Electronic Notes in Theoretical Computer Science 40 (2001)
URL: http://www.elsevier.nl/locate/entcs/volume40.html 12 pages

Weightable quasi-metric semigroups and semilattices

Salvador Romaguera [1,2]

Escuela de Caminos
Departamento de Matemática Aplicada
Universidad Politécnica de Valencia
Apartado 22012, 46071 Valencia, Spain.

Michel Schellekens [3]

Department of Computer Science
National University of Ireland, University College Cork
Cork, Ireland

Abstract

In [Sch00] a bijection has been established, for the case of semilattices, between invariant partial metrics and semivaluations. Semivaluations are a natural generalization of valuations on lattices to the context of semilattices and arise in many different contexts in Quantitative Domain Theory ([Sch00]). Examples of well known spaces which are semivaluation spaces are the Baire quasi-metric spaces of [Mat95], the complexity spaces of [Sch95] and the interval domain ([EEP97]). In [Sch00a], we have shown that the totally bounded Scott domains of [Smy91] can also be represented as semivaluation spaces.

In this extended abstract we explore the notion of a semivaluation space in the context of semigroups. This extension is a natural one, since for each of the above results, an *invariant* partial metric is involved. The notion of invariance has been well studied for semigroups as well (e.g. [Ko82]).

As a further motivation, we discuss three Computer Science examples of semigroups, given by the domain of words ([Smy91]), the complexity spaces ([Sch95],[RS99]) and the interval domain ([EEP97]).

An extension of the correspondence theorem of [Sch00] to the context of semigroups is obtained.

[1] The first named author acknowledges financial support of the Spanish Ministry of Science and Technology, grant BFM2000-1111.

[2] Email: `sromague mat.upv.es`

[3] Email: `m.schellekens@cs.ucc.ie`

1 Background

A function $d: X \times X \to \mathcal{R}_0^+$ is a *quasi-pseudo-metric* iff

1) $\forall x \in X.\, d(x,x) = 0$
2) $\forall x, y, z \in X.\, d(x,y) + d(y,z) \geq d(x,z)$.

A *quasi-pseudo-metric space* is a pair (X, d) consisting of a set X together with a quasi-pseudo-metric d on X.

The topology induced in a natural way by the quasi-pseudo-metric d is denoted by $\mathcal{T}(d)$.

In case a quasi-pseudo-metric space is required to satisfy the T_0-separation axiom, we refer to such a space as a *quasi-metric* space.

In that case, condition 1) and the T_0-separation axiom can be replaced by the following condition:

1′) $\forall x, y.\, d(x,y) = d(y,x) = 0 \Leftrightarrow x = y$.

The *associated partial order* $\leq_d$ of a quasi-metric d is defined by $x \leq_d y$ iff $d(x,y) = 0$.

A quasi-pseudo-metric space is T_0 iff the associated order of the space is a partial order (e.g. [FL82]). We will work under the assumption that all spaces satisfy the T_0 separation axiom; that is we will solely refer to quasi-metric spaces in the following.

The *conjugate* d^{-1} of a quasi-metric d is defined to be the function $d^{-1}(x,y) = d(y,x)$, which is again a quasi-metric (e.g. [FL82]). The conjugate of a quasi-metric space (X, d) is the quasi-metric space (X, d^{-1}). The *metric* d^* induced by a quasi-metric d is defined by $d^*(x,y) = max\{d(x,y), d(y,x)\}$.

A function f from a quasi-metric space (X, d) to a quasi-metric space (X', d') is quasi-uniformly continuous iff $\forall \epsilon > 0\, \exists \delta > 0\, \forall x, y \in X.\, d(x,y) < \delta \Rightarrow d'(f(x), f(y)) < \epsilon$.

We discuss a few examples of quasi-metric spaces.

The function $d_{\mathcal{R}}: \mathcal{R}^2 \to \mathcal{R}_0^+$, defined by $d_{\mathcal{R}}(x,y) = y - x$ when $x < y$ and $d_{\mathcal{R}}(x,y) = 0$ otherwise, and its conjugate are quasi-metrics. We refer to $d_{\mathcal{R}}$ as the "left distance" and to its conjugate as the "right distance". These quasi-metrics correspond to the nonsymmetric versions of the standard metric m on the reals, where $\forall x, y \in \mathcal{R}.\, m(x,y) = |x - y|$.

Note that the right distance has the usual order on the reals as associated order, that is $\forall x, y \in \mathcal{R}.\, x \leq_{d_{\mathcal{R}}^{-1}} y \Leftrightarrow x \leq y$, while for the left distance we have $\forall x, y \in \mathcal{R}.\, x \leq_{d_{\mathcal{R}}} y \Leftrightarrow x \geq y$.

The function $d_c: (\overline{\mathcal{R}} - \{0\})^2 \to \mathcal{R}_0^+$, defined by $d_c(x,y) = \frac{1}{y} - \frac{1}{x}$ when $y < x$ and 0 otherwise, and its conjugate are quasi-metrics.

The *complexity space* (C, d_C) has been introduced in [Sch95] (cf. also [Sch96] and [RS99]. Here

$$C = \{f : \omega \to \overline{\mathcal{R}}^+ | \sum_{n=0}^{\infty} 2^{-n} \frac{1}{f(n)} < +\infty\}$$

and d_C is the quasi-metric on C defined by

$$d_C(f, g) = \sum_{n=0}^{\infty} 2^{-n}[(\frac{1}{g(n)} - \frac{1}{f(n)}) \vee 0]$$

whenever $f, g \in C$. The complexity space (C, d_C) is a quasi-metric space with a maximum $\top$, which is the function with constant value ∞.

The *dual complexity space* is introduced in [RS99] as a pair (C^*, d_{C^*}), where $C^* = \{f : \omega \to \mathcal{R}_0^+ \mid \sum_{n=0}^{\infty} 2^{-n} f(n) < +\infty\}$, and d_{C^*} is the quasi-metric defined on C^* by $d_{C^*}(f, g) = \sum_{n=0}^{\infty} 2^{-n}[(g(n) - f(n)) \vee 0]$, whenever $f, g \in C^*$. We recall that (C, d_C) is isometric to (C^*, d_{C^*}) by the isometry $\Psi : C^* \to C$, defined by $\Psi(f) = 1/f$ (see [RS99]). Via the analysis of its dual, several quasi-metric properties of (C, d_C), in particular Smyth completeness and total boundedness, are studied in [RS99].

2 Semivaluations

We recall some basic facts on semivaluations from [Sch00].

We recall the definition of a valuation on a lattice $(L, \sqsubseteq)$.

A function $f : L \to \mathcal{R}_0^+$ is a *valuation* iff

(1) f is increasing.
(2) $\forall x, y \in L.\, f(x \sqcap y) + f(x \sqcup y) = f(x) + f(y)$.

In case the function f is decreasing and satisfies (2), we refer to f as a *co-valuation*.

If f only satisfies (2) we say that f satisfies the *modularity law*, or also that f is *modular*.

There does not seem to be a consistent terminology in the literature. Valuations, also called evaluations, as used in computer science (e.g. [BS97] or [Jon89]) typically satisfy (1) and (2) above. In the classical mathematical literature a valuation only needs to satisfy (2) (e.g. [Bir84]).

It is convenient for matters of presentation to reserve the definition given above for a valuation in order to state results on connections between partial metrics and valuations as they occur in Computer Science.

Finally, a (co)valuation f on a (semi)lattice $(L, \sqsubseteq)$ is called *strictly increasing* (*strictly decreasing*) if $\forall x, y \in L.\, x \sqsubset y \Rightarrow f(x) < f(y)$ $(f(x) > f(y))$.

If $(X, \preceq)$ is a meet semilattice then a function $f: (X, \preceq) \rightarrow \mathcal{R}_0^+$ is a *meet valuation* iff

$$\forall x, y, z \in X.\, f(x \sqcap z) \geq f(x \sqcap y) + f(y \sqcap z) - f(y)$$

and f is *meet co-valuation* iff

$$\forall x, y, z \in X.\, f(x \sqcap z) \leq f(x \sqcap y) + f(y \sqcap z) - f(y).$$

If $(X, \preceq)$ is a join semilattice then a function $f: (X, \preceq) \rightarrow \mathcal{R}_0^+$ is a *join valuation* iff

$$\forall x, y, z \in X.\, f(x \sqcup z) \leq f(x \sqcup y) + f(y \sqcup z) - f(y)$$

and f is *join co-valuation* iff

$$\forall x, y, z \in X.\, f(x \sqcup z) \geq f(x \sqcup y) + f(y \sqcup z) - f(y).$$

A function is a *semivaluation* if it is either a join valuation or a meet valuation. A join (meet) valuation space is a join (meet) semilattice equipped with a join (meet) valuation. A *semivaluation space* is a semilattice equipped with a semivaluation.

Lemma 2.1 *Semivaluations are increasing.*

Proposition 2.2 *Let L be a lattice.*
1) A function $f : L \rightarrow \mathcal{R}_0^+$ is a join valuation if and only if it is increasing and satisfies join-modularity, i.e.:

$$f(x \sqcup z) + f(x \sqcap z) \leq f(x) + f(z).$$

2) A function $f : L \rightarrow \mathcal{R}_0^+$ is a meet valuation if and only if it is increasing and satisfies meet-modularity, i.e.

$$f(x \sqcup z) + f(x \sqcap z) \geq f(x) + f(z).$$

Corollary 2.3 *A function on a lattice is a valuation iff it is a join valuation and a meet valuation. A function on a lattice is a co-valuation iff it is a join co-valuation and a meet co-valuation.* □

With slight abuse of terminology, we will refer to quasi-metric spaces for which the associated order is a semilattice, simply as *semilattices*, and a similar convention holds for the case of lattices.

The terminology of *quasi-metric (semi)lattice* is reserved for quasi-metric spaces which are (semi)lattices for which the operations are quasi-uniformly continuous. This is in accordance with the terminology used for the theory of uniform lattices (e.g. [Web91] and [Web93]).

A join semilattice (X, d) is *invariant* iff $\forall x, y, z \in X.\, d(x \sqcup z, y \sqcup z) \leq d(x, y)$. In that case we also write that the quasi-metric d is invariant. The notions of an *invariant meet semilattice* and of an *invariant lattice* are defined in the obvious way. One can easily verify that invariant join semilattices are quasi-metric join semilattices and that similar results hold for the case of invariant meet semilattices and for invariant lattices.

It is convenient to present the following alternative characterization of invariance ([Sch00]).

Lemma 2.4 *A join semilattice* (X, d) *is invariant iff* $\forall x, y \in X.\, d(x \sqcup y, y) = d(x, y)$. *A meet semilattice* (X, d) *is invariant iff* $\forall x, y \in X.\, d(x, x \sqcap y) = d(x, y)$.

3 Weightable quasi-metric semigroups

A quasi-metric space (X, d) is *weightable* iff there exists a function $w\colon X \to \mathcal{R}_0^+$ such that $\forall x, y \in X.\, d(x, y) + w(x) = d(y, x) + w(y)$. The function w is called a *weighting function*, $w(x)$ is the *weight* of x and the quasi-metric d is *weightable by the function* w. A *weighted* space is a triple (X, d, w) where (X, d) is a quasi-metric space weightable by the function w.

We recall that the weighting functions of a weightable space are determined by a unique fading weighting ([Sch00], cf. also [KV94]).

A function $f\colon X \to \mathcal{R}_0^+$ is *fading* iff $\inf_{x \in X} f(x) = 0$.

Definition 3.1 A weighted quasi-metric space is of fading weight iff its weighting function is fading.

Proposition 3.2 *The weighting functions of a weightable quasi-metric space are strictly decreasing. The weighting functions are exactly the functions* $f + c$*, where* $c \geq 0$ *and where* f *is the unique fading weighting of the space.*

A quasi-metric semigroup is a triple $(X, d, \star)$ such that (X, d) is a quasi-metric space and $(X, \star)$ is a semigroup such that d is $\star$-invariant, i.e. for all $x, y, z \in X$:

$$d(x \star z, y \star z) \leq d(x, y) \text{ and } d(z \star x, z \star y) \leq d(x, y).$$

Remark 3.3 It immediately follows from the above definition that if $(X, d, \star)$ is a quasi-metric semigroup, then $(X, \leq_d, \star)$ is an ordered semigroup and $(X, \mathcal{T}(d), \star)$ is a topological semigroup.

A *weightable quasi-metric semigroup* is a quasi-metric semigroup $(X, d, \star)$ such that d is a weightable quasi-metric on X.

A *weightable invariant meet semigroup* is a weightable quasi-metric semigroup $(X, d, \star)$ such that (X, d) is an invariant meet semilattice.

We remark that a quasi-metric d on a semigroup $(X, \star)$ is $\star$-invariant if and only the following inequality is satisfied (see also [Ko82]):

$$(1)\ d(a \star b, c \star d) \leq d(a, c) + d(b, d).$$

This follows since: $d(a \star b, c \star d) \leq d(a \star b, c \star b) + d(c \star b, c \star d) \leq d(a, c) + d(b, d)$, where the last inequality follows by $\star$-invariance. The converse is obvious.

In particular, we observe that if (X, d) is an invariant meet semilattice, then for each $a, b, x, y \in X$, we have

$$d(a \sqcap b, x \sqcap y) \leq d(a \sqcap b, x \sqcap b) + d(x \sqcap b, x \sqcap y) \leq d(a, x) + d(b, y).$$

Since on a meet semilattice $(X, \sqsubseteq)$, the operation $\sqcap$ is associative, it follows from the above observation that for each invariant meet semilattice (X, d), the triple $(X, d, \sqcap)$ is obviously a quasi-metric semigroup, and hence, it is a weightable invariant meet semigroup whenever the quasi-metric d is weightable.

The fact that each invariant semilattice gives rise to a quasi-metric semigroup indicates that it is natural to consider extensions of this structure with other semigroup operations.

Indeed, not all semilattices occuring in the literature simply are semigroups with respect to their associated semilattice operation. One example is the complexity space, which when equipped with the natural pointwise addition operation yields a quasi-metric semigroup with respect to this operation as well as with respect to its lattice operations (Example 1 below).

We also explore extending the domain of words (or "streams", cf. [O'N97]) with an operation of addition. We show that a natural operation can be defined on this domain, which on undefined elements yields undefined and for which the domain of words forms a quasi-metric semigroup.

Our interest lies in modeling both the domain of words and the complexity space as weightable invariant meet semigroups in such a way that the semigroup operation naturally given to the corresponding "support" set (the "alphabet" respectively and the interval $(0, \infty]$, respectively) extend to the space.

We also discuss the example of the interval domain, which provides another example of a quasi-metric semigroup.

Example 1 Let Σ be a nonempty alphabet. Let Σ^∞ be the set of all finite and infinite sequences ("words") over Σ, where we adopt the convention that the empty sequence ϕ is an element of Σ^∞.

Denote by $\sqsubseteq$ the prefix order on Σ^∞, i.e. $x \sqsubseteq y \Leftrightarrow x$ is a prefix of y.

Then $(\Sigma^\infty, \sqsubseteq)$ is an algebraic complete partial order which is a Scott domain if Σ is countable (see [Smy91] Example 2.2).

Now, if for each $x, y \in \Sigma^\infty$ we define $x \sqcap y$ as the longest common prefix of x and y.

Now, for each $x \in \Sigma^\infty$ denote by $\ell(x)$ the length of x. Then $\ell(x) \in [1, \omega]$ whenever $x \neq \phi$ and $\ell(\phi) = 0$. For each $x, y \in \Sigma^\infty$ define $\ell(x, y) = \ell(x \sqcap y)$.

Thus, the function d defined on $\Sigma^\infty \times \Sigma^\infty$ by

$$d(x, y) = 2^{-\ell(x,y)} - 2^{-\ell(x)}$$

is a quasi-metric on Σ^∞ and (Σ^∞, d) is a weightable quasi-metric space by the (fading) weighting function w defined on Σ^∞ by $w(x) = 2^{-\ell(x)}$ for all $x \in \Sigma^\infty$ (see [Kün93] Example 8). (We adopt the convention that $2^{-\omega} = 0$.)

Furthermore, the associated order $\leq_d$ of d coincides with the prefix order $\sqsubseteq$ on Σ^∞.

(Σ^∞, d) is clearly a meet semilattice. Since for each $x, y \in \Sigma^\infty$ we have $d(x, x \sqcap y) = d(x, y)$, it follows from Lemma 4 that (Σ^∞, d) is invariant.

Now suppose that there exists an operation $+$ on Σ for which $(\Sigma, +)$ is an (Abelian) semigroup.

We shall prove that, then, Σ^∞ can be endowed with the structure of an (Abelian) semigroup $(\Sigma^\infty, \oplus)$ such that d is $\oplus$-invariant and thus $(\Sigma^\infty, d, \oplus)$ is a weightable invariant meet semigroup.

For each $x, y \in \Sigma^\infty \setminus \phi$, we define $x \oplus y$ as the element of Σ^∞ of length $\ell(x \oplus y) = \min\{\ell(x), \ell(y)\}$ such that for each $k \leq \ell(x \oplus y)$, $(x \oplus y)(k) = x(k) + y(k)$.

It is straightforward to show that for each $x, y, z \in \Sigma^\infty$, $(x \oplus y) \oplus z = x \oplus (y \oplus z)$, and that $x \oplus y = y \oplus x$ if $(\Sigma, +)$ is an Abelian semigroup.

Therefore, $(\Sigma^\infty, \oplus)$ is an (Abelian) semigroup.

Next we prove that the quasi-metric d is $\oplus$-invariant.

Let $x, y, z \in \Sigma^\infty$. If $\ell(z) \leq \ell(x, y)$, then $x \oplus z = y \oplus z$, whence $d(x \oplus z, y \oplus z) = 0 \leq d(x, y)$. If $\ell(z) > \ell(x, y)$, then $\ell(x \oplus z, y \oplus z) \geq \ell(x, y)$. Together with $\ell(x \oplus z) \leq \ell(x)$, this implies $d(x \oplus z, y \oplus z) = 2^{-\ell(x \oplus z, y \oplus z)} - 2^{-\ell(x \oplus z)} \leq 2^{-\ell(x,y)} - 2^{-\ell(x)} = d(x, y)$. The proof of $d(z \oplus x, z \oplus y) \leq d(x, y)$ is analogous.

We conclude that d is $\oplus$-invariant and, consequently, $(\Sigma^\infty, d, \oplus)$ is a weightable invariant meet semigroup.

The empty sequence is of course not a neutral element for the addition but rather an element of absorption. Indeed, from the definition of $\oplus$ it follows that $x \oplus \phi = \phi = \phi \oplus x$.

One can motivate the summation as follows. If we interprete a finite list as an infinite list of which only finitely many elements are defined, then the sum makes sense: adding a defined value to an undefined one should give undefined.

We can not simply choose ϕ as neutral element in our context as shown by the following counter example:

Choose Σ to be the set of integers $\mathcal{Z}$ with the ordinary addition. Let $[a]$ denote the sequence with length 1 and with a as its only element. If ϕ were the neutral element of $\oplus$ then $d([1] \oplus \phi, [1] \oplus [1])$ would be $d([1], [2])$ and d would not be $\oplus$ invariant.

Example 2 As a second example of a weightable invariant join semigroup, we mention the complexity space of [Sch95] (cf. also [RS99], [RS98] and [RS98a]). We leave the verifications to the reader.

Example 3 The interval domain $(I(\mathcal{R}), p)$ consisting of the closed intervals of the reals, ordered by reverse inclusion and equipped with the partial metric p (see [O'N98] and [Hec96]) defined by:

$$p([a,b],[c,d]) = \max\{b,d\} - \min\{a,c\}.$$

One can easily verify that the associated weighted quasi-metric space $(I(\mathcal{R}), d_p)$ is a quasi-metric meet semilattice, where the meet is defined as follows: $[a,b] \sqcap [c,d] = [min\{a,c\}, max\{b,d\}]$.

Hence we obtain an example of a weightable invariant quasi-metric meet semigroup.

In the following we focus on weightable invariant (quasi-metric) meet semigroups. The results generalize in a straightforward way to weightable invariant (quasi-metric) join semigroups.

In the sequel by an *ordered meet semigroup* we mean an ordered semigroup $(X, \sqsubseteq, \star)$ such that $(X, \sqsubseteq)$ is a meet semilattice.

For a nonnegative real valued function f on an ordered meet semigroup $(X, \sqsubseteq, \star)$ consider the following condition

$$(*) \qquad f(a \star b \sqcap x \star y) - f(a \star b) \le f(a \sqcap x) + f(b \sqcap y) - f(a) - f(b).$$

If $f : (X, \sqsubseteq, \star) \to \mathcal{R}_0^+$ is a function on $(X, \sqsubseteq, \star)$ which is a meet semigroup and f is a fading strictly decreasing meet co-valuation satisfying $(*)$ then $(X, \sqsubseteq, \star)$ is an ordered meet semigroup.

Indeed, note that if $a \le x$ and $b \le y$ then $f(a \sqcap x) = f(a)$ and $f(b \sqcap y) = f(b)$, whence by (*) $f((a*b) \sqcap (x*y)) - f(a*b) = 0$. Since $(a*b) \sqcap (x*y)) \le a*b$ and f is strictly decreasing, $(a*b) \sqcap (x*y) = a*b$ follows and thus $a*b \le x*y$.

The following theorem extends Theorem 11 of [Sch00] to the context of quasi-metric semigroups. As such, the first part of the theorem repeats Theorem 11 of [Sch00], while the second part discusses quasi-metric semigroups. For the sake of completeness we provide the entire proof for the theorem.

Theorem 3.4 *For every meet semilattice $(X, \preceq)$, there exists a bijection between invariant co-weighted quasi-metrics d on X with $\le_d = \preceq$ and fading strictly increasing meet valuations $f\colon (X, \preceq) \to (\mathcal{R}_0^+, \le)$. The map $f \mapsto d_f$ is defined by $d_f(x,y) = f(x) - f(x \sqcap y)$. The inverse is the function which to each weighted space (X,d) associates its unique fading co-weighting. Similarly, one can show that for every meet semilattice $(X, \preceq)$, there exists a bijection between invariant weighted quasi-metrics d on X with $\le_d = \preceq$ and fading strictly decreasing meet valuations $f\colon (X, \preceq) \to (\mathcal{R}_0^+, \le)$. The map $f \mapsto d_f$ is defined by $d_f(x,y) = f(x \sqcap y) - f(x)$. The inverse is the function which to each* *weighted space* (X, d) *associates its unique fading weighting. Moreover, on a meet semigroup* $(X, \preceq, \star)$, *the corresponding* d *and* f *described above, interact with* $\star$ *as follows:* $(X, d, \star)$ *is a quasi-metric semigroup* $\Leftrightarrow$ f *satisfies property* $(*)$.

Proof. Let $f : (X, \preceq, \star) \to \mathcal{R}_0^+$ be a function on $(X, \preceq, \star)$ which is a meet semigroup and let f be a fading strictly decreasing meet co-valuation satisfying $(*)$. Then $(X, \preceq, \star)$ is an ordered meet semigroup such that f is a fading strictly decreasing meet co-valuation on $(X, \preceq, \star)$ satisfying $(*)$. Define the function d_f on $X \times X$ by

$$d_f(x, y) = f(x \sqcap y) - f(x)$$

for all $x, y \in X$.

We first show that d_f is a quasi-metric on X :

Indeed, since f is decreasing, $d_f(x, y) \geq 0$ for all $x, y \in X$. On the other hand, if $d(x, y) = d(y, x) = 0$, then $f(x \sqcap y) = f(x) = f(y) = 0$. Since f is strictly decreasing, we deduce that $x \sqcap y = x = y$. Next we show that d_f satisfies the triangle inequality. Let $x, y, z \in X$. Since f is a meet co-valuation,

$$f(x \sqcap z) - f(x) \leq f(x \sqcap y) - f(x) + f(y \sqcap z) - f(y)$$

so,

$$d_f(x, z) \leq d_f(x, y) + d_f(y, z).$$

We have shown that d_f is a quasi-metric on X.

Moreover, f is a weighting for d_f since for each $x, y \in X$:

$$d_f(x, y) + f(x) = f(x \sqcap y) = d_f(y, x) + f(y).$$

Next we show that the order $\leq_{d_f}$ coincides with $\preceq$. Indeed, for $x, y \in X$, one has:

$$x \leq_d y \Leftrightarrow d_f(x, y) = 0 \Leftrightarrow f(x \sqcap y) = f(x) \Leftrightarrow x \sqcap y = x \Leftrightarrow x \preceq y.$$

Hence (X, d_f) is a (quasi-metric) meet semilattice. On the other hand, for $x, y \in X$:

$$d_f(x, x \sqcap y) = f(x \sqcap (x \sqcap y)) - f(x) = f(x \sqcap y) - f(x) = d_f(x, y).$$

So, by Lemma 4, (X, d_f) is invariant. Therefore, (X, d_f) is an invariant meet semilattice. It remains to show that $(X, d, \star)$ is a quasi-metric semigroup. Indeed, let $a, b, x, y \in X$. By condition $(*)$,

$$d(a \star b, x \star y) = f(a \star b \sqcap x \star y) - f(a \star b) \leq$$
$$f(a \sqcap x) + f(b \sqcap y) - f(a) - f(b) = d(a, x) + d(b, y).$$

We conclude that $(X, d_f, \star)$ is a quasi-metric semigroup.

Conversely, let $(X, d, \star)$ be a weightable invariant meet semigroup. By Proposition 6, (X, d) has a unique fading weighting function w. We shall show that w is a strictly decreasing meet co-valuation for $(X, \leq_d, \star)$ satisfying condition $(*)$ (recall that, by assumption, $(X, \leq_d, \star)$ is an ordered meet semigroup).

Suppose $x <_d y$. Then $d(x, y) = 0$ and $d(y, x) > 0$. Since $d(x, y) + w(x) = d(y, x) + w(y)$, we deduce that $w(x) > w(y)$. So, w is strictly decreasing.

Next we show that w is a meet co-valuation. Indeed, let $x, y, z \in X$. Then

$$\begin{aligned} w(y \sqcap z) - w(x \sqcap z) &= d(x \sqcap z, y \sqcap z) - d(y \sqcap z, x \sqcap z) \\ &\geq d(x \sqcap z, y \sqcap z) - d(y, x) \end{aligned}$$

Since,

$$\begin{aligned} d(y, x) + w(y) &= d(y, x \sqcap y) + w(y) \\ &= d(x \sqcap y, y) + w(x \sqcap y) = w(x \sqcap y) \end{aligned}$$

we deduce that

$$\begin{aligned} w(y \sqcap z) - w(x \sqcap z) &\geq d(x \sqcap z, y \sqcap z) + w(y) - w(x \sqcap y) \\ &\geq w(y) - w(x \sqcap y). \end{aligned}$$

Now we show that w satisfies $(*)$. Indeed, let $a, b, x, y \in X$. Then

$$\begin{aligned} & w((a \star b) \sqcap (x \star y)) - w(a \star b) \\ &= d(a \star b, (a \star b) \sqcap (x \star y)) - d((a \star b) \sqcap (x \star y), a \star b) \\ &= d(a \star b, x \star y) \\ &\leq d((a, x) + d(b, y) \\ &= d(a, a \sqcap x) + d(b, b \sqcap y) \\ &= w(a \sqcap x) - w(a) + w(b \sqcap y) - w(b). \end{aligned}$$

Here we used inequality (1) in deducing the inequality.

Finally, we need to verify that the correspondence obtained above is a bijection, i.e. that the maps $f \to d_f$ and $d \to f_d$ are inverse to each other.

a) To verify that $d_{f_d} = d$, we remark that $d(x, y) = d(x \sqcup y, y) = f_d(y) - f_d(x \sqcup y) = d_{f_d}(x, y)$, where the last equality is the definition of d_{f_d} and the rest was already shown in the second part of the above proof.

b) To verify that $f_{d_f} = f$, we remark that by the first part of the above proof, f is a weighting of d_f. By the uniqueness of fading weightings, $f = f_{d_f}$.

References

[Bir84] G. Birkhoff, *Lattice Theory*, AMS Colloquium Publications **25**, Providence, Rhode Island, 1984.

[BS97] M. A. Bukatin, J. S. Scott, Towards computing distances between programs via Scott domains, in: S. Adian, A. Nerode, eds., Logical Foundations of Computer Science, *Lecture Notes in Computer Science*, **1234**, 33-43, Springer, 1997.

[BSh98] M. A. Bukatin, S. Y. Shorina, Partial metrics and Co-continuous Valuations, in M. Nivat, ed., Foundations of Software Science and

Computation Structures, *Lecture Notes in Computer Science* **1378**, 125-139, Springer-Verlag, 1998.

[CD73] P. Crawley, R. P. Dilworth, *Algebraic Theory of Lattices*, Prentice-Hall, Inc., 1973.

[EEP97] A. Edalat, M. H. Escardo, P. J. Potts, Semantics of Exact Real Arithmetic, Proceedins of 12th annual IEEE symposium on Logic in Computer Science, IEEE, 1997, 248-257.

[FL82] P. Fletcher, W. Lindgren, *Quasi-uniform spaces,* Marcel Dekker, Inc., NY, 1982.

[Hec94] R. Heckmann, Spaces of valuations, in: *Proc. 11th Summer Conference on General Topology and Applications.* Ann. New York Acad. Sci. **806**, 1996, 174 - 200.

[Hec96] R. Heckmann, Approximation of Metric Spaces by Partial Metric Spaces, *Applied Categorical Structures* **7**, 1999, 71-83.

[JP89] C. Jones, G. Plotkin, A probabilistic powerdomain of evaluations. In: *LICS '89*, IEEE Computer Society Press, 186 - 195.

[Jon89] C. Jones, Probabilistic Non-determinism, PhD-thesis, University of Edinburgh, 1989.

[Ko82] R. Kopperman, Lengths on semigroups and groups, Semigroup Forum Vol. **25**, 1982, 345 - 360.

[Kün93] H. P. Künzi, Nonsymmetric topology, in: *Proc. Szekszárd Conference,* Bolyai Soc. Math. Studies **4**, 1993, 303-338.

[KV94] H. P. Künzi, V. Vajner, Weighted quasi-metrics, in: *Proc. 8th Summer Conference on General Topology and Applications.* Ann. New York Acad. Sci., **728**, 1994, 64-77.

[Law95] J. D. Lawson, Valuations on continuous lattices. In: Continuous Lattices and Related Topics (R. E. Hoffman, Ed.), *Mathematik Arbeitspapiere* **27**. Universität Bremen.

[Mat94] S. G. Matthews, Partial metric topology, in: *Proc. 8th Summer Conference on General Topology and Applications.* Ann. New York Acad. Sci. **728**, 1994, 183-197.

[Mat95] S. G. Matthews, An extensional treatment of lazy data flow deadlock, *Theoretical Computer Science* **151**, 195 - 205, 1995.

[O'N97] S. J. O'Neill, Partial metrics, valuations and domain theory. In S. Andima et al., eds., Proc. 11th Summer Conference on General Topology and Applications, *Annals New York Academy of Sciences*, **806**, 304-315, New York, 1997.

[O'N98] S. J. O'Neill, A Fundamental Study in to the Theory and Application of the Partial Metric Spaces, PhD Thesis, University of Warwick, 1998.

[RS99] S. Romaguera, M. Schellekens, Quasi-metric properties of Complexity Spaces, *Topology and its Applications*, **98**, 311-322, 1999.

[RS98] S. Romaguera, M. Schellekens, Duality and quasi-normability for complexity spaces, *Applied General Topology*, to appear.

[RS98a] S. Romaguera, M. Schellekens, A characterization of norm-weightable Riesz spaces, preprint.

[Sch95] M. P. Schellekens, The Smyth Completion: A Common Foundation for Denotational Semantics and Complexity Analysis, in: *proc. MFPS 11*, Electronic Notes in Theoretical Computer Science, Vol. I, Elsevier, 1995, 211-232.

[Sch96] M. P. Schellekens, On upper weightable spaces, in: *Proc. 11th Summer Conference on General Topology and Applications.* Ann. New York Acad. Sci. **806**, 1996, 348-363.

[Sch96a] M. P. Schellekens, Complexity Spaces Revisited, extended abstract in: Proc. 8th Prague Topology Conference, Topology Atlas, 1996.

[Sch99] M. P. Schellekens, Complexity Spaces: Lifting & Directedness, *Topology Proceedings* **22**, 403 - 425, 1999.

[Sch00] M. P. Schellekens, The correspondence between partial metrics and semivaluations, *Theoretical Computer Science, to appear.*

[Sch00a] M. P. Schellekens, A characterization of partial metrizability. Domains are quantifiable, *preprint*, 2000.

[Smy91] M. B. Smyth, Totally bounded spaces and compact ordered spaces as domains of computation, in G. M. Reed, A. W. Roscoe and R. F. Wachter, editors, *Topology and Category Theory in Computer Science*, 207-229, Oxford University Press, 1991.

[Web91] H. Weber, Uniform Lattices I: A generalization of topological Riesz spaces and topological Boolean Rings. *Ann. Mat. Pura Appl.* **160**, 1991, 347 - 370.

[Web93] H. Weber, Uniform lattices II: Order continuity and exhaustivity, *Ann. Mat. Pura Appl.* (IV) **165**, 133-158, 1993.