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On SI_2-continuous Spaces

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Abstract

In this paper, we construct a new way below relation from any given T_0 space by making use of the cut operator, and introduce the concepts of SI_2-continuous spaces and SI_2-quasicontinuous spaces. The main results are: (1) a space (X, τ) is SI_2-continuous iff the set X equipped with the SI_2-topology τ_{SI_2} is a C-space; (2) a space (X, τ) is SI_2-quasicontinuous iff (X, τ_{SI_2}) is a locally hypercompact space; (3) a space is SI_2-continuous iff it is a meet SI_2-continuous and SI_2-quasicontinuous space.

Keywords: SI_2-continuous space, SI_2-quasicontinuous space, meet SI_2-continuous space, irreducible set

1 Introduction

The theory of continuous domains, due to its strong background in computer science, general topology and logic, has been extensively studied by people from various areas (see [1,7,8]). An important direction in the study of continuous domains is to extend the theory of continuous domains to that of posets as much as possible, and a lot of work has been done in this area (see [10,11,12,13,14,16]), but it is still rather restrictive, taking into consideration only the case of existing a join. In

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[4,6], Erné introduced s_2-continuous posets and precontinuous posets respectively by making use of the cut operator instead of joins. The notions of s_2-continuity and precontinuity generalize the important characterizations of continuity from dcpos to arbitrary posets and have the advantage that not even the existence of directed joins is required. Recently, based on Erné's work, many interesting continuity of posets were investigated (see [17,21,22,23,24,25]).

As we all known, the non-empty irreducible subsets of a poset with respect to the Alexandroff topology are exactly the directed sets, and the Scott topology $\sigma(P)$ on a poset P is defined by directed suprema, i.e., $U \in \sigma(P)$ if and only if $U = \uparrow U$ and for any directed set D, $\vee D \in U$ implies $D \cap U \neq \emptyset$ whenever $\vee D$ exists. In [20], Zhao and Ho gave a new method of deriving a new topology τ_{SI} out of a given T_0 space which makes use of the replacement of directed sets with irreducible sets, and proved that (X, τ) is SI-continuous if and only if X equipped with irreducible-derived topology τ_{SI} is a C-space. Furthermore, a deep study concerning the replacement of directed sets with irreducible sets can be found in [2,3].

Motivated by the works of Zhao and Ho, we continue to develop domain theory in T_0 spaces. In this paper, we define a new way below relation in T_0 spaces by making use of the cut operator and introduce the concept of SI$_2$-continuous spaces. It is proved that a space (X, τ) is SI$_2$-continuous if and only if the set X equipped with the SI$_2$-topology τ_{SI_2} is a C-space. As a common generalization of both SI$_2$-continuous spaces and quasicontinuous domains, we introduce the concept of SI$_2$-quasicontinuous spaces and prove that a space (X, τ) is SI$_2$-quasicontinuous if and only if (X, τ_{SI_2}) is a locally hypercompact space. Finally, we introduce the concept of meet SI$_2$-continuous spaces and prove that a space is SI$_2$-continuous if and only if it is a meet SI$_2$-continuous and SI$_2$-quasicontinuous space.

We would like to thank the referee for informing us that some properties of SI$_2$-topology are also presented in [18]. It is also proved that a space (X, τ) is SI$_2$-continuous if and only if (X, τ_{SI_2}) is a C-space independently in [18].

2 Preliminaries

In this section, we recall some basic definitions and notations needed in this paper; more detail can be found in [8,20]. For a poset P, $x \in P$ and $A \subseteq P$, let $\downarrow x = \{y \in P : y \leq x\}$, $\downarrow A = \bigcup\{\downarrow x : x \in A\}$; $\uparrow x$ and $\uparrow A$ are defined dually. A subset A is called an *upper set* if $A = \uparrow A$ and a *lower set* is defined dually. $A^{\uparrow}$ and $A^{\downarrow}$ denote the sets of all upper and lower bounds of A, respectively. Let $A^{\delta} = (A^{\uparrow})^{\downarrow}$. We put $P^{(<\omega)} = \{F \subseteq P : F \text{ is finite}\}$ and $\mathrm{Fin}P = \{\uparrow F : F \in P^{(<\omega)}\}$.

For a poset P, the topology generated by the collection of sets $P \setminus \downarrow x$(as subbasic open subsets) is called the *upper topology* and denoted by $\upsilon(P)$; the *lower topology* $\omega(P)$ on P is defined dually. The *Alexandroff topology* $A(P)$ on a poset P is the topology consisting of all its upper subsets. A subset U of a poset P is called *Scott open* if $U = \uparrow U$ and $D \cap U \neq \emptyset$ for all directed sets $D \subseteq P$ with $\vee D \in U$ whenever $\vee D$ exists. The topology formed by all the Scott open sets of P is called the *Scott topology*, written as $\sigma(P)$. For a poset P and $x, y \in P$, we say that x is way-below

y and write $x \ll y$ if for every directed set $D \subseteq P$ with $\vee D$ exists, $x \leq \vee D$ implies $x \leq d$ for some $d \in D$. A poset P is called *continuous* if $\{y \in P : x \ll y\}$ is directed and $x = \vee\{y \in P : x \ll y\}$ for each $x \in P$. We order the collection of nonempty subset of a poset P by $G \leq H$ if $H \subseteq \uparrow G$. We say that a family of sets is directed if given F_1, F_2 in the family, there exists F in the family such that $F_1, F_2 \leq F$, i.e., $F \subseteq \uparrow F_1 \cap \uparrow F_2$.

Let (X, τ) be a topological space. A non-empty subset F of (X, τ) is *irreducible* if whenever $F \subseteq A \cup B$ for closed sets A and B, then $F \subseteq A$ or $F \subseteq B$. The set of all irreducible subsets of (X, τ) is denoted by $Irr_\tau(X)$. A space (X, τ) is called a *C-space* if for any $U \in \tau$, $x \in U$, there exists $u \in U$ such that $x \in int_\tau \uparrow u$. A space (X, τ) is called *locally hypercompact* if for any x and any open set U containing x, there exists a finite set E such that $x \in int_\tau \uparrow E \subseteq \uparrow E \subseteq U$. Obviously, a C-space is locally hypercompact.

Give a topological space (X, τ), denote the interior of a subset $A \subseteq X$ by $int_\tau A$ and the closure of A by $cl_\tau A$. For a T_0 topological space (X, τ), the *specialization order* $\leq_\tau$ on (X, τ) is defined by $x \leq_\tau y$ if and only if $x \in cl_\tau(y)$. Unless otherwise stated, throughout the paper, whenever an order-theoretic concepts is mentioned, it is to be interpreted with respect to the specialization order on (X, τ).

Definition 2.1 ([7,15]) Let P be a poset and $x, y \in P$.

(1) Define a relation $\prec$ on P by $x \prec y$ iff $y \in int_{\upsilon(P)} \uparrow x$.

(2) P is called *hypercontinuous* if for all $x \in P$, $x = \bigvee\{u \in P : u \prec x\}$ and $\{u \in P : u \prec x\}$ is directed.

(3) P is called *quasi-hypercontinuous* if for all $x \in P$ and $U \in \upsilon(P)$ with $x \in U$, there exists $H \in P^{(<\omega)}$ such that $x \in int_{\upsilon(P)} \uparrow H \subseteq \uparrow H \subseteq U$.

it is proven in [5,15] that a poset P is hypercontinuous if and only if for all $x \in P$ and $U \in \upsilon(P)$ with $x \in U$, there exists $y \in P$ such that $x \in int_{\upsilon(P)} \uparrow y \subseteq \uparrow y \subseteq U$, that is, a poset P is hypercontinuous iff P equipped with the upper topology $\upsilon(P)$) is a C-space.

Definition 2.2 ([4]) Let P be a poset.

(1) For any $x, y \in P$, define $\ll_2$ on P by $x \ll_2 y$ if for all directed sets $D \subseteq P$ with $y \in D^\delta$, there exists $d \in D$ such that $x \leq d$. The set $\{y \in P : y \ll_2 x\}$ will be denoted by $\Downarrow x$ and $\{y \in P : x \ll y\}$ denoted by $\Uparrow x$.

(2) P is called s_2-continuous if for all $x \in P$, $x \in (\Downarrow x)^\delta$ and $\Downarrow x$ is directed.

Definition 2.3 ([4,5]) Let P be a poset. A subset $U \subseteq P$ is called weak Scott open if it satisfies

(1) $U = \uparrow U$;

(2) for all directed sets $D \subseteq P$, $D^\delta \cap U \neq \emptyset$ implies $D \cap U \neq \emptyset$.

The collection of all weak Scott open subsets of P forms a topology. This topology will be called the weak Scott topology of P and will be denoted by $\sigma_2(P)$. Obviously, $\sigma_2(P) \subseteq \sigma(P)$ and $\sigma_2(P) = \sigma(P)$ if P is a dcpo.

Theorem 2.4 ([4]) *Let P be a poset. Then the following statements are equivalent.*

(1) *P is s_2-continuous;*

(2) *$(P, \sigma_2(P))$ is a C-space.*

Definition 2.5 ([21]) Let P be a poset and $G, H \subseteq P$.

(1) Define $G \ll_2 H$ if for all directed sets $D \subseteq P$, $\uparrow H \cap D^\delta \neq \emptyset$ implies $\uparrow G \cap D \neq \emptyset$. We write $G \ll_2 x$ for $G \ll_2 \{x\}$ and $y \ll_2 H$ for $\{y\} \ll_2 H$.

(2) P is called *s_2-quasicontinuous* if for each $x \in P$, $w(x) = \{F \subseteq P : F \in P^{(<\omega)}$ and $F \ll_2 x\}$ is directed and $\uparrow x = \bigcap\{\uparrow F : F \in w(x)\ \}$.

Theorem 2.6 ([21]) *Let P be a poset. Then the following statements are equivalent.*

(1) *P is s_2-quasicontinuous;*

(2) *$(P, \sigma_2(P))$ is locally hypercompact.*

Definition 2.7 ([20]) Let (X, τ) be a T_0 space. A subset U of X is called *SI-open* if the following conditions are satisfied:

(1) $U \in \tau$;

(2) For any $F \in Irr_\tau(X)$, $\vee F \in U$ implies $F \cap U \neq \emptyset$ whenever $\vee F$ exists.

The collection of all SI-open sets of (X, τ) is denoted by τ_{SI}. Obviously, $\tau_{SI} \subseteq \tau$.

Definition 2.8 ([20]) Let (X, τ) be a T_0 space.

(1) Define $x \ll_{SI} y$ if for any irreducible set, $y \leq \vee F$ implies $x \in \downarrow F$ whenever $\vee F$ exists. Denote the set $\{x \in X : x \ll_{SI} a\}$ by $\Downarrow_{SI} a$ and the set $\{x \in X : a \ll_{SI} x\}$ by $\Uparrow_{SI} a$.

(2) (X, τ) is called *SI-continuous* if for any $x \in X$, the following conditions hold:
   (i) $\Uparrow_{SI} x$ is open in (X, τ).
   (ii) $\Downarrow_{SI} x$ is directed and $x = \vee \Downarrow_{SI} x$.

It is proven in [2] that the condition "directed" in Definition 2.8(2) can be replaced by "irreducible" condition and one still has an equivalent continuous.

Theorem 2.9 ([20]) *Let (X, τ) be a T_0 space. Then the following statements are equivalent.*

(1) *(X, τ_{SI}) is a C-space;*

(2) *(X, τ) is SI-continuous.*

Lemma 2.10 ([9]) *(Order Rudin Lemma) Let P be a preorder and $\mathcal{F}$ a directed family of finitary upper sets of P. Any lower set L that meets all members of $\mathcal{F}$ has a directed lower subset D that still meets all members of $\mathcal{F}$.*

3 $\mathrm{SI_2}$-continuous spaces

In this section, we introduce a new concept of $\mathrm{SI_2}$-continuous spaces and prove that a space (X,τ) is $\mathrm{SI_2}$-continuous if and only if the set X equipped with the $\mathrm{SI_2}$-topology τ_{SI_2} is a C-space.

Definition 3.1 Let (X,τ) be a T_0 space and $x,y \in X$.

(1) Define $x \ll_{SI_2} y$ if for any $F \in Irr_\tau(X)$, $y \in F^\delta$ implies $x \in \downarrow F$. Denote the set $\{x \in X : x \ll_{SI_2} a\}$ by $\Downarrow_{SI_2} a$, and the set $\{x \in X : a \ll_{SI_2} x\}$ by $\Uparrow_{SI_2} a$.

(2) (X,τ) is called *SI_2-continuous* if for each $x \in X$, the following conditions hold:
  (i) $\Uparrow_{SI_2} x$ is open in (X,τ).
  (ii) $\Downarrow_{SI_2} x$ is directed and $x = \vee \Downarrow_{SI_2} x$.

In fact, we have $x = \vee \Downarrow_{SI_2} x$ iff $x \in (\Downarrow_{SI_2} x)^\delta$ since $\Downarrow_{SI_2} x \subseteq \downarrow x$.

Proposition 3.2 *Let (X,τ) be a T_0 space and $x,y,u,v \in X$. Then*

(1) *If $x \ll_{SI_2} y$, then $x \le y$.*

(2) *If $u \le x \ll_{SI_2} y \le v$, then $u \ll_{SI_2} v$.*

(3) *If a smallest element $\perp$ exists, then $\perp \ll_{SI_2} x$.*

(4) *$x \ll_{SI_2} y$ implies $x \ll_{SI} y$.*

(5) *If (X,τ) is an SI_2-continuous space, then $x \ll_{SI} y \Leftrightarrow x \ll_{SI_2} y$.*

(6) *If (X,τ) is an SI_2-continuous space, then (X,τ) is SI-continuous.*

Proof. The conditions (1)-(4) are straightforward.

(5) Suppose $x \ll_{SI} y$. Since (X,τ) is an $\mathrm{SI_2}$-continuous, $y = \vee \Downarrow_{SI_2} y$ and $\Downarrow_{SI_2} y$ is directed. Since every directed set is irreducible, by the definition of $\ll_{SI}$, we have that $x \in \downarrow (\Downarrow_{SI_2} y) = \Downarrow_{SI_2} y$. Thus $x \ll_{SI_2} y$.

(6) It is straightforward from (5) □

Definition 3.3 Let (X,τ) be a T_0 space. A subset U of X is called *SI_2-open* if the following conditions are satisfied:

(1) $U \in \tau$;

(2) For any $F \in Irr_\tau(X)$, $F^\delta \cap U \neq \emptyset$ implies $F \cap U \neq \emptyset$.

The collection of all $\mathrm{SI_2}$-open subsets of (X,τ) forms a topology. This topology will be called $\mathrm{SI_2}$-topology and denoted by τ_{SI_2}. The complement of an $\mathrm{SI_2}$-open set is called *SI_2-closed*. Recall that an upper set F in a poset P is a filter if every finite subset of F has a lower bounded in F. Let $\mathrm{SOFilt}_\tau(X)$ denote the collection of all $\mathrm{SI_2}$-open filters in (X,τ).

Proposition 3.4 *Let (X,τ) be a T_0 space. Then the following conditions hold.*

(1) *For any $x \in X$, $cl_\tau\{x\} = cl_{\tau_{SI_2}}\{x\}$.*

(2) *The specialization orders of spaces (X,τ) and (X,τ_{SI_2}) coincide.*

(3) *A closed set C in (X, τ) is SI$_2$-closed if and only if for any $F \in Irr_\tau(X)$, $F \subseteq C$ implies $F^\delta \subseteq C$.*

(4) *An open set U in (X, τ) is SI$_2$-open if and only if for any $F \in Irr_\tau(X)$, $F^\delta \cap U \neq \emptyset$ implies $F \cap U \neq \emptyset$.*

(5) $\tau_{SI_2} \subseteq \tau_{SI} \subseteq \tau$.

(6) *A set U is clopen in (X, τ) if and only if it is clopen in (X, τ_{SI_2}).*

(7) *U is co-prime in τ_{SI_2} if and only if $U \in SOFilt_\tau(X)$.*

(8) *If $y \in int_{\tau_{SI_2}} \uparrow x$, then $x \ll_{SI_2} y$.*

Proof. The conditions (1)-(5) are easy to obtained.

(6) Obviously, if U is clopen in (X, τ_{SI_2}), then U is clopen in (X, τ). Without loss of generality, assume that U is a non-trivial clopen set in (X, τ). Let $F \in Irr_\tau(X)$ with $U \cap F^\delta \neq \emptyset$. If $U \cap F = \emptyset$, then $F \subseteq X \backslash U$. Since $X \backslash U$ is clopen, $F^\delta \subseteq X \backslash U$, a contradiction. Hence $U \cap F \neq \emptyset$. So U is SI$_2$-open. Similarly, we can deduce that $X \backslash U$ is SI$_2$-open.

(7) Let U is co-prime in τ_{SI_2}. It is suffices to show that U is a filter. Suppose $x, y \in U$. Then $X \backslash \downarrow x$ and $X \backslash \downarrow y$ are SI$_2$-open and $U \nsubseteq (X \backslash \downarrow x) \bigcup (X \backslash \downarrow y) = X \backslash (\downarrow x \bigcap \downarrow y)$ since U is co-prime in τ_{SI_2}. So there exists $z \in U$ such that $z \notin X \backslash (\downarrow x \bigcap \downarrow y)$, that is, $z \leq x, y$. Thus U is a filter. Conversely, suppose that U is not a co-prime in τ_{SI_2}, then there exist $V, W \in \tau_{SI_2}$ such that $U \subseteq V \bigcup W$ with $U \nsubseteq V$ and $U \nsubseteq W$. Choose $x \in U \backslash V$ and $y \in U \backslash W$. Since U is a filter, there is a $z \in U$ such that $z \leq x$ and $z \leq y$. Then we have $z \notin V \bigcup W$, a contradiction. Hence, (7) holds.

(8) Let $y \in int_{\tau_{SI_2}} \uparrow x$ and $F \in Irr_\tau(X)$. If $y \in F^\delta$, then $int_{\tau_{SI_2}} \uparrow x \cap F^\delta \neq \emptyset$. By Definition 3.3, $int_{\tau_{SI_2}} \uparrow x \cap F \neq \emptyset$. Thus $x \in \downarrow F$ and $x \ll_{SI_2} y$. □

Lemma 3.5 *Let (X, τ) be a locally hypercompact space. If A is an irreducible set in (X, τ), then there exists a directed subset $D \subseteq \downarrow A$ such that $D^\uparrow = A^\uparrow$. Furthermore, we have $D^\delta = A^\delta$, $cl_\tau D = cl_\tau A$.*

Proof. Let $A \in Irr_\tau(X)$. Consider the collection $\mathcal{F} = \{\uparrow F \in \mathrm{Fin}P : A \cap int_\tau \uparrow F \neq \emptyset\}$. Let $H, G \in \mathcal{F}$. Then $int_\tau(\uparrow H) \cap int_\tau(\uparrow G) \cap A \neq \emptyset$ since A is irreducible. Pick x in this intersection. Since (X, τ) is locally hypercompact, there exists $E \in X^{(<\omega)}$ such that $x \in int_\tau(\uparrow E) \subseteq \uparrow E \subseteq int_\tau(\uparrow H) \cap int_\tau(\uparrow G)$. Then $\uparrow E \in \mathcal{F}$ and $E \subseteq \uparrow H \cap \uparrow G$. Thus $\mathcal{F}$ is directed and $\uparrow F \cap \downarrow A \neq \emptyset$ for any $\uparrow F \in \mathcal{F}$. By Lemma 2.10, there is a directed set $D \subseteq \downarrow A$ such that $D \cap \uparrow F \neq \emptyset$ for every $\uparrow F \in \mathcal{F}$. Obviously, $A^\uparrow \subseteq D^\uparrow$ since $D \subseteq \downarrow A$. Let y be an upper bound of D. Assume that $y \notin A^\uparrow$, then there exists $x \in A$ such that $x \nleq y$. Since (X, τ) is a locally hypercompact space and $x \in X \backslash \downarrow y \in \tau$, there is a finite subset F of X such that $x \in int_\tau \uparrow F \subseteq \uparrow F \subseteq X \backslash \downarrow y$. Thus $\uparrow F \in \mathcal{F}$. So $D \cap \uparrow F \neq \emptyset$. Thus $D \cap (X \backslash \downarrow y) \neq \emptyset$, that is, there is a $d \in D$ such that $d \nleq y$, a contradiction. Hence $D^\uparrow \subseteq A^\uparrow$. Therefore, $D^\uparrow = A^\uparrow$, then we have $D^\delta = A^\delta$.

Now we show that $\mathrm{cl}_\tau A = \mathrm{cl}_\tau D$. Obviously, $\mathrm{cl}_\tau D \subseteq \mathrm{cl}_\tau A$. Let $x \in \mathrm{cl}_\tau A$, $U \in \tau$ and $x \in U$. Since (X, τ) is locally hypercompact, there exists $G \in P^{(<\omega)}$ such that

$x \in \mathrm{int}_\tau \uparrow G \subseteq \uparrow G \subseteq U$. Note that $x \in \mathrm{cl}_\tau A$, thus $A \cap \mathrm{int}_\tau \uparrow G \neq \emptyset$. So $\uparrow G \in \mathcal{F}$, which implies $D \cap \uparrow G \neq \emptyset$ and $D \cap U \neq \emptyset$. Thus $x \in \mathrm{cl}_\tau D$. Therefore $\mathrm{cl}_\tau D = \mathrm{cl}_\tau A$.□

Since a C-space is locally hypercompact, by Lemma 3.5, we have the following result.

Lemma 3.6 **([20])** *If F is an irreducible subset of a C-space (X, τ), then there is a directed subset $D \subseteq \downarrow F$ such that $D^\uparrow = F^\uparrow$. In particular, $\vee D = \vee F$, if either exists.*

Lemma 3.7 *Let P be a poset. Then the following conditions hold.*

(1) $\upsilon(P)_{SI_2} = \upsilon(P)_{SI} = \upsilon(P)$.

(2) $A(P)_{SI_2} = \sigma_2(P)$.

(3) *If P is an s_2-continuous poset, then $\sigma_2(P)_{SI_2} = \sigma_2(P)_{SI} = \sigma_2(P)$.*

Proof. (1) By Proposition 3.4(5), $\upsilon(P)_{SI_2} \subseteq \upsilon(P)_{SI} \subseteq \upsilon(P)$. For any $x \in P$, let F be an irreducible set with $F^\delta \cap (P \setminus \downarrow x) \neq \emptyset$. Then there exists $z \in F^\delta$ such that $z \not\leq x$. Suppose $F \cap (P \setminus \downarrow x) = \emptyset$. Then $x \in F^\uparrow$, so $z \leq x$, a contradiction. Hence (1) holds as desired.

(2) Because a non-empty subset $F \subseteq P$ is irreducible with respect to the Alexandoff topology $A(P)$ iff it is a directed set.

(3) Obviously, $\sigma_2(P)_{SI_2} \subseteq \sigma_2(P)_{SI} \subseteq \sigma_2(P)$. Let P be an s_2-continuous poset. By Theorem 2.4, $(P, \sigma_2(P))$ is a C-space. Let $U \in \sigma_2(P)$ and $F \in Irr_\tau(X)$. If $F^\delta \cap U \neq \emptyset$, by Lemma 3.6, there is a directed set $D \subseteq \downarrow F$ such that $D^\delta = F^\delta$. Thus we have that $D \cap U \neq \emptyset$, which implies $F \cap U \neq \emptyset$. Hence $U \in \sigma_2(P)_{SI_2}$. Therefore, $\sigma_2(P)_{SI_2} = \sigma_2(P)_{SI} = \sigma_2(P)$. □

The following example shows that an SI-open set need not be SI$_2$-open.

Example 3.8 ([4]) Consider three disjoint countable sets $A = \{a_n : n \in \mathbf{N_0}\}, B = \{b_n : n \in \mathbf{N_0}\}, C = \{c_n : n \in \mathbf{N}\}$, and the order $\leq$ on $P = A \cup B \cup C$ is defined as follows:

$\downarrow a_0 = \{a_0\} \cup B$,

$\downarrow a_n = \{b_m : m < n\}(n \in \mathbf{N}, n \neq 1, 2)$,

$\downarrow a_1 = \{b_0\} \cup C$,

$\downarrow a_2 = \{b_0, b_1\} \cup C$,

$\downarrow b_n = \{b_n\}(n \in \mathbf{N_0})$,

$\downarrow c_n = \{c_m : m \leq n\}(n \in \mathbf{N})$,

$x \leq y \Leftrightarrow x \in \downarrow y$.

Then $\uparrow b_0$ is open in $\sigma(P)$ but not in $\sigma_2(P)$ since $C = \{c_n : n \in \mathbf{N}\}$ is a directed lower set with $b_0 \in C^\delta \cap \uparrow b_0 \neq \emptyset$ while $C \cap \uparrow b_0 = \emptyset$. Thus $\uparrow b_0 \in A(P)_{SI} = \sigma(P)$, but $\uparrow b_0 \notin A(P)_{SI_2} = \sigma_2(P)$. Therefore, $\uparrow b_0$ is an SI-open set but not SI$_2$-open in $(P, A(P))$.

The following theorem exhibits an important property of relation $\ll_{SI_2}$ on SI$_2$-continuous spaces, i.e., the interpolation property.

Theorem 3.9 *Let (X,τ) be an SI$_2$-continuous. Then the following conditions hold.*

(1) *The relation $\ll_{SI_2}$ satisfies interpolation property, i.e., $x \ll_{SI_2} z$ implies $x \ll_{SI_2} y \ll_{SI_2} z$ for some $y \in X$.*

(2) *If $x \ll_{SI_2} z$ and $z \in F^\delta$ for an irreducible set F in (X,τ), then $x \ll_{SI_2} y$ for some element $y \in F$.*

Proof. (1) Let $x \ll_{SI_2} z$. Since (X,τ) is SI$_2$-continuous, $z \in (\Downarrow_{SI_2} z)^\delta \subseteq (\bigcup\{(\Downarrow_{SI_2} y)^\delta : y \in \Downarrow_{SI_2} z\})^\delta = (\bigcup\{\Downarrow_{SI_2} y : y \in\Downarrow_{SI_2} z\})^\delta$. As the union of a directed family of directed sets is directed, $\bigcup\{\Downarrow_{SI_2} y : y \in\Downarrow_{SI_2} z\}$ is directed. By the definition of $\ll_{SI_2}$, there exist $y \in\Downarrow_{SI_2} z$ and $w \in\Downarrow_{SI_2} y$ such that $x \leq w$. Thus $x \ll_{SI_2} y \ll_{SI_2} z$.

(2) It is straightforward from (1). □

Lemma 3.10 *If a space (X,τ) is SI$_2$-continuous, then all sets $\Uparrow_{SI_2} x$ for $x \in X$ are SI$_2$-open.*

Proof. Since (X,τ) is SI$_2$-continuous, $\Uparrow_{SI_2} x \in \tau$. Let F be an irreducible subset with $F^\delta \cap \Uparrow_{SI_2} x \neq \emptyset$. Then there exists $z \in F^\delta$ such that $x \ll_{SI_2} z$. By Theorem 3.9(2), there is a $y \in F$ such that $x \ll_{SI_2} y$. Thus $F \cap \Uparrow_{SI_2} x \neq \emptyset$. Hence $\Uparrow_{SI_2} x$ is SI$_2$-open. □

Proposition 3.11 *Let (X,τ) be an SI$_2$-continuous space and $x \in X$. Then*

(1) *An upper set U is SI$_2$-open iff for every $x \in U$, there is a $u \in U$ such that $u \ll_{SI_2} x$.*

(2) *The sets of the form $\Uparrow_{SI_2} x, x \in X$ form a basis for the SI$_2$-topology.*

(3) *$int_{\tau_{SI_2}} \uparrow x = \Uparrow_{SI_2} x$.*

(4) *For any subset $A \subseteq X$, $int_{\tau_{SI_2}} A = \bigcup\{\Uparrow_{SI_2} u : \Uparrow_{SI_2} u \subseteq A\}$.*

Proof. (1)Let U be an SI$_2$-open and $x \in U$. Since (X,τ) is SI$_2$-continuous, $x \in (\Downarrow_{SI_2} x)^\delta$ and $\Downarrow_{SI_2} x$ is directed. Thus $U \cap (\Downarrow_{SI_2} x)^\delta \neq \emptyset$. It follows that there exists $u \in U$ such that $u \ll_{SI_2} x$. Conversely, if for any $x \in U$, there is a $u \in U$ such that $u \ll_{SI_2} x$, then $U = \{\Uparrow_{SI_2} u : u \in U\}$, which is SI$_2$-open by Lemma 3.10. Thus U is SI$_2$-open.

(2) It is immediate consequence of (1).

(3) By Proposition 3.4(8), $int_{\tau_{SI_2}} \uparrow x \subseteq \Uparrow_{SI_2} x$. By Lemma 3.10, $\Uparrow_{SI_2} x$ is SI$_2$-open and $\Uparrow_{SI_2} x \subseteq \uparrow x$. Thus $int_{\tau_{SI_2}} \uparrow x = \Uparrow_{SI_2} x$.

(4) This follows directly from (2). □

Lemma 3.12 *In an SI$_2$-continuous space (X,τ) the following hold.*

(1) *If $x \ll_{SI_2} y$, then there is an SI$_2$-open filter U with $y \in U \subseteq \Uparrow_{SI_2} x$.*

(2) *If $y \not\leq z$, then there is an SI$_2$-open filter U containing y but not z.*

Proof. (1) By the interpolation property, we construct inductively a decreasing sequence of elements y_n with $x \ll_{SI_2} ... \ll_{SI_2} y_n \ll_{SI_2} y_{n-1} \ll_{SI_2} ... \ll_{SI_2} y_1 = y$. Set $U = \bigcup\{\Uparrow_{SI_2} y_n : n = 1, 2, ...\}$. Clearly, $y \in U$ and $U \subseteq \Uparrow_{SI_2} x$. Now we show that U is an SI$_2$-open filter. Clearly, U is an upper set. If $x_1, x_2 \in U$, then there are y_{n_1}, y_{n_2} such that $x_1 \in \Uparrow_{SI_2} y_{n_1}, x_2 \in \Uparrow_{SI_2} y_{n_2}$. Without loss of generality, we assume that $n_1 \leq n_2$, then $y_{n_2} \ll_{SI_2} y_{n_1} \ll_{SI_2} x_1$. Thus $y_{n_2} \leq x_1, x_2$. Note that $y_{n_2} \in U$. Thus U is a filter. It is from Proposition 3.11 that U is SI$_2$-open.

(2) Suppose that $y \not\leq z$. Since (X,τ) is SI$_2$-continuous, $y = \vee \Downarrow_{SI_2} y$. It follows that $z \notin (\Downarrow_{SI_2} y)^{\uparrow}$. Thus there exists $x \in \Downarrow_{SI_2} y$ such that $x \not\leq z$. By the condition (1), there is an SI$_2$-open filter U such that $y \in U \subseteq \Uparrow_{SI_2} x$, but $z \notin U$. □

Theorem 3.13 *Let (X,τ) be a T_0 space. Then the following conditions are equivalent.*

(1) *(X,τ) is SI$_2$-continuous;*

(2) *each $\Uparrow_{SI_2} x$ is SI$_2$-open, and if $U \in \tau_{SI_2}$, then $U = \bigcup\{\Uparrow_{SI_2} x : x \in U\}$;*

(3) *$SOFilt_\tau(X)$ is a basis of τ_{SI_2} and $(\tau_{SI_2}, \subseteq)$ is a continuous lattice;*

(4) *τ_{SI_2} has enough co-primes and $(\tau_{SI_2}, \subseteq)$ is a continuous lattices;*

(5) *(X, τ_{SI_2}) is a C-space.*

Proof. (1)⇒(2) By Lemma 3.10 and Proposition 3.11.

(2)⇒(3) Obviously, $\Uparrow_{SI_2} x \in \tau$ is open. By Proposition 3.11 and Lemma 3.12, SOFilt$_\tau(X)$ is a basis of τ_{SI_2}. In order to prove the continuity of τ_{SI_2}, let $U \in \tau_{SI_2}$. For any $x \in U$, there is a $y \in U$ such that $y \ll_{SI_2} x$ by (2). Then $x \in \Uparrow_{SI_2} y \in \tau_{SI_2}$, and we claim that $\Uparrow_{SI_2} y \ll U$. Indeed, if $\mathcal{D}$ is a directed family of SI$_2$-open sets with $U \subseteq \bigcup \mathcal{D}$, then there exists $W \in \mathcal{D}$ such that $y \in W$. Thus $\Uparrow_{SI_2} y \subseteq \uparrow y \subseteq W$. Thus $U = \bigcup\{V : V \ll U\}$.

(3)⇔ (4) Consequence of Proposition 3.4(7).

(3)⇒(5) Let $x \in U \in \tau_{SI_2}$. Since $(\tau_{SI_2}, \subseteq)$ is continuous, there exists $V \in \tau_{SI_2}$ such that $x \in V \ll U$. Since SOFilt$_\tau(X)$ is a basis of τ_{SI_2}, there exists $F \in$ SOFilt$_\tau(X)$ such that $x \in F \subseteq V$. Now we show that there exists $y \in U$ such that $x \in F \subseteq \uparrow y$. If not, then for any $y \in U$, $F \not\subseteq \uparrow y$. Thus $y \in X \setminus \downarrow z_y$ for some $z_y \in F$, and thus there exists $F_y \in$ SOFilt$_\tau(X)$ such that $y \in F_y \subseteq X \setminus \downarrow z_y$. Hence $U \subseteq \bigcup\{F_y : y \in U\}$. Since $V \ll U$, there exists a finite set $\{y_i : i = 1, 2, ..., n\}$ such that $V \subseteq \bigcup\{F_{y_i} \in$ SOFilt$_\tau(X) : i = 1, 2, ..., n\}$. Let $z_{y_i} = z_i$. Then $z_i \in F$. Since F is a filter, there exists $z \in F$ such that $z \leq z_i$ for all i. Notice that $z \in F \subseteq V \subseteq \bigcup\{F_{y_i} \in$ SOFilt$_\tau(X) : i = 1, 2, ..., n\}$. Then there is a $F_{y_k} \in$ SOFilt$_\tau(X)$ such that $z \in F_k \subseteq X \setminus \downarrow z_k$, which contradicts $z \leq z_i$. Thus there exists $y \in U$ such that $x \in F \subseteq \uparrow y$. Since F is SI$_2$-open, $x \in int_{\tau_{SI_2}} \uparrow y \subseteq \uparrow y \subseteq U$. Hence (X, τ_{SI_2}) is a C-space.

(5)⇒(1) Let (X, τ_{SI_2}) be a C-space. For any $x \in X$, let $D_x = \{y \in X : x \in int_{\tau_{SI_2}} \uparrow y\}$. By Proposition 3.4(8), $D_x \subseteq \Downarrow_{SI_2} x$. First we show that D_x is directed and $x = \vee D_x$. For any $d_1, d_2 \in D_x$, $x \in int_{\tau_{SI_2}} \uparrow d_1 \cap int_{\tau_{SI_2}} \uparrow d_2 \in \tau_{SI_2}$. Since (X, τ_{SI_2}) is a C-space, there is a $d \in int_{\tau_{SI_2}} \uparrow d_1 \cap int_{\tau_{SI_2}} \uparrow d_2$ such that $x \in int_{\tau_{SI_2}} \uparrow d \subseteq \uparrow d \subseteq int_{\tau_{SI_2}} \uparrow d_1 \cap int_{\tau_{SI_2}} \uparrow d_2 \subseteq \uparrow d_1 \cap \uparrow d_2$. It follows that $d \in D_x$ and $d_1, d_2 \leq d$. Thus D_x is directed. Obviously, x is an upper bound of D_x. Let y be any upper bound of D_x and $P = (X, \leq_\tau)$. If $x \not\leq y$, then $x \in P \setminus \downarrow y \in \tau_{SI_2}$. Since (X, τ_{SI_2}) is a C-space, there is a $w \in P$ such that $x \in int_{\tau_{SI_2}} \uparrow w \subseteq \uparrow w \subseteq P \setminus \downarrow y$. Thus $w \in D_x$ and $w \not\leq y$, contradicting to the assumption that y is an upper bound of D_x. Thus $x = \vee D_x$.

Since $D_x \subseteq \Downarrow_{SI_2} x$, we have $x = \vee \Downarrow_{SI_2} x$. Now we show that $\Downarrow_{SI_2} x$ is directed. For any $y_1, y_2 \in \Downarrow_{SI_2} x$, $y_1 \ll_{SI_2} x, y_2 \ll_{SI_2} x$. By the definition of $\ll_{SI_2}$ and $x \in D_x^\delta$, there are $d_1, d_2 \in D_x$ such that $y_1 \leq d_1, y_2 \leq d_2$. Since D_x is directed, there exists $d \in D_x$ such that $d_1, d_2 \leq d$. Thus $y_1, y_2 \leq d$. Note that $D_x \subseteq \Downarrow_{SI_2} x$, so $\Downarrow_{SI_2} x$ is directed.

Finally, we can directly check that $\Uparrow_{SI_2} x = \bigcup_{z \in \uparrow x} int_{\tau_{SI_2}}(\uparrow z)$. In fact, for any $y \in \Uparrow_{SI_2} x$, $x \ll_{SI_2} y$. From the above argument we can see that $y \in D_y^\delta$ and D_y is directed, so there is a $z \in D_y$ such that $x \leq z$. Follows from the definition of D_y, we have $y \in int_{\tau_{SI_2}} \uparrow z$ and $z \in \uparrow x$. Thus $y \in \bigcup_{z \in \uparrow x} int_{\tau_{SI_2}}(\uparrow z)$, i.e., $\Uparrow_{SI_2} x \subseteq \bigcup_{z \in \uparrow x} int_{\tau_{SI_2}}(\uparrow z)$. To prove the inverse inclusion, let $y \in \bigcup_{z \in \uparrow x} int_{\tau_{SI_2}}(\uparrow z)$. Then there is a $z \in \uparrow x$ such that $y \in int_{\tau_{SI_2}}(\uparrow z)$, so $y \in int_{\tau_{SI_2}}(\uparrow x)$. By Proposition 3.4(8), $x \ll_{SI_2} y$, i.e., $y \in \Uparrow_{SI_2} x$. Thus $\Uparrow_{SI_2} x = \bigcup_{z \in \uparrow x} int_{\tau_{SI_2}}(\uparrow z)$. Therefore, $\Uparrow_{SI_2} x$ is open in (X, τ). All there show that (X, τ) is SI_2-continuous. □

By Theorem 2.9, Lemma 3.7 and Theorem 3.13, we have the following corollary.

Corollary 3.14 *Let P be a poset. Then the following conditions are equivalent.*

(1) *P is hypercontinuous;*

(2) *$(P, \upsilon(P))$ is SI-continuous;*

(3) *$(P, \upsilon(P))$ is SI_2-continuous.*

Corollary 3.15 *Let P be a poset. Then the following conditions are equivalent.*

(1) *P is an s_2-continuous poset;*

(2) *$(P, \sigma_2(P))$ is a C-space;*

(3) *$(P, \sigma_2(P))$ is SI_2-continuous ;*

(4) *$(P, A(P))$ is SI_2-continuous.*

Proof. (1) $\Leftrightarrow$ (2) By Theorem 2.4.

(2) $\Rightarrow$ (3) Let $(P, \sigma_2(P))$ be a C-space. By Lemma 3.7, $\sigma_2(P)_{SI_2} = \sigma_2(P)$. Thus we have that $(P, \sigma_2(P)_{SI_2})$ is a C-space. From Theorem 3.13, it follows that $(P, \sigma_2(P))$ is SI_2-continuous.

(3) $\Rightarrow$ (1) Suppose that $(P, \sigma_2(P))$ is SI_2-continuous, then for any $x \in P$, $x \in (\Downarrow_{SI_2} x)^\delta$ and $\Downarrow_{SI_2} x$ is directed. Note that $\Downarrow_{SI_2} x \subseteq \{y \in P : y \ll_2 x\}$. Thus P is s_2-continuous.

(2) $\Leftrightarrow$ (4) From Lemma 3.7 and Theorem 3.13, it follows that the condition (3) and (4) are equivalent. □

The following example shows that an SI-continuous space need not be SI_2-continuous.

Example 3.16 ([4]) Consider the Euclidean plane $P = \mathbb{R} \times \mathbb{R}$ under the usual order, then P is a continuous poset, so $(P, \sigma(P))$ is a C-space, which implies that $(P, A(P))$ is an SI-continuous space (By Lemma 5.2 and Theorem 6.4 in [20]). Because every lower half-plane

$$E_a = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y \leq a\}$$

is a directed lower set with $E_a^\delta = \mathbb{R} \times \mathbb{R}$, while $\bigcap\{E_a : a \in \mathbb{R}\} = \emptyset$, thus $\ll$ is empty. Hence P is not s_2-continuous. By Corollary 3.15, $(P, A(P))$ is not an SI_2-continuous space.

4 SI$_2$-quasicontinuous spaces

In this section, we introduce the concept of SI_2-quasicontinuous spaces and prove that a space (X, τ) is SI_2-quasicontinuous if and only if (X, τ_{SI_2}) is a locally hypercompact space.

Definition 4.1 Let (X, τ) be a T_0 space and $F \in Irr_\tau(X)$. $G, H \subseteq X$. Define $G \ll_{SI_2} H$ if $\uparrow H \cap F^\delta \neq \emptyset$ implies $\uparrow G \cap F \neq \emptyset$.

Write $G \ll_{SI_2} x$ for $G \ll_{SI_2} \{x\}$ and $y \ll_{SI_2} H$ for $\{y\} \ll_{SI_2} H$. The set $\{x \in X : G \ll_{SI_2} x\}$ will be denoted by $\Uparrow_{SI_2} G$ and $\{x \in X : x \ll_{SI_2} H\}$ denoted by $\Downarrow_{SI_2} H$. Let $\mathrm{fin}(x) = \{E \in X^{(<\omega)} : E \ll_{SI_2} x\}$.

Definition 4.2 A T_0 space (X, τ) is called *SI_2-quasicontinuous* if for any $x \in X$, the following conditions hold:

(1) for any $E \in X^{(<\omega)}$, $\Uparrow_{SI_2} E$ is open in (X, τ);

(2) $\mathrm{fin}(x)$ is directed;

(3) $\uparrow x = \bigcap\{\uparrow E : E \in \mathrm{fin}(x)\}$.

Remark 4.3 It is verify that the condition (3) in above definition is equivalent to

(3′) for any $x, y \in X$, if $x \not\leq y$, then there exists $E \in \mathrm{fin}(x)$ such that $y \notin \uparrow E$.

It is easy to get the following proposition and we omit the proof.

Proposition 4.4 *Let (X, τ) be a T_0 space and $G, H \subseteq X$. Then*

(1) *$G \ll_{SI_2} H$ iff $G \ll_{SI_2} x$ for all $x \in H$.*

(2) *$G \ll_{SI_2} H \Rightarrow G \leq H$.*

(3) *$A \leq G \ll_{SI_2} H \leq B \Rightarrow A \ll_{SI_2} B$.*

(4) *If $x \in int_{\tau_{SI_2}} \uparrow H$, then $H \ll_{SI_2} x$.*

Lemma 4.5 ([8]) *(Rudin's Lemma) Let $\mathcal{F}$ be a directed family of nonempty finite subsets of a poset P. Then there exists a directed set $D \subseteq \bigcup_{F \in \mathcal{F}} F$ such that $D \cap F \neq \emptyset$ for all $F \in \mathcal{F}$.*

Lemma 4.6 *Let $\mathcal{H}$ be a directed family of nonempty finite sets in a T_0 space. If $G \ll_{SI_2} x$ and $\bigcap_{H \in \mathcal{H}} \uparrow H \subseteq \uparrow x$, then $H \subseteq \uparrow G$ for some $H \in \mathcal{H}$.*

Proof. Suppose not. Then the collection $\{H \setminus \uparrow G : H \in \mathcal{H}\}$ is a directed family of nonempty finite sets. By Lemma 4.5, there exists a directed set $D \subseteq \bigcup\{H \setminus \uparrow G : H \in \mathcal{H}\}$ such that $D \cap (H \setminus \uparrow G) \neq \emptyset$ for all $H \in \mathcal{H}$. Then $D^\uparrow \subseteq \bigcap_{d \in D} \uparrow d \subseteq \bigcap_{H \in \mathcal{H}} \uparrow (H \setminus \uparrow G) \subseteq \bigcap_{H \in \mathcal{H}} \uparrow H \subseteq \uparrow x$. Thus $x \in D^\delta$. Since every directed set is irreducible and $G \ll_{SI_2} x$, there exists $d \in D$ such that $d \in \uparrow G$. But this contradicts $d \in H \setminus \uparrow G$ for some H. $\square$

We now derive the interpolation property for SI$_2$-quasicontinuous spaces.

Theorem 4.7 *Let X be an SI$_2$-quasicontinuous space. If $H \ll_{SI_2} x$, then there exists $E \in X^{(<\omega)}$ such that $H \ll_{SI_2} E \ll_{SI_2} x$.*

Proof. The statements has been proved for quasicontinuous domains in [8], and the similar proof carries over to this setting. $\square$

Proposition 4.8 *Let (X, τ) be an SI$_2$-quasicontinuous space. Then*

(1) *For any nonempty set $H \subseteq X$, $\Uparrow_{SI_2} H = int_{\tau_{SI_2}} \uparrow H$.*

(2) *A subset U of X is SI$_2$-open iff $U = \bigcup\{\Uparrow_{SI_2} E : E \in X^{(<\omega)}$ and $\uparrow E \subseteq U\}$. The set $\{\Uparrow_{SI_2} E : E \in X^{(<\omega)}\}$ form a basis of τ_{SI_2}.*

Proof. (1) From Proposition 4.4(4), we have $int_{\tau_{SI_2}} \uparrow H \subseteq \Uparrow_{SI_2} H$. Obviously, $\Uparrow_{SI_2} H \subseteq \uparrow H$. Now we only need to show that $\Uparrow_{SI_2} H$ is SI$_2$-open. Since (X, τ) is SI$_2$-quasicontinuous, $\Uparrow_{SI_2} H \in \tau$. Let $F \in Irr_\tau(X)$ and $\Uparrow_{SI_2} H \cap F^\delta \neq \emptyset$. Choose $x \in \Uparrow_{SI_2} H \cap F^\delta$, i.e., $H \ll_{SI_2} x$ and $x \in F^\delta$. By Theorem 4.7, there exists $E \in X^{(<\omega)}$ such that $H \ll_{SI_2} E \ll_{SI_2} x$, which implies $\uparrow E \cap F \neq \emptyset$. Notice that $E \subseteq \Uparrow_{SI_2} H$, so $\Uparrow_{SI_2} H \cap F \neq \emptyset$. Thus $\Uparrow_{SI_2} H$ is SI$_2$-open. Therefore $\Uparrow_{SI_2} H = int_{\tau_{SI_2}} \uparrow H$.

(2) The sufficiency follows from the condition (1). To prove the necessity, let $U \in \tau_{SI_2}$ and $x \in U$. From the definition of SI$_2$-topology, we have $U \ll_{SI_2} x$. By Theorem 4.7, there exists $E \in X^{(<\omega)}$ such that $U \ll_{SI_2} E \ll_{SI_2} x$. Thus $x \in \bigcup\{\Uparrow_{SI_2} E : E \in X^{(<\omega)}$ and $\uparrow E \subseteq U\}$. Obviously, the converse inclusion is always true. Thus $U = \bigcup\{\Uparrow_{SI_2} E : E \in X^{(<\omega)}$ and $\uparrow E \subseteq U\}$, and thus the set $\{\Uparrow_{SI_2} E : E \in X^{(<\omega)}\}$ form a basis of τ_{SI_2}. $\square$

Theorem 4.9 *Let (X, τ) be a T_0 space. Then the following conditions are equivalent.*

(1) *(X, τ) is an SI$_2$-quasicontinuous space;*

(2) *(X, τ_{SI_2}) is a locally hypercompact space.*

Proof. (1)$\Rightarrow$ (2) For any x and any SI$_2$-open U containing x, by Proposition 4.8, there exists $E \in X^{(<\omega)}$ such that $x \in int_{\tau_{SI_2}} \uparrow E = \Uparrow_{SI_2} E \subseteq \uparrow E \subseteq U$.

(2)$\Rightarrow$ (1) For any $x \in X$, let $\mathcal{H} = \{H \in X^{(<\omega)} : x \in int_{\tau_{SI_2}} \uparrow H\}$. First, we show that $\mathcal{H}$ is nonempty and $\uparrow x = \bigcap_{H \in \mathcal{H}} \uparrow H$. Since X is SI$_2$-open, it follows from (2) that there exists $H \in X^{(<\omega)}$ such that $x \in int_{\tau_{SI_2}} \uparrow H \subseteq \uparrow H \subseteq U$. Then $H \in \mathcal{H} \neq \emptyset$. Obviously, $\uparrow x \subseteq \bigcap_{H \in \mathcal{H}} \uparrow H$. If $x \not\leq y$, then $x \in X \setminus \downarrow y \in \tau_{SI_2}$. By (2), there exists $H \in X^{(<\omega)}$ such that $x \in int_{\tau_{SI_2}} \uparrow H \subseteq \uparrow H \subseteq X \setminus \downarrow y$. Thus $H \in \mathcal{H}$ and $y \notin \uparrow H$. Thus $\uparrow x = \bigcap_{H \in \mathcal{H}} \uparrow H$.

Now we show that $\mathcal{H}$ is directed. Let $H_1, H_2 \in \mathcal{H}$. Then $x \in int_{\tau_{SI_2}} \uparrow H_1 \cap int_{\tau_{SI_2}} \uparrow H_2$. It follows from (2) that there exists $H \in X^{(<\omega)}$ such that $x \in int_{\tau_{SI_2}} \uparrow H \subseteq \uparrow H \subseteq int_{\tau_{SI_2}} \uparrow H_1 \cap int_{\tau_{SI_2}} \uparrow H_2 \subseteq \uparrow H_1 \cap \uparrow H_2$, so $H \in \mathcal{H}$ and $H_1, H_2 \leq H$.

Thus $\mathcal{H}$ is directed. Obviously, $\mathcal{H} \subseteq \text{fin}(x)$. Then by Lemma 4.6, it is easy to show that $\text{fin}(x)$ is directed, and $\uparrow x \subseteq \bigcap_{H \in \text{fin}(x)} \uparrow H \subseteq \bigcap_{H \in \mathcal{H}} \uparrow H = \uparrow x$.

Finally, we show that $\Uparrow_{SI_2} E$ is open in (X, τ) for any $E \in X^{(<\omega)}$. For any $x \in \Uparrow_{SI_2} E$, $E \ll_{SI_2} x$. Notice that $\uparrow x = \bigcap_{H \in \mathcal{H}} \uparrow H$ and $\mathcal{H}$ is directed, by Lemma 4.6, there exists $H \in \mathcal{H}$ such that $H \subseteq \uparrow E$. Thus $x \in int_{\tau_{SI_2}} \uparrow H \subseteq int_{\tau_{SI_2}} \uparrow E \subseteq \Uparrow_{SI_2} E$, which implies $\Uparrow_{SI_2} E$ is open in (X, τ). Therefore, (X, τ) is SI_2-quasicontinuous. □

By Theorem 3.13 and Theorem 4.9, we have the following corollary.

Corollary 4.10 *If (X, τ) is SI_2-continuous, then (X, τ) is SI_2-quasicontinuous.*

Corollary 4.11 *Let P be a poset. Then the following conditions are equivalent.*

(1) *P is a quasi-hypercontinuous poset;*

(2) *$(P, \upsilon(P))$ is an SI_2-quasicontinuous space.*

Proof. By Lemma 3.7 and Theorem 4.9.

□

Lemma 4.12 *Let P be a poset. If $(P, \sigma_2(P))$ is a locally hypercompact space, then $\sigma_2(P)_{SI_2} = \sigma_2(P)$.*

Proof. Obviously, $\sigma_2(P)_{SI_2} \subseteq \sigma_2(P)$. Let $U \in \sigma_2(P)$ and $F \in Irr_{\sigma_2(P)}(P)$, if $F^\delta \cap U \neq \emptyset$. By Lemma 3.5, there exists directed subset $D \subseteq \downarrow F$ such that $D^\delta = F^\delta$. Thus $D^\delta \cap U \neq \emptyset$. Since $U \in \sigma_2(P)$, $D \cap U \neq \emptyset$, which implies $F \cap U \neq \emptyset$. Therefore $U \in \sigma_2(P)_{SI_2}$. □

Corollary 4.13 *Let P be a poset. Then the following conditions are equivalent.*

(1) *P is an s_2-quasicontinuous poset;*

(2) *$(P, \sigma_2(P))$ is a locally hypercompact space;*

(3) *$(P, \sigma_2(P))$ is an SI_2-quasicontinuous space;*

(4) *$(P, A(P))$ is an SI_2-quasicontinuous space.*

Proof. (1)⇔(2) By Theorem 2.6.

(2)⇒(3) Let $(P, \sigma_2(P))$ be a locally hypercompact space. By Lemma 4.12, we have $\sigma_2(P)_{SI_2} = \sigma_2(P)$. Thus $(P, \sigma_2(P)_{SI_2})$ is a locally hypercompact space. By Theorem 4.9, $(P, \sigma_2(P))$ is an SI_2-quasicontinuous space.

(3)⇒(1) Let $(P, \sigma_2(P))$ be an SI_2-quasicontinuous space. It is easy to see that $\text{fin}(x) \subseteq \{E \in P^{(<\omega)} : E \ll_2 x\}$. Thus P is an s_2-quasicontinuous poset.

(2)⇔(4) By Lemma 3.7(2), we have $A(P)_{SI_2} = \sigma_2(P)$. From Theorem 4.9, it follows that (2) and (4) are equivalent. □

Example 4.14 ([4]) Let $P = \{a\} \cup \{a_n : n \in N\}$. The partial order on P is defined by setting $a_n < a_{n+1}$ for all $n \in N$, and $a_1 < a$. Then P is an s_2-quasicontinuous poset which is not s_2-continuous. By Theorem 3.13 and Corollary 4.13, This poset P equipped with the Alexandroff topology $A(P)$ is an SI_2-quasicontinuous space, but it is not SI_2-continuous.

5 Meet SI$_2$-continuous spaces

In this section, we define a meet SI$_2$-continuous space and prove that (X,τ) is SI$_2$-continuous if and only if it is a meet SI$_2$-continuous and SI$_2$-quasicontinuous space.

Definition 5.1 A T_0 space (X,τ) is called *meet SI$_2$-continuous* if for any $x \in X$ and any $F \in Irr_\tau(X)$ with $x \in F^\delta$, then $x \in cl_{\tau_{SI_2}}(\downarrow x \cap \downarrow F)$.

Proposition 5.2 *Let (X,τ) be a T_0 space. Considering the following statements.*

(1) *For any $x \in X$ and $U \in \tau_{SI_2}$, $\uparrow(U \cap \downarrow x) \in \tau_{SI_2}$.*

(2) *(X,τ) is meet SI$_2$-continuous.*

Then (1)⇒(2). If (X,τ) satisfies $\uparrow(U \cap \downarrow x) \in \tau$ for any $x \in X$ and $U \in \tau_{SI_2}$, then the two conditions are equivalent.

Proof. (1)⇒(2) Let F be an irreducible set in (X,τ) with $x \in F^\delta$. Suppose $x \notin cl_{\tau_{SI_2}}(\downarrow x \cap \downarrow F)$. Then there exists $U \in \tau_{SI_2}$ containing x such that $U \cap (\downarrow x \cap \downarrow F) = \emptyset$, thus $\uparrow(U \cap \downarrow x) \bigcap F = \emptyset$. By hypothesis $\uparrow(U \cap \downarrow x) \in \tau_{SI_2}$, we have $\uparrow(U \cap \downarrow x) \cap F^\delta = \emptyset$. But $x \in \uparrow(U \cap \downarrow x) \cap F^\delta$, a contradiction. Thus (X,τ) is meet SI$_2$-continuous.

(2)⇒(1) For any $F \in Irr_\tau(X)$, if $\uparrow(U \cap \downarrow x) \cap F^\delta \neq \emptyset$, then there exists $y \in F^\delta$ and $u \in U \cap \downarrow x$ such that $u \leq y$, which implies $u \in F^\delta$. By (2), $u \in cl_{\tau_{SI_2}}(\downarrow u \cap \downarrow F)$. Note that $U \in \tau_{SI_2}$ and $u \in U$, thus $U \cap (\downarrow u \cap \downarrow F) \neq \emptyset$. It follows that $U \cap (\downarrow x \cap \downarrow F) \neq \emptyset$. Hence $\uparrow(U \cap \downarrow x) \cap F \neq \emptyset$. Since $\uparrow(U \cap \downarrow x) \in \tau$, we have $\uparrow(U \cap \downarrow x) \in \tau_{SI_2}$. □

Corollary 5.3 *Let P be a poset. The following statements are equivalent.*

(1) *For any $x \in P$ and any $U \in \sigma_2(P)$, $\uparrow(U \cap \downarrow x) \in \sigma_2(P)$;*

(2) *P is meet s_2-continuous;*

(3) *$(P, A(P))$ is meet SI$_2$-continuous.*

Proof. (1)⇔(2) See [5,21].

(1)⇔(3) Since any upper set is open in $A(P)$ and $A(P)_{SI_2} = \sigma_2(P)$, by Proposition 5.2, the condition (1) and (3) are equivalent. □

Lemma 5.4 *If H is a finite set in a meet SI$_2$-continuous space (X,τ), then $int_{\tau_{SI_2}} \uparrow H \subseteq \bigcup\{\Uparrow_{SI_2} x : x \in H\}$.*

Proof. Suppose $y \in U := int_{\tau_{SI_2}} \uparrow H$ but $y \notin \bigcup\{\Uparrow_{SI_2} x : x \in H\}$. Let $H = \{x_1, x_2, ..., x_n\}$. For each i there exists $F_i \in Irr_\tau(X)$ such that $y \in F_i^\delta$ with $x_i \notin \downarrow F_i$. Since (X,τ) is meet SI$_2$-continuous, $y \in cl_{\tau_{SI_2}}(\downarrow y \cap \downarrow F_i)$. Choose $z_1 \in U \cap \downarrow y \cap \downarrow F_1 \neq \emptyset$. Since $y \in F_2^\delta$ and $z_1 \leq y$, we have $z_1 \in F_2^\delta$, which implies that $z_1 \in cl_{\tau_{SI_2}}(\downarrow z_1 \cap \downarrow F_2)$. Choose $z_2 \in U \cap \downarrow z_1 \cap \downarrow F_2 \neq \emptyset$. Thus we can get $z_{i+1} \in U \cap \downarrow z_i \cap \downarrow F_{i+1}, i = 1, 2, \ldots, n-1$, and $z_n \in \bigcap_{i=1}^n \downarrow F_i$. Note that $z_n \in U \subseteq \uparrow H$, so there exists $x_j \in H$ such that $x_j \leq z_n$. Thus $x_j \in \downarrow F_j$, a contradiction to $x_i \notin \downarrow F_i$ for any $i \in \{1, 2, \ldots, n\}$. Hence $int_{\tau_{SI_2}} \uparrow H \subseteq \bigcup\{\Uparrow_{SI_2} x : x \in H\}$. □

Theorem 5.5 *Let (X, τ) be a T_0 space. Then the following conditions are equivalent.*

(1) *(X, τ) is an SI$_2$-continuous space;*

(2) *(X, τ) is an SI$_2$-quasicontinuous and meet SI$_2$-continuous space;*

(3) *(X, τ) is a meet SI$_2$-continuous space, for any $x \in X$, $\Downarrow_{SI_2} x$ is directed with $\Uparrow_{SI_2} x \in \tau$ and whenever $x \not\leq y$ in X, then there are $U \in \tau_{SI_2}$ and $V \in \omega(P)$ such that $x \in U, y \in V$, and $U \cap V = \emptyset$, where $P = (X, \leq_\tau)$.*

Proof. (1)⇒(2) By Corollary 4.10, (X, τ) is SI$_2$-quasicontinuous. Now we prove that (X, τ) is meet SI$_2$-continuous. Consider any $x \in X$ and any $F \in Irr_\tau(X)$ with $x \in F^\delta$. For any $U \in \tau_{SI_2}$ with $x \in U$, by SI$_2$-continuity of (X, τ), $x \in U \cap (\Downarrow_{SI_2} x)^\delta \neq \emptyset$, which implies $U \cap \Downarrow_{SI_2} x \neq \emptyset$, that is, there exists $u \in U$ such that $u \ll_{SI_2} x$. Thus there exists $e \in F$ such that $u \leq e$, so $u \in U \cap \downarrow x \cap \downarrow F$. Hence $x \in cl_{\tau_{SI_2}}(\downarrow x \cap \downarrow F)$.

(2)⇒(3) First, we show that $\Downarrow_{SI_2} x$ is nonempty for any $x \in X$. If not, then $x \notin \bigcup\{\Uparrow_{SI_2} y : y \in X\} \subseteq \bigcup\{\Uparrow_{SI_2} E : E \in X^{(<\omega)}\}$. By Proposition 4.8(1) and Lemma 5.4, $\Uparrow_{SI_2} E = int_{SI_2} \uparrow E \subseteq \bigcup\{\Uparrow_{SI_2} y : y \in E\}$. From Proposition 4.8(2), it is follows that $X = \bigcup\{\Uparrow_{SI_2} E : E \in X^{(<\omega)}\} \subseteq \bigcup\{\bigcup \Uparrow_{SI_2} y : E \in X^{(<\omega)}$ and $y \in E\} = \bigcup\{\Uparrow_{SI_2} y : y \in X\}$, which implies $x \notin X$, a contradiction. Thus $\Downarrow_{SI_2} x$ is nonempty.

Now we show that $\Downarrow_{SI_2} x$ is directed and $\Uparrow_{SI_2} x \in \tau$. For any $x \in X$, let $u, v \in \Downarrow_{SI_2} x$. Since (X, τ) is SI$_2$-quasicontinuous, by Theorem 4.7, there are $E_1, E_2 \in X^{(<\omega)}$ such that $u \ll_{SI_2} E_1 \ll_{SI_2} x, v \ll_{SI_2} E_2 \ll_{SI_2} x$. Thus $E_1, E_2 \in \text{fin}(x)$ and $E_1 \subseteq \uparrow u, E_2 \subseteq \uparrow v$. Since $\text{fin}(x)$ is directed, there exists $E \in \text{fin}(x)$ such that $E_1, E_2 \leq E$, i.e., $E \subseteq \uparrow E_1 \cap \uparrow E_2$. Thus $x \in \Uparrow_{SI_2} E \subseteq \uparrow E \subseteq \uparrow E_1 \cap \uparrow E_2$. By Proposition 4.8(1) and Lemma 5.4, we have $\Uparrow_{SI_2} E = int_{SI_2} \uparrow E \subseteq \bigcup\{\Uparrow_{SI_2} y : y \in E\}$. So there is a $y \in E$ such that $x \in \Uparrow_{SI_2} y$, i.e., $y \in \Downarrow_{SI_2} x$ and $u, v \leq y$. Thus $\Downarrow_{SI_2} x$ is directed. Since (X, τ) is SI$_2$-quasicontinuous, $\Uparrow_{SI_2} E \in \tau$ for any finite E. Therefore, $\Uparrow_{SI_2} x \in \tau$.

Suppose that $x \not\leq y$ in X. Let $P = (X, \leq_\tau)$. Then $x \in P \setminus \downarrow y \in \tau_{SI_2}$. By Theorem 4.9, there exists $E \in X^{(<\omega)}$ such that $x \in int_{\tau_{SI_2}} \uparrow E \subseteq \uparrow E \subseteq P \setminus \downarrow y$. Let $U = int_{\tau_{SI_2}} \uparrow E$ and $V = P \setminus \uparrow E$. Then $x \in U \in \tau_{SI_2}$, $y \in V \in \omega(P)$ and $U \cap V = \emptyset$.

(3)⇒(1) We only have to check that $x = \vee \Downarrow_{SI_2} x$ for all $x \in X$. Let y be any upper bound of $\Downarrow_{SI_2} x$. Assume $x \not\leq y$. By the condition (3), there are $U \in \tau_{SI_2}$ and $V \in \omega(P)$ such that $x \in U, y \in V$ and $U \cap V = \emptyset$. We may assume that V is a basic ω-open set, i.e., there exists $H \in X^{(<\omega)}$ such that $V = P \setminus \uparrow H$. Thus $U \subseteq \uparrow H$. By Lemma 5.4, $x \in U \subseteq int_{\tau_{SI_2}} \uparrow H \subseteq \bigcup\{\Uparrow_{SI_2} : x \in H\}$. Hence, there is a $z \in H$ such that $z \ll_{SI_2} x$. Thus $z \in \Downarrow_{SI_2} x \subseteq \downarrow y$, which implies $y \in \uparrow H$, a contradiction. Thus (1) holds. □

Let P be a poset. By Corollary 3.15, Corollary 4.13 and Theorem 5.5, we have the following corollary.

Corollary 5.6 ([21]) *Let P be a poset. Then the following conditions are equiva-*

lent.

(1) *P is an s_2-continuous poset;*

(2) *P is an s_2-quasicontinuous and meet s_2-continuous poset;*

(3) *P is a meet s_2-continuous space, $\Downarrow x$ is directed for any $x \in P$, and whenever $x \nleq y$ in X, then there are $U \in \sigma_2(P)$ and $V \in \omega(P)$ such that $x \in U, y \in V$, and $U \cap V = \emptyset$.*

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