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On the Iterated Edge-Biclique Operator

Sylvain Legay

France

Leandro Montero[1]

L@bisen, AIDE Lab.
ISEN Nantes - Yncrea Ouest
Carquefou, France

Abstract

A biclique of a graph G is a maximal induced complete bipartite subgraph of G. The edge-biclique graph of G, $KB_e(G)$, is the edge-intersection graph of the bicliques of G. A graph G diverges (resp. converges or is periodic) under an operator H whenever $\lim_{k\to\infty} |V(H^k(G))| = \infty$ (resp. $\lim_{k\to\infty} H^k(G) = H^m(G)$ for some m or $H^k(G) = H^{k+s}(G)$ for some k and $s \geq 2$). The kth-iterated edge-biclique graph of G, $KB_e^k(G)$, is the graph obtained by applying the edge-biclique operator k successive times to G. In this paper we study the iterated edge-biclique operator KB_e. In particular, we give sufficient conditions for a graph to be convergent or divergent under the operator KB_e and we propose some conjectures on the subject.

Keywords: Bicliques, Edge-biclique graphs, Divergent graphs, Iterated graph operators, Graph dynamics

1 Introduction

Intersection graphs of certain special subgraphs of a general graph have been studied extensively. We can mention line graphs (intersection graphs of the edges of a graph), interval graphs (intersection graphs of a family of subpaths of a path), and in particular, clique graphs (intersection graphs of the the family of all cliques of a graph) [4,5,8,11,12,27,29].

The *clique graph* of G is denoted by $K(G)$. Clique graphs were introduced by Hamelink in [19] and characterized in [33]. It was proved in [1] that the clique graph recognition problem is NP-Complete.

The clique graph can be thought as an operator from *Graphs* into *Graphs*. The *iterated clique graph* $K^k(G)$ is the graph obtained by applying the clique operator k successive times. It was introduced by Hedetniemi and Slater in [20]. Much work has

[1] Email: `lpmontero@gmail.com`

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been done in the field of the iterated clique operator, looking at the possible different behaviors. The goal is to decide whether a given graph converges, diverges, or is periodic under the clique operator when k grows to infinity. This question remains open for the general case, moreover, it is not known if it is computable. However, partial characterizations have been given for convergent, divergent and periodic graphs, restricted to some classes of graphs. Some of them lead to polynomial time algorithms to solve the problem.

For the clique-Helly graph class, graphs which are convergent to the trivial graph have been characterized in [3]. Cographs, P_4-tidy graphs, and circular-arc graphs are examples of classes where the different behaviors were also characterized [7,22]. On the other hand, divergent graphs were considered. For example, in [31], families of divergent graphs are given. Periodic graphs were studied in [8,26]. It has been proved that for every integer i, there are graphs with period i and graphs which converge in i steps. More results about iterated clique graphs can be found in [9,10,23,24,25,32].

A *biclique* is a maximal bipartite complete induced subgraph. Bicliques have applications in various fields, for example biology: protein-protein interaction networks [6], social networks: web community discovery [21], genetics [2], medicine [30], information theory [18], etc. More applications (including some of these) can be found in [28]. The *biclique graph* of a graph G, denoted by $KB(G)$, is the intersection graph of the family of all bicliques of G. It was defined and characterized in [16]. However no polynomial time algorithm is known for recognizing biclique graphs. As for clique graphs, the biclique graph construction can be viewed as an operator between the class of graphs.

The *iterated biclique graph* $KB^k(G)$ is the graph obtained by applying to G the biclique operator k times iteratively. It was introduced in [15] and all possible behaviors were characterized. It was proven that a graph is either divergent or convergent, but never periodic (with period bigger than 1). Also, general characterizations for convergent and divergent graphs were given. These results were based on the fact that if a graph G contains a clique of size at least 5, then $KB(G)$ or $KB^2(G)$ contains a clique of larger size. Therefore, in that case G diverges. Similarly if G contains the *gem* or the *rocket* graphs as an induced subgraph, then $KB(G)$ contains a clique of size 5, and again G diverges. Otherwise it was shown that after removing false-twin vertices of $KB(G)$, the resulting graph is a clique on at most 4 vertices, in which case G converges. Moreover, it was proved that if a graph G converges, it converges to the graphs K_1 or K_3, and it does so in at most 3 steps. These characterizations led to an $O(n^4)$ time algorithm (later improved to $O(n+m)$ time [13]) for recognizing convergent or divergent graphs under the biclique operator.

The *edge-biclique graph* of a graph G, denoted by $KB_e(G)$, is the edge-intersection graph of the family of all bicliques of G. We recall that edge-intersection means that $KB_e(G)$ has a vertex for each biclique of G and two vertices are adjacent in $KB_e(G)$ if their corresponding bicliques in G share an edge (and not just a vertex as in $KB(G)$). The edge-biclique graph $KB_e(G)$ was defined in [17] and studied in [14], however there is no characterization so far to recognize edge-biclique graphs.

In this work we study edge-biclique graphs not only because of their mathematical interest but also because in real-life problems, bicliques often represent the relation between two types of entities (each partition of the biclique) therefore if would make sense to study when two objects (bicliques) share a common relationship (an edge) more than just an entity (a vertex). In particular, we focus on the *iterated edge-biclique graph*, denoted by $KB_e^k(G)$, and defined as the graph obtained by applying the edge-biclique operator k successive times to G. We give some non-trivial sufficient conditions for a graph to be convergent or divergent under the KB_e operator and we propose some conjectures that would help to fully characterize the behavior of a graph under the KB_e operator.

This work is organized as follows. In Section 2 the necessary notation is given. In Section 3 and Section 4 we present some results about convergent and divergent graphs, respectively. Finally, in Section 5 we state some conjectures on the subject.

2 Preliminaries

Along the paper we restrict to undirected simple graphs. Let $G = (V, E)$ be a graph with vertex set $V(G)$ and edge set $E(G)$. A *clique* of G is a maximal complete induced subgraph, while a *biclique* is a maximal bipartite complete induced subgraph of G. The *open neighborhood* of a vertex $v \in G$, denoted $N(v)$, is the set of vertices adjacent to v while the *closed neighborhood* of v, denoted by $N[v]$, is $N(v) \cup \{v\}$. A *path* (*cycle*) on k vertices ($k \geq 3$), denoted by P_k (C_k), is a set of vertices $v_1, v_2, ..., v_k \in G$ such that $v_i \neq v_j$ for all $1 \leq i \neq j \leq k$ and v_i is adjacent to v_{i+1} for all $1 \leq i \leq k-1$ (and v_k is adjacent to v_1). A graph is *connected* if there exists a path between each pair of vertices. The *girth* of G is the length of a shortest induced cycle in the graph. We assume that all graphs of this paper are connected.

Given a family of sets $\mathcal{H}$, the *intersection graph* of $\mathcal{H}$ is a graph that has the members of $\mathcal{H}$ as vertices, and there is an edge between two sets $E, F \in \mathcal{H}$ when E and F have non-empty intersection.

A graph G is an *intersection graph* if there exists a family of sets $\mathcal{H}$ such that G is the intersection graph of $\mathcal{H}$. We remark that any graph is an intersection graph [34].

Let H be any graph operator and let G be a graph. The iterated graph under the operator H is defined iteratively as follows: $H^0(G) = G$ and for $k \geq 1$, $H^k(G) = H^{k-1}(H(G))$. We say that G diverges (resp. converges or is periodic) under the operator H whenever $\lim_{k\to\infty} |V(H^k(G))| = \infty$ (resp. $\lim_{k\to\infty} H^k(G) = H^m(G)$ for some m or $H^k(G) = H^{k+s}(G)$ for some k and $s \geq 2$). The study of the behavior of a graph G under the operator H consists of deciding if G converges, diverges or is periodic under H.

We assume that the empty graph is convergent under the operator KB_e, as it is obtained by appyling the edge-biclique operator to a graph that does not contain any bicliques.

3 Convergence

To start this section we have this first easy result.

Lemma 3.1 *For $n \geq 2$, the complete graph K_n converges to the empty graph under the operator KB_e in two steps.*

Proof. Clearly each edge of K_n is a biclique that does not edge-intersect with another one. Then $KB_e(G)$ consists of $\frac{n(n-1)}{2}$ isolated vertices (and no bicliques), therefore $KB_e^2(G)$ is the empty graph. □

Next we show that graphs without induced cycles of length 3 and 4 are convergent.

Theorem 3.2 *If G has girth at least five, then the edge-biclique operator applied to G converges towards the graph induced by the union of all the cycles and paths connecting cycles of G.*

Proof. If G has girth at least five, then every biclique is a star. Moreover G has no triangles, so $N(v)$ is a stable set and thus, for each v of degree more than one, $N[v]$ is a maximal biclique. Notice also that if u is adjacent to v, $N[u]$ and $N[v]$ contain a common edge, therefore the vertices in $KB_e(G)$ corresponding to the bicliques $N[u]$ and $N[v]$ will be adjacent. We can conclude that $KB_e(G)$ is exactly the graph induced by all vertices of degree at least two of G. For k big enough, the only vertices left in $KB_e^k(G)$ are those which belong to cycles or to paths connecting cycles, that is, G converges under the operator KB_e towards the graph induced by the cycles and paths connecting cycles of G. □

As an immediate result of Theorem 3.2, we obtain the following corollary.

Corollary 3.3 *If G has girth at least five and has no vertices of degree one, then $KB_e(G) = G$.*

One natural question that arises from Corollary 3.3 is: Given a graph G such that $KB_e(G) = G$, does G have girth at least five and no vertices of degree one? The answer is no, for instance, the graph $\overline{C_7}$ shown in Figure 1 satisfies that $KB_e(G) = G$ but its girth is three[2].

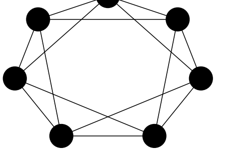

Fig. 1. The graph $\overline{C_7}$ is the smallest graph satisfying $KB_e(G) = G$ with girth less than five.

[2] Found using the computer.

From Theorem 3.2, we also obtain the following result.

Corollary 3.4 *For every $k \geq 1$, there is a graph that converges in k steps under the operator KB_e.*

Proof. Just take any induced cycle C_n, $n \geq 5$, and join one of its vertices to the endpoint of a simple path P_k. Observe that this graph converges to C_n in exactly k steps. □

Corollary 3.5 *Trees converge to the empty graph under the operator KB_e.*

4 Divergence

In this section we study the divergence of the operator KB_e. We start first with the following definition.

Definition 4.1 Let G be a graph and let $C = v_0v_1 \dots v_{n-1}$ be an induced cycle of length $n \geq 5$. We say that C has *good neighbors* whenever for all vertices $v \in G - C$, if $\{v_{i-1}, v_{i+1}\} \subseteq N(v)$ then $v_i \in N(v)$, for $i = 0, \dots, n-1$ and all subindices taken $(mod\ n)$. (see Fig 2).

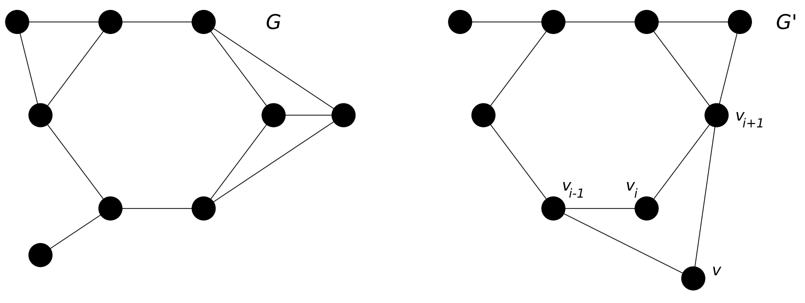

Fig. 2. G has a cycle with *good neighbors* while G' has not, since v is adjacent to v_{i-1} and v_{i+1} but not adjacent to v_i.

Now we present an important proposition that assures that the good neighbors property is invariant through the iterations of the operator KB_e.

Proposition 4.2 *Let G be a graph and let $C = v_0v_1 \dots v_{n-1}$ be an induced cycle of length $n \geq 5$ with good neighbors. Let B_i, $i = 0, \dots, n-1$, be bicliques in G containing the vertices $\{v_{i-1}, v_i, v_{i+1}\}$ $(mod\ n)$, respectively, $B_i \subseteq N[v_i]$, and let b_i, $i = 0, \dots, n-1$, be the vertices in $KB_e(G)$ corresponding to the bicliques $B_i \in G$. Then $C' = b_0b_1 \dots b_{n-1}$ is an induced cycle of $KB_e(G)$. Moreover, C' has good neighbors.*

Proof. As C is an induced cycle in G, let B_i, $i = 0, \dots, n-1$, be bicliques that contain the vertices $\{v_{i-1}, v_i, v_{i+1}\}$ $(mod\ n)$, respectively. Clearly, each B_i intersects B_{i+1} in the edge v_iv_{i+1}, therefore if we call b_i, $i = 0, \dots, n-1$, the corresponding vertices in $KB_e(G)$ to the bicliques B_i, then we have that $b_0b_1 \dots b_{n-1}$ form a cycle C' in $KB_e(G)$. Now, let $v \in G$ be a vertex in $B_i - \{v_{i-1}, v_i, v_{i+1}\}$. As B_i is a biclique of G, either v is adjacent to v_{i-1} and v_{i+1} but not adjacent to v_i, which is not possible because C has good neighbors, or v is adjacent to v_i. Therefore, for all $i = 0, \dots, n-1$, $B_i \subseteq N[v_i]$ and C' is an induced cycle of $KB_e(G)$.

Now, let $b \in KB_e(G) - C'$ be a vertex such that $\{b_{i-1}, b_{i+1}\} \subseteq N(b)$ for some i. If B is the biclique of G corresponding to the vertex $b \in KB_e(G)$, then B contains v_{i-1} and v_{i+1}, since $B_{i-1} \subseteq N[v_{i-1}]$ and $B_{i+1} \subseteq N[v_{i+1}]$. As v_{i-1} and v_{i+1} are not adjacent in G, there exists a vertex $v \in B \cap B_{i-1} \cap B_{i+1}$ such that v is adjacent to both v_{i-1} and v_{i+1}. If $v \neq v_i$, since C has good neighbors, v must also be adjacent to v_i, contradicting the fact that $v \in B_{i-1}$ (or B_{i+1}). Therefore, $v = v_i$ and B and B_i have an edge in common, that is, b is adjacent to b_i in $KB_e(G)$ and thus C' has good neighbors. $\square$

Before the main theorem, we define the following family of graphs.

Definition 4.3 For $n \geq 3$ and $m \geq 1$, the $(n, m)-$*necklace* graph on $n+m$ vertices consists of an induced cycle C_n and a complete graph K_m, such that for an edge $e \in C_n$, every vertex of the K_m is adjacent to both endpoints of e. (see Fig 3).

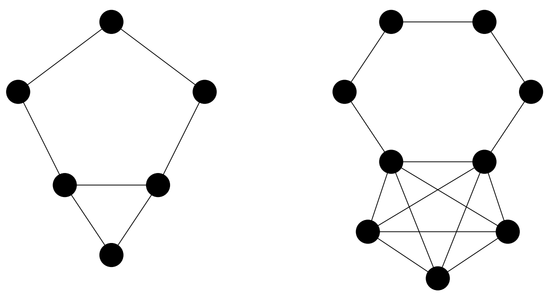

Fig. 3. $(5, 1) -$ *necklace* and $(6, 3) -$ *necklace* graphs.

Now we present the main theorem of this section.

Theorem 4.4 *Let G be a graph that contains an induced $(n, m)-$necklace, $n \geq 5$, $m \geq 1$, such that its cycle has good neighbors. Then, either $KB_e^2(G)$ or $KB_e^3(G)$ contains an induced $(n, m') -$ necklace such that its cycle has good neighbors, and $m' > m$.*

Proof. Let $C_n = v_0 v_1 \dots v_{n-1}$ be the induced cycle and $K_m = \{w_1, \dots, w_m\}$ be the complete graph of the $(n, m) -$ *necklace*, respectively. Let $v_i v_{i+1}$, for some $i \in \{0, \dots, n-1\}$ $(mod\ n)$, be the edge of the C_n such that w_j is adjacent to v_i and v_{i+1} for all $j = 1, \dots, m$. Let B_t, $t = 0, \dots, n-1$, be bicliques that contain the vertices $\{v_{t-1}, v_t, v_{t+1}\}$ $(mod\ n)$, respectively, and let b_t, $t = 0, \dots, n-1$, be the corresponding vertices in $KB_e(G)$ to the bicliques B_t. By Proposition 4.2, $C'_n = b_0 b_1 \dots b_{n-1}$ is an induced cycle in $KB_e(G)$ with good neighbors.

Consider the following two families of bicliques $B^1 = \{B^1_j : \{w_j, v_i, v_{i-1}\} \subseteq B^1_j, j = 1, \dots, m\}$ and $B^2 = \{B^2_j : \{w_j, v_{i+1}, v_{i+2}\} \subseteq B^2_j, j = 1, \dots, m\}$. Clearly, all these $2m$ bicliques are different and moreover, they are different to the bicliques B_t for $t = 0, \dots, n-1$ as C_n has good neighbors. Now we can see that $(\bigcap_{j=1}^m B^1_j) \cap$

$B_{i-1} \cap B_i = \{v_{i-1}, v_i\}$ and $(\bigcap_{j=1}^m B_j^2) \cap B_{i+1} \cap B_{i+2} = \{v_{i+1}, v_{i+2}\}$. Therefore if b_j^1 and b_j^2, $j = 1, \ldots, m$, are the corresponding vertices in $KB_e(G)$ to the bicliques B_j^1 and B_j^2, we have that in $KB_e(G)$, $K_m^1 = \{b_1^1, \ldots, b_m^1\}$ and $K_m^2 = \{b_1^2, \ldots, b_m^2\}$ are two complete graphs such that b_j^1 is adjacent to b_{i-1} and b_i, and b_j^2 is adjacent to b_{i+1} and b_{i+2}, for all $j = 1, \ldots, m$. Notice that as C_n has good neighbors in G, then in $KB_e(G)$ we have $N(b_j^1) \cap C'_n = \{b_{i-1}, b_i\}$ and $N(b_j^2) \cap C'_n = \{b_{i+1}, b_{i+2}\}$, for all $j = 1, \ldots, m$.

Now, let $\widetilde{B}_t$, $t = 0, \ldots, n-1$, be the bicliques of $KB_e(G)$ that contain the vertices $\{b_{t-1}, b_t, b_{t+1}\}$ $(mod\ n)$, respectively, and $\widetilde{b}_t$, $t = 0, \ldots, n-1$, be the corresponding vertices in $KB_e^2(G)$ to the bicliques $\widetilde{B}_t$. Again, by Proposition 4.2, $C''_n = \widetilde{b}_0\widetilde{b}_1 \ldots \widetilde{b}_{n-1}$ is an induced cycle in $KB_e^2(G)$ with good neighbors.

Now for each b_j^1, $j = 1, \ldots, m$, we have that $\{b_j^1, b_i, b_{i+1}\}$ is contained in a biclique $\widetilde{B}_j^1$. Similarly, for each b_j^2, $j = 1, \ldots, m$, $\{b_j^2, b_i, b_{i+1}\}$ is contained in a biclique $\widetilde{B}_j^2$. In the worst case (to minimize the number of bicliques), if there is exactly a perfect matching between K_m^1 and K_m^2, say b_j^1 is adjacent to b_j^2, for each $j = 1, \ldots, m$, then $\widetilde{B}_j^1 = \widetilde{B}_j^2$. We have the following two cases:

Case A: *There is at least one vertex $b_1^1 \in K_m^1$ not adjacent to any vertex of K_m^2.* Clearly, $\widetilde{B}_1^1 \neq \widetilde{B}_j^2$, for all $j = 1, \ldots, m$, and furthermore, these $m+1$ bicliques are different to the bicliques $\widetilde{B}_t$ for $t = 0, \ldots, n-1$. Observe that $(\bigcap_{j=1}^m \widetilde{B}_j^2) \cap B_1^1 = \{b_i, b_{i+1}\}$ and moreover, $\widetilde{B}_i \cap \widetilde{B}_{i+1} = \{b_i, b_{i+1}\}$. Therefore, if $\widetilde{b}_1^1$ and $\widetilde{b}_j^2$, $j = 1, \ldots, m$, are the corresponding vertices in $KB_e^2(G)$ to the bicliques $\widetilde{B}_1^1$ and $\widetilde{B}_j^2$, respectively, we have that in $KB_e^2(G)$, $\{\widetilde{b}_1^1, \widetilde{b}_1^2, \ldots, \widetilde{b}_m^2\}$ is a complete graph on $m+1$ vertices such that, as C'_n has good neighbors, every vertex of this K_{m+1} is only adjacent to $\widetilde{b}_i$ and to $\widetilde{b}_{i+1}$ on the cycle C''_n. That is, $KB_e^2(G)$ contains an induced $(n, m+1)-necklace$ such that its cycle C''_n has good neighbors. See Fig 4 for an example where K_m^1 and K_m^2 have no edge in-between.

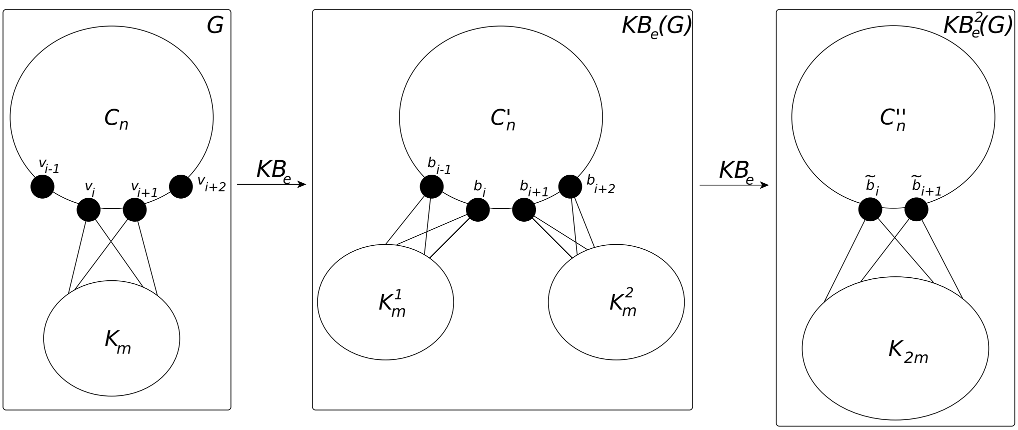

Fig. 4. Graphs G containing an $(n, m) - necklace$, $KB_e(G)$, and $KB_e^2(G)$ containing an $(n, 2m) - necklace$.

Case B: *Every vertex of K_m^1 is adjacent to at least one vertex of K_m^2 (and by symmetry every vertex of K_m^2 is adjacent to at least one vertex of K_m^1).* As explained above, the worst case is when there is a perfect matching between K_m^1 and K_m^2. Without loss of generality, suppose that b_j^1 is adjacent to b_j^2 for each $j = 1, \ldots, m$, otherwise we would obtain at least $m+1$ bicliques having the edge $b_i b_{i+1}$ in common and therefore $KB_e^2(G)$ will contain an induced $(n, m+1) - necklace$ such that its cycle C_n'' has good neighbors. As there is a matching between K_m^1 and K_m^2, let $\widetilde{B}_j'$ be the bicliques that contain the set $\{b_j^1, b_i, b_{i+1}, b_j^2\}$ for each $j = 1, \ldots, m$. These bicliques contain the edge $b_i b_{i+1}$ and they are different to the bicliques $\widetilde{B}_t$ for $t = 0, \ldots, n-1$. Then, if $\widetilde{b}_j'$, $j = 1, \ldots, m$, are the corresponding vertices in $KB_e^2(G)$ to the bicliques $\widetilde{B}_j'$, we have that in $KB_e^2(G)$, $\{\widetilde{b}_1', \ldots, \widetilde{b}_m'\}$ is a complete graph on m vertices such that, as C_n' has good neighbors, every vertex of this K_m is only adjacent to $\widetilde{b}_i$ and to $\widetilde{b}_{i+1}$ on the cycle C_n''.

Now for each b_j^1, $j = 1, \ldots, m$, we have that $\{b_j^1, b_{i-1}, b_{i-2}\}$ is contained in a biclique $\widetilde{B}_j^1$. All these m bicliques have the edge $b_{i-1}b_{i-2}$ in common. In addition, they are clearly different to the bicliques $\widetilde{B}_t$, $t = 0, \ldots, n-1$ and $\widetilde{B}_j'$, $j = 1, \ldots, m$. Suppose now that there is an edge in common between the bicliques, say $\widetilde{B}_1^1$ and $\widetilde{B}_1'$. Then, there must exist a vertex $b \in KB_e(G)$ adjacent to b_{i-2}, b_1^1 and b_{i+1}. This implies that in G, there must exist a biclique B (corresponding to the vertex $b \in KB_e(G)$) that has edges in common with the bicliques B_{i-2}, B_1^1 and B_{i+1}. Therefore, as C_n has good neighbors, there must a vertex $v \in B$ adjacent to the vertices v_{i-2} and v_{i+1}. Finally, as B has an edge in common with the biclique B_1^1, v must be adjacent to either to v_i, or to v_{i-1} and w_1. In both cases we obtain a contradiction as B would contain either the $K_3 = \{v, v_i, v_{i+1}\}$ or the $K_3 = \{v, v_{i-2}, v_{i-1}\}$ which is not possible if B is a biclique. We can conclude then that there are no edges in common between the bicliques $\widetilde{B}_j^1$ and $\widetilde{B}_j'$, for all $j = 1, \ldots, m$. Now let $\widetilde{b}_j^1$ be the vertices in $KB_e^2(G)$ corresponding to the bicliques $\widetilde{B}_j^1$ of $KB(G)$, for $j = 1, \ldots, m$ respectively. Then, these vertices form a K_m in $KB_e^2(G)$ and they are only adjacent to the vertices $\widetilde{b}_{i-2}$ and $\widetilde{b}_{i-1}$ of the cycle C_n''.

Now, let β_t, $t = 0, \ldots, n-1$, be bicliques of $KB_e^2(G)$ that contain the vertices $\{\widetilde{b}_{t-1}, \widetilde{b}_t, \widetilde{b}_{t+1}\}$ $(mod\ n)$, respectively, and $\widetilde{\beta}_t$, $t = 0, \ldots, n-1$, the corresponding vertices in $KB_e^3(G)$ to the bicliques β_t. By Proposition 4.2, $C_n''' = \widetilde{\beta}_0\widetilde{\beta}_1 \ldots \widetilde{\beta}_{n-1}$ is an induced cycle in $KB_e^3(G)$ with good neighbors. To finish, consider the following two families of bicliques: $\beta^1 = \{\beta_j^1 : \{\widetilde{b}_j^1, \widetilde{b}_{i-1}, \widetilde{b}_i\} \subseteq \beta_j^1, j = 1, \ldots, m\}$ and $\beta^2 = \{\beta_j^2 : \{\widetilde{b}_j', \widetilde{b}_{i-1}, \widetilde{b}_i\} \subseteq \beta_j^2, j = 1, \ldots, m\}$. Clearly, all these $2m$ bicliques are different as there are no edges in common between the bicliques $\widetilde{B}_j^1$ and $\widetilde{B}_j'$, for all $j = 1, \ldots, m$, and moreover, they are different to the bicliques β_t for $t = 0, \ldots, n-1$ as C_n'' has good neighbors. Since all these $2m$ bicliques contain the edge $\widetilde{b}_{i-1}\widetilde{b}_i$, then if $\widetilde{\beta}_j^1$ and $\widetilde{\beta}_j^2$, $j = 1, \ldots, m$, are the corresponding vertices in $KB_e^3(G)$ to the bicliques β_j^1 and β_j^2, respectively, we have that in $KB_e^3(G)$, $\{\widetilde{\beta}_1^1, \ldots, \widetilde{\beta}_m^1, \widetilde{\beta}_1^2, \ldots, \widetilde{\beta}_m^2\}$ is a complete graph on $2m$ vertices such that, as C_n''' has good neighbors, every vertex of this K_{2m} is only adjacent to $\widetilde{\beta}_i$ and to $\widetilde{\beta}_{i-1}$ on the cycle C_n'''. That is, $KB_e^3(G)$ contains an induced $(n, 2m) - necklace$ such that its cycle C_n''' has good neighbors. See Fig 5 for a representation of this case. $\square$

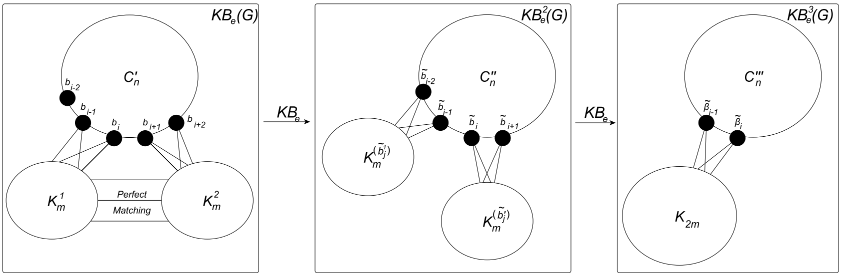

Fig. 5. Graphs $KB_e(G)$ with a perfect matching between K^1_m and K^2_m, $KB^2_e(G)$ with no edges between the complete graphs, and $KB^3_e(G)$ containing an $(n, 2m) - necklace$.

As a corollary, we obtain the following divergence theorem.

Theorem 4.5 *Let G be a graph that contains an induced $(n, m) - necklace$, $n \geq 5$, $m \geq 1$, such that its cycle has good neighbors. Then G diverges under the operator KB_e.*

Proof. Applying Theorem 4.4 several times, we obtain that either $KB^2_e(G)$ or $KB^3_e(G)$ contains an induced $(n, m') - necklace$, and $m' > m$, then that either $KB^4_e(G)$, $KB^5_e(G)$ or $KB^6_e(G)$ contains an induced $(n, m'')-necklace$, and $m'' > m'$, etc, all having its cycles with good neighbors. Therefore, G is divergent under the operator KB_e as $\lim_{k\to\infty} |V(KB^k_e(G))| = \infty$. □

To finish the section, we obtain a second corollary.

Corollary 4.6 *Let G be a graph and let C_n be an induced cycle of length $n \geq 5$ with good neighbors. If there is a vertex $v \in G - C_n$ such that $N(v) \cap C_n$ has at least one edge and not all C_n, then G diverges under the operator KB_e.*

Proof (Sketch) Observe that either G, $KB_e(G)$ or $KB^2_e(G)$, contain an $(n, 1) - necklace$, $n \geq 5$, such that its cycle has good neighbors. Therefore G diverges under the operator KB_e following Theorem 4.5. □

5 Open problems

We propose the following conjectures.

Conjecture 5.1 *A graph G is either divergent or convergent under the KB_e operator but never periodic (with period bigger than 1).*

Conjecture 5.2 *$G = KB_e(G)$ if and only if either $G = \overline{C_7}$, $G = G_9$ (see Fig. 6) or G has girth at least five and has no vertices of degree one.*

Note that Corollary 3.3 along with the fact that $KB_e(\overline{C_7}) = \overline{C_7}$, $KB_e(G_9) = G_9$ prove the “only if” part of Conjecture 5.2.

Conjecture 5.3 *It is computable to decide if a graph diverges or converges under the operator KB_e.*

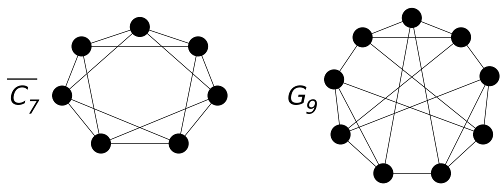

Fig. 6. Graphs $\overline{C_7}$ and G_9 satisfying $KB_e(G) = G$ with girth less than five.

Conjecture 5.4 *A graph G is divergent under the operator KB_e if and only if there exists some k such that $KB_e^k(G)$ contains an induced $(n,m)-$necklace, $n \geq 5$, $m \geq 1$, with its cycle having good neighbors.*

Clearly Theorem 4.5 proves the "only if" part of Conjecture 5.4 and moreover, the "if" part along with Conjecture 5.1 imply Conjecture 5.3.

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