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Coalgebraic Modal Logic Beyond Sets

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Abstract

Polyadic coalgebraic modal logic is studied in the setting of locally presentable categories. It is shown that under certain assumptions, accessible functors admit expressive logics for their coalgebras. Examples include typical functors used to describe systems with name binding, interpreted in nominal sets.

Keywords: coalgebra, modal logic, locally presentable category

1 Introduction

In recent years, coalgebra has received much attention as a unifying abstract approach to transition systems [29,16]. Many kinds of systems considered in theoretical computer science, including labelled, probabilistic and timed ones, are modeled as coalgebras for certain functors (called behaviour functors in this context) on the category **Set** of sets and functions. Other categories have also been considered, for example presheaf categories [11] or the category **Nom** of nominal sets [10] to model process algebras with name binding. The coalgebraic approach provides an abstract view on notions of coinduction and bisimulation.

Properties of transition systems are normally specified with a modal logic. Various logics have been developed to describe properties of different kinds of systems, e.g., Hennessy-Milner logic for labelled transition systems [14], probabilistic modal logic [17] for probabilistic systems, or logics for systems with name binding [24,8]. Importantly, such logics are *expressive*, i.e., they characterize their respective notions of bisimilarity. However, non-expressive fragments of these logics are also often used to characterize other notions of process equivalence, e.g., trace equivalence or testing equivalence [13]. A successful abstract theory of transition systems must provide a general perspective on modal logics and their properties.

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The first abstract approach to logics for coalgebras was *coalgebraic logic* of Moss [25], providing expressive logics for essentially all functors on **Set**. However general, coalgebraic logic is rather difficult to use in practice, as its syntax involve applications of the behaviour functor to formulas, and it does not provide simple and natural modalities like those known from Hennessy-Milner or similar logics. On the other hand, logics developed in [15,19,26,28] are close to their usual concrete presentations, but their expressivity depends on some conditions imposed on the behaviour functor. For example, modalities in [26] are *predicate liftings*, which map predicates on X to predicates on BX, where B is the behaviour functor, and the resulting modal logic is expressive provided enough predicate liftings exist for B. This approach was analyzed and generalized by Schröder [30], who noted that predicate liftings are equivalent to functions $B2 \to 2$ and considered *polyadic modal logic*, where modalities of any arity, such as functions $B(2^n) \to 2$, are allowed. He then proved polyadic modal logics expressive for all accessible behaviour functors.

All results mentioned above apply to functors on **Set**. In particular, Schröder's expressivity proof is set-theoretic in nature and it is not immediately clear how to translate it to other base categories. It is the purpose of this paper to generalize the definition of polyadic modal logic, and the proof of its expressivity, to accessible functors on locally presentable categories that satisfy some additional conditions.

Our approach is inspired by recent work by Kurz and Bonsangue [6,7,20,21], who use Stone dualities to obtain logics for coalgebras on arbitrary categories, and by that by Pavlovic, Mislove and Worrell [27], who exploit logical connections between data and tests to develop an abstract theory of testing. In those works, as in the present paper, contravariant adjunctions provide the infrastructure for linking processes and formulas. In [6,7,20], the adjunctions are assumed to be categorical dualities. This easily implies the existence of expressive logics for all functors, and the main effort is directed towards the nontrivial task of finding concrete presentations of those logics; to that end, in [21] adjunctions that are not dualities were used. In the present, more flexible approach, the duality assumption is not made. This often makes concrete presentations of expressive logics easier to find, and opens a possibility to treat various interesting, but non-expressive logics in a uniform fashion, but it comes for a price: the existence of expressive logics depends on certain conditions, as in [30]. On the other hand, in [27] the duality assumption is not made, and the adjunctions arise from certain cogenerators in the relevant categories. This does not apply to all examples of interest, and in the present paper we work with more general adjunctions. Also, in [27] the main focus is on non-expressive logics, and no expressivity results are provided there.

The paper is structured as follows. After §2 of technical preliminaries, §3 presents a categorical generalization of Schröder's polyadic modal logic, which is proved expressive under some conditions in §4. In §5, a categorical notion of modality is suggested. Examples for functors on three different categories are studied in §6.

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2 Preliminaries

The reader is assumed to be acquainted with basic category theory; [2,22] are good references.

An epimorphism $e : X \to Y$ is *strong* if for every commutative square (i) with m mono there exists a unique diagonal $d : Y \to U$ such that (ii) commutes.

$$\begin{tikzcd} X \arrow[r, two heads, "e"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ U \arrow[r, hook, "m"'] & Z \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, two heads, "e"] \arrow[d, "f"'] & Y \arrow[d, "g"] \arrow[dl, dotted, "d"'] \\ U \arrow[r, hook, "m"'] & Z \end{tikzcd}$$

(i) (ii)

Y is then a *strong quotient* of X. One says that strong epis and monos form a *factorization system* in a category $\mathcal{C}$ if every morphism in $\mathcal{C}$ can be factorized as a strong epi followed by a mono.

A source $\{ f_i : X \to Y_i \mid i \in I \}$ is *jointly monic* if for every $g, h : Z \to X$, one has $g = h$ if $f_i \circ g = f_i \circ h$ for all $i \in I$. An object X in a category $\mathcal{C}$ is a *cogenerator* if for every object Y, the source of all morphisms from Y to X is jointly monic. For example, every set with at least two elements is a cogenerator in **Set**.

A category $\mathcal{D}$ is *filtered* if (i) for every $d, d' \in \mathcal{D}$ there exists a cospan $d \to d'' \leftarrow d'$ in $\mathcal{D}$, and (ii) every parallel pair of morphisms in $\mathcal{D}$ has a coequalizer in $\mathcal{D}$. A *filtered colimit* is a colimit of a diagram whose domain category is nonempty and filtered; the dual notion is that of *cofiltered limit*. An object X of a category $\mathcal{C}$ is *finitely presentable* if the functor $\hom(X, -) : \mathcal{C} \to \mathbf{Set}$ preserves filtered colimits. For example, finitely presentable objects of **Set** are exactly finite sets, and in an equational class of algebras, an algebra is finitely presentable if and only if it can be presented by finitely many generators and finitely many equations. A category $\mathcal{C}$ is *locally finitely presentable* if it is cocomplete and has a set $\mathcal{G}$ of finitely presentable objects such that every object of $\mathcal{C}$ is a filtered colimit of objects in $\mathcal{G}$. For $\mathcal{C}, \mathcal{D}$ finitely presentable, a functor $B : \mathcal{C} \to \mathcal{D}$ is *finitary* if it preserves filtered colimits. In a locally presentable category, an object is *finitely generated* if it is a strong quotient of a finitely presentable object. In **Set**, finitely presentable and finitely generated objects coincide, and an algebra is finitely generated if and only if it is so in the sense of universal algebra.

The above notions can be generalized to κ-filtered colimits, locally κ-presentable categories and κ-accessible functors, for any regular cardinal κ. All definitions, results and proofs given in this paper work for the more general case with no change. For more information and intuition on locally presentable categories, see [3,23].

For an endofunctor L on a category $\mathcal{C}$, an *algebra* is an object X (the *carrier*), with a map $g : LX \to X$ (the *structure*). An algebra morphism from $g : LX \to X$ to $h : LY \to Y$ is a map $f : X \to Y$ in $\mathcal{C}$ such that $f \circ g = h \circ Lf$. Dually, for an endofunctor B, a *coalgebra* is an object X (the *carrier*), with a map $g : X \to BX$ (the *structure*). A coalgebra morphism from $g : X \to BX$ to $h : Y \to BY$ is a map $f : X \to Y$ in $\mathcal{C}$ such that $h \circ f = Bf \circ g$. For example, if $B = \mathcal{P}_\omega(A \times -)$ on **Set**, where $\mathcal{P}_\omega$ is the finite powerset functor and A is a fixed set of labels, then B-coalgebras are finitely branching labelled transition systems (LTSs). For a coalgebra $h : X \to BX$ in **Set**, elements (called *processes* in this context) $x, y \in X$ are *behaviourally equivalent* if they are identified by a coalgebra morphism from h. For LTSs as coalgebras in **Set**, behavioural equivalence coincides with strong bisimilarity. More information and examples of coalgebras can be found in [29,16].

On finitely branching LTSs, bisimilarity is characterized by finitary Hennessy-Milner logic [14], with syntax

$$\phi ::= \top \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \langle a\rangle\phi \tag{1}$$

and with semantics defined, on a given LTS, by

$$x \models \langle a\rangle\phi \iff x \xrightarrow{a} y \text{ s.t. } y \models \phi$$

and the standard interpretation of propositional connectives. Fragments of Hennessy-Milner logic have also been considered (see [13] for a survey). For example, restricted to the grammar

$$\phi ::= \top \mid \langle a\rangle\phi, \tag{2}$$

the logic characterizes *trace equivalence* on LTSs.

Acquaintance with various known approaches aimed at generalizing Hennessy-Milner and other logics to other functors (on **Set**) is not strictly necessary to understand the following technical developments. However, without any knowledge of those approaches it would be hard to put the present work in context. Due to lack of space that related work is not described here; [30] is a good reference, but e.g. [15,19,25,26,28] are also worth reading.

3 Logical Connections

Our generalization of coalgebraic modal logic proceeds along lines similar to those of [27]. To gain momentum, we begin by considering the familiar setting of sets and functions. Typically, the semantics of a logic is some satisfaction relation $\models \subseteq X \times \Phi$ between the set Φ of tests (formulas) and the set X of tested entities (processes), or equivalently a function:

$$\models : X \times \Phi \to 2$$

(here and in the following, 2 denotes the two-element set $\{\mathtt{tt}, \mathtt{ff}\}$). Its two transposes:

$$[\![\text{-}]\!] : \Phi \to 2^X \qquad [\![\text{-}]\!]^\flat : X \to 2^\Phi \tag{3}$$

defining the semantics of processes by sets of formulas that hold for them, and the semantics of formulas by sets of processes that satisfy them. In particular, two processes in X are logically equivalent if they are equated by $[\![\_]\!]^\flat$. This functional presentation is easily generalized to cover logics where another set is used for "truth values"; for example, in some probabilistic logics the continuous interval $[0,1]$ is used instead of 2.

Abstracting from the category of sets, consider any symmetric monoidal closed category $(C, \otimes, \multimap)$ with a chosen object Ω. The contravariant internal hom-functor $- \multimap \Omega$ on $\mathcal{C}$ is self-adjoint, with the bijection

$$\mathcal{C}(X, \Phi \multimap \Omega) \cong \mathcal{C}(X \otimes \Phi, \Omega) \cong \mathcal{C}(\Phi \otimes X, \Omega) \cong \mathcal{C}(\Phi, X \multimap \Omega) \tag{4}$$

obtained from the symmetric monoidal closed structure.[3] Even more generally, we assume any *logical connection*, i.e., any contravariant adjunction

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F", tail] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G", tail] \end{tikzcd} \qquad\qquad \mathcal{C}(X, G\Phi) \cong \mathcal{D}(\Phi, FX) \tag{5}$$

(the contravariance of F and G is marked by the cross arrow tails), where $X \in \mathcal{C}$, $\Phi \in \mathcal{D}$. Slightly abusing notation, we will denote both sides of the bijection in (5) by $-^\flat$. Objects of $\mathcal{C}$ are thought of as sets (or structures) of processes, and objects of $\mathcal{D}$ as sets (or structures) of formulas. The connection (5) provides the infrastructure for relating processes and formulas. It is clear that (4) is a special case of (5), and (3) is a special case of (4).

In any connection, the composite (covariant) functors GF and FG are monads on $\mathcal{C}$ and $\mathcal{D}$, respectively. We denote the units and multiplications of these monads by η^{GF}, η^{FG}, μ^{GF} and μ^{FG}. The bijection (5) can be expressed in terms of these transformations:

$$f^\flat = Ff \circ \eta^{FG}_\Phi \qquad\qquad g^\flat = Gg \circ \eta^{GF}_X, \tag{6}$$

for $f : X \to G\Phi$ in $\mathcal{C}$ and $g : \Phi \to FX$ in $\mathcal{D}$. We will sometimes use the following property of adjunctions:

$$F\eta^{GF} \circ \eta^{FG} F = \mathrm{id}. \tag{7}$$

The following is a central definition in our approach to logics for coalgebras.

Definition 3.1 In the situation of (5), for any endofunctor B on $\mathcal{C}$, a *polyadic coalgebraic modal logic* (or shortly a *logic*) for B-coalgebras is a pair (L, ρ) where L (called the *syntax*) is an endofunctor on $\mathcal{D}$, and $\rho : LF \Longrightarrow FB$ (called the *semantics*) connects L and B along the adjunction.

A connection ρ as above defines the adjoint connection $\rho^\star : BG \Longrightarrow GL$ by

$$\rho^\star = GL\eta^{FG} \circ G\rho G \circ \eta^{GF} BG; \tag{8}$$

in turn, $\rho^\star$ determines ρ by $\rho = (\rho^\star)^\star = FB\eta^{GF} \circ F\rho F \circ \eta^{FG} LF$.

[3] In [27], the object Ω of truth values was assumed to be a cogenerator in $\mathcal{C}$. Here no such assumption is made, and indeed in §6.2 the object of truth values is not a cogenerator. However, we later assume that Ω is an *internal cogenerator*, see Remark 4.6.

If L has an initial algebra $a : L\Phi_L \to \Phi_L$, then Φ_L can be thought of as the object of L-formulas. Given any coalgebra $h : X \to BX$, the semantic interpretation $[\![\_]\!]_h$ of Φ_L in h is defined by L-induction in $\mathcal{D}$:

$$\begin{tikzcd} LFX \arrow[d, "\rho_X"'] & L\Phi_L \arrow[l, dashed, "L[\![\_]\!]_h"'] \arrow[dd, "a"] \\ FBX \arrow[d, "Fh"'] & \\ FX & \Phi_L \arrow[l, dashed, "[\![\_]\!]_h"] \end{tikzcd} \tag{9}$$

and its transpose $[\![\_]\!]_h^\flat : X \to G\Phi_L$ is a map that, intuitively, identifies logically equivalent processes.

Example 3.2 To illustrate the framework described so far on a simple example, consider the logic for trace equivalence on labelled transition systems. To this end, take $\mathcal{C} = \mathcal{D} = \mathbf{Set}$, $F = G = 2^-$ and $B = \mathcal{P}(A\times -)$ for a fixed set A. The syntax (2) is modeled by the functor $L = 1 + A \times -$ with an initial algebra $\Phi_L = A^*$. The connection ρ at X, i.e., a function $\rho_X : L(2^X) \to (BX \to 2)$, is defined by cases:

$$\begin{aligned} \rho_X(\top)(\beta) &= \mathtt{tt} \text{ always} \\ \rho_X(\langle a\rangle\phi)(\beta) &= \mathtt{tt} \iff \exists\,(a,y) \in \beta.\ \phi(y) = \mathtt{tt}, \end{aligned}$$

where $\beta \in BX$ and $\phi \in 2^X$. The similarity of this definition to the usual semantics of (2) is hopefully apparent. Indeed, it is straightforward to check that in any LTS $h : X \to BX$, $[\![x]\!]_h^\flat \in 2^{A^*}$ is (the characteristic function of) the set of traces of $x \in X$, and the kernel of $[\![\_]\!]_h^\flat$ is trace equivalence on h.

We now proceed to formulate and prove that logics (L, ρ) respect behaviour, i.e., that behavioural equivalence implies logical equivalence. This property of logics is usually defined in terms of individual processes, however in the categorical setting a more abstract approach is needed. Since $[\![\_]\!]_h^\flat$ intuitively identifies logically equivalent processes, and coalgebra maps identify behaviourally equivalent processes, the following theorem plausibly captures the right categorical notion:

Theorem 3.3 *Any logic* (L, ρ) respects behaviour*, i.e., for any coalgebra* $h : X \to BX$*, the map* $[\![\_]\!]_h^\flat$ *factorizes through every coalgebra map from* h.

Proof. Consider any other coalgebra $g : Y \to BY$ and a coalgebra map $f : X \to Y$ from h to g. It is enough to show that $[\![\_]\!]_h^\flat = [\![\_]\!]_g^\flat \circ f$, or equivalently that $[\![\_]\!]_h =$

$Ff \circ [\![\_]\!]_g$. This is proved by induction from the definition (9), since in the diagram:

$$
\begin{tikzcd}
LFX \arrow[d, "\rho_X"'] & LFY \arrow[l, "LFf"'] \arrow[d, "\rho_Y"] & L\Phi_L \arrow[l, "L[\![\_]\!]_g"'] \arrow[dd, "a"] \\
FBX \arrow[d, "Fh"'] & FBY \arrow[l, "FBf"'] \arrow[d, "Fg"] & \\
FX & FY \arrow[l, "Ff"] & \Phi_L \arrow[l, "[\![\_]\!]_g"]
\end{tikzcd}
$$

the upper left part commutes by naturality of ρ, the right part by (9), and the lower left part since f is a coalgebra map. $\square$

4 Expressivity

Recall the intuition that for a given logic (L, ρ), with L admitting initial algebras, the interpretation $[\![\_]\!]^\flat_h$ in a coalgebra $h : X \to BX$ identifies logically equivalent processes. Expressivity of a logic means that logical equivalence implies behavioural equivalence, therefore one can say that a logic (L, ρ) is expressive if $[\![\_]\!]^\flat_h$ is a coalgebra morphism from h. This, however, requires a B-coalgebra structure on $G\Phi_L$, which intuitively is an unnecessary strong assumption: for expressivity, it should be sufficient to provide a B-coalgebra on the *image* of $[\![\_]\!]^\flat_h$ in $G\Phi_L$, and a morphism from h to that coalgebra. This leads to the following definition:

Definition 4.1 A logic (L, ρ) for B-coalgebras is *expressive* if for every $h : X \to BX$, the map $[\![\_]\!]^\flat_h$ is a coalgebra morphism from h followed by a mono in $\mathcal{C}$.

The following theorem gives simple conditions sufficient for logic expressivity.

Theorem 4.2 *In the situation of* (5)*, for any* $B : \mathcal{C} \to \mathcal{C}$*, for any logic* (L, ρ) *for* B*-coalgebras, if*

- L *has an initial algebra,*
- $\mathcal{C}$ *has a (StrEpi,Mono)-factorization system,*
- B *preserves monos, and*
- $\rho^\star : BG \Longrightarrow GL$ *is pointwise monic,*

then (L, ρ) *is expressive.*

Proof. The following diagram in $\mathcal{C}$ commutes:

$$\begin{tikzcd}
 & BGFX \arrow[r, "BG\llbracket\text{-}\rrbracket_h"] \arrow[d, hook, "\rho^\star_{FX}"'] & BG\Phi_L \arrow[d, hook, "\rho^\star_{\Phi_L}"] \\
BX \arrow[ur, "B\eta^{GF}_X"] \arrow[dr, "\eta^{GF}_{BX}"'] & GLFX \arrow[r, "GL\llbracket\text{-}\rrbracket_h"] & GL\Phi_L \\
 & GFBX \arrow[u, "G\rho_X"] & \\
X \arrow[uu, "h"] \arrow[r, "\eta^{GF}_X"'] & GFX \arrow[u, "GFh"] \arrow[r, "G\llbracket\text{-}\rrbracket_h"'] & G\Phi_L \arrow[uu, "Ga"']
\end{tikzcd}$$

Indeed, the lower right part is (9) mapped along G, the upper right part is the naturality of $\rho^\star$, the lower left part is the naturality of η^{GF} and the upper left part commutes by (8) and (7). The outer shape of this diagram is

$$\begin{tikzcd}
BX \arrow[r, "\llbracket\text{-}\rrbracket^\flat_h"] & BG\Phi_L \arrow[d, hook, "\rho^\star_{\Phi_L}"] \\
 & GL\Phi_L \\
X \arrow[uu, "h"] \arrow[r, "\llbracket\text{-}\rrbracket^\flat_h"'] & G\Phi_L \arrow[u, "Ga"']
\end{tikzcd}$$

(see (6)). Let $m \circ e$ by the strong epi-mono factorization of $\llbracket\text{-}\rrbracket^\flat_h$. Since B preserves monos, Bm is a mono:

$$\begin{tikzcd}
BX \arrow[r, two heads, "Be"] & BI \arrow[r, hook, "Bm"] & BG\Phi \arrow[d, hook, "\rho^\star_\Phi"] \\
 & & GL\Phi \\
X \arrow[uu, "h"] \arrow[r, two heads, "e"'] & I \arrow[uu, dashed, "i"'] \arrow[r, hook, "m"'] & G\Phi \arrow[u, "Ga"']
\end{tikzcd}$$

and a diagonal morphism $i : I \to BI$ as above exists since e is strong. This makes e a coalgebra morphism from h, and $m \circ e$ satisfies Definition 4.1. □

The first three conditions of Theorem 4.2 hold in most practical examples, and usually the key condition to check is the pointwise monicity of $\rho^\star$. In Example 3.2, for any $\Phi \in \mathcal{D}$, the function

$$\rho^\star_\Phi : B(2^\Phi) \to (L\Phi \to 2)$$

is defined by:

$$\rho^\star_X(\beta)(\top) = \mathtt{tt} \text{ always}$$
$$\rho^\star_X(\beta)(\langle a\rangle\phi) = \mathtt{tt} \iff \exists\,(a,y)\in\beta.\ y(\phi)=\mathtt{tt},$$

where $\beta \in B(2^\Phi) = \mathcal{P}(A\times 2^\Phi)$ and $\phi\in\Phi$, and it is not always pointwise monic: for example, for $\Phi=\{\phi,\psi\}$, it is straightforward to check that

$$\rho^\star_\Phi(\{(a,\{\phi\})\,,(a,\{\psi\})\}) = \rho^\star_\Phi(\{(a,\{\phi,\psi\})\}).$$

Indeed, the logic for traces is not expressive for B-coalgebras. Note, however, that the conditions of Theorem 4.2 are not necessary for (L,ρ) to be expressive.

A natural question arises as to what conditions are sufficient for expressive logics to exist for a given B on $\mathcal{C}$. Assuming $\mathcal{D}$, F and G have been chosen, a promising choice is $L = FBG$, with the canonical

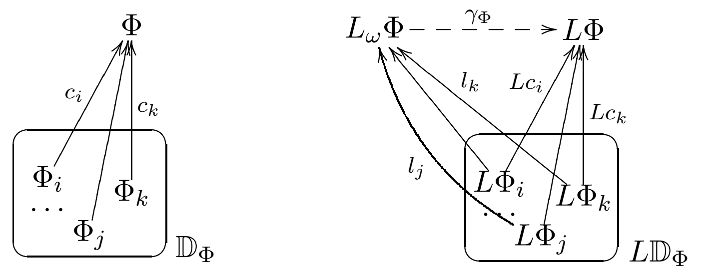
$$\rho = FB\eta^{GF} : LF = FBGF \Longrightarrow FB$$
$$\rho^\star = \eta^{GF}BG : BG \Longrightarrow GFBG = GL$$

and the monad unit η^{GF} is usually pointwise monic (see Remark 4.6). Unfortunately, FBG often fails to have initial algebras. For example, if $\mathcal{C}=\mathcal{D}=\mathbf{Set}$ and $F=G=2^-$, then even for finitary B, such as $B=\mathcal{P}_\omega$, the functor FBG does not have initial algebras for cardinality reasons.

In search for a better candidate for L, note that finitary functors on locally finitely presentable categories have initial algebras [3]. Assuming $\mathcal{D}$ locally finitely presentable, a general technique to restrict any functor L on $\mathcal{D}$ to a finitary L_ω that acts "almost as" L is via left Kan extensions: define

$$L_\omega = \mathrm{Lan}_I(LI)$$

where $I : \mathbf{Pres}_\omega\mathcal{D} \to \mathcal{D}$ is the inclusion functor of the full subcategory of finitely presentable objects. In more elementary terms, to calculate $L_\omega\Phi$, represent Φ as a filtered colimit of a diagram $\mathbb{D}_\Phi$ of finitely presentable objects, map $\mathbb{D}_\Phi$ along L, i.e., form the (filtered) diagram $L\mathbb{D}_\Phi$, and take its colimit as $L_\omega\Phi$.



(10)

The unique mediating morphism γ_Φ extends to a natural transformation $\gamma : L_\omega \Longrightarrow$

L, and L_ω coincides with L on $\mathbf{Pres}_\omega\mathcal{D}$. Moreover, L_ω is finitary even if L is not [23, Prop. 2.4.3].

We may now define $L_\omega = (FBG)_\omega$ with the canonical connections $\rho : L_\omega F \Longrightarrow FB$ and $\rho^\star : BG \Longrightarrow GL_\omega$ defined by:

$$\rho = FB\eta^{GF} \circ \gamma F \qquad \rho^\star = G\gamma \circ \eta^{GF} BG. \tag{11}$$

As before, it is natural to assume that η^{GF} is pointwise monic, but $G\gamma$ almost never is. However, under certain additional conditions their composition is pointwise monic. To spell out those conditions, one more important notion is needed:

Definition 4.3 [Adámek] A locally finitely presentable category is *strongly locally finitely presentable* if for every cofiltered limit cone $\{l_i : Y \to Y_i\}_{i\in I}$, and for any mono $f : X \to Y$ with X finitely generated, there exists $i \in I$ such that $l_i \circ f$ is a mono.

$$\begin{tikzcd} & Y_i & Y_j \cdots & Y_k \\ X \arrow[r, hook, "f"] & Y \arrow[u, "l_i"] \arrow[ur] \arrow[urr, "l_k"'] \end{tikzcd} \tag{12}$$

For example, **Set** and **Pos** are all strongly locally finitely presentable (and the locally countably presentable $\omega\mathbf{Cpo}$ is strongly so). The category **Un** of unary algebras is not strongly locally finitely presentable, even though it is locally finitely presentable (see [1]).

We are now ready to formulate sufficient conditions for $\rho^\star$ in (11) to be pointwise monic.

Theorem 4.4 *In the situation of* (5)*, for a B on $\mathcal{C}$, with L_ω and ρ defined as above, if*

- *$\mathcal{C}$ is strongly locally finitely presentable,*
- *$\mathcal{D}$ is locally finitely presentable,*
- *B is finitary and preserves monos, and*
- *η^{GF} is pointwise monic,*

then $\rho^\star = G\gamma \circ \eta^{GF} BG$ is pointwise monic.

Proof. For an object Φ in $\mathcal{D}$, we shall prove that $\rho^\star_\Phi : BG\Phi \to GL_\omega\Phi$ is a mono. Recall from (10) that $L_\omega\Phi$ is a part of a cocone

$$\{l_i : FBG\Phi_i \to L_\omega\Phi\}_{i\in I}$$

for the diagram $FBG\mathbb{D}_\Phi$, where $\mathbb{D}_\Phi$ is a filtered diagram of finitely presentable objects with Φ as the colimit. To show that $\rho^\star_\Phi$ is a mono it is enough to show that the source

$$\{Gl_i \circ \rho^\star_\Phi : BG\Phi \to GFBG\Phi_i\}_{i\in I} \tag{13}$$

is jointly monic. Further, for any $i \in I$, one has

$$Gl_i \circ \rho^\star_\Phi = \eta^{GF}_{BG\Phi_i} \circ BGc_i\,;$$

indeed, chase the diagram

$$\begin{tikzcd}
BG\Phi \arrow[r, "\eta^{GF}_{BG\Phi}"'] \arrow[d, "BGc_i"'] \arrow[rr, bend left, "\rho^\star_\Phi"] & GFBG\Phi \arrow[r, "G\gamma_\Phi"'] \arrow[dr, "GFBGc_i"] & GL_\kappa\Phi \arrow[d, "Gl_i"] \\
BG\Phi_i \arrow[rr, "\eta^{GF}_{BG\Phi_i}"'] & & GFBG\Phi_i
\end{tikzcd}$$

where the left square is the naturality of η^{GF}, and the triangle commutes by definition of γ in (10). Since η^{GF} is pointwise monic, to prove the joint monicity of (13) it is enough to show that the source

$$\{BGc_i : BG\Phi \to BG\Phi_i\}_{i\in I}$$

is jointly monic.

To this end, consider an object X in $\mathcal{C}$ and maps $f, g : X \to BG\Phi$ such that for each $i \in I$:

$$BGc_i \circ f = BGc_i \circ g\,.$$

We must prove that $f = g$.

Since $\mathcal{C}$ is locally finitely presentable, finitely presentable objects generate it and without loss of generality we may assume that X is finitely presentable. Moreover, $G\Phi$ is a colimit of a filtered diagram $\mathbb{E}$ of finitely presentable objects. Denote the colimiting cocone by

$$\{n_j : Y_j \to G\Phi\}_{j\in J}\,.$$

Since B is finitary, it preserves the colimit, and

$$\{Bn_j : BY_j \to BG\Phi\}_{j\in J}$$

is a colimiting cocone of the filtered diagram $B\mathbb{E}$. By finite presentability of X, there exists a $j \in J$ and two maps $f', g' : X \to BY_j$ such that

$$f = Bn_j \circ f' \qquad \text{and} \qquad g = Bn_j \circ g'\,.$$

Since $\mathcal{C}$ is locally finitely presentable, strong epis and monos form a factorization system [3] and the map $n_j : Y_j \to G\Phi$ factorizes into a strong epi $e : Y_j \to Z$ followed by a mono $m : Z \to G\Phi$. By definition Z is finitely generated.

Recall that Φ is a colimit of a diagram $\mathbb{D}_\Phi$ and denote the colimiting cocone by

$$\{c_i : \Phi_i \to \Phi\}_{i\in I}\,.$$

G, being a contravariant adjoint, maps the cocone to a limiting cone

$$\{Gc_i : G\Phi \to G\Phi_i\}_{i \in I}$$

of the cofiltered diagram $G\mathbb{D}_\Phi$. Now, by strong local finite presentability of $\mathcal{C}$, there exists an index $i \in I$ such that $Gc_i \circ m$ is a mono. Since B preserves monos, also $BGc_i \circ Bm$ is a mono.

Note that $f = Bm \circ Be \circ f'$ and $g = Bm \circ Be \circ g'$. Moreover, by our assumption on f and g,

$$BGc_i \circ Bm \circ Be \circ f' = BGc_i \circ Bm \circ Be \circ g'$$

By monicity of $BGc_i \circ Bm$, one has $Be \circ f' = Be \circ g'$ and finally

$$f = Bm \circ Be \circ f' = Bm \circ Be \circ g' = g\,.$$

□

Corollary 4.5 *In the situation of (5), if $\mathcal{C}$ is strongly locally finitely presentable, $\mathcal{D}$ is locally finitely presentable and η^{GF} is pointwise monic, then every finitary functor on $\mathcal{C}$ that preserves monos, admits an expressive logic.*

Proof. Combine Theorems 4.2 and 4.4. The only non-trivial point to make is that in every locally presentable category, strong epis and monos form a factorization system [3, Prop. 1.61]. □

Remark 4.6 The meaning of the pointwise monicity of η^{GF} becomes clear when the above result is specialized to adjunctions arising from chosen objects in symmetric monoidal closed categories, as in (4). An object Ω is an *internal cogenerator* if for any X, the map $\eta_X : X \to (X \multimap \Omega) \multimap \Omega$ is a mono. For example, for $F = G = \Omega^-$ on **Set**, the pointwise monicity assumption means that the set Ω of logical values must have at least two elements. Corollary 4.5 specializes to:

Corollary 4.7 *If a strongly locally finitely presentable, symmetric monoidal closed category $\mathcal{C}$ has an internal cogenerator, then every finitary functor on $\mathcal{C}$ that preserves monos, admits an expressive logic.*

5 Polyadic Modalities

Results proved in §4 show how to guarantee an expressive logic for B-coalgebras to exist. However, it might not be clear how to present the syntax and semantics of the logic in concrete situations. Moreover, the development presented so far does not suggest any treatment of (possibly non-expressive) fragments of the canonical logic. For example, it would be useful to know whether every logic according to Definition 3.1 is a fragment of an expressive logic. This section addresses these questions. First, we analyze the structure of the canonical logic L_ω and define a logic L^+_ω, with semantics essentially the same as that of L_ω, but with syntax allowing for a simpler presentation in concrete examples. The structure of L^+_ω suggests a general notion of polyadic modality. It is also showed that any logic with finitary syntax is canonically represented in L_ω. These results will considerably simplify the presentation of our main examples in §6.

By definition,

$$L_\omega\Phi = \mathrm{colim}_{(\Psi,\Psi\to\Phi)\in I/\Phi}\, FBG\Psi$$

(see (10)). Replacing the colimit with a coproduct, define

$$L^+_\omega\Phi = \coprod_{(\Psi,\Psi\to\Phi)\in I/\Phi} FBG\Psi = \coprod_{\Psi} \mathcal{D}(\Psi,\Phi)\cdot FBG\Psi \tag{14}$$

where the coproduct on the right side is indexed over a chosen generating set of finitely presentable objects, and $\cdot$ denotes copower. The evident mediating morphism $\delta_\Phi : L^+_\omega\Phi \to L_\omega\Phi$ extends to a natural transformation δ, and is epi. G, being a contravariant adjoint, maps epis to monos, hence the canonical adjoint connection

$$G\delta \circ G\gamma \circ \eta^{GF}BG : BG \Longrightarrow GL^+_\omega$$

is pointwise monic if and only if the corresponding connection (11) for L_ω is pointwise monic. Therefore L^+_ω is expressive if and only if L_ω is, provided that it is finitary and so admits initial algebras. In concrete cases, L^+_ω is slightly easier to present syntactically than L_ω. Its structure also suggests a general notion of polyadic modality: intuitively, in an obvious sense, a modality (or indeed any logical connective) of arity n is an operator mapping n-tuples of formulas to formulas. A finitely presentable object Ψ can be seen as an arity object, and a map $\Psi \to \Phi$ as a tuple indexed by Ψ. This, together with the structure of (14), motivates the following definition:

Definition 5.1 For a finitely presentable object $\Psi \in \mathcal{D}$, the object $FBG\Psi$ is the *object of B-modalities of arity Ψ*.

Examples in §6 will confirm the plausibility of this definition.

We proceed to show that every logic (L,ρ) with finitary syntax can be seen as a fragment of L_ω. We begin with a basic notion of logic morphism:

Definition 5.2 For any B on $\mathcal{C}$, a logic (L,ρ) is *represented* in (L',ρ') by $\theta : L \Longrightarrow L'$ if the equation

$$\rho = \rho' \circ \theta F \tag{15}$$

holds.

Clearly θ preserves the semantics ρ. Moreover, for any L, and for a logic (L',ρ'), a transformation $\theta : L \Longrightarrow L'$ defines a semantics for L by (15). In particular, the semantics of a logic L can be defined by showing how the syntax L is embedded in L_ω. The following representation theorem shows that every logic with a finitary syntax can be defined this way.

Theorem 5.3 *For any B on $\mathcal{C}$, any logic (L,ρ) with L finitary is represented in $\big(L_\omega, FB\eta^{GF} \circ \gamma F\big)$.*

Proof. First, note that any (L,ρ) (with L not necessarily finitary) is canonically represented in $\big(FBG, FB\eta^{GF}\big)$ by $\iota^\rho : L \Longrightarrow FBG$ defined as the transpose of the adjoint connection $\rho^\star$, or more explicitly by $\iota^\rho = \rho G \circ L\eta^{FG}$. Indeed, a straightforward calculation shows that (15) commutes for $\theta = \iota^\rho$. If L is finitary, the representation ι^ρ yields a transformation $\iota^\rho_\omega : L \Longrightarrow L_\omega$ along the bijection

$$\mathrm{Nat}(L, FBG) \cong \mathrm{Nat}(LI, FBGI) = \mathrm{Nat}(LI, L_\omega I) \cong \mathrm{Nat}(L, L_\omega)$$

where $I : \mathbf{Pres}_\omega\mathcal{D} \to \mathcal{D}$ is the inclusion functor, the left and the right bijections hold since (by finitarity) $L = \mathrm{Lan}_I\, LI$, and the middle equation holds since $L_\omega I = FBGI$ by definition of L_ω. Now, the transformation $\gamma \circ \iota^\rho_\omega : L \Longrightarrow FBG$ is also mapped to ι^ρ_ω along the same series of bijections:

$$\gamma \circ \iota^\rho_\omega \mapsto (\gamma \circ \iota^\rho_\omega)I = \gamma I \circ \iota^\rho_\omega I = \iota^\rho_\omega I \mapsto \iota^\rho_\omega$$

hence, by bijectivity, $\iota^\rho = \gamma \circ \iota^\rho_\omega$ and the equation (15)

$$\rho = \iota^\rho F \circ FB\eta^{GF} = \iota^\rho_\omega F \circ \gamma F \circ FB\eta^{GF}$$

holds. □

Together with observations on the structure on L_ω made earlier in this section, the above theorem allows one to give more concrete presentations of expressive and non-expressive logics. Examples shown in the following section illustrate this point.

6 Examples

This section shows how Definitions 4.1 and 5.1 specialize to useful and natural notions in concrete settings, and how Theorems 4.2 and 4.4 can be used to find expressive logics (and to present their non-expressive fragments) for transition systems. In §6.1, the familiar setting of sets and functions is studied. Schröder's polyadic coalgebraic modal logic [30,18] is shown to be a special case of the present approach, hence all examples covered there are examples here a well. However, for completeness we describe the classical example of finitary Hennessy-Milner logic. In §6.2, the case of nominal sets and equivariant functions is studied, and it is shown how Milner-Parrow-Walker logic [24] for late bisimilarity on systems with name binding, is an expressive fragment of our L_ω. Finally, §6.3 illustrates the importance of the technical assumption of strong local presentability in Theorem 4.4, on the example of unary algebras and homomorphisms.

6.1 Sets and Finitary Hennessy-Milner Logic

Let $\mathcal{C} = \mathcal{D} = \mathbf{Set}$, $F = G = 2^-$, and consider any finitary B on $\mathcal{C}$. A finitely presentable set is (isomorphic) to a finite cardinal $n \in \mathbb{N}$, a modality of arity n according to Definition 5.1 is a function

$$\lambda : B(2^n) \to 2,$$

and the syntax L_ω^+ can be described by the grammar:

$$\phi ::= [\lambda](\phi_1, \dots, \phi_n)$$

where $n \in \mathbb{N}$ and $\lambda : B(2^n) \to 2$. The logic L_ω is additionally quotiented by a straightforward equivalence of modalities ensuring that a modality that ignores some of its arguments is equal to a modality of a smaller arity.

Given $h : X \to BX$, the inductive definition (9) of $[\![\_]\!]_h : \Phi_{L_\omega^+} \to 2^X$ translates to:

$$\begin{aligned}
\llbracket [\lambda](\phi_1, \dots, \phi_n) \rrbracket_h &= 2^h(\rho_X([\lambda]([\![\phi_1]\!]_h, \dots, [\![\phi_n]\!]_h))) = \\
&= 2^h(2^{B\eta_X^{GF}}(2^{B2^{([\![\phi_1]\!]_h, \dots, [\![\phi_n]\!]_h)}}(\lambda))) = \\
&= \lambda \circ B2^{([\![\phi_1]\!]_h, \dots, [\![\phi_n]\!]_h)} \circ B\eta_X^{GF} \circ h = \\
&= \lambda \circ B([\![\phi_1]\!]_h, \dots, [\![\phi_n]\!]_h)^\flat \circ h = \\
&= w \circ B\left\langle [\![\phi_1]\!]_h, \dots, [\![\phi_n]\!]_h \right\rangle \circ h.
\end{aligned}$$

Note that $([\![\phi_1]\!]_h, \dots, [\![\phi_n]\!]_h) : n \to 2^X$ is a tuple of functions, and $\langle [\![\phi_1]\!]_h, \dots, [\![\phi_n]\!]_h \rangle : X \to 2^n$ is a function obtained by tupling.

The above syntax and semantics of $\Phi_{L_\omega^+}$ both correspond exactly to the polyadic coalgebraic modal logic of [30,18], which is thus a special case of the present approach. Also the result on the existence of expressive polyadic modal logic in [30] immediately follows from Corollary 4.7. Indeed, **Set** is cartesian closed and strongly locally finitely presentable, and any set with at least two elements is an internal cogenerator. Moreover, all functors on **Set** preserve monos with nonempty domains, and in [4] it was shown how to modify any functor on **Set** so that it preserve all monos, without a substantial change in its category of coalgebras.

For a specific application, consider $B = \mathcal{P}_\omega(A \times -)$ for a fixed set A of labels; B-coalgebras are finitely branching labelled transition systems. A B-modality according to Definition 5.1 is a function $w : \mathcal{P}_\omega(A \times 2^n) \to 2$ for $n \in \mathbb{N}$. Any such function can be presented as an expression built of negations, finite conjunctions, diamond modalities and placeholders, with an interpretation as in Hennessy-Milner logic. For example, the expression $\langle a \rangle(- \wedge \neg -) \wedge \neg \langle b \rangle -$ defines a function $w : \mathcal{P}_\omega(A \times 2^3) \to 2$. It is straightforward to see that any modality w can be described with such an expression. Formulas in L_ω^+ are built of such expressions, and the canonical connection $\rho : L_\omega^+ 2^- \Longrightarrow 2^{\mathcal{P}_\omega(A \times -)}$ is derived from the interpretation of them, for example:

$$\rho_X\left(\langle a \rangle(\phi \wedge \neg\psi) \wedge \neg\langle b \rangle \sigma\right)(\beta) = \mathtt{tt} \iff \begin{cases} \exists\,(a, x) \in \beta.\ \phi(x) = \mathtt{tt}, \psi(x) = \mathtt{ff} \text{ and} \\ \nexists\,(b, x) \in \beta.\ \sigma(x) = \mathtt{tt}. \end{cases}$$

The syntax L_ω^+ obviously relates to finitary Hennessy-Milner logic (1). It is easy to see that given an LTS $h : X \to \mathcal{P}_\omega(A \times X)$, the map $[\![\_]\!]_h$ defined as in (9) is the usual semantics of that logic. Thus finitary Hennessy-Milner logic is a special case of the present approach, and its expressivity follows from Corollary 4.5.

Polyadic modalities used above are admittedly quite complicated, which makes L^+_ω rather awkward, given that it is little more than finitary Hennessy-Milner logic. One can alleviate this problem by choosing a fragment of L^+_ω and using Theorem 4.2 to show that it is still expressive. For example, consider a logic L defined by the grammar

$$\phi ::= \langle a \rangle \bigwedge_{j=1..n} \psi_j \qquad\qquad \psi ::= \phi \mid \neg\phi \tag{16}$$

that is, by the functor $L\Phi = A \times \sum_{n \in N} (2 \times \Phi)^n$ on **Set**. The obvious inclusion of L in L^+_ω determines a connection $\rho : L2^- \Longrightarrow 2^{\mathcal{P}_\omega(A\times -)}$ as shown in §5; explicitly, the adjoint connection $\rho^\star : \mathcal{P}_\omega(A \times 2^-) \Longrightarrow 2^{L-}$ is defined by:

$$\rho^\star_\Phi(\beta)(\langle a \rangle(\psi_1 \wedge \cdots \wedge \psi_n)) = \mathtt{tt} \iff \exists (a,y) \in \beta. \forall i = 1..n. \begin{cases} \psi_i = \phi_i \Rightarrow y(\phi_i) = \mathtt{tt} \\ \psi_i = \neg\phi_i \Rightarrow y(\phi_i) = \mathtt{ff} \end{cases}$$

By Theorem 4.2, to prove L expressive it is enough to show that $\rho^\star$ is pointwise monic. To this end, for any distinct $\beta, \gamma \in B2^\Phi$ one needs to find $a \in A$, $n \in \mathbb{N}$, $\phi_i \in \Phi$ and $\psi_i \in \{\phi_i, \neg\phi_i\}$ such that

$$\rho^\star_\Phi(\beta)(\langle a \rangle(\psi_1 \wedge \cdots \wedge \psi_n)) \neq \rho^\star_\Phi(\gamma)(\langle a \rangle(\psi_1 \wedge \cdots \wedge \psi_n))$$

Without loss of generality assume $\beta \not\subseteq \gamma$ and fix any $(a,x) \in \beta$ such that $(a,x) \notin \gamma$. Define $\delta \subseteq 2^\Phi$ by:

$$\delta = \{y \,:\, (a,y) \in \gamma\}$$

Obviously, δ is finite. Pick $n = |\delta|$. For any $y \in \delta$ we have $y \neq x$, hence one can choose an element $\phi_y \in \Phi$ such that $x(\phi_y) \neq y(\phi_y)$. Define $\phi \in L\Phi$ by:

$$\phi = \langle a \rangle \bigwedge_{y \in \delta} \psi_y$$

where $\psi_i = \phi_i$ iff $x(\phi_y) = \mathtt{tt}$ and $\psi_i = \neg\phi_i$ otherwise. It is straightforward to check that

$$\rho^\star_\Phi(\beta)(\phi) = \mathtt{tt} \qquad \text{and} \qquad \rho^\star_\Phi(\gamma)(\phi) = \mathtt{ff}$$

therefore $\rho^\star_\Phi(\beta) \neq \rho^\star_\Phi(\gamma)$ and $\rho^\star_\Phi$ is pointwise monic.

6.2 Nominal Sets and Systems with Name Binding

We begin by recalling the basics of nominal sets. For more information, see e.g. [12].

Throughout this section, fix a countably infinite set $\mathcal{N} = \{a, b, c, \ldots\}$ of names. An action of the symmetric group $Sym(\mathcal{N})$ (i.e., the group of permutations of $\mathcal{N}$) on a set X is a function $\bullet_X : Sym(\mathcal{N}) \times X \to X$ such that for any $x \in X$ there is $\mathrm{id}_\mathcal{N} \bullet_X x = x$ and, for any $\pi, \sigma \in Sym(\mathcal{N})$, that $(\pi\sigma) \bullet_X x = \pi \bullet_X (\sigma \bullet_X x)$. A set $\mathcal{N}_0 \subseteq \mathcal{N}$ *supports* an $x \in X$ if for all π that fix $\mathcal{N}_0$ there is $\pi \bullet_X x = x$. A tuple $(X, \bullet_X)$, is a *nominal set*, denoted by X, if every element of X is supported by a finite set. In a nominal set every element x has the smallest supporting set, denoted $supp(x)$, and $a\#x$, read "a is fresh in x", means $a \notin supp(x)$. **Nom** is the category of nominal sets with *equivariant maps*, i.e., functions $f : X \to Y$ such that $f(\pi \bullet_X x) = \pi \bullet_Y f(x)$ for all $x \in X$ and $\pi \in \mathit{Sym}(\mathcal{N})$.

The set $\mathcal{N}$ is nominal, with the action defined by $\pi \bullet_{\mathcal{N}} a = \pi(a)$. For any nominal set X, the nominal *abstraction set* $[\mathcal{N}]X$ has the carrier $(\mathcal{N} \times X)/\sim_{[\mathcal{N}]X}$, where $(a, x) \sim_{[\mathcal{N}]X} (b, y)$ if and only if for all $c \in \mathcal{N}$ such that $c\#x$ and $c\#y$ there is $[a \leftrightarrow c] \bullet_X x = [b \leftrightarrow c] \bullet_X y$. This construction extends to a functor $[\mathcal{N}]$ on **Nom**.

Nom has colimits and finite limits calculated as in **Set**. Also the covariant finite powerset functor extends to a functor $\mathcal{P}_\omega$ on **Nom**, with $\mathit{Sym}(\mathcal{N})$-action calculated pointwise. **Nom** is also cartesian closed, and the exponential X^Y is the set of (not necessarily equivariant) functions from Y to X with an action defined by

$$(\pi \bullet_{X^Y} f)(y) = \pi \bullet_X (f(\pi^{-1} \bullet_Y y))$$

for all $\pi \in \mathit{Sym}(\mathcal{N})$ and $y \in Y$, restricted to functions that are finitely supported with respect to this action, i.e., those functions for which there exists a finite $\mathcal{N}_0 \subseteq \mathcal{N}$ such that for all π that fix $\mathcal{N}_0$ there is $f(\pi \bullet_Y y) = \pi \bullet_X f(y)$ for all $y \in Y$.

In the following two particular types of exponentials will be used. First, let 2 be the set $\{\mathtt{tt}, \mathtt{ff}\}$ with the trivial action. For any X, a function $f : X \to 2$ is supported by $\mathcal{N}_0$ if and only if $f(\pi \bullet_X x) = f(x)$ for each $x \in X$ and each π that fixes $\mathcal{N}_0$. The set 2^X consists of functions satisfying this condition for a finite $\mathcal{N}_0$. It is straightforward to check that 2 is an internal cogenerator for the cartesian closed structure of **Nom**. Note that 2 is *not* a cogenerator in **Nom**.

Now consider the nominal set $X^{\mathcal{N}}$ for a given set X. It is not difficult to check that a function $f : \mathcal{N} \to X$ is supported by $\mathcal{N}_0 \subseteq \mathcal{N}$ if and only if:

- for all $a \in \mathcal{N}$, $\mathcal{N}_0 \cup \{a\}$ supports $f(a)$, and
- for all $a, b \in \mathcal{N} \setminus \mathcal{N}_0$, $(a, f(a)) \sim_{[\mathcal{N}]X} (b, f(b))$.

It follows that every function in $X^{\mathcal{N}}$ is uniquely determined by a finite partial function $\bar{f} : \mathcal{N} \xrightarrow{\text{fin}} X$ together with an element $\hat{f} \in [\mathcal{N}]X$. Indeed, given these data, the function $f : \mathcal{N} \to X$ defined by:

$$f(a) = \begin{cases} \bar{f}(a) \text{ if } a \in \mathrm{dom}(\bar{f}) \\ y \in X \text{ s.t. } (a, y) \in \hat{f}, \text{ otherwise} \end{cases}$$

(here y is uniquely determined) is finitely supported, and every finitely supported function can be obtained this way.

The *free nominal set* over a set Z is $\mathit{Sym}(\mathcal{N}) \times Z$ with the evident $\mathit{Sym}(\mathcal{N})$-action. A nominal set is finitely presentable in **Nom** if and only if it is isomorphic to the free nominal set over a finite set, quotiented by a finite set of equations. **Nom** is locally finitely presentable. A nominal set X is finitely generated if and only if there exists a finite $Z \subseteq X$ that *generates* X, i.e., such that for all $x \in X$ there exist $z \in Z$, $\pi \in \mathit{Sym}(\mathcal{N})$ such that $x = \pi \bullet_X z$.

Nom is strongly locally finitely presentable. The proof of this proceeds as follows:

(i) *In every finitely generated nominal set X, every finite $\mathcal{N}_0 \subseteq \mathcal{N}$ supports only finitely many elements.* To prove this, let a finite Z generate X and show that for a fixed $z \in Z$ there are only finitely many elements of the form $\pi \bullet_X z$ supported by $\mathcal{N}_0$. To this end, consider any $\pi \in \mathit{Sym}(\mathcal{N})$ and observe that if $\mathcal{N}_0$ supports $\pi \bullet_X z$ then $\overrightarrow{\pi}(\mathit{supp}(z)) = \mathit{supp}(\pi \bullet_X z) \subseteq \mathcal{N}_0$. Moreover, for any $\sigma \in \mathit{Sym}(\mathcal{N})$, if π and σ agree on $\mathit{supp}(z)$ then $\pi^{-1}\sigma$ fixes $\mathit{supp}(z)$, hence $\pi \bullet_X z = \sigma \bullet_X z$. Altogether, $\pi \bullet_X z \neq \sigma \bullet_X z$ are both supported by $\mathcal{N}_0$ only if π and σ are different maps when restricted to $\mathit{supp}(z)$, and if they both map $\mathit{supp}(z)$ to subsets of $\mathcal{N}_0$. But there are only finitely many such maps.

(ii) *For any X, Y finitely generated, there are only finitely many equivariant maps from X to Y.* To prove this note that for any equivariant $f : X \to Y$, for any $x \in X$ there is $\mathit{supp}(f(x)) \subseteq \mathit{supp}(x)$. This, together with (i) applied to Y, means that any fixed $x \in X$ can be mapped to only finitely many elements of Y with an equivariant map. Since X is finitely generated, an equivariant map from X to Y is determined by how it acts on a finite subset of X, hence there are only finitely many such maps.

(iii) *$\mathit{Sym}(\mathcal{N})$ as a nominal set, i.e., the free nominal set on one generator $\star$, is a generator (in the categorical sense of the word) in* **Nom**. Indeed, take any equivariant $f, g : X \to Y$. If $f \neq g$, take any $x \in X$ such that $f(x) \neq g(x)$ and take the equivariant $h : \mathit{Sym}(\mathcal{N}) \to X$ determined by $h(\star) = x$. Then $f \circ h \neq g \circ h$.

(iv) **Nom** *is strongly locally finitely presentable.* In the situation of (12), consider any $f, g : \mathit{Sym}(\mathcal{N}) \to X$ such that $f \neq g$. Since limiting cones are jointly monic, and m is a mono, there is an $i_{f,g} \in I$ such that $c_{i_{f,g}} \circ m \circ f \neq c_{i_{f,g}} \circ m \circ g$. By (ii), there are only finitely many choices of f and g. Since the diagram is cofiltered, take i to be a common bound of all $i_{f,g}$. Then obviously $c_i \circ m \circ f \neq c_i \circ m \circ g$ for all $f \neq g$. Now take any nominal set Z with two functions $h, k : Z \to X$ such that $c_i \circ m \circ h = c_i \circ m \circ k$. By the previous observation, for any map $l : \mathit{Sym}(\mathcal{N}) \to Z$ there must be $h \circ l = k \circ l$. But $\mathit{Sym}(\mathcal{N})$ is a generator by (iii), therefore $h = k$.

Consider the following functor on **Nom**:

$$BX = \mathcal{P}_\omega(\mathcal{N} \times X^{\mathcal{N}} + \mathcal{N} \times \mathcal{N} \times X + \mathcal{N} \times [\mathcal{N}]X + X).$$

This is the functor for late bisimulation on systems with name binding (see [11,10,9,5] for a comparison), i.e., observational equivalence coincides with late bisimilarity. B is finitary on **Nom**. To apply the framework of polyadic modal logic, choose $\mathcal{C} = \mathcal{D} = \mathbf{Nom}$ and $F = G = 2^{-}$. As we have seen, all assumptions of Theorem 4.4 hold, therefore the canonical finitary logic L_ω^+ is expressive for late bisimilarity. However, modalities used in L_ω^+ are quite complicated; we therefore present a simpler logic L and use Theorem 4.2 to prove its expressivity, as in §6.1. Specifically, we choose

$$L\Phi = \mathcal{N} \times \Sigma_{n \in \mathbb{N}}(\mathcal{N} \times 2 \times \Phi)^n + \mathcal{N} \times \mathcal{N} \times \bar{\Phi} + \mathcal{N} \times [\mathcal{N}]\bar{\Phi} + \bar{\Phi}$$

where $\bar{\Phi}$ is shorthand for $\Sigma_{n\in\mathbb{N}}(2\times\Phi)^n$. It is obvious how to present this functor with the grammar:

$$\begin{aligned}\phi \ ::=\ & \langle a\rangle(\langle b_1\rangle\psi_1 \wedge\cdots\wedge\langle b_m\rangle\psi_m)\\ & \mid \langle\bar{a}b\rangle(\psi_1\wedge\cdots\wedge\psi_m)\\ & \mid \langle\bar{a}(b)\rangle(\psi_1\wedge\cdots\wedge\psi_m)\\ & \mid \langle\tau\rangle(\psi_1\wedge\cdots\wedge\psi_m)\\ \psi \ ::=\ & \phi \mid \neg\phi\end{aligned}$$

where $a, b, b_i \in \mathcal{N}$ and b binds in the ψ_i in the third production.

A connection $\rho : L2^- \Longrightarrow 2^{B-}$ is determined, at a nominal set X, by its transpose $L2^X \times BX \to 2$, i.e. an equivariant relation $\models\ \subseteq BX \times L2^X$ defined by cases as follows. Here for simplicity negations are ignored, but it is obvious how to extend the definition to the full grammar:

$$\begin{aligned}
\beta \models \langle a\rangle(\langle b_1\rangle\phi_1\wedge\cdots\wedge\langle b_m\rangle\phi_m) &\iff \exists\iota_1(a,f)\in\beta.\forall i=1..m.\ \phi_i(f(b_i))=\mathtt{tt}\\
\beta \models \langle\bar{a}b\rangle(\phi_1\wedge\cdots\wedge\phi_m) &\iff \exists\iota_2(a,b,x)\in\beta.\forall i=1..m.\ \phi_i(x)=\mathtt{tt}\\
\beta \models \langle\bar{a}(b)\rangle(\phi_1\wedge\cdots\wedge\phi_m) &\iff \exists\iota_3(a,[(b,x)]_{[\mathcal{N}]X})\in\beta.\forall i=1..m.\ \phi_i(x)=\mathtt{tt}\\
\beta \models \langle\tau\rangle(\phi_1\wedge\cdots\wedge\phi_m) &\iff \exists\iota_4(x)\in\beta.\forall i=1..m.\ \phi_i(x)=\mathtt{tt}
\end{aligned}$$

where $\phi_i \in 2^X$, $f \in X^{\mathcal{N}}$, and the ι_i are the coproduct inclusions in BX.

To prove L expressive, by Theorem 4.4, it is enough to show that $\rho^\star$ is pointwise monic. The proof is much the same as in §6.1: for a nominal set Φ, and for any distinct $\beta,\gamma\in B2^\Phi$, without lost of generality assume $\beta\not\subseteq\gamma$ and pick any $v\in\beta\setminus\gamma$. Assume that $v=\iota_1(a,f)$ with $f\in(2^\Phi)^{\mathcal{N}}=2^{\Phi\times\mathcal{N}}$. Define $\delta\subseteq 2^{\Phi\times\mathcal{N}}$ by:

$$\delta = \{\, g \mid\ \iota_1(a,g)\in\gamma\,\}\,.$$

Obviously δ is finite. For any $g\in\delta$ we have $g\neq f$, hence for some $a_g\in\mathcal{N}$ and $\phi_g\in\Phi$ one has $f(a_g)(\phi_g)\neq g(a_g)(\phi_g)$. Define $\phi\in L\Phi$ by:

$$\phi = \langle a\rangle(\bigwedge_{g\in\delta}\langle a_g\rangle\psi_g)$$

where $\psi_g=\phi_g$ if $f(a_g)(\phi_g)=\mathtt{tt}$ and $\psi_g=\neg\phi_g$ otherwise. It is straightforward to check that $\rho^\star_\Phi(\beta)(\phi)=\mathtt{tt}$ and $\rho^\star_\Phi(\gamma)(\psi)=\mathtt{ff}$, therefore $\rho^\star_\Phi(\beta)\neq\rho^\star_\Phi(\gamma)$.

On the other hand, assume $v=\iota_3(a,[(b,x)]_{[\mathcal{N}]2^\Phi})$ and define $\delta\subseteq[\mathcal{N}]2^\Phi$ by:

$$\delta = \{\, g \mid\ \iota_3(a,g)\in\gamma\,\}\,.$$

Again δ is finite. For any $g\in\delta$ we have $g\neq[(b,x)]_{[\mathcal{N}]2^\Phi}$, hence one can choose some $c\in\mathcal{N}$, $x_g\neq y_g\in 2^\Phi$ such that for all $g\in\delta$ $(b,x)\sim_{[\mathcal{N}]2^\Phi}(c,x_g)$ and $(c,y_g)\in g$.

Further, one can choose a $\phi_g \in \Phi$ such that $x_g(\phi_g) \neq y_g(\phi_g)$. Define $\phi \in L\Phi$ by:

$$\phi = \langle \bar{a}(c) \rangle \bigwedge_{g \in \delta} \psi_g$$

where $\psi_g = \phi_g$ if $x_g(\phi_g) = \mathtt{tt}$ and $\psi_g = \neg\phi_g$ otherwise. It is straightforward to check that $\rho^\star_\Phi(\beta)(\phi) = \mathtt{tt}$ and $\rho^\star_\Phi(\gamma)(\psi) = \mathtt{ff}$, therefore $\rho^\star_\Phi(\beta) \neq \rho^\star_\Phi(\gamma)$. The other two cases of v are easier and altogether show that $\rho^\star_\Phi$ is monic. Expressivity of L follows from Theorem 4.2.

In fact, the logic L can be easily translated to the logic $\mathcal{LM}$ of [24], where it is proved to be expressive for late bisimilarity. The only nontrivial bit of the translation is

$$\langle a \rangle (\langle b_1 \rangle \psi_1 \wedge \cdots \wedge \langle b_m \rangle \psi_m) \quad \mapsto \quad \langle a(c) \rangle^L ([c = b_1]\psi_1 \wedge \cdots \wedge [c = b_m]\psi_m)$$

where c is any variable fresh in $\psi_1, \ldots, \psi_m$. The image of the translation is a proper subset of of $\mathcal{LM}$ (for example, match operators can occur only directly under late input modalities), but by Theorem 4.2 it is an expressive subset. Indeed, a close inspection of the proof of Theorem 1 in [24] shows that only formulas of this form are needed for the expressivity of $\mathcal{LM}$.

6.3 Unary Algebras and Distant Transition Systems

This example shows that the assumption of strong local presentability cannot be dropped from Theorem 4.4.

A unary algebra X is a set, also denoted X and called the carrier, with a function $s_X : X \to X$, called the successor function of the algebra. A homomorphism from X to Y is a function f between the respective carriers such that $f \circ s_X = s_Y \circ f$. The category of unary algebras and their homomorphisms is denoted **Un**.

For a unary algebra X, and a subset $Y \subseteq X$, the subalgebra of X generated by Y is denoted and defined by $\overline{Y} = \{\, s^n_X(y) \mid y \in Y, n \in \mathbb{N} \,\}$ (we omit X in this notation as it will always be clear from the context.) A unary algebra X is finitely presentable if and only if is finitely generated, i.e., if there is a finite subset $Y \subseteq X$ such that $\overline{Y} = X$. **Un** is locally finitely presentable, but *not* strongly locally finitely presentable (see [1]).

Un is cartesian closed, with Y^X an algebra of homomorphisms $f : \mathbb{N} \times X \to Y$ (here $\mathbb{N}$ is the unary algebra of natural numbers and incrementation), with the successor defined by $s_{X^Y}(f)(n, x) = f(n + 1, x)$. However, this closed symmetric monoidal structure is not convenient for our purposes; in particular, the algebra $2 = 1 + 1$, an obvious candidate for the algebra of logical values, is not an internal cogenerator for this structure. We therefore choose another contravariant adjunction on **Un**, not based on any closed symmetric monoidal structure. Define $\mathbb{P} : \mathbf{Un} \to$

$\mathbf{Un}^{op}$ by:

$$\begin{aligned} \mathbb{P}X &= \mathcal{P}X, \\ s_{\mathbb{P}X}(\Phi) &= \{\, x \in X \mid s_X(x) \in \Phi \,\} \text{ for } \Phi \subseteq X, \\ \mathbb{P}f &= \overleftarrow{f} \qquad\qquad\qquad\quad \text{for } f : X \to Y. \end{aligned}$$

To check that $\mathbb{P}f$ is a homomorphism, calculate for $f : X \to Y$, $\Phi \subseteq Y$:

$$\begin{aligned} s_{\mathbb{P}X}(\mathbb{P}f(\Phi)) &= s_{\mathbb{P}X}(\overleftarrow{f}(\Phi)) = \{\, x \in X \mid f(s_X(x)) \in \Phi \,\} = \\ &= \{\, x \in X \mid s_Y(f(x)) \in \Phi \,\} = \overleftarrow{f}\{\, y \in Y \mid s_Y(y) \in \Phi \,\} = \mathbb{P}f(s_{\mathbb{P}Y}(\Phi)). \end{aligned}$$

$\mathbb{P}$ is a contravariant self-adjoint. Indeed, for any homomorphism $f : X \to \mathbb{P}Y$, define $f^\flat : Y \to \mathbb{P}X$ by:

$$f^\flat(y) = \{\, x \in Y \mid y \in f(x) \,\}.$$

To check that $f^\flat$ is a homomorphism, calculate:

$$\begin{aligned} f^\flat(s_Y(y)) &= \{\, x \in X \mid s_Y(y) \in f(x) \,\} = \{\, x \in X \mid y \in s_{\mathbb{P}Y}(f(x)) \,\} = \\ &= \{\, x \in X \mid y \in f(s_X(x)) \,\} = s_{\mathbb{P}X}\{\, x \in X \mid y \in f(x) \,\} = s_{\mathbb{P}X}(f^\flat(y)). \end{aligned}$$

The bijectivity of the construction $f \mapsto f^\flat$ follows from its bijectivity on sets. Maps in $\mathbf{Un}$ are monos if and only if they are injective on carriers, and pointwise monicity of the unit $\eta^{\mathbb{P}\mathbb{P}}$ follows from its pointwise monicity on sets.

Let $\mathcal{P}_\omega : \mathbf{Un} \to \mathbf{Un}$ be the "finitely covered powerset" functor, mapping an algebra to the set of all subsets of finitely generated subalgebras:

$$\begin{aligned} \mathcal{P}_\omega X &= \left\{\, Z \subseteq X \;\middle|\; Z \subseteq \overline{Y} \text{ for some finite } Y \subseteq X \,\right\} \\ s_{\mathcal{P}_\omega X}(Y) &= \{\, s_X(y) \mid y \in Y \,\} \\ \mathcal{P}_\omega f &= \overrightarrow{f} \end{aligned}$$

The above is well defined since for any $f : X \to X'$ and $Z \subseteq \overline{Y} \subseteq X$ there is

$$\mathcal{P}_\omega f(Z) \subseteq \overrightarrow{f}(\overline{Y}) = \overline{\overrightarrow{f}(Y)},$$

hence $\mathcal{P}_\omega f(Z) \in \mathcal{P}_\omega X'$. To check that $\mathcal{P}_\omega f$ is a homomorphism, calculate for $f : X \to Z$, $Y \in \mathcal{P}_\omega X$:

$$\mathcal{P}_\omega f(s_{\mathcal{P}_\omega X}(Y)) = \overrightarrow{f}\{\, s_X(y) \mid y \in Y \,\} = \{\, s_Z(f(y)) \mid y \in Y \,\} = s_{\mathcal{P}_\omega Z}(\mathcal{P}_\omega f(Y)).$$

To check that $\mathcal{P}_\omega$ is finitary on $\mathbf{Un}$, consider any $f : Y \to \mathcal{P}_\omega X$ with Y finitely generated. For each $y \in Y$, let $G_y \subseteq X$ be a finite set such that $f(y) \subseteq \overline{G_y}$, and take G be the (finite) union of all G_y's taken over a set of y's generating Y. Let Z

be the subalgebra of X generated by G. Then for each $y \in Y$, $f(y) \in \mathcal{P}_\omega Z$ and f factorizes through the inclusion $\mathcal{P}_\omega Z \to \mathcal{P}_\omega X$.

We will consider coalgebras $h : X \to BX = \mathcal{P}_\omega(A \times X)$ for a fixed unary algebra A of labels. Such a coalgebra can be seen a labelled transition system $(X, A, \longrightarrow)$ defined on the carriers of X and A, together with successor functions $s_X : X \to X$ and $s_A : A \to A$ such that:

(i) $\forall x, y \in X, a \in A.\ x \xrightarrow{a} y \Longrightarrow s_X(x) \xrightarrow{s_A(a)} s_X(y)$,

(ii) $\forall x, y \in X, a \in A.\ s_X(x) \xrightarrow{a} y \Longrightarrow \exists z \in X, b \in A.\ y = s_X(z), a = s_A(b), x \xrightarrow{b} z$

(iii) $\forall x \in X.\ \exists$finite $A' \subseteq A, X' \subseteq X.\ (\forall y \in X, a \in A.\ x \xrightarrow{a} y \Longrightarrow$
$\exists n \in \mathbb{N}, a' \in A', y' \in Y'.\ a = s_A^n(a'), y = s_X^n(y'), x \xrightarrow{a'} y')$.

These transition systems are introduced here to show the technical importance of strong local presentability assumption, and are not expected to have any practical applications. However, to get some intuition, one might see the elements of X and A as processes and actions observed from some distance, with the action of s_X and s_A corresponding to taking a "step back", which can make some processes or actions appear identical (if, for example, $s_A(a) = s_A(b)$ for $a \neq b$). This intuition explains conditions (i) and (ii) above, and condition (iii) is analogous to the finite branching condition of ordinary LTSs, with the additional possibility of a process moving "away" by a nondeterministically chosen distance with each action.

Note that B is finitary. For a finitely generated algebra Ψ, a B-modality of arity Ψ according to Definition 5.1 is a predicate $\lambda \subseteq B\mathbb{P}\Psi$, and the syntax L^+_ω can be described by the grammar:

$$\phi ::= [\lambda](\phi_1, \ldots, \phi_n)$$

where λ is of arity Ψ, n is the number of generators of Ψ, and for $\phi_i \in \Phi$, the tuple $(\phi_1, \ldots, \phi_n) : n \to \Phi$ represents its unique extension $\overline{(\phi_1, \ldots, \phi_n)} : \Psi \to \Phi$, i.e., a tuple of arity Ψ. Moreover,

$$s_{L^+_\omega}([\lambda](\phi_1, \ldots, \phi_n)) = [s_{\mathbb{P}B\mathbb{P}\Psi}(\lambda)](\phi_1, \ldots, \phi_n).$$

As in §6.1, L_ω is additionally quotiented by a straightforward equivalence of modalities. Given a coalgebra $h : X \to BX$, the inductive definition (9) of $[\![\_]\!]_h : \Phi_{L^+_\omega} \to \mathbb{P}X$ translates to:

$$\begin{aligned} x \in [\![[\lambda](\phi_1, \ldots, \phi_n)]\!]_h \iff{} & x \in \mathbb{P}h(\rho_X([\lambda]([\![\phi_1]\!]_h, \ldots, [\![\phi_n]\!]_h))) \iff \\ \iff{} & h(x) \in \mathbb{P}B\eta_X^{\mathbb{P}\mathbb{P}}(\mathbb{P}B\mathbb{P}\overline{([\![\phi_1]\!]_h, \ldots, [\![\phi_n]\!]_h)}(\lambda)) \iff \\ \iff{} & B\mathbb{P}\overline{([\![\phi_1]\!]_h, \ldots, [\![\phi_n]\!]_h)}(B\eta_X^{\mathbb{P}\mathbb{P}}(h(x))) \in \lambda \iff \\ \iff{} & B\overline{([\![\phi_1]\!]_h, \ldots, [\![\phi_n]\!]_h)}^{\flat}(h(x)) \in \lambda \iff \\ \iff{} & \beta_x^h \in \lambda \end{aligned} \tag{17}$$

where $\beta_x^h \in B\mathbb{P}\Psi$ is defined by:

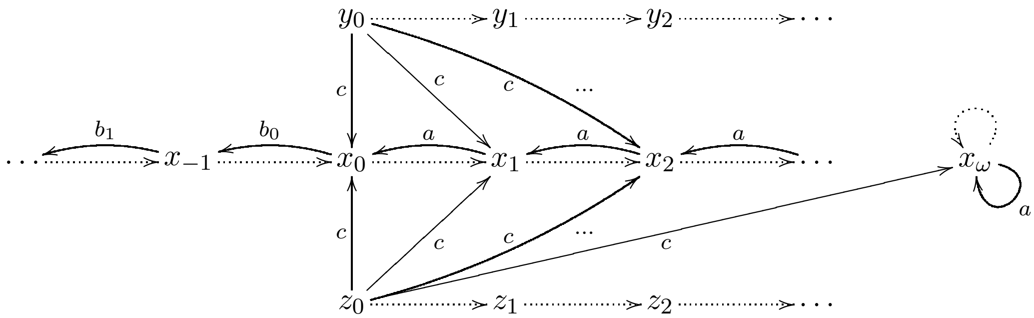
$$\beta_x^h = \left\{ \left(a, \left\{ s_\Psi^k(g_i) \,\middle|\, k \in \mathbb{N}, y \in s_\Phi^k(\phi_i) \right\} \right) \,\middle|\, (a,y) \in h(x) \right\}$$

where $g_i \in \Psi$ is the i'th generator of Ψ, i.e., $\overline{(\phi_1,\ldots,\phi_n)}(g_i) = \phi_i$.

It turns out that all these complicated modalities do not ensure the expressivity of L_ω^+. For a counterexample, consider the following algebra A of labels:

$$\begin{tikzcd}
\cdots \arrow[r, dotted] & b_3 \arrow[r, dotted] & b_2 \arrow[r, dotted] & b_1 \arrow[r, dotted] & b_0 \arrow[r, dotted] & a \arrow[loop right, dotted] & & c \arrow[loop right, dotted]
\end{tikzcd}$$

(the action of s_A is indicated with dotted arrows), and the coalgebra $h : X \to \mathcal{P}_\omega(A \times X)$ described by the graph:



where transitions are indicated with solid arrows, and the transitions of $y_1, y_2, \ldots$ and $z_1, z_2, \ldots$, determined by those of y_0 and z_0 by condition (i) and (ii) above, are omitted for clarity. Note that neither A nor X are finitely generated.

No coalgebra morphism from h identifies y_0 and z_0. To see this, note that no coalgebra morphism identifies x_ω with x_n for any $n \in \mathbb{N}$ (this is easily proved by induction over n). Since z_0 can do a c-labelled step to x_ω and y_0 cannot, the two processes are not behaviourally equivalent. However, no formula from L_ω^+ distinguishes them. The proof of this is similar to the classical proof of the inexpressivity of finitary HML with respect to infinitely branching LTSs, and it relies on the fact that every formula in $\phi \in \Phi_{L_\omega^+}$ is *x-continuous*, meaning that for some $n_\phi \in \mathbb{N}$, for all $m > n_\phi$, $x_m \in [\![\phi]\!]_h \iff x_\omega \in [\![\phi]\!]_h$. Indeed, is is straightforward to show that:

(i) If ϕ is x-continuous than so is $s(\phi)$, using the fact that $[\![\_]\!]_h$ is a homomorphism, and it is enough to take $n_{s(\phi)} = n_\phi$.

(ii) A set Ψ of formulas finitely generated by a set of x-continuous formulas is x-continuous; here take $n_\Psi = \max(n_\phi)$, with ϕ ranging over the set of generators.

(iii) Every formula is x-continuous. This is proved by induction using (17): for $\phi = [\lambda](\phi_1,\ldots,\phi_n)$ with λ of arity Ψ, choose $n_\phi = n_\Psi + 1$ and show that $\beta^h_{x_{n_\phi}} = \beta^h_\omega$.

(iv) For every formula ϕ, $y_0 \in [\![\phi]\!]_h \iff z_0 \in [\![\phi]\!]_h$. This follows from (17), since by (iii) one has $\beta^h_{y_0} = \beta^h_{z_0}$.

This means that L_ω^+ is not expressive for B-coalgebras, hence neither is L_ω. This shows that the assumption of strong local presentability cannot be dropped from Theorem 4.4.

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