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Electronic Notes in Theoretical Computer Science

Electronic Notes in Theoretical Computer Science 167 (2007) 303–324

www.elsevier.com/locate/entcs

Primitive Recursiveness of Real Numbers under Different Representations

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Abstract

In mathematics, various representations of real numbers have been investigated. All these representations are mathematically equivalent because they lead to the same real structure—Dedekind-complete ordered field. Even the effective versions of these representations are equivalent in the sense that they define the same notion of computability of real numbers. However, the primitive recursive (p.r., for short) versions of these representations can lead to different notions of p.r. real numbers. Several interesting results about p.r. real numbers can be found in literatures. In this paper we summarize the known results about the primitive recursiveness of real numbers for different representations as well as show some new relationships. Our goal is to clarify systematically how the primitive recursiveness depends on the representations of the real numbers.

Keywords: Primitive Recursiveness, Representations of Real Numbers, Effectively Computable.

[1] Supported by National Natural Science Foundation of China grants 60496327, 10410638 and 60473004, German Research Foundation (DFG) grant 446 CHV113/240/0-1, KAISI Fund in Sun Yat-sen University, and Australian Research Council grant DP0452628.

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doi:10.1016/j.entcs.2006.08.018

1 Introduction

The computability of real numbers is introduced by Alan Turing in his seminal paper [19]. According to Turing, "*the 'computable' numbers may be described briefly as the real numbers whose expressions as a decimal are calculable by finite means*". In order to define the "finite means" precisely, he introduces the nowadays well-known Turing machines. Since Turing machines compute exactly the computable functions on natural numbers, Turing defines actually the real numbers with computable decimal expansions as computable real numbers. Namely, x is *computable* if there is a computable function $f : \mathbb{N} \to \{0, 1, \cdots, 9\}$ such that $x = \sum_{n=0}^{\infty} f(n) \cdot 10^{-(n+1)}$. Here we consider only the real numbers in the interval $[0, 1]$. As it was pointed out by Robinson [16], Myhill [11], Rice [15] and others, the computability of real numbers can be equivalently defined by means of Cauchy sequences, Dedekind cuts and other representations of real numbers. That is, the computability of reals is independent of their representations. The class of computable reals will be denoted by **EC** (for **E**ffectively **C**omputable).

Besides the computability, the subrecursive real numbers like primitive recursive and polynomial time computable real numbers have also been discussed. The different notions of subrecursive real numbers could be defined if different representations are used. Specker [18] is the first who investigates this problem and he shows that decimal expansions, Dedekind cuts and Cauchy sequences lead to three different versions of p.r. real numbers. Later on, Peter [14], Mostowski [10], and Lehman [9] investigated other versions of p.r. reals and showed some more relations between the notions of p.r. real numbers based on different representations. However, not every important representation of real numbers have been discussed and there is no a systematical overview about the subrecursiveness of real numbers so far.

This paper aims to address the deficit. We summarize the known results about primitive recursiveness of reals which we can find in literatures. We will give some new properties of the p.r. reals and analyze systematically the dependence of primitive recursiveness of reals on the representations.

This paper is organized as follows. Firstly we recall the representations in the computability theory for real numbers in the next section, then we will survey and explore the hierarchy in these representations in section 3, 4, 5, 6 and 7 by nested intervals, Cauchy sequences, b-adic expansion, Dekedind cut and continued fraction, respectively. And we will conclude the paper in the last section.

2 Representations of Real Numbers

In this section, we recall the representations of real numbers which will be discussed in this paper. First we explain the classical form of the representations. Since we are interested in the effectivizations of the representations to different levels, all representations will be defined again in a uniform way such that they depend on some given class $\mathcal{F}$ of functions. According to the choice of the class $\mathcal{F}$, various computability of different levels about real numbers can be defined. These notions depend also on the selected representations.

For simplicity, we consider only the real numbers in the unity interval $[0, 1]$. If a real number x is not in this interval, then there is a $y \in [0, 1]$ and a natural number n such that $x = y + n$ or $x = y - n$. In this case, the real numbers x and y should have the same computability level in any reasonable sense.

Now we recall the representations of real numbers informally.

A sequence (x_s) of rational numbers is called a *Cauchy sequence* if, for any $\epsilon > 0$, there is an N such that $|x_s - x_t| \leq \epsilon$ for all $s, t \geq N$. That is, Cauchy sequences are simply the converging sequences. A Cauchy sequence (x_s) represents a real number x if the sequence converges to x. This representation is called *naive Cauchy representation* (see Weihrauch [20]). In other words, a naive Cauchy representation of a real number x is a sequence (x_s) of rational numbers which converges to x. A more popular representation by Cauchy sequence in computable analysis uses the Cauchy sequence with an effective convergence modulus and we call this representation simply *Cauchy representation*. More precisely, a Cauchy representation of a real number x is a sequence (x_s) of rational numbers which converges to x effectively in the sense that $|x_s - x| \leq 2^{-s}$ for all s. Some variations of Cauchy representation will be discussed in the section 4.

A *Dedekind cut* is a pair (C, D) of sets of rational numbers such that C is closed downward, i.e., if $u \leq v$ and $v \in C$ then $u \in C$, and D is upward, i.e., if $u \leq v$ and $u \in D$ then $v \in D$. A Dedekind cut (C, D) represents a real number x means actually that x is the least upper bound of C. Since a Dedekind cut (C, D) is uniquely determined by the set C, we define usually the (left) Dedekind cut of x as the set $C_x := \{r \in \mathbb{Q} : r < x\}$ of rational numbers and regard the set C_x as the *Dedekind cut representation* of x. Since any set can be described uniquely by its characteristic function, we can also define the Dedekind cut representation of a real number x as a function $f : \mathbb{N}^2 \to \{0, 1\}$ such that $f(n, m) = 1$ if and only if $n/m < x$.

The decimal representation might be the most well-known representation of real numbers. If x is a real number in the interval $[0, 1]$, then x can be denoted by a decimal expansion $x = 0.a_0a_1a_2a_3\cdots$ where $a_s \in \{0, 1, \ldots, 9\}$ such that

$x = \sum_{s=0}^{\infty} a_s \cdot 10^{-(s+1)}$. The sequence (a_s) corresponds to a function $f : \mathbb{N} \to \{0, 1, \ldots, 9\}$. Thus we can define the *decimal representation* of a real number $x \in [0, 1]$ as a function $f : \mathbb{N} \to \{0, 1, \ldots, 9\}$ such that $x = \sum_{s=0}^{\infty} f(s) \cdot 10^{-(s+1)}$. The decimal representation represents the real numbers in base 10. In general, for any natural number $b > 1$, we can also represent real numbers in base b. This is the b-adic expansion. That is, a *b-adic representation* of a real number x is a function $f : \mathbb{N} \to \{0, 1, \ldots, b-1\}$ such that $x = \sum_{s=0}^{\infty} f(s) \cdot b^{-(s+1)}$. If the base $b = 2$, then the b-adic representation is called *binary representation.*

The binary representation of real numbers relates a real number to a set of natural numbers in a very natural way. Let $f : \mathbb{N} \to \{0, 1\}$ be a binary representation of a real number x. Then we have $x = \sum_{s=0}^{\infty} f(s) \cdot 2^{-(s+1)} = \sum_{s \in A} 2^{-(s+1)}$ where $A := \{s \in \mathbb{N} : f(s) = 1\}$. Thus the binary representation is the characteristic function of A. The real number x is usually denoted also by $x = x_A$.

Another representation of real numbers is by sequences of nested rational intervals. A sequence (I_s) of closed intervals with rational endpoints is called nested if $I_{s+1} \subseteq I_s$ for all s. This sequence represents a real number x, if x is the unique common member of all intervals. The interval sequence can be defined by two functions. Therefore, we can define the *nested interval representation* of a real number x as a function pair $f, g : \mathbb{N} \to \mathbb{Q}$ such that $f(s) \leq f(s+1) \leq x \leq g(s+1) \leq g(s)$ for all s and $\lim_{s\to\infty}(g(s) - f(s)) = 0$.

Finally, we explain the continued fraction expansion of real numbers. It is known that every positive real number x has a unique regular continued fraction expansion of the form

$$x = b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \ldots}} \tag{1}$$

where $b_0 \geq 0, b_n \geq 1$ for $n \in \mathbb{N}$. For brevity, we write $x = [b_0, b_1, b_2, \cdots]$ in which the number b_n is called the *partial quotient of order n.* It is obvious that in the case for the rational numbers, the numbers of b_n is finite. That is, if x is rational, then $x = [b_0, b_1, b_2, \cdots, b_n]$ for some n. For convenience, we denote this rational number by $x = [b_0, b_1, b_2, \cdots, b_n, 0, 0, \cdots]$. Thus, a *continued fraction representation* of a real number x is a function $f : \mathbb{N} \to \mathbb{N}$ such that $x = [f(0), f(1), f(2), \cdots]$.

All representations mentioned above are mathematically equivalent because they deduce the same (more precisely, isomorphic) structure which is called Dedekind-complete ordered field. However, if we are interested in the computability of real numbers, the situation is different. In order to explain this more precisely, we look at firstly how the computability notion can be introduced to real numbers. The idea is very simple. As we have seen, every representation of real numbers mentioned above uses functions, either from natural numbers to natural numbers or from natural numbers to rational numbers. Thus, any effectivity notion about these functions can be transfered naturally to the real numbers represented by these functions. To this end, we have to extend the computability (or subcomputability) of the functions on natural numbers to the functions from natural numbers to rational numbers. This extension looks like the following:

Let $\mathcal{F}$ be a class of some functions $f : \mathbb{N} \to \mathbb{N}$. We say that a function $g : \mathbb{N} \to \mathbb{Q}$ belongs to $\mathcal{F}$ means that there are functions $a, b, c : \mathbb{N} \to \mathbb{N}$ in $\mathcal{F}$ such that $g(n) = (a(n) - b(n))/(c(n) + 1)$ for all n.

Now we give the precise definition of the relativization of all above representations to a class $\mathcal{F}$ of functions.

Definition 2.1 Let $\mathcal{F}$ be a class of functions $f : \mathbb{N} \to \mathbb{N}$ or $f : \mathbb{N} \to \mathbb{Q}$, and let $x \in [0, 1]$ be a real number.

(i) x has an $\mathcal{F}$-Cauchy representation ($x \in \mathbf{CS}(\mathcal{F})$) if there is a function $f : \mathbb{N} \to \mathbb{Q}$ in $\mathcal{F}$ such that the sequence $f(s)$ converges to x effectively.

(ii) x has an $\mathcal{F}$-Dedekind cut representation ($x \in \mathbf{DC}(\mathcal{F})$) if there is a function $f : \mathbb{N}^2 \to \{0, 1\}$ in $\mathcal{F}$ such that $f(n, m) = 1$ if and only if $n/(m + 1) < x$

(iii) x has a b-adic representation ($x \in \mathbf{bAE}(\mathcal{F})$) if there is a function $f : \mathbb{N} \to \{0, 1, \cdots, b - 1\}$ in $\mathcal{F}$ such that $x = \sum_{s=0}^{\infty} f(s) \cdot b^{-(s+1)}$. Especially, for $b = 10$ and $b = 2$, they are a decimal and binary representation, respectively.

(iv) x has a continued fraction representation ($x \in \mathbf{CF}(\mathcal{F})$) if there is a function $f : \mathbb{N} \to \mathbb{N}$ in $\mathcal{F}$ such that $x = [f(0), f(1), \cdots]$.

(v) x has a nested interval representation ($x \in \mathbf{NI}(\mathcal{F})$) if there are two functions $f, g : \mathbb{N} \to \mathbb{Q}$ in $\mathcal{F}$ such that $f(s) \leq f(s + 1) \leq x \leq g(s + 1) \leq g(s)$ for all s and $\lim_{s\to\infty}(g(s) - f(s)) = 0$.

When we limit the function class $\mathcal{F}$ to be p.r. functions, it will lead to the definitions of various versions of "p.r. real numbers". Denote by $\mathbf{R}_4, \mathbf{R}_3, \mathbf{R}_2^b, \mathbf{R}_1$ and $\mathbf{R}_0$ the classes of real numbers which have p.r. continued fraction, p.r. Dedekind cut, p.r. b-adic expansion, p.r. Cauchy representation, and p.r. nested interval representations, respectively. That is,

$$\mathbf{R}_4 = \mathbf{CF}(\mathcal{F}),\ \mathbf{R}_3 = \mathbf{DC}(\mathcal{F}),\ \mathbf{R}_2^b = \mathbf{bAE}(\mathcal{F}),\ \mathbf{R}_1 = \mathbf{CS}(\mathcal{F}),\ \mathbf{R}_0 = \mathbf{NI}(\mathcal{F}).$$

We will see that the relationship among these classes is as follows.

$$\mathbf{R}_4 \subsetneq \mathbf{R}_3 \subsetneq \mathbf{R}_2^b \subsetneq \mathbf{R}_1 \subsetneq \mathbf{R}_0 = \mathbf{EC}$$

3 The Nested Interval Representation

In this section we discuss the representation of reals by p.r. nested interval. And we will see that a real number has a p.r. nested interval representation if and only if it is computable.

By definition, a p.r. nested interval representation of a real number x supplies the p.r. upper and lower bounds of x to any precision. This is equivalent to a p.r. approximation to x with a p.r. error estimation which is called a *p.r. approximation* of x by Skordev [17]. More precisely, a p.r. approximation of a real x is a pair (a, e) of p.r. functions $a, e : \mathbb{N} \to \mathbb{Q}$ such that

(i) e is monotonically decreasing and converges to 0; and

(ii) $|a(n) - x| \leq e(n)$ holds for all $n \in \mathbb{N}$.

Lemma 3.1 *A real number has a p.r. nested interval representation if and only if it has a p.r. approximation.*

Proof. Suppose that x has a p.r. nested interval representation (f, g). That is, for any n, we have $f(n) \leq f(n+1) \leq x \leq g(n+1) \leq g(n)$ and $\lim_{n\to\infty}(g(n) - f(n)) = 0$. Define two p.r. functions $a, e : \mathbb{N} \to \mathbb{Q}$ by $a(n) = (g(n) + f(n))/2$ and $e(n) = (g(n) - f(n))/2$ for all n. Then (a, e) is a p.r. approximation of x.

On the other hand, if x has a p.r. approximation (a, e), then $a(n) - e(n) \leq x \leq a(n) + e(n)$ for all $n \in \mathbb{N}$. We can define two p.r. functions f and g by $f(n) := \max\{a(t) - e(t) : t \leq n\}$ and $g(n) := \min\{a(t) + e(t) : t \leq n\}$. The function pair (f, g) is obviously a p.r. nested interval representation of x. □

Remember that any computable number x has a computable approximation with a computable error estimation. The next result shows that computable real numbers have even p.r. approximations.

Theorem 3.2 (Skordev [17]) *A real number is computable if and only if it has a p.r. approximation.*

Proof. If a real number x has a p.r. approximation (a, e), then, for all n, we have $|a(s(n)) - x| \leq 2^{-n}$ for the computable function s defined by $s(n) = \mu i(e(i) \leq 2^{-n})$. That is, x is computable.

¿From the other direction, if x is computable, then there is a computable function $f : \mathbb{N} \to \mathbb{Q}$ such that $|f(n) - x| \leq 2^{-n}$. Let M be a Turing machine which computes the function f. According to the Kleene's predicate [5], there is a p.r. predicate T such that $T(n, y, s)$ holds if and only if the machine M with the input n outputs y in s steps. Therefore, $f(n) = y$ if and only if $T(n, y, s)$ for some s.

Fix an $a \in \mathbb{N}$ and $b \in \mathbb{Q}$ such that $f(a) = b$ and define two p.r. functions $g, h : \mathbb{N} \to \mathbb{N}$ by

$$g(\langle n, y, s\rangle) = \begin{cases} n & \text{if } T(n, y, s); \\ a & \text{otherwise.} \end{cases}$$

$$h(\langle n, y, s\rangle) = \begin{cases} y & \text{if } T(n, y, s); \\ b & \text{otherwise.} \end{cases}$$

Here $\langle \cdot, \cdot, \cdot\rangle : \mathbb{N} \times \mathbb{Q} \times \mathbb{N} \to \mathbb{N}$ is a p.r. pairing function. It is easy to see that $h(n) = f(g(n))$ holds for all n.

Let g' be an unbounded monotonically increasing primitive function defined by

$$g'(n) = \max\{g(i) : 1 \leq i \leq n\}.$$

Then the primitive function

$$h'(n) = f(g'(n)).$$

satisfies

$$|h'(n) - x| = |f(g'(n)) - x| \leq 2^{-g'(n)}.$$

So $(h'(n), 2^{-g'(n)})$ is a p.r. approximation for x. □

Corollary 3.3 *A real number has a p.r. nested interval representation if and only if it is computable. That is,* $\mathbf{R}_0 = \mathbf{EC}$.

4 Cauchy Representation

Although the primitive recursive approximation (a, e) of a real number x mentioned in the Section 3 has a p.r. error estimation e, this estimation converges to 0 not necessarily fast enough. For example, it is not guaranteed that, for any n, a stage m can be found such that $e(m) \leq 1/n$. That is the reason, why the real numbers of p.r. approximations does not form a proper subset of computable real numbers. In order to introduce properly the notion of primitive recursive real number, we should require that also the error estimation converges to 0 primitive recursively. The p.r. Cauchy representation discussed in this section supplies a good approach to this direction.

As mentioned in the Section 2, the p.r. Cauchy representation of a real number x is simply a p.r. function $f : \mathbb{N} \to \mathbb{Q}$ such that $|x - f(n)| \leq 2^{-n}$. That is, it is a p.r. approximation with the error estimation function $e(n) := 2^{-n}$. The choice of the function $\lambda n.2^{-n}$ is not essential. The function $\lambda n.n^{-1}$ is also widely used in literature. Actually, and p.r. function $e : \mathbb{N} \to \mathbb{Q}$ which converges to 0 primitive recursively suffices. In this paper, we use either the function $\lambda n.2^{-n}$ or $\lambda n.n^{-1}$ without further explanation.

Similar to the case of the computable Cauchy representation, not only the error estimation function has different choices, p.r. Cauchy representation of a real number can also be stated in several different but equivalent ways.

Proposition 4.1 *Let x be a real number. Then the following conditions are equivalent.*

(i) *x has a p.r. Cauchy representation f, i.e., $|x - f(n)| \le 2^{-n}$ for all n;*

(ii) *There is a p.r. function $f : \mathbb{N} \to \mathbb{Q}$ and a p.r. function $e : \mathbb{N} \to \mathbb{N}$ such that*

$$(\forall n, m \in \mathbb{N})(m \ge e(n) \Longrightarrow |f(m) - x| \le 2^{-n}). \tag{2}$$

(iii) *There is a p.r. function $f : \mathbb{N} \to \mathbb{Q}$ such that $\lim_{n\to\infty} f(n) = x$ and*

$$(\forall n, m \in \mathbb{N})(n \le m \Longrightarrow |f(n) - f(m)| \le 2^{-n}). \tag{3}$$

(iv) *There is a p.r. function $f : \mathbb{N} \to \mathbb{Q}$ and a p.r. function $e : \mathbb{N} \to \mathbb{N}$ such that $\lim_{n\to\infty} f(n) = x$ and*

$$(\forall n, m \in \mathbb{N})(n, m \ge e(k) \Longrightarrow |f(n) - f(m)| \le 2^{-k}). \tag{4}$$

Proof. "(i) $\Rightarrow$ (ii)" It suffices to define $e(n) := n$ for all n.

"(ii) $\Rightarrow$ (iii)" Let f and e be a p.r. functions which satisfy the condition (2). Define a p.r. function $f_1(n) := f(e(n+1))$ for all n. Then for any $n \le m$ we have

$$\begin{aligned} |f_1(n) - f_1(m)| &\le |f_1(n) - x| + |f_1(m) - x| \\ &= |f(e(n+1)) - x| + |f(e(m+1)) - x| \\ &\le 2^{-(n+1)} + 2^{-(m+1)} \le 2 \cdot 2^{-(n+1)} = 2^{-n} \end{aligned}$$

That is, the primitive function f_1 satisfies the condition (3).

"(iii) $\Rightarrow$ (vi)" Let f be a p.r. function with $\lim_{n\to\infty} f(n) = x$ which satisfies the condition (3). Define $e(n) := n + 1$ and $f_1(n) := f(n + 1)$. Then, for any $n, m \ge e(k)$ we have $|f_1(n) - f_1(m)| \le |f(n+1) - f(k+1)| + |f(m+1) - f(k+1)| \le 2^{-k}$. That is, f_1 and e satisfy (4).

"(vi) $\Rightarrow$ (i)" Let $f : \mathbb{N} \to \mathbb{Q}$ and $e : \mathbb{N} \to \mathbb{N}$ be p.r. functions such that $\lim_{n\to\infty} f(n) = x$ and $|f(n) - f(m)| \le 2^{-k}$ for all $n, m \ge e(k)$. Define $f_1(n) := f(e(n)+1)$ for all n and let $m \to \infty$, we get $|f_1(n) - x| \le 2^{-n}$ for all n. That is, f_1 is a p.r. Cauchy representation of x. □

A real number x is called *Cauchy p.r.* if it has a primitive Cauchy representation. The class of Cauchy p.r. real numbers is denoted by $\mathbf{R}_1$.

Theorem 4.2 *The class of Cauchy p.r. real numbers is closed under arithmetical operations. That is, it is a field.*

Proof. Let x and y be Cauchy p.r. real numbers and let f and g be p.r. Cauchy representations of x and y, respectively. Then the function h defined by $h(n) := f(n+1) + g(n+1)$ is a p.r. Cauchy representation of $x+y$. That is, $x+y$ is Cauchy p.r. too.

Choose a natural number k such that $\max\{|f(n)|, |g(n)|, |x|, |y|\} \leq 2^k$ for all n. Since $|f(n)g(n) - xy| \leq |f(n)||g(n) - y| + |y||f(n) - x| \leq 2^{-(n-k-1)}$, the function h defined by $h(n) := f(n+k+1)g(n+k+1)$ is a p.r. Cauchy representation of xy and hence xy is also Cauchy p.r.

If $y \neq 0$, then there is a constant k such that $\min\{|y|, |g(n)|\} \geq 2^{-k}$ and $\max\{|f(n)|, |g(n)|\} \leq 2^k$ for all n. Since

$$\begin{aligned}\left|\frac{f(n)}{g(n)} - \frac{x}{y}\right| &= \left|\frac{f(n)y - g(n)x}{g(n)y}\right| \\ &\leq \frac{|f(n)||g(n)-y| + |g(n)||f(n)-x|}{|g(n)y|} \leq 2^{-n+3k+1}.\end{aligned}$$

Thus, the function h defined by $h(n) = f(n+3k+1)/g(n+3k+1)$ is a p.r. Cauchy representation of x/y and hence x/y is Cauchy p.r. $\square$

By a simply diagonalization against all p.r. Cauchy representations, it is easy to construct a computable real number which does not have p.r. Cauchy representation. That is we have

Theorem 4.3 *The class of Cauchy p.r. reals is a proper subset of computable reals. That is,* $\mathbf{R}_1 \subsetneq \mathbf{EC}$.

5 Representations by b-adic Expansion

In mathematics real numbers are usually represented by its decimal expansion while in computer science the binary expansions are more popular. Of course, real numbers can also be represented in b-adic expansions for any $b > 1$. We call a real x *b-adic primitive recursive* (or *b-adic p.r.*, in short) if there is a p.r. function f such that $x = \sum_{n=0}^{\infty} f(n) \cdot b^{-(n+1)}$. The class of all b-adic p.r. real numbers is denoted by $\mathbf{R}_2^b$. In this section we can see that the classes $\mathbf{R}_2^b$ are proper subsets of the Cauchy p.r. real number class $\mathbf{R}_1$ for all $b > 1$. Besides, for different b, the classes $\mathbf{R}_2^b$ are not necessarily the same.

Notice that, if x has a p.r. b-adic expansions, i.e., $x = \sum_{n=0}^{\infty} f(n) \cdot b^{-(n+1)}$ for a p.r. function f, then the function g defined by $g(n) = \sum_{i=0}^{n} f(i) \cdot b^{-(i+1)}$ is a p.r. Cauchy representation of x. This observation implies immediately that $\mathbf{R}_2 \subseteq \mathbf{R}_1$.

On the other hand, Specker [18] shows that $\mathbf{R}_2^{10} \neq \mathbf{R}_1$. The proof's idea of Specker for the inequality $\mathbf{R}_2^{10} \neq \mathbf{R}_1$ is to show that $\mathbf{R}_2^{10}$ is not closed under arithmetical operations. Since $\mathbf{R}_1$ is closed under arithmetical operations, they are different. This idea can be extended to the case of other base b. Thus, the primitive recursiveness of real numbers based on Cauchy representation and b-adic expansions are different.

Specker's proof uses the following technical lemma which can be proved by the Kleene normal form theorem [5] for the computable functions.

Lemma 5.1 (Specker [18]) *There are p.r. functions u and v such that the function q defined by*

$$q(n) := u((\mu t \geq n)(v(t) = 0))$$

is not a p.r. function.

Theorem 5.2 (Specker [18]) *There exists a decimal p.r. real number x such that $3x$ is not decimal p.r.*

Proof. We want to find a real number $x := 0.a_0a_1a_2a_3\cdots$ such that the function a defined by $a(i) := a_i$ is p.r., but for $3x = b.b_0b_2b_2b_3\cdots$, the function b defined by $b(i) := b_i$ is not p.r. Let's look first at an example. For simplicity, we restrict that $a_i \in \{1, 3, 5\}$ for all i.

$$\begin{aligned} x = 0\,.\,a_0\,a_1\,a_2\,\cdots \quad &:= 0\,.\,3\,3\,5\,1\,3\,1\,1\,5\,5\,3\,3\,1\,5\,1\,1\cdots \\ 3x = b\,.\,b_0\,b_1\,b_2\,\cdots \quad &:= 1\,.\,0\,0\,5\,3\,9\,3\,4\,6\,5\,9\,9\,4\,5\,3\,?\cdots \end{aligned}$$

where "?" can be 3 or 4 depending on the first non-3 digit of x after this place is equal to 1 or 5.

Check the odd-even property of b_i we can find that a digit b_i is even if and only if the first digit a_j with $j > i$ and $a_j \neq 3$ is 5. That is, for any n, we have

$$b(n) \equiv 0 \mod 2 \iff a((\mu t > n)(a(t) \neq 3)) = 5. \tag{5}$$

Now the problem is reduced to find a p.r. function $a : \mathbb{N} \to \{1, 3, 5\}$ such that the function b satisfying condition (5) is not p.r.. It suffices to find a p.r. function $a : \mathbb{N} \to \{1, 3, 5\}$ such that the function $q : \mathbb{N} \to \{1, 5\}$ defined by

$$q(n) := a((\mu t > n)(a(t) \neq 3)) \tag{6}$$

is not p.r. For any p.r. functions $u : \mathbb{N} \to \{1, 5\}$ and $v : \mathbb{N} \to \mathbb{N}$, if we define the function a by

$$a(n) := 3 \cdot sg(v(n)) + u(v(n)) \cdot \overline{sg}(v(n))$$

where sg and $\overline{sg}$ are signal functions. Then we have $a(n) \in \{1, 3, 5\}$ and $a(n) \neq 3$ if and only if $v(n) = 0$ & $a(n) = u(n)$ for all n. This implies that

$$a((\mu t > n)(a(t) \neq 3)) = u((\mu t > n)(v(t) = 0))$$

Thus, the problem is reduced further to find two p.r. functions $u : \mathbb{N} \to \{1, 5\}$ and $v : \mathbb{N} \to \mathbb{N}$ such that the function defined by

$$q(n) := u((\mu t > n)(v(t) = 0))$$

is not primitive recursive. The existence of such p.r. functions u and v follows from the Lemma 5.1. □

Remark 5.3 By the same kind of construction of Specker's, it is not hard to see that any p.r. b-adic expansion reals is not closed under arithmetical operations.

Corollary 5.4 *The class of b-adic expansions p.r. reals is a proper subset of the class of Cauchy p.r. real numbers, i.e.,* $\mathbf{R}_2^b \subsetneq \mathbf{R}_1$.

Now we explore the relationship among the classes $\mathbf{R}_2^b$ for different b's. Firstly we look at a simple example. Let $b, d > 1$ be bases such that there is a $k > 0$ such that $b^k = d$. If x is a b-adic p.r. real number, i.e., $x := \sum_{n\in\mathbb{N}} f(n) \cdot b^{-(n+1)}$ for a p.r. function f, then the function $g(n) := \sum_{i=0}^{k-1} f(nk+i)$ is also p.r. and hence $x = \sum_{n\in\mathbb{N}} g(n) d^{-(n+1)}$ is d-adic p.r. too. This observation has been extended by Mostowski [10] to the following result.

Theorem 5.5 (Mostowski [10]) *Let $b, d > 1$. If a power of b is divisible by d, then any b-adic p.r. real is also d-adic p.r., i.e.,* $\mathbf{R}_2^b \subseteq \mathbf{R}_2^d$.

Proof. Suppose that $b^k = s \cdot d$ for some $k, s \in \mathbb{N}$ and $x = \sum_{n=0}^{\infty} f(n) b^{-(n+1)}$ is a p.r. b-adic expansion of x. Define a p.r. function g by $g(n) := \sum_{i=0}^{k-1} f(n \cdot k + i) \cdot b^{k-1-i}$. Notice that, $g(n) \leq (1 + b + \cdots + b^{k-1})(b-1) = b^k - 1$, because $f(i) \leq b-1$ for all i. Now the d-adic expansion of x can be obtained as follows.

$$x = \sum_{n=0}^{\infty} f(n) b^{-(n+1)} = \sum_{n=0}^{\infty} g(n) b^{-k(n+1)} \tag{7}$$

$$= \sum_{i=0}^{n} h(i) d^{-(i+1)} + r(n) b^{-k} + \sum_{i=n}^{\infty} g(i) b^{-k(i+1)} = \sum_{n=0}^{\infty} h(n) d^{-(n+1)} \tag{8}$$

where the p.r. functions r and h are defined inductively by

$$\begin{cases} h(0) := qt(g(0), s) \\ r(0) := rs(g(0), s) \\ h(n+1) := qt\left(g(n) + r(n) b^k, s^{n+1}\right) \\ r(n+1) := rs\left(g(n) + r(n) b^k, s^{n+1}\right). \end{cases}$$

Remember that qt and rs are the quotient and rest functions, respectively. By definition, we have $r(n) < s^n$ for all n. This implies immediately the last equality of (8). It remains only to show that $h(n) < d$ for all n which follow from the following inequalities:

$$h(0) \cdot s \leq g(0) \leq b^k - 1 < b^k = d \cdot s$$

$$\begin{aligned} h(n+1) \cdot s^{n+1} &\leq g(n) + r(n)b^k \leq (b^k - 1) + (s^n - 1)b^k = s^n b^k - 1 \\ &= s^{n+1}d - 1 < d \cdot s^{n+1}. \end{aligned}$$

Thus, $h(n) < d$ for all n and $x = \sum_{n=0}^{\infty} h(n)d^{-(n+1)}$ is a p.r. d-adic expansion.□

The inverse of the Theorem 5.5 was an open question in Mostowski [10]. A positive answer to this question was given by Lachlan [8].

Theorem 5.6 (Lachlan [8]) *Let $b, d > 1$, $\mathbf{R}_2^b \subseteq \mathbf{R}_2^d$ if and only if d divides a power of b, i.e., $(\exists k, s)(b^k = s \cdot d)$.*

To prove this theorem, Lachlan shows an equivalent characterization of the class $\mathbf{R}_2^b$ as follows. Let $R(b) := \{m \cdot b^{-n} : n, m \in \mathbb{N}\}$ be the class of all b-adic rational numbers. For any set A of rational numbers, denote by C_A^0 the class of real number x such that the Dedekind cut of x restricted to A (i.e., the intersection $C_x \cap A$) is primitive recursive. Then Lachlan shows that $\mathbf{R}_2^b = C_{R(b)}^0$ for any $b > 1$. That is, a real number x is b-adic p.r. if and only if the set $\{(n, m) : m \cdot b^{-n} < x \;\&\; n, m \in \mathbb{N}\}$ is p.r. Furthermore, Lachlan shows that, $C_A^0 \neq \emptyset$, if A is p.r. dense (roughly, for any rational numbers $x < y$, a rational number z between x and y can be found primitive recursively). For any natural number $b, d > 1$, if no power of b is divisible by d, then $R(b) \setminus R(d)$ is p.r. dense. This implies that $\mathbf{R}_2^b \setminus \mathbf{R}_2^d = C_{R(b)}^0 \setminus C_{R(d)}^0 \neq \emptyset$.

6 The Dedekind Cut Representation

This section discusses the real numbers which have p.r. Dedekind cuts. We will see that, these real numbers can be described equivalently in four different ways. By definition, the (left) Dedekind cut of a real number x is the set $C_x := \{r \in \mathbb{Q} : r < x\}$ of rational numbers. Thus, x has a p.r. Dedekind cut means that the set C_x is a p.r. set, i.e., the characteristic function $\chi_x : \mathbb{Q} \to \{0, 1\}$ is p.r. Here we need a notion of p.r. functions from rational numbers to natural numbers which can be defined by representing rational numbers as integer pairs. However, we can avoid this by considering the relation L_x defined by $L_x(m, n) \iff: m/(n+1) < x$ and say that x has a *p.r. Dedekind cut* if the relation L_x is p.r. The real numbers which have p.r. Dedekind cuts are called *Dedekind p.r.* and the class of all Dedekind p.r. reals is denoted by $\mathbf{R}_3$.

Obviously, for any positive real number x and natural numbers n, m, we have $m/(n+1) < x$ if and only if $m \leq \lfloor (n+1) \cdot x \rfloor$ where $\lfloor y \rfloor$ denotes the integer part of the real number y, i.e., the maximal natural number t such that $t \leq y$. The function $f(n) := \lfloor n \cdot x \rfloor$ is also called *Beatty function* or *Beatty sequence* of x after the Beatty's Theorem which asserts that the set $\{\lfloor nx \rfloor : n \in \mathbb{N}\}$ and $\{\lfloor nx \rfloor : n \in \mathbb{N}\}$ partitions natural numbers, if the positive irrational numbers x, y satisfy $1/x + 1/y = 1$ (see, e.g. [3]).

By the above observation, we have immediately the following description of Dedekind p.r. real numbers.

Theorem 6.1 (Peter [13]) *A real number is Dedekind p.r. if and only if its Beatty function is p.r.*

Another description of Dedekind p.r. real numbers uses the Hurwitz's characteristic of real numbers based on the Farey sequences ([4]). In mathematics, the Farey sequence of order n is the increasing sequence of irreducible fractions between 0 and 1 which have denominators less than or equal to n. For example, the Farey sequence of order four is $F_4 = \{0/1, 1/4, 1/3, 1/2, 2/3, 3/4, 1/1\}$. In general, the Farey sequence F_n of order n is an increasing sequence of irreducible fractions s/t such that $0 \leq s \leq t \leq n$ and $t \neq 0$. One of the most interesting properties, due to Haros (see [2]), of Farey sequence is that, for any three successive terms $s_1/t_1, s_2/t_2$ and s_3/t_3 of F_n, the middle one is always the "mediant" of its neighborhoods, i.e., $s_2/t_2 = (s_1 + s_3)/(t_1 + t_3)$.

By means of Farey sequence, Hurwitz ([4]) describes an irrational real number $x \in (0, 1)$ by a function $\gamma_x : \mathbb{N} \to \{0, 1\}$ which is called *Hurwitz characteristic* of x and is defined as follows.

Initially, let $\gamma_x(0) = 0$. To define $\gamma_x(1)$, notice that x is between two elements $0/1$ and $1/1$ of the Farey sequence of order 1. Comparing x with the mediant (i.e., $(0 + 1)/(1 + 1)$) of these elements and define $\gamma_x(1) := 0$ if $x < 1/2$ and $\gamma_x(1) := 1$ if $x > 1/2$.

Suppose that $\gamma_x(n)$ is defined and x is located between two adjacent fractions in some Farey sequence of the lowest order which have been used sofar, say s_1/t_1 and s_2/t_2. Then defined $\gamma_x(n+1) := 0$ if $x < (s_1 + s_2)/(t_1 + t_2)$, and $\gamma_x(n + 1) := 1$ if $x > (s_1 + s_2)/(t_1 + t_2)$.

The Hurwitz characteristic supplies a simply way to find the continued fraction of a real number. We will explain in the proof of Theorem 7.4 in Section 7. The following theorem gives a new characterization of the Dedekind p.r. real numbers.

Theorem 6.2 (Lehman [9]) *A real number $x \in (0, 1)$ is Dedekind p.r. if and only if its Hurwitz characteristic γ_x is p.r.*

Proof. Suppose that $x \in (0, 1)$ is Dedekind p.r., i.e., L_x is p.r. Define a function e by $e(m, n) = 1$ if $x < m/(n + 1)$ and $e(m, n) = 0$ if $x > m/(n + 1)$.

Then e is p.r. Now the Hurwitz characteristic γ_x can be defined more formally with help of four additional functions s_1, s_2, t_1 and t_2 as follows.

Initially let $\gamma_x(0) := 0$, $s_1(0) := 0$ and $s_2(0) = t_1(0) = t_2(0) := 1$. At the stage n, suppose that x locates between $s_1(n)/t_1(n)$ and $s_2(n)/t_2(n)$. Then define $\gamma_x(n+1) = 0$ or 1 depending on whether $x < \frac{s_1(n)+s_2(n)}{t_1(n)+t_2(n)}$ or not. That is, we have

$$\gamma_x(n+1) = e\left(s_1(n) + s_2(n), t_1(n) + t_2(n) - 1\right). \tag{9}$$

Now x locates between new fractions $\frac{s_1(n+1)}{t_1(n+1)}$ and $\frac{s_2(n+1)}{t_2(n+1)}$ of some Farey sequence which equal to $\frac{s_1(n)}{t_1(n)}$ and $\frac{s_1(n)+s_2(n)}{t_1(n)+t_2(n)}$ if $\gamma_x(n+1) = 0$, or $\frac{s_1(n)+s_2(n)}{t_1(n)+t_2(n)}$ and $\frac{s_2(n)}{t_2(n)}$ otherwise, respectively. That is, we have

$$\begin{cases} s_1(n+1) = s_1(n) + \gamma_x(n+1)s_2(n), \\ t_1(n+1) = t_1(n) + \gamma_x(n+1)t_2(n), \\ s_2(n+1) = s_2(n) + \left(1 - \gamma_x(n+1)\right)s_1(n), \\ t_2(n+1) = t_2(n) + \left(1 - \gamma_x(n+1)\right)t_1(n). \end{cases} \tag{10}$$

Thus, all functions γ_x, s_1, s_2, t_1, t_2 defined by equations (9) and (10) are p.r. and especially, x has a p.r. Hurwitz characteristic.

On the other hand, suppose that x has a p.r. Hurwitz characteristic γ_x. Then we can define four p.r. functions s_1, s_2, t_1 and t_2 according to (10). By a simple induction, we can see that $\max\{t_1(n), t_2(n)\} > n$ for all n. Consequently $\frac{s_1(n)}{t_1(n)}$ and $\frac{s_2(n)}{t_2(n)}$ are adjacent fractions in some Farey series of order greater than n. It follows that there can be no number m/n such that $\frac{s_1(n)}{t_1(n)} < \frac{m}{n} < \frac{s_2(n)}{t_2(n)}$. Hence we have $\lfloor nx \rfloor = \lfloor ns_1(n)/t_1(n) \rfloor = \lfloor ns_2(n)/t_2(n) \rfloor$. So x has a p.r. Dedekind cut by Theorem 6.1. □

Now we discuss the relation between Dedekind p.r. and b-adic real numbers. Notice that, any finite initial segment of a b-adic expansions of a real number x corresponds to a rational number which is less (or equal, if x is rational) than x. Thus, it is possible from a p.r. Dedekind cut of x to find a p.r. b-adic expansions of x. This can be done in a primitive recursive way. The other direction is impossible as shown by Specker [18].

Theorem 6.3 (Specker [18]) *Let $b > 1$. Any Dedekind p.r. real is b-adic p.r. But there is a b-adic p.r. real which is not Dedekind p.r. That is, $\mathbf{R}_3 \subsetneq \mathbf{R}_2^b$.*

Proof. If $x \in [0, 1]$ is Dedekind p.r., then, by Theorem 6.1, the Beatty function $\lfloor n \cdot x \rfloor$ is a p.r. function. The b-adic expansion f of x can be defined recursively by $f(0) := \lfloor x \rfloor$ and, for any n,

$$f(n+1) := \max\left\{t \le b : \sum_{i=0}^{n} f(i) \cdot b^{-(i+1)} + t \cdot b^{-(n+2)} \le x\right\}$$
$$= \max\left\{t \le b : \sum_{i=0}^{n} f(i) \cdot b^{(n+1-i)} + t \le \lfloor b^{(n+2)} x \rfloor\right\}$$

Thus, x is b-adic expansion p.r. and hence $\mathbf{R}_3 \subseteq \mathbf{R}_2^b$.

To prove the inequality $\mathbf{R}_3 \neq \mathbf{R}_2^b$, assume by contradiction that $\mathbf{R}_3 = \mathbf{R}_2^b$ for some $b > 1$. Choose a natural number d such that d does not divide b^k for any $k \in \mathbb{N}$. According to Theorem 5.6, we have $\mathbf{R}_3 = \mathbf{R}_2^b \not\subseteq \mathbf{R}_2^d$. This contradicts the fact $\mathbf{R}_3 \subseteq \mathbf{R}_2^d$. □

By Theorem 6.3, we cannot always get a p.r. Dedekind cut of a real x from a p.r. b-adic expansion of x. However, the situation is different, if x can be represented in b-adic expansion primitive recursively and uniformly in all bases b. Here the uniform dependence of a b-adic expansion to its base b refers to the dependence of each digits to the base. This can be described by a "uniform digits function". Precisely, a function $f : \mathbb{N}^2 \to \mathbb{N}$ is called a *uniform base expansion* of a real number x if, for all $n \in \mathbb{N}$ and any natural number $b \geq 2$,

$$0 \le f(b,n) < b \ \text{ and } \ x = \sum_{n=0}^{\infty} f(b,n) b^{-(n+1)}. \tag{11}$$

If this function f is p.r., then we say that x has a p.r. uniform base expansion.

The following theorem shows the relationship between p.r. uniform base expansions and p.r. Dedekind cuts.

Theorem 6.4 *A real number has a p.r. uniform base expansion if and only if it is Dedekind p.r.*

Proof. Suppose that $x \in [0,1]$ has a p.r. uniform base expansion f_x, i.e., the p.r. function f_x satisfies the condition (11). This means that, for any natural number $b > 1$, we have

$$b \cdot x = f_x(b,0) + \frac{f_x(b,1)}{b^1} + \frac{f_x(b,2)}{b^2} + \cdots = \sum_{i=0}^{\infty} f(b,i) b^{-i}.$$

Thus, we have $\lfloor b \cdot x \rfloor = f_x(b,0)$, if x is not a rational number and hence is not of the form $x = m/b^k$. That is, the Beatty function of x is p.r. By Theorem 6.1, x is a Dedekind p.r. real number. If x is rational, then x is obviously a Dedekind p.r. real number too.

On the other hand, if x is a Dedekind p.r. real number, then its Beatty function $\lfloor n \cdot x \rfloor$ is p.r. The uniform base expansion f_x of x can be obviously defined inductively by

$$\begin{cases} f_x(0) & := \lfloor b \cdot x \rfloor \\ f_x(n+1) & := \lfloor b^{n+1} \cdot x \rfloor - b \cdot \lfloor b^n \cdot x \rfloor . \end{cases}$$

and hence f_x is p.r. That is, x has a p.r. uniform base expansion. □

Between Cauchy p.r. reals and Dedekind p.r. reals Specker [18] has shown the following decomposition theorem.

Theorem 6.5 (Specker [18]) *Every Cauchy p.r. real number is the sum of two Dedekind p.r. real numbers.*

Proof. Let x be a Cauchy p.r. real number. Suppose w.l.o.g. that $x > 1$. It is not difficult to see that there is a p.r. function $f : \mathbb{N} \to \{1, 2, 3\}$ such that $x = \sum_{n=0}^{\infty} f(n)2^{-n}$. Define two p.r. functions h_0, h_1 by

$$h_0(n) := \begin{cases} 0 & \text{if } \lfloor \sqrt{n} \rfloor \text{ is even,} \\ 1 & \text{if } \lfloor \sqrt{n} \rfloor \text{ is odd;} \end{cases}$$

$$h_1(n) := \begin{cases} 1 & \text{if } \lfloor \sqrt{n} \rfloor \text{ is even,} \\ 0 & \text{if } \lfloor \sqrt{n} \rfloor \text{ is odd;} \end{cases}$$

Thus, the real number x can be decomposed into to reals s and t, i.e., $x = s+t$, where s and t are defined by

$$s = \sum_{n=0}^{\infty} h_0(n)f(n) \cdot 2^{-n} \quad \text{and} \quad t = \sum_{n=0}^{\infty} h_1(n)f(n) \cdot 2^{-n}.$$

It remains to show that s and t are Dedekind p.r. We consider here only the real t. The proof for s is similar.

Notice that, for any n and k, if $1 \leq k \leq 4n + 3$, then $4n^2 + 4n + k$ is odd and hence $h_1(4n^2 + 4n + k) = 0$. Thus we have

$$t = \sum_{k=0}^{4n^2+4n} \frac{h_1(k)f(k)}{2^k} + \sum_{k=4n^2+8n+4}^{\infty} \frac{h_1(k)f(k)}{2^k}.$$

Denote two partial sums by $t_1(n)$ and $t_2(n)$, respectively, and let

$$\omega(n) = n \cdot 2^{4n^2+4n} \cdot t_1(n) \quad \text{and} \quad R(n) = n \cdot 2^{4n^2+4n} \cdot t_2(n).$$

Then ω is a p.r. function and $0 \leq R(n) < 1$ for all n. This implies that

$$m/n < t \iff m \cdot 2^{4n^2+4n} < \omega(n) + R(n) \iff m \cdot 2^{4n^2+4n} < \omega(n).$$

That is, t has a p.r. Dedekind cut. □

By Theorem 6.5, the class $\mathbf{R}_1$ of Cauchy p.r. reals is the closure of the class $\mathbf{R}_3$ of Dedekind p.r. reals under arithmetical operations.

7 The Continued Fraction Expansion Representation

The continued fraction is another very interesting representation of real numbers. As we have mentioned in Section 2, any irrational number x can represented as an infinite continued fraction $x = [b_0, b_1, b_2, \cdots]$ where $b_n \geq 1$ for $n \in \mathbb{N}$. In this section, we use "real numbers" to refer to just "irrational numbers" for simplicity since the technical results for the cases of rational numbers are trivial and obvious.

For $x = [b_0, b_1, b_2, \cdots]$ and $n \in \mathbb{N}$, the finite continued fraction $x_n := [b_0, b_1, b_2, \cdots, b_n]$ is a rational number and can be denoted by u_n/v_n. By simple calculation, u_n, v_n can be determined inductively as follows

$$\begin{cases} u_{-1} = 1 \quad v_{-1} = -1, \quad u_0 = b_0, \quad v_0 = 1 \\ u_{n+2} = b_{n+2}u_{n+1} + u_n, \\ v_{n+2} = b_{n+2}v_{n+1} + v_n. \end{cases} \tag{12}$$

Here the terms u_{-1}, v_{-1} are defined for the technical simplicity. The fractions u_n/v_n are called the *convergent of order* n and they are reduced fractions for all n. By a simple induction we can show that $v_n < v_{n+1}$ for all $n \geq 1$ and hence $v_n \geq n$ for all n. Each convergent is nearer to x than the preceding convergent. In addition, the convergents provide the best approximations to x in the following sense: if $n > 1$, $0 < v \leq v_n$, and $u/v \neq u_n/v_n$, then $|x - u_n/v_n| < |x - u/v|$.

About the convergents of a continued fraction expansion, we have the following further properties which will be used in the proofs later on.

Lemma 7.1 (cf. [2,12]) *Let $x = [b_0, b_1, b_2, \cdots]$ and let u_n/v_n be its convergent of order n. Then we have*

(i) $u_n v_{n-1} - v_n u_{n-1} = (-1)^{n-1}$;

(ii) $b_{n+1} = \left\lfloor \frac{u_{n-1} - x v_{n-1}}{x v_n - u_n} \right\rfloor$;

(iii) $\frac{1}{v_n(v_{n+1}+v_n)} < \left| x - \frac{u_n}{v_n} \right| < \frac{1}{v_n v_{n+1}}$;

(iv) $\frac{u_0}{v_0} < \frac{u_2}{v_2} < \frac{u_4}{v_4} < \cdots < x < \cdots < \frac{u_5}{v_5} < \frac{u_3}{v_3} < \frac{u_1}{v_1}$.

Now we are going to investigate the class of real numbers which have a p.r. continued fractions. By (12), if x has a p.r. continued fraction, then the corresponding sequences (u_s) and (v_s) are p.r. too. That is, the sequence (u_s/v_s) is a p.r. approximation of x. According to Lemma 7.1.(iii), it is easy to see that x is Cauchy p.r. Of course, we can do better. From items 3 and 4 of the Lemma 7.1 and the fact that $v_n \geq n$, we have

$$0 < nx - \frac{nu_{2n}}{v_{2n}} < \frac{n}{v_{2n}v_{2n+1}} \leq \frac{1}{v_{2n}},$$

and hence

$$\frac{nu_{2n}}{v_{2n}} < nx < \frac{nu_{2n}}{v_{2n}} + \frac{1}{v_{2n}}.$$

In order to change the integer part of $\frac{nu_{2n}}{v_{2n}}$, we must add at least $\frac{1}{v_{2n}}$ to it. This implies that

$$\left\lfloor \frac{nu_{2n}}{v_{2n}} \right\rfloor \leq \lfloor nx \rfloor < \left\lfloor \frac{nu_{2n}+1}{v_{2n}} \right\rfloor \leq \left\lfloor \frac{nu_{2n}}{v_{2n}} \right\rfloor + 1.$$

That is, the Beatty function $\lfloor nx \rfloor = \lfloor nu_{2n}/v_{2n} \rfloor$ of x is primitive recursive. By Theorem 6.1, x is Dedekind p.r., i.e., $\mathbf{R}_4 \subseteq \mathbf{R}_3$. This result belongs to Lehman [9]. The next natural question is, whether there exists a Dedekind real which does not have a p.r. continued fraction? To answer this question, Lehman shows another characterization of the real numbers which have p.r. continued fraction by means of primitive-recursively irrationality of Péter [13,14].

Roughly speaking, an irrational number x is called *primitive-recursively irrational* (*p.r. irrational* for short) if it is possible to find a primitive-recursively lower bound of the distance between x and any given rational number. More precisely, there is a p.r. function f such that for all positive integers m and n

$$\left| x - \frac{m}{n} \right| > \frac{1}{f(n)}. \tag{13}$$

Péter [13,14] used a slightly different but equivalent definition and she used the name *recursively irrational* because in Péter [13] recursive means actually primitive recursive. The name *primitive-recursively irrational* was used by Goodstein [1] where it is shown that π is p.r. irrational.

Péter [13] shows that a Cauchy p.r. real is continued fraction p.r. if it is p.r. irrational. Lehman [9] shows that this is in fact a necessary and sufficient condition of a real number with p.r. continued fraction.

Theorem 7.2 (Lehman [9]) *A real number x has a p.r. continued fraction expansion if and only if it is Cauchy p.r. and p.r. irrational.*

Proof. Suppose that $x = [b_0, b_1, b_2, \cdots]$ is a p.r. continued fraction. According to (12), the sequence (u_n/v_n) of the convergents of x is obviously p.r. From Lemma 7.1, it is easy to see that x is Cauchy p.r. To show that x is p.r. irrational, it suffices to look at the following inequality.

$$\left|x - \frac{m}{n}\right| \geq \left|x - \frac{u_n}{v_n}\right| > \frac{1}{v_n(v_n + v_{n+1})}$$

for $m, n > 0$. Here the first inequality follows from the fact that (u_n/v_n) is the best approximation to x and $n \leq v_n$ and the second inequality follow from Lemma 7.1.(iii). Thus, the p.r. function $f(n) := v_n(v_n + v_{n+1})$ witnesses that x is p.r. irrational.

For the other direction, suppose that x is a p.r. irrational Cauchy p.r. real number. Then there are p.r. functions c, d, f such that

$$\left|x - \frac{c(n)}{d(n)}\right| \leq \frac{1}{n} \text{ and } \left|x - \frac{m}{n}\right| > \frac{1}{f(n)} \tag{14}$$

for all positive integers m, n. Assume w.l.o.g. that the function f is monotone increasing. Otherwise we can consider the function $f'(n) := \max\{f(m) : m \leq n\}$ instead.

Let $x = [b_0, b_1, b_2, \cdots]$ and u_n/v_n be its convergent of order n. We want to show that (b_n) is a p.r. sequence. Define a p.r. function $h(n) := f^{(n+1)}(v_1)$, i.e., $h(0) := f(v_1)$ and $h(n+1) := fh(n)$. Then, by item (iii) of Lemma 7.1 and the second part of (14), we have $1/f(v_n) < |x - u_n/v_n| < 1/v_n v_{n+1} \leq 1/v_{n+1}$. That is, we have $v_{n+1} \leq f(v_n)$ for all n. By a simple induction we can show that $f(v_{n+1}) \leq h(n)$ which implies immediately that

$$\left|x - \frac{u_{n+1}}{v_{n+1}}\right| > \frac{1}{f(v_{n+1})} \geq \frac{1}{h(n)} \geq \left|x - \frac{ch(n)}{dh(n)}\right| = |x - x_n| \tag{15}$$

where $x_n := \frac{ch(n)}{dh(n)}$. Since x is between $\frac{u_n}{v_n}$ and $\frac{u_{n+1}}{v_{n+1}}$ and it is closer to $\frac{u_{n+1}}{v_{n+1}}$ than to $\frac{u_n}{v_n}$. So x_n must lie between $\frac{u_n}{v_n}$ and $\frac{u_{n+1}}{v_{n+1}}$ too. Since the rational numbers $\frac{u_n}{v_n}$ and $\frac{u_{n+1}}{v_{n+1}}$ have the same partial quotients of order less than $n+1$ which are partial quotients of order less than $n+1$ of the real x and the rational number x_n too (see [12], p. 35). Thus, it suffices to show that the sequence (b_n) can be defined primitive-recursively from the p.r. sequence (x_n). By item (iii) of the Lemma 7.1, this can be realized by the following definition combining with the equations (12).

$$\begin{cases} b_0 = \lfloor x_0 \rfloor \\ b_{n+1} = \left\lfloor \frac{u_{n-1} - x_{n+1} v_{n-1}}{x_{n+1} v_n - u_n} \right\rfloor \end{cases} \tag{16}$$

Therefore (b_n) is a p.r. sequence and hence x has a p.r. continued fraction expansion. □

By means of the characterization of the Theorem 7.2, Lehman [9] can further show not every Dedekind p.r. real has a p.r. continued fraction expansion.

To this end, we need another technical lemma which can be easily proved from the Lemma 5.1.

Lemma 7.3 (Lehman [9]) *There is a p.r. function $\lambda : \mathbb{N} \to \{1, 2, \}$ which takes value 1 infinitely many times such that the function σ defined by $\sigma(n) := \mu m \geq n(\lambda(m) \neq 2)$ is not p.r.*

Theorem 7.4 (Lehman [9]) *There is a Dedekind p.r. real number x which is not primitive recursively irrational.*

Proof. The real number x is given by its Hurwitz characteristic γ_x which is defined by $\gamma_x(0) = 0$ and

$$\gamma_x(n+1) := \begin{cases} \gamma_x(n) & \text{if } \lambda(n) = 2, \\ 1 \dot{-} \gamma_x(n) & \text{if } \lambda(n) \neq 2. \end{cases}$$

where λ is the p.r. function of Lemma 7.3. By Theorem 6.2 x is Dedekind p.r.

As it is shown by Hurwitz [4], the continued fraction $x = [0, b_1, b_2, b_3, ...]$ of x can be easily obtained from the Hurwitz characteristic γ_x of x by simply counting the successive 0's and 1's. Namely, b_1 is the number of leading 0's of the sequence $(\gamma_x(n))$. Then, the next b_2 values are 1, the next b_3 values are 0, etc., where $b_i \geq 1$. That is, the sequence $(\gamma_x(n))$ have the following form.

$$\underbrace{0\ 0\ \cdots\ 0}_{b_1}\ \underbrace{1\ 1\ \cdots\ 1}_{b_2}\ \underbrace{0\ 0\ \cdots\ 0}_{b_3}\ \underbrace{1\ 1\ \cdots\ 1}_{b_4}\ 0 \cdots\cdots$$

By the definition of γ_x, we have $\gamma_x(n) \neq \gamma_x(n+1)$ if and only if $\lambda(n) \neq 2$. Thus, for any n, the first natural number m after n such that $\lambda(m) \neq 2$ is bounded above by $h(n) := \sum_{i \leq n+1} b_i$. If (b_n) is p.r., then h is a p.r. function too. In this case, the function σ of the Lemma 7.3 should be p.r. because

$$\begin{aligned} \sigma(n) &= (\mu m \geq n)(\lambda(m) \neq 2) \\ &= (\mu m \leq h(n))(m \geq n \;\&\; \lambda(m) \neq 2). \end{aligned}$$

This contradicts Lemma 7.3. Hence we conclude that (b_n) is not p.r. and x does not have a p.r. continued fraction expansion. By Theorem 7.2, x is nor p.r. irrational. □

Corollary 7.5 (Lehman [9]) *The class of real numbers which have p.r. continued fraction expansions is a proper subset if the class of Dedekind p.r. real numbers, i.e.,* $\mathbf{R}_4 \subsetneq \mathbf{R}_3$.

8 Conclusion

In this paper we summarize several known results about primitive recursiveness of real numbers under different representations which are scattered in literatures and show some new relations among them as well. We have seen that, the p.r. reals under different representations form a comprehensive hierarchy:

$$\mathbf{CF}(\mathcal{F}) \subsetneq \mathbf{DC}(\mathcal{F}) \subsetneq \mathbf{bAE}(\mathcal{F}) \subsetneq \mathbf{CS}(\mathcal{F}) \subsetneq \mathbf{NI}(\mathcal{F}) = \mathbf{EC}.$$

for the class $\mathcal{F}$ of p.r. functions. Among these classes, it seems that the class $\mathbf{CS}(\mathcal{F})$ might be properly regarded as the class of *primitive recursive reals*. And we also see that there is a hierarchy inside $\mathbf{bAE}(\mathcal{F}) = \mathbf{R}_2^b$ for different b's in the primitive recursive level. Their relation is that $\mathbf{R}_2^b \subseteq \mathbf{R}_2^d$ if and only if d divides a power of b.

It is also very natural to discuss these classes for other function classes $\mathcal{F}$. For example, Ko [6,7] have shown a similar hierarchy if $\mathcal{F}$ is the class of all polynomial time computable functions.

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