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Universal Approach to $Z-$frame Envelopes of Semilattices

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Abstract

In this paper we deal with systems of sets on the category of all semilattices. Based on the result of $Z-$frame freely generated by $Z-$sites studied in [9], we prove that every semilattice admits a $Z-$frame envelope. It is a universal approach for any system $\mathbf{Z}$ of sets. We define and study a new continuity named Z_a-continuty on semilattice and prove that in some specific systems of sets (Low, Fin and Idl), every Z_a-continuous semilattice admits a $Z-$continuous envelope.

Keywords: semilattice; $Z-$frame envelope; admissible $Z-$set; $Z-$continuous semilattice.

1 Introduction

In [1], Gehrke and Van Gool developed a universal construction which associates to arbitrary lattice two distributive lattice envelopes with a Galois connection between them. With such a construction, they establish a topological duality for bounded lattice. Of course, some of the results about this construction can be regarded as finitary versions of the results on injective hull of semilattices of Burns and Lakser [2]. The key notion to these constructions are both admissibility, one is finitely

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version and another is arbitrary version. It is worth noting that the distributive envelope of a semilattice $D^{\wedge}(L)$ in [1] is a kind of $k-$frame and the injective hull of a semilattice in [2] is actually a frame, hence both of them are $Z-$frames in the sense of [9] and [10]. Just as Gehrke and Van Gool have pointed out in [1] that it would be interesting to see whether the similar constructions of distributive envelopes or injective hulls could be obtained when we consider the system of sets **Z** on semilattices.

In this paper, we shall generalize the notion of admissible sets to $Z-$admissible set on semilattices. As a result, the concept of $Z-$frame envelope will be defined. With the relevant results about $Z-$site and freely generated $Z-$frame introduced in [9], We prove that every semilattice admits a $Z-$frame envelope. Actually, our method to $Z-$frame envelopes is a universal approach, and the results obtained here can be applied smoothly to any $Z-$frame, like $k-$frame, frame and preframe (which is also called meet-continuous semilattice) [5, 7]. Some examples will be used to illustrate this point. A new concept of Z_a-continuous semilattice will also be introduced, this continuity is weaker than the notion of $Z-$continuity studied in [10]. We prove that for some specific systems of sets, e.g. Low, Fin and Idl, the $Z-$frame envelopes of a Z_a-continuous semilattice can be characterized in a special way and shown to satisfy $Z-$continuity.

2 Z-frame and Z-continuity

In the following, by a semilattice we shall mean a finite-meet semilattice. A semilattice homomorphism $f : A \to B$ is a function from a semilattice A to a semilattice B which preserves fnite meets. Let **Slat** denote the category of all semilattices and semilattice homomorphisms.

A set system **Z** on **Slat** is a function which assigns to each semilattice A a collection $\mathbf{Z}(A)$ of subsets of A, such that the following conditions are satisfied:

$(Z1)$ every element of $\mathbf{Z}(A)$ is downset, the intersection of two elements of $\mathbf{Z}(A)$ is still in $\mathbf{Z}(A)$ and $\downarrow a \in \mathbf{Z}(A)$ for all $a \in A$;

$(Z2)$ for any $\mathcal{A} \in \mathbf{Z}(\mathbf{Z}(A))$, $\cup\mathcal{A} \in \mathbf{Z}(A)$;

$(Z3)$ for any semilattice homomorphism $f : A \to B$, if $D \in \mathbf{Z}(A)$ then $\downarrow f(D) \in \mathbf{Z}(B)$.

A subset $T \subseteq A$ is called a $Z-$set if $\downarrow T = \{x \in A : x \leq t$ holds for some $t \in T\}$ is an element of $Z(A)$. A semilattice A is said to be $Z-$complete if every element of $Z(A)$ has a join. This is equivalent to that $\vee T$ exists for each $Z-$set T.

A $Z-$complete semilattice A is called a $Z-$frame if the following equation

$$a \wedge \vee T = \vee\{a \wedge x : x \in T\}$$

holds for any $a \in A$ and $Z-$set T of A.

A $Z-$frame homomorphism $f : A \to B$ is a semilattice homomorphism which preserves joins of $Z-$sets, that is,

$$f(\vee T) = \vee f(T)$$

holds for every $Z-$set T of A.

The $Z-$below relation $\ll_z$ on a semilattice A is given by $x \ll_z y$ iff for each $Z-$set T, $a \leq \vee T$ implies the existence of a $t \in T$ such that $x \leq t$. We write $\Downarrow_z x$ for the set $\{y \in A : y \ll_z x\}$.

A semilattice A is called $Z-$continuous if it is $Z-$complete and satisfies the following condition:

$$\text{For each } x \in A, \Downarrow_z x \text{ is a } Z-\text{set}, \vee \Downarrow_z x \text{ exists and } a = \vee \Downarrow_z x.$$

The relation $\ll_z$ in a $Z-$continuous semilattice A is interpolating, that is, if $x \ll_z y$ then there is $z \in A$ with $x \ll_z z \ll_z y$.

Example 2.1 (1) Let Low be the system of sets which assigns to a semilattice A a collection

$$Low(A) = \{\downarrow S : S \subseteq A\}.$$

Then A is a $Low-$frame if and only if it is a frame [2]. And A is $Low-$continuous if and only if it is completely distributive lattice [6].

(2) Let Fin be the system of sets which assigns to a semilattice A a collection

$$Fin(A) = \{\downarrow F : F \subseteq A, F \text{ finite}\}.$$

Then $Fin-$frame are exactly the distributive lattice, i.e., a kind of $k-$frame [1], and the $F-$continuous semilattices are the freely generated join-semilattice (in which every element is a join of finitely many co-primes).

(3) Let Idl be the system of sets which assigns to a semilattice A a collection

$$Idl(A) = \{\downarrow D : D \subseteq A, D \text{ directed}\}.$$

Then A is a $Idl-$frame if and only if it is a preframe in the sense of [7]. Preframe are also called meet-continuous semilattice. The $Idl-$continuous semilattice are the continuous semilattice in the sense of [8].

3 $Z-$frame envelopes

In [11] Johnstone introduced the method of sites and coverages. By this method, he constructed the frame freely generated by a site, and used this theory to describe the structure of coproducts of frames. The method has wider applications. In [9], more generalized notions and results have also been obtained for $Z-$frame.

Given a set system **Z** on **Slat**, by a $Z-$coverage C on a semilattice A we mean a function C which assigns to each element $a \in A$ a collection $C(a)$ of subsets of A satisfying the conditions:

(i) each $S \in C(a)$ is a $Z-$set of A and $S \subseteq \downarrow a$;

(ii) if $S \in C(a)$ and $b \leq a$, then $\{x \wedge b : x \in S\} \in C(b)$.

A $Z-$site is a pair (A, C) with A a semilattice and C a $Z-$coverage on A.

Let (A, C) be a $Z-$site, a function $f : A \to B$ from A to a $Z-$frame B is said to transform covers to joins if for any $a \in A$ and $S \in C(a)$,

$$f(a) = \vee f(S).$$

A $Z-$frame L is freely generated by a $Z-$site (A, C) if there is a semilattice homomorphism $g : A \to L$ which transforms covers to joins and is universal in the sense that for any semilattice homomorphism $f : A \to B$ from A to a $Z-$frame B, if f transforms covers to joins, then there is a unique $Z-$frame homomorphism $\tilde{f} : L \to B$ with $f = \tilde{f} \circ g$, i.e., the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[dr, "f"'] & L \arrow[d, "!\tilde{f}"] \\ & B \end{tikzcd}$$

The next theorem shows that for each $Z-$site (A, C) there is a $Z-$frame freely generated by (A, C). For the details of the theorem, please refer to the thesis [9].

Theorem 3.1 *For any $Z-$site (A, C) there is a $Z-$frame freely generated by (A, C).*

In [2], a subset M of a semilattice is called admissible if $\vee M$ exists, and the equation $\vee\{a \wedge x : x \in M\} = a \wedge \vee M$ holds for any element a of the semilattice. In [1], the finitary version concept of admissible subsets have also been introduced. Now, we define it in a uniform manner and we will see that this concept plays a central role in the whole paper.

Definition 3.2 Let A be a semilattice. A subset M of A is $Z-$admissible if M satisfies the conditions:

(i) M is a $Z-$set and $\vee M$ exists;

(ii) for each $a \in A$, $\vee\{a \wedge m : m \in M\}$ exists and $\vee\{a \wedge m : m \in M\} = a \wedge \vee M$.

We say that a function $f : A \to B$ from a semilattice A to a $Z-$frame B preserves $Z-$admissible joins if for each $Z-$admissible set $M \subseteq A$, we have $f(\vee M) = \vee f(M)$.

Definition 3.3 Let A be a semilattice. An embedding $\eta^{\wedge}_A : A \to E^{\wedge}(A)$ of A into a $Z-$frame $E^{\wedge}(A)$ which preserves meets and $Z-$admissible joins is a $Z-$frame envelope of A if it satisfies the following universal property:

for any semilattice homomorphism $f : A \to B$ from A to a $Z-$frame B, if f preserves $Z-$admissible joins, then there is a unique $Z-$frame homomorphism $\hat{f} : E^{\wedge}(A) \to B$ with $f = \hat{f} \circ \eta^{\wedge}_A$, i.e., the following diagram commutes.

$$\begin{tikzcd} A \arrow[r, "\eta^{\wedge}_A"] \arrow[dr, "f"'] & E^{\wedge}(A) \arrow[d, "!\hat{f}"] \\ & B \end{tikzcd}$$

Lemma 3.4 *(1) If M is a $Z-$admissible set of semilattice A and $a \in A$, then $\{a \wedge m : m \in M\}$ is again a $Z-$admissible set of A.*

(2) If M_1, M_2 are two $Z-$admissible sets of semilattice A, then $\{m_1 \wedge m_2 : m_1 \in M_1, m_2 \in M_2\}$ is again an $Z-$admissible set of A.

Proof. (1) It is clear that $\downarrow a \cap \downarrow M = \downarrow \{a \wedge m : m \in M\}$. Then $\{a \wedge m : m \in M\}$

is a $Z-$set since $\downarrow a$ and $\downarrow M$ are both $Z-$ideals of A. For each $x \in A$,

$$\begin{aligned} x \wedge \vee\{a \wedge m : m \in M\} &= x \wedge (a \wedge \vee M) \\ &= (x \wedge a) \wedge \vee M \\ &= \vee\{x \wedge a \wedge m : m \in M\}, \end{aligned}$$

which shows that $\{a \wedge m : m \in M\}$ is $Z-$admissible.

(2) It is clear that $\downarrow M_1 \cap \downarrow M_2 = \downarrow \{m_1 \wedge m_2 : m_1 \in M_1, m_2 \in M_2\}$. Then $\{m_1 \wedge m_2 : m_1 \in M_1, m_2 \in M_2\}$ is a $Z-$set, since $\downarrow M_1$ and $\downarrow M_2$ are both $Z-$ideals of A. For each $x \in A$,

$$\begin{aligned} x \wedge \vee\{m_1 \wedge m_2 : m_1 \in M_1, m_2 \in M_2\} &= x \wedge \vee_{m_1 \in M_1}(m_1 \wedge \vee M_2) \\ &= x \wedge \vee M_1 \wedge \vee M_2 \\ &= \vee_{m_1 \in M_1}(x \wedge m_1) \wedge \vee M_2 \\ &= \vee_{m_2 \in M_2} \vee_{m_1 \in M_1} (x \wedge m_1 \wedge m_2), \end{aligned}$$

which shows that $\{m_1 \wedge m_2 : m_1 \in M_1, m_2 \in M_2\}$ is $Z-$admissible. □

Theorem 3.5 *Every semilattice admits a $Z-$frame envelop.*

Proof. Let A be a semilattice. For any $a \in A$, set

$$C(a) = \{M \subseteq A : M \text{ is } Z\text{−admissible and } \vee M = a\}.$$

By Lemma 3.4(1), the function C is a $Z-$coverage of A and hence (A, C) is a $Z-$site. Then, by Theorem 3.1, there exists a $Z-$frame L freely generated by the $Z-$site (A, C). Suppose $g : A \to L$ is the universal map which transform covers to joins. We show that $g : A \to L$ is a $Z-$frame envelop of A. For any $Z-$admissible set $M \subseteq A$, let $a = \vee M$, then $M \in C(a)$ and hence

$$g(\vee M) = g(a) = \vee g(M).$$

This means that the semilattice homomorphism g preserves $Z-$admissible joins. Now suppose $f : A \to B$ is a semilattice homomorphism to a $Z-$frame B and that f preserves $Z-$admissible joins. Then f transforms covers to joins since for each $a \in A$ and $M \in C(a)$, $f(a) = f(\vee M) = \vee f(M)$. Thus, by the universal property of $g : A \to L$, there exists a unique $Z-$frame homomorphism $\hat{f} : L \to B$ such that $f = \hat{f} \circ g$. By the Definition 3.3, we can conclude that the embedding $g : A \to L$ is a $Z-$frame envelope of A. □

Example 3.6 (1) Let *Low* be the system of set which assigns to a semilattice A of a collection $Low(A)$ of all downsets of A. Suppose $\eta_A^\wedge : A \to E^\wedge(A)$ is a $Z-$frame envelope of a semilattice A. By Example 2.1(1), $E^\wedge(A)$ is a frame and it is easy to see that $E^\wedge(A)$ is an injective hull of A in the sense of [2].

(2) Let Fin be the system of set which assigns to a semilattice A of a collection $Fin(A) = \{\downarrow F : F \subseteq S, F \text{ finite}\}$. Suppose $\eta_A^\wedge : A \to E^\wedge(A)$ is a $Z-$frame envelope of semilattice A. By Example 2.1(2), $E^\wedge(A)$ is a distributive lattice and $\eta_A^\wedge : A \to E^\wedge(A)$ is a distributive $\wedge-$envelope of A in the sense of [1].

(3) Let Idl be the system of set which assigns to a semilattice A of a collection $Idl(A)$ of all ideals of A. Suppose $\eta_A^\wedge : A \to E^\wedge(A)$ is a $Z-$frame envelope of semilattice A. By Example 2.1(3), we have $E^\wedge(A)$ is a preframe (meet-continuous semilattice).

With Theorem 3.5 and Example 3.6, it is clear that our approach to $Z-$frame envelops of semilattice is indeed a universal approach for any system of sets.

4 Z_a-ideals

The notions of $a-$ideals and $D-$ideals of semilattices are introduced in [1] and [2] respectively. The set of all finite generated $a-$ideal of a semilattice is a distributive lattice and forms a distributive $\wedge-$envelope of the semilattice. Similarly, the set of all $D-$ideal forms a frame and isomorphic to any injective hull of a semilattice. In this section, we will generalize the notions of $a-$ideals and $D-$ideals to a more general concept, named Z_a-ideals and investigate some basic properties of them.

Definition 4.1 Let A be a semilattice. A subset $I \subseteq A$ is a Z_a-ideal if it satisfies the followings:

(i) $I = \downarrow I$, i.e., I is a downset;

(ii) if $M \subseteq I$ is $Z-$admissible then $\vee M \in I$, i.e., I is closed under $Z-$admissible joins.

Each principle ideal $\downarrow a$ is Z_a-ideal. The set of all Z_a-ideals of A denoted by $I_{z,a}(A)$ is a closure system since any intersection of Z_a-ideals is again a Z_a-ideal. Therefore, for any $Z-$set T, there is a smallest Z_a-ideal containing T, denoted by $\langle T\rangle$, which is called the Z_a-ideal generated by T. And the set of all generated Z_a-ideals is denoted by $GI_{z,a}(A)$, i.e.,

$$GI_{z,a}(A) = \{\langle T\rangle : T \in \mathbf{Z}(A)\}.$$

Remark 4.2 (1) Let Low be the system of sets which assigns to a semilattice A of a collection $Low(A)$ of all downsets of A. Then a subset $I \subseteq A$ is a Z_a-ideal iff I is a $D-$ideal of A in the sense of [2]. Therefore $GI_{z,a}(A)$ is the set of all $D-$ideals of A, by Corollary 2 of [2], we have that $GI_{z,a}(A)$ is isomorphism to any injective hull of semilattice A.

(2) Let Fin be the system of sets which assigns to a semilattice A of a collection $Fin(A)$. Then a subset $I \subseteq A$ is a generated Z_a-ideal iff I is a finitely generated a-ideal of A in the sense of [1]. So, by Theorem 3.9 of [1], $GI_{z,a}(A)$ forms a distributive envelope of A.

In general, given a system of sets $\mathbf{Z}$ and a semilattice A, it is difficult to prove that whether $GI_{z,a}(A)$ always forms a $Z-$frame envelope of A. We will try to solve this issue for some specific cases in next section. Here we give some simple facts of the generated Z_a-ideals.

Lemma 4.3 *Let A be a semilattice, M be a $Z-$admissible subset of A, T be a $Z-$set of A. Then $M \subseteq \downarrow T$ and $\vee M = \vee T$ implies T is also $Z-$admissible.*

Proof. For any $a \in A$, $a \wedge \vee T$ is an upper bound of $\{a \wedge t : t \in T\}$. Clearly, it is also the least upper bound of $\{a \wedge t : t \in T\}$ since $a \wedge \vee T = a \wedge \vee M = \vee_{m \in M}(a \wedge m)$.□

Proposition 4.4 *Let A be a semilattice, T be a $Z-$set of A. Then*

$$\langle T \rangle = \{x \in A : x = \vee M \text{ for some } Z\text{−admissible subset } M \subseteq \downarrow T\}.$$

Proof. Let $J = \{x \in A : x = \vee M$ for some $Z-$admissible subset $M \subseteq \downarrow T\}$. Suppose I is a Z_a-ideal containing T. Then $\downarrow T \subseteq I$. Thus $J \subseteq I$ since I is closed under $Z-$admissible joins. So $J \subseteq \langle T \rangle$.

Conversely, we need to show that J is a Z_a-ideal of A. For any $x \in A$, if $x \leq y$ for some $y \in J$ then there exists $Z-$admissible subset $M \subseteq \downarrow T$ such that $x \leq \vee M$. Thus $x = x \wedge \vee M = \vee\{x \wedge m : m \in M\}$. By Lemma 3.4(1), $\{x \wedge m : m \in M\} \subseteq \downarrow T$ is also $Z-$admissible. Thus $x \in J$ and hence J is a downset. Assume that $N \subseteq J$ is $Z-$admissible. Then for any $n \in N$, there is $Z-$admissible subset $M_n \subseteq \downarrow T$ and $n = \vee M_n$. By Lemma 4.3 we can take $M_n = \downarrow n \cap \downarrow T$. Let $M = \cup_{n \in N} M_n$. We show that M is $Z-$admissible. Let $j_T : A \to Z(A)$ with $j_T(x) = \downarrow x \cap \downarrow T$. Then for any $x, y \in A$,

$$j_T(x \wedge y) = \downarrow (x \wedge y) \cap \downarrow T = \downarrow x \cap \downarrow y \cap \downarrow T = (\downarrow x \cap \downarrow T) \cap (\downarrow y \cap \downarrow T) = j_T(x) \cap j_T(y),$$

it means that j_T is a function between semilattices which preserves binary meets. Thus $j_T(N) = \{\downarrow n \cap \downarrow T : n \in N\}$ is a $Z-$set of $Z(A)$. By (Z2),

$$M = \cup_{n \in N} M_n = \cup\{\downarrow n \cap \downarrow T : n \in N\}$$

is a $Z-$set of A. Moreover, for any $a \in A$,

$$\begin{aligned} a \wedge \vee M &= a \wedge \vee N = \vee_{n \in N}(a \wedge n) \\ &= \vee_{n \in N}(a \wedge \vee M_n) = \vee_{n \in N}(\vee_{m \in M_n}(a \wedge m)) = \vee_{m \in M}(a \wedge m), \end{aligned}$$

thus M is $Z-$admissible. Note that $M \subseteq \downarrow T$ and $\vee N = \vee M \in J$. Then J is closed under the $Z-$admissible joins, and hence a Z_a-ideal. □

Proposition 4.5 *Let A be a semilattice. Then $GI_{z,a}(A)$ is again a semilattice.*

Proof. Suppose $\langle T_1 \rangle, \langle T_2 \rangle \in GI_{z,a}(A)$. We claim that $\langle T_1 \rangle \cap \langle T_2 \rangle$ is a Z_a-ideal and generated by the $Z-$set $T = \{t_1 \wedge t_2 : t_1 \in T_1, t_2 \in T_2\}$. Since $\downarrow T = \downarrow \{t_1 \wedge t_2 : t_1 \in T_1, t_2 \in T_2\} = \downarrow T_1 \cap \downarrow T_2 \in Z(A)$, then T is a $Z-$set of A. To prove $\langle T_1 \rangle \cap \langle T_2 \rangle = \langle T \rangle$. It is clear that $\langle T_1 \rangle \cap \langle T_2 \rangle \supseteq \langle T \rangle$.

Conversely, for each $x \in \langle T_1 \rangle \cap \langle T_2 \rangle$, by Proposition 4.4, there exist $Z-$admissible subset $M_1 \subseteq \downarrow T_1$, $M_2 \subseteq \downarrow T_2$ such that $x = \vee M_1 = \vee M_2$. By Lemma 3.4(2), $M = \{m_1 \wedge m_2 : m_1 \in M_1, m_2 \in M_2\}$ is $Z-$admissible. Note that

$$M \subseteq \downarrow M_1 \cap \downarrow M_2 \subseteq \downarrow T_1 \cap \downarrow T_2 = \downarrow T$$

and $\vee M = x$ then $x \in \langle T\rangle$. Thus $\langle T_1\rangle \cap \langle T_2\rangle = \langle \downarrow T_1 \cap \downarrow T_2\rangle = \langle T\rangle$ and hence $GI_{z,a}(A)$ is a semilattice. □

5 Z_a-continuity

In this section, we define a weaker continuity called Z_a-continuity on semilattices. Given a semilattice A, we will prove that in some specific set systems $\mathbf{Z}$ on **Slat**, for example Low, Fin and Idl, the set of all generated Z_a-ideals $GI_{z,a}(A)$ is a $Z-$continuous $Z-$frame and forms a $Z-$frame envelope if A is a Z_a-continuous semilattice.

Definition 5.1 A semilattice is called Z_a-continuous if it satisfies the following condition:

For each $x \in A$, $\Downarrow_{z,a} x$ is $Z-$admissible and $a = \vee \Downarrow_{z,a} x$,

where $\Downarrow_{z,a} x = \{y \in A : y \ll_{z,a} x\}$ and the binary relation $\ll_{z,a}$ is defined by $x \ll_{z,a} y$ iff for each $Z-$admissible set M, $a \le \vee M$ implies the existence of an $m \in M$ such that $x \le m$.

Remark 5.2 (1) For each $x, y \in A$, $y \ll_z x$ implies $y \ll_{z,a} x$, and $y \ll_{z,a} x$ implies $y \le x$.

(2) Each $Z-$continuous semilattice is Z_a-continuous but not conversely.

Example 5.3 Let $L = \{c_0, c_1, \cdots, c_n, \cdots\} \cup \{a, b\}$. The partial order on L is defined as $c_0 \le c_1 \le \cdots \le c_n \le \cdots \le a$ and $c_0 \le b \le a$ (Fig.1). Given $\mathbf{Z} = Idl$, we have L is not $Z-$continuous since L is not meet continuous. And note that the directed set $\{c_0, c_1, \cdots, c_n, \cdots\}$ is not admissible, so, it is readily verified that L is Z_a-continuous .

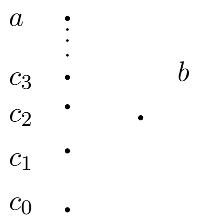

Fig.1

In the remainder of this section, for convenience, we always assume that $\mathbf{Z} = Idl$. And it is clear that the results obtained here for Idl still hold for systems of sets Low and Fin.

Proposition 5.4 *Let A be a Z_a-continous semilattice, $x, z \in A$. Then*

$$x \ll_{z,a} z \Longrightarrow (\exists y \in A)(x \ll_{z,a} y \ll_{z,a} z)$$

Proof. Let $I = \cup\{\Downarrow_a y : y \ll_{z,a} z\}$. Since A is Z_a-continuous, then I is directed and $\vee I = z$. We now prove that I is admissible. Suppose $a \in A$, then

$$\begin{aligned} a \wedge (\vee I) &= a \wedge (\vee \Downarrow_{z,a} z) = \vee_{y \ll_{z,a} z}(a \wedge y) \\ &= \vee_{y \ll_{z,a} z}(a \wedge \vee_{t \ll_{z,a} y} t) = \vee_{y \ll_{z,a} z} \vee_{t \ll_{z,a} y} (a \wedge t) \\ &= \vee_{t \in I}(a \wedge t). \end{aligned}$$

Thus I is directed and admissible with $x \ll_{z,a} z = \vee I$, so there exists $y \in \Downarrow_{z,a} z$ such that $x \leq t \ll_{z,a} y$ for some t, hence $x \ll_{z,a} y \ll_{z,a} z$. □

Proposition 5.5 *Let A be a Z_a-continuous semilattice, T be a $Z-$set (directed subset) of A. Then*

$$\langle T \rangle = \{x \in A : \Downarrow_{z,a} x \subseteq \cup_{t \in T} \Downarrow_{z,a} t\} = \langle \cup_{t \in T} \Downarrow_{z,a} t \rangle.$$

Proof. For the first equal sign, it suffices to show that

$$I = \{x \in A : \Downarrow_{z,a} x \subseteq \cup_{t \in T} \Downarrow_{z,a} t\}$$

is a Z_a-ideal. Clearly, it is a downset. Suppose $M \subseteq I$ is directed and admissible and $y \ll_{z,a} \vee M$, by Proposition 5.4, there is $m_0 \in M \subseteq I$ such that $y \ll_{z,a} m_0$. Thus

$$\Downarrow_{z,a} (\vee M) \subseteq \cup_{m \in M} \Downarrow_{z,a} m \subseteq \cup_{t \in T} \Downarrow_{z,a} t.$$

This means that $\vee M \in I$ and hence I is closed under directed-admissible joins.

The second equal sign is clear, since $\cup_{t \in T} \Downarrow_{z,a} t$ is directed, $\cup_{t \in T} \Downarrow_{z,a} t \subseteq \downarrow T$ and $t = \vee \Downarrow_{z,a} t$ for any $t \in T$. □

Corollary 5.6 *Let A be a Z_a-continuous semilattice, T_1, T_2 be $Z-$admissible (Idl-admissible) subsets of A. Then $\langle T_1 \rangle = \langle T_2 \rangle$ implies $\cup_{t \in T_1} \Downarrow_{z,a} t = \cup_{t \in T_2} \Downarrow_{z,a} t$.*

Proof. Suppose $t_1 \in T_1$, then $t_1 \in \langle T_2 \rangle$ by $T_1 \subseteq \langle T_1 \rangle = \langle T_2 \rangle$. By Proposition 5.5, we have $\Downarrow_{z,a} t_1 \subseteq \cup_{t \in T_2} \Downarrow_{z,a} t$. Since t_1 is arbitrary in T_1 then $\cup_{t \in T_1} \Downarrow_{z,a} t \subseteq \cup_{t \in T_2} \Downarrow_{z,a} t$. The other inclusion can be proved in the same way. □

Proposition 5.7 *Let A be a Z_a-continuous semilattice. Then*

$$GI_{z,a}(A) = \{\langle T \rangle : T \subseteq A \text{ directed } \}$$

is a continuous semilattice.

Proof. By Proposition 4.5, $GI_{z,a}(A)$ is a semilattice. Next we show that $GI_{z,a}(A)$ is continuous. Suppose $D = \{\langle T_i \rangle : i \in I\} \subseteq GI_{z,a}(A)$ is directed. It follows from Corollary 5.6 that $\{\cup_{t \in T_i} \Downarrow_{z,a} t : i \in I\}$ is directed, and hence $\cup_{i \in I}(\cup_{t \in T_i} \Downarrow_{z,a} t)$ is directed. Note that $\langle T_i \rangle \leq \langle \cup_{i \in I}(\cup_{t \in T_i} \Downarrow_{z,a} t) \rangle$ for all $i \in I$, and if $\langle T \rangle \geq \langle T_i \rangle$ for all $i \in I$, then $\cup_{i \in I}(\cup_{t \in T_i} \Downarrow_{z,a} t) \subseteq T$, thus $\langle \cup_{i \in I}(\cup_{t \in T_i} \Downarrow_{z,a} t) \rangle \leq \langle T \rangle$. So we have $\langle \cup_{i \in I}(\cup_{t \in T_i} \Downarrow_{z,a} t) \rangle = \vee D$, which means that $GI_{z,a}(A)$ is a dcpo.

Now we turn to prove the continuity. Suppose $T \subseteq A$ and $D = \{\langle T_i \rangle : i \in I\}$ is directed with $\langle T \rangle \leq \vee D = \langle \cup_{i \in I}(\cup_{t \in T_i} \Downarrow_{z,a} t) \rangle$. For each $y \in \cup_{t \in T} \Downarrow_{z,a} t$, by Corollary 5.6, we have

$$y \in \cup_{t\in T} \Downarrow_{z,a} t \subseteq \cup\{\Downarrow_{z,a} z : z \in \cup_{i\in I}(\cup_{t\in T_i} \Downarrow_{z,a} t)\}.$$

Then there exists $i \in I$ such that $y \in\Downarrow_{z,a} z$ for some $z \in \cup_{t\in T_i} \Downarrow_{z,a} t$, by Proposition 5.5, we have $y \in \langle T_i\rangle$. This means that $\langle y\rangle \ll \langle T\rangle$ in $GI_{z,a}(A)$. Note that $\{\langle y\rangle : y \in \cup_{t\in T} \Downarrow_{z,a} t\}$ is directed in $GI_{z,a}(A)$ and $\vee\{\langle y\rangle : y \in \cup_{t\in T} \Downarrow_{z,a} t\} = \langle \cup_{t\in T} \Downarrow_{z,a} t\rangle = \langle T\rangle$. Thus $GI_{z,a}(A)$ is a continuous dcpo, hence a continuous semilattice. □

Since continuous semilattice is meet-continuous semilattice, then $GI_{z,a}(A)$ a preframe for Z_a-continuous semilattice A. The next theorem will show that $GI_{z,a}(A)$ forms a preframe envelope of Z_a-continuous semilattice A.

Theorem 5.8 *Let A be a Z_a-continuous semilattice. Then the embedding $\eta_A^\wedge : A \to GI_{z,a}(A)$ is a preframe envelope of A, where $\eta_A^\wedge(a) =\downarrow a$ for all $a \in A$.*

Proof. It is clear that $\eta_A^\wedge(a)$ preserves finite meets and directed-admissible joins. It remains to show that it satisfies the universal property. Let B be a preframe and $f : A \to B$ be a function which preserves finite meets and directed-admissible joins. If $g : GI_{z,a}(A) \to B$ is a preframe homomorphism such that $g \circ \eta_A^\wedge = f$, then for any directed subset $T \subseteq A$, we have

$$g(\langle T\rangle) = g(\vee_{t\in T} \downarrow t) = \vee_{t\in T} g(\eta_A^\wedge(t)) = \vee_{t\in T} f(t).$$

This shows that there is at most one preframe homomorphism $g : GI_{z,a}(A) \to B$ satisfying $g \circ \eta_A^\wedge = f$. Let $\hat{f} : GI_{z,a}(A) \to B$ be the function defined by

$$\hat{f}(\langle T\rangle) = \vee_{t\in T} f(t), \text{ for any directed } T \subseteq A.$$

Note that for any directed subset $T \subseteq A$,

$$\hat{f}(\langle T\rangle) = \vee_{t\in T} f(t) = \vee_{t\in T} f(\vee \Downarrow_{z,a} t) = \vee\{f(x) : x \in \cup_{t\in T} \Downarrow_{z,a} t\}.$$

Thus for any directed subset $T_1, T_2 \subseteq A$, if $\langle T_1\rangle = \langle T_2\rangle$, then $\hat{f}(\langle T_1\rangle) = \hat{f}(\langle T_2\rangle)$ by Corollary 5.6. It shows that the function $\hat{f}$ is well-defined and monotone.

Next we show that $\hat{f}$ is a preframe homomorphism. Let $\langle T_1\rangle, \langle T_2\rangle$ be two elements of $GI_{z,a}(A)$. Then

$$\begin{aligned}\hat{f}(\langle T_1\rangle \wedge \langle T_2\rangle) &= \hat{f}(\langle \downarrow T_1 \cap \downarrow T_2\rangle) = \hat{f}(\langle \{t_1 \wedge t_2 : t_1 \in T_1, t_2 \in T_2\}\rangle) \\ &= \vee\{f(t_1 \wedge t_2) : t_1 \in T_1, t_2 \in T_2\} = \vee\{f(t_1) \wedge f(t_2) : t_1 \in T_1, t_2 \in T_2\} \\ &= (\vee_{t_1\in T_1} f(t_1)) \wedge (\vee_{t_2\in T_2} f(t_2)) = \hat{f}(\langle T_1\rangle) \wedge \hat{f}(\langle T_2\rangle).\end{aligned}$$

Let $\{T_i : i \in I\}$ be a directed subset of $GI_{z,a}(A)$. Then

$$\begin{aligned}\hat{f}(\vee_{i\in I}\langle T_i\rangle) &= \hat{f}(\langle \cup_{i\in I}(\cup_{t\in T_i} \Downarrow_{z,a} t)\rangle) = \vee\{f(x) : x \in \cup_{i\in I}(\cup_{t\in T_i} \Downarrow_{z,a} t)\} \\ &= \vee_{i\in I} \vee \{f(x) : x \in \cup_{t\in T_i} \Downarrow_{z,a} t\} = \vee_{i\in I} \hat{f}(\langle T_i\rangle).\end{aligned}$$

We now can conclude that $\eta_a^\wedge : A \to GI_{z,a}(A)$ is a preframe envelope of A. □

If the set system Idl replaced by Fin and Low, it is routine to obtained similar results.

Corollary 5.9 *Each $Z-$frame envelope of a Z_a-continuous semilattice is $Z-$continuous, here $\mathbf{Z} = Idl, Fin$ or Low.*

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