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P_3-Hull Number of Graphs with Diameter Two

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Abstract

Let G be a finite, simple, undirected graph with vertex set $V(G)$. If there is a vertex subset $S \subseteq V(G)$, and every vertex of $V(G)$ with at least two neighbors in S is also a member of S, then S is termed P_3-convex. The P_3-convex hull of of S is the smallest convex set containing S. The P_3-hull number $h(G)$ is the cardinality of a smallest set of vertices whose P_3-convex hull is the entire graph. In this paper we establish some bounds on the P_3-hull number of graphs with diameter two. Particularly, in biconnected C_6-free diameter two graphs the P_3-hull number is at most 4. We also establish the upper bound $h(G) \leq \left\lceil \frac{k}{1+b} \right\rceil + 1$ or alternatively $h(G) \leq \lceil \log_{c+1}(k.c+1) \rceil + 1$, for strongly regular graphs $G(n,k,b,c)$.

Keywords: graph, P_3-convexity, hull number, diameter two

1 Introduction

For a graph G, the vertex set is denoted $V(G)$ and the edge set $E(G)$. We consider finite, simple, and undirected graphs. A set $\mathcal{C}$ of subsets of $V(G)$ is a *convexity* in G if $\emptyset, V(G) \in \mathcal{C}$ and $\mathcal{C}$ is closed under intersection. Each element of $\mathcal{C}$ is a *convex set*. The *convex hull* $H_{\mathcal{C}}(S)$ of a subset $S \subseteq V(G)$ is the smallest convex set containing

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S. If $H_{\mathcal{C}}(S) = V(G)$ then S is a *hull set*. The cardinality $h_{\mathcal{C}}(G)$ of the smallest hull set in G is called the *hull number* of G.

Some convexities in graphs are defined by a set $\mathcal{P}$ of paths in graphs. In this case, a subset $\mathcal{C} \in V(G)$ is convex precisely when $\mathcal{C}$ contains all the vertices belonging to the paths of $\mathcal{P}$ whose extreme vertices are also in $\mathcal{C}$. The *geodetic convexity* considers $\mathcal{P}$ as the set of all shortest paths in G [4,11,12,18]. In the *monophonic convexity* [9,13,17] $\mathcal{P}$ is the set of all induced paths. The *triangle path convexity* [7,15] and P_3 *convexity* consider $\mathcal{P}$ as the set of all paths with three vertices [5,6,8,14].

The concept of hull number was introduced by Everett and Seidman [18] in the geodetic convexity. Considering this convexity, the computation of hull number is NP-hard for bipartite graphs [1], but it can be computed in polynomial time for cographs and split graphs [10], $(q, q-4)$-graphs [1], $\{paw, P5\}$-free graphs [11], and distance-hereditary graphs [21]. The hull number was studied in other graph convexities. In monophonic convexity [13] and triangle path convexity [15], the hull number can be determined in polynomial time for general graphs. In the P_3 convexity the computation of hull number is NP-hard, but it can be determined in polynomial time for cographs and chordal graphs [5], and for complementary prisms [16]. Recently, Penso et al. [22] studied the complexity of computing the hull number restricted to planar graphs, showing that it remains NP-hard on planar graphs with maximum degree three and four.

In this paper we study exclusively the P_3-convexity $\mathcal{C}$ on a graph G. Given a set $S \subseteq V(G)$, the P_3-*interval* $I[S]$ of S is formed by S, together with every vertex outside S with at least two neighbors in S. If $I[S] = S$, then the set S is P_3-*convex*. The P_3-*convex hull* $H_{\mathcal{C}}(S)$ of S is the smallest P_3-convex set containing S. Since a graph G uniquely determines its P_3-convexity $\mathcal{C}$, we may write $H(S)$, instead of $H_{\mathcal{C}}(S)$. The P_3-convex hull $H(S)$ can be formed from the sequence $I^p[S]$, where p is a nonnegative integer, $I^0[S] = S$, $I^1[S] = I[S]$, and $I^p[S] = I[I^{p-1}[S]]$ for $p \geq 2$. When, for some p, we have $I^q[S] = I^p[S]$, for all $q \geq p$, then $I^p[S]$ is a P_3-convex set. If $H(S) = V(G)$ we say that S is a P_3-*hull set* of G. The cardinality $h(G)$ of a minimum P_3-hull set in G is called the P_3-*hull number* of G.

We study the hull number in P_3 convexity in diameter two graphs. We present an algorithm that provides a hull set of maximum cardinality $\lceil \log(\Delta + 1) \rceil + 1$. For C_6-free diameter two graphs we prove that the hull number is at most 4. Also, we show that for a subclass of the graphs with diameter two, the strongly regular graphs, the hull number is at most 3.

Before we present our results and proofs, we summarize some notation and useful definitions.

The *open neighborhood* of a vertex $v \in V(G)$ is $N(v) = \{w \in V(G) \mid vw \in E(G)\}$, and its *closed neighborhood* is $N[v] = N(v) \cup \{v\}$. For a set S of vertices of G, let $N[S] = \bigcup_{u \in S} N[u]$. The degree of a vertex $v \in V(G)$ is denoted by $d(v)$, the *minimum degree* of a graph G is $\delta(G) = min\{d(v) \mid v \in V(G)\}$ and the *maximum degree* of G is $\Delta(G) = max\{d(v) \mid v \in V(G)\}$. If $u, v \in V(G)$, we denote by $d(u, v)$ the distance between u and v, and the *diameter* of G is the greatest distance between all pairs of vertices of G. Therefore, if a graph G has diameter 2, then every pair of non-adjacent vertices has a common neighbor. Figure 1 shows some graphs with diameter two.

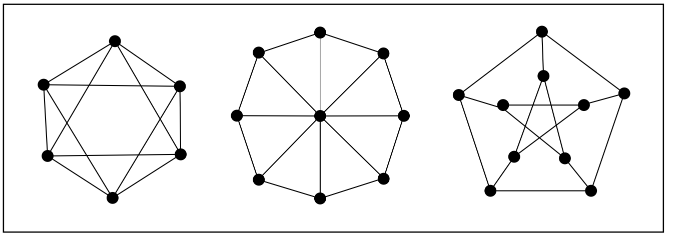

Fig. 1. Examples of graphs with diameter two.

The class of graphs with diameter two is broad and includes a class of interest in this work. A graph $G(n, k, b, c)$, with n vertices, is *strongly regular* if it is k-regular and there are integers b and c such that every two adjacent vertices have b common neighbours and every two non-adjacent vertices have c common neighbours.

A graph G is connected if for any two vertices $u, v \in V(G)$, there is a u-v path in G. A connected graph G is called *biconnected*, if for every vertex $v \in V(G)$, $G - v$ is connected. A cut-vertex v of a connected graph G is such that $G - v$ is disconnected. A graph is *C_6-free* if it contains no induced cycle with 6 vertices. A subset $S \subseteq V(G)$ is a *dominating set* if every vertex not in S is adjacent to at least one member of S. We say that a set T of vertices is *co-convex* if every vertex of T has at most one neighbor outside T. The following fact can be easily verified.

Fact 1.1 *Let S be a hull set of G and let T be a co-convex set of G. Then, S contains a vertex of T .*

To conclude this section, we present some observations about diameter two graphs.

Observation 1 *Let G be a graph of diameter two. Then, the following properties are true:*

(a) If $u, v \in V(G)$, then $N[u] \cap N[v] \neq \emptyset$;

(b) $N(v)$ is a dominating set $\forall v \in V(G)$;

(c) If $\delta(G) = 1$, then $\Delta(G) = n - 1$;

(d) If G has a cut-vertex, then $\Delta(G) = n - 1$.

2 Results

In this section, we present our results. We start by discussing the worst case of hull number for a graph G with diameter two, which occurs when G has a cut-vertex. Note that equality holds for star graphs $K_{1,n}$, where $n \geq 2$.

Proposition 2.1 *Let G be a diameter two graph with a cut-vertex v. Then, $h(G) = \omega(G - v)$, where $\omega(G)$ is the number of connected components of G.*

Proof. Let G be a diameter two graph with a cut-vertex v. Consider $G' = G -$

v. Since v is a cut-vertex, G' is disconnected with $k = \omega(G') \geq 2$ connected components, say $C_1, \ldots, C_k$. Let S be a subset of $V(G')$ containing exactly one vertex of each connected component C_i, $i = 1, \ldots, k$. Note that $|S| = k$ and recall that v is adjacent to every vertex of G. We will show that every vertex in $V(G) - S$ belongs to $H(S)$. First we show that $v \in H(S)$. Let $u_i \in C_i$ and $u_j \in C_j$ such that $u_i, u_j \in S$ and $i \neq j$. Since $u_i, u_j \in N(v)$, $v \in H(\{u_i, u_j\})$. Now, consider a vertex $u \in C_i - S$, for some $i \in \{1, \ldots, k\}$. Since C_i is connected, there exists in C_i a path $P = u, u_1, \ldots, u_\ell$, where $\ell \geq 1$, such that $u_\ell \in S$. Since v is adjacent to every vertex in P and $v \in H(S)$, we have that $u_{\ell-1} \in I^2(S)$, $u_{\ell-2} \in I^3(S)$, and so on, resulting in $u \in I^{|P|-2}$. Therefore, $u \in H(S)$. We can conclude that S is a hull set of G of cardinality $k = \omega(G - v)$ and $h(G) \leq k$.

For the lower bound, observe that each connected component C_i has exactly one neighbor, v, outside C_i, $i = 1, \ldots, k$. So, each C_i is co-convex and follows by Fact 1.1 that $h(G) \geq k$. Therefore, $h(G) = \omega(G - v)$. □

By Observation 1(c) and 1(d), if G has diameter two and $\Delta < n - 1$, then $\delta > 1$ and G has no cut-vertex. These restrictions can be used for establishing conditions for a set S to be a hull set of G.

Lemma 2.2 *Let G be a biconnected diameter two graph and $S' \subseteq V(G)$. If $H(S')$ is a dominating set of G, then for any $v \in V(G) - H(S')$ the set $S = S' \cup \{v\}$ is a hull set of G.*

Proof. Let G be a biconnected diameter two graph and suppose $H(S')$ is a dominating set of G. Note that, every $v \in V(G) - H(S')$ has a single neighbor in $H(S')$. Consider $S = S' \cup \{v\}$, such that $v \in V(G) - H(S')$. We will prove that S is a hull set by showing that every vertex $u \in V(G) - H(S')$ belongs to the convex hull of S.
Claim 1: The graph G' induced by the set of vertices $V(G) - H(S')$ is connected.
Proof of Claim 1: For a proof by contradiction, suppose that G' is disconnected. Let G_1 and G_2 be two connected components of G', $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$. Whereby $d(v_1, v_2) = 2$, there exists $w \in H(S')$ such that $v_1 w, v_2 w \in E(G)$. Furthermore, w is a single neighbor of v_1 and v_2 in $H(S')$. Using the same argument, it follows that all vertices of G_1 and G_2 are adjacent only to w in $H(S')$. Since $H(S')$ is a dominating set the vertex w is unique, otherwise, all vertices of G_1 and G_2 would be in $H(S')$. Hence, w is a cut vertex, which is a contradiction since G is biconnected. □

By Claim 1 consider the path $P = v, v_1, v_2, \ldots, v_k u$ in $H(S')$. As $vv_1 \in E(G)$ and there exists a vertex $v'_1 \in H(S')$, such that $v'_1 v_1 \in E(G)$, it follows that $v_1 \in I[S]$. Analogously, since $v_1 v_2 \in E(G)$ and there exists $v'_2 \in H(S')$, such that $v'_2 v_2 \in E(G)$, it follows that $v_2 \in I^2[S]$. Thus, successively, since $v_n u \in E(G)$ and there exists a vertex $v'_k \in H(S')$, such that $v'_k v_k \in E(G)$, we can conclude that $u \in I^{k+1}[S]$. So, the proof is complete. □

A simple upper bound can be obtained through Lemma 2.2.

Corollary 2.3 *If G is a biconnected diameter two graph, then $h(G) \leq c(G) + 1$, where $c(G)$ is a cardinality of a minimum cut of vertices in G.*

Proof. Consider X a minimum cut of vertices of G. Since G has diameter two, the set X is a dominating set. By Lemma 2.2, $X \cup \{v\}$, for all $v \in V(G) \backslash H(X)$, is a hull set. Therefore, $h(G) \leq c(G) + 1$. □

We can improve on this bound by providing a polynomial time algorithm that constructs a hull set in an iterative way. The algorithm constructs a set S' whose convex hull is dominating. Hence, by Lemma 2.2 the graph G has a hull set $S = S' \cup \{v\}$, where $v \in V(G) - H(S')$. The algorithm uses the following properties of diameter two graphs. In a diameter two graph G, any non-empty set C can be used to partition the remaining set of vertices of G into two other subsets, as follows:

- Set N: subset of $V(G) - C$ containing all vertices adjacent to some vertex of C, such that they do not belong to C, i.e., $N = N(C) - C$.
- Set O: subset of $V(G) - C$ containing all vertices adjacent only to vertices of N, i.e., $O = N[N] - N[C]$.

As shown in Proposition 2.4, for diameter two graphs, the sets C, N and O together partition $V(G)$.

Proposition 2.4 *If G is a diameter two graph, then $V(G) = C \cup N \cup O$.*

Proof. For a proof by contradiction, suppose that there exists a vertex $v \in V(G)$ such that $\{v\} \cap (C \cup N \cup O) = \emptyset$. Consider $u \in N(v)$. In the best case, if $u \in O$, then $d(v, w) = 3$, for $w \in C$, which is a contradiction since G has diameter two. □

Another important property of sets C, N and O is that they can be used to identify when a set is a dominating set.

Proposition 2.5 *Let G be a diameter two graph and $S \subseteq V(G)$. Consider $C = H(S)$, $N = N(C) - C$ and $O = V(G) - N[C]$. Then, the following statements are true:*

(a) If $O = \emptyset$, then $H(S)$ is a dominating set.

(b) If $o \in O$ then $|N(o) \cap N| \geq |H(S)|$.

Proof. Proof of (a). Since $O = \emptyset$, by Proposition 2.4, either $v \in H(S)$ or $v \in N$. By definition of sets C and N, $H(S)$ is a dominating set, completing the proof of (a).

Proof of (b). By contradiction, suppose that $o \in O$ and $|N(o) \cap N| < |H(S)|$. By definition of the set O, the vertex o is not adjacent to any vertex in $H(S)$. Since G is a diameter two graph, every vertex in $H(S)$ is adjacent to at least one vertex in $N(o)$. As $|N(o) \cap N| < |H(S)|$, it follows that there exists at least one vertex $v \in N[C] \cap N[O]$ such that $|N[v] \cap C| \geq 2$. So, $v \in H(S)$ and $o \in N[C]$. This is a contradiction, which proves (b). □

Proposition 2.5 leads to an algorithm which iteratively chooses vertices in $V(G)$ to compose a set C, such that C is a dominating set. Consider a set S' and $C = H(S')$. Observe that $N = N[C] - C$ is the set dominated by $H(S')$ and every vertex v belonging to $O = N[N] - N[C]$ satisfies $|N(v) \cap N| \geq |H(S')|$. When

$|H(S')| > \Delta$, we have $O = \emptyset$ and consequently $H(S')$ will be a dominating set. In this case, by Lemma 2.2, there is a hull set of cardinality $|S'| + 1$. We will prove that the cardinality of S' is at most $\lceil \log(\Delta(G) + 1) \rceil$.

The algorithm starts by including any vertex $v \in V(G)$ in S' and then calculates the sets $H = H(S')$, N and O. While $O \neq \emptyset$ the algorithm selects any vertex of O, adds it to S', and calculates the sets H, N, and O. The iterative process ends when O is empty, in which case the dominating set $H(S')$ is returned. A pseudocode of the process is given as Algorithm 1.

Algorithm 1: HullSet(G)

Data: Diameter two graph $G = (V, E)$
Result: S' such that $N(H(S')) = V(G)$

$S' \leftarrow \emptyset$
$H \leftarrow \emptyset$
$O \leftarrow V$
while $O \neq \emptyset$ **do**
  $S' \leftarrow S' \cup \{v\} \mid v \in O$
  $H \leftarrow H(S')$
  $O \leftarrow V - N[H]$
end
return S'

In the next proofs we represent S'_i, O_i and H_i, respectively as the sets S', O and H in the i-th iteration of the Algorithm 1.

The first property we can observe is that the cardinality of the set H doubles at each iteration.

Lemma 2.6 *Consider the i-th iteration of Algorithm 1. If $O_i \neq \emptyset$, then $|H(S'_i)| \geq 2(|H(S'_{i-1})| + 1)$.*

Proof. Consider the i-th iteration of Algorithm 1 and assume that $N_i = N(H_i) - H_i$. Let $u \in O_{i-1}$, such that $S'_i = S'_{i-1} \cup \{u\}$. Notice that every element in $N(u) \cap N_{i-1}$ has at least two neighbors in H_i, namely u and its neighbor in H_{i-1}. Therefore, $N(u) \cap N_{i-1} \subset H_i$. Hence $|H_i| \geq |H_{i-1}| + |N(u) \cap N_{i-1}| + 1$, where the unit entry comes from the vertex u. But by Proposition 2.5, $|N(u) \cap N_{i-1}| \geq |H_{i-1}|$. Therefore, $|H_i| \geq 2|H_{i-1}| + 1$. □

When $|H(S')| > \Delta$, by Proposition 2.5, we can conclude that $H(S')$ is a dominating set. The next proposition establishes the maximum number of iterations needed for this to occur.

Proposition 2.7 *Algorithm 1 runs in at most $k = \lceil \log(\Delta(G) + 1) \rceil$ iterations.*

Proof. By Lemma 2.6, in each iteration of Algorithm 1, $|H(S'_i)| \geq 2|H(S'_{i-1})| + 1$. There is an exponential growth of $H(S'_i)$. Solving the recurrence, $T(n) \geq 2 \cdot T(n-1) + 1$, the cardinality of the set can be given by the inequality $|H(S'_i)| \geq 2^i - 1$. By Proposition 2.5, the vertices in O_i must have a degree greater than the cardinality of $H(S'_{i-1})$. So, O_i is empty when $H(S'_{i-1}) > \Delta$, i.e., when $2^k \geq \Delta + 1$. Isolating k, we have that $k \geq \log(\Delta + 1)$. Therefore $k = \lceil \log(\Delta(G) + 1) \rceil$. □

Corollary 2.8 *If G is a biconnected diameter two graph, then*

$$h(G) \leq \lceil \log(\Delta(G) + 1) \rceil + 1.$$

Proof. By Proposition 2.7, the Algorithm 1 runs in at most $\lceil \log(\Delta(G) + 1) \rceil$ iterations. At each iteration of the algorithm the set S' is incremented by one. So, $|S'| \leq \lceil \log(\Delta(G) + 1) \rceil$ and $H(S')$ is a dominating set. Since there is a dominating set of cardinality $\lceil \log \Delta + 1) \rceil$, by Lemma 2.2 we have that $h(G) \leq \lceil \log(\Delta + 1) \rceil + 1$. □

If we add some constraints related to the diameter two graph we can establish a sharper bound.

Theorem 2.9 *If G is a biconnected diameter two graph having a pair of true/false twin vertices, then $h(G) \leq 3$.*

Proof. If G has a pair of vertices u, v such that u and v are either true or false twins, then, by definition, $N(u) = N(v)$. Therefore, $H(\{u, v\}) \supset N(v)$. Since G is biconnected with diameter two, it follows that $N(v)$ is a dominating set. Thus, by Lemma 2.2, $\{u, v, w\}$ is a hull set of G, where w (if any) is a vertex that does not belong to $H(\{u, v\})$. □

Theorem 2.10 *If G is a biconnected C_6-free diameter two graph, then $h(G) \leq 4$.*

Proof. Let G be a biconnected C_6-free diameter two graph and let $S = \{a, b, c\}$ be a set of vertices of size three, such that $|H(S)|$ is maximum. If $H(S)$ is a dominating set, by Lemma 2.2, the theorem holds. Otherwise, since $H(S)$ is not a dominating set, it follows that each vertex of S has at least one distinct neighbor w such that $w \notin H(S)$. Otherwise, the neighborhood of each vertex would belong to $H(S)$ and $H(S)$ would be a dominating set. Assume w is such a neighbor of $a \in S$ ($w \notin H(S)$). We have the following cases:

Case 1: *$G[S]$ contains at least one edge.*

Let ab be an edge of $G[S]$. In this case, $H(\{w, b, c\})$ contains a, and thus $H(S) \subseteq H(\{w, b, c\})$. Since $w \notin H(S)$, it follows that $\{w, b, c\}$ is a set of size three with hull of size greater than $|H(S)|$, a contradiction.

Case 2: *S is an independent set.*

Since G has diameter two, it follows that the pairs $\{a, b\}$, $\{a, c\}$ and $\{b, c\}$ have common neighbors.

(i) If there is a vertex z that is neighbor of a, b and c, then $H(\{w, b, c\})$ contains z. Therefore, it also contains a, which implies that $H(S) \subset H(\{w, b, c\})$ ($w \notin H(S)$), contradicting the maximality of $H(S)$.

(ii) Otherwise, G contains a C_6: $a, x_1, b, x_2, c, x_3, a$. Given that $\{a, b, c\}$ is an independent set and G has no induced C_6, $G[\{x_1, x_2, x_3\}]$ induces an edge. Let (x_1, x_2) be an edge of $G[\{x1, x2, x3\}]$. As $H(S)$ is not a dominating set, x_1 has a neighbor u such that $u \notin H(S)$. Consider $H(\{a, c, u\})$. It is easy to see that $H(\{a, c, u\})$ contains x_1 (a neighbor of u and a). Thus, it also contains x_2 (a neighbor of c and x_1), and b (a neighbor of x_1 and x_2). Therefore, $H(\{a, c, u\}) \supset H(S = \{a, b, c\})$ because $u \notin H(S)$, contradicting the maximality of $H(S)$.

Therefore, if G is a biconnected C_6-free graph then G has a set of vertices of size three such that $H(S)$ is a dominating set, which implies, by Lemma 2.2, that $h(G) \leq 4$.□

We now consider strongly regular graphs, a subclass of diameter two graphs. We start by analyzing the strongly regular graphs $G(n, k, b, c)$, when $c = 0$. In this case, each such graph is a union of disjoint complete graphs.

Theorem 2.11 *If G is a strongly regular graph $G(n, k, b, c)$ with $c = 0$, then $h(G) \leq 2\omega(G)$.*

Proof. If G is a strongly regular graph with $c = 0$, then G has $\omega(G)$ (connected) components. Each such connected component is a complete graph with $k + 1 = b + 2$ vertices. A hull set for G can be constructed by joining the hull set of each component. Since the components are complete graphs, the hull number of each component is 2. So, $h(G) \leq 2\omega(G)$. □

A strongly regular graph with $c > 0$ is a connected nontrivial k-regular graph, which is not a complete graph. The result of Theorem 2.10 is valid for biconnected C_6-free diameter two graphs. We can give an upper bound for strongly regular graphs, with $c > 0$, in general.

Theorem 2.12 *If G is a strongly regular graph $G(n, k, b, c)$ with $c > 0$, then there exists a set S' such that $|S'| \leq \lceil \frac{k}{1+b} \rceil$ and $H(S')$ is a dominating set.*

Proof. For any $v \in V(G)$, we have that $|N(v)| = k$ and $N(v)$ is a dominating set. Consider $u \in N(v)$ and $S_1 = \{v, u\}$. Since u is adjacent to v, u and v have b neighbors in common. So, $|H(S_1)| \geq |S_1| + b$. Observe that $H(S_1)$ has at least $b + 1$ neighbors of v. If $N(v) \subset H(S_1)$, then $H(S_1)$ is a dominating set. Otherwise there exists $w \in N(v) \setminus H(S_1)$ and we set $S_2 = \{u, v, w\}$. Again, since there are b neighbors in common between v and w, it follows that $|H(S_2)| \geq |H(S_1)| + (b + 1) \geq (|S_1| + b) + (b + 1)$, that is, $|H(S_2)| \geq 2b + 3$. Note that, because of the assumed structure of S_2 the vertex v belongs to $H(S_2 \setminus \{v\})$. So, by resetting S_2 as $S_2 = \{u, w\}$, it can be seen that $H(S_2)$ has at least $2b + 2$ neighbors of v. This process can be continued, selecting vertices $w_i \in N(v) \setminus H(S'_{i-1})$ and setting $S'_i = S'_{i-1} \cup \{w_i\}$. In the i-th iteration $H(S'_i) \geq i \cdot b + i$. When $i \cdot b + i \geq k$, we have $N(v) \subseteq H(S'_i)$. Therefore, when $i \geq \lceil \frac{k}{1+b} \rceil$, the set $H(S'_i)$ is a dominating set with $|S'_i| = \lceil \frac{k}{1+b} \rceil$. □

Corollary 2.13 *If G is a strongly regular graph $G(n, k, b, c)$ with $c > 0$, then*

$$h(G) \leq \lceil \frac{k}{1+b} \rceil + 1.$$

Proof. By theorem 2.12, G has a dominating set S', with $|S'| \leq \lceil \frac{k}{1+b} \rceil$. Thus, by Lemma 2.2, G has a hull set S with $|S| \leq \lceil \frac{k}{1+b} \rceil + 1$. □

Theorem 2.13 presents a tight bound for strongly regular graphs, but it does not consider the parameter c. There are graphs whose relation between the parameters k and b do not provide a tight bound, for example, the graph $GQ(3, 9)$ whose parameters are $(112, 30, 2, 10)$. With respect to the bound given in Theorem 2.13 the hull number is at most $\lceil \frac{k}{1+b} \rceil + 1 = 11$, but the actual bound, which we computationally confirm, is 2. Another bound for the class of strongly regular graphs can be established using the parameter c and the Algorithm 1.

When the cardinality of $H(S')$ is greater than k, by Proposition 2.5, we can conclude that $H(S')$ is a dominating set. So we want to determine the maximum number of iterations when this occurs. A bound on this number is established in the following results.

Lemma 2.14 *Consider the i-th iteration of Algorithm 1 when it is applied to a strongly regular graph $G(n, k, b, c)$ with $c > 0$. If $O_i \neq \emptyset$, then $|H(S_i)| \geq (c + 1) \times (|H(S_{i-1})| + 1)$.*

Proof. Consider the i-th iteration of the Algorithm 1. Note that each vertex in O_i has only c vertices in common with each vertex of H_{i-1}. The selected vertex $u \in O_i$ will be added to S_i, and all vertices in $N(u) \cap N(H_{i-1})$ will be included in $H_i = H(S_i)$. Note that $|N(u) \cap N(H_{i-1})| = c \times |H_{i-1}|$, that is, $H(S'_i) \subseteq H(S'_{i-1}) \cup (N(u) \cap N(H_{i-1})) \cup \{u\}$. So, $|H(S'_i)| \geq |H(S'_{i-1})| + |(N(u) \cap N(H_{i-1})| + |\{u\}|$. Since $|N(u) \cap N(H_{i-1})| = c \times |H(S'_{i-1})|$, we can conclude that $|H(S'_i)| \geq (c+1)|H(S'_{i-1})| + 1$. □

Proposition 2.15 *If Algorithm 1 is applied to a strongly regular graph G (n, k, b, c), it runs in at most $i = \lceil \log_{c+1}(kc + 1) \rceil$ iterations.*

Proof. By Lemma 2.14, in each iteration of the Algorithm 1, $|H(S_i)| \geq (c + 1) \times |H(S_{i-1})| + 1$. Thus, in the worst case, $H(S_i)$ grows exponentially. By solving the recurrence, $T(n) \geq (c + 1) \cdot T(n - 1) + 1$, it can be shown that the rate of growth is bounded by $|H(S'_i)| \geq \frac{(c+1)^i - 1}{c} + 1$. Furthermore, by Proposition 2.5, $|O_i| > H(S'_{i-1})$ implies that $O_i \neq \emptyset$. So $O_i = \emptyset$ when $H(S'_{i-1}) > k$, i.e. when $\frac{(c+1)^i - 1}{c} + 1 \geq k$. Solving for i, we have that $i \geq \log_{c+1}(kc + 1)$. When $i = \lceil \log_{c+1}(kc + 1) \rceil$ we have that $H(S_i) \geq k$. □

Given the minimum rate of growth of $H(S')$, proved in the Lemma 2.14, we can analyze the growth of the cardinality of the sets H, S and O in each iteration of Algorithm 1. At each iteration a new vertex is added to S', and the cardinality of $H(S')$ is multiplied by $c + 1$. That is, $|H(S_i)| \geq (c + 1) \times |H(S_{i-1})| + 1$.

Theorem 2.16 *If G is a strongly regular graph with $c > 0$, then*

$$h(G) \leq \lceil \log_{c+1}(kc+1) \rceil + 1.$$

Proof. By Lemma 2.14, G has a set S' such that $H(S')$ is a dominating set, and $|S'| = \lceil \log_{c+1}(kc+1) \rceil$. So, by Lemma 2.2, $h(G) \leq \lceil \log_{c+1}(kc+1) \rceil + 1$. □

Table 1
Comparative results for the hull number of some strongly regular graphs

Graph	**h**	**Corol. 2.13**	**Theor. 2.16**
C_5 (5,2,0,1)	3	3	3
Petersen (10,3,0,1)	3	4	3
Clebsch (16,5,0,2)	3	6	3
(4,4)-Hook (16,6,2,2)	2	3	3
Hoffman-Singleton (50,7,0,1)	4	8	4
Gewirtz (56,10,0,2)	3	11	4
Brouwer-Haemers (81,20,1,6)	2	11	3
M_{22} (77,16,0,4)	2	17	4
$H(2,10)$ (100,18,8,2)	2	3	4
Higman-Sims (100,22,0,6)	2	23	4
1^{st} sub. of McLaughlin (112,30,2,10)	2	5	3
Cameron (231,30,9,3)	2	4	4
Berlekamp-van L. (243,22,1,2)	3	12	4
McLaughlin (275,112,30,56)	2	5	3
Grassmann $J_2(6,2)$ (651,90,33,9)	2	4	4
G2(4) (416,100,36,20)	2	4	3
Games (729,112,1,20)	*	57	4

We have made a comparison of the upper bounds on the hull numbers of various graphs that were generated by Algorithm 1 and the use of the theoretical results reported in this work. The results for the strongly regular connected graphs available in [19,23,24] are presented in Table 1.

3 Final Remarks

We have demonstrated how the concept of the P_3-convexity can be used to compute to determine the P_3-hull number of graphs with diameter 2. We have proposed a polynomial-time algorithm to determine a P_3-hull set that has maximum size $\left\lceil \frac{k}{1+b} \right\rceil + 1$. The algorithm also facilitates the identification of a P_3-hull set of maximum size $\lceil \log_{c+1}(k.c+1) \rceil + 1$ for strongly regular graphs. Finally, it was established that the P_3-hull number for biconnected C_6-free diameter two graphs is at most 4, which enables the computation of the P_3-hull number for this subclass in time $O(n^4)$.

For future work, we suggest determining the P_3-hull number of diameter two graphs in general, that is, of graphs with diameter 2 that contain C_6 as an induced subgraph. We also suggest identifying the P_3-hull number of graphs with diameter 3.

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