

# Higher Dimensional Categories: Induction on Extensivity<sup>1</sup>

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## Abstract

In this paper, we explore, enrich, and otherwise mildly generalise a prominent definition of weak  $n$ -category by Batanin, as refined by Leinster, to give a definition of weak  $n$ -dimensional  $\mathcal{V}$ -category, with a view to applications in programming semantics. We require  $\mathcal{V}$  to be locally presentable and to be (infinitarily) extensive, a condition which ensures that coproducts are suitably well-behaved. Our leading example of such a  $\mathcal{V}$  is the category  $\omega\text{-}\mathbf{Cpo}$ ,  $\omega\text{-}\mathbf{Cpo}$ -enriched bicategories already having been used in denotational semantics.

We illuminate the implicit use of recursion in Leinster's definition, generating the higher dimensions by a process of repeated enrichment. The key fact is that if  $\mathcal{V}$  is a locally presentable and extensive category, then so are the categories of small  $\mathcal{V}$ -graphs and small  $\mathcal{V}$ -categories. Iterating, this produces categories of  $n$ -dimensional  $\mathcal{V}$ -graphs and strict  $n$ -dimensional  $\mathcal{V}$ -categories that are also locally presentable and extensive. We show that the free strict  $n$ -dimensional  $\mathcal{V}$ -category monad on the category of  $n$ -dimensional  $\mathcal{V}$ -graphs is cartesian. This, along with results due to Garner, allows us to follow Batanin and Leinster's approach for defining weak  $n$ -categories. In the case that  $\mathcal{V} = \mathbf{Set}$ , the resulting definition of weak  $n$ -dimensional  $\mathcal{V}$ -category agrees with Leinster's definition.

**Keywords:** higher dimensional category, enriched category, extensive category, cartesian monad

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# 1 Introduction

Bicategories have been used for many years to model computational phenomena such as concurrency [6] and binders [18]. The collectivity, **Bicat**, of bicategories has the structure of a tricategory, which have occasionally appeared explicitly [17] and more often implicitly [6] in the programming semantics literature. But what is a weak  $n$ -category in general? Strict  $n$ -categories for arbitrary  $n$  have been used to model concurrency [19] and in connection with rewriting [12]. So it seems only a matter of time until weak  $n$ -categories for arbitrary  $n$  prove to be of value for programming semantics too. Here, we explore, enrich, and otherwise mildly generalise a prominent definition by Batanin [1], as refined by Leinster [16], together with the surrounding theory: they assumed knowledge of sophisticated mathematics, glossed over the inherent recursion, and did not consider the concerns of programming semantics.

Batanin and Leinster’s definition of weak  $n$ -category starts with the free strict  $n$ -category monad  $T^{(n)}$ , induced from the canonical adjunction between the category  $n\text{-}\mathbf{Gph}$  of  $n$ -graphs and the category  $n\text{-}\mathbf{Cat}$  of strict  $n$ -categories. The monad  $T^{(n)}$  is cartesian (see Definition 3.1). For any cartesian monad  $T$  on any category  $\mathcal{C}$  with finite limits, the slice category  $\mathcal{C}/T1$  possesses a canonical monoidal structure (Proposition 5.1). A monoid in that monoidal category is called a  $T$ -operad [16] or sometimes a  $T$ -club [14] or a  $T$ -category [3]. If  $O = ((\text{ar}0 : O \rightarrow T1), m, e)$  is a  $T$ -operad, an  $O$ -algebra consists of an object  $X$  of  $\mathcal{C}$  together with an action of  $O$  on  $X$ . Leinster introduces the notion of *contraction* (Definition 6.1) on  $T^{(n)}$ -operads, and defines a weak  $n$ -category to be an algebra for the initial  $T^{(n)}$ -operad with contraction.

The heart of the induction implicit in Batanin and Leinster’s work is simpler than one might imagine. The key fact is that if  $\mathcal{V}$  is an (infinitarily) extensive category with finite limits (see Definition 2.1), then so are the categories of small  $\mathcal{V}$ -graphs (Definition 2.2) and of small  $\mathcal{V}$ -categories. Iterating, this produces the categories  $\mathcal{V}\text{-}\mathbf{Gph}^{(n)}$  (Definition 2.3, enriching  $n\text{-}\mathbf{Gph}$ ) and  $\mathcal{V}\text{-}\mathbf{Cat}^{(n)}$  (Definition 2.4, enriching  $n\text{-}\mathbf{Cat}$ ) that are also extensive and have finite limits. Extensivity enables one to show that the forgetful functor from  $\mathcal{V}\text{-}\mathbf{Cat}^{(n)}$  to  $\mathcal{V}\text{-}\mathbf{Gph}^{(n)}$  admits a left adjoint, and that the induced monad  $T^{(n)}$  is cartesian.

We adapt Garner’s conceptual formulation [10] of the notion of contraction to the enriched setting. A *weak  $n$ -dimensional  $\mathcal{V}$ -category* is then defined as an algebra for the initial  $T^{(n)}$ -operad with contraction. To ensure that the initial  $T^{(n)}$ -operad with contraction actually exists, we impose a mild size condition on  $\mathcal{V}$ , that of being locally presentable. This additional condition is harmless to our inductive process, since whenever  $\mathcal{V}$  is locally presentable, so is  $\mathcal{V}\text{-}\mathbf{Gph}$ . Letting  $\mathcal{V}$  be the category  $\omega\text{-}\mathbf{Cpo}$  allows the modelling of recursion, see for instance the use of  $\omega\text{-}\mathbf{Cpo}$ -enriched

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bicategories in [18], axiomatising binders [9].

In Section 2, we recall and further develop the notion of infinitary extensivity that we studied broadly in this context in [8], extending the finitary extensivity condition of Lawvere as developed by Carboni, Lack and Walters in [5] and by Cockett in [7]. In Section 3, we use the extensivity condition to show that the forgetful functor from  $\mathcal{V}\text{-Cat}$  to  $\mathcal{V}\text{-Gph}$  generates a cartesian monad on  $\mathcal{V}\text{-Gph}$ , and in Section 4, we extend that result to higher dimensions, yielding a cartesian monad  $T^{(n)}$  on  $\mathcal{V}\text{-Gph}^{(n)}$ . We then explain  $T$ -operads for any cartesian monad  $T$  in Section 5. In Section 6, following but adapting Garner’s homotopy theoretic ideas [10,11], we define contraction in the enriched setting. Then we show in Section 7 that for every  $\mathcal{V}$  which is locally presentable and extensive, the  $\mathcal{V}$ -enriched version of initial  $T^{(n)}$ -operad with contraction exists, generalising Batanin and Leinster’s line of thought to complete a definition of weak  $n$ -dimensional  $\mathcal{V}$ -category.

## 2 Extensivity, $\mathcal{V}$ -graphs and $\mathcal{V}$ -categories

Extensive categories were introduced by Lawvere and further developed in [5,7]. In their work, extensive categories meant categories with finite coproducts satisfying an additional condition. We need to extend that to a corresponding infinitary condition, cf [8].

For any category  $\mathcal{V}$  with small coproducts, and for any set  $I$  and  $I$ -indexed family  $(X_i)_{i \in I}$  of objects of  $\mathcal{V}$ , consider the functor

$$\coprod: \prod_{i \in I} (\mathcal{V}/X_i) \longrightarrow \mathcal{V}/(\coprod_{i \in I} X_i) \quad (1)$$

mapping  $(f_i: A_i \longrightarrow X_i)_{i \in I}$  to  $(\coprod_{i \in I} f_i: \coprod_{i \in I} A_i \longrightarrow \coprod_{i \in I} X_i)$ .

**Definition 2.1** A category  $\mathcal{V}$  with small coproducts is **extensive** if for any set  $I$  and  $I$ -indexed family  $(X_i)_{i \in I}$  of objects of  $\mathcal{V}$ , the functor  $\coprod$  in (1) is an equivalence of categories.

Our leading examples of extensive categories are **Set** and the category  $\omega\text{-Cpo}$  of posets with sups of  $\omega$ -chains, together with, for any extensive category  $\mathcal{V}$  with finite limits, the categories  $\mathcal{V}\text{-Gph}^{(n)}$  and  $\mathcal{V}\text{-Cat}^{(n)}$ , which are defined recursively. In order to define the former, we first need to define the category of  $\mathcal{V}$ -graphs.

**Definition 2.2** Let  $\mathcal{V}$  be a category.

- (i) A **small  $\mathcal{V}$ -graph**  $G$  consists of a set  $\text{ob}(G)$  together with, for each  $x, y \in \text{ob}(G)$ , an object  $G(x, y) \in \mathcal{V}$ .
- (ii) A **map of  $\mathcal{V}$ -graphs** from  $G$  to  $G'$  is a function  $f: \text{ob}(G) \longrightarrow \text{ob}(G')$  together with, for each  $x, y \in \text{ob}(G)$ , a map  $f_{x,y}: G(x, y) \longrightarrow G'(fx, fy)$  in  $\mathcal{V}$ .

We denote the category of all small  $\mathcal{V}$ -graphs and maps by  $\mathcal{V}\text{-Gph}$ . The construction  $(-)\text{-Gph}$  routinely extends to an endo-2-functor on the 2-category of locally small categories.

**Definition 2.3** For each natural number  $n$  and any category  $\mathcal{V}$ , the category  $\mathcal{V}\text{-}\mathbf{Gph}^{(n)}$  is defined as follows:

$$\mathcal{V}\text{-}\mathbf{Gph}^{(0)} = \mathcal{V}; \quad \mathcal{V}\text{-}\mathbf{Gph}^{(n+1)} = (\mathcal{V}\text{-}\mathbf{Gph}^{(n)})\text{-}\mathbf{Gph}.$$

**Definition 2.4** For each natural number  $n$  and any category  $\mathcal{V}$  with finite products, the category  $\mathcal{V}\text{-}\mathbf{Cat}^{(n)}$  is defined as follows (using the cartesian structure for enrichment):

$$\mathcal{V}\text{-}\mathbf{Cat}^{(0)} = \mathcal{V}; \quad \mathcal{V}\text{-}\mathbf{Cat}^{(n+1)} = (\mathcal{V}\text{-}\mathbf{Cat}^{(n)})\text{-}\mathbf{Cat}.$$

When  $\mathcal{V} = \mathbf{Set}$ , we abbreviate  $\mathcal{V}\text{-}\mathbf{Gph}^{(n)}$  by  $n\text{-}\mathbf{Gph}$ , and we abbreviate  $\mathcal{V}\text{-}\mathbf{Cat}^{(n)}$  by  $n\text{-}\mathbf{Cat}$ .

We now show that if  $\mathcal{V}$  is an extensive category with finite limits, then so are  $\mathcal{V}\text{-}\mathbf{Gph}$  and  $\mathcal{V}\text{-}\mathbf{Cat}$ . Actually, to ensure that  $\mathcal{V}\text{-}\mathbf{Gph}$  and  $\mathcal{V}\text{-}\mathbf{Cat}$  are extensive, the much weaker requirement of  $\mathcal{V}$  having a strict initial object suffices. Recall that an initial object  $0$  in a category is called **strict** if every morphism going into  $0$  is an isomorphism. Every extensive category has a strict initial object; consider the case  $I = \emptyset$  in (1).

**Proposition 2.5** *If  $\mathcal{V}$  is a category with a strict initial object  $0$ , then  $\mathcal{V}\text{-}\mathbf{Gph}$  is extensive.*

**Proof.** The coproduct of a set-indexed family  $(G_i)_{i \in I}$  of  $\mathcal{V}$ -graphs is given by  $\text{ob}(\coprod_{i \in I} G_i) = \coprod_{i \in I} \text{ob}(G_i)$  and

$$\left( \coprod_{i \in I} G_i \right) ((i, x), (i', x')) = \begin{cases} G_i(x, x') & \text{if } i = i', \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that the functor  $\coprod: \prod_{i \in I} (\mathcal{V}\text{-}\mathbf{Gph}/G_i) \rightarrow \mathcal{V}\text{-}\mathbf{Gph}/(\coprod_{i \in I} G_i)$  (as in (1)) is full and faithful. For any object  $(f: H \rightarrow \coprod_{i \in I} G_i)$  in  $\mathcal{V}\text{-}\mathbf{Gph}/(\coprod_{i \in I} G_i)$ , define an object  $(f_i: H_i \rightarrow G_i)_{i \in I} \in \prod_{i \in I} (\mathcal{V}/G_i)$  by the pullbacks of  $f$  along the coprojections  $\sigma_i: G_i \rightarrow \coprod_{i \in I} G_i$ ; note that these pullbacks always exist, and  $H_i$  are just the suitable “full sub”  $\mathcal{V}$ -graphs of  $H$ . Since  $0$  is strict,  $(\coprod_{i \in I} f_i: \coprod_{i \in I} H_i \rightarrow \coprod_{i \in I} G_i)$  is isomorphic to  $f$ . Hence  $\coprod$  is also essentially surjective.  $\square$

**Proposition 2.6** *If  $\mathcal{V}$  is a category with a strict initial object and finite products, then  $\mathcal{V}\text{-}\mathbf{Cat}$  is extensive.*

**Proof.** Similar to the above. See [8, Proposition 2.6] for details (note that in a category  $\mathcal{V}$  with an initial object  $0$  and finite products,  $0$  is strict if and only if  $0 \times B \cong 0$  for every  $B \in \mathcal{V}$ ).  $\square$

Finite limits in  $\mathcal{V}\text{-}\mathbf{Gph}$  or  $\mathcal{V}\text{-}\mathbf{Cat}$  are straightforward. We will treat pullbacks as an example, as they will play an important role in Sections 3 and 4.

**Proposition 2.7** *Let  $\mathcal{V}$  have finite limits. A commutative square*

$$\begin{array}{ccc} P & \xrightarrow{k} & B \\ h \downarrow & & \downarrow g \\ A & \xrightarrow{f} & X \end{array}$$

*in  $\mathcal{V}\text{-}\mathbf{Gph}$  or in  $\mathcal{V}\text{-}\mathbf{Cat}$  is a pullback if and only if the square as on the left of the following diagram is a pullback in  $\mathbf{Set}$ , and for any pair  $p_1, p_2 \in \text{ob}(P)$ , writing  $a_i = h(p_i)$ ,  $b_i = k(p_i)$  and  $x_i = f(a_i) = g(b_i)$  for  $i = 1, 2$ , the square as on the right is a pullback in  $\mathcal{V}$ .*

$$\begin{array}{ccc} \text{ob}(P) & \xrightarrow{k} & \text{ob}(B) \\ h \downarrow & & \downarrow g \\ \text{ob}(A) & \xrightarrow{f} & \text{ob}(X) \end{array} \qquad \begin{array}{ccc} P(p_1, p_2) & \xrightarrow{k_{p_1, p_2}} & B(b_1, b_2) \\ h_{p_1, p_2} \downarrow & & \downarrow g_{b_1, b_2} \\ A(a_1, a_2) & \xrightarrow{f_{a_1, a_2}} & X(x_1, x_2) \end{array}$$

We need several results about extensivity in order to prove cartesianness of a monad in later sections, so we collect those results in the remainder of this section, extending work from [8].

**Proposition 2.8** *A category  $\mathcal{V}$  with small coproducts is extensive if and only if for any set  $I$ ,  $I$ -indexed family  $(X_i)_{i \in I}$  of objects of  $\mathcal{V}$ , morphism  $f: A \rightarrow \coprod_{i \in I} X_i$  in  $\mathcal{V}$ , and  $I$ -indexed family of commutative squares*

$$\begin{array}{ccc} A_i & \xrightarrow{\tau_i} & A \\ f_i \downarrow & & \downarrow f \\ X_i & \xrightarrow{\sigma_i} & \coprod_{i \in I} X_i \end{array} \tag{2}$$

*in  $\mathcal{V}$  (in which  $\sigma_i$  is the  $i$ -th coprojection), each square (2) is a pullback square if and only if  $(\tau_i)_{i \in I}$  defines a coproduct (that is,  $A = \coprod_{i \in I} A_i$  with  $\tau_i$  the  $i$ -th coprojection).*

**Proof.** See [8, Proposition 2.3]. □

**Proposition 2.9** *Let  $\mathcal{V}$  be an extensive category. For any set  $I$  and  $I$ -indexed family of pullback squares in  $\mathcal{V}$  as on the left of the following diagram, the square*

as on the right is a pullback.

$$\begin{array}{ccc}
 P_i & \xrightarrow{q_i} & B_i \\
 \downarrow p_i & \lrcorner & \downarrow g_i \\
 A_i & \xrightarrow{f_i} & X_i
 \end{array}
 \qquad
 \begin{array}{ccc}
 \coprod_{i \in I} P_i & \xrightarrow{\coprod_{i \in I} q_i} & \coprod_{i \in I} B_i \\
 \downarrow \coprod_{i \in I} p_i & \lrcorner & \downarrow \coprod_{i \in I} g_i \\
 \coprod_{i \in I} A_i & \xrightarrow{\coprod_{i \in I} f_i} & \coprod_{i \in I} X_i
 \end{array}$$

**Proof.** By the definition of extensivity, the functor  $\coprod: \prod_{i \in I}(\mathcal{V}/X_i) \rightarrow \mathcal{V}/(\prod_{i \in I} X_i)$  is an equivalence of categories and, in particular, it preserves binary products.  $\square$

**Proposition 2.10** *Let  $\mathcal{V}$  be an extensive category with finite products. For any  $B \in \mathcal{V}$ , the functor  $(-) \times B: \mathcal{V} \rightarrow \mathcal{V}$  preserves small coproducts.*

**Proof.** In any category, a square as on the left of the following diagram is always a pullback. Hence for any object  $B \in \mathcal{V}$ , set  $I$ , and  $I$ -indexed family  $(X_i)_{i \in I}$  of objects of  $\mathcal{V}$ , for each  $i \in I$  the square as on the right is a pullback.

$$\begin{array}{ccc}
 A \times B & \xrightarrow{h \times B} & C \times B \\
 \downarrow \pi_1 & & \downarrow \pi_1 \\
 A & \xrightarrow{h} & C
 \end{array}
 \qquad
 \begin{array}{ccc}
 X_i \times B & \xrightarrow{\sigma_i \times B} & (\prod_{i \in I} X_i) \times B \\
 \downarrow \pi_1 & & \downarrow \pi_1 \\
 X_i & \xrightarrow{\sigma_i} & \prod_{i \in I} X_i
 \end{array}$$

Therefore by Proposition 2.8,  $(\prod_{i \in I} X_i) \times B \cong \prod_{i \in I} (X_i \times B)$ .  $\square$

**Proposition 2.11** *Let  $\mathcal{V}$  be an extensive category. For any object  $Y \in \mathcal{V}$ , the slice category  $\mathcal{V}/Y$  is again extensive.*

**Proof.** Clearly  $\mathcal{V}/Y$  has small coproducts given by  $\prod_{i \in I} (f_i: X_i \rightarrow Y) = ([f_i]_{i \in I}: \prod_{i \in I} X_i \rightarrow Y)$ . Also note that for any object  $(f: X \rightarrow Y)$  of  $\mathcal{V}/Y$ , the canonical functor  $(\mathcal{V}/Y)/f \rightarrow \mathcal{V}/X$  which maps  $(h: (g: A \rightarrow Y) \rightarrow f) \in (\mathcal{V}/Y)/f$  to  $(h: A \rightarrow X) \in \mathcal{V}/X$  is an isomorphism of categories. For any set  $I$  and  $I$ -indexed family  $(f_i: X_i \rightarrow Y)_{i \in I}$  of objects of  $\mathcal{V}/Y$ , the diagram

$$\begin{array}{ccc}
 \prod_{i \in I} ((\mathcal{V}/Y)/f_i) & \xrightarrow{\coprod} & (\mathcal{V}/Y)/[f_i]_{i \in I} \\
 \cong \downarrow & & \downarrow \cong \\
 \prod_{i \in I} (\mathcal{V}/X_i) & \xrightarrow{\coprod} & \mathcal{V}/(\prod_{i \in I} X_i)
 \end{array}$$

(in which the vertical arrows are the canonical isomorphisms mentioned above) commutes. Since the lower  $\coprod$  is an equivalence by the assumption, so is the upper one.  $\square$

**Corollary 2.12** *Let  $\mathcal{V}$  be an extensive category with pullbacks.*

- (i) *For any morphism  $g: B \rightarrow X$  in  $\mathcal{V}$ , set  $I$ , and  $I$ -indexed family of pullback squares in  $\mathcal{V}$  as on the left of the following diagram, the square as on the right is a pullback.*

$$\begin{array}{ccc}
 P_i & \xrightarrow{q_i} & B \\
 \downarrow p_i & \lrcorner & \downarrow g \\
 A_i & \xrightarrow{f_i} & X
 \end{array}
 \qquad
 \begin{array}{ccc}
 \coprod_{i \in I} P_i & \xrightarrow{[q_i]_{i \in I}} & B \\
 \downarrow \coprod_{i \in I} p_i & \lrcorner & \downarrow g \\
 \coprod_{i \in I} A_i & \xrightarrow{[f_i]_{i \in I}} & X
 \end{array}$$

- (ii) *For any object  $X \in \mathcal{V}$ , set  $I$ ,  $I$ -indexed family of morphisms  $(f_i: A_i \rightarrow X)_{i \in I}$  in  $\mathcal{V}$ , set  $J$ ,  $J$ -indexed family of morphisms  $(g_j: B_j \rightarrow X)_{j \in J}$  in  $\mathcal{V}$ , and  $(I \times J)$ -indexed family of pullback squares in  $\mathcal{V}$  as on the left of the following diagram, the square as on the right is a pullback.*

$$\begin{array}{ccc}
 P_{i,j} & \xrightarrow{q_{i,j}} & B_j \\
 \downarrow p_{i,j} & \lrcorner & \downarrow g_j \\
 A_i & \xrightarrow{f_i} & X
 \end{array}
 \qquad
 \begin{array}{ccc}
 \coprod_{i \in I, j \in J} P_{i,j} & \xrightarrow{\coprod_{j \in J} ([q_{i,j}]_{i \in I})} & \coprod_{j \in J} B_j \\
 \downarrow \coprod_{i \in I} ([p_{i,j}]_{j \in J}) & \lrcorner & \downarrow [g_j]_{j \in J} \\
 \coprod_{i \in I} A_i & \xrightarrow{[f_i]_{i \in I}} & X
 \end{array}$$

**Proof.**

- (i) By the assumption, the slice category  $\mathcal{V}/X$  has finite products  $\times_X$  (given by pullbacks in  $\mathcal{V}$ ), and is extensive (Proposition 2.11). Hence by Proposition 2.10, binary product by  $(g: B \rightarrow X) \in \mathcal{V}/X$  preserves small coproducts, that is,  $(\coprod_{i \in I} f_i) \times_X g \cong \coprod_{i \in I} (f_i \times_X g)$ .
- (ii) Using (i) iteratively, we obtain  $(\coprod_{i \in I} f_i) \times_X (\coprod_{j \in J} g_j) \cong \coprod_{i \in I, j \in J} (f_i \times_X g_j)$ .  $\square$

### 3 A cartesian monad on $\mathcal{V}\text{-Gph}$

In this section, subject to conditions on  $\mathcal{V}$ , we show that the monad on  $\mathcal{V}\text{-Gph}$  induced by the forgetful functor  $U: \mathcal{V}\text{-Cat} \rightarrow \mathcal{V}\text{-Gph}$  is cartesian, meaning the following.

- Definition 3.1** (i) A natural transformation is called **cartesian** if all the naturality squares of it are pullbacks.
- (ii) For any category  $\mathcal{C}$  with pullbacks, a monad  $T = (T, \mu, \eta)$  on  $\mathcal{C}$  is **cartesian** if the functor  $T$  preserves pullbacks, and if  $\mu$  and  $\eta$  are cartesian natural transformations.

**Proposition 3.2** *If  $\mathcal{V}$  has finite products and small coproducts, and if for any  $B \in \mathcal{V}$  the functor  $(-) \times B: \mathcal{V} \rightarrow \mathcal{V}$  preserves small coproducts, then the forgetful functor  $U: \mathcal{V}\text{-Cat} \rightarrow \mathcal{V}\text{-Gph}$  admits a left adjoint.*

**Proof.** Given a  $\mathcal{V}$ -graph  $G = (\text{ob}(G), (G(x, y))_{x, y \in \text{ob}(G)})$ , the free  $\mathcal{V}$ -category on  $G$  is given by the  $\mathcal{V}$ -graph

$$\left( \text{ob}(G), \left( \coprod_{n \in \mathbb{N}} \coprod_{\substack{w_0, \dots, w_n \in \text{ob}(G) \\ w_0 = x, w_n = y}} G(w_{n-1}, w_n) \times \cdots \times G(w_0, w_1) \right)_{x, y \in \text{ob}(G)} \right)$$

with evident composition laws (defined using the assumption that  $(-) \times B$  preserves small coproducts) and identity elements.  $\square$

Examples of categories  $\mathcal{V}$  satisfying the assumptions of Proposition 3.2 include cartesian closed categories with small coproducts [20] and extensive categories with finite products (by Proposition 2.10).

**Proposition 3.3** *If  $\mathcal{V}$  is an extensive category with finite limits, then the left adjoint to the forgetful functor  $U: \mathcal{V}\text{-Cat} \rightarrow \mathcal{V}\text{-Gph}$  determined by Proposition 3.2 preserves pullbacks.*

**Proof.** This follows from Proposition 2.7, Proposition 2.9, and Corollary 2.12 (ii).  $\square$

**Proposition 3.4** *If  $\mathcal{V}$  has a strict initial object 0 and finite products, then the categories  $\mathcal{V}\text{-Gph}$  and  $\mathcal{V}\text{-Cat}$  admit small coproducts and the forgetful functor  $U: \mathcal{V}\text{-Cat} \rightarrow \mathcal{V}\text{-Gph}$  preserves small coproducts.*

**Proof.** In both  $\mathcal{V}\text{-Gph}$  and  $\mathcal{V}\text{-Cat}$ , small coproducts are given by taking disjoint union of objects and setting the hom-objects between objects from different components to be 0 (cf. the proof of Proposition 2.5).  $\square$

**Proposition 3.5** *If  $\mathcal{V}$  is an extensive category with finite limits, then the unit  $\eta: \text{id}_{\mathcal{V}\text{-Gph}} \Rightarrow UF$  of the adjunction  $F \dashv U$  in Proposition 3.2 is cartesian.*

**Proof.** This follows from Proposition 2.8.  $\square$

**Proposition 3.6** *If  $\mathcal{V}$  is an extensive category with finite limits, then for each natural number  $m$ , the natural transformation*

$$(\mathcal{V}\text{-Gph})^m \xrightarrow{F^m} (\mathcal{V}\text{-Cat})^m \xrightarrow{\prod} \mathcal{V}\text{-Cat} \xrightarrow{\text{id}_{\mathcal{V}\text{-Cat}}} \mathcal{V}\text{-Cat},$$

$\begin{array}{ccc} & U \nearrow & \mathcal{V}\text{-Gph} \xrightarrow{F} \\ & & \Downarrow \varepsilon \\ & & \mathcal{V}\text{-Cat} \end{array}$

where  $\varepsilon$  is the counit of the adjunction  $F \dashv U$  in Proposition 3.2 and  $\prod$  is the  $m$ -ary product functor, is cartesian.

**Proof.** Let  $(f^{(1)}, \dots, f^{(m)}): (G^{(1)}, \dots, G^{(m)}) \rightarrow (G'^{(1)}, \dots, G'^{(m)})$  be a morphism



in  $(\mathcal{V}\text{-}\mathbf{Gph})^m$ . Our aim is to show that the square

$$\begin{array}{ccc} FU(FG^{(1)} \times \dots \times FG^{(m)}) & \xrightarrow{\varepsilon_{FG^{(1)} \times \dots \times FG^{(m)}}} & FG^{(1)} \times \dots \times FG^{(m)} \\ \downarrow FU(Ff^{(1)} \times \dots \times Ff^{(m)}) & & \downarrow Ff^{(1)} \times \dots \times Ff^{(m)} \\ FU(FG'^{(1)} \times \dots \times FG'^{(m)}) & \xrightarrow{\varepsilon_{FG'^{(1)} \times \dots \times FG'^{(m)}}} & FG'^{(1)} \times \dots \times FG'^{(m)} \end{array}$$

in  $\mathcal{V}\text{-}\mathbf{Cat}$  is a pullback. By Proposition 2.7 it suffices to show that for every pair of objects  $(x^{(1)}, \dots, x^{(m)}), (y^{(1)}, \dots, y^{(m)}) \in \text{ob}(FU(FG^{(1)} \times \dots \times FG^{(m)})) = \text{ob}(G^{(1)} \times \dots \times G^{(m)})$ , the square

$$\begin{array}{ccc} FU(\prod_{i=1}^m (FG^{(i)})(x^{(i)}, y^{(i)})) & \longrightarrow & \prod_{i=1}^m (FG^{(i)})(x^{(i)}, y^{(i)}) \\ \downarrow & & \downarrow \\ FU(\prod_{i=1}^m (FG'^{(i)})(f^{(i)}x^{(i)}, f^{(i)}y^{(i)})) & \longrightarrow & \prod_{i=1}^m (FG'^{(i)})(f^{(i)}x^{(i)}, f^{(i)}y^{(i)}) \end{array}$$

in  $\mathcal{V}$  is a pullback. This follows by appropriate applications of Proposition 2.10, Proposition 2.9 and Corollary 2.12 (ii).  $\square$

Let  $T = (T, \mu, \eta)$  be the monad on  $\mathcal{V}\text{-}\mathbf{Gph}$  induced by the adjunction  $F \dashv U: \mathcal{V}\text{-}\mathbf{Cat} \rightarrow \mathcal{V}\text{-}\mathbf{Gph}$ .

**Theorem 3.7** *If  $\mathcal{V}$  is an extensive category with finite limits, then the monad  $T$  on  $\mathcal{V}\text{-}\mathbf{Gph}$  is cartesian.*

**Proof.** This follows from Proposition 3.3, Proposition 3.5, and Proposition 3.6 (take  $m = 1$ ).  $\square$

## 4 A cartesian monad on $\mathcal{V}\text{-}\mathbf{Gph}^{(n)}$

In this section we show that the forgetful functor from the category of strict  $n$ -dimensional  $\mathcal{V}$ -categories to that of  $n$ -dimensional  $\mathcal{V}$ -graphs has a left adjoint. We assume throughout that  $\mathcal{V}$  is extensive and has finite limits. It follows that  $\mathcal{V}\text{-}\mathbf{Gph}$  and  $\mathcal{V}\text{-}\mathbf{Cat}$  are likewise (by Propositions 2.5, 2.6 and 2.7), and so, by induction, for each natural number  $n$ , the categories  $\mathcal{V}\text{-}\mathbf{Gph}^{(n)}$  and  $\mathcal{V}\text{-}\mathbf{Cat}^{(n)}$  are also extensive with finite limits.

Recall that, by Propositions 2.10 and 3.2, the forgetful functor  $U: (\mathcal{V}\text{-}\mathbf{Cat}^{(n)})\text{-}\mathbf{Cat} \rightarrow (\mathcal{V}\text{-}\mathbf{Cat}^{(n)})\text{-}\mathbf{Gph}$  admits a left adjoint  $F$ .

**Definition 4.1** For each natural number  $n$ , we define an adjunction  $F^{(n)} \dashv U^{(n)}: \mathcal{V}\text{-}\mathbf{Cat}^{(n)} \rightarrow \mathcal{V}\text{-}\mathbf{Gph}^{(n)}$  recursively as follows:

- (i)  $F^{(0)} = U^{(0)} = \text{id}_{\mathcal{V}}$ ;

(ii)  $F^{(n+1)} \dashv U^{(n+1)}$  is the composite:

$$(\mathcal{V}\text{-}\mathbf{Gph}^{(n)})\text{-}\mathbf{Gph} \begin{array}{c} \xrightarrow{F^{(n)}\text{-}\mathbf{Gph}} \\ \perp \\ \xleftarrow{U^{(n)}\text{-}\mathbf{Gph}} \end{array} (\mathcal{V}\text{-}\mathbf{Cat}^{(n)})\text{-}\mathbf{Gph} \begin{array}{c} \xrightarrow{F} \\ \perp \\ \xleftarrow{U} \end{array} (\mathcal{V}\text{-}\mathbf{Cat}^{(n)})\text{-}\mathbf{Cat}.$$

**Proposition 4.2** *For each natural number  $n$ ,  $F^{(n)}: \mathcal{V}\text{-}\mathbf{Gph}^{(n)} \rightarrow \mathcal{V}\text{-}\mathbf{Cat}^{(n)}$  preserves pullbacks.*

**Proof.** For  $n = 0$ , the assertion is trivial. Proceeding inductively, if  $F^{(n)}$  preserves pullbacks, so does  $F^{(n)}\text{-}\mathbf{Gph}$  by Proposition 2.7. The functor  $F: (\mathcal{V}\text{-}\mathbf{Cat}^{(n)})\text{-}\mathbf{Gph} \rightarrow (\mathcal{V}\text{-}\mathbf{Cat}^{(n)})\text{-}\mathbf{Cat}$  preserves pullbacks by Proposition 3.3.  $\square$

**Proposition 4.3** *For each natural number  $n$ ,  $U^{(n)}: \mathcal{V}\text{-}\mathbf{Cat}^{(n)} \rightarrow \mathcal{V}\text{-}\mathbf{Gph}^{(n)}$  preserves small coproducts.*

**Proof.** For  $n = 0$ , the assertion is trivial. Proceeding inductively, if  $U^{(n)}$  preserves small coproducts, it preserves initial objects, and so the functor  $U^{(n)}\text{-}\mathbf{Gph}$  preserves small coproducts. The functor  $U: (\mathcal{V}\text{-}\mathbf{Cat}^{(n)})\text{-}\mathbf{Cat} \rightarrow (\mathcal{V}\text{-}\mathbf{Cat}^{(n)})\text{-}\mathbf{Gph}$  also preserves small coproducts by Proposition 3.4.  $\square$

**Proposition 4.4** *For each natural number  $n$ , the unit  $\eta^{(n)}: \text{id}_{\mathcal{V}\text{-}\mathbf{Gph}^{(n)}} \Rightarrow U^{(n)}F^{(n)}$  of the adjunction  $F^{(n)} \dashv U^{(n)}$  is cartesian.*

**Proof.** Observe that adjunctions whose units are cartesian are closed under composition. Proceeding inductively, if  $\eta^{(n)}$  is cartesian, so is  $\eta^{(n)}\text{-}\mathbf{Gph}$  by Proposition 2.7. The unit of the adjunction  $F \dashv U: (\mathcal{V}\text{-}\mathbf{Cat}^{(n)})\text{-}\mathbf{Cat} \rightarrow (\mathcal{V}\text{-}\mathbf{Cat}^{(n)})\text{-}\mathbf{Gph}$  is cartesian by Proposition 3.5.  $\square$

**Proposition 4.5** *For each pair of natural numbers  $n$  and  $m$ , the natural transformation*

$$(\mathcal{V}\text{-}\mathbf{Gph}^{(n)})^m \xrightarrow{(F^{(n)})^m} (\mathcal{V}\text{-}\mathbf{Cat}^{(n)})^m \xrightarrow{\prod} \mathcal{V}\text{-}\mathbf{Cat}^{(n)} \xrightarrow{\text{id}_{\mathcal{V}\text{-}\mathbf{Cat}^{(n)}}} \mathcal{V}\text{-}\mathbf{Cat}^{(n)},$$

$\begin{array}{ccc} & U^{(n)} \nearrow & \mathcal{V}\text{-}\mathbf{Gph}^{(n)} \\ & & \Downarrow \varepsilon^{(n)} \\ & F^{(n)} \searrow & \mathcal{V}\text{-}\mathbf{Cat}^{(n)} \end{array}$

where  $\varepsilon^{(n)}$  is the counit of the adjunction  $F^{(n)} \dashv U^{(n)}$  and  $\prod$  is the  $m$ -ary product functor, is cartesian.

**Proof.** By induction on  $n$ . Suppose the claim is true for  $n = k$  and for all  $m$ . For brevity, we will write the adjunction  $F^{(k)}\text{-}\mathbf{Gph} \dashv U^{(k)}\text{-}\mathbf{Gph}$  as  $F' \dashv U'$ , and whose counit  $\varepsilon^{(k)}\text{-}\mathbf{Gph}$  as  $\varepsilon'$ . We aim to show that for every morphism  $(f^{(1)}, \dots, f^{(m)}): (G^{(1)}, \dots, G^{(m)}) \rightarrow (G'^{(1)}, \dots, G'^{(m)})$  in  $(\mathcal{V}\text{-}\mathbf{Gph}^{(k+1)})^m$ ,

the outer rectangle in the diagram

$$\begin{array}{ccccc}
 FF'U'U(\prod_{i=1}^m FF'G^{(i)}) & \xrightarrow{F\varepsilon'_{U(\prod_{i=1}^m FF'G^{(i)})}} & FU(\prod_{i=1}^m FF'G^{(i)}) & \xrightarrow{\varepsilon_{\prod_{i=1}^m FF'G^{(i)}}} & \prod_{i=1}^m FF'G^{(i)} \\
 \downarrow FF'U'U(\prod_{i=1}^m FF'f^{(i)}) & & \downarrow FU(\prod_{i=1}^m FF'f^{(i)}) & & \downarrow \prod_{i=1}^m FF'f^{(i)} \\
 FF'U'U(\prod_{i=1}^m FF'G'^{(i)}) & \xrightarrow{F\varepsilon'_{U(\prod_{i=1}^m FF'G'^{(i)})}} & FU(\prod_{i=1}^m FF'G'^{(i)}) & \xrightarrow{\varepsilon_{\prod_{i=1}^m FF'G'^{(i)}}} & \prod_{i=1}^m FF'G'^{(i)}
 \end{array}$$

in  $\mathcal{V}\text{-Cat}^{(k+1)}$  is a pullback. The right square is a pullback by Proposition 3.6, so we shall show that the left square is also a pullback. Since  $F$  preserves pullbacks by Proposition 3.3, it suffices to show that the square

$$\begin{array}{ccc}
 F'U'U(\prod_{i=1}^m FF'G^{(i)}) & \xrightarrow{\varepsilon'_{U(\prod_{i=1}^m FF'G^{(i)})}} & U(\prod_{i=1}^m FF'G^{(i)}) \\
 \downarrow F'U'U(\prod_{i=1}^m FF'f^{(i)}) & & \downarrow U(\prod_{i=1}^m FF'f^{(i)}) \\
 F'U'U(\prod_{i=1}^m FF'G'^{(i)}) & \xrightarrow{\varepsilon'_{U(\prod_{i=1}^m FF'G'^{(i)})}} & U(\prod_{i=1}^m FF'G'^{(i)})
 \end{array}$$

in  $(\mathcal{V}\text{-Cat}^{(k)})\text{-Gph}$  is a pullback. By Proposition 2.7, it suffices to show that for every pair of objects  $(x^{(1)}, \dots, x^{(m)}), (y^{(1)}, \dots, y^{(m)}) \in \text{ob}(F'U'U(\prod_{i=1}^m FF'G^{(i)})) = \text{ob}(G^{(1)} \times \dots \times G^{(m)})$ , the square

$$\begin{array}{ccc}
 F^{(k)}U^{(k)}(\prod_{i=1}^m (FF'G^{(i)})(x^{(i)}, y^{(i)})) & \longrightarrow & \prod_{i=1}^m (FF'G^{(i)})(x^{(i)}, y^{(i)}) \\
 \downarrow & & \downarrow \\
 F^{(k)}U^{(k)}(\prod_{i=1}^m (FF'G'^{(i)})(f^{(i)}x^{(i)}, f^{(i)}y^{(i)})) & \longrightarrow & \prod_{i=1}^m (FF'G'^{(i)})(f^{(i)}x^{(i)}, f^{(i)}y^{(i)})
 \end{array}$$

in  $\mathcal{V}\text{-Cat}^{(k)}$  is a pullback. This follows from Proposition 2.10, Proposition 4.3 for  $n = k$ , that the left adjoint  $F^{(k)}$  preserves small coproducts, Proposition 2.9, Corollary 2.12 (i), and the induction hypothesis.  $\square$

For each natural number  $n$ , let  $T^{(n)} = (T^{(n)}, \mu^{(n)}, \eta^{(n)})$  be the monad on  $\mathcal{V}\text{-Gph}^{(n)}$  induced by the adjunction  $F^{(n)} \dashv U^{(n)}: \mathcal{V}\text{-Cat}^{(n)} \rightarrow \mathcal{V}\text{-Gph}^{(n)}$ .

**Theorem 4.6** *For each natural number  $n$ , the monad  $T^{(n)}$  is cartesian.*

**Proof.** This follows from Proposition 4.2, Proposition 4.4, and Proposition 4.5 (take  $m = 1$ ).  $\square$

This theorem will allow us to use the theory of  $T$ -operads, which we now explain.

## 5 $T$ -operads

In this section we shall review  $T$ -operads and their algebras for a cartesian monad  $T$  [16].

**Proposition 5.1** *If  $T$  is a cartesian monad on a category  $\mathcal{C}$  with finite limits, with terminal object  $1$ , the slice category  $\mathcal{C}/T1$ , with objects  $(p: P \rightarrow T1)$ , has a monoidal structure as follows. The monoidal product is*

$$(p: P \rightarrow T1) \otimes (q: Q \rightarrow T1) = (\bar{q} \circ \pi_2: (P, p) * Q \rightarrow T1),$$

where  $(P, p) * Q$  is defined by the pullback below, and  $\bar{q} = \mu_1 \circ Tq$  is the Kleisli extension of  $q$ :

$$\begin{array}{ccccc} (P, p) * Q & \xrightarrow{\pi_2} & TQ & \xrightarrow{\bar{q}} & T1 \\ \pi_1 \downarrow & \lrcorner & \downarrow T! & & \\ P & \xrightarrow{p} & T1 & & \end{array}$$

The unit is  $I = (\eta_1: 1 \rightarrow T1)$ , and the coherence isomorphisms are determined by the universal property of pullbacks.

**Definition 5.2** Let  $T$  be a cartesian monad on a category  $\mathcal{C}$  with finite limits. A  **$T$ -operad** is a monoid in the monoidal category  $(\mathcal{C}/T1, \otimes, I)$ .

The idea is that a cartesian monad  $T$  determines a notion of algebraic theory in which  $T$ -operads are algebraic theories. In that notion of algebraic theory, the object of arities is given by  $T1$ , and the underlying object  $(\text{ar}_O: O \rightarrow T1) \in \mathcal{C}/T1$  of a  $T$ -operad  $O = ((\text{ar}_O: O \rightarrow T1), m, e)$  represents a family of operations indexed by arities.

We denote the category of  $T$ -operads, i.e., the category of monoids in  $(\mathcal{C}/T1, \otimes, I)$ , by  **$T$ -Opd**. Note that we have a canonical forgetful functor

$$V: T\text{-Opd} \rightarrow \mathcal{C}/T1. \quad (3)$$

The objective of an algebraic theory is to define a notion of algebra. In the case of  $T$ -operads, we define an algebra of a  $T$ -operad  $O$  to be an object  $C$  of  $\mathcal{C}$  (the base category of the monad  $T$ ) equipped with an action of  $O$ . We start with a general definition of an action of a monoid in a monoidal category.

Let  $\mathcal{M} = (\mathcal{M}, \otimes, I)$  be a monoidal category and  $\mathcal{A}$  a category. A **pseudo action** of  $\mathcal{M}$  on  $\mathcal{A}$  is a strong monoidal functor from  $\mathcal{M}$  to the strict monoidal category  $([\mathcal{A}, \mathcal{A}], \circ, \text{id}_{\mathcal{A}})$  of endofunctors on  $\mathcal{A}$ . We often transport the structure of a strong monoidal functor along the 2-adjunction  $(-) \times \mathcal{A} \dashv [\mathcal{A}, -]$  and give a pseudo action in terms of a functor  $*$ :  $\mathcal{M} \times \mathcal{A} \rightarrow \mathcal{A}$  together with coherent natural isomorphisms

$$(X \otimes Y) * A \cong X * (Y * A), \quad I * A \cong A$$

where  $X, Y \in \mathcal{M}$  and  $A \in \mathcal{A}$ .

Given a pseudo action  $*$  of  $\mathcal{M}$  on  $\mathcal{A}$ , we can define an **action** of a monoid  $(M, m, e)$  in  $\mathcal{M}$  on an object  $A$  of  $\mathcal{A}$  to be a morphism

$$a: M * A \longrightarrow A$$

in  $\mathcal{A}$  satisfying the associativity and unit axioms of action.

Now we return to the setting of  $T$ -operad. We have a canonical pseudo action of the monoidal category  $(\mathcal{C}/T1, \otimes, I)$  on the category  $\mathcal{C}$  given by the functor  $*$ :  $(\mathcal{C}/T1) \times \mathcal{C} \longrightarrow \mathcal{C}$  which maps  $(p: P \longrightarrow T1) \in \mathcal{C}/T1$  and  $C \in \mathcal{C}$  to  $(P, p) * C \in \mathcal{C}$  defined as a pullback

$$\begin{array}{ccc} (P, p) * C & \xrightarrow{\pi_2} & TC \\ \pi_1 \downarrow & \lrcorner & \downarrow T! \\ P & \xrightarrow{p} & T1. \end{array}$$

An  $O$ -**algebra** for a  $T$ -operad  $O$  is then defined as an action of  $O$ .

**Example 5.3** If we let  $\mathcal{C} = \mathbf{Set}$  and  $T$  be the free monoid monad, then  $T$ -operads coincide with **non-symmetric operads**. The arities are the natural numbers:  $T1 \cong \mathbb{N}$ .

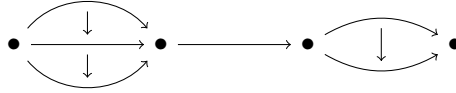
In more detail, the data of a  $T$ -operad in this case consist of a set  $O$ , and functions  $\text{ar}_O: O \longrightarrow \mathbb{N}$ ,  $m: (O, \text{ar}_O) * O \longrightarrow O$  and  $e: 1 \longrightarrow O$ . Unravelling this, we obtain a family of sets  $(O_n)_{n \in \mathbb{N}}$ , a family of functions  $(m_{n, k_1, \dots, k_n}: O_n \times O_{k_1} \times \dots \times O_{k_n} \longrightarrow O_{k_1 + \dots + k_n})_{n, k_1, \dots, k_n \in \mathbb{N}}$ , and an element  $e \in O_1$ . We interpret  $O_n$  as the set of all (derived) operations of arity  $n$ , and  $m_{n, k_1, \dots, k_n}$  maps a tuple  $(\phi, \theta_1, \dots, \theta_n)$  of operations to their substitution  $\phi(\theta_1, \dots, \theta_n)$ . Non-symmetric operads correspond to the subclass of equational theories called *strongly regular equational theories* [4], that is, theories in which the same variables appear in the same order, without repetition, on either side of any equation. Examples of strongly regular equational theories include the theories of monoids and of monoid actions, but not those of commutative monoids or of groups.

An  $O$ -algebra consists of a set  $C$  together with a function  $(O, \text{ar}_O) * C \longrightarrow C$  satisfying the axioms of action. Since  $(O, \text{ar}_O) * C = \coprod_{n \in \mathbb{N}} O_n \times C^n$ , such a function corresponds to a family of functions  $[\![\phi]\!]: C^n \longrightarrow C$  for each  $n \in \mathbb{N}$  and  $\phi \in O_n$  (the interpretation of  $\phi$ ).

**Example 5.4** If we let  $\mathcal{C} = n\text{-Gph} = \mathbf{Set}\text{-Gph}^{(n)}$  and  $T = T^{(n)}$  be the free strict  $n$ -category monad, then  $T^{(n)}$ -operads are known as  **$n$ -globular operads** [16, Section 9.3]. The arities form a  $n$ -graph  $T^{(n)}1$ , whose element can be seen as a *globular pasting scheme*.

More precisely, define for any pair of natural numbers  $n$  and  $k \in \{0, \dots, n\}$ , a  **$k$ -cell** of an  $n$ -graph recursively on  $n$  as follows: a 0-cell of an  $n$ -graph  $G$  is an element of  $\text{ob}(G)$ ; a  $(k+1)$ -cell of  $G$  is a triple  $(x, y, \theta)$  where  $x, y \in \text{ob}(G)$  and  $\theta$  is a  $k$ -cell of  $G(x, y)$ . Then a  $k$ -cell of  $T^{(n)}1$  is a  $k$ -dimensional globular pasting

scheme. We draw the picture of a typical 2-dimensional globular pasting scheme:



The multiplication of an  $n$ -globular operad  $((\text{ar}_O: O \longrightarrow T1), m, e)$  composes a cell  $\phi$  of the  $n$ -graph  $O$  and a family of cells of  $O$  aligned in the shape of the arity of  $\phi$ .

An  $O$ -algebra consists of an  $n$ -graph  $C$  together with a map of  $n$ -graphs  $(O, \text{ar}_O) * C \longrightarrow C$ . A  $k$ -cell of the  $n$ -graph  $(O, \text{ar}_O) * C$  is a  $k$ -cell  $\phi$  of  $O$  together with a family of cells in  $C$  aligned in the shape of the arity of  $\phi$ . See [16] for more details.

**Example 5.5** If we let  $\mathcal{C} = \mathcal{V}\text{-}\mathbf{Gph}^{(n)}$  for an extensive category  $\mathcal{V}$  with finite limits and  $T = T^{(n)}$  be the free strict  $n$ -dimensional  $\mathcal{V}$ -category monad, then the  $T^{(n)}$ -operads are an enriched version of  $n$ -globular operads. They are pretty much the same as  $n$ -globular operads, the only difference being that it no longer makes sense to talk about individual  $n$ -cells; they now form an object of  $\mathcal{V}$ .

## 6 Contractions

In this section we describe the notion of *contraction*, introduced by Leinster [16], and generalise it to the enriched setting. We follow Garner [10] and define contraction as a choice of certain diagonal fillers. The following definition is an example of the construction described in [11, Proposition 3.8].

**Definition 6.1** Let  $\mathcal{C}$  be a category,  $J$  a set, and  $\mathcal{F} = (f_j: A_j \longrightarrow B_j)_{j \in J}$  a  $J$ -indexed family of morphisms in  $\mathcal{C}$ .

- (i) A **contraction** (with respect to  $\mathcal{F}$ ) on a morphism  $g: C \longrightarrow D$  in  $\mathcal{C}$  is a  $J$ -indexed family of functions  $(\kappa_j)_{j \in J}$  such that for each  $j \in J$ ,  $\kappa_j$  assigns to every pair of morphisms  $(h, k)$  in  $\mathcal{V}$  which makes the perimeter of (4) commute, a morphism  $\kappa_j(h, k)$  making the whole diagram (4) commute.

$$\begin{array}{ccc}
 A_j & \xrightarrow{h} & C \\
 f_j \downarrow & \nearrow \kappa_j(h, k) & \downarrow g \\
 B_j & \xrightarrow{k} & D
 \end{array} \tag{4}$$

- (ii) A **map of morphisms with contractions** from  $(g: C \longrightarrow D, (\kappa_j)_{j \in J})$  to  $(g': C' \longrightarrow D', (\kappa'_j)_{j \in J})$  is a map of morphisms  $(u: C \longrightarrow C', v: D \longrightarrow D')$  from  $g$  to  $g'$  which commutes with contractions: for each  $j \in J$  and  $(h, k)$  in

the domain of  $\kappa_j$ ,  $u \circ \kappa_j(h, k) = \kappa'_j(u \circ h, v \circ k)$ .

$$\begin{array}{ccccc}
 A_j & \xrightarrow{h} & C & \xrightarrow{u} & C' \\
 f_j \downarrow & \nearrow \kappa_j(h, k) & \downarrow g & \nearrow \kappa'_j(uh, vk) & \downarrow g' \\
 B_j & \xrightarrow{k} & D & \xrightarrow{v} & D'
 \end{array}$$

In other words, for each  $j \in J$ ,  $\kappa_j$  is a section of the function  $\rho_j$  below, induced by the universality of pullback.

$$\begin{array}{ccc}
 \mathcal{C}(B_j, C) & \xrightarrow{\mathcal{C}(B_j, g)} & \mathcal{C}(B_j, D) \\
 \downarrow \mathcal{C}(f_j, C) & \searrow \rho_j & \downarrow \mathcal{C}(f_j, D) \\
 & P_j & \\
 & \downarrow & \\
 \mathcal{C}(A_j, C) & \xrightarrow{\mathcal{C}(A_j, g)} & \mathcal{C}(A_j, D)
 \end{array}$$

As observed in [10], Leinster’s notion of contraction, for each natural number  $n$ , is a special case of Definition 6.1 where  $\mathcal{C} = n\text{-}\mathbf{Gph}$  and  $\mathcal{F}$  is a certain family  $\mathcal{F}^{(n)} = (f_0^{(n)}, \dots, f_{n+1}^{(n)})$  consisting of  $n + 2$  morphisms in  $n\text{-}\mathbf{Gph}$ . Before giving a precise definition, we wish to give an intuitive idea of them by drawing a suggestive picture. For example, when  $n = 2$  the family can be drawn as

$$\mathcal{F}^{(2)} = \left( \begin{array}{c} \left( \begin{array}{c} \left( \begin{array}{c} \left( \begin{array}{c} \bullet \\ \downarrow f_0^{(2)} \\ \bullet \end{array} \right) \end{array} \right) \end{array} \right) \end{array} \right), \begin{array}{c} \left( \begin{array}{c} \left( \begin{array}{c} \bullet \longrightarrow \bullet \end{array} \right) \end{array} \right) \end{array} \right), \begin{array}{c} \left( \begin{array}{c} \left( \begin{array}{c} \bullet \rightleftarrows \bullet \end{array} \right) \end{array} \right) \end{array} \right), \begin{array}{c} \left( \begin{array}{c} \left( \begin{array}{c} \bullet \rightleftarrows \bullet \\ \downarrow f_3^{(2)} \\ \bullet \end{array} \right) \end{array} \right) \end{array} \right) \end{array} \right).$$

The idea is that an element of  $\mathcal{F}^{(n)}$  is “the inclusion of a sphere into a ball”, although  $f_{n+1}^{(n)}$  is no longer a monomorphism due to lack of cells of dimension greater than  $n$ .

To give a recursive definition of  $\mathcal{F}^{(n)}$  in the enriched setting, we start with auxiliary definitions. For any category  $\mathcal{V}'$  with an initial object  $0$ , define the **suspension** functor  $\Sigma: \mathcal{V}' \rightarrow \mathcal{V}'\text{-}\mathbf{Gph}$  which maps  $X \in \mathcal{V}$  to  $\Sigma X = (\{s, t\}, (\Sigma X(i, j))_{i, j \in \{s, t\}})$  given by  $\Sigma X(s, t) = X$ ,  $\Sigma X(i, j) = 0$  if  $(i, j) \neq (s, t)$ ; cf. [16, Section 9.3]. Also define the **discrete**  $\mathcal{V}'$ -**graph** functor  $D: \mathbf{Set} \rightarrow \mathcal{V}'\text{-}\mathbf{Gph}$  which maps a set  $I$  to  $DI = (I, (0)_{i, j \in I})$ . The functor  $D$  is the left adjoint of  $\text{ob}(-): \mathcal{V}'\text{-}\mathbf{Gph} \rightarrow \mathbf{Set}$ .

**Definition 6.2** Let  $\mathcal{V}$  be an extensive category with finite limits. For each natural number  $n$ , define a family  $\mathcal{F}^{(n)} = (f_0^{(n)}, \dots, f_{n+1}^{(n)})$  of morphisms in  $\mathcal{V}\text{-}\mathbf{Gph}^{(n)}$

recursively as follows.

- (i)  $f_0^{(0)}: 0 \longrightarrow 1$  and  $f_1^{(0)}: 1 + 1 \longrightarrow 1$  are the unique morphisms in  $\mathcal{V}$  into the terminal object 1.
- (ii)  $f_0^{(n)}: D\emptyset \longrightarrow D\{*\}$ , where  $\emptyset$  and  $\{*\}$  are the empty set and a singleton respectively, is the unique morphism in  $\mathcal{V}\text{-}\mathbf{Gph}^{(n)}$  out of the initial object  $D\emptyset$ , and for each  $i \in \{1, \dots, n+1\}$ ,  $f_i^{(n)} = \Sigma f_{i-1}^{(n-1)}$ .

For each object  $X \in \mathcal{V}\text{-}\mathbf{Gph}^{(n)}$ , we have the category  $\mathbf{Contr}_{\mathcal{F}^{(n)}}(X)$  of morphisms into  $X$  with contractions (with respect to  $\mathcal{F}^{(n)}$ ), whose objects are morphisms  $g$  with contractions as in Definition 6.1 (i) such that the codomain of  $g$  is  $X$ , and whose morphisms are maps of morphisms with contractions  $(u, v)$  as in Definition 6.1 (ii) such that  $v = \text{id}_X$ . Note that we have the canonical forgetful functor

$$V': \mathbf{Contr}_{\mathcal{F}^{(n)}}(X) \longrightarrow \mathcal{V}\text{-}\mathbf{Gph}^{(n)}/X. \quad (5)$$

We will in particular be concerned with the case where  $X = T^{(n)}1$ .

## 7 Weak $n$ -categories enriched over locally presentable and extensive categories

Following Leinster, for any extensive category  $\mathcal{V}$  with finite limits (with additional condition; see below) and any natural number  $n$ , we define a *weak  $n$ -dimensional  $\mathcal{V}$ -category* to be an algebra of the *initial  $T^{(n)}$ -operad with contraction*. More precisely, define the category  $T^{(n)}\text{-}\mathbf{OC}$  of  $T^{(n)}$ -operads with contraction to be the following pullback of categories:

$$\begin{array}{ccc} T^{(n)}\text{-}\mathbf{OC} & \longrightarrow & \mathbf{Contr}_{\mathcal{F}^{(n)}}(T^{(n)}1) \\ \downarrow & \lrcorner & \downarrow V' \\ T^{(n)}\text{-}\mathbf{Opd} & \xrightarrow{V} & \mathcal{V}\text{-}\mathbf{Gph}^{(n)}/T^{(n)}1, \end{array}$$

where the forgetful functors  $V$  and  $V'$  are the suitable instances of (3) and (5) respectively. If the category  $T^{(n)}\text{-}\mathbf{OC}$  has an initial object  $((\text{ar}_L: L \longrightarrow T^{(n)}1), m, e, \kappa)$ , then by the initial  $T^{(n)}$ -operad with contraction we mean its underlying  $T^{(n)}$ -operad  $((\text{ar}_L: L \longrightarrow T^{(n)}1), m, e)$  (forgetting the contraction  $\kappa$ ).

Our final task, therefore, is to prove that  $T^{(n)}\text{-}\mathbf{OC}$  actually has an initial object, perhaps under some additional condition on  $\mathcal{V}$ . For this, we have the following theorem.

**Theorem 7.1** *If  $\mathcal{V}$  is a locally presentable and extensive category, then for any natural number  $n$  the category  $T^{(n)}\text{-}\mathbf{OC}$  has an initial object.*

**Proof.** We shall follow the argument in [16, Appendix G] (where  $\mathcal{V} = \mathbf{Set}$  and  $n = \omega$ ) and show that  $\mathcal{V}\text{-}\mathbf{Gph}^{(n)}/T^{(n)}1$  is locally presentable (hence is both complete



and cocomplete), and that  $V$  and  $V'$  are monadic and have ranks (i.e., preserve  $\alpha$ -filtered colimits for some regular cardinal  $\alpha$ ). Then by [13, Theorem 27.1] it follows that the forgetful functor from  $T^{(n)}\text{-OC}$  to  $\mathcal{V}\text{-Gph}^{(n)}/T^{(n)}1$  is also monadic, thus in particular  $T^{(n)}\text{-OC}$  has an initial object, given by the free algebra over the initial object in  $\mathcal{V}\text{-Gph}^{(n)}/T^{(n)}1$ .

If  $\mathcal{V}'$  is locally presentable, then so is  $\mathcal{V}'\text{-Gph}$  (see e.g., [15, Proposition 4.4]). Applying this iteratively, we see that  $\mathcal{V}\text{-Gph}^{(n)}$  is locally presentable. Hence so is  $\mathcal{V}\text{-Gph}^{(n)}/T^{(n)}1$ , being its slice. The functor  $V$  is monadic because it is the forgetful functor from a category of monoids and admits a left adjoint  $G$  (which, incidentally, is of a particularly simple form  $GP = \coprod_{n \in \mathbb{N}} P^{\otimes n}$  thanks to Proposition 4.3). It is routine to show that  $V$  has a rank. That the functor  $V'$  is monadic and has a rank can be deduced easily from [2].  $\square$

The condition of  $\mathcal{V}$  being locally presentable and extensive is an axiomatic reason why Batanin and Leinster's approach works. Of course the category **Set** satisfies this condition, but in their work this fact is used only implicitly, often in the form of concrete set-theoretic manipulation.

**Definition 7.2** Let  $\mathcal{V}$  be a locally presentable extensive category and  $n$  a natural number. A **weak  $n$ -dimensional  $\mathcal{V}$ -category** is an algebra of the initial  $T^{(n)}$ -operad with contraction, where  $T^{(n)}$  is the free strict  $n$ -dimensional  $\mathcal{V}$ -category monad on  $\mathcal{V}\text{-Gph}^{(n)}$ .

We remark that when  $\mathcal{V}(1, -)$  is not conservative, it might be more appropriate to replace (i) of Definition 6.2 by the family of morphisms  $0 \rightarrow X$  and  $X + X \rightarrow X$  (codiagonal) where  $X$  ranges over a set of strong generators of  $\mathcal{V}$  (exists if  $\mathcal{V}$  is locally presentable). We thank an anonymous reviewer for pointing this out.

**Example 7.3** If we let  $\mathcal{V} = \mathbf{Set}$  and  $n = 2$ , then weak 2-categories (weak 2-dimensional **Set**-categories) are equivalent to *unbiased bicategories*, which are a variant of bicategories equipped with for each natural number  $m$ , an  $m$ -ary horizontal composition operation. See [16, Section 9.4] for details.

**Example 7.4** If we let  $\mathcal{V} = \omega\text{-Cpo}$  and  $n = 2$ , then weak 2-dimensional  $\omega\text{-Cpo}$ -categories are the unbiased version of  $\omega\text{-Cpo}$ -enriched bicategories as in [18].

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