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# The Steiner bi-objective shortest path problem

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ABSTRACT

In this paper, we introduce the Steiner Bi-objective Shortest Path Problem. This problem is defined on a directed graph $G = (V, A)$, with a subset $T \subset V$ of terminals. Arcs are labeled with travel time and cost. The goal is to find a complete set of efficient paths between every pair of nodes in $T$. The motivation behind this problem stems from data preprocessing for vehicle routing problems. We propose a solution method based on a labeling approach with a multi-objective A* search strategy guiding the search towards the terminals. Computational results based on instances generated from real road networks show the efficiency of the proposed algorithm compared to state-of-art approaches.

## 1. Introduction

The Vehicle Routing Problem (VRP) can be described as the problem of designing a set of routes that start and end at a depot and that visit a number of geographically dispersed locations, called customers. In the standard version of the problem, the road network of the geographic area at hand is not explicitly considered. Instead, a directed graph $G = (T, D)$ is introduced, where $T$ is composed of the depot and the customers, and $D$ represents all the possible connections between these nodes: $D = \{(i, j) : i \in T, j \in T \setminus \{i\}\}$. A weight $c_{ij}$ is then associated with every arc $(i, j) \in D$ to indicate travel costs (distances) between nodes.

Weights $c_{ij}$ are assumed to be precomputed using the road-network structure, which can very easily be done with shortest path algorithms. This can be performed by applying a goal directed search independently for each arc in $D$ (*e.g.*, algorithm A*, Hart et al. Hart et al. (1968)), by solving one-to-all shortest paths starting from each node in $T$ (*e.g.*, Dijkstra's algorithm, Dijkstra (1959)), or by computing all-to-all shortest paths (*e.g.*, Floyd's algorithm, Floyd (1962)). All these algorithms admit a polynomial-time complexity and generally allow a very fast computation of the data in view of the limited number of customers in VRP applications (rarely more than a few hundreds). A large amount of literature exists to accelerate these algorithms for large-scale networks (see, *e.g.*, Bast et al. (2016)), but none of them are really necessary in this context.

In many real-world routing problems however, and in most variants of the VRP, this model is not accurate enough to determine optimal solutions (Ben Ticha et al., 2018). Indeed, road segments usually have at least two attributes, time and distance, and nodes in $T$ may be connected together using many different paths with different trade-offs in distance and time. The VRP with Time Windows (VRPTW) gives a good illustration of this difficulty. In the VRPTW, vehicle routes are constrained by time windows that define the earliest and latest possible starting times of the service for customers. The objective is to minimize the total traveled distance. The min-distance path between any pair of customers $i$ and $j$ should usually be preferred because it minimizes the impact of traveling from $i$ to $j$ on the cost function. However, depending on the arrival time to $i$, a fastest path could sometimes be required to reach customer $j$ on time. As it depends on the complete sequence of customers in the route, deciding which path should be preferred between two successive customers cannot be determined in advance. The preprocessing approach described above is thus not appropriate.

To handle this situation, a more effective model has been proposed by Garaix et al. (2010). It consists in representing road network information with a multigraph. The node set is still $T$ but the arc set is replaced by an arc multiset $D$ constructed as follows. Given two nodes in $T$, a complete set of efficient paths is computed between these two nodes. Then, an arc is added to $D$ for every path in the complete set. Arc weights (travel distance, travel time) are defined to their value in the corresponding path. An example of a multigraph is depicted on Figure 1. In this small example, pairs of nodes are connected by at most two arcs: in some cases the min-cost and min-time paths coincide, in other cases they are distinct and two parallel arcs are added. More generally, the size of the Pareto front, *i.e.* the number of Pareto optimal objective vectors, can be much larger and many parallel arcs could be inserted. All these efficient solutions are essential because they might all offer the best compromise to minimize costs and satisfy time constraints.

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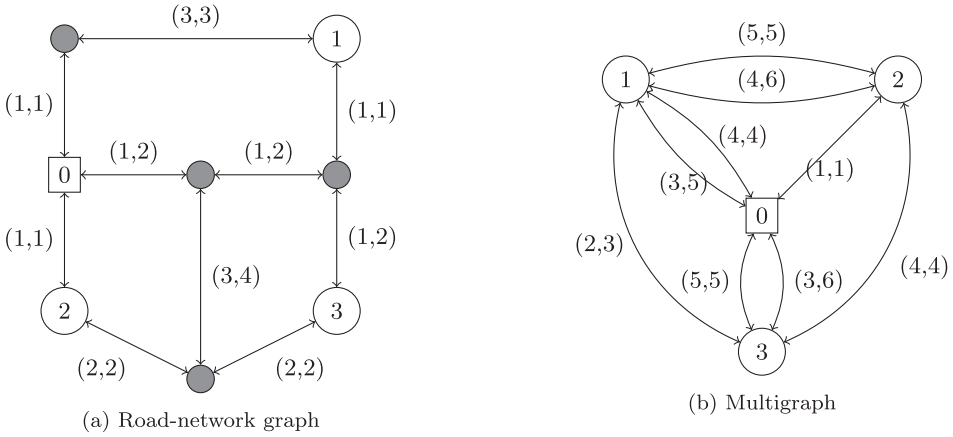


(a) Road-network graph

(b) Multigraph

**Fig. 1.** Road-network graph and multigraph for a simple example with one depot, three customers and two attributes (arcs labeled with (distance,time)).

This new model opens important research perspectives that recently started receiving a strong interest (*e.g.*, Lai et al. (2016), Letchford et al. (2014), Huang et al. (2017), Ben Ticha et al. (2017)). Among others, it opens the question of the tractability of computing the multigraph. Clearly, the arc multiset and their weights cannot be obtained by solving simple shortest path problems. The aim of the Steiner Multi-objective Shortest Path Problem is to deliver this information. It consists in computing complete sets of efficient paths between every pair of nodes in $T$. In what follows, we adopt the standard terminology of Steiner tree problems and call the nodes in $T$ terminals.

In this paper, we introduce the Steiner Multi-objective Shortest Path Problem and propose a solution approach for its solution. We limit our study to the case of two attributes (Steiner Bi-objective Shortest Path Problem, Steiner BSPP). The proposed approach is based on a dynamic programming algorithm with an A* guiding strategy. In addition to the introduction of this new problem, the contributions of the paper are:

- An original implementation of the A* strategy that simultaneously drives the search towards all the terminals.
- An experimental study giving new insights on how efficiently multigraph data can be computed for vehicle routing problems.

The rest of this paper is organized as follows. In Section 2, we present an overview of the related literature. Section 3 formally introduces the Steiner BSPP and useful notation. It also reports some basic properties that will be helpful for the remainder of the paper. In Section 4, we describe our solution method. In Section 5, we propose some enhancements when the problem is addressed in the context of the VRPTW. Finally, numerical results are reported in Section 6.

## 2. Literature review

The Steiner BSPP is related to two types of problems: bi-objective shortest path problems (BSPPs) and "Steiner-like" problems:

- BSPPs are multi-objective extensions of standard shortest path problems (SPPs). Every arc of the graph receives two weights (that correspond to two criteria) and the goal is to generate efficient paths. Three variants can naturally be considered: the one-to-one BSPP, the one-to-all BSPP and the all-to-all BSPP, where efficient paths are searched between one origin and one destination, one origin and all destinations, all origins and all destinations, respectively. All three variants are NP-hard (Serafini, 1987).
- Steiner problems originate from the well-known Steiner Tree Problem. Given a weighted graph $G = (V, A)$ and a set of terminals $T \subset V$, the Steiner tree problem consists in finding a min-cost tree covering all the nodes in $T$. It generalizes the minimum spanning tree problem, where all nodes have to be covered. Following this terminology, the Steiner Traveling Salesman Problem was introduced and investigated by several authors (Cornuéjols et al., 1985; Letchford et al., 2013)). In this problem, a minimum cost cycle visiting all the vertices in $T \subset V$ is sought. It generalizes the Traveling Salesman Problem (case $T = V$). Following the same trend, we called our problem Steiner BSPP because we are only interested in shortest paths between a subset of nodes (the terminals) instead of all-to-all shortest paths. It generalizes the all-to-all shortest path problem ($T = V$) and the one-to-one shortest path problem ($|T| = 2$)

If the relationship with Steiner problems is important to understand the name given to our problem, the Steiner BSPP is essentially a variant of BSPPs. Actually, solving the all-to-all BSPP also solves the Steiner BSPP. Equivalently, solving a one-to-all BSPP starting from each node in $T$ or solving a one-to-one BSPP for each pair of nodes in $T$ also solves the Steiner BSPP. Clearly our point here is to propose a more efficient approach.

BSPPs (and more generally Multi-objective SPPs) have been widely studied in the literature, as acknowledged by several literature reviews (Clímaco and Pascoal, 2012; Ehrgott and Gandibleux, 2000; Skriver, 2000)). A first important matter concerns the size of solution sets, which directly translates to the number of arcs that would have to be added to the multigraph in our VRP context. Hansen (1980) shows that, in the worst case, this number can be exponential in the size of graph $G$. Müller-Hannemann and Weihe (2006) and Mandow and Pérez de la Cruz (2009) however show that much better behaviors can be observed in practice. Evaluating the typical size of the solution sets in our context will be a meaningful output of the paper.

A second important information that can be derived from the literature concerns exact solution methods. The prominent approaches to solve BSPPs are labeling algorithms. These algorithms all follow the same principle (Hansen, 1980; Martins, 1984b). A label represents a partial path in the graph. The most elementary operation in the algorithm is to select a label and extend the associated partial path with an additional arc. It results in a new label that can be dominated or kept for further extensions. Algorithms differ in the order in which these elementary label extensions are performed and in the data structures that are used to efficiently manage label selection and dominance. Labeling algorithms can solve one-to-one and one-to-all BSPPs. Algorithms applied to solve all-to-all SPPs have a very different nature and, as far as we know, have never been extended to the bi-objective or multi-objective case. Also, the presence of nonpositive arc weights can complicate a lot the problems, especially when elementary paths are required (see, *e.g.*, Martins and Santos (1999), Irnich and Desaulniers (2005) or Feillet et al. (2004)). Seeing that we are only interested in positive weights in our context, we focus on this case hereafter.

Regarding the one-to-all BSPP, two main label selection heuristics are employed in the literature. In a node-selection strategy *e.g.*, (Brumbaugh-Smith and Shier, 1989), all labels representing paths ending at a same node are selected and extended to all successors of this

node. In a label-selection strategy, a single label is selected, based on its value. Again this label is extended to all the successors of the ending node of the path that it represents (*e.g.*, Tung and Chew (1992)). Several computational studies evaluated various variants of these two strategies and gave a slight advantage to node-selection techniques (Guerriero and Musmanno, 2001; Paixão and Santos, 2013).

In the one-to-one BSPP, the label-selection strategy is employed but with a destination-driven selection rule, following the principle of A* algorithm (Mandow and Pérez de la Cruz, 2010; Stewart and White III, 1991). Labels are selected according to an optimistic evaluation of the cost that paths reaching the destination from this label could have. In order to efficiently select labels and add new labels to the set of labels in wait, a heap is generally used. In Mandow and Pérez de la Cruz (2010), all label evaluations (which are actually composed of a value for each criterion) are compared and a label with a non-dominated evaluation is selected. In Tung and Chew (1992), the two components of the evaluation are added and the label with the smallest sum is selected. The evaluation measure is given by the actual cost of the label completed, for both components, by the mono-criterion shortest path cost to reach the destination.

Another category of solution methods is based on ranking. They determine paths progressively, in a non-decreasing order of one of the objectives (Climaco and Martins, 1982; Raith and Ehrgott, 2009). Martins (1984a) proposed a ranking method based on path deletion. In this method, the shortest path according to a selected criterion is determined at each iteration. This path is then eliminated from the network for the next iteration. The algorithm stops when the expected number of path is found or when no more efficient paths are found. Huarng et al. (1996) however showed that this ranking method was not competitive compared to labeling approaches.

Another approach was proposed by Mote et al. Mote et al. (1991). It is organized in two phases. In the first phase, extreme solutions in the convex hull of the BSPP solution space are computed by solving the LP relaxation of the problem. In the second phase, an enumerative method is used to determine the set of efficient paths. The main idea is to restrict enumeration thanks to the information extracted from the first phase. Raith and Ehrgott (2009) studied more deeply this two-phase approach. They investigated different combinations of methods for the two phases: a network simplex method and single objective label setting and label correcting algorithms were tested in Phase 1 and, ranking and bi-objective labeling approaches were explored in Phase 2. They compared the two-phase method with purely labeling approaches and a ranking method. Computational experiments carried out on different instance sets showed the competitiveness of the two-phase method with the different configurations. In their conclusions, Raith and Ehrgott (2009) noticed that the efficiency of solution methods depends a lot on network structure.

Besides these exact methods, many heuristic algorithms have also been applied to BSPPs, such as evolutionary algorithms (*e.g.*, Pangilinan and Janssens (2007)) or ant colony optimization algorithms (*e.g.*, Ghoseiri and Nadjari (2010)). These algorithms are however out of the scope of this research as we focus on exact solution.

## 3. Problem definition, notation and basic properties

### 3.1. Problem definition

We consider a directed graph $G = (V, A)$ modeling a road network. Arcs $(i, j) \in A$ represent road segments and are tagged with two positive weights $d_{ij}$ and $t_{ij}$. We define a path $P$ in $G$ as an ordered list of nodes $P = (u_0, u_1, ..., u_p)$ such that $(u_k, u_{k+1}) \in A$ for $k \in \{0, ..., p-1\}$. The cost vector of path $P$ is the sum of its arc weights:

$$(d(P), t(P)) = \left( \sum_{k\in\{0,...,p-1\}} d_{u_k u_{k+1}}, \sum_{k\in\{0,...,p-1\}} t_{u_k u_{k+1}} \right)$$

We introduce the subset $T \subset V$ of terminals. The Steiner BSPP aims at finding a complete set of efficient paths between every pair of terminals.

A few definitions are given below, to clarify what a complete set of efficient paths is. We consider in these definitions that all paths have the same starting and ending points.

**Definition 1.** *Dominance*

A vector $(a_1, a_2)$ dominates a vector $(b_1, b_2)$ if and only if $a_1 \leq b_1$ and $a_2 \leq b_2$ with at least one inequality being strict.

**Definition 2.** *Efficient path*

A path $P_1$ is efficient if there does not exist any path $P_2$ whose cost vector $(d(P_2), t(P_2))$ dominates $(d(P_1), t(P_1))$.

**Definition 3.** *Set of non-dominated cost vectors (also called Pareto front)*

The set of all non-dominated cost vectors is the set of vectors $(d(P), t(P))$ obtained from efficient paths.

**Definition 4.** *Complete set of efficient paths*

A set $\mathcal{P}$ of efficient paths is complete if every non-dominated cost vector $(d, t)$ admits at least one path $P \in \mathcal{P}$ such that $(d, t) = (d(P), t(P))$. A complete set is minimal if none of its subsets is complete.

For the remainder of the paper, we denote $|V| = n$, $|A| = m$ and $|T| = n_T$.

### 3.2. Additional notation and basic properties

We now introduce additional notation and some simple properties that will be useful in next sections. We first recall that a vector $a = (a_1, a_2)$ is lexicographically smaller than a vector $b = (b_1, b_2)$, denoted by $a \underset{lex}{<} b$, if either $a_1 < b_1$ or both $a_1 = b_1$ and $a_2 < b_2$. Similarly, we say that a path $P_1$ is lexicographically smaller than a path $P_2$, denoted by $P_1 \underset{lex}{<} P_2$, if and only if $(d(P_1), t(P_1)) \underset{lex}{<} (d(P_2), t(P_2))$. The lexicographic order defines a total order between paths.

Let $\mathcal{P}(u, v)$ denote the set of all paths linking two nodes $u$ and $v$ in $T$ and let $\mathcal{P}_{opt}(u, v) = (P_1, P_2, ..., P_r)$ be a complete set of efficient paths between $u$ and $v$. Without loss of generality, we assume that $\mathcal{P}_{opt}(u, v)$ is minimal and that paths in $\mathcal{P}_{opt}(u, v)$ are sorted according to the lexicographic order, *i.e.*, $P_1 \underset{lex}{<} P_2 \underset{lex}{<} ... \underset{lex}{<} P_r$. As a consequence:

$$d(P_1) < d(P_2) < ... < d(P_{r-1}) < d(P_r)$$

$$t(P_1) > t(P_2) > ... > t(P_{r-1}) > t(P_r)$$

Using this notation, we can recall the following simple properties:

**Property 1.** *$P_1$ is the shortest path in distance from $u$ to $v$ in $G$:* $d(P_1) = \min_{P\in\mathcal{P}(u,v)} d(P)$.

**Property 2.** *$P_r$ is the shortest path in time from $u$ to $v$ in $G$:* $t(P_r) = \min_{P\in\mathcal{P}(u,v)} t(P)$.

**Proof.** For each non-efficient path $P \in \mathcal{P}(u, v) \setminus \mathcal{P}_{opt}(u, v)$, at least one path $P_k \in \mathcal{P}_{opt}(u, v)$ exists such that $d(P_k) \leq d(P)$ and $t(P_k) \leq t(P)$. Consequently, for each path $P \in \mathcal{P}(u, v)$, we have $d(P_1) \leq d(P)$ and $t(P_r) \leq t(P)$. □

In the remainder of this paper, we denote by $(d^{min}(u, v), t^{max}(u, v))$ the cost vector associated with the shortest path in distance in $\mathcal{P}_{opt}(u, v)$ and we denote by $(d^{max}(u, v), t^{min}(u, v))$ the cost vector associated with the shortest path in time in $\mathcal{P}_{opt}(u, v)$. Note that any efficient path $P \in \mathcal{P}(u, v)$ is such that $d^{min}(u, v) \leq d(P) \leq d^{max}(u, v)$ and $t^{min}(u, v) \leq t(P) \leq t^{max}(u, v)$.

## 4. Solution algorithm

The core mechanism of our solution algorithm is similar to the label setting algorithm proposed, first, by Martins (1984b) which is, in turn, based on Dijkstra's algorithm (Dijkstra, 1959). It essentially follows a

one-to-all solution framework, with the difference that, in our adaptation, we are only interested in the efficient paths arriving to nodes in $T$ and we guide the search towards this direction. To solve the Steiner BSPP, this algorithm is repeated several times, once for each terminal defined as a starting point

In what follows, the starting node is named $v_0$. A label represents a path $P$ from $v_0$ to a certain node $u \in V$ and is defined with the following information:

$$L = (last(L), d(L), t(L), father(L))$$

where $last(L) = u$, $d(L) = d(P)$, $t(L) = t(P)$ and $father(L)$ is the label from which $L$ was extended (Ø for the initial label). Following previous definitions, we say that a label $L_1$ dominates another label $L_2$ if $(d(L_1), t(L_1))$ dominates $(d(L_2), t(L_2))$. We say that $L_1$ is lexicographically smaller than $L_2$, denoted by $L_1 \underset{lex}{<} L_2$, if $(d(L_1), t(L_1)) \underset{lex}{<} (d(L_2), t(L_2))$; we say that the two labels are equal if $(d(L_1), t(L_1)) = (d(L_2), t(L_2))$, even if the father nodes can differ.

In Section 4.1, we detail the solution method. In Section 4.2, we prove that the proposed algorithm correctly provides the expected set of efficient paths. We then give, in Section 4.3, more details on the data structures and report on the complexity.

### 4.1. The multi-destination-A* algorithm

The solution method and its preprocessing are described in Algorithms 1 and 2. We call this algorithm multi-destination-A* algorithm (MDA*). It provides a minimal complete set of efficient paths between $v_0$ and $s$.

**Algorithm 1** Preprocessing

1: compute $d^{min}(v_0, v)$, $t^{min}(v_0, v)$, $d^{max}(v_0, v)$ and $t^{max}(v_0, v)$ for all $v \in V$
2: **for all** $s \in T$ **do**
3: compute $d^{min}(v, s)$, $t^{min}(v, s)$, $d^{max}(v, s)$ and $t^{max}(v, s)$ for all $v \in V$
4: **end for**

**Algorithm 2** Multi-destination-A* algorithm for the Steiner BSPP

1: $L = (v_0, 0, 0, \emptyset)$
2: $allLabels.add(L)$
3: $Labels[v_0].add(L)$
4: **while** $K(allLabels.Min()) \leq \max_{s \in T} K_s^{max}$ **do**
5: $L = allLabels.extractMin()$
6: $u = last(L)$
7: **for all** $(u, v) \in A$ **do**
8: $L' = (v, d(L) + d_{uv}, t(L) + t_{uv}, L)$
9: **if** $L'$ is not dominated nor equal to a label in $Labels[v]$ **then**
10: $allLabels.add(L')$
11: $Labels[v].add(L')$
12: **if** a label $L'' \in Labels[v]$ is dominated by $L'$ **then**
13: $allLabels.remove(L'')$
14: $Labels[v].remove(L'')$
15: **end if**
16: **end if**
17: **end for**
18: **end while**
19: Compute path set $\mathcal{P}_{opt}(v_0, s)$ from all labels in $Labels[s]$, for all $s \in T$
20: **return** $\mathcal{P}_{opt}(v_0, s)$ for all $s \in T$

The preprocessing is important to implement the A* mechanism. It works as follows:

- Using Dijkstra's algorithm, we compute shortest paths in distance and in time from node $v_0$ to all nodes $v \in V$. Four tables are constructed: $d^{min}(v_0, v)$ and $t^{max}(v_0, v)$ indicating distances and times associated with shortest paths in distance and, $d^{max}(v_0, v)$ and $t^{min}(v_0, v)$ indicating distances and times associated with shortest paths in time.
- Using Dijkstra's algorithms in backwards from all nodes $s \in T$ (that is, with the arcs implicitly reversed), we compute shortest paths in distance and in time from all nodes $v \in V$ to destination nodes $s \in T$. Four series of tables are obtained: $d^{min}(v, s)$ and $t^{max}(v, s)$ indicating distances and times associated with shortest paths in distance and, $d^{max}(v, s)$ and $t^{min}(v, s)$ indicating distances and times associated with shortest paths in time.

In both forward and backward Dijkstra's algorithms, label comparisons are based on the lexicographical order $\underset{lex}{<}$. Thanks to that, we have the guarantee that all the computed paths are efficient.

Algorithm MDA* mainly relies on the two following structures:

- Set *allLabels* contains all the labels that have to be extended. It is used for label selection. It is initialized with a single label anchoring future labels to the starting position $v_0$ (Lines 1–2). When a label with a new non-dominated cost vector is created, it is added to this set (Line 10). When a label is selected for extension or dominated, it is removed (Lines 5 and 13)
- Vector *Labels[v]* contains all the efficient paths that are known for nodes $v \in V$. It is used for dominance. It is initially empty for all nodes but $v_0$ (Line 3). Labels with new non-dominated cost vectors are added (Line 11), newly dominated labels are removed (Line 14).

At each iteration, a label $L$ is selected in *allLabels* (Line 5) and extended to all the successors of $last(L)$ (Line 7). If new promising labels are found (Line 9), the different label sets are updated as explained above.

Label selection function *extractMin()* aims at finding the label apt to lead the most quickly to one of the nodes in $T$. This is where the innovation of our algorithm stands. In the standard A* strategy, the search is guided to a single destination. In Dijkstra's algorithm, all destinations are given the same priority. In order to guide the search to a subset of primary destinations (the set $T$ of terminals), we select the label $L = (u, d(L), t(L), father(L))$ that minimizes value:

$$\min_{s \in T}(d(L) + d^{min}(u, s) - d^{min}(v_0, s))$$

We denote $K(L)$ this value and call it the *key* of the label. For a given node $s$, $d(L) + d^{min}(u, s) - d^{min}(v_0, s)$ is the minimum detour that could be achieved when extending label $L$ to $s$, compared to the shortest path in distance between $v_0$ and $s$. The key thus gives the minimal value among detours to all destinations in $T$, that is, it prioritizes labels that could potentially lead to one of the destinations effectively. Actually, with this key, the algorithm will first explore the min-distance paths leading to nodes in $T$ (the key is zero for these paths), then it will progressively deviate from these paths. Note that distances computed in preprocessing are used when computing the key.

The algorithm should terminate when the labels representing min-time paths have been generated for all terminals in $T$. Indeed, any label with a larger detour would necessarily be dominated by these labels: both time and distance would be larger. More formally, the min-time path between $v_0$ and $s$ is given by label $L_s = (s, d^{max}(v_0, s), t^{min}(v_0, s), father(L_s))$. In $L_s$, the detour is $d^{max}(v_0, s) - d^{min}(v_0, s)$. Therefore, the stopping criterion is that the key of the selected label is larger than $\max_{s \in T} K_s^{max}$ with $K_s^{max} = d^{max}(v_0, s) - d^{min}(v_0, s)$. The selected label is given by function *Min()*, which indicates the label with minimal key, *i.e.*, the label that would be returned using function *extractMin()* (Line 4).

### 4.2. Proof of optimality

**Theorem 1.** *Algorithm MDA* provides minimal complete sets of efficient paths from source node $v_0$ to all destination nodes $s \in T$.*

**Proof.** Let us assume that at the end of Algorithm 2 there exists an efficient path $P$ from the source node $v_0$ to a destination node $s \in T$ such that $(d(P), t(P))$ does not belong to the set of non-dominated vector costs returned by the algorithm. Under this assumption, the set of efficient paths between $v_0$ and $s$ is incomplete. We show that it cannot happen.

Let $(u_0, u_1, ..., u_r)$ be the sequence of nodes visited along path $P$ with $u_0 = v_0$ and $u_r = s$. Let us denote by $P_{0i}$ and $P_{is}$ the paths respectively defined by node sequences $(u_0, ..., u_i)$ and $(u_i, ..., u_r)$ for all $i \in \{0, ..., r\}$. Let $L_i$ be the label associated with path $P_{0i}$. Due to the Principle of Optimality Martins (1984b), label $L_i$ can never be dominated by labels in $Labels[u_i]$. Consequently, the reason why $P$ is not found by Algorithm 2 is that:

- either the algorithm stops before $P$ is generated, that is, there exists $j \in \{1, ..., r-1\}$ such that $K(L_j) > \underset{s' \in T}{max} K^{max}_{s'}$;
- or there exists $j \in \{1, ..., r-1\}$ such that a label $L \in Labels[u_j]$ is equal to $L_j$.

In the second case, let $P'$ be the path obtained from $L$ by extending it with path $P_{js}$. Seeing that $(d(P'), t(P')) = (d(P), t(P))$ we fall into a situation similar to that of $P$ and, inductively, we eventually arrive to the first case.

Without loss of generality, let us thus assume that label $L_j$ obtained from path $P$ is such that $K(L_j) > \underset{s' \in T}{max} K^{max}_{s'}$. We know that $d(P) = d(L_j) + d(P_{js})$. From properties 1 and 2, we also know that $d(P) \leq d^{max}(v_0, s)$, *i.e.*:

$$d(L_j) + d(P_{js}) \leq d^{max}(v_0, s)$$

Then,

$$d(L_j) + d^{min}(u_j, s) \leq d^{max}(v_0, s)$$

and, so,

$$d(L_j) + d^{min}(u_j, s) - d^{min}(v_0, s) \leq d^{max}(v_0, s) - d^{min}(v_0, s)$$

Seeing that $d(L_j) + d^{min}(u_j, s) - d^{min}(v_0, s) = K(L_j)$ and that $d^{max}(v_0, s) - d^{min}(v_0, s) = K^{max}_s$, we obtain

$$K(L_j) \leq K^{max}_s \leq \underset{s' \in T}{max} K^{max}_{s'}$$

which contradicts our assumption and proves that the algorithm provides complete sets of efficient paths. In addition, these sets are minimal because the algorithm prevents from having two labels with the same vector cost attached to the same node. □

### 4.3. Complexity analysis

We analyze the complexity of algorithm MDA* with structures $allLabels$ and $Labels[v]$ implemented as follows:

- Structure $allLabels$ is a heap; it embeds the following methods:
  - $Min()$ returns the label $L$ at the top of the heap, *i.e.*, with the smallest $K(L)$ value; its complexity is in $O(1)$;
  - $extractMin()$ extracts the label $L$ at the top of the heap; its complexity is in $O(log(|allLabels|))$;
  - $add(L)$ inserts label $L$ into $allLabels$; its complexity is in $O(log(|allLabels|))$;
  - $remove(L)$ removes label $L$ from $allLabels$. its complexity is in $O(|allLabels|)$.
- Structure $Labels[v]$ is a chained list for each $v \in V$; it gives access to the following methods:
  - $add(L)$ adds label $L$ to $Labels[v]$; the complexity is in $O(1)$;
  - $remove(L)$ removes label $L$ from the list; it is performed in $O(|Labels[v]|)$.

Without loss of generality, we assume that data are integer. If it is not the case, data can be multiplied by a power of ten. Let us introduce $\delta(v) = \underset{s \in T}{max}(d^{max}(v_0, s) - d^{min}(v_0, v) - d^{min}(v, s))$ for every node $v \in V$, and let $\Delta = \underset{v \in V}{max}(\delta(v))$.

**Theorem 2.** *The complexity of algorithm MDA* is in* $O(m\Delta^2 log(n\Delta))$.

**Proof.** Given a node $v \in V$, a label $L$ arriving at node $v$ has a chance to lead to an efficient path at one of the destination nodes $s \in T$ if $d(L) + d^{min}(v, s) \leq d^{max}(v_0, s)$. Therefore every label $L$ maintained at node $v$ during the labeling procedure verifies $d(L) \leq \underset{s \in T}{max}(d^{max}(v_0, s) - d^{min}(v, s))$. In addition, $d^{min}(v_0, v) \leq d(L)$, and, seeing that distances are assumed to be integer and that at most one label is kept for every value of the distance, the number of labels in list $Labels[v]$ is bounded by $\delta(v)$. Equivalently, the total number of labels in heap $allLabels$ is bounded by $\sum_{v \in V} \delta(v) \leq n\Delta$.

The complexities of the main procedures performed during the search are as follows:

- The extraction of the label with smallest key value requires $O(log(n\Delta))$ operations;
- For each new label $L' = (v, d(L'), t(L'), father(L'))$, the dominance check which implies the insertion of the new (non-dominated) label in list $Labels[v]$ and the removal of dominated labels from $Labels[v]$ and from the heap $allLabels$, requires $O(\delta(v) log(n\Delta))$ operations.
- The extension of a selected label $L = (u, d(L), t(L), father(L))$ through all outgoing arcs $(u, v) \in A$ requires $O(\sum_{(u,v) \in A}(1 + log(n\Delta) + \delta(v) log(n\Delta))$ operations: the first term in the sum corresponds to the generation of the new label, the second term corresponds to the insertion of the new label into the heap and the third term corresponds to the dominance check.

Since every label arriving at a node $v$ and that is in the heap is selected once and $O(\sum_{(u,v) \in A}(1 + log(n\Delta) + \delta(v) log(n\Delta))) \leq O(\sum_{(u,v) \in A} \delta(v) log(n\Delta)$, the total complexity of the algorithm is given by:

$$\begin{aligned}
&\text{complexity of MDA*} \\
&= O\left(\sum_{u \in V} \sum_{L \in Labels[u]} \left( log(n\Delta) + \sum_{v \in V; (u,v) \in A} \delta(v) log(n\Delta) \right)\right) \\
&\leq O\left( n\Delta log(n\Delta) + \sum_{L \in Labels[u]} \sum_{(u,v) \in A} \Delta log(n\Delta) \right) \\
&\leq O\left(n\Delta log(n\Delta) + m\Delta^2 log(n\Delta)\right) \\
&\leq O(m\Delta^2 log(n\Delta))
\end{aligned}$$

□

## 5. Steiner BSPP with time windows

As already explained, this work is motivated by the computation of multigraph data in the context of vehicle routing problems. When two attributes (*e.g.*, time and distance) are present, it generally implies additional constraints on routes. For example, in the VRPTW, time windows limit customer visit times. These constraints can be exploited in the computation, as we now show for the VRPTW.

Let $e_s$ and $l_s$ denote respectively the earliest starting service time and the latest starting service time for a terminal $s \in T$. It is not allowed to reach customer $s$ after time $l_s$; arriving before $e_s$ is possible but implies waiting for the opening time $e_s$. For the depot, this window corresponds to the time horizon: starting time from the depot, latest allowed returning time. A path $P$ between two terminals $u$ and $v$ is feasible with respect to time windows if and only if $e_u + t(P) \leq l_v$. Therefore, sets of efficient paths can be limited to paths satisfying this condition.

We consider the computation of efficient paths starting from a given node $v_0 \in T$. We denote by $T^+(v_0)$ the subset of terminals reachable from $v_0$ within their time windows: $T^+(v_0) = \{s \in T : e_{v_0} + t^{min}(v_0, s) \leq l_s\} \subset T$. Only destination nodes in $T^+(v_0)$ should be considered during the search procedure. To do this, we propose the following enhancements:

1. The key of a label $L = (u, d(L), t(L), father(L))$ is evaluated regarding only reachable destination nodes: $K(L) = \underset{s \in T^+(v_0)}{min}(d(L) + d^{min}(u, s) - d^{min}(v_0, s))$;

**Table 1**
Road network characteristics.

| | | | outgoing arcs | |
|---|---|---|---|---|
| network | n | m | min | max |
| AIX-1 | 5437 | 10098 | 1 | 4 |
| AIX-2 | 19500 | 36203 | 1 | 5 |
| DC | 9559 | 29707 | 1 | 6 |
| DE | 49109 | 119838 | 1 | 6 |
| RI | 53658 | 137596 | 1 | 6 |
| AK | 69082 | 155245 | 1 | 6 |

**Table 2**
Number of instances for each road network.

| | number of terminals | | | | |
|---|---|---|---|---|---|
| network | $n_T = 26$ | $n_T = 51$ | $n_T = 101$ | $n_T = 201$ | $n_T = 501$ |
| AIX-1 | 10 | 10 | 10 | 10 | 10 |
| AIX-2 | 10 | 10 | 10 | 10 | 10 |
| DC | 10 | 10 | 10 | 10 | 10 |
| DE | 10 | 10 | 10 | 10 | 10 |
| RI | 10 | 10 | 10 | 10 | 10 |
| AK | 10 | 10 | 10 | 10 | 10 |

2. The algorithm should terminate once labels $L_s = (s, d^{max}(v_0, s), t^{min}(v_0, s), father(L_s))$ have been generated for destination nodes $s \in T^+(v_0)$. Thus, the stopping criterion is that the selected label key is larger than $\max_{s \in T^+(v_0)} K_s^{max}$;
3. Only nodes that are apt to lead to a destination node with a feasible path should be considered. Every node $u$ such that $e_{v_0} + t^{min}(v_0, u) + t^{min}(u, s) > l_s$ for all destination nodes $s \in T^+(v_0)$ is discarded from graph $G$.

Due to the first enhancement, label keys increase more quickly during the labeling procedure. The second enhancement tightens the stopping condition. Combined with the first enhancement, it limits the number of labels generated during the search. The third enhancement is performed in preprocessing and permits reducing the size of the graph.

## 6. Computational experiments

In this section, we present the computational experiments carried out to evaluate the efficiency of the proposed solution method. First, we present the benchmark problems used in the experiments. Then, we report the computational results and we analyse the impact of considering the time windows on the algorithm performance.

All algorithms are implemented in the C++ programming language and tests are run on an Intel Xeon(R) CPU E5-2620v2 2.1 GHz computer with 32GB of memory.

### 6.1. Test problems

Since we are interested in computing paths for transportation problems, we conducted our computational experiments on the basis of two series of real-world road networks:

- Two road networks (AIX-1 and AIX-2) were constructed based on spatial data from the city of *Aix-en-Provence*[1] in France provided by **OpenStreetMap**[2] database. Each road segment is defined by a length, a maximum allowed speed and a travel direction. Travel times were then computed using speeds and lengths.
- Four road networks (DC, DE, RI and AK) were extracted by Schultes (2005) from **US Census**[3] and correspond to *Washington D.C., Delaware, Rhode Island* and *Alaska*, in the United States, respectively. In these networks, each road segment is given with a distance and a travel time. Note that, the road networks in the original data are undirected and we converted them into directed networks by duplicating all arcs.

Table 1 reports the main characteristics of these 6 road networks. For each road network, the first two columns indicate the number of nodes $n$ and the number of arcs $m$, respectively. The last two columns give the minimum and the maximum number of outgoing arcs over all nodes in the road network.

From each road network, we generated 50 instances by randomly selecting terminals among the network nodes. We considered different terminal set sizes and two different configurations:

- Random: terminals are randomly drawn on the whole road network graph.
- Centered: all terminals but one are randomly selected in the center of the network (a fourth of the area covered by the network, centered). The remaining terminal is randomly selected near the south-west corner of the network. It represents the standard situation of urban deliveries from a distant depot.

Table 2 details the number of instances for each road network and for each value of $n_T$. For each configuration, 10 instances are generated: 5 random instances and 5 centered instances. It amounts to a total of 300 instances.

### 6.2. Computational results

In order to evaluate the performance of our algorithm, we compare its results to those of three state-of-the-art algorithms: A*, LSET and LCOR.

- A* follows the classical guided-search mechanism. The core algorithm solves the bi-objective shortest path problem from a single origin to a single destination. Contrary to MDA*, the key of a label is given by the minimal distance to reach the destination. It is repeated for all pairs of terminals.
- LSET is a one-to-all label setting algorithm. The search is not guided. At each iteration, the minimum label according to the lexicographic order is selected. The stopping criterion defined for MDA* is used. The algorithm is repeated for all possible origin points in $T$.
- LCOR is a one-to-all label correcting algorithm. The search is not guided either. At each iteration, a node is selected and the labels associated with this node are extended to successor nodes. Nodes reenter the queue when their label list is modified. The algorithm is repeated for all possible origin points in $T$.

For each instance, MDA* is also applied $n_T$ times, once for each node in $T$ selected as the source node, so that complete sets of efficient paths are obtained between all pairs of terminals.

Results for random instances are presented in Tables 3, 5, 7, 9, 11 and 13, for road networks AIX-1, AIX-2, DC, DE, RI and AK, respectively. Results for centered instances are presented in Tables 4, 6, 8,10, 12 and 14, for road networks AIX-1, AIX-2, DC, DE, RI and AK, respectively. Columns "MDA*", "A*", "LSET" and "LCOR" report total computing times (in seconds) for the four algorithms. The average number of efficient paths between pairs of terminals is presented in column "#p".

A first observation in these tables is that, apart from a few exceptions, results are homogeneous among the 5 instance replications for a given value $n_T$ and a given configuration (random or centered).

Comparing MDA* and A*, the latter is consistently beaten. A* is only competitive for small values of $n_T$, when the number of executions of

[1] Aix-en-Provence is a city-commune in the region of *Provence-Alpes-Cote d'Azur* in the south of France, about 30 km north of *Marseilles*

[2] OpenStreetMap is a collaborative project wich creates and distributes freely available geospatial data. www.openstreetmap.org/

[3] US Census 2000 TIGER/Line Files. U.S. Census Bureau, Washington, DC, Geography Division. http://www.census.gov/geo/www/tiger/tigerua/uatgr2k.html

**Table 3**
Computing times for AIX-1 - Random.

| $n_T$ | | MDA* | A* | LSET | LCOR | #p |
|---|---|---|---|---|---|---|
| 26 | 1 | 0.5 | 1.0 | 0.5 | 1.4 | 3.8 |
| | 2 | 0.6 | 1.2 | 0.6 | 1.6 | 4.3 |
| | 3 | 0.5 | 1.1 | 0.6 | 1.4 | 4.1 |
| | 4 | 0.5 | 1.0 | 0.6 | 1.4 | 4.1 |
| | 5 | 0.5 | 1.2 | 0.5 | 1.4 | 4.0 |
| 51 | 1 | 1.2 | 3.7 | 1.1 | 2.9 | 3.9 |
| | 2 | 1.1 | 3.3 | 1.1 | 2.8 | 4.0 |
| | 3 | 1.1 | 3.5 | 1.0 | 2.8 | 4.1 |
| | 4 | 1.2 | 4.0 | 1.1 | 2.8 | 4.3 |
| | 5 | 1.2 | 4.0 | 1.1 | 3.1 | 4.2 |
| 101 | 1 | 2.6 | 11.5 | 2.2 | 5.4 | 4.1 |
| | 2 | 2.5 | 10.8 | 2.1 | 5.2 | 3.9 |
| | 3 | 2.6 | 13.4 | 2.3 | 5.8 | 4.1 |
| | 4 | 2.6 | 12.9 | 2.1 | 5.2 | 4.0 |
| | 5 | 2.5 | 11.8 | 2.2 | 5.3 | 4.1 |
| 201 | 1 | 6.0 | 43.4 | 4.3 | 10.6 | 4.1 |
| | 2 | 6.3 | 51.8 | 4.4 | 11.0 | 4.1 |
| | 3 | 6.2 | 47.2 | 4.3 | 10.6 | 4.1 |
| | 4 | 6.4 | 54.1 | 4.4 | 11.5 | 4.2 |
| | 5 | 6.3 | 51.8 | 4.2 | 11.2 | 4.2 |
| 501 | 1 | 22.5 | 285.0 | 11.4 | 27.0 | 4.1 |
| | 2 | 22.9 | 278.1 | 10.6 | 28.0 | 4.2 |
| | 3 | 22.8 | 263.6 | 12.4 | 29.1 | 4.2 |
| | 4 | 22.6 | 255.7 | 12.4 | 27.1 | 4.2 |
| | 5 | 23.0 | 275.0 | 12.8 | 30.7 | 4.4 |

**Table 4**
Computing times for AIX-1 - Centered.

| $n_T$ | | MDA* | A* | LSET | LCOR | #p |
|---|---|---|---|---|---|---|
| 26 | 1 | 0.4 | 0.5 | 0.5 | 0.9 | 2.6 |
| | 2 | 0.4 | 0.6 | 0.5 | 0.9 | 3.0 |
| | 3 | 0.4 | 0.5 | 0.5 | 1.0 | 3.1 |
| | 4 | 0.4 | 0.6 | 0.5 | 1.0 | 3.0 |
| | 5 | 0.4 | 0.6 | 0.6 | 1.3 | 3.5 |
| 51 | 1 | 0.8 | 1.3 | 0.8 | 1.7 | 2.6 |
| | 2 | 0.9 | 1.5 | 1.0 | 2.2 | 3.1 |
| | 3 | 0.8 | 1.5 | 1.1 | 2.3 | 3.2 |
| | 4 | 0.8 | 1.4 | 1.0 | 2.0 | 3.0 |
| | 5 | 0.8 | 1.4 | 0.9 | 2.1 | 2.8 |
| 101 | 1 | 1.9 | 3.9 | 1.6 | 3.5 | 2.8 |
| | 2 | 1.8 | 4.0 | 1.7 | 3.7 | 2.7 |
| | 3 | 1.9 | 4.6 | 2.2 | 4.7 | 3.1 |
| | 4 | 1.9 | 4.4 | 1.8 | 4.0 | 3.0 |
| | 5 | 1.9 | 4.3 | 1.8 | 4.0 | 2.8 |
| 201 | 1 | 4.6 | 14.2 | 3.5 | 7.7 | 2.8 |
| | 2 | 4.7 | 12.8 | 3.4 | 7.2 | 2.7 |
| | 3 | 4.7 | 15.8 | 3.5 | 8.1 | 3.0 |
| | 4 | 4.6 | 14.8 | 3.1 | 7.5 | 2.9 |
| | 5 | 4.6 | 14.3 | 3.3 | 7.2 | 2.9 |
| 501 | 1 | 18.2 | 76.8 | 8.5 | 19.0 | 2.9 |
| | 2 | 18.0 | 79.2 | 8.7 | 18.7 | 2.8 |
| | 3 | 18.1 | 77.7 | 7.7 | 19.2 | 2.8 |
| | 4 | 18.2 | 73.9 | 9.0 | 19.2 | 2.8 |
| | 5 | 18.0 | 78.1 | 8.6 | 20.0 | 2.8 |

**Table 5**
Computing times for AIX-2 - Random.

| $n_T$ | | MDA* | A* | LSET | LCOR | #p |
|---|---|---|---|---|---|---|
| 26 | 1 | 9.3 | 24.5 | 23.8 | 169.7 | 15.6 |
| | 2 | 7.4 | 19.0 | 20.7 | 134.3 | 14.3 |
| | 3 | 9.7 | 25.6 | 20.7 | 120.6 | 13.6 |
| | 4 | 9.1 | 27.7 | 22.7 | 152.8 | 16.1 |
| | 5 | 7.7 | 28.7 | 27.1 | 156.1 | 15.9 |
| 51 | 1 | 19.9 | 80.8 | 43.3 | 296.2 | 14.5 |
| | 2 | 21.2 | 91.3 | 41.5 | 276.4 | 14.8 |
| | 3 | 16.6 | 77.3 | 38.5 | 245.5 | 13.3 |
| | 4 | 23.7 | 91.3 | 45.5 | 298.4 | 14.7 |
| | 5 | 23.7 | 113.5 | 45.2 | 342.0 | 15.2 |
| 101 | 1 | 47.6 | 328.5 | 81.3 | 561.8 | 14.5 |
| | 2 | 48.6 | 345.0 | 81.5 | 551.6 | 14.4 |
| | 3 | 45.5 | 319.5 | 79.1 | 534.1 | 13.3 |
| | 4 | 36.0 | 229.2 | 70.8 | 509.4 | 12.6 |
| | 5 | 45.9 | 344.0 | 78.4 | 615.3 | 14.0 |
| 201 | 1 | 105.3 | 1323.6 | 162.8 | 1106.8 | 14.4 |
| | 2 | 94.0 | 1062.8 | 149.1 | 1041.2 | 13.0 |
| | 3 | 100.7 | 1273.8 | 153.9 | 1170.7 | 13.9 |
| | 4 | 87.0 | 960.4 | 143.1 | 1000.1 | 12.9 |
| | 5 | 96.6 | 1069.8 | 163.0 | 1034.5 | 12.6 |
| 501 | 1 | 285.5 | 7365.8 | 375.2 | 2767.1 | 13.8 |
| | 2 | 274.1 | 5831.1 | 384.4 | 2637.6 | 12.9 |
| | 3 | 269.6 | 6200.6 | 399.8 | 2597.3 | 13.0 |
| | 4 | 273.7 | 6405.3 | 399.4 | 2587.4 | 13.3 |
| | 5 | 275.6 | 6664.8 | 395.1 | 2697.6 | 13.6 |

**Table 6**
Computing times for AIX-2 - Centered.

| $n_T$ | | MDA* | A* | LSET | LCOR | #p |
|---|---|---|---|---|---|---|
| 26 | 1 | 2.7 | 5.4 | 12.8 | 106.7 | 8.2 |
| | 2 | 3.5 | 7.6 | 14.9 | 118.0 | 10.6 |
| | 3 | 2.6 | 5.3 | 10.4 | 66.1 | 7.2 |
| | 4 | 3.3 | 6.6 | 14.2 | 95.1 | 8.9 |
| | 5 | 2.8 | 5.5 | 10.6 | 75.9 | 8.2 |
| 51 | 1 | 7.2 | 21.1 | 24.4 | 239.7 | 8.6 |
| | 2 | 7.7 | 25.1 | 28.9 | 225.1 | 10.0 |
| | 3 | 6.5 | 20.6 | 21.0 | 144.3 | 7.7 |
| | 4 | 7.4 | 22.1 | 25.0 | 181.5 | 8.7 |
| | 5 | 6.5 | 20.0 | 25.6 | 165.2 | 8.3 |
| 101 | 1 | 16.0 | 83.4 | 52.2 | 439.7 | 8.8 |
| | 2 | 17.5 | 98.5 | 53.0 | 432.4 | 9.7 |
| | 3 | 15.0 | 80.2 | 43.6 | 292.7 | 7.6 |
| | 4 | 16.8 | 81.2 | 46.3 | 369.0 | 8.9 |
| | 5 | 17.6 | 86.0 | 50.9 | 350.8 | 8.5 |
| 201 | 1 | 35.1 | 295.6 | 100.5 | 824.6 | 8.4 |
| | 2 | 37.7 | 368.2 | 98.8 | 818.7 | 9.1 |
| | 3 | 36.7 | 300.6 | 94.2 | 655.9 | 8.1 |
| | 4 | 38.2 | 341.9 | 99.3 | 792.3 | 8.8 |
| | 5 | 38.5 | 332.4 | 96.9 | 708.0 | 8.5 |
| 501 | 1 | 123.1 | 1689.8 | 255.5 | 1934.3 | 8.4 |
| | 2 | 120.7 | 1821.6 | 257.2 | 1918.7 | 8.8 |
| | 3 | 123.0 | 1809.6 | 244.1 | 1808.1 | 8.4 |
| | 4 | 125.8 | 1560.3 | 238.5 | 1888.2 | 8.1 |
| | 5 | 124.3 | 1838.9 | 250.2 | 2039.8 | 8.7 |

the one-to-one core algorithm remains limited. Clearly, the advantage of driving the search to a single direction is quickly lost when the size of $n_T$ increases and reaches values that are practically meaningful in the context of vehicle routing.

Similarly, except for a few exceptions, LSET is more efficient than LCOR, sometimes very significantly. For that reason, in the remainder of the analysis we focus on MDA* and LSET. Except for the smaller graph (AIX-1), MDA* is always many times faster than LSET, often up to 10 times. Not surprisingly, MDA* is specially interesting when $n_T$ is small. In this case indeed, being able to limit the search to some promising directions really makes a difference, compared to an algorithm that indifferently explores all directions. Globally, the larger the graph and the smaller $n_T$, the larger the benefits achieved with MDA*.

Result tables for Centered instances even strengthen these observations. Let us recall that this setting is particularly relevant in practice, as it typically corresponds to the situation encountered by urban logistics providers. Having a distant depot (south-west corner) with centered customers is indeed particularly beneficial for MDA*. When the source node ($v_0$) is the depot, all destination nodes are clustered and the search is very efficiently guided. When the source node is any other terminal, all destination nodes but one are close; the best paths to these nodes can efficiently be found; the best paths to the last node (the depot) are also efficiently computed thanks to the A* mechanism. On the contrary, LSET always needs to compute best paths to a large part of the network before being able to stop the search.

Tables 3 to 14 also give interesting insights on the size of complete sets of efficient paths in the different networks. For a given network, this value only slightly depends on value $n_T$, as could be expected, but

**Table 7**
Computing times for DC - Random.

| $n_T$ | | MDA* | A* | LSET | LCOR | #p |
|---|---|---|---|---|---|---|
| 26 | 1 | 5.5 | 13.3 | 22.4 | 144.7 | 15.1 |
| | 2 | 6.4 | 21.1 | 25.7 | 131.6 | 15.2 |
| | 3 | 5.9 | 18.7 | 21.0 | 112.2 | 13.2 |
| | 4 | 8.0 | 27.4 | 27.3 | 183.9 | 19.1 |
| | 5 | 5.6 | 15.6 | 21.1 | 137.3 | 15.5 |
| 51 | 1 | 14.7 | 63.8 | 48.4 | 272.8 | 14.7 |
| | 2 | 16.1 | 87.3 | 48.6 | 287.0 | 15.7 |
| | 3 | 15.3 | 64.7 | 46.3 | 301.4 | 16.4 |
| | 4 | 13.0 | 58.6 | 41.0 | 263.9 | 14.3 |
| | 5 | 16.5 | 80.1 | 47.4 | 345.4 | 17.3 |
| 101 | 1 | 34.6 | 300.0 | 96.7 | 556.5 | 15.2 |
| | 2 | 32.2 | 241.9 | 87.4 | 555.5 | 15.1 |
| | 3 | 42.1 | 397.8 | 119.4 | 719.2 | 18.7 |
| | 4 | 37.8 | 295.5 | 86.5 | 686.8 | 17.3 |
| | 5 | 37.7 | 358.6 | 88.5 | 589.4 | 15.3 |
| 201 | 1 | 72.6 | 1064.9 | 185.9 | 1109.4 | 15.3 |
| | 2 | 84.5 | 1334.4 | 215.8 | 1377.4 | 17.7 |
| | 3 | 78.0 | 1359.5 | 188.3 | 1170.9 | 16.0 |
| | 4 | 79.9 | 1281.1 | 185.4 | 1257.4 | 16.5 |
| | 5 | 73.9 | 1129.1 | 210.0 | 1154.7 | 15.6 |
| 501 | 1 | 211.0 | 7961.5 | 527.2 | 3071.1 | 16.3 |
| | 2 | 205.2 | 7722.3 | 539.0 | 2825.3 | 16.1 |
| | 3 | 201.6 | 7444.0 | 520.5 | 2856.1 | 15.4 |
| | 4 | 205.0 | 7602.0 | 485.4 | 3014.1 | 15.9 |
| | 5 | 199.6 | 7307.8 | 542.7 | 2771.6 | 15.3 |

**Table 8**
Computing times for DC - Centered.

| $n_T$ | | MDA* | A* | LSET | LCOR | #p |
|---|---|---|---|---|---|---|
| 26 | 1 | 4.1 | 11.1 | 19.3 | 140.4 | 14.4 |
| | 2 | 3.5 | 7.9 | 27.5 | 143.8 | 12.6 |
| | 3 | 3.7 | 10.3 | 22.1 | 145.6 | 13.9 |
| | 4 | 2.7 | 6.2 | 15.7 | 163.2 | 13.2 |
| | 5 | 2.9 | 6.3 | 19.1 | 132.3 | 13.1 |
| 51 | 1 | 8.3 | 37.1 | 43.1 | 292.6 | 14.2 |
| | 2 | 9.3 | 46.9 | 48.3 | 326.5 | 15.4 |
| | 3 | 8.1 | 36.6 | 49.7 | 306.8 | 14.6 |
| | 4 | 8.0 | 36.4 | 48.6 | 284.0 | 14.0 |
| | 5 | 8.7 | 42.0 | 44.1 | 288.3 | 14.4 |
| 101 | 1 | 17.0 | 122.7 | 83.9 | 565.9 | 13.6 |
| | 2 | 20.1 | 170.0 | 92.2 | 659.9 | 15.7 |
| | 3 | 19.4 | 168.4 | 110.2 | 658.5 | 15.3 |
| | 4 | 17.9 | 147.1 | 92.0 | 590.1 | 14.3 |
| | 5 | 20.4 | 180.4 | 93.6 | 631.8 | 15.4 |
| 201 | 1 | 40.7 | 589.5 | 218.1 | 1119.7 | 14.0 |
| | 2 | 41.6 | 570.6 | 207.3 | 1223.6 | 14.6 |
| | 3 | 41.9 | 603.6 | 174.4 | 1260.4 | 14.7 |
| | 4 | 37.7 | 508.1 | 166.3 | 1160.8 | 13.6 |
| | 5 | 35.9 | 447.6 | 197.4 | 1130.0 | 13.2 |
| 501 | 1 | 117.9 | 3567.4 | 618.0 | 2866.3 | 14.1 |
| | 2 | 115.1 | 3443.5 | 434.3 | 3001.9 | 14.1 |
| | 3 | 111.7 | 3320.5 | 525.6 | 2911.0 | 13.8 |
| | 4 | 114.2 | 3368.3 | 413.2 | 2931.8 | 14.1 |
| | 5 | 115.4 | 3158.2 | 446.1 | 2949.8 | 13.7 |

**Table 9**
Computing times for DE - Random.

| $n_T$ | | MDA* | A* | LSET | LCOR | #p |
|---|---|---|---|---|---|---|
| 26 | 1 | 41.9 | 89.1 | 2436.2 | 1845.4 | 20.8 |
| | 2 | 25.4 | 65.3 | 2003.1 | 1294.0 | 14.6 |
| | 3 | 60.8 | 227.1 | 2338.6 | 1914.2 | 20.9 |
| | 4 | 38.3 | 100.0 | 2319.0 | 2426.0 | 21.8 |
| | 5 | 49.7 | 126.8 | 2091.5 | 1693.1 | 16.3 |
| 51 | 1 | 83.7 | 275.1 | 4350.8 | 2978.5 | 16.9 |
| | 2 | 145.1 | 677.3 | 4702.5 | 4464.7 | 22.5 |
| | 3 | 175.1 | 647.8 | 4397.2 | 5331.1 | 22.9 |
| | 4 | 136.8 | 565.4 | 5353.1 | 4093.4 | 19.4 |
| | 5 | 122.0 | 428.1 | 4609.7 | 3976.9 | 19.0 |
| 101 | 1 | 280.2 | 1795.0 | 8993.5 | 7332.3 | 19.3 |
| | 2 | 391.5 | 2536.7 | 9621.7 | 9492.8 | 21.3 |
| | 3 | 284.9 | 1656.3 | 9547.9 | 7856.1 | 18.7 |
| | 4 | 283.3 | 1511.6 | 9235.2 | 8017.7 | 18.5 |
| | 5 | 299.5 | 1695.9 | 9120.2 | 7324.3 | 17.2 |
| 201 | 1 | 797.8 | 8648.8 | 18258.2 | 16861.5 | 20.2 |
| | 2 | 671.5 | 6257.1 | 18681.4 | 15227.2 | 18.5 |
| | 3 | 640.2 | 5823.0 | 18601.4 | 14473.7 | 16.5 |
| | 4 | 722.6 | 7472.9 | 21660.4 | 16124.0 | 19.0 |
| | 5 | 743.2 | 6655.7 | 17283.4 | 15907.9 | 18.2 |
| 501 | 1 | 2148.6 | 48383.4 | 45874.4 | 39798.4 | 19.0 |
| | 2 | 2086.3 | 42608.7 | 47167.5 | 39141.1 | 18.0 |
| | 3 | 2075.5 | 38102.2 | 52560.1 | 39243.8 | 18.5 |
| | 4 | 2108.1 | 45133.2 | 49693.9 | 37259.4 | 18.6 |
| | 5 | 2404.2 | 51901.8 | 49771.4 | 43776.5 | 20.7 |

**Table 10**
Computing times for DE - Centered.

| $n_T$ | | MDA* | A* | LSET | LCOR | #p |
|---|---|---|---|---|---|---|
| 26 | 1 | 5.3 | 6.6 | 923.5 | 1038.2 | 5.2 |
| | 2 | 6.0 | 8.7 | 854.2 | 967.0 | 6.6 |
| | 3 | 5.2 | 6.6 | 983.4 | 915.5 | 5.3 |
| | 4 | 5.8 | 7.8 | 790.3 | 1130.4 | 5.9 |
| | 5 | 5.4 | 7.9 | 1239.0 | 1305.5 | 6.4 |
| 51 | 1 | 10.8 | 16.2 | 1710.3 | 1860.8 | 5.2 |
| | 2 | 11.1 | 18.8 | 1769.1 | 2296.3 | 5.5 |
| | 3 | 12.2 | 24.5 | 2295.2 | 2460.0 | 6.8 |
| | 4 | 11.2 | 17.5 | 1799.8 | 2217.7 | 6.0 |
| | 5 | 11.5 | 19.4 | 2070.7 | 1925.9 | 6.0 |
| 101 | 1 | 23.6 | 51.4 | 3293.0 | 4086.3 | 5.3 |
| | 2 | 25.3 | 65.9 | 3865.4 | 4740.1 | 6.4 |
| | 3 | 24.7 | 60.8 | 3732.4 | 4201.3 | 6.1 |
| | 4 | 23.3 | 55.8 | 3983.9 | 4000.4 | 5.5 |
| | 5 | 22.2 | 52.0 | 3743.0 | 4566.0 | 5.4 |
| 201 | 1 | 54.6 | 187.6 | 6662.8 | 8491.9 | 5.6 |
| | 2 | 55.5 | 200.7 | 7284.9 | 8216.5 | 5.8 |
| | 3 | 54.0 | 200.5 | 7479.9 | 8329.5 | 5.9 |
| | 4 | 58.7 | 218.9 | 7469.4 | 9338.0 | 6.4 |
| | 5 | 57.9 | 261.6 | 7929.7 | 8876.5 | 6.4 |
| 501 | 1 | 189.4 | 1112.5 | 16953.1 | 21155.7 | 5.7 |
| | 2 | 187.1 | 1333.4 | 19643.9 | 21895.6 | 6.3 |
| | 3 | 188.9 | 1295.8 | 17553.3 | 21035.5 | 6.0 |
| | 4 | 188.7 | 1324.7 | 17387.9 | 21497.7 | 6.2 |
| | 5 | 200.5 | 1305.3 | 18159.4 | 21075.8 | 6.0 |

is different for the Random and Centered configurations. In the Centered configuration, the average distance between terminals is smaller and one could expect a more limited number of efficient paths. This is confirmed by experiments, but, surprisingly, the reduction factor is extremely dependent on the network (almost null for DC, very large for DE or AK). Anyway, one can see that the average size of efficient sets approximately varies between 3 and 40, which can be interpreted in two different manners. On the one hand, it remains very limited compared to the theoretical exponential-size that these sets could have. On the other hand, time and distance are strongly correlated and it is generally admitted that the longer the distance, the longer the traveling time. Values of #p show that it is not completely true and that the number of efficient paths between two terminals in a road-network can be relatively important.

### 6.3. Results in the context of the VRPTW

We now evaluate the impact of the enhancements proposed in Section 5 when time windows are introduced. For the sake of brevity, we limit our experiments to the 25 AIX-2 instances and the 25 DC instances, with nodes randomly drawn. Time windows are added as follows:

1. A node, denoted $s_0$, is randomly selected in $T$ to represent the depot; other terminals represent customers.
2. The depot time window $[0, l_0]$ defines the time horizon and is set such that every node in $T \setminus \{s_0\}$ can be visited on a back-and-forth route.
3. A feasible VRP solution is then constructed with a simple greedy algorithm.

**Table 11**
Computing times for RI - Random.

| $n_T$ | | MDA* | A* | LSET | LCOR | #p |
|---|---|---|---|---|---|---|
| 26 | 1 | 247.6 | 809.6 | 1711.0 | 12453.8 | 35.1 |
| | 2 | 233.3 | 685.1 | 1841.4 | 11315.9 | 31.7 |
| | 3 | 361.6 | 1197.9 | 2099.0 | 15145.3 | 40.0 |
| | 4 | 235.1 | 725.7 | 2327.8 | 13142.7 | 37.3 |
| | 5 | 758.2 | 2566.0 | 3693.9 | 28150.2 | 57.9 |
| 51 | 1 | 596.0 | 2674.4 | 3433.6 | 23524.0 | 32.6 |
| | 2 | 758.7 | 3905.8 | 4168.2 | 26257.5 | 38.0 |
| | 3 | 1166.0 | 8649.9 | 5739.0 | 41957.7 | 49.4 |
| | 4 | 556.5 | 2763.9 | 3448.7 | 22908.7 | 34.2 |
| | 5 | 711.1 | 3796.6 | 4530.1 | 27262.1 | 34.7 |
| 101 | 1 | 1513.8 | 13173.9 | 7454.2 | 49607.5 | 35.3 |
| | 2 | 1943.1 | 24201.7 | 9107.0 | 64238.5 | 42.3 |
| | 3 | 1504.2 | 14482.2 | 8485.1 | 51957.0 | 35.7 |
| | 4 | 1645.6 | 18710.7 | 8534.4 | 54202.1 | 38.4 |
| | 5 | 1073.2 | 8629.2 | 8233.8 | 46816.5 | 33.1 |
| 201 | 1 | 3105.8 | 54667.5 | 16072.7 | 100117.0 | 36.0 |
| | 2 | 3020.0 | 51102.8 | 15736.8 | 96977.3 | 33.8 |
| | 3 | 3125.2 | 53022.5 | 15586.0 | 101538.0 | 36.2 |
| | 4 | 3481.9 | 70638.8 | 18784.7 | 111291.0 | 36.7 |
| | 5 | 3732.5 | 77161.3 | 18149.0 | 116441.0 | 39.5 |
| 501 | 1 | 8220.0 | 333443.0 | 39876.1 | 244181.0 | 34.9 |
| | 2 | 9371.8 | 36338.7 | 46338.7 | 276690.0 | 37.9 |
| | 3 | 8154.8 | 308090.5 | 40890.9 | 26892.7 | 34.1 |
| | 4 | 9027.8 | 328953.4 | 42953.3 | 27954.1 | 37.5 |
| | 5 | 9250.2 | 320034.0 | 40035.0 | 25021.1 | 37.5 |

**Table 12**
Computing times for RI - Centered.

| $n_T$ | | MDA* | A* | LSET | LCOR | #p |
|---|---|---|---|---|---|---|
| 26 | 1 | 148.7 | 660.6 | 1996.1 | 12324.5 | 27.0 |
| | 2 | 111.9 | 409.1 | 1699.6 | 11821.7 | 29.4 |
| | 3 | 87.9 | 218.8 | 1464.7 | 7863.0 | 18.6 |
| | 4 | 98.2 | 375.1 | 1815.6 | 8796.0 | 20.7 |
| | 5 | 58.1 | 157.7 | 1378.8 | 9121.0 | 20.3 |
| 51 | 1 | 269.3 | 1657.9 | 3376.6 | 21111.2 | 24.6 |
| | 2 | 142.2 | 498.3 | 2292.4 | 13476.7 | 16.9 |
| | 3 | 140.5 | 622.1 | 2522.3 | 18777.3 | 21.3 |
| | 4 | 222.6 | 1085.6 | 2537.6 | 20031.5 | 24.2 |
| | 5 | 302.3 | 1916.6 | 3490.5 | 21924.8 | 30.3 |
| 101 | 1 | 463.1 | 3773.2 | 5407.8 | 33954.8 | 21.1 |
| | 2 | 347.5 | 2288.8 | 4789.9 | 36727.3 | 22.2 |
| | 3 | 541.4 | 4926.1 | 6004.2 | 40479.8 | 25.7 |
| | 4 | 398.0 | 2889.1 | 5001.0 | 37140.0 | 21.8 |
| | 5 | 461.4 | 3398.0 | 6625.2 | 40491.0 | 27.3 |
| 201 | 1 | 950.0 | 11324.7 | 9809.7 | 68533.0 | 21.0 |
| | 2 | 979.3 | 12219.7 | 10539.0 | 71759.8 | 22.9 |
| | 3 | 997.9 | 12035.4 | 11843.0 | 76078.0 | 24.8 |
| | 4 | 910.5 | 11935.3 | 11438.4 | 76861.6 | 24.2 |
| | 5 | 693.9 | 6598.2 | 9920.7 | 57901.8 | 18.0 |
| 501 | 1 | 2750.0 | 71994.4 | 24948.2 | 176894.0 | 22.4 |
| | 2 | 2465.4 | 61735.4 | 26412.1 | 170807.0 | 21.5 |
| | 3 | 2350.1 | 72256.9 | 24382.1 | 173560.0 | 22.3 |
| | 4 | 2821.9 | 71924.2 | 22053.0 | 167324.0 | 21.0 |
| | 5 | 2742.2 | 72347.3 | 23655.0 | 164094.0 | 21.0 |

4. Finally, time windows $[e_s, l_s]$ are defined for all customers $s \in T \setminus \{s_0\}$, centered on the visit time in the greedy solution and with a width equal to $\frac{l_0}{5}$ (truncated to make sure that $e_s \geq 0$ and $l_s \leq l_0$).

Results are presented in Tables 15 and 16. In these tables, Column "$|T^+(v_0)|$" reports the average number of reachable destinations; Column "# removed" indicates the average number of nodes that could be removed. Column "CPU %" indicates the average impact on computing time (in %), including preprocessing. Column "#p" indicates the average impact on the number of efficient paths found between two terminals (in %). Note that this gap is not null because it compares the number of paths with or without time windows.

From these tables, we notice that, with our enhancements, the number of considered destinations $|T^+(v_0)|$ is significantly reduced: approxi-

**Table 13**
Computing times for AK - Random.

| $n_T$ | | MDA* | A* | LSET | LCOR | #p |
|---|---|---|---|---|---|---|
| 26 | 1 | 217.6 | 733.1 | 1111.6 | 2731.9 | 34.2 |
| | 2 | 164.0 | 626.8 | 1127.4 | 2303.3 | 35.8 |
| | 3 | 292.1 | 1050.0 | 1309.6 | 3037.7 | 37.3 |
| | 4 | 161.1 | 633.4 | 1053.8 | 2665.7 | 35.9 |
| | 5 | 260.8 | 1553.4 | 1456.1 | 3147.2 | 45.8 |
| 51 | 1 | 423.5 | 3077.9 | 2184.7 | 4971.3 | 36.0 |
| | 2 | 539.5 | 3389.7 | 2322.9 | 5600.6 | 39.2 |
| | 3 | 448.7 | 3295.3 | 2307.0 | 5347.0 | 38.2 |
| | 4 | 402.5 | 2502.3 | 2008.4 | 4635.5 | 33.7 |
| | 5 | 460.5 | 3274.5 | 2001.6 | 5273.7 | 37.2 |
| 101 | 1 | 1109.6 | 12536.9 | 4454.1 | 10095.5 | 37.0 |
| | 2 | 960.3 | 11787.0 | 4285.7 | 10050.4 | 36.0 |
| | 3 | 945.5 | 9965.8 | 3788.1 | 9376.4 | 33.7 |
| | 4 | 1151.9 | 14056.3 | 4720.7 | 10744.0 | 37.5 |
| | 5 | 1057.6 | 9883.7 | 4473.6 | 10041.0 | 34.5 |
| 201 | 1 | 2396.4 | 49467.7 | 8609.2 | 20472.4 | 37.1 |
| | 2 | 2355.0 | 46347.2 | 8381.8 | 19343.5 | 35.3 |
| | 3 | 2503.1 | 47041.6 | 9202.7 | 21304.7 | 37.0 |
| | 4 | 2499.4 | 57734.6 | 8992.6 | 22188.9 | 39.8 |
| | 5 | 2338.8 | 54599.6 | 9636.2 | 20946.5 | 37.4 |
| 501 | 1 | 6584.5 | 291573.0 | 21734.9 | 50337.2 | 35.8 |
| | 2 | 6837.0 | 349924.0 | 25313.8 | 54515.3 | 38.6 |
| | 3 | 6597.9 | 300878.0 | 20646.0 | 50640.3 | 36.2 |
| | 4 | 6316.4 | 305386.0 | 21351.3 | 50216.9 | 36.2 |
| | 5 | 7058.7 | 337687.0 | 23276.6 | 53547.0 | 38.2 |

**Table 14**
Computing times for AK - Centered.

| $n_T$ | | MDA* | A* | LSET | LCOR | #p |
|---|---|---|---|---|---|---|
| 26 | 1 | 109.4 | 138.5 | 1425.1 | 2563.6 | 12.6 |
| | 2 | 13.8 | 19.2 | 1357.8 | 2291.2 | 9.5 |
| | 3 | 14.4 | 20.8 | 1335.7 | 2729.2 | 11.3 |
| | 4 | 145.3 | 195.2 | 1142.8 | 2313.4 | 14.1 |
| | 5 | 209.0 | 282.8 | 1840.7 | 3023.6 | 14.3 |
| 51 | 1 | 231.7 | 357.6 | 2822.4 | 5368.9 | 10.5 |
| | 2 | 441.1 | 630.5 | 2454.0 | 4946.7 | 11.5 |
| | 3 | 453.1 | 663.2 | 2902.1 | 5440.4 | 11.9 |
| | 4 | 227.9 | 358.9 | 2736.7 | 4679.5 | 11.1 |
| | 5 | 27.2 | 58.2 | 3186.7 | 4323.9 | 9.1 |
| 101 | 1 | 476.6 | 781.5 | 5086.1 | 9504.9 | 9.3 |
| | 2 | 662.4 | 1006.1 | 5507.3 | 9885.5 | 10.1 |
| | 3 | 54.8 | 148.5 | 5626.3 | 8105.5 | 7.8 |
| | 4 | 858.7 | 1320.1 | 6437.0 | 10034.1 | 10.1 |
| | 5 | 737.0 | 1177.6 | 5190.0 | 9748.5 | 9.9 |
| 201 | 1 | 1000.5 | 1900.2 | 10170.1 | 20245.3 | 9.0 |
| | 2 | 159.3 | 590.0 | 11884.4 | 17890.7 | 8.1 |
| | 3 | 881.2 | 1728.8 | 9777.2 | 20692.0 | 8.7 |
| | 4 | 802.8 | 1561.9 | 10312.3 | 19110.1 | 8.3 |
| | 5 | 1946.2 | 3110.6 | 10868.0 | 22597.3 | 9.0 |
| 501 | 1 | 1827.2 | 5095.0 | 26988.8 | 46531.6 | 7.8 |
| | 2 | 3432.9 | 7138.1 | 27202.1 | 46359.8 | 8.3 |
| | 3 | 2094.9 | 5431.2 | 29882.9 | 47885.6 | 7.7 |
| | 4 | 2719.3 | 6828.1 | 30291.2 | 44805.7 | 8.2 |
| | 5 | 2738.2 | 6256.0 | 28583.7 | 46866.3 | 7.8 |

**Table 15**
Impact of enhancements on network AIX-2 - Random.

| $n_T$ | $\lvert T^+(v_0)\rvert$ | # removed | CPU % | #p % |
|---|---|---|---|---|
| 26 | 13 | 2881 | -50.3 | -64.0 |
| 51 | 26 | 3766 | -41.3 | -61.1 |
| 101 | 51 | 3373 | -37.6 | -63.1 |
| 201 | 101 | 3311 | -38.4 | -63.6 |
| 501 | 255 | 2902 | -26.3 | -61.8 |

**Table 16**
Impact of enhancements on network DC.

| $n_T$ | $\lvert T^+(v_0)\rvert$ | # removed | CPU % | #p % |
|---|---|---|---|---|
| 26 | 14 | 1245 | -32.4 | -52.8 |
| 51 | 27 | 1778 | -41.8 | -54.2 |
| 101 | 51 | 1821 | -37.7 | -56.6 |
| 201 | 95 | 1984 | -37.4 | -60.8 |
| 501 | 239 | 2000 | -32.2 | -60.6 |

mately divided by 2 on average. We also see that the number of removed nodes is relatively important. Regarding computing times, it clearly appears that the algorithm is faster when the enhancements are used. Regarding the number of paths, one can observe that the size of complete sets of efficient paths is divided by more than 2.

## 7. Conclusion

In this paper, we introduced and investigated the Steiner bi-objective shortest path problem. The particularity of this problem is to seek for complete set of efficient paths linking a subset of nodes in the network, the so-called terminals. We developed an exact solution approach based on a labeling algorithm combined with a modified $A^*$ algorithm. The proposed approach is based on a goal-directed search strategy that guides labels simultaneously towards all terminals and allows stopping the search quickly once all efficient paths have been found. The motivation for this new problem stems from the preprocessing of travel information for vehicle routing problems and we proposed and evaluated several enhancements in the case of the VRPTW.

Computational experiments were carried out on a large panel of instances based on real road networks. Results show that the proposed algorithm performs very well for all tested instances. The MDA* algorithm largely outperforms state-of-the-art algorithms. Additional savings are obtained when taking into account time windows.

Experiments also give insights on the solution of vehicle routing problems with multigraphs. They show that multigraphs of acceptable size should be obtained and that these graphs could be computed in a reasonable amount of time for road-network graphs with dozens of thousand of nodes and hundreds of customers.

Would these computing times be considered too high in practice, several possible adaptations of our algorithms could be possible. For example, computations might be simplified a lot by only keeping the min-time and min-cost paths. Also, some heuristic rules might be introduced in the dynamic programming algorithm to limit the number of labels. Mono-objective shortest path problems with different weighted sums for travel time and cost could also be solved. Of course, with all these approaches, the theoretical guarantee of optimality would be lost for the subsequent vehicle routing problem.

## Declaration of Competing Interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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