A Cappable Almost Everywhere Dominating Computably Enumerable Degree

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Abstract

We show that there exists an almost everywhere (a.e.) dominating computably enumerable (c.e.) degree which is half of a minimal pair.

Keywords: Almost everywhere domination, minimal pairs, Turing degrees, computably enumerable.

1 Introduction

Dobrinen and Simpson [4] defined the notion of *almost everywhere dominating function* and started studying their Turing degrees. They also defined a uniform version of this property (the *almost everywhere uniformly dominating functions*) but this has been shown in [1] to be equivalent to the former notion. So let us settle on the following definition.

Definition 1.1 Let $f, g : \omega \to \omega$. The function f majorizes g if $f(n) \geq g(n)$ for all n. If $f(n) \geq g(n)$ for all but finitely n then f dominates g. We say that f is an almost everywhere (a.e.) dominating if the measure of reals β with the property

$$g \leq_T \beta \Rightarrow f \text{ dominates } g$$

is 1. A Turing degree is a.e. dominating if it contains an a.e. dominating function.

Kurtz [5] showed that $\mathbf{0}'$ is a.e. dominating. The existence of incomplete c.e. degrees with this property was shown in [3] and a high c.e. degree which is not a.e. dominating was constructed in [2]. In order to have a better picture of this class of c.e. degrees it is natural to compare it with known definable classes, the promptly simple degrees for instance. By well known results these are the non-cappable degrees, i.e. the c.e. degrees which are not halves of minimal pairs in this structure. In this work we show that there are cappable c.e. degrees which are a.e. dominating. Thus a.e. dominating degrees lie in both parts of the decomposition of the c.e. degrees into promptly simple and cappable degrees.

Theorem 1.2 *There is an almost everywhere dominating c.e. degree which is a half of a minimal pair.* [4]

Before we start proving the theorem let us say more about what motivated this work. A central question in the study of a.e. dominating c.e. degrees is whether they are contained in a non-trivial upper cone of degrees. The construction in [3] of an incomplete a.e. dominating c.e. degree does not seem to combine with many other constructions in the theory of c.e. degrees and in particular the *cone avoidance*. To be more specific, that construction seems incompatible with the situation where *there is unpredictably large number of repeated restraints for a single requirement, in a way that when a new restraint is imposed the old one still remains in force.*

Although this is a very common feature in constructions of c.e. degrees (and in the cone avoidance strategy it seems unavoidable), by combining the construction in [3] with known techniques from the theory of promptly simple degrees (in particular the use of the recursion theorem in order to trigger prompt diagonalizations) we were able to show that outside any upper cone of promptly simple degrees there is an a.e. dominating c.e. degree. Theorem 1.2 implies this result but is also the first example of an infinite injury argument which constructs a.e. dominating degrees. Indeed, the proof incorporates

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[4] After the submission of this article the first author was able to show the stronger result that there is an almost everywhere dominating computably enumerable degree which is non-cuppable in the c.e. degrees.

new ideas which help to control the restraint imposed on the a.e. dominating function being constructed, although this may be infinitary. For further background, history, motivation and results related to this area we refer to [2,1,3,4].

2 Proof of theorem 1.2

Following [3], there is a Turing functional Φ such that f is a.e. dominating iff it dominates Φ almost everywhere (where Φ is treated here as an operator from reals to reals). We may assume that all functionals considered here (including Φ) have the standard properties: if $\Gamma^\sigma(n)[s] \downarrow$ then n and the (length of the) use of σ are less than s; also, if $\Gamma^\sigma(n)[s] \downarrow$ then $\Gamma^\sigma(i)[s] \downarrow$ for all $i < n$. We construct two functions f, g such that their degrees form a minimal pair. As in [3] f will be an a.e. dominating function which is approximated from below so that it has c.e. degree. In particular we construct a computable double sequence $(f_s(n))$ such that $f_s(n) \le f_{s+1}(n)$ for all s, n. Then we can take $A = \{\langle n, m \rangle \mid m \le f(n)\}$ so that A is c.e. and Turing equivalent to f. In the construction we often use f to mean f_s where s is the current stage. The function g will be the characteristic sequence of a c.e. set and in the construction it will be treated as a c.e. set. First of all we have the domination strategies which ensure that $\lim_s f_s = f$ exists (by restraining f) *and* that f dominates Φ almost everywhere (by increasing f). Then we got the minimal pair strategies. Both of these strategies can restrain f and so, as in [3], they need *backing strategies* which provide them with a good approximation of the measure of the domain of Φ (i.e. $\mu(\text{dom}\Phi)$). Finally we got the simplicity strategies which ensure that g is not computable.

The construction will be a tree argument both because of the Π_2^0 approximation of $\mu(\text{dom}\Phi)$ and the minimal pair strategies which are infinitary. As we want to make f a.e. dominating we will make sure that *at any stage no more than one restraint on f is in force for a single requirement*. According to the uniform labeling of the tree described below this means that at any stage no more than one restraint on f is in force at a single level of the tree. This is a principle which made the argument in [3] work. Interestingly, the proof of theorem 1.2 involves stretching the minimal pair construction rather than the construction in [3] of an a.e. dominating c.e. degree.

The tree argument in [3] naturally defined a $\mathbf{0}''$ computable sequence (N_i) which consisted of the permanent restraints which were imposed on the a.e. dominating sequence f being constructed. The property (3) which was verified for (N_i) was the main tool for showing that f is a.e. dominating. We will follow the same plan, only that (N_i) will be a bit more complex due to the

infinitary nature of the minimal pair requirements. Such a sequence is an important component of the construction of almost everywhere dominating reals. The following proposition says that every possible construction of an a.e. dominating c.e. real defines such a sequence (N_i). Let

$$D_n = \{\beta \mid \Phi^\beta(n) \downarrow\} \tag{1}$$

$$D_{[n,m)}[g] = \{\beta \mid (\forall k \in [n,m)) \ \Phi^\beta(k)[g(k)] \downarrow\}. \tag{2}$$

Note that by the standard conventions on the functionals, if $\beta \in D_{[n,m)}[g]$ then g dominates Φ^β in the interval $[n, m)$.

Proposition 2.1 *If f is an a.e. dominating c.e. real then from a Turing functional Φ we can effectively get a $\mathbf{0}''$ computable sequence (N_i) such that*

$$\mu(dom\Phi - D_{[N_i, N_{i+1})}[f]) < 3^{-i}. \tag{3}$$

We sketch a proof: the domain of a Turing functional as an operator from the Cantor space to itself is a Π_2^0 class. Also it is not hard to show the following.

Lemma 2.2 *The measure of a Π_n^0 or a Σ_n^0 class is computable from $\mathbf{0}^{(n)}$. Moreover the measure of any finite intersection of Π_n^0 and Σ_n^0 classes is computable from $\mathbf{0}^{(n)}$.*

Now one can define (N_i) inductively by using $\mathbf{0}''$ as an oracle and using the fact that f a c.e. real. Note that the 3^{-i} in proposition 2.1 can be replaced with $3^{-g(i)}$ for any $g \leq_T \mathbf{0}''$; however this is not particularly useful since $\mathbf{0}''$ is anyway used in order to define (N_i).

Requirements and tree.

We approximate functions f, g such that f is approximated from below and is a.e. dominating; g will be a c.e. set and the following requirements will be satisfied:

$$Q_e : \Phi_e^f = \Psi_e^g \text{ total } \Rightarrow h_e = \Phi_e^f = \Psi_e^g$$
$$P_e : \quad |W_e| = \infty \Rightarrow W_e \cap g \neq \emptyset$$

where (Φ_e, Ψ_e) runs over all pairs of Turing functionals, (W_e) over all the c.e. sets and the functions h_e are partial computable and constructed by us. The strategy for Q_e will be an adaptation of the minimal pair strategy and its outcomes will be $\mathtt{inf} <_* \mathtt{fin}$ (i.e. *infinite* and *finite*). The strategy for P_e will be the usual strategy for the construction of a simple set: if $W_e \cap g = \emptyset$ wait until some $x > 2e$ appears in W_e such that it does not violate any g-restraint of higher priority and enumerated into g. The outcomes of P_e are $\mathtt{diag} <_* \mathtt{wait}$

(i.e. *diagonalized* and *wait*). The domination and Q strategies can restrain f and so they need backing strategies as explained above. Let M_e be the eth backing strategy and D_e be the eth domination strategy.

The construction will be a tree argument where the level $3e$ belongs to the backing strategy M_e, the levels $3n + 1$ belong alternately to domination and minimal pair strategies (e.g. $3(2e) + 1$ belongs to D_e and $3(2e + 1) + 1$ belongs to Q_e) and level $3e + 2$ belongs to P_e.

Backing strategies.

Before giving the details about how the domination and Q strategies work, we need to describe the backing strategies M_e which provide a good approximation of $\mu(\mathrm{dom}\Phi)$ and which both of them need. An M strategy comes with a *resolution n* and the *error allowance $\delta = 3^{-n}$*; its job is to find a rational q which approximates $\mu(\mathrm{dom}\Phi)$ from below to within δ. An M strategy first divides $[0, 1)$ into 3^n subintervals

$$[q_1, q_0), [q_2, q_1), \ldots, [q_k, q_{k-1})$$

(where $q_0 = 1, q_k = 0$ and $q_i > q_{i+1}$) of length δ. Its outcomes will be the following specially ordered set

$$q_1 <_* \cdots <_* q_k.$$

When M is visited at stage s it looks for the least t such that μD_t has moved into a different interval of the partition above since the last stage s_0 it was visited. If such does not exist (or s_0 does not exist) it outputs q_k. Otherwise it outputs the small endpoint of the interval of the partition in which μD_t currently belongs. That is, if $\mu D_t \in [q_i, q_{i-1})$ then the current outcome is q_i. To get a picture of the approximation of the outcomes note that for every t the measure μD_t is non-decreasing during the stages of the construction (more precisely the stages of the enumeration of Φ). Moreover

$$\mu D_t \geq \mu D_{t+1}$$

for all t at all stages. So we can picture $\mu D_t, t \in \mathbb{N}$ as a collection of markers on the unit interval which move monotonically towards 1 while preserving their order. Now if $\mu(\mathrm{dom}\Phi) \in [q_i, q_{i-1})$ then there will only be finitely many t such that $q_{i-1} < \mu D_t$ and so after some stage no marker will go beyond $[q_i, q_{i-1})$. Also, for almost all t the marker μD_t will settle in $[q_i, q_{i-1})$. Thus q_i will be the leftmost infinitely often visited outcome (and so, the *final outcome* and it can be seen as an approximation of $\mu(\mathrm{dom}\Phi)$ from below within δ).

Domination strategies.

Since the domination strategy also has to make sure that $\lim_s f_s$ exists it will have infinitely many versions (let D_e be the eth domination strategy); each of them will try to fix f on a certain initial segment, while ensuring that it is a.e. dominating. Each (version of the) domination strategy inherits a restraint N which prohibits it from changing $f \upharpoonright N$, and is based on a backing strategy which provides an approximation q of $\mu(\text{dom}\Phi)$ as above. It operates as follows:

(i) Wait for a stage $s + 1$ such there exists $\sigma \in s^{<s}$ with $|\sigma| > N$ and the following properties:
- $\sigma \supset f_s \upharpoonright N$
- $(\forall n < |\sigma|)\ f_s(n) \leq \sigma(n)$
- $\mu(D_{[N,|\sigma|)}[\sigma]) > q - \delta$

(ii) Let $f_{s+1} \supset \sigma$ and restrain $f \upharpoonright |\sigma|$.

Then, if the guess q is correct, the reals β such that Φ^β is not majorized by f in $[N, |\sigma|)$ will have measure less than 2δ. This strategy also contributes to the convergence of (f_s) since it restrains it on a certain segment. Note that in [3] the domination strategies are incorporated into other strategies rather than distinguished as in this construction. If this strategy sits on a node α the restraint it imposes will be r_α and its state/current outcome will be `active` or `inactive` according to whether $r_\alpha \neq 0$ or $r_\alpha = 0$ respectively.
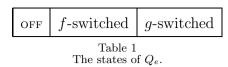
Minimal pair strategy.

Suppose that α works for Q_e. If it works in isolation, it just has to follow the usual minimal pair strategy: it enumerates a partial computable function h_α as follows. At expansionary stages (i.e. stages where the length of agreement

$$\ell_e = \max\{x \mid \Phi_e^f \upharpoonright x = \Psi_e^g \upharpoonright x\}$$

is larger than ever before) it defines $h_\alpha = \Phi_e^f \upharpoonright \ell = \Psi_e^g \upharpoonright \ell$ and defines restraint r_α equal to the (maximum) use of these computations. Also it requires that when one of $f \upharpoonright r_\alpha$, $g \upharpoonright r_\alpha$ changes, then the other one does not change until the next expansionary stage occurs. By the usual inductive argument on the stages one can verify that this strategy succeeds (see [6]).

The strategies for all Q_e can work together with non-computability requirements for f, g on a tree, thus constructing a minimal pair of c.e. degrees.

OFF	f-switched	g-switched

Table 1
The states of Q_e.

However this construction has the feature that unpredictable number of re-
peated restraints may be applied to f for the sake of a single requirement Q_e
in such a way that when a new restraint is imposed, the existing restraints
must remain in force. To visualize this situation in the context of a tree argu-
ment, these are the restraints *on the left of the true path* which can easily be
accumulated and should be respected. As we pointed out earlier this feature
prevents us from making f a.e. dominating.

To overcome this obstacle we do the following: whenever we wish to impose
a restraint on f for the sake of Q_e but there is already such a restraint in force
for Q_e we try to *push the existing restraint to g*. The reason that we can
do this is that *a restraint on g is as good as a restraint on f* given that the
purpose of issuing restraints is to preserve one side of the double computation

$$\Phi_e^f \upharpoonright x = \Psi_e^g \upharpoonright x.$$

Of course this means that the restraint of a minimal pair strategy may increase
due to the request of lower priority strategies (i.e. strategies on the same level
of the tree and to its right) and in general the restraint on g will be much
more than usual. But it will be sufficiently well behaved so that it allows the
P_e requirements to succeed in making g non-computable. However it will be
clear by the end of the proof that this method does not allow us to ask g to
be a.e. dominating as well. In other words, we cannot apply this trick to both
f and g in order to construct a minimal pair of a.e. dominating degrees.

Adapted minimal pair strategy on the tree.

Suppose that α on the tree works for Q_e. As usual, α will have its own
enumerations Φ_α, Ψ_α of the functionals Φ_e, Ψ_e. In particular $\Phi_\alpha[s]$ will be
$\Phi_\alpha[t]$ where t is the largest α-stage $\leq s$ (and similarly for Ψ_α). The length of
agreement ℓ_α is now defined according to Φ_α, Ψ_α.

We look at the minimal pair argument in some detail in order to be able
to make the modifications we need. Every Q_e strategy α has a restraint r_α
which may tend to infinity (we use one restraint for both f, g and direct it to
one of these functions if needed). The restraint r_α (or the strategy α) can be
in one of the following states: OFF, f-switched, g-switched (see the table 1).
The states of Q_e are shown in the following table.

OFF	$\leftrightarrows$	f-switched
OFF	$\leftrightarrows$	g-switched
f-switched	$\rightarrow$	g-switched

Table 2
Possible transitions between the states of Q_e.

Switching of the restraint to different states will be performed explicitly during the construction. The possible transitions are shown in table 2.

The idea is that when we have an expansionary stage we can switch to OFF since it does not matter if one side of the double computation is injured. However if the f side is injured we need to switch the restraint to g (and similarly if the g side is injured). A g-switched or f-switched restraint can return to OFF state if another expansionary stage comes, in which case both computations will be restored and the restraint r_α will possibly increase.

The extra transition which is not a feature of the usual minimal pair argument is the third one in table 2. If a Q strategy wants to start defining h_α and setting up a positive r_α it is actually exposing itself to possibly becoming f-switched since the g side of its computations may later be injured. If there is already an f-switched strategy β on the left of α and on the same level then α cannot take this risk since it would allow the possibility of two f restraints on the same level—something we want to avoid. So in that case α will try to g-switch the restraint on the left before it starts. It does that by waiting for a stage where the length of agreement ℓ_α is larger than ℓ_β. Then it can increase r_β up to the use of $\Psi_e^g \upharpoonright |h_\beta|$ (where Q_e is the requirement that α, β are trying to fulfill), set β to g-switched state and proceed.

Comments.

Before we state formally the set of instructions that a Q node α follows we make some remarks. First, when a Q strategy switches from a restraint on f to no restraint on f (i.e. to OFF or g-switched) then all domination strategies below $\alpha^\frown\texttt{fin}$ must be initialized since they counted a restraint that is not permanent. Second, the fact that we can only pass from f-switched to g-switched and not vice-versa is crucial. This transition causes the restraint r_β to increase for nodes β on the left of the true path when they are not accessible. The fact that this transition is one-way guarantees that the state of the Q nodes on the left of the true path will settle down and so their restraint will reach a limit as well. This is something we need since otherwise the simplicity requirements P for g would not get the chance to be satisfied.

Finally we point out the different attitude of the positive strategies for g and f, i.e. the simplicity strategies P and the domination strategies D. A P strategy will respect *both* the usual restraint from the nodes on the left (it will enumerate no number which is less than the last stage where a node to the left of it was accessible) *and* the g-switched restraints of Q nodes to the left of it or above it. The standard restraint from the nodes on the left prevents the situation where there is an f-switched node on the true path, a strategy β in OFF state at the same level to the left of it and a node on the right of the true path switches β to an f restraint thus creating a second restraint on f at the same level. Domination strategies are more demanding and will respect less restraint: they only listen to f-switched restraints of higher priority (i.e. above or to the left of them) and not the standard restraint.

2.1 Formal strategies

We give the sets of instructions for each strategy which constitute the programs actually run in the construction.

M_e-node α.

These are the backing strategies and they run exactly as described above. In particular, if α is an M_e-node then it works with resolution e. This means that it has $k = 3^e$ outcomes

$$q_1 <_* \cdots <_* q_k.$$

which are rationals such that $q_0 = 1, q_k = 0$ and $q_i > q_{i+1}$.

Q_e-node α.

Let N_α be the largest restraint imposed on f by a higher priority requirement. That is, the largest r_β such that $\beta <_L \alpha$ or $\beta \subset \alpha$ and one of the following holds:

- β is an f-switched Q node.
- β is an `active` D node.

Also, let q_n be the outcome of the predecessor of α (i.e. the outcome of its backing strategy) and $\delta_\alpha = 3^{-e}$ (the distance between the outcomes of the backing strategy). Follow the instructions:

(i) If ℓ_α is larger than ever before and larger than all values that the parameters ℓ_β with $\beta <_L \alpha$ and $|\beta| = |\alpha|$ have ever taken then do the following.

- if there is an f-switched strategy β such that $\beta <_L \alpha$ and $|\beta| = |\alpha|$ then set r_β equal to the use of $\Psi_e^g \upharpoonright |h_\beta|$; also g-switch β. Initialize all D strategies of lower priority than β.
- If $\mu(D_{[N_\alpha, N')}[f]) > q_n - \delta_\alpha$ where $N' = \max\{N_\alpha, u\} + 1$ and u is the use of the computations $\Phi_\alpha^f \upharpoonright (|h_\alpha| + 1)$, $\Psi_\alpha^g \upharpoonright (|h_\alpha| + 1)$, then go to the next step. Otherwise access outcome `fin` and pass control to the construction.
- Define

$$h_\alpha \upharpoonright (|h_\alpha| + 1) = \Phi_\alpha^f \upharpoonright (|h_\alpha| + 1) = \Psi_\alpha^g \upharpoonright (|h_\alpha| + 1)$$

set α to OFF state, define r_α equal to the use of the above computations, access outcome `inf` and pass control to the construction.

(ii) Otherwise access outcome `fin` and pass control to the construction.

(apart from the states of a Q node that we discussed, such a strategy may be **active** *or* **inactive** *according to whether $r_\alpha \neq 0$ or $r_\alpha = 0$. This qualification refers to the potential of the strategy to restrain a segment of f.)*

P_e-node α.

(i) If $W_e \cap g \neq 0$ access outcome `diag` and pass control to the construction.

(ii) Otherwise check if currently there exists x (less than the current stage) such that
- $x > 2e$
- $x \in W_e$
- $x > r$ where r is the least number greater than all stages where a node to the left of α was accessible and all g-switched restraints r_β where β is any node of lower priority (i.e. to the left or below α).

If yes, put the least one into g and pass control to the construction along with a request to terminate the current stage. Otherwise access outcome `wait` and pass control to the construction.

D_e-node α.

Let N_α be the largest restraint imposed on f by a higher priority requirement (as described in the Q-node strategy) and s the current stage. Also,

let q_n be the outcome of the predecessor of α (i.e. the outcome of its backing strategy) and $\delta_\alpha = 3^{-e}$ (the distance between the outcomes of its backing strategy).

  (i) If α is in `active` state access outcome `active` and pass control to the construction.

 (ii) If α is in `inactive` state and there exists $\sigma \in s^{<s}$ with the following properties
- $|\sigma| > N_\alpha$
- $\sigma \supset f_s \restriction N_\alpha$
- $(\forall n < |\sigma|)\ f_s(n) \leq \sigma(n)$
- $\mu(D_{[N_\alpha,|\sigma|)}[\sigma] > q_n - \delta_\alpha$

then define $f \restriction |\sigma| = \sigma$, set $r_\alpha = |\sigma|$ and access the `active` outcome (thus setting α in `active` state). Pass control to the construction along with the request to terminate the current stage.

(iii) Otherwise access outcome `inactive` and pass control to the construction.

2.2   Construction

Start with $f_0(n) = 0$ for all n and $g = \emptyset$. Initialize all strategies α (set $r_\alpha = 0$, $h_\alpha = \emptyset$). At stage s do the following:

(A) Initialize every `active` D or Q strategy which became active at some stage $s_0 < s$ and is based on outcome q_i of its backing strategy such that

$$\mu D_t[s] \geq q_{i-1}$$

where t is the largest number such that $\mu D_t[s_0] < q_{i-1}$. Also if α was initialized, do the same for the strategies of lower priority than α (the ones below it or to the right of it).

    (This cancellation is allowed since the outcomes of the backing strategies considered have been proved wrong. Also, it is necessary since such strategies may impose 'too much' restraint on f, thus preventing it from being a.e. dominating. Note that this initialization does not happen automatically since some of these strategies may lie on the left of the true path.)

(B) Start accessing the nodes of the tree starting from the top node , running the corresponding strategies and determining the successor according to the outcome suggested by the previous strategy. Continue until either stage s is terminated by a request of a strategy we run or we reach a strategy α with $|\alpha| = s$. If α is the last node we access, initialize all nodes to the left of it. If a Q strategy α was in off-state at the beginning of the stage and $g \restriction r_\alpha$ or $f \restriction r_\alpha$ changed, α changes to f or g-state respectively.

2.3   Verification

The following lemma is evident from the construction.

Lemma 2.3 *The state of a Q node α can only change provided that it is* **active** *(i.e. $r_\alpha \neq 0$) and one of the following reasons holds.*

(i) *Initialization of α sets its state to* OFF.

(ii) *A node to the right of α on the same level changes the state of α from f-switched to g-switched.*

(iii) *A domination node of lower priority (i.e. below or to the right of it) changes its state from* OFF *to f-switched.*

(iv) *A simplicity strategy for g below α switches its state from* OFF *to g-switched.*

Lemma 2.4 *The following basic facts are true.*

- *There is an infinite leftmost infinitely often visited path TP through the tree.*

- *The states and restraints of the Q and D nodes to the left of TP reach a limit.*

- *If a node lies on TP then it is initialized finitely many times.*

Proof. We prove these claims by simultaneous induction on TP. Note that every node has finitely many branches and so we can immediately conclude that the outcomes of a node on TP will have limit infimum. For the top node all claims are trivial since it is a backing strategy. Suppose that $\alpha = TP \restriction n$ in an infinitely often visited node such that after some stage s_0 no node to the left of it is visited. Since α has finitely many branches there will be an outcome o such that the node $\alpha^+ = \alpha^\frown o$ is accessed infinitely often and any node to the right of it finitely often. So α^+ belongs to TP and after some stage s_1 no node to the left of it is visited. At s_1 there are only finitely many active Q or D nodes to the left of α^+ and so, after some stage $s_2 > s_1$ there will be no initializations of α^+. According to lemma 2.3 the only state transitions that can happen to nodes on the left of α^+ after s_2 concern active Q nodes (of which there are finitely many) which turn from OFF to f-switched or from f-switched to g-switched. This can only happen finitely often and thus the states of all nodes to the left of α^+ will stabilize at some stage $s_3 > s_2$. At stage s_3 the restraints r_β of the nodes $\beta <_L \alpha^+$ have reached their final values. □

Lemma 2.5 *If α is a Q node and $\alpha^\frown$ **fin** is on TP then r_α reaches a limit.*

Proof. After some stage $\alpha^\frown$ fin will stop being initialized and so, it will stop having expansionary stages. Then r_α can not be increased anymore. □

Lemma 2.6 *If α is a P or D strategy on TP then the restraint imposed on α reaches a limit. Hence all P are satisfied.*

Proof. Suppose that α is a P node. By lemma 2.4 the g restraint coming from nodes to the left of α will reach a limit. The only other restraint that α has to consider come from g-switched Q nodes β such that $\beta^\frown\texttt{fin} \subseteq \alpha$. But according to lemma 2.5 this reaches a limit as well. The same argument applies to the case where α is a Q node, only that now we do not have to consider the restraint "last stage where a node to the left was accessible". The satisfaction of P is evident. $\square$

By construction at any stage there is at most one strategy at each level which restrains f (i.e. which is an f-switched Q strategy or an active D strategy). So at each level of the tree there is at most one strategy with a permanent restraint on f (i.e. a permanently active D node or a permanently f-switched Q node with a final restraint). Let (σ_i) be inductively defined as follows. Set $\sigma_0 = \emptyset$ and let σ_{i+1} be the node β of least length which holds a permanent restraint on f and has $|\beta| > |\sigma_i|$. Then $|\sigma_i| < |\sigma_{i+1}|$ and all these nodes lie to the left or on TP. Also, if N_i is the final value of r_{σ_i} (and $N_0 = 0$) we have $N_i < N_{i+1}$.

Lemma 2.7 *The sequence (σ_i) is infinite.*

Proof. Suppose for a contradiction that σ_i, N_i is the last term of the sequence. Let $\sigma \supset \sigma_i$ be a D node on the true path. If $q_0, \ldots, q_k$ are the outcomes of its backing strategy and $\mu(\text{dom}\Phi) \in [q_i, q_{i-1})$ then σ will be under $q_i = q_\sigma$. Also all markers μD_m will eventually pass the threshold $q_\sigma - \delta_\sigma$. After all states and restraints on the left of σ have stabilized no node on the same level as σ and to the left of it will be active (according to hypothesis). After such a stage and when $\mu D_{r_\sigma} > q_\sigma - \delta_\sigma$ (where $r_\sigma = N_i$) σ will be visited and it will define f such that $\mu D_{[N_i, r_\alpha)}[f] > q_\sigma - \delta_\sigma$ thus becoming $\texttt{active}$. By hypotheses it will remain $\texttt{active}$ and this is a contradiction. So (σ_i) is infinite. $\square$

Lemma 2.8 $f = \lim_s f_s$ *exists.*

Proof. Once σ_i becomes $\texttt{active}$ for the last time thus imposing restraint N_i, the segment $f \upharpoonright N_i$ will not change. $\square$

Lemma 2.9 *For every $i \geq 1$ the measure of the reals β in the domain of Φ such that Φ^β is not dominated by f in $[N_{i-1}, N_i)$ is less than $2 \cdot 3^{-n_i}$, where $3n_i + 1$ is the level of σ_i. Formally,*

$$\mu(\text{dom}\Phi - D_{[N_{i-1}, N_i)}[f]) < 2 \cdot 3^{-n_i}.$$

Also, $n_i \geq i$.

Proof. If $3n_i + 1$ or $3n_i + 2$ is the level of σ_i then its resolution is n_i and $i \leq n_i$. Also the error allowance is $\delta_i = 3^{-n_i}$. Let q_j be the outcome of its backing strategy. Then if σ_i became permanently `active` at stage s_0 and m is least such that $\mu D_m[s_0] \in [q_j, q_{j-1})$, by step A of the construction we have $\mu D_m[s] \in [q_j, q_{j-1})$ for all $s \geq s_0$, so $\mu(\mathrm{dom}\Phi) \in [q_j, q_{j-1})$; we also have that $\mu(D_{[N_{i-1}, N_i)}[f]) > q_j - \delta_i$ (otherwise σ_i would not be `active`). So the reals β for which Φ^β is not majorized by f in $[N_{i-1}, N_i)$ can have measure at most $2\delta_i$ which is $2 \cdot 3^{-n_i}$. $\square$

Lemma 2.10 *The degrees of f, g form a minimal pair.*

Proof. Suppose that $h \leq_T f$ and $h \leq_T g$ for some function h. We show that h is computable. Let Φ_e, Ψ_e be programs such that

$$\Phi_e^f = \Psi_e^g = h. \tag{4}$$

Then there is $\alpha \subset TP$ such that $\Phi_\alpha = \Phi_e$ and $\Psi_\alpha = \Psi_e$. We show that $h = h_\alpha$. By (4) we have that $\alpha^\frown \mathtt{inf} \subset TP$. The whole argument takes place in a final segment of stages where α is not initialized and all nodes to the left of α have finalized their states and restraints. It follows by 4 that there will be no f-switched strategy to the left and on the same level as α.

First we wish to show that outcome `inf` is accessed infinitely often and thus h_α is total. Since we get infinitely many expansionary α-stages (stages where α is accessible) it is enough to show that

$$\mu D_{[N_i, N)}[f] > q_\alpha - \delta_\alpha$$

where $N > N_i$ is a fixed number, q_α, δ_α are the outcome and resolution respectively of the backing strategy of α and i is the maximum such that $|\sigma_i| < |\alpha|$. If $|\alpha| = 3e + 1$ then $\delta_\alpha = 3^{-e}$. The level $3n_{i+1} + 1$ of σ_{i+1} will be larger than the level $3e + 1$ of α. So by lemma 2.9 we have

$$\mu(\mathrm{dom}\Phi - D_{[N_i, \infty)}[f]) \leq \sum_{j > i} \mu(\mathrm{dom}\Phi - D_{[N_{j-1}, N_j)}[f]) \leq \sum_{j > e} 2 \cdot 3^{-j} \leq 3^{-e}.$$

So

$$\mu(D_{[N_i, N)}[f]) \geq \mu(D_{[N_i, \infty)}[f]) > \mu(\mathrm{dom}\Phi) - 3^{-e} \geq q_\alpha - 3^{-e}.$$

So α will be allowed to access outcome `inf` infinitely many times thus increasing $|h_\alpha|$ infinitely often. Now we only need to show $h_\alpha = h$. For a contradiction suppose that for a least x, $h_\alpha(x) \neq h(x)$. When $h_\alpha(x)$ was defined at some expansionary α-stage s_0 we had

$$h_\alpha(x) = \Phi_\alpha^f(x) = \Psi_\alpha^g(x). \tag{5}$$

If s_1 is the largest expansionary stage for ℓ_α at which (5) holds, r_α is larger than the corresponding uses of the computations. Let s_2 be the least stage after s_1 at which one of $f \upharpoonright r_\alpha, g \upharpoonright r_\alpha$ changes (from the value they had at s_1), and s_3 the next expansionary stage after s_1. By construction at each stage at most one of f, g can change. So we have the following cases:

(i) *f changes*: When the change happens α becomes g-switched. Note that before s_3 the outcome $\alpha^\frown\texttt{inf}$ will not be accessible and so α remains g-switched until s_3. So no node to the right of α or below it will change $g \upharpoonright r_\alpha$ before stage s_3. This means that at s_3 (5) will hold again, a contradiction.

(ii) *g changes*: When the change happens α becomes f-switched. If it remains in this state until s_3 by the same argument we get that (5) will hold at stage s_3, a contradiction. Otherwise a node to the right of α turns this strategy to g-switched state at a stage s_* between s_2 and s_3. But at that stage (5) must hold (possibly with new use u) and r_α will increase to cover the use of these (possibly new) computations. From stage s_* to stage s_3 the node α will remain in g-state and as before it will preserve the g-side of the computation (5), thus guaranteeing that this equality survives until stage s_3; this is a contradiction.

This shows that (5) holds for all x.   □

Lemma 2.11 *The function f is almost everywhere dominating.*

Proof. If a real β is such that the function Φ^β is total and is not dominated by f, then it must belong to infinitely many of the sets $\text{dom}\Phi - D_{[N_{i-1},N_i)}[f]$ for $i \geq 1$. So according to lemma 2.9 these reals belong to a set of measure 0.   □

References

[1] Binns, Stephen, Bjørn Kjos-Hanssen, Joseph Miller and Reed Solomon, *Lowness notions, measure and domination*, in preparation.

[2] Binns, Stephen, Bjørn Kjos-Hanssen, Manuel Lerman and Reed Solomon, *On a conjecture of Dobrinen and Simpson concerning almost everywhere domination*, J. Symbolic Logic **71** (2006), no. 1, 119–136.

[3] Cholak, Peter, Noam Greenberg and Joseph Miller, *Uniform almost everywhere domination*, to appear.

[4] Dobrinen, Natasha, and Stephen Simpson, *Almost everywhere domination*, J. Symbolic Logic **69** (2004), no. 3, 914–922.

[5] Kurtz, Stuart, "Randomness and Genericity in the Degrees of Unsolvability", Ph.D. thesis, University of Illinois at Urbana-Champaign, 1981. VII + 131 pages.

[6] Soare, Robert I., "Recursively enumerable sets and degrees", Berlin London: Springer-Verlag, 1987.