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The Meet-continuity of L-semilattices [1]

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Abstract

This paper is devoted to search for Cartesian closed category of quantitative domain categories. Firstly, inspired by the method of constructing L-frames, the meet-continuity on L-semilattices is built and characterized, which generalizes the meet-continuity on crisp semilattices. Then, it is shown that a complete L-lattice is an L-frame iff it is distributive and meet-continuous. In particular, it is shown that the category L-**MC** of meet-continuous L-lattices is cartesian closed.

Keywords: L-semilattice, Meet-continuity, Cartesian closedness, Fuzzy domain theory

1 Introduction

Domain theory is the mathematic foundation for the computer functional programming languages, plays an important role in theoretical computer science, and quantitative research of it is the basis for more extensive application. In domain theory,

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the basic condition of a domain category becoming one kind of semantic model of a language is that the category is Cartesian closed, so the research on Cartesian closedness of categories is a basic problem. Naturally, as the main research system of quantitative domain—fuzzy domain theory, the research on Cartesian closed category on it has become a very important research topic in quantitative domain.

It is well known that in domain theory, the meet-continuous lattices play an important role. For this reason, the primary question is whether it can be extended to many-valued setting. Since Zadeh [14] proposed the concept of fuzzy orders (or L-orders), some basic notions in domain theory have been extended to many-valued setting. Demirci proposed the concept of vague lattices in [3]; Bělohlávek introduced the concept of lattice fuzzy orders in [1,2]. As the two kinds of lattices are not equivalent, Zhao and Zhang [18] introduced the concepts of weak Ω-lattices and Ω-lattices, and showed that the notion of Ω-lattices coincides with that of lattice fuzzy orders and the notion of weak Ω-lattices coincides with that of vague lattices. Recently, Yao [12,13] proposed a notion of fuzzy frame (or, L-frame) and then successfully established a many-valued version of Parper-Parpert-Isbell-adjunction between the category of the related L-locales and the category of stratified L-topological spaces. Based on the above works, this paper is devoted to studying meet-continuity in the many-valued framework.

The content of the paper is arranged as follows. In Section 2, several basic notions that will be used throughout this paper are listed. In Section 3, firstly, the meet-continuity on L-semilattices is built and characterized. Then, the relationship between meet-continuous L-lattices and L-frames are discussed. Moreover, the cartesian closedness of the category of meet-continuous L-lattices are investigated. Finally, several conclusions are presented in Section 4.

2 Preliminaries

For the convenience of the reader, in this section, some basic concepts are reviewed.

2.1 Ordered sets and Lattices

For domain theory, we refer to [5].

Definition 2.1 (Gierz et al. [5]) Let X be a set. An order on X is a binary relation $\leq$ on X such that, for all $x, y, z \in X$,
(1) $x \leq x$ (reflexivity);
(2) $x \leq y$ and $y \leq z$ imply $x \leq z$ (transitivity);
(3) $x \leq y$ and $y \leq x$ imply $x = y$ (antisymmetry).
A set X equipped with an order relation $\leq$ is said to be an ordered set. A subset D of an ordered set X is directed provided it is nonempty and every finite subset of D has a join in D.

Definition 2.2 (Gierz et al. [5]) (1) An ordered set is called directed complete if every directed subset has a join;
(2) An ordered set is called a join semilattice if every finite subset has a join;

(3) An ordered set is called a meet semilattice if every finite subset has a meet;
(4) An ordered set is called a lattice if every finite subset has a join and a meet;
(5) An ordered set is called a complete lattice if every subset has a join and a meet.

Definition 2.3 (Gierz et al. [5]) (1) A lattice is distributive if for all elements x, y, z, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$;
(2) A frame is a complete lattice which satisfies the following infinite distributive law:

$$x \wedge (\bigvee Y) = \bigvee_{y \in Y}(x \wedge y)$$

for all elements x and all subsets Y.

Definition 2.4 (Gierz et al. [5]) Let $f : X \longrightarrow Y$ be a mapping between two ordered sets, then f is called order preserving iff $x \leq y$ always implies $f(x) \leq f(y)$. Moreover, we say that f preserves (1) finite joins, or (2) arbitrary joins, or (3) directed joins, if, whenever $A \subset X$, is (1) finite, or (2) arbitrary, or (3) directed, and $\bigvee A$ exists in X, then $\bigvee f(A)$ exists in Y and equals $f(\bigvee A)$. We often call that f is a Scott continuous mapping whenever it preserves directed joins.

Proposition 2.5 *Let $f : X \longrightarrow Y$ be an ordered preserving mapping between ordered sets. Then the following is equivalent:*
(1) *f preserves directed joins;*
(2) *f preserves ideal joins.*
Moreover, if X is a join semilattice and f preserves finite joins, then (1) *and* (2) *are also equivalent to*
(3) *f preserves arbitrary joins.*

Definition 2.6 (Gierz et al. [5]) A meet semilattice X is called meet-continuous if it is directed complete and satisfies

$$x \wedge (\bigvee D) = \bigvee_{d \in D}(x \wedge d)$$

for all elements $x \in X$ and all directed subsets $D \subset X$.

Theorem 2.7 *(Gierz et al. [5]) Let X be a lattice, then the following conditions are equivalent:*
(1) *X is a frame;*
(2) *X is meet-continuous and distributive.*

A commutative quantale is a pair $(\Omega, *)$, where Ω is a complete lattice and $*$ (called the tensor) is a commutative, associative, and monotone operator $* : \Omega \times \Omega \longrightarrow \Omega$ such that $p * (-)$ has a right adjoint $p \to (-)$ for every $p \in \Omega$, that is $a * b \leq c$ iff $a \leq b \to c$ for all $a, b, c \in \Omega$. A commutative quantale is called unital if the operator $*$ has a unit I , i.e., $p * I = p$ for every $p \in \Omega$. If the unit I equals to the greatest element 1, then a commutative unital quantale $(\Omega, *, I)$ just is a complete residuated lattice. A complete residuated lattice L with $* = \wedge$ just is a complete Heyting algebra, or called a frame.

Next, L in this paper always denotes a complete residuated lattice. Let X be a set. L^X denotes the set of all L-subsets of X, that is, the set of all mappings from

X to L. Obviously, L^X is a complete lattice under the pointwise order. For $A \subseteq X$, χ_A denotes the characteristic function of A.

2.2 L-ordered sets

Definition 2.8 (Bělohlávek [1,2], Fan [4] and Zhang and Fan [16]) An L-ordered set is a pair (X, e) such that X is a set and $e : X \times X \longrightarrow L$ is a mapping, called an L-order, that satisfies for all $x, y, z \in X$,
(1) $e(x, x) = 1$ (reflexivity);
(2) $e(x, y) * e(y, z) \le e(x, z)$ (transitivity);
(3) $e(x, y) = e(y, x) = 1$ implies $x = y$ (antisymmetry).

For all $a, b \in L$, let $e_L(a, b) = a \to b$. Then (L, e_L) becomes an L-ordered set.

Suppose that (X, e) is an L-ordered set. Let $e^{op}(x, y) = e(y, x)$ for all $x, y \in X$. Then (X, e^{op}) is also an L-ordered set, called the opposite of (X, e). If $Y \subseteq X$, let $e_Y(x, y) = e(x, y)$ for all $x, y \in Y$. Then (Y, e_Y) becomes an L-ordered set, called a (full) sub-L-ordered set of (X, e).

Let X be a set. For all $A, B \in L^X$, the subsethood degree of A in B is defined by $sub_X(A, B) = \bigwedge_{x\in X} A(x) \to B(x)$. Then (L^X, sub_X) is an L-ordered set.

Definition 2.9 (Bělohlávek [1,2], Fan [4] and Xie et al. [8]) Let (X, e_X), (Y, e_Y) be L-ordered sets. Then a mapping $f : X \longrightarrow Y$ is said to be
(1) an L-order preserving mapping if $e_X(x, y) \le e_Y(f(x), f(y))$ for all $x, y \in X$;
(2) an L-order embedding if $e_X(x, y) = e_Y(f(x), f(y))$ for all $x, y \in X$;
(3) an L-order isomorphism if it is an L-order embedding which maps X onto Y. In this case, we say that X and Y are isomorphic.

Given L-ordered sets (X, e_X) and (Y, e_Y) , denotes the set of all the L-order preserving mappings from X to Y by $[X, Y]$. Let $e_{[X,Y]}(f, g) = \bigwedge_{x\in X} e_Y(f(x), g(x))$ for all $f, g \in [X, Y]$. Then $[X, Y]$ becomes an L-ordered set.

Suppose that (X, e) is an L-ordered set. Define a binary relation $\le$ on the underlying set X as follows: $x \le y$ iff $e(x, y) = 1$. It is simple to check that $\le$ is an order on the underlying set X. For each L-ordered set, we write X_0 for the ordered set $(X, \le)$.

Given an L-ordered set (X, e). $A \in L^X$ is called an upper L-subset (resp., a lower L-subset) of X if $A(x) * e(x, y) \le A(y)$ (resp., $A(x) * e(y, x) \le A(y)$) for all $x, y \in X$. For $x \in X$, let $\downarrow x(y) = e(y, x)$ and $\uparrow x(y) = e(x, y)$ for all $y \in X$. Then $\downarrow x$ is a lower L-subset of X, and $\uparrow x$ is an upper L-subset of X. For $A \in L^X$, let $\downarrow A(y) = \bigvee_{x\in X} A(x) * e(y, x)$ and $\uparrow A(y) = \bigvee_{x\in X} A(x) * e(x, y)$ for all $y \in X$. Then it is easy checked that $A \in L^X$ is an upper L-subset (resp., a lower L-subset) iff $A = \downarrow A$ (resp., $A = \uparrow A$).

Definition 2.10 (Zhang and Fan [6]) Let (X, e) be an L-ordered set and $A \in L^X$. An element $x_0 \in X$ is called a join (resp., meet) of A, in symbols $x_0 = \sqcup A$ (resp., $x_0 = \sqcap A$), if
(1) for all $x \in X$, $A(x) \le e(x, x_0)$ (resp., $A(x) \le e(x_0, x)$),

(2) for all $y \in X$, $\bigwedge_{x\in X}(A(x) \to e(x,y)) \leq e(x_0,y)$ (resp., $\bigwedge_{x\in X}(A(x) \to e(y,x)) \leq e(y,x_0)$).

Proposition 2.11 *(Bělohlávek [1,2] and Xie et al. [8]) Let (X,e) be an L-ordered set. Then*
(1) $x_0 = \sqcup A$ iff $e(x_0,y) = \bigwedge_{x\in X} A(x) \to e(x,y)$ for all $y \in X$;
(2) $x_0 = \sqcap A$ iff $e(y,x_0) = \bigwedge_{x\in X} A(x) \to e(y,x)$ for all $y \in X$.

$x \in X$ is called the minimal (maximal) element of $A \in L^X$ if $A(x) = 1$ and for any $y \in X$, $A(y) \leq e(x,y)$ ($A(y) \leq e(y,x)$). It is easy checked that $x = \max A$ ($x = \min A$) iff $x = \sqcup A$ ($x = \sqcap A$) and $A(x) = 1$.

Definition 2.12 (Lai and Zhang [6] and Yao [10]) In an L-ordered set (X,e), an L-subset D of X is called a directed L-subset in X if
(1) $\bigvee_{x\in X} D(x) = 1$;
(2) for all $x,y \in X$, $D(x) * D(y) \leq \bigvee_{z\in X} D(z) * e(x,z) * e(y,z)$.
The set of all directed L-subsets in X is denoted by $\mathcal{D}(X)$.

An L-ordered set (X,e) is called directed complete if $\sqcup D$ exists for all $D \in \mathcal{D}(X)$.

A directed L-subset is called an L-ideal if it is a lower L-subset additionally. The set of all L-ideals in X is denoted by $\mathcal{I}(X)$, then $\mathcal{I}(X)$ is an L-ordered set. Clearly, for all $x \in X$, $\downarrow x \in \mathcal{I}(X)$.

Let (X,e_X), (Y,e_Y) be L-ordered sets and $f : X \longrightarrow Y$ be an ordinary mapping. One can define the forward fuzzy powerset operator $f^{\to} : L^X \longrightarrow L^Y$ [14] as follows: $f^{\to}(A)(y) = \bigvee_{x\in X} A(x) * e_Y(y,f(x))$, for all $A \in L^X$, $y \in Y$. The right adjoint to $f^{\to}$ is denoted by $f^{\leftarrow}$ (called the L-backward powerset operator) and given by $f^{\leftarrow}(B) = B \circ f$, for any $B \in L^Y$.

Remark 2.13 Indeed, if f is an L-order preserving mapping between two L-ordered sets (X,e_X) and (Y,e_Y), then $f^{\to}(D) \in \mathcal{I}(Y)$ for all $D \in \mathcal{D}(X)$.

Definition 2.14 (Bělohlávek [1,2] and Xie et al. [8]) Let (X,e_X), (Y,e_Y) be L-ordered sets. A mapping $f : X \longrightarrow Y$ is called join-preserving if it satisfies $f(\sqcup A) = \sqcup f^{\to}(A)$ for all $A \in L^X$. A mapping $f : X \longrightarrow Y$ is called L-Scott continuous (or, directed-join-preserving) if it satisfies $f(\sqcup D) = \sqcup f^{\to}(D)$ for all $D \in \mathcal{D}(X)$.

For convenience, the set of all L-Scott continuous mappings between two directed complete L-ordered sets (X,e_X) and (Y,e_Y) is denoted by $[X \to Y]$. Clearly, $[X \to Y]$ is a sub-L-ordered set of $[X,Y]$.

Remark 2.15 Every L-Scott continuous mapping is L-order preserving.

Definition 2.16 (Lai and Zhang [6] and Yao and Lu [9]) Let $(X,e_X),(Y,e_Y)$ be two L-ordered sets and $f : (X,e_X) \to (Y,e_Y)$, $g : (Y,e_Y) \to (X,e_X)$ two L-order preserving mappings. The pair (f,g) is called an L-adjunction between X and Y provided that

$$e_Y(f(x),y) = e_X(x,g(y))$$

for all $x \in X$ and all $y \in Y$, where f is called the left adjoint of g and dually g the right adjoint of f.

Proposition 2.17 *(Yao and Lu [9] and Zhang [15]) Let $(X, e_X), (Y, e_Y)$ be two L-ordered sets and $f : X \to Y$, $g : Y \to X$ be two L-order preserving mappings. Then the following conditions are equivalent:*
(1) *(f, g) is an L-adjunction;*
(2) *f is an L-order preserving and $g(y)=\max f^{\leftarrow}(\downarrow y)$ for all $y \in Y$;*
(3) *g is an L-order preserving and $f(x)=\min g^{\leftarrow}(\uparrow x)$ for all $x \in X$.*

Proposition 2.18 *(Lai and Zhang [6], Stubbe [7]) Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ be L-order preserving mappings. If (f, g) is an L-adjunction, then $f : X \longrightarrow Y$ preserves joins and $g : Y \longrightarrow X$ preserves meets. Conversely, if (X, e_X) is complete then f has a right adjoint whenever f preserves joins; and if (Y, e_Y) is complete then g has a left adjoint whenever g preserves meets.*

2.3 L-lattice

$F \in L^X$ is said a finite L-subset if the set $\{x \in X : F(x) \neq 0\}$ is finite.

Definition 2.19 (Bělohlávek [1,2], Lai and Zhang [6] and Zhang et al. [16,17]) An L-ordered set (X, e) is called a complete L-lattice if $\sqcup A$ and $\sqcap A$ exist for all $A \in L^X$.

Proposition 2.20 *(Bělohlávek [1,2], Lai and Zhang [6] and Zhang et al. [16,17]) Let (X, e) be an L-ordered set. The following statements are equivalent:*
(1) *(X, e) is a complete L-lattice;*
(2) *for all $A \in L^X$, $\sqcup A$ exists;*
(2) *for all $A \in L^X$, $\sqcap A$ exists.*

By Theorem 4.2 in [6], we obtain the following results.

Remark 2.21 (1) L itself is complete, and for all $A \in L^L$, $\sqcup A = \bigvee_{a\in L} A(a) * a$ and $\sqcap A = \bigwedge_{a\in L} A(a) \to a$.
(2) $[X \to L]$ is complete, and for all $\mathcal{A} \in L^{[X\to L]}$, $(\sqcup\mathcal{A})(x) = \bigvee_{A\in[X\to L]} \mathcal{A}(A) * A(x)$ and $(\sqcap\mathcal{A})(x) = \bigwedge_{A\in[X\to L]} \mathcal{A}(A) \to A(x)$.
(3) Let (X, e_X) be an L-ordered set and (Y, e_Y) a complete L-ordered set. Then $[X \to Y]$ is complete.

Theorem 2.22 *(Zhao and Zhang [18]) Let (X, e) be an an L-ordered set. Then (X, e) is a join L-semilattice iff each finite L-subset has a join; (X, e) is a meet L-semilattice iff each finite L-subset has a meet.*

Definition 2.23 (Zhao and Zhang [18]) An L-lattice is an L-ordered set which is simultaneously a join L-semilattice and a meet L-semilattice.

3 Meet-continuous L-lattices

In this section, we first show serval important results, such as: every mapping between L-ordered sets, preserving joins of directed L-subsets and finite L-subsets, preserves joins of arbitrary L-subsets. Then we propose two special kinds of L-lattices, meet-continuous L-lattices and distributive L-lattices, and show that every meet-continuous and distributive L-lattice is an L-frame. Finally, we show that the category of meet-continuous L-lattices is a cartesian closed category.

Proposition 3.1 *Let (X, e) be an L-ordered set and $A \in L^X$. If every finite L-subset of X has a join, then $D_A(x) = \bigvee_{F \leq_{Fin} A} e(x, \sqcup F)$ is directed, and $\sqcup A = \sqcup D_A$ whenever $\sqcup D_A$ exists, where $F \leq_{Fin} A$ denotes that F is a finite L-subset and it smaller than or equal to A.*

Proof. Firstly, we show that D_A is directed.

$$\bigvee_{x \in X} D_A(x) = \bigvee_{x \in X} \bigvee_{F \leq_{Fin} A} e(x, \sqcup F) = \bigvee_{F \leq_{Fin} A} \bigvee_{x \in X} e(x, \sqcup F) = 1,$$

For all $x_1, x_2 \in X$,

$$\begin{aligned}
D_A(x_1) * D_A(x_2) &= \bigvee_{F_1, F_2 \leq_{Fin} A} e(x_1, \sqcup F_1) * e(x_2, \sqcup F_2) \\
&= \bigvee_{F_1, F_2 \leq_{Fin} A} e(x_1, \sqcup F_1) * e(x_2, \sqcup F_2) * \\
&\qquad e(\sqcup F_1, \sqcup(F_1 \vee F_2)) * e(\sqcup F_2, \sqcup(F_1 \vee F_2)) \\
&\leq \bigvee_{F_1, F_2 \leq_{Fin} A} e(x_1, \sqcup(F_1 \vee F_2)) * e(x_2, \sqcup(F_1 \vee F_2)) \\
&\leq \bigvee_{F \leq_{Fin} A} e(x_1, \sqcup F) * e(x_2, \sqcup F) \\
&\leq \bigvee_{F \leq_{Fin} A} \bigvee_{x \in X} e(x, \sqcup F) * e(x_1, x) * e(x_2, x) \\
&= \bigvee_{x \in X} (\bigvee_{F \leq_{Fin} A} e(x, \sqcup F)) * e(x_1, x) * e(x_2, x) \\
&= \bigvee_{x \in X} D_A(x) * e(x_1, x) * e(x_2, x),
\end{aligned}$$

Secondly, we show that $\sqcup A$ exists and $\sqcup A = \sqcup D_A$. For all $y \in X$,

$$\begin{aligned} e(\sqcup D_A, y) &= \bigwedge_{x \in X} D_A(x) \to e(x, y) \\ &= \bigwedge_{x \in X} \Big(\bigvee_{F \leq_{Fin} A} e(x, \sqcup F)\Big) \to e(x, y) \\ &= \bigwedge_{x \in X} \bigwedge_{F \leq_{Fin} A} e(x, \sqcup F) \to e(x, y) \\ &= \bigwedge_{F \leq_{Fin} A} e(\sqcup F, y) \\ &= \bigwedge_{F \leq_{Fin} A} \bigwedge_{x \in X} F(x) \to e(x, y) \\ &\geq \bigwedge_{F \leq_{Fin} A} \bigwedge_{x \in X} A(x) \to e(x, y) \\ &= \bigwedge_{x \in X} A(x) \to e(x, y). \end{aligned}$$

Moreover, for all $x_0 \in \{x \in X \mid A(x) \neq 0\}$, define $D_{x_0} \in L^X$ as

$$D_{x_0}(x) = \begin{cases} A(x_0) & x = x_0 \\ 0 & x \neq x_0. \end{cases}$$

Then D_{x_0} is finite and $D_{x_0} \leq A$. So $D_A(x_0) \geq e(x_0, \sqcup D_{x_0}) \geq D_{x_0}(x_0) = A(x_0)$. Therefore, it follows that $A(x) \leq D_A(x) \leq e(x, \sqcup D_A)$ for all $x \in X$. □

By Proposition 3.1, we can obtain the following results.

Proposition 3.2 *Let $f : X \longrightarrow Y$ be an L-ordered preserving mapping between L-ordered sets. Then the following is equivalent:*
(1) f preserves joins of directed L-subsets;
(2) f preserves joins of L-ideals.
Moreover, if (X, e_X) is a join L-semilattice and f preserves joins of finite L-subsets, then (1) and (2) are also equivalent to
(3) f preserves joins of arbitrary L-subsets.

Proof. (1)⇒ (2) is obvious.

(2)⇒ (1). For all directed L-subsets D, $\downarrow D$ is an L-ideal and $\sqcup D = \sqcup \downarrow D$. Then we have $f(\sqcup D) = f(\sqcup \downarrow D) = \sqcup f^{\to}(\downarrow D)$. In order to show that $f(\sqcup D) = \sqcup f^{\to}(D)$, we only need to show that $\sqcup f^{\to}(D) = \sqcup f^{\to}(\downarrow D)$. Since $D \leq \downarrow D$, it is simple to check that $e_Y(\sqcup f^{\rightarrow}(D), \sqcup f^{\rightarrow}(\downarrow D)) = 1$. Moreover, we have

$$
\begin{aligned}
& e_Y(\sqcup f^{\rightarrow}(\downarrow D), \sqcup f^{\rightarrow}(D)) \\
&= \bigwedge_{y\in Y} f^{\rightarrow}(\downarrow D)(y) \rightarrow e_Y(y, \sqcup f^{\rightarrow}(D)) \\
&= \bigwedge_{y\in Y} (\bigvee_{x\in X} \bigvee_{x'\in X} D(x') * e_X(x,x') * e_Y(y, f(x))) \rightarrow e_Y(y, \sqcup f^{\rightarrow}(D)) \\
&= \bigwedge_{x\in X} \bigwedge_{x'\in X} D(x') * e_X(x,x') \rightarrow (\bigwedge_{y\in Y} e_Y(y, f(x)) \rightarrow e_Y(y, \sqcup f^{\rightarrow}(D))) \\
&= \bigwedge_{x\in X} \bigwedge_{x'\in X} D(x') * e_X(x,x') \rightarrow e_Y(f(x), \sqcup f^{\rightarrow}(D)) \\
&\geq \bigwedge_{x\in X} \bigwedge_{x'\in X} D(x') * e_Y(f(x), f(x')) \rightarrow e_Y(f(x), \sqcup f^{\rightarrow}(D)) \\
&= \bigwedge_{x\in X} f^{\rightarrow}(D)(f(x)) \rightarrow e_Y(f(x), \sqcup f^{\rightarrow}(D)) \\
&\geq \bigwedge_{y\in Y} f^{\rightarrow}(D)(y) \rightarrow e_Y(y, \sqcup f^{\rightarrow}(D)) \\
&= e_Y(\sqcup f^{\rightarrow}(D), \sqcup f^{\rightarrow}(D)) = 1.
\end{aligned}
$$

Now suppose (X, e_X) is a join L-semilattice and f preserves joins of finite L-subsets. Then it is obvious that (3) implies (1) and (2). So we only need to show that (1) and (2) imply (3). Let $A \in L^X$ have a join. By Proposition 3.1 and (1), we have $f(\sqcup A) = f(\sqcup D_A) = \sqcup f^{\rightarrow}(D_A)$ and $\sqcup f^{\rightarrow}(A) = \sqcup D_{f^{\rightarrow}(A)}$. Next, we show that $\sqcup f^{\rightarrow}(D_A) = \sqcup D_{f^{\rightarrow}(A)}$. For all $y \in Y$,

$$
\begin{aligned}
f^{\rightarrow}(D_A)(y) &= \bigvee_{x\in X} D_A(x) * e_Y(y, f(x)) \\
&= \bigvee_{x\in X} \bigvee_{F\leq_{Fin}A} e_X(x, \sqcup F) * e_Y(y, f(x)) \\
&\leq \bigvee_{x\in X} \bigvee_{F\leq_{Fin}A} e_Y(f(x), f(\sqcup F)) * e_Y(y, f(x)) \\
&\leq \bigvee_{F\leq_{Fin}A} e_Y(y, f(\sqcup F)) \\
&= \bigvee_{F\leq_{Fin}A} e_Y(y, \sqcup f^{\rightarrow}(F)) \\
&\leq \bigvee_{f^{\rightarrow}(F)\leq_{Fin}f^{\rightarrow}(A)} e_Y(y, \sqcup f^{\rightarrow}(F)) \\
&\leq \bigvee_{F\leq_{Fin}f^{\rightarrow}(A)} e_Y(y, \sqcup F) = D_{f^{\rightarrow}(A)}(y),
\end{aligned}
$$

which implies that $e_Y(\sqcup f^{\rightarrow}(D_A), \sqcup D_{f^{\rightarrow}(A)}) = 1$. On the other hand, we have

$$\begin{aligned}
e_Y(\sqcup D_{f^{\rightarrow}(A)}, \sqcup f^{\rightarrow}(D_A)) &= \bigwedge_{z\in Y}(\bigvee_{F\leq_{Fin} f^{\rightarrow}(A)} e_Y(z, \sqcup F)) \to e_Y(z, \sqcup f^{\rightarrow}(D_A)) \\
&= \bigwedge_{z\in Y}\bigwedge_{F\leq_{Fin} f^{\rightarrow}(A)} e_Y(z, \sqcup F) \to e_Y(z, \sqcup f^{\rightarrow}(D_A)) \\
&= \bigwedge_{F\leq_{Fin} f^{\rightarrow}(A)} e_Y(\sqcup F, \sqcup f^{\rightarrow}(D_A)) \\
&= \bigwedge_{F\leq_{Fin} f^{\rightarrow}(A)}\bigwedge_{y\in Y} F(y) \to e_Y(y, \sqcup f^{\rightarrow}(D_A)) \\
&\geq \bigwedge_{y\in Y} f^{\rightarrow}(A)(y) \to e_Y(y, \sqcup f^{\rightarrow}(D_A)) \\
&\geq \bigwedge_{y\in Y} f^{\rightarrow}(D_A)(y) \to e_Y(y, \sqcup f^{\rightarrow}(D_A)) \\
&\qquad (Since\ A \leq D_A, f^{\rightarrow}(A) \leq f^{\rightarrow}(D_A)) \\
&= e_Y(\sqcup f^{\rightarrow}(D_A), \sqcup f^{\rightarrow}(D_A)) = 1.
\end{aligned}$$

Therefore, it follows that $\sqcup f^{\rightarrow}(D_A) = \sqcup D_{f^{\rightarrow}(A)}$. □

Definition 3.3 Let (X, e) be a meet L-semilattice and $\wedge$ be the meet operation on X_0 such that for all $x \in X$, the function $\wedge_x() = x \wedge (), s \mapsto \wedge_x(s) = x \wedge s$ is an L-order preserving mapping, i.e., for any $s, y \in X$, $e(s, y) \leq e(\wedge_x(s), \wedge_x(y))$, then (X, e) is called meet-continuous if it is directed complete, and satisfies

$$(L - MCL) \qquad \wedge_x(\sqcup D) = \sqcup \wedge_x^{\rightarrow}(D)$$

for all $x \in X$ and all $D \in \mathcal{D}(X)$. An L-lattice (X, e) is meet-continuous if it is a complete L-lattice and satisfying (L-MCL).

The condition (L-MCL) could be called the L-meet-continuous law of binary meets over joins of arbitrary directed L-subsets, which is the L-counterpart of (MCL). Clearly, for L=2, (L-MCL)=(MCL). We can easy to check that if (X, e) is a meet-continuous, then X_0 is a crisp one.

Proposition 3.4 *In a directed complete L-semilattice (X, e), the following conditions are equivalent:*
(1) *for two $I_1, I_2 \in \mathcal{I}(X)$, we have*
$$\wedge_{\sqcup I_1}(\sqcup I_2) = \sqcup \wedge_{\sqcup I_1}^{\rightarrow}(I_2)\ (or\ \ \wedge_{\sqcup I_1}(\sqcup I_2) = \sqcup \wedge_{\sqcup I_2}^{\rightarrow}(I_1));$$
(2) *for two $D_1, D_2 \in \mathcal{D}(X)$, we have*
$$\wedge_{\sqcup D_1}(\sqcup D_2) = \sqcup \wedge_{\sqcup D_1}^{\rightarrow}(D_2)\ (or\ \ \wedge_{\sqcup D_1}(\sqcup D_2) = \sqcup \wedge_{\sqcup D_2}^{\rightarrow}(D_1);$$
(3) *(X, e) is meet-continuous.*

Proof. Trivial. □

Proposition 3.5 *Let (X, e) be a directed complete L-semilattice. Then the following conditions are equivalent:*
(1) *(X, e) is meet-continuous;*
(2) *$\sqcup I_1 \wedge \sqcup I_2 = \sqcup(I_1 * I_2)$ for all $I_1, I_2 \in \mathcal{I}(X)$.*

Proof. (1) $\Rightarrow$ (2): We only need to show that $\sqcup\wedge^{\rightarrow}_{\sqcup I_1}(I_2) = \sqcup(I_1 * I_2)$, since (X, e) is meet-continuous. For all $x \in X$, on one hand, since

$$\begin{aligned}\wedge^{\rightarrow}_{\sqcup I_1}(I_2)(x) &= \bigvee_{x' \in X} I_2(x') * e(x, x' \wedge \sqcup I_1)\\ &\geq I_2(x) * e(x, \sqcup I_1) \geq (I_1 * I_2)(x),\end{aligned}$$

we obtain $e(\sqcup(I_1 * I_2), \sqcup\wedge^{\rightarrow}_{\sqcup I_1}(I_2)) = 1$; on the other hand,

$$\begin{aligned}e(\sqcup\wedge^{\rightarrow}_{\sqcup I_1}(I_2), \sqcup(I_1 * I_2)) &= \bigwedge_{x \in X} \wedge^{\rightarrow}_{\sqcup I_1}(I_2)(x) \to e(x, \sqcup(I_1 * I_2))\\ &= \bigwedge_{x' \in X} I_2(x') \to e(x' \wedge \sqcup I_1, \sqcup(I_1 * I_2))\\ &= \bigwedge_{x' \in X} I_2(x') \to e(\sqcup\wedge^{\rightarrow}_{x'}(I_1), \sqcup(I_1 * I_2))\\ &= \bigwedge_{x' \in X} I_2(x') \to \left(\bigwedge_{y \in X} I_1(y) \to e(y \wedge x', \sqcup(I_1 * I_2))\right)\\ &\geq \bigwedge_{x' \in X}\bigwedge_{y \in X} I_2(x' \wedge y) * I_1(x' \wedge y) \to e(y \wedge x', \sqcup(I_1 * I_2))\\ &\geq \bigwedge_{x \in X} (I_1 * I_2)(x) \to e(x, \sqcup(I_1 * I_2)) = 1.\end{aligned}$$

(2) $\Rightarrow$ (1): By (2), we only need to show that $\sqcup(\downarrow x * I) = \sqcup\wedge^{\rightarrow}_{x}(I)$ for all $x \in X$ and $I \in \mathcal{I}(X)$. Indeed,

$$\begin{aligned}e(\sqcup\wedge^{\rightarrow}_{x}(I), \sqcup(\downarrow x * I)) &= \bigwedge_{y \in X} \wedge^{\rightarrow}_{x}(I)(y) \to e(y, \sqcup(\downarrow x * I))\\ &= \bigwedge_{x' \in X} I(x') \to e(x' \wedge x, \sqcup(\downarrow x * I))\\ &= \bigwedge_{x' \in X} I(x') \to e(x' \wedge x, x \wedge \sqcup I))\\ &\geq \bigwedge_{x' \in X} I(x') \to e(x', \sqcup I) = 1.\end{aligned}$$

Conversely, for all $y \in X$,

$$\wedge^{\rightarrow}_{x}(I)(y) = \bigvee_{x' \in X} I(x') * e(y, x' \wedge x) \geq I(y) * e(y, x) = (\downarrow x * I)(y).$$

Then, we have

$$\begin{aligned} e(\sqcup(\downarrow x * I), \sqcup\wedge_x^{\rightarrow}(I)) &= \bigwedge_{y\in X} (\downarrow x * I)(y) \to e(y, \sqcup\wedge_x^{\rightarrow}(I)) \\ &\geq \bigwedge_{y\in X} (\downarrow x * I)(y) \to \wedge_x^{\rightarrow}(I)(y) \\ &\geq \bigwedge_{y\in X} (\downarrow x * I)(y) \to (\downarrow x * I)(y) = 1. \end{aligned}$$

□

Definition 3.6 Let (X, e) be an L-lattice and $\wedge$ be the meet operation on X_0 such that for all $x \in X$, the function $\wedge_x() = x \wedge (), s \mapsto \wedge_x(s) = x \wedge s$ is an L-order preserving mapping, then (X, e) is distributive if

$$(L-DL) \qquad \wedge_x (\sqcup F) = \sqcup\wedge_x^{\rightarrow}(F)$$

for all $x \in X$ and all finite L-subsets F.

The condition $(L\text{-}DL)$ could be called the L-distributive law of binary meets over joins of arbitrary finite L-subsets, which is the L-counterpart of (DL). Clearly, for L=2, $(L\text{-}DL)$=(DL). We can easy to check that if (X, e) is a distributive L-lattice, then X_0 is a crisp one.

Proposition 3.7 *Let (X, e) be an L-lattice. Then the following conditions are equivalent:*
(1) *(X, e) is distributive;*
(2) *for all $x \in X$ and all finite L-subsets F, $\wedge_x(\sqcup\!\downarrow\! F) = \sqcup\wedge_x^{\rightarrow}(\downarrow\! F)$.*
(3) *for two finite L-subsets F_1, F_2, we have*

$$\wedge_{\sqcup F_1} (\sqcup F_2) = \sqcup\wedge_{\sqcup F_1}^{\rightarrow}(F_2) \ (or \ \ \wedge_{\sqcup F_1} (\sqcup F_2) = \sqcup\wedge_{\sqcup F_2}^{\rightarrow}(F_1));$$

(4) *for two finite L-subsets F_1, F_2, we have*

$$\wedge_{\sqcup\downarrow F_1} (\sqcup\!\downarrow\! F_2) = \sqcup\wedge_{\sqcup\downarrow F_1}^{\rightarrow}(\downarrow\! F_2) \ (or \ \ \wedge_{\sqcup\downarrow F_1} (\sqcup\!\downarrow\! F_2) = \sqcup\wedge_{\sqcup\downarrow F_2}^{\rightarrow}(\downarrow\! F_1);$$

(5) *$\sqcup\!\downarrow\! F_1 \wedge \sqcup\!\downarrow\! F_2 = \sqcup(\downarrow\! F_1 * \downarrow\! F_2)$ for all finite L-subsets F_1, F_2.*

Proof. The equivalence of (1), (2), (3) and (4) is obvious. Similar to the proof of Proposition 3.5, we can show the equivalence of (1) and (5). □

Definition 3.8 Let (X, e) be a complete L-lattice and $\wedge$ be the meet operation on X_0 such that for all $x \in X$, the function $\wedge_x() = x \wedge (), s \mapsto \wedge_x(s) = x \wedge s$ is an L-order preserving mapping. We call (X, e) an L-frame if for all $x \in X$, $\wedge_x$ has a right adjoint, or equivalently, the following identity holds:

$$(L-IDL) \qquad \wedge_x (\sqcup A) = \sqcup\wedge_x^{\rightarrow}(A)$$

for all $x \in X$ and all $A \in L^X$.

Remark 3.9 When $\wedge_x^{\rightarrow}$ is replaced by $(\wedge_x)_L^{\rightarrow}$, the definition above is exactly that in [12].

Proposition 3.10 *Let (X, e) be a complete L-lattice. Then the following conditions are equivalent:*
(1) *(X, e) is an L-frame;*

(2) *for all* $x \in X$ *and all* L*-subsets* A, $\wedge_x(\sqcup{\downarrow}A) = \sqcup\wedge_x^{\rightarrow}({\downarrow}A)$.
(3) *for two* L*-subsets* A_1, A_2, *we have*

$$\wedge_{\sqcup A_1}(\sqcup A_2) = \sqcup\wedge_{\sqcup A_1}^{\rightarrow}(A_2) \ (or\ \ \wedge_{\sqcup A_1}(\sqcup A_2) = \sqcup\wedge_{\sqcup A_2}^{\rightarrow}(A_1));$$

(4) *for two lower* L*-subsets* A_1, A_2, *we have*

$$\wedge_{\sqcup A_1}(\sqcup A_2) = \sqcup\wedge_{\sqcup A_1}^{\rightarrow}(A_2) \ (or\ \ \wedge_{\sqcup A_1}(\sqcup A_2) = \sqcup\wedge_{\sqcup A_2}^{\rightarrow}(A_1);$$

(5) $\sqcup A_1 \wedge \sqcup A_2 = \sqcup(A_1 * A_2)$ *for all lower* L*-subsets* A_1, A_2.

Proof. Similar to the proof of Proposition 3.7. □

Proposition 3.11 *Let* (X, e) *be a meet* L*-semilattice and the function* $\wedge_x : X \longrightarrow X, s \mapsto x \wedge s$ *be an* L*-order preserving mapping. Then the following two conditions are equivalent:*
(1) *for all* $x \in X$, *the function* $\wedge_x : X \longrightarrow X, s \mapsto x \wedge s$ *has a right adjoint;*
(2) $max\wedge_x^{\leftarrow}({\downarrow}t)$ *exists for all* $x, t \in X$.
These conditions imply
(3) *for all* $A \in L^X$ *with* $\sqcup A$ *exists and all* $x \in X$, *we have*

$$\wedge_x(\sqcup A) = \sqcup\wedge_x^{\rightarrow}(A).$$

If (X, e) *is an* L*-lattice, then (3) implies the* L*-distributive law (L-DL). If* (X, e) *is a complete* L*-lattice, then (1)-(3) are equivalent to* (X, e) *being an* L*-frame.*

Proof. The equivalence of (1) and (2) follows from Proposition 2.17. Condition (3) follows by Proposition 2.18, and, trivially, (3) implies (L-DL). If (X, e) is a complete L-lattice, then (3) implies (1) by Proposition 2.18 and of course it is just (L-IDL). □

Proposition 3.12 *Let*(X, e) *be a* L*-lattice, the the following conditions are equivalent:*
(1) (X, e) *is a* L*-frame;*
(2) (X, e) *is meet-continuous and distributive.*

Proof. That (1) impies (2) is clear from Proposition 3.11.
(2)⇒(1): By (L-MC), the mapping $\wedge_x : X \longrightarrow X, s \mapsto x \wedge s$ preserves joins of directed L-subsets; by (L-DL) of Definition 3.6, it preserves joins of finite L-subsets. Hence, it preservers joins of arbitrary L-subsets. Thus, Proposition 3.11 (3) holds and so (1) follows. □

Let L-**MC** denote the category of meet-continuous L-lattices with L-Scott continuous mappings as morphisms. Next, we discuss the Cartesian closedness of the category L-**MC**. First of all, we review the following concept:

A category with terminal and finite products is called Cartesian-closed if for each pair (A, B) of objects there exists an object B^A (sometimes be called the function space from A to B) and a morphism $ev : A \times B^A \longrightarrow B$ with the following universal property: for each morphism $f : A \times C \longrightarrow B$ there exists a unique morphism $\overline{f} : C \longrightarrow B^A$ such that $ev \circ (id_A \times \overline{f}) = f$.

Clearly, a terminal object in the category of crisp meet-continuous lattices is certainly terminal in L-**MC** since each crisp meet-continuous lattice can be viewed as a meet-continuous L-lattice. Therefore, we only need to prove the following conclusions: (1) L-**MC** is closed under finite products, i.e., Proposition 3.15; (2) The function space of L-**MC** is closed, i.e., Proposition 3.17; (3) The evaluation morphism ev of L-**MC** has the above universal property, i.e., Proposition 3.18.

Let $\{(X_j, e_j) \mid j \in J\}$ be a family of L-ordered sets. The product of $\{X_j \mid j \in J\}$ is given by

$$\forall x = (x_j)_{j\in J},\ y = (y_j)_{j\in J},\ \textstyle\prod_{j\in J} e_j(x, y) = \bigwedge_{j\in J} e_j(x_j, y_j).$$

Then for all $j \in J$, the jth-projection $p_j : \prod_{j\in J} X_j \longrightarrow X_j$ is an L-order preserving mapping. Thus, by Remark 2.13, $p_j^{\rightarrow}(D) \in \mathcal{I}(X_j)$ holds for all $j \in J$ and $D \in \mathcal{D}(\prod_{j\in J} X_j)$.

Lemma 3.13 *Let $\{X_j \mid j \in J\}$ be a family of L-ordered sets and $X = \prod_{j\in J} X_j$. Then for all $A \in L^X$, $(\sqcup(p_j^{\rightarrow}(A)))_{j\in J}$ is the join of A.*

Proof. The proof is similar to that of Proposition 5.2 in [11]. □

Remark 3.14 Lemma 3.13 implies that p_j preserves the joins of all L-subsets, for all $j \in J$.

By Lemma 3.13, we can obtain the following result.

Proposition 3.15 *Let $\{X_j \mid j \in J\}$ be a family of meet-continuous L-lattices, then $X = \prod_{j\in J} X_j$ is also a meet-continuous L-lattice.*

Proof. By Lemma 3.13, it is obvious that (X, e) is a complete L-lattice. So we only need to show that (X, e) is meet-continuous. Let $x = (x_j)_{j\in J} \in X$, $D \in \mathcal{D}(X)$. By Lemma 3.13 and the meet-continuity of X_j, it follows that

$$\begin{aligned}\wedge_x(\sqcup D) &= (x_j \wedge \sqcup(p_j^{\rightarrow}(D)))_{j\in J} = (\sqcup \wedge_{x_j}^{\rightarrow} p_j^{\rightarrow}(D))_{j\in J}\\ &= (\sqcup(\wedge_{x_j} \circ p_j)^{\rightarrow}(D))_{j\in J}\end{aligned}$$

and

$$\sqcup \wedge_x^{\rightarrow} (D) = (\sqcup(p_j^{\rightarrow}(\wedge_x^{\rightarrow}(D))))_{j\in J} = (\sqcup(p_j \circ \wedge_x)^{\rightarrow}(D))_{j\in J}.$$

Therefore, in order to show that $\wedge_x(\sqcup D) = \sqcup \wedge_x^{\rightarrow} (D)$, we only need to show that $\sqcup(\wedge_{x_j} \circ p_j)^{\rightarrow}(D) = \sqcup(p_j \circ \wedge_x)^{\rightarrow}(D)$ for all $j \in J$. Indeed, for all $y_j \in X_j$,

$$\begin{aligned}(\wedge_{x_j} \circ p_j)^{\rightarrow}(D)(y_j) &= \bigvee_{z=(z_j)_{j\in J}\in X} D(z) * e_j(y_j, (\wedge_{x_j} \circ p_j)(z))\\ &= \bigvee_{z=(z_j)_{j\in J}\in X} D(z) * e_j(y_j, x_j \wedge z_j)\end{aligned}$$

and

$$\begin{aligned}(p_j \circ \wedge_x)^{\rightarrow}(D)(y_j) &= \bigvee_{z=(z_j)_{j\in J}\in X} D(z) * e_j(y_j, (p_j \circ \wedge_x)(z))\\ &= \bigvee_{z=(z_j)_{j\in J}\in X} D(z) * e_j(y_j, x_j \wedge z_j)\end{aligned}$$

□

Lemma 3.16 *Let $(X, e_X), (Y, e_Y)$ be two meet-continuous L-lattices. If f, g are two L-Scott continuous mappings from X to Y, then $f \wedge g$ is also an L-Scott continuous mapping from X to Y.*

Proof. Since f, g are two L-Scott continuous mappings, for all $x, y \in X$, we obtain

$$\begin{aligned} e_X(x, y) &\leq e_Y(f(x), f(y)) \wedge e_Y(g(x), g(y)) \\ &\leq e_Y((f \wedge g)(x), f(y)) \wedge e_Y((f \wedge g)(x), g(y)) \\ &= e_Y((f \wedge g)(x), (f \wedge g)(y)), \end{aligned}$$

which implies that $f \wedge g$ is an L-order preserving mapping. Next, we show that $f \wedge g$ is L-Scott continuous. Indeed, for all $I \in \mathcal{I}(X)$,

$$\begin{aligned} e_Y(\sqcup (f \wedge g)^{\rightarrow}(I), (f \wedge g)(\sqcup I)) &= \bigwedge_{y \in Y} (f \wedge g)^{\rightarrow}(I)(y) \to e_Y(y, (f \wedge g)(\sqcup I)) \\ &= \bigwedge_{x \in X} I(x) \to e_Y((f \wedge g)(x), (f \wedge g)(\sqcup I)) \\ &\geq \bigwedge_{x \in X} I(x) \to e_X(x, \sqcup I) = 1. \end{aligned}$$

Conversely,

$$\begin{aligned} & e_Y((f \wedge g)(\sqcup I), \sqcup (f \wedge g)^{\rightarrow}(I)) = e_Y(\sqcup (f^{\rightarrow}(I) * g^{\rightarrow}(I)), \sqcup (f \wedge g)^{\rightarrow}(I)) \\ & \geq sub_Y((f^{\rightarrow}(I) * g^{\rightarrow}(I)), (f \wedge g)^{\rightarrow}(I)) \\ & = \bigwedge_{y \in Y} (f^{\rightarrow}(I) * g^{\rightarrow}(I))(y) \to (f \wedge g)^{\rightarrow}(I)(y) \\ & = \bigwedge_{y \in Y} (f^{\rightarrow}(I) * g^{\rightarrow}(I))(y) \to \left(\bigvee_{x \in X} I(x) * e_Y(y, (f \wedge g)(x)) \right) \\ & = \bigwedge_{y \in Y} (f^{\rightarrow}(I) * g^{\rightarrow}(I))(y) \to \left(\bigvee_{x \in X} I(x) * (e_Y(y, f(x)) \wedge e_Y(y, g(x))) \right) \\ & \geq \bigwedge_{y \in Y} (f^{\rightarrow}(I) * g^{\rightarrow}(I))(y) \to \left(\bigvee_{x \in X} I(x) * e_Y(y, f(x)) * e_Y(y, g(x)) \right) \\ & \geq \bigwedge_{y \in Y} (f^{\rightarrow}(I) * g^{\rightarrow}(I))(y) \to (f^{\rightarrow}(I) * g^{\rightarrow}(I))(y) = 1. \end{aligned}$$

□

Proposition 3.17 *Suppose $(X, e_X), (Y, e_Y)$ are two meet-continuous L-lattices, then the complete L-lattice $([X \to Y], e_{[X \to Y]})$ is meet-continuous.*

Proof. It is easy to check that the meet operator $\wedge$ is an L-order preserving mapping. Let $\mathcal{I} \in \mathcal{I}([X \to Y])$. We show that $\wedge_f(\sqcup \mathcal{I}) = \sqcup \wedge_f^{\to}(\mathcal{I})$ for all $f \in [X \to Y]$.

$$\begin{aligned}
& e_{[X\to Y]}(\sqcup \wedge_f^{\to}(\mathcal{I}), \wedge_f(\sqcup \mathcal{I})) \\
&= \bigwedge_{g\in[X\to Y]} \wedge_f^{\to}(\mathcal{I})(g) \to e_{[X\to Y]}(g, f \wedge \sqcup \mathcal{I}) \\
&= \bigwedge_{g\in[X\to Y]} (\bigvee_{g'\in[X\to Y]} \mathcal{I}(g') * e_{[X\to Y]}(g, f \wedge g')) \to e_{[X\to Y]}(g, f \wedge \sqcup \mathcal{I}) \\
&= \bigwedge_{g'\in[X\to Y]} \mathcal{I}(g') \to (\bigwedge_{g\in[X\to Y]} e_{[X\to Y]}(g, f \wedge g') \to e_{[X\to Y]}(g, f \wedge \sqcup \mathcal{I})) \\
&= \bigwedge_{g'\in[X\to Y]} \mathcal{I}(g') \to e_{[X\to Y]}(f \wedge g', f \wedge \sqcup \mathcal{I}) \\
&\geq \bigwedge_{g'\in[X\to Y]} \mathcal{I}(g') \to e_{[X\to Y]}(g', \sqcup \mathcal{I}) = 1.
\end{aligned}$$

For all $x \in X$ and $y \in Y$, define $A_x^{\mathcal{I}} \in L^Y$ by $A_x^{\mathcal{I}}(y) = \bigvee_{g\in[X\to Y]} \mathcal{I}(g) * e_Y(y, g(x))$, similar to the proof of Proposition 5.3 in [11], we can show that $A_x^{\mathcal{I}} \in \mathcal{I}(Y)$ and $\sqcup \mathcal{I} = f$, where $f(x) = \sqcup A_x^{\mathcal{I}}$. Then,

$$\begin{aligned}
e_Y((f \wedge \sqcup \mathcal{I})(x), y) &= e_Y(f(x) \wedge \sqcup A_x^{\mathcal{I}}, y) = e_Y(\sqcup \wedge_{f(x)}^{\to}(A_x^{\mathcal{I}}), y) \\
&= \bigwedge_{y'\in Y} \wedge_{f(x)}^{\to}(A_x^{\mathcal{I}})(y') \to e_Y(y', y) \\
&= \bigwedge_{y'\in Y} (\bigvee_{y''\in Y} A_x^{\mathcal{I}}(y'') * e_Y(y', y'' \wedge f(x))) \to e_Y(y', y) \\
&= \bigwedge_{y''\in Y} A_x^{\mathcal{I}}(y'') \to e_Y(y'' \wedge f(x), y) \\
&= \bigwedge_{y''\in Y} (\bigvee_{g\in[X\to Y]} \mathcal{I}(g) * e_Y(y'', g(x))) \to e_Y(y'' \wedge f(x), y) \\
&= \bigwedge_{g\in[X\to Y]} \mathcal{I}(g) \to (\bigwedge_{y''\in Y} e_Y(y'', g(x)) \to e_Y(y'' \wedge f(x), y)) \\
&= \bigwedge_{g\in[X\to Y]} \mathcal{I}(g) \to e_Y(f(x) \wedge g(x), y),
\end{aligned}$$

where the last equation holds since $\bigwedge_{y''\in Y} e_Y(y'', g(x)) \to e_Y(y'' \wedge f(x), y) \leq e_Y(f(x) \wedge g(x), y)$ when $y'' = g(x)$ and $\bigwedge_{y''\in Y} e_Y(y'', g(x)) \to e_Y(y'' \wedge f(x), y) \geq$

$e_Y(f(x)\wedge g(x),y)$ by $e_Y(y'',g(x))\leq e_Y(y''\wedge f(x),g(x)\wedge f(x))$. Moreover,

$$\begin{aligned}
&e_Y(\sqcup\wedge_f^{\rightarrow}(\mathcal{I})(x),y)=\bigwedge_{g\in[X\to Y]}\wedge_f^{\rightarrow}(\mathcal{I})(g)\to e_Y(g(x),y)\\
&=\bigwedge_{g\in[X\to Y]}\Big(\bigvee_{g'\in[X\to Y]}\mathcal{I}(g')*e_{[X\to Y]}(g,f\wedge g')\Big)\to e_Y(g(x),y)\\
&=\bigwedge_{g'\in[X\to Y]}\mathcal{I}(g')\to\Big(\bigwedge_{g\in[X\to Y]}e_{[X\to Y]}(g,f\wedge g')\to e_Y(g(x),y)\Big)\\
&\leq\bigwedge_{g'\in[X\to Y]}\mathcal{I}(g')\to e_Y(g'(x)\wedge f(x),y)\ (\textit{by Lemma } 3.16 \textit{ and let } g=f\wedge g').
\end{aligned}$$

Hence, we have $e_{[X\to Y]}(f\wedge\sqcup\mathcal{I},\sqcup\wedge_f^{\rightarrow}(\mathcal{I}))=1$ and thus $\sqcup\wedge_f^{\rightarrow}(\mathcal{I})=f\wedge\sqcup\mathcal{I}$. □

For a map $f:X\times Z\longrightarrow Y$, define $\overline{f}:Z\longrightarrow Y^X$ by $\overline{f}(z)(x)=f(x,z)(\forall z\in Z,\forall x\in X)$.

Similar to the proof of Proposition 5.6, 5.7 in [11], we can obtain the following results.

Proposition 3.18 *Let* (X,e_X), (Y,e_Y) *and* (Z,e_Z) *be complete* L*-lattices and* L=***frame***. *Then*
(1) *if* $f:X\times Z\longrightarrow Y$ *is an* L-***MC****-morphism, then so is* $\overline{f}:Z\longrightarrow Y^X$;
(2) *the evaluation function* $ev:X\times[X\to Y]\longrightarrow Y$ *is* L*-Scott continuous.*

Hence, by Proposition 3.15, 3.17, 3.18, we can obtain the following result.

Theorem 3.19 L-***MC*** *is cartesian closed when* L *is a frame.*

4 Conclusions and further work

Taking complete residuated lattices as the structure of truth values, we proposed the notions of meet-continuity and distributivity on L-lattices and showed that a complete L-lattice is an L-frame iff it is distributive and meet-continuous. Moreover, we proved that every mapping between L-ordered sets, preserving joins of directed L-subsets and finite L-subsets, preserves joins of arbitrary L-subsets. Additionally, we investigated the cartesian closedness of the category of meet-continuous L-lattices.

Based on the existing results in fuzzy domain theory (e.g.[12,13]], we claim that fuzzy domain is not an isolated field, it unites lattice-valued topology and algebra theory closely. Further attempts can be made to study interrelationships between fuzzy domains, lattice-valued topology, lattice-valued order, and lattice-valued algebra. In this paper, we consider a fuzzification of the well-known notion of meet-continuity for meet semilattices. Although the obtained results are similar to those in classical domain theory, some proofs are different from them. These results also indicate that the coordination between many concepts of classical domain theory still holds in the fuzzy setting.

Some further work can be considered, for example,

(1) In this paper, we just considered that L-**MC** is cartesian closed when L is a frame. Therefore, a more interesting question is whether there exists a complete residuated lattice L such that L is not a frame, but L-**MC** is Cartesian closed.

(2) It should be emphasized that the theory of fuzzy domains is not a straightforward generalization of its classical counterpart. In fact, since the structure of truth values L is much more complicated than that the Boolean algebra **2**, many easy problems in the classical case turn out to be much complicated in the many-valued setting. For example, an ordered set is tensored iff it has a least element, but it is difficult to find such simple description for fuzzy domains. On the other hand, in fuzzy domain theory, many properties depend on the logic property of L, the structure of truth values. Therefore, looking for the new properties is a challenging issue for future research.

(3) Since for classical domain theory, there is a close relationship between continuous lattices and meet-continuous lattices, for example, every complete lattice is a continuous lattice iff it is a meet-continuous lattice and a quasicontinuous lattice, one can study the quasicontinuity of fuzzy domains, and then obtains the similar result.

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