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Representations of Algebraic Domains and Algebraic L-domains by Information Systems [1]

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Abstract

Information systems play an important role in characterizing order structures. In this paper, we introduce the notions of the algebraic information system and algebraic L-information system. They are of the same logic-oriented style as the information system introduced by Scott(1982). But the axioms in this paper are briefer than reported in existing work. We also prove that the two new information systems exactly represent the algebraic domains and algebraic L-domains respectively. Based on the notion of approximable mapping between the algebraic information systems and the algebraic L-information systems, we obtain the result that the corresponding categories of algebraic information systems and algebraic L-information systems are equivalent to the category of algebraic domains and algebraic L-domains respectively.

Keywords: Algebraic domain; Algebraic L-domain; Information system; Category equivalence

1 Introduction

In 1982, Scott [13] introduced the notion of information system and approximable mapping as a logic-oriented approach to denotational semantics of programming languages. Later on, Larsen and Winskel [11] proved the category of information systems is equivalent to that of Scott domains with continuous functions as morphisms. This category is also equivalent to that of algebraic $\cap$-structures with continuous functions as morphisms. In 1993, Hoofman [8] introduced the information systems that represent bounded complete continuous domains. By considering transitive and interpolative relations, Vickers [16] built an information systems representation for continuous posets. Vickers's approach is more general and can be used to represent all continuous domains, but it is not Scott-style. In 2008, Spreen and Xu [14] first introduced the notion of general continuous information system of

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Scott style which exactly captures continuous dcpos. They also gave the notion of general algebraic information system which captures algebraic domains by adding other rules on general continuous information system. But there were many rules in their general algebraic information system. In this paper we will introduce a new class of algebraic information system in which there are only four rules.

L-domains were independently introduced by Coquand [2] and Jung [9]. Jung [9,10] showed that L-domains form a maximal cartesian closed full subcategory of the continuous domains. This is the same for algebraic case. Spreen [14] gave a subclass of continuous information systems which can represent L-domains. But the information system has many conditions so it is not easy to judge. Zhang [18] presented a logic-oriented approach to algebraic L-domains with Gentzen-style proof systems so it is different from Scott's original approach. More articles about information systems can be found in [20,21,17,5,6,7].

Based on our new algebraic information system, we introduce algebraic L-information systems. Our method of constructing algebraic L-information systems is skillful and completely different from Spreen's. Our definition of algebraic L-information systems is brief. The paper is organized as follows: Section 2 recalls some basic notions of domain theory and gives the definition of algebraic information systems. We show that the states of an algebraic information system form an algebraic domain with respect to set inclusion. Algebraic L-information systems are introduced in Section 3. It is shown that algebraic L-information systems and algebraic L-domains correspond to each other under the relationship between algebraic information systems and algebraic domains. In Section 4, we propose the approximable mappings among the algebraic information systems and algebraic L-information systems respectively. It is shown that the corresponding categories of algebraic information systems and algebraic L-information systems are equivalent to the categories of algebraic domains and algebraic L-domains with Scott continuous functions as morphisms respectively.

2 Domains and information systems

Domain theory is the interdiscipline of theories of lattice, topology, category and computer science. The main purpose of the domain theory is to give models for spaces on which to define computable functions. In high-level denotational semantics, the spaces of higher type (e.g. function spaces) and spaces defined recursively (e.g. reflexive domains) are needed. Many special domain constructs (or functors) are also required in order to create the desired structures. In this paper we mainly introduce the algebraic domains and algebraic L-domains.

For any set A, we write $F \sqsubseteq A$ to mean that F is a finite subset of A. Let P be a poset, a nonempty subset D of P is said to be directed if for two arbitrary elements a and b of D, there exists $c \in D$ such that $a \leq c$ and $b \leq c$. We use $X = \bigsqcup_{i\in I} d_i$ to mean that X is the supremum of a directed set. A poset is called a dcpo if every directed subset has a least upper bound. Let $x, y \in P$, x is said to approximate y (in symbol $x \ll y$) if and only if for every directed set $D \subseteq P$, $y \leq \bigsqcup D$ means that there is a $d \in D$ such that $x \leq d$. x is said to be compact if $x \ll x$ (We use $K(P)$ to denote the set of all compact elements of P). Let $\Downarrow x = \{a \mid a \in P; a \ll x\}$ for every $x \in P$. A subset B of P is said to be a basis of P if for every $x \in P$, $\Downarrow x \bigcap B$ is a directed set and $x = \bigsqcup(\Downarrow x \bigcap B)$. A dcpo is called a domain if it has a basis. In particular, a dcpo is called an algebraic domain if all compact elements of it form a basis. A domain D is pointed if it contains a least element $\perp$.

A function f between two domains P_1 and P_2 is called Scott continuous if f preserves all joins of directed sets, that is, for a directed set D of P_1,

$$f(\bigsqcup D) = \bigsqcup f(D).$$

A pointed algebraic domain in which every ideal $\downarrow x$ is a complete lattice(in its induced order) is called an algebraic L-domain.

The algebraic domain category is denoted by **AlGDOM** in which algebraic domains are objects and Scott continuous functions are morphisms. The algebraic L-domain category as the full subcategory of **AlGDOM** is denoted by **AlGLDOM** in which all objects are algebraic L-domains. More results about domain theory can be found in [1,3,4]. The category theory in this paper can be found in [12,19].

In computer science, the notion of information systems was introduced as a logic-oriented approach to denotational semantics of programming languages. With the aid of information systems, the domains of elements are represented set theoretically by Scott[13]. Information systems are very familiar from mathematical logic and can construct concrete domains. The information systems introduced by Scott[13] represented the Scott domains. In 2008, Spreen and Xu [14] introduced the notion of general continuous information system which exactly captures continuous domains. In section 5 of [14], they characterize general algebraic information system by adding another requirement to the definition of a continuous information system. However, we think this is not the best construction since the algebraic domains has its particular properties. The general algebraic information system is defined as follows:

Definition 2.1 [14] Let A be a set, Con is a collection of nonempty finite subsets of A and $\vdash \subseteq Con \times$A, a general algebraic Information system is a triple (A,Con,$\vdash$), and the following rules hold for all sets $x, y \in Con$ and all elements $a \in A$.
(1) $\{a\} \in Con$,
(2) $x \vdash a \Rightarrow x \bigcup \{a\} \in Con$,
(3) $x \subseteq y$ and $x \vdash a \Rightarrow y \vdash a$,
(4) $x \vdash y \vdash a \Rightarrow x \vdash a$,
(5) $x \vdash y \Rightarrow \exists z \in Con$, $x \vdash z \vdash z \vdash y$,
(6) $x \vdash F \sqsubseteq A \Rightarrow \exists z \in Con$, s.t. $F \subseteq z$ and $x \vdash z$.

In this information system, X is a state of $(A, Con, \vdash)$ if X satisfies the next three conditions:
(1) $\overline{X} = X$,
(2) $\forall F \subseteq X, \exists y \in Con$ and $y \subseteq X$, s.t. $F \subseteq y$,

(3) $\forall a \in X$, $\exists y \in Con$ and $y \subseteq X$, s.t. $y \vdash a$.
For a general algebraic information system, with respect to set inclusion, the states of A form a poset for which they denoted $|A|$.

Intuitively, for an information system $(A, Con, \vdash)$, the set A should be thought of as atomic propositions giving information about data and a set Con of finite sets of mutually consistent (i. e. non-contradictory) propositions. Furthermore, the entailment relation then tells us which propositions are derivable from what. The general algebraic information systems is a concrete representation of algebraic domain.

Proposition 2.2 *[14] Let (A,Con,$\vdash$) be a general algebraic information system, then $|A|$ is an algebraic domain.*

Definition 2.3 Let L be an algebraic domain. Then a general algebraic information system $\mathbf{IS}(L)$ can be constructed as follows:
(i) $A = L$,
(ii) $Con = \{x \mid x \sqsubseteq A,\ x$ directed with respect to $\preceq$ $(a \preceq b$ if $a = b$ or $a \ll b)\}$,
(iii) If $x \in Con$, then $x \vdash a \Leftrightarrow (\exists b \in x) a \ll b$.

Proposition 2.4 *Let L be an algebraic domain. Then L is order isomorphic to $|\boldsymbol{IS}(L)|$.*

3 AL information systems

In this section, we introduce a new notion of algebraic information system(AL information system). Our method of constructing algebraic information system is different from Spreen's because we obtain the AL information systems based on the bases of algebraic domains. We also obtain a bijective correspondence between AL information system and algebraic domain.

Definition 3.1 Let A be a set, Con is a collection of nonempty finite subsets of A and $\vdash \subseteq Con \times A$. Then an algebraic Information system (simply call AL information system) is a triple $(A, Con, \vdash)$, and the following rules hold for all sets $x, y \in Con$, elements $a, b \in A$.
(1) $\{a\} \in Con$,
(2) $x \vdash a \Rightarrow x \bigcup \{a\} \in Con$,
(3) $a \in x \Rightarrow x \vdash a$,
(4) $x \vdash y$ and $y \vdash a \Rightarrow x \vdash a$.
Where $x \vdash y$ means that $x \vdash b$ for all $b \in y$.

In an AL information system, for any subset $X \subseteq A$, $\overline{X} := \{a \in A \mid (\exists x \sqsubseteq X) x \vdash a\}$.

Definition 3.2 X is a state of $(A, Con, \vdash)$ if X satisfies the following two conditions:
1.$\overline{X} = X$,
2.$\forall F \sqsubseteq X, \exists y \sqsubseteq X$ and $y \in Con$, such that $y \vdash F$.

Since we define the states of an AL information system, with respect to set inclusion, the states of A form a poset for which we denote $|A|$. Using rules (2) (3) of AL information system, we can easily prove that the condition (2) of state is equivalent to that $\forall F \subseteq X$, $\exists z \sqsubseteq X$ and $z \in Con$, such that $F \subseteq z$. Notice that if $X \in |A|$, $y \in Con$ and $y \sqsubseteq X$, then it is trivial to check that $\overline{y} \subseteq X$.

In the following steps, we prove that the states of an AL information system $(A, Con, \vdash)$ is an algebraic domain.

Proposition 3.3 *$|A|$ is a DCPO.*

Proof. Assume D=$\{X_i\}_{i\in I}$ is a directed subset of $|A|$ and $X = \bigcup_{i\in I} X_i$. If $a \in \overline{X}$, by the definition of state, there exists $x \sqsubseteq X$, such that $x \vdash a$. For $x \sqsubseteq X$ and D is directed, there exists $i_0 \in I$ such that $x \subseteq X_{i_0}$. Since $X_{i_0} \in |A|$, so $a \in X_{i_0} \subseteq X$, we get $\overline{X} = X$. If F is any finite set of X, because $\{X_i\}_{i\in I}$ is directed, then there exists $i_1 \in I$ such that $F \subseteq X_{i_1}$. Since $X_{i_1} \in |A|$, so $\exists y \in Con$ and $y \sqsubseteq X_{i_1}$ such that $y \vdash F$, also $y \sqsubseteq X$. We have completed the proof. □

Proposition 3.4 *Let $(A, Con, \vdash)$ be an AL information system, $x \in Con$. Then*
(1) $\overline{x} \in |A|$.
(2) A subset X of A is a state if and only if there exist a family of $\{x_i\}_{i\in I} \in Con$ such that $\{\overline{x}_i\}_{i\in I}$ is directed and $X = \bigcup_{i\in I} \overline{x}_i$.

Proof. By the rules (4) of AL information system, we can easily get that for any $y \in Con$, $\overline{\overline{y}} = \overline{y}$, so $\overline{x}$ satisfies the condition (1) of the state. Since $\overline{x}$ also satisfies the condition (2) of the state is obvious as $x \vdash a$ for any $a \in \overline{x}$, it implies $\overline{x} \in |A|$.

Next, suppose a set $\{x_i\}_{i\in I} \subseteq Con$, $\{\overline{x_i}\}_{i\in I}$ is directed and $X = \bigcup_{i\in I} \overline{x_i}$. If $y_1 \sqsubseteq X$ and $y_1 \vdash a$, because $\{\overline{x_i}\}_{i\in I}$ is directed, then there exists an $i_1 \in I$, such that $y_1 \subseteq \overline{x_{i_1}}$. It implies $a \in \overline{x_{i_1}} \subseteq X$, so $\overline{X} = X$. If $\forall F \sqsubseteq X$, then there exists an $i_2 \in I$, such that $F \subseteq \overline{x_{i_2}}$, thus $x_{i_2} \vdash F$ and $x_{i_2} \subseteq X$. By the above steps, we get $X \in |A|$.

Conversely, suppose $X \in |A|$, consider the family $\mathscr{A} = \{\overline{x} \mid x \in Con \text{ and } x \subseteq X\}$. From the rule (1) of AL information system, we only need to prove $X = \bigcup \mathscr{A}$. If $\overline{x_{i_1}} \in \mathscr{A}$, $\overline{x_{i_2}} \in \mathscr{A}$,then $x_{i_1} \bigcup x_{i_2} \sqsubseteq X$, by the properties of X, there exists x such that $x \sqsubseteq X$ and $x \vdash x_{i_1} \bigcup x_{i_2}$. This means that $\overline{x}$ is an upper bound of $\overline{x_{i_1}}$, $\overline{x_{i_2}}$, so $\mathscr{A}$ is directed. □

Corollary 3.5 *Let A be an AL information system, then for any states $X, X' \in |A|$,*

$$X \ll X' \Leftrightarrow (\exists x \in Con) X \subseteq \overline{x} \subseteq X'$$

Hence, the compact elements of $|A|$ are $\overline{x}(x \in Con)$.

Proof. Suppose $X, X' \in |A|$, $X \ll X'$. By Proposition 3.4, $X' = \bigcup_{i\in I} \overline{x_i}(x_i \in Con$ and $x_i \sqsubseteq X')$. Since $\{\overline{x_i}\}_{i\in I}$ is directed, there exists $i_0 \in I$ such that $X \subseteq \overline{x_{i_0}}$, thus $X \subseteq \overline{x_{i_0}} \subseteq X'$. Conversely, suppose $X \subseteq \overline{x} \subseteq X'$, and $X' \subseteq \bigsqcup_{j\in J} X_j(X_j \in |A|)$. Since $x \subseteq \overline{x} \subseteq X' \subseteq \bigsqcup_{j\in J} X_j$, there exists $j_0 \in J$, such that $x \subseteq X_{j_0}$, so $X \subseteq \overline{x} \subseteq X_{j_0}$. It implies $X \ll X'$. □

Theorem 3.6 *Let A be an AL information system. Then $|A|$ is an algebraic domain. Conversely, for any algebraic domain $(L, \leq)$, we can construct an AL information system* $\boldsymbol{IS}(L)$ *as follows:*
(i) $A = K(L)$,
(ii) $Con = \{x \mid x \sqsubseteq A$ *and* x *has a greatest element*$\}$,
(iii) If $x \in Con$, *then* $x \vdash a \Leftrightarrow a \in\downarrow x$.
Then L is order isomorphic to $|\boldsymbol{IS}(L)|$.

Proof. We can easily prove that $|A|$ is an algebraic domain using the conclusions above.

Now we prove the second part. Let L be an algebraic domain, consider $\mathbf{IS}(L)$=$(K(L), Con, \vdash)$. By the definition of Con, $\{a\} \in Con$. Suppose $x \in Con$, $x \vdash a$, then by definition, x has a greatest element b and $a \in\downarrow x =\downarrow b$, thus b is the greatest of $x \cup a$, therefore $x \cup \{a\} \in Con$. Suppose $x \in Con$, and $a \in x$, then $a \in\downarrow x$, thus $x \vdash a$. If $x, y \in Con$, $x \vdash b$ for all $b \in y$, this means $y \subseteq\downarrow x$. If $y \vdash a$, then $a \in\downarrow y \subseteq\downarrow x$, hence $x \vdash a$.

Now we only need to prove that $\mathbf{IS}(L)$ is an AL information system. We prove that for any subset X of K(L), it is a state of $\mathbf{IS}(L)$ if and only if there exists some directed subset $\{b_i\}_{i\in I}$ of $(K(L), \leq)$ such that $X = \bigcup_{i\in I}\{\downarrow b_i\}$: On the one hand, suppose X is a state of $\mathbf{IS}(L)$. For any $F \sqsubseteq X$, by the definition of X, there exists a subset x of X, $x \in Con$ and $x \vdash F$. By the definition of $\mathbf{IS}(L)$, x has a greatest element a and $F \subseteq\downarrow x =\downarrow a$, so a is an upper bound of F, this means that X is directed. If $b \leq c \in X$, then $\{c\} \vdash b$, hence $b \in X$, this means that X is a down set, so $X = \bigcup_{b\in X}\{\downarrow b\}$. On the other hand, if $\{b_i\}_{i\in I}$ is a directed set in $K(L)$ and $X = \bigcup_{i\in I}\{\downarrow b_i\}$. For any $x \in Con$, $x \subseteq X$, $x \vdash a \Leftrightarrow a \in\downarrow x \Rightarrow a \in X$, so $\overline{X} = X$. For any $F \sqsubseteq X$, because $\{b_i\}_{i\in I}$ is a directed set, there exists a $b_{i_o} \in \{b_i\}_{i\in I}$ such that $F \subseteq\downarrow b_{i_o}$. By the definition of $\mathbf{IS}(L)$, $\{b_{i_o}\} \in Con$ and $\{b_{i_o}\} \vdash F$, thus X is a state of $\mathbf{IS}(L)$.

The order isomorphisim between $(L, \leq)$ and $|\mathbf{IS}(L)|$ just follows immediately from the definition of algebraic domains. □

Our AL information systems have some benefits:

(1) Brief definitions and rules: Our rules of AL information systems and the conditions of states are briefer than general algebraic information system, so it is easy to construct and operate.

(2) We can directly construct the AL information system based on the basis of an algebraic domain.

(3) Convenient generalization: Basing on our AL information system, we can obtain some brief information systems of subclasses of algebraic domains. Our AL information systems are different from the general algebraic information systems

and have some advantages. We show them in the following examples.

Example 3.7 Let L be an algebraic domain, $L = (a_1, a_2, a_3, \cdots a_n, \cdots, \top)$ and $a_1 < a_2 < a_3 < \cdots < a_n < \cdots < \top$. And the general algebraic information system of L can be constructed as in Definition 2.1, $x = \{\top, a_1\} \in Con$, but $x \nvdash x$, and $x \nsubseteq \overline{x}$, so it is not an AL information system.

Example 3.8 Let $[0,1]$ be an unit interval with order and $B = [0,1]$ be a basis of an algebraic domain L. Then we can construct the AL information system of L directly:
(i) $A = B$,
(ii) $Con = \{x \mid x \sqsubseteq A\}$,
(iii) If $x \in Con$, then $x \vdash a \Leftrightarrow a \in \downarrow x$.
Then L is order isomorphic to $|\mathbf{IS}(L)|$.

We construct the general algebraic information system of L. We know that $L \cong IdB$, so $L = \{xa \mid x \in (0,1], a \in \{1,2\}\} \bigcup \{0\}$, this means L double every element of B except 0, the order of L is defined as follows: $xa \le yb$ if $x < y$ or $x = y$, $a \le b$. The general algebraic information system of L:
(i) $A = L$,
(ii) $Con = \{x \mid x \sqsubseteq A\}$,
(iii) If $x \in Con$, then $x \vdash a \Leftrightarrow (\exists b \in x) a \ll b$.

Because L is more complex than B, the set of Con of Al information system is briefer than the general algebraic information system's. We don't need to consider the way-below relation of L is another advantage of our approach.

Observing this example we can find that If an algebraic domain is complicated, AL information system is simpler than existing ones since our method of constructing information system do not need to consider the way-below relation.

Proposition 3.9 *Let $\boldsymbol{A}$=$(A, Con, \vdash)$ be an AL information system. If $\boldsymbol{A}$ satisfies the condition (SEMI), then $|\boldsymbol{A}|$ is an algebraic semilattice.*

$$x_1, x_2 \in Con, x_1 \vdash F, x_2 \vdash F \Rightarrow (\exists y \in Con) x_1 \vdash y, x_2 \vdash y, y \vdash F \qquad (SEMI)$$

The proof of Proposition 3.9 is obvious by the property of algebraic semilattice.

Proposition 3.9 is a simple example of generalization of AL information systems. By adding the other conditions there are easy to obtain some other information systems of subclasses of algebraic domains (e. g. algebraic lattice, algebraic Scott domain). In the next section, we get a brief representation of information system of algebraic L-domain, it is an important work of information system.

4 An Information systems representation of algebraic L-domains

Algebraic L-domains is very important in Domain theory because they form a maximal Cartesian closed full subcategory of the algebraic continuous domains. However, a brief representation of algebraic L-domain by information system is hard. The main problem in finding an information system description of algebraic L-domains is the characterization of those consistent sets that represent local supremum. In this section a subclass of AL information systems is defined capturing exactly the algebraic L-domains.

Lemma 4.1 *[4] Let P be a poset. For P to be a compete lattice, it is sufficient to assume the existence of sups of finite sets and that of directed sets.*

Definition 4.2 An algebraic information system $(A, Con, \vdash)$ is said to satisfy Property L if $z_1,\ z_2 \in Con$ and $x \vdash z_1 \cup z_2$, there exists z such that for any y, $x \vdash y \vdash z_1 \cup z_2$, then $y \vdash z \vdash z_1 \cup z_2$. We call z that x-sup of z_1, z_2, or we say z_1, z_2 has a x-sup z.

Remark 4.3 We can see that in Definition 4.2 z is not unique because if $z \vdash a$, then $z \cup \{a\}$ and z can $\vdash$ each other, hence z can be replaced by $z \cup \{a\}$. In particular, $z \cup z_1 \cup z_2$ also satisfies the condition.

Lemma 4.4 *Let $(A, Con, \vdash)$ be an algebraic information system satisfying the Property L. If $z_1, z_2 \in Con$, $x \vdash y \vdash z_1 \cup z_2$, then the set of x-sup of z_1, z_2 is the same as that of y-sup of z_1, z_2. In particular, if $X \in |A|$ and $x_1, x_2 \subseteq X$, then the set of x_1-sup of z_1, z_2 is the same as that of x_2-sup of z_1, z_2.*

Proof. Suppose z_x is a x-sup of z_1, z_2 and z_y is a y-sup of z_1, z_2. Then we can deduce $x \vdash y \vdash z_y \vdash z_1 \cup z_2$, it implies $z_y \vdash z_x$, $x \vdash y \vdash z_1 \cup z_2$, thus $y \vdash z_x \vdash z_1 \cup z_2$, it implies $z_x \vdash z_y$. That means the set of x-sup of z_1, z_2 is the same as that of y-sup of z_1, z_2. If $x_1, x_2 \subseteq X$, $x_1 \vdash z_1 \cup z_2, x_2 \vdash z_1 \cup z_2$, then $x_1 \cup x_2 \sqsubseteq X$, hence we can find $x \subseteq X$, such that $x \vdash x_1 \cup x_2$. By the above discussion, z is a x_1-sup of z_1, z_2 iff z is a x-sup of z_1, z_2 iff z is a x_2-sup of z_1, z_2. We have completed the proof. □

Theorem 4.5 *Let $(A, Con, \vdash)$ be an algebraic information system with $\emptyset \in Con$. If $(A, Con, \vdash)$ also satisfies the Property L, then $|A|$ is an algebraic L-domain.*

Proof. Because $\emptyset \in Con$, $\overline{\emptyset}$ is a bottom of $|A|$. Since $|A|$ is an algebraic domain, we only need to prove that for any $X \in |A|$, $\downarrow X$ is a complete lattice. By Lemma 4.1, we only need to prove that for every finite set, there exists sup in $\downarrow X$.

Suppose $X_1, X_2 \in \downarrow X$, if $z_1, z_2 \in Con$ and $z_1 \sqsubseteq X_1$, $z_2 \sqsubseteq X_2$, then $\exists x \subseteq X$ such that $x \vdash z_1 \cup z_2$. By Property L, there exists a y which is a x-sup of z_1, z_2. We consider the set $E = \{y \mid y \text{ is } x\text{-sup of } z_1, z_2 \text{ for some } x \subseteq X, z_1 \subseteq X_1, z_2 \subseteq X_2\}$ and $Y = \bigcup E$. We check $Y \in |A|$. If $y \in E$ and $a \in \overline{y}$, then we can deduce $y \cup a \in E$, it implies $a \in Y$, so $\overline{y} \subseteq Y \Rightarrow Y = \bigcup\{\overline{y} \mid y \in E\}$.

Next, suppose $y_a \in E, y_b \in E$, then y_a is x_a-sup of $z_{1a}, z_{2a}(x_a \sqsubseteq X, z_{1a} \sqsubseteq X_1, z_{2a} \sqsubseteq X_2)$, y_b is x_b-sup of $z_{1b}, z_{2b}(x_b \sqsubseteq X, z_{1b} \sqsubseteq X_1, z_{2b} \sqsubseteq X_2)$. Because $z_{1a} \cup z_{1b} \sqsubseteq X_1$, there exists $z_{1c} \subseteq X_1$ such that $z_{1c} \vdash z_{1a} \cup z_{1b}$, similarly, there exists $z_{2c} \subseteq X_2$ such that $z_{2c} \vdash z_{2a} \cup z_{2b}$. Because $z_{1c} \cup z_{2c} \sqsubseteq X$, there exists $x_c \sqsubseteq X$ such that $x_c \vdash z_{1c} \cup z_{2c}$. By Definition 4.2, we can find y_c is a x_c-sup of z_{1c}, z_{2c}. It is trivial to check $x_c \vdash y_c \vdash z_{1a} \cup z_{1b} \cup z_{2a} \cup z_{2b}$. By Lemma 4.4, we can know y_a is a x_c-sup of z_{1a}, z_{2a} and y_b is a x_c-sup of z_{1b}, z_{2b}. It implies $y_c \vdash y_a \cup y_b$, we obtain that $y_c \cup y_a \cup y_b$ is also x_c-sup of z_{1c}, z_{2c}, so $y_c \cup y_a \cup y_b \in E$. It implies E is a directed set, hence the set $\{\overline{y} \mid y \in E\}$ is also a directed set. We know $\overline{y} \in |A|(y \in Con)$, and $Y = \bigsqcup\{\overline{y} \mid y \in E\}$, so $Y \in |A|$. It is easy to check that Y is an upper bound of X_1, X_2 in $\downarrow X$.

Suppose W is an other upper bound of X_1, X_2 in $\downarrow X$. If $y \in E$, then y is a x-sup of $z_1, z_2(x \subseteq X, z_1 \subseteq X_1, z_2 \subseteq X_2)$. Because $z_1 \cup z_2 \sqsubseteq X_1 \cup X_2 \subseteq W$, there exists $w \sqsubseteq W$ and $w \vdash z_1 \cup z_2$. By Lemma 4.4, we can obtain that y is a w-sup of z_1, z_2 because $w, x \subseteq X$. This means $w \vdash y$, so $y \subseteq W$, it implies $Y \subseteq W$. We prove that $|A|$ is an algebraic L-domain. □

Theorem 4.6 *Let L be an algebraic L-domain. Then the AL information system* ***IS***(L) *constructed in Theorem 3.6 satisfies Property L.*

Proof. If $x \vdash z_1 \cup z_2$, then there exist the greatest elements a, b, c respectively for x, z_1, z_2 and $b, c \in \downarrow a$. Because L is an algebraic L-domain, $\downarrow a$ is a complete lattice(in its induced order). Consider in the complete lattice $\downarrow a$, we have $b \vee c$ and $b \vee c \in K(L)$, $b \vee c$ is a x-sup of z_1, z_2, thus $\mathbf{IS}(L)$ satisfies Property L. □

5 Categorical equivalence

In this section, we introduce an Approximable mapping among the AL information system to discuss the categorical properties.

Definition 5.1 An approximable mapping H among the AL information systems $(A, Con, \vdash)$ and $(A', Con', \vdash')$ written H: $A \Vdash A'$ is a relation between Con and A' satisfying for all $x, x' \in Con, y \in Con'$ and $b \in A'$, and all nonempty finite subsets F of A' satisfy the following condition;
(1) xHy and $y \vdash' b \Rightarrow xHb$,
(2) $x \vdash x'$ and $x'Hb \Rightarrow xHb$,
(3) $xHF \Rightarrow (\exists z \in Con')F \subseteq z$ and xHz.
where xHy means that $x \vdash b$ for all $b \in y$.

Requirements (1) and (2) of Definition 5.1 are equivalent to the following statement: $x \vdash x', x'Hy$ and $y \vdash' b \Rightarrow xHb$.

For an AL information system $(A, Con, \vdash)$, where $x \in Con$ and $a \in A$, we set x Id a if $x \vdash a$. Then Id: $A \Vdash A$ such that for all $A \Vdash A'$, $H \circ Id' = H = Id \circ H$, and for an approximable mapping $H : A \Vdash A'$ and $G : A' \Vdash A$" the composition $H \circ G$: $A \Vdash A$" is defined by

$$x(H \circ G)c \Leftrightarrow (\exists y \in con')xHy, yGc.$$

It is easy to check that $H \circ G$ is an approximable mapping between $(A, Con, \vdash)$ and (A",Con",$\vdash$"). All AL information systems and all approximable mappings between them is a category denoted **ALGINS**.

ALL information system An AL information system which satisfies Property L.

ALGLINS: The category of all ALL information systems and all approximable mappings between them.

In domain theory, **ALGDOM** is the category of all algebraic domains and all scott continuous functions between them. **ALGLDOM** is the category of all algebraic L-domains and all Scott continuous functions between them. In the following, we prove that **ALGINS** and **ALGLINS** are categorical equivalence to **ALGDOM** and **ALGLDOM** respectively.

Let L, L' be algebraic domains, f is a scott continuous function between L and L'. We define the relation $\mathscr{C}(f)$ between **IS**(L) and **IS**(L') as follows:

$$x \in Con, x\mathscr{C}(f)b \Leftrightarrow (\exists a \in x)b \leq f(a).$$

Proposition 5.2 *$\mathscr{C}(f)$ is an approximable mapping between **IS**(L) and **IS**(L').*

Proof. First, we know that **IS**(L)=$(K(L), Con, \vdash)$, if $x \in Con$, then by definition of Con, x has a greatest element, we denote a. If $y \in Con'$, then y has a greatest element b, thus $x\mathscr{C}(f)y \Leftrightarrow y \subseteq \downarrow f(a) \Leftrightarrow b \leq f(a)$. Since $y \vdash' c \Leftrightarrow c \in \downarrow y \Leftrightarrow c \in \downarrow b$, so $c \leq f(a)$, it implies $x\mathscr{C}(f)c$.

Next, it is easy to check that $\mathscr{C}(f)$ satisfies the second condition of the approximable mapping. At last, because $\downarrow f(a) \bigcap K(L')$ is a directed set, we can also check that $\mathscr{C}(f)$ satisfies the third condition of the approximable mapping. Hence, $\mathscr{C}(f)$ is an approximable mapping. □

Let $(A, Con, \vdash)$ and $(A', Con', \vdash')$ be AL information systems, H is an approximable mapping between them, Then we can define a map $\mathscr{G}(H)$ from $|A|$ to $\mathscr{P}(A')$ as follows:

$$(D \in |A|)\mathscr{G}(H)(D) = \bigcup H(x)(x \subseteq D, x \in Con).$$

Proposition 5.3 *$\mathscr{G}(H)$ is a continuous function from $|A|$ to $|A'|$.*

Proof. First, we check $\mathscr{G}(H)(D) \in |A'|$. If $F \sqsubseteq \mathscr{G}(H)(D)$, then for all $b \in F$, $\exists x_b \in Con$ with $x_b \subseteq D$, such that x_bHb. Because $\bigcup_{b\in F} x_b$ is a finite set and $D \in |A|$, there exists $x \in Con$ with $x \subseteq D$, such that $\bigcup_{b\in F} x_b \subseteq x$, it implies xHF. By rule (3) of H, there exists $y \in Con'$ such that xHy and $F \subseteq y$, that means $y \subseteq \mathscr{G}(H)(D)$ and $y \vdash F$. It is easy to check $\overline{\mathscr{G}(H)(D)} = \mathscr{G}(H)(D)$ by the above and rule (1) of H. So $\mathscr{G}(H)(D) \in |A'|$.

Next, we prove $\mathscr{G}(H)(\bigsqcup_{i\in I} D_i) = \bigsqcup_{i\in I} \mathscr{G}(H)(D_i)$. The left is an upper bound of the right, so we only need to check the left is contained in the right, this is trivial because $\forall F \sqsubseteq D(F \in Con)$, F is contained in some $D_{i_0}(i_0 \in I)$. Therefore, $\mathscr{G}(H)$ is a continuous function from $|A|$ to $|A'|$. $\square$

By the above properties, we can define the functor $\mathscr{C}$: **ALGDOM** $\to$ **ALGINS** and $\mathscr{G}$: **ALGINS** $\to$ **ALGDOM**. And we prove **ALGINS** is Categorical equivalent to **ALGDOM**.

Theorem 5.4 *Let L and L' be algebraic domains and $f : L \to L'$ be a continuous function. Then*

$$f_{L'} \circ f = \mathscr{G}(\mathscr{C}(f)) \circ f_L,$$

where f_L is the isomorphism between $(L, \leq)$ and $(|\boldsymbol{IS}(L)|, \subseteq)$, $f_{L'}$ is the isomorphism between $(L', \leq)$ and $(|\boldsymbol{IS}(L')|, \subseteq)$ defined in Theorem 3.6.

$$\begin{array}{ccc} (L,\leq) & \xrightarrow{\quad f \quad} & (L',\leq) \\ {\scriptstyle f_L}\Big\downarrow & & {\scriptstyle f_{L'}}\Big\downarrow \\ (|\mathbf{IS}(L)|,\subseteq) & \xrightarrow{\mathscr{G}(\mathscr{C}(f))} & (|\mathbf{IS}(L')|,\subseteq) \end{array}$$

Proof. For all $a \in L$, we have $f_{L'} \circ f(a) = K(L') \bigcap \downarrow f(a)$. $\mathscr{G}(\mathscr{C}(f)) \circ f_L = \bigcup x\mathscr{C}(f)b(x \in Con_{K(L)}, x \subseteq\downarrow a) =\downarrow f(\downarrow a \bigcap K(L)) \bigcap K(L')$, since f preserves the directed set, $K(L') \bigcap \downarrow f(a) =\downarrow f(\downarrow a \bigcap K(L)) \bigcap K(L')$. $\square$

For any AL information system $(A, Con, \vdash)$, denote the compact elements of $(|A|, \subseteq)$ by $K(|A|)$. In Corollary 3.5, it has been proved $K(|A|) = \{\overline{x} \mid x \in Con\}$, we introduce a relation $H_A \subseteq Con_A \times K(|A|)$ by

$$(x, \overline{y}) \in H_A \Leftrightarrow y \subseteq \overline{x}.$$

We can verify that H_A is an approximable mapping from $(A, Con, \vdash)$ to $(K(|A|), Con, \vdash)$. On the other side, define a relation $I_A \subseteq Con_{K(A)} \times A$ by

$$(\alpha, a) \in I_A \Leftrightarrow a \in \bigcup \alpha.$$

It is trivial to check that I_A is an approximable mapping from $(K(|A|), Con, \vdash)$ to $(A, Con, \vdash)$ and it is easy to check that I_A is an inverse of H_A. Hence, H_A is an isomorphism in **ALGINS**.

Theorem 5.5 *Let $(A, Con, \vdash)$ and $(A', Con', \vdash')$ be AL information systems and $H : (A, Con, \vdash) \to (A', Con', \vdash')$ be an approximable mapping. Then*

$$H \circ H_{A'} = H_A \circ \mathscr{C}(\mathscr{G}(H)),$$

where H_A and $H_{A'}$ are defined above.

$$\begin{array}{ccc}
(A, Con, \vdash) & \xrightarrow{H} & (A', Con', \vdash') \\
{\scriptstyle H_A}\big\downarrow & & {\scriptstyle H_{A'}}\big\downarrow \\
(K(|A|), Con_{|A|}, \vdash_{|A|}) & \xrightarrow{\mathscr{C}(\mathscr{G}(H))} & ((K(|A'|), Con_{|A'|}, \vdash_{|A'|}))
\end{array}$$

Then ***ALGINS*** *is categorical equivalent to* ***ALGDOM****.*

Proof. For $x \in Con$, we have $xH \circ H_{A'}c = (\overline{y'} \mid y' \subseteq \overline{x'} \subseteq xHb)$, and $xH_A \circ \mathscr{C}(\mathscr{G}(H)c = x_{K(|A|)}\mathscr{C}(\mathscr{G}(H))c(x_{K(|A|)} \subseteq \{\overline{y} \mid y \subseteq \overline{x}\}) = x_{K(|A|)}\mathscr{C}(\mathscr{G}(H))c(x_{K(|A|)} \subseteq\downarrow_{K|A|} \overline{x}) = \{\overline{y'} \mid \overline{y'} \subseteq xHb\}$. It is equivalent with $xH \circ H_{A'}c$ because $xHy \Leftrightarrow xH\overline{y}$. With Theorem 5.4, we prove that **ALGINS** is categorical equivalent to **ALGDOM**. □

Theorem 5.6 ***ALGLINS*** *is categorical equivalent to* ***ALGLDOM****.*

Proof. Obvious. □

We know that **ALGDOM** is not a cartesian closed category and **ALGLDOM** is a maximal subcategory of **ALGDOM** which is cartesian closed. So we can easily get the following conclusion.

ALGINS is not cartesian closed, **ALGLINS** is a maximal subcategory of **ALGINS** which is cartesian closed.

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