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From Iterative Algebras to Iterative Theories (Extended Abstract)†

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Abstract

Iterative theories introduced by Calvin Elgot formalize potentially infinite computations as solutions of recursive equations. One of the main results of Elgot and his coauthors is a description of a free iterative theory as the theory of all rational trees. Their algebraic proof of this fact is extremely complicated. In our paper we show that by starting with "iterative algebras", i. e., algebras admitting a unique solution of all systems of flat recursive equations, a free iterative theory is obtained as the theory of free iterative algebras. The (coalgebraic) proof we present is dramatically simpler than the original algebraic one. And our result is, nevertheless, much more general: we describe a free iterative theory on any finitary endofunctor of every locally presentable category $\mathcal{A}$. This allows us, e. g., to consider iterative algebras over any equationally specified class $\mathcal{A}$ of finitary algebras.

Keywords: free iterative theory, rational monad, coalgebra

† The full version of this paper containing all proofs can be found at the URL http://www.iti.cs.tu-bs.de/~milius

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1 Introduction

Iterative theories have been introduced by Calvin C. Elgot [9] as a model of computation formalized as a sequence of instantaneous descriptions of an abstract machine. He and his co-authors then proved that for every signature Σ a free iterative theory on Σ exists [7] and that it consists of all rational Σ-trees [10]. Recall that a Σ-tree (i.e., a tree, possibly infinite, labelled by operation symbols in Σ so that every node with n children is labelled by an n-ary symbol) is *rational* if it has up to isomorphism only finitely many subtrees, see [13].

In the present paper we introduce *iterative algebras* rather than iterative theories, and we show that the theory formed by all free iterative algebras is Elgot's free iterative theory. In the classical case of Σ-algebras, iterativity has been introduced by Evelyn Nelson [16] as follows: given a Σ-algebra A, let us consider an arbitrary system of recursive equations

$$x_i \approx t_i, \qquad i = 1, \dots, n, \tag{1.1}$$

where $X = \{x_1, x_2, \dots, x_n\}$ is a finite set of variables and $t_1, t_2, \dots, t_n$ are terms over $X + A$, none of which is a single variable x_i. The algebra A is called *iterative* provided that for every such system of equations there exists a unique *solution*. That is, there exists a unique n-tuple ${x_1}^\dagger, {x_2}^\dagger, \dots, {x_n}^\dagger$ of elements of A such that each of the formal equations in (1.1) becomes an equality after the substitution ${x_1}^\dagger/x_1, {x_2}^\dagger/x_2, \dots, {x_n}^\dagger/x_n$:

$${x_i}^\dagger = t_i({x_1}^\dagger/x_1, {x_2}^\dagger/x_2, \dots, {x_n}^\dagger/x_n), \qquad i = 1, \dots, n.$$

Example: let Σ consist of a single binary operation symbol, $*$, then the algebra A of all (finite and infinite) binary trees is iterative. For example, the system

$$\begin{aligned} x_1 &\approx x_2 * t \\ x_2 &\approx (x_1 * s) * t \end{aligned} \tag{1.2}$$

where s and t are trees in A has the unique solution ${x_1}^\dagger = \cdots * s) * t) * t) * s) * t) * t$, and analogously for ${x_2}^\dagger$.

Every system (1.1) above can be modified to a *flat system*, i.e., one where each right-hand side is either a *flat term*

$$t_i = \sigma(x_1, \dots, x_k), \quad \text{for } \sigma \in \Sigma_k,\ x_1, \dots, x_k \in X,$$

or an element of A

$$t_i \in A.$$

For example, the above system (1.2) has the following modification to a flat system:

$$x_1 \approx x_2 * x_3 \qquad x_3 \approx t \qquad x_5 \approx s$$

$$x_2 \approx x_4 * x_3 \qquad x_4 \approx x_1 * x_5$$

Therefore, an algebra is iterative iff every flat equation system has a unique solution.

Now Σ-algebras are a special case of algebras for an endofunctor $H : \mathcal{A} \longrightarrow \mathcal{A}$ (which are pairs consisting of an object A of $\mathcal{A}$ and a morphism $\alpha : HA \longrightarrow A$): here $\mathcal{A}$ is the category of sets and $H = H_\Sigma$ is the *polynomial functor* given on objects by $H_\Sigma X = \Sigma_0 + \Sigma_1 \times X + \Sigma_2 \times X^2 + \cdots$. For an algebra (A, α) observe that a flat equation system has its right-hand sides in $H_\Sigma X + A$, thus, it can be represented by a morphism

$$e : X \longrightarrow H_\Sigma X + A, \quad e(x_i) = t_i.$$

A *solution* of e is then a morphism

$$e^\dagger : X \longrightarrow A, \quad e^\dagger(x_i) = {x_i}^\dagger\,,$$

with the property that the following diagram

(1.3)
$$\begin{tikzcd} X \arrow[r, "e^\dagger"] \arrow[d, "e"'] & A \\ H_\Sigma X + A \arrow[r, "H_\Sigma e^\dagger + A"'] & H_\Sigma A + A \arrow[u, "{[\alpha, A]}"'] \end{tikzcd}$$

commutes.

Definition 1.1 A Σ-algebra A is called *iterative* provided that for every flat equation morphism $e : X \longrightarrow H_\Sigma X + A$, where X is a finite set, there exists a unique solution $e^\dagger : X \longrightarrow A$.

"Classical" algebras are seldom iterative. But there are enough interesting iterative algebras. For example, the algebra

$$T_\Sigma$$

of all (finite and infinite) Σ-trees is iterative. And so is its subalgebra

$$R_\Sigma$$

of all rational Σ-trees. In fact, the full subcategory $\mathsf{Alg}_{it}\,\Sigma$ of $\mathsf{Alg}\,\Sigma$ formed by all iterative Σ-algebras is rich enough: a limit or a filtered colimit of iterative algebras is always iterative, thus $\mathsf{Alg}_{it}\,\Sigma$ is reflective in $\mathsf{Alg}\,\Sigma$ (see Reflection Theorem in [6]). From this it follows that every set generates a free iterative algebra, i.e., the forgetful functor $\mathsf{Alg}_{it}\,\Sigma \longrightarrow \mathsf{Set}$ is a right-adjoint. This defines a monad R_Σ on Set. We prove that

(i) R_Σ is a free iterative monad on H_Σ, and

(ii) R_Σ assigns to every set X the algebra $R_\Sigma X$ of all rational Σ-trees on X.

In this way a new proof of the result of Elgot et al. describing a free iterative monad (or theory) is achieved.

In our proof we work with an arbitrary endofunctor H of the category of sets which is *finitary*, i. e., preserves filtered colimits. As in Definition 1.1, an algebra $\alpha : HA \longrightarrow A$ is called *iterative* if for every flat equation morphism $e : X \longrightarrow HX + A$, where X is a finite set, there exists a unique solution, i. e., a unique morphism $e^\dagger : X \longrightarrow A$ with $e^\dagger = [\alpha, A] \cdot (He^\dagger + A) \cdot e$, compare with (1.3). The main technical result is coalgebraic: in order to describe a free iterative algebra on a set Y, we form the diagram Eq_Y of all coalgebras $e : X \longrightarrow HX + Y$ of the endofunctor $H(-) + Y$ on finite sets X. We prove that a colimit of that diagram

$$RY = \operatorname{colim} \mathrm{Eq}_Y$$

carries naturally the structure of an algebra, and that RY is a free iterative algebra on Y. From that we derive that the monad $R(-)$ is a free iterative monad on H. In our proof the fact that H is a finitary endofunctor of Set plays no rôle: the same result holds for finitary endofunctors of all locally finitely presentable categories. Thus, if we start with e. g. an equational class $\mathcal{A}$ of finitary algebras then, again, for every finitary endofunctor H the free iterative algebras RY are constructed as colimits of coalgebras of $H(-) + Y$ on finitely presentable objects of $\mathcal{A}$, and they form a free iterative theory on H.

Related Work. In the classical setting, i. e., for polynomial endofunctors of Set, iterative algebras were introduced by Evelyn Nelson [16] to obtain a short proof of Elgot's free iterative theories. Our paper can be seen as a categorical generalization of that paper with distinctive coalgebraic "flavour". Also Jerzy Tiuryn introduced a concept of iterative algebra in [17] with the same aim as ours: to relate iterative theories of Elgot to properties of algebras. But the approach of [17] is fundamentally different from ours; e.g., the trivial, one-element, algebra is not iterative in the sense of Tiuryn, thus, his iterative algebras are not closed under limits.

The description of the rational monad as a colimit is also presented in [12].

The present paper is a dramatic improvement of our previous description of the rational monad in [3], [4] where we assumed that the endofunctor preserves monomorphisms and the underlying category satisfies three rather technical conditions, and the proof was much more involved. The current approach includes all equationally defined algebraic categories as base categories (whereas in [4] we still needed strong side conditions which only hold in very few algebraic categories). All proofs have been omitted, the reader can find them in the full version of our paper [5].

2 Iterative Algebras

Notation 2.1 Throughout the paper all categories are assumed to have finite coproducts. We denote by inl and inr the coproduct injections of $A + B$.

In order to define the concept of a flat equation morphism as in the introduction (a morphism $e : X \longrightarrow HX + A$ in Set where X is finite) in a general category, we need the appropriate generalization of finiteness. Recall that a functor is called *finitary* provided that it preserves filtered colimits. A set is finite if and only if its hom-functor is finitary. This has inspired Gabriel and Ulmer [11] to the following

Definition 2.2 An object of A a category $\mathcal{A}$ is *finitely presentable* if its hom-functor $\mathcal{A}(A, -) : \mathcal{A} \longrightarrow \mathsf{Set}$ is finitary.

A category $\mathcal{A}$ is called *locally finitely presentable* provided that it has colimits and a (small) set of finitely presentable objects whose closure under filtered colimits is all of $\mathcal{A}$.

Examples 2.3

(i) Set, the category of posets, and every variety of finitary algebras are locally finitely presentable.

(ii) Let H be a finitary endofunctor of a locally finitely presentable category $\mathcal{A}$. Then the category $\mathsf{Alg}\ H$ of H-algebras and homomorphisms is also locally finitely presentable, see [6].

Definition 2.4 Given an endofunctor $H : \mathcal{A} \longrightarrow \mathcal{A}$, by a *finitary flat equation morphism* (later just: *equation morphism*) in an object A we mean a morphism $e : X \longrightarrow HX + A$ of $\mathcal{A}$, where X is a finitely presentable object of $\mathcal{A}$.

Suppose that A is an underlying object of an H-algebra $\alpha : HA \longrightarrow A$. Then by a *solution* of e in the algebra A is meant a morphism $e^\dagger : X \longrightarrow A$ in $\mathcal{A}$ such that the square

$$\begin{tikzcd} X \arrow[r, "e^\dagger"] \arrow[d, "e"'] & A \\ HX + A \arrow[r, "He^\dagger + A"'] & HA + A \arrow[u, "{[\alpha, A]}"'] \end{tikzcd} \tag{2.1}$$

commutes.

An H-algebra is called *iterative* provided that every finitary flat equation morphism has a unique solution.

Example 2.5

(i) Groups, lattices etc. considered as Σ-algebras are seldom iterative. For example, if a group is iterative, then its unique element is the unit element 1, since the recursive equations $x \approx x \cdot y,\ y \approx 1$ have a unique solution. If a lattice is iterative, then it has a unique element: consider $x \approx x \vee x$.

(ii) The algebra of addition on the set

$$\widetilde{\mathbb{N}} = \{\,1, 2, 3, \dots\,\} \cup \{\,\infty\,\}$$

is iterative (and "almost classical"). (Observe that 0 is not included. This is forced by the uniqueness of solutions of $x = x + x$.)

(iii) The algebras T_Σ and R_Σ (see Introduction) are iterative.

Remark 2.6 We denote by

$$\mathsf{Alg}_{it}\ H$$

the category of all iterative algebras and all homomorphisms.

Proposition 2.7 *Iterative algebras are closed under limits and filtered colimits in* $\mathsf{Alg}\ H$*.*

The proof is a rather simple calculation based on the fact that $\mathsf{Alg}\ H$ has limits and filtered colimits formed on the level of the base category. Using the Reflection Theorem of [6] we derive:

Corollary 2.8 *The category* $\mathsf{Alg}_{it}\ H$ *is a reflective subcategory of* $\mathsf{Alg}\ H$*.*

Corollary 2.9 *Every object of* $\mathcal{A}$ *generates a free iterative* H*-algebra.*

In other words, the natural forgetful functor $U : \mathsf{Alg}_{it}\ H \longrightarrow \mathcal{A}$ has a left adjoint.

Definition 2.10 The finitary monad on $\mathcal{A}$ formed by free iterative H-algebras is called the *rational monad* of H and is denoted by $\mathbb{R} = (R, \eta, \mu)$.

Thus, $\mathbb{R}$ is the monad of the above adjunction

$$\begin{tikzcd} \mathsf{Alg}_{it}\ H \arrow[r, phantom, "\perp"] \arrow[r, shift right=1.5ex, "U"'] & \mathcal{A} \arrow[l, shift right=1.5ex, "R"'] \end{tikzcd}$$

More detailed, for every object Z of $\mathcal{A}$ we denote by RZ a free iterative H-algebra on Z with the universal arrow

$$\eta_Z : Z \longrightarrow RZ\,,$$

and the algebra structure

$$\rho_Z : HRZ \longrightarrow RZ\,.$$

Then $\mu_Z : RRZ \longrightarrow RZ$ is the unique homomorphism of H-algebras with $\mu_Z \cdot \eta_{RZ} = id$.

Proposition 2.11 *An initial iterative algebra of the endofunctor $H(-) + Z$ is precisely a free iterative H-algebra on Z.*

Example 2.12 *The rational monad of $H_\Sigma : \mathsf{Set} \longrightarrow \mathsf{Set}$.*

Recall from the introduction the algebra T_Σ of all (finite and infinite) Σ-trees. This algebra is iterative — this is folklore. For every set Z the algebra $T_\Sigma Z$ of all Σ-trees over Z (i. e., trees with nodes having $n > 0$ children labelled by n-ary operation symbols and leaves labelled by constant symbols or variables from Z) is also iterative, since $T_\Sigma Z = T_{\Sigma'}$, where $\Sigma'_0 = \Sigma_0 + Z$, and $\Sigma'_i = \Sigma_i$, for $i > 0$.

As proved in [16] the subalgebra $R_\Sigma Z$ of all rational Σ-trees, i.e., Σ-trees over Z which have only finitely many subtrees (up to isomorphism), is a free iterative Σ-algebra on Z.

Corollary 2.13 *The rational monad $\mathbb{R}_\Sigma$ of the polynomial endofunctor H_Σ of Set is given by the formation of the Σ-algebras $R_\Sigma(Z)$ of all rational Σ-trees over Z.*

Example 2.14 The rational monad of $\mathcal{P}_{\text{fin}} : \mathsf{Set} \longrightarrow \mathsf{Set}$, the finite power-set functor has been described in [2]: it assigns to a set X the algebra $A(X)/\sim$, where $A(X)$ is the algebra of all rational extensional finitely-branching trees (where "extensional" means that every pair of distinct siblings define non-isomorphic subtrees). And $\sim$ is the largest bisimulation of $A(X)$ defined as follows: $t \sim s$ iff the cuttings at level n have the same extensional quotients, for all natural numbers n.

3 A Coalgebraic Construction

The aim of this section is to describe an initial iterative H-algebra as a colimit of a simple diagram Eq in the given base category $\mathcal{A}$. We assume throughout this section that

(a) $\mathcal{A}$ is a locally finitely presentable category, see Definition 2.2,

and

(b) H is a finitary endofunctor of $\mathcal{A}$.

We choose a set $\mathcal{A}_{fp}$ of representatives of finitely presentable objects of $\mathcal{A}$ w.r.t. isomorphism.

The initial iterative algebra is proved to be a colimit of the diagram

$$\mathrm{Eq} : \mathsf{EQ} \longrightarrow \mathcal{A}$$

whose objects are all H-coalgebras carried by finitely presentable objects of $\mathcal{A}$:

$$e : X \longrightarrow HX \quad \text{with } X \text{ in } \mathcal{A}_{fp},$$

with the usual coalgebra homomorphisms as morphisms, and with Eq the obvious forgetful functor $e \longmapsto X$.

A colimit

$$R_0 = \operatorname{colim} \mathrm{Eq}$$

of this diagram (with colimit morphisms $e^\sharp : X \longrightarrow R_0$ for all $e : X \longrightarrow HX$ in EQ) yields a canonical morphism

$$i : R_0 \longrightarrow HR_0$$

Namely, i is the unique morphism such that every $e^\sharp$ becomes a coalgebra homomorphism, i.e., the squares

(3.1)
$$\begin{tikzcd} X \arrow[r, "e"] \arrow[d, "e^\sharp"'] & HX \arrow[d, "He^\sharp"] \\ R_0 \arrow[r, "i"'] & HR_0 \end{tikzcd}$$

commute. (In fact, the forgetful functor $\mathsf{Coalg}\, H \longrightarrow \mathcal{A}$ creates colimits.)

Theorem 3.1 *R_0 is the initial iterative H-algebra. More precisely, the morphism i is an isomorphism and $i^{-1} : HR_0 \longrightarrow R_0$ is an initial iterative H-algebra.*

Sketch of Proof. (a) It is easy to see that the diagram Eq is filtered, and the morphisms $He^\sharp \cdot e$ form a cocone, thus, i is well-defined. We now construct a morphism $j : HR_0 \longrightarrow R_0$ and prove that it is inverse to i. We use the fact that in a locally finitely presentable category the given object HR_0 is a colimit of the diagram of all arrows $p : P \longrightarrow HR_0$ where P is in $\mathcal{A}_{fp}$. More precisely, let $\mathcal{A}_{fp}/HR_0$ denote the comma-category (of all these arrows p), then the forgetful functor $D_{HR_0} : \mathcal{A}_{fp}/HR_0 \longrightarrow \mathcal{A}$ has, in $\mathcal{A}$, the colimit cocone formed by all $p : P \longrightarrow HR_0$. Thus, in order to define j we need to define morphisms $jp : P \longrightarrow R_0$ forming a cocone of the diagram D_{HR_0}. We know that HR_0 is a filtered colimit of $H \cdot \mathrm{Eq}$ and that $\mathcal{A}(P, -)$ preserves this colimit, since P is in $\mathcal{A}_{fp}$. Therefore, p factors through one of the colimit morphisms

(3.2)
$$\begin{tikzcd} P \arrow[r, "p"] \arrow[dr, "p'"'] & HR_0 \\ & HW \arrow[u, "Hg^\sharp"'] \end{tikzcd}$$

for some $g : W \longrightarrow HW$ in EQ. We form a new object

$$e_{p'} \equiv P + W \xrightarrow{[p',g]} HW \xrightarrow{H\mathsf{inr}} H(P + W)$$

of EQ and define j to be the unique morphism such that the following square

(3.3)
$$\begin{tikzcd} P \arrow[r, "\mathsf{inl}"] \arrow[d, "p"'] & P + W \arrow[d, "{e_{p'}}^\sharp"] \\ HR_0 \arrow[r, "j"'] & R_0 \end{tikzcd}$$

commutes for every p in $\mathcal{A}_{fp}/HR_0$. To prove that this is well-defined we need to show that

(i) ${e_{p'}}^\sharp\mathsf{inl}$ is independent of the choice of factorization (3.2),

and

(ii) the morphisms ${e_{p'}}^\sharp \cdot \mathsf{inl}$ form a cocone of $\mathcal{A}_{fp}/HR_0$.

And then we verify $j = i^{-1}$.

(b) To prove that (R_0, i^{-1}) is an iterative algebra we just show existence of solutions, leaving out uniqueness. For every equation morphism

$$e : X \longrightarrow HX + R_0 = \mathrm{colim}(HX + \mathrm{Eq})$$

there exists, since X is finitely presentable, a factorization through the colimit morphism $HX + f^\sharp$ (for some $f : V \longrightarrow HV$ in EQ):

(3.4)
$$\begin{tikzcd} X \arrow[r, "e"] \arrow[dr, "e_0"'] & HX + R_0 \\ & HX + V \arrow[u, "HX+f^\sharp"'] \end{tikzcd}$$

Recall from 2.1 that $\mathsf{can} : HX + HV \longrightarrow H(X + V)$ denotes the canonical morphism. Define a new object, $\overline{e}$, of EQ as follows:

(3.5)
$$\overline{e} \equiv X + V \xrightarrow{[e_0,\mathsf{inr}]} HX + V \xrightarrow{HX+f} HX + HV \xrightarrow{\mathsf{can}} H(X + V)$$

Observe that

(3.6)
$$f^\sharp = \overline{e}^\sharp \cdot \mathsf{inr}$$

because $\mathsf{inr} : V \longrightarrow X + V$ is a coalgebra morphism (in EQ) from f to $\overline{e}$. We define a solution of e by

(3.7)
$$e^\dagger \equiv X \xrightarrow{\mathsf{inl}} X + V \xrightarrow{\overline{e}^\sharp} R_0\,.$$

In fact, in the following diagram

(3.8)
$$
\begin{tikzcd}
X \arrow[rrr, "e^\dagger"] \arrow[d, "e_0"] \arrow[ddd, bend right=60, "e"'] & & & R_0 \\
HX+V \arrow[r, "HX+f"] \arrow[dd, "HX+f^\sharp"] \arrow[ddr, phantom, "\text{(i)}"] & HX+HV \arrow[r, "{[He^\dagger,Hf^\sharp]}"] \arrow[dr, "HX+Hf^\sharp"'] & HR_0 \arrow[ur, "i^{-1}"] & \\
 & & HX+HR_0 \arrow[u, "{[He^\dagger,HR_0]}"'] & \\
HX+R_0 \arrow[rrr, "He^\dagger+R_0"'] \arrow[urr, "HX+i"] & & & HR_0+R_0 \arrow[uuu, "{[i^{-1},R_0]}"']
\end{tikzcd}
$$

all inner parts commute: see (3.4) for the left-hand part, (3.1) for part (i), whereas the lower part commutes trivially (analyze the two components separately) and so does the middle triangle. It remains to verify the upper part: here we use (3.1) and (3.5) to conclude that the following diagram

$$
\begin{tikzcd}
X \arrow[r, "\mathsf{inl}"] \arrow[d, "e_0"'] & X+V \arrow[r, "\overline{e}^\sharp"] \arrow[dl, "{[e_0,V]}"'] \arrow[d, "\overline{e}"] & R_0 \\
HX+V \arrow[d, "HX+f"'] & H(X+V) \arrow[d, "H\overline{e}^\sharp"] & \\
HX+HV \arrow[ur, "\mathsf{can}"] \arrow[r, "{[He^\dagger,Hf^\sharp]}"'] \arrow[r, phantom, shift left=4, "\text{(ii)}"] & HR_0 \arrow[uur, bend right, "i^{-1}"'] &
\end{tikzcd}
$$

commutes. In fact, the left-hand component of (ii) commutes by definition of $e^\dagger$ and the right-hand one does by (3.6). Thus, (3.8) commutes, proving that $e^\dagger$ is a solution of e.

(c) Initiality. Let $\alpha : HA \longrightarrow A$ be an iterative H-algebra. We prove first that there is at most one H-algebra homomorphism from R_0. Let

$$
\begin{tikzcd}
HR_0 \arrow[r, "i^{-1}"] \arrow[d, "Hh"'] & R_0 \arrow[d, "h"] \\
HA \arrow[r, "\alpha"'] & A
\end{tikzcd}
$$

be a homomorphism. For every object $e : X \longrightarrow HX$ of EQ the following diagram

$$
\begin{tikzcd}
X \arrow[r, "e^\sharp"] \arrow[d, "e"'] & R_0 \arrow[rr, "h"] \arrow[d, "i"'] & & A \\
HX \arrow[r, "He^\sharp"'] \arrow[d, "\mathsf{inl}"'] & HR_0 \arrow[r, "Hh"] & HA \arrow[ur, "\alpha"'] & \\
HX+A \arrow[rrr, "H(he^\sharp)+A"'] & & & HA+A \arrow[uu, "{[\alpha,A]}"']
\end{tikzcd}
\tag{3.9}
$$

commutes, see (3.1), which proves that $he^\sharp$ is a solution of $\mathsf{inl}\, e$ in A.

This determines h uniquely, since the $e^\sharp$'s form a colimit cocone of $R_0 = \operatorname{colim} \mathrm{Eq}$.

Conversely, let us define a morphism $h : R_0 \longrightarrow A$ by the above rule

$$he^\sharp = (\mathsf{inl}\, e)^\dagger \quad \text{for all } e : X \longrightarrow HX \text{ in } \mathsf{EQ}$$

where $(-)^\dagger$ is the unique solution in A. This is well-defined since the morphisms $(\mathsf{inl}\, e)^\dagger$ form a cocone of the diagram Eq: in fact, let

$$
\begin{tikzcd}
X \arrow[r, "e"] \arrow[d, "p"'] & HX \arrow[d, "Hp"] \\
Y \arrow[r, "f"'] & HY
\end{tikzcd}
$$

be a morphism of EQ. We prove that $(\mathsf{inl}\, f)^\dagger p$ is a solution of $\mathsf{inl}\, e$ by considering the corresponding diagram:

$$
\begin{tikzcd}
X \arrow[r, "p"] \arrow[d, "e"'] & Y \arrow[r, "(\mathsf{inl}\, f)^\dagger"] \arrow[d, "f"] & A \\
HX \arrow[r, "Hp"] \arrow[d, "\mathsf{inl}"'] & HY \arrow[d, "\mathsf{inl}"] & \\
HX+A \arrow[r, "Hp+A"'] & HY+A \arrow[r, "H(\mathsf{inl}\, f)^\dagger+A"'] & HA+A \arrow[uu, "{[\alpha,A]}"']
\end{tikzcd}
$$

This proves

$$(\mathsf{inl}\, e)^\dagger = (\mathsf{inl}\, f)^\dagger p\,.$$

The morphism h above is a homomorphism of algebras because the diagram (3.9) commutes: the outward square commutes by definition of h, the upper left-hand square by (3.1), and the lower part is obvious. This shows that the upper right-hand part commutes when precomposed with $e^\sharp$, e in EQ. Since the $e^\sharp$'s form a colimit cocone, it follows that h is a homomorphism. $\square$

Corollary 3.2 *A free iterative H-algebra RZ is a colimit,*

$$RZ = \operatorname{colim} \mathrm{Eq}_Z$$

of the diagram

$$\mathrm{Eq}_Z : \mathsf{EQ}_Z \longrightarrow \mathcal{A}$$

where EQ_Z consists of all equation morphisms $e : X \longrightarrow HX + Z$, $X \in \mathcal{A}_{fp}$ (and the connecting morphisms are the coalgebra homomorphisms w.r.t. $H(-) + Z$) and Eq_Z sends e to X.

In fact, this is a consequence of Proposition 2.11 and Theorem 3.1.

Remark 3.3 We denote, again, the colimit morphisms of Eq_Z by

$$e^\sharp : X \longrightarrow RZ$$

for all $e : X \longrightarrow HX + Z$ in EQ_Z. The appropriate isomorphism is denoted by

$$i_Z : RZ \longrightarrow HRZ + Z$$

It is characterized by the fact that the two coproduct injections of $HRZ + Z$ are (in the notation of Definition 2.10)

$$\mathsf{inl} = i_Z \rho_Z \quad \text{and} \quad \mathsf{inr} = i_Z \eta_Z$$

In other words, $i_Z = [\rho_Z, \eta_Z]^{-1}$.

4 An Alternative Definition of Iterativity

In the Introduction we considered non-flat systems (1.1) of recursive equations for Σ-algebras. And we argued that, due to the possibility of flattening such a system, we will just have to consider the flat equation morphism $e : X \longrightarrow H_\Sigma X + A$. We are going to make that statement precise by showing that in iterative algebras (in general, not only in Set) much more general systems of recursive equations are uniquely solvable. This implies that, for polynomial endofunctors of Set, our definition of iterative algebras coincides with that presented by Evelyn Nelson [16]. And as we explain in the next section, it also implies that the rational monad is iterative in the sense of Calvin Elgot [9].

Let us first remark that the condition stated in the Introduction for (1.1), that no right-hand side be a single variable, is substantial: the equation $x \approx x$ has a unique solution only in the trivial terminal algebras. Systems satisfying the above condition are called *guarded*.

We first consider guarded systems where the right-hand sides live in the free H-algebra (i. e., finite trees in case $H = H_\Sigma$). Such systems are called *finitary*.

Since H is finitary we have for every object X in $\mathcal{A}$ a free algebra $\varphi^0_X : HFX \longrightarrow FX$ on X with universal arrow $\eta^0_X : X \longrightarrow FX$. This defines a monad $\mathbb{F} = (F, \eta^0, \mu^0)$ where the component μ^0_X is the unique homomorphism $\mu^0_X : FFX \longrightarrow FX$ with $\mu^0_X \cdot \eta^0_{FX} = id$. It is easy to see that FX is an initial algebra of $H(-) + X$; thus, by Lambek's Lemma [14] the morphism

$$j_X = [\varphi^0_X, \eta^0_X] : HFX + X \longrightarrow FX$$

is an isomorphism. For every H-algebra $\alpha : HA \longrightarrow A$ we have the unique homomorphism

$$\widehat{\alpha} : FA \longrightarrow A \qquad \text{with} \quad \widehat{\alpha} \cdot \eta_A = id$$

(which, in case of H_Σ, is the computation of (finite) terms over A in the Σ-algebra A). This allows us to define solutions of finitary equations morphisms in A as follows:

Definition 4.1

(i) By a *finitary equation morphism* in an object Y (of parameters) is meant a morphism

$$e : X \longrightarrow F(X + Y), \quad X \text{ finitely presentable.}$$

(ii) Given an H-algebra $\alpha : HA \longrightarrow A$ and a morphism $f : Y \longrightarrow A$ (interpreting the parameters in A), we say that the finitary equation morphism e has a *solution* $e^\dagger_f : X \longrightarrow A$, *induced by* f provided that the square

(4.1)
$$\begin{tikzcd} X \arrow[r, "e^\dagger_f"] \arrow[d, "e"'] & A \\ F(X+Y) \arrow[r, "F[e^\dagger_f, f]"'] & FA \arrow[u, "\widehat{\alpha}"'] \end{tikzcd}$$

commutes.

(iii) We call e *guarded* provided that it factors through the summand $HF(X + Y) + Y$ of $F(X+Y) = HF(X+Y)+X+Y$ (see j_{X+Y} above):

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[dr, dotted] & F(X+Y) \\ & HF(X+Y)+Y \arrow[u, "{[\varphi^0, \eta^0 \cdot \mathsf{inr}]}"'] \end{tikzcd}$$

Remark 4.2

(i) The square (4.1) in Definition 4.1 means, for polynomial functors, that the assignment $e^\dagger_f$ of variables $x \in X$ to elements of A has the following property: form the "substitution" mapping $[e^\dagger_f, f] : X + Y \longrightarrow A$ (which interprets the variables as $e^\dagger_f$ does, and the parameters as f does). Extend it to the unique homomorphism

$$\widehat{\alpha} \cdot F[e^\dagger_f, f] : F(X + Y) \longrightarrow A$$

of the free algebra. Then the (formal) equations $x \approx e(x)$ become actual identities in A after the substitution $x \longmapsto e^\dagger_f(x)$ for all $x \in X$.

(ii) The next result states that in an iterative algebra A every finitary guarded equation morphism $e : X \longrightarrow F(X + Y)$ defines unique function $e^\dagger_{(-)} : \mathcal{A}(Y, A) \longrightarrow \mathcal{A}(X, A)$ such that (4.1) commutes for every $f \in \mathcal{A}(Y, A)$.

Theorem 4.3 *An H-algebra A is iterative if and only if every finitary guarded equation morphism has, for any interpretation of the parameters in A, a unique solution.*

Remark 4.4 The proof of Theorem 4.3 follows from the next result, generalizing "finitary" to "rational". That is, let $\alpha : HA \longrightarrow A$ be an iterative algebra. We denote (analogously to $\widehat{\alpha}$ above) by

$$\widetilde{\alpha} : RA \longrightarrow A$$

the unique homomorphism of H-algebras with $\widetilde{\alpha} \cdot \eta_A = id$. We define a *rational equation morphism* on an object Y as a morphism

$$e : X \longrightarrow R(X + Y), \qquad X \text{ finitely presentable.}$$

Given a morphism $f : Y \longrightarrow Z$, the *solution* of e *induced by* f is a morphism $e^\dagger_f : X \longrightarrow A$ such that the square

$$\begin{tikzcd}
X \arrow[r, "e^\dagger_f"] \arrow[d, "e"'] & A \\
R(X+Y) \arrow[r, "{R[e^\dagger_f, f]}"'] & RA \arrow[u, "\widetilde{\alpha}"']
\end{tikzcd}$$

commutes. Finally, e is called *guarded* if it factors through the summand $HR(X + Y) + Y$ of $R(X + Y) = HR(X + Y) + X + Y$ (see Remark 3.3).

Theorem 4.5 *In an iterative algebra, for every guarded rational equation morphism e and every interpretation f of its parameters there exists a unique solution $e^\dagger_f$.*

Sketch of Proof. Let $\alpha : HA \longrightarrow A$ be an iterative algebra. Given a guarded rational equation morphism

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[dr, dotted, "e_0"'] & R(X+Y) \\ & HR(X+Y)+Y \arrow[u, "{[\rho_{X+Y}, \eta_{X+Y} \cdot \mathsf{inr}]}"'] \end{tikzcd}$$

and a morphism $f : Y \longrightarrow A$, we will prove that e has a solution induced by f; we leave out the proof of the uniqueness.

Recall from Corollary 3.2 that $R(X+Y) = \operatorname{colim} \mathrm{Eq}_{X+Y}$ with colimit cocone $g^\sharp : W \longrightarrow R(X+Y)$ for all $g : W \longrightarrow HW + X + Y$ in EQ_{X+Y}. Since this colimit is filtered and H is finitary, this implies that

$$HR(X+Y) + Y = \operatorname{colim} H\mathrm{Eq}_{X+Y} + Y$$

with the colimit cocone formed by all $Hg^\sharp + Y$. Since X is a finitely presentable object, the morphism

$$e_0 : X \longrightarrow \operatorname{colim} H\mathrm{Eq}_{X+Y} + Y$$

factors through the colimit cocone:

$$\begin{tikzcd} X \arrow[r, "e_0"] \arrow[dr, "w"'] & HR(X+Y)+Y \\ & HW+Y \arrow[u, "Hg^\sharp + Y"'] \end{tikzcd}$$

for some object $g : W \longrightarrow HW + X + Y$ of EQ_{X+Y} and some morphism w.

We define a finitary flat equation morphism as follows:

$$\langle e \rangle \equiv W + X \xrightarrow{[g, \mathsf{inm}]} HW + X + Y \xrightarrow{[\mathsf{inl}, w, \mathsf{inr}]} HW + Y \xrightarrow{H\mathsf{inl} + f} H(W+X) + A \tag{4.2}$$

where $\mathsf{inm} : X \longrightarrow HW + X + Y$ is the middle coproduct injection. We obtain a unique solution $\langle e \rangle^\dagger : W + X \longrightarrow A$ and prove that the following morphism

$$e^\dagger \equiv X \xrightarrow{\mathsf{inr}} W + X \xrightarrow{\langle e \rangle^\dagger} A \tag{4.3}$$

is a solution of e induced by f.

Indeed, consider the following diagram:

(4.4)
$$\begin{tikzcd}
X \arrow[rrrr, "e_f^\dagger"] \arrow[drr, "\mathsf{inr}"] \arrow[ddr, "w"] \arrow[dddr, "e_0"'] \arrow[dddd, "e"'] & & & & A \\
& & W+X \arrow[urr, "\langle e\rangle^\dagger"] \arrow[d, "\langle e\rangle"] & & \\
& HW+Y \arrow[r, "H\mathsf{inl}+f"] \arrow[d, "Hg^\sharp+Y"] & H(W+X)+A \arrow[r, "H\langle e\rangle^\dagger+A"] \arrow[d, phantom, "\text{(i)}"] & HA+A \arrow[uur, "{[\alpha,A]}"] & \\
& HR(X+Y)+Y \arrow[rr, "{HR[e_f^\dagger,f]+f}"'] \arrow[dl, "{[\rho,\eta\cdot\mathsf{inr}]}"'] & {} & HRA+A \arrow[u, "H\widetilde{\alpha}+A"'] \arrow[dr, "{[\rho,\eta]}"] & \\
R(X+Y) \arrow[rrrr, "{R[e_f^\dagger,f]}"'] & & & & RA \arrow[uuuu, "\widetilde{\alpha}"']
\end{tikzcd}$$

All of its parts, except, perhaps, for the square (i), commute. The right-hand component of (i) is obvious. To prove the commutativity of the left-hand component of (i), we remove H and show that the equation

$$\langle e\rangle^\dagger \cdot \mathsf{inl} = \widetilde{\alpha} \cdot R[e_f^\dagger, f] \cdot g^\sharp \tag{4.5}$$

holds. To this end observe first that $\widetilde{\alpha} \cdot R[e_f^\dagger, f] : R(X+Z) \longrightarrow A$ is an H-algebra homomorphism between iterative algebras extending $[e_f^\dagger, f]$. Precomposing this homomorphism with the colimit injection $g^\sharp : W \longrightarrow R(X+Z)$ yields the unique solution of an equation morphism $\overline{g}$ in the iterative algebra A. Thus, to establish (4.5) it suffices to show that $\langle e\rangle^\dagger \cdot \mathsf{inl}$ is a solution of $\overline{g}$. In fact, the outward square of the following diagram

$$\begin{tikzcd}
W \arrow[rr, "\mathsf{inl}"] \arrow[d, "g"'] \arrow[ddd, bend right=60, "\overline{g}"'] & & W+X \arrow[r, "\langle e\rangle^\dagger"] \arrow[ddd, "\langle e\rangle"] & A \\
HW+X+Y \arrow[dr, "{[\mathsf{inl},w,\mathsf{inr}]}"] \arrow[dd, "{HW+[e_f^\dagger,f]}"'] & & & \\
& HW+Y \arrow[dl, "HW+f"] \arrow[dr, "H\mathsf{inl}+f"] & & \\
HW+A \arrow[rr, "H\mathsf{inl}+A"'] & & H(W+X)+A \arrow[r, "H\langle e\rangle^\dagger+A"'] & HA+A \arrow[uuu, "{[\alpha,A]}"']
\end{tikzcd}$$

commutes. □

Corollary 4.6 *Every rational guarded equation morphism $e : X \longrightarrow R(X+Y)$ has a unique solution in the algebra RY, i. e., there exists a unique $e^\ddagger$:*

$X \longrightarrow RY$ such that the square

$$\begin{tikzcd} X \arrow[r, "e^\ddagger"] \arrow[d, "e"'] & RY \\ R(X+Y) \arrow[r, "R[e^\ddagger, \eta]"'] & RRY \arrow[u, "\mu"'] \end{tikzcd}$$

commutes.

In fact, apply Theorem 4.5 to the iterative algebra RY and the morphism $\eta_Y : Y \longrightarrow RY$.

5 Free Iterative Monads

Assumptions 5.1 Throughout this section H denotes a finitary endofunctor of a locally finitely presentable category $\mathcal{A}$. We suppose, just for convenience, that coproduct injections in $\mathcal{A}$ are monomorphisms — this assumption can be avoided, see the full version [5], where we work with arbitrary finitary endofunctors H (and with idealized monads, generalizing the ideal monads below) in the last section.

We are going to prove that the rational monad $\mathbb{R}$, introduced in Section 2, is iterative in the sense of C. Elgot, and that it can be characterized as a free iterative monad on H.

5.2. Iterative Monads. This is a concept that C. Elgot has introduced in [9] for the base category $\mathcal{A} = \mathsf{Set}$. He used the language of algebraic theories rather than monads, but we have proved in [1] that the following concepts are equivalent to those of Elgot.

Definition 5.3 By an *ideal monad* is understood a sixtuple

$$\mathbb{S} = (S, \eta, \mu, S', \sigma, \mu')$$

consisting of a monad (S, η, μ), a subfunctor $\sigma : S' \hookrightarrow S$, and a natural transformation $\mu' : S'S \longrightarrow S'$ such that

(i) $S = S' + Id$ with coproduct injections σ and η

and

(ii) μ' is a restriction of μ, i. e., the square

$$\begin{tikzcd} S'S \arrow[r, "\mu'"] \arrow[d, "\sigma S"'] & S' \arrow[d, "\sigma"] \\ SS \arrow[r, "\mu"'] & S \end{tikzcd}$$

commutes.

Examples 5.4

(i) The rational monad $\mathbb{R} = (R, \eta, \mu)$ on an endofunctor H is ideal. Here we consider the subfunctor

$$\rho : HR \hookrightarrow R$$

expressing the H-algebra structure $\rho_Z : HRZ \longrightarrow RZ$ of each RZ, see Definition 2.10. The restriction of μ here is

$$\mu' = H\mu : HRR \longrightarrow HR\,.$$

(ii) The free-algebra monad $\mathbb{F}$ of Section 4 is ideal (again consider $\varphi^0 : HF \longrightarrow F$).

(iii) Classical algebraic theories (groups, lattices, etc.) are usually not ideal.

Definition 5.5 Let $\mathbb{S} = (S, \eta, \mu, S', \sigma, \mu')$ be an ideal monad on $\mathcal{A}$.

(i) By a *finitary equation morphism* is meant a morphism

$$e : X \longrightarrow S(X + Y)$$

in $\mathcal{A}$ where X is a finitely presentable object ("of variables") and Y is any object ("of parameters").

(ii) By a *solution* of e is meant a morphism

$$e^\dagger : X \longrightarrow SY$$

for which the square

$$\begin{tikzcd}
X \arrow[r, "e^\dagger"] \arrow[d, "e"'] & SY \\
S(X+Y) \arrow[r, "S{[e^\dagger, \eta_Y]}"'] & SSY \arrow[u, "\mu_Y"']
\end{tikzcd}$$

commutes.

(iii) The equation morphism e is called *guarded* if it factors through the summand $S'(X + Y) + Y$ of $S(X + Y) = S'(X + Y) + X + Y$:

$$\begin{tikzcd}
X \arrow[r, "e"] \arrow[dr, dotted] & S(X+Y) \\
 & S'(X+Y)+Y \arrow[u, "{[\sigma_{X+Y}, \eta_{X+Y}\mathsf{inr}]}"']
\end{tikzcd}$$

(iv) The ideal monad $\mathbb{S}$ is called *iterative* provided that every guarded finitary equation morphism has a unique solution.

Example 5.6 The rational monad of every finitary endofunctor is iterative, see Corollary 4.6.

Definition 5.7 An *ideal monad morphism* from an ideal monad $(S, \eta, \mu, S', \sigma, \mu')$ to another one $(T, \eta^T, \mu^T, T', \tau, \mu'^T)$ is a monad morphism $\lambda : (S, \eta, \mu) \longrightarrow (T, \eta^T, \mu^T)$ which has a domain-codomain restriction to the ideals. That is, there is a natural transformation $\lambda' : S' \longrightarrow T'$ with $\lambda \cdot \sigma = \tau \cdot \lambda'$.

Given a functor H, a natural transformation $\lambda : H \longrightarrow S$ is called *ideal* provided that it factors through $\sigma : S' \hookrightarrow S$.

Example 5.8 For the rational monad $\mathbb{R}$, the natural transformation

$$\kappa \equiv H \xrightarrow{H\eta} HR \xrightarrow{\rho} R$$

is ideal.

Theorem 5.9 (Rational Monad as a Free Iterative Monad.) *For every iterative monad $\mathbb{S}$ and every ideal natural transformation $\lambda : H \longrightarrow S$ there exists a unique ideal monad morphism $\overline{\lambda} : \mathbb{R} \longrightarrow \mathbb{S}$ with $\lambda = \overline{\lambda} \cdot \kappa$.*

Remark. Let us form the category $\mathsf{Fin}(\mathcal{A}, \mathcal{A})$ of all finitary endofunctors and natural transformations. For the category

$$\mathsf{FIM}(\mathcal{A})$$

of all finitary iterative monads (i.e., iterative monads $(S, \eta, \mu, S', \sigma, \mu')$ with S and S' finitary) and ideal monad morphisms we have a forgetful functor

$$U : \mathsf{FIM}(\mathcal{A}) \longrightarrow \mathsf{Fin}(\mathcal{A}, \mathcal{A}), \qquad \mathbb{S} \longmapsto S'$$

The theorem states that U has a left adjoint, viz, the functor $H \longmapsto \mathbb{R}$.

Sketch of Proof. (1) For every object Z consider SZ as an H-algebra

$$HSZ \xrightarrow{\lambda_{SZ}} SSZ \xrightarrow{\mu_Z} SZ\,.$$

It is iterative. In fact, every equation morphism $e : X \longrightarrow HX + SZ$, X in $\mathcal{A}_{fp}$, yields the following guarded equation morphism w.r.t. $\mathbb{S}$:

$$\overline{e} \equiv X \xrightarrow{e} HX + SZ \xrightarrow{\lambda_X + SZ} SX + SZ \xrightarrow{\mathsf{can}} S(X + Z)\,,$$

and it is not difficult to prove that a morphism $e^\dagger : X \longrightarrow SZ$ is a solution of e in the H-algebra SZ if and only if it is a solution of $\overline{e}$ w.r.t. the iterative monad $\mathbb{S}$.

(2) Denote by $\overline{\lambda}_Z : RZ \longrightarrow SZ$ the unique homomorphism of H-algebras with $\overline{\lambda}_Z \cdot \eta_Z = \eta^S_Z$. Then $\overline{\lambda} : R \longrightarrow S$ is a monad morphism with $\lambda = \overline{\lambda} \cdot \kappa$. And $\overline{\lambda}$

is ideal:

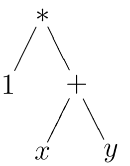
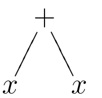
$$\begin{tikzcd}
HRZ \arrow[rr, "\rho_Z"] \arrow[d, "H\overline{\lambda}_Z"'] & & RZ \arrow[ddd, "\overline{\lambda}_Z"] \\
HSZ \arrow[d, "\lambda'_{SZ}"'] \arrow[dr, "\lambda_{SZ}"] & & \\
S'SZ \arrow[r, "\sigma_{SZ}"'] \arrow[d, "\mu'_Z"'] & SSZ \arrow[dr, "\mu_Z"] & \\
S'Z \arrow[rr, "\sigma_Z"'] & & SZ
\end{tikzcd}$$

We see that $\mu'^S \cdot \lambda' S \cdot H\overline{\lambda} : HR \longrightarrow S'$ is the desired restriction of $\overline{\lambda}$.

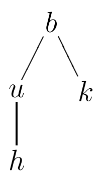
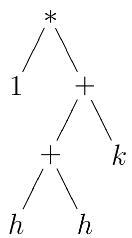
(3) *Uniqueness of* $\overline{\lambda}$. Suppose that $\overline{\lambda} : \mathbb{R} \longrightarrow \mathbb{S}$ is an ideal monad morphism with $\overline{\lambda} \cdot \kappa = \lambda$. For any object Z, $\overline{\lambda}_Z$ is an H-algebra homomorphism extending η^S_Z, thus the freeness of RZ as an iterative H-algebra establishes the desired uniqueness. □

Remark 5.10 For polynomial endofunctors on Set, the freeness of $\mathbb{R}$ specializes to *second order substitution*, see [8], i. e., substitution of rational trees for operation symbols.

For example, consider a signature Σ with a binary operation symbol b, and a unary one u, and another signature Γ with two binary operation symbols $+$ and $*$ and a constant symbol 1. The assignment

(5.1) $b(x, y) \longmapsto$  $u(x) \longmapsto$ 

of operation symbols in Σ to rational trees over Γ gives rise to a natural transformation $\lambda : H_\Sigma \longrightarrow R_\Gamma$. The induced ideal monad morphism $\overline{\lambda} : \mathbb{R}_\Sigma \longrightarrow \mathbb{R}_\Gamma$ replaces, for any set of variables X, the operation symbols in trees of $R_\Sigma X$ according to λ. Example:

$\overline{\lambda}(\{ h, k \}) :$  $\longmapsto$ 

The requirement that λ be an ideal transformation means that no operation symbol of Σ is replaced by a single variable, i.e., that λ is a so-called *non-erasing* substitution.

6 Conclusions and Future Work

Our paper shows that finitary endofunctors H generate free iterative monads without any restriction on H. Our proof, simpler and clearer than any presented before, is based on the concept of an iterative algebra. The main technical result is a description of an initial iterative algebra as a colimit of all H-coalgebras carried by finitely presentable objects. From this result we derived that the algebraic theory formed by all free iterative H-algebras is iterative in the sense of Calvin Elgot. In fact, that theory can be characterized as a free iterative theory on H. For polynomial endofunctors of the category of sets this approach has already been taken by Evelyn Nelson [16], but our proof is independent of hers. It substantially clarifies and simplifies the original proof (which occupies most of the papers [9,7,10]) as well as the coalgebraic proof we have found previously [3,4]. The freeness of the rational monad can be used to formulate clearly the "second-order substitution" described for rational Σ-trees by Bruno Courcelle [8], see Remark 5.10.

Our result can be applied to arbitrary base categories which are locally finitely presentable. For example, to the category of all finitary endofunctors of Set. In the future we intend to use this in an attempt to describe the monad of algebraic trees, see Courcelle [8], categorically.

References

[1] P. Aczel, J. Adámek, S. Milius and J. Velebil, Infinite Trees and Completely Iterative Theories: A Coalgebraic View, *Theoret. Comput. Sci.* 300 (2003), 1–45.

[2] J. Adámek, On a description of terminal coalgebras and iterative theories, *Electron. Notes Theor. Comput. Sci.* 82.1 (2003).

[3] J. Adámek, S. Milius and J. Velebil, Free Iterative Theories — A Coalgebraic View, *Math. Structures Comput. Sci.* 13 (2003), 259–320.

[4] J. Adámek, S. Milius and J. Velebil, On Rational Monads and Free Iterative Theories, *Electron. Notes Theor. Comput. Sci.* 69 (2003).

[5] J. Adámek, S. Milius and J. Velebil, From Iterative Algebras to Iterative Theories, full version, available via http://www.iti.cs.tu-bs.de/~milius/

[6] J. Adámek and J. Rosický, *Locally presentable and accessible categories*, Cambridge University Press, 1994.

[7] S. Bloom and C. C. Elgot, The Existence and Construction of Free Iterative Theories, *J. Comput. System Sci.* 12 (1974), 305–318.

[8] B. Courcelle, Fundamental Properties of Infinite Trees, *Theoret. Comput. Science* 25 (1983), 95–169.

[9] C. C. Elgot, Monadic Computation and Iterative Algebraic Theories, in: *Logic Colloquium '73* (eds: H. E. Rose and J. C. Shepherdson), North-Holland Publishers, Amsterdam, 1975.

[10] C. C. Elgot, S. L. Bloom and R. Tindell, On the Algebraic Structure of Rooted Trees, *J. Comput. System Sci.* 16 (1978), 361–399.

[11] P. Gabriel and F. Ulmer, *Lokal präsentierbare Kategorien*, Lecture N. Math 221, Springer-Verlag, Berlin 1971.

[12] N. Ghani, C. Lüth and F. De Marchi, *Electron. Notes Theor. Comput. Sci* 65(1), 2002.

[13] S. Ginali, Regular trees and the free iterative theory, *J. Comput. System Sci.* 18 (1979), 228–242.

[14] J. Lambek, A fixpoint theorem for complete categories, *Math. Z.* 103 (1968), 151–161.

[15] S. Milius, On Iteratable Endofunctors, *Electron. Notes Theor. Comput. Sci* 69 (2003).

[16] E. Nelson, Iterative Algebras, *Theoret. Comput. Science* 25 (1983), 67–94.

[17] J. Tiuryn, Unique Fixed Points vs. Least Fixed Points, *Theoret. Comput. Sci.* 12 (1980), 229–254.