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A Coalgebraic Perspective on Monotone Modal Logic

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Abstract

The paper has two main parts: First we make the connection between monotone modal logic and the general theory of coalgebras precise by defining functors $\mathsf{Up}\mathcal{P} : \mathsf{Set} \to \mathsf{Set}$ and $\mathbb{U}\mathsf{pV} : \mathsf{Stone} \to \mathsf{Stone}$ such that $\mathsf{Up}\mathcal{P}$- and $\mathbb{U}\mathsf{pV}$-coalgebras correspond to monotone neighbourhood frames and descriptive general monotone frames, respectively. Then we investigate the relationship between the coalgebraic notions of equivalence and monotone bisimulation. In particular, we show that the $\mathsf{Up}\mathcal{P}$-functor does not preserve weak pullbacks, and we prove interpolation for a number of monotone modal logics using results on $\mathsf{Up}\mathcal{P}$-bisimulations.

Keywords: Modal logic, coalgebra, bisimulation, frame.

1 Introduction

There is an obvious connection between coalgebra and modal logic: Coalgebras for an endofunctor T can be seen as abstract dynamic systems or transition systems and modal logic seems to be the natural specification language to talk

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about these systems. The main benefit which can be expected from defining coalgebraic semantics for modal logic is to obtain results for different types of modal logics in a uniform way. Here the type of a modal logic is determined by the endofunctor T. Research in this direction has been carried out for the inductively defined Kripke polynomial functors (cf. [10,13,30]) but also for arbitrary endofunctors T (see [24,19,26,27]).

In normal modal logic, a modal operator $\Box$ is finite meet preserving (expressed by $\Box\top \leftrightarrow \top$ and $\Box(p \wedge q) \leftrightarrow \Box p \wedge \Box q$). Monotone modal logic generalises normal modal logic by weakening this requirement for $\Box$ to monotonicity $(p \to q / \Box p \to \Box q)$. This entails that Kripke frames no longer constitute an adequate semantics; instead (non-normal) monotone modal logics are interpreted over monotone neighbourhood frames of the form $(W, \nu : W \to \mathcal{P}(\mathcal{P}(W)))$ where $\nu(w)$ is upwards closed. Monotone modal logics arise naturally in the modelling of open systems and game-like situations (see [25,2,29]), where the ability of an agent to achieve an outcome where φ holds is expressed as $\Box\varphi$. In such a formalisation, $\Box\varphi \wedge \Box\psi \to \Box(\varphi \wedge \psi)$ need no longer be valid, since the agent may need to use different actions/strategies to achieve φ, respectively ψ. However, monotonicity is clearly a valid principle.

One aim of this paper is to show that monotone modal logics fall under the scope of the above sketched coalgebraic approach to modal logic. This will be achieved in Section 3 by first defining a functor $\mathsf{Up}\mathcal{P}$ on Set such that $\mathsf{Up}\mathcal{P}$-coalgebras correspond to monotone neighbourhood frames. Similar to what has been done for normal modal logic (cf. [17]) we also define a functor $\mathbb{U}\mathsf{pV}$ on the category Stone of Stone spaces such that the category of $\mathbb{U}\mathsf{pV}$-coalgebras is dual to the category BAM of monotone Boolean algebra expansions, which form the algebraic semantics for monotone modal logic.

In Section 4 we then take a look at the different notions of equivalence for $\mathsf{Up}\mathcal{P}$-coalgebras, namely $\mathsf{Up}\mathcal{P}$-bisimilarity as defined by Aczel and Mendler [1] and behavioural equivalence. It is well known that if the functor T is weak pullback preserving, then T-bisimilarity and behavioural equivalence are the same (cf. [31]). It turns out however that the two notions differ for our $\mathsf{Up}\mathcal{P}$-functor. This means that $\mathsf{Up}\mathcal{P}$ does not preserve weak pullbacks despite the fact that it is very similar to the weak pullback preserving filter functor defined in Gumm [11]. The two standard examples for functors with this property are the $(\text{-})^3_2$-functor defined by Aczel and Mendler in [1] and the contravariant powerset functor composed with itself (cf. [31]). However, the $\mathsf{Up}\mathcal{P}$-functor has the advantage that its defintion arises naturally by translating monotone neighbourhood frames into the coalgebraic setting. Furthermore, we argue that behavioural equivalence is the better notion for studying $\mathsf{Up}\mathcal{P}$-coalgebras as it coincides with bisimulation between monotone neighbourhood frames.

In Section 5 we use $\mathsf{Up}\mathcal{P}$-bisimulations to define the bisimulation product of two monotone neighbourhood frames, thereby generalising bisimulation products of Kripke frames (cf. Marx [23]). We then use bisimulation products to prove Craig Interpolation for some monotone modal logics, following the proof idea in [23] for interpolation in normal modal logics.

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2 Preliminaries

2.1 Topological preliminaries

As we are going to define an endofunctor on the category of Stone spaces we will briefly state the definition of a Stone space and the well-known Stone duality. For the basic notions of general topology that are needed we refer the reader to [6]. Given a topological space $\mathbb{X} = (X, \tau)$ we use $K(\mathbb{X})$, $O(\mathbb{X})$ and $\mathsf{Clp}(\mathbb{X})$ to denote the collections of closed, open and clopen subsets, respectively.

Definition 2.1 (Stone) Let $\mathbb{X} = (X, \tau)$ be a topological space. Then $\mathbb{X}$ is a *Stone space* if $\mathbb{X}$ is compact, Hausdorff and has a basis of clopen subsets. With Stone we will denote the category with Stone spaces as objects and continuous maps as morphisms between them.

Definition 2.2 (Vietoris topology) Let $\mathbb{X} = (X, \tau) \in \mathsf{Stone}$. Furthermore we define for a clopen $U \subseteq \mathbb{X}$, $[\ni]U := \{F \in K(\mathbb{X}) \mid F \subseteq U\}$, $\langle\ni\rangle U := \{F \in K(\mathbb{X}) \mid F \cap U \neq \emptyset\}$ and let τ_v be the topology on $K(\mathbb{X})$ generated by $\{[\ni]U \mid U \in \mathsf{Clp}(\mathbb{X})\} \cup \{\langle\ni\rangle U \mid U \in \mathsf{Clp}(\mathbb{X})\}$. Then $\mathbb{V}\mathbb{X} := (K(\mathbb{X}), \tau_v)$ is called the *Vietoris space over* $\mathbb{X}$.

Fact 2.3 (Vietoris functor) *We can define a functor* $\mathbb{V} : \mathsf{Stone} \to \mathsf{Stone}$ *by*

$$\begin{aligned} \mathbb{X} &\mapsto \mathbb{V}\mathbb{X} \\ (f : \mathbb{X} \to \mathbb{Y}) &\mapsto f[\_] : \mathbb{V}\mathbb{X} \to \mathbb{V}\mathbb{Y} \end{aligned}$$

For information about the Vietoris topology and the Vietoris functor we refer the reader to [14], Section III.4.

Fact 2.4 *The categories* BA *of Boolean algebras and* Stone *are dually equivalent. We denote the well-known functors witnessing this fact by* $\mathbb{C}\mathsf{lp} : \mathsf{Stone}^{\mathrm{op}} \to$

BA *and* $\mathbb{S}\mathsf{p} : \mathsf{BA} \to \mathsf{Stone}^{\mathrm{op}}$.

2.2 Coalgebraic preliminiaries

We assume that the reader is familiar with the basic notions of category theory. As a standard reference we refer to [20].

Definition 2.5 ($\mathsf{Coalg}(T)$) Let $\mathbb{C}$ be a category and $T : \mathbb{C} \to \mathbb{C}$ be an endofuntor over $\mathbb{C}$. Then a T-*coalgebra*, or T-*system*, is a pair $(X, \gamma : X \to TX)$ where X denotes an object of $\mathbb{C}$ and γ denotes a morphism in $\mathbb{C}$. An arrow $f : X_1 \to X_2 \in \mathbb{C}$ is a T-*coalgebra morphism* between two T-coalgebras (X_1, γ_1) and (X_2, γ_2) if $\gamma_2 \circ f = Tf \circ \gamma_1$. The category of T-coalgebras and T-coalgebra morphisms is denoted by $\mathsf{Coalg}(T)$.

Coalgebras are closely related to the Kripke semantics of (normal) modal logic.

Example 2.6 (Coalgebras and modal logic) *The category $\mathsf{Coalg}(\mathcal{P})$ of coalgebras for the powerset functor $\mathcal{P} : \mathsf{Set} \to \mathsf{Set}$ is (trivially) isomorphic to the category of Kripke frames and bounded morphisms between them. A slightly more complicated example is the category of coalgebras for the Vietoris functor $\mathbb{V} : \mathsf{Stone} \to \mathsf{Stone}$ (cf. Fact 2.3) which corresponds to the category of descriptive general Kripke frames [17].*

There are two standard notions of equivalence for systems: bisimilarity and behavioural equivalence. It is however a well known fact that for most choices of the functor T these two notions coincide (namely if T preserves weak pullbacks). We will now introduce these two notions and see later on that the functor we are defining is a natural example of a functor for which bisimilarity is stronger than behavioural equivalence.

Definition 2.7 (T-bisimulation) Let $T : \mathsf{Set} \to \mathsf{Set}$ be a functor, and (X_1, γ_1), $(X_2, \gamma_2) \in \mathsf{Coalg}(T)$. Then we call a relation $Z \subseteq X_1 \times X_2$ a T-*bisimulation* if there is a function $\gamma : Z \to TZ$ such that the following diagram commutes

$$\begin{tikzcd}
X_1 \arrow[d, "\gamma_1"] & Z \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, dashed, "\gamma"] & X_2 \arrow[d, "\gamma_2"] \\
TX_1 & TZ \arrow[l, "T\pi_1"] \arrow[r, "T\pi_2"'] & X_2
\end{tikzcd}$$

Points $x_1 \in X_1$ and $x_2 \in X_2$, such that $x_1 Z x_2$ for a T-bisimulation Z, are called T-*bisimilar*.

Definition 2.8 (Behavioural equivalence) Let $T : \mathsf{Set} \to \mathsf{Set}$ be a functor, and (X_1, γ_1), $(X_2, \gamma_2) \in \mathsf{Coalg}(T)$. Then two states $x_1 \in X_1$ and $x_2 \in X_2$

are called *behaviourally equivalent* if there is an (X, γ) and morphisms $f_i : (X_i, \gamma_i) \to (X, \gamma)$ in $\mathsf{Coalg}(T)$ such that $f_1(x_1) = f_2(x_2)$. The T-coalgebras (X_1, γ_1) and (X_2, γ_2) are behaviourally equivalent if there is a T-coalgebra (X, γ) and surjective morphisms $f_i : (X_i, \gamma_i) \twoheadrightarrow (X, \gamma) \in \mathsf{Coalg}(T)$.

2.3 Monotone modal logic

We assume the reader is familiar with normal modal logic [4] and refer to [5,12] for non-normal modal logic. In this subsection, we fix our modal setting and introduce the basic semantic structures.

For simplicity we will work in a modal language $\mathcal{L}$ with only one modal operator. Let PROP be a set of proposition letters. Then the set of well-formed $\mathcal{L}$-formulas is given by

$$\varphi ::= \bot \mid p \mid \neg\varphi \mid \varphi \vee \varphi \mid \Box\varphi \qquad \text{where } p \in \text{PROP}.$$

$\top, \wedge, \to$ and $\leftrightarrow$ are defined as the usual abbreviations, and $\Diamond$ abbreviates $\neg\Box\neg$. A set Λ of $\mathcal{L}$-formulas is a *monotone modal logic* if Λ contains all propositional tautologies, and Λ is closed under the rules *modus ponens* $(p, p \to q/q)$, *uniform substitution* and *monotonicity* $(p \to q/\Box p \to \Box q)$. The smallest monotone modal logic will be called $\mathbf{M}$, and in section 5 we will consider various extensions of $\mathbf{M}$ with one or more of the axioms in the box below. If Σ is a set of $\mathcal{L}$-formulas, then $\mathbf{M}.\Sigma$ denotes the smallest monotone modal logic containing Σ.

N	$\Box\top$	4	$\Box\Box p \to \Box p$	T	$\Box p \to p$
P	$\neg\Box\bot$	4’	$\Box p \to \Box\Box p$	D	$\Box p \to \Diamond p$

The semantics of monotone modal logic is formulated in terms of monotone neighbourhood frames. We will define these structures together with their morphisms as a category MNF.

Definition 2.9 (MNF) A *monotone (neighbourhood) frame* (for the language $\mathcal{L}$) is a pair $\mathbb{F} = (W, \nu)$ where W is a non-empty set (of worlds) and $\nu : W \to \mathcal{P}(\mathcal{P}(W))$ is a neighbourhood function which is upwards closed, i.e., $\forall w \in W,\ \forall X, Y \in \mathcal{P}(W) : X \subseteq Y, X \in \nu(w) \ \Rightarrow\ Y \in \nu(w)$. Let $\mathbb{F} = (W, \nu)$ and $\mathbb{F}' = (W', \nu')$ be monotone frames. A function $f : W \to W'$ is a *bounded morphism* from $\mathbb{F}$ to $\mathbb{F}'$ (notation: $f : \mathbb{F} \to \mathbb{F}'$) if

$$\begin{array}{lll}(\text{BM1}) & X \in \nu(w) & \Longrightarrow\ f[X] \in \nu'(f(w)). \\ (\text{BM2}) & X' \in \nu'(f(w)) & \Longrightarrow\ \exists X \subseteq W : f[X] \subseteq X' \ \&\ X \in \nu(w).\end{array}$$

The category MNF consists of monotone neighbourhood frames and bounded morphisms.

In some cases it is more convenient to work with a definition of bounded morphism formulated for the inverse image map instead of the image map, and it is easy to show that (BM1) and (BM2) are equivalent with the following condition: For all $X' \subseteq W'$,

$$f^{-1}[X'] \in \nu(x) \;\text{ iff }\; X' \in \nu'(f(x)). \tag{1}$$

Similarly to Kripke semantics, a neighbourhood function ν defines a map $m_\nu : \mathcal{P}(W) \to \mathcal{P}(W)$:

$$m_\nu(X) = \{w \in W \mid X \in \nu(w)\}\ . \tag{2}$$

Note that m_ν is monotone whenever ν is upwards closed.

Finally, a *monotone model* is a triple (W, ν, V) where (W, ν) is a monotone frame, and $V : \text{PROP} \to \mathcal{P}(W)$ is a valuation of the proposition letters. We can now define the set of states $[\![\varphi]\!]$ where an $\mathcal{L}$-formula φ is true in a monotone model $\mathbb{M} = (W, \nu, V)$ as follows:

$[\![\bot]\!] = \emptyset\,;\quad [\![p]\!] = V(p)\,;\quad [\![\neg\varphi]\!] = W \setminus [\![\varphi]\!]\,;\quad [\![\varphi \vee \psi]\!] = [\![\varphi]\!] \cup [\![\psi]\!]\,;\quad [\![\Box\varphi]\!] = m_\nu([\![\varphi]\!])$.

2.4 Algebra and duality

We will briefly introduce the main concepts and results from the algebraic duality theory of monotone modal logic. Apart from fixing notation, the purpose of this section is to provide the reader with some insight into the relationship between algebraic duality and the $\mathbb{U}\mathsf{pV}$-functor of section 3.2.

We assume familiarity with the algebraic duality theory of normal modal logic [32,8,9], in particular with descriptive (Kripke) frames, Boolean algebras with operators and Stone representation [15,16]. For a detailed account of the algebraic duality for monotone modal logic we refer to [12].

2.4.1 Basic notions

A *monotone Boolean algebra expansion* (BAM) is an algebraic structure $\mathbb{A} = (A, +, -, 0, f)$ where $Bl\mathbb{A} = (A, +, -, 0)$ is a Boolean algebra and $f : A \to A$ is a monotone map, i.e., $a \leq b$ implies $f(a) \leq f(b)$, for all $a, b \in A$, where $a \leq b$ iff $b = a + b$. Thus a Boolean algebra with operator(s) (BAO) can be seen as a BAM in which f is normal and additive. If $\mathbb{A}_1 = (A_1, +, -, 0, f_1)$ and $\mathbb{A}_2 = (A_2, +, -, 0, f_2)$ are BAMs, then a map $\eta : \mathbb{A}_1 \to \mathbb{A}_2$ is a BAM-*homomorphism* if η is a Boolean homomorphism, and for all $a_1 \in A_1$, $\eta(f_1(a_1)) = f_2(\eta(a_1))$. We define the category of BAMs and BAM-homomorphisms as BAM.

For basic duality we obtain a BAM from a monotone frame $\mathbb{F} = (W, \nu)$ by taking the *full complex algebra* $\mathbb{F}^+ = (\mathcal{P}(W), \cup, \setminus, \emptyset, m_\nu)$. In the other direction, we will use Stone duality. Given a BAM $\mathbb{A}$, we let $\mathbb{S}\mathsf{p}\mathbb{A} = (\mathit{Uf}\,\mathbb{A}, \widehat{A})$ denote the dual Stone space of $\mathit{Bl}\mathbb{A}$, where $\mathit{Uf}\,\mathbb{A}$ is the set of ultrafilters of $\mathit{Bl}\mathbb{A}$, and $\widehat{A}$ is the image of $\mathbb{A}$ under the Stone representation map $r : \mathbb{A} \to \mathcal{P}(\mathit{Uf}\,\mathbb{A})$ defined by $r(a) = \widehat{a} = \{u \in \mathit{Uf}\,\mathbb{A} \mid a \in u\}$. Recall that $\widehat{A}$ is a clopen basis for $\mathbb{S}\mathsf{p}\mathbb{A}$. We now define the *ultrafilter frame* of $\mathbb{A}$ by $\mathbb{A}_+ = (\mathit{Uf}\,\mathbb{A}, \nu_+)$ where $\nu_+(u)$ is defined as follows for the different types of subsets of $\mathbb{S}\mathsf{p}\mathbb{A}$,

(clopen)	$\forall \widehat{a} \in \widehat{A}$:	$\widehat{a} \in \nu_+(u)$	iff	$f(a) \in u$.
(closed)	$\forall C \in K(\mathbb{S}\mathsf{p}\mathbb{A})$:	$C \in \nu_+(u)$	iff	$\forall a \in A : C \subseteq \widehat{a} \to f(a) \in u$.
(arbitrary)	$\forall X \subseteq \mathit{Uf}\,\mathbb{A}$:	$X \in \nu_+(u)$	iff	$\exists C \in K(\mathbb{S}\mathsf{p}\mathbb{A})$: $C \subseteq X \;\&\; C \in \nu_+(u)$.

A *general monotone frame* is a structure $\mathbb{G} = (W, \nu, A)$, where (W, ν) is a monotone frame and $A \subseteq \mathcal{P}(W)$ is a collection of subsets which contains $\emptyset$ and is closed under complementation in W, finite unions and the map m_ν. If $\mathbb{G}_1 = (W_1, \nu_1, A_1)$ and $\mathbb{G}_2 = (W_2, \nu_2, A_2)$ are two general monotone frames, and $\theta : W_1 \to W_2$ a map, then θ is a *general frame bounded morphism* between $\mathbb{G}_1$ and $\mathbb{G}_2$ if θ is a bounded morphism between the monotone frames (W_1, ν_1) and (W_2, ν_2), and θ also satisfies the following condition: $\theta^{-1}[a_2] \in A_1$ for all $a_2 \in A_2$. The category GMF consists of general monotone frames and general frame bounded morphisms.

The basic duality between general monotone frames and BAMs is obtained via the following constructions. Given a general monotone frame $\mathbb{G} = (W, \nu, A)$, we define the *underlying* BAM of $\mathbb{G}$ as $\mathbb{G}^* = (A, \cup, \setminus, \emptyset, m_\nu)$. Given a BAM $\mathbb{A}$, the *general ultrafilter frame* of $\mathbb{A}$ is defined as $\mathbb{A}_* = (\mathbb{A}_+, \widehat{A})$.

Let $\mathbb{A}$ be a BAM. Then we define the *canonical extension of* $\mathbb{A}$ by $(\mathbb{A}_+)^+$. A class K of BAMs is *canonical* if K is closed under taking canonical extensions. A monotone modal logic Λ is canonical if the variety V_Λ defined by Λ is canonical.

2.4.2 Descriptive general frames

In the algebraic duality theory of normal modal logic, descriptive (Kripke) frames were introduced to obtain a categorical equivalence with BAOs [8,9]. The descriptive general monotone frames we are about to define serve the same purpose for the category BAM. More precisely, descriptive general monotone frames will be defined such that $\mathbb{G}$ is descriptive iff $\mathbb{G} \cong (\mathbb{G}^*)_*$. First recall that in a general monotone frame $\mathbb{G} = (W, \nu, A)$, the admissible sets A may be taken as the basis for a topology τ_A on W. We will refer to $\mathbb{W} = (W, \tau_A)$ as the *topological space of* $\mathbb{G}$. Let $\mathbb{G} = (W, \nu, A)$ be a general monotone frame.

Then $\mathbb{G}$ is called *differentiated* if for all $w, v \in W$: $w = v$ iff $\forall a \in A(w \in a \Leftrightarrow v \in a)$; *compact* if for all $A' \subseteq A$, $\bigcap A' \neq \emptyset$ if A' has the finite intersection property; *tight* if for all $w \in W$, all $C \in K(\mathbb{W})$ and all $X \subseteq W$,

$$C \in \nu(w) \text{ iff } \forall a \in A(C \subseteq a \to a \in \nu(w)),$$
$$X \in \nu(w) \text{ iff } \exists C \in K(\mathbb{W})(C \subseteq X \,\&\, C \in \nu(w)).$$

Finally, $\mathbb{G}$ is *descriptive* if $\mathbb{G}$ is differentiated, compact and tight.

For brevity, we will refer to descriptive general monotone frames simply as descriptive monotone frames. Note that the tightness condition is a natural requirement if one wishes to show that $\mathbb{G} \cong (\mathbb{G}^*)_*$ for a general monotone frame $\mathbb{G}$, since the neighbourhood function in the general ultrafilter frame of $\mathbb{G}^*$ will be of this form. Furthermore, the conditions of differentiation and compactness are the same as for general Kripke frames, hence we have the following fact.

Fact 2.10 *Let $\mathbb{G}$ be a general monotone frame. Then $\mathbb{G}$ is differentiated and compact iff $\mathbb{W}$ is a Stone space in which A forms a clopen basis.*

Let DMF be the category of descriptive monotone frames with general frame bounded morphisms. In [12] the following fundamental result is proved.

Theorem 2.11 *The categories DMF and BAM are dually equivalent.*

3 Coalgebras for monotone modal logic

3.1 Monotone frames as $\mathsf{Up}\mathcal{P}$-coalgebras

It is not difficult to see that monotone frames are coalgebras: A monotone frame is a pair of the type $(X, \nu : X \to 2^{2^X})$. However not every $2^{2^{(-)}}$-coalgebra corresponds to a monotone frame. The aim of this section is now to define a functor $\mathsf{Up}\mathcal{P} : \mathsf{Set} \to \mathsf{Set}$ such that the $\mathsf{Up}\mathcal{P}$-coalgebras and monotone frames are the same.

Definition 3.1 ($\mathsf{Up}\mathcal{P}$ on objects) Let X be a set, then the set $\mathsf{Up}\mathcal{P}X$ is defined as the set of all upward closed sets of subsets of X:

$$\mathsf{Up}\mathcal{P}X := \{W \subseteq \mathcal{P}X \mid \forall U_1, U_2 \in \mathcal{P}X.(U_1 \subseteq U_2 \wedge U_1 \in W) \Rightarrow U_2 \in W\}$$

Furthermore we define for an arbitrary $W \subseteq \mathcal{P}X$ its upward closure $\uparrow(W)$ as follows

$$\uparrow(W) := \{U \in \mathcal{P}X \mid \exists U' \in W.U' \subseteq U\}.$$

Lemma and Definition 3.2 *The following mapping defines a functor* $\mathsf{Up}\mathcal{P} : \mathsf{Set} \to \mathsf{Set}$*:*

$$X \mapsto \mathsf{Up}\mathcal{P}X$$
$$(f : X \to Y) \mapsto \mathsf{Up}\mathcal{P}f := (f^{-1})^{-1}[\_]$$

Proof. We first check whether the mapping is well-defined on morphisms. Let $f : X \to Y$ be a function and suppose $W \in \mathsf{Up}\mathcal{P}X$, then we have to show that $V := (f^{-1})^{-1}[W] \in \mathsf{Up}\mathcal{P}Y$. It is easy to see that $V \in \mathcal{P}\mathcal{P}Y$, so it suffices to show that V is upward closed. To that aim take an arbitrary $U_1 \in V$ and a $U_2 \in \mathcal{P}Y$ such that $U_1 \subseteq U_2$. Then by the definition of V we know that $f^{-1}[U_1] \in W$. Furthermore we know that $f^{-1}[U_1] \subseteq f^{-1}[U_2]$ and by the fact that W was upwards closed we get $f^{-1}[U_2] \in W$. But then also $U_2 \in V$. That $\mathsf{Up}\mathcal{P}$ satisfies the functorial laws, is not difficult to see and it follows immediately from the fact that it is a subfunctor of $2^{2^{(-)}}$. □

Before we establish the obvious connection between the categories MNF and $\mathsf{Coalg}(\mathsf{Up}\mathcal{P})$, we first want to get a better understanding of how $\mathsf{Up}\mathcal{P}$ acts on morphisms.

Lemma 3.3 *Let $f : X \to Y$ and suppose $W \in \mathsf{Up}\mathcal{P}X$. Then*

$$\mathsf{Up}\mathcal{P}f(W) = \uparrow(\{f[U'] \mid U' \in W\}).$$

Proof. Easy to check. □

It is clear that monotone frames (W, ν) and $\mathsf{Up}\mathcal{P}$-coalgebras are the same mathematical structures. Furthermore it is easy to show that bounded morphisms are $\mathsf{Up}\mathcal{P}$-coalgebra morphisms and vice versa.

Lemma 3.4 *The categories $\mathsf{Coalg}(\mathsf{Up}\mathcal{P})$ and MNF are isomorphic.*

Proof. For a detailed proof we refer the reader to [12]. □

In the following we will not make any distinction between the notions $\mathsf{Up}\mathcal{P}$-coalgebras and monotone frames and between $\mathsf{Up}\mathcal{P}$-coalgebra morphisms and bounded morphisms.

Remark 3.5 One might ask why we do not consider the category of $2^{2^{(-)}}$-coalgebras as the coalgebraic analogue of monotone frames. Indeed this would work well on objects: Given a $2^{2^{(-)}}$-coalgebra $(W, \nu : W \to 2^{2^W})$ the corresponding monotone frame would be the pair $(W, \nu' := \uparrow \circ \nu)$. Morphisms of the category $\mathsf{Coalg}(2^{2^{(-)}})$ would however no longer correspond to bounded morphisms.

3.2 Descriptive monotone frames as $\mathbb{U}\mathsf{pV}$*-coalgebras*

In [17] it has been argued that coalgebras for a functor $T : \mathsf{Stone} \to \mathsf{Stone}$ are interesting from a normal modal logic perspective. This will also turn out to be true in the case of monotone modal logic. We will first define a functor $\mathbb{U}\mathsf{pV} : \mathsf{Stone} \to \mathsf{Stone}$ and then show that the category of descriptive monotone frames is isomorphic to the category $\mathsf{Coalg}(\mathbb{U}\mathsf{pV})$. As corollaries we obtain a duality between the categories $\mathsf{Coalg}(\mathbb{U}\mathsf{pV})$ and BAM, a representation of BAMs as algebras for a functor and the existence of the final $\mathbb{U}\mathsf{pV}$-coalgebra.

Definition 3.6 ($\mathbb{U}\mathsf{pV}$ on objects) Let $\mathbb{X} = (X, \tau) \in \mathsf{Stone}$, and let $U \subseteq K(\mathbb{X})$, then U is *upwards closed (upc)* if for all $F, F' \in K(\mathbb{X})$ such that $F \subseteq F'$, $F \in U$ implies $F' \in U$. U is called $[\ni]$*-closed* if F belongs to U for all closed sets F satisfying $a \in U$ for all clopens $a \supseteq F$. We define $\mathbb{U}\mathsf{pV}(X, \tau) := \{U \in K(\mathbb{V}\mathbb{X}) \mid U \text{ is upc and } [\ni]\text{-closed } \}$.

Remark 3.7 The requirement for all sets to be $[\ni]$-closed is one half of the tightness condition for a general monotone frame. The terminology $[\ni]$-closed has been chosen because a set U is $[\ni]$-closed iff for all $F \in -U$ there is an $a \in \mathsf{Clp}(\mathbb{X})$ such that $F \in [\ni]a$ and $U \cap [\ni]a = \emptyset$.

It is clear that $\mathbb{U}\mathsf{pV}\mathbb{X} \subseteq \mathbb{V}\mathbb{V}\mathbb{X}$. Thus if we can show that $\mathbb{U}\mathsf{pV}\mathbb{X}$ is a closed subset in $\mathbb{V}\mathbb{V}\mathbb{X}$, then it follows that $\mathbb{U}\mathsf{pV}\mathbb{X}$ is a Stone space with the relative topology.

Lemma 3.8 *Let* $\mathbb{X} = (X, \tau) \in \mathsf{Stone}$*. Then* $\mathbb{U}\mathsf{pV}\mathbb{X}$ *is a closed subset in* $\mathbb{V}\mathbb{V}\mathbb{X}$*.*

Proof. The proof uses only standard techniques from general topology and has to be omitted here for lack of space. $\square$

Definition 3.9 ($\mathbb{U}\mathsf{pV}$-functor) The map $\mathbb{U}\mathsf{pV} : \mathsf{Stone} \to \mathsf{Stone}$ is defined as follows. For a Stone space $\mathbb{X} = (X, \tau)$,

$$\mathbb{U}\mathsf{pV}\mathbb{X} = (\{U \in K(\mathbb{V}\mathbb{X}) \mid U \text{ is upc and } [\ni]\text{-closed } \}, \tau')$$

where τ' is the relative topology induced by the topology on $\mathbb{V}\mathbb{V}\mathbb{X}$. For a continuous function $f : \mathbb{X} \to \mathbb{Y}$,

$$\begin{aligned} \mathbb{U}\mathsf{pV}f : \mathbb{U}\mathsf{pV}\mathbb{X} &\to \mathbb{U}\mathsf{pV}\mathbb{Y} \\ U &\mapsto \mathbb{U}\mathsf{pV}f(U) = (f^{-1})^{-1}[U] = 2^{2^f}(U) \\ &\qquad = \{D \in K(\mathbb{Y}) \mid f^{-1}[D] \in U\} \end{aligned}$$

It follows from Lemma 3.8 that $\mathbb{U}\mathsf{pV}$ is well-defined on objects. The next two lemmas show that $\mathbb{U}\mathsf{pV}f$ is also well-defined on morphisms.

Lemma 3.10 *Let* $f : \mathbb{X} \to \mathbb{Y} \in \mathsf{Stone}$*, where* $\mathbb{X} = (X, \tau_X)$*,* $\mathbb{Y} = (Y, \tau_Y)$*, and* $U \in \mathbb{U}\mathsf{pV}\mathbb{X}$*. Then* $\mathbb{U}\mathsf{pV}f(U) \in \mathbb{U}\mathsf{pV}\mathbb{Y}$*.*

Proof. That $\mathbb{U}\mathsf{pV}f(U)$ is upc is easy to check. It remains to show that $\mathbb{U}\mathsf{pV}f(U) = (f^{-1})^{-1}[U]$ is $[\ni]$-closed. The proof uses a standard compactness argument and is left out due to lack of space. □

Lemma 3.11 *The function* $\mathbb{U}\mathsf{pV}f : \mathbb{U}\mathsf{pV}\mathbb{X} \to \mathbb{U}\mathsf{pV}\mathbb{Y}$ *is continuous.*

Proof. For completeness reasons we provide the rather technical proof. Readers who are not interested in the technical details are advised to skip it. It suffices to show that $(\mathbb{U}\mathsf{pV}f)^{-1}\,[[\ni]b]$ and $(\mathbb{U}\mathsf{pV}f)^{-1}\,[\langle\ni\rangle b]$ is clopen for an arbitrary clopen subset b of $\mathbb{V}\mathbb{Y}$. We are only going to consider the case in which we are dealing with a set of the form $[\ni]b$. The $\langle\ni\rangle$-case can be treated in a similar way. Note that if $b \in \mathsf{Clp}(\mathbb{V}\mathbb{Y})$ then there is a suitable family of clopen sets $a_i^k \in \mathsf{Clp}(\mathbb{Y})$ such that $b = \bigcap_{k=1}^{n} \bigcup_{i=1}^{m_k} M_i^k a_i^k$ where $M_i^k \in \{[\ni], \langle\ni\rangle\}$. Because $[\ni]$ distributes over meets we get: $[\ni]b = \bigcap_{k=1}^{n} [\ni] \bigcup_{i=1}^{m_k} M_i^k a_i^k$. Hence it suffices to look at a set b of the form $\bigcup_{i=1}^{m'} M_i a_i$ ($M_i \in \{[\ni], \langle\ni\rangle\}$). Because $\langle\ni\rangle$ distributes over joins this set is of the form $\bigcup_{i=1}^{m} [\ni]a_i \cup \langle\ni\rangle a$. So to sum it up we have to show that

$$\begin{aligned} Z &:= (\mathbb{U}\mathsf{pV}f)^{-1}\left[[\ni]\bigcup_{i=1}^{m}[\ni]a_i \cup \langle\ni\rangle a\right] \\ &= (\mathbb{U}\mathsf{pV}f)^{-1}\left[\{U \in \mathbb{U}\mathsf{pV}\mathbb{Y} \mid \forall F \in U.(\exists i.F \subseteq a_i) \text{ or } F \cap a \neq \emptyset\}\right] \end{aligned}$$

is clopen for arbitrary $a_i, a \in \mathsf{Clp}(\mathbb{Y})$. First we fix for any $i \in \{1, \ldots, m\}$ a set $U_i \in \{U \in \mathbb{U}\mathsf{pV}\mathbb{Y} \mid \forall F \in U.(\exists i.F \subseteq a_i) \text{ or } F \cap a \neq \emptyset\}$ and a set $F_i \in U_i$ such that $F_i \subseteq a_i$ and $F_i \cap a = \emptyset$. We may assume that this is possible because suppose that for one i' there is no such $U_{i'}$ and $F_{i'}$. Then

$$\begin{aligned} &\{U \in \mathbb{U}\mathsf{pV}\mathbb{Y} \mid \forall F \in U.(\exists i.F \subseteq a_i) \text{ or } F \cap a \neq \emptyset\} = \\ &\{U \in \mathbb{U}\mathsf{pV}\mathbb{Y} \mid \forall F \in U.(\exists i.i \neq i' \& F \subseteq a_i) \text{ or } F \cap a \neq \emptyset\} \end{aligned}$$

and we can forget about the $a_{i'}$. We now claim that

$$\begin{aligned} Z &\overset{(*)}{=} \{V \in \mathbb{U}\mathsf{pV}\mathbb{X} \mid \forall G \in V.(\exists i.f[G] \subseteq a_i) \text{ or } f[G] \cap a \neq \emptyset\} \\ &= [\ni]\bigcup_{i=1}^{m}[\ni]f^{-1}[a_i] \cup \langle\ni\rangle f^{-1}[a] \end{aligned}$$

and the last set is clearly a clopen subset of $\mathbb{U}\mathsf{pV}\mathbb{X}$. It remains to show that $(*)$ is indeed true. The $\subseteq$-part of the equation is easy to check. So we will focus on the $\supseteq$-part. Let $V \in \{V \in \mathbb{U}\mathsf{pV}\mathbb{X} \mid \forall G \in V.(\exists i.f[G] \subseteq a_i) \text{ or } f[G] \cap a \neq \emptyset\}$. We have to prove that

$$\mathbb{U}\mathsf{pV}f[V] \in \{U \in \mathbb{U}\mathsf{pV}\mathbb{Y} \mid \forall F \in U.(\exists i.F \subseteq a_i) \text{ or } F \cap a \neq \emptyset\}.$$

Let $F' \in \mathbb{U}\mathsf{pV} f[V]$, i.e. there is a $G \in V$ such that $f[G] \subseteq F'$.

Case: $f[G] \cap a \neq \emptyset$. Then clearly also $F' \cap a \neq \emptyset$.

Case: $f[G] \subseteq a_i$ for some i. Suppose for a contradiction that for all $i \in \{1, \ldots, m\}$ we have $F' \not\subseteq a_i$ and that $F' \cap a = \emptyset$. We now define $\bar{F} := F' \cup F_i$. Then $\bar{F} \in U_i$ as U_i is upwards closed. Furthermore $\bar{F} \not\subseteq a_{i'}$ for all $i' \in \{1, \ldots, m\}$ (otherwise $F' \subseteq a_{i'}$). As $\bar{F} \in U_i$ we therefore must have $\bar{F} \cap a \neq \emptyset$. But this implies $F' \cap a \neq \emptyset$ as $F_i \cap a = \emptyset$ and we have arrived at a contradiction. □

Now that we have defined the functor $\mathbb{U}\mathsf{pV}$ we are able to prove the main result of this section.

Theorem 3.12 *The category* DMF *of descriptive monotone frames is isomorphic to the category* $\mathit{Coalg}(\mathbb{U}\mathsf{pV})$.

Proof. We just define the functors $\mathbb{C}\mathsf{oa} : \mathsf{DMF} \to \mathsf{Coalg}(\mathbb{U}\mathsf{pV})$ and $\mathbb{D}\mathsf{mf} : \mathsf{Coalg}(\mathbb{U}\mathsf{pV}) \to \mathsf{DMF}$ which induce the isomorphism between the categories. The proof that we have an isomorphism is merely spelling out the definitions and is omitted.
We first define the functor $\mathbb{C}\mathsf{oa}$:

$$\mathbb{C}\mathsf{oa}(W, \nu, A) := (\mathbb{W}, \underline{\nu} : \mathbb{W} \to \mathbb{U}\mathsf{pV}\mathbb{W}) \qquad \mathbb{C}\mathsf{oa} f := f$$

where $\mathbb{W} = (W, \tau_A)$ and $\underline{\nu}(w) := \{F \subseteq W \mid F \in K(\mathbb{W}) \text{ and } F \in \nu(w)\}$. The map $\underline{\nu}$ is well-defined because the tightness condition on (W, ν, A) ensures that $\underline{\nu}(w)$ is $[\ni]$-closed. Fact 2.10 gives us that the functor is well-defined on objects. Let us now define $\mathbb{D}\mathsf{mf}$:

$$\mathbb{D}\mathsf{mf}(\mathbb{W}, \nu) := (W, \overline{\nu}, \mathsf{Clp}(\mathbb{W})) \qquad \mathbb{D}\mathsf{mf} f := f$$

where $\overline{\nu}(w) = \{U \subseteq W \mid \exists F \in \nu(w). F \subseteq U\}$. Again Fact 2.10 gives us that $(W, \overline{\nu}, \mathsf{Clp}(\mathbb{W}))$ is differentiated and compact. Tightness follows from the $[\ni]$-closedness of $\nu(w)$.
Furthermore, it is straightforward to prove that a bounded morphism $f \in \mathsf{DMF}$ is also an $\mathbb{U}\mathsf{pV}$-coalgebra morphism and vice versa, a fact which is needed in the definition both of $\mathbb{C}\mathsf{oa}$ and $\mathbb{D}\mathsf{mf}$. □

To round up the section we state some immediate consequences of the theorem.

Corollary 3.13 *The categories* BAM *and* $\mathit{Coalg}(\mathbb{U}\mathsf{pV})$ *are dually equivalent.*

Proof. Follows directly from the duality between BAM and DMF (cf. Theorem 2.11) and Theorem 3.12. □

Corollary 3.14 *There is a functor* $L : \mathsf{BA} \to \mathsf{BA}$ *such that* $\mathsf{Alg}(L) \cong \mathsf{BAM}$.

Proof. An immediate consequence of $\mathsf{Coalg}(\mathbb{U}\mathsf{pV})^{\mathrm{op}} \cong \mathsf{BAM}$ and Stone duality. □

Corollary 3.15 *The category* $\mathsf{Coalg}(\mathbb{U}\mathsf{pV})$ *has cofree coalgebras, in particular the final* $\mathbb{U}\mathsf{pV}$*-coalgebra exists.*

Proof. Consequence of the duality and the fact that the forgetful functor from BAM to BA has a left adjoint. □

4 Equivalence notions

In section 3 we established the equivalence between $\mathsf{Up}\mathcal{P}$-coalgebras and monotone frames, and we will now investigate the relationship between the various equivalence notions associated with these two perspectives on monotone modal logic.

In subsection 4.1 we will show that coalgebraic $\mathsf{Up}\mathcal{P}$-bisimulation is strictly stronger than the logical notion of monotone bisimulation (Definition 4.1), and in subsection 4.2 the main result states that behavioural equivalence is equivalent with monotone bisimilarity. These results allow us to conclude that the $\mathsf{Up}\mathcal{P}$-functor does not preserve weak pullbacks (Corollary 4.10).

4.1 Bisimulation

The central notion of model equivalence in modal logic is that of bisimulation. Bisimulations for Kripke models were introduced by van Benthem in [3], where one also finds the well-known characterisation result which states that (normal) modal logic is the (Kripke) bisimulation invariant fragment of first-order logic.

For monotone (neighbourhood) models, monotone bisimulations have been presented in Pauly [28], which also contains a number of results on the relationship between Kripke and monotone bisimulations together with a version of the van Benthem characterisation theorem for monotone modal logic. This last result, which is also included in [12], is a strong argument for monotone bisimulation being the correct logical notion of model equivalence.

Definition 4.1 (Monotone bisimulation) Let $(X_1, \nu_1), (X_2, \nu_2) \in \mathsf{MNF}$. A non-empty relation $Z \subseteq X_1 \times X_2$ is a *monotone bisimulation* between (X_1, ν_1) and (X_2, ν_2) if for all $x_1 \in X_1$ and $x_2 \in X_2$ such that $x_1 Z x_2$, the following two conditions are satisfied.

(forth) $\forall C_1 \in \nu_1(x_1).\exists C_2 \in \nu_2(x_2)$ such that $(\forall c_2 \in C_2.\exists c_1 \in C_1 : c_1 Z c_2)$.
(back) $\forall C_2 \in \nu_2(x_2).\exists C_1 \in \nu_1(x_1)$ such that $(\forall c_1 \in C_1.\exists c_2 \in C_2 : c_1 Z c_2)$.

If $dom(Z) = X_1$ and $ran(Z) = X_2$, then we will call Z a *full monotone bisimulation*. Two monotone frames (X_1, ν_1) and (X_2, ν_2) are said to be *(full) monotone bisimilar* if there is a (full) monotone bisimulation between them, and two states $x_1 \in X_1$ and $x_2 \in X_2$ are called *(full) monotone bisimilar states* if there is a (full) monotone bisimulation Z between (X_1, ν_1) and (X_2, ν_2) such that $x_1 Z x_2$.

For the $\mathsf{Up}\mathcal{P}$-functor, $\mathsf{Up}\mathcal{P}$-bisimulation amounts to the following.

Definition 4.2 ($\mathsf{Up}\mathcal{P}$-bisimulation) Let $(X_1, \nu_1), (X_2, \nu_2) \in \mathsf{Coalg}(\mathsf{Up}\mathcal{P})$. A non-empty relation $Z \subseteq X_1 \times X_2$ is an $\mathsf{Up}\mathcal{P}$*-bisimulation* between (X_1, ν_1) and (X_2, ν_2) if there is a function $\mu : Z \to \mathsf{Up}\mathcal{P}Z$ such that: $\nu_1 \circ \pi_1 = \mathsf{Up}\mathcal{P}\pi_1 \circ \mu$ and $\nu_2 \circ \pi_2 = \mathsf{Up}\mathcal{P}\pi_2 \circ \mu$.

$$\begin{tikzcd} X_1 \arrow[d, "\nu_1"'] & Z \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, dashed, "\mu"'] & X_2 \arrow[d, "\nu_2"] \\ \mathsf{Up}\mathcal{P}X_1 & \mathsf{Up}\mathcal{P}Z \arrow[l, "\mathsf{Up}\mathcal{P}\pi_1"'] \arrow[r, "\mathsf{Up}\mathcal{P}\pi_2"] & \mathsf{Up}\mathcal{P}X_2 \end{tikzcd}$$

We will say that (X_1, ν_1) and (X_2, ν_2) are $\mathsf{Up}\mathcal{P}$*-bisimilar via* (Z, μ), if μ makes the above diagram commute. If $\pi_i : Z \twoheadrightarrow X_i$, $i \in \{1, 2\}$, that is, the projections are surjective, then Z is called a *full* $\mathsf{Up}\mathcal{P}$*-bisimulation*.

Thus in order to show that Z is an $\mathsf{Up}\mathcal{P}$-bisimulation between two $\mathsf{Up}\mathcal{P}$-coalgebras (X_1, ν_1) and (X_2, ν_2), we must be able to endow Z with a coalgebraic structure μ in such a way that the projections $\pi_i : (Z, \mu) \to (X_i, \nu_i)$ are bounded morphisms. For $i \in \{1, 2\}$, π_i is a bounded morphism if and only if

$$\forall C_i \subseteq X_i.(C_i \in \nu_i(x_i) \iff \pi_i^{-1}[C_i] \in \mu(x_1, x_2)). \tag{3}$$

Or equivalently, for all $Y \subseteq Z$, $C_i \subseteq X_i$,

(BM1) $Y \in \mu(x_1, x_2)$ implies $\pi_i[Y] \in \nu_i(x_i)$,
(BM2) $C_i \in \nu_i(x_i)$ implies $\pi_i^{-1}[C_i] \in \mu(x_1, x_2)$.

These conditions provide us with concrete constraints that $\mu : Z \to \mathsf{Up}\mathcal{P}Z$ must satisfy in addition to upwards closure. Condition (BM1) tells us when we are *allowed* to add some $Y \subseteq Z$ to $\mu(x_1, x_2)$, namely when both $\pi_1[Y] \in \nu_1(x_1)$

and $\pi_2[Y] \in \nu_2(x_2)$. That is (BM1) gives rise to a *largest* μ. On the other hand, (BM2) tells us when we *must* add some Y to $\mu(x_1, x_2)$, namely when $\pi_i^{-1}[C_i] \subseteq Y$ for some $C_i \in \nu_i(x_i)$ and some $i \in \{1, 2\}$, thus giving rise to a *smallest* μ. We now give a formal definition.

Definition 4.3 (μ_s and μ_l) Let $(X_1, \nu_1), (X_2, \nu_2) \in \mathsf{Coalg}(\mathsf{Up}\mathcal{P})$, and $\emptyset \neq Z \subseteq X_1 \times X_2$ be given. Then we define $\mu_s, \mu_l : Z \to \mathsf{Up}\mathcal{P}Z$ as follows.

$$Y \in \mu_s(x_1, x_2) \text{ iff } \exists C_1 \in \nu_1(x_1).\pi_1^{-1}[C_1] \subseteq Y \text{ or } \exists C_2 \in \nu_2(x_2).\pi_2^{-1}[C_2] \subseteq Y. \tag{4}$$

$$Y \in \mu_l(x_1, x_2) \quad \text{iff} \quad \pi_1[Y] \in \nu_1(x_1) \text{ and } \pi_2[Y] \in \nu_2(x_2). \tag{5}$$

It should be clear from the definition that both μ_s and μ_l are upwards closed, and for any μ satisfying (3), we have $\mu_s(x_1, x_2) \subseteq \mu(x_1, x_2) \subseteq \mu_l(x_1, x_2)$. There is, of course, no guarantee that μ_s and μ_l themselves will satisfy both bounded morphism conditions. The next proposition tells us when this is the case.

Proposition 4.4 *Let $(X_1, \nu_1), (X_2, \nu_2) \in \mathsf{Coalg}(\mathsf{Up}\mathcal{P})$, and $\emptyset \neq Z \subseteq X_1 \times X_2$. Then Z is an $\mathsf{Up}\mathcal{P}$-bisimulation between (X_1, ν_1) and (X_2, ν_2) if and only if Z satisfies the following two conditions for all $(x_1, x_2) \in Z$.*

($\mathsf{Up}\mathcal{P}$-***forth***) $\forall C_1 \in \nu_1(x_1).\exists C_2 \in \nu_2(x_2)$ *such that* $\forall c_2 \in C_2.\exists c_1 \in C_1$ *s.t.* $c_1 Z c_2$ *and* $Z^{-1}[C_2] \cap C_1 \in \nu_1(x_1)$.

($\mathsf{Up}\mathcal{P}$-***back***) $\forall C_2 \in \nu_2(x_2).\exists C_1 \in \nu_1(x_1)$ *such that* $\forall c_1 \in C_1.\exists c_2 \in C_2$ *s.t.* $c_1 Z c_2$ *and* $Z[C_1] \cap C_2 \in \nu_2(x_2)$.

Proof. We only sketch the proof. For the direction from left to right, it is straightforward to check that the ($\mathsf{Up}\mathcal{P}$-**forth**) and ($\mathsf{Up}\mathcal{P}$-**back**) conditions hold for an $\mathsf{Up}\mathcal{P}$-bisimulation. For the direction from right to left, one can show that when Z satisfies the ($\mathsf{Up}\mathcal{P}$-**forth**) and ($\mathsf{Up}\mathcal{P}$-**back**) conditions then (X_1, ν_1) and (X_2, ν_2) are $\mathsf{Up}\mathcal{P}$-bisimilar via both (Z, μ_s) and (Z, μ_l). □

Due to the above characterisation, the next corollary is immediate.

Corollary 4.5 *If Z is an $\mathsf{Up}\mathcal{P}$-bisimulation, then Z is also a monotone bisimulation.*

As we announced at the beginning of this subsection, $\mathsf{Up}\mathcal{P}$-bisimulations are a strict subset of monotone bisimulations. This is shown by the following example.

Example 4.6 Consider the monotone frames $\mathbb{F}_1 = (\{s_1, t_1, u_1, v_1\}, \nu_1)$ where $\nu_1(s_1) = \uparrow(\{\{t_1\}, \{u_1, v_1\}\})$, $\nu_1(u_1) = \uparrow(\{\{u_1\}\})$ and $\nu_1(t_1) = \nu_1(v_1) = \emptyset$; and $\mathbb{F}_2 = (\{s_2, t_2\}, \nu_2)$ where $\nu_2(s_2) = \uparrow(\{\{t_2\}\})$ and $\nu_2(t_2) = \emptyset$. Then $Z = \{(s_1, s_2), (t_1, t_2), (v_1, t_2)\}$ is a monotone bisimulation. In fact, Z is the maximal monotone bisimulation on $\mathbb{F}_1$ and $\mathbb{F}_2$. But Z does not satisfy ($\mathsf{Up}\mathcal{P}$**-forth**) for the neighbourhood $\{u_1, v_1\} \in \nu_1(s_1)$, since $Z^{-1}[\{t_2\}] \cap \{u_1, v_1\} = \{v_1\} \notin \nu_1(s_1)$. This problem will occur for any monotone bisimulation linking s_1 and s_2, since u_1 is not monotone bisimilar with any state in F_2, thus s_1 and s_2 are monotone bisimilar, but not $\mathsf{Up}\mathcal{P}$-bisimilar.

The ($\mathsf{Up}\mathcal{P}$**-forth**)-condition will fail for a monotone bisimulation Z if, for example, $Y_1 \in \nu_1(x_1)$, $Y_1 \nsubseteq dom(Z)$ and for all $X_1 \subsetneq Y_1, X_1 \notin \nu_1(x_1)$, since then $Z^{-1}[Y_2] \cap Y_1 \subsetneq Y_1$ for any Y_2, and hence $Z^{-1}[Y_2] \cap Y_1 \notin \nu_1(x_1)$. This failure can be eliminated if we consider full monotone bisimulations. We leave the easy proof of the following lemma to the reader.

Lemma 4.7 *Let $(X_1, \nu_1), (X_2, \nu_2) \in \mathit{Coalg}(\mathsf{Up}\mathcal{P})$, and $Z \subseteq X_1 \times X_2$. Then the following holds: If Z is a full monotone bisimulation, then Z is an $\mathsf{Up}\mathcal{P}$-bisimulation.*

The next example shows that $\mathsf{Up}\mathcal{P}$-bisimulations need not be full.

Example 4.8 Let $\mathbb{F}_1 = (\{s_1, t_1, u_1\}, \nu_1)$ and $\mathbb{F}_2 = (\{s_2, t_2\})$ where $\nu_i(s_i) = \uparrow(\{\{t_i\}\})$, $\nu_i(t_i) = \emptyset$, for $i \in \{1, 2\}$, and $\nu_1(u_1) = \uparrow(\{\{u_1\}\})$. Then $Z = \{(s_1, s_2), (t_1, t_2)\}$ is a (maximal) $\mathsf{Up}\mathcal{P}$-bisimulation, but Z is clearly not full.

4.2 Behavioural equivalence

As it turns out, the concept of behavioural equivalence ties in better with the frame theoretic notion of bisimulation, and we will now show that two states are monotone bisimilar if and only if they are behaviourally equivalent (Theorem 4.9).

Recall from Definition 2.8 that for two $\mathsf{Up}\mathcal{P}$-coalgebras (X_1, ν_1) and (X_2, ν_2), two states $s_1 \in X_1$ and $s_2 \in X_2$ are behaviourally equivalent if they can be identified via two bounded morphisms $f_i : X_i \to Y$, $i \in \{1, 2\}$, in some $\mathsf{Up}\mathcal{P}$-coalgebra (Y, δ). The behavioural equivalence induces a relation on $X_1 \times X_2$,

$$\mathsf{pb}(f_1, f_2) = \{(x_1, x_2) \in X_1 \times X_2 \mid f_1(x_1) = f_2(x_2)\},$$

and it is well-known that $(\mathsf{pb}(f_1, f_2), \pi_1, \pi_2)$ is the pullback of f_1 and f_2 (in Set), and $\pi_1 \circ f_1 = \pi_2 \circ f_2$. It is straightforward to show that the $\mathsf{pb}(f_1, f_2)$-relation is, in fact, a monotone bisimulation. Thus behavioural equivalence between states implies monotone bisimilarity. The main result of this section states that the other implication holds as well.

Theorem 4.9 (State equivalence) *Let $(X_1, \nu_1), (X_2, \nu_2)$ be in $\mathsf{Coalg}(\mathsf{Up}\mathcal{P})$. Two states $s_1 \in X_1$ and $s_2 \in X_2$ are behaviourally equivalent if and only if s_1 and s_2 are monotone bisimilar.*

Proof. Due to lack of space we only sketch the proof, which may be found in [12] together with the definition of disjoint union and bisimulation quotient which are the two basic constructions needed for the proof. For the direction from left to right one can show that pb-relations are monotone bisimulations. For the other direction, we must construct a monotone frame into which (X_1, ν_1) and (X_2, ν_2) can be embedded. We do so by first forming the disjoint union $(X_1 + X_2, \nu_{1+2})$ of (X_1, ν_1) and (X_2, ν_2). Then the inclusion maps $\kappa_i : X_i \to X_1 + X_2$, $i \in \{1, 2\}$, are bounded morphisms. Furthermore, Z is contained in the maximal bisimulation Z_M on $(X_1 + X_2, \nu_{1+2})$, and by taking the bisimulation quotient $(Y, \gamma) := (X_1 + X_2, \nu_{1+2})/Z_M$, the natural map $\varepsilon : (X_1 + X_2) \to Y = (X_1 + X_2)/Z_M$ is a bounded morphism. Now define $f_i := \varepsilon \circ \kappa_i$,then it follows that $f_i : X_i \to Y$, $i \in \{1, 2\}$, are bounded morphisms and $f_1(s_1) = f_2(s_2)$. □

Rutten [31] shows that for functors T which preserve weak pullbacks, the $\mathsf{pb}(f_1, f_2)$-relations are also T-bisimulations.

Since we know that $\mathsf{Up}\mathcal{P}$-bisimulation really is a stronger concept than monotone bisimulation, we obtain the following corollary from Theorem 4.9.

Corollary 4.10 *The functor $\mathsf{Up}\mathcal{P}$ does not preserve weak pullbacks.*

Example 4.11 As a specific example of a $\mathsf{pb}(f_1, f_2)$-relation which is not an $\mathsf{Up}\mathcal{P}$-bisimulation, consider again the frames $\mathbb{F}_1$ and $\mathbb{F}_2$ from Example 4.6 together with the following isomorphic copy of $\mathbb{F}_2$: $\mathbb{G} = (Y, \mu)$ where $Y = \{x, y\}, \mu(x) = \uparrow(\{\{y\}\}), \mu(y) = \emptyset$. Let $f_i : W_i \to Y$, $i \in \{1, 2\}$, be defined by $f_1(s_1) = f_2(s_2) = x$ and $f_1(u_1) = f_1(v_1) = f_2(t_2) = y$. Then f_1 and f_2 are bounded morphisms and $\mathsf{pb}(f_1, f_2) = Z = \{(s_1, s_2), (t_1, t_2), (v_1, t_2)\}$. But as we already know, there is no $\mathsf{Up}\mathcal{P}$-bisimulation linking s_1 and s_2.

The distinction between behavioural equivalence and $\mathsf{Up}\mathcal{P}$-bisimilarity fades when we look at system equivalence rather than equivalence between states.

Theorem 4.12 (System equivalence) *Let $(X_1, \nu_1), (X_2, \nu_2) \in \mathsf{Coalg}(\mathsf{Up}\mathcal{P})$. Then the following are equivalent.*

(i) (X_1, ν_1) *and* (X_2, ν_2) *are behaviourally equivalent systems.*
(ii) *There exists a full* $\mathsf{Up}\mathcal{P}$*-bisimulation between* (X_1, ν_1) *and* (X_2, ν_2).

Proof. Again, we only provide a sketch.
(i) ⇒ (ii): Behavioural equivalence of two systems requires that the systems

can be *surjectively* mapped onto a third system, hence the obtained $\mathsf{pb}(f_1, f_2)$-relation is a full monotone bisimulation, which is also a full $\mathsf{Up}\mathcal{P}$-bisimulation by Lemma 4.7.
(ii) $\Rightarrow$ (i): This can be proved similarly to Theorem 4.9. All we need to observe is that the constructed $f_i = \varepsilon \circ \kappa_i$ are surjective, since the maximal bisimulation on $(X_1 + X_2, \nu_{1+2})$ is full. $\square$

5 Interpolation

In this section, we will demonstrate how our results on bisimulations can be combined with algebraic duality into a general test for interpolation (via superamalgamation) in monotone modal logics (Lemma 5.4).

Superamalgamation (SUPAP) of varieties has provided algebraic characterizations of the Craig Interpolation Property (CIP) for a large class of modal logics, where it is possible to show that: Λ has CIP iff V_Λ has SUPAP. However, we have found only little in the literature regarding interpolation in monotone modal logics or superamalgamation in BAM-varieties. One of the few sources is Madarász [21] who generalises results for BAO-varieties to varieties of Boolean algebras expanded with an operation f which is non-normal, i.e. $f(0) \neq 0$, but still additive. Madarász [22] also provides some results on the limitations of the CIP-SUPAP relationship.

We start by recalling the definitions of Craig interpolation and superamalgamation. For an $\mathcal{L}$-formula φ, let $\mathrm{FV}(\varphi)$ denote the set of proposition letters occurring in φ. A modal logic Λ over the language $\mathcal{L}$ has the *Craig Interpolation Property (CIP)* if for any $\mathcal{L}$-formulas φ, ψ such that $\varphi \to \psi \in \Lambda$, there is an $\mathcal{L}$-formula θ such that $\mathrm{FV}(\theta) \subseteq \mathrm{FV}(\varphi) \cap \mathrm{FV}(\psi)$ and $\varphi \to \theta \in \Lambda$, $\theta \to \psi \in \Lambda$. θ is called an *interpolant*.

Let K be a class of algebras such that each $\mathbb{A} \in \mathsf{K}$ has a partial ordering. K has the *superamalgamation property (SUPAP)* if, for any $\mathbb{A}_0, \mathbb{A}_1, \mathbb{A}_2 \in \mathsf{K}$ and embeddings e_1, e_2 such that $\mathbb{A}_1 \overset{e_1}{\leftarrowtail} \mathbb{A}_0 \overset{e_2}{\rightarrowtail} \mathbb{A}_2$, there exists an $\mathbb{A} \in \mathsf{K}$ and embeddings g_1, g_2 such that

1. $\mathbb{A}_1 \overset{g_1}{\rightarrowtail} \mathbb{A} \overset{g_2}{\leftarrowtail} \mathbb{A}_2$.
2. $g_1 \circ e_1 = g_2 \circ e_2$.
3. $\forall x_1 \in \mathbb{A}_1, \forall x_2 \in \mathbb{A}_2$,

$$g_1(x_1) \leq g_2(x_2) \Rightarrow \exists x_0 \in \mathbb{A}_0 : x_1 \leq e_1(x_0) \text{ and } e_2(x_0) \leq x_2$$
$$g_2(x_2) \leq g_1(x_1) \Rightarrow \exists x_0 \in \mathbb{A}_0 : x_2 \leq e_2(x_0) \text{ and } e_1(x_0) \leq x_1.$$

Showing that Λ has CIP under the assumption that V_Λ has SUPAP can be done under very general circumstances (see [12]), but we formulate the result for monotone modal logic.

Theorem 5.1 (SUPAP $\Rightarrow$ CIP) *Let Λ be a monotone modal logic over the language $\mathcal{L}$, and V_Λ the variety of* BAM*s defined by Λ. Then Λ has CIP if V_Λ has SUPAP.*

Marx [23] provides sufficient conditions for SUPAP formulated in terms of Kripke frames, and here we will prove a version for monotone frames, or equivalently for $\mathsf{Up}\mathcal{P}$-coalgebras, in Lemma 5.4. The construction involves *bisimulation products*, which we introduce now.

Definition 5.2 (Bisimulation product) Let (X_1, ν_1) and (X_2, ν_2) be $\mathsf{Up}\mathcal{P}$-coalgebras, $Z \subseteq X_1 \times X_2$, and $\mu : Z \to \mathsf{Up}\mathcal{P}Z$. Then (Z, μ) is a *bisimulation product* of (X_1, ν_1) and (X_2, ν_2) if (X_1, ν_1) and (X_2, ν_2) are full $\mathsf{Up}\mathcal{P}$-bisimilar via (Z, μ).

From Proposition 4.4 and Lemma 4.7 we know that when (X_1, ν_1) and (X_2, ν_2) are behaviourally equivalent systems via the surjections f_1 and f_2, then bisimulation products of (X_1, ν_1) and (X_2, ν_2) exist. Namely, by taking $Z = \mathsf{pb}(f_1, f_2)$, then (Z, μ_s) and (Z, μ_l) are both bisimulation products of (X_1, ν_1) and (X_2, ν_2), and for $\mu \in \{\mu_s, \mu_l\}$, the diagram below commutes.

$$\begin{tikzcd}
(Z,\mu) \arrow[r, two heads, "\pi_2"] \arrow[d, two heads, "\pi_1"'] & (X_2,\nu_2) \arrow[d, two heads, "f_2"] \\
(X_1,\nu_1) \arrow[r, two heads, "f_1"'] & (Y,\gamma)
\end{tikzcd}$$

For a class F of monotone frames, we will say that F *has bisimulation products* if for any $\mathbb{F}_1, \mathbb{F}_2 \in \mathsf{F}$, such that $\mathbb{F}_1$ and $\mathbb{F}_2$ are behaviourally equivalent systems via the surjections f_1 and f_2, there is a $\mu : \mathsf{pb}(f_1, f_2) \to \mathsf{Up}\mathcal{P}\mathsf{pb}(f_1, f_2)$ such that $(\mathsf{pb}(f_1, f_2), \mu) \in \mathsf{F}$. That is, we must be able to choose the neighbourhood function μ such that the full $\mathsf{Up}\mathcal{P}$-bisimulation induced by the system equivalence equipped with this μ is in F. Thus, in particular, one can show that μ_s or μ_l turns $\mathsf{pb}(f_1, f_2)$ into a frame of the right kind, and we will say that F *has smallest bisimulation products* if $(\mathsf{pb}(f_1, f_2), \mu_s) \in \mathsf{F}$, and F *has largest bisimulation products* if $(\mathsf{pb}(f_1, f_2), \mu_l) \in \mathsf{F}$.

Remark 5.3 At first glance, bisimulation products look similar to (weak) pullbacks in the category $\mathsf{Coalg}(\mathsf{Up}\mathcal{P})$. However, in general, bisimulation products need not be weak pullbacks. To see this, consider the frames $\mathbb{F}_i = (W_i, \nu_i)$ where $W_i = \{s_i, t_i\}$ and $\nu_i(s_i) = \nu_i(t_i) = \{W_i\}$, $i \in \{1, 2\}$, and let $Z =$

$W_1 \times W_2$. Then both (Z, μ_s) and (Z, μ_l) are bisimulation products of $\mathbb{F}_1$ and $\mathbb{F}_2$, but it can be checked that neither is a (weak) pullback in $\mathsf{Coalg}(\mathsf{Up}\mathcal{P})$.

Before we state the main result of this section, recall the following from subsection 2.4. $\mathbb{F}^+$ denotes the full complex algebra of a monotone frame $\mathbb{F}$, and $\mathbb{A}_+$ denotes the ultrafilter frame of a BAM $\mathbb{A}$.

Lemma 5.4 (Bisimulation product lemma) *Let K be a class of BAMs and F a class of monotone frames. Then K has SUPAP if the following three conditions are satisfied:*

(i) F *has bisimulation products.*
(ii) *For all $\mathbb{F}$ in F: $\mathbb{F}^+ \in \mathsf{K}$.*
(iii) *For all $\mathbb{A}$ in K: $\mathbb{A}_+ \in \mathsf{F}$.*

Proof. The proof of this lemma is virtually identical to that of Lemma 5.2.6 in Marx [23], which relies only on the Kripke version of bisimulation products and the basic duality between Kripke frames and BAOs. These analogues for monotone frames and BAMs are shown in [12]. $\square$

Note that if K is a canonical variety and $\mathsf{F} = \{\mathbb{F} \mid \mathbb{F}^+ \in \mathsf{K}\}$, then conditions (ii) and (iii) in the Bisimulation product lemma always hold, since then $\mathbb{A} \in \mathsf{K}$ implies that the canonical extension $(\mathbb{A}_+)^+$ is in K, and hence $\mathbb{A}_+ \in \mathsf{F}$. From [12] (Theorem 7.13, Corollary 10.35) we have the following canonicity result.

Proposition 5.5 (Canonical logics) *Let $\mathbf{M}$ denote the smallest monotone modal logic. If $\Gamma \subseteq \{$N,P,4',T,D$\}$, then $\Lambda = \mathbf{M}.\Gamma$ is canonical.*

It is straightforward to show the next proposition (a proof may be found in [12]).

Proposition 5.6 *The following frame classes have smallest bisimulation products:*

(M) *The class M of all monotone frames.*
(N) *The class N of monotone frames satisfying (n) $\forall w \in W : W \in \nu(w)$.*
(P) *The class P of monotone frames satisfying (p) $\forall w \in W : \emptyset \notin \nu(w)$.*
(4') *The class $\mathsf{4'}$ of monotone frames satisfying*
(iv') $\forall w \in W\ \forall X \subseteq W : X \in \nu(w) \to m_\nu(X) \in \nu(w)$.
(T) *The class T of monotone frames satisfying*
(t) $\forall w \in W\ \forall X \subseteq W : X \in \nu(w) \to w \in X$.
(D) *The class D of monotone frames satisfying*
(d) $\forall w \in W\ \forall X \subseteq W : X \in \nu(w) \to W \setminus X \notin \nu(w)$.

Together with the Bisimulation product lemma 5.4, Propositions 5.5 and 5.6 provide us with the following theorem.

Theorem 5.7 *If* $\Gamma \subseteq \{\mathrm{N,P,T,4',D}\}$ *then* $\Lambda = \mathbf{M}.\Gamma$ *has CIP.*

Remark 5.8 The algebraic duality presented in this paper is built up around the notion of σ-canonicity [15,7]. However, there is a second (dual) way of constructing canonical extensions, and hence ultrafilter frames and descriptive monotone frames, which is referred to as π-canonicity, for which the same duality with monotone frames hold. Thus the Bisimulation product lemma 5.4 could equally have been formulated for the notion of π-ultrafilter frame. Furthermore, one can show that when $\Gamma \subseteq \{\mathrm{N,P,4}\}$ and $\Lambda = \mathbf{M}.\Gamma$ then V_Λ is π-canonical and, similarly to the proof of Proposition 5.6, the class of frames for Λ has largest bisimulation products. Hence it follows that Λ has CIP. See [12] for details.

6 Conclusions and Further Research

We hope to have demonstrated that coalgebra offers an interesting perspective on monotone modal logic, and vice versa, that monotone modal logic can offer new angles on coalgebraic results. There are a number of directions in which the work of this paper can be extended.

First one can now easily see monotone modal logic as a coalgebraic modal logic in the style of Pattinson and prove soundness and completeness for the basic monotone modal logic in the same way as it was proven for normal modal logic in [27]. It might be possible to obtain further results about normal modal logic and monotone modal logic in a uniform way.

Another question is whether behavioural equivalence is more suitable than T-bisimilarity for studying coalgebras for functors that are not weak pullback preserving. This has already been suggested in Kurz [18, Sec. 1.2] and Wolter [33]. Our $\mathsf{Up}\mathcal{P}$-functor also supports this idea, but one has to provide further arguments.

In Marx [23] the connection between bisimulation products (zigzag products) of Kripke frames, various degrees of amalgamation, and preservation of first-order validities under the bisimulation product construction are investigated. Although our definition of bisimulation products is slightly different than the approach in [23], one could try to generalize these results to monotone modal logic by using our bisimulation products.

References

[1] P. Aczel and N. Mendler. A final coalgebra theorem. In D. H. Pitt et al, editor, *Category Theory and Computer Science*, volume 389 of *LNCS*, pages 357–365. Springer, 1989.

[2] R. Alur, T. A. Henzinger, and O. Kupferman. Alternating-time temporal logic. In *Compositionality: The Significant Difference*, LNCS 1536, pages 23–60. Springer, 1998.

[3] J. van Benthem. *Modal Correspondence Theory*. PhD thesis, University of Amsterdam, 1976.

[4] P. Blackburn, M. de Rijke, and Y. Venema. *Modal Logic*. Cambridge University Press, 2001.

[5] B. F. Chellas. *Modal Logic - An Introduction*. Cambridge University Press, 1980.

[6] R. Engelking. *General Topology*. Heldermann Verlag, 1989.

[7] M. Gehrke and B. Jónsson. Monotone bounded distributive lattice expansions. *Mathematica Japonica*, 52(2):197–213, 2000.

[8] R. Goldblatt. Metamathematics of modal logic, part I. *Reports on Mathematical Logic*, 6:41–78, 1976.

[9] R. Goldblatt. Metamathematics of modal logic, part II. *Reports on Mathematical Logic*, 7:21–52, 1976.

[10] R. Goldblatt. A calculus of terms for coalgebras of polynomial functors. In A. Corradini, M. Lenisa, and U. Montanari, editors, *Coalgebraic Methods in Computer Science (CMCS'01)*, volume 44.1 of *ENTCS*. Elsevier, 2001.

[11] H. Peter Gumm. Functors for coalgebras. *Algebra Universalis*, 45:135–147, 2001.

[12] H. H. Hansen. Monotonic modal logic (Master's thesis). Preprint 2003-24, ILLC, University of Amsterdam, 2003.

[13] B. Jacobs. Many-sorted coalgebraic modal logic: a model-theoretic study. *Theoretical Informatics and Applications*, 35(1):31–59, 2001.

[14] P. Johnstone. *Stone Spaces*. Cambridge University Press, 1982.

[15] B. Jónsson and A. Tarski. Boolean algebras with operators I. *American Journal of Mathematics*, 73:891–939, 1951.

[16] B. Jónsson and A. Tarski. Boolean algebras with operators II. *American Journal of Mathematics*, 74:127–162, 1952.

[17] C. Kupke, A. Kurz, and Y. Venema. Stone coalgebras. In H. P. Gumm, editor, *Coalgebraic Methods in Computer Science (CMCS'03)*, volume 82.1 of *ENTCS*, 2003. Also available as Technical Report PP-2003-15, ILLC, University of Amsterdam.

[18] A. Kurz. *Logics for Coalgebras and Applications to Computer Science*. PhD thesis, Ludwig-Maximilians-Universität München, 2000.

[19] A. Kurz and D. Pattinson. Definability, canonical models, compactness for finitary coalgebraic modal logic. In Lawrence Moss, editor, *Proceedings of CMCS 2002*, volume 65.1 of *ENTCS*. Elsevier, 2002.

[20] S. Mac Lane. *Category Theory for the Working Mathematician*. Springer, 1971.

[21] J. Madarász. Interpolation and amalgamation; pushing the limits. Part I. *Studia Logica*, 61:311–345, 1998.

[22] J. Madarász. Interpolation and amalgamation; pushing the limits. Part II. *Studia Logica*, 62:1–19, 1999.

[23] M. Marx. *Algebraic Relativization and Arrow Logic*. PhD thesis, University of Amsterdam, 1995.

[24] L. Moss. Coalgebraic logic. *Annals of Pure and Applied Logic*, 96:277–317, 1999.

[25] R. Parikh. The logic of games and its applications. In M. Karpinski and J. van Leeuwen, editors, *Topics in the Theory of Computation*, Annals of Discrete Mathematics 24. Elsevier, 1985.

[26] D. Pattinson. Semantical principles in the modal logic of coalgebras. In *Proceedings 18th International Symposium on Theoretical Aspects of Computer Science (STACS 2001)*, volume 2010 of *LNCS*, Berlin, 2001. Springer.

[27] D. Pattinson. Coalgebraic modal logic: soundness, completeness and decidability of local consequence. *Theoretical Computer Science*, 309:177–193, 2003.

[28] M. Pauly. Bisimulation for general non-normal modal logic. Manuscript, 1999.

[29] M. Pauly. *Logic for Social Software*. PhD thesis, University of Amsterdam, 2001.

[30] M. Rößiger. Coalgebras and modal logic. In Horst Reichel, editor, *Coalgebraic Methods in Computer Science (CMCS'00)*, volume 33 of *Electronic Notes in Theoretical Computer Science*, pages 299–320, 2000.

[31] J. J. M. M. Rutten. Universal coalgebra: a theory of systems. *Theoretical Computer Science*, 249:3–80, 2000.

[32] G. Sambin and V. Vaccaro. Topology and duality in modal logic. *Annals of Pure and Applied Logic*, 37:249–296, 1988.

[33] Uwe Wolter. On corelations, cokernels, and coequations. In Horst Reichel, editor, *Coalgebraic Methods in Computer Science (CMCS'00)*, volume 33 of *ENTCS*, 2000.