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Electronic Notes in Theoretical Computer Science 345 (2019) 77–85

Electronic Notes in Theoretical Computer Science

www.elsevier.com/locate/entcs

Some Results on Poset Models Consisting of Compact Saturated Subsets

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Abstract

Given a topological space X, the set $\mathcal{Q}(X)$ of all nonempty saturated compact subsets of X is a poset with respect to the reverse inclusion order. The posets of the form $\mathcal{Q}(X)$ play important roles in several aspects of domain theory. In this paper, we investigate some further properties of such posets, in particular their links to the dcpo models of T_1 topological spaces.

Keywords: Scott topology; maximal point space; dcpo model; compact saturated subset; K-filter defined space; k-space

https://doi.org/10.1016/j.entcs.2019.07.017
1571-0661/

A dcpo model of a topological space X is a dcpo (directed complete poset) P such that X is homeomorphic to the maximal point space of P with the subspace topology of the Scott space of P. Although it has been proved that every T_1 topological space has a dcpo model [11], one may not be able to construct a simply defined dcpo model for a general T_1 space, if it is not a complete metrizable space. In [12], the authors considered the dcpo models of the form $CK(X)$ consisting of all nonempty compact closed subsets of space X with the reverse inclusion order. The key notion employed in [12] is the CK-filter defined topology, described using the nonempty compact closed subsets. They proved that if a T_1 space is CK-filter defined, then the dcpo $(CK(X), \supseteq)$ is a bounded complete dcpo model of X. In particular, for every Hausdorff k-space (namely, compactly generated space), $CK(X)$ is a bounded complete dcpo model of X. In the current paper, we shall make use the set $\mathcal{Q}(X)$ of all nonempty compact saturated subsets of a topological space X in the place of $CK(X)$ to investigate the corresponding problems considered in [12]. The main results include: (1) For any T_1 well-filtered K-filter defined space (to be defined in Section 2) X, $\mathcal{Q}(X)$ is a dcpo model of X; (2) a Hausdorff space is a k-space iff it is CK-filter defined; (3) a first countable, coherent T_1 space is well-filtered if and only if it is sober. The result (2) answers a problem posed in [12] and establishes a new characterization of Hausdorff k-spaces.

1 Preliminaries

In this section, we recall some basic notions and results to be used in the sequel, most of them can be found in [1,2].

Let P be a poset. For $D \subseteq P$, we use $\bigvee D$ (resp., $\bigwedge D$) to denote the supremum (resp., infimum) of D if it exists.

A subset A of a poset P is called an *upper* (resp., a *lower*) set if $A = \uparrow A = \{x \in P \mid a \leq x \text{ for some } a \in A\}$ (resp., $A = \downarrow A = \{x \in P \mid x \leq a \text{ for some } a \in A\}$). A subset D of P is *directed* if it is nonempty and every finite subset of D has an upper bound in D. A poset is called a *directed complete poset* (dcpo, for short) if every directed subset in it has a supremum. A poset is called *bounded complete* if every subset that is bounded above has a supremum. In particular, a bounded complete poset has a bottom element, the supremum of empty set.

For two elements x and y in a poset P, We say that x is *way below* y, written as $x \ll y$ if for any directed set $D \subseteq P$ with existing $\bigvee D$ and $\bigvee D \geq y$, there is some $d \in D$ such that $x \leq d$. An element $x \in P$ is called *compact* if $x \ll x$. The set of all compact elements of P will be denoted by $K(P)$. A poset L is said to be *continuous* (resp., *algebraic*) if every element is the directed supremum of

[1] Supported by the NSF of China (11701500, 11626207, 11671008), the NSF of Project of Jiangsu Province of China (BK20170483)

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(resp., compact) elements that are way below it. A continuous dcpo is often called a *domain.*

A subset $U \subseteq P$ of poset P is *Scott open* iff (i) $U = \uparrow U$, and (ii) for any directed subset $D \subseteq P$, $\bigvee D \in U$ implies $D \cap U \neq \emptyset$ whenever $\bigvee D$ exists. The collection of all Scott open sets of P forms a topology, called the *Scott topology* of P and denoted by $\sigma(P)$. The space $\Sigma P = (P, \sigma(P))$ is called the Scott space of P.

For a topological space X, a nonempty subset $A \subseteq X$ is said to be *irreducible* if for any finite family $\{C_i\}_{i\in F}$ of closed sets, whenever $A \subseteq \cup_{i\in F} C_i$, then $A \subseteq C_i$ for some $i \in F$. A topological space X is called *sober* if for every irreducible closed set C, there exists a unique $x \in X$ such that $\mathrm{cl}(\{x\}) = C$, where $\mathrm{cl}(\{x\})$ means the closure of $\{x\}$. The *specialization order* on a T_0 space X is defined by that $x \leq y$ iff $x \in \mathrm{cl}(\{y\})$. Alternatively $x \leq y$ iff every open set that contains x must also contain y. Thus open sets are upper sets and closed sets are lower sets. A subset of X is called *saturated* if it is an intersection of open subsets, equivalently if it is an upper set with respect to the specialization order. Notice that for a poset L, a subset A is saturated in $(L, \sigma(L))$ iff A is an upper set. A topological space X is called *well-filtered* if for each filter basis $\mathcal{C}$ of compact saturated sets and each open set U with $\bigcap \mathcal{C} \subseteq U$, there is a $K \in \mathcal{C}$ with $K \subseteq U$. It is known by [1, Theorem II-1.21] that if X is sober, then X is well-filtered. A topological space is said to be *coherent* if the intersection of any two compact saturated sets is again compact.

A dcpo L is said to be well-filtered (resp., compact, sober, coherent) if ΣP is a well-filtered (resp., compact, sober, coherent) space.

Proposition 1.1 ([1, Proposition I-1.24.2.]) *Let X be a topological space. If X is well-filtered, then $K = \bigcap \mathcal{C}$ is a nonempty compact saturated set for each filter base $\mathcal{C}$ of nonempty compact saturated sets C.*

2 Upper Vietoris topologies and Scott topologies on $\mathcal{Q}(X)$

In this section, instead of nonempty compact closed subsets of a T_1 space used by Zhao and Xi in [12], we use nonempty compact saturated subsets of a topological space X to form a dcpo model of X.

For any T_0 well-filtered space (X, τ), let $\mathcal{Q}(X, \tau)$ ($\mathcal{Q}(X)$ for short) be the set of all nonempty compact saturated subsets of X. The poset $(\mathcal{Q}(X), \supseteq)$ is directed complete: for any directed subset $\mathcal{D} \subseteq \mathcal{Q}(X)$, $\bigcap \mathcal{D} = \bigvee_{\mathcal{Q}(X)} \mathcal{D}$.

Definition 2.1 An upper set U of a topological space (X, τ) is called K-open if, for any filtered family $\{K_i\}_{i\in I} \subseteq \mathcal{Q}(X)$ with $\bigcap_{i\in I} K_i = \uparrow x \subseteq U$ for some $x \in U$, then $K_i \subseteq U$ for some $i \in I$.

Definition 2.2 An upper set U of a topological space (X, τ) is called K^*-open if, for any filtered family $\{K_i\}_{i\in I} \subseteq \mathcal{Q}(X)$ with $\bigcap_{i\in I} K_i \subseteq U$, then $K_i \subseteq U$ for some $i \in I$.

Let τ_K be the set of all K-open sets of X. Obviously, $\emptyset$ and X are K-open. It is easy to verify that τ_K is indeed a topology on X. We call τ_K the *K-generated topology*.

Clearly, the intersection of two K*-open sets is K*-open. In general, the union of two K*-open sets may not be K*-open. So, all K*-open sets may not form a topology. The topology generated by K*-open sets and $\emptyset$ as a base is denoted by τ_{K^*}, called the *K*-generated topology*.

It is easy to see that every K*-open set is K-open. For a well-filtered space, every open set of X is K*-open. Thus, we have the following result.

Proposition 2.3 *Let (X,τ) be a well-filtered space. Then we have $\tau \subseteq \tau_{K^*} \subseteq \tau_K$.*

Definition 2.4 Let (X,τ) be a T_0 space. Then

i) (X,τ) is called K-filter defined if $\tau_K = \tau$.

ii) (X,τ) is called K*-filter defined if $\tau_{K^*} = \tau$.

Remark 2.5 1) For every well-filtered dcpo L, the space $\Sigma L = (L, \sigma(L))$ is K-filter defined.

2) The well-known non-sober dcpo constructed by Johnstone [3] is K-filter defined but not well-filtered.

Recall that in [7] the upper Vietoris topology has a basis of open sets of the form $\Box U = \{K \in \mathcal{Q}(X) \mid K \subseteq U\}$ where U ranges over the open subsets of X. The specialization order for the upper Vietoris topology on $\mathcal{Q}(X)$ agrees with the Smyth preorder $A \leq B$, i.e., $B \subseteq A$.

Note that, the specialization order on T_1 space (X,τ) reduces to the discrete order. So, for a T_1 space X, compact saturated subsets of X are the same as compact subsets of X. Let $\eta_X : X \to \mathcal{Q}(X)$ be the mapping given by $\eta_X(x) = \{x\}$ for all x.

Proposition 2.6 *Let X be a T_1 well-filtered space and $\mathcal{Q}(X)$ endowed with the upper Vietoris topology. Then $\eta_X : X \to Max(\mathcal{Q}(X))$ is a homeomorphism.*

Denote the upper Vietoris topology (resp., the Scott topology) on $\mathcal{Q}(X)$ by $UV(\mathcal{Q}(X))$ (resp., $\sigma(\mathcal{Q}(X))$). Clearly, for a well-filtered space X, $UV(\mathcal{Q}(X)) \subseteq \sigma(\mathcal{Q}(X))$.

Proposition 2.7 *If (X,τ) is T_1, well-filtered and K-filter defined, then*

$$UV(\mathcal{Q}(X)) \mid_{Max(\mathcal{Q}(X))} = \sigma(\mathcal{Q}(X)) \mid_{Max(\mathcal{Q}(X))} .$$

Proof. It suffices to show that $UV(\mathcal{Q}(X)) \mid_{Max(\mathcal{Q}(X))} \supseteq \sigma(\mathcal{Q}(X)) \mid_{Max(\mathcal{Q}(X))}$. For any $\mathcal{U} \in \sigma(\mathcal{Q}(X))$, let $U = \{x \mid \{x\} \in \mathcal{U} \cap Max(\mathcal{Q}(X))\}$. It is easy to see that $\mathcal{U} \cap Max(\mathcal{Q}(X)) = \Box U \cap Max(\mathcal{Q}(X))$. Next we show that $U \in \tau$. Since $\mathcal{U} \in \sigma(\mathcal{Q}(X))$, for any $x \in U$ and any filtered family $\{K_i\}_{i\in I} \subseteq \mathcal{Q}(X)$, if $\bigvee_{i\in I} K_i = \bigcap_{i\in I} K_i = \{x\} \in \mathcal{U}$, then there exists some $i \in I$ such that $K_i \in \mathcal{U}$, which implies that $K_i \subseteq U$. Thus, $U \in \tau_K$. Since (X,τ) is K-filter defined, we have $U \in \tau_K = \tau$. $\Box$

Theorem 2.8 *Let X be a T_1 well-filtered K-filter defined space. Then $\mathcal{Q}(X)$ is a dcpo model of X.*

Proof. It follows from Propositions 2.6 and 2.7. $\square$

Proposition 2.9 *Let (X,τ) be a T_1 and well-filtered space. If $UV(\mathcal{Q}(X)) \mid_{Max(\mathcal{Q}(X))} = \sigma(\mathcal{Q}(X)) \mid_{Max(\mathcal{Q}(X))}$, then (X,τ) is K^*-filter defined.*

Proof. Let $U \in \tau_{K^*}$. We show that $U \in \tau$. Let $\mathcal{U} = \{K \in \mathcal{Q}(X) \mid K \subseteq U\}$. Then $\{x\} \in \mathcal{U}$ holds for any $x \in U$. Since $U \in \tau_{K^*}$, $\mathcal{U} \in \sigma(\mathcal{Q}(X))$. As $UV(\mathcal{Q}(X)) \mid_{Max(\mathcal{Q}(X))} = \sigma(\mathcal{Q}(X)) \mid_{Max(\mathcal{Q}(X))}$, there exists $V \in \tau$ such that $\{x\} \in \Box V \cap Max(\mathcal{Q}(X)) \subseteq \mathcal{U} \cap Max(\mathcal{Q}(X))$. So, we have $x \in V \subseteq U$, and thus $U \in \tau$. $\square$

Example 2.10 Let $X = \mathbb{R}$ be the set of all real numbers and τ the topology on X, where $U \in \tau$ if and only if $U = V - A$ for some Euclidean open set V and a countable set A. Then, $\mathcal{Q}(X)$ is the family of all nonempty finite subsets of $\mathbb{R}$. Thus, every subset is K-open, and thus τ_K is a discrete topology. So, (X,τ_K) is not K-filter defined. Also $\sigma(\mathcal{Q}(X)) \mid_{Max(\mathcal{Q}(X))}$ is discrete, which reveals that $\mathcal{Q}(X)$ can not be a dcpo model of X.

Proposition 2.11 *Every T_1 first countable and well-filtered space is K-filter defined.*

Proof. Let (X,τ) be a T_1 first countable and well-filtered space. It suffices to show $\tau_K \subseteq \tau$. Let $U \in \tau_K$. For any $x \in U$, we need to show that there exists a $V \in \tau$ such that $x \in V \subseteq U$. Let $\{V_i \mid i \in \mathbb{N}\}$ be a neighborhood base of x with $V_{i+1} \subseteq V_i$. Suppose that $x \in V_i \nsubseteq U$ for all $i \in \mathbb{N}$. Take $x_i \in V_i \setminus U (\forall i \in \mathbb{N})$. Then $\{x_i \mid i \in \mathbb{N}\}$ converges to x. Set $A_i = \{x_j \mid j \in \mathbb{N}, j \geq i\} \cup \{x\}$. Then A_i is compact for all $i \in \mathbb{N}$. Since (X,τ) is well-filtered, $\bigcap_{i=0}^{\infty} A_i = \{x\} \subseteq U$. However, $A_i \nsubseteq U$ for all $i \in \mathbb{N}$, which contradicts $U \in \tau_K$. $\square$

By Theorem 2.8 and Proposition 2.11, we have the following result.

Corollary 2.12 *Let X be a T_1 first countable and well-filtered space. Then $\mathcal{Q}(X)$ is a dcpo model of X.*

3 Compact subsets in K*-generated topological spaces

For a well-filtered space X, the K*-generated topology may contain more open subsets than the original topology. In this section, we show that the compact subsets with respect to these two topologies are the same.

Lemma 3.1 *Let X be a T_1 well-filtered space. For any $K \in \mathcal{Q}(X,\tau)$ and any closed set C in (X,τ_{K^*}), we have $C \cap K \in \mathcal{Q}(X,\tau)$.*

Proof. It suffices to verify the compactness of $C \cap K$ for each $K \in \mathcal{Q}(X,\tau)$ and any closed set C in (X,τ_{K^*}). Suppose that $\{U_i\}_{i\in I}$ is a directed family of open sets in (X,τ) such that $\bigcup_{i\in I} U_i \supseteq C \cap K$ and $U_i \cap (C \cap K) \neq \emptyset$ for any $i \in I$, while $U_i \nsupseteq C \cap K$ for any $i \in I$. Then for any $i \in I$, $(X - U_i) \cap (C \cap K) \neq \emptyset$. Set $D_i = K \cap (X - U_i)(\forall i \in I)$. Then each D_i is a compact saturated subset of K and

$\bigcap_{i\in I} D_i \subseteq K - C$. Since C is closed in (X, τ_{K^*}), there exists some $i_0 \in I$ such that $D_{i_0} = K \cap (X - U_{i_0}) \subseteq K - C$, which contradicts $(X - U_{i_0}) \cap (C \cap K) \neq \emptyset$. □

Let $\mathcal{Q}(X, \tau_{K^*})$ be the set of all compact saturated sets of (X, τ_{K^*}).

Theorem 3.2 *Let X be a T_1 well-filtered space. Then*

$$\mathcal{Q}(X, \tau) = \mathcal{Q}(X, \tau_{K^*}).$$

Proof. It is easy to check that $\mathcal{Q}(X, \tau_{K^*}) \subseteq \mathcal{Q}(X, \tau)$. Conversely, we show that $\mathcal{Q}(X, \tau) \subseteq \mathcal{Q}(X, \tau_{K^*})$. Let $K \in \mathcal{Q}(X, \tau)$ and $\{C_i \mid i \in I\}$ be a family of closed subsets in (X, τ_{K^*}) with $\{K \cap C_i \mid i \in I\}$ satisfying finite intersection property. It suffices to show the compactness of K in τ_{K^*}. Next, we need to check that $\bigcap_{i\in I}(K \cap C_i) \neq \emptyset$.

By Lemma 3.1, we have $K \cap C_i \in \mathcal{Q}(X, \tau)$ for all $i \in I$. Suppose that $\bigcap_{i\in I}(K \cap C_i) = \emptyset$. Since (X, τ) is well-filtered, there exists some $i_0 \in I$ such that $K \cap C_{i_0} = \emptyset$, which contradicts the finite intersection property. Thus, $K \in \mathcal{Q}(X, \tau_{K^*})$, which implies that $\mathcal{Q}(X, \tau) \subseteq \mathcal{Q}(X, \tau_{K^*})$. □

Corollary 3.3 *Let X be a T_1 well-filtered space. Then (X, τ_{K^*}) is K^*-filter defined.*

4 CK-filter defined topologies and k-spaces

In this section, we give a positive answer to a question raised by Zhao and Xi in [12].

For any T_0 space (X, τ), let $CK(X)$ be the set of all nonempty closed compact subsets of X. The poset $(CK(X), \supseteq)$ is directed complete: for any directed subset $\mathcal{D} \subseteq CK(X)$, $\bigcap \mathcal{D} = \bigvee_{CK(X)} \mathcal{D}$.

Definition 4.1 ([12]) A subset U of a topological space (X, τ) is called *CK*-open if, for any filtered family $\mathcal{F} \subseteq CK(X)$ with $|\bigcap \mathcal{F}| = 1$, that is, $\bigcap \mathcal{F}$ is a singleton, and $\bigcap \mathcal{F} \subseteq U$, then $F \subseteq U$ for some $F \in \mathcal{F}$. The topology consisting of all CK-open subsets is called the CK-generated topology.

Definition 4.2 ([12]) A subset U of a topological space (X, τ) is called *CK**-open if, for any filtered family $\mathcal{F} \subseteq CK(X)$ with $\bigcap \mathcal{F} \subseteq U$, then $F \subseteq U$ for some $F \in \mathcal{F}$. The topology generated by all CK*-open subsets as a basis is called the CK*-generated topology.

Obviously, every open set of X is CK*-open, and every CK*-open set is CK-open. It is shown in [12, Lemma 2.9] that a subset of a Hausdorff space is CK-open if and only if it is CK*-open. The next example shows that a CK-open set of a T_1 space need not to be CK*-open.

Example 4.3 Let $(\mathbb{N}, \tau_{cof})$ be the set $\mathbb{N}$ of all positive integers equipped with the co-finite topology τ_{cof}. Then $(\mathbb{N}, \tau_{cof})$ is not well-filtered. However, for any $x \in \mathbb{N}$, any open neighborhood U of x and filtered family of compact subsets $\{K_i\}_{i\in I}$, $\bigcap_{i\in I} K_i = \{x\} \subseteq U$ implies that there exists $i_0 \in I$ such that $K_{i_0} \subseteq U$.

Definition 4.4 ([12]) i) A topological space (X, τ) is called *CK*-filter defined if $\tau_{CK} = \tau$.

ii) A topological space (X, τ) is called CK^*-filter defined if $\tau_{CK^*} = \tau$.

Definition 4.5 ([12]) A space X is a k-space (or, *compactly generated space*) if a subset U of X is open if and only if for any compact set K, $U \cap K$ is open in the subspace K. Equivalently, a subset B is closed if and only if for any compact set K, $B \cap K$ is closed in the subspace K.

Lemma 4.6 ([12, Theorem 2.4]) *Every Hausdorff k-space is CK-filter defined.*

Xi and Zhao in [12] asked if Hausdorff CK-filter defined spaces are k-spaces. We now give a positive answer for more general case as follows.

Theorem 4.7 *Every CK-filter defined space is a k-space.*

Proof. Let (X, τ) be CK-filter defined. Assume $U \subseteq X$ such that $U \cap K$ is open in K for any compact set $K \subseteq X$. Let $\{K_i\}_{i \in I}$ be a filtered family of compact closed sets such that $\bigcap_{i \in I} K_i = \{x\} \subseteq U$. Then there is an $i_0 \in I$ such that $K_i \subseteq K_{i_0}$ for $i \geq i_0$. So, we assume that every K_i is contained in K_{i_0}. Then every K_i is a closed subset of the compact space K_{i_0}. Suppose that $K_i - U = K_i \cap U^C \neq \emptyset$ for all $i \in I$, then, as $U \cap K_i$ is open in K_i, $K_i - U = K_i \cap U^C$ is closed in K_i. Hence, every $K_i - U$ is closed in K_{i_0} and $\{K_i \mid i \in I\}$ satisfies the finite intersection property. Now $\bigcap_{i \in I}(K_i - U) = \bigcap_{i \in I} K_i - U = \emptyset$, which contradicts that K_{i_0} is compact. So, there exists an i' such that $K_{i'} \subseteq U$. Since (X, τ) is CK-filter defined, $U \in \tau_{CK} = \tau$, and thus U is open in X. Therefore, X is a k-space. □

By Theorem 4.7 and Lemma 4.6, we have the following result which gives a new characterization for Hausdorff k-spaces.

Corollary 4.8 *A Hausdorff space is a k-space iff it is CK-filter defined.*

5 First countable spaces with bounded complete dcpo models are sober

It is well known that every sober space is well-filtered. But there exists a well-filtered T_1 space which is non-sober. There are also dcpos whose Scott spaces are well-filtered but non-sober (see [6], [4, Example 2.6.1], [13]). There are even coherent well-filtered spaces which are non-sober.

We now show that if we add the first countability, then coherence and well-filteredness imply sobriety. Using this result we deduce that if a first countable T_1 space has a bounded complete dcpo model, then it is sober.

Lemma 5.1 *Let X be a coherent and first countable space. Then X is well filtered if and only if it is sober.*

Proof. We only need to show that X is sober if it is well-filtered. Suppose that X is well-filtered. If X is not sober, there is an irreducible closed subset $\widetilde{X}$ which is not the closure of any single point. We consider $Max(\widetilde{X})$, the set of maximal points of $\widetilde{X}$ which is also irreducible, although it may not be closed.

Then for any $x, y \in Max(\widetilde{X})$, there exist two countable neighborhood bases $\{V_x^i \mid i \in \mathbb{N}\}$ with $V_x^{i+1} \subseteq V_x^i$ and $\{V_y^j \mid j \in \mathbb{N}\}$ with $V_y^{i+1} \subseteq V_y^i$. Since $Max(\widetilde{X})$ is irreducible, $V_x^i \cap V_y^j \cap Max(\widetilde{X}) \neq \emptyset$ for any $i, j \in \mathbb{N}$. For any $i \in \mathbb{N}$, we take $a_i \in V_x^i \cap V_y^j \cap Max(\widetilde{X})$. By the choice of each a_i, each neighbourhood of x contains all a_i, except finite elements. Thus, $\{a_i \mid i \in \mathbb{N}\} \cup \{x\}$ is compact. Similarly, $\{a_i \mid i \in \mathbb{N}\} \cup \{y\}$ is also compact. By the coherence of X, we have $\{a_i \mid i \in \mathbb{N}\} = (\{a_i \mid i \in \mathbb{N}\} \cup \{x\}) \cap (\{a_i \mid i \in \mathbb{N}\} \cup \{y\})$ is compact. Let

$$A_0 = \{a_i \mid i \geq 0, i \in \mathbb{N}\},$$

$$A_1 = \{a_i \mid i \geq 1, i \in \mathbb{N}\},$$
$$A_2 = \{a_i \mid i \geq 2, i \in \mathbb{N}\}, \cdots,$$
$$A_k = \{a_i \mid i \geq k, i \in \mathbb{N}\}, \cdots.$$

Note that $A_k = (\{a_i \mid i \geq k, i \in \mathbb{N}\} \cup \{x\}) \cap (\{a_i \mid i \geq k, i \in \mathbb{N}\} \cup \{y\})$, we have that A_k is compact by the coherence, while $\bigcap_{k=0}^{\infty} A_k = \emptyset$, which contradicts the well-filteredness. □

Remark 5.2 (1) The set $\mathbb{R}$ of all real numbers equipped with the co-countable topology is coherent (because only finite subsets are compact) and well-filtered, but it is non-sober.

(2) The set $\mathbb{N}$ of all positive integers equipped with the co-finite topology is first countable and coherent (in this case every subset is compact), but it is non-sober.

For any bounded complete dcpo P, the Scott space ΣP is coherent [4, Corollary 4.1.8] and well-filtered [8, Corollary 3.2]. It thus follows that the maximal point space $\mathrm{Max}(P)$ is also coherent and well-filtered. So the above theorem implies the following result.

Corollary 5.3 *If a first countable space has a bounded complete dcpo model, then it is sober.*

In [12, Corollary 2.14], it was proved that every Hausdorff k-space (particularly, every first countable Hausdorff space) has a bounded complete dcpo model. The above result indicates that the sobriety is a necessary condition for a first countable T_1 space to have a bounded complete dcpo model.

However, we do not know the answer to the following problem.

Problem 1. If a k-space has a bounded complete dcpo model, must it be sober?

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