



Variations on Wadge Reducibility

Extended Abstract

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Abstract

Wadge reducibility in the Baire and Cantor spaces is very important in descriptive set theory. We consider Wadge reducibility in so called φ -spaces which are topological counterpart of the algebraic directed-complete partial orderings. It turns out that in many spaces the Wadge reducibility behaves worse than in the classical case but there exist also interesting examples of spaces with a better behaviour.

Keywords: Directed-complete partial ordering, φ -space, retract, Wadge reducibility, Borel set, difference hierarchy.

1 Introduction

Recall that $A \subseteq \omega^\omega$ is *Wadge reducible* to $B \subseteq \omega^\omega$ (in symbols $A \leq_W B$) if $A = f^{-1}(B)$ for some continuous function $f : \omega^\omega \rightarrow \omega^\omega$. Replacing the Baire space ω^ω by arbitrary topological space X , we get the preordering \leq_W on the powerset $P(X)$ called the *Wadge reducibility in X* .

Wadge reducibility in the Baire space and in the Cantor space 2^ω are very important in descriptive set theory because it subsumes many interesting hierarchies and is almost well ordered on the Borel sets (which means it is well founded and for all Borel sets A and B it holds $A \leq_W B$ or $\overline{B} \leq_W A$), see [22,20,7].

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In this paper, we consider Wadge reducibility in arbitrary spaces, with the emphasis to the so called φ -spaces [4] which are topological counterpart of the algebraic directed-complete partial orderings. For general information on domain theory we refer the reader to [4,5].

We will try to understand which properties (or their weaker versions) of the classical Wadge reducibility hold true in the other spaces. E.g., we discuss when some substructures of the Wadge ordering are almost well ordered, which sets have a supremum (or a weak version of supremum) under Wadge reducibility, and consider relationship of the Wadge reducibility to hierarchies from [14] (see also earlier relevant papers [18,19,11,12]).

We consider also the following generalization of the Wadge reducibility which is useful in some situations. Let X be a space, S a non-empty set, and S^X the set of all maps $\nu : X \rightarrow S$. Let \leq be the preordering on S^X defined as follows: $\mu \leq \nu$, if $\mu = \nu \circ f$ for some continuous $f : X \rightarrow X$. The superposition of maps $\nu \circ f$ is usually written simply νf . Note that the preordering $(\{0, 1\}^X; \leq)$ is naturally isomorphic to the preordering $(P(X); \leq_w)$ (the isomorphism identifies subsets of X with their characteristic functions).

We denote spaces by letters X, Y, \dots , elements of spaces (points) by x, y, \dots (for concrete examples of spaces also special notation maybe used), finitary elements of φ -spaces by p, q, \dots , subsets of spaces (pointsets) by A, B, \dots and classes of subsets of spaces (pointclasses) by $\mathcal{A}, \mathcal{B}, \dots$. By \overline{A} we denote the complement of a set $A \subseteq X$, i.e. $\overline{A} = X \setminus A$ and by $co\text{-}\mathcal{A} = \{\overline{A} | A \in \mathcal{A}\}$ — the dual of a pointclass \mathcal{A} .

In Section 2 we make some general remarks on the Wadge reducibility in arbitrary spaces. In Section 3 we establish some facts specific to the φ -spaces. In Sections 4 — 7 we discuss in more details two interesting subclasses of φ -spaces closely related to [9,10] and some concrete examples of φ -spaces. We conclude in Section 8 with a discussion of possible future work. Because of the space bound, we omit the proofs which may be found in the full version of this paper [17].

2 General Remarks

In this section we present some general notions and facts about Wadge reducibility in topological spaces.

We start with a couple of results about the existence of supremums in the Wadge ordering. First we show that for many spaces the structure of Wadge degrees is not an upper semilattice.

Proposition 2.1 *Let X be a space such that any continuous function on X has a fixed point. Then for any $A \subseteq X$ the sets A, \overline{A} have no supremum under*

the Wadge reducibility.

The next result gives a sufficient condition for existence of supremums under Wadge reducibility.

Proposition 2.2 (i) *If the direct sum $X \oplus X$ is equivalent to X in the category of topological spaces then any two elements $\mu, \nu \in S^X$ have a supremum in $(S^X; \leq)$.*

(ii) *If the direct sum of the infinite sequence (X, X, \dots) is equivalent to X in the category of topological spaces then any sequence ν_0, ν_1, \dots of elements of S^X has a supremum in $(S^X; \leq)$.*

We say that a continuous function $c : X \rightarrow Y$ is a *quasiretraction* if for any continuous function $f : Y \rightarrow Y$ there is a continuous function $\tilde{f} : X \rightarrow X$ such that $fc = c\tilde{f}$. A space Y is a *quasiretract* of a space X if there is a quasiretraction $r : X \rightarrow Y$. Recall that a *retraction* is a continuous function $r : X \rightarrow Y$ such that there is a continuous function $s : Y \rightarrow X$ (called *section*) with $rs = id_Y$. Note that a section is always an embedding. A space Y is a *retract* of a space X if there is a retraction $r : X \rightarrow Y$. Any retraction is a quasiretraction (consider $\tilde{f} = sfr$).

The next proposition relates quasiretractions and retractions to the Wadge reducibility.

Proposition 2.3 (i) *If $c : X \rightarrow Y$ is a quasiretraction then the map $\nu \mapsto \nu c$ is a monotone function from $(S^Y; \leq)$ to $(S^X; \leq)$.*

(ii) *If $r : X \rightarrow Y$ is a retraction then $\nu \mapsto \nu r$ is an embedding of $(S^Y; \leq)$ into $(S^X; \leq)$.*

Corollary 2.4 (i) *If $c : X \rightarrow Y$ is a quasiretraction then the map $A \mapsto c^{-1}(A)$ is a monotone function from $(P(Y); \leq_W)$ to $(P(X); \leq_W)$.*

(ii) *If $r : X \rightarrow Y$ is a retraction then $A \mapsto r^{-1}(A)$ is an embedding of $(P(Y); \leq_W)$ into $(P(X); \leq_W)$.*

Let ω_1 be the first non-countable ordinal and let $\{\Sigma_\alpha^0\}_{\alpha < \omega_1}$ be the Borel hierarchy in X (see [14].) For any $0 < \beta < \omega_1$, let $\{D_\alpha(\Sigma_\beta^0)\}_{\alpha < \omega_1}$ denote the difference hierarchy over Σ_β^0 . The difference hierarchy over Σ_1^0 is called simply *difference hierarchy* and is denoted by $\{\Sigma_\alpha^{-1}\}_{\alpha < \omega_1}$. As usual, let Π_α^{-1} denote the dual class for Σ_α^{-1} and $\Delta_\alpha^{-1} = \Sigma_\alpha^{-1} \cap \Pi_\alpha^{-1}$, and similarly for the levels of the Borel hierarchy.

Let us relate the corresponding levels of the hierarchies in different spaces.

Proposition 2.5 *If $f : X \rightarrow Y$ is a continuous function then the map $A \mapsto f^{-1}(A)$ respects all levels of the introduced hierarchies, i.e. $A \in \Sigma_\alpha^0(Y)$ implies $f^{-1}(A) \in \Sigma_\alpha^0(X)$ and similarly for the other levels.*

Proposition 2.6 *Let $s : X \rightarrow Y, r : Y \rightarrow X$ be a section-retraction pair of continuous functions.*

(i) *If $s(X)$ is open then the map $A \mapsto s(A)$ respects the levels $\Sigma_\alpha^{-1}, D_\alpha(\Sigma_\beta^0)$, $\text{co-}D_\alpha(\Sigma_\beta^0)$ ($\alpha, \beta < \omega_1, \beta > 1$).*

(ii) *If $s(X)$ is Σ_2^{-1} then the map $A \mapsto s(A)$ respects the levels Σ_{2n}^{-1} ($n < \omega$), Σ_α^{-1} ($\omega \leq \alpha < \omega_1$) and $D_\alpha(\Sigma_\beta^0), \text{co-}D_\alpha(\Sigma_\beta^0)$ ($\alpha, \beta < \omega_1, \beta > 1$).*

3 φ -Spaces

Here we discuss Wadge reducibility in the φ -spaces which are topological counterpart of the algebraic directed-complete partial orderings.

Let us recall some relevant definitions from [4]. Let X be a T_0 -space. For $x, y \in X$, let $x \leq y$ denote that $x \in U$ implies $y \in U$, for all open sets U . Let $F(X)$ be the set of *finitary elements* of X , i.e. elements $p \in X$ such that the upper cone $O_p = \{x | p \leq x\}$ is open. The space X is called a φ -space if any open set is a union of the sets O_p ($p \in F(X)$). A φ -space X is called a φ_0 -space if $(X; \leq)$ contains a least element (usually denoted \perp).

A φ -space X is *complete* if any nonempty directed set S without greatest element has a supremum $\sup S \in X$, and $\sup S$ is a limit point of S (notice that $\sup S \notin F(X)$ and for any finitary element $p \leq \sup S$ there is $s \in S$ with $p \leq s$). As is well known, any φ -space X is canonically embeddable in a complete φ -space called the *completion* of X . A φ -space X with countable set $F(X)$ is called *countably based*. Main results of the paper additionally assume that X is complete.

Results presented below rely heavily on results from [14]. If $A \in \Sigma_\alpha^{-1} \setminus \Pi_\alpha^{-1}$, then we say that A is a *proper Σ_α^{-1} -set*. Let us consider the question whether such sets can contain the bottom \perp or the top element \top (provided these elements exist).

Proposition 3.1 *Let X be a φ_0 -space. For any $\alpha < \omega_1$, if A is proper Σ_α^{-1} then $\perp \notin A$.*

Proposition 3.2 *Let X be a φ -space with a top element \top . For any number $n < \omega$ and any proper Σ_n^{-1} -set A , $\top \in A$ iff n is odd.*

We conclude this section with a couple of results on Wadge reducibility in φ -spaces.

Proposition 3.3 *Let X be a complete φ -space. For any number $n < \omega$, every Σ_n^{-1} -set is Wadge reducible to every set from $\Delta_2^0 \setminus \Pi_n^{-1}$.*

The next result is an immediate corollary of the last proposition.

Theorem 3.4 *Let X be a complete φ -space having chains of finitary elements of arbitrary finite length.*

(i) *For any number $n < \omega$, the class of proper Σ_n^{-1} -sets forms a Wadge degree.*

(ii) *For any number $n < \omega$, $G_n <_S G_{n+1}$ and $G_n <_S \overline{G}_{n+1}$, where G_n is a proper Σ_n^{-1} -set ($n < \omega$).*

Finally, we state a result showing that in many φ -spaces the structure of Wadge degrees is not an upper semilattice. In Sections 4 and 6 we will see that in many φ -spaces this maybe generalized to the assertion that any two incomparable Wadge degrees do not have a supremum.

Proposition 3.5 *Let X be a complete φ_0 -space. Then $A \not\leq_W \overline{A}$ for any $A \subseteq X$.*

4 Reflective Spaces

Here we consider in more details the Wadge reducibility for a subclass of φ -spaces. We start with a technical definition which is a version of the corresponding notion introduced in [9,10].

Definition 4.1 Let I be a non-empty set. By an I -discrete weak semilattice we mean a structure of the form $(P; \leq, \{P_i\}_{i \in I})$ with the following properties:

- (i) $(P; \leq)$ is a preordering;
- (ii) $P = \cup\{P_i | i \in I\}$;
- (iii) for all $x_0, x_1, \dots \in P$ and $i \in I$ there exists $u_i = u_i(x_0, x_1, \dots) \in P_i$ which is a least upper bound for x_0, x_1, \dots in the set P_i , i.e. $\forall k < \omega (x_k \leq u_i)$ and for any $y \in P_i$ with $\forall k (x_k \leq y)$ it holds $u_i \leq y$;
- (iv) for all $x_0, x_1, \dots \in P$, $i \neq i' \in I$ and $y \in P_{i'}$, if $y \leq u_i(x_0, x_1, \dots)$ then $y \leq x_k$ for some $k < \omega$.

The following properties of the I -discrete weak semilattices are immediate (see also [9,10]).

Proposition 4.2 1) *For any $i \in I$, any ω -sequence x_0, x_1, \dots in P_i has a supremum $u_i(x_0, x_1, \dots) \in P_i$. In particular, $(P_i; \leq)$ is an upper semilattice.*

2) *For all $y, x_0, x_1, \dots \in P$, if $\forall k < \omega (x_k \leq y)$ then $\exists i \in I (u_i(x_0, x_1, \dots) \leq y)$.*

3) *For all $y, x_0, x_1, \dots \in P$, if $\forall i \in I (y \leq u_i(x_0, x_1, \dots))$ then $\exists k < \omega (y \leq x_k)$.*

4) *If I has at least two elements and $(\{x_0, x_1, \dots\}; \leq)$ has no greatest element then the set $\{x_0, x_1, \dots\}$ has no supremum in $(P; \leq)$.*

Now we define spaces which are the main object of this section.

Definition 4.3 By a reflective space we mean a complete φ_0 -space X for which there exist continuous functions $q_0, e_0, q_1, e_1 : X \rightarrow X$ such that $q_0 e_0 = q_1 e_1 = id_X$ and $e_0(X), e_1(X)$ are disjoint open sets.

Next we establish an interesting property of Wadge reducibility in the reflective spaces.

Theorem 4.4 Let X be a reflective space, S a non-empty set and let $\mathcal{P}_s = \{\nu \in S^X \mid \nu(\perp) = s\}$ for any $s \in S$. Then $(S^X; \leq, \{\mathcal{P}_s\}_{s \in S})$ is an S -discrete weak semilattice.

The next result is the main particular case of Theorem 4.4 for $S = \{0, 1\}$.

Corollary 4.5 Let X be a reflective space, $\mathcal{P}_0 = \{A \subseteq X \mid \perp \notin A\}$ and $\mathcal{P}_1 = \{A \subseteq X \mid \perp \in A\}$. Then $(P(X); \leq_W, \mathcal{P}_0, \mathcal{P}_1)$ is a $\{0, 1\}$ -discrete weak semilattice (and, consequently, if a sequence x_0, x_1, \dots in P has no greatest element under \leq_W then it has no supremum under \leq_W).

Theorem 4.4 and Proposition 3.1 immediately imply

Corollary 4.6 In any reflective space, the structures $(\Sigma_\alpha^{-1} \setminus \Pi_\alpha^{-1}; \leq_W)$ and $(\Pi_\alpha^{-1} \setminus \Sigma_\alpha^{-1}; \leq_W)$ are upper semilattices for all $\alpha < \omega$.

Another nice property of reflective spaces is the following.

Theorem 4.7 In any countably based reflective space X , the difference hierarchy does not collapse, i.e. $\Sigma_\alpha^{-1} \neq \Pi_\alpha^{-1}$ for all $\alpha < \omega_1$.

Some additional efforts yield the following.

Theorem 4.8 Let X be a countably based reflective space. For any $\alpha < \omega_1$, there exist least elements in $(\Delta_2^0 \setminus \Pi_\alpha^{-1}; \leq_W)$, and $(\Sigma_\alpha^{-1} \setminus \Pi_\alpha^{-1}; \leq_W)$.

5 Examples of Reflective Spaces

In this section we consider in more detail some concrete examples of the reflective spaces.

We start with defining some spaces. Let $\omega^{\leq \omega}$ be the completion of the partial ordering $(\omega^*; \sqsubseteq)$ (for definition and properties of completions see [4, 5]). Of course, $\omega^{\leq \omega} = \omega^* \cup \omega^\omega$ consists of all finite and infinite strings of natural numbers.

Let $2^{\leq \omega}$ be obtained in the same way from $(2^*; \sqsubseteq)$, and $n^{\leq \omega}$ (for any $2 \leq n < \omega$) be obtained by the same construction with 2 replaced by n . Thus,

$n^{\leq\omega} = n^* \cup n^\omega$ consists of all finite and infinite words over the alphabet $\{0, \dots, n-1\}$.

From the well-known properties of completions follows that $\omega^{\leq\omega}$ and $n^{\leq\omega}$ are complete countably based f_0 -spaces (i.e. Scott domains).

Let ω_\perp^ω be the space of partial functions $g : \omega \rightarrow \omega$ with the usual structure of an f -space (as usual in domain theory, we identify the partial function g with the total function $g : \omega \rightarrow \omega_\perp = \omega \cup \{\perp\}$ where $g(x)$ is undefined iff $g(x) = \perp$, for some 'bottom' element $\perp \notin \omega$). For $2 \leq n < \omega$, let n_\perp^ω be the space of partial functions $g : \omega \rightarrow \{0, \dots, n-1\}$ defined similarly to ω_\perp^ω . As is well known, ω_\perp^ω and n_\perp^ω are complete countably based f_0 -spaces.

Finally, let \mathbf{U} be the space of all open subsets of the Cantor space 2^ω distinct from the biggest open set 2^ω . It is well known that $(\mathbf{U} \subseteq)$ is a complete countably based f_0 -space, finitary elements being exactly the clopen subsets of 2^ω distinct from 2^ω .

Proposition 5.1 *The spaces $\omega^{\leq\omega}$, $n^{\leq\omega}$, ω_\perp^ω , n_\perp^ω and \mathbf{U} are reflective.*

Next we show that the class of reflective spaces has some natural closure properties.

Theorem 5.2 (i) *If X is a reflective space and Y a complete φ_0 -space then $X \times Y$ is a reflective space.*

(ii) *If X is an f_0 -space and Y a reflective f_0 -space then the space Y^X of all continuous functions from X to Y with the topology of pointwise convergence is a reflective f -space. The same holds true for the b -spaces from [4].*

Next we relate the introduced spaces one to another and to some other spaces. First we establish an interesting minimality property of the spaces $\omega^{\leq\omega}$ and $n^{\leq\omega}$.

Theorem 5.3 *The spaces $\omega^{\leq\omega}$ and $n^{\leq\omega}$ ($2 \leq n < \omega$) are retracts of any reflective space X .*

The space \mathbf{U} is in a sense opposite to the spaces $\omega^{\leq\omega}$ and $n^{\leq\omega}$. It is known (see e.g. [1]) that \mathbf{U} is universal in the sense that every complete countably based f_0 -space is a retract of \mathbf{U} . This is of some interest in the context of this paper since from Corollary 2.4 we immediately get

Corollary 5.4 *For any complete countably based f_0 -space X , $(P(X); \leq_W)$ is embeddable into $(P(\mathbf{U}); \leq_W)$.*

Next we relate the introduced spaces to the Baire and Cantor spaces ω^ω and n^ω ($2 \leq n < \omega$).

Proposition 5.5 (i) *n_\perp^ω is a retract of $(n+1)_\perp^\omega$ and ω_\perp^ω .*

- (ii) n^ω is a retract of ω^ω .
- (iii) There exist quasiretractions $d : \omega^\omega \rightarrow \omega^{\leq \omega}$, $d_n : (n+1)^\omega \rightarrow n^{\leq \omega}$, $c : \omega^\omega \rightarrow \omega_\perp^\omega$ and $c_n : \omega^\omega \rightarrow n_\perp^\omega$.

Finally, let us consider Wadge reducibility in some of the introduced spaces. For any space X , let $Bor(X)$ denote the class of Borel sets in X .

Proposition 5.6 *The structures $(Bor(\omega^{\leq \omega}); \leq_W)$ and $(Bor(n^{\leq \omega}); \leq_W)$ are almost well ordered.*

Actually, from the deep Theorem 1 proved in [2,3] it is easy to obtain the following description of the order type of the structures from the last proposition. Let θ denote the ordinal of $(Bor(\omega^{\leq \omega}); \leq_W)$. Since this (big) ordinal was computed by W. Wadge [21,22] we call it the *Wadge ordinal*. Let $\theta \times \{0, 1\}$ denote the order type of the partial ordering obtained from the ordering $(\theta; <)$ by replacing every point by two incomparable points. Hence, $\theta \times \{0, 1\}$ is the order type of non-selfdual Wadge degrees of Borel sets in the Baire (or Cantor) space.

Theorem 5.7 *The order type of the quotient structures $(Bor(\omega^{\leq \omega}); \leq_W)$ and $(Bor(n^{\leq \omega}); \leq_W)$ is $\theta \times \{0, 1\}$, hence these structures are isomorphic.*

From Theorem 5.3 and Corollary 2.4 we immediately obtain

Corollary 5.8 *For any reflective space X , the structure $(Bor(X); \leq_W)$ has a substructure of order type $\theta \times \{0, 1\}$.*

6 2-Reflective Spaces

Here we consider in more details the Wadge reducibility for another class of φ -spaces. This section is in a sense parallel to Section 4. We start with a definition which is a version of the corresponding notion in [9,10].

Definition 6.1 Let I be a non-empty set. By an 2 - I -discrete weak semilattice we mean a structure of the form $(P; \leq, \{P_i^j\}_{i,j \in I})$ with the following properties:

- (i) $(P; \leq)$ is a preordering;
- (ii) $P = \cup \{P_i^j \mid i, j \in I\}$;
- (iii) for all $x_0, x_1, \dots \in P$ and $i, j \in I$ there exists $u_i^j = u_i(x_0, x_1, \dots) \in P_i^j$ which is a least upper bound for x_0, x_1, \dots in the set P_i^j , i.e. $\forall k < \omega (x_k \leq u_i^j)$ and for any $y \in P_i^j$ with $\forall k (x_k \leq y)$ it holds $u_i^j \leq y$;
- (iv) for all $x_0, x_1, \dots \in P$, $i \neq i' \in I$, $j \neq j' \in I$ and $y \in P_{i'}^{j'}$, if $y \leq u_i^j(x_0, x_1, \dots)$ then $y \leq x_k$ for some $k < \omega$.

The following properties of the 2- I -discrete weak semilattices are immediate (see also [9,10]).

Proposition 6.2 1) For any $i, j \in I$, any ω -sequence x_0, x_1, \dots in P_i^j has a supremum $u_i^j(x_0, x_1, \dots) \in P_i^j$.

2) For all $y, x_0, x_1, \dots \in P$, if $\forall k < \omega (x_k \leq y)$ then $\exists i, j \in I (u_i^j(x_0, x_1, \dots) \leq y)$.

3) For all $y, x_0, x_1, \dots \in P$, if $\forall i, j \in I (y \leq u_i^j(x_0, x_1, \dots))$ then $\exists k < \omega (y \leq x_k)$.

4) If I has at least two elements and $(\{x_0, x_1, \dots\}; \leq)$ has no greatest element then the set $\{x_0, x_1, \dots\}$ has no supremum in $(P; \leq)$.

Now we define spaces which are the main object of this section.

Definition 6.3 By a 2-reflective space we mean a complete φ_0 -space X with a top element \top such that there exist continuous functions $q_0, e_0, q_1, e_1 : X \rightarrow X$ and open sets B_0, C_0, B_1, C_1 with the following properties:

- (i) $q_0 e_0 = q_1 e_1 = id_X$;
- (ii) $B_0 \supseteq C_0$ and $B_1 \supseteq C_1$;
- (iii) $e_0(X) = B_0 \setminus C_0$ and $e_1(X) = B_1 \setminus C_1$;
- (iv) $B_0 \cap B_1 = C_0 \cap C_1$.

Now we establish an interesting property of Wadge reducibility in the 2-reflective spaces.

Theorem 6.4 Let X be a 2-reflective space, S a non-empty set and let $\mathcal{P}_s^t = \{\nu \in S^X \mid \nu(\perp) = s \wedge f(\top) = t\}$ for all $s, t \in S$. Then $(S^X; \leq, \{\mathcal{P}_s^t\}_{s,t \in S})$ is a 2- S -discrete weak semilattice.

The next result is the main particular case of Theorem 4.4 for $S = \{0, 1\}$.

Corollary 6.5 Let X be a 2-reflective space, $\mathcal{P}_0^0 = \{A \subseteq X \mid \perp \notin A, \top \notin A\}$, $\mathcal{P}_1^0 = \{A \subseteq X \mid \perp \in A, \top \notin A\}$ and similarly for $\mathcal{P}_0^1, \mathcal{P}_1^1$. Then $(P(X); \leq_w, P_i^j)$ is a 2- $\{0, 1\}$ -discrete weak semilattice (and, consequently, if a sequence x_0, x_1, \dots in P has no greatest element under \leq_w then it has no supremum under \leq_w).

The next result is parallel to Theorem 4.7.

Theorem 6.6 In any 2-reflective space X , the difference hierarchy does not collapse.

We have also the following analog of Theorem 4.8. By a *least pair* of a preordering $(P; \leq)$ we mean a pair x_0, x_1 of incomparable elements of P such that $\forall y \in P (x_0 \leq y \vee x_1 \leq y)$.

Theorem 6.7 *Let X be a countably based 2-reflective space. For any $\omega \leq \alpha < \omega_1$, there is a least pair in $(\Sigma_\alpha^{-1} \setminus \Pi_\alpha^{-1}; \leq_W)$.*

By Theorem 3.4, the structure $(\Sigma_\alpha^{-1} \setminus \Pi_\alpha^{-1}; \leq_W)$ for $\alpha < \omega$ is trivial. The last theorem shows that for $\alpha \geq \omega$ this structure is non-trivial. It turns out that for any countably based 2-reflective space this structure contains an isomorphic copy of the ordering $\omega_1 \times \{0, 1\}$.

7 Examples of 2-Reflective Spaces

In this section which is parallel to Section 5 we consider in more detail some concrete examples of the 2-reflective spaces.

We start with defining some spaces. Let $\omega_\top^{\leq \omega}$ be the completion of the partial ordering $(\omega^* \cup \{\top\}; \sqsubseteq)$ which is obtained from the ordering $(\omega^*; \sqsubseteq)$ by adding a top element $\top \notin \omega^*$ bigger than all the other elements. Let $n_\top^{\leq \omega}$ (for any $2 \leq n < \omega$) be defined in the same way from the partial ordering $(\omega^* \cup \{\top\}; \sqsubseteq)$.

Let $(C_\omega; \leq)$ be the completion of the partial ordering $(A_\omega; \leq)$ defined as follows:

$$\begin{aligned} A_\omega &= \{(0, \sigma), (1, \sigma) \mid \sigma \in \omega^*\}; \\ (0, \sigma) &\leq (0, \tau) \text{ iff } \sigma \sqsubseteq \tau; (1, \sigma) \leq (1, \tau) \text{ iff } \sigma \sqsupseteq \tau; \\ (0, \sigma) &\leq (1, \tau) \text{ iff } \tau \sqsubseteq \tau \vee \sigma \sqsubseteq \sigma; (1, \sigma) \not\leq (0, \tau). \end{aligned}$$

Let the space $(C_n; \leq)$ be defined in the same way from the partial ordering $(A_n; \leq)$ for any $2 \leq n < \omega$ which is defined similarly, only for $\sigma, \tau \in n^*$.

From the properties of completions follows that $\omega_\top^{\leq \omega}$, $n_\top^{\leq \omega}$, $(C_\omega; \leq)$ and $(C_n; \leq)$ are topped complete countably based f_0 -spaces (hence, continuous lattices).

Finally, let $(P\omega; \subseteq)$ be the well known continuous lattice formed by the power set of ω with the Scott topology, hence finitary elements of $P\omega$ are exactly the finite subsets of ω .

Proposition 7.1 *The spaces $(C_\omega; \leq)$, $(C_n; \leq)$ and $P\omega$ are 2-reflective.*

Next we show that the class of 2-reflective spaces has some natural closure properties.

Theorem 7.2 (i) *If X is a 2-reflective space and Y a topped complete φ_0 -space then $X \times Y$ is a 2-reflective space.*

(ii) *If X is an f_0 -space and Y a 2-reflective f -space then the space Y^X of all continuous functions from X to Y with the topology of pointwise convergence is a 2-reflective f -space. The same holds true for the b -spaces from [4].*

Next we relate the spaces introduced above to some other spaces. First we establish an interesting minimality property of the spaces C_ω and C_n .

Theorem 7.3 *The spaces $\omega_{\mp}^{\leq \omega}$, $n_{\mp}^{\leq \omega}$ C_ω and C_n ($2 \leq n < \omega$) are retracts of any 2-reflective space X .*

The space $P\omega$ is in a sense opposite to the spaces C_ω and C_n . It is known [8] that $P\omega$ is universal in the sense that every complete countably based continuous lattice is a retract of $P\omega$. This is of some interest in the context of this paper since the structure of Wadge degrees in $P\omega$ is most complicated in the following sense.

Corollary 7.4 *For any complete countably based continuous lattice X , the structure $(P(X); \leq_W)$ is embeddable into $(P(P\omega); \leq_W)$.*

Next we relate some of the spaces considered above.

Proposition 7.5 (i) $P\omega$ is a retract of ω_\perp^ω .

(ii) There exists a quasiretraction $p : \omega^\omega \rightarrow P\omega$.

By Corollary 2.4, the map $A \mapsto p^{-1}A$ is a monotone function from $(P(P\omega); \leq_W)$ to $(P(\omega^\omega); \leq_W)$. One may wonder about the range of this map (in principle, all sets in the range might be Wadge equivalent). From results in [19] follows that the map $A \mapsto p^{-1}(A)$ respects the classes $A \in \Sigma_\alpha^{-1} \setminus \Pi_\alpha^{-1}$ for all $\alpha < \omega_1$.

The last result implies the following.

Theorem 7.6 *For any infinite ordinal $\alpha < \omega$, the structure $(\Sigma_\alpha^{-1} \setminus \Pi_\alpha^{-1}; \leq_W)$ in ω_\perp^ω has a substructure of order type $\omega_1 \times \{0, 1\}$.*

We conclude this section with an analog of Corollary 5.8.

Theorem 7.7 *For any 2-reflective space X , the structure $(\text{Bor}(X); \leq_W)$ has a substructure of order type $\theta \times \{0, 1\}$.*

8 Concluding Remarks

We see that in some domains the behaviour of the Wadge reducibility resembles its behaviour in the Baire and Cantor spaces (at least within Δ_2^0). Moreover, in Section 5 we observed that the study of Wadge reducibility in φ -spaces subsumes that in the Baire and Cantor space. The paper [15] shows that some results on Wadge reducibility in $P\omega$ have interesting implications for the theory of ω -ary boolean operations.

Hence, the study of Wadge reducibility in φ -spaces (especially in concrete important spaces like $P\omega$) seems to be a natural development of the classical theory of Wadge degrees and deserves, in our opinion, further work.

Many natural questions remain open. E.g., we do not currently know whether the structure $(\text{Bor}(P\omega); \leq_W)$ contain infinite antichains or infinite descending chains. Another interesting open question is the existence (or non-existence) of Wadge complete sets in classes of hierarchies in natural spaces, say in ω_\perp^ω .

Along with Wadge reducibility, people began to consider also some its weaker variants [6,23]. A search for such useful variants and their applications seems also reasonable.

Another possible topic is the study of effective versions of the Wadge reducibility. In fact, this direction started already in the papers [9,10] which contain effective versions of some results of this paper. In [1] a general realizability framework for the (joint) investigation of effective and non-effective topology was suggested. This framework might turn to be of use also in the development of our topic.

In this paper we concentrated mainly on the Wadge reducibility of sets, though its generalization to the case of maps $\nu : X \rightarrow S$ to arbitrary set S is also of interest, even for the Baire or Cantor space X . We plan to report on some our results in this direction in a future paper which is closely related to [16].

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