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A Metrized Duality Theorem for Markov Processes

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Abstract

We extend our previous duality theorem for Markov processes by equipping the processes with a pseudometric and the algebras with a notion of metric diameter. We are able to show that the isomorphisms of our previous duality theorem become isometries in this quantitative setting. This opens the way to developing theories of approximate reasoning for probabilistic systems.

Keywords: Markov processes, Aumann algebras, Stone duality, metrics, isometry, probabilistic reasoning, quantitative reasoning.

1 Introduction

Stone-type dualities are recognized as being ubiquitous, especially in computer science. For example Plotkin [22] and Smyth [23] (see also [21]) emphasized that the duality between state-transformer semantics and predicate-transformer semantics is an instance of Stone-type duality. A similar duality was observed for probabilistic transition systems [14]. Recently several authors, see for example [3], have emphasized the duality between logics and transition systems from a coalgebraic perspective. Mislove et al. [19] have found a duality between labelled Markov processes and C^*-algebras based on the closely related classical Gelfand duality.

In a recent paper [15], a Stone-type duality was developed for Markov processes defined on continuous state spaces. The algebraic counterpart of the Markov processes were called Aumann algebras in honour of Aumann's work on probabilistic reasoning [1]. Aumann algebras capture, in algebraic form, a modal logic in which bounds on probabilities enter into the modalities. This logic can be stripped down to a very spartan core – just the modalities and finite conjunction – and still characterize bisimulation for labelled Markov processes [5,6]. However, to obtain

the strong completeness properties that are implied by the duality theorems, one needs infinitary proof principles [12].

One of the critiques [11] of logics and equivalences used for the treatment of probabilistic systems is that boolean logic is not robust with respect to small perturbations of the real-valued system parameters. Accordingly, a theory of metrics [7, 8, 24, 25] was developed and metric reasoning principles were advocated. In the present paper we extend our exploration of duality theory with an investigation into the role of metrics and exhibit a metric analogue of the duality theory. This opens the way for an investigation into quantitative aspects of *approximate reasoning*.

In the present paper we integrate quantitative information into the duality of [15] by endowing Markov processes with a (pseudo)metric and Aumann algebras with a quantitative "norm-like" structure called a *metric diameter*. The interplay between the pseudometric and the boolean algebra is somewhat delicate and had to be carefully examined for the duality to emerge. The final results have easy proofs but the correct way to impose quantitative structure on Aumann algebras was elusive. The key idea is to axiomatize the notion of metric diameter on the Aumann algebra side. This is a concept more like a norm than a distance, but one can derive a metric from it. The idea comes from a paper by Banaschewski and Pultr [2] on Stone duality for metric spaces. However, our formulation is not quite the same as theirs.

2　Background

Given a relation $\mathfrak{R} \subseteq M \times M$, the set $N \subseteq M$ is $\mathfrak{R}$-closed iff

$$\{m \in M \mid \exists n \in N, (n, m) \in \mathfrak{R}\} \subseteq N.$$

We assume that the reader is familiar with the definitions of field of sets, σ-algebra, measurable function, measurable space and measure.

If (M, Σ) is a measurable space and $\mathfrak{R} \subseteq M \times M$ is a relation on M, then $\Sigma(\mathfrak{R})$ denotes the set of measurable $\mathfrak{R}$-closed subsets of M.

Given a measurable space $\mathcal{M} = (M, \Sigma)$, we view the set $\Delta(M, \Sigma)$ of measures defined on Σ, as itself being a measurable space by endowing it with the σ-algebra $\mathfrak{F}$ generated by the sets $F(S, r) = \{\mu \in \Delta(M, \Sigma) : \mu(S) \geq r\}$ for arbitrary $S \in \Sigma$ and $r > 0$.

Measure theory works most smoothly in conjunction with certain topological assumptions. A *Polish space* is the topological space underlying a complete separable metric space. An *analytic* space is the image of a Polish space X under a continuous function $f : X \to Y$, where Y is also a Polish space. The special properties of analytic spaces were crucial in the proof of the logical characterization of bisimulation [6].

Let M be a set and $d : M \times M \to \mathbb{R}$.

Definition 2.1 We say that d is a *pseudometric* on M if for arbitrary $x, y \in M$,

$$(1): d(x, x) = 0$$

$$(2): d(x, y) = d(y, x)$$

$$(3): d(x, y) \leq d(x, z) + d(z, y)$$

We say that (M, d) is a *pseudometric space*.

Pseudometrics arise as metric analogues of bisimulation [7, 8]. A pseudometric defines an equivalence relation called the *kernel* of the pseudometric, by $x \sim y$ iff $d(x, y) = 0$. The metrics defined in [7, 8] had bisimulation as the kernel.

In a pseudometric space (M, d), the *open ball* with center $x \in M$ and radius $\varepsilon > 0$ is the set $\{y \in M \mid d(x, y) < \varepsilon\}$. The collection of open balls forms a base for a topology called the *metric topology*. We can extend the pseudometric to sets in a manner analogous to the way in which one extends metrics to compact sets.

Definition 2.2 For a pseudometric d on M we define the *Hausdorff pseudometric* d^H on the class of subsets of X by

$$d^H(X, Y) = \max(\sup_{x \in X} \inf_{y \in Y} d(x, y), \sup_{y \in Y} \inf_{x \in X} d(x, y)).$$

We need to verify that these are indeed pseudometrics. The proof of the following lemma is omitted here; it is not too hard.

Lemma 2.3 *If $d : M \times M \to [0, 1]$ is a pseudometric on M, then d^H is a pseudometric on subsets of M.*

Markov processes (MPs) are models of probabilistic systems with a continuous state space and probabilistic transitions [6, 9, 20]. In earlier papers, they were called *labeled* Markov processes to emphasize the fact that there were multiple possible actions, but here we will suppress the labels, as they do not contribute any relevant structure for our results.

Definition 2.4 [Markov process] A *Markov process (MP)* is a tuple $\mathcal{M} = (M, \Sigma, \theta)$, where (M, Σ) is an analytic space and $\theta \in [\![M \to \Delta(M, \Sigma)]\!]$.

In a Markov process $\mathcal{M} = (M, \Sigma, \theta)$, M is the *support set*, denoted by $\text{supp}\mathcal{M}$, and θ is the *transition function*. For $m \in M$, $\theta(m) : \Sigma \to [0, 1]$ is a probability measure on the state space (M, Σ). For $N \in \Sigma$, the value $\theta(m)(N) \in [0, 1]$ represents the probability of a transition from m to a state in N.

The condition that θ is a measurable function $[\![M \to \Delta(M, \Sigma)]\!]$ is equivalent to the condition that for fixed $N \in \Sigma$, the function $m \mapsto \theta(m)(N)$ is a measurable function $[\![M \to [0, 1]]\!]$ (see e.g. Proposition 2.9 of [9]).

Markovian logic (ML) is a multi-modal logic for semantics based on MPs [1, 4, 10, 13, 16, 18, 26]. In addition to the Boolean operators, this logic is equipped with *probabilistic modal operators* L_r for $r \in \mathbb{Q}_0$ that bound the probabilities of transitions. Intuitively, the formula $L_r\varphi$ is satisfied by $m \in \mathcal{M}$ whenever the probability of a transition from m to a state satisfying φ is at least r.

Definition 2.5 [Syntax] The formulas of $\mathcal{L}$ are defined by the grammar

$$\varphi \ ::= \ \bot \mid \varphi \rightarrow \varphi \mid L_r\varphi$$

where r can be any element of $\mathbb{Q}_0$.

The Boolean operators $\vee$, $\wedge$, $\neg$, and $\top$ are defined from $\rightarrow$ and $\bot$ as usual. For $r_1, \ldots, r_n \in \mathbb{Q}_0$ and $\varphi \in \mathcal{L}$, let

$$L_{r_1 \cdots r_n}\varphi = L_{r_1} \cdots L_{r_n}\varphi.$$

The *semantics* for $L_r\varphi$ is defined as follows; the semantics of the other constructs are obvious. For MP $\mathcal{M} = (M, \Sigma, \theta)$ and $m \in M$,

$$\mathcal{M}, m \models L_r\varphi \text{ if } \theta(m)(\llbracket\varphi\rrbracket) \geq r,$$

where $\llbracket\varphi\rrbracket = \{m \in M \mid \mathcal{M}, m \models \varphi\}$.

For this to make sense, $\llbracket\varphi\rrbracket$ must be measurable, this is readily verified. We define negation in the obvious fashion and use the words *valid* and *satisfiable* in the usual way.

This logic can be axiomatized using Hilbert-style axioms. The (strong) completeness of this logic is proved in [12, 18, 26] by assuming Lindenbaum's lemma (every consistent set can be expanded to a maximal consistent set) as a meta-axiom. The duality theorem of Kozen et al. [15] implies strong completeness without needing this assumption.

The logical equivalence induced by ML on the class of MPs coincides with bisimulation equivalence [6, 20]. The proof requires that the state space be an analytic space.

3   Stone Duality for Markov Processes

In this section we briefly summarize the results of our previous duality paper [15].

We introduce an algebraic version of Markovian logic consisting of a Boolean algebra with operators F_r for $r \in \mathbb{Q}_0$ corresponding to the operators L_r of ML. We call these algebras *Aumann algebras*. They are dual to certain Markov processes constructed from zero-dimensional Hausdorff spaces called *Stone–Markov processes* (SMPs).

Definition 3.1 [Aumann algebra] An *Aumann algebra (AA)* is a structure $\mathcal{A} = (A, \to, \bot, \{F_r\}_{r \in \mathbb{Q}_0}, \le)$ where

- $(A, \to, \bot, \le)$ is a Boolean algebra;

- for each $r \in \mathbb{Q}_0$, $F_r : A \to A$ is a unary operator; and

- the axioms in Table 1 hold for all $a, b \in A$ and $r, s, r_1, \ldots, r_n \in \mathbb{Q}_0$.

The Boolean operations $\vee$, $\wedge$, $\neg$, and $\top$, are defined from $\to$ and $\bot$ as usual.

Morphisms of Aumann algebras are Boolean algebra homomorphisms that commute with the operations F_r. The category of Aumann algebras and Aumann algebra homomorphisms is denoted **AA**.

We abbreviate $F_{r_1} \cdots F_{r_n} a$ by $F_{r_1 \cdots r_n} a$.

(AA1) $\top \le F_0 a$

(AA2) $\top \le F_r \top$

(AA3) $F_r a \le \neg F_s \neg a$, $r + s > 1$

(AA4) $F_r(a \wedge b) \wedge F_s(a \wedge \neg b) \le F_{r+s} a$, $r + s \le 1$

(AA5) $\neg F_r(a \wedge b) \wedge \neg F_s(a \wedge \neg b) \le \neg F_{r+s} a$, $r + s \le 1$

(AA6) $a \le b \Rightarrow F_r a \le F_r b$

(AA7) $\left(\bigwedge_{r<s} F_{r_1 \cdots r_n r} a \right) = F_{r_1 \cdots r_n s} a$

Table 1
Aumann algebra

The operator F_r is the algebraic counterpart of the logical modality L_r. The first two axioms state tautologies, while the third captures the way F_r interacts with negation. Axioms (AA4) and (AA5) assert finite additivity, while (AA6) asserts monotonicity.

The most interesting axiom is the infinitary axiom (AA7). It asserts that $F_{r_1 \cdots r_n s} a$ is the greatest lower bound of the set $\{F_{r_1 \cdots r_n r} a \mid r < s\}$ with respect to the natural order $\le$. In SMPs, it implies countable additivity.

As expected, the formulas of Markovian logic modulo logical equivalence form a free countable Aumann algebra. Define $\equiv$ on formulas in the usual way and let $[\varphi]$ denote the equivalence class of φ modulo $\equiv$, and let $\mathcal{L}_\equiv = \{[\varphi] \mid \varphi \in \mathcal{L}\}$.

Theorem 3.2 *The structure*

$$(\mathcal{L}/\equiv, \to, [\bot], \{L_r\}_{r \in \mathbb{Q}_0}, \le)$$

is an Aumann algebra, where $[\varphi] \le [\psi]$ *iff* $\vdash \varphi \to \psi$.

A *Stone–Markov process* (SMP) is a Markov process $(M, \mathcal{A}, \theta)$, where $\mathcal{A}$ is a distinguished countable base of clopen sets that is closed under the set-theoretic Boolean

operations and the operations

$$F_r(A) = \{m \mid \theta(m)(A) \geq r \mid\}, \quad r \in \mathbb{Q}_0.$$

The measurable sets Σ are the Borel sets of the topology generated by $\mathcal{A}$. Morphisms of such spaces are required to preserve the distinguished base; thus a morphism $f : \mathcal{M} \to \mathcal{N}$ is a continuous function such that

- for all $m \in M$ and $B \in \Sigma_{\mathcal{N}}$,

$$\theta_{\mathcal{M}}(m)(f^{-1}(B)) = \theta_{\mathcal{N}}(f(m))(B);$$

- for all $A \in \mathcal{A}_{\mathcal{N}}$, $f^{-1}(A) \in \mathcal{A}_{\mathcal{M}}$.

Unlike Stone spaces, SMPs are not topologically compact, but we do postulate a completeness property that is a weak form of compactness, which we call *saturation*. One can saturate a given SMP by a completion procedure that is reminiscent of Stone–Čech compactification [15]. Intuitively, one adds points to the structure without changing the represented algebra. An MP is *saturated* if it is maximal with respect to this operation.

Definition 3.3 [Stone–Markov Process] A Markov process $\mathcal{M} = (M, \mathcal{A}, \theta)$ with distinguished base is a *Stone–Markov process (SMP)* if it is saturated.

The morphisms of SMPs are just the morphisms of MPs with distinguished base as defined above.

The category of SMPs and SMP morphisms is denoted **SMP**.

Fix an arbitrary countable Aumann algebra

$$\mathcal{A} = (A, \to, \bot, \{F_r\}_{r \in \mathbb{Q}_0}, \leq).$$

Let $\mathcal{U}^*$ be the set of all ultrafilters of $\mathcal{A}$. The classical Stone construction gives a Boolean algebra of sets isomorphic to $\mathcal{A}$ with elements

$$(\!|a|\!)^* = \{u \in \mathcal{U}^* \mid a \in u\}, \qquad (\!|\mathcal{A}|\!)^* = \{(\!|a|\!)^* \mid a \in A\}.$$

The sets $(\!|a|\!)^*$ generate a Stone topology τ^* on $\mathcal{U}^*$, and the $(\!|a|\!)^*$ are exactly the clopen sets of the topology.

Let $\mathcal{F}$ be the set of elements of the form $\alpha^r = F_{t_1 \cdots t_n r} a$ for $a \in A$ and $t_1, \ldots, t_n, r \in \mathbb{Q}_0$. As before, we consider this term as parameterized by r; that is, if $\alpha^r = F_{t_1 \cdots t_n r} a$, then α^s denotes $F_{t_1 \cdots t_n s} a$. The set $\mathcal{F}$ is countable since A is. Axiom (AA7) asserts all infinitary conditions of the form

$$\alpha^s = \bigwedge_{r < s} \alpha^r. \tag{1}$$

for $\alpha^s \in \mathcal{F}$. Let us call an ultrafilter u *bad* if it violates one of these conditions in the sense that for some $\alpha^s \in \mathcal{F}$, $\alpha^r \in u$ for all $r < s$ but $\alpha^s \notin u$. Otherwise, u is called *good*. Let $\mathcal{U}$ be the set of good ultrafilters of $\mathcal{A}$.

Let $\tau = \{B \cap \mathcal{U} \mid B \in \tau^*\}$ be the subspace topology on $\mathcal{U}$, and let

$$(\!| a |\!) = \{u \in \mathcal{U} \mid a \in u\} = (\!| a |\!)^* \cap \mathcal{U} \qquad\qquad (\!| \mathcal{A} |\!) = \{(\!| a |\!) \mid a \in A\}.$$

Then τ is countably generated by the sets $(\!| a |\!)$ and all $(\!| a |\!)$ are clopen in the subspace topology.

We can now form a Markov process $\mathbb{M}(\mathcal{A}) = (\mathcal{U}, \Sigma, \theta)$, where Σ is the σ-algebra generated by $(\!| \mathcal{A} |\!)$ and $\theta : \mathcal{U} \to (\!| \mathcal{A} |\!) \to [0,1]$ is defined on the generators by

$$\theta(u)((\!| a |\!)) = \sup\{r \in \mathbb{Q}_0 \mid F_r a \in u\} = \inf\{r \in \mathbb{Q}_0 \mid \neg F_r a \in u\}.$$

It can be shown that θ extends uniquely to a transition function [15].

Theorem 3.4 *If $\mathcal{A}$ is a countable Aumann algebra, then $\mathbb{M}(\mathcal{A}) = (\mathcal{U}, (\!| \mathcal{A} |\!), \theta)$ is a Stone Markov process.*

Most of the technical difficulties of our earlier paper are in the proof of this theorem.

Let $\mathcal{M} = (M, \mathcal{B}, \theta)$ be a Stone Markov process with distinguished base $\mathcal{B}$. By definition, $\mathcal{B}$ is a field of clopen sets closed under the operations

$$F_r(A) = \{m \in M \mid \theta(m)(A) \geq r\}.$$

Theorem 3.5 *The structure $\mathcal{B}$ with the set-theoretic Boolean operations and the operations F_r, $r \in \mathbb{Q}_0$ is a countable Aumann algebra.*

We denote this algebra by $\mathbb{A}(\mathcal{M})$. Now we have the duality theorem.

Theorem 3.6 (Duality Theorem [15])

(i) *Any countable Aumann algebra $\mathcal{A}$ is isomorphic to $\mathbb{A}(\mathbb{M}(\mathcal{A}))$ via the map $\beta : \mathcal{A} \to \mathbb{A}(\mathbb{M}(\mathcal{A}))$ defined by*

$$\beta(a) = \{u \in \mathrm{supp}(\mathbb{M}(\mathcal{A})) \mid a \in u\} = (\!| a |\!).$$

(ii) *Any Stone Markov process $\mathcal{M} = (M, \mathcal{A}, \theta)$ is homeomorphic to $\mathbb{M}(\mathbb{A}(\mathcal{M}))$ via the map $\alpha : \mathcal{M} \to \mathbb{M}(\mathbb{A}(\mathcal{M}))$ defined by*

$$\alpha(m) = \{A \in \mathcal{A} \mid m \in A\}.$$

In [15], we also give a categorical version of this theorem with the contravariant functors between the two categories given explicitly.

4   Extending the Duality to Metrized Markov Processes

We add quantitative structure to both Markov processes and Aumann algebras. We prove an extended version of the representation theorem for metrized Markov processes versus metrized Aumann algebras. This theorem states that starting from an arbitrary metrized Markov process, we can extend the Aumann algebra constructed in the previous section to a metrized Aumann algebra that preserves the pseudometric and conversely. In other words the natural isomorphisms that arise in the duality of [15] will turn out to be isometries.

We equip Stone Markov processes with a pseudometric that measures distances between the states of the MP. The key condition that we impose is that for a particular state m, the diameters of the clopens containing m converges to 0.

Definition 4.1 [Metrized Markov process] A metrized Markov process is a tuple $(\mathcal{M}, d)$, where $\mathcal{M} = (M, \mathcal{A}, \theta)$ is a Stone Markov process with $\mathcal{A}$ its countable base of clopens and $d : M \times M \to [0, 1]$ is a pseudometric on M satisfying for arbitrary $m \in M$ the property

$$(M) \quad \inf_{c \in \mathcal{A}, m \in c} \sup\{d(n, n') \mid n, n' \in c\} = 0.$$

The following lemma gives a number of conditions equivalent to (M). In particular, it shows the connection between the topology of the Stone Markov space and the pseudometric topology.

Lemma 4.2 *For a metrized MP $(\mathcal{M}, d)$, where $\mathcal{M} = (M, \mathcal{A}, \theta)$, the following are equivalent:*

(i) $\forall m, \quad \inf_{c \in \mathcal{A}, m \in c} \sup\{d(n, n') \mid n, n' \in c\} = 0$

(ii) $\forall m, m' \quad \inf_{c \in \mathcal{A}, m, m' \in c} \sup\{d(n, n') \mid n, n' \in c\} = d(m, m')$

(iii) $\forall m, \ \forall \varepsilon > 0 \ \exists c \in \mathcal{A} \ (m \in c \wedge \forall n, n' \in c \ d(n, n') < \varepsilon)$

(iv) $\forall m \ \forall \varepsilon > 0 \ \exists c \in \mathcal{A} \ (m \in c \wedge \forall n \in c \ d(m, n) < \varepsilon)$

(v) *The topology generated by $\mathcal{A}$ refines the pseudometric topology generated by d.*

(vi) *The pseudometric d is continuous in both arguments with respect to the $\mathcal{A}$-topology.*

Proof. Note that (i) is just (M).

(i) $\Leftrightarrow$ (iii) is immediate from the definitions.

For (iii) $\Rightarrow$ (iv), we can substitute m, n for n, n' in (iii).

For (iv) $\Rightarrow$ (iii), let m and $\varepsilon > 0$ be arbitrary, and let $c \in \mathcal{A}$ such that $m \in c$ and for all $n \in c$, $d(m, n) < \varepsilon/2$ and $d(n, m) < \varepsilon/2$. Then for any $n, n' \in c$, $d(n, n') \leq d(n, m) + d(m, n') < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

For (iv) $\Leftrightarrow$ (v), let $N_\varepsilon(m) = \{x \mid d(m, x) < \varepsilon\}$. Then,

$$\forall m \; \forall \varepsilon > 0 \; \exists c \in \mathcal{A} \; (m \in c \wedge \forall n \in c \; d(m, n) < \varepsilon \wedge d(n, m) < \varepsilon)$$
$$\Leftrightarrow \; \forall m \; \forall \varepsilon > 0 \; \exists c \in \mathcal{A} \; (m \in c \wedge \forall n \in c \; n \in N_\varepsilon(m))$$
$$\Leftrightarrow \; \forall m \; \forall \varepsilon > 0 \; \exists c \in \mathcal{A} \; (m \in c \wedge c \subseteq N_\varepsilon(m)).$$

The last statement says that every basic open neighborhood of the pseudometric topology contains a basic open neighborhood of the $\mathcal{A}$-topology, which says exactly that the $\mathcal{A}$-topology refines the pseudometric topology.

For (iii) $\Rightarrow$ (vi), to show that d is continuous in its second argument, let m, x and $\varepsilon > 0$ be arbitrary and let $c \in \mathcal{A}$ such that $x \in c$ and for all $n, n' \in c$, $d(n, n') < \varepsilon$. Then for all $y \in c$,

$$d(m, y) \leq d(m, x) + d(x, y) < d(m, x) + \varepsilon$$
$$d(m, x) \leq d(m, y) + d(y, x) < d(m, y) + \varepsilon$$

so $d(m, y) \in (d(m, x) - \varepsilon, d(m, x) + \varepsilon)$. That d is continuous in its first argument follows from symmetry.

For the other direction, suppose d is continuous in its second argument. Then for all m and $\varepsilon > 0$, the set $N_\varepsilon(m)$ is open and contains m, thus there exists a basic open set $c \in \mathcal{A}$ such that $m \in c$ and $c \subseteq N_\varepsilon(m)$. Thus the $\mathcal{A}$-topology refines the pseudometric topology of d.

Statement (ii) implies (i) immediately by taking $m' = m$ in (ii).

To show (i) implies (ii), let m, m' and $\varepsilon > 0$ be arbitrary. From (iv), we have $c \in \mathcal{A}$ such that $m \in c$ and from (iii) we have that for all $n, n' \in c$, $d(n, n') < \varepsilon/2$ and we have $c' \in \mathcal{A}$ such that $m' \in c'$ and for all $n, n' \in c'$, $d(n, n') < \varepsilon/2$. We claim that for all $n, n' \in c \cup c'$, $d(n, n') < d(m, m') + \varepsilon$, which will establish (ii).

If $n, n' \in c$, then

$$d(n, n') \leq d(n, m) + d(m, n') < \varepsilon/2 + \varepsilon/2 = \varepsilon \leq d(m, m') + \varepsilon.$$

If $n, n' \in c'$, the argument is the same, replacing m by m'. If $n \in c$ and $n' \in c'$, then

$$d(n', n) = d(n, n') \leq d(n, m) + d(m, m') + d(m', n')$$
$$< \varepsilon/2 + d(m, m') + \varepsilon/2 = d(m, m') + \varepsilon.$$

$\square$

Definition 4.3 [Isometric Markov processes] Given two metrized MPs $(\mathcal{M}_i, d_i)$, where $\mathcal{M}_i = (M_i, \Sigma_i, \theta_i)$ for $i = 1, 2$, an isometry from $\mathcal{M}_1$ to $\mathcal{M}_2$ is a map $f : M_1 \to M_2$ such that for arbitrary $m, n \in M_1$,

$$d_1(m, n) = d_2(f(m), f(n)).$$

Now we introduce the metrized Aumann algebras. Despite their name, the metrized AAs are not directly equipped with a pseudometric structure, but with a concept of metric diameter. Later we will prove that the metric diameter can indeed define a pseudometric.

Definition 4.4 [Metrized Aumann algebra] A *metrized Aumann algebra* is a tuple $(\mathcal{A}, | \ |)$, where $\mathcal{A} = (A, \to, \perp, \{F_r\}_{r \in \mathbb{Q}_0}, \le)$ is an Aumann algebra and $| \ | : A \to [0, 1]$ is a metric diameter on A, which is a map satisfying, for arbitrary $a, b \in A$ and ultrafilter u, the following properties

(A1) $|\perp| = 0$;

(A2) if $a \le b$, then $|a| \le |b|$;

(A3) if $a \wedge b \ne \perp$, then $|a \vee b| \le |a| + |b|$;

(A4) $\inf\{|a| \mid a \in u\} = 0$.

Definition 4.5 [Isometric Aumann Algebras] Given two metrizable Aumann Algebras $(\mathcal{A}_i, | \ |_i)$ for $i = 1, 2$, an isometry from $\mathcal{A}_1$ to $\mathcal{A}_2$ is a map $f : A_1 \to A_2$ such that for any $a \in A_1$,
$$|a|_1 = |f(a)|_2.$$

We can now extend the duality results presented in the previous section to include the metric structure.

Consider a metrizable MP $(\mathcal{M}, d)$, where $\mathcal{M} = (M, \mathcal{A}, \theta)$. As before, let $\mathbb{A}(\mathcal{M})$ be the AA constructed from $\mathcal{M}$. We extend this construction so that $\mathbb{A}(\mathcal{M})$ becomes a metrized AA. For arbitrary $a \in \mathcal{A}$, let

$$|a|_d = \sup\{d(m, n) \mid m, n \in a\},$$

under the assumption that $\sup \varnothing = 0$.

Lemma 4.6 $(\mathbb{A}(\mathcal{M}), | \ |_d)$ *is a metrized Aumann Algebra.*

Proof. (A1) $|\perp|_d = 0$ follows from the assumption that $\sup \varnothing = 0$.

(A2) If $c_1 \subseteq c_2$, then $\sup\{d(m, n) \mid m, n \in c_1\} \le \sup\{d(m, n) \mid m, n \in c_2\}$.

(A3) Suppose that $c_1 \cap c_2 \ne \varnothing$. Let $z \in c_1 \cap c_2$, $x \in c_1$ and $y \in c_2$. Then, the triangle inequality for d guarantees that

$$d(x, y) \le d(x, z) + d(z, y) \le \sup\{d(x, z) \mid x, z \in c_1\} + \sup\{d(z, y) \mid z, y \in c_2\} = |c_1| + |c_2|.$$

Consequently, $\sup\{d(x,y) \mid x \in c_1, y \in c_2\} \leq |c_1| + |c_2|$ and similarly, $\sup\{d(y,x) \mid x \in c_1, y \in c_2\} \leq |c_1| + |c_2|$. Since

$$|c_1 \cup c_2| = \max\{|c_1|, |c_2|, \sup\{d(x,y) \mid x \in c_1, y \in c_2\}, \sup\{d(x,y) \mid x \in c_2, y \in c_1\}\},$$

$$|c_1 \cup c_2| \leq |c_1| + |c_2|.$$

(A4) Using the notation of the previous duality theorem 3.6, we know that for any ultrafilter u of $\mathbb{A}(\mathcal{M})$ and any clopen $c \in \mathcal{A}$, we have that

$$\alpha^{-1}(u) \in c \text{ iff } c \in u.$$

Then, $\inf\{|c|_d \mid c \in u\} = \inf\{|c|_d \mid \alpha^{-1}(u) \in c\}$.

Since α is a bijection, $\alpha^{-1}(u) \in \mathcal{M}$ and $|c|_d = \sup\{d(n,n') \mid n, n' \in c\}$, using (M) we obtain

$$\inf_{c \in \mathcal{A}, c \ni \alpha^{-1}(u)} \sup\{d(n,n') \mid n, n' \in c\} = 0,$$

therefore $\inf\{|c|_d \mid c \in u\} = 0$. $\square$

Consider a metrizable AA $(\mathcal{A}, |\ |)$ and, as before, let $\mathbb{M}(\mathcal{A})$ be the MP constructed from $\mathcal{A}$. We extend this construction so that $\mathbb{M}(\mathcal{A})$ will become a metrizable MP.

For arbitrary ultrafilters u, v of $\mathcal{A}$, let

$$\delta_{|\ |}(u,v) = \inf\{|a| \mid a \in u \cap v\}.$$

Lemma 4.7 $(\mathbb{M}(\mathcal{A}), \delta_{|\ |})$ *is a metrized MP.*

Proof. First, we prove that $\delta_{|\ |}$ is a pseudometric over the space of ultrafilters. From (A4) we can simply infer that $\delta_{|\ |}(u,u) = 0$, while the symmetry of $\delta_{|\ |}$ follows from the definition.

We now prove the triangle inequality: let u, v, w be three arbitrary ultrafilters. Let $a \in u \cap v$, $b \in u \cap w$ and $c \in w \cap v$. Obviously $b \cup c \in u \cap v$. Then,

$$\inf_{a \in u \cap v} |a| \leq |b \cup c|.$$

Since $b \cap c \neq \varnothing$, using (A3) we get $|b \cup c| \leq |b| + |c|$ which guarantees that for any $b \in u \cap w$ and any $c \in w \cap v$,

$$\inf_{a \in u \cap v} |a| \leq |b| + |c|,$$

implying

$$\inf_{a \in u \cap v} |a| \leq \inf_{b \in u \cap w} |b| + \inf_{c \in w \cap v} |c|.$$

Hence,

$$\delta_{|\ |}(u,v) \leq \delta_{|\ |}(u,w) + \delta_{|\ |}(w,v)$$

which proves that $\delta_{|\ |}$ is a pseudometric.

It remains to verify (M). Since from theorem 3.6 we know that $a \in u$ iff $u \in \beta(a)$, (M) follows directly from (A4). $\qquad \square$

Finally, we extend the representation theorem 3.6 to include the metric structure. Essentially, we show that the isomorphisms α and β of the duality theorem are isometries.

Theorem 4.8 (The metric duality theorem)   (i) *Any metrizable countable Aumann algebra $(\mathcal{A}, |\ |)$ is isomorphic to $(\mathbb{A}(\mathbb{M}(\mathcal{A})), |\ |_{\delta_{|\ |}})$ via the map $\beta : \mathcal{A} \to \mathbb{A}(\mathbb{M}(\mathcal{A}))$ defined by*

$$\beta(a) = \{u \in \mathrm{supp}(\mathbb{M}(\mathcal{A})) \mid a \in u\} = (\!|a|\!).$$

Moreover, β is an isometry of metrizable Aumann algebras, i.e., for arbitrary $a \in A$,

$$|a| = |\beta(a)|_{\delta_{|\ |}}.$$

(ii) *Any metrizable Stone Markov process $(\mathcal{M}, d)$, where $\mathcal{M} = (M, \mathcal{A}, \theta)$ is homeomorphic to $(\mathbb{M}(\mathbb{A}(\mathcal{M})), \delta_{|\ |_d})$ via the map $\alpha : \mathcal{M} \to \mathbb{M}(\mathbb{A}(\mathcal{M}))$ defined by*

$$\alpha(m) = \{A \in \mathcal{A} \mid m \in A\}.$$

Moreover, α is an isometry of MPs, i.e., for arbitrary $m, n \in M$,

$$d(m, n) = \delta_{|\ |_d}(\alpha(m), \alpha(n)).$$

Proof. We only need to prove the two isometries.

(i). The isometry of AAs. We need to prove that $|a| = |\beta(a)|_{\delta_{|\ |}}$.

Observe that
$$|\beta(a)|_{\delta_{|\ |}} = \sup_{u,v \in \beta(a)} \delta_{|\ |}(u, v) = \sup_{u,v \in \beta(a)} \inf_{a' \in u \cap v} |a'|.$$

Since $\beta(a)$ is the set of all ultrafilters containing a, a' quantifies over all elements that belong to the intersection of all ultrafilters containing a. But this intersection is nothing else but the principal filter $\uparrow a$ of a. Hence, the previous equality became

$$|\beta(a)|_{\delta_{|\ |}} = \inf_{a' \in \uparrow a} |a'|.$$

Now the monotonicity stated by (A2) guarantees that

$$\inf_{a' \in \uparrow a} |a'| = |a|.$$

(ii). The isometries of MPs. We need to prove that $d(m, n) = \delta_{|\ |_d}(\alpha(m), \alpha(n))$.

From Lemma 4.2(ii) we know that

$$d(m, n) = \inf_{c \in \mathcal{A}, c \ni m, n} |c|_d.$$

From Theorem 3.6 we also know that

$$m, n \in c \text{ iff } c \in \alpha(m) \cap \alpha(n).$$

Consequently,

$$d(m, n) = \inf\{|c|_d \mid c \in \alpha(m) \cap \alpha(n)\} = \delta_{|\ |_d}(\alpha(m), \alpha(n)).$$

$\square$

We have claimed earlier that the metric diameter on an Aumann algebra induces a pseudometric. We now demonstrate this.

Let $(\mathcal{A}, |\ |)$ be a metrized AA. For arbitrary $a, b \in \mathcal{A}$ and $\varepsilon > 0$, let

$$B_\varepsilon(b) = \bigcup\{\beta(b') \mid b' \in \mathcal{A}, |b'| \leq \varepsilon, b \wedge b' \neq \bot\}.$$

Intuitively, $B_\varepsilon(b)$ is a ball that contains all ultrafilters that are at distance at most ε from some ultrafilter containing b. This definition allows us to define a natural distance on $\mathcal{A}$ by

$$d_{|\ |}(a, b) = \inf\{\varepsilon > 0 \mid B_\varepsilon(b) \supseteq \beta(a) \text{ and } B_\varepsilon(a) \supseteq \beta(b)\}.$$

Intuitively, if in the light of the duality we think of the elements of $\mathcal{A}$ as sets of ultrafilters, then the previous distance is just the Hausdorff pseudometric of the distance between ultrafilters.

To prove that the previous construction is not void, we show in the next lemma that for any non-zero element $a \in \mathcal{A}$ the ball $B_\varepsilon(a)$ is not empty for any ε.

Lemma 4.9 *If $a \neq \bot$, then for any $\varepsilon > 0$ there exists $a' \neq \bot$ such that $a \wedge a' \neq \bot$ and $|a'| \leq \varepsilon$.*

Proof. Since $a \neq \bot$, there exists an ultrafilter u such that $a \in u$. For any other $a' \in u$, $a \cap a' \in u$, hence $a \cap a' \neq \bot$. Moreover, using (A4) there exists $a' \in u$ such that $|a'| \leq \varepsilon$. $\square$

Now we prove that $d_{|\ |}$ is indeed a pseudometric: it is the Hausdorff pseudometric of the pseudometric $\delta_{|\ |}$ on ultrafilters.

Lemma 4.10 *The function $d_{|\ |}$ previously defined on the support set of a metrizable Aumann Algebra is a pseudometric. Moreover,*

$$d_{|\ |}(a, b) = \max\{\sup_{u \in \beta(a)} \inf_{v \in \beta(b)} \delta_{|\ |}(u, v), \sup_{u \in \beta(a)} \inf_{v \in \beta(b)} \delta_{|\ |}(v, u)\}.$$

We omit the proof, which is not completely trivial, from this abstract.

4.1   Metric Duality in Categorical Form

We present the previous results in a more categorical format. The categories of metrized Aumann algebras (**MAA**) and metrized Markov processes (**MMP**) are defined as follows.

The objects of **MAA** are metrized AAs and their morphisms are expansive morphisms of AAs, i.e., morphisms $f : \mathcal{A}_1 \to \mathcal{A}_2$ of AAs such that for any $a \in \mathcal{A}_1$,

$$|a|_1 \leq |f(a)|_2.$$

The objects of **MMP** are metrized MPs and their morphisms are non-expansive morphisms of MPs, i.e., morphisms $f : \mathcal{M}_1 \to \mathcal{M}_2$ of SMPs such that for any $m, n \in \mathcal{M}_1$,

$$d_1(m, n) \geq d_2(f(m), f(n)).$$

We define contravariant functors $\mathbb{A} : \mathbf{MMP} \to \mathbf{MAA}^{\mathrm{op}}$ and $\mathbb{M} : \mathbf{MAA} \to \mathbf{MMP}^{\mathrm{op}}$. The functor $\mathbb{A}$ on an object $\mathcal{M}$ produces the Aumann algebra $\mathbb{A}(\mathcal{M})$ defined in Theorem 3.5. On arrows $f : \mathcal{M} \to \mathcal{N}$ we define $\mathbb{A}(f) = f^{-1} : \mathbb{A}(\mathcal{N}) \to \mathbb{A}(\mathcal{M})$. We have previously proved that this is an Aumann algebra homomorphism. To see that it is also expansive, consider a morphism $f : \mathcal{M} \to \mathcal{N}$ such that for any $m, n \in \mathcal{M}$,

$$d_{\mathcal{M}}(m, n) \geq d_{\mathcal{N}}(f(m), f(n)).$$

Observe that for arbitrary $a \in \mathbb{A}(\mathcal{N})$,

$$|a|_{\mathbb{A}(\mathcal{N})} = |a|_{d_{\mathcal{N}}} \text{ and } |f^{-1}(a)|_{\mathbb{A}(\mathcal{M})} = |f^{-1}(a)|_{d_{\mathcal{M}}}$$

and in this context the previous inequality guarantees that

$$|a|_{\mathbb{A}(\mathcal{N})} \leq |f^{-1}(a)|_{\mathbb{A}(\mathcal{M})}.$$

The functor $\mathbb{M} : \mathbf{MAA} \to \mathbf{MMP}^{\mathrm{op}}$ on an object $\mathcal{A}$ gives the Stone–Markov process $\mathbb{M}(\mathcal{A})$ defined in Theorem 3.4. On morphisms $h : \mathcal{A} \to \mathcal{B}$, it maps ultrafilters to ultrafilters by $\mathbb{M}(h) = h^{-1} : \mathbb{M}(\mathcal{B}) \to \mathbb{M}(\mathcal{A})$; that is,

$$\mathbb{M}(h)(u) = h^{-1}(u) = \{A \in \mathcal{A}_{\mathcal{N}} \mid h(A) \in u\}.$$

Another way to view $\mathbb{M}(h)$ is by composition, recalling that an ultrafilter can be identified with a homomorphism $\overline{u} : \mathcal{A} \to 2$ by $u = \{a \mid \overline{u}(a) = 1\}$. In this view,

$$\mathbb{M}(h)(\overline{u}) = \overline{u} \circ h,$$

where $\circ$ denotes function composition.

We have proven in our previous paper [15] that this is a morphism of SMPs.

That it is also non-expansive can be demonstrated as follows.

Let $h : \mathcal{A} \to \mathcal{B}$ be a morphism such that for arbitrary $a \in \mathcal{A}$,

$$|a|_\mathcal{A} \leq |f(a)|_\mathcal{B}.$$

Using the previous results it is not difficult to verify that for arbitrary ultrafilters u, v of $\mathcal{B}$,

$$\delta_{|\ |_\mathcal{B}}(u, v) \geq \delta_{|\ |_\mathcal{A}}(h^{-1}(u), h^{-1}(v)).$$

And since

$$\delta_{|\ |_\mathcal{B}}(u, v) = \delta_{\mathbb{M}(\mathcal{B})}(u, v) \text{ and } \delta_{|\ |_\mathcal{A}}(h^{-1}(u), h^{-1}(v)) = \delta_{\mathbb{M}(\mathcal{A})}(h^{-1}(u), h^{-1}(v)),$$

We obtain that $\mathbb{M}(h)$ is non-expansive.

Theorem 4.11 *The functors $\mathbb{M}$ and $\mathbb{A}$ define a dual equivalence of categories.*

$$\textbf{MMP} \underset{\mathbb{M}}{\overset{\mathbb{A}}{\rightleftarrows}} \textbf{MAA}^{\text{op}}$$

5   Conclusions

We have extended the duality theory of [15] to a quantitative setting. It is important to note that the conditions we have imposed on the pseudometric relate the topology of the Markov process to the pseudometric topology. This can be seen from the fact that the pseudometric topology is refined by the Stone topology. We have defined our Stone Markov processes to be Hausdorff spaces, which was necessary for the duality theory. In effect, this means that the clopens separate points; in other words, one cannot have two states that satisfy exactly the same formulas. In view of the logical characterization of bisimulation, this implies that no two distinct states are bisimilar; that is, the process is already minimal with respect to bisimulation. If we look at a broader class of Markov processes, then we would have possibly nontrivial bisimulations on the space. The Stone Markov processes would be a reflective subcategory with the reflector sending each Markov process to a version of the process with all the bisimulation equivalence classes collapsed to point; this would be a Stone Markov process. How does the topology on a Stone Markov process "know" about the transition structure? Note that the base of clopens is required to be closed under the F_r operations, which are defined in terms of the transition function.

The only work of which we are aware similar to this is a paper by Banaschewski and Pultr [2] called "A Stone duality for metric spaces." They are not working with Markov processes, so there is nothing like the Aumann algebra structure there. They

gave us the idea of using a metric diameter, but their axiomatization is different and the proofs that we developed do not resemble theirs.

The main impact of this work is to put quantitative reasoning about Markov processes on a firmer footing. It has been over a decade since metric analogues of bisimulation were developed, but they have not had the impact that they might have had. One reason is that with ordinary logical reasoning, one has a clear understanding of what completeness means, thus users of these logics have a good understanding of the power of the principles they are using. What does completeness mean for metric reasoning and approximate reasoning in general? The standard Stone-type duality theorem captures the concept of completeness; it is our hope that the present work will pave the way towards a similar understanding of approximate reasoning principles. There is much to be done, however. In a previous paper [17] we began investigating the relationship between the logic and metrics on Markov processes. The results of the present paper could perhaps strengthen and deepen these preliminary results.

Acknowledgments

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