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An Algebra for Directed Bigraphs [⋆]

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Abstract

We study the algebraic structure of *directed bigraphs*, a bigraphical model of computations with locations, connections and resources previously introduced as a unifying generalization of other variants of bigraphs. We give a sound and complete axiomatization of the (pre)category of directed bigraphs. Using this axiomatization, we give an adequate encoding of the Fusion calculus, showing the utility of the added directness.

Keywords: Bigraphical models, categorical meta-models for Concurrency, fusion calculus.

1 Introduction

Bigraphical reactive systems (BRSs) are an emerging graphical meta-model of computation introduced by Milner [9,11] in which both *locality* and *connectivity* are central notions. The key structure of BRSs are *bigraphs*, which are composed by two orthogonal graph structures: a hierarchical *place graph* describing locations, and a *link (hyper-)graph* describing connections. The reaction rules, representing the dynamics of the BRS, may change both these structures. Several process calculi for Concurrency can be represented in bigraphs, such as CCS, pure Mobile Ambients, and (using a mild generalization called *binding bigraphs*), also the π-calculus and the λ-calculus [12]. An important feature of bigraphs is that they support a very general construction, based on the notion of *relative pushout* (RPO) [7], which allows to turn reaction rules into labelled transition systems.

However, Milner's definition of bigraphs is not the only possible one. Sassone and Sobociński have given in [15] an alternative definition, derived from a general categorical construction, the "input-linear cospan" over a particular 2-category of

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place-link graphs. Also this variant enjoys a general construction of RPOs. Interestingly, Milner's and Sassone-Sobociński's variants do not coincide; in fact, these two categories and their respective RPO constructions do not generalize each other.

In previous work [5,4], we have presented *directed bigraphs*, a generalization of both these kinds of bigraphs. Intuitively, the idea of directed bigraphs is to notice that names are not resources on their own, but only a way for denoting (abstract) resources (i.e., edges). A system can "ask" for external resources thorugh the names on its interfaces. Thus, we can identify a "resource request flow" starting from control ports, going through names and terminating in edges. This information is represented in the new notion of *directed link graph*, which replaces the previous notion of link graphs. We have given RPO constructions for this model, generalizing and unifying the constructions independently given by Jensen-Milner and Sassone-Sobociński in their respective variants. Moreover, the very same construction can be used for calculating relative pullbacks as well.

In this paper, we continue this line of investigation. We study the algebraic structure of directed bigraphs, giving a sound and complete axiomatization of this (pre)category. Moreover, we use the operators of this axiomatization for encoding the Fusion calculus, a calculus which was not dealt with by the previous versions of bigraphs. This encoding is adequate, in the sense that congruent processes are represented by exactly the same bigraph, and reduction steps in the original calculus is mimicked one-to-one by steps in the encoding.

Synopsis In Section 2 we briefly recall the main definitions about directed bigraphs and abstract directed bigraphs. In Section 3 we analyze the algebraic structure of directed bigraphs; this analysis is then carried on to the category of abstract directed bigraphs in Section 4. In Section 5 we put directed bigraphs at work, giving an encoding of the Fusion calculus. Conclusions are in Section 6.

2 Directed bigraphs

In this section we recall the definition and some properties of directed bigraphs; for details, we refer to [5,4]. Following Milner's approach, we work in *precategories;* see [8, §3] for an introduction to the theory of supported monoidal precategories. (We prefer precategories to 2-categories, because their concreteness allows for more direct definitions.)

Let $\mathcal{K}$ be a given signature of controls, and $ar : \mathcal{K} \to \omega$ the arity function.

Definition 2.1 *A* polarized interface X *is a pair of disjoint sets of names* $X = (X^-, X^+)$*; the two components are called* downward *and* upward *faces, respectively.*

A directed link graph $A : X \to Y$ *is* $A = (V, E, ctrl, link)$ *where* X *and* Y *are the* inner *and* outer *interfaces,* V *is the set of* nodes*,* E *is the set of* edges*,* $ctrl : V \to \mathcal{K}$ *is the* control map*, and* $link : \mathsf{Pnt}(A) \to \mathsf{Lnk}(A)$ *is the* link map*, where the* ports*, the* points *and the* links *of* A *are defined as follows (where* $+$

denotes disjoint union):

$$\mathsf{Prt}(A) \triangleq \sum_{v\in V} ar(ctrl(v)) \quad \mathsf{Pnt}(A) \triangleq (X^+ + Y^-) \uplus \mathsf{Prt}(A) \quad \mathsf{Lnk}(A) \triangleq (X^- + Y^+) \uplus E$$

The link map cannot connect downward and upward names of the same interface, i.e., the following condition must hold: $(link(X^+) \cap X^-) \cup (link(Y^-) \cap Y^+) = \emptyset$.

Directed link graphs are graphically depicted much like ordinary link graphs, with the difference that edges are explicit objects and points and names are associated to edges (or other names) by (simple) directed arcs. This notation makes explicit the "resource request flow": ports and names in the interfaces can be associated either to locally defined resources (i.e., a local edge) or to resources available from outside the system (i.e., via an outward name).

Definition 2.2 (′DLG) *The precategory of* directed link graphs *has polarized interfaces as objects, and directed link graphs as morphisms.*

Given two directed link graphs $A_i = (V_i, E_i, ctrl_i, link_i) : X_i \to X_{i+1}$ $(i = 0, 1)$, *the composition* $A_1 \circ A_0 : X_0 \to X_2$ *is defined when the two link graphs have disjoint nodes and edges. In this case,* $A_1 \circ A_0 \triangleq (V, E, ctrl, link)$, *where* $V \triangleq V_0 \uplus V_1$, $ctrl \triangleq ctrl_0 \uplus ctrl_1$, $E \triangleq E_0 \uplus E_1$ *and* $link : (X_0^+ + X_2^-) \uplus Pr \to E \uplus (X_0^- + X_2^+)$ *is defined as follows (where* $Pr = \mathsf{Prt}(A_0) \uplus \mathsf{Prt}(A_1)$*):*

$$link(p) \triangleq \begin{cases} link_0(p) & \text{if } p \in X_0^+ \uplus \mathsf{Prt}(A_0) \text{ and } link_0(p) \in E_0 \uplus X_0^- \\ link_1(x) & \text{if } p \in X_0^+ \uplus \mathsf{Prt}(A_0) \text{ and } link_0(p) = x \in X_1^+ \\ link_1(p) & \text{if } p \in X_2^- \uplus \mathsf{Prt}(A_1) \text{ and } link_1(p) \in E_1 \uplus X_2^+ \\ link_0(x) & \text{if } p \in X_2^- \uplus \mathsf{Prt}(A_1) \text{ and } link_1(p) = x \in X_1^-. \end{cases}$$

The identity link graph of X *is* $id_X \triangleq (\emptyset, \emptyset, \emptyset_{\mathcal{K}}, Id_{X^- \uplus X^+}) : X \to X$.

Definition 2.3 *The* support *of a link graph* $A = (V, E, ctrl, link)$ *is the set* $|A| \triangleq V + E$.

Definition 2.4 (idle, lean, open, closed, peer) *Let* $A : X \to Y$ *be a link graph.*

A link $l \in \mathsf{Lnk}(A)$ *is* idle *if it is not in the image of the link map (i.e.,* $l \notin link(\mathsf{Pnt}(A))$*). The link graph* A *is* lean *if there are no idle links.*

A link l *is* open *if it is an inner downward name or an outer upward name (i.e.,* $l \in X^- \cup Y^+$*); it is* closed *if it is an edge.*

A point p *is* open *if* $link(p)$ *is an open link; otherwise it is closed. Two points* p_1, p_2 *are* peer *if they are mapped to the same link, that is* $link(p_1) = link(p_2)$.

Proposition 2.5 *A link graph* $A : X \to Y$ *is epi iff there are no peer names in* Y^- *and no idle names in* Y^+*. Dually,* A *is mono iff there are no idle names in* X^- *and no peer names in* X^+.

A is an isomorphism iff it has no nodes, no edges, and its link map can be decomposed in two bijections $link^+ : X^+ \to Y^+$, $link^- : Y^- \to X^-$.

Definition 2.6 *The* tensor product $\otimes$ *in* ′DLG *is defined as follows. Given two* *objects X, Y, if these are pairwise disjoint then $X \otimes Y \triangleq (X^- \uplus Y^-, X^+ \uplus Y^+)$. Given two link graphs $A_i = (V_i, E_i, ctrl_i, link_i) : X_i \to Y_i$ $(i = 0, 1)$, if the tensor products of the interfaces are defined and the sets of nodes and edges are pairwise disjoint then the tensor product $A_0 \otimes A_1 : X_0 \otimes X_1 \to Y_0 \otimes Y_1$ is defined as $A_0 \otimes A_1 \triangleq (V_0 \uplus V_1, E_0 \uplus E_1, ctrl_0 \uplus ctrl_1, link_0 \uplus link_1)$.*

Finally, we can define the *directed bigraphs* as the composition of standard place graphs (see [8, §7] for definitions) and directed link graphs.

Definition 2.7 *A* (bigraphical) interface *I is composed by a* width *(a finite ordinal, denoted by $width(I)$) and by a polarized interface of link graphs (i.e., a pair of finite sets of names). A* directed bigraph *with signature $\mathcal{K}$ is $G = (V, E, ctrl, prnt, link) : I \to J$, where $I = \langle m, X\rangle$ and $J = \langle n, Y\rangle$ are its inner and outer interfaces respectively; V and E are the sets of nodes and edges respectively, and $prnt$, $ctrl$ and $link$ are the parent, control and link maps, such that $G^P \triangleq (V, ctrl, prnt) : m \to n$ is a place graph and $G^L \triangleq (V, E, ctrl, link) : X \to Y$ is a directed link graph.*

We denote G as combination of G^P and G^L by $G = \langle G^P, G^L\rangle$. In this notation, a place graph and a (directed) link graph can be put together iff they have the same sets of nodes and edges.

Definition 2.8 ('DBig) *The precategory 'DBig of directed bigraph with signature $\mathcal{K}$ has interfaces $I = \langle m, X\rangle$ as objects and directed bigraphs $G = \langle G^P, G^L\rangle : I \to J$ as morphisms. If $H : J \to K$ is another directed bigraph with sets of nodes and edges disjoint from V and E respectively, then their composition is defined by composing their components, i.e.: $H \circ G \triangleq \langle H^P \circ G^P, H^L \circ G^L\rangle : I \to K$..*

The identity directed bigraph of $I = \langle m, X\rangle$ is $\langle id_m, Id_{X^- \uplus X^+}\rangle : I \to I$.

Proposition 2.9 *A directed bigraph G in 'DBig is epi (respectively mono) iff its two components G^P and G^L are epi (respectively mono).*

The isomorphisms in 'DBig are all the combinations $\iota = \langle \iota^P, \iota^L\rangle$ of an isomorphism in 'PLG and an isomorphism in 'DLG.

Definition 2.10 *The* tensor product $\otimes$ *in 'DBig is defined as follows. Given $I = \langle m, X\rangle$ and $J = \langle n, Y\rangle$, where X and Y are pairwise disjoint, then $\langle m, X\rangle \otimes \langle n, Y\rangle \triangleq \langle m + n, (X^- \uplus Y^-, X^+ \uplus Y^+)\rangle$.*

The tensor product of $G_i : I_i \to J_i$ is defined as $G_0 \otimes G_1 \triangleq \langle G_0^P \otimes G_1^P, G_0^L \otimes G_1^L\rangle : I_0 \otimes I_1 \to J_0 \otimes J_1$, when the tensor products of the interfaces are defined and the sets of nodes and edges are pairwise disjoint.

Remarkably, directed link graphs (and bigraphs) have relative pushouts (RPOs) and pullbacks (RPBs), which can be obtained by a general construction, subsuming both Milner's and Sassone-Sobociński's variants. We refer the reader to [5,4].

Actually, in many situations we do not want to distinguish bigraphs differing only on the identity of nodes and edges. To this end, we introduce the category DBig of *abstract directed bigraphs*. The category DBig is constructed from 'DBig forgetting the identity of nodes and edges and any idle edge. More precisely, abstract bigraphs are concrete bigraphs taken up-to an equivalence $\simeq$ (see [8] for details).

Definition 2.11 (abstract directed bigraphs) *Two concrete directed bigraphs G and H are* lean-support equivalent, *written $G \mathrel{\widehat{\simeq}} H$, if they are support equivalent after removing any idle edges.*

The category DBig *of abstract directed bigraphs has the same objects as* ′DBig*, and its arrows are lean-support equivalence classes of directed bigraphs. We denote by $\mathcal{A}$:* ′DBig $\rightarrow$ DBig *the associated quotient functor.*

We remark that DBig is a category (and not only a precategory); moreover, $\mathcal{A}$ enjoys several important properties which we omit here due to lack of space; see [8].

3 Algebraic structure of ′DBig

We begin this section introducing some useful notations.

Remark 3.1 *An interface $\langle 0, (X^-, X^+)\rangle$ is abbreviated as (X^-, X^+); a singleton set $\{x\}$ as x; and $\langle m, (\emptyset, \emptyset)\rangle$ as m. The interfaces $(\emptyset, \emptyset)$ and 0 denote the same interface, the origin ϵ. Hence the identity id_ϵ can be expressed as ϵ, $(\emptyset, \emptyset)$ or 0.*

A bigraph $A : (\emptyset, X^+) \rightarrow (\emptyset, Y^+)$ is defined by a (not necessarily surjective) function $\sigma : X^+ \rightarrow Y^+$, called substitution, *if it has no nodes and no edges and the link map is σ; analogously a bigraph $A : (X^-, \emptyset) \rightarrow (Y^-, \emptyset)$ is defined by a (not necessarily surjective) function $\delta : Y^- \rightarrow X^-$, called* fusion, *if it has no nodes and no edges and the link map is δ. With abuse of notation, we write σ and δ to mean their corresponding bigraphs.*

Let $\vec{x}, \vec{y}$ be two vectors of the same length; we write $(y_0/x_0, y_1/x_1, \dots)$ or $\vartriangle^{\vec{y}}_{\vec{x}}$, where all the x_i are distinct, for the surjective map $x_i \mapsto y_i$; similarly, we write $(y_0/x_0, y_1/x_1, \dots)$ or $\triangledown^{\vec{y}}_{\vec{x}}$, where all y_i are distinct, for the surjective map $y_i \mapsto x_i$.

We denote by $\vartriangle^X : (\emptyset, \emptyset) \rightarrow (\emptyset, X)$ the bigraph defined by the empty substitution $\sigma : \emptyset \rightarrow X$, in the same way we denote $\triangledown_X : (X, \emptyset) \rightarrow (\emptyset, \emptyset)$ for the bigraph defined by the empty fusion $\delta : \emptyset \rightarrow X$.

Note that each substitution σ can be expressed in a unique way as $\sigma = \tau \otimes \vartriangle^X$, where τ is a surjective substitution; while each fusion δ can be expressed in a unique way as $\delta = \zeta \otimes \triangledown_X$, where ζ is a surjective fusion. We denote the renamings by α, i.e. the bijective substitution or bijective fusion.

Finally, we introduce the closure *bigraphs. The* closure $\blacktriangledown\!\!\blacktriangle^x_y : (\emptyset, y) \rightarrow (x, \emptyset)$ *has no nodes, a unique edge e and the link map is $link(x) = e = link(y)$. Two other types of closures are obtained by composing the closure $\blacktriangledown\!\!\blacktriangle^x_y$ and $\vartriangle^x$ or $\triangledown_y$ respectively:*

- *the* up-closure $\blacktriangle_y : (\emptyset, y) \rightarrow (\emptyset, \emptyset)$ *has no nodes, one edge e and $link(y) = e$;*
- *the* down-closure $\blacktriangledown^x : (\emptyset, \emptyset) \rightarrow (x, \emptyset)$ *has no nodes, one edge e and $link(x) = e$.*

Definition 3.2 (wirings) *A* wiring *is a bigraph whose interfaces have zero width (and hence has no nodes). The wirings ω are generated by the composition or tensor product of three base elements: the substitutions $\sigma : (\emptyset, X^+) \rightarrow (\emptyset, Y^+)$; the fusions $\delta : (Y^-, \emptyset) \rightarrow (X^-, \emptyset)$; and the closures $\blacktriangledown\!\!\blacktriangle^x_y : (\emptyset, y) \rightarrow (x, \emptyset)$.*

Definition 3.3 (prime bigraph) *An interface is* prime *if it has width equal to 1. Often we abbreviate a prime interface $\langle 1, (X^-, X^+)\rangle$ with $\langle (X^-, X^+)\rangle$, in particular $\langle (\emptyset, \emptyset)\rangle = 1$. A prime bigraph $P : \langle m, (Y^-, \emptyset)\rangle \rightarrow \langle (\emptyset, X^+)\rangle$ has no upward inner* *names and no downward outer names, and has a prime outer interface.*

An important prime bigraph is $merge_m : m \to 1$, it has no nodes and it maps m sites to an unique root. A bigraph $G : n \to \langle m, (X^-, X^+)\rangle$ without inner names, it can be simply converted in a prime bigraph as follows: $(merge_m \otimes id_{(X^-,X^+)}) \circ G$.

Definition 3.4 (discrete bigraph) *A bigraph D is* discrete *if it has no edges and the link map is a bijection. That means all points are open, there are no peer points and no idle link.*

The discreteness is well-behaved, and preserved by composition and tensor products. It is easy to see that discrete bigraphs form a monoidal sub-precategory of ′DBIG.

Definition 3.5 (ion, atom and molecule) *For any non atomic control K with arity k and a pair of sequence $\vec{x}^-$ and $\vec{x}^+$ of distinct names, whose overall length is k, we define the discrete* ion $K(v)^{\vec{x}^+}_{\vec{x}^-} : \langle(\vec{x}^-, \emptyset)\rangle \to \langle(\emptyset, \vec{x}^+)\rangle$ *as the bigraph with a unique K-node v, whose ports are separately linked to $\vec{x}^-$ or to $\vec{x}^+$. We omit v when it can be understood.*

For a prime discrete bigraph P with outer names $(\emptyset, Y^+)$ the composite $(K^{\vec{x}^+}_{\vec{x}^-} \otimes id_{(\emptyset,Y^+)}) \circ P$ is a discrete molecule*. If K is atomic, we define the discrete* atom $K^{\vec{x}^+}_{\vec{x}^-} : (\vec{x}^-, \emptyset) \to \langle(\emptyset, \vec{x}^+)\rangle$*; it resembles an ion, but has no site.*

An arbitrary (non-discrete) ion, molecule or atom is formed by the composition of $\omega \otimes id_1$ with a discrete one. Often we omit $\cdots \otimes id_I$ in the compositions, when there is no ambiguity; for example we write $merge_m \circ G$ to mean $(merge_m \otimes id_{(X^-,X^+)}) \circ G$ and $K^{\vec{x}^+}_{\vec{x}^-} \circ P$ to mean $(K^{\vec{x}^+}_{\vec{x}^-} \otimes id_{(\emptyset,Y^+)}) \circ P$ (for P prime discrete). Note that every atom and every molecule are prime, furthermore an atom is also ground, but a molecule is not necessarily ground, since it may have sites.

Now, we define some variants of the tensor product, allowing sharing of names. Process calculi often have a parallel product $P \mid Q$, that allows the processes P and Q to share names. In directed bigraphs, this sharing can involve inner downward names and/or outer upword names, as described by the following definitions.

Definition 3.6 (sharing products) *The* outer sharing product*,* inner sharing product *and* sharing product *of two link graphs $A_i : X_i \to Y_i$ $(i = 0, 1)$ are defined as follows:*

$$(X^-, X^+) \curlywedge (Y^-, Y^+) \triangleq (X^- \uplus Y^-, X^+ \cup Y^+)$$
$$(X^-, X^+) \curlyvee (Y^-, Y^+) \triangleq (X^- \cup Y^-, X^+ \uplus Y^+)$$

$$A_0 \curlywedge A_1 \triangleq (V_0 \uplus V_1, E_0 \uplus E_1, ctrl_0 \uplus ctrl_1, link_0 \uplus link_1) : X_0 \otimes X_1 \to Y_0 \curlywedge Y_1$$
$$A_0 \curlyvee A_1 \triangleq (V_0 \uplus V_1, E_0 \uplus E_1, ctrl_0 \uplus ctrl_1, link_0 \uplus link_1) : X_0 \curlyvee X_1 \to Y_0 \otimes Y_1$$
$$A_0 \parallel A_1 \triangleq (V_0 \uplus V_1, E_0 \uplus E_1, ctrl_0 \uplus ctrl_1, link_0 \uplus link_1) : X_0 \curlyvee X_1 \to Y_0 \curlywedge Y_1$$

defined when their interfaces are defined and A_i have disjoint node and edge sets.

The outer sharing product*,* inner sharing product *and* sharing product *of two bigraphs $G_i : I_i \to J_i$ are defined by extending the corresponding products on their* *link graphs with the tensor product on widths and place graphs:*

$$\langle m, X\rangle \curlywedge \langle n, Y\rangle \triangleq \langle n+m, X \curlywedge Y\rangle \qquad \langle m, X\rangle \curlyvee \langle n, Y\rangle \triangleq \langle n+m, X \curlyvee Y\rangle$$

$$\begin{aligned}
G_0 \curlywedge G_1 &\triangleq \langle G_0^P \otimes G_1^P, G_0^L \curlywedge G_1^L\rangle : I_0 \otimes I_1 \to J_0 \curlywedge J_1\\
G_0 \curlyvee G_1 &\triangleq \langle G_0^P \otimes G_1^P, G_0^L \curlyvee G_1^L\rangle : I_0 \curlyvee I_1 \to J_0 \otimes J_1\\
G_0 \parallel G_1 &\triangleq \langle G_0^P \otimes G_1^P, G_0^L \parallel G_1^L\rangle : I_0 \curlyvee I_1 \to J_0 \curlywedge J_1.
\end{aligned}$$

defined when their interfaces are defined and G_i have disjoint node and edge sets.

It is simple to verify that $\curlywedge$, $\curlyvee$ and $\parallel$ are associative, with unit ϵ.

Another way of constructing a sharing product of two bigraphs G_0, G_1 is to disjoin the names of G_0 and G_1, then take the tensor product of the two bigraphs and finally merge the name again:

Proposition 3.7 *Let G_0 and G_1 be bigraphs with disjoint node and edge sets. Then*

$$G_0 \curlywedge G_1 = \sigma(G_0 \otimes \tau G_1) \qquad G_0 \curlyvee G_1 = (G_0 \otimes G_1\zeta)\delta \qquad G_0 \parallel G_1 = \sigma(G_0 \otimes \tau G_1\zeta)\delta$$

where the substitution σ and τ are defined in the following way: if z_i $(i \in n)$ are the upward outer names shared by G_0 and G_1, and w_i are fresh names in bijection with the z_i, then $\tau(z_i) = w_i$ and $\sigma(w_i) = \sigma(z_i) = z_i$ $(i \in n)$. The substitution δ and ζ are defined in a very similar way, but acting on the downward inner names.

Definition 3.8 (prime products) *The* prime outer sharing product *and* prime sharing product *of two bigraphs $G_i : I_i \to J_i$ are defined as follows:*

$$\langle m, (X^-, X^+)\rangle \bar{\curlywedge} \langle n, (Y^-, Y^+)\rangle \triangleq \langle (X^- \uplus Y^-, X^+ \cup Y^+)\rangle$$

$$\begin{aligned}
G_0 \bar{\curlywedge} G_1 &\triangleq merge_{(width(J_0)+width(J_1))} \circ (G_0 \curlywedge G_1) : I_0 \otimes I_1 \to J_0 \bar{\curlywedge} J_1\\
G_0 \mid G_1 &\triangleq merge_{(width(J_0)+width(J_1))} \circ (G_0 \parallel G_1) : I_0 \curlyvee I_1 \to J_0 \bar{\curlywedge} J_1.
\end{aligned}$$

defined when their interfaces are defined and G_i have disjoint node and edge sets.

It is easy to show that $\bar{\curlywedge}$ and $\mid$ are associative, with unit 1 when applied to prime bigraphs. Note that for a wiring ω and a prime bigraph P, we have $\omega \bar{\curlywedge} P = \omega \curlywedge P$ and $\omega \mid P = \omega \parallel P$, because in this case these products have the same meaning.

Now, we can describe *discrete bigraphs,* which complement wirings:

Theorem 3.9 (discrete normal form) (i) *Every bigraph G can be expressed uniquely (up to iso) as: $G = (\omega \otimes id_n) \circ D \circ (\omega' \otimes id_m)$, where D is a discrete bigraph and ω, ω' are two wirings satisfying the following conditions:*

- *in ω, if two outer downward names are peer, then their target is an edge;*
- *in ω' there are no edges, and no two inner upward names are peer (i.e., on inner upward names ω' is a renaming, but outer downward names can be peer).*

(ii) *Every discrete bigraph $D : \langle m, (X^-, X^+)\rangle \to \langle n, (Y^-, Y^+)\rangle$ may be factored uniquely (up to iso) on the domain of each factor D_i, as:*

$$D = \alpha \otimes ((D_0 \otimes \cdots \otimes D_{n-1}) \circ (\pi \otimes id_{dom(\vec{D})}))$$

with α a renaming, each D_i prime and discrete, and π a permutation.

Proof. For the first part, consider a bigraph $G : \langle n, (X^-, X^+)\rangle \to \langle m, (Y^-, Y^+)\rangle$. We divide G in three parts: a discrete $D : \langle n, (Z^-, Z^+)\rangle \to \langle m, (W^-, W^+)\rangle$ and two wirings $\omega : (W^-, W^+) \to (Y^-, Y^+)$ and $\omega' : (X^-, X^+) \to (Z^-, Z^+)$ satisfying the previous conditions. We proceed by cases (where $Pr \triangleq \mathsf{Prt}(G) = \mathsf{Prt}(D)$):

$p \in Pr$, $link_G(p) = e \in E$: we add a fresh name $w_e \in W^+$ and define $link_D(p) = w_e$ and $link_\omega(w_e) = e$;

$p \in Pr$, $link_G(p) = y \in Y^+$: we add a fresh name $w_y \in W^+$ and define $link_D(p) = w_y$ and $link_\omega(w_y) = y$;

$p \in Pr$, $link_G(p) = x \in X^-$: this case is analogous to the previous one;

$y \in Y^-$, $link_G(y) = e \in E$: we define $link_\omega(y) = e$;

$x \in X^+$, $link_G(y) = e \in E$: we add a fresh name $z_e \in Z^+$, a fresh name $w_e \in W^+$ and define $link_{\omega'}(x) = z_e$, $link_D(z_e) = w_e$, $link_\omega(w_e) = e$;

$y \in Y^-$, $link_G(y) = x \in X^-$: we add a fresh name $w_x \in W^-$, a fresh name $z_x \in Z^-$ and define $link_\omega(y) = w_x$, $link_D(w_x) = z_x$ and $link_{\omega'}(z_x) = x$;

$x \in X^+$, $link_G(x) = y \in Y^+$: this case is analogous to the previous one; it is sufficient to invert the direction of links and swap the rule of ω with ω'.

Note that there are no idle names in Z^-, Z^+, W^- and W^+, so those sets are formed only by the fresh names defined in this proof. Furthermore, the three conditions above holds because we create a fresh name every time we need one.

The proof of the second part is easy. Since the outer interface of D has width n, we can decompose D in n discrete and prime parts, obtaining $D_0 \otimes \cdots \otimes D_{n-1}$. The renaming α describe the connections between the inner interface and the outer one. Finally the permutation π gives the right sequence of the sites, so we can take the tensor product of D_i ($i = 0, \ldots. n-1$) in any order. □

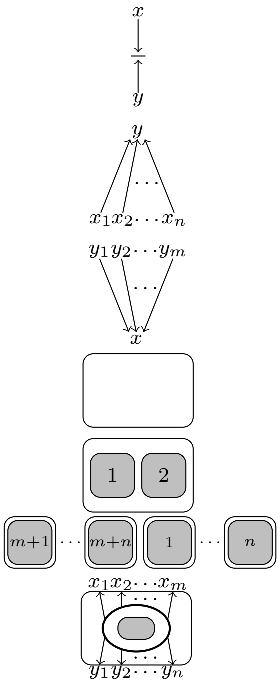
We call this unique factorization *discrete normal form* (DNF). The DNF applies to abstract bigraphs as well, and indeed it will play an important part in the complete axiomatization of DBig, as we will discuss in the next section.

Note that a renaming is discrete but not prime (since it has zero width); this is why the factorization in Theorem 3.9(ii) has such a factor. This unique factorization depends on the fact that the prime bigraphs have no upward inner names and downward outer names. In the special case that D is ground, the factorization in Theorem 3.9(ii) is simply $D = d_0 \otimes \cdots \otimes d_{n-1}$, that is a product of discrete and prime ground bigraphs.

4 Algebraic structure of DBig

In this section we describe a sound and complete axiomatization for directed abstract bigraphs, similarly to that given by Milner for pure bigraphs [10]. Furthermore we give a normal form for discrete bigraphs.

First we introduce the *algebraic signature*, that is a set of elementary bigraphs



$\bowtie^{x}_{y}{:}(\emptyset, y) \to (x, \emptyset)$	closure
$\triangle^{y}_{X}{:}(\emptyset, X) \to (\emptyset, y)$	substitution
$\triangledown^{Y}_{x}{:}(x, \emptyset) \to (Y, \emptyset)$	fusion
$1{:}\epsilon \to 1$	a barren root
$merge{:}2 \to 1$	mapping 2 sites in 1 root
$\gamma_{m,n}{:}m + n \to n + m$	swapping m with n
$K^{\vec{x}}_{\vec{y}}{:}\langle(\vec{y}, \emptyset)\rangle \to \langle(\emptyset, \vec{x})\rangle$	a discrete ion

Fig. 1. Elementary Bigraphs

able to define any other bigraph (Figure 1).

We have to show that all bigraphs can be constructed from these elementary ones by composition and tensor product. Before giving a formal result, we provide an intutive explanation of the meaning of these elementary bigraphs.

- The first three bigraphs build up all wirings, i.e. all the link graphs having no nodes. Indeed, all substitutions (fusions, resp.) can be obtained as tensor products of elementary substitutions $\triangle^{y}_{X}$ (fusions $\triangledown^{Y}_{x}$, resp.); the tensor products of singleton substitutions $\triangle^{y}_{x}$ and/or singleton fusions $\triangledown^{x}_{y}$ give all renamings. The composition and the tensor product of substitutions, fusions and closures give all wirings.
- The next three bigraphs define all placings, i.e. all place graphs having no nodes; for example $merge_m : m \to 1$, merging m sites in a unique root, are defined as:

$$merge_0 \triangleq 1 \qquad merge_{m+1} \triangleq merge \circ (id_1 \otimes merge_m).$$

  Notice that $merge_1 = id$ and $merge_2 = merge$, and that all permutations $\pi : m \to m$ are constructed by composition and tensor from the $\gamma_{m,n}$.
- Finally, for expressing any direct bigraph we need to add only the discrete ions $K^{\vec{x}^+}_{\vec{x}^-}$. In particular, we can express any discrete atoms as $K^{\vec{x}^+}_{\vec{x}^-} \circ 1$.

The following proposition shows that every bigraph can be expressed in a normal form, called (again) *discrete normal form* (DNF). We will use D, Q and N to denote discrete, discrete prime bigraphs, and the discrete molecules respectively.

Proposition 4.1 (discrete normal form) *In* DBig *every bigraph* G*, discrete* D*,* Categorical Axioms

$$A \circ id = A = id \circ A \qquad A \circ (B \circ C) = (A \circ B) \circ C$$
$$A \otimes id_\epsilon = A = id_\epsilon \otimes A \qquad A \otimes (B \otimes C) = (A \otimes B) \otimes C$$
$$\gamma_{I,\epsilon} = id_I \qquad \gamma_{J,I} \circ \gamma_{I,J} = id_{I \otimes J}$$
$$(A_1 \otimes B_1) \circ (A_0 \otimes B_0) = (A_1 \circ A_0) \otimes (B_1 \circ B_0)$$
$$\gamma_{I,K} \circ (A \otimes B) = (B \otimes A) \circ \gamma_{H,J} \qquad (\text{where } A : H \to I, B : J \to K)$$
$$\gamma_{I \otimes J,K} = (\gamma_{I,K} \otimes id_J) \circ (id_I \otimes \gamma_{J,K})$$

Link Axioms

$$\text{⧗}^x_y \circ \triangle^y_z = \text{⧗}^x_z \qquad \triangledown^z_x \circ \text{⧗}^x_y = \text{⧗}^z_y \qquad \triangledown_x \circ \text{⧗}^x_y \circ \triangle^y = id_\epsilon$$
$$\triangle^z_{(Y \uplus y)} \circ (id_{(\emptyset,Y)} \otimes \triangle^y_X) = \triangle^z_{(Y \uplus X)} \qquad (id_{(Y,\emptyset)} \otimes \triangledown^X_y) \circ \triangledown^{(Y \uplus y)}_z = \triangledown^{(X \uplus Y)}_z$$

Place Axioms

$$merge \circ (1 \otimes id_1) = id_1 \qquad merge \circ \gamma_{1,1} = merge$$
$$merge \circ (merge \otimes id_1) = merge \circ (id_1 \otimes merge)$$

Node Axioms

$$(id_1 \otimes \alpha) \circ K^{\vec{x}^+}_{\vec{x}^-} = K^{\alpha(\vec{x}^+)}_{\vec{x}^-} \qquad K^{\vec{x}^+}_{\vec{x}^-} \circ (id_1 \otimes \alpha) = K^{\vec{x}^+}_{\alpha(\vec{x}^-)}$$

Fig. 2. Axiomatization for the abstract directed bigraphs.

discrete and prime Q and discrete molecule N can be described by an expression of the respective following form:

$$G = (\omega \otimes id_n) \circ D \circ (\omega' \otimes id_m) \tag{1}$$

where ω, ω' satisfy the conditions given in Theorem 3.9(i);

$$D = \alpha \otimes ((Q_0 \otimes \cdots \otimes Q_{n-1}) \circ (\pi \otimes id_{dom(\vec{Q})})) \tag{2}$$
$$Q = (merge_{n+p} \otimes id_{\emptyset,Y^+}) \circ (id_n \otimes N_0 \otimes \cdots \otimes N_{p-1}) \circ (\pi \otimes id_{(Y^-,\emptyset)}) \tag{3}$$
$$N = (K^{\vec{x}^+}_{\vec{x}^-} \otimes id_{\emptyset,Y^+}) \circ Q. \tag{4}$$

Furthermore, the expression is unique up to isomorphisms on the parts.

We can use these equations for normalizing any bigraph G as follows; first, we apply equations (1), (2) to G once, obtaining an expression containing discrete and prime bigraphs $Q_0, \ldots, Q_{n-1}$. These are decomposed further using equations (3), (4) repeatedly: each Q_i is decomposed into an expression containing molecules $N_{i,0}, \ldots, N_{i,p_i-1}$, each of which is decomposed in turn into an ion containing another discrete and prime bigraph $Q'_{i,j}$. The last two steps are repeated recursively until the ions are atoms or have only holes as children. Note that the unit 1 is a special case of Q when $n = p = 0$.

In Figure 2 we give a set of axioms which we prove to be sound and complete.

Each of these equations holds only when both sides are defined; in particular, recall that the tensor product of two bigraphs is defined only if the name sets are disjoint. It is important to notice also that for ions only the renaming axiom is needed (because the names are treated positionally).

Theorem 4.2 (Completeness of the axiomatization) *Let us consider two expressions E_0, E_1 constructed from the elementary bigraphs by composition and tensor product. Then, E_0 and E_1 denote the same bigraph in* DBig *if and only if the equation $E_0 = E_1$ can be proved by the axioms in Figure 2.*

Proof. The proof is similar to that of [8, Theorem 10.2]. The "if" direction is simple to prove, since it requires to check that each axiom is valid. The "only if" direction is in two steps. First, we prove by induction on the structure of expressions, that the equality between an expression and its DNF is derivable from the axioms. Next, since DNFs are taken up to iso, we have to show that the equality between isomorphic DNFs is provable from the axioms. This is proved by showing that the axioms can prove the isomorphisms of the components of a DNF, which are ions, discrete and prime bigraphs, and discrete bigraphs. □

5 An Application: the Fusion Calculus

In this section we apply the theory developed in the previous sections to the Fusion calculus [13]. The processes of the finite (monadic) Fusion calculus are defined by the following grammar (sum and fusion prefix can be easily encoded in this syntax):

$$P, Q ::= \mathbf{0} \mid zx.P \mid \bar{z}x.P \mid P|Q \mid (x)P$$

where x, y, z range over a countable set of names $\mathcal{N}$, the processes are taken up to the structural congruence ($\equiv$), that is the least congruence satisfying the abelian monoid laws for composition and the scope laws and scope extension law:

$$(x)\mathbf{0} \equiv \mathbf{0} \qquad (x)(y)P \equiv (y)(x)P \qquad P|(x)Q \equiv (x)(P|Q) \text{ where } x \notin fn(P).$$

In [13], the semantics of the Fusion calculus is given by a labelled transition system for deriving transitions of the form $P \xrightarrow{\varphi} Q$ where φ is a *fusion*, that is a finite equivalence over names of the form $\{x_1{=}y_1, \ldots, x_n{=}y_n\}$. Here we adopt a reaction semantics, similar to that of Explicit Fusion [3]. The configuration of a process is denoted by a pair (P, φ) to mean that P has associated the fusion φ. We define $(P, \varphi) \to (Q, \psi)$ to be the least relation closed under the following rules

$$Com\ \frac{u\varphi v}{(\bar{u}x.P|vy.Q, \varphi) \to (P|Q, \varphi \cup \{x{=}y\})} \qquad \frac{(P, \varphi) \to (Q, \psi)}{(P|R, \varphi) \to (Q|R, \psi)}$$

$$\frac{(P\{z/x\}, \varphi) \to (Q, \psi)}{((x)P, \varphi) \to ((x)Q\{y/z\}, \psi \restriction z)}\ z \notin dom(\phi) \text{ and } y = \begin{cases} w & \text{if } z{=}w \in \psi \\ x & \text{otherwise} \end{cases}$$

where $\psi \restriction z = \psi \cap (\mathcal{N} \setminus \{z\})^2 \cup \{z{=}z\}$. It is easy to check that $(P, \varphi) \to (Q, \varphi \cup \psi)$ iff $P\sigma \xrightarrow{\psi\sigma} Q\sigma$ in the LTS semantics of [13], for any substitution σ which agrees with φ.

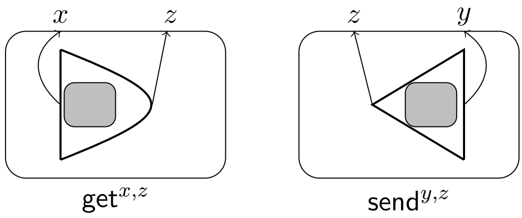


Fig. 3. The controls of the signature for the Fusion calculus.

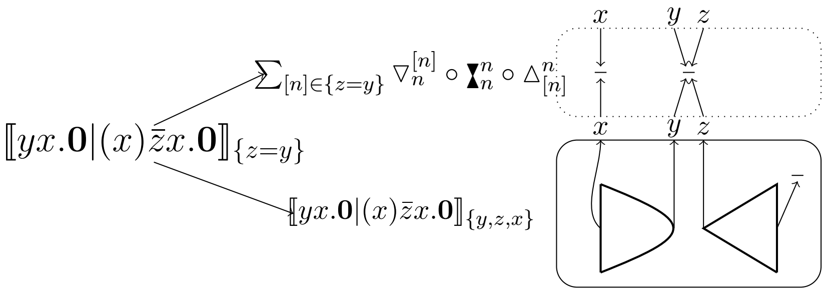


Fig. 4. An example of encoding a fusion process in directed bigraphs.

The signature for representing Fusion processes in directed bigraphs is

$$\mathcal{K}_F \triangleq \{\mathsf{get}{:}2, \mathsf{send}{:}2\}$$

where get and send are passive (Figure 3).

The encoding of processes into bigraphs is based on the idea of representing Fusion names as names on the interfaces, and each name equivalence class by a resource, i.e., an edge. Open names are outer names accessing to internal edges; bound names correspond to edges not accessible from outside.

Formally, a process P is translated to a bigraph of DBig($\mathcal{K}_F$) in two steps. First, for X a finite set of names such that $fn(P) \subseteq X$, we define a bigraph $[\![P]\!]_X : \epsilon \to \langle 1, (\emptyset, X)\rangle$, using the algebraic operators defined in the previous sections:

$$[\![\mathbf{0}]\!]_X = 1 \curlywedge X \qquad [\![P|Q]\!]_X = [\![P]\!]_X \curlywedge [\![Q]\!]_X \qquad [\![(x)P]\!]_X = \blacktriangle_x \circ [\![P]\!]_{X \uplus \{x\}}$$
$$[\![zx.P]\!]_X = \mathsf{get}^{x,z} \circ [\![P]\!]_X \qquad [\![\bar{z}x.P]\!]_X = \mathsf{send}^{x,z} \circ [\![P]\!]_X \qquad \text{where } x, z \in X$$

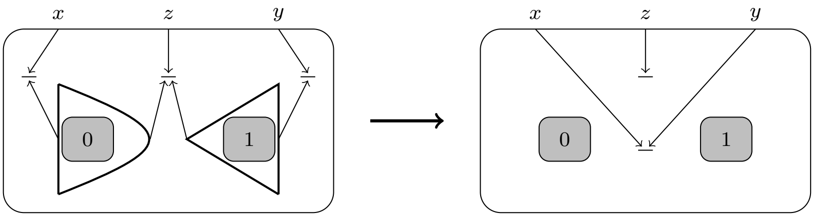
Notice that names in X are represented as outer upward names. In this translation bound names are represented by local (not accessible) edges.

Then, the encoding of a process P under a fusion φ takes the bigraph $[\![P]\!]_{fn(P)}$ and associates to each name in $fn(P)$ an outer accessible edge, according to φ:

$$[\![P]\!]_\varphi = \left(\sum_{[n]_\varphi \in \varphi} \triangledown^{[n]_\varphi}_{n} \circ \text{⧗}^{n}_{n} \circ \triangle^{n}_{[n]_\varphi}\right) \circ \left([\![P]\!]_{fn(P)} \otimes \sum_{m \in Y \setminus fn(P)} \triangle^{m}\right)$$

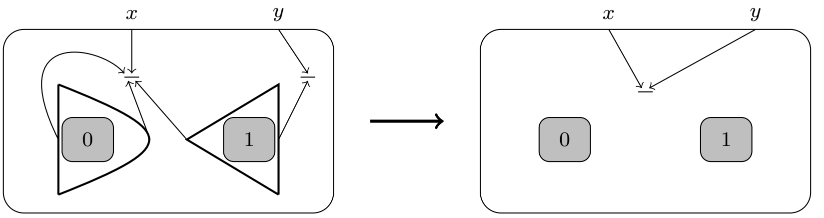
Fusions are represented by linking the fused names (in the outer interface) to the same edge. An example of encoding is given in Figure 4.

The encoding of the syntax is adequate, in the sense that two congruent processes are represented by exactly the same bigraph:



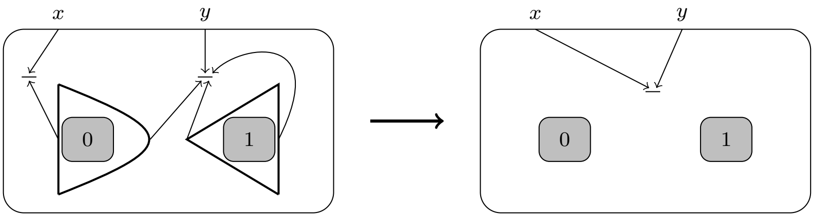
Fuse

$$(\text{⧗}^x_x \otimes \text{⧗}^z_z \otimes \text{⧗}^y_y) \circ (\mathsf{get}^{x,z} \curlywedge \mathsf{send}^{y,z}) \to (\blacktriangledown^z \otimes (\triangledown^{x,y}_w \circ \blacktriangledown^w)) \curlywedge id_1 \curlywedge id_1$$



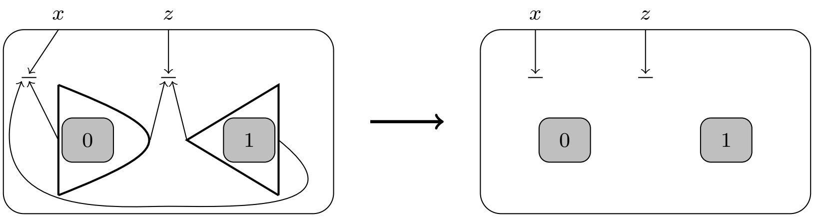
Fuse$_{\mathsf{send}}$

$$(\text{⧗}^x_x \otimes \text{⧗}^y_y) \circ (\mathsf{get}^{x,x} \curlywedge \mathsf{send}^{y,x}) \to (\triangledown^{x,y}_w \circ \blacktriangledown^w) \curlywedge id_1 \curlywedge id_1$$



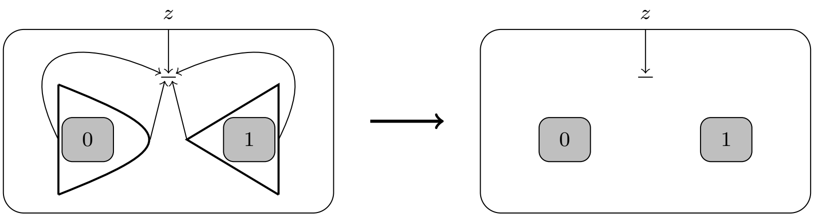
Fuse$_{\mathsf{get}}$

$$(\text{⧗}^x_x \otimes \text{⧗}^y_y) \circ (\mathsf{get}^{x,y} \curlywedge \mathsf{send}^{y,y}) \to (\triangledown^{x,y}_w \circ \blacktriangledown^w) \curlywedge id_1 \curlywedge id_1$$



Sync$_1$

$$(\text{⧗}^x_x \otimes \text{⧗}^z_z) \circ (\mathsf{get}^{x,z} \curlywedge \mathsf{send}^{x,z}) \to (\blacktriangledown^z \otimes \blacktriangledown^x) \curlywedge id_1 \curlywedge id_1$$



Sync$_2$

$$\text{⧗}^z_z \circ (\mathsf{get}^{z,z} \curlywedge \mathsf{send}^{z,z}) \to \blacktriangledown^z \curlywedge id_1 \curlywedge id_1$$

Fig. 5. Reaction rules $\mathcal{R}_F$ for the Fusion calculus.

Proposition 5.1 *Let P and Q be two processes; then $P \equiv Q$ if and only if $[\![P]\!]_\varphi = [\![Q]\!]_\varphi$, for every fusion φ.*

The reaction rules $\mathcal{R}_F$ are shown in Figure 5. The five rules cover the various possibilities of existing fusions between the names involved in the communication rule of the original semantics. As in the case of Milner's bigraphical reactive systems, these rules can be instantiated with only discrete ground bigraphs over the signature $\mathcal{K}_F$; for details see [6, §3.1].

We have the following adequacy result.

Proposition 5.2 $(P, \varphi) \to (P', \varphi') \iff [\![P]\!]_\varphi \longrightarrow [\![P']\!]_{\varphi'}$.

Proof. ($\Rightarrow$) The application of the *Com* rule of the Fusion calculus is encoded by applying one of the rules in $\mathcal{D}_F$ on the correct sub-bigraph, i.e. the one which encodes the right side of the rule.

($\Leftarrow$) If $[\![P]\!]_\varphi \longrightarrow [\![P']\!]_{\varphi'}$, then there is an application of one of the rules in $\mathcal{D}_F$, so we use the *Com* rule of the Fusion on the corresponding P sub-process. $\square$

Working with the abstract bigraphs we obtain the exact match between the Fusion reactions and bigraphic one.

The encoding of the Fusion calculus given in this paper differs from that in [6], where an "explicit fusion" control was used; hence, a single Fusion reaction (communication) had to be mimicked by a sequence of several bigraphical reactions, due to the "execution" of explicit fusions produced by the communication. Instead, in the encoding given here there is a one-to-one correspondence between Fusion and bigraphical reactions. On the other hand, the present reaction system is larger (it has five rules instead of three), and it is not orthogonal in the sense of [6].

6 Conclusions

In this paper we have given a sound and complete axiomatization of the precategory of *directed bigraphs*, a bigraphical model which subsumes and generalizes both Milner's and Sassone-Sobociński variants. We have used this axiomatization for giving an encoding the Fusion calculus, taking advantage of the peculiarities of directed bigraphs; e.g., edges represent equivalence classes of names. Differently from the encoding given in [6], here reactions in the encoding are in one-to-one correspondence with those in the original semantics (at the price of two more rules).

We plan to use this axiomatization for representing other calculi, in particular calculi with resources, locations, etc., which can be represented by edges. An interesting candidate is the ν-calculus [14]; it will be interesting to see which kind of wide transition systems we would obtain.

The new discrete normal form, and associated composition operations, presented in this paper can be useful in view of possible applications and extensions of logics and matching tools for bigraphs, in the line of [1,2]. Another future work is to give a 2-categorical definitions of directed link graphs.

Acknowledgement

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