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On the Difference Hierarchy in Countably Based T_0-Spaces

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Abstract

We establish some results on some variants of the Hausdorff difference hierarchy. In particular, we extend the recently developed theory of difference hierarchy over the open sets in ω-algebraic domains to a similar theory for ω-continuous domains, and prove some analogs of the Hausdorff-Kuratowski theorem for k-partitions. We discuss also a broad class of effective topological spaces closely relevant to our study of the difference hierarchy and to computability in topology.

Keywords: Space, computably enumerable space, difference hierarchy, limit hierarchy, k-partition, ω-continuous domain.

1 Introduction

The aim of this paper is two-fold. First, we continue to study some versions of the classical Hausdorff difference hierarchy (DH). Second, we discuss in some detail a class of effective topological spaces introduced in [12,24] which is relevant to our study of the DH and to computability in topology.

The non-effective DH in Polish spaces is well-understood [10]. Recently, a complete theory of the DH over the open sets in ω-algebraic domains was developed [19]. For the DH over higher levels of Borel hierarchy, many questions remain open. Many basic questions remain open also for the effective version of the DH which is closely relevant to computable analysis [6,7].

Generalizations of the DH from the case of subsets A of a topological space X (identified with the characteristic functions $c_A : X \to \{0,1\}$) to the case of k-partitions $\nu : X \to k$ of X (for any integer $k \geq 2$ identified with the set $\{0, \ldots, k-1\}$) are quite natural and relevant to computable analysis, in particular to degrees of

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discontinuity [8,9,17,18,21,23]. For a set X, let $P(X)$ be the set of subsets of X and k^X the set of k-partitions of X. For any $\mathcal{C} \subseteq P(X)$, let *co*-$\mathcal{C}$ be the set of complements of sets in $\mathcal{C}$ and let $\mathcal{C}_k$ be the set of $\mathcal{C}$-partitions (more exactly, $\mathcal{C}$-measurable k-partitions), i.e., partitions $\nu \in k^X$ such that $\nu^{-1}(i) \in \mathcal{C}$ for each $i < k$. The last notion is naturally extended to partial k-partitions $\nu : X \rightharpoonup k$.

In this paper we establish some new results on the DH and its versions. In particular, we extend the theory of DH over the open sets in ω-algebraic domains to a similar theory for ω-continuous domains, and prove some analogs of the Hausdorff-Kuratowski theorem for k-partitions. In Section 2 we discuss (in a slightly modified and extended form) a natural broad class of effective topological spaces [12] that includes the computable metric spaces [24] and computably enumerable (c.e.) α-spaces [4]. This class is very convenient for development of the effective descriptive set theory. In Section 3 we develop the theory of DH over the open sets in ω-algebraic domains and discuss open problems about the effective DH. In Section 4 we establish some new facts on the so called limit hierarchy of k-partitions previously studied e.g. in [8,5,16,22]. Finally, in Section 5 we prove some new facts (in particular, an analog of the Hausdorff-Kuratowski theorem) for the DH of k-partitions [22] that refines the limit hierarchy.

2 Countably Based T_0-Spaces

Here we briefly discuss some notions related to countably based T_0-spaces and computability over them. Ideas and many of results here are not new. They are slight modifications and/or extensions of those in [12] (see also Exercise 3.2.17 in [24]).

2.1 Spaces

By a *space* we mean here a pair (X, ξ) where X is a nonempty set and $\xi : \omega \to P(X)$ a numbering of subsets of X with the following properties: $\bigcup \xi(\omega) = X$; there is a sequence $\{A_{ij}\}$ of subsets of ω such that $\xi i \cap \xi j = \bigcup \xi(A_{ij})$ for all $i, j \geq 0$; $O_\xi(x) \neq O_\xi(y)$ for all distinct $x, y \in X$ where $O_\xi(x) = \{i \mid x \in \xi i\}$. We denote by $P(X)$ the set of subsets of X, by $\xi i = \xi(i)$ the value of ξ on i and by $\xi(A)$ the image of A under ξ.

Obviously, if (X, ξ) is a space then $\xi(\omega)$ is a basis of the T_0-topology $\{\bigcup \xi(A) \mid A \subseteq \omega\}$ on X and, conversely, if $\mathcal{B}$ is a countable basis of a T_0-topology on X then for any surjection $\xi : \omega \to \mathcal{B}$ the pair (X, ξ) is a space in our sense. Thus, our "spaces" are essentially the countably based T_0-spaces with a fixed numbering of a basis. In particular, we may speak about continuous functions between spaces. If (X, ξ) and (Y, η) are spaces then $f : X \to Y$ is continuous iff there is a sequence $\{A_i\}$ of subsets of ω such that $f^{-1}(\eta i) = \bigcup \xi(A_i)$ for all $i \geq 0$. Note also that if (X, ξ) is a space and Y is a nonempty subset of X then (Y, η), $\eta i = Y \cap \xi i$, is a space, and the topology on Y is induced by the topology on X.

We introduced the technical notions of a space and describe some easy properties of spaces below because this straightforwardly leads to a broad natural class of "effective" topological spaces discussed in the next subsection.

An important example of a space is $(P(\omega), \delta)$ where $\delta i = \check{D}_i = \{A \subseteq \omega \mid D_i \subseteq A\}$ and $\{D_i\}$ is the standard numbering of the finite subsets of ω [13]; we denote the corresponding T_0-space by $P\omega$. The following assertion shows that $P\omega$ may be used to code all spaces.

Proposition 2.1 *Let (X, ξ) and (Y, η) be spaces.*

(i) The map $O_\xi : X \to P\omega$ is a homeomorphic embedding.

(ii) For any continuous function $f : X \to P\omega$ there is a continuous function $\tilde{f} : P\omega \to P\omega$ such that $f = \tilde{f} \circ O_\xi$.

(iii) For any $f : X \to Y$, if the function $O_\eta \circ f : X \to P\omega$ is continuous then f is continuous.

(iv) A function $f : X \to Y$ is continuous iff there is a continuous function $\tilde{f} : P\omega \to P\omega$ such that $O_\eta \circ f = \tilde{f} \circ O_\xi$.

Proof. (i) Since the sets $\check{n} = \{A \subseteq \omega \mid n \in A\}$, $n \geq 0$, form a prebasis in $P\omega$ and $O_\xi^{-1}(\check{n}) = \xi n$, O_ξ is continuous. Since $O_\xi(\xi n) = \check{n} \cap O_\xi(X)$, $O_\xi(\xi n)$ is an open set in the subspace $O_\xi(X)$ for all $n \geq 0$. Therefore, O_ξ is a homeomorphism between X and $O_\xi(X)$.

(ii) Since f is continuous, there is a sequence $\{A_n\}$ of subsets of ω such that $f^{-1}(\check{n}) = \bigcup \xi(A_n)$ for all $n \geq 0$. For any $A \subseteq \omega$, let $\tilde{f}(A) = \{n \mid A_n \cap A \neq \emptyset\}$. Then

$$\begin{aligned} n \in \tilde{f}(O_\xi(x)) &\leftrightarrow A_n \cap O_\xi(x)) \neq \emptyset \leftrightarrow \exists m \in A_n(x \in \xi m) \leftrightarrow x \in \bigcup \xi(A_n) \\ &\leftrightarrow f(x) \in \check{n} \leftrightarrow n \in f(x), \end{aligned}$$

hence $f = \tilde{f} \circ O_\xi$. One easily checks that $\tilde{f}$ is continuous.

The assertion (iii) follows from (i) while (iv) follows from (ii) and (iii). □

For $\xi, \eta : \omega \to P(X)$, let $\xi \leq \eta$ mean that there is a sequence $\{A_n\}$ of subsets of ω such that $\xi n = \bigcup \eta(A_n)$ for all $n \geq 0$. Then $\leq$ is a preorder; we denote the induced equivalence relation by $\equiv$. Obviously, for any spaces (X, ξ) and (X, η), $\xi \leq \eta$ iff the topology induced by η refines that induced by ξ. Clearly, $rng(\xi) \subseteq rng(\eta)$ implies $\xi \leq \eta$. If ξ^* is any numbering of the nonempty sets in $rng(\xi)$ then $\xi^* \equiv \xi$.

We conclude this subsection by a short discussion of two important classes of spaces. The first class is closely related to separable metric spaces, i.e., metric spaces (X, d) having a countable dense set $D \subseteq X$. Fixing a surjection $\nu : \omega \to D$ induces the space (X, ξ) where $\xi = \xi_{\nu,d}$ is defined by $\xi\langle m, n\rangle = \{x \in X \mid d(\nu(m), x) < \nu_{\mathbb{Q}}(n)\}$, where $\langle\, ,\, \rangle$ is a computable Cantor paring function and $\nu : \omega \to \mathbb{Q}$ is a standard computable numbering of the rationals. Obviously, $rng(\nu) = rng(\nu')$ implies $\xi_{\nu,d} \equiv \xi_{\nu',d}$ and the topology induced by ξ coincides with that induced by metric d.

The second class is closely related to the α-spaces [4] defined as follows. On any T_0-space X there is the *specialization partial order* (defined by: $x \leq y$, if $x \in U$ implies $y \in U$, for any open set U in X) and the *approximation* relation (defined by: $x \prec y$, if y is in the interior of the set $\check{x} = \{z \in X \mid x \leq z\}$). The approximation relation is transitive and $x \prec y$ implies $x \leq y$. A T_0-space X is an *α-space* if for any open set $U \subseteq X$ and any $x \in U$ there is a $b \in U$ with $b \prec x$. A *base set* in an α-space X is a subset B of X such that for any open set $U \subseteq X$ and any $x \in U$ there is a $b \in U \cap B$ with $b \prec x$. The class of α-spaces contains several interesting subclasses, including the so called φ- f- and A-spaces, and is closely related to domain theory [4,1]. In particular, the continuous (resp., ω-continuous) domains essentially coincide with the complete α-spaces (resp., complete α-space with a countable base set). An important example of an α-space is $P\omega$.

Let X be an α-space with a countable base set B. Fixing a surjection $\nu : \omega \to B$ induces a space (X, ξ) where $\xi = \xi_\nu$ is defined by $\xi n = \{x \in X \mid \nu n \prec x\}$. Obviously, $rng(\nu) = rng(\nu')$ implies $\xi_\nu \equiv \xi_{\nu'}$ and the topology induced by ξ coincides with the topology on X.

2.2 Computably Enumerable Spaces

Notions of this subsection are obtained by a straightforward "effectivization" of the corresponding notions from the previous subsection. In particular, we require that sequences of subsets of ω discussed above should be (uniformly) c.e.

A space (X, ξ) is called *c.e.* if the set $\{n \mid \xi n \neq \emptyset\}$ is c.e. and there is a c.e. sequence $\{A_{ij}\}$ such that $\xi i \cap \xi j = \bigcup(A_{ij})$ for all $i, j \geq 0$. Note that if (X, ξ) is a c.e. space and $Y \subset X$ then the (Y, ξ_Y) need not be a c.e. space (because the first effectivity condition is not true in general). But in some natural cases (Y, ξ_Y) is a c.e. space. E.g., this holds if Y is effectively open (see below). Many popular spaces (e.g., the space of reals $\mathbb{R}$, the space $P\omega$, the Baire space ω^ω and the Baire domain $\omega^{\leq\omega}$ are c.e. with respect to natural numberings of a basis The c.e. space (ω, ξ), where $\xi n = \{n\}$, is trivial from topological point of view but interesting for computability theory.

For $\xi, \eta : \omega \to P(X)$, let $\xi \leq_c \eta$ mean that there is a c.e. sequence $\{A_n\}$ of subsets of ω such that $\xi n = \bigcup \eta(A_n)$ for all $n \geq 0$. Then $\leq_c$ is a preorder; we denote the induced equivalence relation by $\equiv_c$. For a c.e. space (X, ξ), set $\xi'(n) = \xi g(n)$ where g is a total computable function enumerating the nonempty c.e. set $\{n \mid \xi n \neq \emptyset\}$; then $\xi' \equiv_c \xi$. Set also $\xi_W(n) = \bigcup \xi(W_n)$ where W is the standard numbering of c.e. sets. Then ξ_W is a numbering of the *effectively open* sets, (X, ξ_W) is a c.e. space and $\xi_W \equiv_c \xi$.

Next we notice that two classes of spaces important for the study of computability in analysis and topology are subclasses of the c.e. spaces. Recall that a *computable metric space* [24] is a triple (X, d, ν) where (X, d) is a metric space and $\nu : \omega \to X$ is a numbering of a dense subset $rng(\nu)$ of X such that the set $\{\langle i, j, q, r\rangle \mid \nu_{\mathbb{Q}}(q) < d(\nu i, \nu j) < \nu_{\mathbb{Q}}(r)\}$ is c.e. As follows from [24,12], if (X, d, ν) is a computable metric space then $(X, \xi_{\nu,d})$ is a c.e. space.

A *computable α-space* is a pair (X, ν) where X is an α-space and $\nu : \omega \to X$ is a numbering of a base subset $rng(\nu)$ of X such that the set $\{\langle i, j\rangle \mid \nu i \prec \nu j\}$ is c.e. Similar to a particular case in [12] one checks that if (X, ν) is a computable α-space then (X, ξ_ν) is a c.e. space.

Let (X, ξ) and (Y, η) be c.e. spaces. A function $f : X \to Y$ is called *effectively continuous* if there is a c.e. sequence $\{A_i\}$ of subsets of ω such that $f^{-1}(\eta i) = \bigcup \xi(A_i)$ for all $i \geq 0$. It is easy to check that a function f on $P(\omega)$ is an enumeration operator [13] (i.e., an operator $A \mapsto \{j \mid \exists i(\langle i,j\rangle \in W \wedge D_i \subseteq A)\}$ for some c.e. set W) iff it is an effectively continuous function on $(P(\omega), \delta)$.

Next we establish the effective version of Proposition 2.1.

Proposition 2.2 *Let (X,ξ) and (Y,η) be c.e. spaces.*

(i) The map $O_\xi : X \to P\omega$ is an effective homeomorphic embedding.

(ii) For any effectively continuous function $f : X \to Y$ there is an enumeration operator $\tilde{f} : P(\omega) \to P(\omega)$ such that $f = \tilde{f} \circ O_\xi$.

(iii) For any $f : X \to Y$, if the function $O_\eta \circ f : X \to P\omega$ is effectively continuous then f is effectively continuous.

(iv) A function $f : X \to Y$ is effectively continuous iff there is an enumeration operator $\tilde{f} : P(\omega) \to P(\omega)$ such that $O_\eta \circ f = \tilde{f} \circ O_\xi$.

Proof. More precise formulation of (i) says that the space (Z, δ_Z), where $Z = O_\xi(X)$, is c.e. and both O_ξ and its inverse are effectively continuous. Since $O_\xi^{-1}(\check{D}_n) = \bigcap \xi(D_n)$, and, by the effectivity conditions, $\bigcap \xi(D_n) = \bigcup \xi(B_n)$ for a c.e. sequence $\{B_n\}$, O_ξ is effectively continuous. To show that (Z, δ_Z) is a c.e. space, it suffices to check that the set $\{n \mid \delta n \cap O_\xi(X) \neq \emptyset\}$ is c.e. This follows from

$$\delta n \cap O_\xi(X) \neq \emptyset \leftrightarrow O_\xi^{-1}(\check{D}_n) \neq \emptyset \leftrightarrow \bigcap \xi(D_n) \neq \emptyset \leftrightarrow \exists m \in B_n(\xi m \neq \emptyset).$$

The remaining assertions are checked in the same way as the corresponding non-effective versions in the proof of Proposition 2.1(only now the corresponding sequences are c.e.). □

The next result shows that the effectively continuous functions in the spaces corresponding to computable metric spaces [24] and to α-spaces [1] coincide with the computable functions in the sense of computable analysis and topology. The assertion (i) is proved in [12] while the assertion (ii) is a slight modification of the corresponding fact in [12] and of Theorem 3.2.14 in [24].

Proposition 2.3 *(i) Let (X,d,ν) and (Y,e,μ) be computable metric spaces. A function $f : X \to Y$ is computable w.r.t. the Cauchy representations of (X,d,ν) and (Y,e,μ) iff f is an effectively continuous function from $(X,\xi_{d,\nu})$ to $(Y,\xi_{e,\mu})$.*

(ii) Let (X,ν) and (Y,μ) be c.e. α-spaces. A function $f : X \to Y$ is a computable function from (X,ν) to (Y,μ) (i.e., the relation $\eta m \prec f(\nu n)$ is c.e.) iff f is an effectively continuous function from (X,ξ_ν) to (Y,ξ_μ).

3 Difference Hierarchies of Sets

In this section we improve some previous results on the DH, in particular extend the theory in [19] to the ω-continuous domains.

3.1 Non-Effective Hierarchies

Let us recall [10,19,20] that *Borel hierarchy* in a topological space X is the sequence $\{\mathbf{\Sigma}^0_\alpha\}_{\alpha<\omega_1}$ of subclasses of $P(X)$ defined by induction on α as follows: $\mathbf{\Sigma}^0_0 = \{\emptyset\}$,

$\mathbf{\Sigma}^0_1$ is the class of open sets, $\mathbf{\Sigma}^0_2$ is the class of countable unions of finite Boolean combinations of open sets, and $\mathbf{\Sigma}^0_\alpha$ (for $\alpha > 2$) is the class of countable unions of sets in $\bigcup_{\beta<\alpha}\mathbf{\Pi}^0_\beta$, where $\mathbf{\Pi}^0_\beta = co\text{-}\mathbf{\Sigma}^0_\beta$. Classes $\mathbf{\Sigma}^0_\alpha$, $\mathbf{\Pi}^0_\alpha$ and $\mathbf{\Delta}^0_\alpha = \mathbf{\Sigma}^0_\alpha \cap \mathbf{\Pi}^0_\alpha$ are called *levels* of the Borel hierarchy. If we want to stress the space X in which the levels are considered we can use a more complicated notation like $\mathbf{\Sigma}^0_\alpha(X)$.

The following easy assertion [19] extends a well-known fact for Polish spaces [10]. A class $\mathcal{C}$ has the *σ-reduction property*, if for each countable sequence $C_0, C_1, \ldots$ in $\mathcal{C}$ there is a countable sequence $C'_0, C'_1, \ldots$ in $\mathcal{C}$ (called a *reduct* of $C_0, C_1, \ldots$) such that $C'_i \cap C'_j = \emptyset$ for all $i \neq j$ and $\bigcup_{i<\omega} C'_i = \bigcup_{i<\omega} C_i$.

Proposition 3.1 *For any topological space X and any $\alpha \geq 2$, the class $\mathbf{\Sigma}^0_\alpha$ has the σ-reduction property. The class $\mathbf{\Sigma}^0_1$ does not in general have the reduction property.*

Next we recall definition and basic facts about the DH. An ordinal α is called *even (odd)* if $\alpha = \lambda + n$ where λ is not a successor, $n < \omega$ and n is even (resp., odd). For an ordinal α, let $r(\alpha) = 0$ if α is even and $r(\alpha) = 1$ otherwise. For any ordinal α, define the operation D_α sending sequences of sets $\{A_\beta\}_{\beta<\alpha}$ to sets by

$$D_\alpha(\{A_\beta\}_{\beta<\alpha}) = \bigcup\{A_\beta \setminus \bigcup_{\gamma<\beta} A_\gamma \mid \beta < \alpha,\, r(\beta) \neq r(\alpha)\}.$$

For all ordinals α and classes of sets $\mathcal{C}$, let $D_\alpha(\mathcal{C})$ be the class of all sets $D_\alpha(\{A_\beta\}_{\beta<\alpha})$, where $A_\beta \in \mathcal{C}$ for all $\beta < \alpha$.

Definition 3.2 Let X be a space and $\{\mathbf{\Sigma}^0_\beta\}$ the Borel hierarchy in X. For any β, $0 < \beta < \omega_1$, the sequence $\{D_\alpha(\mathbf{\Sigma}^0_\beta)\}_{\alpha<\omega_1}$ is called the DH over $\mathbf{\Sigma}^0_\beta$. The DH over $\mathbf{\Sigma}^0_1$ is called simply the DH in X and is denoted by $\{\mathbf{\Sigma}^{-1}_\alpha\}_{\alpha<\omega_1}$.

As usual, let $\mathbf{\Pi}^{-1}_\alpha$ denote the dual class for $\mathbf{\Sigma}^{-1}_\alpha$, and $\mathbf{\Delta}^{-1}_\alpha = \mathbf{\Sigma}^{-1}_\alpha \cap \mathbf{\Pi}^{-1}_\alpha$. It is well-known and follows easily from the definitions above (see also [10,20] for details) that $D_\alpha(\mathbf{\Sigma}^0_\beta) \cup co\text{-}D_\alpha(\mathbf{\Sigma}^0_\beta) \subseteq D_\gamma(\mathbf{\Sigma}^0_\beta) \subseteq \mathbf{\Delta}^0_{\beta+1}$ for all α, β and γ with $\alpha < \gamma < \omega_1$ and $0 < \beta < \omega_1$. In particular, $\mathbf{\Sigma}^{-1}_\alpha \subseteq \mathbf{\Delta}^{-1}_\gamma \subseteq \mathbf{\Delta}^0_2$. The following nontrivial properties of the DH's are well-known [10].

Theorem 3.3 *Let X be a Polish (i.e., a countably based complete metrizable) space and $0 < \beta < \omega_1$.*

(i) $\bigcup\{D_\alpha(\mathbf{\Sigma}^0_\beta) | \alpha < \omega_1\} = \mathbf{\Delta}^0_{\beta+1}$ (this assertion is known as the Hausdorff-Kuratowski theorem).

(ii) If X is non-countable then the DH over $\mathbf{\Sigma}^0_\beta$ does not collapse, i.e., $D_\alpha(\mathbf{\Sigma}^0_\beta) \not\subseteq co\text{-}D_\alpha(\mathbf{\Sigma}^0_\beta)$ for each $\alpha < \omega_1$.

Next we explain how to extend the theory of the DH over $\mathbf{\Sigma}^0_1$ in φ-spaces [19] to the α-spaces (X, B) with a fixed base set $B \subseteq X$. Since the proofs are small modifications of those in [19], we concentrate on exact formulations. We use some standard notation and terminology on partially ordered sets (posets) which may be found e.g. in [2]. We apply notions about posets also to preorders, meaning the corresponding quotient-poset of the preorder. A poset $(P; \leq)$ will be often shorter denoted just by P. Any subset of a poset P may be considered as a poset with the induced partial order. In particular, this applies to the "upper cones" $\check{x} = \{y \in P \mid x \leq y\}$ defined by any $x \in P$. A *well preorder* is a preorder P that has neither infinite descending chains nor infinite antichains. For the well-founded preorders there is a canonical rank function rk_P assigning ordinals to the elements of P; the *rank of* P is the supremum of ranks of its elements.

By *a forest* we mean a poset without infinite chains in which every upper cone $\check{x}$ is a chain. *A tree* is a forest having the biggest element (called *the root* of the tree). It is well-known that rank of any countable tree without infinite chains is a countable ordinal, and any countable ordinal is the rank of such a tree.

A *k-poset* is a triple $(P; \leq, c)$ consisting of a poset $(P; \leq)$ and a labeling $c : P \to k$. The *rank* of a k-tree $(T; \leq, c)$ without infinite chains is by definition the rank of $(T; \leq)$. A *homomorphism* $f : (P; \leq, c) \to (P'; \leq', c')$ of k-posets is a monotone function $f : (P; \leq) \to (P'; \leq')$ respecting the labelings, i.e., satisfying $c = c' \circ f$. Let $\widetilde{\mathcal{P}}_k$ ($\widetilde{\mathcal{F}}_k$, $\widetilde{\mathcal{T}}_k$) denote the class of all countable k-posets (resp. k-forests, k-trees) without infinite chains. The h-preorder $\leq$ on $\widetilde{\mathcal{P}}_k$ is defined as follows: $(P, \leq, c) \leq (P', \leq', c')$, if there is a homomorphism from $(P, \leq, c)$ to $(P', \leq', c')$. A k-tree $(T; \leq, c) \in \widetilde{\mathcal{T}}_k$ is *repetition-free* if $c(x) \neq c(y)$ whenever y is an immediate successor of x in $(T; \leq)$. Any tree $\widetilde{\mathcal{T}}_k$ is h-equivalent to a repetition-free tree in $\widetilde{\mathcal{T}}_k$ [21].

Let (X, B) be an α-space, $\nu \in k^X$ and $T = (T, \leq, t) \in \widetilde{\mathcal{T}}_k$. By a *$\nu$-representation* of T we mean a function $g : T \to B$ such that $t = \nu \circ g$ and $x < y$ implies $g(x) \prec g(y)$. The *rank* of a representation g is the rank of T. A k-tree T is *ν-representable* if there exists a ν-representation of T. In the particular case when $k = 2$, $A = \nu \subseteq X$ and T is a repetition-free 2-tree we obtain the notions of *alternating tree* introduced in [19]; such a tree $g : T \to B$ is *1-alternating* if $g(r) \in A$ (where r is the root of T), otherwise it is *0-alternating*. We call a set $A \subseteq X$ *approximable* if for any $x \in A$ there is $b \prec x$ such that $b \in B$ and $\{y \in X \mid b \prec y \prec x\} \subseteq A$.

The next assertions is checked similarly to the corresponding facts in [19] for ω-algebraic domains.

Theorem 3.4 *Let X be an ω-continuous domain and $A \subseteq X$.*

(i) For any $\alpha < \omega_1$, $A \in \mathbf{\Sigma}^{-1}_\alpha$ iff both sets A and $\bar{A}$ are approximable and there is no 1-alternating tree for A of rank α.

(ii) $A \in \mathbf{\Delta}^0_2$ iff both A and $\bar{A}$ are approximable iff $A \in \mathbf{\Sigma}^{-1}_\alpha$ for some $\alpha < \omega_1$.

We conclude this subsection by formulating sufficient conditions for the non-collapse of the DH's. By a *reflective continuous domain* we mean an ω-continuous domain X with a bottom element $\perp$ such that for some continuous functions $q_0, e_0, q_1, e_1 : X \to X$ there hold $q_0 e_0 = q_1 e_1 = id_X$, and $e_0(X), e_1(X)$ are disjoint open sets. By a *2-reflective continuous domain* we mean an ω-algebraic domain X with a top element $\top$ such that there exist continuous functions $q_0, e_0, q_1, e_1 : X \to X$ and open sets B_0, C_0, B_1, C_1 with the following properties: $q_0 e_0 = q_1 e_1 = id_X$; $B_0 \supseteq C_0$ and $B_1 \supseteq C_1$; $e_0(X) = B_0 \setminus C_0$ and $e_1(X) = B_1 \setminus C_1$; $B_0 \cap B_1 = C_0 \cap C_1$. Many natural domains are reflective or 2-reflective [18,20](in particular, $P\omega$ is 2-

reflective). The following fact is again an extension of the corresponding fact for algebraic domains.

Theorem 3.5 *Let X be a reflective or a 2-reflective continuous domain. Then the difference hierarchy does not collapse, i.e., $\mathbf{\Sigma}^{-1}_\alpha \neq \mathbf{\Pi}^{-1}_\alpha$ for all $\alpha < \omega_1$.*

3.2 Effective Hierarchies

Here we briefly discuss effective versions of hierarchies form the previous subsection. Such effective versions may be defined in many topological spaces with some effectivity conditions, in particular in our c.e. spaces. Following a tradition of descriptive set theory, we denote levels of effective hierarchies in the same manner as levels of the corresponding classical hierarchies, using the lightface letters Σ, Π, Δ instead of the boldface $\mathbf{\Sigma}, \mathbf{\Pi}, \mathbf{\Delta}$ in the previous subsection.

First let us sketch a definition of the effective Borel hierarchy in arbitrary c.e. space (X, ξ). We start with the numbering $\xi_W(n) = \bigcup \xi(W_n)$ of the effective open sets in X. Let $\beta : \omega \to P(X)$ be the numbering of finite Boolean combinations of effective open sets induced by ξ_W and the Gödel numbering of Boolean terms. *Finite effective Borel hierarchy* in (X, ξ) is the sequence $\{\Sigma^0_n\}_{n<\omega}$ defined as follows: $\Sigma^0_0 = \{\emptyset\}$; Σ^0_1 is the class of of effective open sets equipped with the numbering ξ_W; Σ^0_2 is the class of sets $\bigcup \beta(W_x)$, $x \geq 0$, equipped with the numbering induced by W; Σ^0_n $(n \geq 3)$ is the class of sets $\bigcup \gamma(W_x)$, $x \geq 0$, equipped with the numbering induced by W, where γ is the numbering of Π^0_{n-1} induced by the numbering of Σ^0_{n-1} (which exists by induction).

The transfinite extension of $\{\Sigma^0_n\}_{n<\omega}$ is also defined in a natural way [13,20]. In place of ω_1 one has to take the first non-computable ordinal ω_1^{CK}. In fact, to obtain reasonable effectivity properties one should enumerate levels $\Sigma^0_{(a)}$ of the transfinite hierarchy not by computable ordinals $\alpha < \omega_1^{CK}$ but rather by their names $|a|_O = \alpha$ in the well-known Kleene notation system $(O; <_O)$ ($a \mapsto |a|_O$ is a surjection from $O \subseteq \omega$ onto ω_1^{CK}). Levels of the transfinite version are defined in the same way as for the finite levels, using the effective induction along the well-founded set $(O; <_O)$ [13]. In this way we obtain the *effective Borel hierarchy* $\{\Sigma^0_{(a)}\}_{a\in O}$ properties of which are effective versions of properties of Borel hierarchy [13,20].

The *effective DH* $\{D_{(a)}(\Sigma^0_{(b)})\}_{a\in O}$, over any level $\Sigma^0_{(b)}$, $|b|_O \geq 1$, is defined by a similar effectivization of the non-effective DH [19,20]. It is well-known and easy to see that the effective DH's satisfy the usual inclusions $D_{(a)}(\Sigma^0_{(b)}) \subseteq D_{(c)}(\Sigma^0_{(b)}) \cap$ *co*-$D_{(c)}(\Sigma^0_{(b)})$ and $D_{(a)}(\Sigma^0_{(b)}) \subseteq \Delta^0_{(2^b)}$ for all $a, c \in O$, $a <_O c$) and do not collapse (at least, in ω, in the Baire space ω^ω and in the Cantor space 2^ω). From [19] it follows that they do not collapse also in all effective reflective and 2-reflective continuous domains.

The effective DH over Σ^0_1 in the space ω, denoted by $\{\Sigma^{-1}_{(a)}\}_{a\in O}$, was introduced and comprehensively studied in [3], in particular we have the effective Hausdorff theorem $\bigcup\{\Sigma^{-1}_{(a)} \mid a \in O\} = \Delta^0_2$. Moreover, $\bigcup\{\Sigma^{-1}_{(a)} \mid a \in O, |a|_O = \omega^2\} = \Delta^0_2$ and ω^2 is the smallest ordinal with this property. In [15] this was extended to the effective DH's over $\Sigma^0_{(b)}$ for each $b \in O, |b|_O > 1$ (i.e., the effective Hausdorff theorem holds in ω) but, when $|b|_O$ increases, one needs bigger and bigger ordinals in place of ω^2. In [16] we have shown that the transfinite effective DH over Σ^0_1 in ω^ω (and in 2^ω) also exhausts Δ^0_2 (i.e., the effective Hausdorff theorem holds) but this time we have the strict inclusion $\bigcup\{\Sigma^{-1}_{(a)} \mid a \in O, |a|_O = \alpha\} \subset \Delta^0_2$ for each $\alpha < \omega_1^{CK}$. In [7] the same fact was obtained for the Euclidean spaces.

We guess that the effective Hausdorff-Kuratowski theorem holds in arbitrary c.e. ω-continuous domain but the proof will be technically much more involved than the proof of Theorem 3.4(ii). Still more problematic is the status of effective analogs of Theorem 3.4(i). Currently we have no idea how a characterization of $\Sigma^{-1}_{(a)}$-sets in c.e. ω-continuous domains could look like.

4 Limit Hierarchies of k-Partitions

It is well-known that the DH is closely related to limiting "computations" [10]. In this section we discuss this relation.

Let X, Y be topological spaces, $f : X \rightharpoonup Y$ a partial function and α, θ countable nonzero ordinals. We say that f is a $\mathbf{\Sigma}^0_\theta$-*function* if $A \in \mathbf{\Sigma}^0_\theta(Y) \to f^{-1}(A) \in \mathbf{\Sigma}^0_\theta(X)$. We say that f is an α-$\mathbf{\Sigma}^0_\theta$-*function* if there is a sequence $\{f_\beta\}_{\beta<\alpha}$ of $\mathbf{\Sigma}^0_\theta$ partial functions from X to Y such that $dom(f) = \bigcup_{\beta<\alpha} dom(f_\beta)$ and $f(x) = f_\beta(x)$ for each $x \in dom(f)$, where β is the least ordinal satisfying $x \in dom(f_\beta)$. Note that the $\mathbf{\Sigma}^0_\theta$-functions coincide with the 1-$\mathbf{\Sigma}^0_\theta$-functions, and domain of any α-$\mathbf{\Sigma}^0_\theta$-function is $\mathbf{\Sigma}^0_\theta$. For a partial function $f : X \rightharpoonup Y$ and an element $y \in Y$, let $f^y : X \to Y$ denote the y-*totalization of* f defined by $f^y(x) = f(x)$ for $x \in dom(f)$ and $f^y(x) = y$ otherwise.

Applying the above definitions to the discrete space $Y = \{0, \ldots, k-1\}$ we obtain the corresponding notions for (partial) k-partitions of X. A simple example of a partial $\mathbf{\Sigma}^0_\theta$-partition is a constant function sending all elements of a $\mathbf{\Sigma}^0_\theta$-set to a fixed $i < k$. For $k \geq 2$, $i < k$ and $\alpha < \omega_1$, let $\mathbf{C}^\mathbf{i}_{\mathbf{k},\alpha}$ be the set of i-totalizations of partial α-$\mathbf{\Sigma}^0_\theta$ k-partitions, and $\mathbf{C}_{\mathbf{k},\alpha}$ be the set of total α-$\mathbf{\Sigma}^0_\theta$ k-partitions. Similar to the proofs of corresponding facts in [14,16], it is easy to show that for any $\alpha < \omega_1$ we have $D_\alpha(\mathbf{\Sigma}^0_\theta) = \mathbf{C}^\mathbf{0}_{\mathbf{2},\alpha}$, $co\text{-}D_\alpha(\mathbf{\Sigma}^0_\theta) = \mathbf{C}^\mathbf{1}_{\mathbf{2},\alpha}$ and $D_\alpha(\mathbf{\Sigma}^0_\theta) \cap co\text{-}D_\alpha(\mathbf{\Sigma}^0_\theta) = \mathbf{C}_{\mathbf{2},\alpha}$. This shows that the sequence $\{\mathbf{C}^\mathbf{i}_{\mathbf{k},\alpha}\}_\alpha$, which we call here the *limit-hierarchy of* k-*partitions over* $\mathbf{\Sigma}^0_\theta$, generalizes the DH over $\mathbf{\Sigma}^0_\theta$.

Proposition 4.1 *Let X be a topological space and θ a countable nonzero ordinal such that $\mathbf{\Delta}^0_{\theta+1} = \bigcup\{D_\alpha(\mathbf{\Sigma}^0_\theta) \mid \alpha < \omega_1\}$. Then $(\mathbf{\Delta}^0_{\theta+1})_k = \bigcup\{\mathbf{C}_{\mathbf{k},\alpha} \mid \alpha < \omega_\mathbf{1}\}$.*

Proof. The inclusion from right to left is easy. Conversely, it suffices to show that any $\nu \in (\mathbf{\Delta}^0_{\theta+1})_k$ is an α-Σ^0_θ-partition for some $\alpha < \omega_1$. Since $\nu^{-1}(i) \in \mathbf{\Delta}^0_{\theta+1}$ for all $i < k$, there is a limit ordinal $\alpha < \omega_1$ such that $\nu^{-1}(i) \in D_\alpha(\mathbf{\Sigma}^0_\theta)$ for all $i < k$. Since α is limit, $\alpha = k \cdot \alpha$. Let $g : \alpha \to \alpha$ be the unique monotone embedding such that any $\beta < \alpha$ is uniquely representable as $\beta = g(\gamma) + i$, for some $\gamma < \alpha$ and $i < k$.

By remarks above, any $\nu^{-1}(i)$ is a total α-$\mathbf{\Sigma}^0_\theta$ 2-partition; let, for any $i < k$, $\{f^i_\beta\}_{\beta<\alpha}$ be a corresponding sequence of $\mathbf{\Sigma}^0_\theta$ partial functions from X to 2. For any

$\beta < \alpha$, let f_β be the constant function with domain $A^i_\gamma = \{x \in X \mid f^i_\gamma(x) = 1\}$ and range $\{i\}$, where $\beta = g(\gamma) + i$. Then the sequence $\{f_\beta\}_{\beta<\alpha}$ of partial constant $\mathbf{\Sigma}^0_\theta$ k-partitions witnesses that ν is an α-$\mathbf{\Sigma}^0_\theta$ k-partition. $\square$

From Proposition 4.1 and Theorems 3.3 and 3.4 we obtain "limit characterizations" of the $\mathbf{\Delta}^0_{\theta+1}$-measurable k-partitions in arbitrary Polish spaces for any nonzero $\theta < \omega_1$, and of $\mathbf{\Delta}^0_2$-measurable k-partitions in arbitrary ω-continuous domain.

It is straightforward to get the effective analog of Proposition 4.1.

5 Difference Hierarchies of k-Partitions

Here we discuss the DH (known also as the Boolean hierarchy) of k-partitions studied for the case of finite k-posets in [11,17] and for the countable case in [21,23].

Let $P = (P; \leq)$ be a countable poset without infinite chains, X a topological space and let $\mathcal{L} \subseteq P(X)$ be closed under finite intersections and countable unions. Functions of the form $S : P \to \mathcal{L}$ are called *P-families* and are denoted also by $\{S_p\}_{p\in P}$. A P-family is *monotone* if it is a monotone function from $(P; \leq)$ into $(\mathcal{L}; \subseteq)$. A P-family S is *admissible* if $\bigcup_p S_p = X$ and $S_p \cap S_q \subseteq \bigcup\{S_r \mid r \leq p, q\}$ for all $p, q \in P$. Note that if P is a forest then P-family S is admissible iff $\bigcup_p S_p = X$ and $S_p \cap S_q = \emptyset$ for all p, q incomparable in P. For any P-family S, define a map $\tilde{S} : P \to P(X)$ by $\tilde{S}_p = S_p \setminus \bigcup_{q<p} S_q$. It is easy to see that if S is admissible then $\{\tilde{S}_p\}_{p\in P}$ is a partition of X. Note that if $\{S_p\}_{p\in P}$ is admissible then the P-family defined by $S^*_p = \bigcup\{S_q \mid q \leq p\}$ is monotone and admissible and the partitions $\{\tilde{S}_p\}_{p\in P}$, $\{\tilde{S}^*_p\}_{p\in P}$ coincide.

We say that an admissible family $\{S_p\}_{p\in P}$ *approximates* a k-partition $\nu \in k^X$ if $\nu = c \circ \tilde{S}$ where $\tilde{S}$ is identified with the function from X to P sending $x \in X$ to the unique $p \in P$ with $x \in \tilde{S}_p$. For a countable k-poset (P, c) without infinite chains, let $\mathcal{L}(P, c)$ be the set of k-partitions approximated by admissible (equivalently, by monotone admissible) P-families. *The difference hierarchy of k-partitions over $\mathcal{L}$* is by definition the family $\{\mathcal{L}(P)\}_{P\in\widetilde{\mathcal{P}}_k}$; by $BH_k(\mathcal{L})$ we denote the collection $\{\mathcal{L}(P) \mid P \in \widetilde{\mathcal{P}}_k\}$ of levels of this hierarchy. We consider also a smaller collection of classes of k-partitions $FBH_k(\mathcal{L}) = \{\mathcal{L}(P) \mid P \in \widetilde{\mathcal{F}}_k\}$ defined by the k-forests. If we omit the condition $\bigcup_p S_p = X$ in the definition of admissible family, we obtain the DH of partial k-partitions over $\mathcal{L}$.

In [11,21] it was observed that levels of the DH are closely related to the h-preorder, namely for all countable k-posets P and Q without infinite chains $P \leq Q$ implies $\mathcal{L}(P) \subseteq \mathcal{L}(Q)$. Since the h-preorder of k-posets is far from being a well preorder, the DH of k-partitions defined by posets do not in general have properties one expects from a hierarchy. The DH of k-partitions defined by forests is in this respect better because $(FBH_k(\mathcal{L}); \subseteq)$ is a well poset. In [17,21] it was shown that over any $\mathcal{L}$ with σ-reduction property we have $BH_k(\mathcal{L}) = FBH_k(\mathcal{L})$, and hence the poset $(BH_k(\mathcal{L}); \subseteq)$ is a well preorder of rank $\leq \omega_1$.

Let $\bigsqcup_i P_i = P_0 \sqcup P_1 \sqcup \cdots$ be the join of a sequence $P_0, P_1, \ldots$ of k-posets. For a

k-forest F and $i < k$, let $p_i(F)$ be the k-tree obtained from F by adjoining a new biggest element and assigning the label i to this element.

It is easy to see that for any topological space X and any nonzero countable ordinal θ we have $\bigcup\{\mathbf{\Sigma}^0_\theta(P) \mid P \in \widetilde{\mathcal{P}}_k\} \subseteq (\mathbf{\Delta}^0_{\theta+1})_k$. The next result shows that in many cases we may improve this relation.

Theorem 5.1 *Let X be a topological space and θ a countable nonzero ordinal such that $\mathbf{\Sigma}^0_\theta$ has the σ-reduction property and $\mathbf{\Delta}^0_{\theta+1} = \bigcup\{D_\alpha(\mathbf{\Sigma}^0_\theta) \mid \alpha < \omega_1\}$. Then $(\mathbf{\Delta}^0_{\theta+1})_k = \bigcup\{\mathbf{\Sigma}^0_\theta(F) \mid F \in \widetilde{\mathcal{F}}_k\}$.*

Proof. We need to show only the inclusion from left to right. We prove this even for partial k-partitions. Let ν be a partial k-partition in $(\Delta^0_{\theta+1})_k$. By Proposition 4.1 there is $\alpha < \omega_1$ and a sequence $\{f_\beta\}_{\beta<\alpha}$ witnessing that ν is a partial α-Σ^0_θ k-partition. Moreover, by the proof of that proposition we way assume that any f_β is a constant function defined on a Σ^0_θ-set A_β and taking some value $i_\beta < k$. Obviously, for any $\gamma < \alpha$ the sequence $\{f_\beta\}_{\beta<\gamma}$ witnesses that the restriction $\nu|_{C_\gamma}$ of ν to the set $C_\gamma = \bigcup\{A_\beta \mid \beta < \gamma\}$ is a partial γ-Σ^0_θ k-partition.

We find $F \in \mathcal{F}_k$ with $\nu \in \Sigma^0_\theta(F)$ by induction on α. For $\alpha = 0, 1$ the assertion is trivial. Let $\alpha = \beta + 1 > 1$. By induction, there is $G \in \mathcal{F}_k$ and an admissible family $\{S_p\}_{p\in G}$ of $\mathbf{\Sigma}^0_\theta$-sets that approximates $\nu|_{C_\beta}$. We set $F = p_{i_\beta}(G)$ and define an admissible family $\{R_q\}_{q\in F}$ of $\mathbf{\Sigma}^0_\theta$-sets by $R_r = A_\beta$ (where r is the root of F) and $R_p = S_p$ for $p \in G$. Then $\{R_q\}_{q\in F}$ approximates ν, hence $\nu \in \mathbf{\Sigma}^0_\theta(F)$.

Finally, let α be a limit ordinal. By induction, for any $\gamma < \alpha$ there is $F_\gamma \in \widetilde{\mathcal{F}}_k$ and an admissible family $\{S^\gamma_p\}_{p\in F_\gamma}$ of $\mathbf{\Sigma}^0_\theta$-sets that approximates $\nu|_{C_\gamma}$. Let $\{A'_\gamma\}_{\gamma<\alpha}$ be a reduct of $\{A_\gamma\}_{\gamma<\alpha}$, $F = \bigsqcup\{F_\gamma \mid \gamma < \alpha\}$ and $\{R_p\}_{p\in F}$ the admissible family of $\mathbf{\Sigma}^0_\theta$-sets defined by $R_p = A'_\gamma \cap S_p$ where γ is the unique ordinal with $p \in F_\gamma$. Then $\{R_q\}_{q\in F}$ approximates ν, hence $\nu \in \mathbf{\Sigma}^0_\theta(F)$. □

Theorem 5.1 and results mentioned in Section 3.1 characterize the classes $(\mathbf{\Delta}^0_{\theta+1})_k$ for $\theta \geq 2$ in any Polish space, and also the class $(\mathbf{\Delta}^0_2)_k$ in the Baire space and Baire domain, in terms of the DH's of k-partitions over k-trees. Currently we do not know whether the equality $\mathbf{\Delta}^0_{\theta+1} = \bigcup\{D_\alpha(\mathbf{\Sigma}^0_\theta) \mid \alpha < \omega_1\}$ implies the equality $(\mathbf{\Delta}^0_{\theta+1})_k = \bigcup\{\mathbf{\Sigma}^0_\theta(P) \mid P \in \widetilde{\mathcal{P}}_k\}$in arbitrary topological space.

Another interesting open question is the non-collapse of the DH of partitions (for the forest-hierarchy this means that for all $S, T \in \widetilde{\mathcal{T}}_k$, $\mathbf{\Sigma}^0_\theta(S) \subseteq \mathbf{\Sigma}^0_\theta(T)$ iff $S \leq T$). In [22] we established the non-collapse of the forest-hierarchy over $\mathbf{\Sigma}^0_1$ in any Polish space and in any reflective or 2-reflective ω-algebraic domain (as above, the last fact holds also for the ω-continuous domains). We guess that the non-collapse property holds also for any $\theta > 1$ but have no proof at hand.

It is straightforward to get the effective analog of Theorem 5.1.

6 Conclusion

Good properties of the DH and its effective versions were obtained so far only for two natural classes of spaces: Polish spaces and ω-continuous domains. To my knowledge, these are the only natural classes of spaces for which natural notions of completeness and completion theorems are known. In this respect these classes are close to each other (though many other basic topological properties of Polish spaces and of ω-continuous domains are very different).

An interesting general open question is to identify broader natural classes of countably based T_0-spaces with good descriptive set theory or find counterexamples clearly demonstrating that outside the two mentioned classes there is no good descriptive set theory.

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