Available online at www.sciencedirect.com
ScienceDirect
Electronic Notes in Theoretical Computer Science 257 (2009) 87–97
Electronic Notes in Theoretical Computer Science
www.elsevier.com/locate/entcs

A Duality Theorem for Quantitative Semantics [1]

Hengyang Wu[a,b,2] Yixiang Chen[b,3]

[a] *Information Engineer College*
Hangzhou Dianzi University
Hangzhou, China

[b] *Shanghai Key Laboratory of Trustworthy Computing*
East China Normal University
Shanghai, China

Abstract

This paper mainly studies quantitative possibility theory in the framework of domain. Using Sugeno's integral and the notion of module a duality theorem is obtained between the extended possibilistic powerdomain over a continuous domain X and the extended fuzzy predicates on X. This duality provides a reassuring link between the spaces of quantitative meaning and the corresponding Scott-topological space.

Keywords: possibility theory; the extended possibilistic powerdomain; module$_{[0,\ \infty)}$; Sugeno's integral

1 Introduction

Possibility theory is an uncertainty theory devoted to the handling of incomplete information (see, for example [2,3,4,10,13], etc.). To a large extent, it is similar to probability theory because it is based on set functions. The name

[1] The work is supported by Research and Development Project of High-Technology (Grant No:2007AA01Z189), Natural Science Foundation of China (No: 90718013), and the Hangzhou Dianzi University fund(No: KYS181507093)

[2] Email: wuhengy_1974@yahoo.com.cn
[3] Email: yxchen@sei.ecnu.edu.cn

doi:10.1016/j.entcs.2009.11.028

"Theory of Possibility" was coined by Zadeh in [13], and was mainly proposed as a framework for translating natural language information into fuzzy constraints on an underlying feature space. The basic building blocks of possibility theory were first synthetized in Dubois and Prade's books [4]. Now, possibility theory has two main research directions: qualitative and quantitative. Quantitative possibility theory has been proposed as a numerical model which could represent quantified uncertainty, whereas qualitative possibility theory centers on ordinal information rather than numerical information. Both approaches share the same basic "maxitivity" axiom. They differ when it comes to conditioning, and to independence notions.

Possibility measures are a concept used in possibility theory. Traditionally, it is defined on a σ-algebra of sets. The set-theoretic operations of such algebras, set complement, for example, cannot be easily reconciled with topological notions. Hence some literature studied measure defined open sets of some topological space(see, for example, [2,6,7], etc.).

We will illustrate our approach with a prime target of quantitative analysis, the completely distributive lattice $[0,\infty]$. Given a dcpo X, we are interested in the function space

$$[X \to [0,\infty]]$$

which could be the carrier of meaning for a cost, or real-time analysis. We stipulated that these functions are continuous. That way we ensure that the process of approximating total element by partial ones in X is consistent with the quantitative evidence provided by some function $f \in [X \to [0,\infty]]$ [6].

The aim of the present paper is to study quantitative possibility theory based on the previous spaces of quantitative meaning in the framework of domain. We first introduce the notion of the extended possibilistic powerdomain which can be used to model possibility computation. Then By using Sugeno's integral and the notion of module we succeed in getting a duality theorem between the extended possibilistic powerdomain over a continuous domain X and the extended fuzzy predicates on X. This duality provides a reassuring link between the spaces of quantitative meaning and space of topological possibility valuations.

2 Preliminaries

In this section we briefly review some basic notions in domain theory. We refer to [5].

Let $(D, \sqsubseteq)$ be a partially ordered set. A subset X of D is directed if it is non-empty and for each pair of elements $a, b \in X$, there is an upper bound $x \in X$ for $\{a, b\}$. A directed complete partial order (dcpo, for short) is a partial order set $(D, \sqsubseteq)$ such that every directed subset X has the least upper bound (or join)$\bigsqcup X$.

A function from dcpo D to E is Scott-continuous iff it preserves the partial order and the least upper bounds of directed sets, i.e., f is Scott-continuous if whenever $x \sqsubseteq y$, then $f(x) \sqsubseteq f(y)$, and for any directed set X, $f(\bigsqcup X) = \bigsqcup_{x \in X} f(x)$. We write $[D \to E]$ for all Scott-continuous mappings from D to E.

O is a Scott-open set iff O is an upper set and for all directed sets X, $\bigsqcup X \in O$ implies there exists $x \in X$ such that $x \in O$. The collection of all Scott-open subsets of D will be called the Scott-topology of D and denoted by $\sigma(D)$.

The relation way-below, written $\ll$, is defined in terms of $\sqsubseteq$ by $x \ll y$ iff for all directed sets X, $y \sqsubseteq \bigsqcup X$, implies there exists d in X such that $x \sqsubseteq d$. A dcpo D is continuous if for any x in D, the set of elements way-below x is directed and has *lub* x, i.e., $\{y : y \ll x\}$ is directed and $x = \bigvee\{y : y \ll x\}$.

3 The Extended Possibility Valuation

In this section, we will introduce the notions of the extended fuzzy predicate, the extended possibility valuation and module$_{[0,\ \infty]}$.

Definition 3.1 Let X be a dcpo. Scott-continuous functions from X into $[0, \infty]$ are called the extended fuzzy predicates. The set of all the extended fuzzy predicates on X is denoted as $\mathcal{F}_{\mathcal{E}}(X)$. The partial order on $\mathcal{F}_{\mathcal{E}}(X)$ is pointwise defined.

Remark 3.2 The extended fuzzy predicate generalizes fuzzy predicate in [2](Definition 3.1, page 2666), which can be seen as an expectation. Since $[0, \infty]$ with their usual linear order and endowed with the Scott topology the only proper open sets of which are the intervals $(a, \infty]$, an equivalent definition is: for any $a \geq 0$, $f^{-1}(a, \infty]$ is a Scott open set of X.

For every open O, the characteristic function χ_O that maps elements of O to 1 and all other elements to 0. Note that χ_U is the extended fuzzy predicate iff U is a Scott open set of X. Scalar multiplication and sup on $\mathcal{F}_{\mathcal{E}}(X)$ are defined as follows:

$$r * f(x) \ := r \times f(x)$$

$$(\sqcup_i f_i)(x) := \sup_i f_i(x),$$

for all $r \in [0, \infty]$ and $f, f_i \in \mathcal{F}_{\mathcal{E}}(X)$. We adopt the convention $0 \times \infty = 0$ and $r \times \infty = \infty$ if $r \neq 0$. It is clear that $\mathcal{F}_{\mathcal{E}}(X)$ is a complete lattice. Moreover, $f = \sup_{\alpha \in [0,\ \infty]}(\alpha \times \chi_{f^{-1}(\alpha,\ \infty]})$ for any $f \in \mathcal{F}_{\mathcal{E}}(X)$.

Definition 3.3 Let X be a dcpo. A function $\Pi : \sigma(X) \longrightarrow [0, \infty]$ is called the extended possibility valuation on X if, Π preserves arbitrary sup, i.e., $\Pi(\sqcup_{i\in I} O_i) = \sup_{i\in I} \Pi(O_i)$ where $\sigma(X)$ is Scott-topology of X and $\{O_i \mid i \in I\}$ is any subset family of $\sigma(X)$. We write $\pi_{\mathcal{E}}(X)$ for the collection of all possibility valuations on X, which will be called the extended possibilistic powerdomain on X, being ordered by the pointwise order $\sqsubseteq$, i.e., $\Pi \sqsubseteq \Pi'$ iff for any $O \in \sigma(X)$, $\Pi(O) \leq \Pi'(O)$. The triple $(X, \sigma(X), \Pi)$ will be said to be a possibility valuation space.

From this definition, we may conclude that Π is strict, i.e., $\Pi(\emptyset) = 0$ since the sup of empty is the least element on the respective space. Intuitively, $\Pi(O)$ expresses someone's subjective evaluation of the statement "y is in O" in a situation in which he guesses whether y is in O.

Remark 3.4 If $[0, \infty]$ is replaced by $[0, 1]$, then this definition is equivalent to Definition 2.1 in [2](page 2663). So the extended possibilistic powerdomain generalizes the possibilistic powerdomian.

Definition 3.5 Let X and Y be dcpos. The denotational semantics of deterministic possibility computation F from X to Y refers to the Scott-continuous mappings $[\![F]\!] : X \to \pi(Y)$.

This definition comes from Definition 2.5 (page 2665)in [2]. The difference is that Definition 2.5 in [2] gives the possibility belonging to the unit interval $[0, 1]$ that the result of computation falls into some Scott open set for some input, but here gives the expectation value belonging to the interval $[0, \infty]$.

For a state x, the point valuation η_x that maps Scott open set O to 1 if x belongs to O otherwise 0. Scalar multiplication and sup on $\pi(X)$ are defined as follows:

$$r * \Pi(O) \quad := r \times \Pi(O)$$

$$(\sqcup_i \Pi_i)(O) := \sup_i \Pi_i(O),$$

for all $r \in [0, \infty]$ and $\Pi, \Pi_i \in \pi(X)$. It is clear that $\pi(X)$ is a complete lattice. The following lemma coming from [6] (Theorem 5.4, page 7) will be used later.

Lemma 3.6 [6] Let X be a continuous domain. Then for any $\Pi \in \pi(X)$, $\Pi = \sqcup\{r \times \eta_x \mid r \times \eta_x \ll \Pi\}$.

Definition 3.7 [6,7] We consider the monoid $([0, \infty], \times, 1)$. A module$_{[0,\ \infty]}$ is a pair $(L; *_L)$, where L is a complete lattice and $*_L : [0, \infty] \times L \to L$ preserves suprema in each coordinate separately, such that for all $a \in L$ and $r, s \in [0, \infty]$,

(1) $1 *_L a = a$;

(2) $(r \times s) *_L a = r *_L (s *_L a)$.

In any module$_{[0,\ \infty]}L$ we have that $0 *_L a$ and $m *_L 0_L$ equal 0_L for all $a \in L$ and $m \in [0,\ \infty]$, since $*_L$ preserves suprema, and 0 and 0_L are least elements (empty suprema), respectively.

A simple example of module$_{[0,\ \infty]}$ is $[0, \infty]$ with its usual linear order, with '$\times$' as $*_{[0,\ \infty]}$. Sup and multiplication, extended to ∞ as usual:

$$\begin{aligned} &x \vee \infty = \infty = \infty \vee x, && x \in [0,\ \infty] \\ &r \times \infty = \infty, && r \in (0,\ \infty] \\ &0 \times \infty = 0. \end{aligned}$$

With this convention, sup and multiplication are sup-preserving on $[0, \infty]$. Other examples are the extended possibilistic powerdomain $\pi_{\mathcal{E}}(X)$ and the space of the extended fuzzy predicates $\mathcal{F}_{\mathcal{E}}(X)$.

Now, we consider the category $\mathbf{Module_{[0,\ \infty]}}$. Morphisms between module$_{[0,\ \infty]}$ $(A; *_A)$ and $(B; *_B)$ are functions $f : A \to B$ such that

(1) f is sup-preserving, i.e., $f(\sup_i a_i) = \sup_i f(a_i)$, where $\{a_i \mid i \in I\}$ is any subset of A;

(2) f is homogeneous, i.e., $f(r *_A a) = r *_B f(a)$, where $r \in [0, \infty]$ and $a \in A$.

Given a module$_{[0,\ \infty]}L$, let $L^\circ = (L,\ [0, \infty])$ be all module$_{[0,\ \infty)}$ morphisms from L to $[0, \infty]$. Then it is easy to verify that L° is a module$_{[0,\ \infty]}$, which will be called the dual module$_{[0,\ \infty]}$ of L. Let A and B be module$_{[0,\ \infty]}$. Then A and B are module$_{[0,\ \infty]}$-isomorphism if there exists a mapping f from A to B such that f is bijective and a module$_{[0,\ \infty]}$ morphism.

4 Duality Theorem

In this section, we will prove duality between the extended possibilistic powerdomain $\pi_{\mathcal{E}}(X)$ over a continuous domain X and all the extended fuzzy predicates $\mathcal{F}_{\mathcal{E}}(X)$ on X. Consider the dual module space of $\mathcal{F}_{\mathcal{E}}(X)$:

$$([X \to [0, \infty]] \to [0, \infty]) \tag{1}$$

could be the domain which, given a property ϕ and some $t \in [0, \infty]$ returns the expectation value satisfying ϕ within time t. We first present the 'Riesz' style represent theorem, i.e.,

$$\pi_{\mathcal{E}}(X) \cong \mathcal{F}_{\mathcal{E}}(X)^\circ. \tag{2}$$

It provides a reassuring link between the spaces of quantitative meaning in (1) and space of topological possibility valuations.

In order to show the isomorphism above, we need to introduce fuzzy integral. Fuzzy integrals of a fuzzy measurable function h over a fuzzy measure space $(X, \mathcal{A}, \mu)$ introduced by Sugeno in his dissertation [9] are well studied (see, for example, [1,8,11,12], etc.). Sugeno's integral is analogous to Lebesgue integral. The difference between them is that addition and multiplication in the definition of Lebesgue integral are replaced respectively by the operations "min" and "max" when Sugeno's integral is considered. Sugeno's integral has been applied in the fields of subjective evaluation, decision system, high level knowledge reasoning and pattern recognition, name a few [12]. The basic style of Sugeno's integral of h over A, a subset of X, with respect to the fuzzy measure μ can be calculated as follows:

$$(S)\int_A h d\mu = \sup_{\alpha\in[0,\ \infty]} [\alpha \wedge \mu(h^{-1}(\alpha,\infty] \cap A)].$$

For our purpose, we change min into product, i.e.,

$$(S)\int_A f du = \sup_{\alpha\in[0,\ \infty]} [\alpha \times \mu(f^{-1}(\alpha,\infty] \cap A)].$$

Similarly, we can define the integral of the extended fuzzy predicate f with respect to the extended possibility valuation Π as follows:

Definition 4.1 Let X be a dcpo, f the extended fuzzy predicate on X, and Π the extended possibility valuation on X. The integral of f with respect to Π over a Scott open set U of X is defined as

$$\int_U f d\Pi = \sup_{\alpha\in[0,\ \infty]} [\alpha \times \Pi(f^{-1}(\alpha,\infty] \cap U)]. \tag{3}$$

Particularly, $\int_X f d\Pi = \sup_{\alpha\in[0,\ \infty]}[\alpha \times \Pi(f^{-1}(\alpha,\infty])]$. We write $\int f d\Pi$ for $\int_X f d\Pi$.

We now collect some properties of the above integral needed in what follows.

Proposition 4.2 *Let X be a dcpo, f, g the extended fuzzy predicates on X and Π, Ξ the extended possibility valuations on X, and let U be a Scott open set of X. Then,*

(1) $f \sqsubseteq g$ implies $\int f d\Pi \le \int g\Pi$;
(2) $\Pi \sqsubseteq \Xi$ implies $\int f d\Pi \le \int f d\Xi$;
(3) $\int \chi_U d\Pi = \Pi(U)$;
(4) $\int \sqcup_{g\in\mathcal{G}} g d\Pi = \sup_{g\in\mathcal{G}} \int g d\Pi$ and $\int g d(\sqcup_i \Pi_i) = \sup_i \int g d\Pi_i$;
(5) $\int (r \times f) d\Pi = r \times \int f d\Pi$ and $\int f d(r \times \Pi) = r \times \int f d\Pi$ where $r \in [0, \infty]$;
(6) $\int f d\eta_x = f(x)$.

Proof. One can refer to the proofs of Proposition 4.2, 4.3, 4.4 and 4.5 (page 2668-2669)in [2]. $\square$

Defining two mappings as follows:

$$\Phi : \pi_{\mathcal{E}}(X) \to \mathcal{F}_{\mathcal{E}}(X)^{\circ}$$

such that $\Phi(\Pi)(f) = \int f d\Pi$ for any $\Pi \in \pi_{\mathcal{E}}(X)$ and $f \in \mathcal{F}_{\mathcal{E}}(X)$.

$$\Psi : \mathcal{F}_{\mathcal{E}}(X)^{\circ} \to \pi_{\mathcal{E}}(X)$$

such that $\Psi(\phi)(O) = \phi(\chi_O)$ for any $\phi \in \mathcal{F}_{\mathcal{E}}(X)^{\circ}$ and $O \in \sigma(X)$.

The following two propositions state that the mappings Φ and Ψ are well-defined.

Proposition 4.3 *Let $\Pi \in \pi_{\mathcal{E}}(X)$ and $\phi \in \mathcal{F}_{\mathcal{E}}(X)^{\circ}$. Then $\Phi(\Pi) \in \mathcal{F}_{\mathcal{E}}(X)^{\circ}$ and $\Psi(\phi) \in \pi_{\mathcal{E}}(X)$.*

Proof. They can be proven by Proposition 4.2(4,5). $\square$

We continue with,

Proposition 4.4 *Φ and Ψ as defined above are module$_{[0,\ \infty]}$ morphisms.*

Proof. They can be proven by Proposition 4.2(4,5). $\square$

Theorem 4.5 *Let X be a dcpo. Then $\Phi \circ \Psi = id_{\mathcal{F}_{\mathcal{E}}(X)^{\circ}}$ and $\Psi \circ \Phi = id_{\pi_{\mathcal{E}}(X)}$.*

Proof. For any $\phi \in \mathcal{F}_{\mathcal{E}}(X)^{\circ}$ and $f \in \mathcal{F}_{\mathcal{E}}(X)$, we have

$$\begin{aligned}
(\Phi \circ \Psi)(\phi)(f) &= \Phi(\Psi(\phi))(f) \\
&= \int f d\Psi(\phi) \\
&= \sup\nolimits_{\alpha \in [0,\ \infty]} \alpha \times \Psi(\phi)(f^{-1}(\alpha,\ \infty]) \\
&= \sup\nolimits_{\alpha \in [0,\ \infty]} \alpha \times \phi(\chi_{f^{-1}(\alpha,\ \infty]}) \\
&= \phi(\sup\nolimits_{\alpha \in [0,\ \infty]} \alpha \times \chi_{f^{-1}(\alpha,\ \infty]}) \text{ (since } \phi \text{ is a module}_{[0,\infty]} \text{ morphism)} \\
&= \phi(f)
\end{aligned}$$

Hence $\Phi \circ \Psi = id_{\mathcal{F}_{\mathcal{E}}(X)^{\circ}}$.

One the other hand, for any $\Pi \in \pi_{\mathcal{E}}(X)$ and $O \in \sigma(X)$, we have

$$\begin{aligned}(\Psi \circ \Phi)(\Pi)(O) &= \Psi(\Phi(\Pi))(O)\\ &= \Phi(\Pi)(\chi_O)\\ &= \int \chi_O d\Pi\\ &= \Pi(O) \text{ (by Proposition 4.2(3))}\end{aligned}$$

So $\Psi \circ \Phi = id_{\pi_{\mathcal{E}}(X)}$. □

Theorem 4.6 *Let X be a dcpo. Then $\pi_{\mathcal{E}}(X)$ and $\mathcal{F}_{\mathcal{E}}(X)^\circ$ are order-isomorphic and module$_{[0,\ \infty]}$-isomorphic.*

Proof. In order to prove that $\pi_{\mathcal{E}}(X)$ and $\mathcal{F}_{\mathcal{E}}(X)^\circ$ are order-isomorphic, we only show that Φ and Ψ are order-preserving. In fact, Φ is order-preserving by Proposition 4.2(2) and Ψ is order-preserving which can directly proven by the definition of Ψ. $\pi_{\mathcal{E}}(X)$ and $\mathcal{F}_{\mathcal{E}}(X)^\circ$ are module$_{[0,\ \infty]}$-isomorphic since the mappings Φ and Ψ are bijective and preserve the module$_{[0,\ \infty]}$ operations by Proposition 4.4. □

The above theorem states that $\mathcal{F}_{\mathcal{E}}(X)^\circ$ can be viewed as the extended possibilistic powerdomain $\pi_{\mathcal{E}}(X)$. If in addition X is a continuous domain we have full duality, meaning that

$$\pi_{\mathcal{E}}(X)^\circ \cong \mathcal{F}_{\mathcal{E}}(X). \tag{4}$$

Defining two mappings as follows:

$$\Gamma : \mathcal{F}_{\mathcal{E}}(X) \to \pi_{\mathcal{E}}(X)^\circ$$

such that $\Gamma(f)(\Pi) = \int f d\Pi$ for any $f \in \mathcal{F}_{\mathcal{E}}(X)$ and $\Pi \in \pi_{\mathcal{E}}(X)$.

$$\Omega : \pi_{\mathcal{E}}(X)^\circ \to \mathcal{F}_{\mathcal{E}}(X)$$

such that $\Omega(\phi)(x) = \phi(\eta_x)$ for any $\phi \in \pi_{\mathcal{E}}(X)^\circ$ and $x \in X$.

The following two propositions state that the mappings Γ and Ω are well-defined.

Proposition 4.7 *Let $f \in \mathcal{F}_{\mathcal{E}}(X)$ and $\phi \in \pi_{\mathcal{E}}(X)^\circ$. Then $\Gamma(f) \in \pi_{\mathcal{E}}(X)^\circ$ and $\Omega(\phi) \in \mathcal{F}_{\mathcal{E}}(X)$.*

Proof. $\Gamma(f) \in \pi_{\mathcal{E}}(X)^\circ$ can be proven by Proposition 4.2(4,5). $\Omega(\phi) \in \mathcal{F}_{\mathcal{E}}(X)$ is because that $\Omega(\phi)$ is Scott-continuous which can be directly verified by the definition of Scott-continuous. □

We continue with,

Proposition 4.8 *Γ and Ω as defined above are module$_{[0,\ \infty]}$ morphisms.*

Proof. Γ is a module$_{[0,\ \infty]}$ morphism which can be proven by Proposition4.2(4,5) and Ω is a module$_{[0,\ \infty]}$ morphism which can be directly verified.$\square$

Theorem 4.9 *Let X be a continuous domain. Then $\Omega \circ \Gamma = id_{\mathcal{F}_{\mathcal{E}}(X)}$ and $\Gamma \circ \Omega = id_{\pi_{\mathcal{E}}(X)^{\circ}}$.*

Proof. First, for any $f \in \mathcal{F}_{\mathcal{E}}(X)$ and $x \in X$, we have

$$\begin{aligned}
(\Omega \circ \Gamma)(f)(x) &= \Omega(\Gamma(f))(x) \\
&= \Gamma(f)(\eta_x) \\
&= \int f d\eta_x \\
&= f(x) \text{ (by Proposition 4.2(6))}
\end{aligned}$$

Hence, $\Omega \circ \Gamma = id_{\mathcal{L}(X)}$.

Second, for any $\phi \in \pi_{\mathcal{E}}(X)^{\circ}$ and $\Pi \in \pi_{\mathcal{E}}(X)$, we have

$$\begin{aligned}
(\Gamma \circ \Omega)(\phi)(\Pi) &= \Gamma(\Omega(\phi))(\Pi) \\
&= \int \Omega(\phi) d\Pi \\
&= \sup_{\alpha \in [0,\ \infty]} \alpha \times \Pi((\Omega(\phi))^{-1}(\alpha,\ \infty]) \\
&= \sup_{\alpha \in [0,\ \infty]} \alpha \times [(\sqcup_{r \times \eta_x \ll \Pi} r \times \eta_x)((\Omega(\phi))^{-1}(\alpha,\ \infty])] \\
&\qquad\qquad \text{(by Lemma 3.6)} \\
&= \sup_{\alpha \in [0,\ \infty]} \alpha \times [\sup_{r \times \eta_x \ll \Pi} r \times \eta_x((\Omega(\phi))^{-1}(\alpha,\ \infty])] \\
&= \sup_{\alpha \in [0,\ \infty]} [\sup_{r \times \eta_x \ll \Pi} \alpha \times r \times \eta_x((\Omega(\phi))^{-1}(\alpha,\ \infty])] \\
&= \sup_{r \times \eta_x \ll \Pi} [\sup_{\alpha \in [0,\ \infty]} \alpha \times r \times \eta_x((\Omega(\phi))^{-1}(\alpha,\ \infty])] \\
&= \sup_{r \times \eta_x \ll \Pi} r \times [\sup_{\alpha \in [0,\ \infty]} \alpha \times \eta_x((\Omega(\phi))^{-1}(\alpha,\ \infty])] \\
&= \sup_{r \times \eta_x \ll \Pi} r \times [\sup_{\alpha \in [0,\ \infty]} \alpha \times \chi_{((\Omega(\phi))^{-1}(\alpha,\ \infty])}(x)] \\
&= \sup_{r \times \eta_x \ll \Pi} r \times \Omega(\phi)(x) \\
&= \sup_{r \times \eta_x \ll \Pi} r \times \phi(\eta_x) \\
&= \phi(\sqcup_{r \times \eta_x \ll \Pi} r \times \eta_x) \ \text{ (since } \phi \text{ is a module}_{[0,\ \infty]} \text{ morphism)} \\
&= \phi(\Pi)
\end{aligned}$$

Hence, $\Gamma \circ \Omega = id_{\pi_{\mathcal{E}}(X)^{\circ}}$. $\square$

Moreover, similar to the proof of Theorem 4.6, the mappings Γ and Ω are order-preserving. Therefore,

Theorem 4.10 *Let X be a continuous domain. Then $\pi_{\mathcal{E}}(X)^{\circ}$ and $\mathcal{F}_{\mathcal{E}}(X)$ are order-isomorphic and module$_{[0,\ \infty]}$-isomorphic.*

Proof. $\pi_{\mathcal{E}}(X)^{\circ}$ and $\mathcal{F}_{\mathcal{E}}(X)$ are module$_{[0,\ \infty]}$-isomorphic since the mappings Γ and Ω are bijective and preserve the module$_{[0,\ \infty]}$ operations by Proposition 4.8. □

5 Conclusion and Further Work

In this paper, we point out that semantic domain of deterministic possibility computation is a module construction. By using Sugeno's integral and the notion of module we obtain a duality theorem, which shows that the extended possibilistic powerdomain over a continuous domain X and the extended fuzzy predicates on X can be represented each other. In the future, we plan to study the semantic domain of nondeterministic possibility computation and set up the corresponding Hoare powerdomain and Smyth powerdomain structures.

Acknowledgement

The authors would like to thank the referees for their invaluable comments and suggestions.

References

[1] Benvenuti and D. Vivona. General theory of the fuzzy integral. *Mathware and Soft Computing*, 3: 199-209,1996.

[2] Yixiang Chen and Hengyang Wu. Domain semantics of possibility computations. *Information Sciences*, 178: 2661-2679, 2008.

[3] D. de Cooman, D. Ruan, and E.E. Kerre (Eds). Foundations and Applications of Possibility Theory. *Advance in Fuzzy Systems-Applications and Theory*.vol. 8, World Scientific, 1995.

[4] D. Dubois and H. Prade. Possibiliry Theory, New York Plenum, 1988.

[5] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, and D.S. Scott. Continuous lattices and domains. Volume 93 of Encyclopedia of Mathemmatics and its Applications, Cambridge University Press, 2003.

[6] M. Huth. A powerdomain of possibility measures. *Electronic Notes in Theoretical Computer Science*, 6: 191-202, 1997.

[7] R. Heckmann and M. Huth. Quantitative semantics, topology, and possibility measures. *Topology and its Applications*, 8: 151-178, 1998.

[8] Y. Narukawa and V. Torra. Fuzzy measures and integrals in evaluation of strategies. *Information Sciences*, 177: 4686-4695, 2007.

[9] M. Sugeno. Theory of fuzzy integrals and its application. PhD thesis, Tokyo Institute of Technology, 1974.

[10] G. J. Wang and H. J. Zhou. Quantitative logic. *Information Sciences*, 179: 226-247, 2009.

[11] C. X. Wu and M. Traore. An extension of Sugeno integral. *Fuzzy Sets and Syestems*, 138: 537-550, 2003.

[12] M. S. Ying. Linguistic quantifiers modeled by Sugeno integrals. *Artificial Intelligence*, 170: 581-606, 2006.

[13] L. A. Zadeh. Fuzzy sets as a basis for a theory of possibility. *Fuzzy Sets and Systems*, 1: 3-28, 1978.