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Semantics for a Basic Relevant Logic with Intensional Conjunction and Disjunction

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Abstract

This paper proposes a basic relevant logic $\mathbf{B}^+_{\sqcap\sqcup}$ with intensional conjunction $\sqcap$ and disjunction $\sqcup$, which are more primitive than those defined by $A \circ B =_{df} \neg(A \to \neg B)$ and $A + B =_{df} \neg A \to B$. $\mathbf{B}^+_{\sqcap\sqcup}$ is a conservative extension of the basic relevant logic $\mathbf{B}^+$. Stronger logics can be obtained by adding axioms or rules to $\mathbf{B}^+_{\sqcap\sqcup}$. Kripke style semantics for $\mathbf{B}^+_{\sqcap\sqcup}$ is given. Three ternary relations R, S_1, S_2 are used to deal with $\to$, $\sqcap$, and $\sqcup$, respectively. We also consider negation-extensions of $\mathbf{B}^+_{\sqcap\sqcup}$. The '$*$' operator is used to model negation.

Keywords: Relevant logics, Intensional conjunction, Intensional disjunction

1 Introduction

In the literature on relevant logics, two binary connectives, fusion $\circ$ and fission $+$, can be defined by $A \circ B =_{df} \neg(A \to \neg B)$ and $A + B =_{df} \neg A \to B$, respectively. They are also called intensional conjunction and disjunction, and share many features classically attributed to extensional analogues, $\wedge$ and $\vee$. The connective $\circ$ can be introduced by way of the rule $A \circ B \to C \Leftrightarrow A \to (B \to C)$, having implication as its residual, which makes it important to algebraic treatment and proof theory of relevant logics. Defined by the above methods, $\circ$ and $+$ are highly related with

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implication. However, two more primitive connectives can be defined: $\sqcap$ and $\sqcup$, which we call intensional conjunction and disjunction also, since further schemes can be added to make them share features of $\circ$ and $+$.

Our idea is inspired by the work of establishing a non-normal conjunction rule in [3] (Chapter 8) for systems denying the law of identity. The rule is as following: $I(A.B, a) = 1$ iff for some $b, c \in W$, $b \ominus c \leq a$ and $I(A, b) = 1 = I(B, c)$, where '.' represents the non-normal conjunction, W is the set of worlds, a, b and c are members of W, I is the assignment function, and $\ominus$ is a two-place operation on W. If we use the expression S_1bca to represent the inequality $b \ominus c \leq a$, where S_1 is a ternary relation, then this evaluation rule is very similar to that for $\circ$ in relational semantics of relevant logics. So, it seems that some intensional conjunction can be defined independent of implication. Parallel to the above work in [3], we design a non-normal disjunction rule, and start our work from these two rules.

In this paper, we propose a base system $\mathbf{B}^+_{\sqcap\sqcup}$, obtained by adding intenstional conjunction and disjunction, denoted by $\sqcap$ and $\sqcup$ respectively, to the minimal positive relevant logic $\mathbf{B}^+$. $\mathbf{B}^+_{\sqcap\sqcup}$ is a conservative extension of $\mathbf{B}^+$. Stronger logics can be obtained by adding axiom or rule schemes to $\mathbf{B}^+_{\sqcap\sqcup}$. As to semantics, we use the ternary relation R to model implication $\rightarrow$. S_1 and S_2 are the other two ternary relations in our semantics. The former is used to deal with $\sqcap$, and the latter is for $\sqcup$. To construct a suitable canonical model, we define dualtheory and anti-dualtheory, and prove priming lemma for dualtheories. In addition, the basic negative system $\mathbf{BM}_{\sqcap\sqcup}$ is obtained by adding contraposition and De Morgan Laws to $\mathbf{B}^+_{\sqcap\sqcup}$ with negation modeled by the Routley $*$-operation.

We concentrate in this paper on the semantics of the basic systems with $\sqcap$ and $\sqcup$. Extensions will be considered in a subsequent draft.

2 The Basic System $\mathbf{B}^+_{\sqcap\sqcup}$

2.1 An Axiom System for $\mathbf{B}^+_{\sqcap\sqcup}$

$\mathbf{B}^+_{\sqcap\sqcup}$ is expressed in a language $\mathcal{L}$, which has the two-place connectives $\rightarrow, \wedge, \vee, \sqcap$ and $\sqcup$, parentheses (and), and a stock of propositional variables $p, q, r, \ldots$ Formulas are defined recursively in the usual manner. Some scope conventions are in force, that is, two-place connectives are ranked $\sqcap$, $\sqcup$, $\wedge$, $\vee$, $\rightarrow$ in order of increasing scope (i.e. $\sqcap$ binds more strongly than $\sqcup$, $\sqcup$ than $\wedge$, etc.), and otherwise association is to the left. $A, B, C, \ldots$ will be used to range over arbitrary formulas.

To define $\mathbf{B}^+_{\sqcap\sqcup}$, let us give an axiom system for $\mathbf{B}^+$ first, which has the following axioms and rules[3]:

Axioms

A1 $A \rightarrow A$

A2 $A \rightarrow A \vee B$, $B \rightarrow A \vee B$

A3 $A \wedge B \rightarrow A$, $A \wedge B \rightarrow B$

[3] Note that this axiom system is as same as that in [4] and [5] except that disjunctive forms of rules are not given distinctivly.

A4 $A \wedge (B \vee C) \to (A \wedge B) \vee C$
A5 $(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$
A6 $(A \to C) \wedge (B \to C) \to (A \vee B \to C)$

Rules
R1 $A, A \to B \Rightarrow B$ (Modus Ponens)
R2 $A, B \Rightarrow A \wedge B$ (Ajunction)
R3 $A \to B, C \to D \Rightarrow (B \to C) \to (A \to D)$ (Affixing).

Thus, $\mathbf{B}^+_{\sqcap\sqcup}$ is obtained by adding the following axioms and rules to $\mathbf{B}^+$:

A7 $(A \vee B) \sqcap C \leftrightarrow (A \sqcap C) \vee (B \sqcap C)$,
$C \sqcap (A \vee B) \leftrightarrow (C \sqcap A) \vee (C \sqcap B)$
A8 $(A \sqcup C) \wedge (B \sqcup C) \leftrightarrow (A \wedge B) \sqcup C$,
$(C \sqcup A) \wedge (C \sqcup B) \leftrightarrow C \sqcup (A \wedge B)$
R4 $A \to B, C \to D \Rightarrow A \sqcap C \to B \sqcap D$
R5 $A \to B, C \to D \Rightarrow A \sqcup C \to B \sqcup D$.

It can be noted that the special cases of **R3** are:

$A \to B \Rightarrow (C \to A) \to (C \to B)$ (Prefixing)
$A \to B \Rightarrow (B \to C) \to (A \to C)$ (Suffixing)
$A \to B, B \to C \Rightarrow A \to C$ (Transitivity).

And the special cases of **R4** and **R5** are, respectively:

$$A \to B \Rightarrow C \sqcap A \to C \sqcap B$$
$$A \to B \Rightarrow A \sqcap C \to B \sqcap C$$
$$A \to B \Rightarrow C \sqcup A \to C \sqcup B$$
$$A \to B \Rightarrow A \sqcup C \to B \sqcup C.$$

We note that this axiomatisation contains slight redundancies. **R4** and **R5**, together with axioms and rules of $\mathbf{B}^+$, suffice to prove each of **A7** and **A8** in one direction.

2.2 *Semantics for* $\mathbf{B}^+_{\sqcap\sqcup}$

Now we define interpretations for $\mathbf{B}^+_{\sqcap\sqcup}$. The semantics is an extension of semantics for $\mathbf{B}^+$ in [6].

A $\mathbf{B}^+_{\sqcap\sqcup}$-*frame* (or *model structure*) is a 6-tuple $< g, O, W, R, S_1, S_2 >$, where W is a set (of worlds); $g \in W$ (the base world); O is a unary relation on W; and R, S_1, and S_2 are ternary relations on W, such that the following definitions apply and postulates hold for all $a, b, c, d \in W$.

d1. $a \leq b =_{df} \exists x(Ox \text{ and } Rxab)$;
p1. Og;
p2. $a \leq a$;
p3. if $Rdbc$ and $a \leq d$ then $Rabc$;
p4. if S_1abd and $d \leq c$ then S_1abc;

p5. if S_2abd and $c \le d$ then S_2abc.

A $\mathbf{B}^+_{\sqcap\sqcup}$-*model* (or *interpretation*) is a 7-tuple $< g, O, W, R, S_1, S_2, I >$, where $< g, O, W, R, S_1, S_2 >$ is a $\mathbf{B}^+_{\sqcap\sqcup}$-frame, and I is a function which assigns to each pair of propositional parameter, p, and world, a, a truth value $I(p, a) \in \{1, 0\}$, satisfying the Atomic Hereditary Condition. Truth values of all formulas at worlds are assigned by the following evaluation rules.

Atomic Hereditary Condition. For a propositional variable p, if $I(p, a) = 1$ and $a \le a'$, then $I(p, a') = 1$.

Evaluation Rules.

- $I(A \wedge B, a) = 1$ iff $I(A, a) = 1$ and $I(B, a) = 1$;
- $I(A \vee B, a) = 1$ iff $I(A, a) = 1$ or $I(B, a) = 1$;
- $I(A \sqcap B, a) = 1$ iff $\exists b, c \in W$, S_1bca, $I(A, b) = 1$ and $I(B, c) = 1$;
- $I(A \sqcup B, a) = 1$ iff $\forall b, c \in W$, if S_2bca then $I(A, b) = 1$ or $I(B, c) = 1$;
- $I(A \to B, a) = 1$ iff $\forall b, c \in W$, if $Rabc$ and $I(A, b) = 1$ then $I(B, c) = 1$.

A $\mathbf{B}^+_{\sqcap\sqcup}$-model is indeed an extension of a $\mathbf{B}^+$-model in [6] by adding S_1, S_2, and evaluation rules for $\sqcap$, $\sqcup$.

We give the following definitions, with $\mathbf{B}^+_{\sqcap\sqcup}$-frame(s) and $\mathbf{B}^+_{\sqcap\sqcup}$-model(s) shortened as frame(s) and model(s), respectively. A is *valid* on a model if $I(A, g) = 1$; A *implies* B on a model if for all $a \in W$: if $I(A, a) = 1$ then $I(B, a) = 1$; A is *valid* on a frame if A is valid on all models based on this frame; A *implies* B on a frame if A implies B on all models based on this frame. At last, A is $\mathbf{B}^+_{\sqcap\sqcup}$-*valid* if A is valid on all frames; A $\mathbf{B}^+_{\sqcap\sqcup}$-*implies* B if A implies B on all frames. For any extension of $\mathbf{B}^+_{\sqcap\sqcup}$, similar definitions can also be given.

The following lemmas will simplify the proof for soundness.

Lemma 2.1 (Hereditary Condition) *For an arbitrary formula A, if $I(A, a) = 1$ and $a \le a'$, then $I(A, a') = 1$.*

Proof. The proof is by an induction on the length of A with Atomic Hereditary Condition as induction basis. Here we give proofs for $\sqcap$ and $\sqcup$.

$\sqcap$. A is of the form $B \sqcap C$. Suppose $I(B \sqcap C, a) = 1$ and $a \le a'$, to show $I(B \sqcap C, a') = 1$. For some $b, c \in W$, S_1bca, and $I(B, b) = 1 = I(C, c)$. By **p4**, S_1bca'. So, $I(B \sqcap C, a') = 1$ as required.

$\sqcup$. A is of the form $B \sqcup C$. Suppose $I(B \sqcup C, a) = 1$ and $a \le a'$, to show $I(B \sqcup C, a') = 1$. Suppose further $b, c \in W$ and S_2bca', to show $I(B, b) = 1$ or $I(C, c) = 1$. By **p5**, S_2bca. So, $I(B, b) = 1$ or $I(C, c) = 1$ as required. □

Lemma 2.2 (Verification Lemma) • *If A implies B on a $\mathbf{B}^+_{\sqcap\sqcup}$-model, then $A \to B$ is valid on this model.*

- *If A implies B on a $\mathbf{B}^+_{\sqcap\sqcup}$-frame, then $A \to B$ is valid on this frame.*
- *A $\mathbf{B}^+_{\sqcap\sqcup}$-implies B iff $A \to B$ is $\mathbf{B}^+_{\sqcap\sqcup}$-valid.*

Proof. For details of proof, please consult [6] (pp. 302-303). □

2.3 Soundness

The soundness of the semantics is demonstrated in this section.

Theorem 2.3 *If A is a theorem of $\mathbf{B}^+_{\sqcap\sqcup}$ then A is $\mathbf{B}^+_{\sqcap\sqcup}$-valid.*

Proof. The proof is by a simple induction over the length of proofs. It suffices to prove that all axioms are $\mathbf{B}^+_{\sqcap\sqcup}$-valid and all rules preserve validity. We give proofs for one of **A8** (in one direction) and **R4**.

For **A8**, suppose for an arbitrary model, $a \in W$ and $I((A \wedge B) \sqcup C, a) \neq 1$. Hence for some $b, c \in W$, S_2bca, $I(A \wedge B, b) \neq 1$ and $I(C, c) \neq 1$. So, $I(A, b) \neq 1$ or $I(B, b) \neq 1$. Hence, $I(A \sqcup C, a) \neq 1$ or $I(B \sqcup C, a) \neq 1$, i.e. $I((A \sqcup C) \wedge (B \sqcup C), a) \neq 1$. By Lemma 2.2, $(A \sqcup C) \wedge (B \sqcup C) \rightarrow (A \wedge B) \sqcup C$ is $\mathbf{B}^+_{\sqcap\sqcup}$-valid.

For **R4**, suppose $A \rightarrow B$ and $C \rightarrow D$ are $\mathbf{B}^+_{\sqcap\sqcup}$-valid, to show that $A \sqcap C \rightarrow B \sqcap D$ is $\mathbf{B}^+_{\sqcap\sqcup}$-valid. Suppose further for an arbitrary model, $a \in W$ and $I(A \sqcap C, a) = 1$. Hence for some $b, c \in W$, S_1bca and $I(A, b) = 1 = I(C, c)$. By Lemma 2.2, $I(B, b) = 1 = I(D, c)$. So, $I(B \sqcap D, a) = 1$. Then, the result follows by Lemma 2.2. □

2.4 Key Notions for Completeness

Completeness is established by the usual way. For any non-theorem A, we design a canonical interpretation which refutes A. Most of techniques come from [6] (Chapter 4) and [3] (Chapter 8). In this section, we give some notions for any logic $\mathbf{L}$ in this paper.

First, where V and U are sets of formulas:

(1) $\vdash_L A$ iff A is a theorem of $\mathbf{L}$.

(1) U is $\mathbf{L}$-*derivable* from V, written $V \vdash_L U$, iff for some $A_1, ..., A_n$ in V and some $B_1, ..., B_m$ in U, $\vdash_L A_1 \wedge ... \wedge A_n \rightarrow B_1 \vee ... \vee B_m$.

(2) An $\mathbf{L}$-*derivation* of A from V, written $V \vdash_L A$, is a finite sequence of formulas $A_1, ..., A_n$, with $A_n = A$ such that each member of the sequence either belongs to V or is obtained from predecessors in the sequence by adjunction or a provable $\mathbf{L}$-implication (i.e. in the latter case A_i is obtained from A_j since $\vdash_L A_j \rightarrow A_i$).

(3) An $\mathbf{L}$-*derivation* of U from V is an $\mathbf{L}$-derivation of some disjunction $B_1 \vee ... \vee B_m$ of formulas $B_1, ..., B_m$ of U from V. Hence, U is $\mathbf{L}$-derivable from V iff there is an $\mathbf{L}$-derivation of U from V.

(4) If Σ is the set of all formulas of the language $\mathcal{L}$, $< V, U >$ is an $\mathbf{L}$-*maximal pair* iff:

- $V \cup U = \Sigma$;
- $V \nvdash_L U$.

Please note that if $< V, U >$ is an $\mathbf{L}$-maximal pair, then $V \cap U = \emptyset$.

Next, it can be noted that if a is a set of formulas, and $b = \Sigma - a$, where Σ is the set of all formulas of the language $\mathcal{L}$, then a satisfies the following **a1**, **a2**, **a3** separately iff b satisfies **b1**, **b2**, **b3** separately.

a1. If $\vdash_L A \rightarrow B$, then if $A \in a$ then $B \in a$;

a2. if $A \in a$ and $B \in a$ then $A \wedge B \in a$;

a3. if $A \vee B \in a$ then $A \in a$ or $B \in a$;
b1. if $\vdash_L A \to B$, then if $B \in b$ then $A \in b$;
b2. if $A \wedge B \in b$ then $A \in b$ or $B \in b$;
b3. if $A \in b$ and $B \in b$ then $A \vee B \in b$.

Then we define, for arbitrary sets of formulas a, b:

(1) a is an **L**-*theory* iff it satisfies **a1** and **a2**;
(2) an **L**-theory a is *prime* iff it satisfies **a3**;
(3) an **L**-theory a is *regular* iff whenever $\vdash_L A$, $A \in a$;
(4) a is an **L**-*anti-dualtheory* iff it satisfies **a1** and **a3**;
(5) an **L**-anti-dualtheory a is *prime* iff it satisfies **a2**;
(6) b is an **L**-*dualtheory* iff it satisfies **b1** and **b3**;
(7) an **L**-dualtheory b is *prime* iff it satisfies **b2**.

So, let a be a set of formulas and $b = \Sigma - a$, then: a is a prime **L**-theory iff a is a prime **L**-anti-dualtheory iff b is a prime **L**-dualtheory; a is an **L**-anti-dualtheory iff b is an **L**-dualtheory.

In following text, if system **L** is obvious, then the subscript '$_L$' and the prefix '**L**-' will simply be omitted.

Now, we define four operations as follows. For arbitrary sets of formulas a, b:

- $a \oplus b = \{B : \exists A \in b, A \to B \in a\}$;
- $a \ominus b = \{C : \exists A \in a, \exists B \in b, \vdash_L A \sqcap B \to C\}$;
- $a \oslash b = \Sigma - \{C : \exists A \notin a, \exists B \notin b, \vdash_L C \to A \sqcup B\}$.

Based on the above definitions, we define three ternary relations R, S_1, S_2 on any set of sets of formulas:

- $Rabc$ iff $a \oplus b \subseteq c$, i.e., for all A, B, if $A \to B \in a$ and $A \in b$, then $B \in c$;
- S_1abc iff $a \ominus b \subseteq c$, i.e., for all A, B, if $A \in a, B \in b$, and $\vdash_L A \sqcap B \to C$, then $C \in c$;
- S_2abc iff $c \subseteq a \oslash b$, i.e., for all A, B, if $A \notin a$, $B \notin b$, and $\vdash_L C \to A \sqcup B$, then $C \notin c$.

2.5 Lemmas about Prime Theories

Our results are based on some lemmas in [6]. First, we list several lemmas, which are proved in [6] (pp. 307-308), or easy to get.

Lemma 2.4 *If $< V, U >$ is an **L**-maximal pair, then V is a prime **L**-theory, and U is a prime **L**-dualtheory.*

Lemma 2.5 (Extension Lemma) *Let V and U be sets of formulas such that $V \nvdash_L U$. Then there is an **L**-maximal pair $< V', U' >$ with $V \subseteq V'$ and $U \subseteq U'$.*

Lemma 2.6 (Priming Lemma 1) *Let V be an **L**-theory, U be closed under disjunction, and $V \cap U = \emptyset$. Then there is an **L**-theory V' such that (1) $V \subseteq V'$; (2) $V' \cap U = \emptyset$; and (3) V' is prime.*

Similar to the Priming Lemma 1, we have the Priming Lemma 2.

Lemma 2.7 (Priming Lemma 2) *Let V be closed under conjunction, U be an $\boldsymbol{L}$-dualtheory, and $V \cap U = \emptyset$. Then there is an $\boldsymbol{L}$-dualtheory U' such that (1) $U \subseteq U'$; (2) $V \cap U' = \emptyset$; and (3) U' is prime.*

Proof. First, $V \nvdash_L U$. Otherwise there would be $A_1, ..., A_n \in V$ such that $A_1 \wedge ... \wedge A_n \in V \cap U$, since U is an **L**-dualtheory. By Lemma 2.5, there are $V' \supseteq V$ and $U' \supseteq U$ such that $< V', U' >$ is an **L**-maximal pair. So, the result follows by Lemma 2.4. □

The following corollary is proved in [6] (pp. 309).

Corollary 2.8 (Corollaries of Priming Lemma 1) ***1.*** *If A is a non-theorem of $\boldsymbol{L}$ then, there is a prime regular $\boldsymbol{L}$-theory c such that $A \notin c$.*

2. *For all $\boldsymbol{L}$-theories a, b', c' if $Rab'c'$ and $C \notin c'$ then, there are prime $\boldsymbol{L}$-theories b, c such that $Rabc$, $b' \subseteq b$ and $C \notin c$.*

Corollary 2.9 (Corollaries of Priming Lemma 1) ***1.*** *For all $\boldsymbol{L}$-theories a', b and prime $\boldsymbol{L}$-theory c, if $S_1a'bc$ then, there is a prime $\boldsymbol{L}$-theory a such that $a' \subseteq a$ and S_1abc.*

2. *For all $\boldsymbol{L}$-theories a, b' and prime $\boldsymbol{L}$-theory c, if $S_1ab'c$ then, there is a prime $\boldsymbol{L}$-theory b such that $b' \subseteq b$ and S_1abc.*

Proof. We only give proof for **1**. The proof for **2** is similar.

1. Set $U = \{A : \exists B \in b, \exists C \notin c, \vdash_L A \sqcap B \to C\}$. Then:

(1) U is closed under disjunction;

(2) a' is disjoint from U.

For (1), suppose $A_1, A_2 \in U$, then $\exists B_1, B_2 \in b$, and $\exists C_1, C_2 \notin c, \vdash_L A_1 \sqcap B_1 \to C_1$ and $\vdash_L A_2 \sqcap B_2 \to C_2$. Since $\vdash_L B_1 \wedge B_2 \to B_1$, by **R4**, $\vdash_L A_1 \sqcap (B_1 \wedge B_2) \to A_1 \sqcap B_1$. Then by **R3**, $\vdash_L A_1 \sqcap (B_1 \wedge B_2) \to C_1$. Similarly, $\vdash_L A_2 \sqcap (B_1 \wedge B_2) \to C_2$. So, $\vdash_L (A_1 \sqcap (B_1 \wedge B_2)) \vee (A_2 \sqcap (B_1 \wedge B_2)) \to C_1 \vee C_2$. Then, by **A7**, $\vdash_L (A_1 \vee A_2) \sqcap (B_1 \wedge B_2) \to (A_1 \sqcap (B_1 \wedge B_2)) \vee (A_2 \sqcap (B_1 \wedge B_2))$. So, $\vdash_L (A_1 \vee A_2) \sqcap (B_1 \wedge B_2) \to C_1 \vee C_2$. Since c is prime, $C_1 \vee C_2 \notin c$. Since b is an **L**-theory, $B_1 \wedge B_2 \in b$. Hence, $A_1 \vee A_2 \in U$, i.e. U is closed under disjunction.

For (2), suppose otherwise $A \in U$ and $A \in a'$. Then for some $B \in b$, $C \notin c$, $\vdash_L A \sqcap B \to C$. But $S_1a'bc$, whence $C \in c$, giving a contradiction.

Hence by (1) and (2), Lemma 2.6 applies to provide a prime **L**-theory a disjoint from U with $a' \subseteq a$. Next, we prove S_1abc. Suppose $A \in a$, $B \in b$ and $\vdash_L A \sqcap B \to C$. Since a is disjoint from U, $C \in c$, i.e., whenever $A \in a$, $B \in b$ and $\vdash_L A \sqcap B \to C$, then $C \in c$. Hence S_1abc. □

Corollary 2.10 (Corollaries of Priming Lemma 2) ***1.*** *For all $\boldsymbol{L}$-anti-dualtheories a', b and prime $\boldsymbol{L}$-anti-dualtheory c, if $S_2a'bc$ then, there is a prime $\boldsymbol{L}$-anti-dualtheory a such that $a \subseteq a'$ and S_2abc.*

2. *For all $\boldsymbol{L}$-anti-dualtheories a, b' and prime $\boldsymbol{L}$-anti-dualtheory c, if $S_2ab'c$ then, there is a prime $\boldsymbol{L}$-anti-dualtheory b such that $b \subseteq b'$ and S_2abc.*

Proof. We only give proof for **1**. The proof for **2** is similar.

1. Set $V = \{A : \exists B \notin b, \exists C \in c, \vdash_L C \to A \sqcup B\}$. Then:

(1) V is closed under conjunction;

(2) $\Sigma - a'$ is disjoint from V.

For (1), suppose $A_1, A_2 \in V$, then $\exists B_1, B_2 \notin b$, and $\exists C_1, C_2 \in c, \vdash_L C_1 \to A_1 \sqcup B_1$ and $\vdash_L C_2 \to A_2 \sqcup B_2$. Since $\vdash_L B_1 \to B_1 \vee B_2$, by **R5**, $\vdash_L A_1 \sqcup B_1 \to A_1 \sqcup (B_1 \vee B_2)$. Then by **R3**, $\vdash_L C_1 \to A_1 \sqcup (B_1 \vee B_2)$. Similarly, $\vdash_L C_2 \to A_2 \sqcup (B_1 \vee B_2)$. So, $\vdash_L C_1 \wedge C_2 \to (A_1 \sqcup (B_1 \vee B_2)) \wedge (A_2 \sqcup (B_1 \vee B_2))$. Then, by **A8**, $\vdash_L (A_1 \sqcup (B_1 \vee B_2)) \wedge (A_2 \sqcup (B_1 \vee B_2)) \to (A_1 \wedge A_2) \sqcup (B_1 \vee B_2)$. So, $\vdash_L C_1 \wedge C_2 \to (A_1 \wedge A_2) \sqcup (B_1 \vee B_2)$. Since c is a prime **L**-anti-dualtheory, $C_1 \wedge C_2 \in c$. Since b is an **L**-anti-dualtheory, $B_1 \vee B_2 \notin b$. Hence, $A_1 \wedge A_2 \in V$, i.e. V is closed under conjunction.

For (2), suppose otherwise $A \in V$ and $A \in \Sigma - a'$, i.e. $A \notin a'$. Then for some $B \notin b$, $C \in c$, $\vdash_L C \to A \sqcup B$. But $S_2a'bc$, whence $C \notin c$, giving a contradiction.

Since a' is an **L**-anti-dualtheory, $\Sigma - a'$ is an **L**-dualtheory. Hence by (1) and (2), Lemma 2.7 applies to provide a prime **L**-dualtheory a'' disjoint from V with $\Sigma - a' \subseteq a''$. Let $a = \Sigma - a''$, then $a \subseteq a'$. Since a'' is a prime **L**-dualtheory, a is a prime **L**-theory, i.e. prime **L**-anti-dualtheory. Next, we prove S_2abc. Suppose $A \notin a$, i.e $A \in a''$, $B \notin b$ and $\vdash_L C \to A \sqcup B$. Since a'' is disjoint from V, $C \notin c$, i.e., whenever $A \notin a$, $B \notin b$ and $\vdash_L C \to A \sqcup B$, then $C \notin c$. Hence S_2abc. □

2.6 Completeness

For any non-theorem A of $\mathbf{B}^+_{\sqcap\sqcup}$, by **1** of Corollary 2.8, there is a a prime regular $\mathbf{B}^+_{\sqcap\sqcup}$-theory g_c such that $A \notin g_c$. Design a *canonical model* for $\mathbf{B}^+_{\sqcap\sqcup}$, $< g_{\mathrm{C}}, O_{\mathrm{C}}, W_{\mathrm{C}}, R, S_1, S_2, I >$, where W_{C} is the class of all prime theories, i.e. the class of all prime anti-dualtheories; O_{C} is defined as the subset of W_{C} such that $a \in O_{\mathrm{C}}$ if a is regular; R, S_1 and S_2 are defined as the above; for every prime theory a in W_{C} and propositional parameter p, $I(p, a) = 1$ iff $p \in a$.

Theorem 2.11 *If A is $\mathbf{B}^+_{\sqcap\sqcup}$-valid, then A is a theorem of $\mathbf{B}^+_{\sqcap\sqcup}$.*

Proof. We prove the contrapositive. Given a non-theorem A, there is a canonical model $< g_{\mathrm{C}}, O_{\mathrm{C}}, W_{\mathrm{C}}, R, S_1, S_2, I >$ for $\mathbf{B}^+_{\sqcap\sqcup}$. We show it is really a $\mathbf{B}^+_{\sqcap\sqcup}$-model. It suffices to show that **p1-5** hold, and I satisfies the Atomic Hereditary Condition. Now, **p1** and Atomic Hereditary Condition are immediate by definitions. By the same proof in [6] (pp. 312), it can be proved $a \le b$ iff $a \subseteq b$. So, we get **p2**. And, **p3-5** are established by definitions of R, S_1 and S_2. Hence, the canonical model is a $\mathbf{B}^+_{\sqcap\sqcup}$-model.

Next, we show that for every world a and formula A, $I(A, a) = 1$ iff $A \in a$. It follows that A is not valid on $< g_{\mathrm{C}}, O_{\mathrm{C}}, W_{\mathrm{C}}, R, S_1, S_2, I >$, and hence that A is not $\mathbf{B}^+_{\sqcap\sqcup}$-valid. The proof is by induction on the complexity of the formulas. The cases for $\wedge$ and $\vee$ are proved by definitions of theory and prime theory. Here, we give proofs for $\to$, $\sqcap$ and $\sqcup$.

$\to$. Suppose $A \to B \in a$, to show $I(A \to B, a) = 1$. Using the induction hypothesis and the definition of R, it follows that for all $b, c \in W_{\mathrm{C}}$, if $Rabc$ and

$I(A, b) = 1$ then $I(B, c) = 1$. Hence, $I(A \to B, a) = 1$ by the evaluation rule for $\to$.

For the converse, suppose $A \to B \notin a$, to show $I(A \to B, a) \neq 1$. By the induction hypothesis and the evaluation rule for $\to$, it suffices to find $b, c \in W_C$ such that $Rabc$, $A \in b$ and $B \notin c$. Define $b' = \{C :\vdash A \to C\}$ and $c' = \{D : \exists C \in b', C \to D \in a\}$. Then b' is a theory by **R1**, **R2**, **R3**, and **A5**. To show c' is a theory, suppose $\vdash D \to E$ and $D \in c'$. So, $\vdash (C \to D) \to (C \to E)$. Since $C \to D \in a$, $C \to E \in a$. Hence, $E \in c'$. Suppose further $D_1, D_2 \in c'$, i.e. for some $C_1, C_2 \in b'$, $C_1 \to D_1, C_2 \to D_2 \in a$. Since $\vdash A \to C_1$ and $\vdash A \to C_2$, using **R3** it follows that $A \to D_1, A \to D_2 \in a$. So $A \to D_1 \wedge D_2 \in a$ by **A5**, i.e. $D_1 \wedge D_2 \in c'$. It follows that c' is a theory. By the definition of R, $Rab'c'$. Further $A \in b'$, but $B \notin c'$. Suppose otherwise, $B \in c'$, then for some $C \in b'$, i.e. $\vdash A \to C$, $C \to B \in a$. Hence $\vdash (C \to B) \to (A \to B)$. So, $A \to B \in a$, contradicting assumptions. Applying Corollary 2.8, there are prime theories b, c such that $Rabc$, $A \in b' \subseteq b$ and $B \notin c$.

$\sqcap$. Suppose $A \sqcap B \in a$, to show $I(A \sqcap B, a) = 1$, i.e. for some $b, c \in W_C$, S_1bca, $I(A, b) = 1 = I(B, c)$. By the induction hypothesis, it suffices to find $b, c \in W_C$ such that $b \ominus c \subseteq a$, $A \in b$ and $B \in c$. Define $b' = \{C :\vdash A \to C\}$ and $c' = \{D :\vdash B \to D\}$. Then b' and c' are theories by **R1**, **R2**, **R3**, and **A5**. It is immediate that $A \in b'$ and $B \in c'$. To show $b' \ominus c' \subseteq a$, suppose $E \in b' \ominus c'$, then for some $C \in b'$ and $D \in c'$, $\vdash C \sqcap D \to E$. Since $\vdash A \to C$ and $\vdash B \to D$, by **R4**, $\vdash A \sqcap B \to C \sqcap D$. By **R3**, $\vdash A \sqcap B \to E$. Since $A \sqcap B \in a$, $E \in a$. Accordingly, $b' \ominus c' \subseteq a$, i.e. $S_1b'c'a$. So, by Corollary 2.9, b' can be primed to b with $b' \subseteq b$, and c' can be primed to c with $c' \subseteq c$ such that S_1bca.

For the converse, suppose $I(A \sqcap B, a) = 1$, i.e. for some $b, c \in W_C$, S_1bca, $I(A, b) = 1 = I(B, c)$. Then using the induction hypothesis, it follows that $A \in b$ and $B \in c$. By **A1**, and the definition of $\ominus$, $A \sqcap B \in b \ominus c$. Hence $A \sqcap B \in a$.

$\sqcup$. Suppose $I(A \sqcup B, a) \neq 1$, i.e. for some $b, c \in W_C$, S_2bca, $I(A, b) \neq 1$ and $I(B, c) \neq 1$. Then using the induction hypothesis, it follows that $A \notin b$ and $B \notin c$. By **A1**, and the definition of $\oslash$, $A \sqcup B \notin b \oslash c$. Hence, $A \sqcup B \notin a$.

For the converse, suppose $A \sqcup B \notin a$, to show $I(A \sqcup B, a) \neq 1$, i.e. for some $b, c \in W_C$, S_2bca, $I(A, b) \neq 1$ and $I(B, c) \neq 1$. By the induction hypothesis, it suffices to find $b, c \in W_C$ such that $a \subseteq b \oslash c$, $A \notin b$ and $B \notin c$. Define $b' = \Sigma - \{C :\vdash C \to A\}$ and $c' = \Sigma - \{D :\vdash D \to B\}$. Then $\{C :\vdash C \to A\}$ and $\{D :\vdash D \to B\}$ are dualtheories by **R1**, **R2**, **R3**, and **A6**. So, b' and c' are anti-dualtheories. It is immediate that $A \notin b'$ and $B \notin c'$. To show $a \subseteq b' \oslash c'$, suppose $F \notin b' \oslash c'$, then for some $C \notin b'$, $D \notin c'$, $\vdash F \to C \sqcup D$. Since $C \notin b'$, $\vdash C \to A$, and since $D \notin c'$, $\vdash D \to B$. By **R5**, $\vdash C \sqcup D \to A \sqcup B$. Since $\vdash F \to C \sqcup D$, by **R3**, $\vdash F \to A \sqcup B$. $A \sqcup B \notin a$, so $F \notin a$ as required. Hence, $a \subseteq b' \oslash c'$, i.e. $S_2b'c'a$. So, by Corollary 2.10, b' can be primed to b with $b \subseteq b'$, and c' can be primed to c with $c \subseteq c'$ such that S_2bca. $\square$

Now, we can see that $\mathbf{B}^+_{\sqcap\sqcup}$ is a conservative extension of $\mathbf{B}^+$, in the sense that it is an extension by adding new notations ($\sqcap$,$\sqcup$), axioms (**A7**,**A8**) and rules (**R4**,**R5**), which has the following feature: let A be a formula in the notation of $\mathbf{B}^+$; then if A is provable in $\mathbf{B}^+_{\sqcap\sqcup}$, then A is also provable in $\mathbf{B}^+$ [1], since every $\mathbf{B}^+_{\sqcap\sqcup}$-valid formula A involving only connectives $\to$, $\wedge$ and $\vee$ is also $\mathbf{B}^+$-valid.

Actually, further features of $\sqcap$ and $\sqcup$ such as commutativity and associativity can be obtained by adding axiom or rule schemes to $\mathbf{B}^+_{\sqcap\sqcup}$. Also, we can introduce $A \circ B \to C \Leftrightarrow A \to (B \to C)$, which makes $\to$ a residual of $\sqcap$, such that S_1 collapses to R. We leave this topic for further discussion.

3 Negation

For basic negation-extension of $\mathbf{B}^+_{\sqcap\sqcup}$, we add De Morgan Laws and contraposition:

A9 $\neg(A \wedge B) \leftrightarrow \neg A \vee \neg B$
A10 $\neg A \wedge \neg B \leftrightarrow \neg(A \vee B)$
R6 $A \to B \Rightarrow \neg B \to \neg A.$

We call this system $\mathbf{BM}_{\sqcap\sqcup}$. Please note that **A9** and **A10** also contain redundancies. By contraposition and positive axioms, we can prove each of **A9** and **A10** in one direction.

A $\mathbf{BM}_{\sqcap\sqcup}$-frame is a 7-tuple $< g, O, W, R, S_1, S_2, * >$, where $*$ is a one-place function from W to W, and the other elements are as before, such that postulate **p6** holds for all $a, b \in W$:

p6. If $a \leq b$ then $b^* \leq a^*$,

which is necessary for the Hereditary Condition.

A $\mathbf{BM}_{\sqcap\sqcup}$-model is a 8-tuple $< g, O, W, R, S_1, S_2, *, I >$, where $< g, O, W, R, S_1, S_2, * >$ is a $\mathbf{BM}_{\sqcap\sqcup}$-frame, and I is as before, with the following evaluation rule for negation:

- $I(\neg A, a) = 1$ iff $I(A, a^*) \neq 1$.

$\mathbf{BM}_{\sqcap\sqcup}$ is sound with respect to the evaluation rule. For completeness, define $*$ on a set of formulas a as: $a^* = \{A | \neg A \notin a\}$. Given a non-theorem A, the canonical interpretation for $\mathbf{BM}_{\sqcap\sqcup}$ is now $< g_{\mathrm{C}}, O_{\mathrm{C}}, W_{\mathrm{C}}, R, S_1, S_2, *, I >$. By De Morgan Laws and contraposition, it can be shown that: if a is an anti-dualtheory, then a^* is a theory; if a is a theory, then a^* is an anti-dualtheory. (For details of proof, please consult [4].) Hence, if a is a prime theory, so is a^*, i.e. $*$ is well-defined. Also, by the definition of a^*, **p6** is easy to verify.

The system $\mathbf{B}_{\sqcap\sqcup}$ is obtained by adding Double Negation, $A \leftrightarrow \neg\neg A$ to $\mathbf{BM}_{\sqcap\sqcup}$. A $\mathbf{B}_{\sqcap\sqcup}$-model is a $\mathbf{BM}_{\sqcap\sqcup}$-model satisfying: for all $a \in W$, $a^{**} = a$.

¿From the definitions of $\mathbf{BM}_{\sqcap\sqcup}$-model and $\mathbf{B}_{\sqcap\sqcup}$-model, we can see that $\mathbf{BM}_{\sqcap\sqcup}$ is a conservative extension of **BM**, and $\mathbf{B}_{\sqcap\sqcup}$ is a conservative extension of **B**.

If $A \to (B \to C) \Rightarrow (B \to \neg A \sqcup C)$ and $B \to A \sqcup C \Rightarrow \neg A \to (B \to C)$ are added to $\mathbf{B}_{\sqcap\sqcup}$, we can get that S_2 is dependent on R. Further, $A \sqcap B \leftrightarrow \neg(A \to \neg B)$ and $A \sqcup B \leftrightarrow (\neg A \to B)$ can be established with some further axiom or rule schemes added.

4 Concluding Remarks

This paper considered a basic relevant logic $\mathbf{B}^{+}_{\sqcap\sqcup}$ with intensional conjunction $\sqcap$ and disjunction $\sqcup$, and some of its negation-extensions. Kripke style semantics were given for these systems. Our semantics extend the traditional relational semantics for relevant logics in [6] by introducing ternary relations S_1 and S_2.

In fact, a wealth of stronger systems can be obtained by adding axioms or rules to these basic systems in this paper. We will consider a range of axiom and rule schemes, and give the corresponding semantical postulates for these schemes in another draft.

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