

Quasi-Nelson Algebras

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Abstract

We introduce a generalization of Nelson algebras having a not-necessarily involutive negation; we suggest to dub this class *quasi-Nelson algebras* in analogy with quasi-De Morgan lattices, these being a non-involutive generalization of De Morgan lattices. We show that, similarly to the involutive case (and perhaps surprisingly), our new class of algebras can be equivalently presented as (1) *quasi-Nelson residuated lattices*, i.e. models of the well-known Full Lambek calculus with exchange and weakening, extended with the Nelson axiom; (2) *non-involutive twist-structures*, i.e. special products of Heyting algebras, which generalize the well-known construction for representing algebraic models of Nelson's constructive logic with strong negation; (3) *quasi-Nelson algebras*, i.e. models of non-involutive Nelson logic viewed as a conservative expansion of the negation-free fragment of intuitionistic logic. The equivalence of the three presentations, and in particular the extension of the twist-structure representation to the non-involutive case, is the main technical result of the paper. We hope, however, that the main impact may be the possibility of opening new ways to (i) obtain deeper insights into the distinguishing feature of Nelson's logic (the Nelson axiom) and its algebraic counterpart; (ii) be able to investigate certain purely algebraic properties (such as 3-potency and (0,1)-congruence orderability) in a more general setting.

Keywords: Nelson logic, quasi-Nelson algebra, quasi-Nelson residuated lattice, semi-De Morgan algebra, non-involutive twist-structure, Nelson identity.

1 Introduction

Nelson's constructive logic with strong negation **N3** [12,15,17,21] can be viewed as either a conservative expansion of the negation-free fragment of intuitionistic logic by a new unary connective of *strong negation* (\sim) or, to within definitional

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equivalence, as the axiomatic extension of \mathbf{FL}_{ew} , the full Lambek calculus with exchange and weakening [4]⁴. Accordingly, the algebraic models of $\mathbf{N3}$ are known as either *Nelson algebras* or as *Nelson residuated lattices* [20]. It is well known that \mathbf{FL}_{ew} is the logic of all commutative integral residuated lattices [4]; adding the axiom of double negation ($\sim\sim x \Rightarrow x$), one obtains the logic \mathbf{NInFL}_{ew} of commutative integral residuated lattices that satisfy the involutive identity ($\sim\sim x \approx x$). Further adding the *Nelson axiom*:

$$\vdash ((x \Rightarrow (x \Rightarrow y)) \wedge (\sim y \Rightarrow (\sim y \Rightarrow \sim x))) \Rightarrow (x \Rightarrow y) \quad (\text{Nelson}_+)$$

one obtains precisely Nelson's constructive logic with strong negation $\mathbf{N3}$, whose algebraic counterpart is the class of Nelson residuated lattices. These are precisely the commutative integral residuated lattices that satisfy the involutive identity and the algebraic counterpart of the Nelson axiom, which is the *Nelson identity*:

$$(x \Rightarrow (x \Rightarrow y)) \wedge (\sim y \Rightarrow (\sim y \Rightarrow \sim x)) \approx x \Rightarrow y. \quad (\text{Nelson})$$

The Nelson axiom/identity is a powerful and somewhat mysterious one. When one adds it to \mathbf{NInFL}_{ew} , one readily obtains the distributive law

$$(x \wedge (y \vee z)) \Rightarrow ((x \wedge y) \vee (x \wedge z))$$

as well as $(3, 2)$ -contraction

$$(x \Rightarrow (x \Rightarrow (x \Rightarrow y))) \Rightarrow (x \Rightarrow (x \Rightarrow y))$$

while neither of them is valid in \mathbf{NInFL}_{ew} . Furthermore, the algebraic models of $\mathbf{N3}$ turn out to admit a nice representation as products (so-called *twist-structures*) of Heyting algebras [18,13], and this construction also uses crucially the Nelson axiom.

In the recent series of papers [10,11,19] we have been taking a closer look at logics in the Nelson family and at the Nelson axiom/identity, focusing on its meaning and consequences in the context of logics extending \mathbf{FL}_{ew} . Specifically, the papers [10,11] study the logic and algebras obtained from $\mathbf{N3}$ by dropping the Nelson axiom while keeping double negation and $(3, 2)$ -contraction; whereas in [19] we identify and investigate a number of conditions (algebraic, syntactic, and order-theoretic) that turn out to be equivalent to the Nelson axiom/identity in the context of \mathbf{NInFL}_{ew} . As mentioned in [19, Problems 8.2 and 8.3], one of the main questions that have been left open is which, if any, of the above-mentioned characterizations could be obtained in a more general context than \mathbf{NInFL}_{ew} , namely (to begin with) that of \mathbf{FL}_{ew} , the logic of all commutative integral residuated lattices.

In other words, we may ask what are the consequences of the Nelson axiom once we drop the double negation law; does any of the above-mentioned results hold, for instance, does the Nelson axiom still allow us to prove $(3, 2)$ -contraction and distributivity? What is the algebraic counterpart of the corresponding logic? Is

⁴ By *extension* (of a given logic) we mean a stronger logic over the same language; by *expansion* we mean a logic obtained by adding new connectives.

there a sensible notion of “non-involutive Nelson residuated lattice”, and do these algebras also admit a product-like representation?

Perhaps surprisingly, the answers to these questions are mostly positive, and the present paper is our first effort at understanding the Nelson axiom in the context of not-necessarily-involutive residuated lattices. Our strategy will be as follows. Mimicking the approach that proved successful in the study of Nelson’s constructive logic with strong negation **N3** and its algebraic models, we shall introduce and view “non-involutive Nelson residuated lattices” under three guises:

- (1) as *quasi-Nelson residuated lattices*, i.e. models of **FL_{ew}** plus the Nelson axiom;
- (2) as *non-involutive twist-structures*, i.e. special products of Heyting algebras, which generalize the well-known construction for representing algebraic models of **N3**;
- (3) as *quasi-Nelson algebras*, i.e. models of non-involutive Nelson logic viewed as an expansion of the negation-free fragment of intuitionistic logic.

The main result of this paper, generalizing the well-known equivalence concerning models of **N3**, is that the above three classes are all term equivalent⁵.

We shall thus begin by introducing quasi-Nelson residuated lattices in Section 2 as commutative integral residuated lattices that additionally satisfy the Nelson identity (1); we then single out certain properties that all quasi-Nelson residuated lattices satisfy (Proposition 2.5), which we shall later on take as basis for our definition of quasi-Nelson algebra (3). Justifying our nomenclature, we shall point out (Proposition 2.7) that, just like every Nelson residuated lattice has a De Morgan algebra reduct, every quasi-Nelson residuated lattice has a quasi-De Morgan algebra reduct [16]. In Section 3 we introduce non-involutive twist-structures (2), showing that every such structure is a quasi-Nelson residuated lattice (Theorem 3.3). It may be worth pointing out that, until the recent papers [7,8], no construction of this kind was known in the literature that would allow one to represent algebras carrying a non-involutive negation. Finally, in Section 4 we close the circle of our equivalences, introducing quasi-Nelson algebras (3) and showing that every such algebra is representable as a non-involutive twist-structure. The overall equivalence is summarized in the final Theorem 4.4.

We would like to point out that our indirect proof strategy is not only unavoidable, for the time being (we do not yet have a direct proof of, e.g., Proposition 2.7), but also insightful, because our representation of quasi-Nelson residuated lattices as twist-structures affords an easy way of constructing concrete examples of such algebras. Lastly, let us stress that, although in this paper we work exclusively with algebras, it is important for the logically-minded reader to keep in mind that we think of algebras as the algebraic counterpart of logics; this strategy is justified because we are dealing with *algebraizable logics* [1], so we can transfer our results back and forth from algebra to logic in a straightforward way.

⁵ Loosely speaking, this means that the three definitions can be seen as alternative presentations of the “same” class of algebras in a different algebraic language, analogous to the presentation of Boolean algebras as Boolean rings or to that of MV-algebras as certain lattice-ordered groups.

2 Quasi-Nelson residuated lattices

We assume familiarity with the rudiments of general algebra and model theory, especially that part of first-order logic known as equational logic. For general algebraic background, see [3,5,9].

Definition 2.1 A *commutative integral bounded residuated lattice* (CIBRL) is an algebra $\mathbf{A} = \langle A; \wedge, \vee, *, \Rightarrow, 0, 1 \rangle$ of type $\langle 2, 2, 2, 2, 0, 0 \rangle$ such that:

- (i) $\langle A; *, 1 \rangle$ is commutative monoid, (Mon)
- (ii) $\langle A; \wedge, \vee, 0, 1 \rangle$ is a bounded lattice (with order \leq), (Lat)
- (iii) $a * b \leq c$ iff $a \leq b \Rightarrow c$ for all $a, b, c \in A$. (Res)

On a CIBRL \mathbf{A} , the presence of the 0 constant allows us to define a *negation* operation (\sim) given by $\sim a := a \Rightarrow 0$ for all $a \in A$. Another abbreviation that we shall need is $a^n := \underbrace{a * \dots * a}_{n \text{ times}}$ (by convention we let $a^0 := 1$ and $a^1 := a$). As we shall

see, all quasi-Nelson residuated lattices satisfy the identity $x^2 \approx x^3$. This is indeed a crucial property for the understanding of such algebras: it is called *3-potency* (or 2-potency by other authors) and is the algebraic counterpart of (3, 2)-contraction.

Using these abbreviations, we list in the following proposition a few well-known properties of CIBRLs that we shall use in the sequel (they are proved in [4, Lemmas 2.6 and 2.8] or are easy consequences thereof).

Proposition 2.2 *Let \mathbf{A} be a CIBRL and $a, b, c, d \in A$. The following properties hold:*

- (i) $a \leq b$ iff $a \Rightarrow b = 1$.
- (ii) $(a * b) \Rightarrow c = a \Rightarrow (b \Rightarrow c)$.
- (iii) $a * b \leq a$.
- (iv) If $a \leq b$ and $c \leq d$, then $a * c \leq b * d$.
- (v) $\sim(a \vee b) = \sim a \wedge \sim b$.
- (vi) $\sim(a * b) = a \Rightarrow \sim b$.
- (vii) $a * (a \Rightarrow b) \leq a \wedge b$.
- (viii) $a \leq \sim \sim a$.
- (ix) $a * (b \vee c) = (a * b) \vee (a * c)$.
- (x) $a * \sim a = 0$.
- (xi) If $a \leq b$, then $\sim b \leq \sim a$.
- (xii) $\sim \sim \sim a = \sim a$.
- (xiii) $a \Rightarrow \sim b = \sim \sim a \Rightarrow \sim b$.
- (xiv) $\sim a \Rightarrow \sim b = \sim \sim (\sim a \Rightarrow \sim b)$.

A non-empty subset F of a CIBRL \mathbf{A} is a *filter* if for all $a, b \in A$ it holds that: (i) $a \leq b$ and $a \in F$ implies $b \in F$; and (ii) $a, b \in F$ implies $a * b \in F$

(one easily verifies that such an F must then be a lattice filter in the usual sense, though the converse may not be true, for (ii) may fail to hold). For $a \in A$, the set $[a] = \{b \in A : a^k \leq b, \text{ for some positive integer } k\}$ is a filter, namely the filter generated by a . Moreover, if \mathbf{A} is $n+1$ -potent, then $[a] = \{b \in A : a^n \leq b\}$. The study of filters on CIBRLs is particularly important because, from a logical point of view, they correspond to logical theories of \mathbf{FL}_{ew} and, algebraically, they are in one-to-one correspondence with congruences. In particular it is well known that, denoting by $\Theta^{\mathbf{A}}(x, 1)$ the principal congruence generated by the pair $(x, 1)$, one has $[a] \subseteq [b]$ if and only if $\Theta^{\mathbf{A}}(a, 1) \subseteq \Theta^{\mathbf{A}}(b, 1)$. We shall use this fact in the proof of Proposition 2.4 below (see [19] for further details).

Definition 2.3 A *quasi-Nelson residuated lattice* is a CIBRL that satisfies the Nelson identity:

$$(x \Rightarrow (x \Rightarrow y)) \wedge (\sim y \Rightarrow (\sim y \Rightarrow \sim x)) \approx x \Rightarrow y \quad (\text{Nelson})$$

By Proposition 2.2 (ii), we can equivalently reformulate the Nelson identity in the following way (which shall be useful in the sequel):

$$(x^2 \Rightarrow y) \wedge ((\sim y)^2 \Rightarrow \sim x) \approx x \Rightarrow y. \quad (\text{Nelson}')$$

We collect below a few useful properties that hold on all quasi-Nelson residuated lattices.

Proposition 2.4 Let \mathbf{A} be a quasi-Nelson residuated lattice and $a, b, c \in A$. The following properties hold:

- (i) If $a^2 \leq b$ and $(\sim b)^2 \leq \sim a$, then $a \leq b$.
- (ii) $a = a^2 \vee (a \wedge \sim a)$.
- (iii) $a^2 = a^3$.
- (iv) $a \leq b$ iff $\Theta^{\mathbf{A}}(b, 1) \subseteq \Theta^{\mathbf{A}}(a, 1)$ and $\Theta^{\mathbf{A}}(a, 0) \subseteq \Theta^{\mathbf{A}}(b, 0)$.
- (v) The lattice reduct of \mathbf{A} is distributive.
- (vi) $a * b = a \wedge b \wedge \sim(a \Rightarrow \sim b)$.
- (vii) $\sim \sim(a * b) = \sim \sim a * \sim \sim b$.
- (viii) $(a * b)^2 = (a \wedge b)^2$.
- (ix) $(\sim \sim(a \wedge b))^2 = (\sim \sim a \wedge \sim \sim b)^2$.
- (x) $(\sim(a \Rightarrow b))^2 = (\sim \sim(a \wedge \sim b))^2$.
- (xi) $(\sim(a^2 \Rightarrow b))^2 = (\sim \sim(a \wedge \sim b))^2$.

Proof. (i). If $a^2 \leq b$, then $a^2 \Rightarrow b = 1$ by Proposition 2.2 (i). Similarly we obtain $(\sim b)^2 \Rightarrow \sim a = 1$. Then using (Nelson) we have $a \Rightarrow b = (a^2 \Rightarrow b) \wedge ((\sim b)^2 \Rightarrow \sim a) = 1 \wedge 1 = 1$, which gives us $a \leq b$ as required.

(ii). Observe that $a^2 \vee (a \wedge \sim a) \leq a$ holds in all CIBRLs, because $a^2 \leq a$ by Proposition 2.2 (iii), and $a \wedge \sim a \leq a$ by the lattice properties. It thus remains to prove that $a \leq a^2 \vee (a \wedge \sim a)$. We shall use item (i) above and check on the one hand

that $a^2 \leq a^2 \vee (a \wedge \sim a)$, which is trivial, and on the other that $(\sim(a^2 \vee (a \wedge \sim a)))^2 \leq \sim a$. We have:

$$\begin{aligned}
 (\sim(a^2 \vee (a \wedge \sim a)))^2 &= (\sim(a^2) \wedge \sim(a \wedge \sim a))^2 && \text{by Prop. 2.2 (v)} \\
 &= ((a \Rightarrow \sim a) \wedge \sim(a \wedge \sim a))^2 && \text{by Prop. 2.2 (vi)} \\
 &\leq (a \Rightarrow \sim a) * \sim(a \wedge \sim a) && (x \wedge y)^2 \leq x * y \text{ by Prop. 2.2 (iv)}.
 \end{aligned}$$

It will thus be sufficient to show that $(a \Rightarrow \sim a) * \sim(a \wedge \sim a) \leq \sim a$. Using (Res), we have $(a \Rightarrow \sim a) * \sim(a \wedge \sim a) \leq \sim a$ iff $a * (a \Rightarrow \sim a) * \sim(a \wedge \sim a) \leq 0$ iff $a * (a \Rightarrow \sim a) \leq \sim \sim(a \wedge \sim a)$. The result then follows, because by Proposition 2.2 (vii) and (viii) we have $a * (a \Rightarrow \sim a) \leq a \wedge \sim a \leq \sim \sim(a \wedge \sim a)$.

(iii). Using item (ii) above, we will show that $(a^2 \vee (a \wedge \sim a))^2 \leq a^3$. Using Proposition 2.2 (ix), we have $(a^2 \vee (a \wedge \sim a)) * (a^2 \vee (a \wedge \sim a)) = (a^2 * (a^2 \vee (a \wedge \sim a))) \vee ((a \wedge \sim a) * (a^2 \vee (a \wedge \sim a))) = a^4 \vee (a^2 * (a \wedge \sim a)) \vee ((a \wedge \sim a) * a^2) \vee (a \wedge \sim a)^2 = a^4 \vee (a^2 * (a \wedge \sim a)) \vee (a \wedge \sim a)^2$. By Proposition 2.2 (x) we have $a * \sim a = 0$ and, as observed in the proof of item (ii) above, $(a \wedge \sim a)^2 \leq a * \sim a = 0$. Hence, $a^4 \vee (a^2 * (a \wedge \sim a)) \vee (a \wedge \sim a)^2 = a^4 \vee (a^2 * (a \wedge \sim a))$. By Proposition 2.2 (iii) $a^4 \leq a^3$, and $a^2 * (a \wedge \sim a) \leq a^3$ by Proposition 2.2 (iv). Thus, $a^4 \vee (a^2 * (a \wedge \sim a)) = a^2 \leq a^3$ as required.

(iv). Assume $a \leq b$ and $(a, 1) \in \theta$ for an arbitrary congruence θ of \mathbf{A} . Then $(a \vee b, 1 \vee b) = (b, 1) \in \theta$ as well. This shows that $\Theta^{\mathbf{A}}(b, 1) \subseteq \Theta^{\mathbf{A}}(a, 1)$. Similarly, $(b, 0) \in \theta$ implies $(a \wedge b, a \wedge 0) = (a, 0) \in \theta$, which means that $\Theta^{\mathbf{A}}(a, 0) \subseteq \Theta^{\mathbf{A}}(b, 0)$. Conversely, assume $\Theta^{\mathbf{A}}(b, 1) \subseteq \Theta^{\mathbf{A}}(a, 1)$ and $\Theta^{\mathbf{A}}(a, 0) \subseteq \Theta^{\mathbf{A}}(b, 0)$. As observed earlier, the first assumption entails $[b] \subseteq [a]$, i.e. $b \in [a]$. By [19, Lemma 2.5] we have $\Theta^{\mathbf{A}}(\sim a, 1) = \Theta^{\mathbf{A}}(a, 0)$ and $\Theta^{\mathbf{A}}(\sim b, 1) = \Theta^{\mathbf{A}}(b, 0)$. Hence the second assumption implies $[\sim a] \subseteq [\sim b]$, i.e. $\sim a \in [\sim b]$. Having proved 3-potency in item (iii) above, we have $[a] = \{c \in A : a^2 \leq c\}$ and $[\sim b] = \{c \in A : (\sim b)^2 \leq c\}$. Thus, $a^2 \leq b$ and $(\sim b)^2 \leq \sim a$. At this point we apply (Nelson') together with Proposition 2.2 (i) to obtain $a \Rightarrow b = (a^2 \Rightarrow b) \wedge ((\sim b)^2 \Rightarrow \sim a) = 1 \wedge 1 = 1$. Hence, again by Proposition 2.2 (i), we have $a \leq b$.

(v). In view of a contradiction, suppose that the lattice reduct of \mathbf{A} is not distributive. Then, by a well-known result in lattice theory, \mathbf{A} contains as a sublattice either the ‘diamond’ or the ‘pentagon’ [3, Theorem I.3.6]. In either case, there exist three distinct elements $a, b, c \in A$ such that $a \wedge c = b \wedge c$ and $a \vee c = b \vee c$. We are going to show that $\Theta^{\mathbf{A}}(a, 1) = \Theta^{\mathbf{A}}(b, 1)$ and $\Theta^{\mathbf{A}}(a, 0) = \Theta^{\mathbf{A}}(b, 0)$ and so, by item (iv) above, we will conclude that $a = b$ against our assumption. Let θ be an arbitrary congruence on \mathbf{A} . Assume $(a, 1) \in \theta$. Then $(a \wedge c, 1 \wedge c) = (b \wedge c, c) \in \theta$ and $(a \vee c, 1 \vee c) = (b \vee c, 1) \in \theta$ as well. From the former (and the absorption law) we have $(b \vee (b \wedge c), b \vee c) = (b, b \vee c) \in \theta$, thus from the latter and transitivity of θ we have $(b, 1) \in \theta$. A symmetric reasoning shows that $(b, 1) \in \theta$ implies $(a, 1) \in \theta$, and so $\Theta^{\mathbf{A}}(a, 1) = \Theta^{\mathbf{A}}(b, 1)$. Next, assume $(a, 0) \in \theta$. Then $(a \wedge c, 0 \wedge c) = (b \wedge c, 0) \in \theta$ and $(a \vee c, 0 \vee c) = (b \vee c, c) \in \theta$ as well. As before, by absorption we get $(b \wedge (b \vee c), b \wedge c) = (b, b \wedge c) \in \theta$, which gives us $(b, 0) \in \theta$ by

transitivity. A symmetric reasoning shows that $(b, 0) \in \theta$ implies $(a, 0) \in \theta$, and so $\Theta^{\mathbf{A}}(a, 0) = \Theta^{\mathbf{A}}(b, 0)$. Thus, by item (iv) above, we conclude that $a = b$, against our hypothesis.

(vi). The inequality $a * b \leq a \wedge b \wedge \sim(a \Rightarrow \sim b)$ holds in any CIBRL. In fact, by (Res) we have $a * b \leq \sim(a \Rightarrow \sim b)$ iff $a * b * (a \Rightarrow \sim b) \leq 0$. The latter inequality holds because, applying Proposition 2.2 (iv), (viii) and (x), we have $a * b * (a \Rightarrow \sim b) = b * a * (a \Rightarrow \sim b) \leq b * (a \wedge \sim b) \leq b * \sim b = 0$. Since $a * b \leq a \wedge b$ by Proposition 2.2 (iii), we obtain $a * b \leq a \wedge b \wedge \sim(a \Rightarrow \sim b)$ by monotonicity of \wedge . To check that $a \wedge b \wedge \sim(a \Rightarrow \sim b) \leq a * b$, we shall use item (iv) above and show that $\Theta^{\mathbf{A}}(a * b, 1) \subseteq \Theta^{\mathbf{A}}(a \wedge b \wedge \sim(a \Rightarrow \sim b), 1)$ and $\Theta^{\mathbf{A}}(a \wedge b \wedge \sim(a \Rightarrow \sim b), 0) \subseteq \Theta^{\mathbf{A}}(a * b, 0)$. Let then θ be an arbitrary congruence of \mathbf{A} . Assume $(a \wedge b \wedge \sim(a \Rightarrow \sim b), 1) \in \theta$. Then, by absorption, $(a \vee (a \wedge b \wedge \sim(a \Rightarrow \sim b)), a \vee 1) = (a, 1) \in \theta$ and similarly $(b, 1) \in \theta$. Then $(a * b, 1 * 1) = (a * b, 1) \in \theta$ as well. Suppose now that $(a * b, 0) \in \theta$. Using Proposition 2.2 (vi), we have $(\sim\sim(a * b), \sim\sim 0) = (\sim(a \Rightarrow \sim b), 0) \in \theta$. Then $(a \wedge b \wedge \sim(a \Rightarrow \sim b), a \wedge b \wedge 0) = (a \wedge b \wedge \sim(a \Rightarrow \sim b), 0) \in \theta$ as required.

(vii). Using item (iv) above, we shall check that $\Theta^{\mathbf{A}}(\sim\sim(a * b), 1) = \Theta^{\mathbf{A}}(\sim\sim a * \sim\sim b, 1)$ and $\Theta^{\mathbf{A}}(\sim\sim(a * b), 0) = \Theta^{\mathbf{A}}(\sim\sim a * \sim\sim b, 0)$. We shall also use the following:

$$\begin{aligned}
 \sim(\sim\sim a * \sim\sim b) &= \sim\sim a \Rightarrow \sim\sim\sim b && \text{by Prop. 2.2 (vi)} \\
 &= \sim\sim a \Rightarrow \sim b && \text{by Prop. 2.2 (xii)} \\
 &= a \Rightarrow \sim b && \text{by Prop. 2.2 (xiii)} \\
 &= \sim(a * b) && \text{by Prop. 2.2 (vi).}
 \end{aligned}$$

Let θ be an arbitrary congruence of \mathbf{A} . Assume $(\sim\sim(a * b), 1) \in \theta$. Since $\sim\sim(a * b) \leq \sim\sim a$ (by items (iii) and (xi) of Proposition 2.2), we have $(\sim\sim(a * b) \vee \sim\sim a, 1 \vee \sim\sim a) = (\sim\sim a, 1) \in \theta$. Similarly we obtain $(\sim\sim b, 1) \in \theta$ and hence $(\sim\sim a * \sim\sim b, 1 * 1) = (\sim\sim a * \sim\sim b, 1) \in \theta$. Conversely, assume $(\sim\sim a * \sim\sim b, 1) \in \theta$. Using $\sim(a * b) = \sim(\sim\sim a * \sim\sim b)$, we immediately obtain $(\sim\sim(\sim\sim a * \sim\sim b), \sim\sim 1) = (\sim\sim(a * b), 1) \in \theta$. Now assume $(\sim\sim(a * b), 0) \in \theta$. Reasoning as before, we have $(\sim\sim(a * b), 0) = (\sim\sim(\sim\sim a * \sim\sim b), 0) \in \theta$. Since $\sim\sim a * \sim\sim b \leq \sim\sim(\sim\sim a * \sim\sim b)$ (by item (viii) of Proposition 2.2), we have $(\sim\sim(\sim\sim a * \sim\sim b) \wedge (\sim\sim a * \sim\sim b), 0 \wedge (\sim\sim a * \sim\sim b)) = (\sim\sim a * \sim\sim b, 0) \in \theta$ as required. Finally, assume $(\sim\sim a * \sim\sim b, 0) \in \theta$. As before, we have $(\sim\sim(\sim\sim a * \sim\sim b), \sim\sim 0) = (\sim\sim(a * b), 0) \in \theta$ as required.

(viii). By Proposition 2.2 (iii) we have $a * b \leq a \wedge b$ and thus $(a * b)^2 \leq (a \wedge b)^2$ by Proposition 2.2 (iv). On the other hand, again by Proposition 2.2 (iv), from $a \wedge b \leq a$ and $a \wedge b \leq b$ we get $(a \wedge b)^2 \leq a * b$ and thus $(a \wedge b)^4 \leq (a * b)^2$. Now applying 3-potency (item (iii) above) we obtain $(a \wedge b)^2 = (a \wedge b)^4 \leq (a * b)^2$ as required.

(ix). It is easy to show that $\sim\sim(a \wedge b) \leq \sim\sim a \wedge \sim\sim b$ using Proposition 2.2 (xi), from which we obtain $(\sim\sim(a \wedge b))^2 \leq (\sim\sim a \wedge \sim\sim b)^2$ by Proposition 2.2 (iv). It remains to check that $(\sim\sim a \wedge \sim\sim b)^2 \leq (\sim\sim(a \wedge b))^2$. By item (viii) above we have $(\sim\sim a \wedge \sim\sim b)^2 = (\sim\sim a * \sim\sim b)^2$, and by item (vii) above, $(\sim\sim a * \sim\sim b)^2 =$

$(\sim\sim(a*b))^2$. By Proposition 2.2 (iii) we have $a*b \leq a \wedge b$, which implies $\sim\sim(a*b) \leq \sim\sim(a \wedge b)$ and thus $(\sim\sim(a*b))^2 \leq (\sim\sim(a \wedge b))^2$ by Proposition 2.2 (xi). Joining these facts, we have $(\sim\sim a \wedge \sim\sim b)^2 = (\sim\sim a * \sim\sim b)^2 = (\sim\sim(a*b))^2 \leq (\sim\sim(a \wedge b))^2$ as required.

(x). Notice that $(\sim\sim(a \wedge \sim b))^2 = (\sim\sim a \wedge \sim b)^2$. This holds because, by item (ix) above, we have $(\sim\sim(a \wedge \sim b))^2 = (\sim\sim a \wedge \sim\sim b)^2$ and, by Proposition 2.2 (xii), we have $(\sim\sim a \wedge \sim\sim b)^2 = (\sim\sim a \wedge \sim b)^2$. Hence, it will be sufficient to check that $(\sim\sim a \wedge \sim b)^2 = (\sim(a \Rightarrow b))^2$. Let us begin with $(\sim\sim a \wedge \sim b)^2 \leq (\sim(a \Rightarrow b))^2$. Observe that $\sim\sim a * \sim b \leq \sim(a \Rightarrow b)$. This holds because, by (Res), we have $\sim\sim a * \sim b \leq \sim(a \Rightarrow b)$ iff $(a \Rightarrow b) * \sim\sim a * \sim b \leq 0$ iff $(a \Rightarrow b) * \sim\sim a \leq \sim\sim b$ iff $a \Rightarrow b \leq \sim\sim a \Rightarrow \sim\sim b = a \Rightarrow \sim\sim b$. The last equality holds by Proposition 2.2 (xiii). By (Res), we have $a \Rightarrow b \leq a \Rightarrow \sim\sim b$ iff $a * (a \Rightarrow b) \leq \sim\sim b$ and the latter holds because, by Proposition 2.2 (vii) and (viii), we have $a * (a \Rightarrow b) \leq b \leq \sim\sim b$. Having shown that $\sim\sim a * \sim b \leq \sim(a \Rightarrow b)$, we can invoke Proposition 2.2 (iv) to obtain $(\sim\sim a * \sim b)^2 \leq (\sim(a \Rightarrow b))^2$. We have shown that $(\sim\sim a * \sim b)^2 = (\sim\sim a \wedge \sim b)^2$ in item (viii) above, so we conclude that $(\sim\sim a \wedge \sim b)^2 = (\sim\sim a * \sim b)^2 \leq (\sim(a \Rightarrow b))^2$ as required. For the converse inequality, let us begin by observing that $\sim a \leq a \Rightarrow b$ and $b \leq a \Rightarrow b$. The latter is an immediate consequence of Proposition 2.2 (iii); as to the former, by (Res) we have $\sim a \leq a \Rightarrow b$ iff $a * \sim a \leq b$, which is true because $a * \sim a = 0$, by Proposition 2.2 (x). Then by monotonicity of \vee we have $\sim a \vee b \leq a \Rightarrow b$. By Proposition 2.2 (xi), we obtain $\sim(a \Rightarrow b) \leq \sim(\sim a \vee b) = \sim\sim a \wedge \sim b$, the last equality holding true by Proposition 2.2 (v). Then we can use once more Proposition 2.2 (iv) to obtain $(\sim(a \Rightarrow b))^2 \leq (\sim\sim a \wedge \sim b)^2$ as required.

(xi). By the preceding item, we have $(\sim(a^2 \Rightarrow b))^2 = (\sim\sim(a^2 \wedge \sim b))^2$. It will thus be sufficient to show that $(\sim\sim(a^2 \wedge \sim b))^2 = (\sim\sim(a \wedge \sim b))^2$. Using items (viii) and (iii) above, we have $(a^2 \wedge \sim b)^2 = (a^2 * \sim b)^2 = a^4 * (\sim b)^2 = a^2 * (\sim b)^2 = (a * \sim b)^2 = (a \wedge \sim b)^2$. Thus $(a^2 \wedge \sim b)^2 = (a \wedge \sim b)^2$, and applying double negation to both sides, we obtain $\sim\sim(a^2 \wedge \sim b)^2 = \sim\sim(a \wedge \sim b)^2$. Now observe that, by item (vii) above, we have $\sim\sim x^2 = (\sim\sim x)^2$, and so $\sim\sim(a^2 \wedge \sim b)^2 = (\sim\sim(a^2 \wedge \sim b))^2$. Similarly we have $\sim\sim(a \wedge \sim b)^2 = (\sim\sim(a \wedge \sim b))^2$, which concludes our proof. \square

Let us briefly comment on the most important properties proven in Proposition 2.4. The identity (ii) was shown in [19, Theorem 6.1] to be equivalent, in the context of compatibly involutive CIBRLs, to the identity (Nelson); we do not know at present whether such an equivalence holds also for arbitrary CIBRLs. Item (iii), i.e. 3-potency, is one of the characterizing features of Nelson residuated lattices [19, Corollary 4.3], and one that greatly simplifies the algebraic study of these structures, for it allows one to prove that they form a variety of *weak Brouwerian semilattices with 1-filter preserving operations*: see [19, Lemma 2.4] for further details. Condition (iv), which we have introduced and dubbed *(0,1)-congruence orderability* in [19], is a generalization of the well-known algebraic property called congruence orderability. In fact, one of the main results of [19, Corollary 7.2] is that a compatibly involutive CIBRL is (0,1)-congruence orderable if and only if it satisfies the identity (Nelson). This equivalence could be proved also for arbitrary CIBRLs, although we will not pursue this here; we refer the reader to [19] for further

discussion on $(0,1)$ -congruence orderability. Finally, item (vi) is worth highlighting, for it entails that it would be possible to give a definition of quasi-Nelson residuated lattices omitting the monoid operation from the algebraic language, for it could be introduced via the term $x \wedge y \wedge \sim(x \Rightarrow \sim y)$. This generalizes a well-known fact about involutive CIBRLs, where one can define $x * y := \sim(x \Rightarrow \sim y)$, and it also explains why we shall not need to introduce $*$ as a primitive operation in our definition of quasi-Nelson algebras (Definition 4.1).

The next proposition will be one of the main ingredients for establishing our equivalent characterization of quasi-Nelson residuated lattices. We shall write $x \rightarrow y$ as a shorthand for $x^2 \Rightarrow y$ (recall that this is also equivalent, in our context, to $x \Rightarrow (x \Rightarrow y)$). As mentioned earlier, one of the main features of Nelson's constructive logic with strong negation **N3** (which is inherited by its algebraic counterpart) is that it can be defined as either an expansion of the negation-free fragment of intuitionistic logic or as an axiomatic extension of **FL_{ew}**. This alternative corresponds to presenting **N3** and the corresponding algebras either in the primitive language $\langle \wedge, \vee, \rightarrow, \sim, 0, 1 \rangle$ or $\langle \wedge, \vee, \Rightarrow, \sim, 0, 1 \rangle$. The above-introduced connective \rightarrow is known as *weak implication*, and coincides with intuitionistic implication on the negation-free fragment of **N3**; the residuated *strong implication* \Rightarrow , which is the one inherited from **FL_{ew}**, can in turn be recovered from the weak by letting $x \Rightarrow y := (x \rightarrow y) \wedge (\sim y \rightarrow \sim x)$. One of main results of the present paper is that this inter-definability can be carried over rather painlessly to the non-involutive setting.

Proposition 2.5 *Let \mathbf{A} be quasi-Nelson residuated lattice. Then:*

- (i) *The reduct $\langle A; \wedge, \vee, 0, 1 \rangle$ is bounded distributive lattice (with order \leq).*
- (ii) *The relation \preceq on A defined for all $a, b \in A$ by $a \preceq b$ iff $a \rightarrow b = 1$ is a quasiorder (i.e. is reflexive and transitive).*
- (iii) *The relation $\equiv := \preceq \cap (\preceq)^{-1}$ is a congruence on the reduct $\langle A; \wedge, \vee, \rightarrow, 0, 1 \rangle$ and the quotient algebra $\mathbf{A}_+ = \langle A; \wedge, \vee, \rightarrow, 0, 1 \rangle / \equiv$ is a Heyting algebra⁶.*
- (iv) *For all $a, b \in A$, it holds that $\sim(a \rightarrow b) \equiv \sim \sim(a \wedge \sim b)$.*
- (v) *For all $a, b \in A$, it holds that $a \leq b$ iff $a \preceq b$ and $\sim b \preceq \sim a$.*
- (vi) *For all $a, b \in A$, it holds that*
 - (vi.1) $\sim \sim(\sim a \rightarrow \sim b) \equiv \sim a \rightarrow \sim b$
 - (vi.2) $\sim(a \vee b) \equiv \sim a \wedge \sim b$
 - (vi.3) $\sim \sim a \wedge \sim \sim b \equiv \sim \sim(a \wedge b)$
 - (vi.4) $\sim a \equiv \sim \sim \sim a$
 - (vi.5) $a \preceq \sim \sim a$
 - (vi.6) $a \wedge \sim a \preceq 0$.

Proof. (i). Certainly $\langle A; \wedge, \vee, 0, 1 \rangle$ is a bounded lattice; distributivity follows from Proposition 2.4 (v). Items (ii) and (iii) are not difficult to prove directly, but one

⁶ Recall that a Heyting algebra \mathbf{A} can be defined as a CIBRL satisfying $a * b = a \wedge b$ for all $a, b \in A$. Since the monoid and meet operations coincide, it is customary to delete the former from the algebraic signature.

rely on the fact that, being 3-potent (Proposition 2.4 (iii)), quasi-Nelson residuated lattices form a variety of *weak Brouwerian semilattices with 1-filter preserving operations* [19, Lemma 2.4]. Then (ii) and (iii) correspond, respectively, to conditions (WBSO2) and (WBSO3). Concerning (iv), notice that the definition of \equiv immediately implies that $x \equiv y$ iff $x^2 = y^2$. Then item (iv) is precisely Proposition 2.4 (xi). The leftward implication of item (v) has been proven in Proposition 2.4 (i). As to the rightward one, if $a \leq b$, then $\sim b \leq \sim a$ by Proposition 2.2 (xi). Then $a \Rightarrow b = \sim b \Rightarrow \sim a = 1$. Thus *a fortiori* $a \rightarrow b = \sim b \rightarrow \sim a = 1$, because $x \Rightarrow y \leq x^2 \Rightarrow y$ in any CIBRL. Item (vi.1) is an easy consequence of Proposition 2.2 (xiv). Proposition 2.2 (v) implies (vi.2). Item (vi.3) is precisely Proposition 2.4 (ix). Proposition 2.2 (xii) implies (vi.4). Item (vi.5) follows from item (v) together with Proposition 2.2 (viii). Finally, regarding (vi.6), notice that, by Proposition 2.4 (viii) and Proposition 2.2 (x), we have $(a \wedge \sim a)^2 = (a * \sim a)^2 = 0$. Then $(a \wedge \sim a) \rightarrow 0 = (a \wedge \sim a)^2 \Rightarrow 0 = 0 \Rightarrow 0 = 1$. \square

We shall see in Section 4 that the above conditions are not only necessary, but also sufficient to provide an equivalent presentation of the class of Nelson residuated lattices. Formally, we shall call an algebra $\mathbf{A} = \langle A; \wedge, \vee, \rightarrow, \sim, 0, 1 \rangle$ of type $\langle 2, 2, 2, 1, 0, 0 \rangle$ that satisfies all the conditions of Proposition 2.5 a *quasi-Nelson algebra* (Definition 4.1). Using this terminology, we can restate the above proposition as follows.

Theorem 2.6 *For every quasi-Nelson residuated lattice \mathbf{A} , the algebra $\langle A; \wedge, \vee, \rightarrow, \sim, 0, 1 \rangle$ is a quasi-Nelson algebra.*

For the reader's convenience, we state here the next proposition without proof; we shall obtain a proof as a corollary of our main result (Theorem 4.4).

Proposition 2.7 *The $\langle \wedge, \vee, 0, 1 \rangle$ -reduct of any quasi-Nelson residuated lattice \mathbf{A} is a lower quasi-de Morgan algebra according to the terminology introduced in [16, Definition 2.2]. That is, for all $a, b, c \in A$, it holds that:*

- (i) $\langle A; \wedge, \vee, 0, 1 \rangle$ is a bounded distributive lattice (with order \leq),
- (ii) $\sim(a \vee b) = \sim a \wedge \sim b$,
- (iii) $\sim \sim(a \wedge b) = \sim \sim a \wedge \sim \sim b$,
- (iv) $\sim \sim \sim a = \sim a$,
- (v) $a \leq \sim \sim a$.

Moreover,

- (vi) $a \wedge \sim a \leq b \vee \sim b$.

Besides explaining the name we have chosen for our non-involutive generalization of Nelson algebras, Proposition 2.7 generalizes the well-known fact that the $\langle \wedge, \vee, 0, 1 \rangle$ -reduct of a Nelson algebra is a De Morgan algebra (i.e. a quasi-De Morgan algebra satisfying $\sim \sim a = a$); or actually a Kleene algebra, that is a De Morgan algebra satisfying item (vi) of our proposition [19, Proposition 4.9].

3 Non-involutive twist-structures

In this section we introduce a construction that will allow us to obtain a better insight into our new class of algebras, as well as to prove equivalence between the alternative presentations. Our definitions obviously generalize/are inspired by the well-known twist-structure constructions used to represent Nelson residuated lattices and related algebras (see e.g. [14]). The crucial part of our construction, however, relies on the non-involutive twist-structures introduced in [7,8], and in particular on the idea of using *two* Heyting algebras related by maps rather than a single Heyting algebra⁷.

Definition 3.1 Let $\mathbf{H}_+ = \langle H_+, \wedge_+, \vee_+, \rightarrow_+, 0_+, 1_+ \rangle$ and $\mathbf{H}_- = \langle H_-, \wedge_-, \vee_-, \rightarrow_-, 0_-, 1_- \rangle$ be Heyting algebras and $n: H_+ \rightarrow H_-$ and $p: H_- \rightarrow H_+$ be maps satisfying the following properties:

- (i) n preserves finite meets, joins and the bounds (i.e., one has $n(x \wedge_+ y) = n(x) \wedge_- n(y)$, $n(x \vee_+ y) = n(x) \vee_- n(y)$, $n(1_+) = 1_-$ and $n(0_+) = 0_-$),
- (ii) p preserves meets, the implication and the bounds (i.e., one has $p(x \wedge_- y) = p(x) \wedge_+ p(y)$, $p(x \rightarrow_- y) = p(x) \rightarrow_+ p(y)$, $p(1_-) = 1_+$ and $p(0_-) = 0_+$),
- (iii) $n \cdot p = Id_{H_-}$ and $Id_{H_+} \leq_+ p \cdot n$.

The algebra $\mathbf{H}_+ \bowtie \mathbf{H}_- = \langle H_+ \times H_-, \wedge, \vee, \rightarrow, \sim, 0, 1 \rangle$ is defined as follows. For all $\langle a_+, a_- \rangle, \langle b_+, b_- \rangle \in H_+ \times H_-$,

$$\begin{aligned} 1 &= \langle 1_+, 0_- \rangle \\ 0 &= \langle 0_+, 1_- \rangle \\ \sim \langle a_+, a_- \rangle &= \langle p(a_-), n(a_+) \rangle \\ \langle a_+, a_- \rangle \wedge \langle b_+, b_- \rangle &= \langle a_+ \wedge_+ b_+, a_- \vee_- b_- \rangle \\ \langle a_+, a_- \rangle \vee \langle b_+, b_- \rangle &= \langle a_+ \vee_+ b_+, a_- \wedge_- b_- \rangle \\ \langle a_+, a_- \rangle \rightarrow \langle b_+, b_- \rangle &= \langle a_+ \rightarrow_+ b_+, n(a_+) \wedge_- b_- \rangle. \end{aligned}$$

A *twist-structure* \mathbf{A} over $\mathbf{H}_+ \bowtie \mathbf{H}_-$ is a $\{\wedge, \vee, \rightarrow, \sim, 0, 1\}$ -subalgebra of $\mathbf{H}_+ \bowtie \mathbf{H}_-$ with carrier set A satisfying $\pi_1(A) = H_+$ and $a_+ \wedge_+ p(a_-) = 0_+$ for all $\langle a_+, a_- \rangle \in A$ ⁸.

Using our terminology, one readily sees that the standard twist-structures corresponding to Nelson algebras (see e.g. [14]) are precisely those where the maps n and p are mutually inverse Heyting algebra isomorphisms.

⁷ The informed reader may recall that the pre-bilattice product construction used e.g. in [2] also employs two different algebras; however, no negation operator is defined on the resulting structure. It may be interesting to notice that the representation of semi-De Morgan algebras introduced in [6] – which has been an inspiration for us too – although not involving a product-like construction, is also based on the idea of representing a semi-De Morgan algebra as a quadruple consisting of two algebras related by maps that satisfy very similar requirements to ours.

⁸ We note that these conditions imply $n(a_+) \wedge_- a_- = 0_-$ for all $\langle a_+, a_- \rangle \in A$ and $\pi_2[A] = H_-$. The first holds because, since $a_+ \wedge_+ p(a_-) = 0_+$, we can apply n to both sides of the equation and, using its properties, we obtain $n(a_+ \wedge_+ p(a_-)) = n(a_+) \wedge_- n(p(a_-)) = n(a_+) \wedge_- a_- = 0_- = n(0_+)$ as required. Likewise, $\pi_2[A] = H_-$ follows from $\pi_1[A] = H_+$. In fact, for all $a_- \in H_-$, we know that $a_- = n(p(a_-))$, where $p(a_-) \in H_+$. Then $\pi_1[A] = H_+$ guarantees that there is $b_+ \in H_+$ such that $\langle p(a_-), b_+ \rangle \in A$, which means that $\sim \langle p(a_-), b_+ \rangle = \langle p(b_+), n(p(a_-)) \rangle = \langle p(b_+), a_- \rangle \in A$. Thus $a_- \in \pi_2[A]$ as required.

Because we want to endow a twist-structure \mathbf{A} with a quasi-Nelson residuated lattice structure, we need to define on \mathbf{A} a pair of operations $\langle *, \Rightarrow \rangle$ satisfying (Res) in Definition 2.1. We let:

$$x \Rightarrow y := (x \rightarrow y) \wedge (\sim y \rightarrow \sim x) \quad x * y := x \wedge y \wedge \sim(x \Rightarrow \sim y).$$

Notice that the definition of \Rightarrow is the same as for standard twist-structures, while that of $*$ is designed to take into account the failure of the identity $x * y = \sim(x \Rightarrow \sim y)$ in a non-involutive context. Consider $\langle a_+, a_- \rangle, \langle b_+, b_- \rangle \in A$. Doing the calculations, we immediately obtain:

$$\langle a_+, a_- \rangle \Rightarrow \langle b_+, b_- \rangle = \langle (a_+ \rightarrow_+ b_+) \wedge_+ (p(b_-) \rightarrow_+ p(a_-)), n(a_+) \wedge_- b_- \rangle.$$

It is more involved to check that the $*$ operation reduces to the following expression:

$$\langle a_+, a_- \rangle * \langle b_+, b_- \rangle = \langle a_+ \wedge_+ b_+, (n(a_+) \rightarrow_- b_-) \wedge_- (n(b_+) \rightarrow_- a_-) \rangle.$$

Proof. Direct calculation tells us that $\langle a_+, a_- \rangle * \langle b_+, b_- \rangle$ is equal to the following expression:

$$\langle a_+ \wedge_+ b_+ \wedge_+ p(n(a_+) \wedge_- n(b_+)), a_- \vee_- b_- \vee_- (n(a_+ \rightarrow_+ p(b_-)) \wedge_- n((pn(b_+) \rightarrow_+ p(a_-)))) \rangle.$$

Regarding the first component, notice that

$$a_+ \wedge_+ b_+ \wedge_+ p(n(a_+) \wedge_- n(b_+)) = a_+ \wedge_+ b_+$$

because, using $Id_{H_+} \leq_+ p \cdot n$ and the fact that n preserves \wedge_- , we obtain $a_+ \wedge_+ b_+ \leq_+ pn(a_+ \wedge_+ b_+) = p(n(a_+) \wedge_- n(b_+))$.

Regarding the second component, first of all observe that $n(a_+ \rightarrow_+ p(b_-)) = n(a_+) \rightarrow_- b_-$ and $n((pn(b_+) \rightarrow_+ p(a_-))) = n(b_+) \rightarrow_- a_-$. The latter is easily shown using the fact that p preserves the implication together with $n \cdot p = Id_{H_-}$, for $n((pn(b_+) \rightarrow_+ p(a_-))) = np(n(b_+) \rightarrow_- a_-) = n(b_+) \rightarrow_- a_-$. As to the former, the inequality $n(a_+ \rightarrow_+ p(b_-)) \leq_- n(a_+) \rightarrow_- b_-$ follows easily using residuation and the fact that n preserves meets: we have $n(a_+ \rightarrow_+ p(b_-)) \leq_- n(a_+) \rightarrow_- b_-$ iff $b_- \geq_- n(a_+) \wedge_- n(a_+ \rightarrow_+ p(b_-)) = n(a_+ \wedge_+ (a_+ \rightarrow_+ p(b_-))) = n(a_+ \wedge_+ p(b_-)) = n(a_+) \wedge_- np(b_-) = n(a_+) \wedge_- b_-$, once again recalling that $n \cdot p = Id_{H_-}$. As to the inequality $n(a_+) \rightarrow_- b_- \leq_- n(a_+ \rightarrow_+ p(b_-))$, we start from $pn(a_+) \rightarrow_+ p(b_-) \leq_+ a_+ \rightarrow_+ p(b_-)$ which holds because $Id \leq p \cdot n$, and since p preserves the implication, we have $pn(a_+) \rightarrow_+ p(b_-) = p(n(a_+) \rightarrow_- b_-) \leq_+ a_+ \rightarrow_+ p(b_-)$. We now apply n to both sides (using monotonicity of n) to get $np(n(a_+) \rightarrow_- b_-) = n(a_+) \rightarrow_- b_- \leq_+ n(a_+ \rightarrow_+ p(b_-))$, where the first equality holds because $n \cdot p = Id$. Putting these facts together, we have that

$$a_- \vee_- b_- \vee_- (n(a_+ \rightarrow_+ p(b_-)) \wedge_- n((pn(b_+) \rightarrow_+ p(a_-))))$$

is equal to

$$a_- \vee_- b_- \vee_- ((n(a_+) \rightarrow_- b_-) \wedge_- (n(b_+) \rightarrow_- a_-)).$$

We proceed to show that $a_- \vee_- b_- \leq_- (n(a_+) \rightarrow_- b_-) \wedge_- (n(b_+) \rightarrow_- a_-)$. Obviously (by integrality) we have $b_- \leq_- n(a_+) \rightarrow_- b_-$, moreover $a_- \leq_- n(a_+) \rightarrow_- b_-$ because the latter is equivalent, by residuation and the requirement that $a_- \wedge_- n(a_+) = 0_-$, to $0_- = a_- \wedge_- n(a_+) \leq_- b_-$. Thus $a_- \vee_- b_- \leq_- n(a_+) \rightarrow_- b_-$. Analogously we have $a_- \vee_- b_- \leq_- n(b_+) \rightarrow_- a_-$ because, on the one hand $a_- \leq_- n(b_+) \rightarrow_- a_-$ by integrality, on the other $b_- \leq_- n(b_+) \rightarrow_- a_-$ by residuation and the requirement that $b_- \wedge_- n(b_+) = 0_-$. Thus

$$a_- \vee_- b_- \vee_- ((n(a_+) \rightarrow_- b_-) \wedge_- (n(b_+) \rightarrow_- a_-))$$

is equal to $(n(a_+) \rightarrow_- b_-) \wedge_- (n(b_+) \rightarrow_- a_-)$ as required. \square

We proceed to check that any twist-structure \mathbf{A} can be endowed with a quasi-Nelson residuated lattice structure.

Proposition 3.2 *Let $\mathbf{A} = \langle A; \wedge, \vee, \rightarrow, \sim, 0, 1 \rangle$ be a twist-structure over $\mathbf{H}_+, \mathbf{H}_-$. Then:*

- (i) $\langle A, *, 1 \rangle$ is a commutative monoid and the pair $\langle *, \Rightarrow \rangle$ satisfies (Res) of Definition 2.1.
- (ii) \mathbf{A} satisfies the Nelson identity $(x \Rightarrow (x \Rightarrow y)) \wedge (\sim y \Rightarrow (\sim y \Rightarrow \sim x)) \approx x \Rightarrow y$.
- (iii) $\langle A; \wedge, \vee, \sim, 0, 1 \rangle$ is a lower quasi-De Morgan algebra and satisfies the Kleene identity $x \wedge \sim x \leq y \vee \sim y$.

Proof. (i). Commutativity of $*$ is immediate. Let $\langle a_+, a_- \rangle, \langle b_+, b_- \rangle \in H_+ \times H_-$. Let us check that $\langle 1_+, 0_- \rangle$ is the neutral element. We have $\langle 1_+, 0_- \rangle * \langle a_+, a_- \rangle = \langle a_+, a_- \rangle * \langle 1_+, 0_- \rangle = \langle a_+ \wedge_+ 1_+, (n(a_+) \rightarrow_- 0_-) \wedge_- (n(1_+) \rightarrow_- a_-) \rangle = \langle a_+, (n(a_+) \rightarrow_- 0_-) \wedge_- (1_- \rightarrow_- a_-) \rangle = \langle a_+, (n(a_+) \rightarrow_- 0_-) \wedge_- a_- \rangle = \langle a_+, a_- \rangle$. The last passage is justified by the fact that $a_- \leq_- n(a_+) \rightarrow_- 0_-$ holds (by residuation) iff $a_- \wedge_- n(a_+) \leq_- 0_-$ which is one of the requirement in the twist-structure definition. For associativity, we have, on the one hand, $(\langle a_+, a_- \rangle * \langle b_+, b_- \rangle) * \langle c_+, c_- \rangle = \langle a_+ \wedge_+ b_+, (n(a_+) \rightarrow_- b_-) \wedge_- (n(b_+) \rightarrow_- a_-) \rangle * \langle c_+, c_- \rangle = \langle a_+ \wedge_+ b_+ \wedge_+ c_+, (n(a_+ \wedge_+ b_+) \rightarrow_- c_-) \wedge_- (n(c_+) \rightarrow_- ((n(a_+) \rightarrow_- b_-) \wedge_- (n(b_+) \rightarrow_- a_-))) \rangle$. On the other hand, $\langle a_+, a_- \rangle * (\langle b_+, b_- \rangle * \langle c_+, c_- \rangle) = \langle a_+, a_- \rangle * \langle b_+ \wedge_+ c_+, (n(b_+) \rightarrow_- c_-) \wedge_- (n(c_+) \rightarrow_- b_-) \rangle = \langle a_+ \wedge_+ b_+ \wedge_+ c_+, (n(a_+) \rightarrow_- ((n(b_+) \rightarrow_- c_-) \wedge_- (n(c_+) \rightarrow_- b_-))) \wedge_- (n(b_+ \wedge_+ c_+) \rightarrow_- a_-) \rangle$. Thus we only need to check that the second components are equal. We have: $(n(a_+ \wedge_+ b_+) \rightarrow_- c_-) \wedge_- (n(c_+) \rightarrow_- ((n(a_+) \rightarrow_- b_-) \wedge_- (n(b_+) \rightarrow_- a_-))) = ((n(a_+) \wedge_- n(b_+)) \rightarrow_- c_-) \wedge_- (n(c_+) \rightarrow_- ((n(a_+) \rightarrow_- b_-) \wedge_- (n(b_+) \rightarrow_- a_-))) = (n(a_+) \rightarrow_- ((n(b_+) \rightarrow_- c_-))) \wedge_- (n(c_+) \rightarrow_- (n(a_+) \rightarrow_- b_-)) \wedge_- (n(c_+) \rightarrow_- (n(b_+) \rightarrow_- a_-)) = (n(a_+) \rightarrow_- ((n(b_+) \rightarrow_- c_-))) \wedge_- (n(a_+) \rightarrow_- (n(c_+) \rightarrow_- b_-)) \wedge_- (n(c_+) \rightarrow_- (n(b_+) \rightarrow_- a_-)) = (n(a_+) \rightarrow_- ((n(b_+) \rightarrow_- c_-) \wedge_- (n(c_+) \rightarrow_- b_-))) \wedge_- (n(c_+) \rightarrow_- (n(b_+) \rightarrow_- a_-)) = (n(a_+) \rightarrow_- ((n(b_+) \rightarrow_- c_-) \wedge_- (n(c_+) \rightarrow_- b_-))) \wedge_- (n(b_+ \wedge_+ c_+) \rightarrow_- a_-).$

Concerning (Res), we have that $\langle a_+, a_- \rangle * \langle b_+, b_- \rangle \leq \langle c_+, c_- \rangle$ means (1) $a_+ \wedge_+ b_+ \leq_+ c_+$ and $c_- \leq_- (n(a_+) \rightarrow_- b_-) \wedge_- (n(b_+) \rightarrow_- a_-)$, that is (2) $c_- \leq_- n(a_+) \rightarrow_- b_-$ and (3) $c_- \leq_- n(b_+) \rightarrow_- a_-$. On the other hand $\langle b_+, b_- \rangle \leq$

$\langle a_+, a_- \rangle \Rightarrow \langle c_+, c_- \rangle$ means $b_+ \leq_+ (a_+ \rightarrow_+ c_+) \wedge_+ (p(c_-) \rightarrow_+ p(a_-))$, that is (1') $b_+ \leq_+ a_+ \rightarrow_+ c_+$ and (3') $b_+ \leq_+ p(c_-) \rightarrow_+ p(a_-)$, and (2') $n(a_+) \wedge_- c_- \leq_- b_-$. The only non-trivial equivalence is the one between (3) and (3'). Assume (3) holds, that is $c_- \leq_- n(b_+) \rightarrow_- a_-$. Then by residuation we have $n(b_+) \leq_- c_- \rightarrow_- a_-$, and applying p to both sides, we get $pn(b_+) \leq_- p(c_- \rightarrow_- a_-) = p(c_-) \rightarrow_+ p(a_-)$. Since $b_+ \leq_+ pn(b_+)$, we have $b_+ \leq_+ pn(b_+) \leq_- p(c_-) \rightarrow_+ p(a_-)$ as required. Conversely, assume (3') holds, that is $b_+ \leq_+ p(c_-) \rightarrow_+ p(a_-)$, which gives us $p(c_-) \leq_+ b_+ \rightarrow_+ p(a_-)$ by residuation. Applying n to both sides, we have $np(c_-) = c_- \leq_- n(b_+ \rightarrow_+ p(a_-))$ because $n \cdot p = Id_{H_-}$. Since n preserves meets, we also have $n(b_+ \rightarrow_+ p(a_-)) \leq_- n(b_+) \rightarrow_- np(a_-) = n(b_+) \rightarrow_- a_-$, from which the desired result immediately follows.

(ii). By item (i) above, we have that A satisfies $x \Rightarrow (x \Rightarrow y) \approx (x * x) \Rightarrow y$. We shall use this fact to check that A satisfies $x \Rightarrow (x \Rightarrow y) \approx x \rightarrow y$, which easily implies the desired result. Let $\langle a_+, a_- \rangle, \langle b_+, b_- \rangle \in H_+ \times H_-$. We have

$$\begin{aligned}
 \langle a_+, a_- \rangle \Rightarrow (\langle a_+, a_- \rangle \Rightarrow \langle b_+, b_- \rangle) &= (\langle a_+, a_- \rangle * \langle a_+, a_- \rangle) \Rightarrow \langle b_+, b_- \rangle \\
 &= \langle a_+ \wedge_+ a_+, (n(a_+) \rightarrow_- a_-) \wedge_- (n(a_+) \rightarrow_- a_-) \rangle \Rightarrow \langle b_+, b_- \rangle \\
 &= \langle a_+, n(a_+) \rightarrow_- a_- \rangle \Rightarrow \langle b_+, b_- \rangle \\
 &= \langle (a_+ \rightarrow_+ b_+) \wedge_+ (p(b_-) \rightarrow_+ p(n(a_+) \rightarrow_- a_-)), n(a_+) \wedge_- b_- \rangle \\
 &= \langle a_+ \rightarrow_+ b_+, n(a_+) \wedge_- b_- \rangle
 \end{aligned}$$

where last passage uses $a_+ \rightarrow_+ b_+ \leq_+ p(b_-) \rightarrow_+ p(n(a_+) \rightarrow_- a_-)$, which we can justify as follows. First observe that $b_- \wedge_- n(b_+) = 0_-$ by definition of twist-structure, hence $b_- \wedge_- n(a_+) \wedge_- (n(a_+) \rightarrow_- n(b_+)) = b_- \wedge_- n(a_+) \wedge_- n(b_+) \leq_- b_- \wedge_- n(b_+) = 0_- \leq_- a_-$. Now $b_- \wedge_- n(a_+) \wedge_- (n(a_+) \rightarrow_- n(b_+)) \leq_- a_-$ is equivalent, by residuation, to $n(a_+) \rightarrow_- n(b_+) \leq_- b_- \rightarrow_- (n(a_+) \rightarrow_- a_-)$. Applying p on both sides (using monotonicity) we get $p(n(a_+) \rightarrow_- n(b_+)) \leq_+ p(b_- \rightarrow_- (n(a_+) \rightarrow_- a_-))$. Thus we have:

$$\begin{aligned}
 a_+ \rightarrow_+ b_+ &\leq_+ pn(a_+ \rightarrow_+ b_+) & Id_{H_+} &\leq_+ p \cdot n \\
 &\leq_+ p(n(a_+) \rightarrow_- n(b_+)) & n \text{ preserves meets} \\
 &\leq_+ p(b_- \rightarrow_- (n(a_+) \rightarrow_- a_-)) \\
 &= p(b_-) \rightarrow_+ p(n(a_+) \rightarrow_- a_-) & p \text{ preserves implication}
 \end{aligned}$$

as required.

(iii) We need to check that \mathbf{A} satisfies the items in Proposition 2.7. It is straightforward to check that $\langle A; \wedge, \vee, 0, 1 \rangle$ is a bounded distributive lattice: just notice that $\langle A; \wedge, \vee, 0, 1 \rangle$ is isomorphic to $\langle H_+, \wedge_+, \vee_+, 0_+, 1_+ \rangle \times \langle H_-, \wedge_-, \vee_-, 0_-, 1_- \rangle^\delta$, where $\langle H_-, \wedge_-, \vee_-, 0_-, 1_- \rangle^\delta$ denotes the order-theoretic dual of $\langle H_-, \wedge_-, \vee_-, 0_-, 1_- \rangle$. Joining this fact with item (i), we have that $\langle A; \wedge, \vee, *, \Rightarrow, 0, 1; \leq \rangle$ is a (distributive) CIBRL. Then items (v), (viii) and (xii) of Proposition 2.2 guarantee that items (ii), (v) and (iv), respectively, of Proposition 2.7 are satisfied. Item (iii) of Proposition 2.7, i.e. $\sim\sim(a \wedge b) = \sim\sim a \wedge \sim\sim b$, can be proven as follows. For

$\langle a_+, a_- \rangle, \langle b_+, b_- \rangle \in H_+ \times H_-$, we have:

$$\begin{aligned}
 \sim \sim (\langle a_+, a_- \rangle \wedge \langle b_+, b_- \rangle) &= \sim \sim \langle a_+ \wedge_+ b_+, a_- \vee_- b_- \rangle \\
 &= \langle pn(a_+ \wedge_+ b_+), np(a_- \vee_- b_-) \rangle \\
 &= \langle pn(a_+ \wedge_+ b_+), a_- \vee_- b_- \rangle & n \cdot p = Id_{H_-} \\
 &= \langle pn(a_+) \wedge_+ pn(b_+), a_- \vee_- b_- \rangle & p \text{ \& } n \text{ pre-} \\
 & & \text{serve meets} \\
 &= \langle pn(a_+) \wedge_+ pn(b_+), np(a_-) \vee_- np(b_-) \rangle & n \cdot p = Id_{H_-} \\
 &= \langle pn(a_+), np(a_-) \rangle \wedge \langle pn(b_+), np(b_-) \rangle \\
 &= \sim \sim \langle a_+, a_- \rangle \wedge \sim \sim \langle b_+, b_- \rangle.
 \end{aligned}$$

Finally, concerning the Kleene identity (last item of Proposition 2.7), we have

$$\begin{aligned}
 \langle a_+, a_- \rangle \wedge \sim \langle a_+, a_- \rangle &= \langle a_+, a_- \rangle \wedge \langle p(a_-), n(a_+) \rangle \\
 &= \langle a_+ \wedge_+ p(a_-), a_- \vee_- n(a_+) \rangle \\
 &= \langle 0_+, a_- \vee_- n(a_+) \rangle & a_+ \wedge_+ p(a_-) = 0_+ \\
 &\leq \langle b_+ \vee_+ p(b_-), 0_- \rangle \\
 &= \langle b_+ \vee_+ p(b_-), b_- \wedge_- n(b_+) \rangle & b_- \wedge_- n(b_+) = 0_- \\
 &= \langle b_+, b_- \rangle \vee \langle p(b_-), n(b_+) \rangle \\
 &= \langle b_+, b_- \rangle \vee \sim \langle b_+, b_- \rangle.
 \end{aligned}$$

□

From the preceding proposition we immediately obtain the following:

Theorem 3.3 *Let $\mathbf{A} = \langle A; \wedge, \vee, \rightarrow, \sim, 0, 1 \rangle$ be a twist-structure over $\mathbf{H}_+, \mathbf{H}_-$. Then, upon defining $x \Rightarrow y := (x \rightarrow y) \wedge (\sim y \rightarrow \sim x)$ and $x * y := x \wedge y \wedge \sim(x \Rightarrow \sim y)$, the algebra $\langle A; \wedge, \vee, *, \Rightarrow, 0, 1 \rangle$ is a quasi-Nelson residuated lattice.*

In turn, from the preceding theorem and Proposition 2.5 we obtain:

Corollary 3.4 *Let $\mathbf{A} = \langle A; \wedge, \vee, \rightarrow, \sim, 0, 1 \rangle$ be a twist-structure over $\mathbf{H}_+, \mathbf{H}_-$. Then $\langle A; \wedge, \vee, \rightarrow, \sim, 0, 1 \rangle$ is a quasi-Nelson algebra.*

4 Quasi-Nelson algebras

In this section we introduce the second abstract presentation for our class of algebras, and show equivalence with the two above-introduced one.

Definition 4.1 A *quasi-Nelson algebra* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \rightarrow, \sim, 0, 1 \rangle$ of type $\langle 2, 2, 2, 1, 0, 0 \rangle$ satisfying the following properties:

- (SN1) The reduct $\langle A; \wedge, \vee, 0, 1 \rangle$ is a bounded distributive lattice (with order \leq).
- (SN2) The relation \preceq on A defined for all $a, b \in A$ by $a \preceq b$ iff $a \rightarrow b = 1$ is a quasiorder on A .

- (SN3) The relation $\equiv := \preceq \cap (\preceq)^{-1}$ is a congruence on the reduct $\langle A; \wedge, \vee, \rightarrow, 0, 1 \rangle$ and the quotient algebra $\mathbf{A}_+ = \langle A; \wedge, \vee, \rightarrow, 0, 1 \rangle / \equiv$ is a Heyting algebra.
- (SN4) For all $a, b \in A$, it holds that $\sim(a \rightarrow b) \equiv \sim(a \wedge \sim b)$.
- (SN5) For all $a, b \in A$, it holds that $a \leq b$ iff $a \preceq b$ and $\sim b \preceq \sim a$.
- (SN6) For all $a, b \in A$,
- (SN6.1) $\sim \sim (\sim a \rightarrow \sim b) \equiv \sim a \rightarrow \sim b$
 - (SN6.2) $\sim(a \vee b) \equiv \sim a \wedge \sim b$
 - (SN6.3) $\sim \sim a \wedge \sim \sim b \equiv \sim \sim (a \wedge b)$
 - (SN6.4) $\sim a \equiv \sim \sim \sim a$
 - (SN6.5) $a \preceq \sim \sim a$
 - (SN6.6) $a \wedge \sim a \preceq 0$.

The above definition is obviously a generalization of Rasiowa's presentation of Nelson algebras [15, Ch. V, p. 68] as well as of Odintsov's definition of *N4-lattices* [13, Definition 5.1]. The careful reader may have guessed that certain items must actually hold in a stronger version (we shall obtain a proof of this as a consequence of the announced Theorem 4.4): for example, the $\langle A; \wedge, \vee, \sim, 0, 1 \rangle$ -reduct any quasi-Nelson algebra \mathbf{A} is a lower quasi-De Morgan algebra, which immediately entails that items (SN6.2), (SN6.3), (SN6.4) could be reformulated as equalities, and (SN6.5) as an inequality (on the other hand, (SN4), (SN6.1) and (SN6.6) would in general fail as equalities). We have avoided doing so here because it would make our subsequent proofs (in particular Proposition 2.5) much more involved.

Let $\mathbf{A} = \langle A, \wedge, \vee, \rightarrow, \sim, 0, 1; \preceq \rangle$ be a quasi-Nelson algebra. Then (SN3) guarantees that we have a Heyting algebra quotient $\mathbf{A}_+ = \langle A_+; \wedge_+, \vee_+, \rightarrow_+, 0_+, 1_+ \rangle$. In order to have all the ingredients for the twist-structure construction, we shall need a second Heyting algebra (to be called \mathbf{A}_-) as well as maps connecting both. Let $[b]$ denote the equivalence class of each $b \in A$ modulo \equiv , and consider the set $A_- := \{[\sim a] : a \in A\} \subseteq A_+$. We can endow A_- with Heyting algebra operations as follows. For all $a, b \in A$, let

$$[\sim a] \wedge_- [\sim b] := [\sim(a \vee b)] \quad (= [\sim a \wedge \sim b] = [\sim a] \wedge_+ [\sim b], \text{ by Definition 4.1 (SN6.2)})$$

$$[\sim a] \vee_- [\sim b] := [\sim(a \wedge b)]$$

$$[\sim a] \rightarrow_- [\sim b] := [\sim \sim (\sim a \rightarrow \sim b)] \quad (= [\sim a \rightarrow \sim b] = [\sim a] \rightarrow_+ [\sim b]), \text{ by Definition 4.1 (SN6.1)}$$

$$0_- := [\sim 1] \quad (= [0] = 0_+)$$

$$1_- := [\sim 0] \quad (= [1] = 1_+).$$

The set A_- is clearly the universe of a $\langle \wedge_+, \rightarrow_+, 0_+, 1_+ \rangle$ -subalgebra of A_+ . In other words, the algebra $\langle A_-, \wedge_-, \rightarrow_-, 0_-, 1_- \rangle$ is a *bounded implicative meet-semilattice*⁹. Notice that \vee_- , though not coinciding with the restriction of \vee_+ to A_- , is nevertheless a join for \wedge_- in \mathbf{A}_- ; hence \mathbf{A}_- forms a Heyting algebra in its

⁹ This is one of the terms used in the literature to designate the class of $\langle \wedge, \rightarrow, 0, 1 \rangle$ -subreducts of Heyting algebras.

own right, though not a Heyting subalgebra of \mathbf{A}_+ . To check this, observe that \vee_- inherits idempotency, associativity and commutativity from \wedge . It remains to verify the absorption law, which we can do as follows: $[\sim a] \vee_- ([\sim a] \wedge_- [\sim b]) = [\sim a] \vee_- [\sim(a \vee b)] = [\sim(a \wedge (a \vee b))] = [\sim a]$ and $[\sim a] \wedge_- ([\sim a] \vee_- [\sim b]) = [\sim a] \wedge_- [\sim(a \wedge b)] = [\sim(a \vee (a \wedge b))] = [\sim a]$.

We proceed to define maps $p: A_- \rightarrow A_+$ and $n: A_+ \rightarrow A_-$ as follows. Let p be the identity map on A_- , and let $n[a] := [\sim \sim a]$ for all $a \in A$. Obviously p preserves the bounds, meets and the implication. On the other hand, p does not necessarily preserve joins, because in general we have $p([\sim a] \vee_- [\sim b]) = p[\sim(a \wedge b)] = [\sim(a \wedge b)] \neq [\sim a \vee \sim b] = [\sim a] \vee_+ [\sim b] = p[\sim a] \vee_+ p[\sim b]$.

The map n preserves the bounds, meets and joins. Indeed, using Definition 4.1 (SN6.3) and (SN6.2), we have $n([a] \wedge_+ [b]) = n[a \wedge b] = [\sim \sim(a \wedge b)] = [\sim \sim a \wedge \sim \sim b] = [\sim(\sim a \vee \sim b)] = [\sim \sim a] \wedge_- [\sim \sim b] = n[a] \wedge_- n[b]$ and $n([a] \vee_+ [b]) = n[a \vee b] = [\sim \sim(a \vee b)] = [\sim(\sim a \wedge \sim b)] = [\sim \sim a] \vee_- [\sim \sim b] = n[a] \vee_- n[b]$. On the other hand, n does not necessarily preserve the implication, because in general $n([a] \rightarrow_+ [b]) = n[a \rightarrow b] = [\sim \sim(a \rightarrow b)] \neq [\sim \sim a \rightarrow \sim \sim b] = [\sim \sim(\sim \sim a \rightarrow \sim \sim b)] = [\sim \sim a] \rightarrow_- [\sim \sim b] = n[a] \rightarrow_- n[b]$.

Let us check that $n \cdot p = Id_{A_-}$ and $Id_{A_+} \leq_+ p \cdot n$. As to the former, by Definition 4.1 (SN6.4), we have $(n \cdot p)[\sim a] = n[\sim a] = [\sim \sim \sim a] = [\sim a]$. As to the latter, observe that $(p \cdot n)[a] = p[\sim \sim a] = [\sim \sim a]$ and we have that $[a] \leq_+ [\sim \sim a]$ iff $[a] \rightarrow_+ [\sim \sim a] = 1_+$ iff $a \preceq \sim \sim a$, which is guaranteed by Definition 4.1 (SN6.5).

Thus we have two Heyting algebras \mathbf{A}_+ , \mathbf{A}_- and maps n, p between them which satisfy the conditions of Definition 3.1. This allows us to build twist-structures over \mathbf{A}_+ , \mathbf{A}_- .

Proposition 4.2 *Every quasi-Nelson algebra \mathbf{A} is isomorphic to a twist-structure over \mathbf{A}_+ , \mathbf{A}_- by the map $\iota(a) := \langle [a], [\sim a] \rangle$.*

Proof. Injectivity of the map ι is guaranteed by Definition 4.1 (SN5). Let us check that ι is a homomorphism, i.e. that each algebraic operation is preserved (we omit the case corresponding to the bounds, which is straightforward). We have:

$$\begin{aligned} \iota(\sim a) &= \langle [\sim a], [\sim \sim a] \rangle \\ &= \langle p[\sim a], n[a] \rangle && \text{by definition of } p \text{ and } n \\ &= \langle [a], [\sim a] \rangle \\ &= \sim \iota(a). \end{aligned}$$

$$\begin{aligned} \iota(a \wedge b) &= \langle [a \wedge b], [\sim(a \wedge b)] \rangle \\ &= \langle [a] \wedge_+ [b], [\sim a] \vee_- [\sim b] \rangle && \text{by definition of } \vee_- \\ &= \langle [a], [\sim a] \rangle \wedge \langle [b], [\sim b] \rangle \\ &= \iota(a) \wedge \iota(b). \end{aligned}$$

$$\begin{aligned}
\iota(a \vee b) &= \langle [a \vee b], [\sim(a \vee b)] \rangle \\
&= \langle [a] \vee_+ [b], [\sim a] \wedge_- [\sim b] \rangle && \text{by definition of } \wedge_- \\
&= \langle [a], [\sim a] \rangle \vee \langle [b], [\sim b] \rangle \\
&= \iota(a) \vee \iota(b).
\end{aligned}$$

$$\begin{aligned}
\iota(a \rightarrow b) &= \langle [a \rightarrow b], [\sim(a \rightarrow b)] \rangle \\
&= \langle [a \rightarrow b], [\sim \sim(a \wedge \sim b)] \rangle && \text{by Definition 4.1 (SN4)} \\
&= \langle [a \rightarrow b], [\sim \sim a \wedge \sim \sim b] \rangle && \text{by Definition 4.1 (SN6.3)} \\
&= \langle [a \rightarrow b], [\sim \sim a] \wedge_- [\sim b] \rangle && \text{by Definition 4.1 (SN6.2)} \\
&= \langle ([a] \rightarrow_+ [b]), n[a] \wedge_- [\sim b] \rangle && \text{by definition of } n \\
&= \langle [a], [\sim a] \rangle \rightarrow \langle [b], [\sim b] \rangle \\
&= \iota(a) \rightarrow \iota(b).
\end{aligned}$$

Obviously we have that $\pi_1(A) = A_+$ and $\pi_2(A) = A_-$. Finally, observe that, by Definition 4.1 (SN6.6), for every $\langle [a], [\sim a] \rangle \in A_+ \times A_-$, we have $[a] \wedge_+ p[\sim a] = [a] \wedge_+ [\sim a] = [a \wedge \sim a] = [0] = 0_+$. Similarly, using also Definition 4.1 (SN6.2), we have $[\sim a] \wedge_- n[a] = [\sim a] \wedge_- [\sim \sim a] = [\sim(a \vee \sim a)] = [\sim a \wedge \sim \sim a] = [0] = 0_-$. Therefore the last requirement of Definition 3.1 is also met. \square

Joining Proposition 4.2 above with Theorem 3.3, we immediately obtain the following.

Corollary 4.3 *Let $\mathbf{A} = \langle A, \wedge, \vee, \rightarrow, \sim, 0, 1 \rangle$ be a quasi-Nelson algebra. Then, upon defining $x \Rightarrow y := (x \rightarrow y) \wedge (\sim y \rightarrow \sim x)$ and $x * y := x \wedge y \wedge \sim(x \Rightarrow \sim y)$, the algebra $\langle A; \wedge, \vee, *, \Rightarrow, 0, 1 \rangle$ is a quasi-Nelson residuated lattice.*

We are thus in a position to state the main result of our paper.

Theorem 4.4 *The following classes of algebras are term equivalent:*

- (i) *Quasi-Nelson residuated lattices, i.e. commutative integral bounded residuated lattices satisfying the (Nelson) identity (Definition 2.3).*
- (ii) *Twist-structures over pairs of Heyting algebras (Definition 3.1).*
- (iii) *Quasi-Nelson algebras (Definition 4.1).*

5 Future work

As mentioned in the abstract, we hope and believe that the main impact of the present paper will lie in the possibility of opening new ways to, in the first place, obtain deeper insights into the distinguishing feature of Nelson's logic – into the meaning of the Nelson axiom/identity – and its algebraic counterpart; in the second place, to investigate certain properties that are of independent algebraic interest in a more general setting than has been done up to now.

For instance, the reader (as well as the authors) of [19] might have been under the impression that a number of key properties of Nelson residuated lattices –

distributivity, 3-potency, (0,1)-congruence orderability, as well as admitting a twist-structure representation – were implied by/equivalent to the Nelson identity only in the presence of the double negation law. We now know that this is not the case, and that the essential features of (the algebraic counterpart of) Nelson’s logic are retained in the non-involutive setting.

These considerations lead to several questions that may be worth exploring in the course of future research. For example, we may ask in which contexts the Nelson identity does *not* imply distributivity, 3-potency, (0,1)-congruence orderability – the most obvious classes to start looking at being, of course, non-commutative and/or non-integral residuated lattices; accordingly, we may ask whether it is possible and worthwhile to introduce and investigate classes such as “non-distributive (non-3-potent, etc.) Nelson residuated lattices”. Also, in the cases where an equivalence does not seem to be preserved (e.g., supposing that in a non-involutive setting the identity $x \approx x^2 \vee (x \wedge \sim x)$ of Proposition 2.4 does not imply the Nelson identity), one would like to obtain some insight into the structural differences between the various resulting classes of algebras.

Finally, another development that will clearly be needed in order to complement the results presented here is to explicitly bring logic into the picture: namely, define a deductive system corresponding to quasi-Nelson algebras, and verify its algebraizability with respect to the intended semantics.

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