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On Products of Transition Systems

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Abstract

For an arbitrary set endofunctor F we give a sufficient and necessary criterium for the existence of products of F-coalgebras. In the case of transition systems, where $F = \mathbb{P}$ is the covariant powerset functor, we introduce impeding paths whose existence impedes the existence of the product. Moreover we show, that the product $\mathcal{A} \otimes \mathcal{A}$ of a finite transition system $\mathcal{A}$ exists if and only if the product $\mathcal{A} \otimes \mathcal{B}$ for each finite transition system $\mathcal{B}$ exists.

Keywords: Coalgebras, transition systems, products

1 Introduction

Given a set functor F, it is well known that the category Set_F of F-coalgebras has arbitrary colimits. In fact (see [4]) the forgetful functor $U : Set_F \to Set$ creates and reflects colimits. The situation is different for limits. Even though equalizers and inverse images always exist in Set_F [2] they are, in general, not created by the forgetful functor.

General products need not exist at all in Set_F. In particular, the product over the empty index set, that is the terminal coalgebra need not exist, unless certain assumptions are made about the functor F. This is mainly due to Lambeks Lemma [3] which states that the structure map of the terminal coalgebra T must be a bijection $\alpha : T \to F(T)$. Unless F is bounded [4], this requirement often leads to set theoretical problems.

A classical case of an unbounded functor is given by the power set functor $\mathbb{P}$ whose coalgebras are the familiar transition systems. Clearly, the terminal $\mathbb{P}$-coalgebra, i.e. the empty product, does not exist since $|X| < |\mathbb{P}(X)|$ for any set X.

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Still, there may be some products of transition systems existing. In [1] examples of finite transition systems were given that show the whole range of situations that may occur. In particular, Gumm and Schröder exhibit pairs $\mathcal{A}$ and $\mathcal{B}$ of nonempty finite transition systems so that $\mathcal{A} \otimes \mathcal{B}$ does not exist, $\mathcal{A} \otimes \mathcal{B}$ exists and is the empty transition system or $\mathcal{A} \otimes \mathcal{B}$ exists and is the largest bisimulation $\sim_{\mathcal{A},\mathcal{B}}$.

In this paper we examine the reasons why a product of two transition system $\mathcal{A}$ and $\mathcal{B}$ might exist, or not. Analysing the critical example in [1], we study certain "bisimilar paths" in a transition system $\mathcal{A}$ whose existence allows or impedes the existence of products $\mathcal{A} \otimes \mathcal{B}$.

As an interesting corollary we obtain the somewhat surprising fact that for any finite transition system $\mathcal{A}$ we have that $\mathcal{A} \otimes \mathcal{A}$ exists in $Set_{\mathbb{P}}$ if and only if $\mathcal{A} \otimes \mathcal{B}$ exists for each finite transition system $\mathcal{B}$.

2 Categorical products of coalgebras

Definition 2.1 (Coalgebra) *Let $F : Set \to Set$ be a functor. A pair (A, α) is called F-coalgebra, if $A \in Set$ and $\alpha : A \to F(A)$. We call A the base set and α the structure of the coalgebra $\mathcal{A}$.*

For the remainder of this section let $\mathcal{A} = (A, \alpha), \mathcal{B} = (B, \beta)$ be F-coalgebras. A map $\varphi : A \to B$ is called homomorphism, if $F\varphi \circ \alpha = \beta \circ \varphi$.

Definition 2.2 (Bisimilarity) *A subset $R \subseteq A \times B$ is called a bisimulation between $\mathcal{A}$ and $\mathcal{B}$, if there exists a structure $\rho : R \to F(R)$, so that $\pi_A : \mathcal{R} \to \mathcal{A}, \pi_B : \mathcal{R} \to \mathcal{B}$ are homomorphisms. Then we say $\mathcal{R} = (R, \rho)$ is a **bisimulation structure** for $\mathcal{A}$ and $\mathcal{B}$. Elements $a \in A, b \in B$ are called **bisimilar** $(a \sim b)$, if there exists a bisimulation R with $(a, b) \in R$.*

There always exists a largest bisimulation $\sim_{\mathcal{A},\mathcal{B}} = \{(a, b) \mid a \in A, b \in B, a \sim b\}$ between $\mathcal{A}$ and $\mathcal{B}$. Note, that bisimilarity is a reflexive and symmetric relation. Moreover it is transitive if the functor F preserve weak pullbacks (see [4], theorem 5.4).

We spell out the categorical product of coalgebras.

Definition 2.3 (Product) *Let $\mathcal{Q} = (Q, \xi)$ be an F-coalgebra and $\varphi_A : \mathcal{Q} \to \mathcal{A}, \varphi_B : \mathcal{Q} \to \mathcal{B}$ homomorphisms. Then $\mathbf{Q} = (\mathcal{Q}, \varphi_A, \varphi_B)$ is called an **$\mathcal{A}$-$\mathcal{B}$-cone**.*

*The **categorical product** of $\mathcal{A}$ and $\mathcal{B}$ is an $\mathcal{A}$-$\mathcal{B}$-cone $\mathbf{P} = (\mathcal{P}, \pi_A, \pi_B)$, so that for all $\mathcal{A}$-$\mathcal{B}$-cones $(\mathcal{Q}, \varphi_A, \varphi_B)$ there exists exactly one homomorphism $\psi : \mathcal{Q} \to \mathcal{P}$ with $\varphi_A = \pi_A \circ \psi$ and $\varphi_B = \pi_B \circ \psi$.*

$$\begin{tikzcd}
& \mathcal{Q} \arrow[dl, "\varphi_A"'] \arrow[dr, "\varphi_B"] \arrow[dd, dotted, "\exists!\psi"'] & \\
\mathcal{A} & & \mathcal{B} \\
& \mathcal{P} \arrow[ul, "\pi_A"] \arrow[ur, "\pi_B"'] &
\end{tikzcd}$$

The base set of the categorical product of $\mathcal{A}$ and $\mathcal{B}$ need not be the cartesian *product $A \times B$, so we write $\mathbf{P} = \mathcal{A} \otimes \mathcal{B}$ for the (categorical) product of $\mathcal{A}$ and $\mathcal{B}$.*

Every bisimulation structure $\mathcal{R}$ together with the projections π_A, π_B yields an $\mathcal{A}$-$\mathcal{B}$-cone.

Whenever we speak about $\mathcal{A}$-$\mathcal{B}$-cones $\mathbf{Q}, \mathbf{Q}'$, then let $\mathbf{Q} = (\mathcal{Q}, \varphi_A, \varphi_B), \mathbf{Q}' = (\mathcal{Q}', \varphi'_A, \varphi'_B)$ and $\mathcal{Q} = (Q, \xi), \mathcal{Q}' = (Q', \xi')$ be the associated coalgebras. If the product $\mathbf{P} = \mathcal{A} \otimes \mathcal{B}$ exists, we denote its corresponding coalgebra by $\mathcal{P} = (P, \eta)$ and the $\mathcal{A}$-$\mathcal{B}$-cone by $\mathbf{P} = (\mathcal{P}, \pi_A, \pi_B)$.

Definition 2.4 *Let $\mathbf{Q}, \mathbf{Q}'$ be $\mathcal{A}$-$\mathcal{B}$-cones and $q \in Q, q' \in Q'$. We write $q \cong_{\mathcal{A},\mathcal{B}} q'$, if there exists an F-coalgebra $\mathcal{R} = (R, \rho)$, homomorphisms $\chi : \mathcal{R} \to \mathcal{Q}, \chi' : \mathcal{R} \to \mathcal{Q}'$ and $r \in R$, so that:*

- *$\chi(r) = q$ and $\chi'(r) = q'$, i.e. $q_1 \sim q_2$*
- *$\varphi_A \circ \chi = \varphi'_A \circ \chi'$ and $\varphi_B \circ \chi = \varphi'_B \circ \chi'$, i.e. the following diagram commutes:*

$$\begin{tikzcd}
& \mathcal{Q} \arrow[r, "\varphi_A"] \arrow[ddr, "\varphi_B"] & \mathcal{A} \\
\mathcal{R} \arrow[ur, "\chi"] \arrow[dr, "\chi'"'] & & \\
& \mathcal{Q}' \arrow[uur, "\varphi'_A"] \arrow[r, "\varphi'_B"'] & \mathcal{B}
\end{tikzcd}$$

*We say q, q' are $\mathcal{A}$-$\mathcal{B}$-**perspective** $q \simeq_{\mathcal{A},\mathcal{B}} q'$, if there exist $n \in \mathbb{N}$, $\mathcal{A}$-$\mathcal{B}$-cones $\mathbf{Q}_1, \ldots, \mathbf{Q}_n$ and $q_i \in Q_i (i = 1, \ldots, n)$, so that $q \cong q_1 \cong \ldots \cong q_n \cong q'$, that is, $\simeq_{\mathcal{A},\mathcal{B}} = (\cong_{\mathcal{A},\mathcal{B}})^*$.*

The relation $\cong_{\mathcal{A},\mathcal{B}}$ is reflexive and symmetric, therefore the transitive closure $\simeq_{\mathcal{A},\mathcal{B}}$ of $\cong_{\mathcal{A},\mathcal{B}}$ is an equivalence relation on the class of all elements of $\mathcal{A}$-$\mathcal{B}$-cones. We introduce the equivalence classes with respect to $\simeq_{\mathcal{A},\mathcal{B}}$. Let $\mathbf{Q}$ be an $\mathcal{A}$-$\mathcal{B}$-cone and $q \in Q$. We define $\mathcal{C}_{\mathbf{Q},q} = \{(\mathbf{Q}', q') \mid \mathbf{Q}' \ \mathcal{A}\text{-}\mathcal{B}\text{-cone}, q' \in Q', q' \simeq_{\mathcal{A},\mathcal{B}} q\}$ the equivalence class of $q \in Q$.

Lemma 2.5 *Assume the product $\mathcal{A} \otimes \mathcal{B}$ to exist. Let $\mathbf{Q}, \mathbf{Q}'$ be $\mathcal{A}$-$\mathcal{B}$-cones, $q \in Q, q' \in Q'$ and $\psi : \mathcal{Q} \to \mathcal{A} \otimes \mathcal{B}, \psi' : \mathcal{Q}' \to \mathcal{A} \otimes \mathcal{B}$ be the unique homomorphisms. Then $q \cong_{\mathcal{A},\mathcal{B}} q' \implies \psi(q) = \psi'(q')$.*

Proof The projections π_A, π_B are jointly mono, otherwise there would be an $\mathcal{A}$-$\mathcal{B}$-cone $\mathbf{Q}$ and homomorphisms $\psi, \psi' : \mathcal{Q} \to \mathcal{P}$ with $\varphi_A = \pi_A \circ \psi$ and $\varphi_B = \pi_B \circ \psi$ but $\psi \neq \psi'$ in contradiction to the uniqueness of the homomorphism $\mathcal{Q} \to \mathcal{P}$ in the definition of the product.

Let $q \cong_{\mathcal{A},\mathcal{B}} q'$. There exists a coalgebra $\mathcal{R} = (R, \rho)$, $r \in R$ and homomorphisms $\chi : \mathcal{R} \to \mathcal{Q}, \chi' : \mathcal{R} \to \mathcal{Q}'$ with $\chi(r) = q$ and $\chi'(r) = q'$. Then $\mathcal{R}$ with the projections $\varphi_A \circ \chi = \varphi'_A \circ \chi'$ and $\varphi \circ \chi = \varphi'_B \circ \chi'$ is an $\mathcal{A}$-$\mathcal{B}$-cone. We compute

$$\begin{aligned}
\pi_A \circ \psi \circ \chi &= \varphi_A \circ \chi \\
&= \varphi'_A \circ \chi' \\
&= \pi_A \circ \psi' \circ \chi' \\
\pi_B \circ \psi \circ \chi &\overset{\text{analogously}}{=} \pi_B \circ \psi' \circ \chi'.
\end{aligned}$$

Hence $\psi\circ\chi = \psi'\circ\chi'$ since π_A, π_B are jointly mono. The following diagram commutes.

$$\begin{tikzcd}
 & & & \mathcal{A} \\
 & \mathcal{Q} \arrow[urr, "\varphi_A"] \arrow[drr, "\psi"] \arrow[dddrr, "\varphi_B"] & & \\
\mathcal{R} \arrow[ur, "\chi"] \arrow[dr, "\chi'"'] \arrow[rrr, dotted, "\psi\circ\chi=\psi'\circ\chi'"] & & & \mathcal{A}\otimes\mathcal{B} \arrow[uu, "\pi_A"'] \arrow[dd, "\pi_B"] \\
 & \mathcal{Q}' \arrow[uuurr, "\varphi'_A"] \arrow[urr, "\psi'"'] \arrow[drr, "\varphi'_B"'] & & \\
 & & & \mathcal{B}
\end{tikzcd}$$

Therefore $\psi(q) = (\psi \circ \chi)(r) = (\psi' \circ \chi')(r) = \psi'(q)$. □

Lemma 2.6 *Assume the product* $\mathbf{P} = \mathcal{A} \otimes \mathcal{B}$ *exists and let* $p, p' \in P$. *Then* $p \simeq_{\mathcal{A},\mathcal{B}} p' \Rightarrow p = p'$.

Proof Assume $p \simeq_{\mathcal{A},\mathcal{B}} p'$. There exists $\mathcal{A}$-$\mathcal{B}$-cones $\mathbf{Q}_i$ and $q_i \in Q_i (i = 1, \dots, n)$, so that $p \cong_{\mathcal{A},\mathcal{B}} q_1 \cong_{\mathcal{A},\mathcal{B}} \dots \cong_{\mathcal{A},\mathcal{B}} q_n \cong_{\mathcal{A},\mathcal{B}} p'$. Let $\psi_i : \mathcal{Q}_i \to \mathcal{A} \otimes \mathcal{B}$ be the unique homomorphisms. Then by lemma 2.5 $p = id_P(p) = \psi_1(q_1) = \dots = \psi_n(q_n) = id_P(p') = p'$. □

Lemma 2.7 *Let* $\mathbf{Q}, \mathbf{Q}'$ *be* $\mathcal{A}$-$\mathcal{B}$*-cones,* $\vartheta : \mathcal{Q} \to \mathcal{Q}'$ *a homomorphism with* $\varphi_A = \varphi'_A \circ \vartheta, \varphi_B = \varphi'_B \circ \vartheta$. *Then* $q \cong_{\mathcal{A},\mathcal{B}} \vartheta(q)$ *for all* $q \in Q$.

Proof This follows immediately from the diagram.

$$\begin{tikzcd}
 & \mathcal{Q} \arrow[r, "\varphi_A"] \arrow[ddr, "\varphi_B"] \arrow[dd, "\vartheta"] & \mathcal{A} \\
\mathcal{Q} \arrow[ur, "id_Q"] \arrow[dr, "\vartheta"'] & & \\
 & \mathcal{Q}' \arrow[uur, "\varphi'_A"] \arrow[r, "\varphi'_B"'] & \mathcal{B}
\end{tikzcd}$$

□

Particularly with regard to the product we get for every $\mathcal{A}$-$\mathcal{B}$-cone $\mathbf{Q}$ with unique homomorphism $\psi : Q \to A \otimes B$ that $q \cong_{A,B} \psi(q)$. We can now prove the following theorem.

Theorem 2.8 *The product $\mathcal{A} \otimes \mathcal{B}$ exists iff the class*

$$M = \{\mathcal{C}_{\mathbf{Q},q} \mid \mathbf{Q}\ \mathcal{A}\text{-}\mathcal{B}\text{-}cone, q \in Q\}$$

of equivalence classes of $\simeq_{\mathcal{A},\mathcal{B}}$ is a set.

Proof First assume the product $\mathbf{P} = \mathcal{A} \otimes \mathcal{B}$ exists. We prove that M is a set by showing $|M| \leq |P|$. Assume there exists an equivalence class $\mathcal{C}_{\mathbf{Q},q}$ with $(\mathbf{P}, p) \notin \mathcal{C}_{Q,q}$ for all $p \in P$. Let $\psi : Q \to P$ be the unique homomorphism with $\varphi_A = \pi_A \circ \psi, \varphi_B = \pi_B \circ \psi$. From lemma 2.7 follows $p = \psi(q) \cong_{A,B} q$ in contradiction to $p \notin \mathcal{C}_{\mathbf{Q},q}$. Hence $M = \{\mathcal{C}_{\mathbf{P},p} \mid p \in P\}$ and therefore $|M| \leq |P|$.

We assume now that M is a set. We shall equip M with a coalgebraic structure so that it becomes the product. For any $\mathcal{A}$-$\mathcal{B}$-cone $\mathbf{Q}$ we define a map $\eta_{\mathbf{Q}} : Q \to M$ by $\eta_{\mathbf{Q}}(q) = \mathcal{C}_{\mathbf{Q},q}$. Then we define a structure map $\mu : M \to FM$ by $\mu(\mathcal{C}_{\mathbf{Q},q}) = (F\eta \circ \xi)(q)$ and denote $\mathcal{M} = (M, \mu)$. Note, that then $\eta_{\mathbf{Q}} : \mathcal{Q} \to \mathcal{M}$ is a homomorphism. We have to show that μ is well-defined. Let $\mathbf{Q}, \mathbf{Q}'$ be $\mathcal{A}$-$\mathcal{B}$-cones, $q \in Q, q' \in Q'$ and $\mathcal{C}_{\mathbf{Q},q} = \mathcal{C}_{\mathbf{Q}',q'}$. We may assume $q \cong_{\mathcal{A},\mathcal{B}} q'$. Then there exists an F-coalgebra $\mathcal{R} = (R, \rho)$, homomorphisms $\chi : R \to Q, \chi' : R \to Q'$ and $r \in R$ with $\varphi_A \circ \chi = \varphi'_A \circ \chi', \varphi_B \circ \chi = \varphi'_B \circ \chi'$ and $q = \chi(r), q' = \chi'(r)$. Let $\mu(\mathcal{C}_{\mathbf{Q},q}) = (F\eta_{\mathbf{Q}} \circ \xi)(q)$. We show $\mu(\mathcal{C}_{\mathbf{Q},q}) = (F\eta_{\mathbf{Q}'} \circ \xi')(q')$:

$$\begin{aligned}
(F\eta_{\mathbf{Q}'} \circ \xi')(q') &= (F\eta_{\mathbf{Q}'} \circ \xi' \circ \chi')(r)\\
&= (F\eta_{\mathbf{Q}'} \circ F\chi' \circ \rho)(r)\\
&= (F\eta_{\mathbf{Q}} \circ F\chi \circ \rho)(r)\\
&= (F\eta_{\mathbf{Q}} \circ \xi \circ \chi)(r)\\
&= (\mu \circ \eta_{\mathbf{Q}} \circ \chi)(r)\\
&= (\mu \circ \eta_{\mathbf{Q}})(q)\\
&= \mu(\mathcal{C}_{\mathbf{Q},q}).
\end{aligned}$$

The projections $\pi_A : \mathcal{M} \to \mathcal{A}, \pi_B : \mathcal{M} \to \mathcal{B}$ are defined by $\pi_A(\mathcal{C}_{\mathbf{Q},q}) = \varphi_A(q)$ and $\pi_B(\mathcal{C}_{\mathbf{Q},q}) = \varphi_B(q)$. We compute $\varphi_A(q) = (\varphi_A \circ \chi)(r) = (\varphi'_A \circ \chi')(r) = \varphi'_A(q)$, hence the projections are well-defined too. We show, that π_A is a homomorphisms (π_B analogously):

$$\begin{aligned}
\alpha \circ \pi_A &= \alpha \circ \varphi_A \circ \eta_{\mathbf{Q}}\\
&= F\varphi_A \circ \xi \circ \eta_{\mathbf{Q}}\\
&= F\varphi_A \circ F\eta_{\mathbf{Q}} \circ \mu\\
&= F(\varphi_A \circ \eta_{\mathbf{Q}}) \circ \mu\\
&= F\pi_A \circ \mu.
\end{aligned}$$

The uniqueness of the homomorphism $\eta : \mathcal{Q} \to \mathcal{M}$ follows from lemma 2.7. Therefore $A \otimes B = (\mathcal{M}, \pi_A, \pi_B)$. □

3 Transition systems and trees

Theorem 2.8 gives an abstract characterisation for the existence of products. If we like to apply this characterisation to coalgebras $\mathcal{A}, \mathcal{B}$, we have to observe whether two elements $q \in \mathbf{Q}, q' \in \mathbf{Q}'$ of $\mathcal{A}$-$\mathcal{B}$-cones are $\mathcal{A}$-$\mathcal{B}$-perspective or not. In this section we will show, that in the case of transition systems we can represent the equivalence classes of $\simeq_{\mathcal{A},\mathcal{B}}$ by roots of trees. This opportunity allows us to give another, more technical characterisation for the existence of products in section 4.

Definition 3.1 *Let $\mathbb{P}$ be the covariant powerset functor. A $\mathbb{P}$-coalgebra (A, α) is called* ***transition system****, where A is interpreted as a set of states and $\alpha : A \to \mathbb{P}(A)$ is the transition function. We write $a \xrightarrow{\alpha} a'$ instead of $a' \in \alpha(a)$. If α is clear from the context, we write also $a \to a'$.*

A transition system (A, α) is called a ***tree****, if there exist $\omega_A \in A$ (root of A) with:*

- $\forall a \in A : \omega_A \notin \alpha(a)$,
- $\forall a \in A : a \neq \omega_A \Rightarrow \exists! v \in A : a \in \alpha(v)$,
- $\forall a \in A. \exists n \in \mathbb{N}. \exists a_0, \ldots, a_n \in A : \omega_A = a_0 \to a_1 \to \ldots \to a_n = a$.

In this section let $(A, \alpha), (B, \beta)$ be transition systems. We will show, that there is a tree in any equivalence class $\mathcal{C}_{Q,q}$. Therefore we can represent the equivalence classes of $\simeq_{A,B}$ by trees.

Lemma 3.2 *Let $a \in A$. Then $\langle a \rangle = (\langle a \rangle, \eta)$ with*

$$\begin{aligned}\langle a \rangle &= \left\{ a_0 a_1 \ldots a_n \in A^+ \mid a = a_0, \forall i : a_i \to a_{i+1} \right\}, \\ \eta(a_0 \ldots a_n) &= \{ a_0 \ldots a_n a_{n+1} \mid a_n \to a_{n+1} \}\end{aligned}$$

is a tree with root $\omega_{\langle a \rangle} = a$ and there exists a homomorphism $\vartheta : \langle a \rangle \to \mathcal{A}$ with $\vartheta(\omega_{\langle a \rangle}) = a$.

Proof It is easy to check, that $\langle a \rangle$ is a tree. We define $\vartheta(a_0 \ldots a_n) = a_n$ and show, that it is a homomorphism:

$$\begin{aligned}(\alpha \circ \vartheta)(a_0 \ldots a_n) &= \alpha(a_n) \\ &= \{ a_{n+1} \mid a_n \to a_{n+1} \} \\ &= \mathbb{P}\vartheta \left(\{ a_0 \ldots a_n a_{n+1} \mid a_n \to a_{n+1} \} \right) \\ &= (\mathbb{P}\vartheta \circ \eta)(a_0 \ldots a_n).\end{aligned}$$

□

If (A, α, λ) is a labeled transition system with label $\lambda : A \to \Lambda$ and $a \in A$, we can define $\kappa = \lambda \circ \vartheta : \langle a \rangle \to \Lambda$. Then ϑ is a homomorphism of $\mathbb{P}(\_) \times \Lambda$-coalgebras.

Corollary 3.3 *Let $\mathbf{Q}$ be an $\mathcal{A}$-$\mathcal{B}$-cone and $q \in Q$. Then $(\langle q \rangle, \varphi_A \circ \vartheta, \varphi_B \circ \vartheta)$ is an $\mathcal{A}$-$\mathcal{B}$-cone and $q \cong_{\mathcal{A},\mathcal{B}} \omega_{\langle q \rangle}$.*

Proof Lemma 3.2 shows the existence of $\langle q \rangle$. Moreover $\vartheta(\omega_{\langle q \rangle}) = q$ and with lemma 2.7 $q \cong_{A,B} \omega_{\langle q \rangle}$. □

Corollary 3.4 *Let $a \in A, b \in B$ and $a \sim b$. Then there exist a tree (T, τ) and homomorphisms $\vartheta_A : T \to A, \vartheta_B : T \to B$ with $\vartheta_A(\omega_T) = a$ and $\vartheta_B(\omega_T) = b$.*

Proof Because a, b are bisimilar, there exist a transition system (R, ρ), homomorphisms $\pi_A : R \to A, \pi_B : R \to B$ and $r \in R$ with $a = \pi_A(r)$ and $b = \pi_B(r)$. Then

$\langle r\rangle$ with the homomorphisms $\vartheta_A = \pi_A \circ \vartheta_R$ and $\vartheta_B = \pi_B \circ \vartheta_R$ is the wanted tree.

$$\begin{tikzcd}
 & \langle r\rangle \arrow[dl, "\vartheta_A"'] \arrow[d, "\vartheta_R"'] \arrow[dr, "\vartheta_B"] & \\
A & R \arrow[l, "\pi_A"] \arrow[r, "\pi_B"'] & B
\end{tikzcd}$$

□

Lemma 3.5 *Let (T, τ) be a tree and $\vartheta : T \to A$ a homomorphism with $\vartheta(\omega_T) = a$. Then there exist a homomorphism $\psi : T \to \langle a\rangle$ with $\psi(\omega_T) = \omega_{\langle a\rangle} = a$.*

Proof We define ψ by induction over the construction of T:

$$\psi(t') = \begin{cases} \omega_{\langle a\rangle} & \text{if } t' = \omega_T \\ \psi(t) \cdot \vartheta(t') & \text{if } t' \neq \omega_T \text{ and } t' \in \tau(t) \end{cases}$$

Note, that for $t \neq \omega_T$ there is a unique element t' with $t \in \tau(t')$, because T is a tree.

We show, that ψ is a homomorphism. Let $s, t \in T$ with $t \in \tau(s)$. (For $t = \omega_T$ let $\psi(s) = \epsilon$ be the empty word.)

$$\begin{aligned}
(\eta \circ \psi)(t) &= \eta\left(\psi(s) \cdot \vartheta(t)\right) \\
&= \left\{\psi(s) \cdot \vartheta(t) \cdot a' \mid a' \in \alpha(\vartheta(t))\right\} \\
&= \left\{\psi(t) \cdot a' \mid a' \in (\mathbb{P}\vartheta \circ \tau)(t)\right\} \\
&= \left\{\psi(t) \cdot \vartheta(t') \mid t' \in \tau(t)\right\} \\
&= \left\{\psi(t') \mid t' \in \tau(t)\right\} \\
&= (\mathbb{P}\psi \circ \tau)(t).
\end{aligned}$$

□

4 Impeding paths

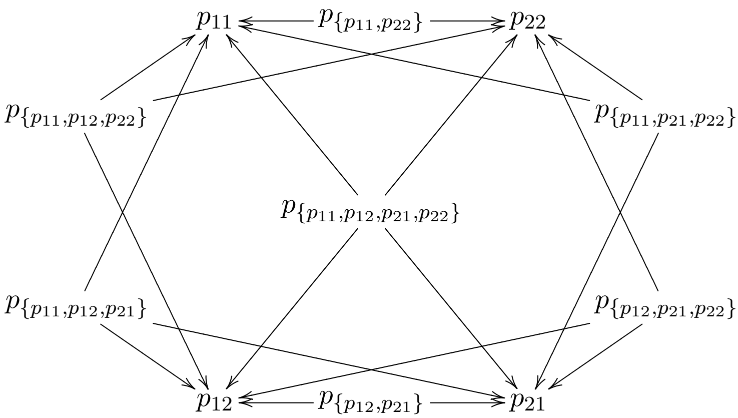
In this section we consider $\mathbb{P}$-coalgebras $\mathcal{A} = (A, \alpha), \mathcal{B} = (B, \beta)$. We introduce impeding paths and prove, that the existence of such a path impedes the existence of the product $\mathcal{A} \otimes \mathcal{B}$. First we look at an example of a transition system whose product with itself exists but is larger than one might expect.

Example 4.1 Consider the following transition system $\mathcal{A} = (A, \alpha)$:

$$\begin{tikzcd}
 & a_0 \arrow[dl] \arrow[dr] & \\
b_1 & & b_2
\end{tikzcd}$$

The largest bisimulation $\sim_{\mathcal{A}\mathcal{A}}$ is the least equivalence relation with $b_1 \sim_{\mathcal{A}\mathcal{A}} b_2$. We construct the product $\mathcal{A} \otimes \mathcal{A} = \mathbf{P} = (\mathcal{P}, \pi_1, \pi_2)$. For all $i, j \in \{1, 2\}$ there is $p_{ij} \in P$ with $\pi_1(p_{ij}) = b_i$, $\pi_2(p_{ij}) = b_j$ and $\eta(p_{ij}) = \emptyset$. For every subset $S \subseteq \{p_{11}, p_{12}, p_{21}, p_{22}\}$ with $\pi_1(S) = \pi_2(S) = \{b_1, b_2\}$ we obtain a state $p_S \in P$ with

$\eta(p_S) = S$ and $\pi_i(p_S) = a_0$. Note, that for every such S the set $Q = \{p_S\} \cup S$ with the structure $\eta|_Q$ and the projections $\pi_1|_Q, \pi_2|_Q$ yields an $\mathcal{A}$-$\mathcal{B}$-cone. We leave it to the reader to show that for $S \neq S'$ the states p_S and $p_{S'}$ are not $\mathcal{A}$-$\mathcal{A}$-perspective. Therefore we get seven states $p \in P$ with $\pi_1(p) = \pi_2(p) = a_0$. The following figure shows the product $\mathcal{P}$:

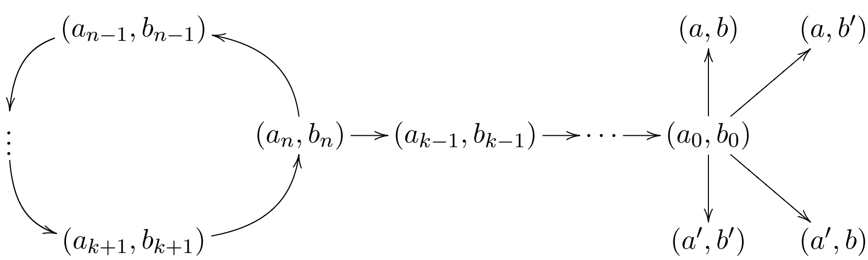


If additionally there is a path $a_n \to \ldots \to a_1 \to a_0$, then the product increases faster than exponentially with respect to n. In fact there would be $2^7 - 1$ states p' with $\pi_1(p') = \pi_2(p') = a_1$ and $2^{2^7-1} - 1$ states p'' with $\pi_1(p'') = \pi_2(p'') = a_2$ in the product. This already suggests that it would be very difficult to construct the product $\mathcal{A} \otimes \mathcal{A}$ if there is a loop in the added path. We will see that indeed in this case the product $\mathcal{A} \otimes \mathcal{A}$ does not exist. The above consideration motivates the following definition:

Definition 4.2 (Impeding path)

- *A bisimilar path in* $A \times B$ *is a sequence* $(a_n, b_n) \ldots (a_0, b_0)$ *with* $a_{i+1} \xrightarrow{\alpha} a_i, b_{i+1} \xrightarrow{\beta} b_i$ *and* $a_i \sim b_i$ *bisimilar for all* i.
- *A bisimilar path is called impeding, if*
  - · *there exist* $0 \leq k < n$ *with* $(a_k, b_k) = (a_n, b_n)$,
  - · *there exist* $a \neq a' \in \alpha(a_0), b \neq b' \in \beta(b_0)$, *so that* $\{a, a'\} \times \{b, b'\} \subseteq \sim_{\mathcal{A},\mathcal{B}}$.
- *An impeding path is called reduced, if* $0 \leq i < j < n \Rightarrow (a_i, b_i) \neq (a_j, b_j)$.

Let $\sigma = (a_n, b_n) \ldots (a_0, b_0)$ be an impeding path. Concatenating any of $(a, b), (a, b'), (a', b), (a', b')$ to σ yields bisimilar paths $(a_n, b_n) \ldots (a_0, b_0)(a, b), \ldots$:



Lemma 4.3 *If there exists an impeding path in $A \times B$, then there exists a reduced impeding path in $A \times B$ too.*

Proof Let $(a_n, b_n) \dots (a_0, b_0)$ be an impeding path in $A \times B$ and $j = \min\{j' \mid \exists i < j' : (a_i, b_i) = (a_{j'}, b_{j'})\}$. Then $(a_j, b_j) \dots (a_0, b_0)$ is a reduced impeding path. □

Theorem 4.4 *If there is an impeding path in $A \times B$ then the product $A \otimes B$ does not exist.*

Proof We prove the theorem by contradiction. Assuming that the product $\mathcal{A} \otimes \mathcal{B}$ exists we construct an $\mathcal{A}$-$\mathcal{B}$-cone $\mathbf{Q}$ which contains pairwise not $\mathcal{A}$-$\mathcal{B}$-perspective states $q_i (i \in \varkappa)$ for an ordinal number $\varkappa$ with $|\varkappa| > |\mathcal{A} \otimes \mathcal{B}|$. This would be a contradiction to theorem 2.8. The proof is structured in the following way:

(i) Construction of $\mathbf{Q}$

(ii) Showing that $\mathbf{Q}$ is an $\mathcal{A}$-$\mathcal{B}$-cone

(iii) Proof that the states q_i are pairwise not $\mathcal{A}$-$\mathcal{B}$-perspective

Construction of Q

Assume that the product $(A \otimes B, \eta, \pi_A, \pi_B)$ exists and that there is an impeding path in $A \times B$. By lemma 4.3 there exists a reduced impeding path $(a_n, b_n) \dots (a_0, b_0)$ in $A \times B$ and

- $k \neq n$ with $(a_k, b_k) = (a_n, b_n)$,
- $a \neq a' \in \alpha(a_0), b \neq b' \in \beta(b_0)$ and $a \sim a' \sim b \sim b'$.

For $\hat{a} \in A, \hat{b} \in B$ with $\hat{a} \sim \hat{b}$ we define

$$S_{\hat{a}\hat{b}} = \left\{ p \in A \otimes B \mid (\pi_A, \pi_B)(p) \in \alpha(\hat{a}) \times \beta(\hat{b}) \right\}$$
$$T_{\hat{a}\hat{b}} = \left\{ p \in A \otimes B \mid (\pi_A, \pi_B)(p) = (\hat{a}, \hat{b}) \right\}$$

Let $\varkappa$ be an ordinal number with $|\varkappa| > |A \otimes B|$. We construct an $\mathcal{A}$-$\mathcal{B}$-cone $\mathbf{Q}$ with $Q = \{q_i \mid i \in \varkappa\} \cup \{q'_j \mid 0 \leq j < k\} \cup A \otimes B$. The projections $\varphi_A : Q \to A, \varphi_B : Q \to B$ are defined by

$$(\varphi_A, \varphi_B)(q) = \begin{cases} (\pi_A, \pi_B)(q) & \text{if } q \in A \otimes B \\ (a_i, b_i) & \text{if } i < k \wedge q \in \{q_i, q'_i\} \\ (a_i, b_i) & \text{if } k \leq i < n \text{ and for some } m \in \mathbb{N}. : q = q_{i+m(n-k)} \\ (a_i, b_i) & \text{if } k \leq i < n \text{ and there exist a limit ordinal number } \varkappa', \\ & \text{so that } q = q_j \text{ for some } j = \varkappa' + (i-k) + m(n-k). \end{cases}$$

Before introducing the transition function, we define some helpful sets for all $i \in \varkappa$

$$P_i = \begin{cases} S_{a_i b_i} \setminus T_{a_{i-1} b_{i-1}} & \text{if } 0 < i < k \text{ or } k < i < n \\ S_{a_k b_k} \setminus \left(T_{a_{k-1} b_{k-1}} \cup T_{a_{n-1} b_{n-1}} \right) & \text{if } i = k \\ P_j & \text{if } (\varphi_A, \varphi_B)(q_j) = (\varphi_A, \varphi_B)(q_i) \end{cases}$$

$$R_i = \{q_j \mid k \le j < i \text{ and } (\varphi_A, \varphi_B)(q_j) = (\varphi_A, \varphi_B)(q_{i+n-k-1})\}\,.$$

Note that P_i is uniquely defined because the impeding path is reduced. We can now define the transition function ξ:

$$\xi(q) = \begin{cases} \eta(q) & q \in A \otimes B \\ S_{a_0 b_0} \setminus (T_{ab'} \cup T_{a'b}) & q = q_0 \\ S_{a_0 b_0} \setminus (T_{ab} \cup T_{a'b'}) & q = q'_0 \\ \{q_{i-1}\} \cup P_i & 0 < i < k \wedge q = q_i \\ \{q'_{i-1}\} \cup P_i & 0 < i < k \wedge q = q'_i \\ \{q_{k-1}, q_{n-1}\} \cup P_k & q = q_k \\ \{q'_{k-1}\} \cup R_i \cup P_k & i = n + m(n-k) \wedge q = q_i \\ \{q'_{k-1}\} \cup R_i \cup P_k & i = \varkappa' + m(n-k) \wedge q = q_i \\ R_i \cup P_i & \text{else} \end{cases}$$

For the special case $k = 0$ we define $\xi(q_0) = \{q_{n-1}\} \cup S_{a_0 b_0} \setminus (T_{ab'} \cup T_{a'b})$ and $\xi(q_{m\cdot n}) = R_{m\cdot n} \cup S_{a_0 b_0} \setminus (T_{ab} \cup T_{a'b'})$. The proof for this special case is very similar to the proof for $k > 0$, so we consider only the case $k > 0$. Figure 1 shows the $\mathcal{A}$-$\mathcal{B}$-cone $\mathbf{Q}$ without the states from $\mathcal{A} \otimes \mathcal{B}$.

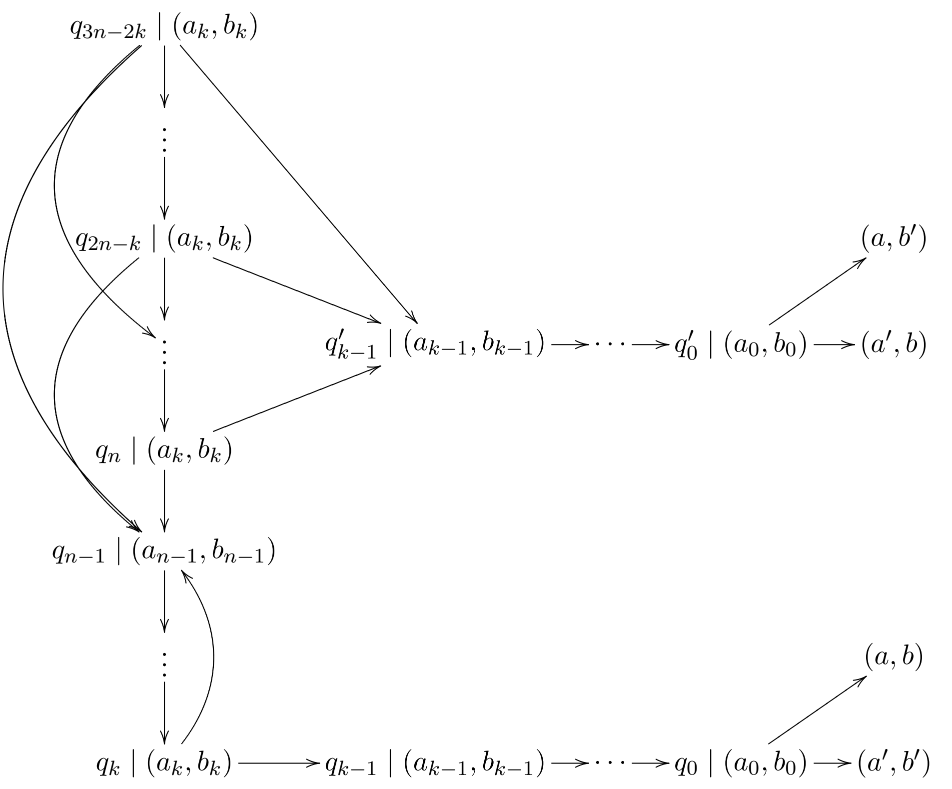


Figure 1. Extract of the $\mathcal{A}$-$\mathcal{B}$-cone $\mathbf{Q}$ without the transitions to $\mathcal{A} \otimes \mathcal{B}$
(in addition to the states q_i we give the image $(\varphi_A, \varphi_B)(q_i)$)

Q is an $\mathcal{A}$-$\mathcal{B}$-cone

We have to show, that φ_A, φ_B are homomorphisms. First we consider some properties:

(i) The largest bisimulation $\sim_{\mathcal{A},\mathcal{B}}$ between $\mathcal{A}$ and $\mathcal{B}$ with a bisimulation structure is an $\mathcal{A}$-$\mathcal{B}$-cone. Hence for $\hat{a} \sim \hat{b}$ there exists a $p \in A \otimes B$ with $(\pi_A, \pi_B)(p) = (\hat{a}, \hat{b})$.

(ii) Therefore, for all $q \in Q$ there exists $p \in A \otimes B$ with $(\hat{a}, \hat{b}) = (\varphi_A, \varphi_B)(q) = (\pi_A, \pi_B)(p)$. Moreover $(\mathbb{P}\pi_A \circ \eta)(p) = (\alpha \circ \pi_A)(p) = \alpha(\hat{a})$, hence $\eta(p) \subseteq S_{\hat{a}\hat{b}} = S_{(\varphi_A,\varphi_B)(q)}$ and

$$\alpha(\hat{a}) = (\mathbb{P}\pi_A \circ \eta)(p) \subseteq \mathbb{P}\pi_A\left(S_{(\varphi_A,\varphi_B)(q)}\right) \subseteq \alpha(\hat{a}) = (\alpha \circ \varphi_A)(q).$$

(iii) From the definition of $T_{\hat{a}\hat{b}}$ it follows immediately that $\mathbb{P}\pi_A(T_{\hat{a}\hat{b}}) = \{\hat{a}\}$, therefore $\mathbb{P}\pi_A\left(T_{ab} \cup T_{a'b'}\right) = \mathbb{P}\pi_A\left(T_{ab'} \cup T_{a'b}\right) = \{a, a'\}$ and $\mathbb{P}\pi_A\left(S_{a_0b_0}\right) = \mathbb{P}\pi_A\left(S_{a_0b_0} \setminus (T_{ab} \cup T_{a'b'})\right)$.

(iv) For q_i with $i > 0$ and $(\varphi_A, \varphi_B)(q_i) = (a_j, b_j) \neq (a_k, b_k)$ we compute

$$\{a_{j-1}\} \cup \mathbb{P}\varphi_A(P_i) = \{a_{j-1}\} \cup \mathbb{P}\varphi_A\left(S_{a_jb_j} \setminus T_{a_{j-1}b_{j-1}}\right) = \alpha(a_j) = (\alpha \circ \varphi_A)(q_i).$$

For q_i with $(\varphi_A, \varphi_B)(q_i) = (a_k, b_k)$ we get

$$\begin{aligned}\{a_{k-1}, a_{n-1}\} \cup \mathbb{P}\varphi_A(P_i) &= \{a_{k-1}, a_{n-1}\} \cup \mathbb{P}\varphi_A\left(S_{a_kb_k} \setminus \left(T_{a_{k-1}b_{k-1}} \cup T_{a_{n-1}b_{n-1}}\right)\right) \\ &= \alpha(a_k).\end{aligned}$$

(v) Let $i > k$ and $\varphi_A(q_i) = a_j$ for some $j \in \{k+1, \ldots, n\}$. Then $\varphi_A(q_{i+n-k-1}) = a_{j-1}$. Furthermore, $j - 1 \geq k$ and therefore $\mathbb{P}\varphi_A(R_i) = \{a_{j-1}\}$.

We divide the proof that φ_A is a homomorphism into cases as in the definition of ξ:

- $q \in A \otimes B$: $(\mathbb{P}\varphi_A \circ \xi)(q) = (\mathbb{P}\pi_A \circ \eta)(q) = (\alpha \circ \pi_A)(q)$,
- $q = q_0$:

$$\begin{aligned}(\mathbb{P}\varphi_A \circ \xi)(q_0) &= \mathbb{P}\varphi_A(S_{a_0b_0} \setminus (T_{ab} \cup T_{a'b'})) \\ &\overset{iii}{=} \mathbb{P}\pi_A(S_{(\varphi_A,\varphi_B)(q_0)}) \\ &\overset{ii}{=} (\alpha \circ \varphi_A)(q_0),\end{aligned}$$

- $q = q_0'$: analogously
- $q = q_i$ for some $0 < i < k$:

$$\begin{aligned}(\mathbb{P}\varphi_A \circ \xi)(q_i) &= \mathbb{P}\varphi_A(\{q_{i-1}\} \cup P_i) \\ &= \{\varphi_A(q_{i-1})\} \cup \mathbb{P}\varphi_A(P_i) \\ &= \{a_{i-1}\} \cup \mathbb{P}\pi_A(P_i) \\ &\overset{iv}{=} (\alpha \circ \varphi_A)(q_i),\end{aligned}$$

- $q = q_i'$ for some $0 < i < k$: analogously
- $q = q_k$:

$$\begin{aligned}(\mathbb{P}\varphi_A \circ \xi)(q_k) &= \mathbb{P}\varphi_A(\{q_{k-1}, q_{n-1}\} \cup P_k)\\ &= \{a_{k-1}, a_{n-1}\} \cup \mathbb{P}\pi_A(P_k)\\ &\overset{iv}{=} (\alpha \circ \varphi_A)(q_k),\end{aligned}$$

- $q = q_i$ and $i = n + m(n-k)$ or $i = \omega' + m(n-k)$: Then $(\varphi_A, \varphi_B)(q_i) = (a_n, b_n)$.

$$\begin{aligned}(\mathbb{P}\varphi_A \circ \xi)(q_i) &= \mathbb{P}\varphi_A(\{q'_{k-1}\} \cup R_i \cup P_k)\\ &= \{a_{k-1}\} \cup \mathbb{P}\varphi_A(R_i) \cup \mathbb{P}\varphi_A(P_k)\\ &\overset{v}{=} \{a_{k-1}\} \cup \{a_{n-1}\} \cup \mathbb{P}\varphi_A(P_k)\\ &\overset{iv}{=} (\alpha \circ \varphi_A)(q_i),\end{aligned}$$

- „else“: Then $q = q_i$ for some $i > k$ and $(\varphi_A, \varphi_B)(q_i) = (a_j, b_j) \neq (a_k, b_k)$.

$$\begin{aligned}(\mathbb{P}\varphi_A \circ \xi)(q_i) &= \mathbb{P}\varphi_A(R_i \cup P_i)\\ &= \mathbb{P}\varphi_A(R_i) \cup \mathbb{P}\varphi_A(P_i)\\ &\overset{v}{=} \{a_{j-1}\} \cup \mathbb{P}\varphi_A(P_i)\\ &\overset{iv}{=} (\alpha \circ \varphi_A)(q_i).\end{aligned}$$

We can prove analogously that φ_B is a homomorphism, so $\mathbf{Q}$ is indeed an $\mathcal{A}$-$\mathcal{B}$-cone.

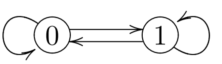
The states $q_i \in Q$ are pairwise not $\mathcal{A}$-$\mathcal{B}$-respective

Let $\psi : Q \to \mathcal{A} \otimes \mathcal{B}$ be the unique homomorphism. We show by contradiction, that $\psi(q_i) \neq \psi(q_j)$ for all $i \neq j$, i.e. q_i, q_j are not $\mathcal{A}$-$\mathcal{B}$-respective ($q_i \not\simeq_{\mathcal{A},\mathcal{B}} q_j$) by theorem 2.8. Assume $\psi(q_i) = \psi(q_j)$ for some $i < j$. We choose i minimal, then $\psi(q_{i'}) \neq \psi(q_j)$ for all $i' < i$ and all $j \in \mathbb{N}$.

- First assume $i < k$. Since the impeding path is reduced $(\varphi_A, \varphi_B)(q_i) \neq (\varphi_A, \varphi_B)(q_j)$ and therefore $\psi(q_i) \neq \psi(q_j)$ for all $j \neq i$.
- Assume now $i > k$. $\psi(q_i) = \psi(q_j)$ implies $(\varphi_A, \varphi_B)(q_i) = (\varphi_A, \varphi_B)(q_j)$. Because $j' = i+n-k = \min\{j'' \mid (\varphi_A, \varphi_B)(q_i) = (\varphi_A, \varphi_B)(q_{j''})\}$ we have $q_{j'-1} \in R_j$. Then we need $q \in \xi(q_i)$ with $\psi(q) = \psi(q_{j'-1})$ since $(\mathbb{P}\psi \circ \xi)(q_i) = (\eta \circ \psi)(q_i) = (\eta \circ \psi)(q_j) = (\mathbb{P}\psi \circ \xi)(q_j)$. Because $(\varphi_A, \varphi_B)(q) = (\varphi_A, \varphi_B)(q_{j'-1})$ we get $q = q_{i'} \in R_i$. Then $\psi(q_{i'}) = \psi(q_{j'-1})$ and $i' < i \leq j' - 1$ in contradiction to i being minimal.
- Let $i = k$. Then $(\varphi_A, \varphi_B)(q_j) = (a_k, b_k)$ and hence $q'_{k-1} \in \xi(q_j)$. We need $q \in \xi(q_k)$ with $\psi(q) = \psi(q'_{k-1})$. The only possibility is $q = q_{k-1}$. Then $\psi(q'_i) = \psi(q_i)$ for all $i < k$ by induction. There exists a state $p \in \mathcal{A} \otimes \mathcal{B}$ with $p \in \xi(q_0)$ and $(\varphi_A, \varphi_B)(p) = (a, b)$. Then we need $p' \in \xi(q'_0)$ with $\psi(p') = \psi(p)$. Otherwise there is no state $q' \in \psi(q'_0)$ with $(\varphi_A, \varphi_B)(q') = (a, b)$.

This proves $\psi(q_i) \neq \psi(q_j)$. Hence $|\varkappa| \leq |\mathcal{A} \otimes \mathcal{B}|$ in contradiction to the choice of $\varkappa$. Consequently, the product $\mathcal{A} \otimes \mathcal{B}$ does not exist. □

Example 4.5 In [1] it is shown, that the product $\mathcal{A} \otimes \mathcal{A}$ of the following transition system $\mathcal{A}$ does not exist.

$\mathcal{A} =$ 

We verify this, using theorem 4.4. $(0,0)(0,0)$ is an impeding path in $\mathcal{A} \times \mathcal{A}$, since we can extend it with $(0,0), (0,1), (1,0), (1,1)$ and $0 \sim 1$. Therefore the product

$\mathcal{A} \otimes \mathcal{A}$ does not exist.

Lemma 4.6 *Let* $\mathbf{Q}, \mathbf{Q}'$ *be* $\mathcal{A}$-$\mathcal{B}$*-cones,* $q \in Q, q' \in Q'$*, so that*

- $(\varphi_A, \varphi_B)(q) = (\varphi'_A, \varphi'_B)(q')$
- *for all bisimilar paths* $(a_0, b_0) \ldots (a_n, b_n)$ *with* $(a_0, b_0) = (\varphi_A, \varphi_B)(q)$ *there does not exist* $a \neq a' \in \alpha(a_0)$ *and* $b \neq b' \in \beta(b_0)$ *with* $a \sim a' \sim b \sim b'$.

Then $q \simeq_{\mathcal{A},\mathcal{B}} q'$.

Proof We define a tree (T, τ) with $T \subseteq (A \times B)^+$ and projections $\vartheta_A : T \to A, \vartheta_B : T \to B$ recursively:

- $\omega_T = (\varphi_A, \varphi_B)(q) = (\vartheta_A, \vartheta_B)(\omega_T)$,
- $\forall t \in T : \tau(t) = \{t \cdot (a, b) \mid a \in (\alpha \circ \vartheta_A)(t), b \in (\beta \circ \vartheta_B)(t)\}, (\vartheta_A, \vartheta_B)(t \cdot (a, b)) = (a, b)$.

We define a map $\chi : \langle q \rangle \to T$ recursively. Note, that the definition yields $\vartheta_A \circ \chi = \varphi_A, \vartheta_B \circ \chi = \varphi_B$.

- $\chi(\omega_{\langle q \rangle}) = \omega_T$,
- for $q_2 \in \xi(q_1)$:

$$\begin{aligned}\chi(q_2) &= \chi(q_1) \cdot ((\varphi_A, \varphi_B)(q_2)) \\ &\in \{\chi(q_1) \cdot (a, b) \mid a \in \alpha(\varphi_A(q_1)), b \in \beta(\varphi_B(q_1))\} = \tau(\chi(q_1)).\end{aligned}$$

Because $(\alpha \circ \varphi_A)(q_1) = (\mathbb{P}\varphi_A \circ \xi)(q_1)$ we have $\varphi_A(q_2) \in \alpha(\varphi_A(q_1)) = (\alpha \circ \vartheta_A)(\chi(q_1)) \subseteq \tau(\chi(q_1))$. Therefore χ is well-defined.

The second condition in the lemma yields

$$\forall p \in \langle q \rangle : a \in \mathbb{P}\varphi_A(\xi(p)) \wedge b \in \mathbb{P}\varphi_B(\xi(p)) \wedge a \sim b \Leftrightarrow (a, b) \in (\varphi_A, \varphi_B)(\xi(p)).$$

We show, that χ is an epimorphism:

$$\begin{aligned}(\tau \circ \chi)(q) &= \{\chi(q) \cdot (a, b) \mid a \in (\alpha \circ \vartheta_A \circ \chi)(q), b \in (\beta \circ \vartheta_B \circ \chi)(q)\} \\ &= \{\chi(q) \cdot (a, b) \mid a \in (\alpha \circ \varphi_A)(q), b \in (\beta \circ \varphi_B)(q)\} \\ &= \{\chi(q) \cdot (a, b) \mid a \in (\mathbb{P}\varphi_A \circ \xi)(q), b \in (\mathbb{P}\varphi_B \circ \xi)(q)\} \\ &= \{\chi(q) \cdot (\varphi_A, \varphi_B)(p) \mid p \in \xi(q)\} \\ &= \{\chi(p) \mid p \in \xi(q)\} \\ &= (\mathbb{P}\chi \circ \xi)(q).\end{aligned}$$

We show, that ϑ_A is a homomorphism (note, that χ is surjective): $\alpha \circ \vartheta_A \circ \chi = \alpha \circ \varphi_A = \mathbb{P}\varphi_A \circ \xi = \mathbb{P}\vartheta_A \circ \mathbb{P}\chi \circ \xi = \mathbb{P}\vartheta_A \circ \tau \circ \chi$. We can define $\chi' : \langle q' \rangle \to T$ analogously. Hence with the lemma 2.7 and corollary 3.3 $q \cong_{\mathcal{A},\mathcal{B}} \omega_{\langle q \rangle} \cong_{\mathcal{A},\mathcal{B}} \omega_T \cong_{\mathcal{A},\mathcal{B}} \omega_{\langle q' \rangle} \cong_{\mathcal{A},\mathcal{B}} q'$. $\square$

Lemma 4.7 *Let* $\mathbf{Q}, \mathbf{Q}'$ *be* $\mathcal{A}$-$\mathcal{B}$*-cones,* $q \in Q, q' \in Q'$*. Then*

$$q \cong_{\mathcal{A},\mathcal{B}} q' \iff \forall p \in \xi(q). \exists p' \in \xi'(q') : p \cong_{\mathcal{A},\mathcal{B}} p' \wedge \forall p' \in \xi'(q'). \exists p \in \xi(q) : p \cong_{\mathcal{A},\mathcal{B}} p'.$$

Proof Let $q \cong_{\mathcal{A},\mathcal{B}} q'$ and $\mathcal{R} = (R, \rho)$ be the $\mathbb{P}$-coalgebra from defintion 2.4, $\chi : \mathcal{R} \to \mathcal{Q}$ and $\chi' : \mathcal{R} \to \mathcal{Q}'$ homomorphisms and $r \in R$ with $\chi(r) = q$ and $\chi'(r) = q'$. Let $p \in \xi(q)$, then there exists $s \in \rho(r)$ with $\chi(s) = p$ and therefore $p \cong_{\mathcal{A},\mathcal{B}} \chi'(s)$.
Assume now, that the right side of the equivalence in the lemma holds. Then for all $p \in \xi(q), p' \in \xi(q')$ there exists an $\mathcal{A}$-$\mathcal{B}$-cone $\mathbf{R}_{pp'}$, homomorphisms $\chi_{pp'} : \mathcal{R}_{pp'} \to \mathcal{Q}, \chi'_{pp'} : \mathcal{R}_{pp'} \to \mathcal{Q}'$ and $r \in R_{pp'}$, so that $\chi_{pp'}(r) = p, \chi'_{pp'}(r) = p'$. We define a coalgebra

$$\mathcal{R} = (R, \rho) = \{r_0\} \oplus \bigoplus_{p \cong_{\mathcal{A},\mathcal{B}} p'} \mathcal{R}_{pp'} \text{ and } \rho(r_0) = \left\{ r \in R_{pp'} \mid \chi_{pp'} = p, \chi'_{pp'} = p' \right\}.$$

Then we can define homomorphisms $\chi : \mathcal{R} \to \mathcal{Q}, \chi' : \mathcal{R} \to \mathcal{Q}'$ with $\chi(r_0) = q, \chi'(r_0) = q'$ and $\chi(r) = \chi_{pp'}(r)$ if $r \in R_{pp'}$. The reader is invited to check, that χ, χ' are homomorphisms and that they commute with the projections. Hence $q \cong_{\mathcal{A},\mathcal{B}} q'$. □

Lemma 4.8 *Let $(A, \alpha), (B, \beta)$ be finite transition systems. Assume there is no impeding path in $A \times B$. Then the product $\mathcal{A} \otimes \mathcal{B}$ exists and is finite.*

Proof Since A, B are finite, there exists $m \in \mathbb{N}$, so that any bisimilar path of length at least m contains a loop. Consider a bisimilar path $(a_1, b_1) \ldots (a_n, b_n)$ of length $n \geq m$. Then $a = a'$ or $b = b'$ for all $a, a' \in \alpha(a_n), b, b' \in \beta(b_0)$ with $a \sim a' \sim b \sim b'$, otherwise the path would be impeding. Then the following situation is impossible:

$$\begin{tikzcd}
 & & & (a,b) & (a,b') \\
(a_1,b_1) \arrow[r] & (a_2,b_2) \arrow[r] & \cdots \arrow[r] & (a_n,b_n) \arrow[u] \arrow[ur] \arrow[d] \arrow[dr] & \\
 & & & (a',b') & (a',b)
\end{tikzcd}$$

We start the construction of the product with elements $(a_1, b_1) \in \sim_{\mathcal{A},\mathcal{B}}$, so that no such bisimilar path exists for any $n \in \mathbb{N}$. Then by lemma 4.6 there is only one equivalence class $\mathcal{C}$ with $(\pi_A, \pi_B)(\mathcal{C}) = (a_1, b_1)$. Take $(a_1, b_1) \in A \times B$, so that for all $a_2 \in \alpha(a_1), b_2 \in \beta(b_1)$ there exist only finite many classes $\mathcal{C}$ with $(\pi_A, \pi_B)(\mathcal{C}) = (a_2, b_2)$. Then by lemma 4.7 there are only finite many classes $\mathcal{C}$ with $(\pi_A, \pi_B)(\mathcal{C}) = (a_1, b_1)$. After at most m steps we have considered all bisimilar pairs (a, b). Then for every bisimilar pair (a, b) there exist only finite many equivalence classes $\mathcal{C}$ with $(\pi_A, \pi_B)(\mathcal{C}) = (a, b)$. Therefore $M = \{\mathcal{C}_{\mathbf{Q},q} \mid \mathbf{Q}\ \mathcal{A}\text{-}\mathcal{B}\text{-cone}, q \in Q\}$ is finite and with theorem 2.8 the product $\mathcal{A} \otimes \mathcal{B}$ exists. Furthermore, M is the base set of the product and therefore $\mathcal{A} \otimes \mathcal{B}$ is finite. □

Corollary 4.9 *Let $\mathcal{A} = (A, \alpha), \mathcal{B} = (B, \beta)$ be finite transition systems. If the product $\mathcal{A}^2 = \mathcal{A} \otimes \mathcal{A}$ exists, then $\mathcal{A} \otimes \mathcal{B}$ exists too.*

Proof Assume the product $\mathcal{A} \otimes \mathcal{B}$ does not exist. Then there exists an impeding path $(a_n, b_n) \ldots (a_0, b_0)$ in $\mathcal{A} \times \mathcal{B}$. Let $(a, b), (a', b')$ with $a \sim a' \sim b \sim b'$ be the possible continuations of the impeding path. Then $(a_n, a_n) \dots (a_0, a_0)$ is a bisimilar path in $\mathcal{A} \times \mathcal{A}$ and $(a, a), (a, a'), (a', a), (a', a')$ are possible continuations of this path. Hence $(a_n, a_n) \dots (a_0, a_0)$ is an impeding path in $\mathcal{A} \times \mathcal{A}$ and by lemma 4.4 $\mathcal{A}^2$ does not exist. □

5 Conclusions and Further Work

For arbitrary F-coalgebras $\mathcal{A}, \mathcal{B}$ we have introduced an equivalence relation $\simeq_{\mathcal{A},\mathcal{B}}$ on the class of all elements of $\mathcal{A}$-$\mathcal{B}$-cones. We have seen that, if the class of all equivalence classes is a set, then it is a base set of the product $\mathcal{A} \otimes \mathcal{B}$. Otherwise the product does not exists (theorem 2.8).

The invention of impeding paths led us to a more technical criterium for the existence of products of transition systems in theorem 4.4. It followed in corollary 4.9 that, if the product $\mathcal{A} \otimes \mathcal{A}$ exists, then $\mathcal{A} \otimes \mathcal{B}$ exists for any transition system $\mathcal{B}$. It would be interesting to generalise this result to arbitrary F-coalgebras.

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