of generalized Scott topology on rpos, for instance the topology restricted to trivial pre-order family is the Scott topology defined by ω-chains in general sense, and the generalized Scott topology is finer than the generalized Alexander topology. The specialization pre-order and comparison of topologies on rpos are discussed in section 3. In this section, some interesting examples are displayed. It is the main contribution of this work that the pre-order family is shown by figures in a concise description. In section 4 we investigate topology limits of $\mathcal{R}$-chain and prove that an $\mathcal{R}$-monotone function is $\mathcal{R}$-continuous iff it's continuous w.r.t the generalized Scott topology. Finally, in section 5 some related work is discussed and some future work is indicated.

As for prerequisites, the reader is expected to be familiar with the partial orders and topology. The notations and terminologies will be mostly standard in domain theory. We refer the readers to [1], [5] and [11] for those having no detailed explanations.

Definition 1.6 Let $P = (P, \sqsubseteq)$ be a poset. $D \subseteq S$ is directed if it's nonempty and every finite subset of D has an upper bound in D. Denote it briefly by $D \subseteq_{dir} P$. If $D \subseteq_{dir} P$ and supremum of D exists, then the supremum of D is written as $\sqcup D$. If every directed subset of P has the supremum, then call P a dcpo (directed complete partial orders). A dcpo with bottom is said to be a cpo. A poset $(P, \sqsubseteq)$ is an ω-complete partial order (ω-cpo) if it has bottom, and every ω-chain $x_0 \sqsubseteq x_1 \sqsubseteq \cdots \sqsubseteq x_n \sqsubseteq \cdots$ in P has a supremum in P.

Definition 1.7 Let $P = (P, \sqsubseteq)$ be a poset, $\mathcal{A} = \{A \subseteq P | A = \uparrow A\}$ is called Alexander topology on P.

Proposition 1.8 *A function $f : P \to Q$ between posets P and Q is continuous* *w.r.t Alexander topology if and only if f is monotone.*

Definition 1.9 [Scott topology, Scott-continuous] Let $(P, \sqsubseteq)$ be a poset. A subset $U \subseteq P$ is called Scott-open if (i) it is upper, i.e., if $x \in U$, $x \sqsubseteq y$ then $y \in U$, and (ii) if $D \subseteq_{dir} P$ for which $\sqcup D \in U$(if $\sqcup D$ exists), then $D \cap P \neq \emptyset$. The collection of Scott open sets forms a topology on P and we call it Scott topology. A function between partially ordered sets is Scott-continuous if and only if it is continuous w.r.t the Scott topology.

Theorem 1.10 *Let P and Q be posets. A function $f : P \to Q$ is Scott-continuous if and only if f is monotone and $f(\sqcup D) = \sqcup f(D)$ for any $D \subseteq_{dir} P$ (if $\sqcup D$ exists).*

Remark 1.11 If '$D \subseteq_{dir} P$' is replaced with 'ω-chain $(x_n)_{n\in\omega} \subseteq P$' in Theorem 1.10, then f is said to be ω-continuous when $f(\sqcup x_n) = \sqcup f(x_n)$. Obviously, the ω-continuity of f coincides with continuity of f on Scott topology defined by replacing '$D \subseteq_{dir} P$' with 'ω-chain $(x_n)_{n\in\omega} \subseteq P$' in Definition 1.9. Though we knew ω-chains does not have (but arbitrarily long chains have, cf.[7]) the full power of directed sets, it is not our focus in this paper.

Notation 1.12 *For an* rpos $(P, \sqsubseteq; (\sqsubseteq_n)_{n\in\omega}), z \in P$ *and* $U \subseteq P$, *we introduce the following notations. For each $n \in \omega$, we use $=_n$ to denote the equivalent relation determined by $\sqsubseteq_n$, i.e., $x =_n y$ if and only if $x \sqsubseteq_n y, y \sqsubseteq_n x$; denote the bottom (if it exists) of $(P, \sqsubseteq_n)$ by $\perp_n$;*

$$\uparrow^n z = \{y \in A \mid z \sqsubseteq_n y\};$$

$$\uparrow^n U = \{x' \in A \mid \exists x \in U, x \sqsubseteq_n x'\};$$

$\downarrow_n z, \downarrow_n U$ is defined dually. If D is directed in $(P, \sqsubseteq_n)$, then then we denote its supremum (if it exists) by $\sqcup^n D$.

2 Generalized Scott Topology on $\mathcal{R}$-posets

For the purpose of completion, the following results in [15] from Definition 2.1 to Theorem 2.3 will be announced without proofs.

Definition 2.1 [$\mathcal{R}$-upper sets] Let $(P, \sqsubseteq; (\sqsubseteq_n)_{n\in\omega})$ be an rpos. $V \subseteq P$ is an $\mathcal{R}$-upper set if for every $x \in V$, there exists an $n \in \omega$ such that $\uparrow^n x \subseteq V$. In particular, for each $x \in P$ and each $n \in \omega, \uparrow^n x$ is an $\mathcal{R}$-upper set.

It is clear that $\mathcal{R}$-upper sets represent a kind of closed property associated with finite simulations.

Theorem 2.2 *The collection of $\mathcal{R}$-upper sets forms a topology on P. The topology is called generalized Alexander topology.*

In an rpos $(P, \sqsubseteq; (\sqsubseteq_n)_{n\in\omega})$, the open sets of generalized Alexander topology are called gA-open sets. We use $\mathcal{O}_{gA}$ to denote the collection of all gA-open sets on P. A function between rpos's is said to be gA-continuous if and only if it is continuous w.r.t the gA-topology. A gA-continuous function can be intrinsically characterized by orders in $(\sqsubseteq_n)_{n\in\omega}$.

Theorem 2.3 *Let P and Q be* rpos*'s, $f : P \to Q$ is gA-continuous iff for arbitrary $p \in \omega$, there exists an $n \in \omega$ such that $\forall s, t \in P,\ s \sqsubseteq_n t$ implies $f(s) \sqsubseteq_p f(t)$.*

Definition 2.4 [$\mathcal{R}$-chain, $\mathcal{R}$-complete] Let $(P, \sqsubseteq; (\sqsubseteq_n)_{n\in\omega})$ be an rpos and let $(x_n)_{n\in\omega}$ be a sequence in P. $(x_n)_{n\in\omega}$ is called an $\mathcal{R}$-chain if $x_n \sqsubseteq_n x_{n+1}$ for every $n \in \omega$. If $x \in P$ satisfies (i)$\forall n \in \omega, x_n \sqsubseteq_n x$, (ii) $x \sqsubseteq y$ for any $y \in P$ satisfying (i), then x is called the least $\mathcal{R}$-upper bound or $\mathcal{R}$-limit of $(x_n)_{n\in\omega}$. We denote it by $x = \bigsqcup^{\mathcal{R}}_{n\in\omega} x_n$. In this case the sequence is said to be convergent (to x). P is an $\mathcal{R}$-complete $\mathcal{R}$-poset (crpos for short) if every $\mathcal{R}$-chain in P is convergent. In an rpos, it can easily be verified that the $\mathcal{R}$-limit of an $\mathcal{R}$-chain is unique if it exists. All rpos's displayed in Fig A.1-A.6 are crpos's, while a non-crpos can be easily obtained, for instance, by removing node 'T' and retaining pre-order family in Fig A.5.

Remark 2.5 The following assumption, which is originally proposed for cartesian-closed categories of crpos and $\mathcal{R}$-continuous functions (cf. [12]), will be needed throughout the paper. It is required that $\mathcal{R}$-limits preserve every order in $(\sqsubseteq_n)_{n\in\omega}$, i.e., $\bigsqcup^{\mathcal{R}}_{n\in\omega} x_n \sqsubseteq_m \bigsqcup^{\mathcal{R}}_{n\in\omega} y_n$ holds if $x_n \sqsubseteq_m y_n$ for some $m \in \omega$ and all $n \geq n_0$ where $n_0 \in \omega$. In particular, if $x \sqsubseteq_m y$ then clearly, we have $x \sqsubseteq_n x \sqsubseteq_{n+1} x \sqsubseteq_{n+2} \cdots$ and $y \sqsubseteq_n y \sqsubseteq_{n+1} y \sqsubseteq_{n+2} \cdots$. It is not surprising that the assumption only means that $\bigsqcup^{\mathcal{R}}_{n\in\omega} x = x \sqsubseteq_m y = \bigsqcup^{\mathcal{R}}_{n\in\omega} y$ holds.

Definition 2.6 [gS-open] Let $(P, \sqsubseteq; (\sqsubseteq_n)_{n\in\omega})$ be an rpos. $U \subseteq P$ is said to be a generalized Scott open set (gS-open set for short), if for any $\mathcal{R}$-chain $(x_n)_{n\in\omega}$ and $\bigsqcup^{\mathcal{R}}_{n\in\omega} x_n \in U$, then there exists $n \in \omega$ such that $\uparrow^n x_n \subseteq U$. We use $\mathcal{O}_{gS}$ to denote the collection of all gS-open sets on P.

Theorem 2.7 *$\mathcal{O}_{gS}$ is a topology on $(P, \sqsubseteq; (\sqsubseteq_n)_{n\in\omega})$. We call it generalized Scott topology (gS-topology for short).*

Proof. Empty set and P are gS-open sets. Suppose $U_1, U_2 \in \mathcal{O}_{gS}$, $(x_n)_{n\in\omega}$ is an $\mathcal{R}$-chain and $\bigsqcup^{\mathcal{R}}_{n\in\omega} x_n \in U_1 \cap U_2$. Then

$$\bigsqcup\nolimits^{\mathcal{R}}_{n\in\omega} x_n \in U_1 \text{ implies } \exists n_1 \in \omega, \uparrow^{n_1} x_{n_1} \subseteq U_1$$
$$\bigsqcup\nolimits^{\mathcal{R}}_{n\in\omega} x_n \in U_2 \text{ implies } \exists n_2 \in \omega, \uparrow^{n_2} x_{n_2} \subseteq U_2$$

Let $n = \max\{n_1, n_2\}$, then $\uparrow^n x_n \subseteq \uparrow^{n_1} x_{n_1} \cap \uparrow^{n_2} x_{n_2} \subseteq U_1 \cap U_2$. Hence $U_1 \cap U_2 \in \mathcal{O}_{gS}$. Let $(U_i)_{i\in\Gamma}$ be gS-open sets (Γ is an index set), if $(x_n)_{n\in\omega}$ is an $\mathcal{R}$-chain and $\bigsqcup^{\mathcal{R}}_{n\in\omega} x_n \in \cup_{i\in\Gamma} U_i$, then $\bigsqcup^{\mathcal{R}}_{n\in\omega} x_n$ is in some U_i ($i \in \Gamma$), so there exists $n \in \omega$ such that $\uparrow^n x_n \subseteq U_i \subseteq \cup_{i\in\Gamma} U_i$. It concludes that gS-open sets are closed for finite intersections and arbitrary unions. □

Theorem 2.8 *$\mathcal{O}_{gS}$ on a trivial* rpos *$(P, \sqsubseteq; (\sqsubseteq)_{n\in\omega})$ is exactly the Scott topology defined by ω-chains on $(P, \sqsubseteq)$.*

Proof. An $\mathcal{R}$-chain on trivial rpos $(P, \sqsubseteq; (\sqsubseteq)_{n\in\omega})$ is exactly an ω-chain in $(P, \sqsubseteq)$ (cf. Definition 2.6). □

Definition 2.9 Let P be an rpos, $b \in P$. b is finite if $\uparrow^n b$ is gS-open for all $n \in \omega$. A subset B of finite elements of an rpos P is a basis for P if every element in P is an $\mathcal{R}$-limit of an $\mathcal{R}$-chain in B. An rpos is algebraic if there exists a basis.

Theorem 2.10 *Let P be an* rpos*. If P is algebraic with basis B, then $\mathcal{B} = \{\uparrow^n b | n \in \omega,\ b \in B\}$ forms a basis for $\mathcal{O}_{gS}$.*

Proof. Let $x \in V$. Since P is algebraic, there is an $\mathcal{R}$-chain $(b_n)_{n\in\omega}$ in B with $x = \bigsqcup^{\mathcal{R}}_{n\in\omega} b_n$. Because V is gS-open and $\bigsqcup^{\mathcal{R}}_{n\in\omega} b_n = x \in V$,

$$\exists n_x \in \omega \text{ such that } \uparrow^{n_x} b_{n_x} \subseteq V \text{ and } x \in \uparrow^{n_x} b_{n_x}.$$

Therefore $V \subseteq \cup_{x\in V} \uparrow^{n_x} b_{n_x}$. Since the other inclusion trivially holds we have $V = \cup_{x\in V} \uparrow^{n_x} b_{n_x}$. Evidently $P = \cup_{B\in\mathcal{B}} B$. If $\uparrow^{n_1} b$ and $\uparrow^{n_2} b_2$ are both in $\mathcal{B}$, and $\uparrow^{n_1} b_1 \cap \uparrow^{n_2} b_2 \neq \emptyset$, then $\uparrow^{n_1} b_1 \cap \uparrow^{n_2} b_2 = \cup\{\uparrow^n b | b \in B,\ \uparrow^n b \subseteq \uparrow^{n_1} b_1 \cap \uparrow^{n_2} b_2\}$. Hence $\exists b \in B,\ n \in \omega,$ such that $\uparrow^n b \subseteq \uparrow^{n_1} b_1 \cap \uparrow^{n_2} b_2$. It concludes that $\mathcal{B} = \{\uparrow^n b | n \in \omega,\ b \in B\}$ forms a basis for $\mathcal{O}_{gS}$. □

Theorem 2.11 (gS-closed) *Let $(P, \sqsubseteq; (\sqsubseteq_n)_{n\in\omega})$ be an* rpos*. $C \subseteq P, C$ is generalized Scott closed (gS-closed for short) iff for any $\mathcal{R}$-chain $(x_n)_{n\in\omega}$ such that $\uparrow^n x_n \cap C \neq \emptyset$ for each $n \in \omega$, then $\bigsqcup^{\mathcal{R}}_{n\in\omega} x_n \in C$.*

Proof. If C is gS-closed, then $P \setminus C$ is gS-open. Let $(x_n)_{n\in\omega}$ be an $\mathcal{R}$-chain, assume that $\bigsqcup^{\mathcal{R}}_{n\in\omega} x_n \notin C$ when $\uparrow^n x_n \cap C \neq \emptyset$ for each $n \in \omega$, then $\bigsqcup^{\mathcal{R}}_{n\in\omega} x_n \in P \setminus C$. Because $P \setminus C$ is gS-open, so there exists $n \in \omega$ such that $\uparrow^n x_n \subseteq P \setminus C$ which contradicts with $\uparrow^n x_n \cap C \neq \emptyset$. Conversely, we need to prove that $P \setminus C$ is gS-open. If $\bigsqcup^{\mathcal{R}}_{n\in\omega} x_n \in P \setminus C$ for an $\mathcal{R}$-chain $(x_n)_{n\in\omega}$ and suppose that $\uparrow^n x_n \nsubseteq P \setminus C$ for all $n \in \omega$, then $\uparrow^n x_n \cap C \neq \emptyset$, hence $\bigsqcup^{\mathcal{R}}_{n\in\omega} x_n \in C$ which contradicts with $\bigsqcup^{\mathcal{R}}_{n\in\omega} x_n \in P \setminus C$. □

Proposition 2.12 *Let $(P, \sqsubseteq; (\sqsubseteq_n)_{n\in\omega})$ be an* rpos*, $x \in P$. Then $\downarrow_m x$ is gS-closed for $m \in \omega$. Consequently, $\downarrow x$ is gS-closed.*

Proof. Let $(x_n)_{n\in\omega}$ be an $\mathcal{R}$-chain. For $m \in \omega$, if $\uparrow^n x_n \cap \downarrow_m x \neq \emptyset$ for each $n \in \omega$, then there exists $y \in \uparrow^n x_n \cap \downarrow_m x$. It means that $x_n \sqsubseteq_n y \sqsubseteq_m x$, so $x_n \sqsubseteq_m x$ when $n \geq m$, hence $\bigsqcup^{\mathcal{R}}_{n\in\omega} x_n \sqsubseteq_m x$ (Remark 2.5). So $\downarrow_m x$ is gS-closed. $\downarrow x$ is gS-closed since $\downarrow x = \cap_{n\in\omega} \downarrow_n x$. □

3 Specialization Pre-order and Comparison of Topologies on $\mathcal{R}$-posets

Let $(P, \mathcal{O})$ be a topological space, $x,\ y \in P$. We define $x \leq_{\mathcal{O}} y$ if and only if for arbitrary $V \in \mathcal{O}$, $x \in O$ implies $y \in O$. $\leq_{\mathcal{O}}$ is called the specialization pre-order and clearly a pre-order on P. It is easy to see that P satisfies T_0 axiom iff $\leq_{\mathcal{O}}$ is a partial order. The generalized Scott topology provides us all information about the underlying pre-order underlying P. The following theorem tells us that $\sqsubseteq = \cap_{n\in\omega} \sqsubseteq_n$ in an rpos can be reconstructed from its generalized Scott topology.

Theorem 3.1 *Let $(P, \sqsubseteq; (\sqsubseteq_n)_{n\in\omega})$ be an* rpos*, $x, y \in P$. Then $x \sqsubseteq y$ if and only if $x \sqsubseteq_{\mathcal{O}_{gS}} y$.*

Proof. If $x \sqsubseteq y, U \in \mathcal{O}_{gS}$, and $x \in U$, then U is gA-open, so there exists $n \in \omega$ such that $\uparrow^n x \subseteq U$. It follows that $y \in U$. Hence $x \sqsubseteq_{\mathcal{O}_{gS}} y$. If $x \sqsubseteq_{\mathcal{O}_{gS}} y$ and $x \not\sqsubseteq y$, then there exists $n \in \omega$ such that $x \not\sqsubseteq_n y$, so $x \in P \backslash \downarrow_n y$. By Proposition 2.11, $\downarrow_n y$ is gS-closed, so $P \backslash \downarrow_n y$ is gS-open. It contradicts with $x \sqsubseteq_{\mathcal{O}_{gS}} y$ since $y \notin P \backslash \downarrow_n y$. □

The following conclusion is a direct corollary of Theorem 3.1 and the general fact that a topological space is T_1 if and only if the specialization order is discrete.

Corollary 3.2 *Let $(P, \sqsubseteq; (\sqsubseteq_n)_{n\in\omega})$ be an* rpos*, then (i) $\mathcal{O}_{gS}$ is T_0, (ii)$\mathcal{O}_{gS}$ is T_1 iff $\sqsubseteq$ is a discrete order.*

Proposition 3.3 *Let $(P, \sqsubseteq; (\sqsubseteq_n)_{n\in\omega})$ be an* rpos*, $U \subseteq P$. If U is gS-open,then U is gA-open, but conversely not, see Example 3.4.*

Proof. If U is gS-open and $x \in U$, then $\exists n \in \omega, \uparrow^n x \subseteq U$ since $x = \bigsqcup^{\mathcal{R}}_{n\in\omega} x$. Hence U is gA-open. □

Example 3.4 Let $P = \omega \cup \{T_n | n \geq 1\} \cup \{S, T\}$ be an rpos and its order family shown in Fig A.1. For $V = \{S, T\} \cup \{T_n | n \geq 1\}$, then V is gA-open. $\omega \subseteq P$ is an $\mathcal{R}$-chain in P, and $\bigsqcup^{\mathcal{R}}_{n\in\omega} \omega = S \in V$. Note that $\uparrow^n n \not\subseteq V$ for any $n \in \omega$, so V is not gS-open.

Intuitively, for the same type of topology induced by orders, the topologies induced by orders in $(\sqsubseteq_n)_{n\in\omega}$ should be a subfamily of the topology induced by $\sqsubseteq = \bigcap_{n\in\omega} \sqsubseteq_n$. In particular, when we consider the discrete order '=' (equality relation), which is a special case of pre-order, the topology induced by '=' is the largest topology - discrete topology. Any topology on P is certainly a subfamily of discrete topology. But this case is not a general case. In the sequel, we investigate the comparability of gS-topology and Scott topologies induced by $\sqsubseteq_n$ and by $\sqsubseteq$.

Proposition 3.5 *Let $(P, \sqsubseteq; (\sqsubseteq_n)_{n\in\omega})$ be an* rpos*, $U \subseteq P$. If U is gS-open,then U is Scott open on $(P, \sqsubseteq)$, but conversely not, see Example 3.6.*

Proof. If U is gS-open, then U is gA-open by Proposition 3.3, hence $\uparrow U = U$. Let $(x_n)_{n\in\omega}$ be an ω-chain in $(P, \sqsubseteq)$ and $\sqcup_{n\in\omega} x_n \in U$, then $(x_n)_{n\in\omega}$ is an $\mathcal{R}$-chain and $\sqcup_{n\in\omega} x_n = \bigsqcup^{\mathcal{R}}_{n\in\omega} x_n \in U$. So there exists $n \in \omega$ such that $\uparrow^n x_n \subseteq U$, which means that $(x_n)_{n\in\omega} \cap U \neq \emptyset$. Therefore U is Scott open in $(P, \sqsubseteq)$. □

Example 3.6 Scott open sets in $(P, \sqsubseteq)$ need not to be gS-open in $(P, \sqsubseteq; (\sqsubseteq_n)_{n\in\omega})$. Let $P = \omega \cup \{T\}$ be an rpos and its order family shown in Fig A.6. Let $V = \{1, T\}$, we have $\uparrow V = V$ and V is Scott open in $(P, \sqsubseteq)$. $2 \sqsubseteq_1 3 \sqsubseteq_2 4 \sqsubseteq_3 5 \sqsubseteq_4 \cdots \sqsubseteq_{n-1} n+1 \sqsubseteq_n \cdots$ is an $\mathcal{R}$-chain with $\mathcal{R}$-limit $T \in V$, but $\uparrow^n x_n = \uparrow^n (n+1) \not\subseteq V$ for any $n \geq 2$.

Example 3.7 gS-open sets in $(P, \sqsubseteq; (\sqsubseteq_n)_{n\in\omega})$ need not to be Scott open in $(P, \sqsubseteq_n)$. Let $P = \omega \cup \{T\}$ be an rpos and its order family shown in Fig A.5. Let $V = \{n | n \geq 1\}$, then V is gS-open, but V not Scott open in any $(P, \sqsubseteq_n)$ since it's not an upper set on $\sqsubseteq_n$.

Scott open sets on some $(P, \sqsubseteq_n)$ need not to be gS-open in $(P, \sqsubseteq; (\sqsubseteq_n)_{n\in\omega})$. In the rpos $(P, \sqsubseteq; (\sqsubseteq_n)_{n\in\omega})$ shown in Fig A.1, let $V = \{S\} \cup \{T_n | n \geq 1\}$, then V is Scott open in $(P, \sqsubseteq_1)$. Let $x_n = n$, $n \geq 1$, then $(x_n)_{n\in\omega}$ is an $\mathcal{R}$-chain with $\bigsqcup^{\mathcal{R}}_{n\in\omega} x_n = S \in V$, but $\uparrow^n x_n \not\subseteq_n V$ for any $n \in \omega$. So V is not gS-open.

The final part of this section is devoted to investigating the comparability between Scott topologies induced by $\sqsubseteq_n$ and those induced by $\sqsubseteq_{n+1}$ and $\sqsubseteq$. Consider the rpos P in Fig A.1, $\{T_n | n \geq 1\}$ is Scott open in $(P, \sqsubseteq_1)$, but it's not Scott open in $(P, \sqsubseteq_2)$ because $\sqcup^2 (n)_{n\in\omega} = T_1$ and $T_1 \in \{T_n | n \geq 1\}$ but $\{T_n | n \geq 1\} \cap (n)_{n\in\omega} = \emptyset$. We propose a condition to make Scott topologies induced by $\sqsubseteq_n$ and $\sqsubseteq_{n+1}$ comparable in the following theorem.

Theorem 3.8 *Let $(P, \sqsubseteq; (\sqsubseteq_n)_{n\in\omega})$ be an* rpos. *For each $n \in \omega$, (i) if $\sqcup^{n+1} D =_n \sqcup^n D$ for every $D \subseteq_{dir} (P, \sqsubseteq_{n+1})$, then Scott open sets in $(P, \sqsubseteq_n)$ are Scott open in $(P, \sqsubseteq_{n+1})$; (ii) if $\sqcup D =_n \sqcup^n D$ for every $D \subseteq_{dir} (P, \sqsubseteq)$, then Scott open sets in $(P, \sqsubseteq_n)$ are Scott open in $(P, \sqsubseteq)$.*

Proof. Only prove (i), similarly for (ii). Suppose $\sqcup^{n+1} D =_n \sqcup^n D$ for every $D \subseteq_{dir} (P, \sqsubseteq_n)$. Let U be Scott open in $(P, \sqsubseteq_n)$ and $D \subseteq_{dir} (P, \sqsubseteq_{n+1})$, then $\uparrow^n U = U$ which implies $\uparrow^{n+1} U = U$. If $\sqcup^{n+1} D \in U$, then $\sqcup^{n+1} D =_n \sqcup^n D$ which implies $\sqcup^n D \in U$. It concludes that $U \cap D \neq \emptyset$, and U is Scott open in $(P, \sqsubseteq_{n+1})$. □

Remark 3.9 The condition '$\sqcup^{n+1} D =_n \sqcup^n D$' is stronger than '$\sqsubseteq_{n+1} \subseteq \sqsubseteq_n$' since, for $x \sqsubseteq_{n+1} y, \sqcup^{n+1}\{x, y\} =_{n+1} y$, the former only means that $\sqcup^{n+1}\{x, y\} =_n \sqcup^n \{x, y\} =_n y$, i.e., $x \sqsubseteq_n y$. As we see from Fig A.1, '$\sqcup^{n+1} D =_n \sqcup^n D$' does not hold in that rpos, for example, ω is directed in every $(P, \sqsubseteq_n)$ but its supremum is changing ($\sqcup^1 \omega =_1 T, \sqcup^2 \omega =_2 T_1, \cdots, \sqcup^n \omega =_n T_{n-1}, \cdots, \sqcup \omega = S$) along with the decreasing of pre-orders in $\mathcal{R}$.

4 Topological Limits in $\mathcal{R}$-posets and Functions Between Generalized Scott Topologies

An element x is a topological limit of a sequence $(x_n)_{n\in\omega}$ in a topology $\mathcal{O}$, denoted by $x \in \lim_{\mathcal{O},n} x_n$, if, for all $V \in \mathcal{O}$ with $x \in V$, there exists $N \in \omega$, such that $\forall n \geq N, x_n \in V$.

Proposition 4.1 *Let $(P, \sqsubseteq; (\sqsubseteq_n)_{n\in\omega})$ be an* rpos *and $y \in P$. For each $\mathcal{R}$-chain $(x_n)_{n\in\omega} \subseteq P$ and with $x = \bigsqcup^{\mathcal{R}}_{n\in\omega} x_n$, then (i) $y \in \lim_{\mathcal{O}_{gS},n} x_n$ iff $y \sqsubseteq_{\mathcal{O}_{gS}} x$, that is to say, $\lim_{\mathcal{O}_{gS},n} x_n = \downarrow x$; (ii) $\lim_{\mathcal{O}_{gA},n} x_n \subseteq \lim_{\mathcal{O}_{gS},n} x_n$.*

Proof. (i) By Theorem 3.1, if $y \sqsubseteq_{\mathcal{O}_{gS}} x$, then $y \sqsubseteq x$. For all $V \in \mathcal{O}_{gS}$ with $y \in V$, then $x \in V$ since V is gA-open (Proposition 3.3). Note that $x = \bigsqcup^{\mathcal{R}}_{n\in\omega} x_n$, so $\exists n \in \omega, \uparrow^n x_n \subseteq V$. Therefore, $y \in \lim_{\mathcal{O}_{gS},n} x_n$. For the converse, let $y \in \lim_{\mathcal{O}_{gS},n} x_n$. Assume $y \not\sqsubseteq x$, then $\exists n \in \omega, y \not\sqsubseteq_n x$, i.e., $y \in P \setminus \downarrow_n x$. By Proposition 2.12, $P \setminus \downarrow_n x$ is gS-open. Because y is a topological limit of $(x_n)_{n\in\omega}$, so there exists $N \in \omega$, such that $m \geq N, x_m \in P \setminus \downarrow_n x$. Then $x_k \in P \setminus \downarrow_n x$ where $k = m + n + 1$.

But $x_k \sqsubseteq_k x$, which implies $x_k \sqsubseteq_n x$. It gives a contradiction to $x_k \in P \setminus \downarrow_n x$. Therefore, $y \sqsubseteq_{\mathcal{O}_{gS}} x$.

(ii) If $y \in \lim_{\mathcal{O}_{gA},n} x_n$, and for all $n \in \omega$, we take $V = \uparrow^n y$, then $V \in \mathcal{O}_{gA}$ and $y \in \uparrow^n y$. Hence, $\exists N \in \omega$ such that $m \geq N, x_m \in \uparrow^n y$. Note that n is arbitrary, we have $y \in \downarrow x$. Consequently, $\lim_{\mathcal{O}_{gA},n} x_n \subseteq \downarrow x = \lim_{\mathcal{O}_{gS},n} x_n$. But the inverse inclusion does not necessarily have to hold. Consider the rpos $P = \omega \cup T$ shown in Fig A.6, $\bigsqcup^{\mathcal{R}}_{n\in\omega}(n+2) = T$, $1 \in \downarrow T$ while $1 \notin \lim_{\mathcal{O}_{gA},n}(n+2)$, because there does not exists any $N \in \omega$ such that $n \geq N, n+2 \in \{1, T\}$ even though 1 is in the gA-open set $\{1, T\}$. □

However, not all topologically convergent sequences are $\mathcal{R}$-chains. For example, let $P = \omega \cup \{T\}$ be an rpos and its order family shown in Fig A.4. For $x_n = n+1, n \in \omega$, then the sequence $(x_n)_{n\in\omega}$ topologically converges to T but $(x_n)_{n\in\omega}$ is not an $\mathcal{R}$-chain.

Definition 4.2 [$\mathcal{R}$-monotone, $\mathcal{R}$-continuous, gS-continuous] Let $f : P \to Q$ be a function between rpos's P and Q. f is $\mathcal{R}$-monotone if f is monotone on every $\sqsubseteq_n$ in $\mathcal{R}$. f is $\mathcal{R}$-continuous if f is $\mathcal{R}$-monotone and $f(\bigsqcup^{\mathcal{R}}_{n\in\omega} x_n) = \bigsqcup^{\mathcal{R}}_{n\in\omega} f(x_n)$ for every $\mathcal{R}$-chain $(x_n)_{n\in\omega}$ in P. A function between rpos's is said to be gS-continuous iff it is continuous w.r.t the gS-topology.

Proposition 4.3 *Let P and Q be crpos's and $f : P \to Q$ be a function. If f is $\mathcal{R}$-monotone, then $\bigsqcup^{\mathcal{R}}_{n\in\omega} f(x_n) \sqsubseteq f(\bigsqcup^{\mathcal{R}}_{n\in\omega} x_n)$ for any $\mathcal{R}$-chain $(x_n)_{n\in\omega}$ in P.*

Proof. Let $(x_n)_{n\in\omega}$ be an $\mathcal{R}$-chain in P and $\bigsqcup^{\mathcal{R}}_{n\in\omega} x_n = x$. If f is $\mathcal{R}$-monotone,then $(f(x_n))_{n\in\omega}$ is an $\mathcal{R}$-chain in Q, and $f(x_n) \sqsubseteq_n \bigsqcup^{\mathcal{R}}_{n\in\omega} f(x_n)$ for every $n \in \omega$, then by Definition 2.4, $\bigsqcup^{\mathcal{R}}_{n\in\omega} f(x_n) \sqsubseteq f(\bigsqcup^{\mathcal{R}}_{n\in\omega} x_n)$ holds. □

Theorem 4.4 *Let P and Q be rpos's and $f : P \to Q$ an $\mathcal{R}$-monotone function. Then f is $\mathcal{R}$-continuous if and only if it's gS-continuous.*

Proof. If a function f is $\mathcal{R}$-continuous, $V \subseteq Q$ is gS-open, and $\bigsqcup^{\mathcal{R}}_{n\in\omega} x_n \in f^{-1}(V)$ where $(x_n)_{n\in\omega}$ is an $\mathcal{R}$-chain in P, then $f(\bigsqcup^{\mathcal{R}}_{n\in\omega} x_n) = \bigsqcup^{\mathcal{R}}_{n\in\omega} f(x_n) \in V$. For V is gS-open, so there exists $n \in \omega$ such that $\uparrow^n f(x_n) \subseteq V$, hence $\uparrow^n x_n \subseteq f^{-1}(\uparrow^n f(x_n)) \subseteq f^{-1}(V)$. It follows that $\uparrow^n x_n \subseteq f^{-1}(V)$. Therefore, $f^{-1}(V) \subseteq P$ is gS-open when $V \subseteq Q$ is gS-open. Conversely, because f is gS-monotone, so $\bigsqcup^{\mathcal{R}}_{n\in\omega} f(x_n) \sqsubseteq f(\bigsqcup^{\mathcal{R}}_{n\in\omega} x_n)$ by Proposition 4.3. Additionally, when f is gS-continuous, we assume that $f(\bigsqcup^{\mathcal{R}}_{n\in\omega} x_n) \not\sqsubseteq_m \bigsqcup^{\mathcal{R}}_{n\in\omega} f(x_n)$ for some $m \in \omega$, then $f(\bigsqcup^{\mathcal{R}}_{n\in\omega} x_n) \in Q \setminus (\downarrow_m \bigsqcup^{\mathcal{R}}_{n\in\omega} f(x_n))$. Since $Q \setminus (\downarrow_m \bigsqcup^{\mathcal{R}}_{n\in\omega} f(x_n))$ is gS-open, $f^{-1}(Q \setminus (\downarrow_m \bigsqcup^{\mathcal{R}}_{n\in\omega} f(x_n)))$ is also gS-open, and $\bigsqcup^{\mathcal{R}}_{n\in\omega} x_n \in f^{-1}(Q \setminus (\downarrow_m \bigsqcup^{\mathcal{R}}_{n\in\omega} f(x_n)))$, hence there exists $n \in \omega$ such that $\uparrow^n x_n \subseteq f^{-1}(Q \setminus (\downarrow_m \bigsqcup^{\mathcal{R}}_{n\in\omega} f(x_n)))$. For all $k \in \omega$ such that $k \geq n+m, \uparrow^k x_k \subseteq \uparrow^n x_n \subseteq f^{-1}(Q \setminus (\downarrow_m \bigsqcup^{\mathcal{R}}_{n\in\omega} f(x_n)))$. Therefore $f(x_k) \in f(\uparrow^k x_k) \subseteq Q \setminus (\downarrow_m \bigsqcup^{\mathcal{R}}_{n\in\omega} f(x_n))$, i.e., $f(x_k) \not\sqsubseteq_m \bigsqcup^{\mathcal{R}}_{n\in\omega} f(x_n)$. But $f(x_k) \sqsubseteq_k \bigsqcup^{\mathcal{R}}_{n\in\omega} f(x_n)$ means that $f(x_k) \sqsubseteq_m \bigsqcup^{\mathcal{R}}_{n\in\omega} f(x_n)$. It's a contradiction. □

5 Conclusions

The analogous 'generalized' topologies has built up in different ways (cf. [2,10,17]), but as we see in this work, the definition of generalized Scott topology on rpos is simple but representative, for instance it retains many fundamental properties of Scott topology on posets, and impressively, an $\mathcal{R}$-monotone function between rpos's is continuous w.r.t generalized Scott topology iff it preserves all $\mathcal{R}$-limits(Definition 2.4). As a type of simple but without-loss-of-generality quantitative domain, rpos reflects the ideas of stepwise comparison and stepwise strengthening comparison conditions. The generalized Scott topology and its properties on rpos's is exactly a demonstration of well-behaved respects of this structure with countable order 'levels'. We will go on to investigate the other structures (algebraic domain, continuous domain, powerdomain, etc.) on the domain based on sets with families of pre-orders.

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A Some Figures

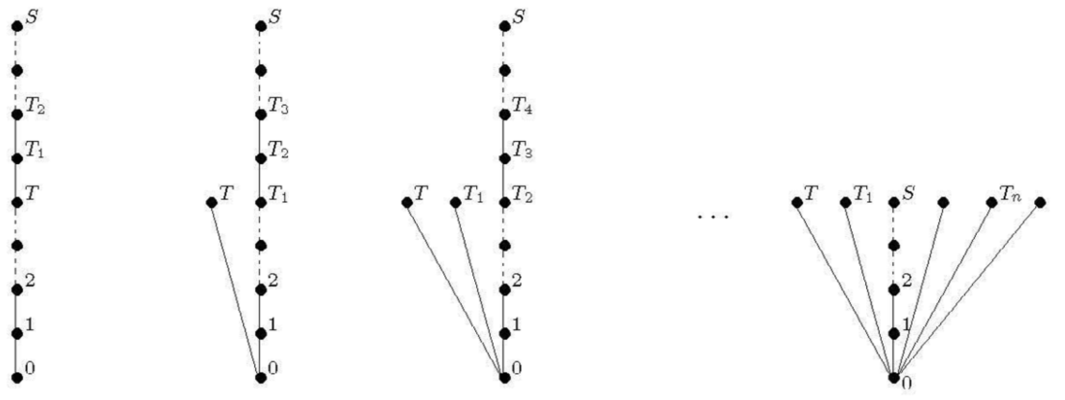

Fig. A.1. Overflowing

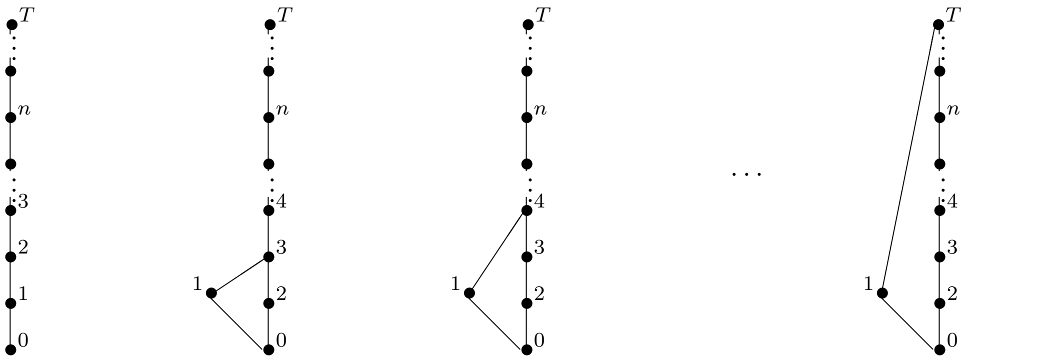

Fig. A.2. Climbing.

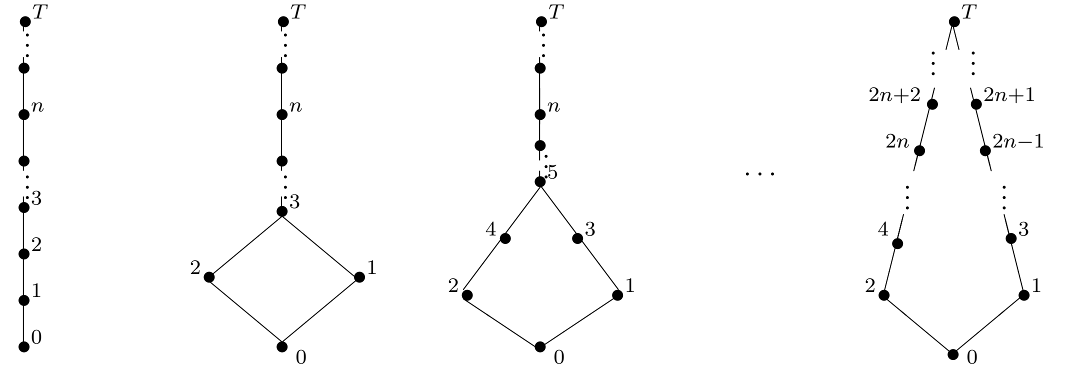

Fig. A.3. Zip.

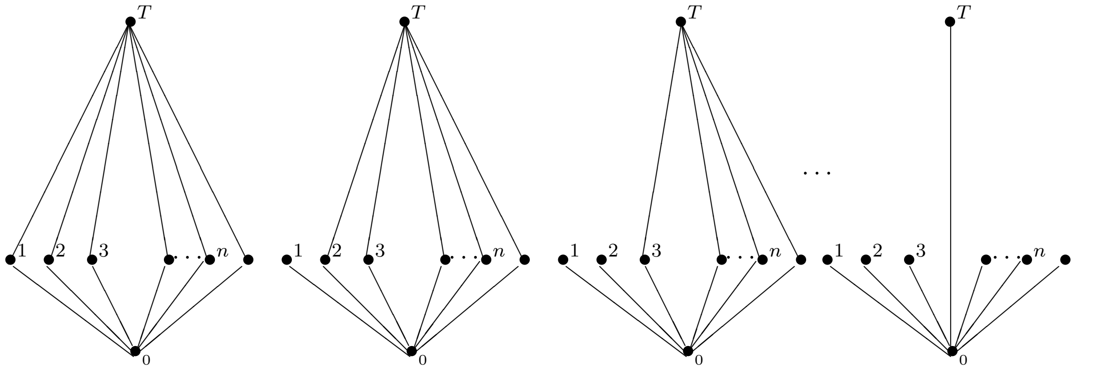

Fig. A.4. Umbrella.

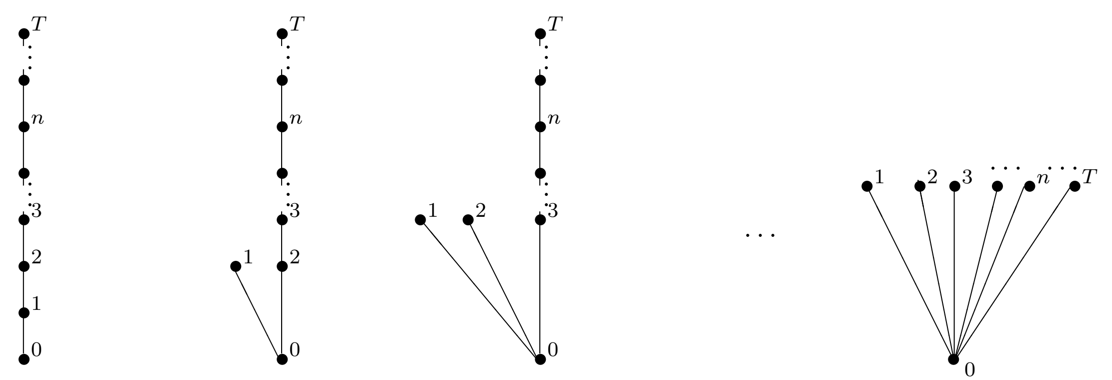

Fig. A.5. Flower.

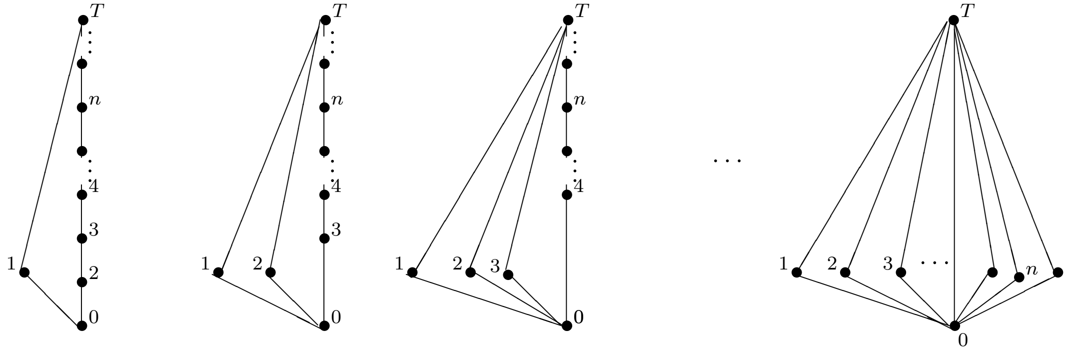

Fig. A.6. Sailing.