

# A Characterization of Claw-free CIS Graphs and New Results on the Order of CIS Graphs

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## Abstract

A graph is CIS if every maximal clique intersects every maximal stable set. Currently, no good characterization or recognition algorithm for the CIS graphs is known. We characterize graphs in which every maximal matching saturates all vertices of degree at least two and use this result to give a structural, efficiently testable characterization of claw-free CIS graphs. We answer in the negative a question of Dobson, Hujdurović, Milanič, and Verret [*Vertex-transitive CIS graphs*, European J. Combin. 44 (2015) 87–98] asking whether the number of vertices of every CIS graph is bounded from above by the product of its clique and stability numbers. On the positive side, we show that the question of Dobson et al. has an affirmative answer in the case of claw-free graphs.

**Keywords:** CIS graph, maximal clique, maximal stable set, maximal independent set, randomly internally matchable graph, claw-free graph.

## 1 Introduction

Many graph classes can be defined in terms of properties of cliques or stable sets in a graph (see, e.g., [7, 16]). In this paper we continue the investigation of *CIS graphs*, defined as graphs in which every maximal clique intersects every maximal stable set. Here, ‘maximality’ refers to maximality under inclusion. CIS graphs were studied in a series of papers [1, 2, 7, 8, 11, 12, 14, 25, 27] under different names; the name CIS (Cliques Intersect Stable sets) was suggested by Andrade et al. [2]. Currently, no good characterization or recognition algorithm for the CIS graphs is known. Recognizing CIS graphs was believed to be co-NP-complete [27], conjectured to be co-NP-complete [28], and conjectured to be polynomial [2]. The difficulty of understanding the structure of CIS graphs is perhaps related to the fact that the class of CIS graphs is not closed under vertex deletion. For example, the *bull*, that is, is the graph with vertex set  $\{v_1, \dots, v_5\}$  and edge set  $\{v_1v_2, v_2v_3, v_3v_4, v_2v_5, v_3v_5\}$ , is a CIS graph, while deleting vertex  $v_5$  from the bull yields the 4-vertex path, which is not CIS.

Some partial results are known regarding the CIS property in particular graph classes. The class of CIS graphs generalizes the class of  $P_4$ -free graphs, also known as *cographs* [8, 10]. Polynomially testable characterizations of the CIS property in the classes of planar graphs and of line graphs were given by Sun and Hu [25] and by Boros et al. [7], respectively. Vertex-transitive CIS graphs were characterized by Dobson et al. [12] and by Hujdurović [15]. Furthermore, Dobson et al. proved that vertex-transitive CIS graphs share the well-known property of perfect graphs [20] stating that the number of vertices of the graph is bounded from above by the product of its clique number and stability number. They asked whether the property holds for all CIS graphs.

A notion closely related to CIS graphs is that of a strong clique. A clique in a graph  $G$  is said to be *strong* if it has non-empty intersection with every maximal stable set of  $G$ . Thus, a graph is CIS if and only if every maximal clique is strong. A clique is *simplicial* if it consists of some vertex and all its neighbors. It is not difficult to see that every strong clique is maximal and every simplicial clique is strong. Hujdurović et al. [16] showed that a clique in a  $C_4$ -free graph is strong if and only if it is simplicial, which leads to a polynomially testable characterization of CIS  $C_4$ -free graphs. The concept of strong clique gives rise to several other interesting graph properties studied in the literature (see, e.g., [7, 16, 17, 23]).

**Our results.** Our results consist of two interrelated parts. First, we give a structural characterization of claw-free CIS graphs, by proving a composition theorem for this class of graphs (Theorem 4.5). This leads to a polynomial-time recognition algorithm for the CIS property in the class of claw-free graphs (Corollary 4.7). The result is derived using a characterization of graphs in which every maximal matching saturates all vertices of degree at least two (Theorem 3.3), a result related to Sumner's characterization of randomly matchable graphs [24] that might be of some independent interest.

Second, we answer in the negative the question of Dobson et al. [12] asking whether the number of vertices of a CIS graph  $G$  is necessarily bounded from above by the product of its stability number,  $\alpha(G)$ , and clique number,  $\omega(G)$ . More precisely, using triangle-free graphs of small stability number [18], we construct a sequence of CIS graphs showing that even the relation  $|V(G)| = \mathcal{O}(\alpha(G)\omega(G))$  fails for general CIS graphs (Theorem 5.1). On the positive side, we show that the question of Dobson et al. has an affirmative answer in the case of claw-free graphs (Theorem 5.4).

**Structure of the paper.** In Section 2 we collect the necessary notations and preliminary results. In Section 3 we characterize graphs in which every maximal matching saturates all vertices of degree at least two. In Section 4 we prove the structural characterization of claw-free CIS graph. In Section 5 we construct a family of counterexamples to the question of Dobson et al. and study the question in the case of claw-free graphs. We conclude the paper in Section 6 by posing a question left open by our work, namely whether the Erdős-Hajnal property holds for the class of CIS graphs. Due to lack of space, proofs of Lemma 4.2 and Corollary 4.7 are omitted.

## 2 Preliminaries

We consider finite, undirected, and non-null graphs only. Unless specified otherwise by using the term *multigraph*, all our graphs will be simple, that is, without loops or multiple edges. A graph  $G = (V, E)$  has vertex set  $V(G) = V$  and edge set  $E(G) = E$ . The *order* of  $G$  is  $|V|$ . Given  $S \subseteq V(G)$ , the *subgraph induced by*  $S$  in  $G$  is denoted by  $G[S]$  and defined as the graph with vertex set  $S$  and edge set  $\{\{x, y\} \mid \{x, y\} \in E; x, y \in S\}$ . The *complement*  $\overline{G}$  of a graph  $G = (V, E)$  is the graph with vertex-set  $V(\overline{G}) = V$  and the edge-set  $E(\overline{G}) = \{\{x, y\} \mid x, y \in V, x \neq y, \text{ and } \{x, y\} \notin E\}$ . We say that  $G$  is *co-connected* if its complement is connected. A *co-component* of  $G$  is the subgraph of  $G$  induced by the vertex set of a (connected) component of  $\overline{G}$ . The *neighborhood* of a vertex  $v \in V(G)$  is the set  $N_G(v)$  of vertices adjacent to  $v$ ; its *closed neighborhood* is the set  $N_G[v]$ , defined as  $N_G[v] = N_G(v) \cup \{v\}$ . The cardinality of  $N_G(v)$  is the *degree* of  $v$ , denoted by  $d_G(v)$ . A *universal vertex* in a graph  $G$  is a vertex of degree  $|V(G)| - 1$ . We denote by  $\delta(G)$  the minimum degree of a vertex in  $G$ . For a set  $S \subseteq V(G)$ , we let  $N_G(S)$  be the set of all vertices not in  $S$  having a neighbor in  $S$ .

As usual, we denote the  $n$ -vertex complete graph, path graph, and cycle graph by  $K_n$ ,  $P_n$ , and  $C_n$ , respectively. The graph  $K_3$  will be also referred to as the *triangle*. By  $K_{m,n}$  we denote the complete bipartite graph with parts of the bipartition of sizes  $m$  and  $n$ . The *claw* is the complete bipartite graph  $K_{1,3}$ . The fact that a graph  $G$  is isomorphic to a graph  $H$  will be denoted by  $G \cong H$ . We say that  $G$  is *H-free* if no induced subgraph of  $G$  is isomorphic to  $H$ . Furthermore, given a set  $\mathcal{F}$  of graphs, we say that a graph  $G$  is  *$\mathcal{F}$ -free* if  $G$  is  $F$ -free for all  $F \in \mathcal{F}$ . Given two vertex-disjoint graphs  $G$  and  $H$ , we denote by  $G+H$  their *disjoint union*, that is, the graph with vertex set  $V(G) \cup V(H)$  and edge set  $E(G) \cup E(H)$ . For a non-negative integer  $k$ , we denote by  $kG$  the graph consisting of  $k$  disjoint copies of  $G$ .

A *clique* in a graph is a set of pairwise adjacent vertices; a *stable set* (or *independent set*) is a set of pairwise non-adjacent vertices. We say that a clique (resp., stable set) is *maximal* if it is inclusion-maximal, that is, if it is not contained in any larger clique (resp., stable set). Given a graph  $G$ , its *stability number* (or *independence number*) is denoted by  $\alpha(G)$  and defined as the maximum size of a stable set in  $G$ ; furthermore, its *clique number* is denoted by  $\omega(G)$  and defined as the maximum size of a clique in  $G$ . A *matching* in a graph  $G$  is a set of pairwise disjoint edges. Given a matching  $M$  and a vertex  $v$ , we say that  $M$  *saturates*  $v$  if  $M$  contains an edge having  $v$  as an endpoint. We will sometimes abuse this terminology and simply say that “ $v$  is in  $M$ ” if  $M$  saturates  $v$ . A matching is *perfect* if it saturates all vertices of the graph. An *internal vertex* in a graph  $G$  is a vertex of degree at least two. We say that a matching  $M$  in a graph  $G$  is a *perfect internal matching* if it saturates all internal vertices of  $G$ , that is, if every vertex not in  $M$  is of degree at most 1. Perfect internal matchings were studied in a series of papers, see, e.g., [3–6]. For undefined graph terminology and notation, we refer to [26].

## 2.1 Preliminaries on line graphs of multigraphs

Given a multigraph  $H$ , its *line graph* is the simple graph  $L(H)$  with vertex set  $E(H)$  in which two distinct vertices  $e$  and  $e'$  are adjacent if and only if  $e$  and  $e'$  have a common endpoint as edges in  $H$ . Clearly, if  $G$  is the line graph of a multigraph  $H$ , then there exists a multigraph  $H'$  without loops such that  $G$  is (isomorphic to) the line graph of  $H'$ . Such a multigraph  $H'$  can be obtained from  $H$  by replacing every loop in  $H$  joining  $v$  with itself with a pendant edge joining  $v$  with a new vertex. Hence, we may assume without loss of generality that all line graphs considered are line graphs of loopless multigraphs. Any loopless multigraph  $H$  can be equivalently represented with a simple graph and a multiplicity function on the edges. Indeed, for every edge  $e \in E(H)$ , let us denote by  $w(e)$  the *multiplicity* of  $e$  in  $H$ , that is, the number of edges of  $H$  with the same endpoints as  $e$ . Then,  $H$  can be equivalently represented with the simple graph  $\tilde{H}$  having vertex set  $V(H)$  and edge set obtained from  $E(H)$  by keeping only one representative edge from each class of multiple edges, together with the restriction of the multiplicity function  $w$  to the edges of  $\tilde{H}$ .

A *weighted graph* is a pair  $(H, w)$  where  $H = (V, E)$  is a graph and  $w : E \rightarrow \mathbb{N}$  is a weight function.<sup>1</sup> Interpreting  $w$  as the multiplicity function of the edges, we see that every weighted graph  $(H, w)$  corresponds to a loopless multigraph. Accordingly, we let  $L(H, w)$  denote the *line graph* of  $(H, w)$ , this is the line graph of the multigraph obtained from  $H$  by replacing each edge  $e \in E(H)$  with  $w(e)$  parallel edges.

## 2.2 Preliminaries on CIS graphs

For  $k \geq 2$ , a  $k$ -*comb* is a graph  $F_k$  with  $2k$  vertices  $v_1, \dots, v_k, w_1, \dots, w_k$  such that  $C = \{v_1, \dots, v_k\}$  is a clique,  $v_i$  is adjacent to  $w_i$  for all  $i \in \{1, \dots, k\}$ , and there are no other edges. In particular,  $S = \{w_1, \dots, w_k\}$  is a stable set, which shows that  $F_k$  is a split graph with a unique split partition  $(C, S)$ ; moreover,  $F_k$  is not a CIS graph since  $(C, S)$  is a disjoint pair of a maximal clique and maximal stable set. A  $k$ -*anticomb* is the graph  $\overline{F_k}$ , the complement of a  $k$ -comb.

An induced  $k$ -comb  $F_k$  in a graph  $G$  is said to be *settled* if there exists a vertex  $v \in V(G) \setminus V(F_k)$  that is adjacent to every vertex of  $C$  and non-adjacent to every vertex of  $S$ , where  $(C, S)$  is the unique split partition of  $F_k$ . Similarly, an induced  $k$ -anticomb  $\overline{F_k}$  in a graph  $G$  with the split partition  $(C, S)$  is said to be *settled* if there exists a vertex  $v \in V(G) \setminus V(\overline{F_k})$  that is adjacent to every vertex of  $C$  and non-adjacent to every vertex of  $S$ . The following lemma describes a necessary (though in general not sufficient) condition for CIS graphs.

**Lemma 2.1 (Andrade et al. [2])** *If  $G$  is CIS, then every  $k$ -comb is settled and every  $k$ -anticomb is settled.*

Two vertices  $x, y$  in a graph  $G$  are said to be *true twins* if  $N_G[x] = N_G[y]$ . Consider the equivalence relation  $\sim$  defined on the vertex set of  $G$  by the rule  $x \sim y$

<sup>1</sup> We denote by  $\mathbb{N}$  the set of all (strictly) positive integers.

if and only if  $x$  and  $y$  are true twins. The *true-twin reduction* of  $G$  is the graph obtained from  $G$  by contracting each equivalence class of the equivalence relation  $\sim$  (which is a clique) into a single vertex. A graph is said to be *true-twin-free* if it coincides with its true-twin reduction. For later use, we recall the following useful property of CIS graphs (see, e.g., [2, 8]).

**Lemma 2.2** *A graph  $G$  is CIS if and only if the true-twin reduction of each component of  $G$  is CIS.*

Next, we recall a characterization of CIS line graphs (of simple graphs) due to Boros et al. [7]. The characterization relies on the following concept related to perfect internal matchings. We say that a maximal matching  $M$  in a graph  $H$  is *absorbing* if every vertex not in  $M$  sees at most one edge of  $M$ , or, more formally, if for every vertex  $v \in V(H)$  that is not saturated by  $M$ , there exists an edge  $e$  in  $M$  such that every neighbor of  $v$  is an endpoint of  $e$ . (In particular, this implies that  $v$  is of degree at most two in  $H$ .) Note that if  $H$  has an edge, then every maximal matching that is a perfect internal matching is absorbing.

**Theorem 2.3 (Boros et al. [7])** *Let  $H$  be a graph without isolated vertices and let  $G = L(H)$ . Then  $G$  is CIS if and only if  $H$  has no subgraph isomorphic to a bull and every maximal matching in  $H$  is absorbing.*

### 3 Randomly internally matchable graphs

A graph  $G$  is *randomly matchable* if every matching of  $G$  can be extended to a perfect matching, or, equivalently, if every maximal matching of  $G$  is perfect. Clearly, a graph  $G$  is randomly matchable if and only if each component of  $G$  is randomly matchable. Therefore, the following theorem due to Sumner completely characterizes the randomly matchable graphs.

**Theorem 3.1 (Sumner [24])** *A connected graph  $G$  is randomly matchable if and only if  $G \cong K_{2n}$  or  $G \cong K_{n,n}$  for some  $n \geq 1$ .*

The concept of perfect internal matchings naturally leads to the following generalization of randomly matchable graphs. We say that a graph  $G$  is *randomly internally matchable* if every matching of  $G$  can be extended to a perfect internal matching, or, equivalently, if every maximal matching of  $G$  is a perfect internal matching. Using this terminology, we note, for later use, the following consequence of Theorem 2.3.

**Corollary 3.2** *Let  $H$  be a triangle-free graph without isolated vertices and let  $G = L(H)$ . Then  $G$  is CIS if and only if  $H$  is randomly internally matchable.*

**Proof.** Immediately from Theorem 2.3, using the fact that if  $H$  is triangle-free, then  $H$  has no subgraph isomorphic to a bull and a maximal matching in  $H$  is absorbing if and only if it is a perfect internal matching.  $\square$

Clearly, a graph  $G$  is randomly internally matchable if and only if each component of  $G$  is randomly internally matchable. In the next theorem, we

characterize the connected randomly internally matchable graphs. A *leaf* in a graph is a vertex of degree one. A *leaf extension* of a graph  $G$  is any graph obtained from  $G$  by adding for each vertex  $v \in V(G)$  a non-empty set  $L_v$  of pairwise non-adjacent new vertices joined to  $v$  by an edge.

**Theorem 3.3** *A connected graph  $G$  is randomly internally matchable if and only if  $G \cong K_{2n}$  for some  $n \geq 1$ ,  $G \cong K_{n,n}$  for some  $n \geq 1$ , or  $G$  is a leaf extension of some graph.*

**Proof.** If  $G \cong K_{2n}$  or  $G \cong K_{n,n}$  for some  $n \geq 1$ , then  $G$  is randomly matchable and therefore also randomly internally matchable. Suppose now that  $G$  is a leaf extension of a graph  $G'$ . Let  $M$  be a maximal matching in  $G$  and let  $v \in V(G)$  be an internal vertex of  $G$ . Then,  $v \in V(G')$  and there exists a vertex  $v'$  of degree one in  $G$  such that  $vv' \in E(G)$ . If  $v$  is not  $M$ -saturated, then neither is  $v'$  and hence  $M \cup \{vv'\}$  is a matching properly containing  $M$ , contrary to the maximality of  $M$ . Therefore,  $v$  is in  $M$  and, since  $v$  and  $M$  were arbitrary,  $G$  is randomly internally matchable.

Suppose now that  $G$  is a connected randomly internally matchable graph. Clearly,  $G$  has at least two vertices and hence  $\delta(G) \geq 1$ . Suppose first that  $\delta(G) \geq 2$ . Then all vertices of  $G$  are internal and hence  $G$  is randomly matchable. By Theorem 3.1,  $G \cong K_{2n}$  or  $G \cong K_{n,n}$  for some  $n \geq 2$ . Suppose now that  $\delta(G) = 1$ . We may assume that  $G$  has at least three vertices, since otherwise  $G \cong K_2$  and we are done. In particular, for every leaf of  $G$ , its unique neighbor is an internal vertex. Let  $L$  denote the set of all leaves in  $G$ , let  $S = N_G(L)$  denote the set of all neighbors of leaves in  $G$ , and let  $R = V(G) \setminus (L \cup S)$ . If  $R = \emptyset$ , then  $G$  is a leaf extension of  $G[S]$  and we are done. So we may assume that  $R \neq \emptyset$ . For every  $s \in S$ , fix a vertex  $s' \in L$  adjacent to  $s$  and let  $M_S = \{ss' \mid s \in S\}$ . Let  $M_R$  be a maximal matching of the graph  $G[R]$ . Then,  $M_R \cup M_S$  is a maximal matching in  $G$ . Since the set of internal vertices of  $G$  is precisely  $S \cup R$  and  $G$  is randomly internally matchable,  $M_R \cup M_S$  saturates all vertices in  $S \cup R$ . Consequently,  $M_R$  saturates all vertices in  $R$ . Since  $M_R$  was an arbitrary maximal matching of  $G[R]$ , we infer that  $G[R]$  is randomly matchable. In particular, every connected component of  $G[R]$  has even order. (Theorem 3.1 exactly characterizes the structure of  $G[R]$  but we will not need this characterization in the rest of the proof.) Since  $G$  is connected, it has an edge of the form  $rv$  where  $r \in R$  and  $v \notin R$ . Since no vertex in  $R$  has a neighbor in  $L$ , we have  $v \in S$ . Then, the set  $M' = (M_S \setminus \{vv'\}) \cup \{rv\}$  is a matching in  $G$ . Extend  $M'$  to a maximal matching  $M$  in  $G$ . Since  $G$  is randomly internally matchable,  $M$  saturates all vertices in  $S \cup R$ . Let  $C$  be the component of  $G[R]$  containing  $r$  and let  $M_C$  be the set of edges of  $M$  fully contained in  $C$ . By construction, every edge in  $M$  saturating a vertex in  $C \setminus \{r\}$  belongs to  $M_C$ . However, this implies that  $C$  is of odd order, a contradiction with the fact that every component of  $G[R]$  has even order. This completes the proof.  $\square$

## 4 Claw-free CIS graphs

In this section we develop a structural characterization of claw-free CIS graphs. We start with several lemmas giving necessary conditions for a claw-free graph to be a CIS graph. The *gem* is the graph obtained from the 4-vertex path  $P_4$  by adding to it a universal vertex.

**Lemma 4.1** *Let  $G$  be a claw-free CIS graph. Then  $G$  is gem-free.*

**Proof.** Suppose for a contradiction that  $G$  is a claw-free CIS graph containing an induced copy of a gem, say on vertex set  $\{s, t, u, v, w\}$  where  $(s, t, u, v)$  is a path and  $w$  is adjacent to all vertices in  $\{s, t, u, v\}$ . Since the subgraph of  $G$  induced by  $\{s, t, u, v\}$  is isomorphic to an induced 2-comb and  $G$  is CIS, Lemma 2.1 implies that there exists a vertex  $x \in V(G) \setminus \{s, t, u, v\}$  such that  $\{tx, ux\} \subseteq E(G)$  and  $\{sx, vx\} \cap E(G) = \emptyset$ . Clearly  $x \neq w$ . Furthermore,  $xw \notin E(G)$ , since otherwise  $\{w, s, v, x\}$  would induce a claw in  $G$ . This implies that  $\{s, t, u, v, w, x\}$  induces a 3-anticomb in  $G$ . (Indeed, the complement of  $G$  contains a comb  $F_3$  having a clique  $\{s, v, x\}$  and a stable set  $\{t, u, w\}$ .) By Lemma 2.1,  $G$  contains a vertex  $y \in V(G) \setminus \{s, t, u, v, w, x\}$  adjacent to every vertex in the clique  $\{t, u, w\}$  and non-adjacent to every vertex in the stable set  $\{s, v, x\}$ . But now, the vertex set  $\{s, t, x, y\}$  induces a claw in  $G$ , a contradiction.  $\square$

We denote by  $W_4$  the 4-wheel, that is, the graph obtained from the 4-vertex cycle  $C_4$  by adding to it a universal vertex.

**Lemma 4.2** *Every connected and co-connected  $\{\text{claw}, \text{gem}\}$ -free graph is  $W_4$ -free.*

Kloks et al. showed in [19] that the class of  $\{\text{claw}, \text{gem}, W_4\}$ -free graphs is exactly the class of *dominoes*, that is, graphs in which each vertex is contained in at most two maximal cliques. Furthermore, the dominoes are precisely the line graphs of triangle-free multigraphs; see also [22]. This result together with Lemmas 4.1 and 4.2 implies the following.

**Corollary 4.3** *Every connected and co-connected claw-free CIS graph is the line graph of a connected triangle-free multigraph.*

A lemma similar to Lemma 2.2 holds for claw-free graphs. The lemma follows immediately from the definitions.

**Lemma 4.4** *A graph  $G$  is claw-free if and only if the true-twin reduction of each component of  $G$  is claw-free.*

Lemmas 2.2 and 4.4 imply that when studying CIS claw-free graphs, we may restrict our attention to connected true-twin-free graphs. Thus, the following theorem gives a complete structural characterization of claw-free CIS graphs. Given a graph  $G$ , the *corona* of  $G$  (with  $K_1$ ) is the graph  $G \circ K_1$  obtained from  $G$  by adding for each vertex  $v \in V(G)$  a new vertex  $v'$  and making  $v'$  adjacent to  $v$ . Note that the corona of  $G$  is a particular leaf extension of it.



**Theorem 4.5** *Let  $G$  be a connected true-twin-free claw-free graph. Then  $G$  is CIS if and only if one of the following holds:*

- (i)  $G \cong \overline{pK_2 + qK_1}$  for some  $p \geq 0$  and  $q \in \{0, 1\}$ .
- (ii)  $G \cong L(K_{n,n})$  for some  $n \geq 1$ .
- (iii)  $G \cong L(G' \circ K_1)$  for some triangle-free graph  $G'$ .

**Proof.** Let  $G$  be a connected true-twin-free CIS claw-free graph. Suppose first that  $G$  is not co-connected and let  $G_1, \dots, G_k$  (with  $k \geq 2$ ) be the co-components of  $G$ . Since  $k \geq 2$  and  $G$  is claw-free, each co-component  $G_i$  has stability number at most two. Since by Lemma 4.1  $G$  is gem-free, each co-component  $G_i$  is  $P_4$ -free. Since every  $P_4$ -free graph with at least two vertices is either disconnected or its complement is disconnected (see, e.g., [10]), each  $G_i$  is either  $K_1$  or is disconnected. Since  $\alpha(G_i) \leq 2$ , we infer each  $G_i$  is either  $K_1$  or the disjoint union of two complete graphs. Moreover, since  $G$  is true-twin-free, at most one  $G_i$  is the single-vertex graph, and each  $G_i$  with  $\alpha(G_i) = 2$  is isomorphic to  $2K_1$ . Consequently,  $G \cong \overline{pK_2 + qK_1}$  for some  $p \geq 0$  and  $q \in \{0, 1\}$ .

Suppose now that  $G$  is co-connected. By Corollary 4.3 there exists a triangle-free multigraph  $H$  such that  $G = L(H)$ . Clearly, we may assume that  $H$  has no isolated vertices. Since  $G$  is true-twin-free,  $H$  is a simple graph. By Corollary 3.2,  $H$  is randomly internally matchable. Theorem 3.3 implies that  $H \cong K_{n,n}$  for some  $n \geq 1$ , or  $H$  is a leaf extension of some (triangle-free) graph  $H'$ . In the former case,  $G \cong L(K_{n,n})$  for some  $n \geq 1$ . In the latter case, the fact that  $G$  is true-twin-free implies that  $H$  is the corona of  $H'$ , that is,  $G \cong L(H' \circ K_1)$ .

For the converse direction, suppose first that  $G \cong \overline{pK_2 + qK_1}$  for some  $p \geq 0$  and  $q \in \{0, 1\}$ . Since the class of CIS graphs is closed under complementation, it suffices to show that the graph  $pK_2 + qK_1$  is CIS. This follows easily from the definition and Lemma 2.2. Suppose next that  $G \cong L(K_{n,n})$  for some  $n \geq 1$  or  $G \cong L(G' \circ K_1)$  for some triangle-free graph  $G'$ . In this case, Theorem 3.3 implies that  $G \cong L(H)$  where  $H$  is a randomly internally matchable triangle-free graph without isolated vertices. Hence, by Corollary 3.2,  $G$  is CIS.  $\square$

Theorem 4.5 has the following structural and algorithmic consequences.

**Corollary 4.6** *A graph  $G$  is claw-free and CIS if and only if the true-twin reduction of each component of  $G$  is of the form  $\overline{pK_2 + qK_1}$  for some  $p \geq 0$  and  $q \in \{0, 1\}$ ,  $L(K_{n,n})$  for some  $n \geq 1$ , or  $L(G' \circ K_1)$  for some triangle-free graph  $G'$ .*

**Proof.** Immediate from Lemmas 2.2 and 4.4, Theorem 4.5, and the fact that all graphs of the form  $\overline{pK_2 + qK_1}$ ,  $L(K_{n,n})$ , or  $L(G' \circ K_1)$  are claw-free.  $\square$

**Corollary 4.7** *There is a polynomial-time algorithm for recognizing claw-free CIS graphs.*



## 5 Bounding the order of CIS graphs by the product of stability and clique numbers

### 5.1 Answering a question of Dobson et. al

We answer the following question of Dobson et al. [12] in the negative.

**Question 1** Does every CIS graph  $G$  satisfy  $|V(G)| \leq \alpha(G) \cdot \omega(G)$ ?

In fact, as we show in the next theorem, the order of CIS graphs cannot even be bounded from above by any linear function of the product  $\alpha(G)\omega(G)$ .

**Theorem 5.1** For every positive integer  $k$  there exists a CIS graph  $G_k$  such that  $|V(G_k)| > k \cdot \alpha(G_k) \cdot \omega(G_k)$ .

**Proof.** Kim proved in [18] that there exists a positive integer  $n_0$  such that for all  $n \geq n_0$ , there exists an  $n$ -vertex triangle-free graph  $H_n$  such that  $\alpha(H_n) \leq 9\sqrt{n \log n}$ . We may assume that  $H_n$  has no isolated vertices, since otherwise we can add, as long as necessary, for each isolated vertex  $v$ , an edge joining  $v$  with some other vertex. Note that modifying  $H_n$  this way does not create any triangles, it does not change the number of vertices, and it does not increase the stability number.

For a positive integer  $k$ , let  $n_k$  be the smallest positive integer such that  $n_k \geq n_0$  and  $n_k \geq 54k\sqrt{n_k \log n_k}$ . Let  $G_k$  be the graph obtained from  $H_{n_k}$  by the following two-step procedure:

- (i) First, construct a graph  $H'_k$  by gluing a triangle along each edge of  $H_{n_k}$ . Formally,  $V(H'_k) = V(H_{n_k}) \cup \{v^e : e \in E(H_{n_k})\}$  and  $E(H'_k) = E(H_{n_k}) \cup \{uv^e : u \text{ is an endpoint of } e \text{ in } H_{n_k}\}$ .
- (ii) Second, let  $p = 6kn_k$  and for each vertex  $v \in V(H_{n_k})$  in the graph  $H'_k$ , substitute  $pK_p$  (the disjoint union of  $p$  copies of  $K_p$ ) for  $v$ .<sup>2</sup> Call the resulting graph  $G_k$ .

Since  $H_{n_k}$  is triangle-free and without isolated vertices, its maximal cliques are its edges. It follows that the maximal cliques of  $H'_k$  are the triangles consisting of the two endpoints of an edge  $e$  of  $H_{n_k}$  together with the new vertex  $v^e$  associated with that edge. In particular, every maximal clique of  $H'_k$  is simplicial, and hence the graph  $H'_k$  is CIS.

Clearly,  $pK_p$  is a CIS graph. Since the class of CIS graphs is closed under substitution [2], we infer that the graph  $G_k$  is also CIS. Its clique and stability numbers can be estimated as follows:

- $\omega(G_k) = 2p + 1 \leq 3p$ .

A clique in  $G_k$  of size  $2p + 1$  can be obtained by choosing any edge  $e$  of  $H_{n_k}$ , taking two cliques of size  $p$ , one from each copy of  $pK_p$  replacing an endpoint of  $e$  in  $G_k$ , and vertex  $v^e$ . It is not difficult to see that there are no larger cliques in  $G_k$ .

<sup>2</sup> Given two graphs  $G$  and  $H$  and a vertex  $x \in V(G)$ , the operation of substituting  $H$  for  $x$  in  $G$  results in the graph obtained from the disjoint union of graphs  $G - x$  and  $H$  by adding all possible edges between  $N_G(v)$  and  $V(H)$ .

- $\alpha(G_k) < 9p\sqrt{n_k \log n_k} + n_k^2$ .

The stability number of the graph obtained from a graph  $F$  by substituting a graph  $H_v$  into every vertex  $v$  of  $F$  equals the maximum total weight of a stable set in the graph  $F$  in which each vertex  $v \in V(F)$  has weight equal to the stability number of  $H_v$  (see, e.g., [21]). Therefore, the stability number of  $G_k$  equals the maximum total weight of a stable set in the graph  $H'_k$  in which each vertex  $v \in V(H_{n_k})$  has weight  $p$  and all other vertices have unit weight. Let  $S$  be a corresponding maximum-weight stable set of  $H'_k$ . Writing  $S = S_p \cup S_1$  where  $S_p = S \cap V(H_{n_k})$  and  $S_1 = S \setminus V(H_{n_k})$ , we see that the total weight of  $S_p$  is at most  $p \cdot \alpha(H_{n_k})$ , while the total weight of  $S_1$  is at most  $|V(H'_k) \setminus V(H_{n_k})| = |E(H_{n_k})|$ . It follows that  $\alpha(G_k) \leq p \cdot \alpha(H_{n_k}) + |E(H_{n_k})| < 9p\sqrt{n_k \log n_k} + n_k^2$ , as claimed.

Consequently, we have  $k \cdot \alpha(G_k) \cdot \omega(G_k) < k \cdot (9p \cdot \sqrt{n_k \log n_k} + n_k^2) \cdot 3p = p \cdot (27k\sqrt{n_k \log n_k} \cdot p + 3kn_k^2) \leq p \cdot (pn_k/2 + pn_k/2) = p^2 n_k < |V(G_k)|$ , where the second inequality follows from  $n_k \geq 54k\sqrt{n_k \log n_k}$  and  $p = 6kn_k$ , and the last one from  $|V(G_k)| = p^2 n_k + |E(H_{n_k})|$ . This completes the proof.  $\square$

## 5.2 A positive answer for claw-free graphs

In order to show that Question 1 has a positive answer for claw-free CIS graphs, we first show a property of weighted randomly internally matchable graphs, which follows easily from the characterization of randomly internally matchable graphs given by Theorem 3.3. For this we need some definitions.

Given a weighted graph  $(H, w)$  and a set  $X \subseteq E(H)$ , we denote by  $w(X)$  the total weight of edges in  $X$ , that is,  $w(X) = \sum_{e \in X} w(e)$ . Given a vertex  $v \in V$ , we denote by  $E(v)$  the set of all edges having  $v$  as endpoint, and define its *weighted degree* as  $d_w(v) = \sum_{e \in E(v)} w(e)$ . We denote by  $\Delta_w(H)$  the maximum weighted degree of a vertex in  $H$ . The maximum size of a matching in a graph  $H$  is its *matching number*, denoted by  $\nu(H)$ . A *maximum matching* in  $H$  is a matching of a size  $\nu(H)$ .

**Lemma 5.2** *Let  $(H, w)$  be a weighted graph such that  $H$  is a connected randomly internally matchable graph. Then  $w(E(H)) \leq \Delta_w(H) \cdot \nu(H)$ .*

**Proof.** By Theorem 3.3,  $H$  satisfies one of the following conditions: (i)  $H \cong K_{2n}$  for some  $n \geq 1$ , (ii)  $H \cong K_{n,n}$  for some  $n \geq 1$ , or (iii)  $H$  is a leaf extension of some  $n$ -vertex graph  $H'$ . It is not difficult to see that in either case, the matching number of  $H$  equals  $n$ . Hence, we want to show the inequality  $w(E(H)) \leq n \cdot \Delta_w(H)$ . If  $H \cong K_{2n}$ , then  $2w(E(H)) = \sum_{x \in V(H)} w(E(x)) \leq 2n \cdot \Delta_w(H)$ . If  $H \cong K_{n,n}$ , with a bipartition  $\{X, Y\}$ , then  $w(E(H)) = \sum_{x \in X} w(E(x)) \leq n \cdot \Delta_w(H)$ . Finally, if  $H$  is a leaf extension of some  $n$ -vertex graph  $H'$ , then  $w(E(H)) \leq \sum_{x \in V(H')} w(E(x)) \leq n \cdot \Delta_w(H)$ . Thus, in either case, the desired inequality holds.  $\square$

From Lemma 5.2 we derive one more intermediate lemma.

**Lemma 5.3** *Let  $H$  be a connected triangle-free multigraph and let  $G = L(H)$ . If  $G$  is CIS, then  $|V(G)| \leq \alpha(G) \cdot \omega(G)$ .*

**Proof.** Let  $(H, w)$  be a weighted graph such that  $H$  is a connected triangle-free graph, and assume that  $G = L(H, w)$  is CIS (cf. Section 2.1). Let  $G' = L(H)$  be the usual line graph of  $H$ . It is not difficult to see that  $G$  can be obtained from  $G'$  by substituting, for each vertex  $v \in V(G')$ , a clique of size  $w(e_v)$  where  $e_v$  is the edge of  $H$  corresponding to  $v$ . Since  $G$  is CIS if and only if  $G'$  is CIS (see, e.g., [8]), we infer that  $G'$  is CIS. By Corollary 3.2,  $H$  is randomly internally matchable. Since  $H$  is connected, Lemma 5.2 implies that  $w(E(H)) \leq \Delta_w(H) \cdot \nu(H)$ .

Clearly, the stability number of  $G$  is given as  $\alpha(G) = \alpha(L(H, w)) = \alpha(L(H)) = \nu(H)$ . Moreover, the clique number of  $G$  is the maximum weight of a clique in  $L(H)$  with respect to the weight function assigning to each vertex  $v \in L(H)$  weight  $w(e_v)$ , where  $e_v$  is the edge of  $H$  corresponding to  $v$ . Since  $H$  is triangle-free, every clique in  $L(H)$  corresponds to a set of edges in  $H$  with a fixed common endpoint. It follows that the clique number of  $G$  equals the maximum weighted degree  $\Delta_w(H)$ . Since also  $|V(G)| = w(E(H))$ , the inequality  $w(E(H)) \leq \Delta_w(H) \cdot \nu(H)$  implies that  $|V(G)| \leq \alpha(G) \cdot \omega(G)$ , as claimed.  $\square$

**Theorem 5.4** *If  $G$  is a CIS claw-free graph, then  $|V(G)| \leq \alpha(G) \cdot \omega(G)$ .*

**Proof.** Suppose for a contradiction that the theorem fails and let  $G$  be a counterexample with the smallest possible number of vertices. That is,  $G$  is a claw-free CIS graph such that  $|V(G)| > \alpha(G) \cdot \omega(G)$ , but every smaller claw-free CIS graph  $G'$  satisfies  $|V(G')| \leq \alpha(G') \cdot \omega(G')$ .

First we show that  $G$  is connected. Suppose not, and let  $G$  be the disjoint union of two graphs  $G_1$  and  $G_2$ . Then each of  $G_1$  and  $G_2$  is a smaller claw-free CIS graph, therefore  $|V(G_i)| \leq \alpha(G_i) \cdot \omega(G_i)$  holds for  $i \in \{1, 2\}$  by the minimality of  $G$ . We have  $\omega(G) = \max\{\omega(G_1), \omega(G_2)\}$ . Without loss of generality we may assume that  $\omega(G) = \omega(G_1) \geq \omega(G_2)$ . We thus have  $|V(G)| = |V(G_1)| + |V(G_2)| \leq \alpha(G_1) \cdot \omega(G_1) + \alpha(G_2) \cdot \omega(G_2) \leq (\alpha(G_1) + \alpha(G_2)) \cdot \omega(G_1) = \alpha(G) \cdot \omega(G)$ . Hence,  $G$  is not a counterexample. This contradiction shows that  $G$  is connected.

Next, we show that  $G$  is co-connected. If not, then  $\overline{G}$  is the disjoint union of two smaller graphs  $H_1$  and  $H_2$ . Since  $\overline{G}$  is CIS, we have that both  $H_1$  and  $H_2$  are CIS. Therefore  $\overline{H_1}$  and  $\overline{H_2}$  are CIS claw-free graphs of smaller order than  $G$ , which implies  $|V(\overline{H_i})| \leq \alpha(\overline{H_i}) \cdot \omega(\overline{H_i})$  for  $i \in \{1, 2\}$ . Similar arguments as above show that  $|V(G)| = |V(\overline{H_1})| + |V(\overline{H_2})| \leq \alpha(\overline{G}) \cdot \omega(\overline{G}) = \alpha(G) \cdot \omega(G)$ , a contradiction. Thus,  $G$  is co-connected.

Now, since  $G$  is a connected and co-connected claw-free CIS graph, Corollary 4.3 implies that  $G$  is the line graph of a connected triangle-free multigraph. By Lemma 5.3, we have  $|V(G)| \leq \alpha(G) \cdot \omega(G)$ , implying that  $G$  is not a counterexample. This contradiction completes the proof.  $\square$

## 6 An open question

Question 1 has been answered in the negative. The following relaxation of it is still open.

**Question 2** *Is there an integer  $k$  such that every CIS graph  $G$  satisfies  $|V(G)| \leq (\alpha(G) \cdot \omega(G))^k$ ?*

A graph class  $\mathcal{G}$  is said to satisfy the *Erdős-Hajnal property* if there exists some  $\varepsilon > 0$  such that  $\max\{\alpha(G), \omega(G)\} \geq |V(G)|^\varepsilon$  holds for all graphs  $G \in \mathcal{G}$ . The well-known Erdős-Hajnal Conjecture [13] asks whether for every graph  $F$ , the class of  $F$ -free graphs has the Erdős-Hajnal property. The conjecture is still open (see [9] for a survey). It is not difficult to see that, using this terminology, Question 2 can be equivalently phrased as follows.

**Question 3** *Does the class of CIS graphs have the Erdős-Hajnal property?*

Note, however, that it is possible that Question 3 bears only superficial resemblance with the Erdős-Hajnal Conjecture. This is because every graph is an induced subgraph of a CIS graph (see, e.g., [2]); furthermore, for a graph  $F$ , the class of  $F$ -free graphs is a subclass of the class of CIS graphs if and only if  $F$  is an induced subgraph of  $P_4$ . It is thus in principle possible that the Erdős-Hajnal Conjecture is true, while Question 3 has a negative answer, or vice versa.

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## References

- [1] Andrade, D. V., E. Boros and V. Gurvich, *Not complementary connected and not CIS  $d$ -graphs form weakly monotone families*, Discrete Math. **310** (2010), pp. 1089–1096.
- [2] Andrade, D. V., E. Boros and V. Gurvich, *On graphs whose maximal cliques and stable sets intersect*, in: B. Goldengorin, editor, *Optimization Problems in Graph Theory: In Honor of Gregory Z. Gutin's 60th Birthday*, Springer International Publishing, Cham, 2018 pp. 3–63, see also RRR 17-2006, RUTCOR Research Reports, Rutgers University.
- [3] Bartha, M. and E. Gombás, *A structure theorem for maximum internal matchings in graphs*, Information Processing Letters **40** (1991), pp. 289–294.
- [4] Bartha, M. and M. Krész, *Tutte type theorems for graphs having a perfect internal matching*, Information Processing Letters **91** (2004), pp. 277–284.
- [5] Bartha, M. and M. Krész, *Deciding the deterministic property for soliton graphs*, Ars Mathematica Contemporanea **2** (2009), pp. 121–136.
- [6] Boros, E., N. Chiarelli and M. Milanič, *Equistable bipartite graphs*, Discrete Math. **339** (2016), pp. 1960–1969.
- [7] Boros, E., V. Gurvich and M. Milanič, *On equistable, split, CIS, and related classes of graphs*, Discrete Appl. Math. **216** (2017), pp. 47–66.
- [8] Boros, E., V. Gurvich and M. Milanič, *On CIS circulants*, Discrete Math. **318** (2014), pp. 78–95.
- [9] Chudnovsky, M., *The Erdős-Hajnal conjecture—a survey*, J. Graph Theory **75** (2014), pp. 178–190.

- [10] Corneil, D. G., H. Lerchs and L. S. Burlingham, *Complement reducible graphs*, Discrete Appl. Math. **3** (1981), pp. 163–174.
- [11] Deng, X., G. Li and W. Zang, *Proof of Chvátal’s conjecture on maximal stable sets and maximal cliques in graphs*, J. Combin. Theory Ser. B **91** (2004), pp. 301–325.
- [12] Dobson, E., A. Hujdurović, M. Milanić and G. Verret, *Vertex-transitive CIS graphs*, European J. Combin. **44** (2015), pp. 87–98.
- [13] Erdős, P. and A. Hajnal, *Ramsey-type theorems*, Discrete Appl. Math. **25** (1989), pp. 37–52, combinatorics and complexity (Chicago, IL, 1987).
- [14] Gurvich, V., *Decomposing complete edge-chromatic graphs and hypergraphs. Revisited*, Discrete Appl. Math. **157** (2009), pp. 3069–3085.
- [15] Hujdurović, A., *Strong cliques in vertex-transitive graphs* (2018), arXiv:1808.09534 [math.CO].
- [16] Hujdurović, A., M. Milanić and B. Ries, *Detecting strong cliques* (2018), arXiv:1808.08817 [math.CO].
- [17] Hujdurović, A., M. Milanić and B. Ries, *Graphs vertex-partitionable into strong cliques*, Discrete Math. **341** (2018), pp. 1392–1405.
- [18] Kim, J. H., *The Ramsey number  $R(3, t)$  has order of magnitude  $t^2 / \log t$* , Random Structures Algorithms **7** (1995), pp. 173–207.
- [19] Kloks, T., D. Kratsch and H. Müller, *Dominoes*, in: *Graph-theoretic concepts in computer science (Herrsching, 1994)*, Lecture Notes in Comput. Sci. **903**, Springer, Berlin, 1995 pp. 106–120.
- [20] Lovász, L., *A characterization of perfect graphs*, J. Combinatorial Theory Ser. B **13** (1972), pp. 95–98.
- [21] Lozin, V. V. and M. Milanić, *A polynomial algorithm to find an independent set of maximum weight in a fork-free graph*, J. Discrete Algorithms **6** (2008), pp. 595–604.
- [22] Metelsky, Y. and R. Tyshkevich, *Line graphs of Helly hypergraphs*, SIAM J. Discrete Math. **16** (2003), pp. 438–448.
- [23] Milanić, M. and N. Trotignon, *Equistable graphs and counterexamples to three conjectures on equistable graphs*, J. Graph Theory **84** (2017), pp. 536–551.
- [24] Sumner, D. P., *Randomly matchable graphs*, J. Graph Theory **3** (1979), pp. 183–186.
- [25] Sun, J. and Z.-q. Hu, *Proof of Ding’s conjecture on maximal stable sets and maximal cliques in planar graphs*, Acta Math. Appl. Sin. Engl. Ser. **26** (2010), pp. 473–480.
- [26] West, D. B., “Introduction to Graph Theory,” Prentice Hall, Inc., Upper Saddle River, NJ, 1996.
- [27] Zang, W., *Generalizations of Grillet’s theorem on maximal stable sets and maximal cliques in graphs*, Discrete Math. **143** (1995), pp. 259–268.
- [28] Zverovich, I. and I. Zverovich, *Bipartite bihypergraphs: a survey and new results*, Discrete Math. **306** (2006), pp. 801–811.