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Electronic Notes in Theoretical Computer Science 345 (2019) 233–247

Electronic Notes in Theoretical Computer Science

www.elsevier.com/locate/entcs

Lattices of Irreducibly-derived Closed Sets

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Abstract

This paper pursues an investigation on the lattices of irreducibly-derived closed sets initiated by Zhao and Ho (2015). This time we focus the closed set lattice arising from the irreducibly-derived topology of Scott topology. For a poset X, the set $\Gamma_{SI}(X)$ of all irreducibly-derived Scott-closed sets (for short, SI-closed sets) ordered by inclusion forms a complete lattice. We introduce the notions of C_{SI}-continuous posets and C_{SI}-prealgebraic posets and study their properties. We also introduce the SI-dominated posets and show that for any two SI-dominated posets X and Y, $X \cong Y$ if and only if the SI-closed set lattices above them are isomorphic. At last, we show that the category of strong complete posets with SI-continuous maps is Cartesian-closed.

Keywords: Irreducible sets; Strong complete posets; C_{SI}-continuity; C_{SI}-prealgebra; Cartesian-closed

1 Introduction

Domain theory or the more general theory of posets embodies the combination of an order and a topology. One of the basic and important results is that a poset is continuous if and only if the lattice of the Scott-closed sets above it is a completely distributive lattice. However, for a non-continuous poset, little is known about the order-theoretic properties of the Scott-closed set lattice. To study the Scott-closed set lattice on a general non-continuous poset, Ho and Zhao introduced the concept of a C-continuous poset in [4], and proved that the Scott-closed set lattice on a poset is a C-prealgebraic lattice, especially a C-continuous lattice. With the help of the C-continuity, they obtained some good results, such as: (i) a complete lattice L is isomorphic to the Scott-closed set lattice for a complete semilattice P if and only if L is weak-stably C-algebraic; (ii) for any two complete semilattices X and

[1] This work is supported by the National Natural Science Foundation of China (NO. 11861006 and NO. 11561002)

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https://doi.org/10.1016/j.entcs.2019.07.026
1571-0661/

Y, $X \cong Y$ if and only if the Scott-closed set lattices above them are isomorphic. In an invited presentation at the 6th International Symposium in Domain Theory, J. D. Lawson emphasized the need to develop the core of domain theory in the wider context of T_0 spaces instead of restricting to posets. Just as directed sets are to domains, irreducible sets play an important role in T_0 spaces. A nonempty subset F of a T_0 space X is irreducible if for any closed two sets A and B with $F \subseteq A \bigcup B$ one has either $F \subseteq A$ or $F \subseteq B$. Zhao and Ho [9] developed the core of domain theory in the context of T_0 spaces by choosing the irreducible sets. They studied the irreducibly-derived topology constructed from any given topology on a set. Moreover, they obtained an interesting result that Scott topology of a poset is exactly the irreducibly-derived topology of its Alexandroff topology. Then Zhao and Xu established some classes of SCL-faithful dcpos and identified some classes of C_σ-unique dcpos based on Scott-closed set lattices.[10,11] Inspired by the works mentioned above, this paper is devoted to investigate the closed set lattices arising from the irreducibly-derived topology of Scott topology.

This paper is arranged as follows. In Section 2, we recall some basic materials which will be used throughout this paper, and by the definition of a irreducibly-derived topology, we obtain the irreducibly-derived topology of Scott topology. In Section 3, we define the C_{SI}-continuity on posets by SI-closed sets, and then study its properties. In Section 4, we define SI-dominated posets and study the lattice of SI-closed sets on the SI-dominated posets. We obtain a conclusion similar to the literature [4], that is, for any two SI-dominated posets X and Y, $X \cong Y$ if and only if the SI-closed set lattices above them are isomorphic. In Section 5, we show that the category of strong complete posets with SI-continuous maps is Cartesian-closed.

2 Preliminaries

Given a topological space (X, τ), a nonempty subset F of X is called an irreducible set if for closed sets $A, B \subseteq X$ with $F \subseteq A \cup B$ one has either $F \subseteq A$ or $F \subseteq B$. The set of all irreducible sets of X will be denoted by $Irr_\tau(X)$. On irreducible sets, here are some elementary properties:

Proposition 2.1 *(Gierz, et al. [2]) For any topological space (X, τ), it holds that:*
(1) *Every singleton is irreducible;*
(2) *A set is in $Irr_\tau(X)$ if and only if its closure is in $Irr_\tau(X)$;*
(3) *The continuous image of an irreducible set is again irreducible;*
(4) *Every directed set of X is irreducible.*

A subset U of a poset X is called Scott-open if (i) $U = \uparrow U$, and (ii) for any directed set D in X, $\bigvee D \in U$ implies $U \bigcap D \neq \varnothing$ whenever $\bigvee D$ exists. The set of all Scott-open sets of X forms a topology (called Scott topology) on X, denoted by $\sigma(X)$.

A complement of a Scott-open set is called a Scott-closed set. We use $\Gamma(X)$ to denote the set of all Scott-closed sets of X. Thus a subset $C \subseteq X$ is Scott-closed if and only if (i) $C = \downarrow C$, and (ii) for any directed set $D \subseteq C$, if $\bigvee D$ exists then

$\bigvee D \in C$. $\sigma(X)$ and $\Gamma(X)$ are both complete and distributive lattices with respect to the inclusion relation.

A subset A of a poset X is called convex if for any $x < y < z$, $x, z \in A$ implies $y \in A$.

Let X be a poset. The way-below relation $\ll$ on X is defined by $x \ll y$ for $x, y \in X$ if for any directed set D with $\bigvee D$ existing, $b \leq \bigvee D$ implies $a \leq d$ for some $d \in D$.

A poset X is said to be continuous if for any $x \in X$, the set $\twoheaddownarrow x = \{y \in X \mid y \ll x\}$ is directed and $x = \bigvee \twoheaddownarrow x$.

According the definition of the irreducibly-derived topology in [9], we can obtain the irreducibly-derived topology from Scott topology on a poset, as described next.

Definition 2.2 Let X be a poset. A subset C of X is called SI-closed if the following conditions are satisfied:
(1) $C = \downarrow C$;
(2) For any $F \in Irr_\tau(X)$, $F \subseteq C$ implies $\bigvee F \in C$ whenever $\bigvee F$ exists.

The set of all SI-closed sets of X is denoted by $\Gamma_{SI}(X)$. A complement of a SI-closed set is called a SI-open set and the set of all SI-open sets of (X, τ) is denoted by $\sigma_{SI}(X)$. By Theorem 3.2 in [9], $\sigma_{SI}(X)$ is a topology on X and is called a irreducibly-derived topology of σ, a SI-topology for short. Obviously, for any poset X, $\Gamma_{SI}(X) \subseteq \Gamma(X)$.

Definition 2.3 (Ho, et al. [3]) A poset X is strongly complete if every irreducible set of X has a supremum. In this case we also say that X is a scpo.

Obviously, every scpo is a dcpo.

Definition 2.4 The SI-topology on a poset X is called lower hereditary if for every SI-closed set C, the SI-topology on C agrees with the relative SI-topology from X.

Proposition 2.5 *(Zhao and Ho [9]) A function $f : X \longrightarrow Y$ between posets X and Y is SI-continuous iff it is order-preserving and $f(\bigvee F) = \bigvee f(F)$ for any $F \in Irr_\sigma(X)$ with $\bigvee F$ existing.*

Lemma 2.6 *(Xu and Mao [6]) Let X be a poset and $A \subseteq X$. If s, t are two upper bounds of A with $s \leq t$ and A has a supremum in $\downarrow t$, denoted $\bigvee_t A$, then A has a supremum in $\downarrow s$, in this case $\bigvee_s A = \bigvee_t A$.*

Lemma 2.7 *Let X be a poset. The following statements are equivalent:*
(1) The SI-topology on X is lower hereditary;
(2) For any $x \in X$, the inclusion map from $\downarrow x$ into X is SI-continuous;
(3) Any minimal upper bound of any irreducible set of X is the supremum for that irreducible set;
(4) For any $x \in X$ and any irreducible set F, $x = \bigvee_x F$ implies $x = \bigvee F$.

Proof. (1)⇒(2): Follows from that any principal ideal is a SI-closed set.

(2)⇒(3): Let F be an irreducible set with a minimal upper bound b. Then b is the supremum of F in $\downarrow b$. Since the inclusion map from $\downarrow b$ into X is continuous, it follows from Proposition 2.5 that b is the supremum of F in X .

(3)⇒(1): Let E be a SI-closed set. A subset B that is SI-closed in E is easily verified to be SI-closed in X. Conversely suppose that A is SI-closed in X . Then $A \cap E \in \Gamma(E)$. To show that $A \cap E$ is SI-closed in E, let F be an irreducible set in $A \cap E$ that has the supremum b in E. Then $\bigvee_E F = b \in E$ and $b = \bigvee F$ by (3). It follows from the SI-closedness of A that $b \in A$. Hence $b \in A \cap E$. This shows that $A \cap E$ is closed in the SI-topology of E.

(3)⇒(4): Suppose that F is an irreducible set with $x = \bigvee_x F$. Then x is a minimal upper bound of F. By (3), we have $\bigvee F = x$.

(4)⇒(3): Let y be a minimal upper bound of an irreducible set F. Then $y = \bigvee_y F$. By (4), we have $y = \bigvee F$. □

Applying Lemmas 2.7, we obtain the following corollary.

Corollary 2.8 *Every scpo has a lower hereditary SI-topology.*

Lemma 2.9 *(Mao and Xu [5]) Let X be a poset, $x \in X$ and $A \subseteq \downarrow x = C$. Then $\bigvee_C A = \bigvee A$ whenever $\bigvee A$ exists; If $(X, \leq)$ is a semilattice, then $\bigvee A = \bigvee_C A$ whenever $\bigvee_C A$ exists, where $\bigvee_C A$ denotes the supremum of A in the principal ideal $\downarrow x = C$.*

Applying Lemmas 2.7 and Lemma 2.9, we obtain the following corollary.

Corollary 2.10 *Every semilattice has a lower hereditary SI-topology.*

Proposition 2.11 *Let X and Y be posets with lower hereditary SI-topologies. Then*
(1) *Every convex subset A of X in the inherited order has also a lower hereditary SI-topology. In particular, every SI-closed set of X has a lower hereditary SI-topology;*
(2) *The product $X \times Y$ has also a lower hereditary SI-topology.*

Proof. (1) Let $F \subseteq A$ be an irreducible set with a minimal upper bound $z \in A$. Then z is also a minimal upper bound of F in X because of the convexity of A and hence the supremum of F in X by Lemma 2.7.

(2) Let $F \subseteq X \times Y$ be an irreducible set with a minimal upper bound $z = (u, v)$. Then it is easy to see that u is a minimal upper bound of $p_1(F)$ and v is a minimal upper bound of $p_2(F)$, where $p_1 : X \times Y \longrightarrow X$ and $p_2 : X \times Y \longrightarrow Y$ are the projection maps. Since X and Y have the lower hereditary SI-topologies, we have $\bigvee p_1(F) = u$ and $\bigvee p_2(F) = v$. Thus $\bigvee F = (\bigvee p_1(F), \bigvee p_2(F)) = (u, v) = z$. Therefore, $X \times Y$ has the lower hereditary SI-topology by Lemma 2.7. □

3 C_{SI}-continuous posets

In this section, we define a new auxiliary relation on a poset X by SI-closed sets. Then we introduce the notions of C_{SI}-continuous and C_{SI}-algebraic posets, which is important for us to study the properties of SI-closed set lattices.

Definition 3.1 Let X be a poset and $x, y \in X$. We say that x is beneath y, denoted by $x \prec_{SI} y$, if for every nonempty SI-closed set $C \subseteq X$ with $\bigvee C$ existing, the relation $\bigvee C \geq y$ always implies that $x \in C$. We denote the set $\{x \in X : x \prec_{SI} y\}$ by $\Downarrow_{SI} y$ and the set $\{x \in X : y \prec_{SI} x\}$ by $\Uparrow_{SI} y$.

Remark 3.2 Obviously, for any poset $(X, \leq)$, $\prec_{SI}$ and $\prec$ defined in [4] are quite different. As $\Gamma_{SI}(X) \subseteq \Gamma(X)$, $x \prec y$ implies that $x \prec_{SI} y$. However, the converse is not true. If $(X, \leq)$ is k-bounded sober defined in [8], then $\prec_{SI} = \prec$.

Example 1. Let $X = (N \times (N \bigcup \{\infty\})) \bigcup \{\top\}$ and define $x \leq y$ if one of the following conditions holds: (1) $y = \top$; (2) $x = (m_1, n_1)$ and $y = (m_2, n_2)$, where $m_1 = m_2, n_1 \leq n_2 (\leq \infty)$ or $n_2 = \infty, n_1 \leq m_2$. Then $(X, \leq)$ is a poset. By [7], we know that $C = N \times (N \bigcup \{\infty\})$ is an irreducible Scott closed set and $\bigvee C = \top$. Hence it is easily checked that $\top \not\prec \top$ and $\top \prec_{SI} \top$.

Example 2. Let $X = N \times (N \bigcup \{\infty\})$ and define $(m, n) \leq (m_0, n_0)$ if $m = m_0, n \leq n_0$, or $n_0 = \infty$ and $n \leq m_0$. As was pointed out in [7], the Scott space $(X, \sigma(X))$ is k-bounded sober. Hence $\prec_{SI} = \prec$.

Remark 3.3 Let X be a poset. Then (1) $x \prec_{SI} y \Rightarrow x \leq y$; (2) $x \leq y \prec_{SI} z \leq w$ implies $x \prec_{SI} w$; (3) For all $x \in X$, $0 \prec_{SI} x$ whenever X is a pointed poset.

Proposition 3.4 *For a poset X, the following conditions are equivalent:*
(1) $x \prec_{SI} y$;
(2) *$x \in C$ for every SI-closed set C with $y \leq \bigvee C$ whenever $\bigvee C$ exists;*
(3) *$x \in \bigcap J(y)$, where $J(y) = \{C \in \Gamma_{SI}(X) : y \leq \bigvee C\}$.*

Corollary 3.5 *Let X be a poset and $x \in X$, then $\Downarrow_{SI} x$ is a SI-closed set of X.*

Proof. By Proposition 3.4, we have $\Downarrow_{SI} x = \bigcap J(x)$, and thus $\Downarrow_{SI} x$ is a SI-closed set of X. □

Definition 3.6 A poset X is said to be C_{SI}-continuous if it satisfies $x = \bigvee \Downarrow_{SI} x$ for all $x \in X$.

A C_{SI}-continuous poset which is also a complete lattice is called a C_{SI}-continuous lattice. Obviously, we have $\Downarrow x \subseteq \Downarrow_{SI} x \subseteq \downarrow x$, $\forall x \in X$ by Remark 3.2, and thus every C-continuous poset is also C_{SI}-continuous.

Proposition 3.7 *For a complete lattice X, the following are equivalent:*
(1) *(X, τ) is C_{SI}-continuous;*
(2) *For each $x \in X$, the set $\Downarrow_{SI} x$ is the smallest SI-closed set C with $x \leq \bigvee C$;*
(3) *For each $x \in X$, there is a smallest nonempty SI-closed set C such that $x \leq \bigvee C$;*
(4) *The $\bigvee$ map $r = (C \mapsto \bigvee C) : \Gamma_{SI}(X) \longrightarrow X$ has a lower adjoint;*
(5) *The $\bigvee$ map $r : \Gamma_{SI}(X) \longrightarrow X$ preserves all existing infs;*
(6) *For any collection $\{C_i : i \in I\}$ of SI-closed sets of X, the following equation holds:*

$$\bigwedge_{i \in I} \bigvee C_i = \bigvee \bigcap_{i \in I} C_i.$$

Proof. (1) ⇒ (2): Condition (1) holds if and only if for each $x \in X$, $\Downarrow_{SI} x \in J(x)$ by Definition 3.6. Thus (2) follows.

Condition (2) trivially implies (3).

(3) ⇒ (1): If $J(x)$ has a smallest element M, then $M \subseteq C$ for all $C \in J(x)$ and thus $M \subseteq \bigcap J(x) \subseteq M$. Hence $M = \bigcap J(x) = \Downarrow_{SI} x$.

Thus (1),(2) and (3) are equivalent.

(3)⇔(4): By the definition of Galois connection, the map r has a lower adjoint if and only if $minr^{-1}(\uparrow x)$ exists for all x. But $minr^{-1}(\uparrow x)$ is precisely the smallest element of $J(x)$.

(4) ⇒ (5): The $\bigvee$ map preserves infs by Theorem O-3.3 in [2].

(5) ⇒ (4): Obviously, for any $x \in X$, $r^{-1}(\uparrow x) = J(x) \neq \varnothing$, which implies that the $\bigvee$ map is cofinal. Thus the $\bigvee$ map has a lower adjoint by O-3.4 in [2].

(5) ⇔ (6): Obviously. □

Proposition 3.8 *A poset X is C_{SI}-continuous if and only if $X_\perp$ is C_{SI}-continuous, where $X_\perp$ is the poset obtained from X by adjoining a new bottom element.*

Proof. Suppose that X is C_{SI}-continuous and $x \in X_\perp$. If $x \in X$, then there is a smallest nonempty SI-closed set C_x of X with $\bigvee C_x \geq x$ by Proposition 3.7. Thus $C'_x = C_x \bigcup \{\perp\}$ is the smallest nonempty SI-closed set of $X_\perp$ with $\bigvee C'_x \geq x$. If $x =\perp$, then by Remark 3.3, $C_\perp = \{\perp\}$ is the smallest nonempty SI-closed set of $X_\perp$ with $\bigvee C_\perp \geq \perp$. Thus $X_\perp$ is C_{SI}-continuous by Proposition 3.7. Conversely, suppose that $X_\perp$ is C_{SI}-continuous and $x \in X$. By Proposition 3.7, there is a smallest nonempty SI-closed subset C'_x of $X_\perp$ with $\bigvee C'_x \geq x$. Then $C_x = C'_x \backslash \{\perp\}$ is the smallest nonempty SI-closed set C_x of X with $\bigvee C_x \geq x$. Thus X is C_{SI}-continuous by Proposition 3.7 again. □

Proposition 3.9 *Let X and Y be posets and $(X, \sigma(X)) \cong (Y, \sigma(Y))$. If X is C_{SI}-continuous, then Y is also C_{SI}-continuous.*

Proof. Obviously. □

Proposition 3.10 *Let X be a C_{SI}-continuous semilattice. Then every principal ideal of X is C_{SI}-continuous.*

Proof. Suppose that X is a C_{SI}-continuous semilattice. Then we claim that for all $x \in X$ and $u \in \downarrow x = C$, $\Downarrow_{SI} u \subseteq \Downarrow^C_{SI} u$ holds, where $\Downarrow^C_{SI} u$ denotes the subset $\Downarrow_{SI} u$ in C. In fact, for all $v \in \Downarrow_{SI} u$ and all $A \in \Gamma_{SI}(C)$ with $\bigvee_C A \geq u$, we have $A \in \Gamma_{SI}(X)$ by Definition 2.2 and Corollary 2.10. It follows from Lemma 2.9 that $x \geq \bigvee_C A = \bigvee A \geq u$. Since $v \in \Downarrow_{SI} u$, there is a $v \in A$. This shows that $v \in \Downarrow^C_{SI} u$. By the C_{SI}-continuity of X and Lemma 2.9, $u = \bigvee \Downarrow_{SI} u = \bigvee_C \Downarrow_{SI} u$. Clearly, u is an upper bound of $\Downarrow^C_{SI} u$ in the principal ideal $C = \downarrow x$. Suppose that t is any upper bound of $\Downarrow^C_{SI} u$ in C. Then t is an upper bound of $\Downarrow_{SI} u$ in C. Thus $u = \bigvee \Downarrow_{SI} u \leq t$. This shows that $\bigvee_C \Downarrow^C_{SI} u = u$. Thus for all $x \in X$, the principal ideal $C = \downarrow x$ is C_{SI}-continuous. □

Proposition 3.11 *Let X be a semilattice. If every principal ideal of X is C_{SI}-continuous, then X is C_{SI}-continuous.*

Proof. Suppose that for all $x \in X$, the principal ideal $C = \downarrow x$ is C_{SI}-continuous. We claim that $\Downarrow^C_{SI} x \subseteq \Downarrow_{SI} x$. In fact, for all $y \in \Downarrow^C_{SI} x$ and all $A \in \Gamma_{SI}(X)$ with $\bigvee A = z \geq x$, the principal ideal $B = \downarrow z$ is C_{SI}-continuous by the hypothesis. By Corollary 3.5 and 2.10, $\Downarrow^C_{SI} x \in \Gamma_{SI}(B)$ and thus $\Downarrow^C_{SI} x \in \Gamma_{SI}(C)$. By Lemma 2.9, we have $\bigvee_B \Downarrow^B_{SI} x = \bigvee_C \Downarrow^B_{SI} x = x$. Since $y \in \Downarrow^C_{SI} x$, there is a $u \in \Downarrow^B_{SI} x$ such that $y \leq u$. Since $A \in \Gamma_{SI}(X)$ with existing $\bigvee A = z \geq x$, we have $A \in \Gamma_{SI}(B)$ and $\bigvee_B A = \bigvee A = z \geq x$ by Corollary 3.5 and 2.10 again. It follows from $u \in \Downarrow^B_{SI} x$ that there is a $a \in A$ such that $a \geq u \geq y$. This shows that $y \in \Downarrow_{SI} x$ and thus $\Downarrow^C_{SI} x \subseteq \Downarrow_{SI} x$. By the C_{SI}-continuity of $C = \downarrow x$ and Lemma 2.9, we have $x = \bigvee_B \Downarrow^B_{SI} x = \bigvee \Downarrow^B_{SI} x$. Clearly x is an upper bound of $\Downarrow^B_{SI} x$ in X. Suppose that t is any upper bound of $\Downarrow_{SI} x$ in X. Then t is an upper bound of $\Downarrow^B_{SI} x$ in X. Thus $x = \bigvee \Downarrow^B_{SI} x \leq t$. This shows that $\bigvee \Downarrow^B_{SI} x = x$ and hence X is C_{SI}-continuous. □

By Propositions 3.10 and 3.11, we immediately have the following characterization of C_{SI}-continuous posets by principal ideals.

Theorem 3.12 *Let X be a semilattice. Then X is C_{SI}-continuous if and only if every principal ideal of X is C_{SI}-continuous.*

Proposition 3.13 *Let X be a C_{SI}-continuous lattice. The for any collection $\{F_i : i \in I\}$ of finite subsets of X the following equation holds:*

$$\bigwedge_{i \in I} \bigvee F_i = \bigvee_{f \in \prod_{i \in I} F_i} \bigcap_{i \in I} f(i).$$

In particular, every C_{SI}-continuous lattice space is distributive.

Proof. Denote the left hand side (respectively, the right hand side) of the equation by a (respectively, b). It suffices to prove that $a \leq b$. Let $a = \bigwedge_{i \in I} \bigvee F_i$ and $x \prec_{SI} a$. For each $i \in I$, the set $\downarrow F_i = \bigcup_{y \in F_i} \downarrow y$ is a SI-closed set by $\downarrow y \in \Gamma_{SI}(X)$, and $x \prec_{SI} a \leq \bigvee F_i$. Thus there is a $d_i \in F_i$ with $x \leq d_i$. Let $f \in \prod_{i \in I} F_i$ be defined by $f(i) = d_i, i \in I$. Then $x \leq \prod_{i \in I} f(i) \leq b$. Therefore $a = \bigvee \Downarrow_{SI} a$ by the C_{SI}-continuity of X and thus $a \leq b$. □

Theorem 3.14 *Let X be a complete lattice. Then the following are equivalent:*
(1) *X is C_{SI}-continuous and continuous;*
(2) *X is completely distributive.*

Proof. It suffices to show that (1) ⇒ (2). Assume that X is both C_{SI}-continuous and continuous. Since X is continuous, for each $a \in X$, $a = \bigvee \twoheaddownarrow a$. Now for each $x \ll a$, $x = \bigvee \Downarrow_{SI} x$. It follows that $a = \bigvee \{y \in X : \exists x.\ y \prec_{SI} x \ll a\}$. Next suppose $y \prec_{SI} x \ll a$, we shall show that $y \lhd a$. Let $A \subseteq X$ with $a \leq \bigvee A$. Construct the se $E = \{\bigvee B : B\ is\ a\ finite\ subset\ of\ A\}$. Then E is directed and thus is irreducible and $\bigvee E = \bigvee A \geq a$. Since $x \ll a$, there is a finite subset $B \subseteq A$ such that $x \leq \bigvee B = \bigvee \downarrow B$. Note that the last set $\downarrow B$ is SI-closed. So it follows from $y \prec_{SI} x$ that $y \leq d$ for some $d \in B \subseteq A$. This implies that $y \lhd a$. Hence X is completely distributive. □

Lemma 3.15 *For any poset X, if $\mathcal{C} \in \Gamma(\Gamma_{SI}(X))$, then $\bigcup \mathcal{C} \in \Gamma(X)$.*

Proof. Let $\mathcal{C}^* = \{C \in \Gamma(X) \mid \exists V \in \mathcal{C}\ s.t.\ C \subseteq V\}$, then $\mathcal{C} \subseteq \mathcal{C}^*$ and $\bigcup\mathcal{C} = \bigcup\mathcal{C}^*$. Therefore, we only need to show $\mathcal{C}^* \in \Gamma(\Gamma(X))$ by Proposition 4.1 in [4]. Obviously, $\mathcal{C}^*$ is a lower set of $\Gamma(X)$. For any directed set $\mathcal{D} \subseteq \Gamma(X)$ with $\mathcal{D} \subseteq \mathcal{C}^*$, $\mathcal{D}^* = \{cl_{\sigma_{SI}}(D) \mid D \in \mathcal{D}\}$ is a directed set of $\Gamma_{SI}(X)$. Moreover, for any $D \in \mathcal{D}$, there is a $V \in \mathcal{C}$ such that $D \subseteq V$, thus $cl_{\sigma_{SI}}(D) \subseteq V$. Since $\mathcal{C}$ is a lower set, $cl_{\sigma_{SI}}(D) \in \mathcal{C}$. Therefore, $\mathcal{D}^* \subseteq \mathcal{C}$, thus we have $\bigvee_{\Gamma_{SI}(X)}\mathcal{D}^* \in \mathcal{C}$. As $\bigvee_{\Gamma(X)}\mathcal{D} \subseteq \bigvee_{\Gamma(X)}\mathcal{D}^* \subseteq \bigvee_{\Gamma_{SI}(X)}\mathcal{D}^*$, $\bigvee_{\Gamma(X)}\mathcal{D} \in \mathcal{C}^*$. To sum up, $\mathcal{C}^* \in \Gamma(\Gamma(X))$. □

Lemma 3.16 *For any poset X, if $F \in Irr_\sigma(X)$, then $\mathcal{F} = \{\downarrow d : d \in F\}$ is an irreducible set of $\Gamma_{SI}(X)$.*

Proof. Let $\mathcal{A}, \mathcal{B} \in \Gamma(\Gamma_{SI}(X))$ such that $\mathcal{F} \subseteq \mathcal{A}\bigcup\mathcal{B}$, then for any $d \in F$, $\downarrow d \in \mathcal{A}$ or $\downarrow d \in \mathcal{B}$ and thus $F \subseteq (\bigcup\mathcal{A})\bigcup(\bigcup\mathcal{B})$. By Lemma 3.15, $\bigcup\mathcal{A}, \bigcup\mathcal{B} \in \Gamma(X)$, and so $F \subseteq (\bigcup\mathcal{A})$ or $F \subseteq (\bigcup\mathcal{B})$. Without loss of generality, we assume that $F \subseteq \bigcup\mathcal{A}$ and thus for any $d \in F$, there is a $A \in \mathcal{A}$ such that $d \in A$. Because A is a lower set, $\downarrow d \subseteq A$, and thus $\downarrow d \in \mathcal{A}$. Therefore, $\mathcal{F} = \{\downarrow d : d \in F\}$ is an irreducible set of $\Gamma_{SI}(X)$. □

Proposition 3.17 *Let X be a poset and $\mathcal{C} \in \Gamma_{SI}(\Gamma_{SI}(X))$. Then $\bigvee_{\Gamma_{SI}(X)}\mathcal{C} = \bigcup\mathcal{C}$.*

Proof. Note that member of $\mathcal{C}$ is a SI-closed set of $\Gamma_{SI}(X)$. In order to prove the equation, it suffices to show that $\bigcup\mathcal{C} \in \Gamma_{SI}(X)$. Obviously $\bigcup\mathcal{C} \in \Gamma(X)$ by Lemma 3.15. Now let F be any irreducible set of X contained in $\bigcup\mathcal{C}$ such that $\bigvee F$ exists in X. We want to prove that $\bigvee F \in C$ for some $C \in \mathcal{C}$. By Lemma 3.16, we have that $\mathcal{F} = \{\downarrow d : d \in F\}$ is an irreducible set of $\Gamma_{SI}(X)$. Moreover, $\mathcal{F} \subseteq \mathcal{C}$ because $\mathcal{C}$ is a lower set in $\Gamma_{SI}(\Gamma_{SI}(X))$. Since $\mathcal{C}$ is a SI-closed set of $\Gamma_{SI}(X)$, $\bigvee_{\Gamma_{SI}(X)}\mathcal{F} \in \mathcal{C}$. Note that $\bigvee_{\Gamma_{SI}(X)}\mathcal{F}$ is precisely $\downarrow\bigvee F$. Hence $\bigvee F \in C$ for some $C \in \mathcal{C}$. □

Definition 3.18 An element x of a poset X is called C_{SI}-compact if $x \prec_{SI} x$. We use $\kappa_{SI}(X)$ to denote the set of all the C_{SI}-compact elements of X.

Obviously, every C-compact element defined in [4] is C_{SI}-compact, thus $\kappa(X) \subseteq \kappa_{SI}(X)$.

Proposition 3.19 *Let X be a lattice.*
(1) If $r \in \kappa_{SI}(X)$, then r is co-prime;
(2) If X is a completely distributive lattice, then every co-prime element is C_{SI}-compact;
(3) If X is a complete lattice and $\kappa_{SI}(X) \neq \varnothing$, then $\kappa_{SI}(X)$ is a pointed scpo with respect to the order inherited from X.

Proof. (1) Suppose $r \in \kappa_{SI}(X)$ and $r \leq x \vee y$. Let $C = \downarrow\{x, y\}$. Then C is SI-closed and $\bigvee C = x \vee y$. Hence $r \in C$ which implies $r \leq x$ or $r \leq y$, thus r is co-prime.

(2) Let X be a completely distributive lattice and $r \in X$ be co-prime. Note that the set $\beta(r) = \{x \in X : x \lhd r\}$ is a directed set and thus $\beta(r)$ is an irreducible set and $\bigvee\beta(r) = r$. Since $x \lhd r$ implies $x \prec_{SI} r$, it follows from Corollary 3.5 that $\bigvee\beta(r) \prec_{SI} r$, i.e., $r \prec_{SI} r$. Hence r is C_{SI}-compact.

(3) Let F be an irreducible set in $\kappa_{SI}(X)$. It suffices to show that $\bigvee F \prec_{SI} \bigvee F$. So let $C \in \Gamma_{SI}(X)$ with $\bigvee F \leq \bigvee C$. Thus $d \leq \bigvee C$ for all $d \in F$. Since $F \subseteq \kappa_{SI}(X)$,

it follows that $d \prec_{SI} d$ for all $d \in F$ and so $F \subseteq C$. Because $C \in \Gamma_{SI}(X)$, this implies that $\bigvee F \in C$ and so $\bigvee F \prec_{SI} \bigvee F$, i.e., $\bigvee F \in \kappa_{SI}(X)$. Also $0 \prec_{SI} 0$ implies that $0 \in \kappa_{SI}(X)$. Hence $\kappa_{SI}(X)$ is a pointed scpo with respect to the order inherited from X. □

Proposition 3.20 *Let X be a poset and C be a nonempty SI-closed set of X. Then for each $x \in C$, $\downarrow x \prec_{SI} C$ holds in $\Gamma_{SI}(X)$.*

Proof. Let $x \in C$. Suppose $\mathcal{C} \in \Gamma_{SI}(\Gamma_{SI}(X))$ with $C \subseteq \bigvee_{\Gamma_{SI}(X)} \mathcal{C}$. Then $C \subseteq \bigcup \mathcal{C}$ by Proposition 3.17. Hence there is a $A \in \mathcal{C}$ such that $x \in A$. This means $\downarrow x \subseteq A$, and thus $\downarrow x \in \mathcal{C}$.

Corollary 3.21 *Let X be a poset. Then for each $x \in X$, it holds that $\downarrow x \in \kappa(\Gamma_{SI}(X))$.*

Definition 3.22 A poset X is said to be C_{SI}-prealgebraic if for each $a \in X$, $a = \bigvee(\downarrow a \bigcap \kappa_{SI}(X))$. A C_{SI}-prealgebraic poset X is C_{SI}-algebraic if for any $a \in X$, $\downarrow(\downarrow a \bigcap \kappa_{SI}(X)) \in \Gamma_{SI}(X)$.

Remark 3.23 (1) Obviously every C_{SI}-prealgebraic poset is C_{SI}-continuous, since $\downarrow a \bigcap \kappa_{SI}(X) \subseteq \Downarrow_{SI} a \subseteq \downarrow a$ and $a = \bigvee(\downarrow a \bigcap \kappa_{SI}(X)) = \bigvee \downarrow a$. In this case, we call a C_{SI}-(pre)algebraic poset which is also a complete lattice a C_{SI}-(pre)algebraic lattice.
(2) Every C_{SI}-prealgebraic poset is C-prealgebraic.

Proposition 3.24 *Let X be a poset and $F \in Irr_\sigma(X)$ with $\bigvee F$ existing. If $x \prec_{SI} y$ for all $x \in F$, then $\bigvee F \prec_{SI} y$.*

Proof. Let $C \in \Gamma_{SI}(X)$ be nonempty such that $\bigvee C$ exists with $y \leq \bigvee C$. Since $x \prec_{SI} y$ for all $x \in F$, it follows that $F \subseteq C$. Because C is SI-closed and F is irreducible, we have $\bigvee F \in C$. Thus $\bigvee F \prec_{SI} y$. □

Proposition 3.25 *In a poset X, the following statements are equivalent:*
(1) X is C_{SI}-algebraic;
(2) X is C_{SI}-continuous, and $x \prec_{SI} y$ if and only if there is a $k \in \kappa_{SI}(X)$ with $x \leq k \leq y$.

Proof. (1) ⇒ (2): Assume (1) and $x, y \in X$ with $x \prec_{SI} y$. Since $y = \bigvee C$ with the SI-closed set $C = \downarrow(\downarrow y \bigcap \kappa_{SI}(X))$ by (1), there is a $k \in \downarrow y \bigcap \kappa_{SI}(X)$ with $x \leq k$, and thus $x \leq k \leq y$. Conversely, let $x \leq k \prec_{SI} k \leq y$, then we have $x \prec_{SI} y$.

The C_{SI}-continuity of X follows directly from Remark 3.23.

(2) ⇒ (1): Assume (2) and let $y \in X$. Then $y = \Downarrow_{SI} y$ and $\Downarrow_{SI} y$ is SI-closed. For every $x \prec_{SI} y$, there is a C_{SI}-compact element k such that $x \leq k \leq y$ by (2), so we conclude that $y = \bigvee(\downarrow y \bigcap \kappa_{SI}(X))$. Further, $\downarrow(\downarrow y \bigcap \kappa_{SI}(X)) \in \Gamma_{SI}(X)$. In fact, we only show that for any irreducible set F with $F \subseteq \downarrow y \bigcap \kappa_{SI}(X)$, $\bigvee F \in \kappa_{SI}(X)$. Let $C \in \Gamma_{SI}(X)$ with $\bigvee F \leq \bigvee C$. As, for any $x \in F$, $x \prec_{SI} x$ and $x \leq \bigvee F$, $x \prec_{SI} \bigvee F$ and thus $F \subseteq C$. Then we have $\bigvee F \in C$ by $C \in \Gamma_{SI}(X)$. □

Proposition 3.26 *For any poset X, the lattice $\Gamma_{SI}(X)$ is C_{SI}-prealgebraic.*

Proof. This follows from Corollary 3.20 and the fact that $C = \bigcup_{x \in C} \downarrow x = \bigvee_{\Gamma_{SI}(X)} \{\downarrow x : x \in C\}$ holds for every $C \in \Gamma_{SI}(X)$. □

Corollary 3.27 *For any poset X, the following statements are equivalent:*
(1) $\Gamma_{SI}(X)$ *is a continuous lattice;*
(2) $\Gamma_{SI}(X)$ *is a completely distributive lattice.*

Proof. (1) $\Leftrightarrow$ (2): Clear from Theorem 3.14 and Proposition 3.26. □

4 SI-closed set lattices of SI-dominated poset

For two dcpos X and Y, $\Gamma(X) \cong \Gamma(Y) \nRightarrow X \cong Y$, Ho et al. [3] exhibit a new class domDCPO of dominated dcpos for which the result holds. Similarly, we first define the SI-dominated posets and then study some properties of the lattice $(\Gamma_{SI}(X), \subseteq)$ on it.

Definition 4.1 Given $C', C \in \Gamma_{SI}(X)$, we write $C' \lhd C$ if there is a $x \in C$ such that $C' \subseteq \downarrow x$. We write $\nabla_{SI} C$ for the set $\{C' \in \Gamma_{SI}(X) \mid C' \lhd C\}$

Clearly, an element C of $\Gamma_{SI}(X)$ is of the form $\downarrow x$ if and only if $C \lhd C$ holds and $C = \bigvee \nabla_{SI} C$.

Proposition 4.2 *$C' \lhd C$ implies $C' \prec_{SI} C$ for all $C', C \in \Gamma_{SI}(X)$.*

Proof. This holds by Proposition 3.20. □

Next, we give a condition for a poset X which guarantees the reverse implication.

Definition 4.3 A poset X is called SI-dominated if for every SI-closed set C of X, the collection $\nabla_{SI} C$ is SI-closed in $\Gamma_{SI}(X)$.

Lemma 4.4 *A poset X is SI-dominated if and only if $C' \prec_{SI} C$ implies $C' \lhd C$ for all $C', C \in \Gamma_{SI}(X)$.*

Proof. If: Let $C \in \Gamma_{SI}(X)$. By assumption, we have $\nabla_{SI} C = \{C' \in \Gamma_{SI}(X) \mid C' \lhd C\} = \{C' \in \Gamma_{SI}(X) \mid C' \prec_{SI} C\}$ and in Corollary 3.5 we showed that the latter is always SI-closed.

Only if: Let $C' \prec_{SI} C$. We know that $\nabla_{SI} C$ is SI-closed and $C = \bigvee \nabla_{SI} C$. Therefore, it must be the case that $C' \in \nabla_{SI} C$. □

Proposition 4.5 *Let X be a SI-dominated poset. Then $C \in \kappa_{SI}(\Gamma_{SI}(X))$ if and only if C is a principal ideal, i.e., $C = \downarrow x$ for some $x \in X$. Also, the principal ideal mapping $\downarrow : X \longrightarrow \kappa_{SI}(\Gamma_{SI}(X)), x \mapsto \downarrow x$ is an order-isomorphism.*

Proof. By Corollary 3.21, it suffices to prove the "only if" part. Suppose $C \in \kappa(\Gamma_{SI}(X))$, then $\nabla_{SI} C$ is a SI-closed set of $\Gamma_{SI}(X)$ and $\bigvee_{\Gamma_{SI}(X)} \nabla_{SI} C = C$. Since $C \prec_{SI} C$, it follows that $C \in \nabla_{SI} C$. Thus $C \subseteq \downarrow x$ for some $x \in C$. This means that $C = \downarrow x$. □

Theorem 4.6 *For any two SI-dominated posets X and Y, the following statements are equivalent:*

(1) $X \cong Y$;
(2) $\sigma_{SI}(X) \cong \sigma_{SI}(Y)$;
(3) $\Gamma_{SI}(X) \cong \Gamma_{SI}(Y)$.

Similar to the proof of Lemma 5.1 in [4], we can obtain that for any complete semilattice and any $C \in \Gamma_{SI}(X)$, $\nabla_{SI}C \in \Gamma(\Gamma_{SI}(X))$. Thus the following result can be checked easily.

Lemma 4.7 *Let X be a complete semilattice and $C \in \Gamma_{SI}(X)$. If $\mathcal{E}$ is an irreducible family in $\nabla_{SI}C$, then $F = \{\bigvee E \mid E \in \mathcal{E}\}$ is an irreducible set of X.*

Proof. Let $B, D \in \Gamma(X)$ such that $F \subseteq B \bigcup D$, then for any $E \in \mathcal{E}$, $\bigvee E \in B$ or $\bigvee E \in D$ and so $\mathcal{E} \subseteq \nabla_{SI}B \bigcup \nabla_{SI}D$. As $\mathcal{E}$ is irreducible, $\mathcal{E} \subseteq \nabla_{SI}B$ or $\mathcal{E} \subseteq \nabla_{SI}D$, which implies $F \subseteq B$ or $F \subseteq D$. □

Proposition 4.8 *All complete semilattices and all complete lattices are SI-dominated.*

Proof. If $C' \lhd C$ for SI-closed sets of a complete semilattice X, then by definition $C' \subseteq \downarrow x$ for some $x \in C$ and hence $\bigvee C' \in C$. If $\mathcal{E}$ is an irreducible family in $\nabla_{SI}C$, then $\bigvee_{\Gamma_{SI}(X)} \mathcal{E} \subseteq \downarrow \bigvee_{E \in \mathcal{E}}(\bigvee E)$, and the element $\bigvee_{E \in \mathcal{E}}(\bigvee E) \in C$ because C is a SI-closed set. □

Corollary 4.9 *For any SI-dominated poset X, the lattice $\Gamma_{SI}(X)$ is C_{SI}-algebraic.*

Corollary 4.10 *Every completely distributive lattice is C_{SI}-prealgebraic.*

5 The category of strong completed posets

In this section, we show that the category of scpos is Cartesian-closed.

Let $SCPO$ be the category whose objects are scpos and morphisms the SI-continuous maps (i.e., the monotone maps preserving sups of irreducible sets). Obviously, every single point set is a terminal object of $SCPO$. Let $CPALG_{SI}$ be the category whose objects are the C_{SI}-prealgebraic lattices and morphisms the lower adjoints which preserve the relation $\prec_{SI}$.

A category with terminal and finite products is called Cartesian-closed if for each pair (A, B) of objects there exists an object B^A (sometimes be called the function space from A to B) and a morphism $ev : A \times B^A \longrightarrow B$ with the following universal property: for each morphism $f : A \times C \longrightarrow B$ there exists a unique morphism $\overline{f} : C \longrightarrow B^A$ such that $ev \circ (id_A \times \overline{f}) = f$.

Let $\{X_j | j \in J\}$ be a nonempty family of scpos and $X = \prod_{j \in J} X_j$ the product of $\{X_j | j \in J\}$. Then for any $j \in J$, the jth-projection $p_j : X \longrightarrow X_j$ is a full Scott-continuous open mapping and $p_j(F) \in Irr_\sigma(X_j)$ for any $j \in J$ and $F \in Irr_\sigma(X)$. Moreover, it is easy to check that $\bigvee F = (\bigvee p_j(F))_{j \in J}$ for any $F \in Irr_\sigma(X)$. Therefore, we have that the category $SCPO$ is closed under finite products.

Let X and Y be two scpos and $[X \to Y]_{SI}$ denote the set of all SI-continuous maps from X to Y and define the order $\leq$ by

$$\forall f, g \in [X \to Y]_{SI},\ f \leq g \Leftrightarrow f(x) \leq g(x), \forall x \in X.$$

Then $[X \to Y]_{SI}$ is a poset and we can have the following results.

Lemma 5.1 *Let X and Y be two scpos and $B \in \Gamma(Y)$, then for any $x \in X$, $\mathcal{B} = \{g \in [X \to Y]_{SI} \mid g(x) \in B\} \in \Gamma([X \to Y]_{SI})$.*

Proof. Obviously, $\mathcal{B}$ is a lower subset of $[X \to Y]_{SI}$. Let $\mathcal{D}$ be a directed set of $[X \to Y]_{SI}$ and $\mathcal{D} \subseteq \mathcal{B}$. As $[X \to Y]_{SI} \subseteq [X \to Y]$ and $[X \to Y]$ is a dcpo, $\mathcal{D}$ is also a directed set of $[X \to Y]$. Then $\bigvee \mathcal{D}$ exists and $\bigvee \mathcal{D}(x) = \bigvee_{g \in \mathcal{D}} g(x)$. It is easy to check that $\{g(x) \mid g \in \mathcal{D}\}$ is a directed set of Y and $\{g(x) \mid g \in \mathcal{D}\} \subseteq B$. Now, let $f(x) = \bigvee \mathcal{D}(x)$ for any $x \in X$. As $B \in \Gamma(Y)$, $f(x) \in B$. Moreover, for any $F \in Irr_\sigma(X)$,

$$\begin{aligned} f(\bigvee F) &= \bigvee_{g \in \mathcal{D}} g(\bigvee F) = \bigvee_{g \in \mathcal{D}} (\bigvee g(F)) = \bigvee_{g \in \mathcal{D}} (\bigvee_{x \in F} g(x)) \\ &= \bigvee_{x \in F} (\bigvee_{g \in \mathcal{D}} g(x)) = \bigvee_{x \in F} f(x) = \bigvee f(F), \end{aligned}$$

which implies that $f \in \mathcal{B}$. To sum up, $\mathcal{B} \in \Gamma([X \to Y]_{SI})$. □

Proposition 5.2 *For any two scpos X and Y, $[X \to Y]_{SI}$ is a scpo.*

Proof. Let $\mathcal{F} \in Irr_\sigma([X \to Y]_{SI})$ and $A = \{g(x) \mid g \in \mathcal{F}\}$ for any $x \in X$. Suppose that $A \subseteq B \bigcup C$ for any $B, C \in \Gamma(Y)$, then $\mathcal{B} = \{g \in [X \to Y]_{SI} \mid g(x) \in B\}, \mathcal{C} = \{g \in [X \to Y]_{SI} \mid g(x) \in C\} \in \Gamma([X \to Y]_{SI})$ and $\mathcal{F} \subseteq \mathcal{B} \bigcup \mathcal{C}$. As $\mathcal{F} \in Irr_\sigma([X \to Y]_{SI})$, $\mathcal{F} \subseteq \mathcal{B}$ or $\mathcal{F} \subseteq \mathcal{C}$. Without losing generality, we might as well assume $\mathcal{F} \subseteq \mathcal{B}$ and so $A \subseteq B$, which implies that $A = \{g(x) \mid g \in \mathcal{F}\} \in Irr_\sigma(Y)$. Thus $\bigvee_{g \in \mathcal{F}} g(x)$ exists. Now let $f(x) = \bigvee_{g \in \mathcal{F}} g(x)$, then for any $F \in Irr_\sigma(X)$,

$$\begin{aligned} \bigvee f(F) &= \bigvee_{x \in F} f(x) = \bigvee_{x \in F} \bigvee_{g \in \mathcal{F}} g(x) = \bigvee_{g \in \mathcal{F}} (\bigvee g(F)) \\ &= \bigvee_{g \in \mathcal{F}} g(\bigvee F) = f(\bigvee F). \end{aligned}$$

To sum up, for any $\mathcal{F} \in Irr_\sigma([X \to Y]_{SI})$, $\bigvee \mathcal{F} = f$ and $\bigvee \mathcal{F} \in [X \to Y]_{SI}$, therefore $[X \to Y]_{SI}$ is a scpo. □

Lemma 5.3 *Suppose that F (resp., E) is an irreducible set of X (resp., Z).*
(1) For $z_0 \in Z$, define $\widetilde{F}_{z_0} \in X \times Z$ by $\widetilde{F}_{z_0} = \{(x, z) \mid x \in F,\ z = z_0\}$. Then $\widetilde{F}_{z_0}$ is an irreducible set of $X \times Z$ and $\bigvee \widetilde{F}_{z_0} = (\bigvee F, z_0)$.
(2) For $x_0 \in X$, define $\widehat{E}_{x_0} \in X \times Z$ by $\widehat{E}_{x_0} = \{(x, z) \mid z \in E,\ x = x_0\}$. Then $\widehat{E}_{x_0}$ is an irreducible set of $X \times Z$ and $\bigvee \widehat{E}_{x_0} = (x_0, \bigvee E)$.

Proof. We only show (1) here, (2) is similar. Obviously,

$$\bigvee \widetilde{F}_{z_0} = (\bigvee p_1(\widetilde{F}_{z_0}), \bigvee p_2(\widetilde{F}_{z_0})) = (\bigvee F, z_0).$$

Let $\widetilde{F}_{z_0} \subseteq A \bigcup B$, where $A, B \in \Gamma(X \times Z)$, then $F \subseteq p_1(A) \bigcup p_1(B)$ and $z_0 \in p_2(A) \bigcup p_2(B)$. As $F \in Irr_\sigma(X)$ and $p_1(A), p_1(B) \in \Gamma(X)$, we have $F \subseteq p_1(A)$ or $F \subseteq p_1(B)$. Suppose that $F \subseteq p_1(A)$, then

$$\widetilde{F}_{z_0} \subseteq \begin{cases} A & z_0 \in p_2(A) \\ B & z_0 \notin p_2(A). \end{cases}$$

This means that $\widetilde{F}_{z_0}$ is an irreducible set of $X \times Z$. □

Proposition 5.4 *If $f : X \times Z \longrightarrow Y$ is a SI-continuous mapping, then so is $\overline{f} : Z \longrightarrow [X \to Y]_{SI}$, where $\overline{f}(z)(x) = f(x, z)$ for any $z \in Z$ and any $x \in X$.*

Proof. Step 1: $\overline{f}$ is a well-defined mapping. We need to show that for all $z_0 \in Z$, $\overline{f}(z_0) = f(\cdot, z_0) : X \longrightarrow Y$ is a SI-continuous mapping. Obviously, $\overline{f}(z_0)$ is monotone. Suppose that $F \in Irr_\sigma(X)$. It is routine to show that

$$\overline{f}(z_0)(\bigvee F) = f(\bigvee F, z_0) = f(\bigvee \widetilde{F}_{z_0}) = \bigvee f(\widetilde{F}_{z_0}) = \bigvee \overline{f}(z_0)(F).$$

These show that $\overline{f}$ is a mapping.

Step 2: $\overline{f}$ is a SI-continuous mapping. Let $F \in Irr_\sigma(Z)$, then

$$\overline{f}(\bigvee F)(x) = f(x, \bigvee F) = \bigvee f(x, F) = \bigvee \{\overline{f}(z)(x) \mid z \in F\} = (\bigvee \overline{f}(F))(x).$$

□

Proposition 5.5 *The evaluation map $ev : X \times [X \to Y]_{SI} \longrightarrow Y$ is SI-continuous.*

Proof. Let $F \in Irr_\sigma(X \times [X \to Y]_{SI})$ and $\bigvee F = (x, f)$, then

$$x = \bigvee p_1(F), f = \bigvee p_2(F) = \bigvee \{g \mid (a, g) \in F\}$$

and

$$f(x) = f(\bigvee p_1(F)) = \bigvee f(p_1(F)) = \bigvee (f \circ p_1)(F).$$

Next, we only need to show $f(x) = ev(\bigvee F) = \bigvee ev(F)$. In fact, $\bigvee ev(F) = \bigvee \{g(a) \mid (a, g) \in F\}$ implies that for all $(a, g) \in F$, we have $(a, g) \leq \bigvee F = (x, f)$, i.e., $a \leq x$, $g \leq f$ and so $g(a) \leq f(x)$. This means that $f(x)$ is an upper bound of $ev(F)$, i.e., $\bigvee ev(F) \leq f(x)$. Let $y \in Y$ be an upper bound of $ev(F)$, then for all $(a, g) \in F$, $g(a) \leq y$ and so $f(a) \leq y$. Therefore

$$f(x) = \bigvee (f \circ p_1)(F) = \bigvee \{f(a) \mid (a, g) \in F\} \leq y.$$

To sum up, we have $f(x) = ev(\bigvee F) = \bigvee ev(F)$. □

Theorem 5.6 *SCPO is Cartesian-closed.*

Next, similar to Ho and Zhao's approach in [4], we obtain an adjunction between SCPO and $CPALG_{SI}$. We remove the proofs of results since they are similar to the proofs in [4].

Lemma 5.7 *Let (g, d) be a Galois connection between two complete lattices M and N, where $d : M \longrightarrow N$ and $g : N \longrightarrow M$. Then for any SI-closed set C of N, $\downarrow g(C)$ is a SI-closed set of M.*

Lemma 5.8 *Let (g, d) be a Galois connection between two complete lattices M and N, where $d : M \longrightarrow N$ and $g : N \longrightarrow M$. If g preserves the sups of SI-closed sets, then d preserves the relation $\prec_{SI}$. If M is C_{SI}-continuous, then the converse conclusion is also true.*

By Proposition II-2.1 in [2], if $f : P \longrightarrow Q$ is a morphism in $SCPO$ then f is continuous with respect to the irreducibly-derived topology. Thus the mapping $f^{-1} : \Gamma_{SI}(Q) \longrightarrow \Gamma_{SI}(P)$ is well-defined and preserves arbitrary infs, so it is an upper adjoint. Similar to [4], we can obtain the following results

Lemma 5.9 *Let $f : P \longrightarrow Q$ be a morphism in SCPO and $h : \Gamma_{SI}(P) \longrightarrow \Gamma_{SI}(Q)$ a lower adjoint of f^{-1}. Then h preserves the relation $\prec_{SI}$.*

Proposition 5.10 (1) *For all $P, Q \in ob(SCPO)$ and all $f \in Hom_{SCPO}(P, Q)$, let $\mathbb{C}(P) = \Gamma_{SI}(P)$ and $\mathbb{C}(f) = g$, where g is a lower adjoint of f^{-1}, then $\mathbb{C}$ is a functor from the category $SCPO$ to the category $CPAlg_{SI}$, i.e., the following diagram.*

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "\mathbb{C}"'] & Q \arrow[d, "\mathbb{C}"] \\ \Gamma_{SI}(P) \arrow[r, "\mathbb{C}(f)=g"] & \Gamma_{SI}(Q) \end{tikzcd}$$

(2) *For all $P, Q \in ob(CPAlg_{SI})$ and all $f \in Hom_{CPAlg_{SI}}(P, Q)$, let $\mathbb{K}(P) = \kappa_{SI}(P)$ and $\mathbb{K}(f) = f|_{\kappa_{SI}(P)}$, then $\mathbb{K}$ is a functor from the category $CPAlg_{SI}$ to the category $SCPO$, i.e., the following diagram.*

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "\mathbb{K}"'] & Q \arrow[d, "\mathbb{K}"] \\ \kappa_{SI}(P) \arrow[r, "\mathbb{C}(f)=f|_{\kappa_{SI}(P)}"] & \kappa_{SI}(Q) \end{tikzcd}$$

Theorem 5.11 *The functor $\mathbb{K}$ is a right adjoint to the functor $\mathbb{C}$.*

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