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Innermost Termination of Rewrite Systems by Labeling[1]

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Abstract

Semantic labeling is a powerful transformation technique for proving termination of term rewrite systems. The semantic part is given by a model or a quasi-model of the rewrite rules. A variant of semantic labeling is predictive labeling where the quasi-model condition is only required for the usable rules. In this paper we investigate how semantic and predictive labeling can be used to prove innermost termination. Moreover, we show how to reduce the set of usable rules for predictive labeling even further, both in the termination and the innermost termination case.

Keywords: Innermost Termination, Predictive Labeling, Semantic Labeling, Term Rewriting, Termination

1 Introduction

We start our discussion by illustrating the limitations of existing versions of semantic and predictive labeling on a concrete example. Consider the following rewrite system $\mathcal{R}$ where $x \div y$ generates a number between 0 and $\lfloor \frac{x}{y} \rfloor$:

$$
\begin{aligned}
x \geqslant \mathsf{0} &\to \mathsf{true} && (1) & \mathsf{id\text{-}inc}(x) &\to x && (7)\\
\mathsf{0} \geqslant \mathsf{s}(y) &\to \mathsf{false} && (2) & \mathsf{id\text{-}inc}(x) &\to \mathsf{s}(x) && (8)\\
\mathsf{s}(x) \geqslant \mathsf{s}(y) &\to x \geqslant y && (3) & x \div y &\to \mathsf{if}(y \geqslant \mathsf{s}(\mathsf{0}), x \geqslant y, x, y) && (9)\\
x - \mathsf{0} &\to x && (4) & \mathsf{if}(\mathsf{false}, b, x, y) &\to \mathsf{div\text{-}by\text{-}zero} && (10)\\
\mathsf{0} - y &\to \mathsf{0} && (5) & \mathsf{if}(\mathsf{true}, \mathsf{false}, x, y) &\to \mathsf{0} && (11)\\
\mathsf{s}(x) - \mathsf{s}(y) &\to x - y && (6) & \mathsf{if}(\mathsf{true}, \mathsf{true}, x, y) &\to \mathsf{id\text{-}inc}((x - y) \div y) && (12)
\end{aligned}
$$

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Proving termination of $\mathcal{R}$ is a difficult task. Consider the recursive calls of $\div$ and if in rules (9) and (12). Essentially, one has to find a well-founded order such that the argument x of if is larger than the argument $x - y$ of $\div$. To this end, one can use the fact that in the previous recursive call the terms $y \geqslant \mathsf{s}(\mathsf{0})$ and $x \geqslant y$ are both reducible to true. This knowledge is important as for $x = \mathsf{0}$ or $y = \mathsf{0}$ the term $x - y$ can be reduced to x. However, when using term orders one generates one separate constraint for each rule of $\mathcal{R}$. Thus, the knowledge of a previous recursive call is not directly available when building the constraint for rule (12). For example, polynomial interpretations with negative coefficients [5] are not expressive enough to solve the constraints of rules (9) and (12).

To solve this problem one can use the technique of *semantic labeling* [9]. We can take an algebra $\mathcal{A}$ over natural numbers $\mathbb{N}$ where we use the natural interpretation for the symbols $-$, s, $\mathsf{0}$, false, true, and $\geqslant$, i.e., $x -_{\mathcal{A}} y = \max(x-y, 0)$, $\mathsf{s}_{\mathcal{A}}(x) = x+1$, $\mathsf{0}_{\mathcal{A}} = \mathsf{false}_{\mathcal{A}} = 0$, $\mathsf{true}_{\mathcal{A}} = 1$, and $x \geqslant_{\mathcal{A}} y = 1$ if $x \geqslant y$, and 0 otherwise. Now, we can also provide *labeling functions* ℓ_f which define how to label the function symbol f in a term $f(t_1, \ldots, t_n)$, depending on the value of their arguments. E.g., we can choose $\ell_{\div}(n, m) = n$, $\ell_{\mathsf{if}}(b_1, b_2, n, m) = b_1 b_2 + \max(n-m, 0)$, and we do not label the remaining symbols. Then by labeling we get the (infinite) TRS $\mathsf{lab}(\mathcal{R})$ consisting of (1)–(8) together with the following rules, for all $i \geqslant j \geqslant 0$:

$$x \div_i y \to \mathsf{if}_j(y \geqslant \mathsf{s}(\mathsf{0}), x \geqslant y, x, y) \tag{13}$$

$$\mathsf{if}_i(\mathsf{false}, b, x, y) \to \mathsf{div\text{-}by\text{-}zero} \tag{14}$$

$$\mathsf{if}_i(\mathsf{true}, \mathsf{false}, x, y) \to \mathsf{0} \tag{15}$$

$$\mathsf{if}_{i+1}(\mathsf{true}, \mathsf{true}, x, y) \to \mathsf{id\text{-}inc}((x - y) \div_i y) \tag{16}$$

Termination of $\mathsf{lab}(\mathcal{R})$ is easily proved by LPO with precedence $\cdots \sqsupset \div_n \sqsupset \mathsf{if}_n \sqsupset \cdots \sqsupset \div_1 \sqsupset \mathsf{if}_1 \sqsupset \div_0 \sqsupset \mathsf{if}_0 \sqsupset \mathsf{id\text{-}inc} \sqsupset - \sqsupset \geqslant \sqsupset \mathsf{s} \sqsupset \mathsf{0} \sqsupset \mathsf{true} \sqsupset \mathsf{false}$. The result of semantic labeling is that if the algebra $\mathcal{A}$ is a model of $\mathcal{R}$ then termination of $\mathsf{lab}(\mathcal{R})$ implies termination of $\mathcal{R}$. However, it is impossible to give an interpretation $\mathsf{id\text{-}inc}_{\mathcal{A}}$ such that $\mathcal{A}$ is a model of $\mathcal{R}$, since there is a conflict between the rules (7) and (8).

One solution is to work with *quasi-models* where it is only required that the interpretation of each left-hand side of a rule is greater than or equal to the interpretation of the corresponding right-hand side. In [4] semantic labeling with quasi-models is extended to *predictive labeling* where $\mathcal{A}$ only has to be a quasi-model of the *usable rules*, the rules which define the function symbols that are needed to perform the labeling. In our example the usable rules are (1)–(6). And indeed $\mathcal{A}$ is a (quasi-) model of these rules. The problem when using quasi-models is the requirement that all interpretations have to be weakly monotone in all arguments. As $-_{\mathcal{A}}$ is not weakly monotone ($1 \geqslant 0$, but $3 -_{\mathcal{A}} 1 = 2 \ngeqslant 3 = 3 -_{\mathcal{A}} 0$) one cannot use the algebra $\mathcal{A}$ to prove termination of $\mathcal{R}$.

As a matter of fact, $\mathcal{R}$ is not terminating:

$$\mathsf{s}(\mathsf{0}) \div \mathsf{id\text{-}inc}(\mathsf{0}) \to \ \mathsf{if}(\mathsf{id\text{-}inc}(\mathsf{0}) \geqslant \mathsf{s}(\mathsf{0}), \mathsf{s}(\mathsf{0}) \geqslant \mathsf{id\text{-}inc}(\mathsf{0}), \mathsf{s}(\mathsf{0}), \mathsf{id\text{-}inc}(\mathsf{0}))$$

$$\begin{aligned}
&\to^2 \ \mathsf{if}(\mathsf{s}(0) \geqslant \mathsf{s}(0), \mathsf{s}(0) \geqslant \mathsf{s}(0), \mathsf{s}(0), \mathsf{id\text{-}inc}(0))\\
&\to^4 \ \mathsf{if}(\mathsf{true}, \mathsf{true}, \mathsf{s}(0), \mathsf{id\text{-}inc}(0))\\
&\to \ \ \mathsf{id\text{-}inc}((\mathsf{s}(0) - \mathsf{id\text{-}inc}(0)) \div \mathsf{id\text{-}inc}(0))\\
&\to^2 \ (\mathsf{s}(0) - 0) \div \mathsf{id\text{-}inc}(0) \to \mathsf{s}(0) \div \mathsf{id\text{-}inc}(0) \to \cdots
\end{aligned}$$

So there cannot be a version of predictive labeling with models and arbitrary interpretations. [3] Nevertheless, $\mathcal{R}$ is *innermost* terminating. Therefore we investigate whether one can use predictive labeling with models for innermost termination, where one can freely choose interpretations and where the algebra only has to be a model of the usable rules. As the previous results on predictive labeling only work for quasi-models, one cannot reuse them for innermost rewriting, e.g., Example 2.3 below shows that the main theorem of predictive labeling [4, Theorem 18] does not hold for innermost rewriting.

The remainder of this paper is organized as follows. In Section 2 we start the formal developments by recalling the basic definitions related to semantic labeling. We show that with respect to innermost termination semantic labeling is incomplete for both models and quasi-models and unsound for quasi-models. Soundness for models does hold and is shown in Section 3. By adapting the idea of predictive labeling to the innermost case we show that the model requirement is only needed for the usable rules induced by the labeling. The next contribution (Section 4) is the integration of an *argument filter*, i.e., a mapping from function symbols to sets of argument positions, to obtain even less usable rules than in [4] for innermost termination. This idea was already used in [3] where argument filters are employed to increase the power of term orders. In the context of semantic labeling, argument filters are used to express which arguments are ignored in interpretation and labeling functions. In Section 5 we return to termination. We show how to integrate argument filters with predictive labeling, resulting in a result that is strictly more powerful than the main theorem of [4]. Concluding remarks are given in Section 6.

2 Semantic Labeling for Innermost Termination

We assume that the reader is familiar with term rewriting [2]. Below we recall the basic definitions related to semantic labeling.

An algebra $\mathcal{A}$ over $\mathcal{F}$ is a pair $(A, \{f_{\mathcal{A}}\}_{f \in \mathcal{F}})$ consisting of a carrier A and, for every n-ary function symbol $f \in \mathcal{F}$, an interpretation function $f_{\mathcal{A}}\colon A^n \to A$. Given an assignment $\alpha\colon \mathcal{V} \to A$ we write $[\alpha]_{\mathcal{A}}(t)$ for the interpretation of the term t. An algebra $\mathcal{A}$ is a model of a rewrite system if $[\alpha]_{\mathcal{A}}(l) = [\alpha]_{\mathcal{A}}(r)$ for all rules $l \to r \in \mathcal{R}$ and all assignments α. If additionally, the carrier A is equipped with a well-founded order $>_A$ then $\mathcal{A}$ is a quasi-model if $[\alpha]_{\mathcal{A}}(l) \geqslant_A [\alpha]_{\mathcal{A}}(r)$ for all $l \to r \in \mathcal{R}$ and all assignments α.

For each function symbol f there also is a corresponding set $L_f \subseteq A$ of labels for f and if L_f is non-empty there also is a labeling function $\ell_f\colon A^n \to L_f$. The

[3] This answers a question raised in [4].

labeled signature $\mathcal{F}_{\mathsf{lab}}$ consists of n-ary function symbols f_a for every n-ary function symbol $f \in \mathcal{F}$ and label $a \in L_f$ together with all function symbols $f \in \mathcal{F}$ such that $L_f = \varnothing$. The labeling function ℓ_f determines the label of the root symbol f of a term $f(t_1, \ldots, t_n)$ based on the values of the arguments $t_1, \ldots, t_n$. For every assignment $\alpha \colon \mathcal{V} \to A$ the mapping $\mathsf{lab}_\alpha \colon \mathcal{T}(\mathcal{F}, \mathcal{V}) \to \mathcal{T}(\mathcal{F}_{\mathsf{lab}}, \mathcal{V})$ is inductively defined as follows:

$$\mathsf{lab}_\alpha(t) = \begin{cases} t & \text{if } t \text{ is a variable,} \\ f(\mathsf{lab}_\alpha(t_1), \ldots, \mathsf{lab}_\alpha(t_n)) & \text{if } t = f(t_1, \ldots, t_n) \text{ and } L_f = \varnothing, \\ f_a(\mathsf{lab}_\alpha(t_1), \ldots, \mathsf{lab}_\alpha(t_n)) & \text{if } t = f(t_1, \ldots, t_n) \text{ and } L_f \neq \varnothing \end{cases}$$

where a denotes the label $\ell_f([\alpha]_{\mathcal{A}}(t_1), \ldots, [\alpha]_{\mathcal{A}}(t_n))$. The labeled TRS $\mathsf{lab}(\mathcal{R})$ over the signature $\mathcal{F}_{\mathsf{lab}}$ consists of the rules $\mathsf{lab}_\alpha(l) \to \mathsf{lab}_\alpha(r)$ for all $l \to r \in \mathcal{R}$ and $\alpha \colon \mathcal{V} \to A$. Moreover, if one uses quasi-models then one needs the set

$$\mathcal{D}\mathrm{ec} = \{ f_a(x_1, \ldots, x_n) \to f_b(x_1, \ldots, x_n) \mid a, b \in L_f, a >_A b \}$$

of decreasing rules. In this case every interpretation function $f_{\mathcal{A}}$ and every labeling function ℓ_f has to be weakly monotone, i.e., if $a \geqslant_A a'$ then $f_{\mathcal{A}}(a_1, \ldots, a, \ldots, a_n) \geqslant_A f_{\mathcal{A}}(a_1, \ldots, a', \ldots, a_n)$ and similarly for ℓ_f.

Zantema [9] obtained the following results for semantic labeling.

Lemma 2.1 *Let $\mathcal{R}$ be a TRS and $\mathcal{A}$ a non-empty algebra.*

(i) *If $\mathcal{A}$ is a model of $\mathcal{R}$ then $t \to_{\mathcal{R}} u$ implies $\mathsf{lab}_\alpha(t) \to_{\mathsf{lab}(\mathcal{R})} \mathsf{lab}_\alpha(u)$.*

(ii) *If $\mathcal{A}$ is a quasi-model of $\mathcal{R}$ then $t \to_{\mathcal{R}} u$ implies $\mathsf{lab}_\alpha(t) \to^{+}_{\mathsf{lab}(\mathcal{R}) \cup \mathcal{D}\mathrm{ec}} \mathsf{lab}_\alpha(u)$.*

From Lemma 2.1 one obtains that $\mathcal{R}$ is terminating if and only if $\mathsf{lab}(\mathcal{R})\,(\cup\, \mathcal{D}\mathrm{ec})$ is terminating when $\mathcal{A}$ is a (quasi-)model of $\mathcal{R}$. Completeness is achieved by removing the labels of a possible infinite rewrite sequence of the labeled TRS. Soundness is proved by transforming a presupposed infinite rewrite sequence in $\mathcal{R}$ into an infinite rewrite sequence in $\mathsf{lab}(\mathcal{R})\,(\cup\, \mathcal{D}\mathrm{ec})$. This transformation is achieved by applying the labeling function $\mathsf{lab}_\alpha(\cdot)$ (for an arbitrary assignment α) to all terms in the infinite rewrite sequence of $\mathcal{R}$. Hence, semantic labeling is sound and complete for termination with respect to both models and quasi-models.

As first new contribution we show that semantic labeling is incomplete for innermost termination (Example 2.2) and that it is not even sound when using quasi-models (Example 2.3). We write $\xrightarrow{\mathsf{i}}_{\mathcal{R}}$ for the innermost rewrite relation of $\mathcal{R}$.

Example 2.2 Consider the TRS $\mathcal{R}$:

$$\begin{aligned} \mathsf{if}(\mathsf{true}, x) &\to \mathsf{if}(\mathsf{test\text{-}ab}(x), x) & \qquad \mathsf{test\text{-}ab}(\mathsf{a}(x)) &\to \mathsf{test\text{-}b}(x) \\ \mathsf{a}(\mathsf{b}) &\to \mathsf{c} & \mathsf{test\text{-}b}(\mathsf{b}) &\to \mathsf{true} \end{aligned}$$

Note that $\mathcal{R}$ is innermost terminating. The reason is that $\mathsf{test\text{-}ab}(x)$ can only be evaluated to true if x is instantiated with $\mathsf{a}(\mathsf{b})$. But this is not allowed as $\mathsf{a}(\mathsf{b})$ is not in normal form. We choose the algebra $\mathcal{A}$ with carrier $A = \{0, 1\}$, interpretations

$\mathsf{if}_{\mathcal{A}}(x,y) = 0$, $\mathsf{b}_{\mathcal{A}} = \mathsf{c}_{\mathcal{A}} = \mathsf{true}_{\mathcal{A}} = 1$, $\mathsf{test\text{-}ab}_{\mathcal{A}}(x) = \mathsf{test\text{-}b}_{\mathcal{A}}(x) = \mathsf{a}_{\mathcal{A}}(x) = x$, and order $>_A = \varnothing$. Then $\mathcal{A}$ is a model (and thus also a quasi-model) of $\mathcal{R}$. Choosing $L_{\mathsf{a}} = A$, $\ell_{\mathsf{a}}(x) = x$, and $L_f = \varnothing$ for all other function symbols f we get the following labeled TRS $\mathsf{lab}(\mathcal{R})$:

$$\begin{array}{lll} \mathsf{if}(\mathsf{true}, x) \to \mathsf{if}(\mathsf{test\text{-}ab}(x), x) & \mathsf{test\text{-}ab}(\mathsf{a}_0(x)) \to \mathsf{test\text{-}b}(x) & \mathsf{test\text{-}b}(\mathsf{b}) \to \mathsf{true} \\ \mathsf{a}_1(\mathsf{b}) \to \mathsf{c} & \mathsf{test\text{-}ab}(\mathsf{a}_1(x)) \to \mathsf{test\text{-}b}(x) & \end{array}$$

There are no decreasing rules. The following reduction shows that $\mathsf{lab}(\mathcal{R})$ is not innermost terminating:

$$\begin{aligned} \mathsf{if}(\mathsf{true}, \mathsf{a}_0(\mathsf{b})) &\to_{\mathsf{lab}(\mathcal{R})} \mathsf{if}(\mathsf{test\text{-}ab}(\mathsf{a}_0(\mathsf{b})), \mathsf{a}_0(\mathsf{b})) \to_{\mathsf{lab}(\mathcal{R})} \mathsf{if}(\mathsf{test\text{-}b}(\mathsf{b}), \mathsf{a}_0(\mathsf{b})) \\ &\to_{\mathsf{lab}(\mathcal{R})} \mathsf{if}(\mathsf{true}, \mathsf{a}_0(\mathsf{b})) \to_{\mathsf{lab}(\mathcal{R})} \cdots \end{aligned}$$

So semantic labeling is incomplete in the innermost case. The next example shows that semantic labeling with quasi-models is unsound in the innermost case.

Example 2.3 The TRS $\mathcal{R} = \{\mathsf{f}(\mathsf{a}(\mathsf{b})) \to \mathsf{f}(\mathsf{a}(\mathsf{b}))\}$ is obviously not innermost terminating. We choose the algebra $\mathcal{A}$ with carrier $A = \{0,1\}$, interpretations $\mathsf{b}_{\mathcal{A}} = \mathsf{f}_{\mathcal{A}}(x) = 1$, $\mathsf{a}_{\mathcal{A}}(x) = x$, and $>_A = >$, which is a (quasi-)model of $\mathcal{R}$. By taking $L_{\mathsf{b}} = L_{\mathsf{f}} = \varnothing$, $L_{\mathsf{a}} = A$, and $\ell_{\mathsf{a}}(x) = x$, we obtain the TRS $\mathsf{lab}(\mathcal{R}) \cup \mathcal{D}\mathrm{ec}$

$$\mathsf{f}(\mathsf{a}_1(\mathsf{b})) \to \mathsf{f}(\mathsf{a}_1(\mathsf{b})) \qquad\qquad \mathsf{a}_1(x) \to \mathsf{a}_0(x)$$

This TRS is innermost terminating because the second rule prohibits an innermost rewrite step with the first rule.

The previous example does not show that semantic labeling with models is unsound for innermost termination because there are no decreasing rules when using models. Indeed, in the next section we show the soundness of semantic labeling with models for innermost termination. Actually, we prove a stronger results by incorporating usable rules.

3 Predictive Labeling for Innermost Termination

Semantic labeling requires that the algebra is a model of all rules. This is in contrast to *predictive* labeling where the model condition only has to be satisfied for the *usable rules*, a concept introduced in [1]. We slightly modify the definition of usable rules by integrating the labeling. Here, $\mathcal{F}\mathrm{un}(t)$ denotes the set of all function symbols occurring in the term t.

Definition 3.1 Let $\mathcal{R}$ be a TRS and ℓ a labeling. We define the set of *usable symbols* $\mathcal{US}_\ell(t) \subseteq \mathcal{F}$ of a term t inductively. If $t \in \mathcal{V}$ then $\mathcal{US}_\ell(t) = \varnothing$. If $t = f(t_1, \ldots, t_n)$ then $\mathcal{US}_\ell(t)$ is the least set such that

(i) $\mathcal{US}_\ell(t_1) \cup \cdots \cup \mathcal{US}_\ell(t_n) \subseteq \mathcal{US}_\ell(t)$,

(ii) if $L_f \neq \varnothing$ then $\mathcal{F}\mathrm{un}(t_1) \cup \cdots \cup \mathcal{F}\mathrm{un}(t_n) \subseteq \mathcal{US}_\ell(t)$, and

(iii) if $l \to r \in \mathcal{R}$ and $\mathrm{root}(l) \in \mathcal{US}_\ell(t)$ then $\mathcal{F}\mathrm{un}(r) \subseteq \mathcal{US}_\ell(t)$.

The usable symbols of $\mathcal{R}$ are defined as

$$\mathcal{US}_\ell(\mathcal{R}) = \bigcup_{l \to r \in \mathcal{R}} \mathcal{US}_\ell(r)$$

and the *usable rules* of $\mathcal{R}$ are defined as

$$\mathcal{U}_\ell(\mathcal{R}) = \{l \to r \in \mathcal{R} \mid \mathrm{root}(l) \in \mathcal{US}_\ell(\mathcal{R})\}.$$

It can be shown that $\mathcal{US}_\ell(t) = \mathcal{G}_\ell(t)$ for the corresponding definition of $\mathcal{G}_\ell$ in [4, Definition 5]. However, there is a difference in the definition of $\mathcal{US}_\ell(\mathcal{R})$ and $\mathcal{G}_\ell(\mathcal{R})$ as in [4] both sides of a rule are considered, i.e., $\mathcal{G}_\ell(r)$ and $\mathcal{G}_\ell(l)$ are added for a rule $l \to r$. The difference is illustrated in the following example.

Example 3.2 Consider the TRS $\mathcal{R} = \{\mathsf{a} \to \mathsf{f}(\mathsf{g}(\mathsf{b})),\ \mathsf{g}(\mathsf{a}) \to \mathsf{c}\}$. Assuming $L_\mathsf{f} \neq \varnothing$ and $L_\mathsf{g} \neq \varnothing$, in [4] one obtains $\mathcal{G}_\ell(\mathcal{R}) = \{\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{f}, \mathsf{g}\}$ and thus both rules are usable. This is in contrast to Definition 3.1 where $\mathcal{US}_\ell(\mathcal{R}) = \{\mathsf{b}, \mathsf{c}, \mathsf{g}\}$ and hence $\mathcal{U}_\ell(\mathcal{R}) = \{\mathsf{g}(\mathsf{a}) \to \mathsf{c}\}$. The advantage of our definition is obvious: we get less usable rules. However, the property in [4] that one only needs interpretations for the symbols in $\mathcal{US}_\ell(\mathcal{R})$ is not valid anymore. To check the model condition for $\mathsf{g}(\mathsf{a}) \to \mathsf{c}$ and to label $\mathsf{g}(\mathsf{a})$ we need an interpretation $\mathsf{a}_\mathcal{A}$ for a.

From now on we assume a fixed TRS $\mathcal{R}$ and just write $\mathcal{US}_\ell$ instead of $\mathcal{US}_\ell(\mathcal{R})$ and $\mathcal{U}_\ell$ instead of $\mathcal{U}_\ell(\mathcal{R})$. Essentially, the aim of predictive (resp. semantic) labeling is to find a model for the usable (resp. all) rules and then try to prove innermost termination of $\mathsf{lab}(\mathcal{R})$ to ensure innermost termination of $\mathcal{R}$. As argued between Lemma 2.1 and Example 2.2, soundness of semantic labeling is proved by transforming an infinite reduction $t_1 \to_\mathcal{R} t_2 \to_\mathcal{R} \dots$ into an infinite reduction $\mathsf{lab}_\alpha(t_1) \to_{\mathsf{lab}(\mathcal{R})} \mathsf{lab}_\alpha(t_2) \to_{\mathsf{lab}(\mathcal{R})} \dots$ by using Lemma 2.1(i). However, in the predictive case this lemma does not hold if the algebra is not a model of all rules. To this end we consider a variant in which only reduction steps $t\sigma \xrightarrow{\mathsf{i}}_\mathcal{R} u$ are regarded where t satisfies $\mathcal{US}_\ell(t) \subseteq \mathcal{US}_\ell$ and where σ is a *normalized* substitution, i.e., where $\sigma(x)$ is in normal form for all $x \in \mathcal{V}$.

Lemma 3.3 *Let $\mathcal{A}$ be a model of $\mathcal{U}_\ell$, let $\mathcal{US}_\ell(t) \subseteq \mathcal{US}_\ell$, and let σ be a normalized substitution. If $t\sigma = C[l\sigma] \xrightarrow{\mathsf{i}}_\mathcal{R} C[r\sigma] = u$ is a reduction with rule $l \to r \in \mathcal{R}$ then*

(i) $\mathsf{lab}_\alpha(t\sigma) \xrightarrow{\mathsf{i}}_{\mathsf{lab}(\mathcal{R})} \mathsf{lab}_\alpha(u)$,

(ii) *there is a term t' such that $u = t'\sigma$ and $\mathcal{US}_\ell(t') \subseteq \mathcal{US}_\ell$, and*

(iii) *$\mathcal{F}\mathrm{un}(t) \subseteq \mathcal{US}_\ell$ implies both $\mathcal{F}\mathrm{un}(t') \subseteq \mathcal{US}_\ell$ and $[\alpha]_\mathcal{A}(t\sigma) = [\alpha]_\mathcal{A}(u)$.*

Note that Lemma 3.3(i) and (ii) will allow us to transform innermost reductions of $\mathcal{R}$ into infinite innermost reductions of $\mathsf{lab}(\mathcal{R})$. This is needed for the proof of the main theorem of this section (Theorem 3.4). Property (iii) is only needed to prove Lemma 3.3.

Proof. We perform structural induction on t. As σ is a normalized substitution t is not a variable, so let $t = f(t_1, \ldots, t_n)$. We first consider a root reduction, i.e., $t\sigma = l\sigma \xrightarrow{\mathsf{i}}_{\mathcal{R}} r\sigma = u$. Let σ_{lab} be the substitution $\mathsf{lab}_\alpha \circ \sigma$ and let α_σ be the assignment $[\alpha]_{\mathcal{A}} \circ \sigma$. We have $\mathsf{lab}_\alpha(l\sigma) = \mathsf{lab}_{\alpha_\sigma}(l)\sigma_{\mathsf{lab}}$ and therefore obtain (i):

$$\mathsf{lab}_\alpha(t\sigma) = \mathsf{lab}_\alpha(l\sigma) = \mathsf{lab}_{\alpha_\sigma}(l)\sigma_{\mathsf{lab}} \xrightarrow{\mathsf{i}}_{\mathsf{lab}(\mathcal{R})} \mathsf{lab}_{\alpha_\sigma}(r)\sigma_{\mathsf{lab}} = \mathsf{lab}_\alpha(r\sigma) = \mathsf{lab}_\alpha(u).$$

Note that labeling does not introduce new redexes and hence the above reduction step is really an innermost step. The reason is that there are no decreasing rules as in Example 2.3. To obtain (ii) we choose $t' = r$. Then $u = t'\sigma$ is obviously satisfied and $\mathcal{US}_\ell(t') = \mathcal{US}_\ell(r) \subseteq \mathcal{US}_\ell$ follows by definition of $\mathcal{US}_\ell$. To prove (iii) let $\mathcal{F}\mathrm{un}(t) \subseteq \mathcal{US}_\ell$. Then $f \in \mathcal{US}_\ell$ and thus $l \to r \in \mathcal{U}_\ell$. Moreover, by the closure property in Definition 3.1(iii) we conclude $\mathcal{F}\mathrm{un}(r) \subseteq \mathcal{US}_\ell$. As the rule is usable we know that $\mathcal{A}$ is a model of this rule. Hence we can finish the root reduction case:

$$[\alpha]_{\mathcal{A}}(t\sigma) = [\alpha]_{\mathcal{A}}(l\sigma) = [\alpha_\sigma]_{\mathcal{A}}(l) = [\alpha_\sigma]_{\mathcal{A}}(r) = [\alpha]_{\mathcal{A}}(r\sigma) = [\alpha]_{\mathcal{A}}(u).$$

Now we consider a reduction below the root: $t_i\sigma = C'[l\sigma] \xrightarrow{\mathsf{i}}_{\mathcal{R}} C'[r\sigma] = u_i$ and $u = f(t_1\sigma, \ldots, u_i, \ldots, t_n\sigma)$. By Definition 3.1(i) we have $\mathcal{US}_\ell(t_i) \subseteq \mathcal{US}_\ell(t) \subseteq \mathcal{US}_\ell$. Hence, we can use the induction hypothesis for t_i. To prove (i) we consider two cases. First, if $L_f = \varnothing$ then

$$\begin{aligned}\mathsf{lab}_\alpha(t\sigma) = f(\mathsf{lab}_\alpha(t_1\sigma), \ldots, \mathsf{lab}_\alpha(t_i\sigma), \ldots, \mathsf{lab}_\alpha(t_n\sigma)) &\xrightarrow{\mathsf{i}}_{\mathsf{lab}(\mathcal{R})} \\ f(\mathsf{lab}_\alpha(t_1\sigma), \ldots, \mathsf{lab}_\alpha(u_i), \ldots, \mathsf{lab}_\alpha(t_n\sigma)) &= \mathsf{lab}_\alpha(u)\end{aligned}$$

directly proves (i). Otherwise, if $L_f \neq \varnothing$ then

$$\begin{aligned}\mathsf{lab}_\alpha(t\sigma) = f_a(\mathsf{lab}_\alpha(t_1\sigma), \ldots, \mathsf{lab}_\alpha(t_i\sigma), \ldots, \mathsf{lab}_\alpha(t_n\sigma)) &\xrightarrow{\mathsf{i}}_{\mathsf{lab}(\mathcal{R})} \\ f_a(\mathsf{lab}_\alpha(t_1\sigma), \ldots, \mathsf{lab}_\alpha(u_i), \ldots, \mathsf{lab}_\alpha(t_n\sigma))&\end{aligned}$$

where $a = \ell_f([\alpha]_{\mathcal{A}}(t_1\sigma), \ldots, [\alpha]_{\mathcal{A}}(t_i\sigma), \ldots, [\alpha]_{\mathcal{A}}(t_n\sigma))$. It remains to show that $a = \ell_f([\alpha]_{\mathcal{A}}(t_1\sigma), \ldots, [\alpha]_{\mathcal{A}}(u_i), \ldots, [\alpha]_{\mathcal{A}}(t_n\sigma))$. To this end it suffices to prove $[\alpha]_{\mathcal{A}}(t_i\sigma) = [\alpha]_{\mathcal{A}}(u_i)$ which directly follows from the induction hypothesis (iii) since $\mathcal{F}\mathrm{un}(t_i) \subseteq \mathcal{US}_\ell(t_i) \subseteq \mathcal{US}_\ell$ by Definition 3.1(ii).

To show (ii) we first get a term t'_i with $t'_i\sigma = u_i$ and $\mathcal{US}_\ell(t'_i) \subseteq \mathcal{US}_\ell$ by induction. We choose $t' = f(t_1, \ldots, t'_i, \ldots, t_n)$ and directly obtain $t'\sigma = u$. To prove $\mathcal{US}_\ell(t') \subseteq \mathcal{US}_\ell$ we define $\mathcal{US}^k_\ell(t')$ to be like $\mathcal{US}_\ell(t')$ where we only apply closure (iii) in Definition 3.1 at most k times. Then it suffices to prove $\mathcal{US}^k_\ell(t') \subseteq \mathcal{US}_\ell$ for all $k \in \mathbb{N}$ which we do by an inner induction on k. We first consider closure (i). Here, we use $\mathcal{US}_\ell(t) \subseteq \mathcal{US}_\ell$ and Definition 3.1(i) to obtain $\mathcal{US}_\ell(t_1) \cup \cdots \cup \mathcal{US}_\ell(t_n) \subseteq \mathcal{US}_\ell$. Thus, $\mathcal{US}^k_\ell(t_1) \cup \cdots \cup \mathcal{US}^k_\ell(t'_i) \cup \cdots \cup \mathcal{US}^k_\ell(t_n) \subseteq \mathcal{US}_\ell$ is also satisfied. For closure (ii) we only have to consider the case $L_f \neq \varnothing$. From $\mathcal{US}_\ell(t) \subseteq \mathcal{US}_\ell$ and Definition 3.1(ii) we conclude $\mathcal{F}\mathrm{un}(t_j) \subseteq \mathcal{US}_\ell$ for all $1 \leqslant j \leqslant n$. As $\mathcal{F}\mathrm{un}(t'_i) \subseteq \mathcal{US}_\ell$ by induction hypothesis (iii), we are done. For closure (iii) let $f \in \mathcal{US}^k_\ell(t')$. If $f \in \mathcal{US}^{k-1}_\ell(t')$ then we only have to apply the inner induction hypothesis. Otherwise, there is a rule $l \to r$ with $\mathrm{root}(l) \in \mathcal{US}^{k-1}_\ell(t')$ and $f \in \mathcal{F}\mathrm{un}(r)$. From the inner induction hypothesis we obtain $\mathrm{root}(l) \in \mathcal{US}_\ell = \bigcup_{l' \to r'} \mathcal{US}_\ell(r')$. Thus, for some r' we have $\mathrm{root}(l) \in \mathcal{US}_\ell(r')$ and by Definition 3.1(iii) we know $f \in \mathcal{US}_\ell(r')$. But then $f \in \mathcal{US}_\ell$ as well.

To finally prove (iii) we assume $\mathcal{F}\mathrm{un}(t) \subseteq \mathcal{US}_\ell$. Then obviously $\mathcal{F}\mathrm{un}(t_i) \subseteq \mathcal{US}_\ell$. Thus, by induction hypothesis (iii) we know $\mathcal{F}\mathrm{un}(t_i') \subseteq \mathcal{US}_\ell$. So $\mathcal{F}\mathrm{un}(t') \subseteq \mathcal{US}_\ell$ is a consequence of $\mathcal{F}\mathrm{un}(t) \subseteq \mathcal{US}_\ell$. Moreover, we also obtain $[\alpha]_\mathcal{A}(t_i\sigma) = [\alpha]_\mathcal{A}(u_i)$ from the induction hypothesis (iii). Hence, we can finally prove (iii):

$$\begin{aligned}[\alpha]_\mathcal{A}(t\sigma) &= f_\mathcal{A}([\alpha]_\mathcal{A}(t_1\sigma), \ldots, [\alpha]_\mathcal{A}(t_i\sigma), \ldots, [\alpha]_\mathcal{A}(t_n\sigma)) \\ &= f_\mathcal{A}([\alpha]_\mathcal{A}(t_1\sigma), \ldots, [\alpha]_\mathcal{A}(u_i), \ldots, [\alpha]_\mathcal{A}(t_n\sigma)) = [\alpha]_\mathcal{A}(u).\end{aligned}$$

□

Theorem 3.4 *If $\mathcal{A}$ is a model of $\mathcal{U}_\ell$ then innermost termination of* $\mathsf{lab}(\mathcal{R})$ *implies innermost termination of $\mathcal{R}$.*

Proof. Suppose $\mathcal{R}$ is not innermost terminating. Then there is a minimal non-terminating term s which is not innermost terminating. By renaming the variables of the rules used for the reductions we can assume that for every rewrite step in this infinite reduction the corresponding rule is instantiated by the same normalized substitution σ. By minimality of s, after a number of reductions there must be a root step, i.e., $s \xrightarrow{\mathsf{i}}{}^*_\mathcal{R} l\sigma \xrightarrow{\mathsf{i}}_\mathcal{R} r\sigma$ for some rule $l \to r \in \mathcal{R}$ where $r\sigma$ is not innermost terminating. By definition of $\mathcal{US}_\ell$ we know $\mathcal{US}_\ell(r) \subseteq \mathcal{US}_\ell$. Hence, starting the infinite reduction with $r\sigma$ we can now simulate every reduction step with the corresponding labeled term $\mathsf{lab}_\alpha(r\sigma)$ using the labeled TRS $\mathsf{lab}(\mathcal{R})$. If $r\sigma \xrightarrow{\mathsf{i}}_\mathcal{R} r_1 \xrightarrow{\mathsf{i}}_\mathcal{R} r_2 \xrightarrow{\mathsf{i}}_\mathcal{R} \cdots$ then by Lemma 3.3(ii) we obtain terms $t_1, t_2, \ldots$ such that $r_i = t_i\sigma$ and $\mathcal{US}_\ell(t_i) \subseteq \mathcal{US}_\ell$. Using Lemma 3.3(i) we can finally prove that $\mathsf{lab}(\mathcal{R})$ is not innermost terminating:

$$\mathsf{lab}_\alpha(r\sigma) \xrightarrow{\mathsf{i}}_{\mathsf{lab}(\mathcal{R})} \mathsf{lab}_\alpha(r_1) = \mathsf{lab}_\alpha(t_1\sigma) \xrightarrow{\mathsf{i}}_{\mathsf{lab}(\mathcal{R})} \mathsf{lab}_\alpha(r_2) = \mathsf{lab}_\alpha(t_2\sigma) \xrightarrow{\mathsf{i}}_{\mathsf{lab}(\mathcal{R})} \cdots$$

□

With Theorem 3.4 it is now possible to prove innermost termination of the leading example with the specified algebra and the specified LPO.

4 Improved Labeling for Innermost Termination

We first modify the leading example to show a limitation of predictive labeling. Afterwards we present an improvement to overcome this limitation.

Example 4.1 We consider a reformulated version of the TRS in the leading example which uses an accumulator. Let $\mathcal{R}$ consist of the rules (1)–(8) together with the following rules:

$$\mathsf{quot}(x, y) \to \div(x, y, \mathsf{0}) \tag{17}$$
$$\div(x, y, z) \to \mathsf{if}(y \geqslant \mathsf{s}(\mathsf{0}), x \geqslant y, x, y, z) \tag{18}$$
$$\mathsf{if}(\mathsf{false}, b, x, y, z) \to \mathsf{div\text{-}by\text{-}zero} \tag{19}$$
$$\mathsf{if}(\mathsf{true}, \mathsf{false}, x, y, z) \to z \tag{20}$$
$$\mathsf{if}(\mathsf{true}, \mathsf{true}, x, y, z) \to \div(x - y, y, \mathsf{id\text{-}inc}(z)) \tag{21}$$

The problem is that we cannot apply Theorem 3.4 with the given algebra $\mathcal{A}$; because id-inc now occurs below the labeled symbol $\div$, the problematic rules (7) and (8) are usable and $\mathcal{A}$ is not a model of these rules. However, the labeling function $\ell_{\div}$ ignores its third argument and thus, we do not need semantics for id-inc to compute the label for $\div$. Therefore, we would like to remove the id-inc-rules from the set of usable rules. How this can be achieved is shown in the remainder of this section.

First, we need a notion to express which arguments of a function symbol should be ignored. To this end we use an *argument filter* which maps every symbol to the set of arguments that are not ignored. We further need a notion to express that an argument filter is suitable for an algebra and a labeling function. Argument filters were introduced in [1] and have been recently [3] used to reduce the usable rules in connection with the dependency pair method.

Definition 4.2 An *argument filter* is a mapping $\pi\colon \mathcal{F} \to 2^{\mathbb{N}}$ such that $\pi(f)$ is a subset of $\{1, \ldots, n\}$ for all $f \in \mathcal{F}$ with arity n. The application of an argument filter π to a term t is denoted by $\pi(t)$ and defined as follows:

$$\pi(t) = \begin{cases} t & \text{if } t \text{ is a variable} \\ f(\pi(t_{i_1}), \ldots, \pi(t_{i_k})) & \text{if } t = f(t_1, \ldots, t_n) \text{ and } \pi(f) = \{i_1, \ldots, i_k\} \end{cases}$$

An algebra $\mathcal{A}$ is π*-conform* if $f_{\mathcal{A}}$ may depend on the i-th argument only if $i \in \pi(f)$. Similarly, a labeling function ℓ_f is π*-conform* if ℓ_f may depend on the i-th argument only if $i \in \pi(f)$.

From now on it is assumed that all algebras and labeling functions are π-conform. We refine Definition 3.1 to get less usable rules when regarding the argument filter.

Definition 4.3 Let $\mathcal{R}$ be a TRS, ℓ a labeling, and π an argument filter. We define the set $\mathcal{US}_{\ell,\pi}(t) \subseteq \mathcal{F}$ of *usable symbols with respect to* π of a term t inductively. If $t \in \mathcal{V}$ then $\mathcal{US}_{\ell,\pi}(t) = \varnothing$. If $t = f(t_1, \ldots, t_n)$ then $\mathcal{US}_{\ell,\pi}(t)$ is the least set such that

(i) $\mathcal{US}_{\ell,\pi}(t_1) \cup \cdots \cup \mathcal{US}_{\ell,\pi}(t_n) \subseteq \mathcal{US}_{\ell,\pi}(t)$,

(ii) if $L_f \neq \varnothing$ and $i \in \pi(f)$ then $\mathcal{F}\mathrm{un}(\pi(t_i)) \subseteq \mathcal{US}_{\ell,\pi}(t)$, and

(iii) if $l \to r \in \mathcal{R}$ and $\mathrm{root}(l) \in \mathcal{US}_{\ell,\pi}(t)$ then $\mathcal{F}\mathrm{un}(\pi(r)) \subseteq \mathcal{US}_{\ell,\pi}(t)$.

The usable symbols $\mathcal{US}_{\ell,\pi}(\mathcal{R})$ and the usable rules $\mathcal{U}_{\ell,\pi}(\mathcal{R})$ with respect to π are defined as

$$\mathcal{US}_{\ell,\pi}(\mathcal{R}) = \bigcup_{l \to r \in \mathcal{R}} \mathcal{US}_{\ell,\pi}(r)$$

and

$$\mathcal{U}_{\ell,\pi}(\mathcal{R}) = \{l \to r \in \mathcal{R} \mid \mathrm{root}(l) \in \mathcal{US}_{\ell,\pi}(\mathcal{R})\}.$$

As before, we assume a fixed TRS $\mathcal{R}$ and therefore just write $\mathcal{US}_{\ell,\pi}$ and $\mathcal{U}_{\ell,\pi}$ for $\mathcal{US}_{\ell,\pi}(\mathcal{R})$ and $\mathcal{U}_{\ell,\pi}(\mathcal{R})$. We now show how innermost termination of the TRS in Example 4.1 can be proved if one only has to find a model for the usable rules with respect to π.

Example 4.4 We choose $\pi(\div) = \{1, 2\}$ and $\pi(\mathsf{if}) = \{1, 2, 3, 4\}$ in Example 4.1. Then $\mathcal{A}$ and the labeling functions are π-conform and the usable rules are (1)–(6) as in the leading example. We obtain a similar labeled TRS and termination is proved by a similar LPO. One only has to extend the precedence for the new symbol quot by demanding $\mathsf{quot} \sqsupset \div_i$ for all $i \in \mathbb{N}$.

The only missing step is to extend the results of Lemma 3.3 and Theorem 3.4 to the refined version of usable rules in Definition 4.3.

Lemma 4.5 *Let $\mathcal{A}$ be a model of $\mathcal{U}_{\ell,\pi}$, let $\mathcal{US}_{\ell,\pi}(t) \subseteq \mathcal{US}_{\ell,\pi}$, and let σ be a normalized substation such that $t\sigma = C[l\sigma] \xrightarrow{\mathsf{i}}_{\mathcal{R}} C[r\sigma] = u$ for a rule $l \to r \in \mathcal{R}$. Then the following properties are satisfied:*

(i) $\mathsf{lab}_\alpha(t\sigma) \xrightarrow{\mathsf{i}}_{\mathsf{lab}(\mathcal{R})} \mathsf{lab}_\alpha(u)$,

(ii) *there is a term t' such that $u = t'\sigma$ and $\mathcal{US}_{\ell,\pi}(t') \subseteq \mathcal{US}_{\ell,\pi}$, and*

(iii) *$\mathcal{F}\mathrm{un}(\pi(t)) \subseteq \mathcal{US}_{\ell,\pi}$ implies both $\mathcal{F}\mathrm{un}(\pi(t')) \subseteq \mathcal{US}_{\ell,\pi}$ and $[\alpha]_{\mathcal{A}}(t\sigma) = [\alpha]_{\mathcal{A}}(u)$.*

Proof. The proof is completely similar to the proof of Lemma 3.3 where one replaces $\mathcal{US}_\ell$ by $\mathcal{US}_{\ell,\pi}$, $\mathcal{F}\mathrm{un}(t)$ by $\mathcal{F}\mathrm{un}(\pi(t))$, and $\mathcal{U}_\ell$ by $\mathcal{U}_{\ell,\pi}$. Therefore, we only give the three additional cases which arise when considering reductions below the root. First, to prove (i) one has to show $\ell_f([\alpha]_{\mathcal{A}}(t_1\sigma), \ldots, [\alpha]_{\mathcal{A}}(t_i\sigma), \ldots, [\alpha]_{\mathcal{A}}(t_n\sigma)) = \ell_f([\alpha]_{\mathcal{A}}(t_1\sigma), \ldots, [\alpha]_{\mathcal{A}}(u_i), \ldots, [\alpha]_{\mathcal{A}}(t_n\sigma))$ as before. If $i \in \pi(f)$ then one can conclude $\mathcal{F}\mathrm{un}(\pi(t_i)) \subseteq \mathcal{US}_{\ell,\pi}$ and proceed as in the proof of Lemma 3.3. Otherwise, $i \notin \pi(f)$ and thus, the equality is valid as ℓ_f ignores its i-th argument. Second, to prove (ii) one has to show $\mathcal{US}_{\ell,\pi}(t') \subseteq \mathcal{US}_{\ell,\pi}$ by looking at the closure properties (i) and (ii) of Definition 4.3. When considering (ii) one cannot conclude $\mathcal{F}\mathrm{un}(\pi(t_i)) \subseteq \mathcal{US}_{\ell,\pi}$ if $i \notin \pi(f)$. However, in that case $\mathcal{F}\mathrm{un}(\pi(t'_i)) \subseteq \mathcal{US}_{\ell,\pi}$ is not required to satisfy (ii). Finally, to prove (iii) one gets the additional case $i \notin \pi(f)$. Then $\mathcal{F}\mathrm{un}(\pi(t')) = \mathcal{F}\mathrm{un}(\pi(t)) \subseteq \mathcal{US}_{\ell,\pi}$ as $\pi(t) = \pi(t')$. Moreover, using the fact that $f_{\mathcal{A}}$ ignores its i-th argument immediately yields $[\alpha]_{\mathcal{A}}(t\sigma) = [\alpha]_{\mathcal{A}}(u)$. $\square$

We are now ready to present the result about improved predictive labeling where under the assumption of π-conformity one only has to find a model for the usable rules with respect to π. As demonstrated in Example 4.1 and Example 4.4 this clearly extends Theorem 3.4.

Theorem 4.6 *Let π be an argument filter. If $\mathcal{A}$ is a model of $\mathcal{U}_{\ell,\pi}$ and if both $\mathcal{A}$ and all labeling functions are π-conform then innermost termination of $\mathsf{lab}(\mathcal{R})$ implies innermost termination of $\mathcal{R}$.*

Proof. Just replace Lemma 3.3 by Lemma 4.5 in the proof of Theorem 3.4. $\square$

A possible extension of Theorem 4.6 is to redefine Definition 4.3 such that $\mathcal{US}_{\ell,\pi}(t_i) \subseteq \mathcal{US}_{\ell,\pi}(t)$ is only required if $i \in \pi(f)$. However the following example shows that this extension is unsound.

Example 4.7 Consider the TRS $\{\mathsf{f}(\mathsf{g}(\mathsf{a})) \to \mathsf{f}(\mathsf{g}(\mathsf{b})),\ \mathsf{b} \to \mathsf{a}\}$. We choose the algebra with carrier $A = \{0,1\}$ and interpretations $\mathsf{f}_{\mathcal{A}}(x) = \mathsf{g}_{\mathcal{A}}(x) = \mathsf{a}_{\mathcal{A}} = 0$ and $\mathsf{b}_{\mathcal{A}} = 1$. For the labeling we use $L_{\mathsf{f}} = L_{\mathsf{a}} = L_{\mathsf{b}} = \varnothing$, $L_{\mathsf{g}} = A$, and $\ell_{\mathsf{g}}(x) = x$. Then both the algebra and the labeling functions are π-conform for the argument filter π defined by $\pi(\mathsf{f}) = \pi(\mathsf{a}) = \pi(\mathsf{b}) = \varnothing$ and $\pi(\mathsf{g}) = \{1\}$. However, using the alternative definition of $\mathcal{US}_{\ell,\pi}(t)$ we get $\mathcal{US}_{\ell,\pi} = \varnothing$ and hence, $\mathcal{A}$ is a model for the usable rules. Thus, the extension cannot be sound as the labeled TRS $\{\mathsf{f}(\mathsf{g}_0(\mathsf{a})) \to \mathsf{f}(\mathsf{g}_1(\mathsf{b})),\ \mathsf{b} \to \mathsf{a}\}$ is terminating while $\mathcal{R}$ is not innermost terminating.

In the next section we combine the idea of usable rules with respect to an argument filter with predictive labeling for full rewriting.

5 Improved Predictive Labeling for Termination

As in [4], for improved predictive labeling in the termination case we do not allow arbitrary algebras but one has to use a so-called $\sqcup$-*algebra* ([4, Definition 8]).

Definition 5.1 Let $\mathcal{A}$ be an algebra and let $>_A$ be a well-founded order on the carrier A. We say that $(\mathcal{A}, >_A)$ is a $\sqcup$-*algebra* if for all finite subsets $X \subseteq A$ there exists a least upper bound $\bigsqcup X$ of X in A.

In the remainder of this section we assume that $\mathcal{R}$ is a *finitely branching* TRS, π an argument filter, and $(\mathcal{A}, >_A)$ a $\sqcup$-algebra such that all interpretations $f_{\mathcal{A}}$ and all labeling functions ℓ_f are weakly monotone and π-conform, and $\mathcal{U}_{\ell,\pi} \subseteq \geqslant_A$.

As in the previous sections we cannot directly achieve the result of Lemma 2.1(ii) to transform infinite $\mathcal{R}$ reductions into infinite reductions of $\mathsf{lab}(\mathcal{R}) \cup \mathcal{D}\mathrm{ec}$ since $\mathcal{A}$ is not a quasi-model of all rules in $\mathcal{R}$. Therefore, we introduce an alternative interpretation function $[\alpha]^*_{\mathcal{A}}(\cdot)$ for all terminating terms ($\mathcal{SN}$) similar to [4, Definition 9]. However, one has to perform a minor modification due to the difference between $\mathcal{US}_\ell$ and $\mathcal{G}_\ell$, cf. Example 3.2.

Definition 5.2 Let $t \in \mathcal{SN}$ and α an assignment. We define the interpretation $[\alpha]^*_{\mathcal{A}}(t)$ inductively as follows where $t' = f_{\mathcal{A}}([\alpha]^*_{\mathcal{A}}(t_1), \dots, [\alpha]^*_{\mathcal{A}}(t_n))$:

$$[\alpha]^*_{\mathcal{A}}(t) = \begin{cases} \alpha(x) & \text{if } t \text{ is a variable,} \\ t' & \text{if } t = f(t_1,\dots,t_n) \text{ and } f \in \mathcal{US}_{\ell,\pi}, \\ \bigsqcup \{[\alpha]^*_{\mathcal{A}}(u) \mid t \to_{\mathcal{R}} u\} \cup \{t'\} & \text{if } t = f(t_1,\dots,t_n) \text{ and } f \notin \mathcal{US}_{\ell,\pi}. \end{cases}$$

Note that the recursion in the definition of $[\alpha]^*_{\mathcal{A}}(\cdot)$ terminates because the union of $\to_{\mathcal{R}}$ and the proper superterm relation $\rhd$ is a well-founded relation on $\mathcal{SN}$.

Further note that the operation $\bigsqcup$ is applied only to finite sets as $\mathcal{R}$ is assumed to be finitely branching.

The induced labeling function [4, Definition 10] can be defined for terminating and for minimal non-terminating terms ($\mathcal{T}^\infty$) but not for arbitrary terms in $\mathcal{T}(\mathcal{F},\mathcal{V})$.

Definition 5.3 Let $t \in \mathcal{SN} \cup \mathcal{T}^\infty$ and α an assignment. We define the labeled term $\mathsf{lab}^*_\alpha(t)$ inductively as follows:

$$\mathsf{lab}^*_\alpha(t) = \begin{cases} t & \text{if } t \text{ is a variable,} \\ f(\mathsf{lab}^*_\alpha(t_1),\dots,\mathsf{lab}^*_\alpha(t_n)) & \text{if } t = f(t_1,\dots,t_n) \text{ and } L_f = \varnothing, \\ f_a(\mathsf{lab}^*_\alpha(t_1),\dots,\mathsf{lab}^*_\alpha(t_n)) & \text{if } t = f(t_1,\dots,t_n) \text{ and } L_f \neq \varnothing \end{cases}$$

where $a = \ell_f([\alpha]^*_\mathcal{A}(t_1),\dots,[\alpha]^*_\mathcal{A}(t_n))$.

The following lemma compares the predicted semantics of an instantiated terminating term to the original semantics of the uninstantiated term, in which the substitution becomes part of the assignment.

Definition 5.4 Given an assignment α and a substitution σ such that $\sigma(x) \in \mathcal{SN}$ for all variables x, the assignment α^*_σ is defined as $[\alpha]^*_\mathcal{A} \circ \sigma$ and the substitution $\sigma_{\mathsf{lab}^*_\alpha}$ as $\mathsf{lab}^*_\alpha \circ \sigma$.

Lemma 5.5 *If $t\sigma \in \mathcal{SN}$ then $[\alpha]^*_\mathcal{A}(t\sigma) \geqslant_A [\alpha^*_\sigma]_\mathcal{A}(t)$. If in addition $\mathcal{F}\mathrm{un}(\pi(t)) \subseteq \mathcal{US}_{\ell,\pi}$ then $[\alpha]^*_\mathcal{A}(t\sigma) = [\alpha^*_\sigma]_\mathcal{A}(t)$.*

Proof. We use structural induction on t. If $t \in \mathcal{V}$ then

$$[\alpha]^*_\mathcal{A}(t\sigma) = ([\alpha]^*_\mathcal{A} \circ \sigma)(t) = [\alpha^*_\sigma]_\mathcal{A}(t).$$

Suppose $t = f(t_1,\dots,t_n)$. We distinguish two cases.

(i) If $f \in \mathcal{US}_{\ell,\pi}$ then

$$\begin{aligned} [\alpha]^*_\mathcal{A}(t\sigma) = f_\mathcal{A}([\alpha]^*_\mathcal{A}(t_1\sigma),\dots,[\alpha]^*_\mathcal{A}(t_n\sigma)) &\geqslant_A \\ f_\mathcal{A}([\alpha^*_\sigma]_\mathcal{A}(t_1),\dots,[\alpha^*_\sigma]_\mathcal{A}(t_n)) &= [\alpha^*_\sigma]_\mathcal{A}(t) \end{aligned}$$

where the inequality follows from the induction hypothesis (note that $t_i\sigma \in \mathcal{SN}$ for all $i = 1,\dots,n$) and the weak monotonicity of $f_\mathcal{A}$. If $\mathcal{F}\mathrm{un}(\pi(t)) \subseteq \mathcal{US}_{\ell,\pi}$ and $i \in \pi(f)$ then $\mathcal{F}\mathrm{un}(\pi(t_i)) \subseteq \mathcal{US}_{\ell,\pi}$ and thus $[\alpha]^*_\mathcal{A}(t_i\sigma) = [\alpha^*_\sigma]_\mathcal{A}(t_i)$ according to the induction hypothesis. Since $f_\mathcal{A}$ is π-conform, the inequality is turned into an equality.

(ii) If $f \notin \mathcal{US}_{\ell,\pi}$ then

$$\begin{aligned} [\alpha]^*_\mathcal{A}(t\sigma) &= \bigsqcup \{\cdots\} \cup \{f_\mathcal{A}([\alpha]^*_\mathcal{A}(t_1\sigma),\dots,[\alpha]^*_\mathcal{A}(t_n\sigma))\} \\ &\geqslant_A f_\mathcal{A}([\alpha]^*_\mathcal{A}(t_1\sigma),\dots,[\alpha]^*_\mathcal{A}(t_n\sigma)) \geqslant_A [\alpha^*_\sigma]_\mathcal{A}(t) \end{aligned}$$

again using weak monotonicity of $f_\mathcal{A}$ and the induction hypothesis. As in this case $\mathcal{F}\mathrm{un}(\pi(t)) \not\subseteq \mathcal{US}_{\ell,\pi}$, we have already proved the second part of the lemma.

□

The next lemma does the same for labeled terms. Since the label of a function symbol only depends on the semantics of its arguments, we can only deal with terminating and minimal non-terminating terms.

Lemma 5.6 *If* $t\sigma \in \mathcal{SN} \cup \mathcal{T}^\infty$ *then* $\mathsf{lab}^*_\alpha(t\sigma) \to^*_{\mathcal{D}\mathrm{ec}} \mathsf{lab}_{\alpha^*_\sigma}(t)\sigma_{\mathsf{lab}^*_\alpha}$. *If in addition* $\mathcal{US}_{\ell,\pi}(t) \subseteq \mathcal{US}_{\ell,\pi}$ *then* $\mathsf{lab}^*_\alpha(t\sigma) = \mathsf{lab}_{\alpha^*_\sigma}(t)\sigma_{\mathsf{lab}^*_\alpha}$.

Proof. We use structural induction on t. If t is a variable then $\mathsf{lab}^*_\alpha(t\sigma) = t\sigma_{\mathsf{lab}^*_\alpha} = \mathsf{lab}_{\alpha^*_\sigma}(t)\sigma_{\mathsf{lab}^*_\alpha}$. Otherwise $t = f(t_1,\dots,t_n)$. Note that $t_1,\dots,t_n \in \mathcal{SN}$. The induction hypothesis yields $\mathsf{lab}^*_\alpha(t_i\sigma) \to^*_{\mathcal{D}\mathrm{ec}} \mathsf{lab}_{\alpha^*_\sigma}(t_i)\sigma_{\mathsf{lab}^*_\alpha}$ for all $i = 1,\dots,n$. Moreover, whenever $\mathcal{US}_{\ell,\pi}(t) \subseteq \mathcal{US}_{\ell,\pi}$ then by Definition 4.3(i) $\mathcal{US}_{\ell,\pi}(t_i) \subseteq \mathcal{US}_{\ell,\pi}$ for every i and thus $\mathsf{lab}^*_\alpha(t_i\sigma) = \mathsf{lab}_{\alpha^*_\sigma}(t_i)\sigma_{\mathsf{lab}^*_\alpha}$ by the induction hypothesis. We distinguish three cases.

(i) If $L_f = \varnothing$ then

$$\begin{aligned}\mathsf{lab}^*_\alpha(t\sigma) &= f(\mathsf{lab}^*_\alpha(t_1\sigma),\dots,\mathsf{lab}^*_\alpha(t_n\sigma))\\ &\to^*_{\mathcal{D}\mathrm{ec}} f(\mathsf{lab}_{\alpha^*_\sigma}(t_1)\sigma_{\mathsf{lab}^*_\alpha},\dots,\mathsf{lab}_{\alpha^*_\sigma}(t_n)\sigma_{\mathsf{lab}^*_\alpha})\\ &= f(\mathsf{lab}_{\alpha^*_\sigma}(t_1),\dots,\mathsf{lab}_{\alpha^*_\sigma}(t_n))\sigma_{\mathsf{lab}^*_\alpha}\\ &= \mathsf{lab}_{\alpha^*_\sigma}(f(t_1,\dots,t_n))\sigma_{\mathsf{lab}^*_\alpha}.\end{aligned}$$

Of course, if $\mathcal{US}_{\ell,\pi}(t) \subseteq \mathcal{US}_{\ell,\pi}$ then there are no reduction steps.

(ii) If $L_f \neq \varnothing$ and $\mathcal{US}_{\ell,\pi}(t) \nsubseteq \mathcal{US}_{\ell,\pi}$ then

$$\begin{aligned}\mathsf{lab}^*_\alpha(t\sigma) &= f_a(\mathsf{lab}^*_\alpha(t_1\sigma),\dots,\mathsf{lab}^*_\alpha(t_n\sigma))\\ &\to^*_{\mathcal{D}\mathrm{ec}} f_a(\mathsf{lab}_{\alpha^*_\sigma}(t_1)\sigma_{\mathsf{lab}^*_\alpha},\dots,\mathsf{lab}_{\alpha^*_\sigma}(t_n)\sigma_{\mathsf{lab}^*_\alpha})\end{aligned}$$

and

$$\begin{aligned}\mathsf{lab}_{\alpha^*_\sigma}(t)\sigma_{\mathsf{lab}^*_\alpha} &= f_b(\mathsf{lab}_{\alpha^*_\sigma}(t_1),\dots,\mathsf{lab}_{\alpha^*_\sigma}(t_n))\sigma_{\mathsf{lab}^*_\alpha}\\ &= f_b(\mathsf{lab}_{\alpha^*_\sigma}(t_1)\sigma_{\mathsf{lab}^*_\alpha},\dots,\mathsf{lab}_{\alpha^*_\sigma}(t_n)\sigma_{\mathsf{lab}^*_\alpha})\end{aligned}$$

with $a = \ell_f([\alpha]^*_{\mathcal{A}}(t_1\sigma),\dots,[\alpha]^*_{\mathcal{A}}(t_n\sigma))$ and $b = \ell_f([\alpha^*_\sigma]_{\mathcal{A}}(t_1),\dots,[\alpha^*_\sigma]_{\mathcal{A}}(t_n))$. Lemma 5.5 yields $[\alpha]^*_{\mathcal{A}}(t_i\sigma) \geqslant_A [\alpha^*_\sigma]_{\mathcal{A}}(t_i)$ for all $i = 1,\dots,n$. Because the labeling function ℓ_f is weakly monotone in all its coordinates, $a \geqslant_A b$. If $a >_A b$ then $\mathcal{D}\mathrm{ec}$ contains the rewrite rule $f_a(x_1,\dots,x_n) \to f_b(x_1,\dots,x_n)$ and thus (also if $a = b$) $f_a(\mathsf{lab}_{\alpha^*_\sigma}(t_1)\sigma_{\mathsf{lab}^*_\alpha},\dots,\mathsf{lab}_{\alpha^*_\sigma}(t_n)\sigma_{\mathsf{lab}^*_\alpha}) \to^*_{\mathcal{D}\mathrm{ec}} \mathsf{lab}_{\alpha^*_\sigma}(t)\sigma_{\mathsf{lab}^*_\alpha}$. We conclude that $\mathsf{lab}^*_\alpha(t\sigma) \to^*_{\mathcal{D}\mathrm{ec}} \mathsf{lab}_{\alpha^*_\sigma}(t)\sigma_{\mathsf{lab}^*_\alpha}$.

(iii) If $L_f \neq \varnothing$ and $\mathcal{US}_{\ell,\pi}(t) \subseteq \mathcal{US}_{\ell,\pi}$ then

$$\begin{aligned}\mathsf{lab}^*_\alpha(t\sigma) &= f_a(\mathsf{lab}^*_\alpha(t_1\sigma),\dots,\mathsf{lab}^*_\alpha(t_n\sigma))\\ &= f_a(\mathsf{lab}_{\alpha^*_\sigma}(t_1)\sigma_{\mathsf{lab}^*_\alpha},\dots,\mathsf{lab}_{\alpha^*_\sigma}(t_n)\sigma_{\mathsf{lab}^*_\alpha})\\ &= f_a(\mathsf{lab}_{\alpha^*_\sigma}(t_1),\dots,\mathsf{lab}_{\alpha^*_\sigma}(t_n))\sigma_{\mathsf{lab}^*_\alpha}\end{aligned}$$

and

$$\mathsf{lab}_{\alpha^*_\sigma}(t)\sigma_{\mathsf{lab}^*_\alpha} = f_b(\mathsf{lab}_{\alpha^*_\sigma}(t_1), \dots, \mathsf{lab}_{\alpha^*_\sigma}(t_n))\sigma_{\mathsf{lab}^*_\alpha}$$

with $a = \ell_f([\alpha]^*_{\mathcal{A}}(t_1\sigma), \dots, [\alpha]^*_{\mathcal{A}}(t_n\sigma))$ and $b = \ell_f([\alpha^*_\sigma]_{\mathcal{A}}(t_1), \dots, [\alpha^*_\sigma]_{\mathcal{A}}(t_n))$. We need to show that $a = b$. Because ℓ_f is π-conform, this amounts to showing $[\alpha]^*_{\mathcal{A}}(t_i\sigma) = [\alpha^*_\sigma]_{\mathcal{A}}(t_i)$ for $i \in \pi(f)$. If we can show that $\mathcal{F}\mathrm{un}(\pi(t_i)) \subseteq \mathcal{US}_{\ell,\pi}$, this follows from Lemma 5.5. (Note that $t_i\sigma \in \mathcal{SN}$ as $t\sigma \in \mathcal{SN} \cup \mathcal{T}^\infty$.) But this can directly be concluded from $\mathcal{F}\mathrm{un}(\pi(t_i)) \subseteq \mathcal{US}_{\ell,\pi}(t) \subseteq \mathcal{US}_{\ell,\pi}$ by closure property (ii) of Definition 4.3.

□

We further need to know that the predicted semantics decreases when rewriting.

Lemma 5.7 *Let $t, u \in \mathcal{SN}$. If $t \to_{\mathcal{R}} u$ then $[\alpha]^*_{\mathcal{A}}(t) \geqslant_A [\alpha]^*_{\mathcal{A}}(u)$.*

Proof. We perform structural induction on t. Obviously, t is not a variable, so let $t = f(t_1, \dots, t_n)$. If $f \notin \mathcal{US}_{\ell,\pi}$ then

$$[\alpha]^*_{\mathcal{A}}(t) = \bigsqcup \{[\alpha]^*_{\mathcal{A}}(v) \mid t \to_{\mathcal{R}} v\} \cup \{\cdots\} \geqslant_A [\alpha]^*_{\mathcal{A}}(u)$$

since $[\alpha]^*_{\mathcal{A}}(u) \in \{[\alpha]^*_{\mathcal{A}}(v) \mid t \to_{\mathcal{R}} v\}$. Thus, for the remaining proof we may assume $f \in \mathcal{US}_{\ell,\pi}$. We consider two cases.

(i) First we consider a root reduction $t = l\sigma \to_{\mathcal{R}} r\sigma = u$. As $\mathrm{root}(l) = \mathrm{root}(t) = f \in \mathcal{US}_{\ell,\pi}$ we know $l \to r \in \mathcal{U}_{\ell,\pi}$ and $\mathcal{F}\mathrm{un}(\pi(r)) \subseteq \mathcal{US}_{\ell,\pi}$ due to closure property (iii) in Definition 4.3. From the assumption $\mathcal{U}_{\ell,\pi} \subseteq\ \geqslant_A$ we infer $l \geqslant_A r$. Using Lemma 5.5 we obtain

$$[\alpha]^*_{\mathcal{A}}(t) = [\alpha]^*_{\mathcal{A}}(l\sigma) \geqslant_A [\alpha^*_\sigma]_{\mathcal{A}}(l) \geqslant_A [\alpha^*_\sigma]_{\mathcal{A}}(r) = [\alpha]^*_{\mathcal{A}}(r\sigma) = [\alpha]^*_{\mathcal{A}}(u).$$

(ii) Next assume a reduction $t \to_{\mathcal{R}} f(t_1, \dots, u_i, \dots, t_n) = u$ below the root where $t_i \to_{\mathcal{R}} u_i$. The induction hypothesis yields $[\alpha]^*_{\mathcal{A}}(t_i) \geqslant_A [\alpha]^*_{\mathcal{A}}(u_i)$ and thus

$$\begin{aligned}[\alpha]^*_{\mathcal{A}}(t) =& f_{\mathcal{A}}([\alpha]^*_{\mathcal{A}}(t_1), \dots, [\alpha]_{\mathcal{A}}(t_i), \dots, [\alpha]_{\mathcal{A}}(t_n)) \geqslant_A \\ & f_{\mathcal{A}}([\alpha]^*_{\mathcal{A}}(t_1), \dots, [\alpha]_{\mathcal{A}}(u_i), \dots, [\alpha]_{\mathcal{A}}(t_n)) = [\alpha]^*_{\mathcal{A}}(u)\end{aligned}$$

by weak monotonicity of $f_{\mathcal{A}}$.

□

We are now ready for the key lemma, which states that rewrite steps between terminating and minimal non-terminating terms can be labeled.

Lemma 5.8 *Let $t, u \in \mathcal{SN} \cup \mathcal{T}^\infty$. If $t \to_{\mathcal{R}} u$ then $\mathsf{lab}^*_\alpha(t) \to^+_{\mathsf{lab}(\mathcal{R}) \cup \mathcal{D}\mathsf{ec}} \mathsf{lab}^*_\alpha(u)$.*

Proof. We use structural induction on t. Obviously $t = f(t_1, \dots, t_n)$. For a root reduction $t = l\sigma \to_{\mathcal{R}} r\sigma = u$ we infer $\mathsf{lab}^*_\alpha(t) = \mathsf{lab}^*_\alpha(l\sigma) \to^*_{\mathcal{D}\mathsf{ec}} \mathsf{lab}_{\alpha^*_\sigma}(l)\sigma_{\mathsf{lab}^*_\alpha} \to_{\mathsf{lab}(\mathcal{R})} \mathsf{lab}_{\alpha^*_\sigma}(r)\sigma_{\mathsf{lab}^*_\alpha} = \mathsf{lab}^*_\alpha(r\sigma) = \mathsf{lab}^*_\alpha(u)$ by Lemma 5.6. Otherwise, we have $u =$

$f(t_1,\ldots,u_i,\ldots,t_n)$ with $t_i \to_{\mathcal{R}} u_i$. We obtain $\mathsf{lab}^*_\alpha(t_i) \to^+_{\mathsf{lab}(\mathcal{R})\cup\mathcal{D}\mathsf{ec}} \mathsf{lab}^*_\alpha(u_i)$ from the induction hypothesis. We distinguish two cases.

(i) If $L_f = \varnothing$ then

$$\begin{aligned}\mathsf{lab}^*_\alpha(t) =& f(\mathsf{lab}^*_\alpha(t_1),\ldots,\mathsf{lab}^*_\alpha(t_i),\ldots,\mathsf{lab}^*_\alpha(t_n)) \to^+_{\mathsf{lab}(\mathcal{R})\cup\mathcal{D}\mathsf{ec}}\\ & f(\mathsf{lab}^*_\alpha(t_1),\ldots,\mathsf{lab}^*_\alpha(u_i),\ldots,\mathsf{lab}^*_\alpha(t_n)) = \mathsf{lab}^*_\alpha(u).\end{aligned}$$

(ii) If $L_f \neq \varnothing$ then

$$\begin{aligned}\mathsf{lab}^*_\alpha(t) =& f_a(\mathsf{lab}^*_\alpha(t_1),\ldots,\mathsf{lab}^*_\alpha(t_i),\ldots,\mathsf{lab}^*_\alpha(t_n)) \to^+_{\mathsf{lab}(\mathcal{R})\cup\mathcal{D}\mathsf{ec}}\\ & f_a(\mathsf{lab}^*_\alpha(t_1),\ldots,\mathsf{lab}^*_\alpha(u_i),\ldots,\mathsf{lab}^*_\alpha(t_n))\end{aligned}$$

with $a = \ell_f([\alpha]^*_{\mathcal{A}}(t_1),\ldots,[\alpha]^*_{\mathcal{A}}(t_i),\ldots,[\alpha]^*_{\mathcal{A}}(t_n))$ and

$$\mathsf{lab}^*_\alpha(u) = f_b(\mathsf{lab}^*_\alpha(t_1),\ldots,\mathsf{lab}^*_\alpha(u_i),\ldots,\mathsf{lab}^*_\alpha(t_n))$$

with $b = \ell_f([\alpha]^*_{\mathcal{A}}(t_1),\ldots,[\alpha]^*_{\mathcal{A}}(u_i),\ldots,[\alpha]^*_{\mathcal{A}}(t_n))$. Because $t_i \in \mathcal{SN}$ we can use Lemma 5.7 to obtain $[\alpha]^*_{\mathcal{A}}(t_i) \geqslant_A [\alpha]^*_{\mathcal{A}}(u_i)$. Hence, $a \geqslant_A b$ by weak monotonicity of ℓ_f and thus $f_a(\mathsf{lab}^*_\alpha(t_1),\ldots,\mathsf{lab}^*_\alpha(u_i),\ldots,\mathsf{lab}^*_\alpha(t_n)) \to^*_{\mathcal{D}\mathsf{ec}} \mathsf{lab}^*_\alpha(u)$.

□

We now have all the ingredients to prove the soundness of improved predictive labeling for termination.

Theorem 5.9 *Let $\mathcal{R}$ be a TRS, let π be an argument filter, and let $(\mathcal{A},>_A)$ be a $\sqcup$-algebra such that $\mathcal{A}$ is a quasi-model of $\mathcal{U}_{\ell,\pi}$ and all interpretation and labeling functions are weakly monotone and π-conform. If $\mathsf{lab}(\mathcal{R}) \cup \mathcal{D}\mathsf{ec}$ is terminating then so is $\mathcal{R}$.*

Proof. Note that for every term $t \in \mathcal{T}^\infty$ there exist a rewrite rule $l \to r \in \mathcal{R}$, a substitution σ, and a subterm u of r such that $t \xrightarrow{>\epsilon}{}^* l\sigma \xrightarrow{\epsilon} r\sigma \trianglerighteq u\sigma$ and $l\sigma, u\sigma \in \mathcal{T}^\infty$. Let α be an arbitrary assignment. We will apply lab^*_α to the terms in the above sequence. From Lemma 5.8 we obtain $\mathsf{lab}^*_\alpha(t) \to^*_{\mathsf{lab}(\mathcal{R})\cup\mathcal{D}\mathsf{ec}} \mathsf{lab}^*_\alpha(l\sigma)$. Since $r\sigma$ need not be an element of $\mathcal{T}^\infty$, we cannot apply Lemma 5.8 to the step $l\sigma \xrightarrow{\epsilon} r\sigma$. Instead we use Lemma 5.6 to obtain $\mathsf{lab}^*_\alpha(l\sigma) \to^*_{\mathcal{D}\mathsf{ec}} \mathsf{lab}_{\alpha^*_\sigma}(l)\sigma_{\mathsf{lab}^*_\alpha}$. Since $\mathsf{lab}_{\alpha^*_\sigma}(l) \to \mathsf{lab}_{\alpha^*_\sigma}(r) \in \mathsf{lab}(\mathcal{R})$, $\mathsf{lab}_{\alpha^*_\sigma}(l)\sigma_{\mathsf{lab}^*_\alpha} \to_{\mathsf{lab}(\mathcal{R})} \mathsf{lab}_{\alpha^*_\sigma}(r)\sigma_{\mathsf{lab}^*_\alpha}$. Because u is a subterm of r, $\mathsf{lab}_{\alpha^*_\sigma}(r)\sigma_{\mathsf{lab}^*_\alpha} \trianglerighteq \mathsf{lab}_{\alpha^*_\sigma}(u)\sigma_{\mathsf{lab}^*_\alpha}$. From closure property (i) of Definition 4.3 we infer $\mathcal{US}_{\ell,\pi}(u) \subseteq \mathcal{US}_{\ell,\pi}(r)$. Since r is a right-hand side of a rewrite rule of $\mathcal{R}$, $\mathcal{US}_{\ell,\pi}(r) \subseteq \mathcal{US}_{\ell,\pi}$. Hence $\mathcal{US}_{\ell,\pi}(u) \subseteq \mathcal{US}_{\ell,\pi}$. Lemma 5.6 now yields $\mathsf{lab}_{\alpha^*_\sigma}(u)\sigma_{\mathsf{lab}^*_\alpha} = \mathsf{lab}^*_\alpha(u\sigma)$. Putting everything together, we obtain $\mathsf{lab}^*_\alpha(t) \to^+_{\mathsf{lab}(\mathcal{R})\cup\mathcal{D}\mathsf{ec}} \cdot \trianglerighteq \mathsf{lab}^*_\alpha(u\sigma)$. Now suppose that $\mathcal{R}$ is non-terminating. Then $\mathcal{T}^\infty$ is non-empty and thus there is an infinite sequence $t_1 \xrightarrow{>\epsilon}{}^* \cdot \xrightarrow{\epsilon} \cdot \trianglerighteq t_2 \xrightarrow{>\epsilon}{}^* \cdot \xrightarrow{\epsilon} \cdot \trianglerighteq \cdots$ By the above argument, this sequence is transformed into

$$\mathsf{lab}^*_\alpha(t_1) \to^+_{\mathsf{lab}(\mathcal{R})\cup\mathcal{D}\mathsf{ec}} \cdot \trianglerighteq \mathsf{lab}^*_\alpha(t_2) \to^+_{\mathsf{lab}(\mathcal{R})\cup\mathcal{D}\mathsf{ec}} \cdot \trianglerighteq \cdots$$

By introducing appropriate contexts, the latter sequence gives rise to an infinite reduction in $\mathsf{lab}(\mathcal{R}) \cup \mathcal{D}\mathrm{ec}$, contradicting the assumption that $\mathcal{R}$ is terminating. $\square$

We conclude this section with an example.

Example 5.10 Consider the TRS $\mathcal{R}$ consisting of (7), (8), and

$$\begin{aligned} \mathsf{nonZero}(\mathsf{0}) &\to \mathsf{false} && (22) & \mathsf{random}(x) &\to \mathsf{rand}(x, \mathsf{0}) && (26)\\ \mathsf{nonZero}(\mathsf{s}(x)) &\to \mathsf{true} && (23) & \mathsf{rand}(x,y) &\to \mathsf{if}(\mathsf{nonZero}(x), x, y) && (27)\\ \mathsf{p}(\mathsf{s}(x)) &\to x && (24) & \mathsf{if}(\mathsf{false}, x, y) &\to y && (28)\\ \mathsf{p}(\mathsf{0}) &\to \mathsf{0} && (25) & \mathsf{if}(\mathsf{true}, x, y) &\to \mathsf{rand}(\mathsf{p}(x), \mathsf{id\text{-}inc}(y)) && (29) \end{aligned}$$

Here, $\mathsf{random}(x)$ generates a random number between 0 and x. We use the algebra $\mathcal{A}$ with carrier $\mathbb{N}$ and natural interpretations $\mathsf{p}_{\mathcal{A}}(x) = \max(x-1, 0)$, $\mathsf{s}_{\mathcal{A}}(x) = x+1$, $\mathsf{0}_{\mathcal{A}} = \mathsf{false}_{\mathcal{A}} = 0$, $\mathsf{true}_{\mathcal{A}} = 1$, and $\mathsf{nonZero}_{\mathcal{A}}(x) = 0$ if $x = 0$, and 1 otherwise. If one takes the standard order $>$ on $\mathbb{N}$ then $\mathcal{A}$ is a $\sqcup$-algebra and a quasi-model for rules (22)–(25). Moreover, for the labeling with $L_{\mathsf{rand}} = L_{\mathsf{if}} = \mathbb{N}$, $\ell_{\mathsf{rand}}(n,m) = n$, $\ell_{\mathsf{if}}(b,n,m) = b + \max(n-1, 0)$, and $L_f = \varnothing$ for all other function symbols, both $\mathcal{A}$ and the labeling functions are monotone. Consider the argument filtering π defined by $\pi(\mathsf{rand}) = \{1\}$ and $\pi(f) = \{1, \ldots, n\}$ for all other function symbols f where n is the arity of f. Note that $\mathcal{A}$ and all labeling functions are π-conform. We have $\mathcal{U}_{\ell,\pi} = \{(22)\text{–}(25)\}$. According to Theorem 5.9, termination of $\mathcal{R}$ follows from termination of $\mathsf{lab}(\mathcal{R}) \cup \mathcal{D}\mathrm{ec}$. The rules (for all $j > i \geqslant 0$)

$$\begin{aligned} \mathsf{id\text{-}inc}(x) &\to x & \mathsf{nonZero}(\mathsf{0}) &\to \mathsf{false}\\ \mathsf{id\text{-}inc}(x) &\to \mathsf{s}(x) & \mathsf{nonZero}(\mathsf{s}(x)) &\to \mathsf{true}\\ \mathsf{p}(\mathsf{s}(x)) &\to x & \mathsf{random}(x) &\to \mathsf{rand}_i(x, \mathsf{0})\\ \mathsf{p}(\mathsf{0}) &\to \mathsf{0} & \mathsf{rand}_i(x,y) &\to \mathsf{if}_i(\mathsf{nonZero}(x), x, y)\\ \mathsf{if}_i(\mathsf{false}, x, y) &\to y & \mathsf{if}_{i+1}(\mathsf{true}, x, y) &\to \mathsf{rand}_{\max(i-1,0)}(\mathsf{p}(x), \mathsf{id\text{-}inc}(y))\\ \mathsf{rand}_j(x,y) &\to \mathsf{rand}_i(x,y) & \mathsf{if}_j(b, x, y) &\to \mathsf{if}_i(b, x, y) \end{aligned}$$

of the latter TRS are oriented by LPO with precedence

$$\mathsf{random} \sqsupset \cdots \sqsupset \mathsf{rand}_1 \sqsupset \mathsf{if}_1 \sqsupset \mathsf{rand}_0 \sqsupset \mathsf{if}_0 \sqsupset \mathsf{nonZero} \sqsupset \mathsf{id\text{-}inc} \sqsupset \mathsf{p} \sqsupset \mathsf{s} \sqsupset \mathsf{true} \sqsupset \mathsf{false}.$$

6 Conclusion

We have analyzed how the powerful technique of semantic labeling can be used to prove innermost termination. It turned out that semantic labeling can be used for models but not for quasi-models. We extended our results to predictive labeling such that one only has to find a model for the usable as opposed to all rules. This approach was further improved by incorporating argument filters. The latter extension was finally integrated with predictive labeling for termination.

The results presented in this paper should be implemented in order to test their effectiveness and combined with dependency pairs [1] to increase their applicability.

Semantic [9] and predictive [4] labeling with infinite (quasi-)models for termination have been implemented in the automatic termination prover TPA [6]. The underlying theory is worked out in [8] and [7]. In the latter paper predictive labeling for termination is combined with dependency pairs. Modifying these results to cover innermost termination is straightforward. Incorporating argument filterings will increase the search space but otherwise poses no challenge. We anticipate that the power of TPA and other termination provers will be increased by the results of this paper.

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