Available online at www.sciencedirect.com

ScienceDirect

Electronic Notes in Theoretical Computer Science 346 (2019) 677–684

Electronic Notes in Theoretical Computer Science

www.elsevier.com/locate/entcs

Graphs with Girth at Least 8 are b-continuous [1]

Allen Ibiapina [2] Ana Silva[3]

ParGO Group - Paralelism, Graphs and Optimization
Centro de Cincias - Departamento de Matemática
Universidade Federal do Ceará
Fortaleza, CE - Brazil

Abstract

A b-coloring of a graph is a proper coloring such that each color class has at least one vertex which is adjacent to each other color class. The b-spectrum of G is the set $S_b(G)$ of integers k such that G has a b-coloring with k colors and $b(G) = \max S_b(G)$ is the b-chromatic number of G. A graph is b-continous if $S_b(G) = [\chi(G), b(G)] \cap \mathbb{Z}$. An infinite number of graphs that are not b-continuous is known. It is also known that graphs with girth at least 10 are b-continuous. In this work, we prove that graphs with girth at least 8 are b-continuous, and that the b-spectrum of a graph G with girth at least 7 contains the integers between $2\chi(G)$ and $b(G)$. This generalizes a previous result by Linhares-Sales and Silva (2017), and tells that graphs with girth at least 7 are, in a way, almost b-continuous.

Keywords: b-chromatic number; b-continuity; girth; bipartite graphs.

1 Introduction

Let G be a simple graph (for basic terminology on graph theory, we refer the reader to [4]). A function $\psi\colon V(G) \to \mathbb{N}$ is a *proper k-coloring of G* if $|\psi(V(G))| = k$ and $\psi(u) \neq \psi(v)$ whenever $uv \in E(G)$. Because we only deal with proper colorings in this text, from now on we refer to them as simply a coloring. We call the elements of $\psi(V(G))$ *colors*. Given a color $i \in \psi(V(G))$, the set $\psi^{-1}(i)$ is called *color class* i. We say that $u \in V(G)$ is a *b-vertex in* ψ(of color $\psi(u)$) if $\psi(N[u]) = \psi(V(G))$. If for some color $c \in \psi(V(G))$, the color class c does not contain b-vertices, we can obtain a $(k-1)$-coloring by changing the color of each vertex $v \in \psi^{-1}(c)$ to another color in $\psi(V(G)) \setminus \psi(N[v])$. We say that this new coloring is obtained from the first

[1] Partially supported by CNPq Projects Universal no. 401519/2016-3 and Produtividade no. 304576/2017-4, and by FUNCAP/CNPq project PRONEM no. PNE-0112-00061.01.00/16.

[2] Email: allenr.roossim@gmail.com

[3] Email: anasilva@mat.ufc.br

https://doi.org/10.1016/j.entcs.2019.08.059
1571-0661/© 2019 The Author(s). Published by Elsevier B.V.

one by *cleaning color c*. In a coloring such that we cannot apply this procedure, all color classes have at least one b-vertex. Such a coloring is called a *b-coloring of G*. Observe that an optimal coloring cannot have the number of colors decreased by the described algorithm; therefore every optimal coloring is also a b-coloring. In [8], the authors define the *b-chromatic number of G*, denoted by $b(G)$, as the largest natural k for which G has a b-coloring with k colors. In the same article, the authors demonstrated that the problem of finding $b(G)$ is NP-complete in general.

Another interesting aspect about b-colorings concerns its existence for every possible value between $\chi(G)$ and $b(G)$. In [8], the authors observe that the cube has a b-coloring using 2 colors and 4 colors, but has no b-coloring using 3 colors. Inspired by this result, in [9] it is shown that for any integer $n \geq 4$ the graph obtained from the complete bipartite graph $K_{n,n}$ by deleting the edges from a perfect matching has a b-coloring using 2 and n colors, but has no b-coloring using a number of colors between 2 and n. This motivates the definition of the *b-spectrum of G*, that is the set $S_b(G)$ containing every integer k such that G has a b-coloring with k colors. A graph G is *b-continous* if $S_b(G) = [\chi(G), b(G)] \cap \mathbb{Z}$. In [2], they prove that for each finite subset $S \subset \mathbb{N} - \{1\}$, there exists a graph G such that $S_b(G) = S$, and also that deciding if a graph is b-continuous is NP-complete even if colorings with $\chi(G)$ and $b(G)$ colors are given.

Now, given a b-coloring with k colors, since each b-vertex has at least $k-1$ neighbors, there exists k vertices with degree at least $k-1$ (this would be a subset of k b-vertices of the k colors). So if we define $m(G)$ as the largest positive integer k such that there exist at least k vertices with degree at least $m(G)-1$ in G, we have that $b(G) \leq m(G)$. This upper bound was introduced in [8], where the authors show that one can find $m(G)$ in polynomial time using the degree list of the graph. Also, they prove that if G is a tree, then $b(G) \geq m(G)-1$, and that one can decide if $b(G) = m(G)$ in polynomial time. Their result was later generalized for graphs with girth at least 7 [6] (the *girth of G* is the minimum lenght of a cycle in G). We also mention that there are many results that say that regular graphs with large girth have high b-chromatic number [3,5,13,3]. Indeed, the following conjecture is still open.

Conjecture 1.1 *If G is a d-regular graph with girth at least 5 and G is not the Petersen graph, then $b(G) = d+1$.*

Because of these results, it makes sense to investigate the b-continuity of graphs with large girth. Indeed, in [1] the authors prove that regular graphs with girth at least 6 and without cycles of length 7 are b-continuous, and in [11], they prove that every graph with girth at least 10 are b-continuous. Here, we improve their result to graphs with girth at least 8.

Theorem 1.2 *If G is a graph with girth at least 8, then G is b-continuous.*

In addition, we prove that graphs with girth at least 7 are, in way, almost b-continuous.

Theorem 1.3 *If G is graph with girth at least 7, then $[2\chi(G), b(G)] \cap \mathbb{Z} \subseteq S_b(G)$.*

Given a graph G and a b-coloring of G with k colors, $k \geq \chi(G) + 1$, the proof of Theorems 1.2 and 1.3 consists in trying to obtain a b-coloring with $k - 1$ colors using simple recoloring procedures; when this is not possible, we get that the graph has a special structure and apply non-constructive arguments to obtain the desired b-coloring. We mention that the coloring problem is NP-complete for graphs with girth at least k, for every fixed $k \geq 3$ [12]. This is why any proof of a result like Theorem 1.2 is expected to have a non-constructive part. In the next section, we present the basic definitions and results, in Section 3 we present our proofs, and in Section 4, we make some further comments on the proof and state some open questions.

2 Preliminaries

In [1], a vertex $u \in V(G)$ is called a *k-iris* if there exists $S \subset N(u)$ such that $|S| \geq k - 1$ and $d(v) \geq k - 1$ for every $v \in S$ (observe Figure 1).

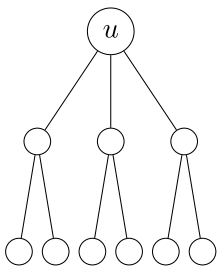

Fig. 1. In the figure, we presente a 4-iris.

This definition is important because of the following important lemma. Observe that the lemma also implies that if G and k satisfies the conditions, then $b(G) \geq k$.

Lemma 2.1 ([1]) *Let G be a graph with girth at least 6 and without cycles of lengh 7. If G has a k-iris with $k \geq \chi(G)$, then G has a b-coloring with k colors.*

As we said before, given a b-coloring of G with k colors, $k > \chi(G) + 1$, we try to obtain a b-coloring of G with $k - 1$ colors. However, this is not always possible, and when this happens, it is because we have a k-iris. Our theorem then follows from the lemma above. We mention that the constraint about not having cycles of length 7 appears only in the above lemma, but not on our proof. We now introduce the further needed definitions.

From now on, let G be a simple graph and ψ be a b-coloring of G with $k > \chi(G) + 1$ colors. We say that u *realizes color* i if $\psi(u) = i$ and u is a b-vertex. We also say that color i is *realized by* u. For $x \in V(G)$ and $i \in \{1, \ldots, k\}$, let $N^{\psi,i}(x)$ be the set of vertices of color i in the neighborhood of x, i.e., $N^{\psi,i}(x) = N(x) \cap \psi^{-1}(i)$; in fact, we omit ψ in the superscript since it is always clear from the context. This is also done in the next definitions. For a subset $X \subseteq V(G)$, let $N^i(X) = (\bigcup_{x \in X} N^i(x)) \setminus X$. Let $B(\psi)$ denote the set of b-vertices in ψ and, for each $i \in \{1, \ldots, k\}$, let $B_i = B(\psi) \cap \psi^{-1}(i)$ be the set of b-vertices in color class i.

Given a set K such that $K \subseteq \psi^{-1}(i)$ for some $i \in \{1, \ldots, k\}$, we say that a color $j \in \{1, \ldots, k\} \setminus \{i\}$ is *dependent on* K if $N^i(B_j) \subseteq K$; denote by $U(K)$ the set of colors depending on K. If $K = \{x\}$, we write simply $U(x)$. Given $x \in V(G) \setminus B(\psi)$, if $|U(x)| \geq 2$ we call x a *useful* vertex; otherwise, we say that x is *useless*. For $j \in \{1, \ldots, k\}$, we say that $x \in V(G)$ is *j-mutable* if x is useless and there exists a color c such that we can change the color of x to c without creating any b-vertex of color j; we also say that color c is *safe for* (x, j). If there is no safe color for (x, j), we say that x is j-imutable.

3 Proofs

The next lemma is the main ingredient in our proof. Combined with Lemma 2.1, it immediatly implies Theorem 1.2.

Lemma 3.1 *Let $G = (V, E)$ be a graph with girth at least 7. If G has b-coloring with k colors where $k \geq \chi(G) + 1$, then either G has a b-coloring with $k - 1$ colors, or G contains a $(k-1)$-iris.*

Proof. Our proof is similar to that made in [11], but we concentrate in one color that we want to eliminate.

Suppose that G does not have a b-coloring with $k - 1$ colors; we prove that G has a $(k-1)$-iris. For this, let ψ be a b-coloring with k colors that minimizes $|B_1|$ and then minimizes $|\psi^{-1}(1)|$ (i.e., it firstly minimizes the number of b-vertices of color 1, then it minimizes the number of vertices of color 1). First, we prove that every $x \in \psi^{-1}(1) \setminus B_1$ is useful. Suppose otherwise and let x be a useless vertex in color class 1, i.e., $|U(x)| \leq 1$. If $U(x) = \emptyset$, then we can recolor x without losing any b-vertex, a contradiction since ψ minimizes $|\psi^{-1}(1)|$. And if $U(x) = \{d\}$, then we can obtain a b-coloring with $k - 1$ by recoloring x and cleaning d, again a contradiction. Therefore, the following holds:

(i) Every $x \in \psi^{-1}(1) \setminus B_1$ is useful.

Now, we choose any $u \in B_1$ and analyse its vicinity in order to obtain the desired $(k-1)$-iris. For this, the following two claims are essential.

Claim 3.2 *Let $j \in \{2, \ldots, k\}$. If every $x \in N^j(u) \setminus B_j$ is 1-mutable, then one of the following holds:*

(ii) $N(u) \cap B_j \neq \emptyset$; or

(iii) There exists a color $d \in \{2, \ldots, k\} \setminus \{j\}$ such that d depends on $N^j(u)$, i.e.,

$$N^j(B_d) \subseteq N^j(u).$$

Proof of claim: Suppose that neither (ii) nor (iii) holds, and let ψ' be obtained from ψ by changing the color of each $x \in N^j(u)$ to a color c safe for $(x, 1)$. Because (*iii*) does not hold, we get that $U(N^j(u)) \subseteq \{1\}$. Therefore, at most one color loses all of its b-vertices, namely color 1, and since every $x \in N^j(u)$ is 1-mutable, no b-vertices of color 1 is created. But because u is not a b-vertex in ψ' (it is not adjacent to color j anymore) and ψ minimizes $|B_1|$, we get that ψ' cannot be a b-coloring, which means that we can obtain a b-coloring with $k-1$ colors by cleaning color 1.$\diamondsuit$

The following claim tells us that (ii) or (iii) actually always hold.

Claim 3.3 *(iv) Every $x \in N^j(u) \setminus B_j$ is 1-mutable, for every $j \in \{2, \dots, k\}$.*

Proof of claim: Suppose, without loss of generality, that $d \in \{2, \dots, k\}$ is such that the colors in $\{d+1, \dots, k\}$ are exactly the colors that contains some 1-imutable vertex. We count the number of colors with b-vertices in the vicinity of u to get that in fact $d \geq k$. So, for each $i \in \{d+1, \dots, k\}$, let $w_i \in N^i(u)$ be a 1-imutable vertex. By definition, this means that, for each $i \in \{d+1, \dots, k\}$, there exists some neighbor of w_i that would be turned into a b-vertex of color 1 in case we change the color of w_i; let v_i be such a vertex. We then know that $v_i \in \psi^{-1}(1) \setminus B_1$, which by (i) gets us that $|U(v_i)| \geq 2$. By the definition of $U(x)$ and the fact that every $x \in \{v_{d+1}, \dots, v_k\}$ is colored with color 1, we get:

$$U(v_i) \cap U(v_\ell) = \emptyset, \text{ for every } i, \ell \in \{d+1, \dots, k\}, i \neq \ell. \tag{1}$$

Now, we investigate the b-vertices around colors $\{2, \dots, d\}$. By Claim 3.2, suppose, without loss of generality, that $p \in \{2, \dots, d\}$ is such that (ii) holds for colors in $\{2, \dots, p\}$, while (iii) holds for colors in $\{p+1, \dots, d\}$. For each $i \in \{p+1, \dots, d\}$, let $c_i \in \{2, \dots, k\} \setminus \{i\}$ be a color depending on $N^i(u)$, which means that $B_{c_i} \subseteq N(N^i(u))$. Observe that, since G has no cycles of length 3, we get:

$$\{2, \dots, p\} \cap \{c_{p+1}, \dots, c_d\} = \emptyset \tag{2}$$

Also, because G has no cycles of length 4, we get $c_i \neq c_\ell$ for every $i \neq \ell$, i.e.:

$$|\{c_{p+1}, \dots, c_d\}| = d - p \tag{3}$$

Finally, because G has no cycles of length smaller than 6, we get that:

$$\{2, \dots, p, c_{p+1}, \dots, c_d\} \cap \bigcup_{i=d+1}^{k} U(v_i) = \emptyset. \tag{4}$$

Now, recall that $\psi(v_i) = 1$ for every $i \in \{d+1, \dots, k\}$, and that $c_i \neq 1$ for every $i \in \{p+1, \dots, d\}$. This means that $1 \notin \{c_{p+1}, \dots, c_d\} \cup \bigcup_{i=d+1}^{p} U(v_i)$. By combining Equations (1) through (4), we get the following, which implies $d \geq k$ as desired:

$$\begin{aligned} k-1 &\geq |\{2, \dots, p\} \cup \{c_{p+1}, \dots, c_d\} \cup U(v_{d+1}) \cup \ldots \cup U(v_k)| \\ &= d - 1 + \textstyle\sum_{i=d+1}^{k} |U(v_i)| \\ &\geq d - 1 + 2(k-d). \end{aligned} \qquad \diamondsuit$$

Now, let $N = (N(u) \cup N(N(u))) \setminus \{u\}$. Observe that because (ii) or (iii) holds for every color $\ell \in \{2, \dots, k\}$, we get that $B(\psi) \subseteq N$. Suppose that $N[u]$ does not contain a $(k-1)$-iris, otherwise the proof is done. This means that at least one color in $\{2, \dots, k\}$, say k, is such that (ii) does not hold for k, which by Claim 3.2 implies that (iii) holds, i.e., that there exists a color in $\{2, \dots, k-1\}$, say 2, such that $N^2(B_k) \subseteq N^2(u)$ (Observe Figure 2). Now, let $w \in N^1(B_k)$; it exists since the vertices in B_k are b-vertices. By (i), there exists at least two colors in $\{2, \dots, k\}$

that depend on w. But because $B(\psi) \subseteq N$, we get a cycle of length at most 6, a contradiction.

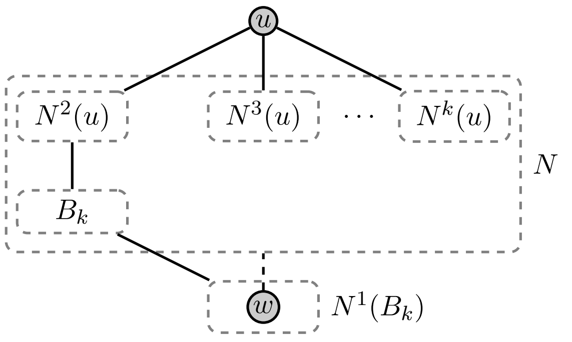

Fig. 2. Structure around u when color k does not satisfy Claim 3.2.(ii).

□

Now, to prove Theorem 1.3, we apply Lemma 3.1 and the next lemma. A *star* is a tree that has at most one vertex with degree bigger than 1, and the *diameter* of a graph G is the maximum number of edges in a shortest path of G. Here, as happens in Lemma 2.1, we get that the existence of a k-iris in G implies $b(G) \geq k$.

Lemma 3.4 *If a graph G has girth at least 7 and a k-iris where $k \geq 2\chi(G)$, then G has a b-coloring with k colors.*

Proof. Let $u \in V(G)$ be a k-iris with $k \geq 2\chi(G)$. Let $u_2, \ldots, u_k$ be neighbors of u such that $d(u_i) \geq k-1$ for every $i \in \{2, \ldots, k\}$; let N_i be a subset of $k-2$ neighbors of u_i different from u. Start by coloring u with 1 and, for each $i \in \{2, \ldots, k\}$, give color i to u_i and colors $\{2, \ldots, k\} \setminus \{i\}$ to N_i. Denote by T the set $\{u, u_2, \ldots, u_k\} \cup \bigcup_{i=2}^{k} N_i$, i.e., T denotes the set of colored vertices. Observe Figure 3.

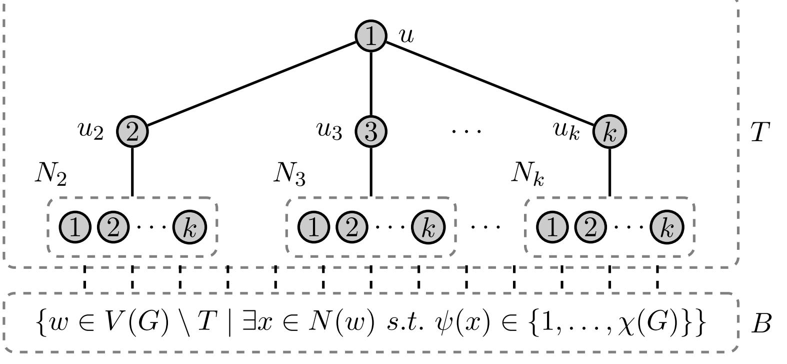

Fig. 3. Subset of vertices around u. The label inside a vertex denotes its color.

Observe that the coloring can be easily done since $G[T] - u$ is a forest formed by $k-1$ stars. Also, note that we already have k b-vertices of distinct colors, and thus it only remains to extend the partial coloring to the rest of the graph. For this, let B be the set of vertices adjacent to some color class in $\{1, \ldots, \chi(G)\}$. We claim that

$|N(w) \cap T| \leq 1$ for every $w \in B$; indeed, because T induces a tree of diameter 4, if this was not true, then we would get a cycle of length at most 6. By the definition of B, we then get that every $w \in B$ has no neighbors in color classes $\chi(G)+1, \dots, k$. Thus, since $k \geq 2\chi(G)$, we can color $G[B]$ with colors $\chi(G)+1, \dots, 2\chi(G)$. Finally, by the definition of B, we know that every $w \in V(G) \setminus (B \cup T)$ has no neighbors of color 1 through $\chi(G)$, which means that we can color $G - T - B$ with these colors.□

4 Conclusion

We have proved that every graph with girth at least 8 is b-continuous, and that graphs with girth at least 7 are in way almost b-continuous. This improves the result presented in [11], where they prove that graphs with girth at least 10 are b-continuous. There, the authors also pose the following questions:

Question 1 *What is the minimum $\hat{g}$ such that G is b-continuous whenever G is a graph with girth at least $\hat{g}$?*

Question 2 *Are bipartite graphs with girth at least 6 b-continuous?*

Recall that the graph obtained from the complete bipartite graph $K_{n,n}$ by removing a perfect matching is not b-continuous, for every $n \geq 4$ [9]. Hence, by our result we get:

$$5 \leq \hat{g} \leq 8.$$

We believe that the same techniques might improve this bound to 7, but not further. In particular, we mention that Lemma 3.1 works for graphs with girth 7 and that the bound is 8 because of Lemma 2.1. Therefore, if the following question is answered "yes", then we get $\hat{g} \leq 7$.

Question 3 *Let G be a graph with girth at least 7 such that G has a k-iris, with $k \geq \chi(G) + 1$. Does G admit a b-coloring with k colors?*

As for the case of bipartite graphs, we think it is worth mentioning a known conjecture about their b-chromatic number. Recall the upper bound $m(G)$ for the b-chromatic number $b(G)$, which is the maximum value k for which there exist k vertices with degree at least $k - 1$. The set of all vertices with degree at least $m(G) - 1$ is denoted by $D(G)$, and a graph is said to be *tight* if $|D(G)| = m(G)$; this means that there is only one candidate set for the b-vertices of a b-coloring of G with $m(G)$ colors. Deciding if $b(G) = m(G)$ is NP-complete even for bipartite tight graphs [9]. In [7], the authors define the class $\mathcal{B}_m$ that contains every bipartite graph $G = (A \cup B, E)$ such that $m(G) = m$, $D(G) = A$ and G has girth at least 6. They conjecture the following:

Conjecture 4.1 *[7] For every $m \geq 3$, and every $G \in \mathcal{B}_m$, we have that:*

$$b(G) \geq m(G) - 1.$$

We mention that, if G is a bipartite graph with girth at least 6 and a b-coloring of G with k colors is given, $k \geq \chi(G)+1$, then, with a little more work, one can get from the proof of Lemma 3.1 that G contains an induced subgraph H that has a structure similar to the structure of a graph in $\mathcal{B}_k$. Trying to use this structure to obtain a b-coloring of H with $k-1$ colors could translate into proving Conjecture 4.1. And on the other way around, we believe that a strategy to prove Conjecture 4.1 could help coloring these graphs, which would imply that the answer to Question 2 is "yes". This means that answering Question 2 seems as hard as proving Conjecture 4.1. We also mention that in [10], it is proved that Conjecture 4.1 is a consequence of the famous Erdős-Faber-Lovász Conjecture, which remains open since 1972 and which is largely believed to hold. This is strong evidence that Conjecture 4.1 holds.

Finally, because of the difficulties in obtaining b-continuity already for bipartite graphs with girth at least 6, maybe a good bet would be also to see if the lower bound for $\hat{g}$ is tight. So, we propose one additional question:

Question 4 *Does there exist a graph with girth 5 that is not b-continuous?*

References

[1] R. Balakrishnan and T. Kavaskar. b-coloring of kneser graphs. *Discrete Appl. Math.*, 160:9–14, 2012.

[2] D. Barth, J. Cohen, and T. Faik. On the b-continuity property of graphs. *Discrete Appl. Math.*, 155:1761–1768, 2007.

[3] M. Blidia, F. Maffray, and Z. Zemir. On b-colorings in regular graphs. *Discrete Appl. Math.*, 157:1787–1793, 2009.

[4] A. Bondy and U.S.R. Murty. *Graph Theory.* Spring-Verlag Press, 2008.

[5] S. Cabello and M. Jakovac. On the b-chromatic number of regular graphs. *Discrete Appl. Math.*, 159:1303–1310, 2011.

[6] V. Campos, C. Lima, and A. Silva. Graphs with girth at least 7 have high b-chromatic number. *European Journal of Combinatorics*, 48:154–164, 2015.

[7] F. Havet, C. Linhares-Sales, and L. Sampaio. b-coloring of tight graphs. *Discrete Appl. Math.*, 160(18):2709–2715, 2012.

[8] R.W. Irving and D.F. Manlove. The b-chromatic number of a graph. *Discrete Appl. Math.*, 91:127–141, 1999.

[9] J. Kratochvíl, Zs. Tuza, and M. Voigt. On the b-chromatic number of graphs. In *WG 2002 - Int. Workshop on Graph-Theoretic Concepts in Comp. Sc.*, 2002.

[10] W.-H. Lin and G.J. Chang. b-coloring of tight bipartite graphs and the erdos–faber–lovász conjecture. *Discrete Appl. Math.*, 161(7-8):1060–1066, 2013.

[11] C. Linhares-Sales and A. Silva. The b-continuity of graphs with large girth. *Graphs and Combinatorics*, 33(5):1139–1146, 2017.

[12] V.V. Lozin and M. Kaminski. Coloring edges and vertices of graphs without short or long cycles. *Contributions do Discrete Mathematics*, 2(1), 2007.

[13] A. El Sahili and H. Kouider. About b-colouring of regular graphs. *Utilitas Math.*, 80:211–215, 2009.