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Three New Genuine Five-valued Logics Intended to Model Non-trivial Concepts

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Abstract

We introduce three 5-valued paraconsistent logics that we name FiveASP1, FiveASP2 and FiveASP3. Each of these logics is genuine and paracomplete. The new value is called e attempting to model the notion of ineffability. If one drops e from any of these logics one obtains a well known 4-valued logic introduced by Avron. If, on the other hand one drops the "implication" connective from any of these logics, one obtains Priest logic FDEe. We present some properties of these logics.

Keywords: Argumentation semantics, stage argumentation semantics, logic programming semantics.

1 Introduction

Belnap claims that a 4-valued logic is a suitable framework for computerized reasoning [19]. Avron in [4,3,5,2] supports this thesis. He shows that a 4-valued logic naturally express true, false, inconsistent or uncertain information. Each of these concepts is represented by a particular logical value. Furthermore in [4] he presents a sound and complete axiomatization of a family of 4-valued logics.

On the other hand, Priest argues in [26] that a 4-valued logic models very well the four possibilities explained before, but here in the context of Buddhist meta-physics, see for instance [31]. This logic is called FDE, but such logic fails to satisfy the well known Modus Ponens inference rule. If one removes the implication connective in this logic, it corresponds to the corresponding fragment of any of the logics studied

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by Avron. Priest then extends FDE to a 5-valued logic named FDEe, see [27]. This new logic has a new valued e with the aim to represent the notion of ineffability, but FDEe lacks of an implication connective.

Some authors claim that many arguments formulated in Buddhist texts correspond to such well recognized rules of inference as Modus Ponens, constructive dilemma and categorical syllogism (also known as hypothetical syllogism) among other rules of inference, see [25,17]. Having an implication that does not satisfy Modus Ponens or removing the implication connective, can be consider as a kind of weakness of FDEe, see [25].

However Priest makes a remarkable work by studying in great detail the work of the buddhist texts from the Pali Canon to the MMK by Nāgārjuna and beeing able to model their way of reasoning in terms of modern non-classical logics [26,27,28,29].

A main issue is to represent the notion of ineffability. The fifth value, e, then, is the value of ineffable.

There is a complex but well known phenomenon that often arises when a philosophy argues that there are limits to thought/language, and tries to justify this view by giving reasons as to why there are things about which one cannot think/talk—in the process appearing to give the lie to the claim. In poetry we also find a similar situation: the need to talk about extreme situations, which somehow we can not talk [9,21]. Currently there is much written in literature (poetry, stories) and philosophy related to apparently illogical concepts such as the paradoxical and the ineffable, we enumerate some examples now. The theory of the two truths began twenty-five centuries ago. It started in the sixth century BCE India with the Siddhārtha Gautama, who became a buddha "awakened one" because he understood: the meaning of the two truths and the reality of all the objects of knowledge is exhaustively comprised of the two truths. According to the Samādhirāja-sūtra, the theory of the two truths is a contribution made by the Buddha towards Indian philosophy. Nāgārjuna, in his Mūlamadhyamakakārikā, attributes the two truths to the Buddha as follows: "the Dharma taught by the buddhas is precisely based on the two truths: a truth of mundane conventions and a truth of the ultimate" [32]. The theory of the two truths is the heart of the Buddha's philosophy according to what the Madhyamaka philosophers indicate. The knowledge of the ultimate truth informs us of how things really are ultimately, and thus takes our minds beyond the limits of conceptual and linguistic conventions [32]. Another example is [12], where the author indicates: "My paper will focus on the ways Woolf disfigures and refigures literature through the image of ruins. Writing about the paradoxical nature of decay, Woolf captures the ineffable quality of a time that inexorably passes yet is shaped by surviving reminiscences." More evidence is the Strange Loop which is a paradoxical construction, a shift from one level of abstraction to another that somehow gives rise to a closed, eternal cycle. The author of [7] indicates that this paradoxical model is prevalent in Jorge Luis Borges's short stories and that we are able to better explore Borges's belief in literature's unique power to create spatiotemporal paradoxes. In [7], the author also analyzes how Borges creates Strange Loops in impossible linkages between distinct narrative frames and he also demonstrate how Borges composes an architectural Strange Loop. We also can find that formerly, eastern thinkers and western mystics were keenly aware of the predicative ineffability of the ultimate. Heidegger thought and wrote about it in terms of different ideas for example "non-truth as the precondition of truth". Chinese side it is found that Confucius was highly sensitive to the problem and gave his own response to it. In [34] is analyzed the relation between Heidegger and analytic philosophers on this issue, and the inquire comparatively how Heidegger and Confucius recognized the limit of language, and how this awareness helped them understand the thinking role of the poetry experience. On the other hand, it is known the connections between the work of Martin Heidegger (1889 – 1976) and Chan/Zen Buddhism – a school of Buddhism originating in China around the 6th Century. In [11], is explored one aspect of that connection, drawing on the work of the Japanese Zen philosopher Dōgen Kigen (1200 – 1253). Heidegger held that being is ineffable, and Dōgen held that ultimate reality is ineffable. More evidence is presented in [16] where is mentioned: "The ineffable is after all beyond words, and those who have an ineffable experience may feel so overwhelmed by the very ineffability of their experience, or emotion, that they resign themselves to saying nothing"; and "Words will always be saying too much or too little... Oh to be silent! Oh to be a painter!". Finally, we can see that even in the context of the Christian religion, the Bible also refers to the concept of ineffable in the book of Romans, verse 8.

Despite all the interest in the subject, we can see that little has been attempted to formalize these concepts in a logic. Priest is concerned with that phenomenon. According to him, Buddhist philosophy has resources to address this kind of issue much less present in Western traditions. Buddhist logicians consider that there are four possibilities: only true, only false, both true and false, and finally neither true or false. Later developments add a fifth possibility: ineffability [3]. Of course, one might be skeptical that such ideas can be made logically respectable. Priest shows how to accomplish this task with some tools from contemporary non-classical logic. His work is impeccable, but as stated earlier, we consider prudent to extend FDEe logic with an "implication" connective that at least satisfies Modus Ponens.

For the nature of this work is desirable to consider the use of paraconsistent logics [14]. In addition recent work on these logics considers also some useful relative new properties, namely genuineness and paracompleteness, see [8,18,24,22,5]. Arguments in favor of rejecting the law of non-contradiction have been supported more recently by the research done on paraconsistent logics and the applications they have encountered, in particular, in artificial intelligence. Paraconsistent logics accept inconsistencies without presenting the problem that once a contradiction has been derived, then any proposition follows as a consequence, as is the case of classical logic.

We introduce three paraconsistent (genuine and paracomplete) logics that are constructed based on the combination of two logics: FDEe (by Priest) and a version of Four due to Avron that we call it $BL_{\supset}$. The main point is to add an implication

[3] As far as the authors know, buddhist texts never talk explicitly about five possibilities, as they actually mention four cases. However, Priest shows that buddhist narratives assume this sort of incommensurable fifth, see [26,27,28,29].

to FDEe that satisfies (at least) Modus Ponens.

Finally, we would like to point out that adding a "implication" connective to non-classical logics is not always a direct task. Corcoran has distinguished twelve uses of the term "implies" [13]. There is nothing to prevent "implies" to be interpreted in any conventional way in order to convey any pragmatical use. We could name it differently, but since we expect it to obey the Modus Ponens rule of inference we decided to call it "implication".

In section...

2 Background

In this section: we define a logic from the semantical point of view, particularly via muti-valued systems; we present one axiomatic formal system for logic $BL_{\supset}$, provided by Avron in [6]; we review the system HBL, a formal axiomatic theory for $BL_{\supset}$ [4]; and finally we present a summary of some material from [27] that we need to borrow for the definition of our contribution.

2.1 Multi-valued logics

A way to define a logic is by means of truth values and interpretations. Multi-valued systems generalize the idea of using the truth tables that are used to determine the validity of formulas in classical logic. It has been suggested that multi-valued systems should not count as logics; on the other hand pioneers such as Łukasiewicz considered such multi-valued systems as alternatives to the classical framework. Like other authors do, we prefer to give to multi-valued systems the benefit of the doubt about their status as logics. The core of a multi-valued system is its *domain* of values D, where some of such values are special and identified as *designated*. Connectives (e.g. $\wedge$, $\vee$, $\rightarrow$, $\neg$) are then introduced as operators over D according to the particular definition of the logic. An *interpretation* is a function $I\colon \mathcal{L} \rightarrow D$ that maps atoms to elements in the domain. The application of I is then extended to arbitrary formulas by mapping first the atoms to values in D, and then evaluating the resulting expression in terms of the connectives of the logic. A formula is said to be a *tautology* if, for every possible interpretation, the formula evaluates to a designated value. Given a set of formulas Γ, and a formula α, the notation $\Gamma \models \alpha$ means that any interpretation that models all formulas of Γ also models α. The most simple example of a multi-valued logic is classical logic where: $D = \{0, 1\}$, 1 is the unique designated value, and connectives are defined through the usual basic truth tables. From now on, we refer to all multi-valued systems as multi-valued logics.

Not all multi-valued logics must have the four connectives mentioned before, in fact classical logic can be defined in terms of two of those connectives $\neg$, $\wedge$ (primitive connectives), and the other two (non-primitive) can be defined in terms of $\neg$, $\wedge$. In case of a logic having the implication connective, it is desirable that it preserves tautologies, in the sense that if $\alpha, \alpha \rightarrow \beta$ are tautologies, then β is also a tautology. This restriction enforces the validity of Modus Ponens in the logic.

Objects 0, 1, 2, 3 and 4 are part of the semantics of logics studied in this paper and were chosen only for convenience, they do not correspond to natural numbers.

A logic satisfies the principle of explosion (EFQ) if $\alpha, \neg\alpha \models \beta$. A logic is paraconsistent if it rejects the principle of explosion. A logic satisfies the principle of non-contradiction (PNC) if $\models \neg(\alpha \wedge \neg\alpha)$. A logic is genuine if it rejects the principle of non-contradiction. A logic satisfies the law of excluded middle if $\models \alpha \vee \neg\alpha$. A logic is paracomplete if it rejects the law of excluded middle.

2.1.1 Logic $BL_{\supset}$.

This logic is a 4-valued logic with truth values in the domain $D = \{0, 1, 2, 3\}$ where 2 and 3 are the designated values [4]. The connectives $\wedge$ and $\vee$, as usually, correspond to the *greatest lower bound* (Glb) and the *least upper bound* (Lub), respectively. The connectives $\to$ and $\neg$ are defined as: $\alpha \to \beta = 3$ if α is not designated, $\alpha \to \beta = \beta$ if α is designated; $\neg(0) = 3$, $\neg(3) = 0$, $\neg(\alpha) = \alpha$ if $\alpha = 1$ or $\alpha = 2$.

The logic $BL_{\supset}$ is induced by the partial order $0 < 1$, $0 < 2$, $1 < 3$, $2 < 3$. Note that if we consider only the values 0, 1 and 3, we obtain the Kleene's logic while if we take the values 0, 2 and 3 we have the PAC logic. The Kleene and PAC logics [6] defined by 3-truth values and which have been studied in detail are sublogics of $BL_{\supset}$.

As A. Avron mentions in [4], $BL_{\supset}$ is *interlaced* [4] and hence satisfies $1 \wedge 2 = 0$ and $1 \vee 2 = 3$.

2.2 The system HBL

Let us consider HBL, a formal axiomatic theory for $BL_{\supset}$ [4] formed by the primitive logical connectives: $\neg, \to, \wedge$ and $\vee$. We also consider one logical connective defined in terms of the primitive ones:

$$\alpha \leftrightarrow \beta := (\alpha \to \beta) \wedge (\beta \to \alpha)$$

the well-formed formulas are constructed as usual, the axiom schemas are:

I1 $\alpha \to (\beta \to \alpha)$

I2 $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$

I3 $((\alpha \to \beta) \to \alpha) \to \alpha$

C1 $(\alpha \wedge \beta) \to \alpha$

C2 $(\alpha \wedge \beta) \to \beta$

C3 $\alpha \to (\beta \to (\alpha \wedge \beta))$

D1 $\alpha \to (\alpha \vee \beta)$

D2 $\beta \to (\alpha \vee \beta)$

D3 $(\alpha \to \gamma) \to ((\beta \to \gamma) \to (\alpha \vee \beta \to \gamma))$

N1 $\neg(\alpha \vee \beta) \leftrightarrow \neg\alpha \wedge \neg\beta$

N2 $\neg(\alpha \wedge \beta) \leftrightarrow \neg\alpha \vee \neg\beta$

N3 $\neg\neg\alpha \leftrightarrow \alpha$

N4 $\neg(\alpha \to \beta) \leftrightarrow \alpha \wedge \neg\beta$

[4] This means that each one of $\wedge$, and $\vee$ is monotonic with respect to both $\leq_t$ and $\leq_k$ [4]

and as the only inference rule: Modus Ponens

$$\frac{\alpha \quad \alpha \to \beta}{\beta}$$

Logic $BL_{\supset}$ is sound and complete with respect to this axiomatization.

$\Gamma \models_X \alpha$ means that α can be proved in logic X from formulas in Γ.

Theorem 2.1 *[4][Soundness and Completeness]*

$$\Gamma \vdash_{BL_{\supset}} \alpha \text{ } if \text{ } and \text{ } only \text{ } if \text{ } \Gamma \models_{BL_{\supset}} \alpha.$$

2.3 First Degree Entailment system

This subsection is a summary of some material from [27] that we need to borrow for the definition of our logics.

First Degree Entailment (FDE) is a system of logic defined by Priest that can be set up in many ways, but one of these is as a 4-valued logic whose values are t (true only), f (false only), b (both), and n (neither). Negation maps t to f, vice versa, n to itself, and b to itself. Conjunction is Glb, and disjunction is Lub. The set of designated values, D, is $\{b, t\}$ The four corners of truth and the FDE logic seem like a correct match. From now on, we will use the four values 0, 1, 2, 3 that correspond to f, n, b, t respectively; this in order to make notation uniform in terms of previously introduced logics.

FDE can be characterised by the following sound and complete rule system, where a double line indicates a two-way rule, and overlining indicates discharging an assumption [5]:

$$\frac{A, B}{A \wedge B} \qquad \frac{A \wedge B}{A(B)} \qquad \frac{A \vee B \qquad \overline{A} \ldots C \qquad \overline{B} \ldots C}{C}$$

$$\frac{\neg(A \wedge B)}{\overline{\overline{\neg A \vee \neg B}}} \qquad \frac{\neg(A \vee B)}{\overline{\overline{\neg A \wedge \neg B}}} \qquad \frac{\neg\neg A}{\overline{\overline{A}}}$$

Now we move to FDEe, a 5-valued logic that incorporates the notion of ineffability. According to Priest, technically, the obvious thought is to add a new value (which in [27] appears as e), 4, to our existing four $\{0, 1, 2, 3\}$, expressing this new status.

Since 4 is the status of claims such that neither they nor their negations should be accepted, it should obviously not be designated. Thus, we still have that same designated values. Priest addresses the following major question: How are the connectives to behave with respect to 4?

Both 4 and 1 are the values of things that are, in some sense, neither true nor false, but they need to behave differently if the two are to represent distinct

[5] The paper [27] downloaded from the Priest's home page has a typo. It says: $\frac{\neg(A \vee B)}{\neg A \vee \neg B}$ instead of: $\frac{\neg(A \vee B)}{\neg A \wedge \neg B}$.

alternatives. The simplest suggestion is to take 4 to be such that whenever any input has the value 4, so does the output: 4-in/4-out.

The logic that results by modifying FDE in this way is obviously a sub-logic of it. It is a proper sub-logic. It is not difficult to check that all the rules of FDE are designation-preserving except the rule for disjunction-introduction, which is not, as an obvious counter-model shows. However, replace this with the rules:

$$\frac{\varphi(A)\ \ C}{A \vee C} \qquad \frac{\varphi(A)\ \ C}{\neg A \vee C} \qquad \frac{\varphi(A)\ \ \psi(B)\ \ C}{(A \wedge B) \vee C}$$

where $\varphi(A)$ and $\psi(B)$ are any sentences containing the atoms that occur in A and B respectively. For example, for rule $\frac{\varphi(A)\ C}{A \vee C}$ if A is the atom a and $\varphi(A)$ is $a \wedge b$ then some possible instances of this rule are: $\frac{a \wedge b\ C}{a \vee C}$ or $\frac{a\ C}{a \vee C}$. Call these the φ *Rules*, and call this system FDE_{φ}. FDE_{φ} is sound and complete with respect to the semantics.

3 Our 5-valued logics

Classical logic is a logic that obeys the law of excluded middle (or the principle of excluded middle) which states that for any proposition, either that proposition is true (t) or its negation is true. In this case we say that the original proposition is false (f). The law is also known as the law (or principle) of the excluded third, in Latin principium tertii exclusi. Another Latin designation for this law is tertium non datur: “no third (possibility) is given”.

A 3-valued logic is any of several many-valued logic systems in which there are three truth values indicating true, false and some indeterminate third value. We could say that this new third valued is neither false nor true (n). The conceptual construction of 3-valued logics and its basic ideas were initially created by Jan Łukasiewicz and C. I. Lewis.

We can push further the paradigm and accept that we can gave both true and false propositions. We can use a four value (b) to represent this fact. A major example in mathematics come from set theory. It is known from Russell’s paradox that the first-order axiomatization of the naive set theory of Cantor and Frege is inconsistent in classical logic. More precisely, some “peculiar” sets lead to triviality if the underlying logic is classical. The most popular of those is the so-called Russell set, $R = \{x | x \notin x\}$, for by the law of excluded middle one immediately gets ‘’, an apparent contradiction. To handle such a contradictory set, a possible idea is to take the Russell just for granted, namely as a set that does belong and does not belong to itself [20].

It is interesting to note that non-well-founded sets (also called hypersets), which are no more indispensable for the foundations of mathematics, have subsequently found useful applications in modeling circular phenomena, particularly in computer science [1].

Going beyond these four values (a kind of final frontier) is to accept ineffable propositions and hence accepting five logical values. This a particularly a natural situation when we are already facing contradictions. As Russel writes in his introduction to the English translation of the Tractatus [33].

Everything, therefore, which is involved in the very idea of the expressiveness of language must remain incapable of being expressed in language, and is, therefore, inexpressible in a perfectly precise sense. This inexpressible contains, according to Mr Wittgenstein, the whole of logic and philosophy. ... It is not this that causes some hesitation in accepting Mr Wittgenstein's position, in spite of the very powerful arguments which he brings to its support. What causes hesitation is the fact that, after all, Mr Wittgenstein manages to say a good deal about what cannot be said, thus suggesting to the sceptical reader that possibly there may be some loophole through a hierarchy of languages, or by some other exit.

We are not claiming in favor or against the point of Wittegestein in his Tractatus. It is a fact that a number of well known books of philosophy, literature and/or religion care about this paradoxical concept [32,12,7,34,11,10,30,16]. It is worth to challenge modern logic to represent the above 5 notions. Priest already did the job. However he left the "implication" connective out of his approach. We present some possible alternatives to include such connective in a logic that he proposed.

We present three logics that are constructed based on the combination of two logics: FDEe (by Priest) and $BL_{\supset}$.

The core of the three logics is based on the following assumptions.

We have 5 values: $\{0, 1, 2, 3, 4\}$. FDEe also uses $\{0, 1, 2, 3, 4\}$. The designated values are $\{2, 3\}$ as FDEe. $\{0, 1, 2, 3\}$ defines a lattice where $0 < 1$, $0 < 2$, $1 < 3$, $2 < 3$.

The connective $\vee$ is the Lub, while $\wedge$ is the Glb. Since 4 is interpreted as ineffable then $X \ op \ 4 = 4 \ op \ X = 4$, where $X \in \{0, 1, 2, 3, 4\}$. With respect to negation ($\neg$), we have $\neg 0 = 3$, $\neg 3 = 0$, $\neg 1 = 1$, $\neg 2 = 2$, $\neg 4 = 4$. Implication is as defined by Avron for the subdomain $\{0, 1, 2, 3\}$, namely: $X \to Y = 3$ when X is not designated, $X \neq 4$, $Y \neq 4$. While $X \to Y = Y$ when X is designated, $Y \neq 4$. The next two expressions are yet undefined $4 \to X$ and $4 \to X$ for X in $\{0, 1, 2, 3, 4\}$. Table 1 presents the truth tables of connectives $\vee$, $\wedge$, $\to$ and $\neg$.

Notice that the sublogic defined in the subdomain $\{0, 1, 2, 3\}$ corresponds exactly to $BL_{\supset}$ logic. Also the sublogic in the domain $\{0, 1, 2, 3, 4\}$ but eliminating the implication connective correspond exactly to FDEe.

3.1 FiveASP1

This logic tries to stay very close to FDEe, we name this logic as FiveASP1. The connective $\to$ is defined according to the truth table given in Table 2.

Theorem 3.1 *FiveASP1 is a paraconsistent, genuine and paracomplete logic.*

Proof. [sketch] Directly using truth tables. For example, to prove that it is paracomplete, it is enough to evaluate the formulas with $val(X) = 1$, for every atom X. $\square$

$\vee$	0	1	2	3	4
0	0	1	2	3	4
1	1	1	2	3	4
2	2	2	2	3	4
3	3	3	3	3	4
4	4	4	4	4	4

$\wedge$	0	1	2	3	4
0	0	0	0	0	4
1	0	1	1	1	4
2	0	1	2	2	4
3	0	1	2	3	4
4	4	4	4	4	4

$\to$	0	1	2	3	4
0	3	3	3	3	?
1	3	3	3	3	?
2	0	1	2	3	?
3	0	1	2	3	?
4	?	?	?	?	?

x	$\neg x$
0	3
1	1
2	2
3	0
4	4

Table 1
Truth tables of connectives $\vee$, $\wedge$, $\to$ and $\neg$.

$\to_{FiveASP1}$	0	1	2	3	4
0	3	3	3	3	4
1	3	3	3	3	4
2	0	1	2	3	4
3	0	1	2	3	4
4	4	4	4	4	4

Table 2
Truth table of connective $\to$ in logic FiveASP1.

FDEe admits no tautologies. FiveASP1 is faithful in this aspect to FDEe and hence we have the following result.

Theorem 3.2 *FiveASP1 admits no tautologies.*

Proof. [sketch] Evaluating each atom in any formula with 4, then the final evaluation is 4 which is not designated. $\square$

Theorem 3.3 *FiveASP satisfies:*

(i) *Modus ponens and Hipothetical sylogism.*

(ii) *All inference rules of FDEe.*

Proof. [sketch] (case 1) They are proven by contradiction using truth tables. (case 2) They are proven by construction, since the three logics behave as logic FDEe

$\to_{FiveASP2}$	0	1	2	3	4
0	3	3	3	3	4
1	3	3	3	3	4
2	0	1	2	3	4
3	0	1	2	3	4
4	4	4	4	4	3

Table 3
Truth table of connective $\to$ in logic FiveASP2.

regarding the connectives $\wedge$, $\vee$ and $\neg$, and that logic satisfies such inference rules.□

3.2 FiveASP2

This logic is somehow the "middle way" between logics FiveASP1 and FiveASP3 (to be introduced soon), we name this logic as FiveASP2. It only changes $4 \to 4 = 4$ in FiveASP1 to $4 \to 4 = 3$ in FiveASP2, in order to allow some basic tautologies. The connective $\to$ is defined according to the truth table given in Table 3. Recall that the notion of ineffable is to some extend paradoxical, we can not talk about something ineffable but actually we do it in order to convey a given major message (at least partially). Here, we have that: if X is ineffable and is true and only true that $X \to Y$, then (our logic claims that) Y is ineffable.

Theorem 3.4 *FiveASP2 is a paraconsistent, genuine and paracomplete logic.*

Proof. [sketch] Directly using truth tables. □

We can observe that FiveASP2 satisfies some well known tautologies, as $X \to X$ and De Morgan laws, among some of them. Hence, we have the following theorem.

Theorem 3.5 *FiveASP2 admits some tautologies.*

Proof. [sketch] FiveASP2 accept the mentioned tautologies in the phrase previous to this theorem and they are proven directly. □

Theorem 3.6 *FiveASP2 satisfies:*

(i) *Modus Ponens and Hipothetical Sylogism.*

(ii) *All inference rules of FDEe.*

Proof. [sketch] (case 1) They are proven by contradiction using truth tables. (case 2) They are proven by construction, since the three logics behave as logic FDEe regarding the connectives $\wedge$, $\vee$ and $\neg$, and that logic satisfies such inference rules.□

3.3 FiveASP3

This logic is kind of pragmatic and the connective "implication" is a kind of metalinguistic connective [13]. We name this logic as FiveASP3. We do not allow to have

$\to_{FiveASP3}$	0	1	2	3	4
0	3	3	3	3	3
1	3	3	3	3	3
2	0	1	2	3	1
3	0	1	2	3	1
4	3	3	3	3	3

Table 4
Truth table of connective $\to$ in logic FiveASP3.

two values X, Y such that $X \to Y = 4$, and hence FiveASP3 is the other extreme case with respect to FiveASP1 logic. Note that this logic gains many well know tautologies. This logic complies with the tautologies **I1-I3**, **C1-C3**, **D3**, **N1-N4** (see section 2.2). We consider this logic potentially useful in Artificial Intelligence applied to art (literature). The connective $\to$ is defined according to the truth table given in Table 4.

Theorem 3.7 *FiveASP3 is a paraconsistent, genuine and paracomplete logic.*

Proof. [sketch] Directly using truth tables. □

We can observe that FiveASP3 satisfies many well known tautologies, as $X \to X$ and De Morgan laws, the standard two implication rules for positive logic among some of them. Hence, we have the following theorem.

Theorem 3.8 *FiveASP3 admits some tautologies.*

Proof. [sketch] FiveASP3 accepts the mentioned tautologies in the phrase previous to this theorem and they are proven directly. □

Theorem 3.9 *FiveASP3 satisfies:*

(i) *Modus Ponens and Hipothetical Sylogism.*

(ii) *All inference rules of FDEe.*

Proof. [sketch] (case 1) They are proven by contradiction using truth tables. (case 2) They are proven by construction, since the three logics behave as logic FDEe regarding the connectives $\wedge$, $\vee$ and $\neg$, and that logic satisfies such inference rules.□

4 A Hilbert system aiming FiveASP3

In this section we investigate Hilberts systems aiming to axiomatize the idea of our FiveASP3 logic.

The system called HFive3 has the following 5 constant symbols: ze, on, tw, tr, i. In addition we have a unary connective symbol for negation ($\neg$) and two binary primitive function symbols ($\wedge$, $\to$). We have two defined binary connectives

($\vee$, $\leftrightarrow$), defined in standard forms, namely:

$$\alpha \vee \beta := \neg(\neg\alpha \wedge \neg\beta) \qquad \alpha \leftrightarrow \beta := (\alpha \to \beta) \wedge (\beta \to \alpha)$$

We have the following axioms. From Logic HBL presented in section 2.2 we borrow axioms even axioms: **I1-I3**, **C1-C3**, **D3**, **N1-N4**. In addition we add the following set of new axioms.

$(\beta \leftrightarrow \neg\beta) \vee (\alpha \to \beta) \vee \neg(\alpha \to \beta)$

$((\alpha \to \alpha) \to \beta) \vee (\alpha \to \alpha)$ $\qquad$ $((\alpha \wedge \beta) \vee \gamma) \to (\beta \vee \gamma)$

$(\alpha \vee (\beta \wedge \gamma)) \to (\beta \vee \alpha)$ $\qquad$ $\neg(\alpha \wedge \beta) \to \neg(\beta \wedge \alpha)$

$\neg ze \leftrightarrow tr$ $\qquad$ $ze \leftrightarrow \neg tr$

$\neg on \leftrightarrow on$ $\qquad$ $\neg tw \leftrightarrow tw$

$ze \vee on \to \alpha$ $\qquad$ $\alpha \to tw \wedge tr$

We also have two meta axioms:

M1) $\alpha \to (\beta^+ \to (\alpha \vee \beta))$ where β^+ is any formula that contains all parameters that occur in formula β.

M2) $\alpha \to \gamma \vee \alpha$ where γ is any formula constructed using only constant symbols in $\{ze, tr, on, tw\}$.

The only rule of inference that we have is Modus Ponens:

$$\alpha, \alpha \to \beta \vdash \beta$$

Clearly HFive3 satisfies the deduction theorem.

Theorem 4.1 *FiveASP3 is sound w.r.t HFive3.*

Theorem 4.2 *Is false that* $\vdash_{HFive3} \alpha \to (\alpha \vee \beta)$. *Is false that* $\vdash_{HFive3} \alpha \to (\beta \vee \alpha)$

We say that a logic L is a suitable extension of HFive3 (hence it includes all axioms and the rule of inference of HFive3) if the formula $i \vee (\alpha \to \alpha)$ cannot be proven in L (recall that i is a constant of logic HFive3).

Claim: The unique 5-valued logic that is sound with respect to any suitable extension L of HFive3 up to isomorphism with exactly two designated values is FiveASP3.

Here we must observe that clasp[6] provides 12 isomorphic logics when permuting the role of the constants, however there is only one "intended model" for which the name of each constant coincides with the desired value for such constant.

[6] https://potassco.org/clasp/

5 Conclusions

We introduce three 5-valued logics by combining FDEe and $BL_{\supset}$. The three of them satisfy the following properties: they are paraconsistent, genuine and paracomplete logics. They satisfy Modus Ponens and Hypothetical Syllogism as well as all inference rules of FDEe. More research needs to be done to understand these logics and to find further mathematical properties of each of them. We also believe that our logics or some extensions of them could be used to represent/understand complex poems where inconsistent, uncertain and/or ineffable beliefs are considered.

It is also interesting to consider extending these logics by defining a possibilistic version of each of our three logics, see [15,23]. In this way we could (perhaps) gain a new dimension of expressibility of some notions.

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