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On Adjacent-vertex-distinguishing Total Colourings of Powers of Cycles, Hypercubes and Lattice Graphs [1]

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Abstract

Given a graph G and a vertex $v \in G$, the chromatic neighbourhood of v is the set of colours of v and its incident edges. An adjacent-vertex-distinguishing total colouring (AVDTC) of a graph G is a proper total colouring of G which every two adjacent vertices on G have different chromatic neighbourhood. It was conjectured in 2005 that the minimum number of colours that guarantees the existence of an AVDTC of a graph G with these colours, $\chi''_a(G)$, is bounded from above by $\Delta(G)+3$ for any graph G. In this work we prove the validity of this conjecture for hypercubes, lattice graphs and powers of cycles C_n^k when either (i) $k=2$ and $n \geq 6$, or (ii) $n \equiv 0 \pmod{k+1}$ through the construction of an explicit AVDTC which shows that $\chi''_a(G) = \Delta(G)+2$ for each of the preceding graph classes.

Keywords: Adjacent-vertex-distinguishing total colouring, adjacent-vertex-distinguishing total chromatic number, lattice graphs, hypercubes, powers of cycles.

1 Introduction

Given a graph $G = (V, E)$, a total colouring of G is a colouring of vertices and edges of G which every two adjacent vertices of G have different colours, every two adjacent edges of G have different colours and every vertex has a colour different from the colours of its incident edges. A famous statement concerning total colourings is the *Total Colouring Conjecture*:

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Conjecture 1.1 (TCC) *Every simple graph G admits a total colouring using at least $\Delta(G) + 1$ and at most $\Delta(G) + 2$ colours.*

The validity of the TCC is known for important classes of graphs, for instance bipartite graphs and most planar graphs. A particular kind of total colourings is the adjacent-vertex-distinguishing total colourings. This class of colourings was introduced by Zhang, Chen, Li, Yao, Lu and Wang in 2005 (see [9]) motivated by the adjacent-vertex-distinguishing proper edge colourings presented in [10] and the vertex-distinguishing proper edge colourings studied by Burris and Schelp in 1997, by Bazgan in 1999, and by Balister, Bollobás and Schelp in 2002 (see [1,2] and [3]).

In order to explain the notion of an adjacent-vertex-distinguishing total colouring, we introduce the concept of chromatic neighbourhood of a vertex v, which is the set of colours assigned to v and all its incident edges. An adjacent-vertex-distinguishing total colouring (AVDTC) of a graph G is a total colouring of G which satisfies the property that every two adjacent vertices have different chromatic neighbourhood. It is important to notice that if two adjacent vertices have different degrees, their chromatic neighbourhood are different for any total colouring. In this context, regular graphs are the most difficult graphs to be coloured with an AVDTC. A natural question to ask is what is the minimal number of colours that guarantees the existence of an AVDTC on G with these colours. The answer to this question is the adjacent-vertex-distinguishing total chromatic number $\chi''_a(G)$. Zhang-Chen-Li-Yao-Lu-Wang [9] stated the following conjecture:

Conjecture 1.2 (AVDTC Conjecture, [9]) *For any simple connected graph G with order at least 2, we have*

$$\chi''_a(G) \leq \Delta(G) + 3,$$

where $\Delta(G)$ stands for the maximum degree among the vertices of G.

The idea of the AVDTC conjecture is that, assuming that the TCC is true, it should be expected that adding at most one colour, one could obtain an AVDTC for the same graph using some nice strategy such as recolouring arguments.

In 2007, Wang proved that the AVDTC Conjecture holds for graphs G with $\Delta(G) = 3$ (see [8]). Chen showed in 2008 that $\chi''_a(G) \leq \Delta(G) + 2$ for bipartite graphs using König's Theorem without showing an AVDTC construction (see [11]). Four years later, Coker and Johannson used the probabilistic method to show that the ADVTC Conjecture is true up to a constant, that is, $\chi''_a(G) \leq \Delta(G) + \mathcal{O}(1)$ (see [5]). In 2014, Papaioannou and Raftopoulou described an algorithmic procedure to prove the AVDTC Conjecture for 4-regular graphs (see [7]). This result was extended by Lu, Li, Luo and Miao for graphs G with $\Delta(G) = 4$. Their approach relied on the *polynomial method* (see [6]).

In this paper we prove that the AVDTC conjecture [9] holds for k-th powers of cycles of length n (C_n^k) when either $k = 2$ and $n \geq 6$, or $n \equiv 0 \pmod{k + 1}$, for square grid lattice graphs and for hypercubes in any dimension. For all these graphs, we show the exact value $\chi''_a(G) = \Delta(G) + 2$ by constructing an explicit

AVDTC with $\Delta(G) + 2$ colours.

We organize the paper in the following way: in Section 2 we introduce some definitions and notation concerning adjacent-vertex-distinguishing total colourings and we state our results. Section 3 is reserved to the proof of Theorem 2.1, where we show that $\chi''_a(C_n^2) = 6$ for every $n \geq 6$. We remark that Theorem 2.1 in an improvement of the upper bound obtained using the result of [7]. In Section 4 we prove Theorem 2.3 constructing an AVDTC with $2k + 2$ colours for k-th powers of cycles of length n when $n \equiv 0 \pmod{k+1}$, and finally, in Section 5, we prove Theorem 2.5 showing that the d-dimensional hypercube has its adjacent-vertex-distinguishing total chromatic number equal to $d+2$ and that d-dimensional square grid lattices have their adjacent-vertex-distinguishing total chromatic number equal to $2d + 2$. Finally, in Section 6, we present our conclusions.

2 Notation and results

2.1 Adjacent-vertex-distinguishing total colouring on graphs

Let $V = V(G)$ be the set of vertices and $E = E(G)$ the set of edges of a graph $G = (V, E)$. We will say that the vertices v and w are *adjacent* whenever they have a common incident edge $e_{v,w}$ in E, and we will denote it by $v \sim w$.
A *total colouring* of G is a function $c : V \cup E \to \mathbb{N}$ that satisfies the following properties:

i) $c(v) \neq c(w), \forall\, v \sim w$ (vertex colouring condition);

ii) $c(e_{v,w}) \neq c(e_{x,y})$, whenever $\{v, w\} \cap \{x, y\} \neq \emptyset$ (edge colouring condition);

iii) $c(e_{v,w}) \neq c(x), \forall\, x \in \{v, w\}$,

that is, adjacent vertices have different colours, so do adjacent edges, and every vertex is coloured differently of its incident edges. Now, given a graph $G = (V, E)$, a colouring $c : V \cup E \to \mathbb{N}$ on G, and a vertex $v \in V$, we define the *chromatic neighbourhood* of v with respect to c as the set

$$C(v) := \{m \in \mathbb{N}; m = c(v) \text{ or } m = c(e_{v,w}) \text{ for some } w \sim v\}\,.$$[3]

We will say that the vertices v and w are *distinguishable* whenever their chromatic neighbourhood differ by at least one colour, that is, $C(v) \neq C(w)$. Furthermore, if a colouring $c : V \cup E \to \mathbb{N}$ has the property that all adjacent vertices are distinguishable, then we will say that the colouring c is an *adjacent-vertex-distinguishing total colouring* of G, which we abbreviate as *AVDTC*.

The *adjacent-vertex-distinguishing total chromatic number* $\chi''_a(G)$ of G is defined as the minimum number of colours such that there exists an AVDTC of G with these colours, that is,

$$\chi''_a(G) := \min_{m \in \mathbb{N}} \Big\{\exists\ c : V \cup E \to \{0, 1, \ldots, m-1\}\,; c \text{ is an AVDTC on } G\Big\}.$$

[3] Observe that the sets $C(\cdot)$ defined above depend on the chosen colouring c; We will not explicit this dependence in the notation.

2.2 Powers of cycles

Let C_n be the cycle of length n. We can associate the vertex set of C_n to the group $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z} = \{0, 1, \ldots, n-1\}$. In this structure, x and y are adjacent if and only if $x = y \pm 1 \pmod{n}$. Analogously, we define the *k-th power of the cycle C_n* as the graph C_n^k with vertex set $\mathbb{Z}_n$ and edges $e_{x,y}$ for $x, y \in \mathbb{Z}/n\mathbb{Z}$ with $x = y \pm j \pmod{n}$ for some $j \in \{1, 2, \ldots, k\}$. Notice that C_n^k is a *$2k$-regular graph*, that is, every vertex in C_n^k has exactly $2k$ adjacent vertices (or incident edges).

2.3 Hypercubes

As we have seen in this section, one can associate C_n to the graph whose vertex set is the group $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ with x and y adjacent if and only if $|x - y| \equiv 1$ mod n. We define the *1-dimensional hypercube* $\mathbb{H}_1$ as the two-vertex path (which we call C_2) seen under this modular structure and recursively, we define the *d-dimensional hypercube* $\mathbb{H}_d$ as the graph whose vertex set is given by $\mathbb{Z}_2^d$ and with $x = (x_1, \ldots, x_d) \sim y = (y_1, \ldots, y_d)$ if and only if $\sum_{i=1}^{d} |x_i - y_i| = 1$. This structure will be useful in Section 5 where we will see $\mathbb{H}_{d+1}$ as the cartesian product of $\mathbb{H}_d$ and $\mathbb{H}_1$. Notice that $\mathbb{H}_d$ is a d-regular graph.

2.4 Lattice graphs

In the same way that we have done so far, we will define the *d-dimensional square grid* $\mathbb{L}_\square^d$ as the graph whose vertex set is $\mathbb{Z}^d$ and with $x = (x_1, \ldots, x_d) \sim y = (y_1, \ldots, y_d)$ if and only if $\sum_{i=1}^{d} |x_i - y_i| = 1$. The subgraph $\mathbb{G}_n^d$ of $\mathbb{L}_\square^d$ induced by the vertex set $\mathbb{Z}_n^d \subset \mathbb{Z}^d$ is the *d-dimensional box of length n*. Notice that $\mathbb{L}_\square^d$ is $2d$-regular but $\mathbb{G}_n^d$ is not. In order to obtain a regular graph with the same vertex set, one can define $x = (x_1, \ldots, x_d) \sim y = (y_1, \ldots, y_d)$ if and only if there exists $i \in \{1, \ldots, d\}$ such that $x_i = y_i \pm 1 \pmod{n}$ and $x_j = y_j, \forall j \neq i$. The resulting graph $\mathbb{T}_n^d$ is the *d-dimensional torus of length n*. The graphs $\mathbb{L}_\square^d$, $\mathbb{G}_n^d$ and $\mathbb{T}_n^d$ defined above are called *lattice graphs*.

2.5 Results

All our results prove the AVDTC conjecture for power of cycles when either $k = 2$ and $n \geq 6$, or $n \equiv 0 \pmod{k+1}$, square grid lattice graphs and hypercubes, without the help of König's Theorem, i.e., we exhibit an explicit AVDTC using the nice arithmetic properties of the finite cyclic groups $\mathbb{Z}_N$.

Our first result says that, for $n \geq 6$, AVDTC conjecture is true if we consider G as the second power of a cycle of length n. Moreover, we give an explicit formula to $\chi''_a\left(C_n^2\right)$.

Theorem 2.1 *For each $n \geq 6$, we have*

$$\chi''_a\left(C_n^2\right) = 6.$$

Remark 2.2 With the group structure defined before for second power of cycles, we establish the orientation on the edges of G which for every edge $e_{i,j}$, we have $0 \leq i < j < n-1$ and $j - i \leq 2$, except for the edges $e_{n-1,0}, e_{n-1,1}$ and $e_{n-2,0}$.

Our second result extends the validity of Conjecture 1.2 for arbitrary powers of cycles C_n^k provided n is multiple of $k+1$.

Theorem 2.3 *If $n \equiv 0 \pmod{k+1}$, then*

$$\chi''_a\left(C_n^k\right) = 2k+2.$$

Remark 2.4 When $n \not\equiv 0 \pmod{k+1}$, the colouring used to prove Theorem 2.3 does not help us (see Section 4). In order to make it useful, we do some modifications on it. This strategy was first used by Campos and de Mello in [4], in the context of total colouring (and n even), where König's Theorem plays an important role. However, this colouring is not an AVDTC. By a recolouring argument, we prove the AVDTC Conjecture for k-th powers of cycles of length n whenever n is even and, $n \equiv k-1 \pmod{k+1}$ or $n \equiv k \pmod{k+1}$. The remaining cases are open.

Theorem 2.5 *If $d \geq 1$, $k > 1$ and $G \in \left\{\mathbb{H}_d, \mathbb{L}_{\square}^d, \mathbb{G}_n^d, \mathbb{T}_{2k}^d\right\}$, then*

$$\chi''_a(G) = \Delta(G) + 2.$$

3 Zhang-Chen-Li-Yao-Lu-Wang theorem for the second power of a cycle

Now we present a proof for Theorem 2.1. We consider $G = C_n^2$ throughout the section. By Lemma 1.1. in [9], we have that $\chi''_a\left(C_n^2\right) \geq 6$. Therefore, it is enough to find an AVDTC of C_n^2 with 6 colours. Indeed, let $r = r(n)$ be the unique element of $\mathbb{Z}_3$ such that $n \equiv r \pmod 3$. For each r we will exhibit an AVDTC $c_r : V \cup E$ that satisfies this requirement. Before defining the colourings c_r, we introduce an auxiliary function $f : \mathbb{Z}_6 \times \mathbb{Z}_6 \to \mathbb{Z}_6 \cup \{\perp\}$ whose image can be seen in the entrances of the 6×6 matrix F given by

$$F = \begin{pmatrix} F_{0,0} & F_{0,1} & F_{0,2} & F_{0,3} & F_{0,4} & F_{0,5} \\ F_{1,0} & F_{1,1} & F_{1,2} & F_{1,3} & F_{1,4} & F_{1,5} \\ F_{2,0} & F_{2,1} & F_{2,2} & F_{2,3} & F_{2,4} & F_{2,5} \\ F_{3,0} & F_{3,1} & F_{3,2} & F_{3,3} & F_{3,4} & F_{3,5} \\ F_{4,0} & F_{4,1} & F_{4,2} & F_{4,3} & F_{4,4} & F_{4,5} \\ F_{5,0} & F_{5,1} & F_{5,2} & F_{5,3} & F_{5,4} & F_{5,5} \end{pmatrix} = \begin{pmatrix} \perp & 5 & 4 & \perp & \perp & \perp \\ 2 & \perp & 3 & \perp & 2 & \perp \\ 1 & 0 & \perp & \perp & 5 & \perp \\ \perp & \perp & \perp & \perp & \perp & \perp \\ 3 & 0 & \perp & \perp & \perp & \perp \\ \perp & \perp & \perp & \perp & \perp & \perp \end{pmatrix}.$$

In order to facilitate the notation, the lines and the columns of the matrix are labelled from 0 to 5. We define the function $f : \mathbb{Z}_6 \times \mathbb{Z}_6 \to \mathbb{Z}_6 \cup \{\perp\}$ by

$$f(i,j) = F_{i,j}.$$

Here, the reader may assume that elements of $\mathbb{Z}_6$ are colours and $\perp$ is a 7-th colour. Given the function f above, it is straightforward to verify the validity of the following proposition.

Proposition 3.1 *Consider $f : \mathbb{Z}_6 \times \mathbb{Z}_6 \to \mathbb{Z}_6 \cup \{\perp\}$ defined above. The following properties hold:*

(i) If $i, j \in \{0, 1, 2\}$ and $f(i, j) = \perp$, then $i = j$.

(ii) If $(i_1, j_1) \neq (i_2, j_2)$ and $f(i_1, j_1) = f(i_2, j_2) \neq \perp$, then

$$\{(i_1, j_1), (i_2, j_2)\} \in \{\{(1, 4), (1, 0)\}, \{(2, 4), (0, 1)\}, \{(4, 0), (1, 2)\}, \{(4, 1), (2, 1)\}\}.$$

(iii) $f(i, j) \notin \{i, j\}$.

(iv) If $i, j \in \{0, 1, 2\}$, then $f(i, j) \notin \{i + 3, j + 3\}$.

(v) For each $i, j \in \mathbb{Z}_6$, we have $f(i, 1) \neq 4$ and $f(1, j) \neq 4$.

Now, we define the AVDTC colourings $c_r : V \cup E \to \mathbb{Z}_6 \cup \{\perp\}$ that will be used to colour C_n^2. We start with $r = 0$. Define

$$c_0(x) = \begin{cases} x \pmod 3, & \text{if } x \in V; \\ f(c_0(i), c_0(j)), & \text{if } x = e_{i,j} \in E. \end{cases}$$

Now for each $\ell \in \{0, 1\}$, we use c_ℓ to define $c_{\ell+1}$ in the following way:

$$c_1(x) = \begin{cases} c_0(x), & \text{if } x \in V \text{ and } x \leq n - 2; \\ 4, & \text{if } x \in V \text{ and } x = n - 1; \\ f(c_1(i), c_1(j)), & \text{if } x = e_{i,j} \in E. \end{cases}$$

$$c_2(x) = \begin{cases} c_1(x), & \text{if } x \in V \text{ and, } x \leq n - 6 \text{ or } x = n - 1; \\ 4, & \text{if } x \in V \text{ and } x = n - 5; \\ 4 - m, & \text{if } x \in V \text{ and } x = n - m \text{ for } 2 \leq m \leq 4; \\ f(c_2(i), c_2(j)), & \text{if } x = e_{i,j} \in E. \end{cases}$$

Our next lemma says that, for any $r \in \mathbb{Z}_3$, the colour $\perp$ does not belong to the image of c_r.

Lemma 3.2 *For any $n \in \mathbb{N}$ we have $c_{r(n)}^{-1}(\perp) = \emptyset$.*

Proof. By the definition of c_r, $c_r(i) \in \mathbb{Z}_6$ for any $i \in V$ and any $r \in \mathbb{Z}_3$. Therefore, we just need to verify that no edge $e \in E$ is coloured with $\perp$. We start with the case $r = 0$. Suppose that there exists $e_{i,j} \in E$ such that $c_0(e_{i,j}) = \perp$, that is, $f(i \pmod 3, j \pmod 3) = \perp$. Thus, by property *(i)* in Proposition 3.1, we have that $i \equiv j \pmod 3$, that is, 3 divides $j - i$. Recalling Remark 2.2, we conclude that (i, j) can only belong to the set $\{(n-2, 0), (n-1, 0), (n-1, 1)\}$, which is a contradiction since $n \equiv 0 \pmod 3$. Therefore, $c_0^{-1}(\perp) = \emptyset$. Moreover, by the validity of the lemma for $r = 0$ and the definition of c_1, in order to prove it for

$r = 1$, it is enough to consider the edges $e_{n-3,n-1}, e_{n-2,n-1}, e_{n-1,0}$ and $e_{n-1,1}$, which follows by the definition of f. The case $r = 2$ is similar. $\square$

In Lemma 3.2, we proved that for any $n \in \mathbb{N}$, $c_{r(n)}$ is a colouring that assumes 6 possible values. Now, we state another lemma, which says that c_r is a total colouring of C_n^2, and we finally prove Theorem 2.1, showing that c_r is indeed an AVDTC of C_n^2.

Lemma 3.3 *For any $n \in \mathbb{N}$, the function $c_{r(n)}$ is a total colouring of C_n^2.*

Proof. By property *(iii)* in Proposition 3.1, we must only show that c_r is a vertex colouring and an edge colouring of C_n^2. Indeed, suppose that $c_r(i) = c_r(j)$ for some $i, j \in C_n^2$ incident to the edge $e_{i,j}$. By definitions of c_r and f, $c_r(e_{i,j}) = \perp$, which contradicts Lemma 3.2. Therefore, c_r is a vertex colouring of C_n^2. Similarly, suppose that $c_r(e_{i,j}) = c_r(e_{i,m})$ for some $i, j, m \in C_n^2$. By property *(ii)* in Proposition 3.1, $c_r(i) = 1$ and, $\{c_r(j), c_r(m)\} \subset \{\{0,4\}, \{2,4\}\}$. Now, using the definition of c_r for each case of $r \in \mathbb{Z}_3$, it is straightforward to verify that both of these colourings on the vertices $c_r(j)$ and $c_r(m)$ imply a contradiction. Hence, c_r is an edge colouring of C_n^2. $\square$

Proof of Theorem 2.1. For each vertex $i \in \{0, 1, \ldots, n-1\}$ we define $\overline{C}_r(i)$ the subset of colours that do not appear on i neither on its incident edges, that is,

$$\overline{C}_r(i) = \left\{ m \in \mathbb{Z}_6;\ c_r^{-1}(\{m\}) \neq \{i\} \cup \bigcup_{j \sim i} \{e_{i,j}\} \right\}$$

Once again, we start with the case $r = 0$. Recall that c_0 colours the vertices of C_n^2 with exactly 3 colours, which are $0, 1$ and 2. Look at the sub-matrix of F associated to the colours $0, 1$ and 2. Since $|\overline{C}_0(i)| = 1$ for every $i \in \{0, 1, \ldots, n-1\}$, property *(iv)* in Proposition 3.1 implies that $\overline{C}_0(i) = \{c_0(i) + 3\}$. Thus, if $\overline{C}_0(i) = \overline{C}_0(j)$, then $c_0(i) = c_0(j)$, which contradicts Lemma 3.3. Therefore, c_0 is an AVDTC of C_n^2. Now suppose that $r = 1$ and recall the definition of c_1. Notice that the colouring c_1 colours the vertices of C_n^2 with the colours $0, 1$ and 2, except the vertex $n-1$ which is coloured with the colour 4. The addition of this fourth colour creates the colours $f(1,4), f(2,4), f(4,0)$ and $f(4,1)$ on the edges of $n-1$ instead of $f(1,0)$ and $f(2,1)$. Thus, we obtain

$$\overline{C}_0(i) = \begin{cases} \{c_1(i) + 3\}, & \text{if } 1 \leq i \leq n-3; \\ \{0\}, & \text{if } i = n-2; \\ \{1\}, & \text{if } i = n-1; \\ \{2\}, & \text{if } i = 0. \end{cases}$$

Therefore, if $\overline{C}_1(i) = \overline{C}_1(j)$, then $c_1(i) = c_1(j)$ and $i, j \in \{1, 2, \ldots, n-3\}$ and once again we obtain a contradiction to Lemma 3.3. Hence, c_1 is an AVDTC (see Fig. 1 for an illustration). The case $r = 2$ is similar. $\square$

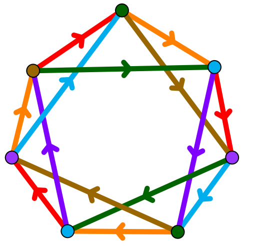

Fig. 1. AVDTC of C_7^2 with 6 colours.

4 Extending the result to larger powers of cycles

In this section we prove Theorem 2.3. The proof relies in some ideas taken from [4]. Here we denote by $G = (V, E)$ the graph C_n^k, in particular $V = \mathbb{Z}_n$. If $n > 2k$, we say that an edge of the form $e_{x,x+t} \in E$, $t \in [k]$ [4], is even (resp. odd) if and only if t is even (resp. odd). Let G_P (G_I) be the spanning sub-graph of G whose edge set E_P (E_I) is formed by all the even (odd) edges of G. To each even edge $e_{x,y}$, with even distance $2t$ between x and y in $\mathbb{Z}_n$, we associate its *central vertex* $x + t \pmod n$. Analogously, to each odd edge $e_{x,y}$, with odd distance $2t + 1$ between x and y in $\mathbb{Z}_n$, we associate its *central edge* $e_{x+t(\text{mod } n),x+t+1(\text{mod } n)}$. As in the previous section, we use finite cyclic groups to colour the graph G. In order to do that we will need disjoint copies $\mathbb{Z}'_N := \{0', \ldots, (N-1)'\}$ of the groups $\mathbb{Z}_N = \{0, \ldots, N-1\}$.

Proof of Theorem 2.3. We start the proof defining a vertex colouring of G. Define $c : V \to \mathbb{Z}_{k+1}$ by $c(x) := x \pmod{k+1}$. Now, we extend c to the domain $V \cup E_P$ by inducing, to each even edge, the colour of its central vertex. Now, we define the edge colouring $c' : E(C_n) \to \mathbb{Z}'_{k+1}$ by $c\left(e_{x,(x+1)(\text{mod } n)}\right) := (c(x))'$. Again, we extend c' to the domain E_I by inducing, to each odd edge, the colour of its central edge. We claim that $c \cup c'$ is an AVDTC of G. It is clear, since $n \equiv 0 \pmod{k+1}$, that $c \cup c'$ is a total colouring of G. Furthermore, we can easily check that

$$\overline{C}(x) = \begin{cases} x + \frac{k+1}{2} \pmod{k+1}, & \text{if } k \text{ is odd;} \\ \left(x + \frac{k}{2} \pmod{k+1}\right)', & \text{if } k \text{ is even.} \end{cases}$$

Therefore, $\overline{C}(x) = \overline{C}(y)$ implies $x = y$ and it completes the proof of the claim.

Since the colouring $c \cup c'$ used $2k+2$ colours, we have shown that $\chi''_a(G) \leq 2k+2$, and as before, by Lemma 1.1. in [9], we conclude that $\chi''_a(G) = \Delta(G) + 2$.

□

[4] For any $k \in \mathbb{N}$ we define the set $[k] := \{1, \ldots, k\}$.

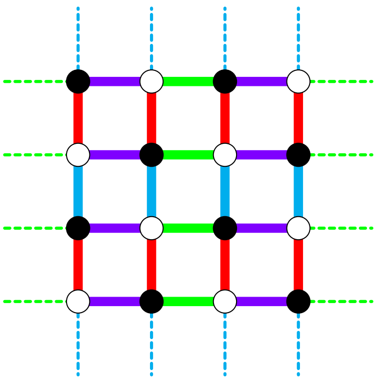

Fig. 2. An AVDTC of $\mathbb{L}^2_\square$ with 6 colours.

5 On the adjacent-vertex-distinguishing total chromatic number of hypercubes and lattice graphs

In this section we prove Theorem 2.5. We divide the proof in two propositions:

Proposition 5.1 *If $d \geq 1$, $k > 1$ and $G \in \left\{\mathbb{L}^d_\square, \mathbb{G}^d_n, \mathbb{T}^d_{2k}\right\}$, then $\chi''_a(G) = 2d+2$.*

Proposition 5.2 *If $d \geq 1$, then $\chi''_a\Big(\mathbb{H}_d\Big) = d+2$.*

By Lemma 1.1. in [9], we have that $\chi''_a(G) \geq \Delta(G)+2$ for every $G \in \left\{\mathbb{H}_d, \mathbb{L}^d_\square, \mathbb{G}^d_n, \mathbb{T}^d_{2k}\right\}$. Therefore, it is enough to obtain an AVDTC $c_d : V(G) \cup E(G) \to \mathbb{N}$ for G with exactly $\Delta(G)+2$ colours.

The idea is to colour all the vertices of G with only two colours (it is possible since these graphs are bipartite) and see that the number of colours needed to colour the edges increases as fast as the dimension does. Since its argument is more delicate, we leave the case of hypercubes to the end of this section.

The argument to obtain an AVDTC for the graphs $\mathbb{L}^d_\square, \mathbb{G}^d_n, \mathbb{T}^d_{2k}$, $d \geq 1$ and $k > 1$, is simpler:

Proof of Proposition 5.1. Since the graphs $\mathbb{L}^d_\square, \mathbb{G}^d_n, \mathbb{T}^d_{2k}$ are bipartite, we can colour all its vertices with the colours $\{0,1\}$. Now, given the set $E_i = \{e_{x,y} \in E(G) : x_j = y_j, \forall j \neq i \text{ and } y_i = x_i + 1\}$, for each $i \in \{1, \ldots, d\}$, we define the sets

$$E^1_i = \{e_{x,y} \in E_i : x_i \text{ is even }\}$$

and

$$E^2_i = \{e_{x,y} \in E_i : x_i \text{ is odd }\}.$$

We colour the edges in E^1_i with the colour i+1 and the edges on E^2_i with the colour $d+i+1$. Since this resulting colouring is an AVDTC of G, Proposition 5.1 is proved. For a illustrative proof, see Fig. 2. $\square$cyan

Finally, we adapt the idea above to colour the hypercubes.

Proof of Proposition 5.2. The proof is by induction. We will show that for every $d \geq 1$, there exists a total colouring $c_d : V(\mathbb{H}_d) \cup E(\mathbb{H}_d) \to \mathbb{Z}_{d+2}$ of $\mathbb{H}_d$ with the following properties:

(i) $c_d(v) \in \{0, 1\}$ for every $v \in \mathbb{H}_d$ and $c_d(v) \neq c_d(w)$ for every $w \sim v$;

(ii) $|c_d(E(\mathbb{H}_d))| = d$, that is, $c_d(e) = d$, $\forall\, e \in E(\mathbb{H}_d)$;

(iii) $\overline{C}_d(x) = \{1 - c_d(x)\}$, for all $x \in V(\mathbb{H}_d)$.

Notice that properties (i) and (iii) imply that c_d is an AVDTC of $\mathbb{H}_d$.

Firstly, consider the 1-dimensional hypercube $\mathbb{H}_1$ whose vertex set is $V(\mathbb{H}_1) = \{0, 1\}$ and edge set is $E(\mathbb{H}_1) = \{e_{0,1}\}$ with the colouring $c_1 : V(\mathbb{H}_1) \cup E(\mathbb{H}_1) \to \mathbb{Z}_3$ given by

$$c_1(x) = \begin{cases} x, & \text{if } x \in V(\mathbb{H}_1), \\ 2, & \text{if } x \in E(\mathbb{H}_1). \end{cases}$$

Thus, c_1 is an AVDTC of $\mathbb{H}_1$ with the required properties above. Now, suppose that there exists $d \in \mathbb{N}$ and an AVDTC $c_d : V(\mathbb{H}_d) \cup E(\mathbb{H}_d) \to \mathbb{Z}_{d+2}$ of $\mathbb{H}_d$ satisfying the properties above. We will obtain an AVDTC $c_{d+1} : V(\mathbb{H}_{d+1}) \cup E(\mathbb{H}_{d+1}) \to \mathbb{Z}_{d+3}$ of $\mathbb{H}_{d+1}$ that also satisfies those properties. Indeed, take $m = 2^d$ and recall that $\mathbb{H}_{d+1}$ is the cartesian product of $\mathbb{H}_d$ and $\mathbb{H}_1$. Hence, if we consider a copy $\mathbb{H}'_d$ of $\mathbb{H}_d$ and label the vertices of $\mathbb{H}_d$ and $\mathbb{H}'_d$ by $\{v_1, v_2, \ldots, v_m\}$ and $\{v'_1, v'_2, \ldots, v'_m\}$, respectively, then $\mathbb{H}_{d+1}$ is just the graph obtained from $\mathbb{H}_{d+1}$ and $\mathbb{H}'_{d+1}$ after joining each pair of vertices v_i and v'_i with an edge e_i, $i \in \{1, \ldots, m\}$. Now we use the induction hypothesis and we colour $\mathbb{H}_d$ with a colouring satisfying the required properties above and use the same colouring on $\mathbb{H}'_d$ changing the colours of the vertices in a way that the first property still holds on $\mathbb{H}_{d+1}$, that is, if we call c the resulting colouring on $V(\mathbb{H}_{d+1})$, then $c(v'_i) = 1 - c(v_i)$, for every $i \in \{1, \ldots, m\}$. Now we define the colouring c_{d+1} on $\mathbb{H}_{d+1}$ by

$$c_{d+1}(x) = \begin{cases} c\,(x), & \text{if } x \in V(\mathbb{H}_{d+1}), \\ c\,(x), & \text{if } x \in E(\mathbb{H}_d) \cup E(\mathbb{H}'_d), \\ d+2, & \text{if } x \in \{e_1, \ldots, e_m\}. \end{cases}$$

Notice that, by the definition of c_{d+1}, the first two properties hold. Moreover, since the chromatic neighbourhood of each vertex gains only the colour $d + 2$ due to the addition of the edges $\{e_1, \ldots, e_m\}$, the third property also holds. Therefore, c_{d+1} is an AVDTC of $\mathbb{H}_{d+1}$.

□

Remark 5.3 If one verifies carefully the proof of Proposition 5.2, they can see that, for every $d \geq 2$, the colouring c_d associates exactly 2^{d-1} elements of $\mathbb{H}_d$ (vertices or edges) to each of the $d + 2$ colours (see Fig. 3).

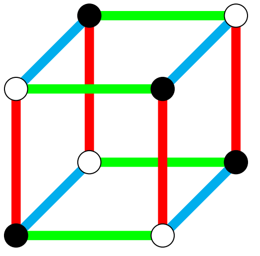

Fig. 3. An AVDTC of $\mathbb{H}_3$ with 5 colours.

6 Conclusion

In this work we prove the validity of the AVDTC conjecture [9] for hypercubes, lattice graphs and powers of cycles C_n^k when either (i) $k = 2$ and $n \geq 6$, or (ii) $n \equiv 0 \pmod{k+1}$, by constructing an explicit AVDTC that yields $\chi''_a(G) = \Delta(G) + 2$, for each of these graph classes. For the later cases (i) and (ii), we remark that this AVDTC also presents explicit total colourings for $r = 0$ of Theorem 10, and $r = k - 1$ or $r = k$ of Theorem 16 of [4]. For the lattice graphs, the same idea that we used to colour square grids can be used to colour other lattices such as hexagonal and triangular grids.

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