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Electronic Notes in Theoretical Computer Science

Electronic Notes in Theoretical Computer Science 73 (2004) 213–215

www.elsevier.com/locate/entcs

Uniform Domains and Uniform Spaces (Abstract)

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Abstract

In a previous work, the current author showed that every compact metric space X can be represented in an omega-algebraic domain D so that X is the retract of the set L(D) of limit (i.e. non-finite) elements of D. This means that every infinite strictly increasing sequence in the set K(D) of finite elements of D can be considered as identifying one point of X, and thus this domain structure can be used to define computation over the space X. In this article, we show a condition on an omega-algebraic domain D which ensures that L(D) has a (separable complete) metric space as its retract. We introduce the notion of a uniform domain, and explain that it corresponds to a uniform space with countable weight. Here, we use the word domain for an omega-algebraic dcpo.

Keywords: Uniform space, domain representation, stratified domain

In [3], the current author showed that every compact metric space X can be represented in an ω-algebraic domain D so that X is the retract of the set $L(D)$ of limit (i.e. non-finite) elements of D. This means that every infinite strictly increasing sequence in the set $K(D)$ of finite elements of D can be considered as identifying one point of X, and thus this domain structure can be used to define computation over the space X. In this article, we show a condition on an ω-algebraic domain D which ensures that $L(D)$ has a (separable complete) metric space as its retract. Following [2], we introduce the notion of a *uniform domain*, and explain that it corresponds to a uniform space with countable weight [1]. Here, we use the word *domain* for an ω-algebraic dcpo.

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doi:10.1016/j.entcs.2004.08.011

Definition 0.1 When P is a poset, we define the *level* of $d \in P$ as the maximal length of a chain $\bot_P = a_0 \lneq a_1 \lneq \ldots \lneq a_n = d$, when it exists, and we write $K_n(P)$ for the set of level-n elements of P. A poset P is *stratified* if each $e \in P$ has a level, that is, if we have $P = K_0(P) \cup K_1(P) \cup \ldots$. A domain D is *stratified* if $K(D)$ is a stratified poset, and D is *evenly stratified* if all the paths to d have the same length for every $d \in K(D)$. We call $K_n(D) \cap K_x$ the set of *level-n approximations* of x.

Definition 0.2 Let D be a stratified domain and $d \in K_m(D)$. We denote by $d^* \subset K_m(D)$ the set of elements of $K_m(D)$ which are compatible with d. If, for each $d \in K_m(D)$, there exists a lower bound of d^* in $K_n(D)$, we define that $n <^* m$. When D be a stratified domain and for each $n \in \mathbb{N}$, there is a $m \in \mathbb{N}$ such that $n <^* m$, we say that D is a *uniform domain*.

Here, a is compatible with b means that a and b have an upper bound in D (which also implies that a and b have an upper bound in $K(D)$).

Definition 0.3 Let P be a poset.
1) $x \in P$ is a *minimal element* if $y \leq x$ implies $y = x$ for all $y \in P$.
We write M_P for the set of all minimal elements of P.
2) We say that P has *enough minimal elements* if, for all $y \in P$, there exists $x \in M_P$ such that $x \leq y$.

Theorem 0.4 *Let D be a uniform domain.*
1) $L(D)$ has enough minimal elements.
2) $M_{L(D)}$ is a retract of $L(D)$.
3) $M_{L(D)}$ is a Hausdorff space.

Note that many of the domains studied in computer science, for example, $P_\omega = \{a \mid a \subseteq N\}$ and Plotkin's T^ω do not have enough minimal elements. The proof for this theorem is analogous to that of the existence of a minimal Cauchy filter in a uniform space. When $d \in K(D)$, we define $\hat{d}$ as the subset $\uparrow d \cap M_{L(D)}$ of $M_{L(D)}$.

Theorem 0.5 *When D is a uniform domain, D induces a complete uniformity μ of countable weight on $M_{L(D)}$, defined through the base consisting of the coverings $\mathcal{B} = \{\mathcal{V}_0, \mathcal{V}_1, \ldots\}$ defined as $\mathcal{V}_i = \{\hat{d} \mid d \in K_i(D)\}$.*

Since the weight of the uniformity constructed in Theorem 0.5 is countable, we have the following.

Corollary 0.6 *When D is a uniform domain, $M_{L(D)}$ is metrizable.*

On the other hand, we have the following.

Theorem 0.7 *Let (X, μ) be a complete uniform space with a countable weight, and $\mathcal{U}_0 \succ \mathcal{U}_1 \succ \ldots$ be a sequence of open coverings which forms a base of μ. From this sequence, we can form a evenly-stratified uniform domain D such that X is homeomorphic to $M_{L(D)}$.*

Thus, a uniform domain can be considered as a uniform space with a selection of a base.

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