Electronic Notes in Theoretical Computer Science 40 (2001)
URL: http://www.elsevier.nl/locate/entcs/volume40.html 21 pages

Uniform Completion versus Ideal Completion of Posets with Projections

Ralph Kummetz [1]

Institut für Algebra
Technische Universität Dresden
D-01062 Dresden, Germany [2]

Abstract

Posets with $(I, \leq)$-indexed projections are triples $\mathcal{D} = (D, \leq, (p_i)_{i \in I})$. They consist of a partially ordered set $(D, \leq)$ and a monotone net $(p_i)_{i \in I}$ of projections on D with respect to a fixed directed index set $(I, \leq)$. In the present paper we prove that there are natural "completions" $\mathsf{C}(\mathcal{D}) = (\mathsf{C}(D), \widehat{\leq}, (\widehat{p_i})_{i \in I})$ and $\mathsf{J}(\mathcal{D}) = (\mathsf{J}(D), \widetilde{\leq}, (\widetilde{p_i})_{i \in I})$ of $\mathcal{D}$. They are complete with respect to the uniformity induced by the kernels of all $\widehat{p_i}$ and $\widetilde{p_i}$, respectively. Moreover, they satisfy a universal property concerning the extension of special mappings ("homomorphisms"). If $\sup_{i \in I} p_i = \mathsf{id}_D$, then $\mathcal{D}$ can be viewed as a substructure of $\mathsf{C}(\mathcal{D})$ and of $\mathsf{J}(\mathcal{D})$. Further, D is dense in $\mathsf{C}(D)$; whence $\mathsf{C}(D)$ appears as the uniform completion of D. On the other hand, $\mathsf{J}(D)$ can be obtained as the ideal completion of a suitable subset of D. A comparison shows that the completion $\mathsf{C}(\mathcal{D})$ may be seen as a substructure of the completion $\mathsf{J}(\mathcal{D})$. We also investigate under which conditions both completions coincide.

1 Introduction

Completions of mathematical structures emerge in many branches of mathematics and theoretical computer science. Well-known are completions of metric spaces, uniform spaces, the ideal completion of partially ordered sets, and so forth. In a recent paper, Bonsangue, van Breugel, and Rutten [3] investigated the completion of generalized metric spaces. This yields a generalization both of the chain completion of (pre)ordered sets and of the metric Cauchy completion.

In a certain sense, a completion of a structure is unique and "small". It is the "smallest" structure satisfying certain nice properties such that the original structure is contained in it as a substructure.

[1] This work was supported by the German Research Foundation (DFG).
[2] Present affiliation: 3SOFT GmbH, Frauenweiherstraße 14, D-91058 Erlangen, Germany.

The mathematical objects we deal with concern both partially ordered sets and uniform spaces induced by some family of projections. Such structures appear in mathematics and in theoretical computer science.

For instance, Plotkin introduced SFP-domains in [14]. They are inverse limits of ω-chains of finite pointed posets and form a convenient mathematical model for the semantics of nondeterministic programming languages. Gunter [6] and Jung [7] extended SFP-domains to bifinite domains (also called profinite ([6]) or FB-domains ([7])). These are limits of directed systems of finite posets. They can be characterized as dcpo's admitting a monotone approximating net of Scott-continuous and image-finite projections (cf. [6,7]). Moreover, bifinite domains turn out to form a maximal cartesian closed category of algebraic domains. Such maximal categories were completely classified by Jung [7].

On the other hand, Baier and Majster-Cederbaum [2,12] investigated pseudo rank ordered posets. They are posets $(D, \leq)$ together with a monotone sequence of projections. These projections define a canonical pseudo-ultrametric on D. In some situations the projections are induced by a weight. This is a special mapping from D into the set of natural numbers with infinity. Intuitively, the elements of D are regarded as "processes" and the weight of $d \in D$ is the maximal number of steps that is needed for an execution of d. In [12] a comparison of the metric completion with the ideal completion of a weighted poset is established for finitary weights.

Recently, Spreen [15] used rank orderings on dI-domains to obtain several models of the untyped λ-calculus.

A general approach comprising both bifinite domains and pseudo rank ordered posets can be found in [9,10], where posets with projections (pop's) are introduced. These posets carry a directed family of projections. The kernels of the projections form a basis for a uniformity: the pop uniformity. This uniformity and its induced topology are closely related to the poset structure, see [9,10] for details.

In the present paper we consider completions of posets with projections. Since we study posets and uniform spaces at the same time, we can form either the ideal completion of the poset (or a suitable subset of it) to obtain an algebraic domain or the uniform completion with respect to the pop uniformity to get a Hausdorff and complete uniform space that contains the original one as a dense subspace. We investigate how both completions are related.

The reader is assumed to have some basic knowledge in topology, especially in the theory of uniform spaces. In the books by Bourbaki [4] and Kelley [8] he or she will find much more than we actually need here. For notions from and a survey on domain theory we recommend Abramsky and Jung [1].

2 Basic facts on posets with projections

In this section we recall some facts concerning posets with projections. Throughout this paper let $(D, \leq)$ be a partially ordered set.

For $A \subseteq D$ let $A{\downarrow}_D := A{\downarrow} := \{d \in D \mid \exists a \in A\colon d \leq a\}$. A *lower* set is a subset $A \subseteq D$ with $A = A{\downarrow}$. We shorten $d{\downarrow} := \{d\}{\downarrow}$ for any $d \in D$. A set $A \subseteq D$ is *directed* provided that it is non-empty and for all $x, y \in D$ there is some $z \in D$ with $z \geq x, y$. A non-empty subset $A \subseteq D$ is *bounded* if there is some $d \in D$ with $a \leq d$ for all $a \in A$. A *dcpo* (*directed complete partial order*) is a poset in which all directed subsets have a supremum. Similarly, a *bcpo* (*bounded complete partial order*) is a poset where each bounded set admits a supremum. An *ideal* is a directed, lower subset of D. Let $(\mathsf{Id}(D), \subseteq)$ be the ideal completion of $(D, \leq)$.

An element $d \in D$ is called *compact* if for all directed subsets $A \subseteq D$ with $\sup A \geq d$ there is some $a \in A$ with $a \geq d$. Let $K(D)$ denote the set of all compact elements of $(D, \leq)$. The poset $(D, \leq)$ is *algebraic* provided that for all $d \in D$ the set $K(D) \cap d{\downarrow}$ is directed and $d = \sup(K(D) \cap d{\downarrow})$.

Let $f : D \longrightarrow E$ be a mapping between posets $(D, \leq)$ and $(E, \leq)$. Then f is *Scott-continuous* if it preserves suprema of directed sets (in case these suprema exist). A mapping $p : D \longrightarrow D$ is a *projection* if p is monotone, idempotent, and $p \leq \mathsf{id}_D$ with respect to the pointwise ordering of mappings, i.e. $p(d) \leq d$ for all $d \in D$. The *kernel* of p is the set $\ker p := \{(d, e) \in D^2 \mid p(d) = p(e)\}$.

Next, we define the objects we deal with. Recall that a net is a mapping whose domain is a directed set.

Definition 2.1 Let $(I, \leq)$ be a directed index set, let $(D, \leq)$ be a poset, and let $(p_i)_{i \in I}$ be a monotone net of projections on D. Then we call the triple $\mathcal{D} = (D, \leq, (p_i)_{i \in I})$ a *poset with $(I, \leq)$-indexed projections* or *$(I, \leq)$-pop*. We say that $\mathcal{D}$ is *approximating* provided that $\sup_{i \in I} p_i(d) = d$ for all $d \in D$. The net $(p_i)_{i \in I}$ is the *projection net* of $\mathcal{D}$. It is *Abelian* if $p_i \circ p_j = p_j \circ p_i$ for all $i, j \in I$.

For any $(I, \leq)$-pop $\mathcal{D} = (D, \leq, (p_i)_{i \in I})$, the kernels $\ker p_i$ form a basis for a uniformity $\mathcal{U}_{\mathcal{D}}$ on D, see [9,10] for details. We call $\mathcal{U}_{\mathcal{D}}$ the *pop uniformity* of $\mathcal{D}$. The induced topology $\tau_{\mathcal{D}}$ is the *pop topology* of $\mathcal{D}$. A neighbourhood basis of an element $d \in D$ is given by the sets $\{e \in D \mid p_i(d) = p_i(e)\}$ with $i \in I$. We say that $\mathcal{D}$ is *Hausdorff, complete, ...* if this is true for the underlying uniform space $(D, \mathcal{U}_{\mathcal{D}})$ (topological space $(D, \tau_{\mathcal{D}})$, respectively).

We list some known facts on $(I, \leq)$-pop's that will be used subsequently without further reference:

Proposition 2.2 *Let $\mathcal{D} = (D, \leq, (p_i)_{i \in I})$ be an $(I, \leq)$-pop, let $(d_n)_{n \in N}$ be a net in D, and let $d \in D$.*

(1) *For all $i, j \in I$ we have $p_i \leq p_j$ if and only if $p_i = p_i \circ p_j$ if and only if*

$p_i = p_j \circ p_i$ (see e.g. [9], Lemma 2.1).

(2) *$(d_n)_{n \in N}$ converges to an element $d \in D$ in the pop topology if and only if, for all $i \in I$, there is an index $n_i \in N$ such that $p_i(d_n) = p_i(d)$ for all $n \geq n_i$.*

(3) *$(d_n)_{n \in N}$ is a Cauchy net with respect to the pop uniformity if and only if, for all $i \in I$, there is an index $n_i \in N$ such that $p_i(d_n) = p_i(d_{n_i})$ for all $n \geq n_i$.*

(4) *The net $(p_i(d))_{i \in I}$ converges to d with respect to the pop topology. This follows from [9], Prop. 2.8(1).*

(5) *$\mathcal{D}$ is approximating if and only if $\leq$ is closed in D^2 (Lemma 2.9 in [9]). Clearly, $\mathcal{D}$ is Hausdorff in this case.*

(6) *If $(D, \leq)$ is a dcpo and p_i is Scott-continuous for all $i \in I$, then D is complete with respect to the pop uniformity (see Prop. 3.1 in [9]).* □

Substructures of $(I, \leq)$-pop's are defined as follows:

Definition 2.3 Let $\mathcal{D} = (D, \leq, (p_i)_{i \in I})$ be an $(I, \leq)$-pop. Let $X \subseteq D$. We say that X *induces a subpop* of $\mathcal{D}$ if $p_i[X] \subseteq X$ for all $i \in I$. Then, together with the induced order and the restricted projections, we call $\mathcal{X} = (X, \leq, (p_i|_X)_{i \in I})$ a *subpop* of $\mathcal{D}$. Moreover, $\mathcal{X}$ is a *full subpop* of $\mathcal{D}$ provided that $p_i[D] \subseteq X$ for all $i \in I$.

Clearly, any subpop $\mathcal{X} = (X, \leq, (p_i|_X)_{i \in I})$ of an $(I, \leq)$-pop is an $(I, \leq)$-pop itself. If $\mathcal{D}$ is approximating, then $\mathcal{X}$ is also approximating.

Example 2.4 Let $\mathcal{D} = (D, \leq, (p_i)_{i \in I})$ be an $(I, \leq)$-pop. The set $\bigcup_{i \in I} p_i[D]$ induces a full subpop of $\mathcal{D}$. Obviously, this is the least full subpop of $\mathcal{D}$. Note that it is approximating. We denote this subpop by $\bigcup_{i \in I} p_i[\mathcal{D}]$.

Full subpop's of $\mathcal{D}$ are precisely those subpop's that are dense in D with respect to the pop topology. This is the subject of the next lemma. Its easy proof is left to the reader.

Lemma 2.5 *Let $\mathcal{D} = (D, \leq, (p_i)_{i \in I})$ be an $(I, \leq)$-pop and let $\mathcal{X} = (X, \leq, (p_i|_X)_{i \in I})$ be a subpop of $\mathcal{D}$. Then $\mathcal{X}$ is a full subpop if and only if X is dense in $(D, \tau_{\mathcal{D}})$.* □

Next, we define structure preserving mappings between $(I, \leq)$-pop's.

Definition 2.6 Let $\mathcal{D} = (D, \leq, (p_i)_{i \in I})$ and $\mathcal{E} = (E, \leq, (q_i)_{i \in I})$ be $(I, \leq)$-pop's and let $f : D \longrightarrow E$.

(1) The mapping f *commutes with all projections* if $q_i \circ f = f \circ p_i$ for all $i \in I$.

(2) We call f a *(pop) homomorphism* provided that f is monotone and commutes with all projections.

(3) We say that f is a *(pop) embedding* if f is an order embedding and a pop homomorphism.

(4) Finally, f is a *(pop) isomorphism* if f is both an order isomorphism and a pop homomorphism. The $(I, \leq)$-pop's $\mathcal{D}$ and $\mathcal{E}$ are said to be *(pop) isomorphic* if there exists a pop isomorphism between them.

Intuitively, the image $p_i(d)$ can be seen as the "i-th approximation" of the element $d \in D$. Hence, the property of $f : D \longrightarrow E$ to commute with all projections means that f preserves all levels of approximation: the i-th approximation of $f(d)$ coincides with the image of the i-th approximation of d.

Clearly, any homomorphism is uniformly continuous with respect to the pop uniformities. Further, there is an obvious connection between subpop's and pop embeddings: let $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ be an $(I, \leq)$-pop. If $\mathcal{X} = (X, \leq, (p_i|_X)_{i\in I})$ is a subpop of $\mathcal{D}$, then the inclusion map $\mathsf{id}_{X,D}$ is a pop embedding. Conversely, let $\mathcal{E} = (E, \leq, (q_i)_{i\in I})$ be an $(I, \leq)$-pop and let $f : D \longrightarrow E$ be a pop embedding. Then $f[D]$ induces a subpop of $\mathcal{E}$ that is pop isomorphic to $\mathcal{D}$.

A natural question is when all projections p_i of an $(I, \leq)$-pop $(D, \leq, (p_i)_{i\in I})$ are also homomorphisms. Abelian projection nets give the answer:

Lemma 2.7 *Let $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ be an $(I, \leq)$-pop. Then the following are equivalent:*

(i) *$(p_i)_{i\in I}$ is Abelian.*

(ii) *p_i is a homomorphism for all $i \in I$.*

(iii) *$p_i[D]$ induces a subpop of $\mathcal{D}$ for all $i \in I$.* □

Again, the easy proof is left to the reader. Note that the composition of projections need not be Abelian, whence in general the projection net $(p_i)_{i\in I}$ is not Abelian. However, if $(I, \leq)$ is a chain, then $(p_i)_{i\in I}$ is always Abelian because of Proposition 2.2(1).

3 Existence and uniqueness of the pop completion

This section is devoted to a universal object that we call "pop completion". We show that each $(I, \leq)$-pop $\mathcal{D}$ has such a completion $\mathsf{C}(\mathcal{D})$ and, furthermore, that it is uniquely determined up to a unique pop isomorphism. Any homomorphism from $\mathcal{D}$ to a complete approximating $(I, \leq)$-pop $\mathcal{E}$ can be extended uniquely to a homomorphism from $\mathsf{C}(\mathcal{D})$ to $\mathcal{E}$. If $\mathcal{D}$ is approximating, then we can pop embed $\mathcal{D}$ as a full approximating subpop into its completion. The pop uniformity of $\mathsf{C}(\mathcal{D})$ is the uniformity of the uniform completion of $(D, \mathcal{U}_\mathcal{D})$.

The following $(I, \leq)$-pop turns out to be the proper candidate for the pop completion:

Proposition 3.1 *Let $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ be an $(I, \leq)$-pop. Let*

$$D_\infty := \{(d_i)_{i\in I} \in \prod_{i\in I} p_i[D] \mid \forall i, j \in I : \ i \leq j \Rightarrow d_i = p_i(d_j)\}$$

be equipped with the product order. For all $i \in I$ define $r_i : D_\infty \longrightarrow D_\infty$ by $r_i((d_j)_{j\in I}) := (p_j(d_i))_{j\in I}$. Then $\mathcal{D}_\infty := (D_\infty, \leq, (r_i)_{i\in I})$ is a complete approximating $(I, \leq)$-pop. If $(p_i)_{i\in I}$ is Abelian, then $(r_i)_{i\in I}$ is Abelian and $r_i((d_j)_{j\in I}) = (p_i(d_j))_{j\in I}$ for all $i \in I$ and all $(d_j)_{j\in I} \in D_\infty$. If p_i is Scott-continuous for all $i \in I$, then r_i is Scott-continuous for all $i \in I$.

Proof (Sketch). For all $i \in I$ let $f_i : p_i[D] \longrightarrow D_\infty$ be defined by $f_i(d) := (p_j(d))_{j\in I}$. Note that f_i is well-defined. Let $g_i : D_\infty \longrightarrow p_i[D]$ be the canonical projection from D_∞ onto $p_i[D]$. Recall that (f_i, g_i) is an embedding projection pair. In particular, f_i is Scott-continuous. By Theorem 3.3(2) in [9] we obtain $(D_\infty, \leq, (f_i \circ g_i)_{i\in I})$ to be a complete approximating $(I, \leq)$-pop. Further, $(f_i \circ g_i)((d_j)_{j\in I}) = f_i(d_i) = (p_j(d_i))_{j\in I} = r_i((d_j)_{j\in I})$ for all $i \in I$.

Now let $(p_i)_{i\in I}$ be Abelian and let $i, j \in I$. Choose some $k \in I$ with $k \geq i, j$ and recall that $d_i = p_i(d_k)$ and $d_j = p_j(d_k)$. Then $p_j(d_i) = p_j(p_i(d_k)) = p_i(p_j(d_k)) = p_i(d_j)$. As a consequence, $(r_i)_{i\in I}$ is Abelian.

Finally, let p_i be Scott-continuous for all $i \in I$. It is well-known that then g_i is also Scott-continuous for all $i \in I$. Consequently, $r_i = f_i \circ g_i$ is Scott-continuous for all $i \in I$. □

Next, we formulate our existence and uniqueness theorem of the "pop completion". The approach to obtain the completion using an inverse limit construction is similar to the one given by Ehrig et al. [5], Theorem 1.14. This result states the existence and uniqueness of a universal completion of so-called projection spaces. These are sets together with a sequence of idempotent self-maps satisfying certain conditions. They do not carry any order relation. For instance, it is easy to see that if $(D, \leq, (p_n)_{n\in\mathbb{N}})$ is an $(\mathbb{N}, \leq)$-pop, then $(D, (p_n)_{n\in\mathbb{N}})$ is a projection space. For further connections between projection spaces and $(\mathbb{N}, \leq)$-pop's the reader is referred to [10], Section 4.2.

Theorem 3.2 *Let $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ be an $(I, \leq)$-pop.*

(1) *There exist a complete approximating $(I, \leq)$-pop $\mathsf{C}(\mathcal{D}) = (\mathsf{C}(D), \widehat{\leq}, (\widehat{p}_i)_{i\in I})$ and a pop homomorphism $\psi : D \longrightarrow \mathsf{C}(D)$ with the following universal property:*
For any complete approximating $(I, \leq)$-pop $(E, \leq, (q_i)_{i\in I})$ and any homomorphism $f : D \longrightarrow E$ there is a unique homomorphism $\overline{f} : \mathsf{C}(D) \longrightarrow E$ with $\overline{f} \circ \psi = f$.

(2) *Let $\mathcal{D}' = (D', \leq', (p'_i)_{i\in I})$ be a complete approximating $(I, \leq)$-pop and let $\phi : D \longrightarrow D'$ be a homomorphism such that $(\mathcal{D}', \phi)$ fulfils the universal property of (1). Then there exists a unique pop isomorphism $\Phi : \mathsf{C}(D) \longrightarrow D'$ with $\Phi \circ \psi = \phi$.*

(3) *$\psi[D]$ induces a full subpop of $\mathsf{C}(\mathcal{D})$ and is dense in $(\mathsf{C}(D), \tau_{\mathsf{C}(\mathcal{D})})$.*

(4) (a) *Let $d, e \in D$. Then $\psi(d) \mathrel{\widehat{\leq}} \psi(e)$ if and only if $p_i(d) \leq p_i(e)$ for all $i \in I$.*
    (b) *$\psi|_{p_i[D]}$ is an order isomorphism from $p_i[D]$ onto $\widehat{p_i}[\mathsf{C}(D)]$ for all $i \in I$.*

(5) *ψ is a pop embedding of $\mathcal{D}$ into $\mathsf{C}(\mathcal{D})$ if and only if $\mathcal{D}$ is approximating.*

(6) (a) *If $(p_i)_{i\in I}$ is Abelian, then so is $(\widehat{p_i})_{i\in I}$.*
    (b) *If p_i is Scott-continuous for all $i \in I$, then ψ and $\widehat{p_i}$ are Scott-continuous for all $i \in I$.*

(7) *$\mathsf{C}(\mathcal{D})$ is pop isomorphic to $\mathcal{D}_\infty$ (cf. Proposition 3.1). More precisely, there is a unique pop isomorphism $\Psi : \mathsf{C}(D) \longrightarrow \mathcal{D}_\infty$ with $(\Psi \circ \psi)(d) = (p_i(d))_{i\in I}$ for all $d \in D$.*

Proof. Let $\mathsf{C}(\mathcal{D}) := \mathcal{D}_\infty$, i.e. $(\mathsf{C}(D), \widehat{\leq}) = (D_\infty, \leq)$ and $\widehat{p_i} = r_i$ for all $i \in I$. Proposition 3.1 tells us that $\mathsf{C}(\mathcal{D})$ is a complete approximating $(I, \leq)$-pop.

Let $\psi : D \longrightarrow \mathsf{C}(D)$ be defined by $\psi(d) := (p_i(d))_{i\in I}$. Clearly, ψ is monotone. Let $\mathcal{V}$ be the uniformity on D_∞ that is induced by the product uniformity of the family $(p_i[D], \mathcal{U}_{\mathsf{dis}})_{i\in I}$, where $\mathcal{U}_{\mathsf{dis}}$ is the discrete uniformity on $p_i[D]$. Let $i \in I$ and let $g_i : D_\infty \longrightarrow p_i[D]$ be the canonical projection from D_∞ onto $p_i[D]$. One easily sees that $(g_i \times g_i)^{-1}[\mathsf{id}_{p_i[D]}] = \ker \widehat{p_i}$. We conclude that $\mathcal{V} = \mathcal{U}_{\mathsf{C}(\mathcal{D})}$. Therefore, we can apply Lemma 3.2 in [9] to deduce that $\psi[D]$ is dense in $\mathsf{C}(D)$. Given $i \in I$, we have $\widehat{p_i}(\psi(d)) = r_i((p_j(d))_{j\in I}) = (p_j(p_i(d)))_{j\in I} = \psi(p_i(d))$; whence ψ is a homomorphism. In particular, $\psi[D]$ induces a subpop of $\mathsf{C}(\mathcal{D})$. It is full in view of Lemma 2.5, which proves (3).

Furthermore, let $\psi(d) \mathrel{\widehat{\leq}} \psi(e)$. Then, by definition of ψ, we have $p_i(d) \leq p_i(e)$ for all $i \in I$. Conversely, if $p_i(d) \leq p_i(e)$ for all $i \in I$, then $\widehat{p_i}(\psi(d)) = \psi(p_i(d)) \mathrel{\widehat{\leq}} \psi(p_i(e)) = \widehat{p_i}(\psi(e))$ for all $i \in I$. As $\mathsf{C}(\mathcal{D})$ is approximating, we conclude that $\psi(d) \mathrel{\widehat{\leq}} \psi(e)$. This proves (4)(a). To verify (4)(b), let $i \in I$. Since ψ is a homomorphism, we have $\psi[p_i[D]] = \widehat{p_i}[\psi[D]] \subseteq \widehat{p_i}[\mathsf{C}(D)]$. On the other hand, let $\widehat{d} \in \mathsf{C}(D)$. Then $\widehat{p_i}(\widehat{d}) \in \psi[D]$ by (3), whence $\widehat{p_i}(\widehat{d}) \in \widehat{p_i}[\psi[D]] = \psi[p_i[D]]$. We obtain $\psi[p_i[D]] = \widehat{p_i}[\mathsf{C}(D)]$. As $\psi|_{p_i[D]}$ is monotone and, in addition, order-reflecting by (4)(a), it is an order isomorphism from $p_i[D]$ onto $\widehat{p_i}[\mathsf{C}(D)]$.

Let ψ be a pop embedding, let $d, e \in D$, and let $p_i(d) \leq e$ for all $i \in I$. Then $\widehat{p_i}(\psi(d)) = \psi(p_i(d)) \mathrel{\widehat{\leq}} \psi(e)$ for all $i \in I$; whence $\psi(d) \mathrel{\widehat{\leq}} \psi(e)$ since $\mathsf{C}(\mathcal{D})$ is approximating. Then $d \leq e$ and thus $\mathcal{D}$ is approximating. Conversely, if $\mathcal{D}$ is approximating, then ψ is an order embedding by Lemma 3.2 in [9] and therefore a pop embedding. This shows (5).

(6) results from Proposition 3.1 (the proof that ψ be Scott-continuous is deferred to Proposition 3.7 below).

To prove the universal property, let $\mathcal{E} = (E, \leq, (q_i)_{i\in I})$ be a complete approximating $(I, \leq)$-pop and let $f : D \longrightarrow E$ be a homomorphism. Let $(d_i)_{i\in I} \in \mathsf{C}(D)$ and let $i, j \in I$ with $j \geq i$. Then $q_i(f(d_j)) = q_i(q_i(f(d_j))) = q_i(f(p_i(d_j))) = q_i(f(d_i))$. Consequently, $(f(d_i))_{i\in I}$ is a Cauchy net in $(E, \mathcal{U}_{\mathcal{E}})$. As $\mathcal{E}$ is complete Hausdorff, let $\overline{f}((d_i)_{i\in I}) := \lim_{i\in I} f(d_i)$. Now let $(d_i)_{i\in I}$,

$(\widetilde{d}_i)_{i\in I} \in \mathsf{C}(D)$ with $d_i \leq \widetilde{d}_i$ for all $i \in I$. Then $f(d_i) \leq f(\widetilde{d}_i)$ for all $i \in I$ and thus $\overline{f}((d_i)_{i\in I}) \leq \overline{f}((\widetilde{d}_i)_{i\in I})$ because the partial order of $\mathcal{E}$ is closed. Hence, $\overline{f}$ is monotone. Note that since $d_i \in p_i[D]$, we have $d_i = p_j(d_i)$ for all $j \geq i$. Consequently, $q_i(f(d_j)) = f(p_i(d_j)) = f(d_i) = f(p_j(d_i))$ for all $i, j \in I$, $j \geq i$. We conclude $q_i(\overline{f}((d_j)_{j\in I})) = \overline{f}((p_j(d_i))_{j\in I}) = \overline{f}(\widehat{p}_i((d_j)_{j\in I}))$; whence $\overline{f}$ commutes with all projections and thus is a homomorphism. Furthermore, $\overline{f}(\psi(d)) = \overline{f}((p_i(d))_{i\in I}) = \lim_{i\in I} f(p_i(d)) = f(d)$ because $(p_i(d))_{i\in I} \longrightarrow d$ and f is continuous. As $\mathcal{E}$ is Hausdorff, we infer from (3) that $\overline{f}$ is the unique homomorphism with $\overline{f} \circ \psi = f$. This proves (1).

Uniqueness of $(\mathsf{C}(\mathcal{D}), \psi)$ (i.e. (2)) can be shown by exploiting the universal property in the usual way. Finally, (7) is true by definition and (2). $\square$

Definition 3.3 For any $(I, \leq)$-pop $\mathcal{D}$ we call the complete approximating $(I, \leq)$-pop $\mathsf{C}(\mathcal{D})$ of Theorem 3.2 the *pop completion* of $\mathcal{D}$. The mapping $\psi : D \longrightarrow \mathsf{C}(D)$ is the *canonical homomorphism*.

Notice that $(\mathsf{C}(D), \mathcal{U}_{\mathsf{C}(\mathcal{D})})$ is the uniform completion of the space $(D, \mathcal{U}_{\mathcal{D}})$.

Corollary 3.4 *Let $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ be an $(I, \leq)$-pop with pop completion $\mathsf{C}(\mathcal{D}) = (\mathsf{C}(D), \widehat{\leq}, (\widehat{p}_i)_{i\in I})$ and canonical homomorphism $\psi : D \longrightarrow \mathsf{C}(D)$. Then $\psi|_{\bigcup_{i\in I} p_i[D]}$ is a pop isomorphism from the subpop $\bigcup_{i\in I} p_i[\mathcal{D}]$ onto the subpop $\bigcup_{i\in I} \widehat{p}_i[\mathsf{C}(\mathcal{D})]$. If, furthermore, $(p_i)_{i\in I}$ is Abelian, then $\psi|_{p_i[D]}$ is a pop isomorphism from the subpop induced by $p_i[D]$ onto the subpop induced by $\widehat{p}_i[\mathsf{C}(D)]$.*

Proof. By Theorem 3.2(1) and (4), ψ is a homomorphism with $\psi[\bigcup_{i\in I} p_i[D]] = \bigcup_{i\in I} \widehat{p}_i[\mathsf{C}(D)]$. Recall that $\psi(d) \mathrel{\widehat{\leq}} \psi(e)$ if and only if $p_i(d) \leq p_i(e)$ for all $i \in I$ (3.2(4)). Thus, given $i, j, k \in I$ with $k \geq i, j$, we infer that $\psi(p_i(d)) \mathrel{\widehat{\leq}} \psi(p_j(e))$ implies $p_i(d) = p_k(p_i(d)) \leq p_k(p_j(e)) = p_j(e)$.

Now let $(p_i)_{i\in I}$ be Abelian. Lemma 2.7 and Theorem 3.2(6)(a) tell us that $p_i[D]$ and $\widehat{p}_i[\mathsf{C}(D)]$ induce subpop's for all $i \in I$. They are isomorphic in virtue of 3.2(4). $\square$

Let $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ be an approximating $(I, \leq)$-pop. In light of Theorem 3.2(5) we may identify $\mathcal{D}$ with the subpop induced by $\psi[D]$. Thus, we assume that $D \subseteq \mathsf{C}(D)$, $\widehat{\leq}|_D = \leq$, $\psi = \mathsf{id}_{D,\mathsf{C}(D)}$, and $\widehat{p}_i|_D = p_i$ for all $i \in I$. Note that for all $i \in I$ we have $\widehat{p}_i[\mathsf{C}(D)] = p_i[D]$ by 3.2(4). Furthermore, $\mathcal{D}$ is a full subpop of $\mathsf{C}(\mathcal{D})$ and D is dense in $\mathsf{C}(D)$. Summing things up, we obtain:

Corollary 3.5 *Let $\mathcal{D}$ be an $(I, \leq)$-pop. Then the following are equivalent:*

(i) *$\mathcal{D}$ is approximating.*

(ii) *$\mathcal{D}$ is a full subpop of a complete approximating $(I, \leq)$-pop.* $\square$

Corollary 3.6 *Let $\mathcal{D}$ be an approximating $(I, \leq)$-pop. Then $\mathcal{D} = C(\mathcal{D})$ (more precisely: ψ is a pop isomorphism) if and only if $\mathcal{D}$ is complete in its pop uniformity.*

Proof. This results from the fact that both D is dense in $(\mathsf{C}(D), \tau_{\mathsf{C}(\mathcal{D})})$ and $(D, \mathcal{U}_{\mathcal{D}})$ is complete; whence D is closed in $(\mathsf{C}(D), \tau_{\mathsf{C}(\mathcal{D})})$ because the latter is Hausdorff. Alternatively: the canonical homomorphism ψ is an order isomorphism due to Lemma 3.2 in [9] and thus a pop isomorphism. A further alternative is to apply Theorem 3.3(1) in [9] and Theorem 3.2(7). $\square$

Let $(D, \leq, (p_i)_{i\in I})$ be an approximating $(I, \leq)$-pop. We show that whenever a subset A has a supremum or an infimum in $(D, \leq)$ that is preserved by each projection p_i, then it has a supremum or an infimum in $(\mathsf{C}(D), \widehat{\leq})$ which coincides with the one formed in $(D, \leq)$. More precisely:

Proposition 3.7 *Let $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ be an $(I, \leq)$-pop and let $\mathsf{C}(\mathcal{D}) = (\mathsf{C}(D), \widehat{\leq}, (\widehat{p_i})_{i\in I})$ be its pop completion. Let $\psi : D \longrightarrow \mathsf{C}(D)$ be the canonical homomorphism. Let $A \subseteq D$ such that $\sup A$ exists and $p_i(\sup A) = \sup p_i[A]$ for all $i \in I$. Then we have $\psi(\sup A) = \sup \psi[A]$ and $\widehat{p_i}(\psi(\sup A)) = \sup \widehat{p_i}[\psi[A]]$ for all $i \in I$. In particular, if $\mathcal{D}$ is approximating, then $\sup_D A = \sup_{\mathsf{C}(D)} A$. Similarly for the infimum.*

Proof. We may assume $\mathsf{C}(\mathcal{D}) = \mathcal{D}_\infty$ and $\psi(d) = (p_i(d))_{i\in I}$ for all $d \in D$, cf. Theorem 3.2(7). Clearly, $\psi[A] \leq \psi(\sup A)$. Let $(e_i)_{i\in I} \in D_\infty$ with $\psi[A] \leq (e_i)_{i\in I}$. Then $p_i(a) \leq e_i$ for all $a \in A$ and all $i \in I$; whence $p_i(\sup A) = \sup p_i[A] \leq e_i$ for all $i \in I$. Therefore, $\psi(\sup A) = \sup \psi[A]$ and $\widehat{p_i}(\psi(\sup A)) = \psi(p_i(\sup A)) = \sup \psi[p_i[A]] = \sup \widehat{p_i}[\psi[A]]$ for all $i \in I$. $\square$

We mention the following two facts without proof:

Proposition 3.8 *Let $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ be any $(I, \leq)$-pop with pop completion $\mathsf{C}(\mathcal{D}) = (\mathsf{C}(D), \widehat{\leq}, (\widehat{p_i})_{i\in I})$. Let $\mathcal{E} = (E, \leq, (q_i)_{i\in I})$ be a complete approximating $(I, \leq)$-pop. Let $f : D \longrightarrow E$ be a homomorphism with unique extension $\overline{f} : \mathsf{C}(D) \longrightarrow E$. If f, p_i, and q_i are Scott-continuous for all $i \in I$, then so is $\overline{f}$.* $\square$

Proposition 3.9 *Let $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ be an $(I, \leq)$-pop and let $\mathcal{X} = (X, \leq, (p_i|_X)_{i\in I})$ be a full subpop of $\mathcal{D}$. Let $\mathcal{E} = (E, \leq, (q_i)_{i\in I})$ be a complete approximating $(I, \leq)$-pop and let $f : X \longrightarrow E$ be a homomorphism. Then there exists a unique homomorphism $\overline{f} : D \longrightarrow E$ with $\overline{f}|_X = f$. If, furthermore, f, p_i, and q_i are Scott-continuous for all $i \in I$, then so is $\overline{f}$.* $\square$

Proposition 3.10 *Let $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ be an $(I, \leq)$-pop and let $\mathcal{X}$ be a full subpop of $\mathcal{D}$. Then the pop completions $\mathsf{C}(\mathcal{X})$ and $\mathsf{C}(\mathcal{D})$ are isomorphic. In particular, $\mathsf{C}(\mathcal{D})$ is the pop completion of the subpop $\bigcup_{i\in I} p_i[\mathcal{D}]$.*

Proof. Let $\mathcal{X} =: (X, \leq, (p_i|_X)_{i\in I})$. Since $\mathcal{X}$ is a full subpop of $\mathcal{D}$, we obtain $p_i[X] = p_i[D]$ for all $i \in I$. Theorem 3.2(7) yields the assertion. $\square$

Especially, if $\mathcal{D}$ is a complete approximating $(I, \leq)$-pop with full subpop $\mathcal{X}$, then $\mathcal{D}$ is (pop isomorphic to) the pop completion $\mathsf{C}(\mathcal{X})$ of $\mathcal{X}$ (Corollary 3.6).

Proposition 3.11 *Let $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ be an $(I, \leq)$-pop. Let E be a "property" that is invariant under monotone mappings (such as e.g. bounded or* *directed). Suppose all subsets of D with property E have a supremum that is preserved by each p_i. Then all subsets of $\mathsf{C}(D)$ with property E have a supremum which is preserved by each $\widehat{p_i}$.*

Proof (Sketch). Let $\widehat{A} \subseteq \mathsf{C}(D)$ have property E and let $i \in I$. Recall that $\psi|_{p_i[D]}$ is an order isomorphism from $p_i[D]$ onto $\widehat{p_i}[\mathsf{C}(D)]$. Let $A_i := (\psi|_{p_i[D]})^{-1}[\widehat{p_i}[\widehat{A}]]$. Then A_i has property E; whence $\sup_D A_i$ exists, $\sup_D A_i = \sup_{p_i[D]} A_i =: \sup A_i$, and $p_i(\sup A_j) = \sup p_i[A_j]$ for all $i, j \in I$. Moreover, one can show that $\widehat{p_i}(\sup \widehat{p_j}[\widehat{A}]) = \widehat{p_i}(\sup \widehat{p_i}[\widehat{A}])$ for all $j \geq i$. Therefore, $(\sup \widehat{p_i}[\widehat{A}])_{i\in I}$ is a Cauchy net. Let $\widehat{d} \in \mathsf{C}(D)$ with $(\sup \widehat{p_i}[\widehat{A}])_{i\in I} \longrightarrow \widehat{d}$. Then it turns out that $\widehat{d} = \sup \widehat{A}$ and $\widehat{p_i}(\widehat{d}) = \sup \widehat{p_i}[\widehat{A}]$. □

Corollary 3.12 *Let $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ be an $(I, \leq)$-pop whose underlying poset is a bcpo such that for all $i \in I$ the projection p_i preserves suprema of bounded sets. Let $\mathsf{C}(\mathcal{D}) = (\mathsf{C}(D), \widehat{\leq}, (\widehat{p_i})_{i\in I})$ be the pop completion of $\mathcal{D}$. Then $(\mathsf{C}(D), \widehat{\leq})$ is a bcpo and $\widehat{p_i}$ preserves suprema of bounded sets for all $i \in I$.* □

If property E means directed, then we can improve Proposition 3.11:

Proposition 3.13 *Let $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ be an $(I, \leq)$-pop whose underlying poset is a dcpo and whose projections p_i are Scott-continuous for all $i \in I$. Let $A(D) := \{\sup_{i\in I} p_i(d) \mid d \in D\}$. Then $A(D)$ induces a full subpop $A(\mathcal{D}) = (A(D), \leq, (p_i|_{A(D)})_{i\in I})$ of $\mathcal{D}$ such that $A(\mathcal{D})$ is approximating, $(A(D), \leq)$ is a dcpo, and $p_i|_{A(D)}$ is Scott-continuous for all $i \in I$. Moreover, $A(\mathcal{D})$ is the pop completion of $\mathcal{D}$. It coincides with $\mathcal{D}$ if and only if $\mathcal{D}$ is approximating.*

Proof. Clearly, the pointwise supremum $\xi := \sup_{i\in I} p_i$ is a projection with $p_i \circ \xi = \xi \circ p_i = p_i$ for all $i \in I$. Hence, $A(\mathcal{D})$ is an approximating full subpop of $\mathcal{D}$. Now $A(D) = \xi[D]$ and $(A(D), \leq)$ is therefore a dcpo with $\sup_{A(D)} C = \sup_D C$ for any directed subset $C \subseteq A(D)$. Thus, $p_i|_{A(D)}$ is Scott-continuous for all $i \in I$. As a consequence, $A(\mathcal{D})$ is complete in its pop uniformity (2.2(6)). Therefore, $A(\mathcal{D}) = \mathsf{C}(A(\mathcal{D})) = \mathsf{C}(\mathcal{D})$ by Corollary 3.6 and Proposition 3.10. It is obvious that $A(\mathcal{D}) = \mathcal{D}$ if and only if $\mathcal{D}$ is approximating. □

Suppose that $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ is an approximating $(I, \leq)$-pop such that $(D, \leq)$ is a dcpo or even a complete lattice and each p_i preserves suprema of all (directed) subsets of D. Then, due to the previous proposition, $\mathsf{C}(\mathcal{D}) = \mathcal{D}$. This does not hold anymore if the projections p_i do not preserve suprema of directed sets, cf. the example given in [9] after Prop. 3.1.

Let $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ and $\mathcal{E} = (E, \leq, (q_i)_{i\in I})$ be $(I, \leq)$-pop's such that $(q_i)_{i\in I}$ is Abelian. Let $[D \longrightarrow E]^{\text{hom}} := \{f : D \longrightarrow E \mid f \text{ is a homomorphism}\}$. For all $i \in I$ define $Q_i : [D \longrightarrow E]^{\text{hom}} \longrightarrow [D \longrightarrow E]^{\text{hom}}$ by $Q_i(f) := q_i \circ f$. Notice that Q_i is well-defined because $q_j \circ Q_i(f) = q_j \circ q_i \circ f = q_i \circ q_j \circ f = q_i \circ f \circ q_j = Q_i(f) \circ p_j$. Clearly, Q_i is a projection. Endow $[D \longrightarrow E]^{\text{hom}}$ with the pointwise order. Then $[\mathcal{D} \longrightarrow \mathcal{E}]^{\text{hom}} := ([D \longrightarrow E]^{\text{hom}}, \leq, (Q_i)_{i\in I})$ is an $(I, \leq)$-pop. It is easy to see that $[\mathcal{D} \longrightarrow \mathcal{E}]^{\text{hom}}$ is approximating if and only if $\mathcal{E}$ is approximating.

If, moreover, q_i is Scott-continuous for all $i \in I$, then let $[D \longrightarrow E]^{\mathrm{Shom}} := \{f : D \longrightarrow E \mid f \text{ is a Scott-continuous homomorphism}\}$ and observe that $Q_i(f) \in [D \longrightarrow E]^{\mathrm{Shom}}$ for all $f \in [D \longrightarrow E]^{\mathrm{Shom}}$ and all $i \in I$. Hence, $[\mathcal{D} \longrightarrow \mathcal{E}]^{\mathrm{Shom}} := ([D \longrightarrow E]^{\mathrm{Shom}}, \leq, (Q_i)_{i \in I})$ is also an $(I, \leq)$-pop. (By abuse of language we write $Q_i : [D \longrightarrow E]^{\mathrm{Shom}} \longrightarrow [D \longrightarrow E]^{\mathrm{Shom}}$.) Clearly, Q_i is Scott-continuous for all $i \in I$.

Now we calculate the pop completions of $[\mathcal{D} \longrightarrow \mathcal{E}]^{\mathrm{hom}}$ and $[\mathcal{D} \longrightarrow \mathcal{E}]^{\mathrm{Shom}}$, respectively:

Theorem 3.14 *Let $\mathcal{D} = (D, \leq, (p_i)_{i \in I})$ be an $(I, \leq)$-pop and let $\mathcal{E} = (E, \leq, (q_i)_{i \in I})$ be an approximating $(I, \leq)$-pop with Abelian projection net. Let $\mathsf{C}(\mathcal{D}) = (\mathsf{C}(D), \widehat{\leq}, (\widehat{p_i})_{i \in I})$ and $\mathsf{C}(\mathcal{E}) = (\mathsf{C}(E), \widehat{\leq}, (\widehat{q_i})_{i \in I})$ be the pop completions of $\mathcal{D}$ and $\mathcal{E}$, respectively. Then*

(1) *$[\mathcal{D} \longrightarrow \mathsf{C}(\mathcal{E})]^{\mathrm{hom}}$, $[\mathsf{C}(\mathcal{D}) \longrightarrow \mathsf{C}(\mathcal{E})]^{\mathrm{hom}}$, and the pop completion of $[\mathcal{D} \longrightarrow \mathcal{E}]^{\mathrm{hom}}$ are pairwise pop isomorphic.*

(2) *If p_i and q_i are Scott-continuous for all $i \in I$, then an analogous result holds for the pop completion of $[\mathcal{D} \longrightarrow \mathcal{E}]^{\mathrm{Shom}}$.*

Proof. We only show (1); (2) is proven similarly. (Use Theorem 3.2(6)(b) and Proposition 3.8 for (2).) As usual, we view $\mathcal{E}$ as a subpop of $\mathsf{C}(\mathcal{E})$. Then, clearly, $[\mathcal{D} \longrightarrow \mathcal{E}]^{\mathrm{hom}}$ can be seen as a subpop of $[\mathcal{D} \longrightarrow \mathsf{C}(\mathcal{E})]^{\mathrm{hom}}$. For all $i \in I$ and all $g \in [D \longrightarrow \mathsf{C}(E)]^{\mathrm{hom}}$ we have $\widehat{q_i} \circ g \in [D \longrightarrow E]^{\mathrm{hom}}$; that is, $[\mathcal{D} \longrightarrow \mathcal{E}]^{\mathrm{hom}}$ is a full subpop of $[\mathcal{D} \longrightarrow \mathsf{C}(\mathcal{E})]^{\mathrm{hom}}$.

In order to show that $[\mathcal{D} \longrightarrow \mathsf{C}(\mathcal{E})]^{\mathrm{hom}}$ is complete, let $(f_n)_{n \in N}$ be a Cauchy net in $[\mathcal{D} \longrightarrow \mathsf{C}(\mathcal{E})]^{\mathrm{hom}}$. Let $i \in I$ and let $n_i \in N$ such that $q_i \circ f_n = q_i \circ f_{n_i}$ for all $n \geq n_i$. Let $d \in D$. Since $q_i(f_n(d)) = q_i(f_{n_i}(d))$ for all $n \geq n_i$, we obtain $(f_n(d))_{n \in N}$ to be a Cauchy net in $\mathsf{C}(\mathcal{E})$. Hence, it is convergent. Let $f(d) := \lim_{n \in N} f_n(d)$. Then $q_i(f(d)) = q_i(\lim_{n \in N} f_n(d)) = \lim_{n \in N} q_i(f_n(d)) = \lim_{n \in N} q_i(f_{n_i}(d)) = q_i(f_{n_i}(d))$. We conclude that $q_i(f_n(d)) = q_i(f(d))$ for all $n \geq n_i$ and all $d \in D$, i.e. $Q_i(f_n) = Q_i(f)$ for all $n \geq n_i$. Hence, $(f_n)_{n \in N}$ converges to f with respect to the pop topology. Moreover, $q_i(f(d)) = q_i(\lim_{n \in N} f_n(d)) = \lim_{n \in N} q_i(f_n(d)) = \lim_{n \in N} f_n(p_i(d)) = f(p_i(d))$; that is, f commutes with all projections. Let $d \leq e$. Then $f_n(d) \mathrel{\widehat{\leq}} f_n(e)$ for all $n \in N$ and thus $f(d) \mathrel{\widehat{\leq}} f(e)$ because $\widehat{\leq}$ is closed. Therefore, f is a homomorphism. As a consequence, $[\mathcal{D} \longrightarrow \mathsf{C}(\mathcal{E})]^{\mathrm{hom}}$ is complete.

Since $\mathsf{C}(\mathcal{E})$ is approximating, $[\mathcal{D} \longrightarrow \mathsf{C}(\mathcal{E})]^{\mathrm{hom}}$ is also approximating. We infer from Corollary 3.6 and Proposition 3.10 that $[\mathcal{D} \longrightarrow \mathsf{C}(\mathcal{E})]^{\mathrm{hom}}$ is the pop completion of $[\mathcal{D} \longrightarrow \mathcal{E}]^{\mathrm{hom}}$.

Let $\psi : D \longrightarrow \mathsf{C}(D)$ be the canonical homomorphism from $\mathcal{D}$ to $\mathsf{C}(\mathcal{D})$. For each $g \in [D \longrightarrow \mathsf{C}(E)]^{\mathrm{hom}}$ let $\overline{g}$ be the unique element of $[\mathsf{C}(D) \longrightarrow \mathsf{C}(E)]^{\mathrm{hom}}$ with $\overline{g} \circ \psi = g$ (Theorem 3.2(1)). Clearly, if $g_1, g_2 \in [D \longrightarrow \mathsf{C}(E)]^{\mathrm{hom}}$ with $g_1 \mathrel{\widehat{\leq}} g_2$, then $\overline{g_1}(\psi(d)) \mathrel{\widehat{\leq}} \overline{g_2}(\psi(d))$ for all $d \in D$. As $\psi[D]$ is dense in $\mathsf{C}(D)$ and $\mathsf{C}(\mathcal{E})$ is approximating, we deduce that $\overline{g_1} \mathrel{\widehat{\leq}} \overline{g_2}$. Let $i \in I$. Then $(\widehat{q_i} \circ \overline{g}) \circ \psi = \widehat{q_i} \circ g$. By uniqueness, $\widehat{q_i} \circ \overline{g} = \overline{\widehat{q_i} \circ g}$. We conclude that the mapping $g \longmapsto \overline{g}$

is a homomorphism from $[\mathcal{D} \longrightarrow \mathsf{C}(\mathcal{E})]^{\text{hom}}$ to $[\mathsf{C}(\mathcal{D}) \longrightarrow \mathsf{C}(\mathcal{E})]^{\text{hom}}$. If $\overline{g_1} \mathrel{\widehat{\leq}} \overline{g_2}$ for any $g_1, g_2 \in [D \longrightarrow \mathsf{C}(E)]^{\text{hom}}$, then $g_1 = \overline{g_1} \circ \psi \mathrel{\widehat{\leq}} \overline{g_2} \circ \psi = g_2$; whence we have an embedding. Obviously, if $h \in [\mathsf{C}(D) \longrightarrow \mathsf{C}(E)]^{\text{hom}}$ and $g := h \circ \psi$, then we have $h \circ \psi = g = \overline{g} \circ \psi$ and thus $h = \overline{g}$ by uniqueness. Consequently, $[\mathcal{D} \longrightarrow \mathsf{C}(\mathcal{E})]^{\text{hom}}$ and $[\mathsf{C}(\mathcal{D}) \longrightarrow \mathsf{C}(\mathcal{E})]^{\text{hom}}$ are isomorphic. $\square$

Remark 3.15 Let $\mathcal{D} = (D, \leq, (p_i)_{i \in I})$ and $\mathcal{E} = (E, \leq, (q_i)_{i \in I})$ be $(I, \leq)$-pop's such that $\mathcal{E}$ is approximating, $(E, \leq)$ is a dcpo, $(q_i)_{i \in I}$ is Abelian, and q_i is Scott-continuous for all $i \in I$. Then it is easy to see that $[D \longrightarrow E]^{\text{Shom}}$ is a dcpo with respect to the pointwise order. The supremum of any directed subset of $[D \longrightarrow E]^{\text{Shom}}$ is taken pointwise. Moreover, the projection net of $[\mathcal{D} \longrightarrow \mathcal{E}]^{\text{Shom}}$ consists of Scott-continuous projections. As a consequence, $[\mathcal{D} \longrightarrow \mathcal{E}]^{\text{Shom}}$ is complete in its pop uniformity by 2.2(6) and thus coincides with its pop completion (Corollary 3.6).

4 Domain completion and ideal completion

In general the underlying poset of the pop completion is not a dcpo. This section deals with another completion of a given $(I, \leq)$-pop: the "domain completion". We show that each $(I, \leq)$-pop $\mathcal{D}$ admits an approximating $(I, \leq)$-pop $\mathsf{J}(\mathcal{D})$ whose partial order is a directed complete partial order and whose projection net consists of Scott-continuous projections. The completion $\mathsf{J}(\mathcal{D})$ also satisfies a universal property with regard to the extension of homomorphisms. Similarly to the pop completion we can pop embed $\mathcal{D}$ into $\mathsf{J}(\mathcal{D})$ provided that $\mathcal{D}$ is approximating.

Since the ideal completion of a suitable subset of D will be the candidate for the domain completion, we need some facts on the ideal completion of a poset. We cite the following well-known statement from MARKOWSKY and ROSEN [13], Theorem 2.7 (cf. also LAWSON [11], Section I; ABRAMSKY and JUNG [1], Prop. 2.2.24.):

Proposition 4.1 *Let $(D, \leq)$ be a poset and let $(\mathsf{Id}(D), \subseteq)$ be the ideal completion of $(D, \leq)$. Let $(E, \leq)$ be a dcpo and let $f : D \longrightarrow E$ be a monotone mapping. Let $\varphi_D : D \longrightarrow \mathsf{Id}(D)$, $\varphi_D(d) := d\!\downarrow$, be the canonical order embedding of $(D, \leq)$ into $(\mathsf{Id}(D), \subseteq)$. Then $f^* : \mathsf{Id}(D) \longrightarrow E$, defined by $f^*(A) := \sup f[A]$, is the unique Scott-continuous mapping such that $f^* \circ \varphi_D = f$.* $\square$

For any $(I, \leq)$-pop $(D, \leq, (p_i)_{i \in I})$ we endow the ideal completion $(\mathsf{Id}(D), \subseteq)$ of $(D, \leq)$ with a natural $(I, \leq)$-pop structure:

Proposition 4.2 *Let $\mathcal{D} = (D, \leq, (p_i)_{i \in I})$ be an $(I, \leq)$-pop. Let $(\mathsf{Id}(D), \subseteq)$ be the ideal completion of $(D, \leq)$.*

(1) *For all $i \in I$ define a mapping $\widetilde{p_i} : \mathsf{Id}(D) \longrightarrow \mathsf{Id}(D)$ by $\widetilde{p_i}(A) := p_i[A]\!\downarrow$ for all $A \in \mathsf{Id}(D)$. Then $\mathsf{Id}(\mathcal{D}) := (\mathsf{Id}(D), \subseteq, (\widetilde{p_i})_{i \in I})$ is a complete $(I, \leq)$-pop with Scott-continuous projections $\widetilde{p_i}$.*

(2) *The mapping* $\varphi_D : D \longrightarrow \mathsf{Id}(D)$, *defined by* $\varphi_D(d) := d{\downarrow}$, *is a pop embedding of* $\mathcal{D}$ *into* $\mathsf{Id}(\mathcal{D})$.

(3) *Let* $(E, \leq, (q_i)_{i\in I})$ *be an* $(I, \leq)$*-pop such that* $(E, \leq)$ *is a dcpo and* q_i *is Scott-continuous for all* $i \in I$. *Let* $f : D \longrightarrow E$ *be a homomorphism and define* $f^* : \mathsf{Id}(D) \longrightarrow E$ *by* $f^*(A) := \sup f[A]$. *Then* f^* *is the unique Scott-continuous homomorphism with* $f^* \circ \varphi_D = f$.

(4) $\mathsf{Id}(\mathcal{D})$ *is approximating if and only if* $D = \bigcup_{i\in I} p_i[D]$.

Proof. (1) Let $i \in I$. Clearly, $\widetilde{p_i}$ is Scott-continuous. Let $A \in \mathsf{Id}(D)$ and let $j \in I$ with $i \leq j$. Then $p_i[p_i[A]{\downarrow}]{\downarrow} = p_i[p_i[A]]{\downarrow} = p_i[A]{\downarrow}$; whence $\widetilde{p_i}$ is idempotent. Obviously, $p_i[A]{\downarrow} \subseteq A$. As $p_i[A]{\downarrow} \subseteq p_j[A]{\downarrow}$, we have that $\mathsf{Id}(\mathcal{D})$ is an $(I, \leq)$-pop. By recalling that $(\mathsf{Id}(D), \subseteq)$ is an (algebraic) dcpo, we infer that $\mathsf{Id}(\mathcal{D})$ is complete with respect to its pop uniformity by 2.2(6).

(2) We know that φ_D is an order embedding of $(D, \leq)$ into $(\mathsf{Id}(D), \subseteq)$. Moreover, $p_i(d){\downarrow} = p_i[d{\downarrow}]{\downarrow} = \widetilde{p_i}(d{\downarrow})$ for all $i \in I$ and all $d \in D$.

(3) We already know that f^* is the unique Scott-continuous mapping such that $f^* \circ \varphi_D = f$ (Proposition 4.1). Let $A \in \mathsf{Id}(D)$ and let $i \in I$. Then $q_i(f^*(A)) = q_i(\sup f[A]) = \sup q_i[f[A]] = \sup f[p_i[A]] = \sup f[p_i[A]{\downarrow}] = f^*(\widetilde{p_i}(A))$.

(4) Observe that $\mathsf{Id}(\mathcal{D})$ is approximating if and only if $A = \bigcup_{i\in I} \widetilde{p_i}(A)$ for all $A \in \mathsf{Id}(D)$. Consequently, if $\mathsf{Id}(\mathcal{D})$ is approximating, then $d{\downarrow} = \bigcup_{i\in I} p_i[d{\downarrow}]{\downarrow} = \bigcup_{i\in I} p_i(d){\downarrow}$ for all $d \in D$. Thus, there is some $i \in I$ with $d \leq p_i(d)$, i.e. $d = p_i(d)$. Conversely, if $D = \bigcup_{i\in I} p_i[D]$ and $A \in \mathsf{Id}(D)$, then we certainly obtain $A = \bigcup_{i\in I} p_i[A]{\downarrow}$. □

Definition 4.3 We call $\mathsf{Id}(\mathcal{D})$ the *ideal completion* of $\mathcal{D}$.

Next, we formulate the existence and uniqueness theorem of the "domain completion". Observe the analogy to Theorem 3.2.

Theorem 4.4 *Let* $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ *be an* $(I, \leq)$*-pop.*

(1) *There exist an approximating* $(I, \leq)$*-pop* $\mathsf{J}(\mathcal{D}) = (\mathsf{J}(D), \widetilde{\leq}, (\widetilde{p_i})_{i\in I})$ *with* $(\mathsf{J}(D), \widetilde{\leq})$ *a dcpo and* $\widetilde{p_i}$ *Scott-continuous for all* $i \in I$ *and a pop homomorphism* $\iota : D \longrightarrow \mathsf{J}(D)$ *with the following universal property:*
For any approximating $(I, \leq)$*-pop* $(E, \leq, (q_i)_{i\in I})$ *with* $(E, \leq)$ *a dcpo and* q_i *Scott-continuous for all* $i \in I$ *and any homomorphism* $f : D \longrightarrow E$ *there is a unique Scott-continuous homomorphism* $f^* : \mathsf{J}(D) \longrightarrow E$ *with* $f^* \circ \iota = f$.

(2) *Let* $\mathcal{D}' = (D', \leq', (p'_i)_{i\in I})$ *be an approximating* $(I, \leq)$*-pop such that* $(D', \leq')$ *is a dcpo and* p'_i *is Scott-continuous for all* $i \in I$. *Let* $\phi : D \longrightarrow D'$ *be a homomorphism such that* $(\mathcal{D}', \phi)$ *fulfils the universal property of (1). Then there exists a unique pop isomorphism* $\Phi : \mathsf{J}(D) \longrightarrow D'$ *with* $\Phi \circ \iota = \phi$.

(3) $(\mathsf{J}(D), \widetilde{\leq})$ *is algebraic with* $K(\mathsf{J}(D)) = \iota[\bigcup_{i\in I} p_i[D]] = \bigcup_{i\in I} \widetilde{p_i}[\iota[D]]$. *Moreover,* $\mathsf{J}(\mathcal{D})$ *is complete in its pop uniformity.*

(4) (a) *Let* $d, e \in D$*. Then* $\iota(d) \mathrel{\widetilde{\leq}} \iota(e)$ *if and only if* $p_i(d) \leq p_i(e)$ *for all* $i \in I$*.*

(b) $\iota|_{p_i[D]}$ *is an order embedding of* $p_i[D]$ *into* $\widetilde{p_i}[\mathsf{J}(D)]$ *for all* $i \in I$*.*

(5) ι *is a pop embedding of* $\mathcal{D}$ *into* $\mathsf{J}(\mathcal{D})$ *if and only if* $\mathcal{D}$ *is approximating.*

(6) $\mathsf{J}(\mathcal{D})$ *is pop isomorphic to* $\mathsf{Id}(\bigcup_{i\in I} p_i[\mathcal{D}])$*. More precisely, there is a unique pop isomorphism* $\Psi : \mathsf{J}(D) \longrightarrow \mathsf{Id}(\bigcup_{i\in I} p_i[\mathcal{D}])$ *with* $(\Psi \circ \iota)(d) = \bigcup_{i\in I} p_i(d){\downarrow}$ *for all* $d \in D$*.*

Proof. Let $\mathsf{J}(\mathcal{D}) := \mathsf{Id}(\bigcup_{i\in I} p_i[\mathcal{D}])$. Then $\mathsf{J}(\mathcal{D})$ is approximating by Proposition 4.2(4). Let $\iota : D \longrightarrow \mathsf{J}(D)$ be defined by $\iota(d) := \bigcup_{i\in I} p_i(d){\downarrow}$. Clearly, ι is (well-defined and) monotone. Let $i \in I$ and let $d \in D$. Then $\widetilde{p_i}(\iota(d)) = p_i[\bigcup_{j\in I} p_j(d){\downarrow}]{\downarrow} = \bigcup_{j\geq i} p_i(p_j(d)){\downarrow} = p_i(d){\downarrow} = \iota(p_i(d))$. Therefore, ι is a homomorphism.

Let $\iota(d) \mathrel{\widetilde{\leq}} \iota(e)$ and let $i \in I$. By definition of ι we obtain $p_i(d) \in \bigcup_{j\geq i} p_j(e){\downarrow}$, whence $p_i(d) \leq p_j(e)$ for some $j \geq i$. This implies $p_i(d) \leq p_i(p_j(e)) = p_i(e)$. Conversely, if $p_i(d) \leq p_i(e)$ for all $i \in I$, then $\widetilde{p_i}(\iota(d)) = \iota(p_i(d)) \mathrel{\widetilde{\leq}} \iota(p_i(e)) = \widetilde{p_i}(\iota(e))$ for all $i \in I$. As $\mathsf{J}(\mathcal{D})$ is approximating, we obtain $\iota(d) \mathrel{\widetilde{\leq}} \iota(e)$. This proves (4).

Let ι be a pop embedding and let $d, e \in D$ with $p_i(d) \leq e$ for all $i \in I$. Then $p_i(d){\downarrow} = \iota(p_i(d)) \subseteq \iota(e) = \bigcup_{j\in I} p_j(e){\downarrow}$ for all $i \in I$; whence $\bigcup_{i\in I} p_i(d){\downarrow} \subseteq \bigcup_{i\in I} p_i(e){\downarrow}$. As ι is an order embedding, we conclude $d \leq e$. Thus, $d = \sup_{i\in I} p_i(d)$ and $\mathcal{D}$ is approximating. Conversely, if $\mathcal{D}$ is approximating, then ι is an order embedding by (4)(a) or by [9], Prop. 4.7. This shows (5).

Now we prove the universal property. Let $\mathcal{E} = (E, \leq, (q_i)_{i\in I})$ and f be as in (1). Let f^* be as in Proposition 4.2(3) applied to $\mathsf{Id}(\bigcup_{i\in I} p_i[D])$, i.e. $f^*(A) := \sup f[A]$ for all $A \in \mathsf{Id}(\bigcup_{i\in I} p_i[D])$. Then f^* is a Scott-continuous homomorphism (4.2(3)). Let $i \in I$ and let $d \in D$. Scott-continuity implies that $f^*(\bigcup_{j\geq i} p_j(d){\downarrow}) = \sup_{j\geq i} f^*(p_j(d){\downarrow}) = \sup_{j\geq i} q_j(f^*(d{\downarrow})) = \sup_{j\geq i} q_j(f(d))$. Hence $q_i(f^*(\iota(d))) = q_i(\sup_{j\geq i} q_j(f(d))) = q_i(f(d))$. As $\mathcal{E}$ is approximating, we conclude that $f^*(\iota(d)) = f(d)$.

Let $g : \mathsf{J}(D) \longrightarrow E$ be a Scott-continuous homomorphism with $g \circ \iota = f$. Since $\iota|_{\bigcup_{i\in I} p_i[D]}$ coincides with the canonical embedding $d \longmapsto d{\downarrow}$ of $\bigcup_{i\in I} p_i[D]$ into $\mathsf{Id}(\bigcup_{i\in I} p_i[D]) = \mathsf{J}(D)$, we infer that $f^* = g$ (Proposition 4.2(3)). This proves (1).

(2) follows from the universal property and (6) holds by definition and (2). It is well-known that $(\mathsf{J}(D), \widetilde{\leq})$ is algebraic with $K(\mathsf{J}(D)) = \{p_i(d){\downarrow} \mid i \in I, d \in D\} = \{\iota(p_i(d)) \mid i \in I,\ d \in D\} = \iota[\bigcup_{i\in I} p_i[D]] = \bigcup_{i\in I} \widetilde{p_i}[\iota[D]]$. Finally, $\mathsf{J}(\mathcal{D})$ is complete in its pop uniformity by Proposition 4.2(1); whence (3) is true. $\square$

Definition 4.5 We call $\mathsf{J}(\mathcal{D})$ the *domain completion* of $\mathcal{D}$. By abuse of language, we call the mapping $\iota : D \longrightarrow \mathsf{J}(D)$ the *canonical homomorphism*, too.

The advantage of the domain completion $\mathsf{J}(\mathcal{D})$ as opposed to the pop completion $\mathsf{C}(\mathcal{D})$ is that $\mathsf{J}(\mathcal{D})$ always yields an algebraic domain. However, $\mathsf{J}(\mathcal{D})$ has some disadvantages compared to $\mathsf{C}(\mathcal{D})$. For instance, $\iota[D]$ need not induce a full subpop of $\mathsf{J}(\mathcal{D})$ and $\iota|_{p_i[D]}$ need not be an isomorphism from $p_i[D]$ onto $\widetilde{p_i}[\mathsf{J}(D)]$. In Theorem 5.6 we will investigate pop's where these problems do not occur.

Moreover, $\mathsf{J}(\mathsf{J}(\mathcal{D}))$ need not be isomorphic to $\mathsf{J}(\mathcal{D})$ whereas $\mathsf{C}(\mathsf{C}(\mathcal{D}))$ is always isomorphic to $\mathsf{C}(\mathcal{D})$ (cf. Corollary 3.6). For example, let $\mathcal{D} = (\mathbb{N}_0, \leq, (\mathsf{id})_{i\in I})$. Then, with the abbreviation $\omega = (\mathbb{N}_0, \leq)$ and the usual ordinal number arithmetic, the underlying posets of $\mathsf{J}(\mathcal{D})$ and $\mathsf{J}(\mathsf{J}(\mathcal{D}))$ are $\omega + 1$ and $\omega + 2$, respectively.

In contrast to the pop completion, the canonical homomorphism of the domain completion need not preserve suprema. Instead, we have the following result, whose proof is left to the interested reader.

Proposition 4.6 *Let $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ be an $(I, \leq)$-pop with domain completion $\mathsf{J}(\mathcal{D}) = (\mathsf{J}(D), \widetilde{\leq}, (\widetilde{p_i})_{i\in I})$. Let $\iota : D \longrightarrow \mathsf{J}(D)$ be the canonical homomorphism. Then the following are equivalent:*

- (i) *ι is Scott-continuous.*
- (ii) *$p_i[D] \subseteq K(D)$ for all $i \in I$.*
- (iii) *For all $i \in I$ we have that p_i is Scott-continuous and $p_i[A]$ has a greatest element for all directed subsets $A \subseteq D$ that have a supremum.*
- (iv) *For any approximating $(I, \leq)$-pop $(E, \leq, (q_i)_{i\in I})$ with $(E, \leq)$ a dcpo and q_i Scott-continuous for all $i \in I$ and any homomorphism $f : D \longrightarrow E$ we have that f is Scott-continuous.* $\square$

Remark 4.7 Let $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ be an approximating $(I, \leq)$-pop. Corollary 3.6 tells us that the canonical homomorphism $\psi : D \longrightarrow \mathsf{C}(D)$ is an isomorphism if and only if $\mathcal{D}$ is complete in its pop uniformity. The question arises when the canonical homomorphism $\iota : D \longrightarrow \mathsf{J}(D)$ is surjective. The answer is given in Theorem 4.8 in [9] (together with Theorem 4.4(5) above): ι is an isomorphism if and only if $(D, \leq)$ is a dcpo with $K(D) = \bigcup_{i\in I} p_i[D]$. In this case $\mathcal{D}$ is complete in its pop uniformity; whence $\mathcal{D}$, $\mathsf{J}(\mathcal{D})$, and $\mathsf{C}(\mathcal{D})$ are pairwise isomorphic. A more detailed analysis when $\mathsf{J}(\mathcal{D})$ and $\mathsf{C}(\mathcal{D})$ are pop isomorphic is given in Section 5.

Remark 4.8 Analogously to Proposition 3.10 we have the following. Let $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ be an $(I, \leq)$-pop and let $\mathcal{X}$ be a full subpop of $\mathcal{D}$. Then the domain completions $\mathsf{J}(\mathcal{X})$ and $\mathsf{J}(\mathcal{D})$ are pop isomorphic. In particular, $\mathsf{J}(\mathcal{D})$ is the domain completion of the subpop $\bigcup_{i\in I} p_i[\mathcal{D}]$.

5 Comparison of the completions

It is natural to ask how these completions are related. This is the subject of the present section. We demonstrate how the pop completion embeds into the domain completion and give some criteria to get equality. Finally, we show when the completions yield a bifinite domain.

We need the following well-known lemma to extend monotone mappings between posets to mappings between their respective ideal completions.

Lemma 5.1 *Let $(D,\leq)$ and $(E,\leq)$ be posets and let $f : D \longrightarrow E$ be monotone. Define $\widetilde{f} : \mathsf{Id}(D) \longrightarrow \mathsf{Id}(E)$ by $\widetilde{f}(A) := f[A]{\downarrow}$. Then $\widetilde{f}$ is a Scott-continuous mapping. In addition, we have the following:*

(1) *f is an order embedding if and only if $\widetilde{f}$ is an order embedding.*

(2) *f is an order isomorphism if and only if $\widetilde{f}$ is an order isomorphism. In this case, $\widetilde{f}^{-1}(C) = f^{-1}[C]{\downarrow} = \widetilde{f^{-1}}(C)$ for all $C \in \mathsf{Id}(E)$.* □

Lemma 5.2 *Let $\mathcal{D} = (D,\leq,(p_i)_{i\in I})$ and $\mathcal{E} = (E,\leq,(q_i)_{i\in I})$ be $(I,\leq)$-pop's and let $f : D \longrightarrow E$ be a monotone mapping. Let $\widetilde{f} : \mathsf{Id}(D) \longrightarrow \mathsf{Id}(E)$ be the Scott-continuous map defined by $\widetilde{f}(A) := f[A]{\downarrow}$. Then f is a homomorphism if and only if $\widetilde{f}$ is a homomorphism.*

Proof (Sketch). The "only-if"-part follows from Proposition 4.2(3) because $\widetilde{f} = (\varphi_E \circ f)^*$, where φ_E is the canonical order embedding $e \longmapsto e{\downarrow}$ of $(E,\leq)$ into $(\mathsf{Id}(E),\subseteq)$. To prove the converse, use principal ideals. □

For any poset $(D,\leq)$ let $\varphi_D : D \longrightarrow \mathsf{Id}(D)$, $d \longmapsto d{\downarrow}$, be the canonical order embedding.

Proposition 5.3 *Let $\mathcal{D} = (D,\leq,(p_i)_{i\in I})$ be an $(I,\leq)$-pop. Let $\psi : D \longrightarrow \mathsf{C}(D)$ be the canonical homomorphism. Then we have the following:*

(1) *$\mathsf{J}(\mathcal{D})$ is isomorphic to the subpop of $\mathsf{Id}(\mathcal{D})$ induced by $\{\bigcup_{i\in I}\widetilde{p_i}(A) \mid A \in \mathsf{Id}(D)\}$.*

(2) *$\mathsf{Id}(\mathcal{D})$ is isomorphic to $\mathsf{J}(\mathcal{D})$ if and only if $D = \bigcup_{i\in I} p_i[D]$.*

(3) *The mapping $\widetilde{\psi} : \mathsf{Id}(D) \longrightarrow \mathsf{Id}(\mathsf{C}(D))$, defined by $\widetilde{\psi}(A) := \psi[A]{\downarrow}$, is a Scott-continuous pop homomorphism. It is a pop embedding if and only if $\mathcal{D}$ is approximating. Furthermore, $\varphi_{\mathsf{C}(D)} \circ \psi = \widetilde{\psi} \circ \varphi_D$. Hence, we have commutative diagrams as in Figure 1.*

Proof (Sketch). (1) Let $A(\mathsf{Id}(D)) := \{\bigcup_{i\in I}\widetilde{p_i}(A) \mid A \in \mathsf{Id}(D)\}$. Since the inclusion map $\mathsf{id}_{\bigcup_{i\in I} p_i[D],D}$ is a pop embedding of $\bigcup_{i\in I} p_i[\mathcal{D}]$ into $\mathcal{D}$, we deduce that the mapping $\phi : \mathsf{Id}(\bigcup_{i\in I} p_i[D]) \longrightarrow \mathsf{Id}(D)$, defined by $\phi(A) := A{\downarrow}_D$, is also a pop embedding (Lemmas 5.1 and 5.2). We leave it to the reader to show that ϕ is in fact a pop isomorphism from $\mathsf{J}(\mathcal{D}) = \mathsf{Id}(\bigcup_{i\in I} p_i[\mathcal{D}])$ onto $A(\mathsf{Id}(\mathcal{D}))$.

(2) If $\mathsf{Id}(\mathcal{D})$ is isomorphic to $\mathsf{J}(\mathcal{D})$, then $\mathsf{Id}(\mathcal{D})$ is approximating and thus $D = \bigcup_{i\in I} p_i[D]$ by Proposition 4.2(4). The converse is clear by Theorem 4.4(6).

(3) The mapping $\widetilde{\psi}$ is a Scott-continuous homomorphism by Lemma 5.2. Furthermore, $\mathcal{D}$ is approximating if and only if ψ is a pop embedding if and only if $\widetilde{\psi}$ is a pop embedding (cf. Theorem 3.2(5) and Lemma 5.1).

$$\begin{tikzcd} \mathsf{C}(D) \arrow[rr, hook, "\varphi_{\mathsf{C}(D)}"] & & \mathsf{Id}(\mathsf{C}(D)) \\ & = & \\ D \arrow[u, "\psi"] \arrow[rr, hook, "\varphi_D"'] & & \mathsf{Id}(D) \arrow[u, "\widetilde{\psi}"'] \\ & = & \\ \bigcup_{i\in I} p_i[D] \arrow[u, hook] \arrow[rr, hook, "\varphi_{\bigcup_{i\in I} p_i[D]}"'] & & \mathsf{Id}(\bigcup_{i\in I} p_i[D]) = \mathsf{J}(D) \arrow[u, hook] \end{tikzcd}$$

Fig. 1. Canonical embeddings.

Let $d \in D$. Then $\varphi_{\mathsf{C}(D)}(\psi(d)) = \psi(d){\downarrow} = \psi[d{\downarrow}]{\downarrow} = \widetilde{\psi}(d{\downarrow}) = \widetilde{\psi}(\varphi_D(d))$. $\square$

At first sight, the following result might surprise the reader – especially in consideration of the previous proposition. It states that the pop completion $\mathsf{C}(\mathcal{D})$ can be embedded into $\mathsf{J}(\mathcal{D})$ and thus into $\mathsf{Id}(\mathcal{D})$. Moreover, it turns out that the domain completion $\mathsf{J}(\mathcal{D})$ can actually be obtained as the pop completion of $\mathsf{Id}(\mathcal{D})$:

Theorem 5.4 *Let $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ be an $(I, \leq)$-pop. Let $\psi : D \longrightarrow \mathsf{C}(D)$ and $\iota : D \longrightarrow \mathsf{J}(D)$ be the canonical homomorphisms.*

(1) *The following $(I, \leq)$-pop's are pairwise pop isomorphic: $\mathsf{J}(\mathcal{D})$, $\mathsf{J}(\mathsf{C}(\mathcal{D}))$, $\mathsf{C}(\mathsf{J}(\mathcal{D}))$, $\mathsf{C}(\mathsf{Id}(\mathcal{D}))$, and $\mathsf{C}(\mathsf{Id}(\mathsf{C}(\mathcal{D})))$.*

(2) *Let $\overline{\iota} : \mathsf{C}(D) \longrightarrow \mathsf{J}(D)$ be the unique homomorphism with $\overline{\iota} \circ \psi = \iota$. Then $\overline{\iota}$ is a pop embedding of $\mathsf{C}(\mathcal{D})$ into $\mathsf{J}(\mathcal{D})$. Assuming $\mathsf{J}(\mathcal{D}) = \mathsf{Id}(\bigcup_{i\in I} p_i[\mathcal{D}])$, we have $\overline{\iota}(\widehat{d}) = \bigcup_{i\in I}(\psi|_{p_i[D]})^{-1}(\widehat{p_i}(\widehat{d})){\downarrow}$ for all $\widehat{d} \in \mathsf{C}(D)$.*

$$\begin{tikzcd} & \mathsf{Id}(D) \\ & \mathsf{J}(D) \arrow[u, hook] \\ D \arrow[ur, "\iota"] \arrow[r, "\psi"'] \arrow[ur, phantom, shift right=4, "="] & \mathsf{C}(D) \arrow[u, hook, "\overline{\iota}"'] \end{tikzcd}$$

Fig. 2. The pop completion embeds into the domain completion.

Proof. (1) By Corollary 3.4, $\psi_0 := \psi|_{\bigcup_{i\in I} p_i[D]}$ is a pop isomorphism from $\bigcup_{i\in I} p_i[\mathcal{D}]$ onto $\bigcup_{i\in I} \widehat{p_i}[\mathsf{C}(\mathcal{D})]$. Therefore, $\widetilde{\psi_0} : \mathsf{J}(D) = \mathsf{Id}(\bigcup_{i\in I} p_i[D]) \longrightarrow$

$\mathsf{Id}(\bigcup_{i\in I}\widehat{p_i}[\mathsf{C}(D)]) = \mathsf{J}(\mathsf{C}(\mathcal{D}))$, $A \longmapsto \psi_0[A]{\downarrow}$, is a pop isomorphism, cf. Lemmas 5.1 and 5.2.

Since $\mathsf{J}(\mathcal{D})$ is complete in its pop uniformity (Theorem 4.4(3)), the canonical homomorphism $\widehat{\psi} : \mathsf{J}(D) \longrightarrow \mathsf{C}(\mathsf{J}(\mathcal{D}))$ is a pop isomorphism by Corollary 3.6.

Let $A(\mathsf{Id}(D)) := \{\bigcup_{i\in I}\widetilde{p_i}(A) \mid A \in \mathsf{Id}(D)\}$. Then we infer from Proposition 3.13 that $\mathsf{C}(\mathsf{Id}(\mathcal{D}))$ is isomorphic to the subpop induced by $A(\mathsf{Id}(D))$. Since the latter is isomorphic to $\mathsf{J}(\mathcal{D})$ by Proposition 5.3(1), we obtain $\mathsf{J}(\mathcal{D})$ to be isomorphic to $\mathsf{C}(\mathsf{Id}(\mathcal{D}))$. When switching from $\mathcal{D}$ to $\mathsf{C}(\mathcal{D})$ we get that $\mathsf{J}(\mathsf{C}(\mathcal{D}))$ is isomorphic to $\mathsf{C}(\mathsf{Id}(\mathsf{C}(\mathcal{D})))$.

(2) Consider again the pop isomorphism $\widetilde{\psi_0} : \mathsf{J}(D) \longrightarrow \mathsf{J}(\mathsf{C}(\mathcal{D}))$ from (1). Its inverse is given by $B \longmapsto \psi_0^{-1}[B]{\downarrow}$ (where $B \in \mathsf{Id}(\bigcup_{i\in I}\widehat{p_i}[\mathsf{C}(D)])$). Let $\widehat{\iota} : \mathsf{C}(D) \longrightarrow \mathsf{Id}(\bigcup_{i\in I}\widehat{p_i}[\mathsf{C}(D)]) = \mathsf{J}(\mathsf{C}(\mathcal{D}))$ be the canonical homomorphism from $\mathsf{C}(D)$ to $\mathsf{J}(\mathsf{C}(\mathcal{D}))$. With regard to Theorem 4.4(5), $\widehat{\iota}$ is a pop embedding. Thus, $\widetilde{\psi_0}^{-1} \circ \widehat{\iota}$ is a pop embedding. Given $d \in D$, we deduce $\widetilde{\psi_0}^{-1}(\widehat{\iota}(\psi(d))) = \psi_0^{-1}[\bigcup_{i\in I}\widehat{p_i}(\psi(d)){\downarrow}]{\downarrow} = \bigcup_{i\in I}\psi_0^{-1}[\psi(p_i(d))]{\downarrow} = \bigcup_{i\in I}p_i(d){\downarrow} = \iota(d)$. Hence, $(\widetilde{\psi_0}^{-1} \circ \widehat{\iota}) \circ \psi = \iota$. By uniqueness we infer $\overline{\iota} = \widetilde{\psi_0}^{-1} \circ \widehat{\iota}$. □

Consider an $(I,\leq)$-pop $(D,\leq,(p_i)_{i\in I})$ with $D = \bigcup_{i\in I}p_i[D]$. Then we know by Proposition 5.3 that $\mathsf{Id}(\mathcal{D}) = \mathsf{J}(\mathcal{D})$. Therefore, the pop embedding $\varphi_D : d \longmapsto d{\downarrow}$ of $\mathcal{D}$ into $\mathsf{Id}(\mathcal{D})$ coincides with the embedding ι and thus extends to a pop embedding of $\mathsf{C}(\mathcal{D})$ into $\mathsf{Id}(\mathcal{D})$ by the previous theorem. Hence, the following corollary generalizes Theorem 3.13 in Majster-Cederbaum and Baier [12], where a similar statement is proven for pointed posets with weight functions. Notice that weight functions induce $(\mathbb{N}_0,\leq)$-pop's whose projections map ideals to ideals, see [10], Section 4.2, for details. Note further that only metric completions and isometric embeddings are considered in [12], whereas we deal with pop completions and pop embeddings.

Corollary 5.5 *Let $\mathcal{D} = (D,\leq,(p_i)_{i\in I})$ be an $(I,\leq)$-pop such that $D = \bigcup_{i\in I}p_i[D]$. Then the pop embedding $\varphi_D : d \longmapsto d{\downarrow}$ of $\mathcal{D}$ into $\mathsf{Id}(\mathcal{D})$ extends uniquely to a pop embedding $\overline{\varphi_D}$ of $\mathsf{C}(\mathcal{D})$ into $\mathsf{Id}(\mathcal{D})$.* □

Next, we investigate when $\mathsf{C}(\mathcal{D})$ coincides with $\mathsf{J}(\mathcal{D})$.

Theorem 5.6 *Let $\mathcal{D} = (D,\leq,(p_i)_{i\in I})$ be an $(I,\leq)$-pop with pop completion $\mathsf{C}(\mathcal{D}) = (\mathsf{C}(D),\widehat{\leq},(\widehat{p_i})_{i\in I})$ and domain completion $\mathsf{J}(\mathcal{D}) = (\mathsf{J}(D),\widetilde{\leq},(\widetilde{p_i})_{i\in I})$. Let $\psi : D \longrightarrow \mathsf{C}(D)$ and $\iota : D \longrightarrow \mathsf{J}(D)$ be the respective canonical homomorphisms. Let $\overline{\iota} : \mathsf{C}(D) \longrightarrow \mathsf{J}(D)$ be the unique homomorphism with $\overline{\iota} \circ \psi = \iota$. Then the following are equivalent:*

(i) *Each monotone net in D is a Cauchy net with respect to the pop uniformity.*

(ii) *For all directed subsets $A \subseteq D$ and for all $i \in I$ we have that $p_i[A]$ has a greatest element.*

(iii) *$\iota[D]$ induces a full subpop of $\mathsf{J}(\mathcal{D})$.*

(iv) $\iota|_{p_i[D]}$ *is an order isomorphism from* $p_i[D]$ *onto* $\widetilde{p_i}[\mathsf{J}(D)]$ *for all* $i \in I$.

(v) $\iota|_{\bigcup_{i\in I} p_i[D]}$ *is a pop isomorphism from* $\bigcup_{i\in I} p_i[\mathcal{D}]$ *onto* $\bigcup_{i\in I} \widetilde{p_i}[\mathsf{J}(\mathcal{D})]$.

(vi) $\overline{\iota}$ *is a pop isomorphism from* $\mathsf{C}(\mathcal{D})$ *onto* $\mathsf{J}(\mathcal{D})$.

In this case we obtain $K(\mathsf{J}(D)) = \bigcup_{i\in I} \widetilde{p_i}[\mathsf{J}(D)]$. *In other words,* $(\mathsf{C}(D), \widehat{\leq})$ *is an algebraic dcpo with* $K(\mathsf{C}(D)) = \bigcup_{i\in I} \widehat{p_i}[\mathsf{C}(D)]$.

Proof. (i)→(ii): Let $A \subseteq D$ be directed. Then $(a)_{a\in A}$ is a monotone net, whence Cauchy by (i). Therefore, for given $i \in I$ we find some $a_i \in A$ such that $p_i(a) = p_i(a_i)$ for all $a \in A$ with $a \geq a_i$. Let $b \in A$ and choose an element $a \in A$ such that $a \geq b, a_i$. Then $p_i(b) \leq p_i(a) = p_i(a_i)$. As a consequence, $p_i[A] \leq p_i(a_i)$ and $p_i(a_i)$ is the greatest element of $p_i[A]$.

(ii)→(vi): Theorem 5.4(2) tells us that $\overline{\iota}$ is a pop embedding. Let $\widehat{\iota} : \mathsf{C}(D) \longrightarrow \mathsf{Id}(\bigcup_{i\in I} \widehat{p_i}[\mathsf{C}(D)]) = \mathsf{J}(\mathsf{C}(\mathcal{D}))$ be the canonical homomorphism from $\mathsf{C}(D)$ into $\mathsf{J}(\mathsf{C}(\mathcal{D}))$. We know that $\widehat{\iota}$ is a pop embedding and that $\overline{\iota} = \phi \circ \widehat{\iota}$ for some pop isomorphism ϕ from $\mathsf{J}(\mathsf{C}(\mathcal{D}))$ onto $\mathsf{J}(\mathcal{D})$, see the proof of Theorem 5.4(2). Therefore, it suffices to show that $\widehat{\iota}$ is surjective. In order to prove this, let $i \in I$ and let $\widehat{A} \subseteq \mathsf{C}(D)$ be directed. Then $\widehat{p_i}[\widehat{A}]$ is a directed subset of $\widehat{p_i}[\mathsf{C}(D)]$. Since $\psi|_{p_i[D]}$ is an order isomorphism from $p_i[D]$ onto $\widehat{p_i}[\mathsf{C}(D)]$ (Theorem 3.2(4)), the set $B := (\psi|_{p_i[D]})^{-1}[\widehat{p_i}[\widehat{A}]]$ is a directed subset of $p_i[D]$; whence $B = p_i[B]$ and $p_i[B{\downarrow}]$ has a greatest element b by condition (ii). Therefore, $\psi(b)$ is the greatest element of $\widehat{p_i}[\widehat{A}]$. We have thus proven that $\widehat{p_i}[\widehat{A}]$ has a greatest element for all directed subset $\widehat{A} \subseteq \mathsf{C}(D)$ and all $i \in I$. Since $\mathsf{C}(\mathcal{D})$ is complete and approximating, we can apply Theorem 4.8 in [9] to obtain that $\widehat{\iota}$ is surjective. Moreover, the same result tells us that $(\mathsf{C}(D), \widehat{\leq})$ is an algebraic dcpo and $K(\mathsf{C}(D)) = \bigcup_{i\in I} \widehat{p_i}[\mathsf{C}(D)]$. This yields $K(\mathsf{J}(D)) = \overline{\iota}[K(\mathsf{C}(D))] = \bigcup_{i\in I} \overline{\iota}[\widehat{p_i}[\mathsf{C}(D)]] = \bigcup_{i\in I} \widetilde{p_i}[\overline{\iota}[\mathsf{C}(D)]] = \bigcup_{i\in I} \widetilde{p_i}[\mathsf{J}(D)]$.

(vi)→(iii): $\psi[D]$ induces a full subpop of $\mathsf{C}(\mathcal{D})$ (Theorem 3.2(3)). By (vi), $\iota[D] = \overline{\iota}[\psi[D]]$ induces a full subpop of $\mathsf{J}(\mathcal{D})$.

(iii)→(iv): Let $i \in I$, We already know that $\iota|_{p_i[D]}$ is an order embedding of $p_i[D]$ into $\widetilde{p_i}[\mathsf{J}(D)]$ (Theorem 4.4(4)(b)). Let $A \in \mathsf{J}(D)$. By (iii) there is an element $d \in D$ such that $\widetilde{p_i}(A) = \iota(d)$; whence $\widetilde{p_i}(A) = \widetilde{p_i}(\iota(d)) = \iota(p_i(d)) \in \iota[p_i[D]]$. Thus, $\iota|_{p_i[D]}$ is surjective.

(iv)→(v): Because of (iv) it is enough to show that $\iota|_{\bigcup_{i\in I} p_i[D]}$ is order reflecting. But this follows as in the proof of Corollary 3.4.

(v)→(i): For the following conclusions let ${\downarrow} := {\downarrow}_{\bigcup_{i\in I} p_i[D]}$. Let $(d_n)_{n\in N}$ be a monotone net in D and let $A := \{p_i(d_n) \mid i \in I,\ n \in N\}{\downarrow}$. Clearly, $A \in \mathsf{Id}(\bigcup_{i\in I} p_i[D])$. Let $i \in I$. Using (v) we find an element $d \in p_i[D]$ such that $\{p_i(d_n) \mid n \in N\}{\downarrow} = p_i[A]{\downarrow} = \widetilde{p_i}(A) = \iota(d) = d{\downarrow}$. Hence, $d = p_i(d_{n_i})$ for some $n_i \in N$. Let $n \geq n_i$. Then $p_i(d_{n_i}) \leq p_i(d_n)$ because $(d_n)_{n\in N}$ is monotone. On the other hand, $p_i(d_n) \leq d = p_i(d_{n_i})$. Therefore, $p_i(d_n) = p_i(d_{n_i})$ for all $n \geq n_i$. We conclude that $(d_n)_{n\in N}$ is Cauchy. □

Corollary 5.7 *Let* $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ *be an* $(I, \leq)$*-pop such that* $p_i[A]$ *has a greatest element for all* $A \in \mathsf{Id}(D)$ *and all* $i \in I$. *Then the pop completion*

$\mathsf{C}(\mathcal{D}) = (\mathsf{C}(D), \widehat{\leq}, (\widehat{p_i})_{i\in I})$ *is isomorphic to the ideal completion* $\mathsf{Id}(\mathcal{D})$ *if and only if* $D = \bigcup_{i\in I} p_i[D]$. *In this case,* $(\mathsf{C}(D), \widehat{\leq})$ *is an algebraic dcpo with* $K(\mathsf{C}(D)) = D$.

Proof. The assertions follow from Theorem 5.6, Proposition 5.3(2), and Proposition 4.2(4). □

Note that the previous corollary generalizes Theorem 3.16 in Majster-Cederbaum and Baier [12]. This theorem states for pointed posets $(D, \leq)$ with a weight function $\|.\|$ that the metric completion is isometric to the ideal completion if $D = \bigcup_{n\in\mathbb{N}_0} q_n^{\|.\|}[D]$ and if $q_n^{\|.\|}[A]$ is finite for all $A \in \mathsf{Id}(D)$, where $q_n^{\|.\|}$ is the projection induced by $\|.\|$ (see [12], cf. also [10], Section 4.2). The metric under consideration is induced by the projections (see also [9], Theorem 2.6).

Finally, we have a look at the completions of $(I, \leq)$-pop's whose projections are image-finite. They are closely related to bifinite domains.

Theorem 5.8 *Let* $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ *be an* $(I, \leq)$*-pop such that* p_i *has finite range for all* $i \in I$. *Then* $\mathsf{C}(\mathcal{D})$ *and* $\mathsf{J}(\mathcal{D})$ *are pop isomorphic. Moreover,* $\mathsf{C}(\mathcal{D})$ *is compact in its pop topology and* $(\mathsf{C}(D), \widehat{\leq})$ *is a bifinite domain with* $K(\mathsf{C}(D)) = \bigcup_{i\in I} \widehat{p_i}[\mathsf{C}(D)]$.

Proof. Let $\psi : D \longrightarrow \mathsf{C}(D)$ be the canonical homomorphism. Since $\widehat{p_i}[\mathsf{C}(D)] = \psi[p_i[D]]$ is finite for all $i \in I$, the pop completion $\mathsf{C}(\mathcal{D})$ is totally bounded in its pop uniformity (Prop. 2.7 in [9]) and thus compact. Moreover, $(\mathsf{C}(D), \widehat{\leq})$ is a bifinite domain by Proposition 3.1 and Theorem 3.2(7). As finite directed sets have a greatest element, we may apply Theorem 5.6 to obtain that $\mathsf{C}(\mathcal{D})$ and $\mathsf{J}(\mathcal{D})$ are pop isomorphic and $K(\mathsf{C}(D)) = \bigcup_{i\in I} \widehat{p_i}[\mathsf{C}(D)]$. □

Theorem 5.8 and Corollary 5.7 yield:

Corollary 5.9 *Let* $\mathcal{D} = (D, \leq, (p_i)_{i\in I})$ *be an* $(I, \leq)$*-pop such that* p_i *has finite range for all* $i \in I$. *Then the pop completion* $\mathsf{C}(\mathcal{D})$ *is isomorphic to the ideal completion* $\mathsf{Id}(\mathcal{D})$ *if and only if* $D = \bigcup_{i\in I} p_i[D]$. *In this case,* $(\mathsf{C}(D), \widehat{\leq})$ *is a bifinite domain and* $K(\mathsf{C}(D)) = D$. □

Acknowledgement

The author wishes to thank A.K. Seda for the invitation to this conference.

References

[1] S. Abramsky and A. Jung. Domain theory. In S. Abramsky, D.M. Gabbay, and T.S.E. Maibaum, editors, *Handbook of Logic in Computer Science*, volume 3, pages 1–168. Clarendon Press, 1994.

[2] C. Baier and M. Majster-Cederbaum. Metric semantics from partial order semantics. *Acta Informatica*, 34:701–735, 1997.

[3] M.M. Bonsangue, F. van Breugel, and J.J.M.M. Rutten. Generalized metric spaces: Completion, topology, and powerdomains via the Yoneda embedding. *Theoretical Computer Science*, 193:1–51, 1998.

[4] N. Bourbaki. *General Topology.* Elements of Mathematics. Hermann, Paris, 1966.

[5] H. Ehrig, F. Parisi-Presicce, P. Boehm, C. Rieckhoff, C. Dimitrovici, and M. Große-Rhode. Combining data type and process specifications using projection algebras. *Theoretical Computer Science*, 71:347–380, 1990.

[6] C. Gunter. *Profinite Solutions for Recursive Domain Equations.* PhD thesis, University of Wisconsin, Madison, 1985.

[7] A. Jung. *Cartesian Closed Categories of Domains*, volume 66 of *CWI Tracts.* Centrum voor Wiskunde en Informatica, Amsterdam, 1989.

[8] J.L. Kelley. *General Topology.* The University Series in Higher Mathematics. Van Nostrand, Princeton, New Jersey, 1955.

[9] R. Kummetz. From partial orders with projections to domains. Extended abstract. In S. Brookes, A. Jung, M. Mislove, and A. Scedrov, editors, *Mathematical Foundations of Programming Semantics, Fifteenth Conference*, volume 20 of *Electronic Notes in Theoretical Computer Science*, 1999. 12 pages.

[10] R. Kummetz. *Partially Ordered Sets with Projections and their Topology.* PhD thesis, Technische Universität Dresden, 2000.

[11] J.D. Lawson. The versatile continuous order. In M. Main, A. Melton, M. Mislove, and D. Schmidt, editors, *Mathematical Foundations of Programming Semantics, Third Conference*, volume 298 of *Lecture Notes in Computer Science*, pages 134–160. Springer Verlag, 1988.

[12] M. Majster-Cederbaum and C. Baier. Metric completion versus ideal completion. *Theoretical Computer Science*, 170:145–171, 1996.

[13] G. Markowsky and B.K. Rosen. Bases for chain-complete posets. *IBM Journal of Research and Development*, 20:138–147, 1976.

[14] G.D. Plotkin. A powerdomain construction. *SIAM Journal on Computing*, 5:452–487, 1976.

[15] D. Spreen. On functions preserving levels of approximation: A refined model construction for various lambda calculi. *Theoretical Computer Science*, 212:261–303, 1999.