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L-fuzzy Scott Topology and Scott Convergence of Stratified L-filters on Fuzzy Dcpos [1]

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Abstract

On a fuzzy dcpo with a frame L as its valued lattice, we define an L-fuzzy Scott topology by means of graded convergence of stratified L-filters. It is a fuzzy counterpart of the classical Scott topology on a crisp dcpo. The properties of L-fuzzy Scott topology are investigated. We establish Scott convergence theory of stratified L-filters. We show that for an L-set, its degree of Scott openness equals to the degree of Scott continuity from the underlying fuzzy dcpo to the lattice L (also being viewed as a fuzzy poset). We also show that a fuzzy dcpo is continuous iff for any stratified L-filter, Scott convergence coincides with topological convergence (w.r.t. the L-fuzzy Scott topology).

Keywords: fuzzy dcpo, stratified L-filter, L-fuzzy Scott topology, Scott convergence, topological convergence.

1 Introduction

Domain Theory, a formal basis for the semantics of programming languages, originates in work by D. Scott [17,18] in the mid-1960s. Domain models for various types of programming languages, including imperative, functional, nondeterministic and probabilistic languages, have been extensively studied.

Quantitative Domain Theory that provides a model of concurrent systems forms a new branch of Domain Theory and has undergone active research in the past three decades. Rutten's generalized (ultra)metric spaces [16], Flagg's

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continuity spaces [5] and Wagner's Ω-categories [21] are examples of frameworks of quantitative domain theory.

Recently, based on complete Heyting algebras, Fan and Zhang [3,29] studied quantitative domains via fuzzy set theory. In their approach, a fuzzy partial order, namely a degree function, is defined on a nonempty set firstly. Then fuzzy directed subsets and (continuous) fuzzy dcpos are defined and studied. Also in [1,2], in order to study fuzzy relational systems, Bělohlávek defines and studies an **L**-order on a set. In fact, we will show that a fuzzy partial order in the sense of Fan-Zhang and an **L**-order in the sense of Bělohlávek are equivalent to each other. In [13,14], Lai and Zhang studied directed complete Ω-categories, where Ω is a commutative unital quantale. Roughly speaking, each Ω-category could be regarded as a fuzzy preordered set in sense of [3,29,30].

In [13,14,29,30] many ideal and nice results have been obtained, but there are still some problems which should be put forward:

(1) The definition of fuzzy directed subsets in [29,30] (Definition 2.6 in [29] and Definition 2.3 in [30]), which is based on the wedge below relation is relatively complex in some sense. [3]

(2) In [29,30], for two fuzzy posets (X, e_X) and (Y, e_Y) and a monotone map $f : X \longrightarrow Y$. The lift $\tilde{f} : L^X \longrightarrow L^Y$ of f is defined by

$$\forall A \in L^X,\ y \in Y,\ \tilde{f}(A)(y) = \bigvee_{x \in X} A(x) \wedge e_Y(y, f(x)).$$

Then it is shown that for any fuzzy directed subset ϕ of X, $\tilde{f}(\phi)$ is also a fuzzy directed subset of Y (Theorem 2.11 in [29], Theorem 2.7 in [30] and more earlier Lemma 12 in [5]). By the criterion of extension from crisp setting to fuzzy setting, when $L = \{0,\ 1\}$, $\tilde{f}$ should be the same as f. While, for two crisp posets X and Y, a monotone map $f : X \longrightarrow Y$ and any directed subset D of X, the image of D under $\tilde{f}$ is not equal to $f(D)$ in general, but to $\downarrow f(D) := \{y \in Y \mid \exists x \in D,\ s.t.\ y \le f(x)\}$. Also, the *lift* in [14] is the same as that in [29,30] by replacing $\wedge$ with the tensor $*$. For this reason, the lift of a map in [13,14,29,30] is not a good extension.

(3) The category of crisp dcpos is cartesian closed. It is natural to ask that, whether the category of fuzzy dcpos is also cartesian closed or not?

(4) In [29,30], a crisp topology, namely the generalized Scott topology, is defined on a given fuzzy dcpo. Can we naturally construct an L-topology on a fuzzy dcpo just like a crisp topology on a crisp dcpo?

[3] In fact, in [13,14] Lai and Zhang have already given a simple definition of fuzzy directed subsets (cf. Definition 5.1 in [14]).

(5) Many important and nice results in [29,30], especially the definition and results of the generalized Scott topology, is based on a completely distributive complete lattice with the top element $\vee$-irreducible and the well below relation multiplicative (Seeing [29,30] for detail). In my opinion such a condition is too strong. In fact, we can show that any finite lattice with a multiplicative well below relation must be a chain. Thus many canonical finite lattices, for example the simplest nontrivial Boolean algebra M_2, can not be supplied as an evaluating lattice in [29,30].

In [25,26,27], for an arbitrary frame L, we call a map between two fuzzy dcpos fuzzy Scott continuous if it is order-preserving and preserves joins of fuzzy directed subsets. Such a category of fuzzy dcpos is cartesian closed. Also, a natural L-topology, called a fuzzy Scott topology, on fuzzy dcpos are defined and studied.

The aim of this paper is to define and study an L-fuzzy topology on fuzzy dcpos. We shall call it an L-fuzzy Scott topology. And then to establish Scott convergence theory for stratified L-filters. Likewise the fuzzy Scott topology, an L-fuzzy Scott topology is an L-fuzzy counterpart of a crisp one.

2 Preliminaries

In this section, L always stands for a complete residuated lattice with $*$ as its tensor, i.e., a complete lattice equipped with a pari of binary operations ($*$, $\rightarrow$) that forms a Galois connection, for detail please refer to [20].

2.1 Fuzzy Sets and Fuzzy Posets

For a set X and an element a in a lattice L, we use $\overline{a}$ to stand for the constant map from X to L with the value a.

For each map $f : X \longrightarrow Y$, we have a map $f_L^{\rightarrow} : L^X \longrightarrow L^Y$ (called L-forward powerset operator, cf. [15]) defined by $f_L^{\rightarrow}(A)(y) = \bigvee_{f(x)=y} A(x)$ ($\forall y \in Y$, $\forall A \in L^X$). The right adjoint to $f_L^{\rightarrow}$ is denoted by $f_L^{\leftarrow}$ (called L-backward powerset operator, cf. [15]) and given by $\forall B \in L^Y$, $f_L^{\leftarrow}(B) = B \circ f$.

An L-relation E on X is an L-subset of $X \times X$. An L-relation E on X is called an L-preorder if

(Ref) $\forall x \in X,\ E(x,x) = 1$;

(Tran) $\forall x,\ y,\ z \in X,\ E(x,y) * E(y,z) \leq E(x,z)$.

An L-preorder E on X is called an L-equivalence if

(Sym) $\forall x,\ y \in X,\ E(x,y) = E(y,x)$.

An L-equivalence E is called an L-equality if $E(x,y) = 1$ implies $x = y$. Let

E be an L-equivalence and R an L-relation on X. R is said to be compatible with E [1,2] if for all $x_1, x_2, y_1, y_2 \in X$, $E(x_1, y_1) * E(x_2, y_2) * R(x_1, x_2) \leq R(y_1, y_2)$.

For L a frame, in order to study quantitative domain theory via fuzzy sets, a kind of fuzzy posets is introduced by L. Fan and Q.Y Zhang [3,29]. Almost at the same time, in order to study fuzzy relational systems, for L a complete residuated lattice, another kind of fuzzy posets are introduced by R. Bělohlavek [1,2].

Definition 2.1 (1) (A fuzzy partial order in sense of Bělohlávek [1,2]) A Bělohlávek-fuzzy partial order (a B-fpo for short) on a set X with an L-equality $\approx$ is an L-preorder $e : X \times X \longrightarrow L$ which is compatible w.r.t. $\approx$ and satisfying

(B) $\forall x,\ y \in X,\ e(x, y) \wedge e(y, x) \leq (x \approx y)$.

(2) (A fuzzy partial order in sense of Fan and Zhang [3,29]) A Fan-Zhang-fuzzy partial order (an FZ-fpo for short) on a set X is an L-preorder $e : X \times X \longrightarrow L$ satisfying

(FZ) $\forall x,\ y \in X,\ e(x, y) = e(y, x) = 1$ implies $x = y$.

By Lemma 4 in [2], we know that if e is a B-fpo on X which is compatible w.r.t. an L-equality $\approx$, then $(x \approx y) = e(x, y) \wedge e(y, x)$ for all $x, y \in X$. Hence there's a unique L-equality compatible with a B-fpo.

Obviously, a B-fpo is always an FZ-fpo since (B) can imply (FZ). Conversely, let e be an FZ-fpo on X. Define $\approx_e : X \times X \longrightarrow L$ by

$$(x \approx_e y) = e(x, y) \wedge e(y, x)\ (\forall x, y \in X).$$

It can be shown straightforwardly that e satisfies (Ref), (Tran), (B) and $\approx_e$ is an L-equality on X and e is a B-fpo on X which is compatible w.r.t. $\approx_e$. Thus a B-fpo and an FZ-fpo are equivalent concepts. The definition of an FZ-fpo is formally simpler than that of a B-fpo. From now on, a B-fpo or an FZ-fpo will be just called a fuzzy partial order. For a fuzzy partial order e on X, we call the pair $(X,\ e)$ (or just be denoted by X if there's no confusion) a fuzzy poset.

Example 2.2 Let L be a residuated lattice.

(1) Define $e_L : L \times L \longrightarrow L$ by $e_L(x, y) = x \to y$ for all $x, y \in L$. Then e_L is a fuzzy order on L.

(2) $\forall A, B \in L^X$, the subsethood degree [7] of A in B is define by $sub_X(A, B) = \bigwedge\limits_{x \in X} A(x) \to B(x)$. Then $sub_X : L^X \times L^X \longrightarrow L$ a fuzzy order on L^X (sometimes the subscript X is omitted).

The following definitions and propositions can be found in [1,2,3,29,30], etc.

Definition 2.3 Let (X, e) be a fuzzy poset and $A \in L^X$. $A^u \in L^X$ (resp., $A^l \in L^X$) is defined by $\forall x \in X$,

$$A^u(x) = \bigwedge_{y \in X} A(y) \to e(y, x) \text{ (resp., } \forall x \in X,\ A^l(x) = \bigwedge_{y \in X} A(y) \to e(x, y)).$$

$A^u(x)$ and $A^l(x)$ can be considered as the degree of x to be an upper bound and a lower bound of A, respectively.

Definition 2.4 Let (X, e) be a fuzzy poset, $x_0 \in X$, $A \in L^X$. The element x_0 is called a join (resp., meet) of A, in symbols $x_0 = \bigsqcup A$ (resp., $x_0 = \bigsqcap A$), if

(1) $\forall x \in X,\ A(x) \le e(x, x_0)$ (resp., $A(x) \le e(x_0, x)$);

(2) $\forall y \in X,\ \bigwedge\limits_{x \in X} A(x) \to e(x, y) \le e(x_0, y)$ (resp., $\bigwedge\limits_{x \in X} A(x) \to e(y, x) \le e(y, x_0)$).

It's easy to verified by (FZ), x_1, x_2 are two joins (resp., meets) of A, then $x_1 = x_2$. That is each $A \in L^X$ has at mostly one join (resp., one meet).

Proposition 2.5 *(1) $x_0 = \bigsqcup A$ iff for all $y \in X$, $e(x_0,\ y) = \bigwedge\limits_{x \in X} A(x) \to e(x, y)$.*

(2) $x_0 = \bigsqcap A$ iff for all $y \in X$, $e(y, x_0) = \bigwedge\limits_{x \in X} A(x) \to e(y, x)$.

A fuzzy poset $(X,\ e)$ is called complete if for all $A \in L^X$, $\bigsqcup A$ and $\bigsqcap A$ exist. For example, $(L,\ e_L)$ is a complete fuzzy poset, where $\bigsqcup A = \bigvee\limits_{x \in X} A(x) * x$ and $\bigsqcap A = \bigwedge\limits_{x \in X} A(x) \to x$ for all $A \in L^X$.

Definition 2.6 $A \in L^X$ is called a fuzzy upper set (resp., a fuzzy lower set) if $\forall x,\ y \in X,\ A(x) * e(x, y) \le A(y)$ (resp., $A(x) * e(y, x) \le A(y)$).

Definition 2.7 For $x \in X$, $\downarrow x \in L^X$ (resp., $\uparrow x \in L^X$) is defined as $\forall y \in X$, $\downarrow x(y) = e(y, x)$ (resp., $\downarrow x(y) = e(x, y)$).

Let (X, e_X) and (Y, e_Y) be two fuzzy posets. We call a map $f : X \longrightarrow Y$ order-preserving or (L-fuzzy) monotone (resp., antitone) if $e_X(x,\ y) \le e_Y(f(x),\ f(y))$ (resp., $e_X(x, y) \le e_Y(f(y), f(x))$) for all $x,\ y \in X$.

Definition 2.8 Let (X, e_X), (Y, e_Y) be two fuzzy posets and $f : X \longrightarrow Y$, $g : Y \longrightarrow X$ two order-preserving maps. (f, g) is called a fuzzy Galois connection between X and Y if

$$e_Y(f(x), y) = e_X(x, g(y))$$

for all $x \in X,\ y \in Y$, where f is called the left adjoint of g and dually g the right adjoint of f.

Remark 2.9 For two order-preserving maps $f : (X, e_X) \longrightarrow (Y, e_Y)$ and $g : (Y, e_Y) \longrightarrow (X, e_X)$ between two fuzzy posets, (f, g) is called a fuzzy Galois connection if for all $x \in X, y \in Y$, we have $e_Y(f(x), y) = e_X(x, g(y))$. Fuzzy Galois connections are extensively studied in [24].

(1) $(f_L^{\rightarrow}, f_L^{\leftarrow})$ is a fuzzy Galois connection between (L^X, sub_X) and (L^Y, sub_Y).

(2) In [11,14,21], for two Ω-categories A and B, a pair of Ω-functors $f : A \longrightarrow B$ and $g : B \longrightarrow A$ is said to be an Ω-adjunction if

$$B(f(a), b) = A(a, g(b))$$

for all $a \in A,\ b \in B$ (cf. Definition 2.9 in [14]). A fuzzy Galois connection in Definition 4.1 is an L-adjunction in sense of [11,14,21].

2.2 Fuzzy Dcpos and Their Continuity

Let (X, e) be a fuzzy poset. We call an L-subset $D \in L^X$ fuzzy directed if

(FD1) $\bigvee\limits_{x \in X} D(x) = 1$;

(FD2) $\forall x, y \in X,\ D(x) * D(y) \leq \bigvee\limits_{z \in X} D(z) * e(x, z) * e(y, z)$.

A fuzzy ideal is a lower fuzzy directed subset. We denote the set of all fuzzy directed subsets and all fuzzy ideals on X by $\mathcal{D}_L(X)$ and $\mathcal{I}_L(X)$, respectively. A fuzzy poset is called a fuzzy dcpo if every fuzzy directed subset has a join, or equivalently every fuzzy ideal has a join.

A map $f : X \longrightarrow Y$ between two fuzzy dcpos is called Scott continuous if for any directed subset $D \in L^X$, $f(\bigsqcup D) = \bigsqcup f_L^{\rightarrow}(D)$. All fuzzy dcpos and Scott continuous maps forms a cartesian closed category [26].

It's easy to verify that $D \in L^X$ iff $\downarrow D$ is a fuzzy ideal and $f : X \longrightarrow Y$ is fuzzy Scott continuous iff for any fuzzy directed subset D of X, $f(\bigsqcup D) = \bigsqcup f_L^{\rightarrow}(D)$. In the following discussion, one will see that for many statements, it is valid for any fuzzy subsets iff it is valid for any fuzzy ideals.

Example 2.10 (Example 5.5 in [25]) For X a nonempty set, a family $\delta \subseteq L^X$ is called an L-topology if (T1) $0_X, 1_X \in \delta$; (T2) $A, B \in \delta$ implies $A * B \in \delta$; (T3) $\{A_j |\ j \in J\} \subseteq L^X$ implies $\bigvee_j A_j \in \delta$. For an L-topology δ on X, put $Lpt(\delta) = \{p : \delta \longrightarrow L |\ p$ preserves arbitrary joins and $p(A * B) \geq p(A) * p(B)$

for all $A, B \in L^X\}$. Define $e_{Lpt} : Lpt(\delta) \times Lpt(\delta) \longrightarrow L$ by

$$\forall f,\ g \in Lpt(\delta),\ e_{Lpt}(f,g) = \bigwedge_{U\in\delta} f(U) \to g(U).$$

Then $(Lpt(\delta), e_{Lpt})$ is a fuzzy dcpo.

Definition 2.11 Let (X, e) be a fuzzy dcpo. For any $x \in X$, define $\Downarrow x \in L^X$ by

$$\forall y \in X,\ \Downarrow x(y) = \bigwedge_{I\in\mathcal{I}_L(X)} e(x, \bigsqcup I) \to I(y).$$

A fuzzy dcpo is called continuous or a fuzzy domain if $\Downarrow x \in \mathcal{I}_L(X)$ and $x = \bigsqcup \Downarrow x$ for all $x \in X$.

If (X, e) is a fuzzy domain, then the map $\Downarrow$ has the property of interpolation, i.e., $\Downarrow y(x) = \bigvee_{z\in X} \Downarrow z(x) * \Downarrow y(z)$ for all $x, y \in X$ (cf. Theorem 4.6 in [13]). A fuzzy dcpo (X, e) is continuous iff $(\Downarrow, \bigsqcup)$ is a fuzzy Galois connection between (X, e) and $(\mathcal{I}_L(X), sub_{\mathcal{I}_L(X)})$ (cf. Theorem 5.10 in [25]).

2.3 An L-fuzzy Topology Induced by an L-generalized Convergence Spaces

Definition 2.12 A stratified L-filter on X is a map $\mathcal{F} : L^X \longrightarrow L$ satisfying

$(LF1)$ $\mathcal{F}(\emptyset) = 0,\ \mathcal{F}(X) = 1$;

$(LF2)$ $\forall A, B \in L^X,\ A \le B$ implies $\mathcal{F}(A) \le \mathcal{F}(B)$;

$(LF3)$ $\forall A, B \in L^X$, $\mathcal{F}(A * B) \ge \mathcal{F}(A) * \mathcal{F}(B)$;

(SF) $\forall a \in L$, $\mathcal{F}(\overline{a}) \ge a$ or $\mathcal{F}(aA) \ge \mathcal{F}(A)$.

Remark 2.13 (1) The conditions (LF2) and (LF3) in Definition 2.12 can be equivalently replaced by

(LF4) $\forall A,\ B \in L^X,\ \mathcal{F}(A \to B) \le \mathcal{F}(A) \to \mathcal{F}(B)$.

(2) For a stratified L-filter $\mathcal{F}$ on X and all $A, B \in L^X$, $sub_X(A, B) \le \mathcal{F}(A \to B)$ since $A \to B \ge sub_X(A, B)$.

The set of all stratified L-filters on X will be denoted by $\mathbb{F}_L^s(X)$.

Example 2.14 (1) For $x \in X$, the map $[x] : L^X \longrightarrow L$ defined by $[x](A) = A(x)$ is a stratified L-filter, called the principal L-filter of x.

(2) Let (X, δ) be an L-topological space. For all $x \in X$, define $\mathcal{U}_\delta^x : L^X \longrightarrow L$ by $\mathcal{U}_\delta^x(A) = A^\circ(x)$ for all $A \in L^X$, where $^\circ : L^X \longrightarrow L^X$ is the L-interior operator of (X, δ). Then $\mathcal{U}_\delta^x$ is an L-filter and if δ is stratified then so is $\mathcal{U}_\delta^x$.

(3) Let (X, τ) be an L-fuzzy topological space. For all $x \in X$, define

$\mathcal{U}_\tau^x : L^X \longrightarrow L$ by

$$\mathcal{U}_\tau^x(A) = \bigwedge_{B \leq A} B(x) * \tau(B)$$

for all $A \in L^X$. Then $\mathcal{U}_\tau^x$ is an L-filter and if τ is enriched then $\mathcal{U}_\tau^x$ is stratified.

Definition 2.15 ([10,23]) A stratified L-generalized convergence structure on X is map $R : \mathbb{F}_L^s(X) \times X \longrightarrow L$ satisfying that

$(LFC1)$ $\forall x \in X,\ R([x], x) = 1.$

$(LFC2)$ $\forall x \in X,\ \forall \mathcal{F},\ \mathcal{G} \in \mathbb{F}_L^s(X),\ \mathcal{F} \leq \mathcal{G}$ implies $R(\mathcal{F}, x) \leq R(\mathcal{G}, y)$.

Theorem 2.16 *([10,23]) Each stratified L-generalized fuzzy convergence structure R on X induces an enriched L-fuzzy topology (cf. Subsection 4.2 in [9]) τ_R on X given by*

$$\forall A \in L^X,\ \tau_R(A) = \bigwedge_{(\mathcal{F},x) \in \mathbb{F}_L^s(X) \times X} (A(x) * R(\mathcal{F}, x)) \rightarrow \mathcal{F}(A)$$

and a stratified L-topology $\delta_R = \{A \in L^X |\ \tau_R(A) = 1\}$.

3 L-fuzzy Scott Topology and Scott Convergence of Stratified L-filters

In the rest of this paper, L stands for a frame, i.e., a complete residuated lattice with $* = \wedge$.

Let (X, e) be a fuzzy dcpo and $\mathcal{F}$ a stratified L-filter on X. Define $\mathcal{F}^l \in L^X$ by

$$\forall x \in X,\ \mathcal{F}^l(x) = \bigvee_{A \in L^X} \mathcal{F}(A) \wedge A^l(x).$$

Proposition 3.1 *(1) $\mathcal{F} \leq \mathcal{G}$ implies $\mathcal{F}^l \leq \mathcal{G}^l$;*

(2) $\forall x \in X,\ [x]^l = \downarrow x$.

Proof. Straightforward. $\square$

Let (X, e) be a fuzzy dcpo. Define a map $S : \mathbb{F}_L^s(X) \times X \longrightarrow L$ by

$$\forall (\mathcal{F}, x) \in \mathbb{F}_L^s(X) \times X,\ S(\mathcal{F}, x) = \bigvee_{D \in \mathcal{D}_L(X)} sub_X(D, \mathcal{F}^l) \wedge e(x, \bigsqcup D).$$

By Proposition 3.1, it's easy to see that S is a stratified L-generalized convergence structure on X. Thus it induces an enriched L-fuzzy topology $\sigma_{LF}(X, e)$

given by

$$\forall A \in L^X,\ \sigma_{LF}(X,e)(A) = \bigwedge_{(\mathcal{F},x)\in\mathbb{F}_L^s(X)\times X} (A(x) \wedge S(\mathcal{F},x)) \to \mathcal{F}(A)$$

and a stratified L-topology

$$\sigma_L(X,e) = \{A \in L^X |\ \forall(\mathcal{F},x) \in \mathbb{F}_L(X) \times X,\ A(x) \wedge S(\mathcal{F},x) \leq \mathcal{F}(A)\}.$$

We call $\sigma_{LF}(X,e)$ ($\sigma_{LF}(X)$ for short) and $\sigma_L(X,e)$ ($\sigma_L(X)$ for short) an L-fuzzy Scott topology and a fuzzy Scott topology on (X,e), respectively. ($\sigma_L(X)$ is studied in [27].)

Lemma 3.2 *(1)* $e(\bigsqcup A, x) = sub(A, \downarrow x)$.
(2) $e(x,y) \wedge sub(\uparrow x, A) \leq sub(\uparrow y, A)$.
(3) $S([y],x) = e(x,y)$.

Proof. Straightforward. □

For a fuzzy directed subset $D \in L^X$, define $\mathcal{F}_D : L^X \longrightarrow L$ by

$$\forall A \in L^X,\ \mathcal{F}_D(A) = \bigvee_{x\in X} D(x) \wedge sub(\uparrow x, A).$$

Proposition 3.3 *(1)* $\mathcal{F}_D$ *is a stratified* L*-filter;*
(2) $\mathcal{F}_D^l = \downarrow D$.
(3) $S(\mathcal{F}_D, \bigsqcup D) = 1$.

Define $\nabla(X) : L^X \longrightarrow L$ by

$$\nabla(X)(A) = \bigwedge_{x,y\in X} (e(x,y) \wedge A(x)) \to A(y).$$

$\nabla(X)(A)$ can be interpreted as the degree of A to be a fuzzy upper set [4].

Define $F(X) : L^X \longrightarrow L$ by

$$\forall A \in L^X,\ F(X)(A) = \bigwedge_{D\in\mathcal{D}_L(X)} A(\bigsqcup D) \to \mathcal{F}_D(A).$$

Lemma 3.4 $F(X) \leq \nabla(X)$.

Proof. $\forall A \in L^X$. $\forall x, y \in X$,

$$F(X)(A) \leq A(\bigsqcup \downarrow x) \to \mathcal{F}_{\downarrow x}(A) = A(x) \to \mathcal{F}_{\downarrow x}(A)$$

[4] $\nabla(X)$ just is the L-fuzzy Alexandrov topology for $L = [0,1]$ in [4].

and by Lemma 3.2(2),

$$\mathcal{F}_{\downarrow x}(A) = \bigvee_{a\in X} \downarrow x(a) \wedge sub(\uparrow a, A) = \bigvee_{a\in X} e(a,x) \wedge sub(\uparrow a, A).$$

Then $F(X)(A) \leq \bigwedge_{x,y\in X} A(x) \to (e(x,y) \to A(y)) = \bigwedge_{x,y\in X} A(x) \to sub(\uparrow x, A) = \nabla(X)(A)$. □

Proposition 3.5 $\sigma_{LF}(X) = F(X)$.

Proof. $\forall A \in L^X$. On one hand,

$$\begin{aligned}\sigma_{LF}(X)(A) &\leq \bigwedge_{D\in\mathcal{D}_L(X)} (A(\bigsqcup D) \wedge S(\mathcal{F}_D, \bigsqcup D)) \to \mathcal{F}_D(A)\\ &= \bigwedge_{D\in\mathcal{D}_L(X)} A(\bigsqcup D) \to \mathcal{F}_D(A)\\ &= F(X)(A).\end{aligned}$$

On the other hand, for all $(\mathcal{F}, x) \in \mathbb{F}_L(X) \times X$,

$$\begin{aligned}&A(x) \wedge S(\mathcal{F}, x) \wedge F(X)(A)\\ &= A(x) \wedge S(\mathcal{F}, x) \wedge F(X)(A) \wedge F(X)(A)\\ &= \bigvee_{E\in\mathcal{D}_L(X)} \bigwedge_{D\in\mathcal{D}_L(X)} A(x) \wedge (sub(E, \mathcal{F}^l) \wedge e(x, \bigsqcup E)) \wedge (A(\bigsqcup D) \to\\ &\qquad \mathcal{F}_D(A)) \wedge F(X)(A)\\ &\leq \bigvee_{E\in\mathcal{D}_L(X)} A(x) \wedge (sub(E, \mathcal{F}^l) \wedge e(x, \bigsqcup E)) \wedge (A(\bigsqcup E) \to\\ &\qquad \mathcal{F}_E(A)) \wedge ((A(x) \wedge e(x, \bigsqcup E)) \to A(\bigsqcup E))\\ &\leq \bigvee_{E\in\mathcal{D}_L(X)} sub(E, \mathcal{F}^l) \wedge A(\bigsqcup E) \wedge (A(\bigsqcup E) \to \mathcal{F}_E(A))\\ &\leq \bigvee_{E\in\mathcal{D}_L(X)} sub(E, \mathcal{F}^l) \wedge \mathcal{F}_E(A)\\ &= \bigvee_{E\in\mathcal{D}_L(X)} \bigvee_{y\in X} sub(E, \mathcal{F}^l) \wedge (E(y) \wedge sub(\uparrow y, A))\\ &\leq \bigvee_{E\in\mathcal{D}_L(X)} \bigvee_{y\in X} (E(y) \to \mathcal{F}^l(y)) \wedge (E(y) \wedge sub(\uparrow y, A))\\ &\leq \bigvee_{E\in\mathcal{D}_L(X)} \bigvee_{y\in X} \mathcal{F}(\uparrow y)) \wedge sub(\uparrow y, A)\\ &\leq \mathcal{F}(A).\end{aligned}$$

Then $F(X)(A) \leq (A(x) \wedge S(\mathcal{F}, x)) \to \mathcal{F}(A)$ and by the arbitrariness of $(\mathcal{F}, x)$, we have $\sigma_{LF}(X)(A) \geq F(X)(A)$. □

For an L-filter $\mathcal{F}$, we call $\mathcal{F}$ Scott converges to x if there exists a fuzzy ideal $I \in \mathcal{I}_L(X)$ such that $I \leq \mathcal{F}^l$ and $x \leq \bigsqcup I$ (this means $e(x, \bigsqcup I) = 1$). For an L-fuzzy topology τ on X, we call $\mathcal{F}$ topologically convergent to x if $\mathcal{U}_\tau^x \leq \mathcal{F}$ (cf. Subsection 6.2 in [8] and Definition 8.2.3 in [9]).

Define $d(X) : L^X \longrightarrow L$ by

$$d(X)(A) = \bigwedge_{D \in \mathcal{D}_L(X)} A(\bigsqcup D) \to \bigsqcup A_L^{\to}(D).$$

$d(X)(A)$ as the degree of A to be Scott continuous.

Lemma 3.6 $\bigsqcup A_L^{\to}(D) = \bigvee_{x \in X} A(x) \wedge D(x)$, *where* $\bigsqcup A_L^{\to}(D)$ *is the join of* $A_L^{\to}(D)$ *taken in the fuzzy poset* (L, e_L). *Thus*

$$d(X)(A) = \bigwedge_{D \in \mathcal{D}_L(X)} A(\bigsqcup D) \to (\bigvee_{x \in X} D(x) \wedge A(x)).$$

Proof. Straightforward. □

Theorem 3.7 $\sigma_{LF}(X) = \nabla(X) \wedge d(X)$.

Proof. $\forall A \in L^X$. On one hand,

$$\sigma_{LF}(X)(A) \leq \bigwedge_{x,y \in X} (A(x) \wedge S([y], x)) \to [y](A) = \nabla(X)(A)$$

and

$$\begin{aligned}
\sigma_{LF}(X)(A) &= F(X)(A) \\
&= \bigwedge_{D \in \mathcal{D}_L(X)} A(\bigsqcup D) \to \mathcal{F}_D(A) \\
&= \bigwedge_{D \in \mathcal{D}_L(X)} A(\bigsqcup D) \to (\bigvee_{x \in X} I(x) \wedge sub(\uparrow x, A)) \\
&\leq \bigwedge_{D \in \mathcal{D}_L(X)} A(\bigsqcup D) \to (\bigvee_{x \in X} I(x) \wedge A(x)) \\
&= d(X)(A).
\end{aligned}$$

On the other hand, we only need to show that $\forall D \in \mathcal{D}_L(X)$,

$$(A(\bigsqcup D) \to (\bigvee_{x \in X} I(x) \wedge A(x))) \wedge \nabla(X)(A) \leq (A(\bigsqcup D) \to \mathcal{F}_D(A)$$

and we only need to show that

$$\nabla(X)(A) \wedge (\bigvee_{x\in X} I(x) \wedge A(x)) \leq \bigvee_{x\in X} I(x) \wedge sub(\uparrow x, A)$$

and we only need to show

$$A(x) \wedge \nabla(X)(A) \leq sub(\uparrow x, A).$$

In fact, $A(x)\wedge\nabla(X)(A) = A(x)\wedge(\bigwedge_{y\in X}(A(x)\wedge e(x,y)) \to A(y)) \leq \bigwedge_{y\in X} e(x,y) \to A(y) = sub(\uparrow x, A)$. $\square$

Theorem 3.8 *[27] (1) $A \in \sigma_L(X)$ iff $A : X \longrightarrow L$ preserves joins of fuzzy directed subsets (when L is being viewed as the complete fuzzy poset (L, e_L)).*

(2) If (X, e) is continuous, then for all $x \in X$, $\Uparrow x \in \sigma_L(X)$.

(3) If (X, e) is continuous, then $\{\overline{a}\wedge \Uparrow x|\ a \in L,\ x \in L\}$ forms a basis of $\sigma_L(X)$.

Proof. (1) is a corollary of Theorem 3.7. (2) can be inferred from the property of interpolation of the fuzzy way-below relation $\Downarrow$. (3) By Theorem 3.7, we can show that $U = \bigvee_{y\in X} \overline{U(y)}\wedge \Uparrow y$ for any $U \in \sigma_L(X)$. $\square$

Proposition 3.9 *For $x \in X$, we have $(\mathcal{U}^x)^l \leq\Downarrow x$. If (X, e) is continuous then $(\mathcal{U}^x)^l = \Downarrow x$.*

Proof. $\forall y \in X,\ B \leq\uparrow y,\ I \in \mathcal{I}_L(X)$,

$$\sigma_{LF}(X)(B) \leq (B(\bigsqcup I) \wedge S(\mathcal{F}_I, \bigsqcup I)) \to \mathcal{F}_I(B)$$

$$= B(\bigsqcup I) \to \mathcal{F}_I(B) \leq B(\bigsqcup I) \to \mathcal{F}_I(\uparrow y) = B(\bigsqcup I) \to I(y)$$

and

$$\begin{aligned}
&B(x) \wedge \sigma_{LF}(X)(B) \wedge e(x, \bigsqcup I) \\
=\ &B(x) \wedge e(x, \bigsqcup I) \wedge \sigma_{LF}(X)(B) \wedge \sigma_{LF}(X)(B) \\
\leq\ &B(x) \wedge e(x, \bigsqcup I) \wedge (B(\bigsqcup I) \to I(y)) \wedge ((B(x) \wedge e(x, \bigsqcup I)) \to B(\bigsqcup I)) \\
\leq\ &B(\bigsqcup I) \wedge (B(\bigsqcup I) \to I(y)) \\
\leq\ &I(y)
\end{aligned}$$

and then $B(x) \wedge \sigma_{LF}(X)(B) \leq e(x, \bigsqcup I) \to I(y)$. By the arbitrariness of $I \in \mathcal{I}_L(X)$, we have $B(x) \wedge \sigma_{LF}(X)(B) \leq\Downarrow x(y)$. Hence $(\mathcal{U}^x)^l(y) = \mathcal{U}^x(\uparrow y) \leq\Downarrow x(y)$ and then $(\mathcal{U}^x)^l \leq\Downarrow x$.

If (X, e) is continuous, then $\sigma_{LF}(X)(\Uparrow x) = 1$ and $(\mathcal{U}^x)^l(y) = \mathcal{U}^x(\uparrow y) \geq \sigma_{LF}(A)(\Uparrow x) \wedge \Uparrow y(x) = \Downarrow x(y)$. Thus $(\mathcal{U}^x)^l \geq \Downarrow x$. $\Box$

Proposition 3.10 *For all* $(\mathcal{F}, x) \in \mathbb{F}^s_L(X) \times X$, $S(\mathcal{F}, x) \leq sub(\mathcal{U}^x, \mathcal{F})$. *Then Scott convergence always implies topological convergence.*

Proof. It's easy to show that $sub(\mathcal{U}^x, \mathcal{F}) = \bigwedge_{A \in L^X} (A(x) \wedge \sigma_{LF}(X)(A)) \to \mathcal{F}(A)$ and for all $A \in L^X$, $\sigma_{LF}(X)(A) \leq (A(x) \wedge S(\mathcal{F}, x)) \to \mathcal{F}(A)$. It follows that $S(\mathcal{F}, x) \leq sub(\mathcal{U}^x, \mathcal{F})$. If $\mathcal{F}$ Scott converges to x, then $S(\mathcal{F}, x) = 1$ and $sub(\mathcal{U}^x, \mathcal{F}) = 1$. Thus $\mathcal{U}^x \leq \mathcal{F}$, i.e., $\mathcal{F}$ topologically converges to x. $\Box$

The following theorem is the main result of this paper which establishes a connection between the continuity of a fuzzy dcpo and the Scott convergence of L-filters.

Theorem 3.11 *Consider the followings:*

(1) (X, e) *is continuous;*

(2) for any stratified L-filter, Scott convergence coincides with topological convergence with respect to the L-fuzzy Scott topology;

(3) $\forall x \in X$, $\mathcal{U}^x$ *is Scott convergent to* x.

(4) $S(\mathcal{F}, x) = sub(\mathcal{U}^x, \mathcal{F})$.

We have (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3) $\Longrightarrow$ (4) *and if* $1 \lhd 1$ [5], *then (4) is equivalent to (1-3).*

Proof. (2) $\Rightarrow$ (3) is obvious. (1) $\Rightarrow$ (2) : Suppose that $\mathcal{F}$ is topological convergent to x, that is $\mathcal{U}^x \leq \mathcal{F}$. Since (X, e) is continuous, we have $x = \bigsqcup \Downarrow x$ and $\Downarrow x$ is a fuzzy ideal and $\Downarrow x = (\mathcal{U}^x)^l \leq \mathcal{F}^l$. It follows that $\mathcal{F}$ is Scott convergent to x.

(3) $\Rightarrow$ (1) : If $\mathcal{U}^x$ is Scott convergent to x, then there exists a fuzzy ideal $I \in \mathcal{I}_L(X)$ such that $x \leq \bigsqcup I$ and $I \subseteq (\mathcal{U}^x)^l \leq \Downarrow x \leq \downarrow x$. It's easy to show that $x = \bigsqcup \Downarrow x$. To show $\Downarrow x$ is fuzzy directed. In fact, $\bigvee_{y \in X} \Downarrow x(y) \geq \bigvee_{y \in X} I(y) = 1$ and for all $a, b \in X$, $\Downarrow x(a) \wedge \Downarrow x(b) \leq (e(x, \bigsqcup I) \to I(a)) \wedge (e(x, \bigsqcup I) \to I(b)) = I(a) \wedge I(b) \leq \bigvee_{c \in X} I(c) \wedge e(a, c) \wedge e(b, c) \leq \bigvee_{c \in X} \Downarrow x(c) \wedge e(a, c) \wedge e(b, c)$.

(1) $\Rightarrow$ (4) : we only need to show $S(\mathcal{F}, x) \geq sub(\mathcal{U}^x, \mathcal{F})$. In fact, since (X, e) is continuous, then $\Downarrow x \in \mathcal{D}_L(X)$, $\bigsqcup \Downarrow x = x$, $(\mathcal{U}^x)^l = \Downarrow x$ and $\Uparrow x$ is fuzzy Scott open. Then

$$S(\mathcal{F}, x) \geq e(x, \bigsqcup \Downarrow x) \wedge sub(\Downarrow x, \mathcal{F}^l) = \bigwedge_{y \in X} \Downarrow x(y) \to \mathcal{F}^l(y)$$

[5] $\lhd$ is the wedge-below relation. Two elements $x \lhd y$ iff for all $y \leq \bigvee A$, there exists $a \in A$ such that $x \leq a$.

$$= \bigwedge_{y\in X} (\mathcal{U}^x)^l(y) \to \mathcal{F}^l(y) \geq \bigwedge_{A\in L^X} \mathcal{U}^x(A) \to \mathcal{F}(A) = sub(\mathcal{U}^x, \mathcal{F}).$$

If $1 \lhd 1$, then (4) implies (3). □

4 Relations between Fuzzy Scott Continuous and Topological Continuous

In this section, for a map, we will define the fuzzy Scott continuity (which is a counterpart of Scott continuity) and topological continuity w.r.t the L-fuzzy topology and then study relations among them and the continuity of fuzzy dcpos.

Define $\Delta(X),\ c(X) : L^X \longrightarrow L$ by

$$\begin{aligned}
\Delta(X)(A) &= \bigwedge_{x,y\in X} (e(x,y) \wedge A(y)) \to A(x),\\
c(X)(A) &= \bigwedge_{D\in\mathcal{D}_L(X)} \bigsqcap A_L^{\rightarrow}(D) \to A(\bigsqcup D),\\
\sigma_{LF}^c(X) &= \Delta(X) \wedge c(X).
\end{aligned}$$

$\sigma_{LF}^c(X)(A)$ can be condisered the degree of A to be Scott closed. We call $A \in L^X$ a fuzzy Scott closed set if $\sigma_{LF}^c(X)(A) = 1$ and denote the set of all fuzzy Scott closed sets by $\sigma_L^c(X)$. Obviously, for any $x \in X$, $\downarrow x \in \sigma_L^c(X)$.

Proposition 4.1 *For $A \in L^X$, $c(A) = \bigwedge_{D\in\mathcal{D}_L(X)} sub(D, A) \to A(\bigsqcup D)$.*

Proof. Trivial since it is easy to show that $\bigsqcap A_L^{\rightarrow}(D) = sub(D, A)$. □

Lemma 4.2 *Let $f : (X, e_1) \longrightarrow (Y, e_2)$ be a fuzzy continuous map and $B \in L^Y,\ D \in \mathcal{D}_L(X)$. Then $\bigsqcup (f_L^{\leftarrow}(B))_L^{\rightarrow}(D) = \bigsqcup B_L^{\rightarrow}(f_L^{\rightarrow}(D))$ (taken in (L, e_L)).*

Proof.

$$\bigsqcup B_L^{\rightarrow}(f_L^{\rightarrow}(D)) = \bigvee_{y\in Y} B(y) \wedge f_L^{\rightarrow}(D)(y) = \bigvee_{y\in Y} B(y) \wedge (\bigvee_{y=f(x)} D(f(x))$$

$$= \bigvee_{x\in X} B(f(x)) \wedge D(x) = \bigvee_{x\in X} f_L^{\leftarrow}(B)(x) \wedge D(x) = \bigsqcup (f_L^{\leftarrow}(B))_L^{\rightarrow}(D).$$

□

Proposition 4.3 *If L is a complete Boolean algebra, then for all $A \in L^X$, $\sigma_{LF}(X)(A) = \sigma_{LF}^c(X)(\neg A)$, $\nabla(X)(A) = \Delta(X)(\neg A)$, $d(X)(A) = c(X)(\neg A)$ and $\sigma_L(X) = \{\neg A |\ A \in \sigma_L(X)\}$, where $(\neg A)(x) = \neg(A(x))$ $(\forall x \in X)$.*

Proof. Straightforward. □

Theorem 4.4 *Let $f : (X, e_1) \longrightarrow (Y, e_2)$ be a map between two fuzzy dcpos. Consider the following:*

(1) f is fuzzy Scott continuous.

(2) $\forall B \in L^X$, $\sigma^c_{LF}(X)(f^{\leftarrow}_L(B)) \geq \sigma^c_{LF}(Y)(B)$.

(3) $\forall B \in \sigma^c_L(Y)$, $f^{\leftarrow}_L(B) \in \sigma^c_L(X)$.

(4) $\forall B \in L^X$, $\sigma_{LF}(X)(f^{\leftarrow}_L(B)) \geq \sigma_{LF}(Y)(B)$.

(5) $\forall B \in \sigma_L(Y)$, $f^{\leftarrow}_L(B) \in \sigma_L(X)$.

We have (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3) $\Longrightarrow$ (4) $\Longrightarrow$ (5) *and if L is a Boolean algebra additinally, then the above five are equivalent to each other.*

Proof. (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5) are obvious. (1) $\Rightarrow$ (2) is trivial since $\Delta(X)(f^{\leftarrow}_L(B)) \geq \Delta(Y)(B)$ and $c(X)(f^{\leftarrow}_L(B)) \geq c(Y)(B)$. (1) $\Rightarrow$ (4) is analogous to (1) $\Rightarrow$ (2).

(3)$\Rightarrow$ (1). $\forall x_1, x_2 \in X$, $f^{\leftarrow}_L(\downarrow x_2) \in \sigma^c_L(X)$ and

$$e_1(x_1, x_2) \leq f^{\leftarrow}_L(\downarrow f(x_2))(x_2) \to f^{\leftarrow}_L(\downarrow f(x_2))(x_1) = \downarrow f(x_2)(f(x_1)) = e_2(f(x_1, x_2)).$$

Thus f is order-preserving. $\forall D \in \mathcal{D}_L(X)$,

$$\begin{aligned}
\bigwedge_{z \in Y} f^{\rightarrow}_L(D)(z) \to e_2(z, y) &= \bigwedge_{x \in X} D(x) \to e_1(f(x), y) \\
&= \bigwedge_{x \in X} D(x) \to f^{\leftarrow}_L(\downarrow y)(x) \\
&= sub(D, f^{\leftarrow}_L(\downarrow y)) \\
&\leq f^{\leftarrow}_L(\downarrow y)(\bigsqcup D) \\
&= \downarrow y(f(\bigsqcup D)) \\
&= e_2(f(\bigsqcup D), y).
\end{aligned}$$

Hence $f(\bigsqcup D) = \bigsqcup f^{\rightarrow}_L(D)$.

If L is a Boolean algebra additionally, then (5) $\Rightarrow$ (3). □

5 A Special Case: The Fuzzy Scott Topology on Crisp Dcpos

There are many kinds of fuzzy filter, such as L-filter of the form $L^{(L^X)}$, L-filter of crisp degree of the form $2^{(L^X)}$ and L-filter of ordinary sets of the form $L^{(2^X)}$. There are also three kinds of fuzzy topology: the L-topology, the L-fuzzifying topology and the L-fuzzy topology. Using a proper kind of fuzzy filters (to define deferent kinds of Scott convergence) on fuzzy dcpos and on crisp ones, we get many kinds of fuzzy versions of Scott topology.

For example, we choose L-filters to define a fuzzy Scott topology on crisp dcpo. A fuzzy Scott topology on crisp dcpo is not only a special case of $\sigma_{LF}(X)$ for a crisp dcpo X in Section 3, but also arises naturally by following three additional origins:

(1) Recall the definition of Scott topology [6] on a crisp dcpo X, a subset is Scott open iff it is a Scott continuous map from X to two elements lattice $\{0, 1\}$. Extending $\{0, 1\}$ to some complete lattice L (sometimes equipped with some necessary conditions), the family of all Scott continuous maps from X to L forms an L-topology on X. In addition, by the slogan that *a predicate and an open set are the same thing* [19], we also get that the Scott continuous (i.e., structure-preserving) maps from X to L consists of an L-topology on X.

(2) In [22], it is shown that for a continuous frame L, the family of all continuous functions from a topological space $(X, \mathcal{T})$ to $(L, \sigma(L))$ is an L-topology on X. If X itself is a dcpo and $\mathcal{T}$ is the Scott topology on X, then a continuous map from $(X, \mathcal{T})$ to $(L, \sigma(L))$ is exactly a Scott continuous map from X to L and the results in [22] still hold. In the case, the condition that L is continuous frame could perhaps be weaken. In fact, Theorem II-4.19 in [6] points out that L is a frame is sufficient.

(3) For a crisp topology $\mathcal{T}$ on X, by using the Lowen functor ω_L (see in [12] for detailed discussion), we can construct an L-topology, namely $\omega_L(\mathcal{T})$, the L-topology generated by $\mathcal{T}$. In fact, $\omega_L(\mathcal{T})$ is a set of all continuous maps $(X, \mathcal{T})$ to $(L, \nu(L))$, where $\nu(L)$ is the upper topology on L. If L is completely distributive, then $\nu(L) = \sigma(L)$. Thus in this case, $\omega_L(\mathcal{T})$ is the set of all continuous maps from $(X, \mathcal{T})$ to $(L, \sigma(L))$. Also, if X is a dcpo and $\mathcal{T}$ is the Scott topology on X, then $\omega_L(\mathcal{T})$ is the set of all Scott continuous maps from X to L.

The four origins come to the same object—a fuzzy Scott topology on crisp dcpos, which has been studied in [28] and the following main results are obtained:

(1) The fuzzy Scott topology is just the L-topology generated by the classical Scott topology.

(2) If the dcpo is continuous, then the fuzzy Scott topological space is ι-sober (an L-topology is called ι-sober if $\iota_L(\delta)$ is sober, where ι_L is the right adjoint of ω_L).

(3) The fuzzy Scott topology is completely distributive iff L is completely distributive and the dcpo is continuous.

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