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On Rough Concept Lattices [1]

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Abstract

Formal concept analysis and rough set theory provide two different methods for data analysis and knowledge processing. Given a context $\mathbb{K}$, one can get the concept lattice $L(\mathbb{K})$ in Wille's sense and the object-oriented rough concept lattice RO-$L(\mathbb{K})$ (resp., attribute-oriented RA-$L(\mathbb{K})$). We study relations of the three kinds of lattices and their properties from the domain theory point of view. The concept of definable sets is introduced. It is proved that the family Def $(\mathbb{K})$ of the definable sets in set-inclusion order is a complete sublattice of RO-$L(\mathbb{K})$ and is a complete field of sets under some reasonable conditions. A necessary and sufficient condition for Def $(\mathbb{K})$ to be equal to RO-$L(\mathbb{K})$ is given. A necessary and sufficient condition is also given for the complete distributivity of RO-$L(\mathbb{K})$. We also study algebraicity of RO-$L(\mathbb{K})$ and several sufficient conditions are given for RO-$L(\mathbb{K})$ to be algebraic.

Keywords: Rough, Concept, Galois connection, Algebraic lattice, Definable set

1 Introduction

With the development of computer science, more and more attention is paid to the research of its mathematical foundations which have been the common field of mathematicians and computer scientists. Domain theory (DT), formal concept analysis (FCA) and rough set theory (RST) are three important crossing fields based on relations (orders) and simultaneously related to topology, algebra, logic, etc., and provide mathematical foundations for computer science and information science.

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Domain theory [1,5] was introduced by Scott for the denotational semantics of programming languages. The fundamental idea of domain theory is partial information and successive approximation. It deals with various posets, approximate orders and operational models of computing. The theory provides mathematical foundation for the design, definition, and implementation of programming languages.

FCA is an order-theoretic method for the mathematical analysis of scientific data, proposed by Wille and others [4] in 1982. Concept lattices are the core of the mathematical theory of FCA. A concept lattice is a partially ordered set consisting of formal concepts, each of which represents a subset of objects called extent and a subset of attributes called intent. Over the past several decades, FCA has become a powerful tool for clustering, data analysis, information retrieval and knowledge discovery.

RST is originated by Pawlak [8,9], which is a new mathematical tool to deal with inexact, uncertain or vague knowledge. The basic concepts of RST are approximation spaces and approximation operators. In an approximation space, every subset can be approximated by two subsets, called the lower and upper approximations of the given subset. Using methods of lower and upper approximations, knowledge hidden in given information can be expressed in the form of decision rules. With the development of RST, it has attracted worldwide attention of researchers and practitioners and has achieved a lot of real applications such as medicine, information analysis, data mining and industry control.

FCA, RST and DT are closely related. Many efforts have been made to compare and combine the three theories [2,6,7], [13]–[17]. Zhang and Shen [16,17] established relationships among FCA, Chu spaces and DT. They discussed the algebraicity of classical concept lattices and introduced the notion of approximable concepts and showed that approximable concept lattices represent algebraic lattices. Düntsch [2] and Yao [13,15] introduced attribute-oriented concepts and object-oriented concepts respectively for contexts in terms of approximation operators. In [7], Lei and Luo called the complete lattices of object-oriented concepts [13,15] rough concept lattices, introduced the notion of rough approximable concepts and showed that rough approximable concept lattices also represent algebraic lattices. With the necessity operator $\Box$ and the possibility operator $\diamond$, a rough concept gives prediction of membership of an object based on its attributes in data analysis [2,13].

In this paper, we will go deeper to discuss properties of rough concept lattices from the domain theory point of view. Some sufficient conditions for a rough concept lattice to be algebraic are given. A sufficient and necessary condition for rough concept lattices to be completely distributive is obtained and some subtle examples are constructed. We also introduce the concept of definable sets for a context and study the special sublattice of the rough concept lattice consisting of definable sets.

2 Preliminaries

In this section we recall some terminologies and facts used in the sequel. For a set X, $\mathcal{P}(X)$ denotes the powerset of X and $F \subseteq_{fin} X$ means that F is a finite subset of X. If $A \subseteq X$, then A^c denotes the complement of A in X. For non-explicitly stated notions please refer to [3]–[5].

2.1 Galois Connections

Definition 2.1 *[5]* Let S and T be two posets. A pair (g, d) of monotone functions $g : S \to T$ and $d : T \to S$ is called a Galois connection between S and T if for all $(s, t) \in S \times T$,

$$g(s) \geq t \Leftrightarrow s \geq d(t).$$

where g, d are called the upper adjunction and lower adjunction, respectively.

Galois connections are efficient tools in dealing with ordered sets. They appeared in the literature in two equivalent versions. The version we adopt here uses order-preserving maps, which is more popular in computer science, and the other version uses order-reversing maps, which occurs in FCA [4], etc.

A *closure operator* on a poset L is a monotone function $c : L \to L$ which satisfies

(1) $x \leq c(x)$ for all $x \in L$, and

(2) $c(c(x)) = c(x)$ for all $x \in L$.

A *kernel operator* on L is dually defined.

For a closure (kernel) operator, the following result is well-known.

Lemma 2.2 [5] *Let $\varphi : L \to L$ be a closure (resp., kernel) operator. Then the set of fixed points of φ is precisely the image of φ, i.e.,*

$$\{x \in L |\ \varphi(x) = x\} = \{\varphi(x) |\ x \in L\}.$$

The next well-known fact shows that closure and interior operators can be derived from Galois connections in a natural way.

Lemma 2.3 [5] *Let the pair (g, d) with $g : S \to T$ and $d : T \to S$ be a Galois connection. Then $g \circ d : T \to T$ is a closure operator on T and $d \circ g : S \to S$ is a kernel operator on S.*

2.2 Algebraic Lattices and Completely Distributive Lattices

Let L be a poset. For $X \subseteq L$, set $\downarrow X = \{y \in L \mid y \leq x \text{ for some } x \in X\}$ and $\uparrow X = \{y \in L \mid x \leq y \text{ for some } x \in X\}$. For a singleton $\{x\}$, we use $\downarrow x$ for $\downarrow \{x\}$ and $\uparrow x$ for $\uparrow \{x\}$. Say that X is a *lower set* if $X = \downarrow X$ and that X is an *upper set* if $X = \uparrow X$. A subset D of L is called *directed* if it is nonempty and every finite subset of D has an upper bound in D. An element $k \in L$ is called *compact* if for all directed subsets $D \subseteq L$ for which $\sup D$ exists, $k \leq \sup D$ always implies the existence of $d \in D$ with $k \leq d$. The set of all compact elements of L is denoted by $K(L)$. A complete lattice L is called *algebraic* iff

$$(\forall x \in L) \quad x = \bigvee(\downarrow x \cap K(L)).$$

Lemma 2.4 [5] *Let L be a poset. Then L is an algebraic lattice if and only if for some set X, L is isomorphic to the image of some closure operator $\varphi : \mathcal{P}(X) \to \mathcal{P}(X)$ which preserves directed unions.*

A complete lattice L is called a *completely distributive lattice* if and only if for any family $\{x_{j,k} \mid j \in J, k \in K(j)\}$ in L the identity

$$\bigwedge_{j \in J} \bigvee_{k \in K(j)} x_{j,k} = \bigvee_{f \in M} \bigwedge_{j \in J} x_{j,f(j)}$$

holds, where M is the set of choice functions defined on J with values $f(j) \in K(j)$.

Let $\emptyset \neq \mathcal{L} \subseteq \mathcal{P}(X)$. When we say that $\mathcal{L}$ is a *complete ring of sets* we mean that $\mathcal{L}$ is closed under arbitrary unions and intersections. We say that $\mathcal{L}$ is a *complete field of sets* if $\mathcal{L}$ is a complete ring of sets and $A \in \mathcal{L}$ implies $A^c \in \mathcal{L}$.

It is well-known that a complete ring of sets is a completely distributive algebraic lattice and a complete field of sets is in addition a complete Boolean algebra.

2.3 Concept Lattices and Rough Concept Lattices

In FCA, a *formal context* $\mathbb{K}$ is a triple (U, V, R), where U is a set of objects, V is a set of attributes and R is a binary relation between U and V with xRy reading as "object x has attribute y".

Definition 2.5 *[4,16]* Let (U, V, R) be a formal context. Define two maps:

$$\alpha : \mathcal{P}(U) \to \mathcal{P}(V) \text{ with } \alpha(A) = \{y \in V \mid \forall a \in A, aRy\},$$

$$\omega : \mathcal{P}(V) \to \mathcal{P}(U) \text{ with } \omega(B) = \{x \in U \mid \forall b \in B, xRb\}.$$

A pair of sets (A, B) is called a (formal) concept, if $A \subseteq U$, $B \subseteq V$, $\alpha(A) = B$ and $\omega(B) = A$, where A is called the extent and B is called the intent of concept (A, B).

To avoid confusion, here we adopt the notations α and ω from Zhang [16,17], which correspond to $'$ in [4] and $*$ in [13]-[15].

Let $\mathbb{K} = (U, V, R)$ be a context. The set of all concepts of $\mathbb{K}$ is ordered by $(A_1, B_1) \leq (A_2, B_2) \Leftrightarrow A_1 \subseteq A_2$ (which is equivalent to $B_2 \subseteq B_1$). The set of all concepts of $\mathbb{K}$ with order $\leq$ is called the concept lattice of $\mathbb{K}$ and denoted as $L(\mathbb{K})$.

Theorem 2.6 [4] *For a context $\mathbb{K}$, the concept lattice $L(\mathbb{K})$ is a complete lattice and the infimum and supremum are given by*

$$\bigwedge_{i\in I}(A_i, B_i) = (\bigcap_{i\in I} A_i,\ \alpha(\omega(\bigcup_{i\in I} B_i))), \quad \bigvee_{i\in I}(A_i, B_i) = (\omega(\alpha(\bigcup_{i\in I} A_i)),\ \bigcap_{i\in I} B_i).$$

Given a formal context $\mathbb{K} = (U, V, R)$, for $x \in U$ and $y \in V$, let $R(x) = \{y \in V |\ xRy\}$ and $R^{-1}(y) = \{x \in U |\ xRy\}$. For $A \subseteq U$ and $B \subseteq V$, let $R(A) = \{y \in V |\ \exists a \in A, aRy\}$ and $R^{-1}(B) = \{x \in U |\ \exists b \in B, xRb\}$. The following definition is imported from RST.

Definition 2.7 *[2,7,13]* Let $\mathbb{K} = (U, V, R)$ be a context. Then the following approximation operators are defined:

$$\begin{aligned}
&{}^{\square} : \mathcal{P}(U) \to \mathcal{P}(V), A \mapsto \{y \in V |\ R^{-1}(y) \subseteq A\},\\
&{}^{\diamond} : \mathcal{P}(V) \to \mathcal{P}(U), B \mapsto \{x \in U |\ R(x) \cap B \neq \emptyset\};\\
&{}^{\square} : \mathcal{P}(V) \to \mathcal{P}(U), B \mapsto \{x \in U |\ R(x) \subseteq B\},\\
&{}^{\diamond} : \mathcal{P}(U) \to \mathcal{P}(V), A \mapsto \{y \in V |\ R^{-1}(y) \cap A \neq \emptyset\}.
\end{aligned}$$

It is easy to check that

(1) ${}^{\square}$ preserves arbitrary intersections and ${}^{\diamond}$ preserves arbitrary unions and thus they are order-preserving;

(2) pairs $({}^{\square}\colon \mathcal{P}(U) \to \mathcal{P}(V),\ {}^{\diamond} : \mathcal{P}(V) \to \mathcal{P}(U))$ and $({}^{\square}\colon \mathcal{P}(V) \to \mathcal{P}(U),\ {}^{\diamond} : \mathcal{P}(U) \to \mathcal{P}(V))$ are Galois connections and thus ${}^{\diamond\square}$ is a closure operator and ${}^{\square\diamond}$ is a kernel operator;

(3) ${}^{\square} : \mathcal{P}(U) \to \mathcal{P}(V)$ and ${}^{\diamond} : \mathcal{P}(U) \to \mathcal{P}(V)$ are dual, i.e., for any $A \in \mathcal{P}(U)$, $A^{c\square c} = A^{\diamond}$. Similarly, ${}^{\square} : \mathcal{P}(V) \to \mathcal{P}(U)$ and ${}^{\diamond} : \mathcal{P}(V) \to \mathcal{P}(U)$ are dual.

A pair of sets (A, B) is called an object-oriented concept [13,15] of $\mathbb{K} = (U, V, R)$, if $A \subseteq U$, $B \subseteq V$ and $A^{\square} = B$, $B^{\diamond} = A$, where A is called the extent of (A, B) and B is called the intent. An attribute-oriented concept [2] $(B, A) \in (\mathcal{P}(V) \times \mathcal{P}(U))$ is dually defined.

The set of all object-oriented concepts of $\mathbb{K}$ is ordered by $(A_1, B_1) \leq (A_2, B_2) \Leftrightarrow A_1 \subseteq A_2$ (which is equivalent to $B_1 \subseteq B_2$). The set of all object-oriented concepts of $\mathbb{K}$ with order $\leq$ is called the rough concept lattice of $\mathbb{K}$ and denoted as RO-$L(\mathbb{K})$. Similarly, we denote the rough concept lattice of attribute-oriented concepts as RA-$L(\mathbb{K})$.

Theorem 2.8 [15] *For a context $\mathbb{K} = (U, V, R)$, the rough concept lattice RO-$L(\mathbb{K})$ is a complete lattice and the infimum and supremum are given by*

$$\bigwedge_{i\in I}(A_i, B_i) = ((\bigcap_{i\in I} A_i)^{\Box\Diamond},\ \bigcap_{i\in I} B_i), \quad \bigvee_{i\in I}(A_i, B_i) = (\bigcup_{i\in I} A_i,\ (\bigcup_{i\in I} B_i)^{\Diamond\Box}).$$

Let $\mathbb{K} = (U, V, R)$ be a context. $\mathbb{K}^c$ denotes context (U, V, R^c), where R^c is the complement relation of R. Yao [15] has discussed relationships between RO-$L(\mathbb{K})$, RA-$L(\mathbb{K})$ and the classical concept lattice $L(\mathbb{K}^c)$ of context $\mathbb{K}^c$ and obtained the conclusion that they are isomorphic to each other. Strictly speaking, we have

Theorem 2.9 *Let $\mathbb{K} = (U, V, R)$ be a context. Then RO-$L(\mathbb{K})$ is dually isomorphic to RA-$L(\mathbb{K})$ and RO-$L(\mathbb{K})$ is also dually isomorphic to $L(\mathbb{K}^c)$.*

In the sequel, we mainly discuss properties of object-oriented concept. For the rough concepts RO-$L(\mathbb{K})$, the set of all extents ordered with set-inclusion relation is denoted as RO-$L_U(\mathbb{K})$ and the set of all intents ordered with set-inclusion relation is denoted as RO-$L_V(\mathbb{K})$. Then we have

$$\text{RO-}L(\mathbb{K}) \cong \text{RO-}L_U(\mathbb{K}) \cong \text{RO-}L_V(\mathbb{K}).$$

To study properties of RO-$L(\mathbb{K})$, we may consider those of RO-$L_U(\mathbb{K})$ or RO-$L_V(\mathbb{K})$ instead. Since $(^\Box, ^\Diamond)$ is a Galois connection, by Lemmas 2.2 and 2.3, we have

$$\text{RO-}L_U(\mathbb{K}) = \{B^\Diamond \mid B \subseteq V\} = \{A^{\Box\Diamond} \mid A \subseteq U\} = \{A \subseteq U \mid A = A^{\Box\Diamond}\},$$

$$\text{RO-}L_V(\mathbb{K}) = \{A^\Box \mid A \subseteq U\} = \{B^{\Diamond\Box} \mid B \subseteq V\} = \{B \subseteq V \mid B = B^{\Diamond\Box}\}.$$

The infimum and supremum in RO-$L_U(\mathbb{K})$ are given by

$$\bigwedge_{i\in I} A_i = (\bigcap_{i\in I} A_i)^{\Box\Diamond}, \quad \bigvee_{i\in I} A_i = \bigcup_{i\in I} A_i.$$

The infimum and supremum in RO-$L_V(\mathbb{K})$ are given by

$$\bigwedge_{i\in I} B_i = \bigcap_{i\in I} B_i, \quad \bigvee_{i\in I} B_i = (\bigcup_{i\in I} B_i)^{\Diamond\Box}.$$

Example 2.10 Let $(X, \mathcal{T})$ be a topological space and $\mathbb{K} = (X, \mathcal{T}, R)$ be a context, where $R = \in$. For any $x \in X$, $R(x) = \{U \in \mathcal{T} | \ x \in U\}$ and for any $U \in \mathcal{T}$, $R^{-1}(U) = \{x \in X | \ x \in U\} = U$. Then for any $A \subseteq X$ and $\mathcal{U} \subseteq \mathcal{T}$, we have

$$\begin{aligned} A^{\Box} &= \{U \in \mathcal{T} | \ R^{-1}(U) \subseteq A\} = \{U \in \mathcal{T} | \ U \subseteq A\}, \\ \mathcal{U}^{\diamond} &= \{x \in X | \ R(x) \cap \mathcal{U} \neq \emptyset\} = \{x \in X | \ \exists U \in \mathcal{U}, x \in U\}, \\ A^{\Box\diamond} &= \{x \in X | \ \exists U \in \mathcal{T}, x \in U \subseteq A\} = A^{\circ}, \end{aligned}$$

where A° is the interior of A. Then

$$\text{RO-}L_U(\mathbb{K}) = \{A \subseteq X | \ A^{\Box\diamond} = A\} = \{A \subseteq X | \ A^{\circ} = A\} = \mathcal{T}$$

is the lattice of open sets of $(X, \mathcal{T})$. So, the rough concept lattice of $\mathbb{K}$ is a frame.

In [11], Vickers introduced the notion of topological systems. A topological system is a triple $(X, A, \models)$ with X being a nonempty set, A being a frame and $\models \subseteq X \times A$, where $\models$ matches the logic of finite observations :

- If S is finite subset of A, then $x \models \bigwedge S \Leftrightarrow x \models a$ for all $a \in S$.
- If S is any subset of A, then $x \models \bigvee S \Leftrightarrow x \models a$ for some $a \in S$.

For each $a \in A$, its extent in $(X, A, \models)$ is $\mathrm{e}(a) = \{x \in X | \ x \models a\}$.

Example 2.11 Let $(X, A, \models)$ be a topological system. Consider the context $\mathbb{K} = (X, A, \models)$. For each $B \subseteq A$, $B^{\diamond} = \{x \in X | \ \exists b \in B, x \models b\} = \bigcup_{b \in B} \mathrm{e}(b) = \mathrm{e}(\bigvee B)$. Then Ro-$L_U(\mathbb{K}) = \{\mathrm{e}(b) | b \in A\}$ is the topology of the specialization of $(X, \mathcal{T}, \models)$.

Example 2.12 Consider the context $\mathbb{K}_0 = (U, V, R)$ shown in Fig.1, which appeared in *[15]* originally. The standard concept lattice $L(\mathbb{K}_0)$ and the rough concept lattice RO-$L(\mathbb{K}_0)$ of $\mathbb{K}_0$ are shown in Fig.2 and Fig.3 respectively. They have different lattice structures and give different data analyses for the context.

	a	b	c	d	e
1	×		×	×	×
2	×		×		
3		×			×
4		×			×
5	×				
6	×	×			×

Fig. 1. Context $\mathbb{K}_0$

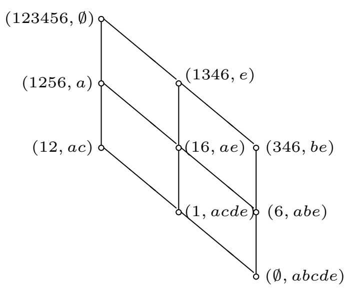

Fig. 2. $L(\mathbb{K}_0)$

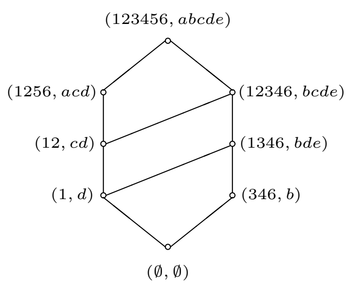

Fig. 3. RO-$L(\mathbb{K}_0)$

3 The Definable Sets

We study a special complete sublattice of RO-$L(\mathbb{K})$ by introducing the notion of definable sets of a context, a generalization of the notion of definable sets in RST.

First we recall the notion of definable sets in a generalized approximation space.

A generalized approximation space is a pair (U, R), where U is a nonempty set called "universe" and R is a binary relation on U. For $X \subseteq U$, the lower and upper approximations of X in (U, R) are respectively defined as

$$\underline{R}X = \{x \in U |\ R(x) \subseteq X\}, \quad \overline{R}X = \{x \in U |\ R(x) \cap X \neq \emptyset\},$$

where $R(x) = \{y \in U |\ xRy\}$. The operators $\underline{R}, \overline{R} : \mathcal{P}(U) \to \mathcal{P}(U)$ are respectively called the lower and upper approximation operators in (U, R). If a subset $X \subseteq U$ satisfies that $\underline{R}X = \overline{R}X$, then X is called a definable set of (U, R).

Similarly, we can define definable sets in a context.

Definition 3.1 Let $\mathbb{K} = (U, V, R)$ be a context and $X \subseteq U$. If $X^{\Box} = X^{\diamond}$, then X is called an object-definable set of $\mathbb{K}$. The family of all object-definable sets of $\mathbb{K}$ is denoted as $\mathrm{Def}_U(\mathbb{K})$.

By the definitions of $\Box, \diamond : \mathcal{P}(U) \to \mathcal{P}(V)$, it is easy to see that $\emptyset^{\diamond} = \emptyset$ and $U^{\Box} = V$. And the following lemma is easy to check.

Lemma 3.2 *For a context $\mathbb{K} = (U, V, R)$, the following statements are equivalent.*

(1) *$\mathbb{K}$ has no empty columns, i.e., for each $y \in V$, $R^{-1}(y) \neq \emptyset$.*

(2) *$\emptyset^{\Box} = \emptyset$.*

(3) *$U^{\diamond} = V$.*

(4) *$X^{\Box} \subseteq X^{\diamond}$ for all $X \subseteq U$.*

Proposition 3.3 *For a context $\mathbb{K} = (U, V, R)$, $\mathrm{Def}_U(\mathbb{K}) \neq \emptyset$ if and only if $\mathbb{K}$ has no empty columns.*

Proof. Suppose $\mathrm{Def}_U(\mathbb{K}) \neq \emptyset$. Pick $X \in \mathrm{Def}_U(\mathbb{K})$. If there is $y \in V$ such that $R^{-1}(y) = \emptyset$, then $R^{-1}(y) \subseteq X$ and $y \in X^{\Box}$. Since $X^{\Box} = X^{\diamond}$, we have $y \in X^{\diamond}$ and hence $R^{-1}(y) \cap X \neq \emptyset$. This contradicts $R^{-1}(y) = \emptyset$. Therefore there is no $y \in V$ such that $R^{-1}(y) = \emptyset$ and $\mathbb{K}$ has no empty columns.

Conversely, if $\mathbb{K}$ has no empty column, then by Lemma 3.2, $\emptyset, U \in \mathrm{Def}_U(\mathbb{K})$. $\Box$

An attribute-definable set of a context $\mathbb{K} = (U, V, R)$ is defined to be a set $Y \subseteq V$ such that $Y^{\Box} = Y^{\diamond}$. The family of attribute-definable sets of $\mathbb{K}$ is denoted by $\mathrm{Def}_V(\mathbb{K})$. Then $\mathrm{Def}_V(\mathbb{K}) \neq \emptyset$ if and only if $\mathbb{K}$ has no empty rows, i.e., for each $x \in U$, $R(x) \neq \emptyset$.

Next we discuss properties of the set of all object-definable sets of a context.

Theorem 3.4 *If context $\mathbb{K} = (U, V, R)$ has no empty columns, then $Def_U(\mathbb{K})$ is a complete field of sets.*

Proof. (1) $\emptyset, U \in \mathrm{Def}_U(\mathbb{K})$ is the smallest and biggest element respectively.

(2) Suppose $X \in \mathrm{Def}_U(\mathbb{K})$. Then $X^{\Box} = X^{\diamond}$. By the duality of $^{\Box}$ and $^{\diamond}$ we have $(X^c)^{\Box} = (X^{\diamond})^c = (X^{\Box})^c = (X^c)^{\diamond}$ and thus $X^c \in \mathrm{Def}_U(\mathbb{K})$.

(3) Let $\{X_i | \ i \in I\} \subseteq \mathrm{Def}_U(\mathbb{K})$. Then for each $i \in I$, $X_i^{\Box} = X_i^{\diamond}$. Since $\mathbb{K}$ has no empty columns, $(\bigcup_{i\in I} X_i)^{\Box} \subseteq (\bigcup_{i\in I} X_i)^{\diamond}$. On the other hand, since $^{\diamond}$ preserves arbitrary unions and $^{\Box}$ is order-preserving, we have $(\bigcup_{i\in I} X_i)^{\diamond} = \bigcup_{i\in I} X_i^{\diamond} = \bigcup_{i\in I} X_i^{\Box} \subseteq (\bigcup_{i\in I} X_i)^{\Box}$. Thus $(\bigcup_{i\in I} X_i)^{\Box} = (\bigcup_{i\in I} X_i)^{\diamond}$, that is, $\bigcup_{i\in I} X_i \in \mathrm{Def}_U(\mathbb{K})$.

(4) Let $\{X_i | \ i \in I\} \subseteq \mathrm{Def}_U(\mathbb{K})$. By (2), (3) and De Morgan's Law, $\bigcap_{i\in I} X_i \in \mathrm{Def}_U(\mathbb{K})$.

All the above shows that $\mathrm{Def}_U(\mathbb{K})$ is a complete field of sets. □

Corollary 3.5 *If context $\mathbb{K} = (U, V, R)$ has no empty columns, then ($Def_U(\mathbb{K})$, $\subseteq$) is both a completely distributive algebraic lattice and a complete Boolean algebra.*

Proof. Straightforward by Theorem 3.4. □

Proposition 3.6 *If context $\mathbb{K} = (U, V, R)$ has neither empty columns nor empty rows, then for each $X \subseteq U$, the following statements are equivalent:*

(1) $X \in Def_U(\mathbb{K})$; (2) $X = X^{\diamond\diamond}$; (3) $X = X^{\Box\Box}$.

Proof. (1) ⇒ (2): Suppose $X \in \mathrm{Def}_U(\mathbb{K})$. Then $X^{\diamond} = X^{\Box}$. Let $x \in X^{\diamond\diamond}$. Then $R(x) \cap X^{\diamond} \neq \emptyset$, i.e., there is $y \in X^{\diamond}$ such that xRy. Since $X^{\diamond} = X^{\Box}$, we have $y \in X^{\Box}$ and $x \in R^{-1}(y) \subseteq X$. Thus $X^{\diamond\diamond} \subseteq X$. On the other hand, since $\mathbb{K}$ has no empty rows, it is easy to check that $X \subseteq R^{-1}(R(X)) = X^{\diamond\diamond}$. Therefore $X = X^{\diamond\diamond}$.

(2) ⇒ (1): Suppose that $X = X^{\diamond\diamond}$. Since $^{\diamond\Box}$ is a closure operator, we have $X^{\Box} = X^{\diamond\diamond\Box} \supseteq X^{\diamond}$. Since $\mathbb{K}$ has no empty columns, $X^{\Box} \subseteq X^{\diamond}$ and $X^{\Box} = X^{\diamond}$. Therefore $X \in \mathrm{Def}_U(\mathbb{K})$.

(2) ⇒ (3): Suppose $X = X^{\diamond\diamond}$. Then by (2)⇒ (1), $X \in \mathrm{Def}_U(\mathbb{K})$. Then by Theorem 3.4, $X^c \in \mathrm{Def}_U(\mathbb{K})$. And by (1)⇒ (2) and the duality of $^{\Box}$ and $^{\diamond}$, we have $X^c = (X^c)^{\diamond\diamond} = X^{\Box c\diamond} = (X^{\Box\Box})^c$ and $X = X^{\Box\Box}$.

(3) ⇒ (2): Suppose $X = X^{\Box\Box}$. Then $X^c = (X^{\Box\Box})^c = (x^c)^{\diamond\diamond}$ and $X^c \in \mathrm{Def}_U(\mathbb{K})$. Therefore, we have $X \in \mathrm{Def}_U(\mathbb{K})$ and thus $X = X^{\diamond\diamond}$. □

Corollary 3.7 *If context $\mathbb{K} = (U, V, R)$ has neither empty columns nor empty rows, then for each $X \in Def_U(\mathbb{K})$, $(X, X^\diamond)$ is an object-oriented concept of $\mathbb{K}$.*

By Theorem 3.4 and Corollary 3.7, we have

Corollary 3.8 *If context $\mathbb{K} = (U, V, R)$ has neither empty columns nor empty rows, then $Def_U(\mathbb{K}) \subseteq RO\text{-}L_U(\mathbb{K})$. Furthermore, $(Def_U(\mathbb{K}), \subseteq)$ is a complete sublattice of $RO\text{-}L_U(\mathbb{K})$.*

For the context $\mathbb{K}_0$ in Example 2.12, it is easy to check that $\mathrm{Def}_U(\mathbb{K}_0) = \{\emptyset, U\} \neq \mathrm{RO}\text{-}L_U(\mathbb{K}_0)$. The next lemma is useful in finding conditions for Def $_U(\mathbb{K})$ to be equal to RO-$L_U(\mathbb{K})$ for a given context $\mathbb{K}$.

Lemma 3.9 *Let (U, V, R) be a context. Then the following two statements are equivalent.*

(1) *For all $B \subseteq V$, $B^{\diamond\diamond\diamond} \subseteq B^\diamond$.*

(2) *For all $x_1, x_2 \in U$, if $R(x_1) \cap R(x_2) \neq \emptyset$, then $R(x_1) = R(x_2)$.*

Proof. (1) $\Rightarrow$ (2) Let $x_1, x_2 \in U$ and $R(x_1) \cap R(x_2) \neq \emptyset$. Pick $y \in R(x_1) \cap R(x_2)$. Then x_1Ry and x_2Ry. For any $z \in R(x_1)$, take $B = \{z\}$. Then $x_1 \in B^\diamond$, $y \in B^{\diamond\diamond}$ and $x_2 \in B^{\diamond\diamond\diamond}$. By (1), $x_2 \in B^\diamond = R^{-1}(z)$ and thus $z \in R(x_2)$. So $R(x_1) \subseteq R(x_2)$. By the same argument, $R(x_2) \subseteq R(x_1)$. Thus $R(x_1) = R(x_2)$.

(2) $\Rightarrow$ (1) Let $B \subseteq V$ and $x \in B^{\diamond\diamond\diamond}$. Then there is $y \in B^{\diamond\diamond}$ such that xRy and then there is $x_1 \in B^\diamond$ such that x_1Ry and then there is $y_1 \in B$ such that x_1Ry_1. Since $y \in R(x) \cap R(x_1) \neq \emptyset$, we have $R(x) = R(x_1)$. Thus $y_1 \in R_1(x) \cap B = R(x) \cap B \neq \emptyset$ and $x \in B^\diamond$. So $B^{\diamond\diamond\diamond} \subseteq B^\diamond$. $\square$

Theorem 3.10 *If context $\mathbb{K} = (U, V, R)$ has no empty columns, then $Def_U(\mathbb{K}) = RO\text{-}L_U(\mathbb{K})$ iff for all $x_1, x_2 \in U$, $R(x_1) \cap R(x_2) \neq \emptyset$ implies $R(x_1) = R(x_2)$.*

Proof. Since $\mathbb{K}$ has no empty columns, for all $Y \subseteq V$ we have $Y \subseteq R(R^{-1}(Y)) = Y^{\diamond\diamond}$ and $Y^\diamond \subseteq Y^{\diamond\diamond\diamond}$. Then by the proof of Proposition 3.6, we have

$$\begin{aligned}\mathrm{Def}_U(\mathbb{K}) = \mathrm{RO}\text{-}L_U(\mathbb{K}) &\Leftrightarrow \forall Y \subseteq V,\ Y^\diamond \in \mathrm{Def}_U(\mathbb{K})\\ &\Leftrightarrow \forall Y \subseteq V,\ Y^\diamond = Y^{\diamond\diamond\diamond}\\ &\Leftrightarrow \forall Y \subseteq V,\ Y^{\diamond\diamond\diamond} \subseteq Y^\diamond.\end{aligned}$$

By Lemma 3.9, this is equivalent to that for all $x_1, x_2 \in U$, $R(x_1) \cap R(x_2) \neq \emptyset$ implies $R(x_1) = R(x_2)$. $\square$

Since object definable sets and attribute definable sets are defined dually, all the properties of object definable sets investigated above have the corresponding results for attribute definable sets. For $\mathrm{Def}_U(\mathbb{K})$ and $\mathrm{Def}_U(\mathbb{K})$ we also have the following proposition.

Theorem 3.11 *If context $\mathbb{K} = (U, V, R)$ has neither empty columns nor empty rows, then $(Def_U(\mathbb{K}), \subseteq)$ and $(Def_V(\mathbb{K}), \subseteq)$ are isomorphic.*

Proof. First, for each $X \in \mathrm{Def}_U(\mathbb{K})$, we have $X = X^{\diamond\diamond}$ and $X^\diamond = X^{\diamond\diamond\diamond}$ and thus $X^\diamond \in \mathrm{Def}_V(\mathbb{K})$. By the same argument, for all $Y \in \mathrm{Def}_V(\mathbb{K})$, we have $Y^\diamond \in \mathrm{Def}_U(\mathbb{K})$.

Define maps as follows:

$$f : \mathrm{Def}_U(\mathbb{K}) \to \mathrm{Def}_V(\mathbb{K}), \quad X \mapsto X^\diamond \in \mathrm{Def}_V(\mathbb{K}),$$

$$g : \mathrm{Def}_V(\mathbb{K}) \to \mathrm{Def}_U(\mathbb{K}), \quad Y \mapsto Y^\diamond \in \mathrm{Def}_U(\mathbb{K}).$$

It is easy to check that f and g are both order-preserving and $g \circ f(X) = X^{\diamond\diamond} = X$ for $X \in \mathrm{Def}_U(\mathbb{K})$ and $f \circ g(Y) = Y^{\diamond\diamond} = Y$ for $Y \in \mathrm{Def}_V(\mathbb{K})$. So, $(\mathrm{Def}_U(\mathbb{K}), \subseteq) \cong (\mathrm{Def}_V(\mathbb{K}), \subseteq)$. □

4 Algebraicity of Rough Concept Lattices

In this section, we give some sufficient conditions for algebraicity of rough concept lattices.

Proposition 4.1 *Let $\mathbb{K} = (U, V, R)$ be a context. If for all $x_1, x_2 \in U$, $R(x_1) \cap R(x_2) \neq \emptyset$ implies $R(x_1) = R(x_2)$, then RO-$L_U(\mathbb{K})$ is an algebraic lattice.*

Proof. First we show that for any $F \subseteq_{fin} R(U)$, $F^\diamond$ is a compact element of RO-$L_U(\mathbb{K})$. Suppose $\mathcal{D}$ is directed in RO-$L_U(\mathbb{K})$ and $F^\diamond \subseteq \bigvee \mathcal{D} = \bigcup \mathcal{D}$. Since $F \subseteq R(U)$, we have $F \subseteq R(R^{-1}(F)) = F^{\diamond\diamond} \subseteq (\bigcup \mathcal{D})^\diamond = \bigcup\{D^\diamond |\ D \in \mathcal{D}\}$. It is easy to see that $\{D^\diamond |\ D \in \mathcal{D}\}$ is directed. So there exists $D \in \mathcal{D}$ such that $F \subseteq D^\diamond$ and $F^\diamond \subseteq D^{\diamond\diamond}$. Since $D \in$ RO-$L_U(\mathbb{K})$, there is $B_0 \subseteq V$ such that $D = B_0^\diamond$ and $F^\diamond \subseteq B_0^{\diamond\diamond\diamond}$. By Lemma 3.9, $B_0^{\diamond\diamond\diamond} \subseteq B_0^\diamond = D$ and $F^\diamond \subseteq D$. So, $F^\diamond$ is a compact element. Now we show that RO-$L_U(\mathbb{K})$ is an algebraic lattice. For all $B \subseteq V$, we have $B^\diamond = R^{-1}(B) = R^{-1}(B \cap R(U)) = (B \cap R(U))^\diamond = \bigcup\{F^\diamond |\ F \subseteq_{fin} B \cap R(U)\} = \bigvee\{F^\diamond |\ F \subseteq_{fin} B \cap R(U)\}$. Therefore RO-$L_U(\mathbb{K})$ is an algebraic lattice. □

A relation $R \subseteq U \times V$ is said to be single-rooted if for any $y \in R(U)$, there is a unique $x \in U$ such that xRy. It is easy to check that if R is a single-rooted relation or a function, then the condition in Proposition 4.1 is automatically satisfied, so we immediately have

Corollary 4.2 *Let $\mathbb{K} = (U, V, R)$ be a context. If R is single-rooted relation or a function, then the rough concept lattice of $\mathbb{K}$ is an algebraic lattice.*

A function $f : S \to T$ between posets is said to be Scott continuous if it preserves directed unions.

Proposition 4.3 *Let $\mathbb{K} = (U, V, R)$ be a context. If ${}^{\diamond\Box} : \mathcal{P}(V) \to \mathcal{P}(V)$ is Scott continuous, then the rough concept lattice of $\mathbb{K}$ is algebraic.*

Proof. Since $({}^{\Box}, {}^{\diamond})$ is a Galois connection, ${}^{\diamond\Box} : \mathcal{P}(V) \to \mathcal{P}(V)$ is a closure operator. Then RO-$L_V(\mathbb{K}) = \{B^{\diamond\Box} | \ B \subseteq V\}$ is the image of closure operator ${}^{\diamond\Box} : \mathcal{P}(V) \to \mathcal{P}(V)$ which preserves directed unions. By Lemma 2.4, RO-$L_V(\mathbb{K})$ is an algebraic lattice. $\Box$

Noticing that the family of compact elements of $\mathcal{P}(V)$ is $K(\mathcal{P}(V)) = \{F | \ F \subseteq_{fin} V\}$. It follows from Proposition 4.3 and Proposition I-4.13 in [5] that if ${}^{\diamond\Box} : \mathcal{P}(V) \to \mathcal{P}(V)$ is Scott continuous, then $K(\text{RO-}L_V(\mathbb{K})) = \{F^{\diamond\Box} | \ F \subseteq_{fin} V\}$.

The following counterexample shows that the continuity of ${}^{\diamond\Box}$ is not necessary for the rough concept lattice of a context to be algebraic.

Example 4.4 Let L be the complete lattice shown in Fig.4.

$L = \mathbb{N} \cup \{\infty\}$:

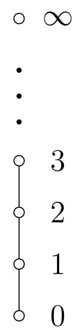

Fig.4.

It is clear that L is an algebraic lattice. Consider formal context $(\sigma(L), L, R)$, where $\sigma(L)$ is the Scott topology on L and $R = \in^{-1}$. For all $B \subseteq L$, we have

$$
\begin{aligned}
B^{\diamond} &= \{U \in \sigma(L) | \ R(U) \cap B \neq \emptyset\} = \{U \in \sigma(L) | \ U \cap B \neq \emptyset\}, \\
B^{\diamond\Box} &= \{x \in L | \ R^{-1}(x) \subseteq B^{\diamond}\} \\
&= \{x \in L | \ \forall U \in \sigma(L), \text{if } x \in U \text{ then } U \cap B \neq \emptyset\} = B^{-},
\end{aligned}
$$

where B^{-} is the Scott closure of B. Then $B \in$ RO-$L_V(\mathbb{K})$ if and only if $B = B^{\diamond\Box}$ if and only if $B = B^{-}$ if and only if B is a Scott closed set of L. Denote the family of Scott closed set of L as $\mathcal{F}$. Then RO-$L_V(\mathbb{K}) = (\mathcal{F}, \subseteq) \cong L$ is an algebraic lattice.

But here the operator ${}^{\diamond\Box} : \mathcal{P}L \to \mathcal{P}L$ is not continuous. In fact, let $\mathcal{D} = \{F \subseteq \mathbb{N} | \ F \text{ is finite}\} \subseteq \mathcal{P}(L)$. Then $\mathcal{D}$ is directed and $\bigcup \mathcal{D} = \mathbb{N}$. It is easy to see that $(\bigcup \mathcal{D})^{\diamond\Box} = (\mathbb{N})^{\diamond\Box} = \mathbb{N}^{-} = L$, but $\bigcup_{F\in\mathcal{D}} F^{\diamond\Box} = \bigcup_{F\in\mathcal{D}} F^{-} = \bigcup_{F\in\mathcal{D}} \downarrow F = \mathbb{N} \neq L$. Thus $^{\diamond\Box}$ does not preserve directed unions and is not continuous.

The following two propositions give some conditions under which $^{\diamond\Box}$ is Scott continuous.

Proposition 4.5 *For a context $\mathbb{K} = (U, V, R)$, if for each $y \in V$, $R^{-1}(y)$ is finite, then $^{\diamond\Box} : \mathcal{P}(V) \to \mathcal{P}(V)$ is Scott continuous.*

Proof. Suppose $\mathcal{D} \subseteq \mathcal{P}(V)$ is directed. Since $^{\diamond\Box}$ is order-preserving, we have

$$\bigcup_{D\in\mathcal{D}} D^{\diamond\Box} \subseteq (\bigcup \mathcal{D})^{\diamond\Box}.$$

Now we show the converse containment. Suppose $y \in (\bigcup \mathcal{D})^{\diamond\Box}$. Then $R^{-1}(y) \subseteq (\bigcup \mathcal{D})^{\diamond} = \bigcup_{D\in\mathcal{D}} D^{\diamond}$. Since $\{D^{\diamond} |\ D \in \mathcal{D}\}$ is directed and $R^{-1}(y)$ is finite, there exists $D \in \mathcal{D}$ such that $R^{-1}(y) \subseteq D^{\diamond}$. Therefore $y \in D^{\diamond\Box} \subseteq \bigcup_{D\in\mathcal{D}} D^{\diamond\Box}$ and thus we have

$$(\bigcup \mathcal{D})^{\diamond\Box} \subseteq \bigcup_{D\in\mathcal{D}} D^{\diamond\Box}.$$

So $(\bigcup \mathcal{D})^{\diamond\Box} = \bigcup_{D\in\mathcal{D}} D^{\diamond\Box}$ and $^{\diamond\Box}$ is Scott continuous. $\Box$

Corollary 4.6 *For a context $\mathbb{K} = (U, V, R)$, if for each $y \in V$, $R^{-1}(y)$ is finite, then the rough concept lattice of $\mathbb{K}$ is an algebraic lattice.*

Theorem 4.7 *For a context $\mathbb{K} = (U, V, R)$, $^{\diamond\Box} : \mathcal{P}(V) \to \mathcal{P}(V)$ is Scott continuous if and only if for each $y \in V$ and each $B \subseteq V$, $y \in B^{\diamond\Box}$ implies $y \in F^{\diamond\Box}$ for some $F \subseteq_{fin} B$.*

Proof. $\Rightarrow$: Suppose $y \in V$ and $B \subseteq V$. Let $\mathcal{D} = \{F |\ F \subseteq_{fin} B\}$. Then $\mathcal{D}$ is directed and $B = \bigcup \mathcal{D}$. Since $^{\diamond\Box}$ is continuous, we have

$$B^{\diamond\Box} = (\bigcup \mathcal{D})^{\diamond\Box} = \bigcup\{F^{\diamond\Box} |\ F \subseteq_{fin} B\}.$$

Then $y \in B^{\diamond\Box}$ implies $y \in F^{\diamond\Box}$ for some $F \subseteq_{fin} B$.

$\Leftarrow$: Suppose $\mathcal{D} \subseteq \mathcal{P}(V)$ is directed. Since $^{\diamond\Box}$ is order-preserving, we have $\bigcup_{D\in\mathcal{D}} D^{\diamond\Box} \subseteq (\bigcup \mathcal{D})^{\diamond\Box}$. Conversely, let $y \in (\bigcup \mathcal{D})^{\diamond\Box}$. Then by the assumption, there is $F \subseteq_{fin} \bigcup \mathcal{D}$ with $y \in F^{\diamond\Box}$. Since $\mathcal{D}$ is directed and F is finite, there is $D \in \mathcal{D}$ such $F \subseteq D$ and thus $y \in F^{\diamond\Box} \subseteq D^{\diamond\Box} \subseteq \bigcup_{D\in\mathcal{D}} D^{\diamond\Box}$ and then $(\bigcup \mathcal{D})^{\diamond\Box} \subseteq \bigcup_{D\in\mathcal{D}} D^{\diamond\Box}$. Therefore $(\bigcup \mathcal{D})^{\diamond\Box} = \bigcup_{D\in\mathcal{D}} D^{\diamond\Box}$ and $^{\diamond\Box} : \mathcal{P}(V) \to \mathcal{P}(V)$ is continuous. $\Box$

By Proposition 4.3, Theorem 4.7 and the definitions of $\diamond$ and $\Box$, we have

Corollary 4.8 *Let $\mathbb{K} = (U, V, R)$ be a context. If for each $y \in V$ and $B \subseteq V$, $R^{-1}(y) \subseteq R^{-1}(B)$ implies $R^{-1}(y) \subseteq R^{-1}(F)$ for some $F \subseteq_{fin} B$, then the rough concept lattice of $\mathbb{K}$ is an algebraic lattice.*

Theorem 4.9 *Let $\mathbb{K} = (U, V, R)$ be a context. If for each $x \in U$ and $A \subseteq U$, $x \in A^{\Box\Diamond}$ implies $x \in F^{\Box\Diamond}$ for some $F \subseteq_{fin} A$, then the rough concept lattice of $\mathbb{K}$ is algebraic.*

Proof. Consider RO-$L_U(\mathbb{K}) = \{A^{\Box\Diamond} | \ A \subseteq U\}$. First we show that for each $F \subseteq_{fin} V$, $F^{\Box\Diamond}$ is a compact element of RO-$L_U(\mathbb{K})$. Let $\mathcal{D}$ be a directed subset of RO-$L_U(\mathbb{K})$ and $F^{\Box\Diamond} \subseteq \bigvee \mathcal{D}$. Since $^{\Box\Diamond}$ is a kernel operator, we have $F^{\Box\Diamond} \subseteq F$ and thus $F^{\Box\Diamond}$ is finite. Then by the directedness of $\mathcal{D}$ there is $D \in \mathcal{D}$ such that $F^{\Box\Diamond} \subseteq D$. Therefore $F^{\Box\Diamond}$ is compact. For any $A \subseteq U$, we have $A = \bigcup\{F | \ F \subseteq_{fin} A\}$. Since $^{\Box\Diamond}$ is order-preserving, $A^{\Box\Diamond} \supseteq \bigcup\{F^{\Box\Diamond} | \ F \subseteq_{fin} A\}$. On the other hand, by the assumption, for each $x \in A^{\Box\Diamond}$, there is $F \subseteq_{fin} A$ such that $x \in F^{\Box\Diamond}$. Thus $A^{\Box\Diamond} \subseteq \bigcup\{F^{\Box\Diamond} | \ F \subseteq_{fin} A\}$. There $A^{\Box\Diamond} = \bigcup\{F^{\Box\Diamond} | \ F \subseteq_{fin} A\} = \bigvee\{F^{\Box\Diamond} | \ F \subseteq_{fin} A\}$. So RO-$L_U(\mathbb{K})$ is algebraic. $\Box$

Remark 4.10 *It is easy to show by the similar argument of the proof of Theorem 4.7 that the condition in Theorem 4.9 is equivalent to that $^{\Box\Diamond} : \mathcal{P}(U) \to \mathcal{P}(U)$ is continuous. However, we cannot get the algebraicity of RO-$L_U(\mathbb{K})$ directly from the continuity of $^{\Box\Diamond}$ since here $^{\Box\Diamond}$ is merely a kernel operator instead of a closure operator.*

5 Completely Distributivity of Rough Concept Lattices

In this section we discuss the complete distributivity of rough concept lattices and give some useful examples.

Definition 5.1 *[12]* A relation $R \subseteq U \times V$ ia called regular, if there exists a relation $\sigma \subseteq V \times U$ such that $R = R \circ \sigma \circ R$.

By the definition of regular relations, we have immediately the following lemma.

Lemma 5.2 *A relation $R \subseteq U \times V$ is regular iff its inverse relation R^{-1} is regular.*

Example 5.3 *[12]* (1) For any set X, relation $\in \subseteq X \times \mathcal{P}(X)$ is regular.

(2) Every function $f : U \to V$ is a regular relation.

Lemma 5.4 [12] Let $R \subseteq U \times V$ be a relation and $\Phi_R(U) = \{R(A) | \ A \subseteq U\}$. Then the following three conditions are equivalent:

(1) R is regular;

(2) $\forall (x,y) \in U \times V,\ xRy \ \Rightarrow\ \exists (u,v) \in U \times V$ such that

(a) $xRv,\ uRy$, and

(b) $\forall (s,t) \in U \times V, sRv$ and $uRt \ \Rightarrow\ sRt$;

(3) $(\Phi_R(U), \subseteq)$ is completely distributive.

Applying Lemmas 5.2 and 5.4 to rough concept lattices, we have the following significant theorem.

Theorem 5.5 *Let $\mathbb{K} = (U, V, R)$ be a context. Then the rough concept lattice of $\mathbb{K}$ is completely distributive if and only if relation R is regular.*

Proof. By Lemmas 5.2 and 5.4, RO-$L_U(\mathbb{K}) = \{B^\diamond |\ B \subseteq V\} = \{R^{-1}(B) |\ B \subseteq V\}$ is completely distributive if and only if R^{-1} is regular if and only if R is regular. □

By Example 5.3 and Theorem 5.5, we have the following corollary.

Corollary 5.6 (1) *For any set X, the rough concept lattice of $(X, \mathcal{P}(X), \in)$ is a completely distributive lattice.*

(2) *For a context $\mathbb{K} = (U, V, R)$, if R is a function from U to V, then the rough concept lattice RO-L$(\mathbb{K})$ is completely distributive.*

By Theorem 3.10 and Theorem 5.5, we have immediately the following

Corollary 5.7 *If context $\mathbb{K} = (U, V, R)$ has no empty columns and for all $x_1, x_2 \in U,\ R(x_1) \cap R(x_2) \neq \emptyset$ implies $R(x_1) = R(x_2)$. Then the relation R is regular.*

Proof. By Corollary 3.5 and Theorem 3.10, we have that RO-$L_U(\mathbb{K}) = Def_U(\mathbb{K})$ is a completely distributive lattice. Then by Theorem 5.5, R is regular. □

Now we give some examples of completely distributive rough concept lattices.

Example 5.8 Let $(X, \mathcal{T})$ be a topological space and $\mathbb{K} = (X, \mathcal{T}, R)$ a context, where $R = \notin$. For any $x \in X$, $R(x) = \{U \in \mathcal{T} |\ x \notin U\} = \{U \in \mathcal{T} |\ x \in U^c\}$ and for any $U \in \mathcal{T}$, $R^{-1}(U) = \{x \in X |\ x \notin U\} = U^c$. Then for any $A \subseteq X$ and $\mathcal{U} \subseteq \mathcal{T}$, we have

$$A^\Box = \{U \in \mathcal{T} |\ R^{-1}(U) \subseteq A\} = \{U \in \mathcal{T} |\ U^c \subseteq A\},$$

$$\mathcal{U}^\diamond = \{x \in X |\ R(x) \cap \mathcal{U} \neq \emptyset\} = \{x \in X |\ \exists U \in \mathcal{U}, x \in U^c\} = \bigcup_{U \in \mathcal{U}} U^c.$$

So, RO-$L_U(\mathbb{K}) = \{\mathcal{U}^\diamond |\ \mathcal{U} \subseteq \mathcal{T}\} = \{\bigcup_{U \in \mathcal{U}} U^c |\ \mathcal{U} \subseteq \mathcal{T}\} = \{\bigcup \mathcal{A} |\ \mathcal{A} \subseteq \mathcal{F}\}$, where $\mathcal{F}$ is the family of all closed sets of $(X, \mathcal{T})$. It is easy to see that RO-$L_U(\mathbb{K})$ is closed under arbitrary intersections and unions in $\mathcal{P}(X)$. So, RO-$L_U(\mathbb{K})$ is a complete ring of sets and a completely distributive algebraic lattice. It follows from Theorem 5.5 that relation $\notin \subseteq X \times \mathcal{T}$ is regular.

Example 5.9 Let $(L, \leq)$ be a poset and $\mathbb{K}$ be context (L, L, R) where $R = \leq$. For $x, y \in L$, $R(x) = \{y \in L | \ x \leq y\} = \uparrow x$, $R^{-1}(y) = \{x \in L | \ x \leq y\} = \downarrow y$. For any $A, B \subseteq L$, we have

$$A^{\square} = \{y \in L | \ R^{-1}(y) \subseteq A\} = \{y \in L | \ \downarrow y \subseteq A\},$$
$$B^{\diamond} = \{x \in L | \ R(x) \cap B \neq \emptyset\} = \{x \in L | \ \exists y \in B, x \leq y\} = \downarrow B.$$

So, RO-$L_U(\mathbb{K}) = \{B^{\diamond} | \ B \subseteq L\} = \{\downarrow B | \ B \subseteq L\} = \{B \subseteq L | \ B$ is a lower set of $L\}$. Thus, RO-$L_U(\mathbb{K})$ is a complete ring of sets and hence a completely distributive algebraic lattice. By Theorem 5.5, every partial order is a regular relation.

6 Concluding Remarks

Rough set theory and formal concept analysis capture different aspects of data. Combining the two theories, one gets the rough concept lattice of a given context. In this paper, we discussed properties of rough concept lattices from the domain theory point of view. We mainly investigated algebraicity and completely distributivity of rough concept lattices. We also introduced the notions of definable sets for a context and discussed algebraic properties of them. The work makes new links between FCA, RST and domain theory which may improve our understandings of the three theories and provides more research topics.

The close links of the three theories may also provide us topics for further research in information systems and formal topology. By combining them together, we may establish relationships between RST, FCA, Domain theory, formal topology and information systems.

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