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Reductions and Saturation Reductions of (Abstract) Knowledge Bases

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Abstract

Rough set theory is a useful tool for dealing with fuzzyness and uncertainty of knowledge. In rough set theory, knowledge reductions and generatings are important research topics and critical steps of knowledge acquisition. This paper generalize knowledge bases to abstract knowledge bases and study (abstract) knowledge bases on infinite universe by considering the problem of existence of finite reductions of infinite knowledge bases. For abstract knowledge bases, the concept of saturations and saturation reductions are introduced. Global properties of saturations and saturation reductions of abstract knowledge bases are investigated. It is proved that for a given abstract knowledge base which is closed w.r.t. arbitrary unions on a finite universe U, its saturation augmented the unverse U forms a topology, whereas a counterexample is constructed to show that this may not be true if U is infinite. Making use of the saturation of an abstract knowledge base, some sufficient and/or necessary conditions for existence of finite reductions of an infinite abstract knowledge base are given. It is proved that for an abstract knowledge base on finite universe, there is one and only one saturation reduction. Some examples are constructed to reveal various cases of existence of knowledge reductions. Simple applications of saturation reductions are also given.

Keywords: semilattice; poset; topology; abstract knowledge base; saturation reduction; existence; core

1 Introduction

Rough set theory [8,13,14] is an important tool for dealing with fuzzyness and uncertainty of knowledge, and has become an active branch of information sciences. Meanwhile it has been successfully applied in medical science, material science, management science and so on. Basic opinion in rough set theory is that knowledge (human intelligence) is the ability to classify elements [5,7,13]. Also, one can say that knowledge is a family of classification patterns in some interesting fields,

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providing us some facts from which one can deduce new ones [6,10,12]. Early rough set [8,13] theory mainly consider equivalence relations on U which are subsets of $U \times U$ and determine partitions on U. One deals with not only a single classification (knowledge or partition) on U, but also a family of classifications [5,6]. This leads to the definition of a knowledge base. Specifically, given a universe U and a family of equivalence relations on U, the pair $K = (U, \mathcal{P})$ is called a knowledge base.

As the developing of rough set theory, one considers more general families of sets, such as the lower and upper approximation families determined by binary relations, image families of some operators (called rough operators) obtained by axiomatic methods. These kinds of families often are closed w.r.t. unions or intersections. So one can consider very general families of sets as abstract knowledge bases. Mathematically, let $U \neq \emptyset$ be a set of elements we are interested in, called a universe, any subset $X \subseteq U$ is called a concept or knowledge on U. Every concept family (the family of subsets on U) is called an abstract knowledge base on U.

On one hand, one can derive more knowledge from a given knowledge base. For example, from "tall", "played basketball in NBA" and "a Chinese man born in Shanghai" one can figure out by taking intersections of suitable sets that the man may be Ming Yao. Mathematically we can derive new families of sets from a given family. This paper considers the new abstract knowledge base formed by nonempty finite intersections of an abstract knowledge base, called the saturation (in Topology, an intersection of open sets is called a saturated set, hence we give the name). Given a family $\mathcal{P}$ on U, the family $\mathcal{P}^* = \{\cap\mathcal{F} \mid \emptyset \neq \mathcal{F} \subseteq \mathcal{P}, \mathcal{F} \text{ is finite}\}$ is called the saturation of $\mathcal{P}$, where finite intersections is nonempty intersections. Properties of the saturation will be discussed in this paper.

On the other hand, it is well-know that elements in a knowledge base are not of the same importance, some even are redundant. So we often consider reductions of a knowledge base by deleting unrelated or unimportant elements with the requirement of keeping the ability of classification. In classic rough set theory, the universe one deals with normally is a nonempty finite set. In this case a knowledge base is finite and reductions always exist. For infinite universe, this is not the case. This paper will study (abstract) knowledge bases on infinite universe by considering the problem of existence of finite reductions of infinite knowledge bases. Some sufficient and/or necessary conditions for existence of finite reductions of an abstract knowledge base are given. Some examples are constructed to reveal various cases of existence of knowledge reductions.

Since usual reductions of a knowledge base K involve only the special intersection $\mathrm{ind}K = \cap_{R \in K} R$, and since reductions on a finite universe exist but generally not unique, usual knowledge reductions lose much information and bring new uncertainty. In view of this, we will define a new kind of reductions of abstract knowledge bases, called saturation reductions. It turns out that saturation reductions of an abstract knowledge base not only have special significance in dealing with information of knowledge bases, but also have their own rights to research. We also give examples in the last section to show possible applications with topology and ordered structures.

2 Preliminaries

We give some basic concepts and results which will be used in the sequel. Most of them come from [4,11,13]. For other unstated concepts please refer to [2,3].

Definition 2.1 *Let $(X, \mathcal{T})$ be a topological space, $\mathcal{B} \subseteq \mathcal{T}$. If every open set of X is a union of some elements of $\mathcal{B}$, that is for each $U \in \mathcal{T}$, there exists $\mathcal{B}_1 \subseteq \mathcal{B}$ such that $U = \cup_{B \in \mathcal{B}_1} B$, then we call $\mathcal{B}$ a base of $\mathcal{T}$, or a base of topological space X.*

Lemma 2.2 (see [11, Th.2.6.3]) *Let $\mathcal{B}$ be a family of subsets of X. If $\mathcal{B}$ is closed w.r.t. finite intersections (include empty intersection), then there is a unique topology on X having $\mathcal{B}$ as a base.*

Let $(X, \mathcal{T})$ be a topological space, $\mathcal{F}$ the closed sets of X. Since complements of open sets are closed sets, $\mathcal{F}$ is closed w.r.t. arbitrary intersections and finite unions. If $\mathcal{B}$ is a base of $\mathcal{T}$, then $\mathcal{B}^c = \{X - B | B \in \mathcal{B}\} \subseteq \mathcal{F}$ is called a closed base. In this case, each element of $\mathcal{F}$ can be expressed as an intersection of some elements of $\mathcal{B}^c$.

Intervals in the real line have 9 classes: open intervals (a, b), $(a, +\infty)$, $(-\infty, b)$, $(-\infty, +\infty)$; closed intervals $[a, b]$; half open and half closed intervals $[a, b)$, $(a, b]$, $[a, +\infty)$ and $(-\infty, b]$. We will not take a singleton as an interval.

Definition 2.3 *Let G be an open set of the real line. If an open interval $(a, b) \subseteq G$ with endpoints a, $b \notin G$, then (a, b) is called a structure interval of G.*

Lemma 2.4 (See [1, Th.1 (open set structure theorem)]) *Each nonempty open set of the real line can be expressed as a union of finite or countable mutually disjoint structure intervals. If an open set is expressed as a union of mutually disjoint open intervals, then these intervals must be the structure intervals of the open set.*

The following two lemmas are easy to prove and will be used in the sequel.

Lemma 2.5 *Let $U = \mathbb{R}$ be the real line. If $X \subseteq U$ is a union of some intervals, then X can be expressed as a union of some mutually disjoint intervals.*

Lemma 2.6 *If $\{A_i\}_{i \in I}$ is a family of mutually disjoint intervals of real line, then $\{A_i\}_{i \in I}$ is a countable family.*

Definition 2.7 (1) *Let $U \neq \emptyset$ be a set and $\mathcal{P} \neq \emptyset$ a family of equivalence relations on U. Then the pair $K = (U, \mathcal{P})$ is called* a knowledge base*, sometimes we say $\mathcal{P}$ a knowledge base, and U the universe of K or $\mathcal{P}$. Set $ind(\mathcal{P}) = \cap_{R \in \mathcal{P}} R$, then $ind(\mathcal{P})$ is still an equivalence relation on U, and is called the indiscernible relation of K.*

(2) *Let $\mathcal{P} \neq \emptyset$ be a family of subsets of U. Then $\mathcal{P}$ is called* an abstract knowledge base*, U is called the universe of $\mathcal{P}$. Set $ind(\mathcal{P}) = \cap_{A \in \mathcal{P}} A$, then $ind(\mathcal{P})$ is a subset of U, called the indiscernible set of $\mathcal{P}$.*

(3) *Say $A \in \mathcal{P}$ to be not necessary if $ind(\mathcal{P}) = ind(\mathcal{P} - \{A\})$. Otherwise, A is said to be* necessary*. The set $core(\mathcal{P}) = \{A \in \mathcal{P} | A$ is necessary$\}$ is called the core of $\mathcal{P}$. If every element in $\mathcal{P}$ is necessary, then we say that $\mathcal{P}$ is* independent.

It is easy to see that any knowledge base $\mathcal{P}$ in the sense of Definition 2.7(1) can be viewed as an abstract knowledge base $\mathcal{P}$ on $U \times U$ by taking $R \in \mathcal{P}$ as a subset of $U \times U$. So, a knowledge base is a special abstract knowledge base.

Let L be a poset. A subset $D \subseteq L$ is *directed* if each finite subset of D has an upper bound in D. A poset is called a *directed complete poset* (briefly, *dcpo*) if each directed subset has a supremum. A poset is said to be a *semilattice* if every pair of elements of L has an infimum. If $(L, \leqslant)$ is a poset, then $(L, \geqslant)$ is a poset, called the dual poset of L, denoted by L^{op}. If $\forall a, b \in L$, $a \leqslant b$ or $b \leqslant a$ holds, then "$\leqslant$" is said a linear order, and L is called a *total-ordered set or a chain.* If $\forall a, b \in L$, neither $a \leqslant b$ nor $b \leqslant a$ holds, then L is called an *anti-chain.* A nonempty subset $X \subseteq L$ is said to be *filtered* if every pair of elements a, $b \in X$, there is $c \in X$ such that $c \leqslant a$ and $c \leqslant b$. It is easy to see that every semilattice itself is filtered.

Definition 2.8 *Let P be a poset, x, $y \in P$. We say that x* approximates *y, written $x \ll y$, if whenever D is directed with $\sup D \geqslant y$, then $x \leqslant d$ for some $d \in D$. If $x \ll x$, then we say x a compact element.*

3 Saturations of Abstract Knowledge Bases

In this section we introduce the concept of saturations of abstract knowledge bases and investigate their properties.

Definition 3.1 *Let $\mathcal{P}$ be an abstract knowledge base. Then $\mathcal{P}^*$ is called the saturation of $\mathcal{P}$ if $\mathcal{P}^*$ consists of all the nonempty finite intersections of $\mathcal{P}$. If $\mathcal{P} = \mathcal{P}^*$, then $\mathcal{P}$ is called saturated.*

Notice that generally $\mathcal{P} \subseteq \mathcal{P}^*$. However, even if $(\mathcal{P}, \subseteq)$ is a semilattice, one can easily construct examples with $\mathcal{P} \neq \mathcal{P}^*$. If an abstract knowledge base is a semilattice, then itself must be filtered and $ind\mathcal{P} = ind\mathcal{P}^*$. Moreover, saturations are closed w.r.t. nonempty finite intersections. So, by Lemma 2.2, $\mathcal{P}^* \cup \{U\}$ is a base of some topology on U. Further more, we have the following theorem.

Theorem 3.2 *Let U be a given nonempty finite set, $\mathcal{P}$ a family of subsets on U which is closed w.r.t. arbitrary unions. Then $\mathcal{P}^* \cup \{U\}$ is a topology on U.*

Proof. Since U is finite, any topology on U must be closed w.r.t. arbitrary unions and intersections. Thus, the family $\mathcal{F}$ of closed sets of a topology $\mathcal{T}$ on U is also closed w.r.t. arbitrary unions and intersections. Note that $\mathcal{P}$ is closed w.r.t. arbitrary unions, we see that the family $\mathcal{B} = \{U - X | X \in \mathcal{P}\}$ is closed w.r.t. arbitrary intersections. By Lemma 2.2, there is a unique topology $\mathcal{T}$ on U having $\mathcal{B}$ as a base. In this case, the family $\mathcal{F}$ of closed sets of $\mathcal{T}$ is exactly the family $\mathcal{P}^* \cup \{U\}$, that is, $\mathcal{F} = \mathcal{P}^* \cup \{U\}$. So, $\mathcal{P}^* \cup \{U\}$ is closed w.r.t. arbitrary unions and intersections. Thus, $\mathcal{P}^* \cup \{U\}$ is a topology on U. □

For an infinite universe U, even if $\mathcal{P}$ is closed w.r.t. arbitrary unions, $\mathcal{P}^* \cup \{U\}$ may not be a topology. See the following example.

Example 3.3 *Let $U = \mathbb{R}$ be the reals, $\mathcal{P}$ the family of subsets expressed as unions of some mutually disjoint intervals of $\mathbb{R}$. Then empty union is empty and $\emptyset \in \mathcal{P}$. Noticing that unions of some unions of intervals are still unions of intervals* *and by Lemma 2.5, these unions of intervals can also be expressed as unions of some mutually disjoint intervals, we have that $\mathcal{P}$ is closed w.r.t. arbitrary unions. However, $\mathcal{P}^* \cup \{U\}$ is not a topology.*

To see this, let $Y = \cap \cup Q_{i,j}$ be a nonempty finite intersection of elements in $\mathcal{P}$, where for $i = 1, 2, \cdots, n$, $Q_{i,j} (j \in J)$ is a family of mutually disjoint intervals. By Lemma 2.6, J is a countable set. By the completely distributive law, we have that $Y = \cup_{f \in M} (\cap Q_{i,f(i)})$, where $M = \{f | f : \{1, \cdots, n\} \longrightarrow J$ is a map$\}$. As a finite product of the countable set J, M is also a countable set. It is clear that the finite intersection $\cap Q_{i,f(i)}$ is either $\emptyset$, or a singleton, or an interval. If cardinalities $|\cap Q_{i,f(i)}| \leq 1$ for all $i \in J$, then Y is a countable set. If there is an $i \in J$ such that $\cap Q_{i,f(i)}$ is an interval, then Y has a nonempty interior. So, any finite intersection Y of elements in $\mathcal{P}$ is either countable, or has a nonempty interior.

Every irrational singleton is a finite intersection $\{\xi\} = (-\infty, \xi] \cap [\xi, +\infty)$ in $\mathcal{P}$ and thus is in $\mathcal{P}^*$. So, the uncountable set of all irrationals is a union of elements of $\mathcal{P}^*$. Thus, the set of all irrationals is not in $\mathcal{P}^* \cup \{U\}$, for it clearly has empty interior. That is to say, $\mathcal{P}^* \cup \{U\}$ is not closed w.r.t. unions, let along a topology.

Definition 3.4 *Let $\mathcal{P}$ be an abstract knowledge base. If $\forall P \in \mathcal{P}$, $(\mathcal{P} - \{P\})^* \neq \mathcal{P}^*$, then $\mathcal{P}$ is said to be minimally saturated.*

By Definition 3.4, it is easy to show the following

Proposition 3.5 *If no element $A \in \mathcal{P}$ can be expressed as a finite intersection of elements in $\mathcal{P} - \{A\}$, then $\mathcal{P}$ is minimally saturated.*

Example 3.6 *A chain or an anti-chain $\mathcal{P}$ is saturated and minimally saturated.*

4 Reductions of Infinite Abstract Knowledge Bases

This section will give existence conditions of finite reductions of infinite abstract knowledge bases. We have first the following definition.

Definition 4.1 *Let $\mathcal{P}$ be an abstract knowledge base on a universe U, $\mathcal{Q} \subseteq \mathcal{P}$. We say $\mathcal{Q}$ a* reduction *of $\mathcal{P}$ if $ind(\mathcal{Q}) = ind(\mathcal{P})$ and $\forall A \in \mathcal{Q}$, $ind(\mathcal{Q}) \neq ind(\mathcal{Q} - \{A\})$. Sometimes, we also say that $\mathcal{Q}$ is a* reduction *of $K = (U, \mathcal{P})$.*

Definition 4.2 *For an abstract knowledge base $\mathcal{P}$ on U, if $\mathcal{P}$ is finite, then $\mathcal{P}$ is called* a finite knowledge base*; if $\mathcal{P}_0 \subseteq \mathcal{P}$ is a reduction of $\mathcal{P}$ and has only finite elements, then $\mathcal{P}_0$ is called a* finite reduction *of $\mathcal{P}$.*

Lemma 4.3 (see [13, Theorem 1.9]) *A (finite) reduction always exists for any knowledge base on a finite universe.*

Corollary 4.4 *Every finite knowledge base on an infinite universe has (finite) reductions. If $\mathcal{P}$ has the least element, then $\mathcal{P}$ has a finite reduction.*

Let $\mathcal{P}$ be an abstract knowledge base. If $(\mathcal{P}, \subseteq)$ is a chain (resp., an anti-chain, a filtered set, a semilattice), then we say that $\mathcal{P}$ is a *chain*(resp., an anti-chain, a filtered set, a semilattice).

The following three propositions are not difficult, and their proofs are omitted.

Proposition 4.5 *If $\mathcal{P}$ is a chain, then $\mathcal{P}$ has a finite reduction if and only if $\mathcal{P}$ has the least element.*

Proposition 4.6 *If $\mathcal{P}$ is filtered, then $\mathcal{P}$ has a finite reduction iff $\mathcal{P}$ has the least element. Particularly, if $\mathcal{P}$ is a semilattice, then $\mathcal{P}$ has a finite reduction iff $\mathcal{P}$ has the least element.*

Proposition 4.7 *If $\mathcal{P}$ is an abstract knowledge base and $\mathcal{P}^*$ is the saturation of $\mathcal{P}$, then $\mathcal{P}$ has a finite reduction iff $\mathcal{P}^*$ has a finite reduction.*

Theorem 4.8 *If $\mathcal{P}$ is an abstract knowledge base, then $\mathcal{P}$ has a finite reduction iff the saturation $\mathcal{P}^*$ of $\mathcal{P}$ has the least element.*

Proof. $\Rightarrow$: Let $\mathcal{P}_0$ be a finite reduction of $\mathcal{P}$, $\mathcal{P}^*$ the saturation of $\mathcal{P}$ and $\mathcal{P}_0^*$ the saturation of $\mathcal{P}_0$. Then $\mathcal{P}_0^* \subseteq \mathcal{P}^*$, and by Proposition 4.6, we conclude that $\mathcal{P}_0^*$ has the least element $R_0 \in \mathcal{P}_0^*$. So, $ind\mathcal{P} = ind\mathcal{P}^* \subseteq ind\mathcal{P}_0^* = ind\mathcal{P}_0 = ind\mathcal{P}$. Thus $R_0 = ind\mathcal{P}_0^* = ind\mathcal{P}_0 = ind\mathcal{P}^*$ and R_0 is the least element of $\mathcal{P}^*$. We conclude that $\mathcal{P}^*$ has a finite reduction $\{R_0\}$.

$\Leftarrow$: Let $\mathcal{P}^*$ be the saturation of $\mathcal{P}$ with a finite reduction. Then $(\mathcal{P}^*, \subseteq)$ is a semilattice. By Proposition 4.6, we conclude that $\mathcal{P}^*$ has the least element R_0. So, there are $R_1, \cdots, R_n \in \mathcal{P}$ such that $R_0 = R_1 \cap \cdots \cap R_n$. For the finite knowledge base $\{R_1, \cdots, R_n\}$, by Corollary 4.4, there is a finite reduction $\mathcal{P}_0 \subseteq \{R_1, \cdots, R_n\}$ such that $ind\mathcal{P}_0 = R_1 \cap \cdots \cap R_n = ind\mathcal{P}^* = ind\mathcal{P}$. Since $\mathcal{P}_0 \subseteq \mathcal{P}$ is independent, $\mathcal{P}_0$ is a finite reduction of $\mathcal{P}$. $\square$

Corollary 4.9 *If $\mathcal{P}$ is an abstract knowledge base, then $\mathcal{P}$ has a finite reduction iff there are finite elements $R_1, \cdots, R_n \in \mathcal{P}$ such that $R_1 \cap \cdots \cap R_n = ind\mathcal{P}$.*

Proposition 4.10 *Let $\mathcal{P}$ be an abstract knowledge base with no infinite anti-chains. If every maximal chain in $\mathcal{P}$ has the least element, then $\mathcal{P}$ has a reduction.*

Proof. Since the set of all the least elements of maximal chains in $\mathcal{P}$ forms an anti-chain which by the assumption is finite. The anti-chain has a finite reduction and the intersection of the anti-chain is exactly $ind\mathcal{P}$. Clearly, the finite reduction is also a reduction of $\mathcal{P}$. $\square$

Proposition 4.11 *Let $\mathcal{P}^{op}$ be the dual of an abstract knowledge base $\mathcal{P}$. If in the set inclusion order $\mathcal{P}^{op}$ is a dcpo and max($\mathcal{P}^{op}$) is finite, then $\mathcal{P}$ has a finite reduction.*

Proof. Every maximal chain of $\mathcal{P}$ has an element in $\max(\mathcal{P}^{op})$. This maximal element in $\mathcal{P}^{op}$ is just the least element of maximal chain of $\mathcal{P}$ and thus a minimal element of $\mathcal{P}$. So, we have that $\cap max(\mathcal{P}^{op}) = ind\mathcal{P}$. By the finiteness of $\max(\mathcal{P}^{op})$, we see that $\max(\mathcal{P}^{op})$ has a finite reduction which is also a finite reduction of $\mathcal{P}$. $\square$

Example 4.12 *Let $\mathcal{P}_0$ be an infinite knowledge base which is independent in the sense of Definition 2.7(3). Let $\mathcal{P} = \mathcal{P}_0 \cup \{ind(\mathcal{P}_0)\}$. Then $\{ind(\mathcal{P}_0)\}$ is a finite reduction of $\mathcal{P}$. However, $ind(\mathcal{P}_0)$ is not a compact element of $(\mathcal{P}^*)^{op}$, showing that elements in a finite reduction of $\mathcal{P}$ needn't be compact in $(\mathcal{P}^*)^{op}$.*

Proposition 4.13 *Let $\mathcal{P}$ be an infinite abstract knowledge base. If $ind(\mathcal{P})$ is a compact element in $(\mathcal{P}^*)^{op}$, then $\mathcal{P}$ has and only has finite reductions. In this case,*

$core(\mathcal{P}) = \cap red(\mathcal{P})$.

Proof. Since $ind(\mathcal{P})$ can be expressed as a filtered intersection of finite intersections of elements in $\mathcal{P}$ and $ind(\mathcal{P})$ is compact in $(\mathcal{P}^*)^{op}$, $ind(\mathcal{P})$ is a finite intersection of elements in $\mathcal{P}$. A part of these finite elements forms a finite reduction of $\mathcal{P}$, showing that $\mathcal{P}$ has finite reductions. If $\mathcal{P}$ has another infinite reduction $\mathcal{P}'$, then the intersection of this reduction is $ind(\mathcal{P})$ and can be expressed as a filtered intersection of finite intersections of elements in the reduction $\mathcal{P}'$. Since $ind(\mathcal{P})$ is compact in $(\mathcal{P}^*)^{op}$, $ind(\mathcal{P})$ is a finite intersection of elements in the reduction $\mathcal{P}'$, showing that $\mathcal{P}'$ is not independent, a contradiction. So, $\mathcal{P}$ has no infinite reductions.

Let $P \in \cap red(\mathcal{P})$. If $P \notin core(\mathcal{P})$, then $ind(\mathcal{P} - \{P\}) = ind(\mathcal{P})$. Since $ind(\mathcal{P})$ is compact in $(\mathcal{P}^*)^{op}$, $ind(\mathcal{P})$ is a finite intersection of elements in $\mathcal{P} - \{P\}$. And a part of these finite elements forms a finite reduction without P, a contradiction! So, $\cap red(\mathcal{P}) \subseteq core(\mathcal{P})$. The other direction of inclusion is globally true. Thus, $core(\mathcal{P}) = \cap red(\mathcal{P})$. □

5 Saturation Reduction of Knowledge Bases

This section will give another reduction concept of abstract knowledge bases, called saturation reductions. It turns out that every abstract knowledge base on a finite universe has a unique saturation reduction.

Definition 5.1 *Let $\mathcal{P}$ be an abstract knowledge base, $\mathcal{P}_0 \subseteq \mathcal{P}$. If $\mathcal{P}_0$ is minimally saturated and $\mathcal{P}_0^* = \mathcal{P}^*$, then $\mathcal{P}_0$ is called a saturation reduction of $\mathcal{P}$.*

Obviously, $\mathcal{P}$ is minimally saturated iff $\mathcal{P}$ is a saturation reduction of $\mathcal{P}$.

The following four propositions are not difficult and their proofs are omitted.

Proposition 5.2 *Let $\mathcal{P}_0$, $\mathcal{P}_1$ be saturation reductions of $\mathcal{P}$ and $\mathcal{P}_0 \subseteq \mathcal{P}_1$, then $\mathcal{P}_0 = \mathcal{P}_1$.*

Proposition 5.3 *Let $\mathcal{P}$ be an abstract knowledge base, $\mathcal{P}^*$ the saturation of $\mathcal{P}$. Then saturation reductions of $\mathcal{P}$ are all saturation reductions of $\mathcal{P}^*$.*

Proposition 5.4 *Let $\mathcal{P}$ and $\mathcal{P}'$ be abstract knowledge bases. Let $\mathcal{P}_0$ be a saturation reduction of $\mathcal{P}$ and $max(\mathcal{P})$ the set of all maximal elements of $\mathcal{P}$. Then $max(\mathcal{P}) \subseteq \mathcal{P}_0$. If $\mathcal{P}_0 \subseteq \mathcal{P}' \subseteq \mathcal{P}^*$, then $\mathcal{P}_0$ is a saturation reduction of both $\mathcal{P}'$ and $\mathcal{P}^*$.*

Proposition 5.5 *Let $\mathcal{P}$ be an abstract knowledge base, $\mathcal{P}_1$ the saturation reduction of $\mathcal{P}$. Then for every $P \in \mathcal{P}_1$, there are no finite elements $K_1, \cdots, K_m \in \mathcal{P}_1 \setminus \{P\}$ such that $P = \cap_{i=1}^m K_i$.*

When the universe U is finite, an abstract knowledge base $\mathcal{P}$ on U in the set inclusion order is a finite poset and then a *dcpo*. By the Zorn's Lemma, we know that $max(\mathcal{P}) \neq \emptyset$. With this observation, we can prove the following theorem which reveals the existence of saturation reduction.

Theorem 5.6 *If $\mathcal{P}$ is an abstract knowledge base on a finite universe U, then $\mathcal{P}$ has at least a saturation reduction.*

Proof. Inductively construct a family $K_0, K_1, \cdots, K_n, \cdots$ of subsets of $\mathcal{P}$ such that

$K_0 = max(\mathcal{P}) \subseteq \mathcal{P}$,
$K_1 = max(\mathcal{P} \setminus K_0^*) \subseteq \mathcal{P}$,
$K_2 = max(\mathcal{P} \setminus (K_0 \cup K_1)^*) \subseteq \mathcal{P}$,
$\cdots, \cdots, \cdots,$
$K_{n-1} = max(\mathcal{P} \setminus (K_0 \cup K_1 \cup \cdots \cup K_{n-2})^*) \subseteq \mathcal{P}$,
$K_n = max(\mathcal{P} \setminus (K_0 \cup K_1 \cup \cdots \cup K_{n-1})^*)$,
$\cdots, \cdots, \cdots.$

Noticing that $max(\mathcal{P}) \neq \emptyset$, we know that the family $\mathcal{P} \setminus (K_0 \cup K_1 \cup \cdots \cup K_i)^*$ is strictly decreased. Since U and $\mathcal{P}$ are finite, there is some n such that $\mathcal{P} \setminus (K_0 \cup K_1 \cup \cdots \cup K_{n-1})^* = \emptyset$ and then $K_n = \emptyset$. Set $\mathcal{P}_0 = \cup_{i=0}^n K_i \subseteq \mathcal{P}$. We have that

$$\mathcal{P}_0^* = (\cup K_i)^* = (K_0 \cup K_1 \cup \cdots \cup K_{n-1})^* \supseteq \mathcal{P},$$

$$(K_0 \cup K_1 \cup \cdots \cup K_{n-1})^* \supseteq \mathcal{P}^* \supseteq (K_0 \cup K_1 \cup \cdots \cup K_{n-1})^*.$$

And $\mathcal{P}^* = \mathcal{P}_0^*$.

Let $P \in \mathcal{P}_0$. Then there is $i_0 < n$ such that $P \in K_{i_0}$. We assert that $P \notin (\mathcal{P}_0 \setminus \{P\})^*$. In fact, if $P \in (\mathcal{P}_0 \setminus \{P\})^* \subseteq \mathcal{P}^*$, then there is a nonempty finite set $\mathcal{A} \subseteq \cup_{i=0}^n K_i$ such that $P = \bigcap\{A | A \in \mathcal{A}\}$. Set

$$\mathcal{A}_1 = \mathcal{A} \cap \cup_{i=0}^{i_0-1} K_i, \qquad \mathcal{A}_2 = \mathcal{A} \cap \cup_{i=i_0}^{n-1} K_i.$$

Then $\mathcal{A}_1 \cup \mathcal{A}_2 = \mathcal{A}$ and $P = (\cap \mathcal{A}_1) \cap (\cap \mathcal{A}_2)$.

Notice that $\mathcal{A}_2 \subseteq \cup_{i=i_0}^{n-1} K_i \subseteq \mathcal{P} \setminus (\cup_{i=0}^{i_0-1} K_i)^*$. If $\mathcal{A}_2 \neq \emptyset$, then for each $A \in \mathcal{A}_2$, since $P \in K_{i_0}$ is the maximal element and $K_i \cap K_j = \emptyset$ $(\forall i \neq j)$, we have $P \nsubseteq A$, a contradiction to the above equation of P. So $\mathcal{A}_2 = \emptyset$. Then $P = \cap \mathcal{A}_1 \notin K_{i_0}$, this is a contradiction. Then the assertion is proved. By the assertion, we have $(\mathcal{P}_0 \setminus \{P\})^* \neq \mathcal{P}^*$, showing that $\mathcal{P}_0$ is a saturation reduction of $\mathcal{P}$. □

Proposition 5.7 *Let $\mathcal{P}$ be an abstract knowledge base on a finite universe U, $\mathcal{P}_0$ the saturation reduction constructed in Theorem 5.6 and $\mathcal{P}_1$ another saturation reduction of $\mathcal{P}$. Then $\mathcal{P}_0 \subseteq \mathcal{P}_1$. So, $\mathcal{P}_0$ is the unique saturation reduction of $\mathcal{P}$.*

Proof. We need to prove $K_i \subseteq \mathcal{P}_1$ $(i = 0, \cdots, n)$. To this end, we use the mathematical induction.

(1) For $K_0 = max(\mathcal{P})$ and $\forall A \in K_0$, there exist $P_s \in \mathcal{P}_1 (s = 1, \cdots, m)$ such that $A = \cap P_i$. Since A is a maximal element, $A = P_s \in \mathcal{P}_1$ and $K_0 \subseteq \mathcal{P}_1$.

(2) Assume that when $i \leqslant j$, $K_i \subseteq \mathcal{P}_1$. Then

(3) $\cup_{i=0}^j K_i \subseteq \mathcal{P}_1$. Since $\mathcal{P}_1$ is a saturation reduction of $\mathcal{P}$, we have that $\mathcal{P}_1 \setminus \cup_{i=0}^j K_i = \mathcal{P}_1 \setminus (\cup_{i=0}^j K_i)^*$. Thus, for K_{j+1} and $A \in K_{j+1}$, we have that $A \notin (\cup_{i=0}^j K_i)^*$ and $\mathcal{P}_1 \setminus \cup_{i=0}^j K_i = \mathcal{P}_1 \setminus (\cup_{i=0}^j K_i)^* \subseteq \mathcal{P} \setminus (\cup_{i=0}^j K_i)^* \subseteq \downarrow K_{j+1}$. Since $\mathcal{P}_1$ is a saturation reduction, there is a finite $\mathcal{A} \subseteq \mathcal{P}_1$ such that $A = (\cap \mathcal{A}_1) \cap (\cap \mathcal{A}_2)$, $\mathcal{A}_1 \subseteq \mathcal{P}_1 \setminus \cup_{i=0}^j K_i$, $\mathcal{A}_2 \subseteq \cup_{i=0}^j K_i$. We assert that $\mathcal{A}_2 = \emptyset$. In fact, if $\mathcal{A}_2 \neq \emptyset$, then by $\mathcal{A}_1 \subseteq \downarrow K_{j+1}$ and that A is a maximal element of $\downarrow K_{j+1}$, we have that $\mathcal{A}_1 = \emptyset$ or $\mathcal{A}_1 = \{A\}$. Thus $A = \cap \mathcal{A}_2 \in (\cup_{i=0}^j K_i)^*$, a contradiction! So, $\mathcal{A}_2 = \emptyset$ and $A = \cap \mathcal{A}_1 = A$. So $\mathcal{A}_1 = \{A\} \subseteq \mathcal{P}_1$ and $A \in \mathcal{P}_1$. Thus, $K_{j+1} \subseteq \mathcal{P}_1$. By the

principle of mathematical induction, we have that $K_i \subseteq \mathcal{P}_1$ $(i = 0, \cdots, n)$. Then $\mathcal{P}_0 = \cup_{i=0}^{n} K_i \subseteq \mathcal{P}_1$. Since $\mathcal{P}_0$ and $\mathcal{P}_1$ are both minimal saturated, $\mathcal{P}_0 = \mathcal{P}_1$. □

Corollary 5.8 *Let $\mathcal{P}$ be an abstract knowledge base on a finite universe U, $\mathcal{P}^*$ the saturation of $\mathcal{P}$. Then a saturation reduction of $\mathcal{P}^*$ is also a saturation reduction of $\mathcal{P}$ and is included in $\mathcal{P}$.*

Proof. Let $\mathcal{P}_0$ be a saturation reduction of $\mathcal{P}$. Since a saturation reduction of $\mathcal{P}$ is also a saturation reduction of $\mathcal{P}^*$, and by Proposition 5.7, $\mathcal{P}_0$ is also the unique saturation reduction of $\mathcal{P}^*$ and is included in $\mathcal{P}$. □

By the construction process in Theorem 5.6, we can give an algorithm to compute the saturation reduction of an abstract knowledge base $\mathcal{P}$ on a finite universe as follows.

(1) Initially, calculate $K_0 = max(\mathcal{P})$, this can be realized by comparison procedure, then set $K := K_0$ and $S := \emptyset$.

(2) Recursively calculate $S := S \cup K$, S^*, $\mathcal{P} \setminus S^*$ and $K := max(\mathcal{P} \setminus S^*)$ under the control of the Boolean condition "$K \neq \emptyset$". To calculate S^*, one can make use of the algorithm constructed in [12]; to calculate a complement of a set, one can use search-deleting procedure. The Boolean condition "$K \neq \emptyset$" means that if the condition is fulfilled, then the recursive procedure continues, otherwise stops and goes to the next step.

(3) $\mathcal{P}_0 := S$ is what we need, the saturation reduction of $\mathcal{P}$.

(4) Stop. The procedure is completed.

Proposition 5.9 *Let $\mathcal{P} \neq \emptyset$ be an abstract knowledge base on U, $\mathcal{P}_0$ the saturation reduction of $\mathcal{P}$, $V \subseteq U$. Set $\mathcal{P}|_V = \{P \cap V | P \in \mathcal{P}\}$. If $\mathcal{P}|_V \neq \{\emptyset\}$, then $\mathcal{P}_0|_V \neq \{\emptyset\}$ and $(\mathcal{P}|_V)^* = (\mathcal{P}_0|_V)^*$.*

Proof. First we need to prove $\mathcal{P}_0|_V \neq \{\emptyset\}$. Let $P \cap V \in \mathcal{P}|_V$ and $P \cap V \neq \emptyset$. Since $\mathcal{P}_0$ is the saturation reduction of $\mathcal{P}$, there are $P_1, \cdots, P_n \in \mathcal{P}_0$ such that $\emptyset \neq P = \cap_{i=1}^{n} P_i$ and $\emptyset \neq P \cap V = \cap_{i=1}^{n} P_i \cap V$. Thus $P_i \cap V \neq \emptyset$ and $P_i \cap V \in \mathcal{P}_0|_V$. So, $\mathcal{P}_0|_V \neq \{\emptyset\}$.

Since $\mathcal{P}_0$ is the saturation reduction of $\mathcal{P}$, we have that $\mathcal{P}_0|_V \subseteq \mathcal{P}|_V$ and $(\mathcal{P}_0|_V)^* \subseteq (\mathcal{P}|_V)^*$. Let $P \in (\mathcal{P}|_V)^*$. Then there is $P_i \cap V \in P|_V (i = 1, \cdots, n)$ such that $P = \cap_{i=1}^{n} (P_i \cap V)$ with $P_i \in \mathcal{P}$. For each $P_i \in \mathcal{P}$, there exists $P_i^{(j)} \in \mathcal{P}_0$ $(j = 1, \cdots, k)$ such that $P_i = \cap_{j=1}^{k} P_i^{(j)}$. Then $P = \cap_{i=1}^{n} \cap_{j=1}^{k} (P_i^{(j)} \cap V)$. Since $P_i^{(j)} \cap V \in \mathcal{P}_0|_V$, we have $P \in (\mathcal{P}_0|_V)^*$. To sum up, we have $(\mathcal{P}_0|_V)^* = (\mathcal{P}|_V)^*$. □

Normally, $\mathcal{P}_0|_V$ is not the saturation reduction of $\mathcal{P}|_V$ by the following example.

Example 5.10 *Let $U = \{1, 2, 3, 4\}$, $\mathcal{P} = \{\{1\}, \{1, 2, 3\}, \{1, 2, 4\}\}$ on U and $V = \{1, 3, 4\} \subseteq U$. Then it is easy to show that $\mathcal{P}$ is minimally saturated, i.e, $\mathcal{P}$ is the saturation reduction of itself. But $\mathcal{P}|_V = \{\{1\}, \{1, 3\}, \{1, 4\}\}$ is not minimally saturated. So, $\mathcal{P}|_V$ can't be a saturation reduction of $\mathcal{P}|_V$.*

Let $\mathcal{P}_0$ be a saturation reduction of $\mathcal{P}$, $V \subseteq U$, and $\mathcal{P}_0|_V = \{P_1 \cap V, \cdots, P_n \cap V\}$. Then one can get a minimally saturated subfamily $\mathcal{P}_1|_V \subseteq \mathcal{P}_0|_V$ of $\mathcal{P}_0|_V$ such that $(\mathcal{P}_1|_V)^* = (\mathcal{P}_0|_V)^*$, then $\mathcal{P}_1|_V \subseteq \mathcal{P}_0|_V$ is the saturation reduction of $\mathcal{P}|_V$.

6 Some Examples

Firstly, we are intend to construct special examples of equivalence relations to show that there is knowledge base which is a chain but has no reductions, there is an independent knowledge base which is an infinite anti-chain and that there is a knowledge base which has not only infinite reductions but also finite reductions. The following four examples respectively reflect these situations.

Example 6.1 *Let $U = \mathbb{N}$ be a universe and $\mathcal{P} = \{R_0, R_1, \cdots, R_i, \cdots\}$ a family of equivalence relations on U, where $R_i = \{(0,0), (1,1), \cdots, (i,i)\} \cup \{(k,j) | k, j \geq i+1\}$. Then $\mathcal{P}$ is decreased and thus a chain with no least element. By Theorem 4.5, $K = (U, \mathcal{P})$ has no finite reductions. More over K has no reduction.*

To see these, assume $\mathcal{P}$ has a reduction $\mathcal{P}_0$. Then $\mathcal{P}_0$ must be infinite and $ind\mathcal{P} = ind\mathcal{P}_0$. Let t be the least index of R_i such that $R_i \in \mathcal{P}_0$. Then $\mathcal{P}_0 - \{R_t\} \neq \emptyset$. Since $\mathcal{P}$ is decreased, $ind\mathcal{P}_0 = ind(\mathcal{P}_0 - \{R_t\})$ and $\mathcal{P}_0$ is not independent, a contradiction to $\mathcal{P}_0$ being a reduction. Thereby this contradiction shows $\mathcal{P}$ has no reduction.

Example 6.2 *Let $U = \mathbb{N}^+ = \{1, 2, \cdots\}$ be a universe, $\mathcal{P} = \{R_1, \cdots, R_n, \cdots\}$ a family of equivalence relations on U, where $R_n = \{(1,n), (n,1), (1,1), \cdots, (n,n)\} \cup \{(k,j) | k, j \geq n+1\}$. Then R_n is an equivalence relation on U. We assert that $K = (U, \mathcal{P})$ has no reduction.*

In fact, $ind\mathcal{P} = \{(x,x) | x \in U\} = \triangle$, the identity relation on U. Any finite meets of $\mathcal{P}$ cannot be $\triangle$. By Corollary 4.9, we see that $\mathcal{P}$ has no finite reduction. Let $R_{i_1}, \cdots, R_{i_k}, \cdots$ be an infinite sequence of $\mathcal{P}$. For every pair (i,n) with $i \neq n$, pick $i_k > max\{i,n\}$. Then $(i,n) \notin R_{i_k}$. With this fact, we see that $R_{i_1} \cap \cdots \cap R_{i_k} \cap \cdots = \triangle$. This implies that for any infinite sequence $\mathcal{P}'$ of $\mathcal{P}$ and any $R \in \mathcal{P}'$, one has that $ind\mathcal{P}' = ind\mathcal{P} = ind(\mathcal{P}' - \{R\}) = \triangle$ and $\mathcal{P}'$ is not independent. So, $\mathcal{P}$ has no infinite reduction, either.

In this example, $\forall i < n$, we have that

$$R_i = \{(1,i), (i,1), (1,1), \cdots, (i,i)\} \cup \{(k,j) | k, j \geq i+1\},$$

$$R_n = \{(1,n), (n,1), (1,1), \cdots, (n,n)\} \cup \{(k,j) | k, j \geq n+1\}$$

and $(1,n) \in R_n$, $(1,n) \notin R_i$, $(i+1, n+1) \in R_i$, $(i+1, n+1) \notin R_n$. So, $R_n \nsubseteq R_i$ and $R_i \nsubseteq R_n$. By this fact, we see that $\mathcal{P}$ is an anti-chain.

Example 6.3 *Let $U = \mathbb{N}^+$ be a universe, $\mathcal{P} = \{R_1, \cdots, R_n, \cdots\}$ a family of equivalence relations on U such that $U/R_1 = \{\{1\}, \{2, 3, \cdots\}\}$, $U/R_2 = \{\{1,2\}, \{3,4,\cdots\}\}$, $U/R_3 = \{\{1,2,3\}, \{4,5,\cdots\}\}$, $\cdots$, $U/R_n = \{\{1,2,\cdots,n\}, \{n+1, n+2, \cdots\}\}$, $\cdots$. R_n is indeed an equivalence relation. We will show that the knowledge base $K = (U, \mathcal{P})$ has itself as a reduction.*

In fact, $\forall i < n$, we have

$$U/R_i = \{\{1, 2, \cdots, i\}, \{i + 1, i + 2, \cdots\}\},$$

$$U/R_n = \{\{1, 2, \cdots, n\}, \{n + 1, n + 2, \cdots\}\}$$

and $[1]_{R_i} = \{1, 2, \cdots, i\}$, $[1]_{R_n} = \{1, 2, \cdots, n\}$, $[n+1]_{R_i} = \{i+1, i+2, \cdots\}$, $[n+1]_{R_n} = \{n + 1, n + 2, \cdots\}$. So, $[1]_{R_n} \nsubseteq [1]_{R_i}$ and $[n + 1]_{R_i} \nsubseteq [n + 1]_{R_n}$. This reveals that $\mathcal{P}$ is an anti-chain.

Let $R_m \in P$. Then $(m, m + 1) \notin R_m$ and when $i \neq m$, $(m, m + 1) \in R_i$. So $(m, m + 1) \in ind(P - \{R_m\})$. It is easy to see that $ind\mathcal{P} = \{(x, x)|x \in U\} = \triangle \neq ind(\mathcal{P} - \{R_m\})$. So, $\mathcal{P}$ is independent and $\mathcal{P}$ is a reduction of itself.

Example 6.4 *Add another equivalence relation $R_0 = \{(x, x)|x \in U\} = \triangle$ to $\mathcal{P}$ in Example 6.3, one gets a new knowledge base $K' = (U, \mathcal{P} \cup \{R_0\})$. It is easy to see that $\mathcal{P}$ and $\{R_0\}$ are the two reductions of K', one infinite and the other finite.*

To sum up, we see that families $\mathcal{P}$ of equivalence relations in Examples 6.1-6.4 are respectively a chain with no reduction, an anti-chain with no reduction, an independent anti-chain having only itself as a reduction, and a knowledge base having not only a finite reduction but also an infinite reduction.

It is easy to show that $core(K) \subseteq \cap red(K)$ for any (abstract) knowledge base. However, for an infinite knowledge base, the following example shows that its core needn't be the intersection of all its reductions.

Example 6.5 *Let $K = (U, \mathcal{P})$ be the infinite knowledge base in Example 6.2 with no reductions such that $ind(K) = \triangle$. Let $K' = (U, \mathcal{P}')$, where $\mathcal{P}' = \mathcal{P} \cup \{\triangle\}$. Then $\{\triangle\}$ is a reduction of K' and this is the only one reduction of K'. Intersection of all reductions of K' is $\triangle$. However, every element in K' is not necessary and core $(K') = \emptyset \neq \{\triangle\}$.*

Secondly, we give an example to show that an abstract knowledge base on an infinite universe U needn't have saturation reductions.

Example 6.6 *Let U be an infinite set, $\mathcal{P} = 2^U$ an abstract knowledge base. Then $\mathcal{P}$ has no saturation reduction.*

In fact, if $\mathcal{P}_0$ is a saturation reduction of $\mathcal{P}$, then it is easy to see that $U \in \mathcal{P}_0$ and $\forall x \in U$, $U - \{x\} \in \mathcal{P}_0$. By Proposition 5.5, when $F \subseteq U$ is finite and $|F| \geqslant 2$, we have that $U \setminus F \notin \mathcal{P}_0$. Let $A \subseteq U$ be an infinite set. Then $\forall x \in A$, we have $(U - A) \cup \{x\} \in \mathcal{P} = \mathcal{P}_0^*$. So there are finite elements $P_1, \cdots, P_n \in \mathcal{P}_0$ such that $(U-A)\cup\{x\} = \cap_{i=1}^n P_i$. Then $U-A = (U-A)\cup\{x\})\cap(U-\{x\}) = (U-\{x\})\cap\cap_{i=1}^n P_i$ is an intersection of finite elements deferent from $U - A$ in $\mathcal{P}_0$. By Proposition 5.5, $U - A \notin \mathcal{P}_0$. So, $\mathcal{P}_0 = \{U - \{x\}|x \in U\} \cup \{U\}$. Since subsets of $\mathcal{P}_0^*$ are all infinite, $\mathcal{P}_0^* \neq 2^U$, a contradiction! So, $\mathcal{P}$ has indeed no saturation reductions.

Thirdly, we give some examples of simple applications of saturation reductions.

Example 6.7 *Figure out the number of different sub-semilattices of* $2^{\{a,b\}}$.

Let $\mathcal{P}$ be a subsemilattice of 2^U with $U = \{a, b\}$. Then $\mathcal{P}$ has only one saturation reduction $\mathcal{P}_0 \subseteq \mathcal{P}$ which is a family of subsets of U such that each $P \in \mathcal{P}_0$, P is not an intersection of other elements of $\mathcal{P}_0$. By Proposition 5.4, $\mathcal{P}_0$ is also the saturation reduction of $\mathcal{P}_0$. So each subsemilattice uniquely determines a family of subsets $\mathcal{P}_0$ on U such that $\mathcal{P}_0$ is the saturation reduction of itself. Conversely, a minimally saturated family uniquely determines a subsemilattice $\mathcal{P}_0^*$. So, to figure out the number of different subsemilattices of 2^U, we need to figure out the number of different minimally saturated families. If $U = \{a, b\}$, then $2^U = \{\emptyset, \{a\}, \{b\}, U\}$, the number of different minimally saturated families having 1 element is $C_4^1 = 4$, the number of different minimally saturated families having 2 elements is $C_4^2 = 6$ (chain or anti-chain), the number of different minimally saturated families having 3 elements is $C_2^1 + 1 = 3$, for in this case the families must contain U and if they contain $\emptyset$, they do not contain $\{a\}$ and $\{b\}$ at the same time; if they do not contain $\emptyset$, the other three elements are indeed minimally saturated. Since 2^U is not minimally saturated, the number of different minimally saturated families having 3 elements is 0. So the number of different subsemilattices is $4 + 6 + 2 + 1 = 13$.

Example 6.8 *Let* $U = \{a, b, c\}$ *be a universe. Caculate the number of different topologies on* U.

We first prove that when $|U| \leqslant 3$, the saturation reduction $\mathcal{P}_0$ of a topology $\mathcal{P}$ on U is closed w.r.t. nonempty unions. In fact, when $|U| = 1, 2$, it is easy to check. When $|U| = 3$, then for any pair of two elements $A, B \in \mathcal{P}_0$, if $A \cup B = U$ or A, B can be compared, obviously $A \cup B \in \mathcal{P}_0$. If A, B cannot be compared, when $|A \cup B| = 1$, then $A = B$ or one of A, B is $\emptyset$ and $A \cup B$ is A or B in $\mathcal{P}_0$; when $|A \cup B| = 2$, then $A \cup B \in max(\mathcal{P} \setminus \{U\}) \subseteq \mathcal{P}_0$. So, when $|U| \leq 3$, the saturation reduction of each topology on U is closed w.r.t. nonempty unions.

With this result, we come to consider the example.

Let $U = \{a, b, c\}$. Then $2^U = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, U\}$. Each topology on U uniquely determines its saturation reduction. The saturation reduction is minimally saturated, contains U and is closed w.r.t. nonempty unions. So, to figure out the number of different topologies on U is to figure out the number of families $\mathcal{P}$ which satisfy conditions that (1) minimally saturated, (2) include U, (3) closed w.r.t. nonempty unions and (4) saturations containing $\emptyset$.

Let $\mathcal{P}_0$ be a family of this kind. Then

(i) If $\mathcal{P}_0$ has 1 element, then $\mathcal{P}_0^*$ is not a topology, the number of $\mathcal{P}_0$ is 0;

(ii) If $\mathcal{P}_0$ has 2 elements, then $\mathcal{P}_0 = \{\emptyset, U\}$, the number of $\mathcal{P}_0$ is 1;

(iii) If $\mathcal{P}_0$ has 3 elements, then (a) when $\emptyset \in \mathcal{P}_0$, the number of $\mathcal{P}_0$ is $C_6^1 = 6$, (b) when $\emptyset \notin \mathcal{P}_0$, the number of $\mathcal{P}_0$ is $C_3^1 = 3$;

(iv) If $\mathcal{P}_0$ has 4 elements, then (a) when $\emptyset \in \mathcal{P}_0$, the number of $\mathcal{P}_0$ is $C_3^2 + C_3^1 C_2^1 = 9$, (b) when $\emptyset \notin \mathcal{P}_0$, the number of $\mathcal{P}_0$ is $C_3^3 + C_3^1 C_2^1 + C_3^2 = 10$;

(v) If $\mathcal{P}_0$ has 5 elements, then $\mathcal{P}_0$ is not minimally saturated.

So the number of different topologies on U is just the number of $\mathcal{P}_0$ which is now $0 + 1 + 6 + 3 + 9 + 10 + 0 = 29$.

Note that for a finite U with $|U| > 3$, the saturation reduction of a topology $\mathcal{P}$ on U needn't be closed w.r.t. nonempty unions. For example, let $U = \{1, 2, 3, 4\}$ and $\mathcal{P} = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 4\}, U\}$. Then $\mathcal{P}_0 = \{\{1\}, \{2\}, \{1, 2, 3\}, \{1, 2, 4\}, U\}$ is the saturation reduction of $\mathcal{P}$. Clearly, $\mathcal{P}_0$ is not closed w.r.t. nonempty unions.

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