On the Relationship between Filter Spaces and Weak Limit Spaces

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Abstract

Countably based filter spaces have been suggested in the 1970's as a model for recursion theory on higher types. Weak limit spaces with a countable limit base are known to be the class of spaces which can be handled by the Type-2 Model of Effectivity (TTE). We prove that the category of countably based filter spaces is equivalent to the category of weak limit spaces with a countable limit base. As a consequence we obtain that filter spaces form yet another category from which the category QCB of quotients of countably based topological spaces inherits its cartesian closed structure.

Keywords: Filter Spaces, Weak Limit Spaces, QCB-Spaces, Higher Type Computability

1 Introduction

A category of spaces designed for modelling higher type computation should have the property of being cartesian closed. Cartesian closedness means that finite products and function spaces can be constructed inside a category. Computation on non-discrete spaces requires to deal with approximations. The usual mathematical tool for modelling approximations are topological spaces. Unfortunately, the category Top of topological spaces lacks the property of being cartesian closed. However, there exist several relevant cartesian closed subcategories and supercategories of Top which can be used as an alternative. Examples of cartesian closed supercategories of Top are the category of equilogical spaces [4] and the category of *filter spaces* [9].

In the 1970's, M. Hyland established the relevance of filter spaces for computation on the continuous functionals over $\mathbb{N}$, see [9]. At least three notions of *filter spaces* exist in the literature. All notions endow a set X with a convergence relation between filters on X and points of X subject to certain axioms (cf. Section 2.1).

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The most general notion is the one considered in [9]. In this paper we are interested in the slightly less general notion of *canonical* filter spaces: They enjoy the property that convergence of a filter to a point x depends solely on those sets in the filter that contain x. This matches with the intuition that a filter should converge to x if it is viewed to contain enough properties of x providing sufficient information about x. A third notion of filter spaces and a comparison of the three notions can be found in [7]. Note that some authors call filter spaces *convergence spaces*. We prefer the name "filter space", because the name "convergence space" is also used for other kinds of spaces.

From the perspective of computability theory, we are particularly interested in filter spaces with a countable basis. In these spaces, a countable set of properties suffice to describe the elements of the space. The presence of a countable basis provides a handle to define computable functions between countably based filter spaces by means of enumeration operators.

Weak limit space [12] are a generalisation of limit spaces [10,11] and thus of sequential topological spaces [5]. The constitutive structure of a weak limit space is a convergence relation between sequences and points (cf. Section 2.2). Weak limit spaces play an important role in Weihrauch's representation-based approach to Computable Analysis, the *Type-2 Theory of Effectivity* (TTE) [16]: The class of spaces admitting an *admissible* (i.e. continuously well-behaved) multi-representation is exactly the class of weak limit space with a countable limit base (cf. [13,12]).

In Section 3 we construct an embedding of the weak limit spaces into the category of canonical filter spaces which preserves countable products. In Section 4 we present and prove our main result. It states that the category ωWLim of weak limit spaces with a countable limit base is equivalent to the category ωCFil of countably based canonical filter spaces. Thus countably based canonical filter spaces are already characterised by the apparently simpler concept of sequence convergence: they are simply weak limit spaces in different guise. The existence of countable basis is crucial for this result.

An important cartesian closed subcategory of Top is the category $\mathsf{QCB_0}$ of qcb-spaces satisfying the T_0-property. A *qcb-space* is defined to be a *q*uotient of a *c*ountably *b*ased topological space. Remarkably, $\mathsf{QCB_0}$ has the property of inheriting its cartesian closed structure from many interesting cartesian closed supercategories relevant to higher type computability. Examples are the categories of: equilogical spaces, compactly generated spaces, weak limit spaces, Baire space representations, cf. [3,6,11,12]. The category of countably based equilogical spaces exemplifies the domain-theoretic approach to Computable Analysis, whereas the category of Baire space representations describes the TTE approach. Hence $\mathsf{QCB_0}$ qualifies as a convenient category for modelling higher type computation [2]. Our main theorem implies that the aforementioned categories of filter spaces belong to this list. This answers positively a question in [2,14]. Moreover, at least in topological terms, the approach to higher type computation via countably based (canonical) filter spaces agrees with the TTE approach.

Since this is an extended abstract, most of the proofs are omitted.

2 Preliminaries

In this section we repeat the definitions of filter spaces and weak limit spaces together with some known facts about these concepts.

2.1 Filter Spaces and Related Notions

Let X and Y be non-empty sets. By $\mathcal{P}(X)$ we denote the powerset of X. A *filter* $\mathcal{F}$ on X is a non-empty family of non-empty subsets of X which is closed under finite intersection and extension to supersets. For $x \in X$, we write $[x]$ for the *principal filter* $\{A \subseteq X \mid x \in A\}$. A *filter base* Φ is a non-empty subset of $\mathcal{P}(X) \setminus \{\emptyset\}$ such that for all $A, B \in \Phi$ there is some $C \in \Phi$ with $\emptyset \neq C \subseteq A \cap B$. For a filter base Φ we denote by $[\Phi]$ the smallest filter containing Φ, i.e. $\{A \subseteq X \mid \exists B \in \Phi. B \subseteq A\}$. Given a function $f \colon X \to Y$, $f^+\Phi$ denotes the filter base $\{f(A) \mid A \in \Phi\}$. Note that $[f^+[\Phi]]$ is equal to $[f^+\Phi]$.

There are several notions of filter spaces in the literature (which are sometimes also called *convergence spaces*). Generally, a *filter space* is a pair $(X, \downarrow)$, where X is a set and $\downarrow$ is a *convergence relation* between the filters on X and the points of X satisfying certain axioms. If $\mathcal{F} \downarrow x$ holds, then one says that the filter $\mathcal{F}$ *converges to* x or that x is a *limit of* $\mathcal{F}$. For a filter base Φ, it is convenient to write $\Phi \downarrow x$ for $[\Phi] \downarrow x$ and to say also in this case that Φ converges to x. Since any filter $\mathcal{F}$ is equal to $[\mathcal{F}]$, this convention does not cause ambiguity.

The least restrictive notion of filter spaces [9] requires a filter space $(X, \downarrow)$ to fulfil the axioms (F1) and (F2):

(F1) $[x] \downarrow x$;

(F2) if $\mathcal{F} \downarrow x$ and $\mathcal{F} \subseteq \mathcal{G}$ then $\mathcal{G} \downarrow x$,

where $\mathcal{F}, \mathcal{G}$ are filters on X and $x \in X$.

A function $f \colon \mathfrak{X} \to \mathfrak{Y}$ between filter spaces $\mathfrak{X}$ and $\mathfrak{Y}$ is called *filter-continuous*, if $\mathcal{F} \downarrow x$ in $\mathfrak{X}$ implies $f^+\mathcal{F} \downarrow f(x)$ in $\mathfrak{Y}$ for every filter $\mathcal{F}$ of $\mathfrak{X}$ and every $x \in \mathfrak{X}$. Obviously, composition of functions preserves filter continuity. We denote the category of filter spaces satisfying Axioms (F1), (F2) and of filter-continuous functions by Fil. In the following, we shall use Gothic letters $\mathfrak{X}, \mathfrak{Y}, \ldots$ for filter spaces. The carrier set of a space $\mathfrak{X}$ will often be denoted by $\mathfrak{X}$ as well. As long as no confusion can occur, we shall use the symbol '$\downarrow$' without decorations to denote the convergence relation of any filter space.

In this paper, we are only interested in *canonical* filter spaces $(X, \downarrow)$. They are defined to satisfy the axioms (F1), (F2) and

(F3) if $\mathcal{F} \downarrow x$ then $\mathcal{F} \cap [x] \downarrow x$.

In the following, we will often use the term "filter space" to mean spaces satisfying Axioms (F1), (F2), (F3). We denote the category whose objects are the canonical filter spaces and whose morphisms are the filter-continuous functions by CFil.

The idea behind canonical filter spaces is the following: A filter $\mathcal{F}$ is defined to converge to a point x, if $\mathcal{F}$ is deemed to contain enough properties of x, where a

property of x is any set containing x. The fact that our definition allows a converging filter to contain properties that do not belong to the limit slightly mismatches with this idea. A more fitting definition of filter spaces is set up by the Axioms (F1), (F2'), (F3'):

(F2') if $\mathcal{F} \downarrow x$ and $\mathcal{F} \subseteq \mathcal{G} \subseteq [x]$ then $\mathcal{G} \downarrow x$;

(F3') if $\mathcal{F} \downarrow x$ then $\mathcal{F} \subseteq [x]$.

However, the corresponding category PFil can easily be verified to be equivalent to CFil.

The category CFil has countable products and is cartesian closed (see [7], where CFil is denoted by FIL^b). Given a sequence of filter spaces $(\mathfrak{X}_i)_i$, the carrier set of the product $\bigotimes_{i \in \mathbb{N}} \mathfrak{X}_i$ is the cartesian product of the carrier sets and its convergence relation is given by

$$\mathcal{F} \downarrow x \iff \forall i \in \mathbb{N}. \, \mathrm{pr}_i^+ \mathcal{F} \downarrow \mathrm{pr}_i(x) \text{ in } \mathfrak{X}_i \,,$$

where pr_i denotes the respective set-theoretic projection function. The exponential $\mathfrak{Y}^{\mathfrak{X}}$ in CFil for canonical filter spaces $\mathfrak{X}$ and $\mathfrak{Y}$ has the set $\mathcal{C}(\mathfrak{X}, \mathfrak{Y})$ of filter-continuous functions as its carrier set and its convergence relation is defined as follows: a filter $\mathcal{F}$ on $\mathcal{C}(\mathfrak{X}, \mathfrak{Y})$ converges to a function $f \in \mathcal{C}(\mathfrak{X}, \mathfrak{Y})$ iff

$$\mathcal{A} \downarrow x \text{ implies } [\{F \cdot A \,|\, F \in \mathcal{F} \cap [f], \, A \in \mathcal{A}\}] \downarrow f(x)$$

for every filter $\mathcal{A}$ on $\mathfrak{X}$ and every $x \in X$, where $F \cdot A := \{g(a) \,|\, g \in F, a \in A\}$. Note that the function space construction of CFil is different from the construction for Hyland's larger category Fil. However, for CFil-spaces satisfying the Merging Axiom

(F4) if $\mathcal{F} \downarrow x$ and $\mathcal{G} \downarrow x$ then $\mathcal{F} \cap \mathcal{G} \downarrow x$

the function space constructions in Fil and CFil agree, cf. [7]. We denote the full subcategory of filter spaces satisfying Axioms (F1)–(F4) by MFil. R. Heckmann analyses in [7] the relationship between equilogical spaces (which is a cartesian closed supercategory of Top) and the three notions of filter spaces.

The importance of filter spaces lies in the fact that CFil presents a cartesian closed supercategory of the non cartesian closed category Top of topological spaces [7,9]. The corresponding embedding functor $\mathcal{I}_{\mathsf{Top}} \colon \mathsf{Top} \hookrightarrow \mathsf{MFil}$ maps a topological space Z to the filter space which has the same carrier set and whose convergence relation is given by: $\mathcal{F} \downarrow z$ iff $\mathcal{F}$ contains the neighbourhood filter $[\{U \text{ open} \,|\, z \in U\}]$ of z.

A family $\mathcal{B}$ of subsets of $\mathfrak{X}$ is called a *basis* for a filter space $\mathfrak{X}$, if for every filter $\mathcal{F}$ converging to some $x \in \mathfrak{X}$ the family $\mathcal{F} \cap \mathcal{B}$ is a filter base converging to x. Note that in contrast to topological bases, filter space bases do not characterise filter spaces; indeed, the powerset $\mathcal{P}(Y)$ of Y is a basis for every filter space with carrier set Y. By ωCFil, we denote the category of countably based canonical filter spaces as objects and filter-continuous functions as morphisms, and by ωMFil the full subcategory of those filter space that satisfy the Merging Axiom (F4). Filter spaces bases relate to topological bases in the following way:

Lemma 2.1 *Any topological base for a topological space Z is a filter space basis for*

$\mathcal{I}_{\mathsf{Top}}(Z)$. *If $\mathcal{B}$ is a basis for $\mathcal{I}_{\mathsf{Top}}(Z)$, then $\{\mathrm{Int}(B)\,|\,B \in \mathcal{B}\}$ is a topological base for Z, where $\mathrm{Int}(B)$ denotes the interior of B.*

Proof. Omitted. □

The categories $\omega\mathsf{CFil}$ and $\omega\mathsf{MFil}$ are cartesian closed as well, because forming countable products and forming expontentials in CFil preserve the existence of a countable basis. Given bases $\mathcal{A}$ and $\mathcal{B}$ for canonical filter spaces $\mathfrak{X}$ and $\mathfrak{Y}$, one can show similar to [9] that

$$\mathcal{D} := \big\{A \times B \,\big|\, A \in \mathcal{A},\ B \in \mathcal{B}\big\}$$

is a countable basis for the product $\mathfrak{X} \otimes \mathfrak{Y}$ in CFil and that the closure under finite intersection of the family

$$\mathcal{E} := \big\{\{f \in \mathcal{C}(\mathfrak{X},\mathfrak{Y})\,|\,f(A) \subseteq B\}\,\big|\,A \in \mathcal{A},\ B \in \mathcal{B}\big\}$$

is a countable basis for the exponential $\mathfrak{Y}^{\mathfrak{X}}$ in CFil.

For functions between countably based filter spaces one can introduce a reasonable notion of computability by considering numberings of the respective bases. We modify slightly the tentitive definition in [9] and call a triple $(X, \downarrow, \alpha)$ a *coded filter space*, if $(X, \downarrow)$ is a canonical filter space and α is a numbering of a basis for $(X, \downarrow)$. A function f between two coded filter spaces $(X, \downarrow, \alpha)$ and $(Y, \downarrow, \beta)$ is defined to be *computable* iff there is a computable function $g\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ such that

$$x \in \bigcap_{i \in I} \alpha_i \,\wedge\, \{\alpha_i \,|\, i \in I\} \downarrow x \quad \text{implies} \quad f(x) \in \bigcap_{j \in g(I)} \beta_j \,\wedge\, \{\beta_j \,|\, j \in g(I)\} \downarrow f(x)$$

for all $x \in X$ and all $I \subseteq \mathbb{N}$. It is not surprising that computable functions between coded filter spaces are filter-continuous.

2.2 Weak Limit Spaces and Related Notions

Let X and Y be sets. We write sequences $x\colon \mathbb{N} \to X$ over X as $(x_n)_n$ or $(x_n)_{n \in \mathbb{N}}$ and *generalised* sequences $x\colon \mathbb{N}^+ \to X$ as $(x_n)_{n \leq \infty}$ or $(x_n)_{n \in \mathbb{N}^+}$, where $\mathbb{N}^+ := \mathbb{N} \cup \{\infty\}$. By $[(x_n)_n]$ we denote the *Fréchet filter* $\big\{M \subseteq X \,\big|\, \exists n \in \mathbb{N}.\{x_n, x_{n+1}, \dots\} \subseteq M\big\}$ and by $[(x_n)_n, y]$ the filter $[(x_n)_n] \cap [y]$ which is a Fréchet filter as well.

A *weak limit space* ([13,12]) equips a set X with a *convergence relation* $\to$ between sequences $(x_n)_n$ and points x of X. If $(x_n)_n \to x_\infty$, then we say that $(x_n)_n$ converges to x_∞ in the space $(X, \to)$ or that x_∞ is a limit of the sequence $(x_n)_n$. The convergence relation of a weak limit space $(X, \to)$ is required to satisfy the following axioms:

(L1) $(x)_n \to x$;

(L4) if $(x_n)_n \to x_\infty$ and $(\xi_n)_n \to \infty$ in $(\mathbb{N}^+, \tau_{\mathbb{N}^+})$ then $(x_{\xi_n})_n \to x_\infty$;

(L5) if $(x_{n+1})_n \to x_\infty$ and $x_0 \in X$ then $(x_n)_n \to x_\infty$.

Here $\tau_{\mathbb{N}^+}$ denotes the standard topology $\big\{U \subseteq \mathbb{N}^+ \,\big|\, \infty \in U \implies U \in [(n)_n]\big\}$ on $\mathbb{N}^+$.

Weak limit spaces are a generalisation of Kuratowski's *limit spaces* [10] (called *L-spaces* in [9]). The convergence relation of a limit space $(X, \to)$ is subject to the axioms (L1), (L2), (L3):

(L2) if $(x_n)_n \to x_\infty$, then $(y_n)_n \to x_\infty$ for every subsequence $(y_n)_n$ of $(x_n)_n$;

(L3) if $(x_n)_n \not\to x_\infty$, then $(x_n)_n$ has a subsequence $(y_n)_n$ such that $(z_n)_n \not\to x_\infty$ for every subsequence $(z_n)_n$ of $(y_n)_n$.

Any weak limit space fulfils Axiom (L2), but not necessarily Axiom (L3). The convergence relation of a topological space Z satisfies the axioms of a limit space. We denote the corresponding weak limit space by $\mathcal{L}_{\mathsf{Top}}(Z)$. A weak limit space is called *topological*, if it lies the image of $\mathcal{L}_{\mathsf{Top}}$.

A function f between two weak limit spaces X and Y is called *sequentially continuous*, if it preserves convergent sequences [2]. By WLim we denote the category whose objects are the weak limit spaces and whose morphisms are the sequentially continuous functions.

Given a weak limit space X, we call a family $\Phi \subseteq \mathcal{P}(\mathsf{X})$ a *witness of convergence* for an element x, if $\Phi \subseteq [x]$ and, for every sequence $(y_n)_n$, $\Phi \subseteq [(y_n)_n]$ implies $(y_n)_n \to x$ in X. For example, any neighbourhood base of a point z in a topological space is a witness of convergence for z.

Lemma 2.2 *In a weak limit space, a sequence $(x_n)_n$ converges to an element x_∞ if and only if the Fréchet filter $[(x_n)_n, x_\infty]$ is a witness of convergence for x_∞.*

Proof. This is an easy consequence of the axioms of weak limit spaces. □

One can show that a relation $\leadsto \ \subseteq Y^{\mathbb{N}} \times Y$ satisfies the axioms of a weak limit space if and only if it satisfies Axiom (L1) and the equivalence in Lemma 2.2.

A *limit base* for a weak limit space X is a family $\mathcal{B} \subseteq \mathcal{P}(\mathsf{X})$ such that for every element $x \in \mathsf{X}$ and every sequence $(y_n)_n$ converging to x, $\mathcal{B}$ contains a witness of convergence Φ for x such that $\Phi \subseteq [(y_n)_n]$. This implies that for every convergent sequence $(x_n)_{n \leq \infty}$ of X and for every sequence $(z_n)_n$ that does not converge to x_∞ there is some $B \in \mathcal{B}$ such that $x_\infty \in B$, $x_n \in B$ for almost all n and $z_m \notin B$ for infinitely many m. Similar to bases of filter spaces, the powerset of a set Y is a limit base for every weak limit space with carrier set Y. By $\omega\mathsf{WLim}$ we denote the full subcategory of WLim consisting of all weak limit spaces admitting a countable limit base.

Both categories WLim and $\omega\mathsf{WLim}$ have countable products and are cartesian closed [12,13]. The product $\bigotimes_{i \in \mathbb{N}} \mathsf{X}_i$ of a sequence $(\mathsf{X}_i)_i$ of weak limit spaces is constructed as one expects. The exponential Y^{X} is obtained by equipping the set $\mathcal{C}(\mathsf{X}, \mathsf{Y})$ of sequentially continuous functions from X to Y with the convergence relation $\Rightarrow$ of *continuous convergence* defined by: $(f_n)_n \Rightarrow f_\infty$ iff $(f_{\xi_n}(x_n))_n \to f(x_\infty)$ holds in Y for all $(x_n)_n \to x_\infty$ in X and all $(\xi_n)_n \to \infty$ in $(\mathbb{N}^+, \tau_{\mathbb{N}^+})$. Given countable limit bases $\mathcal{B}_i$ for weak limit spaces X_i closed under finite intersection, countable limit bases for the product $\bigotimes_{i \in \mathbb{N}} \mathsf{X}_i$ and the exponential $\mathsf{X}_2^{\mathsf{X}_1}$ are constructed by

$$\mathcal{D} := \left\{ \prod_{i \in \mathbb{N}} B_i \,\middle|\, k \in \mathbb{N}, \, B_k \in \mathcal{B}_k, \, \forall i \neq k. \, B_i = \mathsf{X}_i \right\} \tag{1}$$

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[2] i.e., $(x_n) \to x_\infty$ in X implies $(f(x_n))_n \to f(x_\infty)$ in Y

and

$$\mathcal{E} := \big\{ \{ f \in \mathcal{C}(\mathsf{X}_1, \mathsf{X}_2) \,|\, f(\bigcap_{i=1}^k A_i) \subseteq B \} \,\big|\, \{A_1, \ldots, A_k\} \subseteq \mathcal{B}_1,\, B \in \mathcal{B}_2 \big\} \qquad (2)$$

respectively. Proofs can be found in [12,13].

3 Embedding Weak Limit Spaces into Filter Spaces

We define an embedding $\mathcal{I}$ of the category of weak limit spaces WLim into the category CFil of canonical filter spaces. It preserves countable products and maps countably based weak limit spaces to countably based filter spaces.

Let $\mathsf{X} = (X, \to)$ be a weak limit space. We define a filter convergence relation $\downarrow_\mathsf{X}$ on X by

$$\mathcal{F} \downarrow_\mathsf{X} x \;:\Longleftrightarrow\; \mathcal{F} \cap [x] \text{ contains a countable witness of convergence for } x.$$

Clearly, $\mathcal{I}(\mathsf{X}) := (X, \downarrow_\mathsf{X})$ is a canonical filter space. We call $\mathcal{I}(\mathsf{X})$ the filter space *associated to* X. From Lemma 2.2 we can deduce the following characterisation of convergence of sequences in a weak limit space in terms of filter convergence in the associated filter space.

Lemma 3.1 *Let* X *be a weak limit space. Then a sequence* $(y_n)_n$ *converges to a point* x *in* X *if and only if the Fréchet filter* $[(y_n)_n]$ *converges to* x *in* $\mathcal{I}(\mathsf{X})$.

By setting $\mathcal{I}(f) := f$ for every morphism f in WLim, we obtain an embedding functor from WLim to CFil.

Proposition 3.2 *Let* X *and* Y *be weak limit spaces. Then a function* $f \colon \mathsf{X} \to \mathsf{Y}$ *is sequentially continuous if and only if* f *is a filter-continuous function from* $\mathcal{I}(\mathsf{X})$ *to* $\mathcal{I}(\mathsf{Y})$.

Proof. *Only-if-part:* Let f be sequentially continuous. Let $\mathcal{F}$ be a filter that converges to some x_∞ in $\mathcal{I}(\mathsf{X})$. There is a sequence $(F_i)_i$ of sets in $\mathcal{F} \cap [x_\infty]$ such that $\{F_i \,|\, i \in \mathbb{N}\}$ is a witness of convergence for x. We define $G_j := f(\bigcap_{i=0}^j F_i) \in [f^+\mathcal{F}] \cap [f(x_\infty)]$. Let $(y_n)_n$ be a sequence with $\{G_j \,|\, j \in \mathbb{N}\} \subseteq [(y_n)_n]$. Then there are natural numbers $m_0 < m_1 < m_2 \ldots$ such that $y_n \in G_j$ for all $j \in \mathbb{N}$ and $n \geq m_j$. For $j \in \mathbb{N}$ and $n \in \{m_j, \ldots, m_{j+1} - 1\}$ we choose some $x_n \in \bigcap_{i=0}^j F_i$ with $f(x_n) = y_n$. Then $(x_n)_n$ converges x_∞, because we have $\{F_i \,|\, i \in \mathbb{N}\} \subseteq [(x_n)_n]$. By sequential continuity, $(y_n)_n$ converges to $f(x_\infty)$. Hence the family $\{G_j \,|\, j \in \mathbb{N}\}$ constitutes a countable witness of convergence for $f(x_\infty)$ contained in $f^+\mathcal{F}$, implying $f^+\mathcal{F} \downarrow_\mathsf{Y} f(x_\infty)$. Therefore f is filter-continuous.
If-part: Let f be filter-continuous. Let $(x_n)_{n \leq \infty}$ be a convergent sequence of X. By Lemma 3.1 the filter $\mathcal{F} := [(x_n)_n] \cap [x_\infty]$ converges to x_∞ in $\mathcal{I}(\mathsf{X})$. Hence $f^+\mathcal{F}$ converges to $f(x_\infty)$. Since $f^+\mathcal{F} \subseteq [(f(x_n))_n]$, the Fréchet filter $[(f(x_n))_n]$ converges to $f(x_\infty)$ by Axiom (F2). Hence $(f(x_n))_n$ converges to $f(x_\infty)$ by Lemma 3.1. $\quad\square$

The functor $\mathcal{I}$ gives rise to an alternative embedding of the topological spaces into CFil, namely by the functor $\mathcal{I} \circ \mathcal{L}_\mathsf{Top}$. We characterise the class of topological spaces on which $\mathcal{I} \circ \mathcal{L}_\mathsf{Top}$ agrees with the standard embedding $\mathcal{I}_\mathsf{Top}$ from Section 2.1.

Proposition 3.3 *A topological space Z satisfies $\mathcal{IL}_{\mathsf{Top}}(Z) = \mathcal{I}_{\mathsf{Top}}(Z)$ if and only if Z is first-countable (i.e. every element has a countable neighbourhood base).*

Proof. Omitted.   □

From Lemma 2.1 we know:

Corollary 3.4 *A topological space Z satisfies $\mathcal{I}_{\mathsf{Top}}(Z) \in \omega\mathsf{CFil}$ if and only if Z has a countable topological base.*

By Proposition 3.5 the functor $\mathcal{I}$ preserves the existence of a countable base. We denote the arising functor from $\omega\mathsf{WLim}$ to $\omega\mathsf{CFil}$ by $\mathcal{I}_\omega$.

Proposition 3.5 *Any countable limit base $\mathcal{B}$ for a weak limit space X is a basis for the filter space $\mathcal{I}(\mathsf{X})$, provided that $\mathcal{B} \cup \{\emptyset\}$ is closed unter finite intersection.*

Proof. Omitted.   □

We obtain from Propositions 3.2 and 3.5:

Theorem 3.6 *The functor $\mathcal{I}$ is embeds WLim into CFil. Its restriction $\mathcal{I}_\omega$ embeds $\omega\mathsf{WLim}$ into $\omega\mathsf{CFil}$.*

The functor $\mathcal{I}$ preserves countable products, but not exponentials. By contrast, the restriction $\mathcal{I}_\omega$ preserves exponentials, as we will see in Section 4.

Proposition 3.7 *For a sequence $(\mathsf{X}_i)_i$ of weak limit spaces, we have $\bigotimes_{i\in\mathbb{N}} \mathcal{I}(\mathsf{X}_i) = \mathcal{I}(\bigotimes_{i\in\mathbb{N}} \mathsf{X}_i)$.*

Proof. Omitted.   □

Example 3.8 The functor $\mathcal{I}$ does not preserve exponentials. As an example we consider an uncountable discrete [3] space $\mathsf{D} \in \mathsf{WLim}$ as domain space and as codomain space the two point discrete space 2 with carrier set $\{0,1\}$. Since $[x]$ is the only filter to converge to x in $\mathcal{I}(\mathsf{D})$, the filter

$$\mathcal{F} := \left[\left\{F \subseteq \mathcal{C}(\mathsf{D}, 2) \,\middle|\, \exists E \subseteq \mathsf{X} \text{ finite.} \,\forall f \in F.\, E \subseteq f^{-1}\{0\}\right\}\right]$$

converges in $\mathcal{I}(2)^{\mathcal{I}(\mathsf{D})}$ to the constant zero function $\mathbf{0}$. Assume that $\mathcal{F}$ contains a countable witness $\{F_i \,|\, i \in \mathbb{N}\}$ for $\mathbf{0}$ in 2^{D}. Then for every $i \in \mathbb{N}$ there is a finite set E_i with $\left\{f \colon \mathsf{D} \to \{0,1\} \,\middle|\, E \subseteq f^{-1}\{0\}\right\} \subseteq F_i$. We define the sequentially continuous, hence filter-continuous function $f_n \colon \mathsf{D} \to \{0,1\}$ by $f_n(x) = 0 \; :\Longleftrightarrow \; x \in \bigcup_{i=0}^{n} E_i$. Then $\{F_i \,|\, i \in \mathbb{N}\} \subseteq [(f_n)_n]$. By uncountability of D, there exists an element $x \in \mathsf{D} \setminus \bigcup_{i\in\mathbb{N}} E_i$. Since $(f_n(x))_n \not\to \mathbf{0}(x)$ in 2, $(f_n)_n$ does not converge to $\mathbf{0}$ in 2^{D}, a contradiction. Therefore $\mathcal{F}$ does not converge to $\mathbf{0}$ in $\mathcal{I}(2^{\mathsf{D}})$.

4   Equivalence of $\omega\mathsf{WLim}$ and $\omega\mathsf{CFil}$

We prove in this section our main result that the categories $\omega\mathsf{CFil}$ and $\omega\mathsf{WLim}$ are equivalent. Actually we prove the stronger result that the object-part of $\mathcal{I}_\omega$

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[3] i.e., $(x_n)_n \to x_\infty$ if and only if $x_n = x_\infty$ for almost all $n \in \mathbb{N}$

is a bijection between the weak limit spaces with a countable limit base and the countably based canonical filter spaces.

In order to obtain the inverse of $\mathcal{I}_\omega$, we define at first a retraction $\mathcal{W}$ from the filter spaces back to the weak limit spaces. Any filter space $\mathfrak{X} = (X, \downarrow)$ induces a natural convergence relation $\to_{\mathfrak{X}}$ on X. It is defined by

$$(x_n)_n \to_{\mathfrak{X}} x_\infty \ :\Longleftrightarrow\ [(x_n)_n] \cap [x_\infty] \text{ converges to } x_\infty \text{ in } \mathfrak{X}.$$

By the next proposition, $\to_{\mathfrak{X}}$ satisfies the axioms of a weak limit space. Furthermore any filter-continuous function is sequentially continuous w.r.t. the induced convergence relations. So setting $\mathcal{W}(\mathfrak{X}) := (X, \to_{\mathfrak{X}})$ and $\mathcal{W}(f) := f$ for any morphism in CFil defines a functor from the category of canonical filter spaces to the category of weak limit spaces.

Proposition 4.1 *For any canonical filter space $\mathfrak{X}$, $\mathcal{W}(\mathfrak{X})$ is a weak limit space. Moreover, any filter-continuous function f from $\mathfrak{X}$ to a canonical filter space $\mathfrak{Y}$ is a sequentially continuous function from $\mathcal{W}(\mathfrak{X})$ to $\mathcal{W}(\mathfrak{Y})$.*

Proof. Omitted. □

By the next lemma, $\mathcal{W}$ preserves the existence of a countable basis. We denote the arising functor from ωCFil to ωWLim by $\mathcal{W}_\omega$.

Lemma 4.2 *Any basis $\mathcal{B}$ for a canoncial filter space $\mathfrak{X}$ is a limit base for the weak limit space $\mathcal{W}(\mathfrak{X})$.*

Proof. Omitted. □

Lemma 3.1 implies that $\mathcal{W}$ is a left-inverse to $\mathcal{I}$.

Lemma 4.3 *Every weak limit space X satisfies $\mathcal{W}\mathcal{I}(\mathsf{X}) = \mathsf{X}$.*

By contrast, canonical filter spaces $\mathfrak{X}$ do not necessarily satisfy $\mathcal{I}\mathcal{W}(\mathfrak{X}) = \mathfrak{X}$. The filter space $\mathcal{I}(2)^{\mathcal{I}(\mathsf{D})}$, where D and 2 are the discrete weak limit spaces from Example 3.8, yields a counterexample. Surprisingly, $\mathcal{I}\mathcal{W}(\mathfrak{X}) = \mathfrak{X}$ does hold true, if $\mathfrak{X}$ has countable basis.

Proposition 4.4 *Any canonical filter space $\mathfrak{X}$ with a countable basis satisfies $\mathcal{I}\mathcal{W}(\mathfrak{X}) = \mathfrak{X}$.*

Proof. Let $\mathcal{B}$ be a countable of basis of $\mathfrak{X}$. Let $\mathcal{F}$ be a filter on $\mathfrak{X}$ and $x \in \mathfrak{X}$. First, let $\mathcal{F}$ converge to x in $\mathfrak{X}$. Then the countable filter base $\Phi := \mathcal{F} \cap [x] \cap \mathcal{B}$ converges to x in $\mathfrak{X}$. By Axiom (F2), every sequence $(y_n)_n$ with $\Phi \subseteq [(y_n)_n]$ converges to x in $\mathcal{W}(\mathfrak{X})$. Hence Φ is a witness of convergence for x contained in $\mathcal{F}$. Therefore $\mathcal{F}$ converges to x in $\mathcal{I}\mathcal{W}(\mathfrak{X})$.
Conversely, let $\mathcal{F}$ converge to x in $\mathcal{I}\mathcal{W}(\mathfrak{X})$. Then $\mathcal{F}$ contains a countable witness of convergence $\{F_i \,|\, i \in \mathbb{N}\}$ for x in $\mathcal{W}(\mathfrak{X})$. Let $\mathcal{D} := \{\emptyset\} \cup \{B \in \mathcal{B} \,|\, \forall n. \bigcap_{i=0}^n F_i \nsubseteq B\}$, and let $i \mapsto \beta_i$ be a numbering of $\mathcal{D}$ with $\forall i \in \mathbb{N}. \exists j > i. \beta_j = \beta_i$. We choose for every n some $y_n \in \bigcap_{i=0}^n F_i \setminus \beta_n$. Since $\{F_i \,|\, i \in \mathbb{N}\} \subseteq [(y_n)_n]$, $(y_n)_n$ converges to x in $\mathcal{W}(\mathfrak{X})$. Hence the filter $[(y_n)_n] \cap [x]$ and the filter base $\Phi := [(y_n)_n] \cap [x] \cap \mathcal{B}$ converge to x in $\mathfrak{X}$. Since $\mathcal{D} \cap [(y_n)_n] = \emptyset$, for every $B \in \Phi$ there is some $n \in \mathbb{N}$ with

$\bigcap_{i=0}^{n} F_i \subseteq B$, implying $[\Phi] \subseteq \mathcal{F}$. Hence $\mathcal{F}$ converges to x in $\mathfrak{X}$ by Axiom (F2). □

By Proposition 4.4 and Lemma 4.3, the object-part of the functor $\mathcal{I}_\omega$ is a bijection between the countably based weak limit spaces and the countably based canonical filter spaces. By Proposition 3.2, for all $\mathsf{X}, \mathsf{Y} \in \omega\mathsf{WLim}$ the morphism part of $\mathcal{I}_\omega$ constitutes a bijection between the morphisms from X to Y and the morphisms between from $\mathcal{I}_\omega(\mathsf{X})$ and $\mathcal{I}_\omega(\mathsf{Y})$. We obtain our main result:

Theorem 4.5 *The category $\omega\mathsf{CFil}$ of countably based canonical filter spaces is equivalent to the category $\omega\mathsf{WLim}$ of weak limit spaces with a countable limit base.*

From [13] we know that $\omega\mathsf{WLim}$ is locally cartesian closed and has all countable limits and all countable colimits. Hence:

Corollary 4.6 *The category $\omega\mathsf{CFil}$ is locally cartesian closed and has all countable limits and all countable colimits.*

It is not difficult to verify that $\mathcal{I}_\omega$ maps a space $\mathsf{X} \in \omega\mathsf{WLim}$ to a filter space satisfying the Merging Axiom (F4) if and only if X fulfils the Merging Axiom for weak limit spaces:

(L6) if $(y_{2n})_n \to x$ and $(y_{2n+1})_n \to x$ then $(y_n)_n \to x$.

We obtain:

Theorem 4.7 *The category $\omega\mathsf{MFil}$ is equivalent to the category of countably based weak limit spaces satisfying Axiom (L6).*

A *qcb-space* [14] is a topological space Z that is a topological quotient of a countably based topological space[4]. The category QCB_0 of qcb-spaces satisfying the T_0-property and of continuous functions as morphisms is cartesian closed [13]. It is a full subcategory of several categories in such a way that the respective embeddings preserve finite products and exponentials. Examples are the categories of: equilogical spaces (as an example of a cartesian closed supercategory of Top), sequential spaces, compactly generated spaces, quotients of core compact spaces (as examples of cartesian closed subcategories of Top), Baire space representations (as an example of an effective category), limit spaces and weak limit spaces with a countable limit base. The corresponding proofs are contributed by several authors, see [3,6,11,12]. Together with the fact that topological spaces satisfy the Merging Axiom (L6), Theorems 4.5 and 4.7 imply that QCB_0 also lives inside the filter space categories CFil and MFil and inherits the respective constructions of products and exponentials. The same holds true for the filter space category Fil in [9], because functions spaces in MFil are constructed as in Fil, cf. [7]. This gives the expected positive answer to a question in [2,14].

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[4] i.e., there is a surjection $q \colon A \to Z$ from a countably based space A onto Z satisfying: V open in $Z \iff q^{-1}[V]$ open in A.

5 Discussion

We have seen that countably based canonical filter spaces are basically the same mathematical objects as weak limit spaces. In [13] computability for functions between countably based weak limits spaces is introduced by endowing the spaces with multi-representations obtained from numberings of the respective limit bases. One can show that the induced computability notion is equivalent to the one generated by coded filter spaces (see Section 2.1). However, the corresponding category of coded filter spaces as objects and computable functions as morphisms does not have all finite colimits. In [13], an effective cartesian closed category, EffWeakLim, with finite limits and colimits is defined by imposing an effectivity condition on the used multi-representations. It would be interesting to know whether this effectivity notion has connections to effectivity properties on filter space bases similar to the ones considered in [9].

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