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Remarks on an Edge-coloring Problem

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Abstract

We consider a multicolored version of a problem that was originally proposed by Erdős and Rothschild. For positive integers n and r, we look for n-vertex graphs that admit the maximum number of r-edge-colorings with no copy of a triangle where exactly two colors appear. It turns out that for $2 \leq r \leq 12$ colors and n sufficiently large, the complete bipartite graph on n vertices with balanced bipartition (the n-vertex Turán graph for the triangle) yields the largest number of such colorings, and this graph is unique with this property.

Keywords: Edge-colorings, Turán Problem, Erdős-Rothschild Problem

1 Introduction and main results

This paper is concerned with a multicolored version of a problem that was originally proposed by Erdős and Rothschild [9]. The motivation for their problem lies in the well-known Turán problem, where, given an integer n and a graph F, we look for the maximum number $\mathrm{ex}(n, F)$ of edges in an n-vertex graph G such that G does not contain F as a subgraph. A graph G that does not contain F as a subgraph is said to be F-*free* and an F-free n-vertex graph with $\mathrm{ex}(n, F)$ edges is called F-*extremal.*

Turán [22] solved this problem for all n whenever $F = K_{\ell+1}$ is a complete graph on $(\ell + 1)$ vertices. He showed that, for all positive integers n and $\ell \geq 2$, any

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$K_{\ell+1}$-extremal graph is isomorphic to the *Turán graph* $T_\ell(n)$, the complete ℓ-partite graph on n-vertices whose partition $\mathcal{V} = \{V_1, \ldots, V_\ell\}$ is *balanced*, that is, such that $|V_i| \leq |V_j| + 1$ for all $i, j \in [\ell] = \{1, \ldots, \ell\}$. In particular, $\mathrm{ex}(n, K_3) = \lfloor n/2 \rfloor \cdot \lceil n/2 \rceil$. There is a vast literature about the Turán problem, we refer to [12] (and to the references therein) for more information.

The question of Erdős and Rothschild involves *r-edge-colorings* of n-vertex graphs with the property that *every color class is F-free.* They wondered whether any n-vertex graph would admit more such colorings than the corresponding F-extremal graph. Note that n-vertex F-extremal graphs admit $r^{\mathrm{ex}(n,F)}$ colorings, as their edge set may be colored arbitrarily. Precisely, Erdős and Rothschild conjectured that the number of $K_{\ell+1}$-free 2-colorings is maximized by $T_\ell(n)$. Yuster [23] verified this conjecture for $\ell = 2$ and $n \geq 6$. Alon, Balogh, Keevash and Sudakov [2] showed that, for $r \in \{2, 3\}$ and $n \geq n_0$, where n_0 is a constant depending on r and ℓ, the Turán graph $T_\ell(n)$ is also optimal for the number of $K_{\ell+1}$-free r-colorings. However, they also provided a construction showing that $T_\ell(n)$ is not optimal for any $r \geq 4$, but did not characterize the graphs that achieve extremality. Pikhurko and Yilma [20] determined the extremal graphs for $r = 4$ and $\ell \in \{2, 3\}$. Together with Staden [19], they also generalized the original Erdős-Rothschild problem and showed that it always admits an extremal solution that is a complete multipartite graph (however, their proof does not settle whether it is necessarily balanced). Moreover, they defined an optimization problem whose solution produces a complete multipartite graph for which the number of colorings approximates the maximum.

Balogh [3] was the first to consider r-colorings that avoid a copy of a graph F colored in a non-monochromatic way. A similar problem was investigated by Hoppen and Lefmann [15] and by Benevides, Hoppen and Sampaio [6], who considered edge-colorings of a graph avoiding a copy of F with a *prescribed pattern.* Given a number $r \geq 1$ of colors and a graph F, an *r-pattern* P of F is a partition of its edge set into at most r classes, and an edge-coloring of a graph G is said to be *(F, P)-free* if G does not contain a copy of F in which the partition of the edge set induced by the coloring is isomorphic to P. For instance, if $F = K_3$, there are three possible patterns: the monochromatic pattern P_M (where all edges lie in the same class), the rainbow pattern P_R (where each class is a singleton) and the 2-colored pattern P_2 (where there are two classes, one singleton and one with cardinality two). It is clear that the original Erdős-Rothschild problem is precisely the problem of finding the largest number of (F, P_M)-free colorings in an n-vertex graph.

For a formal statement of this multicolored version of the Erdős-Rothschild problem, fix a positive integer r and a graph F, and let P be a pattern of F. Let $\mathcal{C}_{r,F,P}(G)$ be the set of all (F, P)-free r-colorings of a graph G. We write

$$c_{r,F,P}(n) = \max \{ |\mathcal{C}_{r,F,P}(G)| \colon |V(G)| = n \},$$

and we say that an n-vertex graph G is *(F, P)-extremal* if $|\mathcal{C}_{r,F,P}(G)| = c_{r,F,P}(n)$. In this paper, our main objective is to study (K_3, P_2)-extremal graphs for the 2-colored pattern P_2.

Regarding patterns P of K_3, the following is known. As mentioned above, the

Turán graph $T_2(n)$ is the single (K_3, P_M)-extremal graph for $r = 2$ and $n \geq 6$ (see [23]) and for $r = 3$ and $n \geq n_0$ (see [2] and [14]). Moreover, the graph $T_4(n)$ is the single (K_3, P_M)-extremal graph for $r = 4$ and $n \geq n_0$ (see [20]). To the best of our knowledge, the extremal graphs for $r \geq 5$ are not known. For the rainbow pattern P_R, the complete graph K_n is trivially the single (K_3, P_R)-extremal graph for $r = 2$. If $r \geq 5$, Odermann and the current authors [16] have proved that the Turán graph $T_2(n)$ is the single (K_3, P_R)-extremal graph for $n \geq n_0$ (and for $r \geq 10$ and $n \geq 5$). Very recently, Balogh and Li [4] have proved that the complete graph K_n and the Turán graph $T_2(n)$ are the single (K_3, P_R)-extremal graphs for $r = 3$ and $r = 4$, respectively. Approximate results had also been obtained in [5,6,10,16].

Less is known for the pattern P_2. The work of [3] implies that the Turán graph $T_2(n)$ is (K_3, P_2)-extremal for $r = 2$ and $n \geq n_0$. Results from [6] prove that, for any $r \geq 3$, one of the extremal configurations is always a complete multipartite graph (see [6, Theorem 1.1]) and that the Turán graph $T_2(n)$ is (K_3, P_2)-extremal for $r = 3$ and $n \geq n_0$ (see [6, Theorem 1.3]). On the other hand, let $r = 27$ and consider a partition of the set of colors into three sets C_1, C_2, C_3, where $|C_1| = |C_2| = |C_3| = 9$. We shall color the 4-partite graph $T_4(n)$ whose vertex set is partitioned $V_1 \cup \cdots \cup V_4$ as follows (for simplicity, assume that n is divisible by 4). Edges between V_1 and V_2, and between V_3 and V_4 are assigned colors in C_1; edges between V_1 and V_3, and between V_2 and V_4 are assigned colors in C_2; edges between V_1 and V_4, and between V_2 and V_3 are assigned colors in C_3. Clearly, any triangle in $T_4(n)$ must be rainbow, so that this produces colorings in $\mathcal{C}_{27,K_3,P_2}(T_4(n))$. Moreover, the number of colorings produced in this way is equal to

$$9^{6\cdot\frac{n^2}{16}} = 27^{\frac{n^2}{4}} = 27^{\mathrm{ex}(n,K_3)}.$$

Since there are many other ways of coloring $T_4(n)$ (for instance, changing the choice of the sets C_1, C_2, C_3), we conclude that $c_{r,K_3,P_2}(n) > r^{\mathrm{ex}(n,K_3)}$, so that the Turán graph is not (K_3, P_2)-extremal for $r = 27$. Actually, a similar analysis shows that $T_4(n)$ admits more (K_3, P_2)-free colorings than $T_2(n)$ for all $r \geq 27$. We believe that $T_2(n)$ is (K_3, P_2)-extremal for all $r \leq 26$, at least for $n \geq n_0$. In this note, we offer a partial result in this direction by using the regularity lemma combined with a linear programming approach.

Theorem 1.1 *Let P_2 be the pattern of K_3 with exactly two classes and let $2 \leq r \leq 12$. Then there exists n_0 such that, for every $n \geq n_0$ and every n-vertex graph G, we have*

$$|\mathcal{C}_{r,K_3,P_2}(G)| \leq r^{\mathrm{ex}(n,K_3)}.$$

Moreover, equality holds in this equation if and only if G is isomorphic to the bipartite Turán graph $T_2(n)$.

As we shall see below, in light of [17, Lemma 3.1], to prove Theorem 1.1, it suffices to prove the following stability result, which states that any n-vertex graph with a 'large' number of colorings must be 'almost bipartite'. For a graph $G = (V, E)$ and a subset $W \subset V$, we write $e_G(W)$ to denote the number of edges of G with both endpoints in W. We simply write $e(W)$ if the graph G under consideration is obvious from the context.

Lemma 1.2 *Let $2 \leq r \leq 12$ be fixed. For all $\delta > 0$, there exists n_0 with the following property. If $G = (V, E)$ is a graph on $n > n_0$ vertices which has at least $r^{\mathrm{ex}(n,K_3)}$ distinct (K_3, P_2)-free r-colorings, then there is a partition $V = W_1 \cup W_2$ of its vertex set such that $e_G(W_1) + e_G(W_2) \leq \delta n^2$.*

Note that Lemma 1.2 immediately implies that $|\mathcal{C}_{r,K_3,P_2}(G)| \leq r^{\mathrm{ex}(n,K_3)+o(n^2)}$ for $r \in \{2, \ldots, 12\}$.

In the remainder of the paper, we shall discuss the main ingredients in our proof of Lemma 1.2. We should mention that, in the last few years, there has been a lot of activity on the Erdős-Rothschild problem for several combinatorial structures, such as set systems, the power lattice and sum-free-sets [7,8,13].

2 Main ingredients

We first observe that, because of previous results by Hoppen, Lefmann and Odermann [17], the stability of Lemma 1.2 implies Theorem 1.1. The authors of [17] defined the following notion of stability.

Definition 2.1 Let F be a graph with chromatic number $\chi(F) = \ell + 1 \geq 3$ and let P be a pattern of F. The pair (F, P) satisfies the Color Stability Property for a positive integer r if, for every $\delta > 0$, there exists n_0 with the following property. If $n > n_0$ and G is an n-vertex graph such that $|\mathcal{C}_{r,F,P}(G)| \geq r^{\mathrm{ex}(n,F)}$, then there exists a partition $V(G) = V_1 \cup \cdots \cup V_\ell$ such that $\sum_{i=1}^{\ell} e_G(V_i) \leq \delta n^2$.

Then they showed that the Turán graph $T_\ell(n)$ is the only n-vertex graph that maximizes $c_{r,K_{\ell+1},P}(n)$ for a class of patterns of complete graphs that satisfy the Color Stability Property, namely patterns for which there is a vertex v such that all edges incident with v lie in different classes. Patterns of this type are called *locally rainbow*. Note that the 2-colored triangle is locally rainbow.

Lemma 2.2 *[17, Lemma 3.1] Let $\ell \geq 2$ and let P be a locally rainbow pattern of $K_{\ell+1}$ such that $(K_{\ell+1}, P)$ satisfies the Color Stability Property of Definition 2.1 for a positive integer $r > e(\ell + 1)$. Then there is n_0 such that every graph of order $n > n_0$ has at most $r^{\mathrm{ex}(n,K_{\ell+1})}$ distinct $(K_{\ell+1}, P)$-free r-edge colorings. Moreover, the only graph on n vertices for which the number of such colorings is $r^{\mathrm{ex}(n,K_{\ell+1})}$ is the Turán graph $T_\ell(n)$.*

They also remarked that, in the case where the forbidden graph is a triangle, the lower bound $r > e(\ell + 1)$ in the statement of this lemma may be replaced by $r \geq 3$. Since Lemma 1.2 establishes that the 2-colored triangle satisfies the Color Stability Property for $3 \leq r \leq 12$, Theorem 1.1 follows.

2.1 Regularity Lemma

Our proof of Lemma 1.2 is based on the Szemerédi Regularity Lemma [21]. Let $G = (V, E)$ be a graph, and let A and B be two disjoint subsets of $V(G)$. If A and B are non-empty, define the edge-density between A and B by

$$d(A, B) = \frac{e(A, B)}{|A||B|},$$

where $e(A, B)$ is the number of edges with one endpoint in A and the other in B. For $\varepsilon > 0$ the pair (A, B) is called *ε-regular* if, for every $X \subseteq A$ and $Y \subseteq B$ satisfying $|X| > \varepsilon|A|$ and $|Y| > \varepsilon|B|$, we have

$$|d(X, Y) - d(A, B)| < \varepsilon.$$

An *equitable partition* of a set V is a partition of V into pairwise disjoint classes $V_1, \ldots, V_m$ of almost equal size, i.e., $||V_i| - |V_j|| \leq 1$ for all $i, j \in [m]$. An equitable partition of the vertex set V of G into the classes $V_1, \ldots, V_m$ is called *ε-regular* if at most $\varepsilon\binom{m}{2}$ of the pairs (V_i, V_j) are not ε-regular.

We now state a (colored) version of the Regularity Lemma, which may be found in [18].

Lemma 2.3 *For every $\varepsilon > 0$ and every integer r, there exists an $M = M(\varepsilon, r)$ such that the following property holds. If the edges of a graph G of order $n > M$ are r-colored $E(G) = E_1 \cup \cdots \cup E_r$, then there is a partition of the vertex set $V(G) = V_1 \cup \cdots \cup V_m$, with $1/\varepsilon \leq m \leq M$, which is ε-regular simultaneously with respect to the graphs $G_i = (V, E_i)$ for all $i \in [r]$.*

A partition $V_1 \cup \cdots \cup V_m$ of $V(G)$ as in Lemma 2.3 will be called a *multicolored ε-regular partition*. For $\eta > 0$, we may define a *multicolored cluster graph* $H(\eta)$ associated with this partition: the vertex set is $[m]$ and $e = \{i, j\}$ is an edge of $H(\eta)$ if $\{V_i, V_j\}$ is a regular pair in G *for every* color $c \in [r]$ and is η-dense for some color $c \in [r]$. Each edge e is assigned the list L_e containing all colors for which it is η-dense, so that $|L_e| \geq 1$ for every edge in the multicolored cluster graph $H(\eta)$.

Given a colored graph F, we say that a multicolored cluster graph H contains F if H contains a copy of F for which the color of each edge of F is contained in the list of the corresponding edge in H. More generally, if F is a graph with color pattern P, we say that H contains (F, P) if it contains some colored copy of F with pattern isomorphic to P. In connection with this definition, we may obtain the following embedding result. The proof of this result follows from arguments such as in the proof of the Key Lemma [18].

Lemma 2.4 *For every $\eta > 0$ and all positive integers k and r, there exist $\varepsilon = \varepsilon(r, \eta, k) > 0$ and a positive integer $n_0(r, \eta, k)$ with the following property. Suppose that G is an r-colored graph on $n > n_0$ vertices with a multicolored ε-regular partition $V = V_1 \cup \cdots \cup V_m$ which defines the multicolored cluster graph $H = H(\eta)$. Let F be a fixed k-vertex graph with a prescribed color pattern P on $t \leq r$ classes. If H contains (F, P), then the graph G also contains (F, P).*

2.2 Stability

Another basic tool in our paper are stability results for graphs.

It will be convenient to use the following recent theorem by Füredi [11].

Theorem 2.5 *Let $G = (V, E)$ be a K_{k+1}-free graph on m vertices. If $|E| = \mathrm{ex}(m, K_{k+1}) - t$, then there exists a partition $V = V_1 \cup \ldots \cup V_k$ with $\sum_{i=1}^{k} e(V_i) \leq t$.*

We also use the following simple lemma due to Alon and Yuster [1].

Lemma 2.6 *Let G be a bipartite graph on m vertices with partition $V(G) = U_1 \cup U_2$ and with at least $\mathrm{ex}(m, K_3) - t$ edges. If we add at least $3t$ new edges to G, then in the resulting graph there is a copy of K_3 with exactly one new edge, which connects two vertices of K_3 in the same class U_i.*

3 Proving Lemma 1.2

In this section, we provide a sketch of the proof of Lemma 1.2. Fix the number $r \in \{2, \ldots, 12\}$ of colors and $\delta > 0$. To avoid case analysis, we concentrate on the case $r \geq 6$ [4].

With foresight, we consider auxiliary constants $\xi > 0$ and $\eta > 0$ such that

$$\xi < \frac{\delta}{14}, \quad r^{r\eta + h(r\eta)} < \left(\frac{r}{M(r)}\right)^{\xi} \text{ and } \eta < \frac{\delta}{2r}, \tag{1}$$

where $M(r)$ is defined in (11) and $h(x) = -x \log_2 x - (1-x)\log_2(1-x)$, with $h(0) = h(1) = 0$, is the *entropy function* in $[0, 1]$. It is well known that

$$\binom{n}{\alpha n} \leq 2^{H(\alpha)n}$$

for any $0 \leq \alpha \leq 1/2$.

Let $\varepsilon = \varepsilon(r, \eta, 3) > 0$ satisfy the assumption in Lemma 2.4, and assume without loss of generality that $\varepsilon < \eta/2$. Fix $M = M(r, \varepsilon)$ given by Lemma 2.3.

Let Δ be an r-edge coloring of $G = (V, E)$ that contains no 2-colored triangle. By Lemma 2.3, there is a multicolored ε-regular partition $V = V_1 \cup \cdots \cup V_m$ of the colored graph, where $1/\varepsilon \leq m \leq M$.

For each color, there are at most $\varepsilon\binom{m}{2}$ irregular pairs with respect to the partition $V = V_1 \cup \cdots \cup V_m$, hence at most

$$r \cdot \varepsilon \cdot \binom{m}{2} \cdot \left(\frac{n}{m}\right)^2 \leq \frac{r\varepsilon}{2} \cdot n^2 \leq \frac{r\eta}{4} \cdot n^2 \tag{2}$$

edges of G are contained in an irregular pair with respect to some color. Moreover, there are at most

$$m \cdot \left(\frac{n}{m}\right)^2 = \frac{n^2}{m} \leq \varepsilon n^2 \leq \frac{\eta}{2} \cdot n^2 \tag{3}$$

[4] The cases $2 \leq r \leq 5$ may be proved with similar arguments.

edges with both ends in some class V_i, where $m \geq 1/\varepsilon$. Finally, the number of edges e with ends in different classes V_i and V_j such that the color of e has density less than η between V_i and V_j is at most

$$r \cdot \eta \cdot \binom{m}{2} \cdot \left(\frac{n}{m}\right)^2 \leq \frac{r\eta}{2} \cdot n^2. \tag{4}$$

Using (2), (3) and (4) gives at most $r\eta n^2$ edges of these three types, which may be chosen in at most $\binom{n^2}{r\eta n^2}$ ways. Note that this set of edges could be colored in at most $r^{r\eta n^2}$ different ways.

Let $H = H(\eta)$ be the multicolored cluster graph associated with the partition $V = V_1 \cup \cdots \cup V_m$. Let $E_j(H) = \{e \in E(H) : |L_e| = j\}$ and $e_j(H) = |E_j(H)|$, $j \in [r]$. The number of r-edge colorings of G that give rise to the partition $V = V_1 \cup \cdots \cup V_m$ and to the multicolored cluster graph H is bounded above by

$$\binom{n^2}{r\eta n^2} \cdot r^{r\eta n^2} \cdot \left(\prod_{j=1}^{r} j^{e_j(H)}\right)^{\left(\frac{n}{m}\right)^2} \leq 2^{h(r\eta)n^2} \cdot r^{r\eta n^2} \cdot \left(\prod_{j=1}^{r} j^{e_j(H)}\right)^{\left(\frac{n}{m}\right)^2}. \tag{5}$$

Recall that ξ is a constant defined in (1).

Claim 3.1 *There must be a multicoloured cluster graph H such that*

$$e_{r-3}(H) + \cdots + e_r(H) \geq \mathrm{ex}(m, K_3) - \xi m^2.$$

Before proving this claim, we show that it implies the desired result. Let H' be the subgraph of H with edge-set $E_{r-3} \cup \cdots \cup E_r$. By Theorem 2.5 there is a partition $U_1 \cup U_2 = [m]$ with

$$e_{H'}(U_1) + e_{H'}(U_2) \leq \xi m^2.$$

Let $\widehat{H}$ be a bipartite subgraph of H' with bipartition $U_1 \cup U_2$ and the maximum number of edges. Note that by Theorem 2.5 we have

$$e(\widehat{H}) \geq \mathrm{ex}(m, K_3) - 2\xi m^2.$$

We claim that $e_1(H) + \cdots + e_{r-4}(H) \leq 6\xi m^2$. Otherwise, by Lemma 2.6, the graph obtained by adding the edges of $E_1 \cup \cdots \cup E_{r-4}$ to $\widehat{H}$ would contain a triangle such that exactly one of the edges f_1 is in $E_1 \cup \cdots \cup E_{r-4}$. Let f_2 and f_3 be the other two edges of the triangle, which lie in $E_{r-3} \cup \cdots \cup E_r$. If there is a color $\alpha \in L_{f_1} \cap (L_{f_2} \cup L_{f_3})$, say $\alpha \in L_{f_1} \cap L_{f_2}$, we may choose a color $\beta \neq \alpha$ in L_{f_3}, as $|L_{f_3}| \geq r - 3 \geq 3$. Otherwise, let $\beta \in L_{f_1}$ and note that $|L_{f_2} \cup L_{f_3}| \leq r - 1$, while $|L_{f_2}| + |L_{f_3}| \geq 2r - 6 \geq r$. So there is a color $\alpha \in L_{f_2} \cap L_{f_3}$, where $\alpha \neq \beta$. In both cases, this would lead to a 2-colored triangle in G by Lemma 2.4, a contradiction.

As a consequence, the number of edges of H with both ends in the same set U_i is at most $7\xi m^2$. Let $W_i = \cup_{j \in U_i} V_j$ for $i \in \{1, 2\}$. Then, by our choice of η and ξ, we have

$$e_G(W_1) + e_G(W_2) \leq r\eta n^2 + (n/m)^2(e_H(U_1) + e_H(U_2)) < \delta n^2, \tag{6}$$

as required.

To conclude the proof of Lemma 1.2, we need to prove Claim 3.1.

Proof. (Proof of Claim 3.1) Suppose for a contradiction that any coloring of G avoiding a 2-colored triangle leads to a multicolored cluster graph H for which

$$e_{r-3}(H) + \cdots + e_r(H) < \mathrm{ex}(m, K_3) - \xi m^2. \tag{7}$$

Given a 2-element set $S \subset [r]$ and $j \in \{2, \ldots, r-4\}$, let $E_j(S, int_{\geq 1}; H)$ be the set of all edges $e' \in E_j(H)$ that satisfy $|L_{e'} \cap S| \geq 1$, and let $e_j(S, int_{\geq 1}; H) = |E_j(S, int_{\geq 1}; H)|$.

Proposition 3.2 *Consider a multicolored cluster graph H with no 2-colored triangle.*

(a) For all 2-element subsets $S \subseteq [r]$ of colors, the subgraph H' of the multicolored cluster graph H with edge set $\bigcup_{j=2}^{r-4} E_j(S, int_{\geq 1}; H) \cup \bigcup_{\ell=r-3}^{r} E_\ell(H)$ is triangle-free.

(b) Moreover, there exists a 2-element subset $S \subseteq [r]$ such that

$$\left| \bigcup_{j=2}^{r-4} E_j(S, int_{\geq 1}; H) \right| \geq \sum_{j=2}^{r-4} \frac{\binom{r}{2} - \binom{r-j}{2}}{\binom{r}{2}} \cdot |E_j(H)|. \tag{8}$$

Before proving Proposition 3.2, note that it leads to the following inequality:

$$\sum_{j=2}^{r-4} \frac{\binom{r}{2} - \binom{r-j}{2}}{\binom{r}{2}} \cdot e_j(H) + \sum_{\ell=r-3}^{r} e_\ell(H) \leq \mathrm{ex}(m, K_3). \tag{9}$$

Proof. We first argue that $\bigcup_{j=2}^{r-4} E_j(S, int_{\geq 1}; H) \cup \bigcup_{\ell=r-3}^{r} E_\ell(H)$ is triangle-free. For a contradiction suppose that there is a triangle with edges f_1, f_2, f_3. Note that the lists of these edges have size at least two.

If one of the edges lies in $\bigcup_{\ell=r-3}^{r} E_\ell(H)$, we may sum the sizes of the lists L_{f_1}, L_{f_2} and L_{f_3} to obtain at least $(r-3)+4 = r+1$. In particular, two of the lists must have a color α in common, and the third list contains $\beta \neq \alpha$, which produces a 2-colored triangle, a contradiction. Next, assume that $f_1, f_2, f_3 \in \bigcup_{j=2}^{r-4} E_j(S, int_{\geq 1}; H)$. If we sum the sizes of $L_{f_1} \cap S$, $L_{f_2} \cap S$ and $L_{f_3} \cap S$, we obtain at least three, so that two of the lists must contain the same color $\alpha \in S$, and the third list contains an element $\beta \neq \alpha$, which proves part (a)

For part (b), we claim that

$$\sum_{S \in \binom{[r]}{2}} \sum_{j=2}^{r-4} |E_j(S, int_{\geq 1}; H)| = \sum_{j=2}^{r-4} \left(\binom{r}{2} - \binom{r-j}{2} \right) \cdot e_j(H).$$

Indeed, for $j = 2, \ldots, r-4$, every edge $e \in E_j(H)$ is counted on the left hand side for all sets $S \in \binom{[r]}{2}$ such that $|S \cap e| \geq 1$, which amounts to $\binom{r}{2} - \binom{r-j}{2}$ times.

r	6	7	8	9	10	11	12
$M(r)$	$2^{5/3} \approx 3.17$	$3^{7/5} \approx 4.65$	$4^{14/11} \approx 5.84$	$5^{6/5} \approx 6.90$	$4^{3/2} = 8$	$4^{55/34} \approx 9.42$	$3^{11/5} \approx 11.21$

Table 1
Approximate values of $M(r)$.

By averaging, as there are $\binom{r}{2}$ distinct 2-element subsets in $[r]$, there exists a 2-element subset $S \subseteq [r]$ such that

$$\left|\bigcup_{j=2}^{r-4} E_j(S, int_{\geq 1}; H)\right| \geq \sum_{j=2}^{r-4} \frac{\binom{r}{2} - \binom{r-j}{2}}{\binom{r}{2}} \cdot e_j(H).$$

□

There are at most M^n partitions on $m \leq M$ classes. Thus, using (5), summing over all partitions and all corresponding multicolored cluster graphs H, the number of r-edge-colorings of G avoiding a 2-colored triangle is bounded above by

$$M^n \cdot \sum_H 2^{h(r\eta)n^2} \cdot r^{r\eta n^2} \cdot \left(\prod_{j=1}^{r} j^{e_j(H)}\right)^{\left(\frac{n}{m}\right)^2}. \tag{10}$$

Note that finding the maximum $M(r)$ of $\prod_{j=1}^{r} j^{e_j(H)}$ in this equation is equivalent to maximizing

$$e_2 \ln 2 + e_3 \ln 3 + \cdots + e_r \ln r,$$

which is a linear objective function with respect to the variables $e_2, \ldots, e_r \geq 0$. Together with linear constraints in (14) and (9), we obtain a linear program as follows. Given H, set $\zeta(H) = (\text{ex}(m, K_3) - e_{r-3}(H) - \cdots - e_r(H))/m^2$, so that $\zeta(H) \geq \xi$ by (7). The inequalities (14) with $j = 2$ and (9) tell us that to find an upper bound on (10), we may consider the linear program

$$\begin{aligned} &\max \quad x_2 \ln 2 + x_3 \ln 3 + \cdots + x_{r-4} \ln(r-4) \\ &\sum_{j=2}^{r-4} \frac{\binom{r}{2} - \binom{r-j}{2}}{\binom{r}{2}} \cdot x_j \leq 1 \\ &x_2, \ldots, x_{r-4} \geq 0, \end{aligned} \tag{11}$$

where x_i plays the role of $e_i(H)/(\zeta(H)m^2)$. As it turns out, for $r \in \{6, \ldots, 12\}$, if $y(r)$ is the optimum of the linear program, the value of $M(r) = e^{y(r)}$ is given in Table 1.

Clearly, for any multicolored cluster graph H, we have

$$\begin{aligned} \prod_{j=1}^{r} j^{e_j(H)} &= \left(\prod_{j=1}^{r-4} j^{e_j(H)}\right)\left(\prod_{j=r-3}^{r} j^{e_j(H)}\right) \\ &\leq \left(\prod_{j=1}^{r-4} j^{e_j(H)}\right) r^{e_{r-3}(H)+\cdots+e_r(H)} \\ &\leq M(r)^{\zeta(H)m^2} r^{\text{ex}(m,K_3)-\zeta(H)m^2} \leq M(r)^{\xi m^2} r^{\text{ex}(m,K_3)-\xi m^2}, \end{aligned} \tag{12}$$

as $M(r) < r$.

Since, for each partition, there are at most $2^{rM^2/2}$ choices for the multicolored cluster graph H, with (12), equation (10) is at most

$$\begin{aligned} & M^n \cdot 2^{h(r\eta)n^2} \cdot r^{r\eta n^2} \cdot 2^{\frac{rM^2}{2}} \cdot \left(\frac{M(r)}{r}\right)^{\xi n^2} \cdot r^{\mathrm{ex}(m,K_3)} \\ \overset{n \gg 1}{\leq} \; & 2^{\frac{3}{2}h(r\eta)n^2} \cdot r^{r\eta n^2} \cdot \left(\frac{M(r)}{r}\right)^{\xi n^2} \cdot r^{\mathrm{ex}(m,K_3)} \\ \leq \; & r^{(r\eta+h(r\eta))n^2} \cdot \left(\frac{M(r)}{r}\right)^{\xi n^2} \cdot r^{\mathrm{ex}(m,K_3)} \overset{n \gg 1}{\ll} r^{\mathrm{ex}(m,K_3)}. \end{aligned} \tag{13}$$

This implies that G has fewer than $r^{\mathrm{ex}(n,K_3)}$ colorings, a contradiction that proves Claim 3.1. □

4 Concluding Remarks

We proved that the Turán graph for K_3 is the unique n-vertex graph maximizing the number of r-edge-colorings with no copy of a triangle where exactly two colors appear. Of course, one may try to apply the same approach to $r \geq 13$, but the optimum value $M(r)$ of the linear program (11) satisfies $M(r) > r$, so that (12) fails to hold.

A possible way to circumvent this problem might be to include additional linear constraints to the linear program (11), in order to decrease the optimum value $M(r)$.

For instance, for $j = 2, \ldots, \lfloor r/3 \rfloor$, let H'_j be the subgraph of the multicolored cluster graph H (defined in the proof of Lemma 1.2) with edge set $E_j \cup \cdots \cup E_{r-2j}$, and fix a bipartite subgraph B'_j of H'_j with the maximum number of edges, so that $|E(B'_j)| > |E(H'_j)|/2 = (e_j(H) + \cdots + e_{r-2j}(H))/2$ (this is a well-known fact about the maximum cut of a graph). Let H''_j be the subgraph of H with edge set $E(B'_j) \cup E_{r-2j+1} \cup \cdots \cup E_r$. Note that H''_j is triangle-free, as any such triangle would have three edges f_1, f_2, f_3 such that $|L_{f_i}| \geq j \geq 2$ for all i and such that $\max_i |L_{f_i}| \geq r - 2j + 1 \geq 2$. By the pigeonhole principle, two of the lists would have a common color α, and the third list has a color $\beta \neq \alpha$. For $j = 2, \ldots, \lfloor r/3 \rfloor$, this implies that

$$\frac{1}{2} \cdot (e_j(H) + \cdots + e_{r-2j}(H)) + e_{r-2j+1}(H) + \cdots + e_r(H) \leq \mathrm{ex}(m, K_3). \tag{14}$$

So far we have not been successful in achieving an optimum that is less than r for some $r \geq 13$. It is conceivable that such an approach could be extended for all $r \leq 26$, as we already know that the bipartite Turán graph cannot be optimal for $r \geq 27$.

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