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On Coloring a Class of Claw-free Graphs

To the memory of Frédéric Maffray

Yingjun Dai Angèle M. Foley Chính T. Hoàng

Department of Physics and Computer Science
Wilfrid Laurier University
Waterloo, Canada

Abstract

Given a set L of graphs, a graph G is L-free if G does not contain any graph in L as an induced subgraph. Recently, Frédéric Maffray and co-authors showed that the problem of coloring $\{claw, 4K_1, K_5 \setminus e\}$-free graphs can be solved in polynomial time. In this paper, we investigate a related class of graphs. A *hole* is an induced cycle of length at least 4. Two vertices x, y of a graph G are *twins* if for any vertex z different from x and y, xz is an edge if and only if yz is an edge. A *hole-twin* is the graph obtained from a hole by adding a vertex that forms a twin with some vertex of the hole. Hole-twins, and $K_5 \setminus e$, are interesting in their connection with line-graphs. They are among the forbidden subgraphs in the characterization of line-graphs. In this paper, we show there is a polynomial time algorithm to color ($claw$, $4K_1$, hole-twin)-free graphs.

Keywords: Graph coloring, Line-Graph, *Claw*, Hole-twin, C_6-*twin*, C_5-*twin*, C_4-*twin*, P_5-twin

1 The result

Motivated by results in a recent paper of Frédéric Maffray and co-authors [9], we investigate a graph coloring problem on ($claw$, $4K_1$, hole-twin)-free graphs (definitions not given here will be given later). Graph coloring is one of the most fundamental problems in graph theory, with many theoretical and practical results. We can define *k-coloring*, for some integer k, of a graph G, as a mapping $f : V(G) \rightarrow \{1, \dots, k\}$ such that $f(u) \neq f(v)$ for any two adjacent vertices $u, v \in V(G)$. The key question in graph coloring is: what is the smallest integer k for which such a coloring exists? This number k is called the *chromatic number*, $\chi(G)$, of the graph. Determining the chromatic number of a graph is called the VERTEX COLORING problem and is known to be NP-hard in general [14,10]. However, for some specific graph classes, the problem can be solved in polynomial time.

We say a graph G is $\mathcal{H}$-free for some set $\mathcal{H}$ of graphs if G does not contain any member of $\mathcal{H}$ as an induced subgraph. The papers [9] and [17] study $\mathcal{H}^4$-free graphs, where $\mathcal{H}^4$ is any set of four vertex graphs. For any $\mathcal{H}^4$, VERTEX COLORING for

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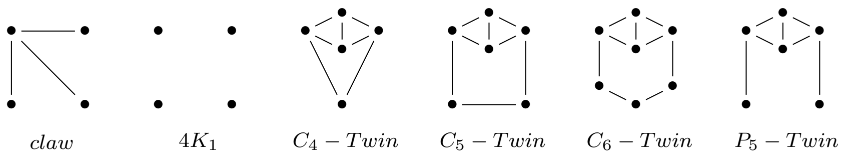

Fig. 1. The graphs discussed in this paper

$\mathcal{H}^4$-free is known to be NP-complete or solvable in polynomial time, with three exceptions. One of the exceptions is the class of $\{claw, 4K_1\}$-free graphs. The result of [9] shows that VERTEX COLORING is solvable in polynomial time for (claw, $4K_1$, $K_5 \setminus e$)-free graphs. This class was further significant as it contains the $4K_1$-free line-graphs.

Given a graph G, the *line-graph* $L(G)$ of G is defined to be the graph whose vertices are the edges of G, and two vertices of $L(G)$ are adjacent if their corresponding edges in G are incident. Beineke [1] derived a characterization of line-graphs in terms of a set of nine forbidden induced subgraphs, two of which are the claw and $K_5 \setminus e$. Thus given the connection of (claw, $4K_1$, $K_5 \setminus e$)-free graphs to $4K_1$-free line-graphs, it is natural to ask about polynomial time solvability of VERTEX COLORING of graphs where $K_5 \setminus e$ is replaced by one of the other nine forbidden induced subgraphs. In fact we go further and pose the question for a more general class, which we introduce and call *hole-twin*. A *hole* is an induced cycle of length at least 4. Two vertices x, y of a graph G are *twins* if xy is an edge, and for any vertex z different from x and y, xz is an edge if and only if yz is an edge. A *hole-twin* is the graph obtained from a hole by adding a vertex that form a twin with some vertex of the hole. Figure 1 shows the three smallest hole-twins: C_4-twins, C_5-twins, and C_6-twins.

The motivation behind defining the hole-twin class is the observation that three of Beineke's nine forbidden graphs (C_4-twin, C_5-twin and P_5-twin (see Figure 1)) are hole-twins or an induced subgraph of a hole-twin. Thus the hole-twin generalizes this idea by allowing holes of arbitrary length. It is now natural to wonder about the complexity of coloring (*claw*, $4K_1$, hole-twin)-free graphs.

The purpose of this paper is to prove the following theorem.

Theorem 1.1 *There is a polynomial time algorithm to color (claw, $4K_1$, hole-twin)-free graphs.*

In Section 2, we discuss the background results needed to prove our main theorem. In Section 3, we prove Theorem 1.1 along with necessary lemmas and ancillary results. And finally, in Section 4, we discuss the significance of our work, along with some open problems.

2 Definitions and Background

In this section, we discuss the background results needed to prove our main theorem. The section has two subsections. In the first subsection, we discuss claw-free graphs.

In the second section, we discuss clique widths.

2.1 Claw-free graphs

Let $\alpha(G)$ denote the number of vertices in the largest stable set of G. Let $\chi(G)$ denotes the chromatic number of G, and $\omega(G)$ denote the number of vertices in a largest clique of G. A k-hole is a hole with k vertices.

Our results rely on known theorems on perfect claw-free graphs, and we discuss these results now. A graph is *Berge* if it does not contain as induced subgraph an odd hole, or an odd antihole (complement of a hole). A graph G is *perfect* if for each induced subgraph H of G, we have $\chi(H) = \omega(H)$. Parthasarathy and Ravindra [18] proved that claw-free Berge graphs are perfect. Chvátal and Sbihi [7] showed that claw-free perfect graphs can be recognized in polynomial time. Hsu [13] showed that these graphs can be colored in polynomial time. We note that Chudnovsky, Robertson, Seymour and Thomas [6] proved that a graph is perfect if and only if it is Berge, solving a long standing conjecture of Berge [2]. Perfect graphs can be recognized in polynomial time (Chudnovsky, Cornuejols, Liu, Seymour and Vušković [5]), and they can be optimally colored in polynomial time (Grötschel, Lovász and Schrijver [11]). For more information on perfect graphs, see Berge and Chvátal [3]. In the paper of Chvátal and Sbihi [7], the following result ("Ben Rebea's Lemma"), crucial to our algorithm, is established (in this paper, "contains" means "contains as induced subgraph").

Lemma 2.1 *[7] Let G be a connected claw-free graph with $\alpha(G) \geq 3$. If G contains an odd antihole then G contains a C_5.*

It is well known that VERTEX COLORING is polynomial time solvable for graphs G with $\alpha(G) = 2$. In the next section, we discuss the notion of the clique widths of graphs.

2.2 Clique widths of graphs

Consider the following operations to build a graph.

- Create a vertex u labeled by integer ℓ.
- Make the disjoint union of several graphs.
- For some pair of distinct labels i and j, add **all** edges between vertices with label i and vertices with label j.
- For some pair of distinct labels i and j, relabel **all** vertices of label i by label j.

In regards to the four operations above, when a new vertex is created, it must be assigned a label. The *clique-width* [8] of a graph is the minimum number of labels needed to build the graph with the above four operations. Given sets of vertices X, Y, we write $X \; 0 \; Y$ to mean there are no edges between any vertex in X and any vertex in Y; this structure is called the *co-join*. We write $X \; 1 \; Y$ to mean there are all edges between X and Y, this structure is referred to as the *join* of X and Y.

The following three observations are folklore (see [4]).

Observation 2.1 *[Folklore] Let G be a graph and let X be a set of vertices of G of bounded size. Then G has bounded clique width if and only if $G - X$ have bounded clique width.*

Observation 2.2 *[Folklore] Let G be a graph such that G is the join of two graphs G_1 and G_2. Then G has bounded clique width if and only if both G_i have bounded clique widths.*

Observation 2.3 *[Folklore] Let G be a graph such that G is the co-join of two graphs G_1 and G_2. Then G has bounded clique width if and only if both G_i have bounded clique widths.*

Rao [19], improving a result of Koblera and Rotics [15], proved the following theorem.

Theorem 2.2 ([19]) *For any constant c,* VERTEX COLORING *is polynomial-time solvable for the class of graphs with clique-width at most c.*

Our main result will rely on the fact that a special class of graphs (in the Lemma below) has bounded clique width.

Lemma 2.3 *Let G be a graph such that $V(G)$ can be covered by k (disjoint) cliques $X_1, \dots, X_k$. For a vertex x, let X_{i_x} be the clique containing x, and let $N_F(x)$ be the set of neigbours y of x such that $y \in X_j$ for $j \neq i_x$. Suppose G satisfies the following conditions: (i) for every vertex x and any set X_j with $j \neq i_x$, x has at most one neighbor in X_j, and (ii) for any vertex x, $N_F(x)$ is a clique. Then G has clique width at most $2k$.*

Proof. By (i) and (ii), if some two vertices x, y are adjacent with $x \in X_i, y \in X_j, i \neq j$, then we have $N_F(x) - \{y\} = N_F(y) - \{x\}$; that is, x and y have the same neighbourhood in $V(G) - (X_i \cup X_j)$. It follows that we can partition the vertices of G into pairwise disjoint sets $Y_1, Y_2, \dots, Y_t$, $Z = V(G) - (Y_1 \cup Y_2 \cup \ldots \cup Y_t)$, such that the following holds: (1) each Y_s is a clique with at least two vertices, (2) if two vertices x, y are adjacent with $x \in X_i, y \in X_j, i \neq j$, then x and y belong to some clique Y_s, and (3) every edge of G belongs to a clique X_i, or a clique Y_s.

The vertices of a set X_i will be associated with two labels $\ell_{i,new}, \ell_{i,old}$. We will label the vertices of G one by one. Suppose we are about to label a vertex x.

(i) If there is a vertex with a label $\ell_{i,new}$, re-label it with label $\ell_{i,old}$ for all i.

(ii) Label x with label $\ell_{i_x,new}$ (X_{i_x} is the set containing x)

(iii) For each neighbour y of x in a set X_{i_y}, label y with label $\ell_{i_y,new}$

(iv) Add edges between vertices with *new* labels (building the clique Y_s)

(v) Add edges between vertices of label $\ell_{i,new}$ and label $\ell_{i,old}$ for all i (building the cliques X_i).

(vi) Re-label all vertices of label $\ell_{i,new}$ with label $\ell_{i,old}$ for all i.

We repeat the above steps until all vertices are labeled. We will use $2k$ labels.

This proves the lemma.

In the next section, we will establish a number of intermediate results before proving Theorem 1.1.

3 The proofs

In this section, we will assume that G is a connected (*claw*, $4K_1$, hole-twin)-free graph, and we will focus on what happens when G contains an odd hole. We know G contains no hole C_k with $k \geq 8$ since G is $4K_1$-free. So we assume G contains a C_7 or a C_5.

Lemma 3.1 *Let G be a connected (claw, $4K_1$, hole-twin)-free graph. If G contains a C_7, then G has at most 21 vertices.*

Proof. Suppose that G contains a 7-hole H, with vertices $h_1, ..., h_7$ and edges $h_i h_{i+1}$, with the subscripts taken modulo 7. A vertex in $G - H$ is a *k-vertex* if it is adjacent to k vertices in H.

Let Y_i denote the set of 4-vertices adjacent to h_i, h_{i+1}, h_{i+2}, h_{i+3}. Let Z_i denote the sets of 4-vertices adjacent to h_i, h_{i+1}, h_{i+3}, h_{i+4}. It is easy to see that a 4-vertex must be of type Y_i, or Z_i.

Observation 3.1 *G has no k-vertex $\forall\, k \in \{0, 1, 2, 3, 5, 6, 7\}$.*

Proof. If G has a k-vertex, for $k \in \{0,\ 1,\ 2\}$, then G contains a $4K_1$. If G has a k-vertex, for $k \in \{5, 6, 7\}$, then G contains a claw. If there exists a 3-vertex, then G contains a C_7-twin, or a claw.

From the above observations, it follows that a vertex in $G - H$ must be of type Y_i, or Z_i.

Observation 3.2 *Y_i is a clique.*

Proof. If Y_i contains non-adjacent vertices u,v, then $\{h_{i+3}, h_{i+4}, u, v\}$ induces a claw.

Observation 3.3 $|Y_i| \leq 1$ *for any i.*

Proof. Suppose some Y_i has at least two vertices u, v. Then uv is an edge by Observation 3.2, and so $\{h_{i+3}, h_{i+4}, h_{i+5}, h_{i+6}, h_i, u, v\}$ induces a C_6-twin.

Observation 3.4 *Z_i is a clique.*

Proof. If Z_i contains non-adjacent vertices u,v, then $\{h_i, h_{i+6}, u, v\}$ induces a claw.

Observation 3.5 $|Z_i| \leq 1$ *for any i.*

Proof. Suppose some Z_i has at least two vertices u, v. Then uv is an edge by Observation 3.4, and so $\{h_{i+4}, h_{i+5}, h_{i+6}, h_i, u, v\}$ induces a C_5-twin.

From the above observations, we have $|V(G)| = |V(H)| + \sum_{i=1}^{7} |Z_i| + \sum_{i=1}^{7} |Y_i| \leq 21$. We have established Lemma 3.1.

Lemma 3.2 *Let G be a connected (claw, $4K_1$, hole-twin)-free graph. If G contains a C_5, then either $\alpha(G) = 2$, or G has bounded clique width, or both.*

Proof. Suppose G contains a 5-hole H, with vertices $h_1, ..., h_5$, and edges $h_i h_{i+1}$ with the subscripts taken modulo 5. We begin with an observation.

Observation 3.6 *G has no k-vertex $\forall\, k \in \{1, 3\}$.*

Proof. Suppose G has 1-vertex, then G contains a claw. If there exists some 3-vertex, then G contains a C_5-twin or a claw.

Next, we define the following sets, for each $i \in \{1, ..., 5\}$.

- Let X_i be the set of 2-vertices adjacent to h_{i-2} and h_{i+2}.
- Let Y_i be the set of 4-vertices not adjacent to h_i.
- Let R be the set of 0-vertices.
- Let T be the set of 5-vertices.

Let $Y = Y_1 \cup \ldots \cup Y_5$, and $X = X_1 \cup \ldots \cup X_5$. From Observation 3.6, we have $V(G) = Y \cup X \cup R \cup T$.

We will need a number of observations below.

Observation 3.7 *We have T 0 R.*

Proof. If there is an edge between a vertex $t \in T$ and a vertex $r \in R$, then G has a claw with t, r, and some two vertices in H.

Observation 3.8 *We have T 0 X.*

Proof. If there is an edge between a vertex $t \in T$ and a vertex $x \in X$, then G has a claw with t, x, and some two vertices (that are non-neighbors of x) in H.

Observation 3.9 *We have T 1 Y.*

Proof. Suppose a vertex $t \in T$ is not adjacent to some vertex $y \in Y_i$ for some i. Then the set $\{y, h_{i-1}, t, h_{i+1}, h_i\}$ induces a C_4-twin.

Observation 3.10 *We have R 0 Y.*

Proof. If there is an edge between a vertex $r \in R$ and a vertex $y \in Y$, then G has a claw with r, y, and some two vertices in H.

Observation 3.11 *X_i is a clique.*

Proof. Let $u, v \in X_i$ and $uv \notin E$. Then $\{u, v, h_{i+1}, h_{i+2}\}$ induces a claw.

Observation 3.12 *Y_i is a clique.*

Proof. Let $u, v \in Y_i$ and $uv \notin E$. Then $\{u, v, h_i, h_{i+1}\}$ induces a claw.

Observation 3.13 *$|Y_i| \leq 1$ for $i = 1, 2, \ldots, 5$.*

Proof. Suppose some Y_i contains two vertices u, v. By Observation 3.12, uv is an edge of G. Now, $\{h_{i-1}, h_i, h_{i+1}, u, v\}$ induces a C_4-twin.

Observation 3.14 *R is a clique.*

Proof. If R is not a clique, then some two non-adjacent vertices of R and some two non-adjacent vertices of H induce a $4K_1$.

Observation 3.15 *A vertex u of X_i cannot be adjacent to two vertices in X_{i+1},* *and by symmetry, u cannot be adjacent to two vertices of X_{i-1}.*

Proof. Let $u \in X_i$, $v, k \in X_{i+1}$ and $uv \in E$, $uk \in E$. Then $\{u, v, h_{i-1}, h_i, h_{i+1}, h_{i+2}, k\}$ induces a C_6-twin.

Observation 3.16 *A vertex u of X_i cannot be adjacent to two vertices in X_{i+2}; and by symmetry, u cannot be adjacent to two vertices of X_{i-2}.*

Proof. Let $u \in X_i$, $v, k \in X_{i+2}$ and $uv \in E$, $uk \in E$. Then $\{u, v, h_i, h_{i+1}, h_{i+2}, k\}$ induces a C_5-twin.

For a vertex $x \in X_i$ for some i, define $N_F(x)$ to be the set of vertices y such that xy is an edge, and $y \in X_j$ for some $j \neq i$. By Observations 3.15 and 3.16, for each $x \in X_i$, we have $|N_F(x)| \leq 4$.

Observation 3.17 *For any i and any vertex $x \in X_i$, the set $N_F(x)$ is a clique.*

Proof. We will prove the Observation by contradiction. Let x be a vertex in X_i for some i. Suppose $N_F(x)$ is not a clique, and so there are non-adjacent vertices $y, z \in N_F(x)$. First, let us suppose $y \in X_{i+1}$. If $z \in X_{i+2} \cup X_{i-2}$, then the set $\{x, y, z, h_{i+2}\}$ induces a claw. Thus, z belongs to X_{i-1}, but now $\{h_{i+1}, h_i, h_{i-1}, y, x, z, h_{i-2}\}$ induces a C_6-twin. So we know $\{y, z\} \cap (X_{i+1} \cup X_{i-1}) = \emptyset$. Thus, we may assume $y \in X_{i+2}$ and $z \in X_{i-2}$. Now, the set $\{x, y, z, h_{i+2}\}$ induces a claw. We have established the observation.

We now continue the proof of Lemma 3.2. We know $\alpha(T) \leq 2$ for otherwise G has a claw with one vertex in H and some three vertices in T. Suppose T contains two non-adjacent vertices t_1, t_2. Then X has to be empty, for otherwise the set $\{h, x, t_1, t_2\}$ induces a claw, where x is a vertex in X, and h is a neighbour of x in H (by Observation 3.8, X has no neighbours in T). Now, R has to be empty, for otherwise there is an edge rz with $r \in R$ and $z \in Y \cup T$ (since G is connected); and this is a contradiction to Observations 3.10 and 3.7. Now, G is the join of T and $H \cup Y$ by Observation 3.9. The set $Y \cup H$ cannot contain a stable set S on three vertices, for otherwise S and a vertex in T induce a claw. It follows that $\alpha(G) = 2$, and we are done.

So we may assume T is a clique. Note that cliques have clique width 2.

Let G_1 be the subgraph of G obtained by removing all vertices in $H \cup Y$. Since the set Y is finite (by Observations 3.13), by Observation 2.1, we only need to prove G_1 has bounded clique width. In G_1, there are no edges between T (if it is not empty) and $X \cup R$ by Observations 3.7 and 3.8. So, by Observation 2.3, we only need to prove the graph G_2 induced by $X \cup R$ has bounded clique width.

There is an edge between any vertex $r \in R$ and any vertex $x \in X$, for otherwise there is a $4K_1$ containing r, x, and some two vertices of H. So, G_2 is the join of R and X. By Observation 2.2, we only need to prove $G_3 = G_2 - R = X$ has bounded clique width. Recall Observation 3.17 that for each $x \in X_i$, $N_F(x)$ is a clique. Thus, G_3 satisfies the hypothesis of Lemma 2.3, and so it has bounded clique width. The proof of Lemma 3.2 is completed.

Now, we can prove the main theorem.

Proof of Theorem 1.1. Let G be a (*claw*, $4K_1$, hole-twin)-free graph. We may assume that G is connected and has $\alpha(G) \geq 3$. We may assume that G is not perfect, for otherwise we may use the algorithm of Hsu [13] to color a claw-free perfect graph in polynomial time. Thus G contains an odd hole or odd antihole. By Lemma 2.1, we know G must contain an odd hole H. Since $\alpha(G) < 4$, H is a 7-hole or a 5-hole. If H is a 7-hole, then by Lemma 3.1, G has a bounded number of vertices and we are done. So H is a 5-hole. By Lemma 3.2, G has bounded clique width and we are done.

4 Conclusions and open problems

In this paper, we prove that VERTEX COLORING can be solved in polynomial time for the class of ($claw$, $4K_1$, hole-twin)-free graphs. The problem is NP-hard for claw-free graphs [12], and for $4K_1$-free graphs [16]. The complexity of VERTEX COLORING is unknown for the class of $\{claw, 4K_1\}$-free graphs. See [9] and [17] for the background of this problem. We pose this as an open problem to conclude our paper.

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