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A Duality Between Ω-categories and Algebraic Ω-categories [1,2]

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Abstract

In this paper, we propose a definition of algebraic Ω-categories. Let Ω-**POID** denote the category of Ω-categories with Ω-functors between them such that inverse image of ideals are also ideals, and let Ω-$\mathbf{AlgDom}_G$ denote the category of algebraic Ω-categories with Scott continuous functors between them having left Ω-adjoints. We show that Ω-$\mathbf{AlgDom}_G$ and Ω-**POID** are dual equivalent to each other.

Keywords: Duality, (Algebraic) Ω-category, Ω-adjunction, Ideals.

1 Introduction

The Stone duality and Stone representation come from the classical Stone representation of Boolean algebras [19], and lead to locale theory as 'pointless topology' [2]. Abramsky related the important application of Stone duality in Theoretical Computer Science, particularly in Domain Theory of denotational semantics of computer programming languages [1]. It provides the right framework for understanding the relationship between denotational semantics and program logic. Study of dualities between categories of certain domains were originated by Hofmann, Mislove and

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Stralka [7] and Lawson [13]. Therein, there are two basic dualities in domain theory: the first is the duality between the category of posets and the category of algebraic domains, many other dualities can be induced by this one; the second one is the duality between the category of domains (i.e., continuous dcpos) and the category of completely distributive lattices.

Quantitative Domain Theory, which models concurrent systems, forms a new branch of Domain Theory, and has undergone active research in the past three decades. Rutten's generalized (ultra)metric spaces [17], Flagg's continuity spaces [6] and Wagner's Ω-categories [24] are examples of quantitative domain theory frameworks. Therein, the Ω-category approach has been payed more and more attentions, including Waszkiewicz [25], Hofmann and Waszkiewicz [8] and Lai and Zhang [12]. And a kind of Lawson duality in framework of Ω-categories has been studied by Hofmann and Waszkiewicz [9].

Ω-categories are interesting objects for mathematicians and theoretical computer scientists. Firstly, Ω-categories are a special kind of enriched categories, so they can be studied as categories. In 1973, Lawvere [14] observed that the theory of Ω-categories unifies preordered sets ($\Omega = \{0, 1\}$, the two point lattice), generalized metric spaces ($\Omega = [0, \infty)^{op}$), and many other mathematical structures into one framework. Secondly, due to the adjunction $a * b \leq c \Leftrightarrow b \leq a \to c$ in the quantale Ω, if we interpret the complete lattice as a set of truth values, the operators $*$ and $\to$ can be interpreted as the logic connectives conjunction and implication respectively. Therefore, the theory of Ω-categories has a many-valued logic flavor [17]. This feature also leads to the point that Ω-categories can be regarded as generalized preordered sets, or Ω-valued preordered sets. For instance, we can interpret the $A(a, b)$ as the degree to which a is smaller than or equal to b, that is, the connection between two points is measured by an element in Ω. Thirdly, Ω-categories are closely related to topology. This can be roughly explained as follows. Generalized metric spaces and many-valued preordered sets are special kinds of Ω-categories, and conversely general Ω-categories can also be studied as Ω-valued quasi-metric spaces or many-valued preordered sets.

The aim of this paper is to study the first duality mentioned in the first paragraph in framework of Ω-categories, that is the duality between the category of Ω-categories and of algebraic Ω-categories. This paper is organized as follows: in Section 2, we recall some basic materials related to Ω-category theory and some preparations are made; in Section 3, we firstly give a definition of an algebraic Ω-categories and then establish a duality between the category of Ω-categories and the category of algebraic Ω-categories.

2 Preliminaries and preparations

We refer to [15] for general category theory, to [10] for enriched category theory, to [16] for quantales, and to [12] for Ω-categories.

A commutative quantale is a pair $(\Omega, *)$, where Ω is a complete lattice and $*$ is a commutative, associative, and monotone operation $* : \Omega \times \Omega \longrightarrow \Omega$ such that

$p * (-)$ has a right adjoint for every $p \in \Omega$. The right adjoint of $p * (-)$ is denoted $p \to (-)$. A commutative quantale is called unital if $*$ has a unit I, i.e. $p * I = p$ for every $p \in \Omega$. It should be noted that the unit I need not be the greatest element of Ω. Throughout this paper, $(\Omega, *, I)$, or just Ω, will always denote a commutative, unital quantale if not otherwise specified.

Proposition 2.1 *Suppose that $(\Omega, *, I)$ is a commutative unital quantale, then*

(I1) $p * \bigvee_i q_i = \bigvee_i (p * q_i)$.
(I2) $I \le p \to q \Leftrightarrow p \le q$;
(I3) $I \to p = p$;
(I4) $(p \to q) * (q \to r) \le p \to q$;
(I5) $(\bigvee_i p_i) \to q = \bigwedge_i (p_i \to q)$;
(I6) $p \to (\bigwedge_i q_i) = \bigwedge_i (p \to q_i)$;
(I7) $(r \to p) \to (r \to q) \ge p \to q$;
(I8) $(p \to r) \to (q \to r) \ge q \to p$;
(I9) $p \to (q \to r) = (p * q) \to r$.

Categorically speaking, a commutative unital quantale $(\Omega, *, I)$ is just a symmetric, monoidal closed category with the underlying category being a complete lattice. Therefore, we can develop a theory of categories enriched over Ω [10,14].

A category enriched over Ω [14], or an Ω-category, is a set A together with an assignment of an element $A(a, b) \in \Omega$ to every ordered pair of $(a, b) \in A \times A$, such that

(1) $I \le A(a, a)$ for every $a \in A$;
(2) $A(a, b) * A(b, c) \le A(a, c)$ for all $a, b, c \in A$.

For all $a, b \in \Omega$, let $\Omega(a, b) = a \to b$. Then $(\Omega, \to)$ becomes an Ω-category [14]. The **L**-preordered sets [3] for **L** a complete residuated lattice, the generalized metric space [14,23] and the $\mathcal{V}$-continuity space in [5,6] are special cases of Ω-categories.

Suppose that A is an Ω-category. Let $A^{op}(a, b) = A(b, a)$ for all $a, b \in A$. Then A^{op} is also an Ω-category, called the opposite of A. If B is a subset of A, let $B(x, y) = A(x, y)$ for all $x, y \in B$. Then B becomes an Ω-category, called a (full) subcategory of A. An Ω-functor between Ω-categories A and B is a map $f : A \longrightarrow B$ such that $A(a, b) \le B(f(a), f(b))$ for all $a, b \in A$. An Ω-functor $f \in [A^{op}, \Omega]$ (resp., $f \in [A, \Omega]$) is always called a lower set (resp., an upper set) in A.

Given two Ω-categories A and B, denote the set of all the Ω-functors from A to B by $[A, B]$. For all $f, g \in [A, B]$, let $[A, B](f, g) = \bigwedge\limits_{x \in A} B(f(x), g(x))$. Then $[A, B]$ becomes an Ω-category, called the functor category from A to B [10]. All Ω-categories and Ω-functors form an ordinary category, denoted by Ω-**Cat**.

For an ordinary set X, Ω^X the set of all maps from X to Ω, the members are called Ω-sets of X. The family Ω^X is also an Ω-category, which is the same to $[X, \Omega]$ by regarding X as a discrete Ω-category. That is to say, $\Omega^X(f, g) = \bigwedge\limits_{x \in X} f(x) \to g(x)$ $(\forall f, g \in \Omega^X)$

Definition 2.2 A pair of Ω-functors $f \in [A, B], g \in [B, A]$ is said to be an Ω-

adjunction, in symbols $f \dashv g : A \rightharpoonup B$, if $B(f(a), b) = A(a, g(b))$ for all $a \in A,\ b \in B$. In this case, we say f is a left Ω-adjoint of g and g is a right Ω-adjoint of f. Sometimes we also say that (f, g) is an Ω-adjunction between A and B.

Theorem 2.3 *[12] Suppose $f : A \longrightarrow B$ and $g : B \longrightarrow A$ are two maps (need not be Ω-functors) between Ω-categories. Then the following conditions are equivalent:*

(1) (f, g) is an Ω-adjunction.

(2) For all $a \in A, b \in B$, $A(a, g(b)) = B(f(a), b)$.

(3) f and g are functors and $I \leq A(a, gf(a)), I \leq B(fg(b), b)$ $(\forall a \in A, b \in B)$.

Example 2.4 A fundamental example of Ω-adjunctions are that induced by Kan extension. Let $f : A \longrightarrow B$ be an Ω-functor. For each $\psi \in [B, \Omega]$, define $f^{\leftarrow}(\psi) = \psi \circ f$. Then we obtain a functor $f^{\leftarrow} : [B, \Omega] \longrightarrow [A, \Omega]$, which has a left Ω-adjoint $f^{\rightarrow} : [A, \Omega] \longrightarrow [B, \Omega]$ given by $f^{\rightarrow}(\phi)(y) = \bigvee_{x \in A} \phi(x) * B(f(x), y)$ $(\forall y \in B)$ for each $\phi \in [A, \Omega]$. That is $f^{\rightarrow} \dashv f^{\leftarrow} : [A, \Omega] \rightharpoonup [B, \Omega]$ is an Ω-adjunction. Since if $f : A \longrightarrow B$ is an Ω-functor then so is $f : A^{op} \longrightarrow B^{op}$, we have $f^{\rightarrow} \dashv f^{\leftarrow} : [A^{op}, \Omega] \rightharpoonup [B^{op}, \Omega]$ is an Ω-adjunction.

Let A be an Ω-category. For $\phi \in \Omega^A$, define $\mathbf{y}(\phi)(x) = \bigvee_{a \in A} A(x, a) * \phi(a)$ $(\forall x \in A)$. For $x \in A$, by $\mathbf{y}(x)$ we mean the Ω-set $\mathbf{y}(I_x)$, where I_x is the Ω-set sending x to the unit I and others to 0. In fact, $\mathbf{y}(x)(y) = A(y, x)$ for any $x, y \in A$.

Definition 2.5 ([12,26]) In an Ω-category A, an Ω-set ϕ of A is called a directed set in A if

(1) $\bigvee_{x \in A} \phi(x) \geq I$;

(2) $\forall x, y \in A,\ \phi(x) * \phi(y) \leq \bigvee_{z \in A} \phi(z) * A(x, z) * A(y, z)$.

A directed set is called an ideal if it is a lower set additionally. The set of all ideals in A is denoted by $\mathcal{I}(A)$, then $\mathcal{I}(A)$ is a subcategory of $[A^{op}, \Omega]$. Clearly, for each $x \in A$, $\mathbf{y}(x) \in \mathcal{I}(A)$.

Proposition 2.6 *(1) For any $x \in A, J \in \mathcal{I}(A)$, $\mathcal{I}(A)(\mathbf{y}(x), J) = J(x)$.*

(2) Let $f : A \longrightarrow B$ be an Ω-functor, then $f^{\leftarrow}(J) \in \mathcal{I}(A)$ for any $J \in \mathcal{I}(B)$.

Proof. Straightforward. □

Lemma 2.7 *For $\phi \in \Omega^A$, we have*

(1) for any $\psi \in \Omega^A$, $\Omega^A(\phi, \psi) * \phi \leq \psi$.

(2) for any $x, y \in A$, $A(x, y) * \Omega^A(\phi, \mathbf{y}(x)) \leq \Omega^A(\phi, \mathbf{y}(y))$.

(3) $\mathbf{y}(\phi)$ is the smallest lower set which is larger than or equal to ϕ under point-wise order in Ω^A;

(4) if ϕ is directed then $\mathbf{y}(\phi)$ is an ideal;

(5) for an Ω-functor $f \in [A, B]$, if ϕ is directed set in A then $f^{\rightarrow}(\phi) \in \mathcal{I}(B)$.

Proof. (1), (2) and (3) are straightforward.

(4) Suppose that ϕ is directed. Then

(i) $\bigvee_{x\in A} \mathbf{y}(\phi)(x) \geq \bigvee_{x\in A} \phi(x) \geq I$.

(ii) For any $x, y \in A$,

$$\begin{aligned}
\mathbf{y}(\phi)(x) * \mathbf{y}(\phi)(y) &= \bigvee_{a,b\in A} A(x,a) * \phi(a) * A(y,b) * \phi(b) \\
&\leq \bigvee_{a,b,c\in A} \phi(c) * A(a,c) * A(b,c) * A(x,a) * A(y,b) \\
&\leq \bigvee_{c\in A} \phi(c) * A(x,c) * A(y,c) \\
&\leq \bigvee_{c\in A} \bigvee_{z\in A} \phi(c) * A(z,c) * A(x,z) * A(y,z) \\
&= \bigvee_{z\in A} (\bigvee_{c\in A} \phi(c) * A(z,c)) * A(x,z) * A(y,z) \\
&= \bigvee_{z\in A} \mathbf{y}(\phi)(z) * A(x,z) * A(y,z).
\end{aligned}$$

Then $\mathbf{y}(\phi)$ is directed.

(5) By Proposition 5.3 in [26], we know that $f_\Omega^{\rightarrow}(\phi)$ is directed, and by (4), $f^{\rightarrow}(\phi) = \mathbf{y}(f_\Omega^{\rightarrow}(\phi))$ is an ideal. □

Let A be an Ω-category. An element $b \in A$ is called a colimit [10] of a functor $f \in [K, A]$ weighted by $\phi \in [K^{op}, \Omega]$ if for each $y \in A$, $A(b, y) = \bigwedge_{k\in K} \phi(k) \to A(f(k), y)$. Weighted colimits, when they exist, are unique up to isomorphism. It is written by $b = \mathrm{colim}_\phi f$ if b is a colimit of f weighted by ϕ.

Consider an Ω-category A as an Ω-preordered set, an element $b \in A$ is called a join of $\phi : A \longrightarrow \Omega$, in symbols $b = \sqcup\phi$, if $A(b, x) = \bigwedge_{y\in A} \phi(y) \to A(y, x)$ for any $x \in A$. In fact, if ϕ is a lower set in A, then $\sqcup\phi = \mathrm{colim}_\phi \mathrm{id}$, where $\mathrm{id} : A \longrightarrow A$ is the identical functor (cf. Example 3.2(4) and Proposition 3.3(2) in [12]).

Proposition 2.8 *In an Ω-category A, for $\phi \in \Omega^A$, if $\sqcup\phi$ exists then so does $\sqcup\mathbf{y}(\phi)$ and $\sqcup\phi = \sqcup\mathbf{y}(\phi)$.*

Proof. Suppose that $a = \sqcup\phi$, we only need to show that for any $x \in A$, $\bigwedge_{y\in A} \mathbf{y}(\phi)(y) \to A(y, x) = \bigwedge_{y\in A} \phi(y) \to A(y, x)$. In fact, $\bigwedge_{y\in A} \mathbf{y}(\phi)(y) \to A(y, x) = \bigwedge_{y\in A} \bigwedge_{z\in A} (\phi(z) * A(y, z)) \to A(y, x) = \bigwedge_{z\in A} \phi(z) \to \bigwedge_{y\in A} (A(y, z) \to A(y, x)) = \bigwedge_{y\in A} \phi(y) \to A(y, x)$. □

Proposition 2.9 *[4,10,12,20] If $f : A \longrightarrow B$ has a right Ω-adjoint, that is f is a left Ω-adjunction. Then f preserves the existing joins, that is $f(\sqcup\phi) = \sqcup f^{\rightarrow}(\phi)$.*

Proof. Easily following from Theorem 3.11 in [12] and Proposition 2.9 above. See also Theorem 4.5 in [26]. □

By *a class of weights* [2,10,11] is meant a functor $\Phi : \Omega\text{-}\mathbf{Cat} \longrightarrow \Omega\text{-}\mathbf{Cat}$ such that (1) for every Ω-category A, $\Phi(A) \subseteq [A^{op}, \Omega]$; (2) $\Phi(A)$ contains the image of the Yoneda embedding $\mathbf{y} : A \longrightarrow [A^{op}, \Omega]$; (3) $\Phi(f) = f^{\rightarrow}$ for every Ω-functor $f : A \longrightarrow B$. The class of weights $\mathcal{P}$ given by $\mathcal{P}(A) = [A^{op}, \Omega]$ is the largest class of weights. The class of weights $\mathcal{Y}$ given by $\mathcal{Y}(A) = \{\mathbf{y}(a) |\ a \in A\}$ is the smallest class of weights. The correspondence $\mathcal{I} : A \longrightarrow \mathcal{I}(A)$ is a class of weights (Lemma 5.3 in [12]).

Let Φ be a class of weights. An Ω-category is call Φ-cocomplete if for any $\phi \in \Phi(K)$ and any functor $f \in [K, A]$, $\mathrm{colim}_\phi f$ always exists. Let Φ be a class of wights. A functor $f \in [A, B]$ between Φ-cocomplete Ω-categories is called Φ-cocontinuous if it preserves colimits weights in Φ, that is $\mathrm{colim}_\phi g = \mathrm{colim}_\phi(fg)$ for all $\phi \in \Phi(K)$ and $g \in [K, A]$.

Proposition 2.10 *[2,12] An Ω-category A is Φ-cocomplete iff $\sqcup\phi$ exists for any $\phi \in \Phi(A)$. A functor $f \in [A, B]$ is Φ-cocontinuous iff $f(\sqcup\phi) = \sqcup f^{\rightarrow}(\phi)$ for any $\phi \in \Phi(A)$.*

Proof. This proposition can be implied by using Proposition 3.5, Corollary 3.5 and Corollary 4.6 in [12]. □

Corollary 2.11 *An Ω-category A is $\mathcal{I}$-cocomplete iff $\sqcup I$ exists for any $I \in \mathcal{I}(A)$. A functor $f \in [A, B]$ is $\mathcal{I}$-cocontinuous iff $f(\sqcup I) = \sqcup f^{\rightarrow}(I)$ for any $I \in \mathcal{I}(A)$.*

An $\mathcal{I}$-cocontinuous functor is called Scott continuous in some papers, e.g. [26], it is a counterpart of a Scott continuous map in domain theory.

3 Algebraic Ω-category and its dual to Ω-category

Let L be an $\mathcal{I}$-cocomplete Ω-category. Define $\mathbf{w} : L \times L \longrightarrow \Omega$ by

$$\mathbf{w}(a, b) = \bigwedge_{J \in \mathcal{I}(L)} L(b, \sqcup J) \rightarrow J(a)\ (\forall a, b \in L).$$

We call $\mathbf{w}$ the way below relation on L (which is denoted by $\Downarrow$ in [26]). For $x \in L$, if $\mathbf{w}(x, x) \geq I$, then we call x a compact element in L and denote by $K(L)$ the set of all compact elements in L.

Let L be an $\mathcal{I}$-cocomplete Ω-category and $x \in L$. Define a map $k_x : L \longrightarrow \Omega$ by $k_x = \mathbf{y}(x)|_{K(L)}$, $\mathbf{y}(x)$ restricted on $K(L)$, that is $k_x(y) = e(y, x)$ if $y \in K(L)$ and otherwise 0. If k_x is directed in L (or equivalently, $k_x \in \mathcal{I}(K(L))$) and $x = \sqcup k_x$ for any $x \in L$, then we call L an algebraic Ω-category. The algebraic Ω-category of a generalization of the algebraic fuzzy dcpos in [26] for Ω a complete residuated lattice and that in [22] for Ω a complete Heyting algebra.

The aim of this section is to establish a duality between the following two categories:

The one is Ω-**POID**: objects are Ω-categories, morphisms are maps between them such that inverse image of ideals are still ideals (maps like that the one $f : A \longrightarrow B$ between Ω-categories such that $f^{\leftarrow}(I) \in \mathcal{I}(A)$ for all $I \in \mathcal{I}(B)$, it is routine to show that such a map is automatically an Ω-functor).

The other is Ω-$\mathbf{AlgDom}_G$: objects are algebraic Ω-categories, morphisms are Scott continuous maps between them which having left Ω-adjoints.

The duality between Ω-$\mathbf{POID}$ and Ω-$\mathbf{AlgDom}_G$ will show the reasonableness of the definition of algebraicness of Ω-categories.

3.1 A functor Ω-$\mathbf{POID}$ *from to* Ω-$\mathbf{AlgDom}_G^{op}$

Proposition 3.1 *For any Ω-category A, $\mathcal{I}(A)$ is $\mathcal{I}$-cocomplete as a full subcategory of $[A^{op}, \Omega]$.*

Proof. Suppose that $\Phi \in \mathcal{I}(\mathcal{I}(A))$. We will show that $\sqcup\Phi = \bigvee_{J\in\mathcal{I}(A)} \Phi(J) * J$. Put $\bigvee_{J\in\mathcal{I}(A)} \Phi(J) * J = \phi$.

Step 1. $\phi \in \mathcal{I}(A)$. In fact,

(i) ϕ is a lower set. For any $x, y \in A$,

$$\phi(x) \to \phi(y) \geq \bigwedge_{J\in\mathcal{I}(A)} (\Phi(J)*J(x)) \to (\Phi(J)*J(y)) \geq \bigwedge_{J\in\mathcal{I}(A)} J(x) \to J(y) \geq A(y, x).$$

(ii) $\bigvee_{x\in A} \phi(x) = \bigvee_{x\in A} \bigvee_{J\in\mathcal{I}(A)} \Phi(J)*J(x) = \bigvee_{J\in\mathcal{I}(A)} \Phi(J)*(\bigvee_{x\in A} J(x)) \geq \bigvee_{J\in\mathcal{I}(A)} \Phi(J) \geq I$.

(iii) For any $x, y \in A$,

$$\begin{aligned}
&\phi(x) * \phi(y) \\
= &\bigvee_{J_1,J_2\in\mathcal{I}(A)} \Phi(J_1) * J_1(x) * \Phi(J_2) * J_2(y) \\
\leq &\bigvee_{J_1,J_2,J\in\mathcal{I}(A)} \Phi(J) * \mathcal{I}(A)(J_1, J) * \mathcal{I}(A)(J_2, J) * J_1(x) * \Phi(J_2) * J_2(y) \\
\leq &\bigvee_{J\in\mathcal{I}(A)} \Phi(J) * J(x) * J(y) \\
\leq &\bigvee_{J\in\mathcal{I}(A)} \bigvee_{z\in A} \Phi(J) * J(z) * A(x, z) * A(y, z) \\
= &\bigvee_{z\in A} (\bigvee_{J\in\mathcal{I}(A)} \Phi(J) * J(z)) * A(x, z) * A(y, z) \\
= &\bigvee_{z\in A} \phi(z) * A(x, z) * A(y, z).
\end{aligned}$$

Step 2. $\sqcup\Phi = \phi$. In fact, for any $\phi_1 \in \mathcal{I}(A)$,

$$\begin{aligned}\mathcal{I}(A)(\phi,\phi_1) &= [A^{op},\Omega](\phi,\phi_1)\\ &= \bigwedge_{x\in A} \phi(x) \to \phi_1(x)\\ &= \bigwedge_{x\in A}\bigwedge_{J\in\mathcal{I}(A)} (\Phi(J) * J(x)) \to \phi_1(x)\\ &= \bigwedge_{J\in\mathcal{I}(A)}\bigwedge_{x\in A} \Phi(J) \to (J(x) \to \phi_1(x))\\ &= \bigwedge_{J\in\mathcal{I}(A)} \Phi(J) \to (\bigwedge_{x\in A} J(x) \to \phi_1(x))\\ &= \bigwedge_{J\in\mathcal{I}(A)} \Phi(J) \to \mathcal{I}(A)(J,\phi_1).\end{aligned}$$

□

Corollary 3.2 *Suppose that $\Phi \in \mathcal{I}(\mathcal{I}(A))$. Then for any $x \in A$, $(\sqcup\Phi)(x) = \Phi(\mathbf{y}(x))$.*

Proof. By Proposition 3.1,

$$(\sqcup\Phi)(x) = \bigvee_{J\in\mathcal{I}(A)} \Phi(J) * J(x) = \bigvee_{J\in\mathcal{I}(A)} \Phi(J) * \mathcal{I}(A)(\mathbf{y}(x), J) \leq \Phi(\mathbf{y}(x))$$

since Φ is a lower set. For the other direction,

$$(\sqcup\Phi)(x) = \bigvee_{J\in\mathcal{I}(A)} \Phi(J) * J(x) \geq \Phi(\mathbf{y}(x)) * \mathbf{y}(x)(x) \geq \Phi(\mathbf{y}(x)).$$

□

Proposition 3.3 *In the $\mathcal{I}$-cocomplete Ω-category $\mathcal{I}(A)$, for any $J \in \mathcal{I}(A)$, we have*

$$\mathbf{w}(J,J) = \bigvee_{x\in A} J(x) * \mathcal{I}(A)(J,\mathbf{y}(x)).$$

It follows that for each $x \in X$, $\mathbf{y}(x)$ is a compact element in $\mathcal{I}(A)$.

Proof. By the definition of way below relation $\mathbf{w}$, for any $J \in \mathcal{I}(A)$,

$$\mathbf{w}(J,J) = \bigwedge_{\Phi\in\mathcal{I}(\mathcal{I}(A))} \mathcal{I}(A)(J,\sqcup\Phi) \to \Phi(J).$$

On the one hand, for any $x \in A,\ \Phi \in \mathcal{I}(\mathcal{I}(A))$, we have

$$\begin{aligned}
& J(x) * \mathcal{I}(A)(J, \sqcup\Phi) * \mathcal{I}(A)(J, \mathbf{y}(x)) \\
\leq\ & \mathcal{I}(A)(J, \mathbf{y}(x)) * J(x) * (J(x) \to (\sqcup\Phi)(x)) \\
\leq\ & \mathcal{I}(A)(J, \mathbf{y}(x)) * (\sqcup\Phi)(x) \\
=\ & \mathcal{I}(A)(J, \mathbf{y}(x)) * \Phi(\mathbf{y}(x)) \\
\leq\ & \Phi(J).
\end{aligned}$$

This shows that $J(x) * \mathcal{I}(A)(J, \mathbf{y}(x)) \leq \mathcal{I}(A)(J, \sqcup\Phi) \to \Phi(J)$. By the arbitrariness of $x \in A$ and Φ, we have $\mathbf{w}(J, J) \geq \bigvee_{x\in A} J(x) * \mathcal{I}(A)(J, \mathbf{y}(x))$.

On the other hand, define

$$\Phi_J(\phi) = \bigvee_{x\in A} J(x) * \mathcal{I}(A)(\phi, \mathbf{y}(x)) \ (\forall \phi \in \mathcal{I}(A)).$$

If $\Phi_J \in \mathcal{I}(\mathcal{I}(A))$ and $J = \sqcup\Phi_J$, then $\mathbf{w}(J, J) \leq \Phi_J(J) = \bigvee_{x\in A} J(x) * \mathcal{I}(A)(J, \mathbf{y}(x))$.

In fact, (i) for any $\phi_1, \phi_2 \in \mathcal{I}(A)$,

$$\Phi_J(\phi_1) \to \Phi_J(\phi_2) \geq \bigwedge_{x\in A} (J(x) * \mathcal{I}(A)(\phi_1, \mathbf{y}(x))) \to (J(x) * \mathcal{I}(A)(\phi_2, \mathbf{y}(x)))$$

$$\geq \bigwedge_{x\in A} \mathcal{I}(A)(\phi_1, \mathbf{y}(x)) \to \mathcal{I}(A)(\phi_2, \mathbf{y}(x)) \geq \mathcal{I}(A)(\phi_2, \phi_1).$$

Then Φ_J is a lower set.

(ii) $\bigvee_{\phi\in\mathcal{I}(A)} \Phi_J(\phi) = \bigvee_{\phi\in\mathcal{I}(A)} \bigvee_{x\in A} \phi(x) * \mathcal{I}(A)(\phi, \mathbf{y}(x)) \geq \mathbf{y}(x)(x) * \mathcal{I}(A)(\mathbf{y}(x), \mathbf{y}(x)) \geq I$.

(iii) $\forall \phi_1, \phi_2 \in \mathcal{I}(A)$,

$$\begin{aligned}
& \Phi_J(\phi_1) * \Phi_J(\phi_2) \\
=\ & \bigvee_{x,y\in A} J(x) * \mathcal{I}(A)(\phi_1, \mathbf{y}(x)) * J(y) * \mathcal{I}(A)(\phi_2, \mathbf{y}(x)) \\
\leq\ & \bigvee_{x,y,z\in A} J(z) * A(x, z) * A(y, z) * \mathcal{I}(A)(\phi_1, \mathbf{y}(x)) * \mathcal{I}(A)(\phi_2, \mathbf{y}(x)) \\
\leq\ & \bigvee_{z\in A} J(z) * \mathcal{I}(A)(\phi_1, \mathbf{y}(z)) * \mathcal{I}(A)(\phi_2, \mathbf{y}(z)) \\
\leq\ & \bigvee_{z\in A} \bigvee_{\phi\in\mathcal{I}(A)} J(z) * \mathcal{I}(A)(\phi, \mathbf{y}(z)) * \mathcal{I}(A)(\phi_1, \phi) * \mathcal{I}(A)(\phi_2, \phi) \\
=\ & \bigvee_{\phi\in\mathcal{I}(A)} \Phi_J(\phi) * \mathcal{I}(A)(\phi_1, \phi) * \mathcal{I}(A)(\phi_2, \phi).
\end{aligned}$$

In (iii), the fact that $\mathcal{I}(A)(-, \mathbf{y}(z)) = \mathbf{y}(\mathbf{y}(z))$ is an ideal in $\mathcal{I}(A)$ for any $z \in A$ is used.

By (i)-(iii), Φ_J is an ideal in $\mathcal{I}(A)$.

(iv) It is easy to show that $\bigvee_{\phi\in\mathcal{I}(A)} \mathcal{I}(A)(\phi, \mathbf{y}(x)) * \phi = \mathbf{y}(x)$ for all $x \in A$. By Proposition 3.1,

$$\begin{aligned}\sqcup\Phi_J &= \bigvee_{\phi\in\mathcal{I}(A)} \Phi_J(\phi) * \phi\\ &= \bigvee_{\phi\in\mathcal{I}(A)} \bigvee_{x\in A} J(x) * \mathcal{I}(A)(\phi, \mathbf{y}(x)) * \phi\\ &= \bigvee_{x\in A} J(x) * (\bigvee_{\phi\in\mathcal{I}(A)} \mathcal{I}(A)(\phi, \mathbf{y}(x)) * \phi)\\ &= \bigvee_{x\in A} J(x) * \mathbf{y}(x)\\ &= \mathbf{y}(J) = J.\end{aligned}$$

Note that $\mathbf{y}(x) = \sqcup\mathbf{y}(\mathbf{y}(x)) = \bigvee_{\phi\in\mathcal{I}(A)} \mathcal{I}(A)(\phi, \mathbf{y}(x)) * \phi$. □

Proposition 3.4 *For any $J \in \mathcal{I}(A)$, k_J is directed and $\sqcup k_J = J$. Thus $\mathcal{I}(A)$ is an algebraic Ω-category.*

Proof. For any $\phi \in K(\mathcal{I}(A))$, $k_J(\phi) = \mathcal{I}(A)(\phi, J)$ and especially for $x \in A$, $k_J(\mathbf{y}(x)) = \mathcal{I}(A)(\mathbf{y}(x), J) = J(x)$.

(1) $\bigvee_{\phi\in\mathcal{I}(A)} k_J(\phi) \geq \bigvee_{x\in X} \mathcal{I}(A)(\mathbf{y}(x), J) = \bigvee_{x\in A} J(x) \geq I$.

(2) For any $\phi_1, \phi_2 \in K(\mathcal{I}(A))$, we have $\mathbf{w}(\phi_i, \phi_i) \geq I$ $(i = 1, 2)$, by Proposition 3.3, $\bigvee_{x\in A} \mathcal{I}(A)(\mathbf{y}(x), \phi_i) * \mathcal{I}(A)(\phi_i, \mathbf{y}(x)) \geq I$ $(i = 1, 2)$ (note that $\phi_i (i = 1, 2)$ need not be equal to $\mathbf{y}(x)$ for some $x \in X$),

$$\begin{aligned}& k_J(\phi_1) * k_I(\phi_2)\\ \leq\ & \bigvee_{x,y\in A} \mathcal{I}(A)(\phi_1, J) * \mathcal{I}(A)(\phi_2, J) * \mathcal{I}(A)(\mathbf{y}(x), \phi_1) * \mathcal{I}(A)(\phi_1, \mathbf{y}(x)) * \mathcal{I}(A)(\mathbf{y}(y), \phi_2)\\ & * \mathcal{I}(A)(\phi_2, \mathbf{y}(y))\\ \leq\ & \bigvee_{x,y\in A} \mathcal{I}(A)(\mathbf{y}(x), J) * \mathcal{I}(A)(\mathbf{y}(y), J) * \mathcal{I}(A)(\phi_1, \mathbf{y}(x)) * \mathcal{I}(A)(\phi_2, \mathbf{y}(y))\\ =\ & \bigvee_{x,y\in A} J(x) * J(y) * \mathcal{I}(A)(\phi_1, \mathbf{y}(x)) * \mathcal{I}(A)(\phi_2, \mathbf{y}(y))\\ \leq\ & \bigvee_{x,y,z\in A} J(z) * A(x, z) * A(y, z) * \mathcal{I}(A)(\phi_1, \mathbf{y}(x)) * \mathcal{I}(A)(\phi_2, \mathbf{y}(y))\\ \leq\ & \bigvee_{z\in A} J(z) * \mathcal{I}(A)(\phi_1, \mathbf{y}(z)) * \mathcal{I}(A)(\phi_2, \mathbf{y}(z))\\ =\ & \bigvee_{z\in A} k_J(\mathbf{y}(z)) * \mathcal{I}(A)(\phi_1, \mathbf{y}(z)) * \mathcal{I}(A)(\phi_2, \mathbf{y}(z))\\ \leq\ & \bigvee_{\phi\in\mathcal{I}(A)} k_J(\phi) * \mathcal{I}(A)(\phi_1, \phi) * \mathcal{I}(A)(\phi_2, \phi).\end{aligned}$$

(3) By Proposition 3.1, $\sqcup k_J = \bigvee_{\phi\in\mathcal{I}(A)} k_J(\phi) * \phi \geq \bigvee_{x\in A} J(x) * \mathbf{y}(x) = J$ and

$\sqcup k_J = \bigvee_{\phi\in\mathcal{I}(A)} k_J(\phi) * \phi = \bigvee_{\phi\in\mathcal{I}(A)} \mathcal{I}(A)(\phi, J) * \phi \le J$. Hence $\sqcup k_J = J$. □

Theorem 3.5 *Suppose that $f : A \longrightarrow B$ is a morphism in Ω-**FPOID**. Define $\mathbf{Id}(f) = f^{\leftarrow}|_{\mathcal{I}(B)} : \mathcal{I}(B) \longrightarrow \mathcal{I}(A)$ by $\mathbf{Id}(f)(J) = f^{\leftarrow}(J)$ $(\forall J \in \mathcal{I}(B))$. Then $\mathbf{Id}(f)$ is a morphism in Ω-$\mathbf{AlgDom}_G$.*

Proof. $\mathbf{Id}(f) : \mathcal{I}(B) \longrightarrow \mathcal{I}(A)$ is a map since $f^{\leftarrow}(J) \in \mathcal{I}(A)$ for any $J \in \mathcal{I}(B)$. Since $f^{\rightarrow} \rightharpoonup f^{\leftarrow} : [A^{op}, \Omega] \longrightarrow [B^{op}, \Omega]$ is an Ω-adjunction, by Proposition 2.10, $f^{\leftarrow} : [B^{op}, \Omega] \longrightarrow [A^{op}, \Omega]$ preserves arbitrary joins and then $\mathbf{Id}(f) = f^{\leftarrow}|_{\mathcal{I}(B)} : \mathcal{I}(B) \longrightarrow \mathcal{I}(A)$ preserves joins of ideals and so is Scott continuous.

Define $g : \mathcal{I}(A) \longrightarrow \mathcal{I}(B)$ by $g(J') = f^{\rightarrow}(J')$ $(\forall J' \in \mathcal{I}(A))$, that is $g = f^{\rightarrow}|_{\mathcal{I}(A)}$. By Lemma 2.5(5), g is a map and for any $J' \in \mathcal{I}(A),\ J \in \mathcal{I}(B)$,

$$\mathcal{I}(B)(g(J'), J) = [B^{op}, \Omega](f^{\rightarrow}(J'), J) \ = \mathcal{I}(A)(J', f^{\leftarrow}(J)) = \mathcal{I}(A)(J', \mathbf{Id}(f)(J)).$$

Thus $(g, \mathbf{Id}(f))$ is an Ω-adjunction by Theorem 2.3. □

Proposition 3.4 and Theorem 3.5 show that $\mathbf{Id}$:Ω-**POID**$\longrightarrow \Omega$-$\mathbf{AlgDom}_G^{op}$ is a functor which transfers $\mathbf{Id}(A) = \mathcal{I}(A)$ for any Ω-category A and $\mathbf{Id}(f) = f^{\leftarrow}|_{\mathcal{I}(B)} : \mathcal{I}(B) \longrightarrow \mathcal{I}(A)$ for any Ω-functor $f \in [A, B]$.

3.2 A functor from Ω-$\mathbf{AlgDom}_G$ to Ω-$\mathbf{POID}^{op}$

For any algebraic Ω-category L, $K(L)$ is an Ω-category as a full subcategory of L.

Lemma 3.6 *Suppose that $g : L \longrightarrow M$ is a Scott continuous functor between two algebraic Ω-categories which has a left Ω-adjoint $g^{\dashv} : M \longrightarrow L$. Then $g^{\dashv}(K(M)) \subseteq K(L)$.*

Proof. For any $a \in K(M)$, we need to show $g^{\dashv}(a) \in K(L)$, that is $L(g^{\dashv}(a), \sqcup J) \le J(g^{\dashv}(a))$ for all $J \in \mathcal{I}(L)$. In fact,

$$L(g^{\dashv}(a), \sqcup J) = M(a, g(\sqcup J)) = M(a, \sqcup g^{\rightarrow}(J)) \le g^{\rightarrow}(J)(a)$$

$$= \bigvee_{b\in B} J(b) * L(a, g(b)) = \bigvee_{b\in B} J(b) * M(g^{\dashv}(a), b) \le J(g^{\dashv}(a)).$$

□

Lemma 3.7 *For $J \in \mathcal{I}(K(L))$, consider J as an Ω-set of L, we have $\mathbf{y}(J) \in \mathcal{I}(L)$, where*

$$\mathbf{y}(J)(x) = \bigvee_{a\in K(L)} J(a) * L(x, a) \ (\forall x \in L)$$

is that defined in the paragraph above Definition 2.5.

Proof. By Lemma 2.5(3), $\mathbf{y}(J)$ is a lower set and for any $x \in A$, and $\bigvee_{x\in L} \mathbf{y}(J)(x) \ge$

$\bigvee\limits_{x\in L} J(x) \geq I$. For any $x_1, x_2 \in L$,

$$\begin{aligned}
\mathbf{y}(J)(x_1) * \mathbf{y}(J)(x_2) &= \bigvee_{a_1,a_2\in K(L)} J(a_1) * L(x_1,a_1) * J(a_2) * L(x_2,a_2) \\
&\leq \bigvee_{a_1,a_2,a\in K(L)} J(a) * L(a_1,a) * L(x_1,a_1) * L(a_2,a) * L(x_2,a_2) \\
&\leq \bigvee_{a\in K(L)} J(a) * L(x_1,a) * L(x_2,a) \\
&\leq \bigvee_{a\in L} \mathbf{y}(J)(a) * L(x_1,a) * L(x_2,a).
\end{aligned}$$

□

Proposition 3.8 *Let L be an algebraic Ω-category. Then $\mathbf{y}(x)|_{K(L)} \in \mathcal{I}(K(L))$ for any $x \in L$.*

Proof. Clearly $\mathbf{y}(x)|_{K(L)} = k_x \in \mathcal{I}(K(L))$.

(1) $\bigvee\limits_{a\in K(L)} \mathbf{y}(x)|_{K(L)}(x) = \bigvee\limits_{a\in K(L)} k_x(a) \geq I$.

(2) For any $a_1, a_2 \in K(L)$,

$$\mathbf{y}(x)|_{K(L)}(a_2) * K(L)(a_1,a_2) = L(a_2,x) * L(a_1,a_2) \leq L(a_1,x) = \mathbf{y}(x)|_{K(L)}(a_1),$$

thus $\mathbf{y}(x)|_{K(L)}$ is a lower set in $K(L)$.

(3) For any $a_1, a_2 \in K(L)$,

$$\begin{aligned}
&\mathbf{y}(x)|_{K(L)}(a_1) * \mathbf{y}(x)|_{K(L)}(a_2) \\
=\ & k_x(a_1) * k_x(a_2) \\
\leq\ & \bigvee_{a\in L} k_x(a) * L(a_1,a) * L(a_2,a) \\
=\ & \bigvee_{a\in K(L)} k_x(a) * K(L)(a_1,a) * K(L)(a_2,a) \\
=\ & \bigvee_{a\in K(L)} \mathbf{y}(x)|_{K(L)}(a) * *K(L)(a_1,a) * K(L)(a_2,a).
\end{aligned}$$

□

Theorem 3.9 $\mathbf{K} : \Omega\text{-}\mathbf{AlgDom}_G \longrightarrow \Omega\text{-}\mathbf{POID}^{op}$ $(L \mapsto K(L),\ g \mapsto g^{\dashv})$ *is a functor.*

Proof. Suppose that $g : L \longrightarrow M$ is a morphism in Ω-$\mathbf{AlgDom}_G$, we need to show that $g^{\dashv} : K(M) \longrightarrow K(L)$ is a morphism in Ω-**POID**. Suppose that $J \in \mathcal{I}(K(L))$, by Lemma 2.7(4), $\mathbf{y}(J) \in \mathcal{I}(L)$, by Proposition 2.8, $\sqcup J = \sqcup \mathbf{y}(J)$.

Put $c = \sqcup J$, then we have $J(a) = \mathbf{y}(c)(a)$ for all $a \in K(L)$ and then $J = \mathbf{y}(c)|_{K(L)}$. In fact, $J(a) \leq L(a,c) = \mathbf{y}(c)(a)$ since $c = \sqcup J$. Conversely, since $a \in K(L)$, we have

$$I \leq \mathbf{w}(a,a) \leq L(a, \sqcup \mathbf{y}(J)) \to \mathbf{y}(J)(a) = L(a,c) \to \mathbf{y}(J)(a)$$

and

$$\mathbf{y}(c)(a) = L(a,c) \leq \mathbf{y}(J)(a) = \bigvee_{x \in K(L)} J(x) * L(a,x) \leq J(a)$$

since J is a lower set in $K(L)$.

We will show that $(g^{\dashv})^{\leftarrow}(J) = \mathbf{y}(g(c))|K(M)$. For any $b \in L(M)$,

$$(g^{\dashv})^{\leftarrow}(J)(b) = J(g^{\dashv}(b)) = \mathbf{y}(c)(g^{\dashv}(b)) = L(g^{\dashv}(b), c) = M(b, g(c)) = \mathbf{y}(g(c))(b).$$

Hence $(g^{\dashv})^{\leftarrow}(J) \in \mathcal{I}(K(M))$ by Proposition 3.8. $\square$

By the proof of Theorem 3.9, we have

Proposition 3.10 *For any algebraic Ω-category L, all ideals in $K(L)$ has the form $\mathbf{y}(x)|_{K(L)}$ for some $x \in L$.*

Proof. Let $id : L \longrightarrow L$ be the identical functor. Then the left Ω-adjoint of id is still id, thus for any ideal J in $K(L)$, $J = id^{\leftarrow}(J) = \mathbf{y}(x)|_{K(L)}$, where x is the join of J in L. $\square$

3.3 *Duality between Ω-$\mathbf{AlgDom}_G$ and Ω-$\mathbf{POID}$*

For any Ω-category A, define $\eta_A : A \longrightarrow K(\mathcal{I}(A)),\ x \mapsto \mathbf{y}(x)$ $(\forall x \in X)$.

Theorem 3.11 *$\eta : id_{\Omega\text{-}\mathbf{POID}} \longrightarrow \mathbf{K} \circ \mathbf{Id}$ is a natural transformation.*

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f"'] & K(\mathcal{I}(A)) \arrow[d, "f^{\rightarrow}"] \\ B \arrow[r, "\eta_B"'] & K(\mathcal{I}(B)) \end{tikzcd}$$

Figure 1

Proof. For $f : A \longrightarrow B$ a morphism in Ω-$\mathbf{POID}$, $\mathbf{Id}(f) = f^{\leftarrow}|_{\mathcal{I}(B)} : \mathcal{I}(B) \longrightarrow \mathcal{I}(A)$. By Theorem 3.5, the left Ω-adjoint of $\mathbf{Id}(f)$ is $\mathbf{K}\circ\mathbf{Id}(f) = f^{\rightarrow}|_{\mathcal{I}(A)} : \mathcal{I}(A) \longrightarrow \mathcal{I}(B)$.

We need to show that $f^{\rightarrow} \circ \eta_A = \eta_B \circ f$. In fact for any $x \in A$, for any $y \in B$,

$$f^{\rightarrow}(\eta_A(x))(y) = f^{\rightarrow}(\mathbf{y}(x))(y) = \bigvee_{a \in B} \mathbf{y}(x)(a) * B^{op}(f(a), y) = \bigvee_{a \in A} B(y, f(a)) * A(a, x).$$

On one hand,

$$\begin{aligned} \bigvee_{a \in A} B(y, f(a)) * A(a,x) &\leq \bigvee_{a \in A} B(y, f(a)) * B(f(a), f(x)) \\ &\leq B(y, f(x)) = \mathbf{y}(f(x))(y) = \eta_B(f(x)(y); \end{aligned}$$

on the other hand,

$$\bigvee_{a \in A} B(y, f(a)) * A(a,x) \geq B(y, f(x)) * A(x,x) \geq \eta_B(f(x)(y).$$

Hence $f^{\rightarrow}(\eta_A(x))(y) = \eta_B(f(x)(y)$. Therefore $f^{\rightarrow} \circ \eta_A = \eta_B \circ f$. □

Proposition 3.12 *Define a transformation* $\varepsilon : \mathbf{Id} \circ \mathbf{K} \longrightarrow id_{\Omega\text{-}\mathbf{AlgDom}_G}$ *by for any* $L \in \Omega$-$\mathbf{AlgDom}_G$, $\varepsilon_L : \mathcal{I}(K(L)) \longrightarrow L$, $J \mapsto \sqcup J$ $(\forall J \in \mathcal{I}(K(L))$. *Then* ε *is a natural isomorphism. The inverse of* ε *of given by* $\varepsilon_L^{-1}(x) = \mathbf{y}(x)|_{K(L)}$.

Proof. $\varepsilon(\varepsilon^{-1}(x)) = \sqcup(\mathbf{y}(x)|_{K(L)}) = \sqcup k_x = x$ and $\varepsilon^{-1}(\varepsilon(J)) = \varepsilon^{-1}(\sqcup J) = \mathbf{y}(\sqcup J)|_{K(L)} = J$. □

By Propositions 3.11 and 3.12,

Theorem 3.13 **Id** *is the left adjoint of* **K**.

In order to show the isomorphism between **Id** and **K**, we need two additional conditions for the quantale Ω:

(Q1) $I \leq \bigvee A$ implies $I \leq x$ for some $x \in A \subseteq \Omega$;

(Q2) $I \leq x * y$ implies $I \leq x$ or $I \leq y$ for any $x, y \in \Omega$.

The following example gives such a quantale which is nontrivial, $* \neq \wedge$ and $I \neq 1$.

Example 3.14 Let $\Omega = \{0, a, b, 1\}$ be the diamond lattice, that is $0 \leq a, b \leq 1$ and $a \not\leq b, b \not\leq a$. Define $* : \Omega \times \Omega \longrightarrow \Omega$ by

*	0	a	b	1
0	0	0	0	0
a	0	a	b	1
b	0	b	b	b
1	0	1	b	1

Clearly, $*$ is monotone and a is the unit and the conditions (Q1) and (Q2) are satisfied. We now only need to show that $x*(a \vee b) = (x*a) \vee (x*b)$ or $x*1 = x \vee (x*b)$ for any $x \in \Omega$. In fact, if $x = 0$ or $x = a$, then it holds; if $x = 1$, it holds since $1*1 = 1$; if $x = b$, then $x*1 = b = b \vee b = x \vee (x*b)$. Then $(\Omega, *, a)$ is a commutative unital quantale (furhtermore, $*$ is idempotent).

Proposition 3.15 *If (Q1) and (Q2) hold for* Ω, *then the compact elements in* $\mathcal{I}(A)$ *have the form* $\mathbf{y}(x)$ $(x \in A)$. *In this case,* Ω-$\mathbf{AlgDom}_G$ *is dual to* Ω-$\mathbf{POID}$.

Proof. Let A be an Ω-category. Suppose that ϕ is a compact element in $\mathcal{I}(A)$, by Proposition 3.3, we have $\bigvee\limits_{x \in A} \mathcal{I}(A)(\phi, \mathbf{y}(x)) * \mathcal{I}(A)(\mathbf{y}(x), \phi) \geq I$. By (Q1), we have $\mathcal{I}(A)(\phi, \mathbf{y}(x)) * \mathcal{I}(A)(\mathbf{y}(x), \phi) \geq I$ for some $x \in A$. By (Q2) $\mathcal{I}(A)(\phi, \mathbf{y}(x)) \geq I$, $\mathcal{I}(A)(\mathbf{y}(x), \phi) \geq I$, which implies $\phi = \mathbf{y}(x)$. □

4 Conclusions

By introducing a definition of algebraicity of Ω-categories, we show that the category of algebraic Ω-categories (with certain morphisms) and the category of Ω-functors (with certain morphisms) are dual equivalent to each other. The transformation from an Ω-category to an algebraic Ω-category exactly is the ideal completion (i.e., $\mathcal{I}$-completion), and the that from an algebraic Q-category to an Ω-category just is the restriction to the compact objects of an algebraic Ω-category.

Such a duality could be generalized to one between Ω-categories and Φ-algebraic Ω-categories for Φ is a (saturated) class of weights. For Φ being $\mathcal{I}$, an $\mathcal{I}$-algebraic Ω-category just is an algebraic Ω-category in this paper. For Φ is the maximal class $\mathcal{P}$, a $\mathcal{P}$-algebraic Ω-category just is a totally algebraic cocomplete $\mathcal{Q}$-categories in [21]. There are also many interesting examples of other classes of weights in framework of metric spaces studied in [18].

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