Hierarchical Hybrid Logic

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Abstract

We introduce $\mathcal{HHL}$, a hierarchical variant of hybrid logic. We study first order correspondence results and prove a Hennessy-Milner like theorem relating (hierarchical) bisimulation and modal equivalence for $\mathcal{HHL}$. Combining hierarchical transition structures with the ability to refer to specific states at different levels, this logic seems suitable to express and verify properties of hierarchical transition systems, a pervasive semantic structure in Computer Science.

Keywords: Hybrid logic, Hierarchical systems.

1 Introduction

From D. Harel's *statecharts* [6] to the mobile *ambients* [4] proposed by A. Gordon and L. Cardelli, models of hierarchical systems are pervasive in Computer Science. In practice, hierarchical, multi-level transitions often coexist with local ones. The ability to represent both and reason uniformly about them is essential to such models, for example in specific applications such as coordination protocols in the context of distributed systems [1], or to handle software which operates in different modes of execution and is able to commute between them. The global transition structure defines how such systems evolve from a mode (or configuration) to another [8].

This paper introduces a hierarchical variant of hybrid logic [2,3] that adds to the modal description of hierarchical transition structures the ability to refer to specific states at any level of description. As discussed by the authors in [8], hybrid logic, which allows one to refer to specific states in a system, became the specification *lingua franca* for reconfigurable systems. The hierarchical variant proposed here sets the ground for a uniform framework to express and verify properties of any kind of hierarchical transition system.

The paper is organised as follows: after a section on preliminaries, Section 3 introduces hierarchical hybrid logic, $\mathcal{HHL}$. The relevant first-order correspondences are discussed in Section 4. Section 5 discusses bisimulation for this sort of systems and proves a Hennessy-Milner like theorem relating, under the usual conditions of image-finiteness, bisimulation and modal equivalence for $\mathcal{HHL}$. Finally, Section 6 concludes and briefly discusses future work.

2   Hybrid logic

The qualifier *hybrid* [2,3] applies to extensions of modal languages with symbols, called *nominals*, that explicitly refer to individual states in the underlying Kripke frames. A *hybrid signature* is a pair (Prop, Nom), where Prop and Nom are disjoint sets of symbols of *propositional variables* and *nominals*, respectively. The set of *hybrid formulas over* (Prop, Nom) extends the corresponding modal language with formulas i, which only hold at the state named by i, and $@_i\rho$, which asserts that formula ρ holds in the state named by i, for $i \in \text{Nom}$. Formally, the set of formulas, denoted by $\text{Fm}^{\mathcal{HL}}(\text{Prop}, \text{Nom})$, is defined by the grammar

$$\rho ::= p \mid i \mid @_i\rho \mid \diamond\rho \mid \neg\rho \mid \rho \wedge \rho,$$

for $i \in \text{Nom}$ and $p \in \text{Prop}$.

Note that the remaining Boolean connectives and the box modality are introduced as abbreviations. The set $\text{BFm}^{\mathcal{HL}}(\text{Prop}, \text{Nom})$ of *basic formulas* is defined by

$$\text{Prop} \cup \text{Nom} \cup \{\diamond\rho : \rho \in \text{Fm}^{\mathcal{HL}}(\text{Prop}, \text{Nom})\}$$
$$\cup \{@_i\rho : \rho \in \text{Fm}^{\mathcal{HL}}(\text{Prop}, \text{Nom}), i \in \text{Nom}\}$$

Note that considering the restriction of formulas to the basic ones does not reduce the expressibility power of the logic, since we get again these formulas with the boolean connectives in the upper level (cf. Definition 3).

Models of $\mathcal{HL}$ for a signature (Prop, Nom) are Kripke structures with named states, i.e., structures $M = (W, R, V)$ where W is a set of *states*, $R \subseteq W \times W$ is the *accessibility relation*, and $V : \text{Prop} \cup \text{Nom} \to \mathcal{P}(W)$ is a function that interprets propositions and nominals, such that for any $i \in \text{Nom}$, $V(i)$ is a singleton. The set of all models over a signature (Prop, Nom) is denoted by $\text{Mod}^{\mathcal{HL}}(\text{Prop}, \text{Nom})$.

The *satisfaction* relation between a model $M = (W, R, V)$ in $\text{Mod}^{\mathcal{HL}}(\text{Prop}, \text{Nom})$ and a formula $\rho \in \text{Fm}^{\mathcal{HL}}(\text{Prop}, \text{Nom})$ at state $w \in W$, is recursively defined as

follows:

- $M, w \vDash^{\mathcal{HL}} \rho$ iff $w \in V(\rho)$, $\rho \in \text{Nom} \cup \text{Prop}$;

- $M, w \vDash^{\mathcal{HL}} @_i \varphi$ iff $M, V(i) \vDash^{\mathcal{HL}} \varphi$;

- $M, w \vDash^{\mathcal{HL}} \Diamond \varphi$ iff there is a $v \in W$ such that $(w, v) \in R$ and $M, v \vDash^{\mathcal{HL}} \varphi$;

- $M, w \vDash^{\mathcal{HL}} \neg \varphi$ iff it is false that $M, w \vDash^{\mathcal{HL}} \varphi$ (in symbols, $M, w \nvDash^{\mathcal{HL}} \varphi$);

- $M, w \vDash^{\mathcal{HL}} \varphi \wedge \varphi'$ iff $M, w \vDash^{\mathcal{HL}} \varphi$ and $M, w \vDash^{\mathcal{HL}} \varphi'$.

As usual, we write $M \vDash^{\mathcal{HL}} \rho$ when, for any $w \in W$, $M, w \vDash^{\mathcal{HL}} \rho$, and $\vDash^{\mathcal{HL}} \rho$ when $M \vDash^{\mathcal{HL}} \rho$ for all $M \in \text{Mod}^{\mathcal{HL}}(\text{Prop}, \text{Nom})$.

Applications often justify the introduction of a distinguished state in the underlying Kripke structure, regarded as the initial point of evaluation. As discussed in the sequel, such is the case of hierarchical transition systems representing software configurations: each configuration 'starts' at a specific entry point, or initial state. Models for such *pointed* versions of $\mathcal{HL}$ are pairs $((W, R, V), s)$ where $s \in W$. Accordingly, the satisfaction relation is defined by

$$((W, R, V), s) \vDash \rho \ \text{ iff } \ (W, R, V), s \vDash^{\mathcal{HL}} \rho \ .$$

3 Hierarchical hybrid logic

A signature in *hierarchical hybrid logic*, $\mathcal{HHL}$-signature in short, is a tuple $(\text{Prop}, \text{Nom}, \text{PROP}, \text{NOM})$ where Prop, Nom, PROP and NOM are four disjoint sets of *propositions* and *nominals* corresponding to the two levels of assertion, called the 'lower' and the 'upper' level, respectively.

The set of formulas for a signature $(\text{Prop}, \text{Nom}, \text{PROP}, \text{NOM})$ is organised in a two-levels hierarchy.

Definition 3.1 ($\mathcal{HHL}$-formulas) *Let* $\Delta = (\text{Prop}, \text{Nom}, \text{PROP}, \text{NOM})$ *be a $\mathcal{HHL}$-signature. The set* $\text{Fm}^{\mathcal{HHL}}(\Delta)$ *of $\mathcal{HHL}$-formulas is the smallest set such that:*

- $\text{BFm}^{\mathcal{HL}}(\text{Prop}, \text{Nom}) \subseteq \text{Fm}^{\mathcal{HHL}}(\Delta)$;

- $\text{PROP} \subseteq \text{Fm}^{\mathcal{HHL}}(\Delta)$;

- $\text{NOM} \subseteq \text{Fm}^{\mathcal{HHL}}(\Delta)$;

- $@_{\mathtt{i}}\rho \in \text{Fm}^{\mathcal{HHL}}(\Delta)$, *for any* $\mathtt{i} \in \text{NOM}$ *and* $\rho \in \text{Fm}^{\mathcal{HHL}}(\Delta)$;

- $\circledast \rho \in \text{Fm}^{\mathcal{HHL}}(\Delta)$, *for any* $\rho \in \text{Fm}^{\mathcal{HHL}}(\Delta)$;

- $\neg \rho \in \text{Fm}^{\mathcal{HHL}}(\Delta)$, *for any* $\rho \in \text{Fm}^{\mathcal{HHL}}(\Delta)$;

- $\rho \wedge \rho' \in \text{Fm}^{\mathcal{HHL}}(\Delta)$, *for any* $\rho, \rho' \in \text{Fm}^{\mathcal{HHL}}(\Delta)$.

As usual, Boolean connectives and the box modality are defined by abbreviation. Note also that $\text{Fm}^{\mathcal{HL}}(\text{Prop}, \text{Nom}) \subseteq \text{Fm}^{\mathcal{HHL}}(\Delta)$.

Definition 3.2 ($\mathcal{HHL}$-models) *Let* $\Delta = (\text{Prop}, \text{Nom}, \text{PROP}, \text{NOM})$ *be a $\mathcal{HHL}$*

signature. A Kripke Δ-model is a tuple

$$M = (W, R, V, (M_w)_{w \in W}),$$

where

- *W is a non empty set of the so called* upper, *or* super, *states;*
- *$R \subseteq W \times W$ is a binary relation called the* upper accessibility relation*;*
- *$V : \mathrm{PROP} \cup \mathrm{NOM} \to \mathcal{P}(W)$ is a function where, for any $\mathtt{i} \in \mathrm{NOM}$, $V(\mathtt{i})$ is a singleton. When it is implicitly clear, the element $w \in V(\mathtt{i})$ will be identified as the set $V(\mathtt{i})$ itself.*
- *For any $w \in W$, M_w is a $\mathcal{HL}$-pointed model $M_w = (H_w, s_w)$, where $H_w = (W_w, R_w, V_w) \in \mathrm{Mod}^{\mathcal{HL}}(\mathrm{Prop}, \mathrm{Nom})$ and $s_w \in W_w$.*
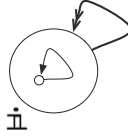

Fig. 1. An almost trivial $\mathcal{HHL}$ model.

Definition 3.3 ($\mathcal{HHL}$-Satisfaction) *Let $\Delta = (\mathrm{Prop}, \mathrm{Nom}, \mathrm{PROP}, \mathrm{NOM})$ be a $\mathcal{HHL}$-signature and $M = (W, R, V, (M_w)_{w \in W})$ be a Δ-model. The* satisfaction rela-tion *between formulas, models and points is recursively defined as follows:*

(i) *$M, w \vDash \rho$ iff $H_w, s_w \vDash^{\mathcal{HL}} \rho$, for $\rho \in \mathrm{BFm}^{\mathcal{HL}}(\mathrm{Prop}, \mathrm{Nom})$;*

(ii) *$M, w \vDash \mathtt{p}$ iff $w \in V(\mathtt{p})$, for $\mathtt{p} \in \mathrm{PROP}$;*

(iii) *$M, w \vDash \mathtt{i}$ iff $V(\mathtt{i}) = \{w\}$, for $\mathtt{i} \in \mathrm{NOM}$;*

(iv) *$M, w \vDash @_{\mathtt{i}}\rho$ iff $M, V(\mathtt{i}) \vDash \rho$;*

(v) *$M, w \vDash \Diamond\rho$ iff there is a $w' \in W$ such that $(w, w') \in R$ and $M, w' \vDash \rho$;*

(vi) *$M, w \vDash \neg\rho$ iff it is not the case that $M, w \vDash \rho$;*

(vii) *$M, w \vDash \rho \wedge \rho'$ iff $M, w \vDash \rho$ and $M, w \vDash \rho'$*

As in the standard case we write $M \vDash \rho$ when, for any $w \in W$, $M, w \vDash \rho$, and $\vDash \rho$ when $M \vDash \rho$ for all $M \in \mathrm{Mod}^{\mathcal{HHL}}(\Delta)$. These definitions extend to sets of formulas as expected. Finally, for $\Gamma \cup \{\rho\} \subseteq \mathrm{Fm}^{\mathcal{HHL}}(\Delta)$, ρ is said to be a global consequence of Γ, $\Gamma \vDash \rho$, if for any model $M \in \mathrm{Mod}^{\mathcal{HHL}}(\Delta)$, $M \vDash \Gamma$ implies $M \vDash \rho$.

4   First-order correspondences

As usual in the introduction of a modal language, this section discusses how for-mulas in hierarchical hybrid logic can be transformed into first-order ones. This is done through the introduction of two possible correspondences: the first one follows the well-known recipe used in the standard translation of modal logic; the second

entails a different, less common perspective taking explicitly into account definability in each possible world, in other words, by defining which substates belong to a superstate. Beyond the theoretical interest of these correspondences, they pave the way to the effective use of a number of proof assistants.

4.1 The standard translation

Definition 4.1 Let $\Delta = (\mathrm{Prop}, \mathrm{Nom}, \mathrm{PROP}, \mathrm{NOM})$ be a $\mathcal{HHL}$-signature. We define the two-sorted first-order signature $\Delta^* = (S, F, P)$ as follows:

- the set of sorts $S = \{W, U\}$, where W is the sort of super-states and U the sort of sub-states.

- the set of operation symbols $F = \{i : W \rightarrow U \mid i \in \mathrm{Nom}\} \cup \{\mathtt{i} : \rightarrow W \mid \mathtt{i} \in \mathrm{NOM}\} \cup \{Init : W \rightarrow U\}$;

- the set of predicate symbols $P = \{R : W \times W,\ r : W \times U \times U,\ Sub : W \times U\} \cup \{p : W \times U \mid p \in \mathrm{Prop}\} \cup \{\mathtt{p} : W \mid \mathtt{p} \in \mathrm{PROP}\}$.

The purpose of operation symbol Sub is to explicitly relate (sub)states to super-states, defining the inhabitants of each possible super-state. Although this construction is not required for defining the standard translation, it plays a role in the alternative translation introduced in the next section.

Definition 4.2 Let $\Delta = (\mathrm{Prop}, \mathrm{Nom}, \mathrm{PROP}, \mathrm{NOM})$ be a $\mathcal{HHL}$-signature. Given a model $M = (W, R, (M_w)_{w \in W}, V) \in \mathrm{Mod}(\Delta)$, we define the model M^* as follows: sorts are realized by the carrier sets $M_W^* = W$ and $M_U^* = \bigcup_{w \in W} W_w$. The definition for functions and predicates, respectively, is given by

$$
\begin{aligned}
&M_i^*(w) = V_w(i) \text{ for } i \in \mathrm{Nom} \\
&M_\mathtt{i}^* = V(\mathtt{i}) \text{ for } \mathtt{i} \in \mathrm{NOM} \\
&M_{Init}^*(w) = s_w \\
&M_R^*(w, w') \text{ iff } (w, w') \in R \\
&M_r^*(w, u, v) \text{ iff } (u, v) \in R_w \\
&M_{Sub}^*(w, u) \text{ iff } u \in W_w \\
&M_p^*(w, u) \text{ iff } u \in V_w(p), p \in \mathrm{Prop} \\
&M_\mathtt{p}^*(w) \text{ iff } w \in V(\mathtt{p}), \mathtt{p} \in \mathrm{PROP}
\end{aligned}
$$

Finally, we obtain the translation of formulas as follows:

Definition 4.3 [Standard translation] The standard translation ST consists of the map

$$
\mathrm{ST} : \mathrm{Fm}^{\mathcal{HHL}}(\Delta) \longrightarrow \mathrm{Fm}^{FOL}(\Delta^*)
$$

recursively defined as follows:

$$\mathrm{ST}_{X,u}(p) \quad = \quad p(X,u) \qquad\qquad\qquad\qquad p \in \mathrm{Prop}$$

$$\mathrm{ST}_{X,u}(i) \quad = \quad u = i(X) \qquad\qquad\qquad\qquad i \in \mathrm{Nom}$$

$$\mathrm{ST}_{X,u}(@_i\rho) \quad = \quad \mathrm{ST}_{X,i(X)}(\rho) \qquad\qquad\qquad i \in \mathrm{Nom} \text{ and}$$
$$\rho \in \mathrm{Fm}^{\mathcal{HL}}(\mathrm{Prop},\mathrm{Nom})$$

$$\mathrm{ST}_{X,u}(\diamond\rho) \quad = \quad (\exists v : U)(r(X,u,v) \wedge \mathrm{ST}_{X,v}(\rho)) \qquad \rho \in \mathrm{Fm}^{\mathcal{HL}}(\mathrm{Prop},\mathrm{Nom})$$

$$\mathrm{ST}_{X,u}(\mathbb{p}) \quad = \quad \mathbb{p}(X) \qquad\qquad\qquad\qquad \mathbb{p} \in \mathrm{PROP}$$

$$\mathrm{ST}_{X,u}(\mathbb{i}) \quad = \quad X = \mathbb{i} \qquad\qquad\qquad\qquad \mathbb{i} \in \mathrm{NOM}$$

$$\mathrm{ST}_{X,u}(@_{\mathbb{i}}\rho) \quad = \quad \mathrm{ST}_{X,u}(\rho)[X \mapsto \mathbb{i}, u \mapsto Init(\mathbb{i})] \qquad \mathbb{i} \in \mathrm{NOM}$$

$$\mathrm{ST}_{X,u}(\circledast\rho) \quad = \quad (\exists Y : W)(R(X,Y) \wedge \mathrm{ST}_{Y,Init(Y)}(\rho))$$

$$\mathrm{ST}_{X,u}(\neg\rho) \quad = \quad \neg\mathrm{ST}_{X,u}(\rho)$$

$$\mathrm{ST}_{X,u}(\rho \wedge \rho') \quad = \quad \mathrm{ST}_{X,u}(\rho) \wedge \mathrm{ST}_{X,u}(\rho')$$

Notation $\mathrm{ST}_{\mathbb{i},Init(\mathbb{i})}(\rho)$ is used for $\mathrm{ST}_{X,u}(\rho)[X \mapsto \mathbb{i}, u \mapsto Init(\mathbb{i})]$, when clear from context.

Lemma 4.4 *Let* $\Delta = (\mathrm{Prop},\mathrm{Nom},\mathrm{PROP},\mathrm{NOM})$ *be a* $\mathcal{HHL}$-*signature and* $M = (W, R, (M_w)_{w \in W}, V)$ *a* Δ-*model and* $\rho \in \mathrm{Fm}^{\mathcal{HL}}(\mathrm{Prop},\mathrm{Nom})$. *Then, for any* $w \in W$, $z \in H_w$,

$$H_w, z \models^{\mathcal{HL}} \rho \;\; iff \;\; M^* \models^{FOL} \mathrm{ST}_{X,u}(\rho)[X \mapsto w, u \mapsto z]$$

where $M_w = (H_w, s_w)$.

Proof. By induction on the structure of formulas.

for $\rho = p,\ p \in \mathrm{Prop}$
$$H_w, z \models^{\mathcal{HL}} p$$
$$\Leftrightarrow \qquad \{\text{defn. of } \models^{\mathcal{HL}}\}$$
$$z \in V_w(p)$$
$$\Leftrightarrow \qquad \{\text{defn. of } M^* \text{ and } z \in W_w\}$$
$$M_p^*(w,z)$$
$$\Leftrightarrow \qquad \{\text{defn of } \models^{FOL}\}$$
$$M^* \models^{FOL} p(X,u)[X \mapsto w, u \mapsto z]$$
$$\Leftrightarrow \qquad \{\text{defn of ST}\}$$
$$M^* \models^{FOL} \mathrm{ST}_{X,u}(p)[X \mapsto w, u \mapsto z]$$

for $\rho = i,\ i \in \mathrm{Nom}$
$$H_w, z \models^{\mathcal{HL}} i$$
$$\Leftrightarrow \qquad \{\text{defn. of } \models^{\mathcal{HL}}\}$$
$$z = V_w(i)$$

$\Leftrightarrow$ $\quad$ $\{$defn. of $M^*\}$

$\quad M_i^*(w) = z$

$\Leftrightarrow$ $\quad$ $\{$defn of $\vDash^{FOL}\}$

$\quad M^* \vDash^{FOL} i(X) = u[X \mapsto w, u \mapsto z]$

$\Leftrightarrow$ $\quad$ $\{$defn of ST$\}$

$\quad M^* \vDash^{FOL} \mathrm{ST}_{X,u}(i)[X \mapsto w, u \mapsto z]$

for $\rho = @_i\varphi$, $i \in \mathrm{Nom}$

$\quad H_w, z \vDash^{\mathcal{HL}} @_i\varphi$

$\Leftrightarrow$ $\quad$ $\{$defn. of $\vDash^{\mathcal{HL}}\}$

$\quad H_w, V_w(i) \vDash^{\mathcal{HL}} \varphi$

$\Leftrightarrow$ $\quad$ $\{$I.H.$\}$

$\quad M^* \vDash^{FOL} \mathrm{ST}_{X,u}(\varphi)[X \mapsto w, u \mapsto V_w(i)]$

$\Leftrightarrow$ $\quad$ $\{$since $V_w(i) = M_i^*(w)\}$

$\quad M^* \vDash^{FOL} \mathrm{ST}_{X,i(X)}(\varphi)[X \mapsto w, u \mapsto z]$

$\Leftrightarrow$ $\quad$ $\{$defn of ST$\}$

$\quad M^* \vDash^{FOL} \mathrm{ST}_{X,u}(@_i\varphi)[X \mapsto w, u \mapsto z]$

for $\rho = \diamond\varphi$

$\quad H_w, z \vDash^{\mathcal{HL}} \diamond\varphi$

$\Leftrightarrow$ $\quad$ $\{$defn. of $\vDash^{\mathcal{HL}}\}$

$\quad H_w, v \vDash^{\mathcal{HL}} \varphi$, for some $v \in W_w$ such that $(z, v) \in R_w$

$\Leftrightarrow$ $\quad$ $\{$I.H + defn of $M^*\}$

$\quad M^* \vDash^{FOL} \mathrm{ST}_{X,u}(\varphi)[X \mapsto w, u \mapsto v]$

$\quad$ for some v such that $M_r^*(w, z, v)$

$\Leftrightarrow$ $\quad$ $\{$defn of $\vDash^{FOL}\}$

$\quad M^* \vDash^{FOL} (\exists v)\, r(X, u, v) \wedge \mathrm{ST}_{X,v}(\varphi)[X \mapsto w, u \mapsto z]$

$\Leftrightarrow$ $\quad$ $\{$defn of ST$\}$

$\quad M^* \vDash^{FOL} \mathrm{ST}_{X,u}(\diamond\varphi)[X \mapsto w, u \mapsto z]$

The cases dealing with conjunction and negation follow directly from the induction hypothesis. $\qquad \Box$

Theorem 4.5 *Let* $\Delta = (\mathrm{Prop}, \mathrm{Nom}, \mathrm{PROP}, \mathrm{NOM})$ *be a* $\mathcal{HHL}$-*signature,* $M = (W, R, (M_w)_{w \in W}, V)$ *a* Δ-*model and* $\rho \in \mathrm{Fm}^{\mathcal{HHL}}(\Delta)$. *Then, for* $w \in W$,

$$M, w \vDash \rho \ \textit{iff}\ M^* \vDash^{FOL} \mathrm{ST}_{X,u}(\rho)[X \mapsto w, u \mapsto M_{Init}^*(w)]$$

Proof. The proof proceeds by induction on the structure of formulas. Thus,

for $\rho \in \mathrm{BFm}^{\mathcal{HL}}(\mathrm{Prop}, \mathrm{Nom})$

$$M, w \vDash \rho$$

$\Leftrightarrow$ $\quad \{\text{defn. of } \vDash\}$

$$H_w, s_w \vDash^{\mathcal{HL}} \rho$$

$\Leftrightarrow$ $\quad \{\text{since } M^*_{Init}(w) = s_w \text{ and Lemma 4.4}\}$

$$M^* \vDash^{FOL} \mathrm{ST}_{X,u}(\rho)[X \mapsto w, u \mapsto M^*_{Init}(w)]$$

for $\rho = @_{\mathtt{i}}\varphi, \ \mathtt{i} \in \mathrm{NOM}$

$$M, w \vDash @_{\mathtt{i}}\varphi$$

$\Leftrightarrow$ $\quad \{\text{defn. of } \vDash\}$

$$M, V(\mathtt{i}) \vDash \varphi$$

$\Leftrightarrow$ $\quad \{\text{I.H. }\}$

$$M^* \vDash^{FOL} \mathrm{ST}_{\mathtt{i},u}(\varphi)[X \mapsto w, u \mapsto M^*_{Init}(V(\mathtt{i}))]$$

$\Leftrightarrow$ $\quad \{\text{since } V(\mathtt{i}) = M^*_{\mathtt{i}}\}$

$$M^* \vDash^{FOL} \mathrm{ST}_{\mathtt{i},Init(\mathtt{i})}(\varphi)$$

$\Leftrightarrow$ $\quad \{\text{defn. of ST}\}$

$$M^* \vDash^{FOL} \mathrm{ST}_{X,u}(@_{\mathtt{i}}\varphi)[X \mapsto w, u \mapsto M^*_{Init}(w)]$$

for $\rho = \diamondsuit\varphi$

$$M, w \vDash \diamondsuit\varphi$$

$\Leftrightarrow$ $\quad \{\text{defn. of } \vDash\}$

$$M, w' \vDash \varphi, \text{ for some } w' \in W \text{ such that } R(w, w')$$

$\Leftrightarrow$ $\quad \{\text{I.H. } + \text{ defn. of } M^*\}$

$$M^* \vDash^{FOL} \mathrm{ST}_{Y,v}(\varphi)[X \mapsto w', u \mapsto M^*_{Init}(w')], \text{ for some } w' \in W \text{ with } M^*_R(w, w')$$

$\Leftrightarrow$ $\quad \{\text{defn. of } \vDash^{FOL}\}$

$$M^* \vDash^{FOL} (\exists Y : W) R(X, Y) \wedge \mathrm{ST}_{Y,Init(Y)}(\varphi)[X \mapsto w]$$

$\Leftrightarrow$ $\quad \{\text{defn. of ST}\}$

$$M^* \vDash^{FOL} \mathrm{ST}_{X,u}(\diamondsuit\varphi)[X \mapsto w, u \mapsto M^*_{Init}(w)]$$

Again, the cases dealing with conjunction and negation follow directly from the induction hypothesis. □

4.2 A different perspective

As mentioned above, an alternative translation, not standard in modal logic, will be considered now.

Let $\Delta = (\mathrm{Prop}, \mathrm{Nom}, \mathrm{PROP}, \mathrm{NOM})$ be a $\mathcal{HHL}$-signature. Then $\mathrm{Mod}_D^{FOL}(\Delta^*)$ denotes the class of all models of $M \in \mathrm{Mod}(\Delta^*)$ such that for each $w \in W$, $M_{Init}(w)$ and $M_i(w)$, for any nominal i, belong to the universe associated to w, that is $M_{Sub}(w, M_{Init}(w))$ and $M_{Sub}(w, M_i(w))$ for any nominal i. If Nom is finite, we

denote the formula

$$(\forall X : W)\Big(Sub(X, Init(X)) \wedge \bigwedge_{i \in \text{Nom}} Sub(X, i(X))\Big)$$

by $D(\Delta)$. And we have, $\text{Mod}_D^{FOL}(\Delta^*) = \{M \in \text{Mod}(\Delta^*) \mid M \vDash^{FOL} D(\Delta)\}$.

In what follows in this section we will assume that Nom is finite. Note also that we are allowing that different local models, W_{w_1} and W_{w_2}, may have elements in common.

Definition 4.6 Let $\Delta = (\text{Prop}, \text{Nom}, \text{PROP}, \text{NOM})$ be a $\mathcal{HHL}$-signature. The operator

$$\_^\circ : \text{Mod}_D^{FOL}(\Delta^*) \to \text{Mod}^{\mathcal{HHL}}(\Delta)$$

is defined as follows: given a model $M \in \text{Mod}_D^{FOL}(\Delta^*)$, we construct the model $M^\circ = (W^\circ, R^\circ, (M_w^\circ)_{w \in W^\circ}, V^\circ)$ as follows:

- $W^\circ = M_W$
- $R^\circ = M_R$
- for any $\mathbb{p} \in \text{PROP}$, $V^\circ(\mathbb{p}) = M_{\mathbb{p}}$
- for any $\mathbb{i} \in \text{NOM}$, $V^\circ(\mathbb{i}) = \{M_{\mathbb{i}}\}$

and for any $w \in W^\circ$, $M_w^\circ = (H_w^\circ, s_w^\circ)$ where $H_w^\circ = (W_w^\circ, R_w^\circ, V_w^\circ)$ is such that

- $W_w^\circ = \{a \mid M_{Sub}(w, a)\}$
- $R_w^\circ = \{(a, b) \mid M_r(w, a, b) \text{ and } M_{Sub}(w, a) \text{ and } M_{Sub}(w, b)\}$
- for any $p \in \text{Prop}$, $V_w^\circ(p) = \{a \mid M_p(w, a) \text{ and } M_{Sub}(w, a)\}$
- for any $i \in \text{Nom}$, $V_w^\circ(i) = M_i(w)$
- $s_w^\circ = M_{Init}(w)$

Observe the role of $D(\Delta)$ in asserting the definability of the local valuations V_w with respect to its functionality over Nom, as well as with respect to definability of the (local) initial states. In order to obtain a translation that is compatible with the operator $^\circ$ the standard translation defined above has to be constrained, leading to the following definition:

Definition 4.7 [(constrained) standard translation]

$\text{ST}_{X,u}^\circ(p) \quad = Sub(X, u) \wedge p(X, u),\ p \in \text{Prop}$

$\text{ST}_{X,u}^\circ(i) \quad = Sub(X, u) \wedge u = i(X),\ i \in \text{Nom}$

$\text{ST}_{X,u}^\circ(\diamond\rho) = (\exists v : U)(Sub(X, v) \wedge r(X, u, v) \wedge \text{ST}_{X,v}^\circ(\rho)),\ \rho \in \text{Fm}^{\mathcal{HL}}(\text{Prop}, \text{Nom})$

and it is defined as in ST for the remaining cases.

Since we will require that our FOL-models satisfy $D(\Delta)$, we may omit the condition $Sub(X, u)$ in $\text{ST}_{X,u}^\circ(i)$.

Lemma 4.8 *Let* $\Delta = (\text{Prop}, \text{Nom}, \text{PROP}, \text{NOM})$ *be a* $\mathcal{HHL}$*-signature,* $M \in$

$\mathrm{Mod}_D^{FOL}(\Delta^*)$ *and* $\rho \in \mathrm{Fm}^{\mathcal{HL}}(\mathrm{Prop}, \mathrm{Nom})$. *Then, for any* $w \in W^\circ$ *and* $z \in H_w^\circ$,

$$H_w^\circ, z \vDash^{\mathcal{HL}} \rho \text{ iff } M \vDash^{FOL} \mathrm{ST}_{X,u}^\circ(\rho)[X \mapsto w, u \mapsto z]$$

Proof. The proof is by induction on the structure of formulas. Thus,

for $\rho = p,\ p \in \mathrm{Prop}$

$\qquad M \vDash^{FOL} \mathrm{ST}_{X,u}^\circ(p)[X \mapsto w, u \mapsto z]$

$\quad \Leftrightarrow \qquad \{\text{defn. of } \mathrm{ST}^\circ\}$

$\qquad M \vDash^{FOL} (Sub(X, u) \wedge p(X, u))[X \mapsto w, u \mapsto z]$

$\quad \Leftrightarrow \qquad \{\text{defn. of } \vDash\}$

$\qquad M_{Sub}(w, z) \text{ and } M_p(w, z)$

$\quad \Leftrightarrow \qquad \{\ \text{defn. of } M^\circ\}$

$\qquad z \in V_w^\circ(p)$

$\quad \Leftrightarrow \qquad \{\text{defn. of } \vDash^{\mathcal{HL}}\}$

$\qquad H_w^\circ, z \vDash^{\mathcal{HL}} p$

for $\rho = i,\ i \in \mathrm{Nom}$

$\qquad M \vDash^{FOL} \mathrm{ST}_{X,u}^\circ(i)[X \mapsto w, u \mapsto z]$

$\quad \Leftrightarrow \qquad \{\text{defn. of } \mathrm{ST}^\circ\}$

$\qquad M \vDash^{FOL} (Sub(X, u) \wedge u = i(x))[X \mapsto w, u \mapsto z]$

$\quad \Leftrightarrow \qquad \{\text{defn. of } \vDash\}$

$\qquad M_{Sub}(w, z) \text{ and } z = M_i(w))$

$\quad \Leftrightarrow \qquad \{\text{defn. of } M^\circ\}$

$\qquad z = V_w^\circ(i)$

$\quad \Leftrightarrow \qquad \{\text{defn. of } \vDash^{\mathcal{HL}}\}$

$\qquad H_w^\circ, z \vDash^{\mathcal{HL}} p$

for $\rho = @_i\varphi,\ i \in \mathrm{Nom}$

$\qquad M \vDash^{FOL} \mathrm{ST}_{X,u}^\circ(@_i\varphi)[X \mapsto w, u \mapsto z]$

$\quad \Leftrightarrow \qquad \{\text{defn. of } \mathrm{ST}^\circ\}$

$\qquad M \vDash^{FOL} \mathrm{ST}_{X,i(X)}^\circ(\varphi)[X \mapsto w, u \mapsto z]$

$\quad \Leftrightarrow \qquad \{\text{substitution}\}$

$\qquad M \vDash^{FOL} \mathrm{ST}_{X,u}^\circ(\varphi)[X \mapsto w, u \mapsto M_i(w)]$

$\quad \Leftrightarrow \qquad \{\ \text{I.H.}\}$

$\qquad H_w^\circ, M_i(w) \vDash^{\mathcal{HL}} \varphi$

$\quad \Leftrightarrow \qquad \{\text{defn. of } \vDash^{\mathcal{HL}}\}$

$\qquad H_w^\circ, z \vDash^{\mathcal{HL}} @_i\varphi$

for $\rho = \diamond\varphi$

$$M \vDash^{FOL} \mathrm{ST}^{\circ}_{X,u}(\diamond\varphi)[X \mapsto w, u \mapsto z]$$

$\Leftrightarrow$ $\quad$ {defn. of ST°}

$$M \vDash^{FOL} (\exists v : U)(Sub(X,v) \wedge r(X,u,v) \wedge \mathrm{ST}^{\circ}_{X,v}(\varphi))[X \mapsto w, u \mapsto z]$$

$\Leftrightarrow$ $\quad$ {defn. of $\vDash$}

there is $a \in M_U$ such that $M_{Sub}(w,a)$ and $M_r(w,z,a)$ and

$$M \vDash^{FOL} \mathrm{ST}^{\circ}_{X,v}(\varphi)[X \mapsto w, v \mapsto a]$$

$\Leftrightarrow$ $\quad$ {I.H.}

$H^{\circ}_{w}, a \vDash^{\mathcal{HL}} \varphi,$ for some $a \in M_U$ such that $M_{Sub}(w,a)$ and $M_r(w,z,a)$

$\Leftrightarrow$ $\quad$ {by def. of $\vDash^{\mathcal{HL}}$}

$$H^{\circ}_{w}, z \vDash^{\mathcal{HL}} \diamond\varphi \qquad\qquad \square$$

Theorem 4.9 *Let* $\Delta = (\mathrm{Prop}, \mathrm{Nom}, \mathrm{PROP}, \mathrm{NOM})$ *be a* $\mathcal{HHL}$-*signature,* $M \in \mathrm{Mod}^{FOL}_{D}(\Delta^*)$ *and* $\rho \in \mathrm{Fm}^{\mathcal{HHL}}(\Delta)$. *Then, for every* $w \in W$

$$M^{\circ}, w \vDash \rho \text{ iff } M \vDash^{FOL} \mathrm{ST}^{\circ}_{X,u}(\rho)[X \mapsto w, u \mapsto M_{Init}(w)]$$

Proof.

for $\rho \in \mathrm{BFm}^{\mathcal{HL}}(\mathrm{Prop}, \mathrm{Nom})$

$$M \vDash^{FOL} \mathrm{ST}^{\circ}_{X,u}(\rho)[X \mapsto w, u \mapsto M_{Init}(w)]$$

$\Leftrightarrow$ $\quad$ {Lemma 4.8 }

$$H^{\circ}_{w}, s^{\circ}_{w} \vDash^{\mathcal{HL}} \rho$$

$\Leftrightarrow$ $\quad$ {defn. of $\vDash$}

$$M^{\circ}, w \vDash \rho$$

for $\rho = @_{\mathtt{i}}\varphi,$ $\mathtt{i} \in \mathrm{NOM}$

$$M \vDash^{FOL} \mathrm{ST}^{\circ}_{X,u}(@_{\mathtt{i}}\varphi)[X \mapsto w, u \mapsto M_{Init}(w)]$$

$\Leftrightarrow$ $\quad$ {defn. of ST°}

$$M \vDash^{FOL} \mathrm{ST}_{X,u}(\varphi)[X \mapsto \mathtt{i}, u \mapsto Init(\mathtt{i})]$$

$\Leftrightarrow$ $\quad$ {I.H.}

$$M^{\circ}, \mathtt{i} \vDash \varphi$$

$\Leftrightarrow$ $\quad$ {defn. of $\vDash$}

$$M^{\circ}, w \vDash @_{\mathtt{i}}\varphi$$

for $\rho = \circledast\varphi$

$$M \vDash^{FOL} \mathrm{ST}^{\circ}_{X,u}(\circledast\varphi)[X \mapsto w, u \mapsto M_{Init}(w)]$$

$\Leftrightarrow$ $\quad$ {defn. of ST°}

$$M \vDash^{FOL} (\exists Y : W)R(X,Y) \wedge \mathrm{ST}^{\circ}_{Y,Init(Y)}(\varphi))[X \mapsto w, u \mapsto M_{Init}(w)]$$

$\Leftrightarrow$ $\{$defn. of $\vDash\}$

there is $w' \in M_W$ st $M_R(w, w')$ and

$M \vDash^{FOL} \mathrm{ST}^{\circ}_{Y, Init(Y)}(\varphi))[Y \mapsto w', u \mapsto M_{Init}(w)]$

$\Leftrightarrow$ $\{$substitution$\}$

there is $w' \in M_W$ st $M_R(w, w')$ and

$M \vDash^{FOL} \mathrm{ST}^{\circ}_{Y, u}(\varphi))[Y \mapsto w', u \mapsto M_{Init}(w')]$

$\Leftrightarrow$ $\{$I.H.$\}$

there is $w' \in M_W$ st $M_R(w, w')$ and $M^{\circ}, w' \vDash \varphi$

$\Leftrightarrow$ $\{$defn. of $\vDash^{\mathcal{HL}}\}$

$M^{\circ}, w \vDash \diamondsuit \varphi$

The cases of conjunction and negation are dealt similarly, easily achieved by direct application of induction hypothesis. □

The operator $\circ$ is not in general injective. However it is surjective. Actually, the surjectivity of $\circ$ together with previous theorem guarantees that to prove that a modal formula is valid, it is enough to show that its translation is a valid FOL formula. This is a very useful property, since it allows us to use FOL theorem provers (well developed) to check modal validities. Moreover, given an $M \in \mathrm{Mod}^{\mathcal{HHL}}(\Delta)$ it is not difficult to see that $M = (M^*)^{\circ}$, and $M^* \in \mathrm{Mod}^{FOL}_D(\Delta^*)$.

Corollary 4.10 *Let* $\Delta = (\mathrm{Prop}, \mathrm{Nom}, \mathrm{PROP}, \mathrm{NOM})$ *be a* $\mathcal{HHL}$*-signature with finite sets* Nom *and* NOM. *Then, for any* $\Gamma \cup \{\rho\} \subseteq \mathrm{Fm}^{\mathcal{HHL}}(\Delta)$, *we have*

$$\Gamma \vDash \rho \ \ iff \ \ \Gamma^* \cup D(\Delta) \vDash^{FOL} (\forall X : W) \mathrm{ST}^{\circ}_{X, Init(X)}(\rho)$$

where $\Gamma^* = \{(\forall X : W), \mathrm{ST}^{\circ}_{X, Init(X)}(\rho) \,|\, \rho \in \Gamma\}$.

Proof. Suppose that $\Gamma \vDash \rho$. Let M be a Δ^* first order model of $\Gamma^* \cup D(\Delta)$. Then, by Theorem 4.9, $M^{\circ} \vDash^{\mathcal{HL}} \Gamma$. Hence, $M^{\circ} \vDash^{\mathcal{HL}} \rho$. That is, for all $w \in W$ $M^{\circ}, w \vDash^{\mathcal{HL}} \rho$. Again, by Theorem 4.9 (in the opposite direction), $M \vDash^{FOL} (\forall X : W) \mathrm{ST}^{\circ}_{X, Init(X)}(\rho)$.

Conversely, suppose $\Gamma^* \cup D(\Delta) \vDash^{FOL} (\forall X : W) \mathrm{ST}^{\circ}_{X, Init(X)}(\rho)$. Let N be a Δ-model such that $N \vDash \Gamma$. Since $\circ$ is surjective there is an $M \in \mathrm{Mod}^{FOL}_D(\Delta^*)$ such that $N = M^{\circ}$. Since $N \vDash \Gamma$, by Theorem 4.9, M is a model of $\Gamma^* \cup D(\Delta)$. Therefore $M \vDash^{FOL} (\forall X : W) \mathrm{ST}^{\circ}_{X, Init(X)}(\rho)$. Again, by Theorem 4.9, $M^{\circ} = N \vDash \rho$. □

5 Hennessy-Milner Theorem for $\mathcal{HHL}$

Bisimulation is a main tool for the study of transition systems which, on their turn, are pervasive structures in computational phenomena. It is also a good example of the fruitful interaction between modal logic and Computer science. This section characterises a notion of hierarchical bisimulation for models of $\mathcal{HHL}$ and proves

a corresponding Hennessy-Milner result relating hybrid equivalence between two models with the existence of a bisimulation relating them.

Definition 5.1 Let Δ = (Prop, Nom, PROP, NOM) be a $\mathcal{HHL}$-signature. An *hierarchical bisimulation* between two Δ-models $M = (W, R, (M_w)_{w \in W}, V)$ and $M' = (W', R', (M'_w)_{w \in W'}, V')$ consists of a relation $\mathbb{B} \subseteq W \times W'$ such that,

(NOM) for any $\mathtt{i} \in \mathrm{NOM}$, $V(\mathtt{i}) \mathbb{B} V'(\mathtt{i})$

- for any $w \in W, w' \in W'$, $w \mathbb{B} w'$ implies:
  (ATOMS) for any $\mathtt{a} \in \mathrm{PROP} \cup \mathrm{NOM}$, $w \in V(\mathtt{a})$ iff $w' \in V'(\mathtt{a})$;
  (ZIG) for any $v \in W$ such that $(w, v) \in R$ there is a $v' \in W'$ such that $v \mathbb{B} v'$ and $(w', v') \in R'$;
  (ZAG) for any $v' \in W'$ such that $(w', v') \in R'$ there is a $v \in W$ such that $v \mathbb{B} v'$ and $(w, v) \in R$.

(LOCAL) M_w and $M'_{w'}$ are bisimilar, *i.e.*, there is a relation $\mathrm{B}^w_{w'} : H_w \times H_{w'}$ such that
  (init) $s_w \mathrm{B}^w_{w'} s'_{w'}$;
  (nom) for any $i \in \mathrm{Nom}$, $V_w(i) \mathrm{B}^w_{w'} V'_{w'}(i)$;
  - for any $u \in H_w, u' \in H'_w$ such that $u \mathrm{B}^w_{w'} u'$,
    (atoms) for any $p \in \mathrm{Prop} \cup \mathrm{Nom}$, $u \in V_w(p)$ iff $u' \in V'_{w'}(p)$;
    (zig) for any $v \in H_w$ such that $(u, v) \in R_w$ there is a $v' \in H'_{w'}$ such that $v \mathrm{B}^w_{w'} v'$ and $(u', v') \in R'_{w'}$;
    (zag) for any $v' \in H'_{w'}$ such that $(u', v') \in R'_{w'}$ there is a $v \in H_w$ such that $v \mathrm{B}^w_{w'} v'$ and $(u, v) \in R_w$.

An example in depicted in Fig. 2. The reader may easily notice the existence of local bisimulations relating the transition systems inside each of the two states of the system in the left with the one in the right, plus a global bisimulation relating precisely those (global) states.
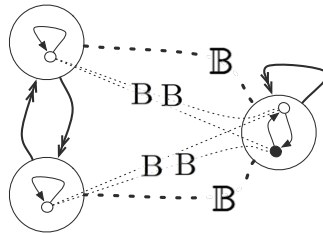

Fig. 2. A $\mathcal{HHL}$-bisimulation.

Lemma 5.2 *Let M and M' be two $\mathcal{HHL}$-models over the same signature. The set of hierarchical bisimulations between M and M' is closed under unions.*

Proof. Let $\mathbb{B}^1, \mathbb{B}^2 \subseteq |W| \times |W'|$ be two bisimulations between models M and M'. Their union $\mathbb{B} = \mathbb{B}^1 \cup \mathbb{B}^2$ is also an hierarchical bisimulation because

• Clearly, all points named by nominals are related by $\mathbb{B}$ as they are related both by $\mathbb{B}^1$ and $\mathbb{B}^2$. Moreover, for any pair (w, w') such that $w \mathbb{B} w'$ either $w \mathbb{B}^1 w'$

or $w\mathbb{B}^2 w'$. As both $\mathbb{B}^1$ and $\mathbb{B}^2$ are hierarchical bisimulations, clauses of (i.) in Definition. 5.1 hold for $\mathbb{B}$.

• A similar argument applies to both (ZIG) and (ZAG) conditions. For clause (iv) let $(w, v) \in R$ and $w\mathbb{B}w'$. Then, either $w\mathbb{B}^1 w'$ or $w\mathbb{B}^2 w'$. Since, $\mathbb{B}^1$ and $\mathbb{B}^2$ are bisimulations, we have that there is a $v' \in W'$ such that $v\mathbb{B}^1 v'$ or $v\mathbb{B}^2 v'$. Hence $v\mathbb{B}v'$. The condition (ZAG) condition is proved similarly.

$\square$

Similarly, one may prove that hierarchical bisimulations are closed for composition as well. Bisimulation invariance is, on the other hand, a main, expected result.

Theorem 5.3 (Bisimulation invariance) *Let M and M' be two $\mathcal{HHL}$-models over the same signature $\Delta = (\mathrm{Prop}, \mathrm{Nom}, \mathrm{PROP}, \mathrm{NOM})$ and $\mathbb{B}$ a bisimulation between them. Then, for any $w\mathbb{B}w'$ and for any $\rho \in \mathrm{Fm}^{\mathcal{HHL}}(\Delta)$,*

$$M, w \vDash \rho \ \textit{iff} \ M', w' \vDash \rho$$

Proof. The proof is by induction on the structure of the sentences.

$\rho \in \mathrm{BFm}^{\mathcal{HL}}(\mathrm{Prop}, \mathrm{Nom})$

$\qquad M, w \vDash \rho$

$\quad \Leftrightarrow \qquad \{\text{defn. of } \vDash\}$

$\qquad (H_w, s_w) \vDash^{\mathcal{HL}} \rho$

$\quad \Leftrightarrow \qquad \{\star\}$

$\qquad (H'_{w'}, s'_{w'}) \vDash^{\mathcal{HL}} \rho$

$\quad \Leftrightarrow \qquad \{\text{defn. of } \vDash\}$

$\qquad M', w' \vDash \rho$

Step $\star$ comes from the (init) clause in Definition 5.1, $s_w \, \mathbb{B}^w_{w'} s_{w'}$, and the standard *bisimulation invariance* of (propositional)-hybrid logic (e.g. [14]). However, this proof can be achieved in a complete analogy with the (top-level) cases proved above. For instance, in order to proof the invariance of $\rho = i$, for $i \in \mathrm{Nom}$, we take the bisimilar initial states s_w and $s'_{w'}$ (by (init)) and we reproduce exactly the same steps of the $\rho = \mathring{\imath}$ proof, but considering the condition (nom) in the place of (NOM). The other cases are obtained exactly in the same way.

$\rho = \mathring{\imath}$, $\mathring{\imath} \in \mathrm{NOM}$

$\qquad M, w \vDash \mathring{\imath}$

$\quad \Leftrightarrow \qquad \{\text{defn. of } \vDash\}$

$\qquad V(\mathring{\imath}) = w$

$\quad \Leftrightarrow \qquad \{\text{ATOMS of Defn. 5.1}\}$

$\qquad V'(\mathring{\imath}) = w'$

$\quad \Leftrightarrow \qquad \{\text{defn. of } \vDash\}$

$$M', w' \vDash \mathtt{i}$$

$\rho = \mathtt{p}$, $\mathtt{p} \in \mathrm{PROP}$

$$M, w \vDash \mathtt{p}$$

$\Leftrightarrow$ $\quad \{\text{defn. of } \vDash\}$

$$w \in V(\mathtt{p})$$

$\Leftrightarrow$ $\quad \{\text{ATOMS of Defn. } 5.1\}$

$$w' \in V'(\mathtt{p})$$

$\Leftrightarrow$ $\quad \{\text{defn. of } \vDash\}$

$$M', w' \vDash \mathtt{p}$$

$\rho = @_{\mathtt{i}}\varphi$

$$M, w \vDash @_{\mathtt{i}}\varphi$$

$\Leftrightarrow$ $\quad \{\text{defn. of } \vDash\}$

$$M, V(\mathtt{i}) \vDash \varphi$$

$\Leftrightarrow$ $\quad \{\text{I.H. + NOM of Defn. } 5.1\}$

$$M', V'(\mathtt{i}) \vDash \varphi$$

$\Leftrightarrow$ $\quad \{\text{defn. of } \vDash\}$

$$M', w' \vDash @_{\mathtt{i}}\varphi$$

$\rho = \diamondsuit\varphi$

$$M, w \vDash \diamondsuit\varphi$$

$\Leftrightarrow$ $\quad \{\text{defn. of } \vDash\}$

$$M, v \vDash \varphi \text{ for some } v \in W \text{ such that } (w, v) \in R$$

$\Leftrightarrow$ $\quad \{\text{I.H. + ZIG for } \Rightarrow + \text{ ZAG for } \Leftarrow \}$

$$M', v' \vDash \varphi \text{ for some } v' \in W' \text{ such that } (w', v') \in R'$$

$\Leftrightarrow$ $\quad \{\text{defn. of } \vDash\}$

$$M', w' \vDash \diamondsuit\varphi \qquad\qquad\qquad\qquad \square$$

A $\mathcal{HHL}$-model M is *image-finite* if for each state $w \in W$, the set $\{v : (w, v) \in R\}$ and the sets $\{v : (u, v) \in R_w, w \in W\}$, $u \in W_w$, are finite. Note that no condition is imposed on the cardinality of W.

Theorem 5.4 *Let Δ be a $\mathcal{HHL}$-signature and M and M' two* image-finite Δ-*models, respectively. Then, for every $w \in W$ and $w' \in W'$, the following conditions are equivalent:*

(i) $M, w \vDash \rho$ *iff* $M', w' \vDash \rho$, *for any formula* $\rho \in \mathrm{Fm}^{\mathcal{HHL}}(\Delta)$

(ii) *There is a bisimulation* $\mathbb{B}$ *between M and M' such that* $w \, \mathbb{B} \, w'$.

Proof. We have just to prove that (i) implies (ii). Let us show that

$$\mathbb{Z} = \left\{(w, w') \in W \times W' : \text{ for any } \rho \in \mathrm{Fm}^{\mathcal{HHL}}(\Delta), M, w \vDash \rho \text{ iff } M', w' \vDash \rho\right\}$$

is a bisimulation. The conditions (ATOM) and (NOM) follow directly from the invariance of the sentences $\rho \in \text{NOM} \cup \text{PROP}$. Since the image-finiteness of $\mathcal{HHL}$-models entails the image-finiteness of its local $M_w, w \in W$, we have that the condition (ATOMS) corresponds to the standard Hennessy-Milner result of the propositional hybrid logic (e.g. [14]).

For the (ZIG) condition, assume that $w \, \mathbb{Z} \, w'$ and let $u \in W$ such that $(w, u) \in R$. To obtain a contradiction, suppose that there is no $u' \in W'$ with $(w', u') \in R'$ and $u \, \mathbb{Z} \, u'$. As in the standard case the image-finite condition makes $S' = \{u' : (w', u') \in R'\}$ finite. Moreover, S' cannot be empty since in such a case $M, w \vDash \neg \Diamond (@_{\mathtt{i}} \mathtt{i})$, which is incompatible with the fact that $M, w \vDash \Diamond(@_{\mathtt{i}} \mathtt{i})$ (since $(w, u) \in R$). By assumption, for every $z \in S'$ there is a formula ψ_z such that $M, u \vDash \psi_z$ and it is false that $M', z \vDash \psi_z$. Consider now the conjunction

$$\psi = \bigwedge_{z \in S'} \psi_z$$

of all of these formulas. Hence, we have that $M, w \vDash \Diamond \psi$ and $M', w' \nvDash \Diamond \psi$, which contradicts $w \, \mathbb{Z} \, w'$. $\square$

6   Discussion and future work

In this paper we introduced $\mathcal{HHL}$ – a hierarchical variant of hybrid logic. We presented first order correspondence results and proved a Hennessy-Milner like theorem relating (hierarchical) bisimulation and modal equivalence for $\mathcal{HHL}$.

On the more practical side, it is clear that $\mathcal{HHL}$ is appropriate to reason about hierarchical transition systems, as they appear in, e.g. reconfigurable programs. The logic, however, is unable to express arbitrary multi-level transitions, thus enforcing a particular specification discipline. Actually, there are some variants (e.g. [9]) whose motivation stems directly from Computer Science applications which may require more complex features. For example, statecharts, already mentioned in the Introduction, comprise different forms of inter-level transitions, including multiple-source and multiple-target ones as well as simultaneous firing of non-conflicting transitions and their prioritisation, which cannot be captured in $\mathcal{HHL}$.

The process of constructing $\mathcal{HHL}$ on top of standard propositional hybrid logic can be made generic through *hybridisation*, a procedure introduced in [12] that consists of taking an arbitrary logic and to systematically develop on top of it the syntax and semantic features of hybrid logic. To be completely general, this is framed in the context of the institution theory of Goguen and Burstall [5], each logic (base and hybridised) being treated abstractly as an *institution*. Actually, $\mathcal{HHL}$ can be obtained through hybridisation of propositional hybrid logic. The latter, however, can be replaced by other logics resulting from the same process being applied to whatever logics are found interesting to specify configurations (states) at the lower level of the hierarchy — e.g., equational, first-order, fuzzy, etc. The application of this idea on the rigorous development of reconfigurable systems was discussed in [7,10,11].

Concerning strictly logical properties, we would like to discuss decidability and completeness properties of $\mathcal{HHL}$. For this reason, we intend to explore, among other things, the finite model property for this logic, as well complete proof calculi, resorting to our previous work [13].

Acknowledgement

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