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Topological Construction of Parameterized Bisimulation Limit [1]

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Abstract

In this paper, we mainly discuss several closure constructions of *parameterized limit bisimulation* and establish a family of *parameterized bisimulation limit* topologies. These topological structures are useful for us to understand and analyze the infinite evolution of parameterized bisimulation.

Keywords: process algebra; $\varepsilon-$parameterized bisimulation; $\varepsilon-$parameterized limit bisimulation; topology

1 Introduction

As one of the most important and mathematically developed models of communication and concurrency, CCS (Communication and Concurrency System) introduced by R. Milner [1,12,13] proposes various behavior equivalences, such as strong (weak) bisimulation equivalence, observation equivalence and so on. These equivalences are useful for relating process description to different levels of abstraction. In [6,7], K. G. Larsen presents parameterized bisimulation equivalence in order to obtain more flexible hierarchic development methods. In Larsen's work, bisimulation equivalence is parameterized with information

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about context called environment. Specially, in CCS model, strong bismulation is generalized by parameterized bisimulation equivalence.

On the other hand, in the actual application of process algebra, specification and implementation are considered as two processes. If there exists a kind of behavioral equivalence between them, then the program is treated as correctness. So, it is the key to establish a certain behavioral equivalence between specification and implementation in order to show the correctness of programs. However, the implementation at the first step can not be completely equivalent to the specification. Hence the implementation should be modified step by step and an evolution sequence of software is produced. Sometimes, the procedure of modification of software might be parallel. In this case, steps of modification might not form a chain but a kind of partial order which appeals to the notion of nets.

In order to understand and analyze the infinite evolution of concurrency program, Mingsheng Ying proposes the strong (weak) bisimulation limits in [8,9,10,11], proves some topological properties and establishes the strong(weak) bisimulation limit topology. The strong (weak) bisimulation limit describes the mechanism that implementation approximates its specification step by step.

As we know, the execution of a program is dependent on the environment. So, the execution environment of the program should be considered when we verify the correctness of one program. Thus, parameterized bisimulation equivalence is an appropriate choice at this point. Just like the strong bisimulation, we still consider the infinite evolution of parameterized bisimulation equivalence. In order to describe this infinite evolution mechanism, in the paper [14], parameterized limit bisimulation and parameterized bisimulation limit are proposed.

In this paper, we mainly extend the strong bisimulation topology to a family of parameterized bisimulation limit topologies. Bases on parameterized limit bisimulation, we construct some natural and reasonable topological structure which are useful for us to understand and analyze the infinite evolution. These topological structures are determined by behaviors of processes and so are completely extensional and observable. We mainly discuss the subnet closure, tail closure, natural extension and iteration structure of parameterized limit bisimulation.

The paper is organized as follows: in section 2, we review some basic concepts and results which include the syntax of CCS, parameterized bisimulation, parameterized limit bisimulation and parameterized bisimulation limit. In section 3, subnet closure, tail closure, natural extension and iteration structure of parameterized limit bisimulation are proposed. The family of parameterized bisimulation limit topologies is established in section 4. Section 5 states some conclusions.

2 Preliminaries

In this section, we review some basic notions which are related to this paper.

First, we recall some concepts of CCS which mainly come from [1,2]. We introduce the *names* $\mathcal{A}$, the *co-namea* $\bar{\mathcal{A}}$ and *labels* $\Gamma = \mathcal{A} \cup \bar{\mathcal{A}}$. Define that $a, b, \cdots$ range over $\mathcal{A}$ and $\bar{a}, \bar{b}, \cdots$ range over $\bar{\mathcal{A}}$; also that $l, l', \cdots$ range over Γ. We also introduce the silent or perfect action τ and define $Act = \Gamma \cup \{\tau\}$ to the set of *actions*; α, β range over Act. Further, we introduce a set $\aleph$ of processes variables and a set $\mathcal{K}$ of processes constants. Let $X, Y, \cdots$ range over $\aleph$, and $A, B, \cdots$ over $\mathcal{K}$. We define ε, the set of process expressions, is the smallest set which includes $\aleph$, $\mathcal{K}$ and the following expressions: $\alpha.E$; $\sum\limits_{i \in I} E_i$; $E_1 \mid E_2$; $E \backslash L (L \subseteq \Gamma)$; and $E[f](f : Act \to Act$ is a relabeling function).

In this paper, we only focus on the process expressions without process variables, called processes, denoted by $\mathcal{P}$. The semantics of processes is given by the labeled transition system $(S, T, \{\xrightarrow{t}: t \in T\})$ which consists of a set S of states, a set T of transition labels, and a transition relation $\xrightarrow{t} \subseteq S \times S$ for each $t \in T$. In our transition system, we shall take S to be $\mathcal{P}$, and T to be Act, $\xrightarrow{\alpha} \subseteq \mathcal{P} \times \mathcal{P}$. $P \xrightarrow{\alpha} P'$ means P can execute the action α and afterward behaves like P'. Then strong bisimulation in CCS model is defined as follows:

Definition 2.1 (Strong bisimulation [1]) A strong bisimulation R is a binary relation on $\mathcal{P}$ such that whenever PRQ and $\alpha \in Act$, then

(i) $P \xrightarrow{\alpha} P'$ implies $\exists Q' \in \mathcal{P}$ such that $Q \xrightarrow{\alpha} Q'$ and $P'RQ'$;

(ii) $Q \xrightarrow{\alpha} Q'$ implies $\exists P' \in \mathcal{P}$ such that $P \xrightarrow{\alpha} P'$ and $P'RQ'$.

Two processes P and Q are said to be strong bisimulation if and only if there exists a strong bisimulation R such that PRQ. We write $P \sim Q$.

From the definition of strong bisimulation, we can see that two processes are considered strong bisimulation equivalent if they have the same set of potential first actions and can remain having potentiality during the course of execution.

Next, we will introduce the definition of parameterized bisimulation. The motivation for parameterizing bisimulation is to parameterize the bisimulation equivalence with a special type information about context called environment. In the papers [6,7], an environment e is considered as a "process" which consumes the actions produced by a process. Similar to the assumption that a process may change after having performed an action, an environment may change after having consumed an action. Thus, the environment transition system can be defined as follows:

Definition 2.2 (Environment Transition System[6]) A labeled transition system $\varepsilon = (E, A, \Rightarrow)$ is called environment transition system, if E is the set of environments, A is the set of actions (identical to the set of actions Act) and $\Rightarrow$ is a subset of $E \times A \times E$ called the *consumption relation*, where $e \overset{\alpha}{\Rightarrow} e'$ is to be read " e may consume the action α and after doing so become the environment e' ".

Definition 2.3 ($\varepsilon-$parameterized bisimulation [6]) Let $\varepsilon = (E, A, \Rightarrow)$ be an environment transition system. Then a $\varepsilon-$parameterized bisimulation, R, is an $E-$ indexed family of binary relation, $R_e \subseteq \mathcal{P} \times \mathcal{P}$ for $e \in E$, s.t. whenever PR_eQ and $e \overset{\alpha}{\Rightarrow} f$ then

(i) if $P \overset{\alpha}{\to} P'$, then there exists Q', s.t. $Q \overset{\alpha}{\to} Q'$ and $P'R_fQ'$;

(ii) if $Q \overset{\alpha}{\to} Q'$, then there exists P', s.t. $P \overset{\alpha}{\to} P'$ and $P'R_fQ'$.

Two processes P and Q are said to be bisimulation equivalent in the environment $e \in E$ if and only if there exists a $\varepsilon-$ parameterized bisimulation R such that PR_eQ. We write $P \sim_e Q$.

Now we come to introduce parameterized limit bisimulation which describes the infinite evolution of parameterized bisimulation. A key notion in the definition is net which is a generalization of sequences. The definition of net and related results mainly come from the paper [11].

Definition 2.4 (Directed set) Let D be a nonempty set. $\leq$ is a binary relation on D. $(D, \leq)$ is called a directed set if $\leq$ satisfies the following conditions:

(i) if $m \in D$, then $m \leq m$;

(ii) if m, n and p are members of D such that $p \leq n$, $n \leq m$, then $p \leq m$;

(iii) if m and n are members of D, then there is p in D such that $m \leq p$ and $n \leq p$.

Definition 2.5 (Cofinality) Let C, D be directed sets. A pair (C, N) is called a cofinality of D, if $N : C \to D$ is a mapping such that for any $n \in D$, there is $m \in C$ with $N_p \geq n$ for any $p \geq m$.

Definition 2.6 (Cofinal subset) Let C, D be directed sets and $C \subseteq D$. If for any $n \in D$, there is $m \in C$ such that $m \geq n$, i.e.,(C, in_C) is a cofinality of D, then C is called a cofinal subset of D.

Definition 2.7 (Net) Let $(D, \leq)$ be a directed set, $U \neq \emptyset$. Then a mapping S from D into U is called a net in U over D.

Usually, a net S in U over D is expressed as $\{S_n : n \in D\}$, where $S_n = S(n) \in U$ for every $n \in D$.

Definition 2.8 (Subnet) Let $\{S_n : n \in D\}$ and $\{T_m : m \in C\}$ be nets. If there exists a mapping $N : C \to D$ such that

(i) $T_m = S_{N_m}$ for every $m \in C$;

(ii) (C, N) is a cofinality of D,

then $\{T_m : m \in C\}$ is called a subnet of $\{S_n : n \in D\}$.

If S is a net over D and C is a cofinal subset of D, then it is easy to see that the restriction $S \mid C = \{S_n : n \in C\}$ is a subnet of S.

We write $\mathcal{P}_N$ for the class of all nets on $\mathcal{P}$. Now we can define the key definition in this section.

Definition 2.9 (Parameterized limit bisimulation [14]) Let $\varepsilon = (E, A, \Rightarrow)$ be an environment transition system. Then a $\varepsilon-$parameterized limit bisimulation, S, is an $E-$indexed family of binary relation, $S_e \subseteq \mathcal{P} \times \mathcal{P}_N$ for $e \in E$, s.t., whenever $(P, \{Q_n : n \in D\}) \in S_e$ and $e \overset{\alpha}{\Rightarrow} f$, then

(i) if $P \xrightarrow{\alpha} P'$, then there exist $\{Q'_n : n \in D\}$ and $n_0 \in D$ such that $Q_n \xrightarrow{\alpha} Q'_n$ for all $n \geq n_0$ and $(P', \{Q'_n : n \in D\}) \in S_f$;

(ii) if C is a cofinal subset of D, and $Q_m \xrightarrow{\alpha} Q'_m$ for all $m \in C$, then there exist $P' \in \mathcal{P}$ and a cofinal subset B of C such that $P \xrightarrow{\alpha} P'$ and $(P', \{Q'_k : k \in B\}) \in S_f$.

From this definition, we can see that $\varepsilon-$parameterized limit bisimulation is the dynamic counterpart of $\varepsilon-$parameterized bisimulation.

Definition 2.10 (Parameterized bisimulation limit[14]) Let $\varepsilon = (E, A, \Rightarrow)$ be an environment transition system. If there exist a $\varepsilon-$parameterized limit bisimulation S and $e \in E$ such that $(P, \{Q_n : n \in D\}) \in S_e \subseteq \mathcal{P} \times \mathcal{P}_N$, then P is called $\varepsilon-$parameterized bisimulation limit of $\{Q_n : n \in D\}$ in the environment e and we write $P \sim_e \lim\limits_{n \in D} Q_n$.

Let

$$\sim_e \lim = \{(P, \{Q_n : n \in D\}) : P \sim_e \lim_{n \in D} Q_n\},$$

for each $e \in E$. Then $\sim \lim$ is an $E-$indexed family of binary relation between processes and nets of processes, and it is the greatest $\varepsilon-$parameterized limit bisimulation.

3 Closure Structures of Parameterized Limit Bisimulation

In this section, we will focus on the topological properties of parameterized limit bisimulation. Subnet closure, tail closure, natural extension and iteration structure are discussed. They are useful for us to understand the infinite evolution of parameterized limit bisimulation from a mathematical point.

Since we shall be dealing with $E-$indexed families and operations, we adopt the following convenient notation. Let $\varepsilon = (E, Act, \Rightarrow)$ be an environment transition system. For $E-$indexed families R and S, let

- $R \subseteq S$ if and only if for all $e \in E$, $R_e \subseteq S_e$.
- $R \cap S$ is an $E-$indexed family with $(R \cap S)_e = R_e \cap S_e$.
- $R \cup S$ is an $E-$indexed family with $(R \cup S)_e = R_e \cup S_e$.

In order to introduce the closure structures of parameterized limit bisimulation, the following notions are essential. Let $t \in Act^* = \bigcup\limits_{n=0}^{\infty} Act^n$ be an action sequence, i.e., $t = \alpha_1\alpha_2 \cdots, \alpha_n$, where $\alpha_i \in Act$, $n \in N$. Then $P \xrightarrow{t} P'$ means that there exist $P_1, P_2, \cdots, P_n$, such that $P \xrightarrow{\alpha_1} P_1 \xrightarrow{\alpha_2} \cdots P_{n-1} \xrightarrow{\alpha_n} P_n = P'$. Similarly, $e \overset{t}{\Rightarrow} f$ means that there exist $e_1, e_2, \cdots, e_n$, such that $e \overset{\alpha_1}{\Rightarrow} e_1 \overset{\alpha_2}{\Rightarrow} \cdots e_{n-1} \overset{\alpha_n}{\Rightarrow} e_n = f$.

According to the definition of parameterized bisimulation equivalence, it is easy to notice that $Iden$, an $E-$indexed family of the identity relation $Iden_e \subseteq \mathcal{P} \times \mathcal{P}$ for $e \in E$, is a parameterized bisimulation. Naturally, we want to generalize $Iden$ to parameterized limit bisimulation, i.e., a dynamic counterpart of it is that the relation in the environment e linking each process with the constant net of this process is $\varepsilon-$parameterized limit bisimulation. But, this is not true because of the nondeterminism of process. For example, let the environment transition system ε and the process P be given by the diagrams below:

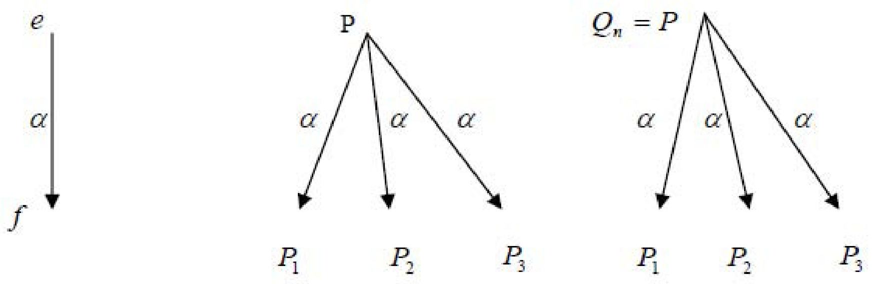

Fig. 1. The example of nondeterminism of process.

Let the net of processes $\{Q_n : n \in D\}$ be the constant net of the process P, i.e., $\{Q_n = P : n \in D\}$. If $P \xrightarrow{\alpha} P_1$, then we should find some net of processes $\{Q'_n : n \in D\}$ and $n_0 \in D$ such that $Q_n \xrightarrow{\alpha} Q'_n$ for $n \geq n_0$ and $(P_1, \{Q'_n : n \in D\})$ belongs to some parameterized limit bisimulation. But because of the nondeterministic choice of process P, there are three possibilities of Q'_n, i.e., $Q'_n = P_1$ for some n, $Q'_n = P_2$ for some n, or $Q'_n = P_3$ for some n. Thus, it is difficult to obtain some constant net of processes $\{Q'_n : n \in D\}$ and $n_0 \in D$ such that $Q_n \xrightarrow{\alpha} Q'_n$ for $n \geq n_0$ and $(P_1, \{Q'_n : n \in D\})$ belongs to some parameterized limit bisimulation. In order to to extend $Iden$ to parameterized limit bisimulation, we have to impose a certain determinacy on the involved processes. Because the execution of a process is dependent on his environment, it is necessary to introduce the determinism of process in its environment.

3.1 $\lambda-$determinate in the environment

Definition 3.1 (The base in the environment e) Let $\varepsilon = (E, A, \Rightarrow)$ be an environment transition system. $\Omega \subseteq \mathcal{P}$, $\Theta \subseteq \Omega$, $e \in E$. If for any $P \in \Omega$, there exists $Q \in \Theta$ such that $P \sim_e Q$, then we say that Θ is the base of Ω in the environment e.

Definition 3.2 (t-derivative in the environment e) Let $\varepsilon = (E, A, \Rightarrow)$ be an environment transition system. e is the environment of P. If $e \overset{t}{\Rightarrow} f$ and there exists $P' \in \mathcal{P}$ such that $P \xrightarrow{t} P'$, then P' is called a $t-$derivative of P in the environment e. If for some $t \in Act^*$ and $e \in E$, P' is a $t-$derivative of P in the environment e, then P' is called a derivative of P in the environment e.

Definition 3.3 ($\lambda-$determinate in the environment e) Let $\varepsilon = (E, A, \Rightarrow)$ be an environment transition system. λ is a cardinal number and $P \in \mathcal{P}$. If, for every derivative Q of P in the environment e (let f be the environment of Q and $f \overset{\alpha}{\Rightarrow} h$), the set $\{Q' : Q \xrightarrow{\alpha} Q'\}$ has a base Θ in the environment h with $|\Theta| < \lambda$, then we say that P is $\lambda-$determinate in the environment e.

The following proposition states that the $\lambda-$determinacy is closed under derivation.

Proposition 3.4 *If P is $\lambda-$determinate in the environment e, $e \overset{t}{\Rightarrow} f$ and $P \xrightarrow{t} P'$, then P' is also $\lambda-$determinate in the environment f.*

The following proposition tells us the $\lambda-$determinacy is preserved by $\varepsilon-$parameterized bisimulation equivalent.

Proposition 3.5 *Let $\varepsilon = (E, A, \Rightarrow)$ be an environment transition system. P is $\lambda-$determinate in the environment e and $P \sim_e Q$, then Q is also $\lambda-$determinate in the environment e.*

Let $cf(D) = \inf\{\ |D'| : D'$ is the cofinal subset of $D\}$. $|D'|$ means the cardinality of D'. Roughly speaking, the following $\varepsilon-$parameterized limit bisimulation consists of the pairs of $cf(D)-$ determinate processes and their constant nets. This example extends the parameterized bisimulation $Iden$.

Proposition 3.6 *Let $\varepsilon = (E, A, \Rightarrow)$ be an environment transition system. $Ilim$ is an $E-$indexed family between processes and the nets of processes, for $e \in E$, let*

$$\begin{aligned} Ilim_e = \{(P, \{Q_n : n \in D\}) : P \in \mathcal{P} \ \ &is\, cf(D) - determinate \\ &in\ the\ environment\, e, \{Q_n : n \in D\} \in \mathcal{P}_N,\ there\ exists \\ &n_0 \in D \ \ such\ that\, Q_n \sim_e P\ for\ any\, n \geq n_0\} \end{aligned}$$

Then $Ilim$ is $\varepsilon-$parameterized limit bisimulation.

In the following three subsections, some attendant results are given. These results are convenient tools for establishing some useful parameterized limit bisimulation.

3.2 Subnet Closure

Let $\varepsilon = (E, A, \Rightarrow)$ be an environment transition system. S is an $E-$indexed family of binary relations between processes and nets of processes, for $e \in E$, $S_e \subseteq \mathcal{P} \times \mathcal{P}_N$. Then $sub(S)$ is also an $E-$indexed family of binary relations, for any $e \in E$,

$$\begin{aligned} sub(S)_e = \{(P, \{Q_n : n \in D\}) : &\text{there is } (P, \{R_m : m \in C\}) \in S_e \\ &\text{such that } \{Q_n : n \in D\} \text{is a subnet of } \{R_m : m \in C\}\} \end{aligned}$$

The following proposition states the necessary and sufficient condition of $sub(S)$ being $\varepsilon-$parameterized limit bisimulation.

Proposition 3.7 *Let $\varepsilon = (E, A, \Rightarrow)$ be an environment transition system. S is an $E-$indexed family of binary relations on $\mathcal{P} \times \mathcal{P}_N$. Then $sub(S)$ is $\varepsilon-$parameterized limit bisimulation if and only if for any $e \in E$, $e \overset{\alpha}{\Rightarrow} f$, for any $(P, \{Q_n : n \in D\}) \in S_e$,*

(i) *if $P \overset{\alpha}{\rightarrow} P'$, then there exist $\{Q'_n : n \in D\} \in \mathcal{P}_N$ and $n_0 \in D$ such that $Q_n \overset{\alpha}{\rightarrow} Q'_n$ for any $n \geq n_0$ and $(P', \{Q'_n : n \in D\}) \in sub(S)_f$.*

(ii) *if C is a cofinal subset of D, $Q_m \overset{\alpha}{\rightarrow} Q'_m$ for all $m \in C$, then there exist $P' \in \mathcal{P}$ and B which is a cofinal subset of C such that $P \overset{\alpha}{\rightarrow} P'$ and $(P', \{Q'_k : k \in B\}) \in sub(S)_f$.*

Proof. " $\Rightarrow$ " If $sub(S)$ is $\varepsilon-$ parameterized bisimulation, then for any $e \in E$, $e \overset{\alpha}{\Rightarrow} f$, and $(P, \{Q_n : n \in D\}) \in S_e$, the fact $S_e \subseteq sub(S)_e$ for any $e \in E$ lead to the conditions of Definition 2.9 hold.

" $\Leftarrow$ " Suppose that $e \overset{\alpha}{\Rightarrow} f$, $e \in E$, $(P, \{R_m : m \in C\}) \in sub(S)_e$. Then there exists $(P, \{Q_n : n \in D\}) \in S_e$ and $\{R_m : m \in C\}$ is the subnet of $\{Q_n : n \in D\}$. So there is a map $N : C \to D$, (C, N) is cofinal of D and $R_m = Q_{N_m}$ for all $m \in C$. Without loss of generality, we can assume that N is increasing.

If $P \xrightarrow{\alpha} P'$, then since $(P, \{Q_n : n \in D\}) \in S_e$, there exist $\{Q'_n : n \in D\}$ and $n_0 \in D$ such that $Q_n \xrightarrow{\alpha} Q'_n$ for any $n \geq n_0$ and $(P', \{Q'_n : n \in D\}) \in sub(S)_f$. Since (C, N) is cofinal of D, so there is $p_0 \in C$ such that $N_{p_0} \geq n_0$ for all $p \geq p_0$. Let $R'_p = Q'_{N_p}$ for each $p \in C$, then $R_p = Q_{N_p} \xrightarrow{\alpha} Q'_{N_p} = R'_p$ for all $p \geq p_0$ and $\{R'_p : p \in C\}$ is a subnet of $\{Q_n : n \in D\}$. So, $(P', \{R'_p : p \in C\}) \in sub(S)_f$.

If F is a cofinal subset of C and $R_q \xrightarrow{\alpha} R'_q$ for all $q \in F$, then $N(F)$ is also a cofinal subset of D. For any $q \in F$, let $Q'_{N_q} = R'_q$. Then $Q_{N_q} = R_q \xrightarrow{\alpha} R'_q = Q'_{N_q}$ for all $q \in F$. Thus there is $P' \in \mathcal{P}$ and G which is a cofinal subset of $N(F)$ such that $P \xrightarrow{\alpha} P'$ and $(P', \{Q'_r : r \in G\}) \in sub(S)_f$. Since N is increasing, so $N^{-1}(G)$ is also a cofinal subset of F, furthermore, $\{Q'_r : r \in G\} = \{R'_q : q \in N^{-1}(G)\}$ and $(P', \{R'_q : q \in N^{-1}(G)\}) \in sub(S)_f$. □

Lemma 3.8 *If S is a $\varepsilon-$parameterized limit bisimulation, then $sub(S)$ is also a $\varepsilon-$parameterized limit bisimulation.*

Proof. It is immediate from Proposition 3.7.

□

3.3 Tail Closure

Definition 3.9 (Tail [11]) Let D be a directed set and $n \in D$. Then we denote that $D[n) = \{m \in D : m \geq n\}$ and $D[n)$ is a cofinal subset of D.

Definition 3.10 Suppose that $\varepsilon = (E, Act, \Rightarrow)$ is an environment transition system. S is an $E-$ indexed family between processes and nets of processes, i.e., for any $e \in E$, $S_e \subseteq \mathcal{P} \times \mathcal{P}_N$. We define that $tail(S)$ is also an $E-$ indexed family between processes and nets of processes, i.e. for any $e \in E$,

$$tail(S)_e = \{(P, \{Q_n : n \in D\}) : (P, \{Q_n : n \in D[n_0)\}) \in S_e \text{ for some } n_0 \in D\}$$

Proposition 3.11 *$tail(S)$ is a parameterized limit bisimulation if and only if for any $e \in E$, $(P, \{Q_n : n \in D\}) \in S_e$, $e \overset{\alpha}{\Rightarrow} f$,*

(i) *if $P \xrightarrow{\alpha} P'$, then there exist that $\{Q'_n : n \in D\}$ and $n_0 \in D$ such that $Q_n \xrightarrow{\alpha} Q'_n$ for any $n \geq n_0$ and $(P', \{Q'_n : n \in D\}) \in tail(S)_f$.*

(ii) *if C is any cofinal subset of D and $Q_m \xrightarrow{\alpha} Q'_m$ for any $m \in C$, then there exist $P' \in \mathcal{P}$ and B which is a cofinal subset of C such that $P \xrightarrow{\alpha} P'$ and*

$(P, \{Q'_k : k \in B\}) \in tail(S)_f$.

Lemma 3.12 *If S is a $\varepsilon-$parameterized limit bisimulation, then $tail(S)$ is also a $\varepsilon-$parameterized limit bisimulation.*

Proof. It can be proved according to the Proposition 3.11. □

3.4 Natural Extension

Definition 3.13 (Natural extension [11]) Let $\{P_m : m \in C\}$ and $\{Q_n : n \in D\}$ be nets of processes. If (C, N) is a cofinal of D and for any $n \in D$, $Q_n = P_{m_n}$ for some $m_n \in C$ with $N_{m_n} \geq n$, then $\{Q_n : n \in D\}$ is called a natural extension of $\{P_m : m \in C\}$.

Definition 3.14 Let $\varepsilon = (E, Act, \Rightarrow)$ be an environment transition system. S is an $E-$indexed family between processes and nets of processes, i.e., for any $e \in E$, $S_e \subseteq \mathcal{P} \times \mathcal{P}_N$. Then we define $ext(S)$ is also an $E-$indexed family between processes and nets of processes, i.e., for any $e \in E$,

$$\begin{aligned} ext(S)_e = \{(P, \{Q_n : n \in D\}) :\ & \text{there exists } \{P_m : m \in C\} \in \mathcal{P}_N \text{ such that} \\ & (P, \{P_m : m \in C\}) \in S_e \text{ and } \{Q_n : n \in D\} \text{ is a natural} \\ & \text{extenstion of } \{P_m : m \in C\}\} \end{aligned}$$

Proposition 3.15 *If S is a $\varepsilon-$parameterized limit bisimulation, then $ext(S)$ is also a $\varepsilon-$parameterized limit bisimulation.*

3.5 Iteration

In this subsection, we will present a special structure of parameterized limit bisimulation. This construction indicates that the composition of two parameterized limit bisimulation is also a parameterized limit bisimulation. This proposition is important in the compositive execution of some programs. First, we should introduce some useful notions. For simplicity, the definition of product directed set mainly come from [11].

Let $(D_i, \leq_i)$ be a directed set for each $i \in I$. Then the product of $\{(D_i, \leq_i) : i \in I\}$ is defined as

$$\times_{i \in I}(D_i, \leq_i) = (\times_{i \in I} D_i, \leq),$$

where $\leq$ is as follows: for any $d, e \in \times_{i \in I} D_i$, $d \leq e$ if and only if $d(i) \leq e(i)$ for every $i \in I$. It is easy to show $\times_{i \in I}(D_i, \leq_i)$ is also a directed set. Let D be a directed set, let E_m be a directed set for every $m \in D$ and let $F = D \times \times_{m \in D} E_m$. If for any $m \in D$, $\{R(m, n) : n \in E_m\}$ is a net over E_m, then the iteration $\prod_{m\in D}\{R(m,n) : n \in E_m\}$ of $\{R(m,n) : n \in E_m\}(m \in D)$ is the net $\{R(m, f(m)) : (m, f) \in F\}$ over F.

Now let S be an $E-$indexed family between processes and nets of processes and T_m be an $E-$ indexed family between processes and nets of processes for every $m \in D$. Then the composition $S \circ \{T_m : m \in D\}$ of S and $\{T_m : m \in D\}$ is also defined as an $E-$indexed family between processes and nets of processes, i.e., for each $e \in E$,

$$\begin{aligned}(S \circ \{T_m : m \in D\})_e = \{(P, \textstyle\prod_{m\in D}\{R(m,n) : n \in E_m\}) :\ & \text{there exist } Q_m \in \mathcal{P} \\ & (m \in D) \text{ such that } (P, \{Q_m : m \in D\}) \in S_e \text{ and for} \\ & \text{each } m \in D, (Q_m, \{R(m,n) : n \in E_m\}) \in (T_m)_e\}\end{aligned}$$

The following proposition tells us iteration of parameterized limit bisimulations is also parameterized limit bisimulation.

Proposition 3.16 *Let $\varepsilon = (E, Act, \Rightarrow)$ be an environment transition system. If S and $T_m(m \in D)$ are $\varepsilon-$parameterized limit bisimulation, then $S \circ \{T_m : m \in D\}$ is also a $\varepsilon-$parameterized limit bisimulation.*

Proposition 3.17 *Let $\varepsilon = (E, Act, \Rightarrow)$ be an environment transition system. If S_i is a $\varepsilon-$ parameterized limit bisimulation for each $i \in I$, then $\bigcup_{i\in I} S_i$ is also a $\varepsilon-$parameterized limit bisimulation.*

3.6 Fixed point

Now, let S be an $E-$indexed family between processes and nets of processes, $\eta(S)$ be the $E-$indexed family between processes and nets of processes such that $\eta(S)_e$ is a subset of $\mathcal{P} \times \mathcal{P}_N$, i.e.

$$\begin{aligned}\eta(S)_e = \{(P, \{Q_n : n \in D\}) :\ & \text{for all } \alpha \in Act, e \overset{\alpha}{\Rightarrow} f \text{ condition (i) and (ii)} \\ & \text{in Definition 2.9 hold}\}\end{aligned}$$

Then $\varepsilon-$parameterized limit bisimulations are pre-fixed points of the increasing functional η.

Proposition 3.18 *An $E-$indexed family S is a $\varepsilon-$parameterized limit bisimulation if and only if $S \subseteq \eta(S)$, i.e. for all $e \in E$, $S_e \subseteq \eta(S)_e$.*

Proposition 3.19 *η is a monotonic map, i.e., S_1 and S_2 are $E-$indexed families between processes and nets of processes, and $S_1 \subseteq S_2$ implies $\eta(S_1) \subseteq \eta(S_2)$.*

Proposition 3.20 *$\sim \lim$ is the greatest fixed point of η, that is $\eta(\sim \lim) = \sim \lim$, and $\eta(S) = S$ imples $S \subseteq \sim \lim$.*

The following proposition (i) states if the nets of processes $\{P_n : n \in D\}$ and $\{Q_n : n \in D\}$ are eventually parameterized bisimulation in the environment e, i.e., there exists $n_0 \in D$ such that $P_n \sim_e Q_n$ for all $n \geq n_0$, then they have the same parameterized bisimulation limit in the environment e. The proposition (ii) indicates the parameterized bisimulation limit of any net of processes is unique up to parameterized bisimulation.

Proposition 3.21 *Let $\varepsilon = (E, A, \Rightarrow)$ be an environment system, $e \in E$.*

(i) *If for some $n_0 \in D$, $P_n \sim_e Q_n$ for each $n \geq n_0$, and $P \sim_e \lim\limits_{n \in D} P_n$, then $P \sim_e \lim\limits_{n \in D} Q_n$.*

(ii) *If $P \sim_e \lim\limits_{n \in D} P_n$, $Q \sim_e \lim\limits_{n \in D} P_n$, then $P \sim_e Q$.*

Proposition 3.22 *Let $\varepsilon = \{E, Act, \Rightarrow\}$ be an environment system. If P_n is $\lambda-$determinate for each $n \geq n_0 \in D$ and $P \sim_e \lim\limits_{n \in D} P_n$, then P is also $\lambda-$determinate.*

Proof. It can be proved according to the definition of $\lambda-$determinate in the environment e and Proposition 3.21.

□

4 Parameterized Bisimulation Limit Topology

As is well known, a topology can be constructed from an existing convergence structure. In this section, we mainly construct a family of parameterized bisimulation limit topologies based on parameterized bisimulation limit. One of the main results in this section is Proposition 4.2, which states that the parameterized bisimulation limit does give rise to a convergence class, furthermore, it yields a topology on processes.

Let $\varepsilon = \{E, A, \Rightarrow\}$ be an environment transition system, S be an $E-$indexed family of binary relation between processes and nets of processes, i.e., for any $e \in E$, $S_e \subseteq \mathcal{P} \times \mathcal{P}_N$. Let $p(S)$ stands for the property of S that for each $e \in E$, $(P, \{P_m : m \in C\}) \notin S_e$ implies that there exists a subnet $\{Q_n : n \in D\}$ of $\{P_m : m \in C\}$ with $(P, \{R_p : p \in E\}) \notin S_e$ for any subnet $(R_p : p \in E)$ of $\{Q_n : n \in D\}$.

Following lemma shows that the η image of the natural extension of any relation between processes and nets of processes satisfies property p.

Lemma 4.1 *Let $\varepsilon = \{E, A, \Rightarrow\}$ be an environment transition system and S be an indexed set between processes and nets of processes, i.e., for any $e \in E$, $S_e \subseteq \mathcal{P} \times \mathcal{P}_N$, then $p(\eta(ext(S)))$ holds.*

Proof. The proof is similar to Lemma 3.1.12 in [11]. But for the integrity of the paper, we show the proof as follows:

If $(P, \{P_m : m \in C\}) \notin \eta(ext(S))_e$, then for some $\alpha \in Act$, $e \overset{\alpha}{\Rightarrow} f$, we have one of the following two cases: we want to find some subnet $\{Q_n : n \in D\}$ of $\{P_m : m \in C\}$ with $(P, \{R_p : p \in E\}) \notin \eta(ext(S))_e$ for any subnet $\{R_p : p \in E\}$ of $\{Q_n : n \in D\}$ in both of these cases.

case 1. There is $P' \in \mathcal{P}$ such that $P \xrightarrow{\alpha} P'$ and for any $\{P'_m : m \in C\} \in \mathcal{P}_N$ and for any $k \in C$,$\{P', \{P'_m : m \in C\}\} \in ext(S)_f$ implies that $P_n \overset{\alpha}{\nrightarrow} P'_n$ for some $n \geq k$.

Let

$$\Omega = \{\{U_m : m \in C\} \in \mathcal{P}_N : (P', \{U_m : m \in C\}) \in ext(S)_f\}.$$

For any $M = (\{U_m : m \in C\}, k), M' = (\{U'_m : m \in C\}, k') \in \Omega \times C$, we define $M \leq N$ if and only if $k \leq k'$. Then $(\Omega \times C, \leq)$ is a directed set. Now, for any $M = (\{U_m : m \in C\}, k) \in \Omega \times C$, we know that there is $n_M \geq k$ with $P_{n_M} \overset{\alpha}{\nrightarrow} P'_{n_M}$ because we are working in Case 1. We set $Q_M = P_{n_M}$ for each $M \in \Omega \times C$. Then $\{Q_M : M \in \Omega \times C\}$ is a subnet of $\{P_m : m \in C\}$, and it suffices to show that for any subnet $\{R_p : p \in E\}$ of $\{Q_M : M \in \Omega \times C\}$, $(P, \{R_p : p \in E\}) \notin \eta(ext(S))_e$. In fact, if $(P, \{R_p : p \in E\}) \in \eta(ext(S))_e$, then $P \xrightarrow{\alpha} P'$ leads to $\{R'_p : p \in E\} \in \mathcal{P}_N$ and $p_0 \in E$ such that $R_p \xrightarrow{\alpha} R_p$ for any $p \geq p_0$ and $(P', \{R'_p : p \in E\}) \in ext(S)_f$. Suppose that $N : E \to \Omega \times C$ is increasing, $R_p = Q_{N_p}$ for every $p \in E$, and (E, N) is a cofinality of $\Omega \times C$. Then $(E, proj_C \circ N)$ is a cofinality of C, where $proj_C$ is the projection from $\Omega \times C$ onto C,i.e., $proj_C(T, m) = m$ for any $T \in \Omega$ and $m \in C$. Furthermore, we find a natural extension $\{V'_m : m \in C\}$ of $\{R'_p : p \in E\}$ with $V'_{n_p} = R'_p$ for every $p \in E$ by using the Axiom of Choice. Thus, we set $T_0 = \{V'_m : m \in C\}$ and obtain $(P', T_0) \in ext(ext(S)_f) = ext(S)_f$. Let $m_0 = proj_C(N_{p_0})$. Then $(T_0, m_0) \in \Omega \times C$, and there is $p \in E$ with $N_p \geq (T_0, m_0)$. Since N is increasing, we have $p \geq p_0$. On the other hand, from $(P', \{V'_m : m \in C\}) \in ext(S)_f$, we know that $R_p = Q_{N_p} = P_{n_{N_p}} \overset{\alpha}{\nrightarrow} R'_p$. This contradicts that $R_p \xrightarrow{\alpha} R'_p$ for all $p \geq p_0$.

Case 2, There is a cofinal subnet D of C such that $P_n \xrightarrow{\alpha} P'_n$ for all $n \in D$, and for any $P' \in \mathcal{P}$ and for any cofinal subset B of D, $P \overset{\alpha}{\nrightarrow} P'$ or $(P', \{P'_k : k \in B\}) \notin ext(S)_f$.

In this case, $\{P_n : n \in D\}$ is subnet of $\{P_m : m \in C\}$. So we need to show only that for any subnet $\{R_p : p \in E\}$ of $\{P_n : n \in D\}$, $(P, \{R_p : p \in E\}) \notin \eta(ext(S))_e$. Suppose that $N : E \to D$ is increasing, $R_p = P_{N_p}$ for every $p \in E$, and (E, N) is a cofinality of D. If $(P, \{R_p : p \in E\}) \in \eta(ext(S)_e$, then from $R_p = P_{N_p} \xrightarrow{\alpha} P'_{N_p}$ for all $p \in E$, we know that there are $P' \in \mathcal{P}$ and a cofinal subset of F of E with $P \xrightarrow{\alpha} P'$ and $(P', \{P'_{N_q} : q \in F\}) \in ext(S)_f$. On the other hand, $N(F)$ is a cofinal subset of D because $N(E)$ is increasing. This leads to $(P', \{P'_{N_q} : q \in F\}) \in ext(S)_f$, contradictorily. □

Proposition 4.2 *Let $\varepsilon = (E, A, \Rightarrow)$ be an environment transition system and $e \in E$.*

(i) *If P is $cf(D)-$determinate in the environment e and there exists $n_0 \in D$ such that $Q_n \sim_e P$ for each $n \geq n_0$, then $P \sim_e \lim_{n \in D} Q_n$.*

(ii) *If $\{Q_n : n \in D\}$ is a subnet of $\{P_m : m \in C\}$ and $P \sim_e \lim_{m \in C} P_m$, then $P \sim_e \lim_{n \in D} Q_n$.*

(iii) *If $P \sim_e \lim_{m \in C} P_m$ does not hold, then there exists a subnet $\{Q_n : n \in D\}$ of $\{P_m : m \in C\}$ such that for any subnet $\{R_p : p \in E\}$ of $\{Q_n : n \in D\}$, $P \sim_e \lim_{p \in E} R_p$ does not hold.*

(iv) *Let D be a directed set, let E_m be a directed set for each $m \in D$, let $F = D \times \times_{m \in D} E_m$, and let $R(m, f) = (m, f(m))$ for each $(m, f) \in F$. If for any $m \in D$, $Q_m \sim_e \lim_{n \in E_m} P(m, n)$, and $Q \sim_e \lim_{m \in D} Q_m$, then $Q \sim_e \lim_{(m,f) \in F} (P \circ R)(m, f)$.*

Proof. (i),(ii) and (iv) are immediate from Proposition 3.6, Lemma 3.8 and Proposition 3.16.

(iii) According to the the definition of $ext(\circ)$, it holds that $\sim \lim \subseteq ext(\sim \lim)$. Since $\sim \lim$ is the greatest $\varepsilon-$parameterized limit bisimulation, Proposition 3.15 asserts that $ext(\sim \lim)$ is also a $\varepsilon-$parameterized limit bisimulation, and we can get that $ext(\sim \lim) \subseteq \sim \lim$ and therefore $ext(\sim \lim) = \sim \lim$. Thus, Proposition 3.20 leads to $\sim \lim = \eta(\sim \lim) = \eta(ext(\sim \lim))$. So, $(P, \{P_m : m \in C\}) \notin \sim_e \lim$, i.e.,$(P, \{P_m : m \in C\}) \notin \eta(ext(\sim \lim))$. According to Lemma 4.1, we can obtain that there exists a subnet $\{Q_n : n \in D\}$ of $\{P_m : m \in C\}$ such that for any subnet $\{R_p : p \in E\}$ of $\{Q_n : n \in D\}$, $(P, \{R_p : p \in E\}) \notin \eta(ext(\sim \lim)) = \sim \lim$, i.e., $P \sim_e \lim_{p \in E} R_p$ does not hold, and the proof is completed according.

□

Next, we will try to establish a family of parameterized bisimulation limit topologies. This is carried out in a standard way in point-set topology. For each $e \in E$, we writes $(\mathcal{P}_D)_e$ for the class of determinate processes in the environment e. For any $U \subseteq (\mathcal{P}_D)_e$, we define

$$cl_e(U) = \{P \in \mathcal{P} : \text{ there exists a net } \{Q_n : n \in D\} \text{ such that } P \sim_e \lim_{n \in D} Q_n$$
$$\text{and for all } n \in D, Q_n \in U\}$$

Then Proposition 3.22 guarantees that $cl_e(U) \subseteq (\mathcal{P}_D)_e$. According to Theorem 2.9 in [15] and Proposition 4.2, we know that cl_e is a closure operator on $(\mathcal{P}_D)_e$ for each $e \in E$, i.e., cl_e is a mapping from $2^{(\mathcal{P}_D)_e}$ into itself and fulfills Kuratowski's axioms of closure:

- $cl_e(\emptyset) = \emptyset$
- $U \subseteq cl_e(U)$
- $cl_e(cl_e(U)) = cl_e(U)$;and
- $cl_e(U \cup V) = cl_e(U) \cup cl_e(V)$.

The closure operator cl_e determined by parameterized bisimulation limits induces a topology for each $e \in E$,

$$(\mathcal{J}_{PB})_e = \{U \in (\mathcal{P}_D)_e : cl_e((\mathcal{P}_D)_e - U) = (\mathcal{P}_D)_e - U\}$$

on $(\mathcal{P}_D)_e$, called the parameterized bisimulation topology in the environment e. Thus, for each $e \in E$, we have a topology $(\mathcal{J}_{PB})_e$. So, we can define $\mathcal{J}_{PB}$ is an $E-$indexed family of parameterized bisimulation limit topology, i.e., for each $e \in E$, $(\mathcal{J}_{PB})_e$ is a topology. Suppose (X, τ) is a topological space, $U \subseteq X$ and $x \in X$. If there is an open set $V \in \tau$ such that $x \in V \subseteq U$, then U is called a neighborhood of x. Let $S = \{S_n : n \in D\}$ eventually in each neighborhood of x;i.e. for each neighborhood U of x, there is $n_0 \in D$ such that $S_n \in U$ for all $n \geq n_0$. In this case, we write $x = (\tau) \lim_{n \in D} S_n$. From Theorem 2.9 [15] and Proposition 4.2, we also know that convergence determined by this topology coincides with parameterized bisimulation limit; more explicitly, for any process $P \in (\mathcal{P}_D)_e$ and for any net $\{P_n : n \in D\}$ of processes in $(\mathcal{P}_D)_e$, $P \sim_e \lim_{n \in D} P_n$ if and only if $P = (\mathcal{J}_{PB})_e \lim_{n \in D} P_n$.

According to Larsen's work, if e is an environment such that for all $\alpha \in Act$, $e \overset{\alpha}{\Rightarrow} e$, then $\sim_e = \sim$. Thus, we can see that the family of parameterized bisimulation limit topologies generalize strong bisimulation topology.

5 Conclusion

In this paper, we mainly discuss the topological proposition of parameterized limit bisimulation in order to characterize the infinite evolution of parameterized bisimulation and establish a family of parameterized bisimulation limit topology based on parameterized bisimulation limit. In the future, we will try to establish the continuous property about some composition operators which are useful in modular design and hierarchic developments.

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