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On a Condition for Semirings to Induce Compact Information Algebras

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Abstract

In this paper we study the relationship between ordering structures on semirings and semiring-induced valuation algebras. We show that a semiring-induced valuation algebra is a complete (resp. continuous) lattice if and only if the semiring is complete (resp. continuous) lattice with respect to the reverse order-relation on semirings. Furthermore, a semiring-induced information algebra is compact, if the dual of the semiring is an algebraic lattice.

Keywords: Semiring-induced valuation algebras, Semirings, Way-below relation, Continuous information algebras.

1 Introduction

The valuation algebra, put forward in [1,2], is an abstract inference tool for treating uncertainty and local computation. In [3], it showed that many instances in other research areas, such as soft constraint systems, probability potentials, propositional logic, etc, can be seen as valuation algebras induced by some semirings. This paper concerns ordering structures on this kind of systems called semiring-induced valuation algebras.

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We can see that, this order relation on semiring-induced valuation algebras defined here is related to the operations multiplication and addition on these underlying semirings. While, the ordering on a semiring is defined directly from the addition operation. Naturally it is necessary to study the relationship between ordering structures on semirings and their corresponding semiring-induced valuation algebras. The study on order relations over semiring-induced information algebras can be founded in [2,3]. However, the research on the connection between two orderings described above did not begin. Here we give the necessary and sufficient condition for semirings to induce information algebras. Moreover, we show constructively that the relationship between the continuity of two order relations over semirings and semiring-induced information algebras respectively. Further, we are concerned that how to construct semiring-induced compact information algebras, which are introduced in the study of representation of information. Then some conditions for semirings to induce continuous or compact information algebras will be provided in this paper.

The rest of this paper is organized as follows. In Section 2 some basic notions in the study of semiring-induced valuation algebras are presented. Sufficient and necessary conditions for a semiring-induced valuation algebra to be an information algebra and then complete are shown respectively in Section 3. Section 4 demonstrates the correspondence between the continuity of the order relations on semiring-induced information algebras and semirings. A sufficient condition for semiring-induced information algebras to be compact is given finally.

2 Notations and Preliminaries

2.1 Valuation algebras

In this paper, if L is a partially ordered set, we write $\vee A$ and $\wedge A$ for the least upper bound (also called a supremum) and the greatest lower bound of A (also called an infimum) in L respectively if they exist. A lattice is a partially ordered set in which any two elements have a supremum and an infimum. If $\vee A$ exists for each subset $A \subseteq L$, we call L a complete lattice. The completeness of L is equivalent to the existence of $\wedge A$ for each subset A of L.

For any ordered set L we can form a new ordered set L^∂(the dual of L) by defining $x \leq_\partial y$ to hold in L^∂ if and only if $y \leq x$ holds in L(see [12]). We call this new order relation $\leq_\partial$ on L^∂ the reverse order-relation of L. If L is a complete lattice, then L^∂ is also complete and $\wedge^\partial A = \vee A$ holds for each subset A of L, where $\wedge^\partial A$ means the supremum of A in L^∂.

A subset A of an ordered set L is said to be directed, if for all $a, b \in A$, there is a $c \in A$ such that $a, b \leq c$. For $a, b \in L$, we call a way-below b, in symbols $a \ll b$, if and only if for all directed subsets $X \subseteq L$, if $\vee X$ exists and $b \leq \vee X$, then there exists an $x \in X$ such that $a \leq x$. An element $a \in L$ satisfying $a \ll a$ is said to be a finite element. We denote the set of all finite elements of L by $K(L)$.

Let D be a lattice. (Ψ, D) is a tuple with two operations defined as follows:

1. Combination $\otimes$: $\Psi \times \Psi \to \Psi, (\phi, \psi) \mapsto \phi \otimes \psi$;

2. Focusing $\Rightarrow$: $\Psi \times D \to \Psi, (\psi, x) \mapsto \psi^{\Rightarrow x}$.
The tuple (Ψ, D) is called a domain-free valuation algebra(see [2]), or simply called a valuation algebra hereafter, if it satisfies the following axioms:

(1) Semigroup: Ψ is associative and commutative under combination. There is $e \in \Psi$ called a neutral element such that for all $\psi \in \Psi$ with $e \otimes \psi = \psi \otimes e = \psi$.

(2) Transitivity: For $\psi \in \Psi$ and $x, y \in D, (\psi^{\Rightarrow y})^{\Rightarrow x} = \psi^{\Rightarrow x \wedge y}$.

(3) Combination: For $\phi, \psi \in \Psi, x \in D$, $(\phi^{\Rightarrow x} \otimes \psi)^{\Rightarrow x} = \phi^{\Rightarrow x} \otimes \psi^{\Rightarrow x}$.

(4) Neutrality: For $x \in D$, $e^{\Rightarrow x} = e$.

(5) Support: For $\psi \in \Psi$, there is an $x \in D$ such that $\psi^{\Rightarrow x} = \psi$.

If a valuation algebra (Ψ, D) also satisfies the idempotency axiom, we call it an information algebra:

(6) Idempotency: For $\psi \in \Psi$ and $x \in D, \psi \otimes \psi^{\Rightarrow x} = \psi$.

An order relation $\leq$ on a valuation algebra is defined as: $\phi \leq \psi$, if $\phi \otimes \psi = \psi$. The following lemma gives some basic properties about this order relation.

Lemma 2.1 [2] If (Ψ, D) is an information algebra, then for $\phi, \psi \in \Psi$ and $x, y \in D$,

(1) $\phi^{\Rightarrow x} \leq \phi$;

(2) $\phi \otimes \psi = \sup\{\phi, \psi\}$;

(3) $\phi \leq \psi$ implies $\phi^{\Rightarrow x} \leq \psi^{\Rightarrow x}$;

(4) $x \leq y$ implies $\phi^{\Rightarrow x} \leq \phi^{\Rightarrow y}$;

(5) If $\phi_i, \psi_i \in \Psi (i = 1, 2)$, then $\phi_1 \leq \phi_2$ and $\psi_1 \leq \psi_2$ imply $\phi_1 \otimes \psi_1 \leq \phi_2 \otimes \psi_2$.

2.2 Semiring-induced valuation algebras

Let A be a set with two binary operations $+$ and $\times$, where $0, 1 \in A$. We call the tuple $\langle A, +, \times, 0, 1\rangle$ a semiring, if

(i) both operations $+$ and $\times$ are commutative and associative;

(ii) $\times$ distributes over $+$;

(iii) $a + 0 = a$ and $a \times 0 = 0$ for all $a \in A$;

(iv) $a \times 1 = a$ for all $a \in A$.

It should be noted that all semirings in the paper are often called a commutative unital semirings in the classical terminology.

If A is a semiring and if furthermore for all $a \in A$, $a + 1 = 1$, then we call A a c-semiring. An order relation $\leq$ on A is defined by: $a \leq b$, if and only if $a + b = b$.

Lemma 2.2 [3,10] Let A be a semiring with the idempotency of $+$, then

(1) $a \leq a + b$;

(2) $a \leq a'$ and $b \leq b'$ imply $a + b \leq a' + b', a \times b \leq a' \times b'$;

(3) $a + b = sup\{a, b\}$;

(4) $\leq$ is a partial order on A.

If A is a c-semiring, then $a \times b \leq a$. Moreover, if A is a c-semiring and $\times$ is idempotent, then A is a distributive lattice and $a \times b = \inf\{a, b\}$.

Proposition 2.3 *Let A be a semiring. Then A is a c-semiring with the idempotent* *operation* $\times$*, if and only if for all* $a, b \in A$*,*

$$a \times (a + b) = a. \tag{1}$$

Proof. Let A be a c-semiring and the operation $\times$ be idempotent, then by Lemma 2.2, we have

$$a \times (a + b) = a \times a + a \times b = a + a \times b = a.$$

Conversely, we assume the Equation (1) is true. Let $a = 1$, then we have $1 \times (1 + b) = 1$ for all $b \in A$. That is, $1 + b = 1$. So A is a c-semiring. Again, $a \times (a+0) = a$ holds for all $a \in A$, if we take $b = 0$ for this equation. Thus $a \times a = a$. Therefore the conclusion is proven. □

Next we introduce a kind of valuation algebras induced by semirings. Here variables will be designated by capital letters like $X, Y, \cdots$. Let Ω_X be the finite set of possible values of X, and it is called the frame of X. Lower-case letters such as $s, t, \cdots$, denote sets of variables. For a nonempty set s of variables, let Ω_s denote the Cartesian product of the frames Ω_X of the variables $X \in s$, i.e., $\Omega_s = \prod_{X \in s} \Omega_X$, and Ω_s is called the frame of s. If s is empty, for convenience, we denote $\Omega_\emptyset = \{\diamond\}$. The elements of Ω_s are called configurations of s, and we use lower-case, bold-faced letters such as $\mathbf{x}, \mathbf{y}, \cdots$ to designate the configurations. If $\mathbf{x}$ is a configurations with domain s and $t \subseteq s$, then $\mathbf{x}^{\downarrow t}$ denotes the projection of $\mathbf{x}$ to the subdomain t. For a tuple $\mathbf{x} \in \Omega_s$, it can be expressed as $\mathbf{x} = (\mathbf{x}^{\downarrow t}, \mathbf{x}^{\downarrow s-t})$, where $t \subseteq s$.

We consider a nonempty finite set r of variables with finite frames, and assume that there exist at least one variable $X \in r$ such that its frame Ω_X contains at least two elements. Let $D = \mathcal{P}(r)$ be the lattice of subset of r. For $x, y \in D$, we denote $x \leq y$ if $x \subseteq y$. A semiring valuation ϕ with domain $s \subseteq r$ is defined to be a function $\phi : \Omega_s \to A$ that associates a value from a semiring A with configuration $\mathbf{x} \in \Omega_s$. Especially let $e_s : \Omega_s \to A$ be a mapping such that $e_s(\mathbf{x}) = 1$ for all $\mathbf{x} \in \Omega_s$. The symbol $d(\phi)$ means the domain of ϕ, and d is called a labeling operation of valuations. We denote the set of all valuations with domain s by Φ_s and let $\Phi = \bigcup_{s \subseteq r} \Phi_s$.

The operations in the pair (Φ, D) are defined as follows:

1. Combination: $\otimes : \Phi \times \Phi \to \Phi$, for $\phi, \psi \in \Phi$ with $d(\phi) = s, d(\psi) = t$ and $\mathbf{x} \in \Omega_{s \cup t}$, we define

$$\phi \otimes \psi(\mathbf{x}) = \phi(\mathbf{x}^{\downarrow s}) \times \psi(\mathbf{x}^{\downarrow t}).$$

2. Marginalization: $\downarrow: \Phi \times D \to \Phi$ is defined for all $\phi \in \Phi$ with $d(\phi) = s$, $t \subseteq s$ and $\mathbf{x} \in \Omega_t$ by

$$\phi^{\downarrow t}(\mathbf{x}) = \sum_{\mathbf{z} \in \Omega_s : \mathbf{z}^{\downarrow t} = \mathbf{x}} \phi(\mathbf{z}).$$

If the operation $+$ on the semiring A is idempotent, then $e_y^{\downarrow x} = e_x$ for $x \leq y$. Furthermore, let A be a c-semiring with an idempotent operation $\times$, then the idempotency of (Φ, D) is established, i.e., $\phi \otimes \phi^{\downarrow s} = \phi$ for all $\phi \in \Phi$ and $s \subseteq d(\phi)$.[3,9]

A congruence relation σ relative to the operations combination and marginalization on (Φ, D), which is induced by a semiring A with an idempotent operation

$+$, is defined as (see [2]):

$$\phi \equiv \psi \ (\mathrm{mod}\ \sigma) \text{ if, and only if } \phi \otimes e_y = \psi \otimes e_x,$$

where $x = d(\phi), y = d(\psi)$. Let $\Psi = \{[\phi]_\sigma : \phi \in \Phi\}$. Two operations on the object (Ψ, D) are defined:

$$\text{Combination: } [\phi]_\sigma \otimes [\psi]_\sigma = [\phi \otimes \psi]_\sigma;$$
$$\text{Focusing: } [\phi]_\sigma^{\Rightarrow x} = [(\phi \otimes e_x)^{\downarrow x}]_\sigma.$$

It has shown (Ψ, D) is a system satisfying all axioms in valuation algebras. We call it a semiring-induced (domain-free) valuation algebra.

At the final part of this section, we give some conclusions about the order relation on the set Φ. For $\phi, \psi \in \Phi$, we define $\phi \leq \psi$ as before if $\phi \otimes \psi = \psi$. According to the definition of combination, $d(\phi) \leq d(\psi)$ if $\phi \leq \psi$. By Lemma 2.2, the following conclusion can be obtained easily.

Lemma 2.4 *Assume $\phi, \psi \in \Phi$ with $d(\phi) = s, d(\psi) = t$ and $s \leq t$. Then $\phi \leq \psi$ if and only if $\psi(\mathbf{x}) \leq \phi(\mathbf{x}^{\downarrow s})$ for all $\mathbf{x} \in \Omega_t$.*

Lemma 2.5 *Let A be a c-semiring with the idempotency of $\times$, and $\phi, \psi \in \Phi_s$. Define a semiring valuation $\eta : \Omega_s \to A$ by $\eta(\mathbf{x}) = \phi(\mathbf{x}) \vee \psi(\mathbf{x})$ for all $\mathbf{x} \in \Omega_s$. Then $\eta = \phi \wedge \psi$.*

Proof. First, by Lemma 2.4 we have $\eta \leq \phi$ and $\eta \leq \psi$.

Suppose that $\gamma \in \Phi$ is a lower bound of ϕ and ψ. We write $d(\gamma) = t$. For all $\mathbf{x} \in \Omega_s$, we have $\phi(\mathbf{x}) \leq \gamma(\mathbf{x}^{\downarrow t})$ and $\psi(\mathbf{x}) \leq \gamma(\mathbf{x}^{\downarrow t})$ by Lemma 2.4. Then $\eta(\mathbf{x}) \leq \gamma(\mathbf{x}^{\downarrow t})$. So $\gamma \leq \eta$ by Lemma 2.4 again. We obtain now that $\eta = \phi \wedge \psi$. $\square$

3 Conditions for the Completeness of the Ordering

In this part a condition for the completeness of the ordering on these semiring-induced valuation algebras is considered. Here we always assume that the representative ϕ of the class $[\phi]_\sigma$ is taken from the set Φ_r, since $\phi \equiv \phi \otimes e_r (\mathrm{mod}\ \sigma)$.

Theorem 3.1 *Let A be a semiring with an idempotent operation $+$. The valuation algebra (Ψ, D) induced by A is an information algebra if, and only if, A is a c-semiring with the idempotency of $\times$.*

Proof. Assume that A is a c-semiring with the idempotency of $\times$. For all $[\phi]_\sigma \in \Psi$ and $x \in D$, we have

$$[\phi]_\sigma \otimes [\phi]_\sigma^{\Rightarrow x} = [\phi]_\sigma \otimes [(\phi \otimes e_x)^{\downarrow x}]_\sigma = [\phi \otimes (\phi \otimes e_x)^{\downarrow x}]_\sigma.$$

Now we show that $\phi \otimes (\phi \otimes e_x)^{\downarrow x} = \phi \otimes e_x$. In fact, by the idempotency of (Φ, D), we have

$$[\phi \otimes (\phi \otimes e_x)^{\downarrow x}] \otimes (\phi \otimes e_x) = \phi \otimes [(\phi \otimes e_x)^{\downarrow x} \otimes \phi \otimes e_x] = \phi \otimes (\phi \otimes e_x) = \phi \otimes e_x.$$

Thus $\phi \otimes (\phi \otimes e_x)^{\downarrow x} \leq \phi \otimes e_x$. On the other hand, since $e_x \leq (\phi \otimes e_x)^{\downarrow x}$ and therefore by Lemma 2.1, $\phi \otimes e_x \leq \phi \otimes (\phi \otimes e_x)^{\downarrow x}$. Thus we obtain $\phi \otimes (\phi \otimes e_x)^{\downarrow x} = \phi \otimes e_x$, that is, $\phi \otimes (\phi \otimes e_x)^{\downarrow x} \equiv \phi (\mathrm{mod}\ \sigma)$. Therefore,

$$[\phi]_\sigma \otimes [\phi]_\sigma^{\Rightarrow x} = [\phi \otimes (\phi \otimes e_x)^{\downarrow x}]_\sigma = [\phi]_\sigma.$$

So the idempotency is true. Hence (Ψ, D) is an information algebra.

Conversely, suppose that (Ψ, D) induced by the semiring A is an information algebra. For all $a, b \in A$, let Y be a variable with $\Omega_Y = \{y_1, y_2, \cdots, y_n\}(n \geq 2)$ and define $\phi : \Omega_Y \to A$ as follows:

$$\phi(y) = \begin{cases} a, \text{ if } \ y = y_1; \\ b, \text{ if } \ y = y_2; \\ 0, \text{ otherwise.} \end{cases}$$

The idempotency gives $[\phi]_\sigma \otimes [\phi]_\sigma^{\Rightarrow \emptyset} = [\phi]_\sigma$. So $\phi \otimes \phi^{\downarrow \emptyset} = \phi$. Then we obtain that $(\phi \otimes \phi^{\downarrow \emptyset})(y_1) = \phi(y_1)$, that is, $a \times (a + b) = a$. Thus A is a c-semiring and $\times$ is idempotent from Theorem 2.3. □

Lemma 3.2 *Let (Ψ, D) be the information algebra induced by a c-semiring A with an idempotent operation $\times$. For $\phi, \psi \in \Phi$ with $d(\phi) = s, d(\psi) = t$, $[\phi]_\sigma \leq [\psi]_\sigma$ if and only if $\psi(\mathbf{x}^{\downarrow t}) \leq \phi(\mathbf{x}^{\downarrow s})$ holds for all $\mathbf{x} \in \Omega_{s \cup t}$.*

Proof. If $[\phi]_\sigma \leq [\psi]_\sigma$, then $\phi \otimes \psi = \psi \otimes e_s$. So $\phi(\mathbf{x}^{\downarrow s}) \times \psi(\mathbf{x}^{\downarrow t}) = \psi(\mathbf{x}^{\downarrow t})$ holds for all $\mathbf{x} \in \Omega_{s \cup t}$. By Lemma 2.2, we have $\psi(\mathbf{x}^{\downarrow t}) \leq \phi(\mathbf{x}^{\downarrow s})$.

The converse also can be proved similarly. □

Proposition 3.3 *If A is a c-semiring with an idempotent operation $\times$, then $(\Psi, \leq)$ is a distribute lattice.*

Proof. First, we show that Ψ is a lattice: Let $[\phi_1]_\sigma, [\phi_2]_\sigma \in \Psi$. Suppose that $\phi_1, \phi_2 \in \Phi_r$ and $\phi : \Omega_r \to A$ is defined as $\phi(\mathbf{x}) = \phi_1(\mathbf{x}) \vee \phi_2(\mathbf{x})$. Similar as the proof in Lemma 2.5, we can show that $[\phi]_\sigma = [\phi_1]_\sigma \wedge [\phi_2]_\sigma$. Thus Ψ is a lattice by Lemma 2.1.

Let $[\phi]_\sigma, [\psi]_\sigma, [\eta]_\sigma \in \Psi$, where $\phi, \psi, \eta \in \Phi_r$. Clearly $[\phi]_\sigma \vee ([\psi]_\sigma \wedge [\eta]_\sigma)$ is a lower bound of $[\phi]_\sigma \vee [\psi]_\sigma$ and $[\phi]_\sigma \vee [\eta]_\sigma$. Suppose that $[\varphi]_\sigma$ is another lower bound of $[\phi \otimes \psi]_\sigma$ and $[\phi \otimes \eta]_\sigma$. Then for all $\mathbf{x} \in \Omega_r$, we conclude

$$\begin{aligned} \varphi(\mathbf{x}) &\geq (\phi \otimes \psi)(\mathbf{x}) \vee (\phi \otimes \eta)(\mathbf{x}) \\ &= (\phi(\mathbf{x}) \wedge \psi(\mathbf{x})) \vee (\phi(\mathbf{x}) \wedge \eta(\mathbf{x})) \\ &= \phi(\mathbf{x}) \wedge (\psi(\mathbf{x}) \vee \eta(\mathbf{x})) \\ &= (\phi \vee (\psi \wedge \eta))(\mathbf{x}) \end{aligned}$$

from Lemma 3.2, Lemma 2.2 and Lemma 2.5 successively. Now by Lemma 3.2 again we obtain $[\varphi]_\sigma \leq [\phi]_\sigma \vee ([\psi]_\sigma \wedge [\eta]_\sigma)$. By the definition of infimum, the conclusion $[\phi]_\sigma \vee ([\psi]_\sigma \wedge [\eta]_\sigma) = ([\phi]_\sigma \vee [\psi]_\sigma) \wedge ([\phi]_\sigma \vee [\eta]_\sigma)$ is shown. □

The following theorem demonstrates that the completeness of these orderings on semiring-induced information algebras and semirings are equivalent.

Theorem 3.4 *If (Ψ, D) is the valuation algebra induced by a semiring A with an idempotent operation $+$, then Ψ is a complete lattice if and only if $(A, \leq)$ is a complete lattice.*

Proof. Suppose that A is a complete lattice. Let $\Upsilon^* = \{[\phi]_\sigma : \phi \in \Upsilon \subseteq \Phi_r\}$ be a subset of Ψ. Define $\psi : \Omega_r \to A$ by $\psi(\mathbf{x}) = \bigwedge_{\phi \in \Upsilon} \phi(\mathbf{x})$ for all $\mathbf{x} \in \Omega_r$. Next we show that $[\psi]_\sigma = \vee\Upsilon^*$. Clearly $[\psi]_\sigma$ is an upper bound of Υ^* by Lemma 3.2. Let $[\eta]_\sigma$ be another upper bound of Υ^* and $\eta \in \Phi_r$. Lemma 3.2 demonstrates that $\eta(\mathbf{x}) \leq \phi(\mathbf{x})$ for all $\mathbf{x} \in \Omega_r$ and $\phi \in \Upsilon$. Then $\eta(\mathbf{x}) \leq \bigwedge_{\phi \in \Upsilon} \phi(\mathbf{x}) = \psi(\mathbf{x})$. By Lemma 3.2 again we obtain that $[\psi]_\sigma \leq [\eta]_\sigma$. Hence $[\psi]_\sigma$ is the supremum of Υ^*. The necessity is obtained.

Conversely, suppose that B is a nonempty subset of A. For each $b \in B$, let a valuation $\phi_b : \Omega_r \to A$ be a constant function that takes value with b. We denote $\bigvee_{b \in B} [\phi_b]_\sigma = [\psi]_\sigma$, since Ψ is complete. Let $\psi \in \Phi_r$. We verify that $\inf B = \psi(\mathbf{y})$ for any $\mathbf{y} \in \Omega_r$:

(1) For any $b \in B$, we have $[\phi_b]_\sigma \leq [\psi]_\sigma$. By Lemma 3.2 we obtain that $\psi(\mathbf{y}) \leq b$.

(2) Suppose that $a \in A$ is also a lower bound of B. Let $\eta : \Omega_r \to A$ be a constant valuation that takes value with a. Then by Lemma 3.2 again, $[\phi_b]_\sigma \leq [\eta]_\sigma$ for all $b \in B$. Immediately we have $[\psi]_\sigma \leq [\eta]_\sigma$. Thus $a \leq \psi(\mathbf{y})$ for $\mathbf{y} \in \Omega_r$.

The proof above shows that $\inf B = \psi(\mathbf{y})$ for $\mathbf{y} \in \Omega_r$. Therefore, A is a complete lattice. □

4 Conditions for the Compactness of the Ordering

Continuous information algebras defined here are popularized from the concept of compact information algebras [2]. In this section we study conditions for semiring-induced information algebras to be continuous or compact.

Definition 4.1 A system (Ψ, Γ, D), where (Ψ, D) is an information algebra, $\Gamma \subseteq \Psi$ is closed under combination and contains the empty information e, satisfying the following axioms of convergence and density, is called a continuous information algebra.

1. Convergency: If $X \subseteq \Gamma$ is directed, then the supremum $\vee X \in \Psi$ exists.
2. Density: For all $\phi \in \Psi$ and $x \in D$, $\phi^{\Rightarrow x} = \vee\{\psi \in \Gamma : \psi = \psi^{\Rightarrow x} \ll \phi\}$.

Moreover, if a continuous information algebra (Ψ, Γ, D) satisfies the axiom of compactness, then we call (Ψ, Γ, D) a compact information algebra [2].

3. Compactness: If $X \subseteq \Gamma$ is a directed set and $\phi \in \Gamma$ such that $\phi \leq \vee X$, then there exists a $\psi \in X$ such that $\phi \leq \psi$.

Definition 4.2 [5] Let L be a complete lattice.

If for all $a \in L$, $a = \vee\{b \in L : b \ll a\}$, L is called a continuous lattice. If for all $a \in L$, $a = \vee\{b \in K(L) : b \leq a\}$, L is called an algebraic lattice.

An equivalent definition of continuous(resp. compact) information algebras is given as follows.

Theorem 4.3 [6,11] Let (Ψ, D) be an information algebra. Then

(1) (Ψ, D) is continuous if and only if $(\Psi, \leq)$ is a complete lattice and for all $\phi \in \Psi$, $x \in D$, $\phi^{\Rightarrow x} = \vee\{\psi \in \Psi : \psi = \psi^{\Rightarrow x} \ll \phi\}$.

(2) (Ψ, D) is compact if and only if $(\Psi, \leq)$ is a complete lattice and for all $\phi \in \Psi$, $x \in D$, $\phi^{\Rightarrow x} = \vee\{\psi \in \Psi : \psi \ll \psi, \psi = \psi^{\Rightarrow x} \leq \phi\})$.

Theorem 4.4 *Let A be a c-semiring with an idempotent operation $\times$. Then (Ψ, D) induced by A is a continuous information algebra if and only if A^∂ is a continuous lattice.*

Proof. “If” part: Theorem 3.4 shows that Ψ is a complete lattice.

The density of (Ψ, D) can be proven as follows: Let $\phi \in \Phi_r$, $x \in D$ and $\Upsilon = \{\psi \in \Phi_r : [\psi]_\sigma = [\psi]_\sigma^{\Rightarrow x} \ll [\phi]_\sigma\}$. A mapping $\eta : \Omega_r \to A$ is defined as $\eta(\mathbf{x}) = \bigwedge_{\psi \in \Upsilon} \psi(\mathbf{x})$ for all $\mathbf{x} \in \Omega_r$. Similar as the proof presented in Theorem 3.4, we obtain that $[\eta]_\sigma$ is the supremum of $\{[\psi]_\sigma : [\psi]_\sigma = [\psi]_\sigma^{\Rightarrow x} \ll [\phi]_\sigma\}$. While, Lemma 2.1 implies that $[\psi]_\sigma = ([\psi]_\sigma^{\Rightarrow x})^{\Rightarrow x} \leq [\phi]_\sigma^{\Rightarrow x}$ for $\psi \in \Upsilon$. Then we have $[\eta]_\sigma \leq [\phi]_\sigma^{\Rightarrow x}$. Thus, in order to prove the continuity

$$[\phi]_\sigma^{\Rightarrow x} = \vee\{[\psi]_\sigma : [\psi]_\sigma = [\psi]_\sigma^{\Rightarrow x} \ll [\phi]_\sigma\} = [\eta]_\sigma,$$

it suffices to show $[\phi]_\sigma^{\Rightarrow x} \leq [\eta]_\sigma$. Since, for all $\mathbf{z} \in \Omega_r$,

$$\phi^{\downarrow x}(\mathbf{z}^{\downarrow x}) = \textstyle\sum_{\mathbf{y} \in \Omega_{r-x}} \phi(\mathbf{z}^{\downarrow x}, \mathbf{y}) = \bigwedge^{\partial}_{\mathbf{y} \in \Omega_{r-x}} \phi(\mathbf{z}^{\downarrow x}, \mathbf{y}),$$

by Lemma 3.2 we only need to prove the claim $\bigwedge^{\partial}_{\mathbf{y} \in \Omega_{r-x}} \phi(\mathbf{z}^{\downarrow x}, \mathbf{y}) \leq_\partial \eta(\mathbf{z})$.

Next we use the continuity of A^∂ to prove the claim above. Take any $a \in A$ such that $a \ll_\partial \bigwedge^{\partial}_{\mathbf{y} \in \Omega_{r-x}} \phi(\mathbf{z}^{\downarrow x}, \mathbf{y})$. We define $\gamma : \Omega_r \to A$ as follows: For $\mathbf{w} \in \Omega_r$, if $\mathbf{w}^{\downarrow x} = \mathbf{z}^{\downarrow x}$, then $\gamma(\mathbf{w}) = a$; otherwise $\gamma(\mathbf{w}) = 1$. Now we show that $\gamma \in \Upsilon$:

(1) $[\gamma]_\sigma = [\gamma]_\sigma^{\Rightarrow x}$: Let $\mathbf{w} \in \Omega_r$. If $\mathbf{w}^{\downarrow x} = \mathbf{z}^{\downarrow x}$, then $\gamma(\mathbf{w}^{\downarrow x}, \mathbf{y}) = a$ for all $\mathbf{y} \in \Omega_{r-x}$. So $\gamma^{\downarrow x}(\mathbf{w}^{\downarrow x}) = \bigvee_{\mathbf{y} \in \Omega_{r-x}} \gamma(\mathbf{w}^{\downarrow x}, \mathbf{y}) = a = \gamma(\mathbf{w})$. Otherwise, we have $\gamma^{\downarrow x}(\mathbf{w}^{\downarrow x}) = \bigvee_{\mathbf{y} \in \Omega_{r-x}} \gamma(\mathbf{w}^{\downarrow x}, \mathbf{y}) = 1 = \gamma(\mathbf{w})$. Then $\gamma^{\downarrow x}(\mathbf{w}^{\downarrow x}) = \gamma(\mathbf{w})$ for all $\mathbf{w} \in \Omega_r$. Thus $\gamma^{\downarrow x} \otimes e_r = \gamma$, i.e., $[\gamma]_\sigma = [\gamma]_\sigma^{\Rightarrow x}$.

(2) $[\gamma]_\sigma \ll [\phi]_\sigma$: Assume that $[\phi]_\sigma \leq \bigvee_{i \in I} [\phi_i]_\sigma$, where $\{[\phi_i]_\sigma : i \in I\}$ is a directed subset of Ψ. For all $\mathbf{y} \in \Omega_{r-x}$, by Lemma 3.2 we have $\phi(\mathbf{z}^{\downarrow x}, \mathbf{y}) \leq_\partial \bigvee^{\partial}_{i \in I} \phi_i(\mathbf{z}^{\downarrow x}, \mathbf{y})$. Thus $\gamma(\mathbf{z}^{\downarrow x}, \mathbf{y}) = a \ll_\partial \phi(\mathbf{z}^{\downarrow x}, \mathbf{y}) \leq_\partial \bigvee^{\partial}_{i \in I} \phi_i(\mathbf{z}^{\downarrow x}, \mathbf{y})$. By the definition of way-below, there exists an $i_{\mathbf{y}} \in I$ such that $a \leq_\partial \phi_{i_{\mathbf{y}}}(\mathbf{z}^{\downarrow x}, \mathbf{y})$. By the finiteness of Ω_{r-x} and the directness of $\{[\phi_i]_\sigma : i \in I\}$, there exists a $j \in I$ such that $\phi_{i_{\mathbf{y}}} \leq \phi_j$ for all $\mathbf{y} \in \Omega_{r-x}$. Then the order relation $[\gamma]_\sigma \leq [\phi_j]_\sigma$ is shown. Let $\mathbf{w} \in \Omega_r$. If $\gamma(\mathbf{w}) = 1$, then $\phi_j(\mathbf{w}) \leq \gamma(\mathbf{w})$ is clearly. If $\mathbf{w}^{\downarrow x} = \mathbf{z}^{\downarrow x}$, then there exists a $\mathbf{y} \in \Omega_{r-x}$ such that $\mathbf{w} = (\mathbf{z}^{\downarrow x}, \mathbf{y})$. Thus

$$\phi_j(\mathbf{w}) = \phi_j(\mathbf{z}^{\downarrow x}, \mathbf{y}) \leq \phi_{i_{\mathbf{y}}}(\mathbf{z}^{\downarrow x}, \mathbf{y}) \leq a = \gamma(\mathbf{w}).$$

By Lemma 3.2 again we have $[\gamma]_\sigma \leq [\phi_j]_\sigma$. Thus $[\gamma]_\sigma \ll [\phi]_\sigma$.

We have shown that $\gamma \in \Upsilon$. Then $a = \gamma(\mathbf{z}) \leq_\partial \bigvee^{\partial}_{\psi \in \Upsilon} \psi(\mathbf{z}) = \eta(\mathbf{z})$. Since A^∂ is a continuous lattice and a is arbitrary, we have

$$\bigwedge^{\partial}_{\mathbf{y} \in \Omega_{r-x}} \phi(\mathbf{z}^{\downarrow x}, \mathbf{y}) = \vee\{a \in A : a \ll_\partial \bigwedge^{\partial}_{\mathbf{y} \in \Omega_{r-x}} \phi(\mathbf{z}^{\downarrow x}, \mathbf{y})\} \leq_\partial \eta(\mathbf{z}).$$

So the claim is shown and then the density of (Ψ, D) is obtained. According to Theorem 4.3 this proves (Ψ, D) is continuous.

"Only if" part: Theorem 3.4 implies that A is a complete lattice. Then A^∂ is complete too.

Next we prove the continuity of A^∂, that is $a = \vee^\partial\{b \in A : b \ll_\partial a\}$ for all $a \in A$. Let $\phi_a : \Omega_r \to A$ be a mapping satisfying $\phi_a(\mathbf{x}) = a$ for all $\mathbf{x} \in \Omega_r$. Since (Ψ, D) is continuous, then $[\phi_a]_\sigma = \vee\{[\psi]_\sigma : [\psi]_\sigma \ll [\phi_a]_\sigma\}$. Thus, for all $\mathbf{x} \in \Omega_r$ we have $a = \vee^\partial\{\psi(\mathbf{x}) : [\psi]_\sigma \ll [\phi_a]_\sigma\}$. Let $[\psi]_\sigma \ll [\phi_a]_\sigma$. Assume that D is a directed subset of A^∂ and $a \leq_\partial \vee^\partial D$. Then $[\phi_a]_\sigma \leq \bigvee_{d\in D}[\phi_d]_\sigma$, where ϕ_d is defined as $\phi_d(\mathbf{z}) = d$ for all $\mathbf{z} \in \Omega_r$. Since $[\psi]_\sigma \ll [\phi_a]_\sigma$, then there exists a $d \in D$ such that $[\psi]_\sigma \leq [\phi_d]_\sigma$. It implies that $\psi(\mathbf{x}) \leq_\partial d$. By the definition of way-below we obtain that $\psi(\mathbf{x}) \ll_\partial a$. Thus Equation

$$\{\psi(\mathbf{x}) : [\psi]_\sigma \ll [\phi_a]_\sigma\} \subseteq \{b \in A : b \ll_\partial a\}. \tag{2}$$

holds. It follows that

$$a = \bigvee^\partial\{\psi(\mathbf{x}) : [\psi]_\sigma \ll [\phi_a]_\sigma\} \leq_\partial \bigvee^\partial\{b \in A : b \ll_\partial a\} \leq_\partial a.$$

Then $a = \bigvee^\partial\{b \in A : b \ll_\partial a\}$. Hence A^∂ is a continuous lattice. □

A characteristic of finite elements in information algebras induced by semirings is the following.

Lemma 4.5 *Let A be a c-semiring with an idempotent operation $\times$. If $\phi : \Omega_s \to A$ is a mapping with $\{\phi(\mathbf{x}) : \mathbf{x} \in \Omega_s\} \subseteq K(A^\partial)$, then $[\phi]_\sigma \in K(\Psi)$.*

Proof. Let $\{[\psi]_\sigma : \psi \in \Theta \subseteq \Phi_r\}$ be a directed subset of Ψ and $[\phi]_\sigma \leq \bigvee_{\psi\in\Theta}[\psi]_\sigma$. Suppose that $\Omega_r = \{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n\}$. For all $\mathbf{x}_i \in \Omega_r$, we have $\phi(\mathbf{x}_i^{\downarrow s}) \leq_\partial \bigvee^\partial_{\psi\in\Theta}\psi(\mathbf{x}_i)$. Since $\phi(\mathbf{x}_i^{\downarrow s}) \in K(A^\partial)$, there exists a $\psi_i \in \Theta$ such that $\phi(\mathbf{x}_i^{\downarrow s}) \leq_\partial \psi_i(\mathbf{x}_i)$, i.e., $\psi_i(\mathbf{x}_i) \leq \phi(\mathbf{x}_i^{\downarrow s})$. By the directness of $\{[\psi]_\sigma : \psi \in \Theta\}$, there exists a $\gamma \in \Theta$ such that $[\psi_i]_\sigma \leq [\gamma]_\sigma$ for all $i \in \{1, 2, \cdots, n\}$. By Lemma 3.2, we have $\gamma(\mathbf{x}_j) \leq \psi_j(\mathbf{x}_j) \leq \phi(\mathbf{x}_j^{\downarrow s})$ for any $\mathbf{x}_j \in \Omega_r$. Then $[\phi]_\sigma \leq [\gamma]_\sigma$. Hence $[\phi]_\sigma$ is a finite element of Ψ. □

A necessary condition for semiring-induced information algebras to be compact is given by the following conclusion. This provides a theoretical basis for constructing examples of compact information algebras.

Theorem 4.6 *Let A be a c-semiring with an idempotent operation $\times$. If A^∂ is an algebraic lattice, then (Ψ, D) induced by A is a compact information algebra.*

Proof. By Theorem 3.4, we know Ψ is a complete lattice. Let $\Gamma = K(\Psi)$. According to the selection of the elements of Γ, the compactness in (Ψ, Γ, D) is obvious.

In what follows, we show the density of (Ψ, Γ, D): For $\phi \in \Phi_r$ and $x \in D$,

$$[\phi]_\sigma^{\Rightarrow x} = \vee\{[\psi]_\sigma \in \Gamma : [\psi]_\sigma = [\psi]_\sigma^{\Rightarrow x} \leq [\phi]_\sigma\}.$$

Let $\Upsilon = \{\psi \in \Phi_r : [\psi] \in \Gamma, [\psi]_\sigma = [\psi]_\sigma^{\Rightarrow x} \leq [\phi]_\sigma\}$ and $\eta : \Omega_r \to A$ be defined as $\eta(\mathbf{x}) = \bigwedge_{\psi\in\Upsilon}\psi(\mathbf{x})$ for all $\mathbf{x} \in \Omega_r$. Similar as the proof in Theorem 4.4, it is suffice to show that $\bigwedge^\partial_{\mathbf{y}\in\Omega_{r-x}}\phi(\mathbf{z}^{\downarrow x}, \mathbf{y}) \leq_\partial \eta(\mathbf{z})$ for all $\mathbf{z} \in \Omega_r$. Take any $k \in K(A^\partial)$ such that

$k \leq_\partial \bigwedge^\partial_{\mathbf{y}\in\Omega_{r-x}} \phi(\mathbf{z}^{\downarrow x}, \mathbf{y})$. We define $\gamma : \Omega_r \to A$ as follows: If $\mathbf{w}^{\downarrow x} = \mathbf{z}^{\downarrow x}, \gamma(\mathbf{w}) = k$; otherwise $\gamma(\mathbf{w}) = 1$. By Lemma 4.5, $[\gamma]_\sigma \in \Gamma$. Meanwhile, $[\gamma]_\sigma = [\gamma]_\sigma^{\Rightarrow x} \leq [\phi]_\sigma$ holds if we take the proof as shown in Theorem 4.4. Hence $\gamma \in \Upsilon$. Then $k = \gamma(\mathbf{z}) \leq_\partial \bigvee^\partial_{\psi\in\Upsilon} \psi(\mathbf{z}) = \eta(\mathbf{z})$. Since A^∂ is an algebraic lattice, we obtain that

$$\bigwedge\nolimits^\partial_{\mathbf{y}\in\Omega_{r-x}} \phi(\mathbf{z}^{\downarrow x}, \mathbf{y}) = \bigvee\nolimits^\partial\{k \in K(A^\partial) : k \leq_\partial \bigwedge\nolimits^\partial_{\mathbf{y}\in\Omega_{r-x}} \phi(\mathbf{z}^{\downarrow x}, \mathbf{y})\} \leq_\partial \eta(\mathbf{z}).$$

Then

$$\eta(\mathbf{z}) \leq \bigvee\nolimits_{\mathbf{y}\in\Omega_{r-x}} \phi(\mathbf{z}^{\downarrow x}, \mathbf{y}) = \phi^{\downarrow x}(\mathbf{z}^{\downarrow x}).$$

By Lemma 3.2, we have $[\phi]_\sigma^{\Rightarrow x} \leq [\eta]_\sigma$. So $[\phi]_\sigma^{\Rightarrow x} = [\eta]_\sigma$. By the proof above, (Ψ, D) is a compact information algebra. □

5 Conclusions

We have studied the ordering structures on semiring-induced valuation algebras. We showed that the valuation algebra (Ψ, D) induced by a semiring A is an information algebra if and only if A is a c-semiring with an idempotent operation $\times$. Then, we gave the equivalence between the continuity of orderings over semirings and their corresponding semiring-induced valuation algebras. Moreover, we proposed a method for constructing semiring-induced compact information algebras. We have proved that (Ψ, D) is a compact information algebra, if A^∂ is an algebraic lattice.

References

[1] Shenoy, P. P., *A valuation-based language for expert systems*, Int. J. Approx. Reasoning **3**(1989), 383–411.

[2] Kohlas, J., "Information Algebras: Generic Structures for Inference," Springer-Verlag, 2003.

[3] Kohlas, J., and N. Wilson, *Semiring induced valuation algebras: Exact and approximate local computation algorithms*, Artifical Intelligence **172**(2008), 1360–1399.

[4] Li, S. J., and M. Ying, *Soft constraint abstraction based on semiring homomorphism*, Theor. Comput. Sci. **403**(2008), 192–201.

[5] Gierz, G., et al., "Continuous Lattices and Domains: Encyclopedia of Mathematics and its Applications," Cambridge University Press, 2003.

[6] Kohlas, J., Lecture Notes on The Algebraic Theory of Information, 2010, URL: http://diuf.unifr.ch/drupal/tns/sites/diuf.unifr.ch.drupal.tns/files/file/kohlas/main.pdf.

[7] Haenni, R., *Ordered valuation algebras: a generic framework for approximating inference*, Int. J. Approx. Reasoning **37**(2004), 1–41.

[8] Pouly, M., "A Generic Framework for Local Computation," Ph.D. Thesis, University of Fribourg, Switzerland, 2008.

[9] Shafer, G., *An axiomatic study of computation in hypertrees*, Working Paper No.232, School of Business, University of Kansas.

[10] Bistarelli, S., Codognet, P., and Rossi, F., *Abstracting soft constraints: Framework, properties, examples*, Artificial Intelligence **139**(2002), 175–211.

[11] Guan, X. C., and Y. M. Li, *On Two Types of Continuous Information Algebras*, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems **20**(2012), 655–671.

[12] Davey, B. A., and H. A. Priestley, "Introduction to lattices and order,"2nd Ed., Cambridge University Press, 2002.