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Computability of Solutions of Operator Equations

Volker Bosserhoff[1,2]

Institut für Theoretische Informatik und Mathematik
Universität der Bundeswehr
Munich, Germany

Abstract

We study operator equations within the Turing machine based framework for computability in analysis. Is there an algorithm that maps pairs (T, u) (where T is given in form of a program) to good approximate solutions of $Tx = u$? Here we consider the case when T is a bounded linear mapping of Hilbert spaces. We are in particular interested in computing the *generalized inverse* $T^\dagger u$ which is the standard concept of solution in the theory of inverse problems. Typically, $T^\dagger$ is discontinuous (i.e. the equation $Tx = u$ is *ill-posed*) and hence no computable mapping. However, we will use effective versions of theorems from the theory of *regularization* to show that the mapping $(T, T^*, u, \|T^\dagger u\|) \mapsto T^\dagger u$ is computable. We then go on to study the computability of *average-case solutions* with respect to Gaussian measures which have been considered in *information based complexity*. Here $T^\dagger$ is considered as an element of an L^2-space. We define suitable representations for such spaces and use the results from the first part of the paper to show that $(T, T^*, \|T^\dagger\|_{L^2}) \mapsto T^\dagger$ is computable.

Keywords: computable functional analysis, operator equations, regularization, Gaussian measures

1 Introduction

1.1 Ill-posed operator equations

We investigate the following question: Given two computable (real or complex) normed spaces X and Y and a program which computes a bounded linear mapping $T : X \to Y$, can we effectively find a programm which computes

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[2] Email: volker.bosserhoff@unibw.de

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T^{-1}? The model of computability on continuous objects on which we will found our considerations shall be the Turing machine based approach of [20] and especially its extension to separable normed spaces (as described e.g. in the introductory sections of [3]). We assume familiarity with these concepts. [3]

Of course, T^{-1} is only well-defined for injective T. In the case of X and Y being Banach spaces and T being bijective, Brattka's computable version of Banach's Inverse Mapping Theorem applies: T^{-1} is computable if T is computable, but there is no effective way to transform a program for T into a program for T^{-1} (see [3] or [4]).

If we restrict ourselves to X and Y being Hilbert spaces, there is a broader concept of solution for $Tx = u$ which is well-established in the theory of inverse problems and allows us to also handle non-injective T – the Moore-Penrose *generalized inverse* $T^\dagger$. It is defined as a *best approximate solution* which is a minimizer of the residual $\|Tx - u\|$ and minimizes $\|x\|$ among all minimizers of the residual; $T^\dagger u$ is well-defined for all $u \in \operatorname{range} T \oplus (\operatorname{range} T)^\perp$ and is a closed linear mapping. A detailed treatment of generalized inverses is given in [9]; the most important facts can also be found in [8, Section 2.1].

$T^\dagger$ is bounded exactly when $\operatorname{range} T$ is closed. In many important applications however – especially when T is a compact [4] operator with infinite dimensional range – this is not fulfilled. Some examples are given in [11,8]. The problem of approximating unbounded linear mappings $S :\subseteq Y \to X$ is considered as an *ill-posed problem*, a term going back to Hadamard.

It is a fundamental fact in computable analysis that computable mappings are necessarily continuous. So there is no hope for $T^\dagger$ to be computable in the ill-posed case. Pour-El and Richards' *First Main Theorem* [14] even states the following (here we give the formulation of Brattka [2]):

Theorem 1.1 *Let Y, X be computable Banach spaces and let $S :\subseteq Y \to X$ be a closed and unbounded linear operator. Suppose there is a computable sequence $(e_n)_{n\in\mathbb{N}}$ in $\operatorname{dom} S$ such that $\operatorname{span}\{e_n\}_{n\in\mathbb{N}}$ is dense in Y and $(Se_n)_{n\in\mathbb{N}}$ is a computable sequence in X. Then there exists a computable point $u_0 \in Y$ such that $Su_0 \in X$ is not computable.* □

It is easy to construct a computable compact operator $T : l^2 \to l^2$ such that $T^\dagger$ fulfills the assumptions of the theorem [5].

General linear ill-posed problems have also been studied in the context

[3] In particular, we will make use of the representations δ_X, $\nu_{\mathbb{N}}$, ρ and $\rho_>$ as defined in [20,3].

[4] B bounded $\Rightarrow \overline{T(B)}$ compact

[5] For example the diagonal operator mapping each unit vector e_i to $\frac{1}{i}e_i$.

of *information-based complexity* (IBC)[6]. In this framework, one question is whether the "information" on some $u \in D$ which is retrieved by applying a finite collection of continuous linear real functions to it already determines the element Su up to a finite precision. Werschulz [21, Theorem 2.1] obtained the following negative result:

Theorem 1.2 *If Y and X are normed spaces and $S : D \to X$ is a linear unbounded mapping defined on some linear subspace D of Y, then for any $f_0, \ldots, f_{n-1} \in \mathcal{L}(Y, \mathbb{R})$ and any $C > 0$ there are $u_1, u_2 \in D$, $\|u_1\|, \|u_2\| \le 1$, such that*

$$f_i(u_1) = f_i(u_2) \text{ for all } 0 \le i \le n$$

and

$$\|S(u_1) - S(u_2)\| > C.$$

□

Traub and Werschulz (in Chapter 6 of [17]) put this result in analogy to Pour-El and Richards' First Main Theorem.

1.2 Regularization

In numerical mathematics, methods have been developed to partly overcome the difficulties related to solving ill-posed operator equations $Tx = u$. A standard approach (which goes back to ideas of Tikhonov and Phillips) is to substitute the original equation by a sequence of "near by" well-posed equations, so called *regularized versions*. The solutions of these equations then converge to the solution of the original equation. In general however, one does not know how close an approximation obtained via regularization is to the actual solution, unless a-priori information on the solution is available. There is a vast literature on regularization methods; see for example [8].

The first part of the paper is devoted to an effective version of a general regularization method: In Section 2 we will first give some background on the Spectral Theorem for bounded self-adjoint operators and on operator calculus. Then, in Section 3, we give an effective version of a theorem of Groetsch and Jacobs, which says that we can compute $T^\dagger u$ from u, T, T^* and $\|T^\dagger u\|$.

1.3 Average case solutions

In information based complexity, Werschulz and others have also studied ill-posed linear approximation problems $S :\subseteq Y \to X$ in an *average case setting*.

[6] Introductions to IBC can be found e.g. in the monographs [15,22,13,17].

In this setting, one additionally assumes that $D := \operatorname{dom} S$ and S are measurable and considers a measure μ on X such that $\mu(D) = 1$. The question now is if for any given precision ϵ there are elements $f_0, \ldots, f_n \in \mathcal{L}Y, \mathbb{R}$ and a mapping (a so called "algorithm") $\Phi : \mathbb{R}^n \to X$ such that the *expected error*

$$\int_D \|\Phi(f_0(u), \ldots, f_n(u)) - Su\|^2 \, \mu(du)$$

is smaller than ϵ.

An important positive result in this context (see [21,12,22,18] and the survey [16]) is that for Y and X being real separable Banach spaces and γ being a centered Gaussian measure on Y, every γ-measurable linear mapping (see below for the definition) is in $L^p(Y, \gamma; X)$, and the finite rank mappings of the form

$$[y \mapsto f_0(y)x_0 + f_1(y)x_1 + \ldots + f_{n-1}(y)x_{n-1}], \quad f_i \in Y^*, \, x_i \in X$$

are L^p-dense among these for all $p \geq 1$. This clearly implies that linear ill-posed problems are solvable on the average with respect to Gaussian measures. In [22, Section 7.5.1] this is shown for X being a Hilbert space and $p = 2$; in [18] the version just stated is implicit. For completeness we give a proof in Section 5 after having collected prerequisites on Gaussian measures in Section 4.

We will see below that $T^\dagger$ is measurable, so the just stated IBC result fully applies to it. From the point of view of computability theory the question arises under what circumstances suitable functionals f_i and elements x_i can be found effectively. More precisely, we ask the following question:

Let X and Y be computable real Hilbert spaces and let γ be a Gaussian measure on Y. Is there an effective procedure which transforms every $m \in \mathbb{N}$ and every program for some $T : X \to Y$ into a number $n \in \mathbb{N}$ and a vector $(a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1}) \in Y^n \times X^n$ such that

$$\|T^\dagger - \langle a_0, .\rangle b_0 + \ldots + \langle a_{n-1}, .\rangle b_{n-1}\|_{L^2(Y,\gamma;X)} \leq 2^{-m}?$$

In order to study this question properly we will define an effectivity structure on $L^2(Y, \gamma; X)$ in Section 6. If we allow both a program for T^* and a list of all rational upper bounds of $\|T^\dagger\|_{L^2(Y,\gamma;X)}$ as additional inputs, then we can construct an algorithm for the above task. This is our main result and will be proved in Section 7. Our algorithm relies in an essential way on the effective regularization methods from the first part of the paper.

1.4 Acknowledgement

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2 The Spectral Theorem and operator calculus for bounded self-adjoint operators

For normed spaces X and Y, let $\mathcal{L}(X,Y)$ denote the space of bounded linear mappings[7] from X into Y. For convenience: $\mathcal{L}(X) := \mathcal{L}(X,X)$.

For a (real or complex) Hilbert space X and any self-adjoint operator $T \in \mathcal{L}(X)$ let $\sigma(T)$ denote the spectrum and let m_T and M_T be the *spectral bounds* of T, i.e. m_T is the smallest and M_T the largest element of $\sigma(T)$ (cf. [10, p. 117]). Remember $\|T\| = \max\{|m_T|, |M_T|\}$.

We now state (a reduced version of) the Spectral Theorem (cf. [10, Theorem 5.2.7]):

Theorem 2.1 *Let X be a Hilbert space and let $T \in \mathcal{L}(X)$ be self-adjoint. Let $(E_\lambda)_{\lambda\in\mathbb{R}} \in (\mathcal{L}(X))^{\mathbb{R}}$ be the spectral family*[8] *generated by T. Then we have*

(i) $E_{\lambda_2} - E_{\lambda_1}$ is non-negative self-adjoint for all $\lambda_2 \geq \lambda_1$.

(ii)
$$T = \int_a^b \lambda \, dE_\lambda$$
for all $a < m_T$ and $b \geq M_T$.

□

Here the integral is to be interpreted as an *operator valued Riemann-Stieltjes integral*: For every $f \in C[a,b]$

$$\int_a^b f(\lambda)\, dE_\lambda$$

is defined in analogy to the classical Riemann-Stieltjes integral with Riemann sums of the form
$$\sum_{i=1}^{k} f(\hat{\lambda}_i)(E_{\lambda_i} - E_{\lambda_{i-1}})$$
with $a = \lambda_0 < \ldots < \lambda_n = b$ and $\hat{\lambda}_i \in [\lambda_{i-1}, \lambda_i]$.

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[7] It is more common to denote this space by $\mathcal{B}(X,Y)$, but we prefer to use the symbol $\mathcal{B}$ in connection with Borel σ-algebras.

[8] See [10, p. 182]. The definition of the spectral family and its further properties will not be important in this paper.

The following facts can be found in Section 3.3 of [10]:

Theorem 2.2 *Let T and $(E_\lambda)_{\lambda\in\mathbb{R}}$ be as in the Spectral Theorem. Let f be a real function which is continuous on the interval $[a, M_T]$ for some $a < m_T$. Then $\int_a^b f(\lambda)\, dE_\lambda =: f(T)$ exists for all $b \geq M_T$ (and does not depend on the choice of a, b). The following properties hold:*

(i) *The operator $f(T)$ is self-adjoint.*

(ii) *The mapping $f \mapsto f(T)$ is linear.*

(iii) $(fg)(T) = f(T)g(T)$.

(iv) *For any real polynomial $p(x) = \sum_{i=1}^n a_i x^i$, we have $p(K) = \sum_{i=1}^n a_i K^i$.*

(v) $\|f(T)\| \leq \max\{|f(x)| \ : \ x \in [a, M_T]\}$.

□

The next lemma will be very useful below.

Lemma 2.3 *Let T and $(E_\lambda)_{\lambda\in\mathbb{R}}$ be as in the Spectral Theorem. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of continuous functions such that $f_n \in C[a_n, M_T]$ for some $a_n < m_T$. Further assume that*

$$(\forall\, t \in [\max\{a_n, a_m\}, M_T])\ [n \leq m \ \Rightarrow \ f_n(t) \leq f_m(t)].$$

We then have:

(i)

$$(\forall\, h \in X)\ \ [n \leq m \ \Rightarrow \ \|f_n(T)h\| \leq \|f_m(T)h\|]\,.$$

(ii) *Under the additional assumption that*

$$(\forall\, n \in \mathbb{N})(\forall\, t \in [a_n, M_T])\ f_n(t) \geq 0$$

and that

$$\lim_{n\to\infty} f_n(T)h =: u$$

exists for some $h \in X$, we have

$$(\forall\, n \in \mathbb{N})\ \|u - f_n(T)h\|^2 \leq \|u\|^2 - \|f_n(T)h\|^2.$$

Proof. Let $h \in X$ be arbitrary and $n \leq m$. We consider Riemann sums over

$[\max\{a_n, a_m\}, M_T]$.

$$\left\langle \sum_{i=1}^{k} f_n(\hat{\lambda}_i)^2 (E_{\lambda_i} - E_{\lambda_{i-1}})h, h \right\rangle = \sum_{i=1}^{k} f_n(\hat{\lambda}_i)^2 \underbrace{\langle (E_{\lambda_i} - E_{\lambda_{i-1}})h, h\rangle}_{\in \mathbb{R}, \geq 0}$$
$$\leq \sum_{i=1}^{k} f_m(\hat{\lambda}_i)^2 \langle (E_{\lambda_i} - E_{\lambda_{i-1}})h, h\rangle = \left\langle \sum_{i=1}^{k} f_m(\hat{\lambda}_i)^2 (E_{\lambda_i} - E_{\lambda_{i-1}})h, h \right\rangle.$$

Taking a limit over Riemann sums we get

$$\langle f_n(T)^2 h, h\rangle \leq \langle f_m(T)^2 h, h\rangle.$$

As $f_n(T)$ and $f_m(T)$ are self-adjoint we get

$$\|f_n(T)h\|^2 \leq \|f_m(T)h\|^2.$$

For the proof of (ii) we proceed similarly:

$$\begin{aligned} &\left\langle \sum_{i=1}^{k} f_n(\hat{\lambda}_i)^2 (E_{\lambda_i} - E_{\lambda_{i-1}})h, h \right\rangle \\ &= \sum_{i=1}^{k} f_n(\hat{\lambda}_i) \underbrace{f_n(\hat{\lambda}_i)\langle (E_{\lambda_i} - E_{\lambda_{i-1}})h, h\rangle}_{\geq 0} \\ &\leq \sum_{i=1}^{k} f_m(\hat{\lambda}_i) f_n(\hat{\lambda}_i)\langle (E_{\lambda_i} - E_{\lambda_{i-1}})h, h\rangle \\ &= \left\langle \sum_{i=1}^{k} f_m(\hat{\lambda}_i) f_n(\hat{\lambda}_i)(E_{\lambda_i} - E_{\lambda_{i-1}})h, h \right\rangle. \end{aligned}$$

Taking a limit over Riemann sums we get

$$\langle f_n(T)^2 h, h\rangle \leq \langle f_m(T) f_n(T) h, h\rangle$$

and hence by self-adjointness

$$\|f_n(T)h\|^2 \leq \langle f_n(T)h, f_m(T)h\rangle.$$

For $m \to \infty$ this yields

$$\|f_n(T)h\|^2 \leq \langle f_n(T)h, u\rangle.$$

¿From this we get

$$\|u - f_n(T)h\|^2 = \|u\|^2 - 2\langle u, f_n(T)h\rangle + \|f_n(T)h\|^2 \leq \|u\|^2 - \|f_n(T)h\|^2.$$

□

3 Approximation of $T^\dagger$ by computable sequences

The following theorem (going back to Groetsch and Jacobs) can be found in [8, Theorem 4.1]. It is the key to our further investigations.

Theorem 3.1 *Let X, Y be Hilbert spaces and $T \in \mathcal{L}(X,Y)$. Let $(f_n)_{n\in\mathbb{N}} \in (C[0,\|T\|^2])^{\mathbb{N}}$ be a family of continuous functions such that*

$$(\forall\, t \in (0, \|T\|^2])\ \lim_{n\to\infty} f_n(t) = 1/t,$$

and

$$\sup\{|tf_n(t)| \ :\ t \in [0,\|T\|^2],\, n \in \mathbb{N}\} < \infty.$$

Then

$$\lim_{n\to\infty} f_n(T^*T)T^*g = T^\dagger g$$

for every $g \in \operatorname{dom} T^\dagger$. If $g \notin \operatorname{dom} T^\dagger$ then

$$\lim_{n\to\infty} \|f_n(T^*T)T^*g\| = \infty.$$

□

There are of course many possible choices for the f_k in the above theorem. We will now fix one (which leads to the method of *Tikhonov regularization*, cf. [8, Chapter 5]):

Corollary 3.2 *Let X, Y be Hilbert spaces. Let $T \in \mathcal{L}(X,Y)$. Set*

$$f_k(t) := \frac{1}{t + \frac{1}{k+1}}$$

for $k \in \mathbb{N}$, $t > -(k+1)^{-1}$. Then for any $g \in \operatorname{dom} T^\dagger$

(i) $\lim_{k\to\infty} f_k(T^*T)T^*g = T^\dagger g$,

(ii) $\|f_k(T^*T)T^*g\|$ *grows monotonously in k,*

(iii) $\|T^\dagger g - f_k(T^*T)T^*g\|^2 \leq \|T^\dagger g\|^2 - \|f_k(T^*T)T^*g\|^2$.

Proof. *It is easy to verify that the f_k fulfill the assumptions of Theorem 3.1 which yields (i). (ii) and (iii) follow from Lemma 2.3: One just has* *to remember that* $\sigma(T^*T) \subseteq [0, \|T\|^2]$ *and observe that each* f_k *is continuous on some compact interval* $[a_k, \|T\|^2]$, $a_k < 0$. □

Corollary 3.3 *Let* X, Y *be Hilbert spaces and* $T \in \mathcal{L}(X, Y)$. *Then* $\operatorname{dom} T^\dagger$ *and* $T^\dagger$ *are measurable.*

Proof. By Theorem 3.1 and Corollary 3.2 there is a sequence of continuous mappings such that $\operatorname{dom} T^\dagger$ is its domain of convergence and $T^\dagger$ is its pointwise limit. □

Our next aim is an effective version of Corollary 3.2. We introduce computable Hilbert spaces:

Definition 3.4 A computable normed space $(X, \|.\|, \alpha)$ (see [3]) is a *computable Hilbert space* if $(X, \|.\|)$ is complete and $\|.\|$ is induced by an inner product.

[6] contains effective versions of many classical results on Hilbert spaces. Computable Hilbert spaces are also considered in [5].

Remark 3.5 The inner product of a computable Hilbert space is always computable (by the polarization identity).

The operator calculus from the previous section can be made effective. This fact has already been used (in a non-uniform way) by Pour-El and Richards [14] to prove their Second Main Theorem. A detailed derivation of uniform versions appears in [7]. We give a brief proof for the special case we will need.

Theorem 3.6 *Let* X *be a computable Hilbert space with Cauchy representation* δ_X *(see [3]). Let* f_k *be as in Corollary 3.2. The mapping*

$$\{(T, k, h) \;:\; T \in \mathcal{L}(X) \text{ non-negative self-adjoint},\ k \in \mathbb{N},\ h \in H\} \to H,$$
$$(T, k, h) \mapsto f_k(T)h$$

is $([\delta_X \to \delta_X], \nu_{\mathbb{N}}, \delta_X)$*-computable.*

Proof. It suffices to demonstrate how to effectively find a 2^{-m}-approximation to $f_k(T)h$ given m, k, a rapidly converging sequence $(h_i)_{i\in\mathbb{N}}$ (i.e. $\|h - h_i\| \le 2^{-i}$) of approximations for $h \in X$, and a $[\delta_X \to \delta_X]$-name of T.

By [3, Theorem 9.10] we can compute a number $s \in \mathbb{Q}$ such that $\|T\| < s$, hence $\sigma(T) \subset [0, s]$. f_k can be evaluated on $I_k := [-\frac{1}{k+2}, s]$. Remember that $\|f_k(T)\| \le \sup f_k(I_k) =: r$. r is simply $(k+1)(k+2)$, so we can effectively choose some $i \in \mathbb{N}$ such that $r2^{-i} \le 2^{-m-2}$. Then it is possible to effectively choose an upper bound $q \in \mathbb{Q}$ for $\|h_i\|$. By the Effective Weierstrass Theorem (see [20]), we can effectively find a polynomial p such that $\sup_I |f_k - p| \leq q^{-1}2^{-m-2}$. So $\|f_k(T) - p(T)\| = \|(f_k - p)(T)\| \leq q^{-1}2^{-m-1}$. We have

$$\begin{aligned}\|p(T)h_i - f_k(T)h\| &\leq \|p(T)h_i - f(T)h_i\| + \|f_k(T)h_i - f_k(T)h\| \\ &\leq \|(f_k - p)(T)\|\|h_i\| + \|f_k(T)\|\|h_i - h\| \\ &\leq q^{-1}2^{-m-2}q + r2^{-i} \\ &\leq 2^{-m-1}.\end{aligned}$$

$p(T)h_i$ can be approximated effectively; we compute a 2^{-m-1}-approximation, say y. Then we finally have $\|y - f_k(T)h\| \leq 2^{-m}$. □

It is well-known that the adjoint T^* of a bounded linear mapping is itself bounded. The mapping $T \mapsto T^*$ however is not computably invariant and hence not computable (see [6]). In view of this fact the following definition makes sense:

Definition 3.7 Let X, Y be computable Hilbert spaces with Cauchy representation δ_X, δ_Y, respectively. Define a representation $\Delta^{\rightarrow}_{X,Y}$ of $\mathcal{L}(X,Y)$ by

$$\Delta^{\rightarrow}_{X,Y}\langle p, q\rangle = T \;:\Longleftrightarrow\; [\delta_X \to \delta_Y](p) = T \;\wedge\; [\delta_Y \to \delta_X](q) = T^*.$$

This is the weakest representation of $\mathcal{L}(X,Y)$ which allows the evaluation of both the mapping and its adjoint. This fact will be used implicitly from now on.

Remark 3.8 Let $\mathcal{K}(X,Y) \subseteq \mathcal{L}(X,Y)$, $\mathcal{K}(Y,X) \subseteq \mathcal{L}(Y,X)$ be the subspaces of all compact mappings. There are representations of $\mathcal{K}(X,Y)$, $\mathcal{K}(Y,X)$ that are stronger than $[\delta_X \to \delta_Y]|^{\mathcal{K}(X,Y)}$, $[\delta_Y \to \delta_X]|^{\mathcal{K}(Y,X)}$ and with respect to which $K \mapsto K^*$ is computable. (This is the *Computable Schauder Theorem* proved in [6].) That representation of $\mathcal{K}(X,Y)$ is hence stronger than $\Delta^{\rightarrow}_{X,Y}|^{\mathcal{K}(X,Y)}$.

The next theorem is obtained as a direct combination of Corollary 3.2, Theorem 3.6 and the definition of $\Delta^{\rightarrow}_{X,Y}$:

Theorem 3.9 *Let X, Y be computable Hilbert spaces. Define a set*

$$A_1 := \{(T, g) \;:\; T \in \mathcal{L}(X,Y),\; g \in \operatorname{dom} T^\dagger\}.$$

There exists a $([\Delta^{\rightarrow}_{X,Y}, \delta_Y]|^{A_1}, \nu_{\mathbb{N}}, \delta_X)$-computable mapping

$$\mathrm{GI}_1 : A_1 \times \mathbb{N} \to X$$

such that

(i) $\lim_{k\to\infty} \mathrm{GI}_1(T, g, k) = T^\dagger g$,

(ii) $\|\mathrm{GI}_1(T,g,k)\|$ *grows monotonously in* k,

(iii) $\|T^\dagger g - \mathrm{GI}_1(T,g,k)\|^2 \leq \|T^\dagger g\|^2 - \|\mathrm{GI}_1(T,g,k)\|^2$.

□

Corollary 3.10 *Let* X, Y *be computable Hilbert spaces. Define a set*

$$A_2 := \{(T,g,c) \ : \ T \in \mathcal{L}(X,Y),\ g \in \operatorname{dom} T^\dagger, c = \|T^\dagger g\|\}.$$

The mapping

$$\mathrm{GI}_2 : A \to X,\ (T,g,c) \mapsto T^\dagger g$$

is $([\Delta^{\rightarrow}_{X,Y}, \delta_Y, \rho_>]|^{A_2}, \delta_X)$*-computable.*

Proof. As we are given T and g in suitable form, we can use GI_1 from the previous theorem and compute a sequence converging to $T^\dagger g$. We additionally have a list of all rational upper bounds of $\|T^\dagger g\|$; so for every $m \in \mathbb{N}$ we can effectively (by exhaustive search) find some k_m with $\|T^\dagger g\|^2 - \|\mathrm{GI}_1(T,g,k_m)\|^2 \leq 2^{-2m}$. Item (iii) of the previous theorem yields that $\mathrm{GI}_1(T,g,k_m)$ then is a 2^{-m}-approximation for $T^\dagger g$. □

4 Prerequisites on Gaussian measures

In this section we collect some definitions and facts from the theory of Gaussian measures. Details can be found in [1]. We also point the reader to [19] which is a comprehensive treatment of general probability distributions on infinite dimensional vector spaces.

Note that while the results of the previous sections hold for real or complex spaces, we will from now on restrict ourselves to real spaces.

Definition 4.1 A Borel probability measure γ on $\mathbb{R}$ is called *Gaussian* if there is some $a \in \mathbb{R}$ such that γ is either the Dirac measure δ_a at a or has density

$$t \mapsto \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(t-a)^2}{2\sigma^2}\right)$$

for some $\sigma > 0$. a is called the *mean*, σ^2 the *variance* of γ.

This definition can be generalized to a wide class of topological vector spaces: A *locally convex space (l.c.s.)* X is a real topological vector space whose topology is generated by a family $\{p_\alpha\}_{\alpha \in A}$ of seminorms separating[9] the points in X. There is a smallest σ-algebra on X with respect to which

[9] A family $\{f_\alpha\}$ of functions on a space X is said to separate the points in X if for any $x, y \in X$, $x \neq y$, there is an α such that $f_\alpha(x) \neq f_\alpha(y)$.

all elements of X^* are[10] measurable; this σ-algebra is called the *cylindrical σ-algebra* on X and is denoted by $\mathcal{E}(X)$. $\mathcal{E}(X)$ coincides with the σ-algebra $\mathcal{B}(X)$ of Borel sets (i.e. the σ-algebra generated by all open sets) if X is complete and metrizable.

Definition 4.2 Let X be a l.c.s. A probability measure γ on $\mathcal{E}(X)$ is called *Gaussian* if, for any $f \in X^*$, the induced measure $\gamma \circ f^{-1}$ on $\mathbb{R}$ is Gaussian. Here the *mean* a_γ of γ is the element of the algebraic dual $(X^*)'$ to X^* defined by

$$a_\gamma(f) = \int_X f(x)\,\gamma(dx),$$

and the *covariance operator* $R_\gamma : X^* \to (X^*)'$ is defined by the formula

$$R_\gamma(f)(g) = \int_X [f(x) - a_\gamma(f)]\,[g(x) - a_\gamma(g)]\ \gamma(dx).$$

Let $\mathcal{G}(X)$ denote the set of all Gaussian measures on X, and let $\mathcal{G}_0(X) := \{\gamma \in \mathcal{G}(X) \ : \ a_\gamma = 0\}$ be the set of all *centered* Gaussian measures on X.

A Gaussian measure γ is uniquely defined by its covariance operator and mean. It has strong order p for every $p \geq 1$, i.e.

$$\int_X \|x\|^p\,\gamma(dx) < \infty.$$

The *reproducing kernel Hilbert space* $X^*_\gamma \subseteq L^2(\gamma)$ of some $\gamma \in \mathcal{G}(X)$ is the closure of the set

$$\{f - a_\gamma(f) \ : \ f \in X^*\}$$

embedded into $L^2(\gamma)$. For centered γ, we will not distinguish between an element f of X^* and its equivalence class in X^*_γ. The elements of X^*_γ are real Gaussian random variables. The covariance operator extends to X^*_γ:

$$R_\gamma(f)(g) := \int_X f(x)\,[g(x) - a_\gamma(g)]\ \gamma(dx),\ f \in X^*_\gamma, g \in X^*.$$

It is an important feature of Gaussian measures that a collection $V \subseteq X^*_\gamma$ of Gaussian random variables is independent exactly when its elements are pairwise uncorrelated, i.e., $R_\gamma(f)(g) = 0$ for all $f, g \in V$, $f \neq g$.

¿From now on we only consider centered Gaussian measures on separable Banach spaces.

[10] In this paper, X^* shall refer to the topological and X' to the algebraic dual of a (topological) linear space X.

Lemma 4.3 *Let X be a separable Banach space and $\gamma \in \mathcal{G}_0(X)$.*

(i) For every $f \in X_\gamma^*$, there is a unique point $x_f \in X$ such that $R_\gamma(f)(g) = g(x_f)$ for all $g \in X^*$. We will from now on consider R_γ as a mapping into X.

(ii) The image $R_\gamma(X_\gamma^*)$ is the intersection of all linear subspaces of X which have full γ measure.

□

Remark 4.4 For a Hilbert space X we can identify X^* with X, so we can define R_γ on X. If X is additionally separable then, by the previous lemma, R_γ maps X into itself. For all $x, y \in X$ we have the formula

$$\langle R_\gamma x, y\rangle = \int_X \langle x, \omega\rangle\langle y, \omega\rangle\, \gamma(d\omega).$$

R_γ – considered as an operator on X – is self-adjoint, non-negative and nuclear [11] .

Lemma 4.5 *Let X be a separable Banach space and let $\gamma \in \mathcal{G}_0(X)$. Let $\{f_n\}_{n\in\mathbb{N}} \subset X^*$ be a family of functions separating the points in X. (Such a family always exists; this is a consequence of the Hahn-Banach-Theorem.) Then X_γ^* coincides with the closure of the linear span of $\{f_n\}_{n\in\mathbb{N}}$ in the space $L^2(\gamma)$.* □

Definition 4.6 Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. We denote the *completion* of $\mathcal{A}$ with respect to μ by $\mathcal{A}_\mu$, i.e.

$$\mathcal{A}_\mu := \{A \cup N \ :\ A \in \mathcal{A},\ (\exists\ J \in \mathcal{A})\ \mu(J) = 0,\ N \subset J\}.$$

$\mathcal{A}_\mu$ is a σ-algebra.

Definition 4.7 Let X, Y be locally convex spaces and let μ be a measure on $\mathcal{E}(X)$. A $(\mathcal{E}(X)_\mu, \mathcal{E}(Y))$-measurable mapping $F : X \to Y$ is called a *μ-measurable linear mapping* if there is a linear (in the usual sense) mapping $\widetilde{F} : X \to Y$ such that $F = \widetilde{F}$ μ-a.e. For $Y = \mathbb{R}$ one also speaks of *μ-measurable linear functionals.*

Lemma 4.8 *Let X be a separable Banach space. Then f is a γ-measurable linear functional if and only if $f \in X_\gamma^*$.* □

[11] An self-adjoint operator A on a separable Hilbert space is nuclear, if for every orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$ the sum $\sum\langle Ae_i, e_i\rangle$ converges.

Lemma 4.9 *Let X, Y be locally convex spaces and $\gamma \in \mathcal{G}_0(X)$. Let $F : X \to Y$ be a linear and $(\mathcal{E}(X)_\gamma, \mathcal{E}(Y))$-measurable mapping. Then $\gamma \circ F^{-1} \in \mathcal{G}_0(Y)$.* □

Lemma 4.10 *Let X, Y be locally convex spaces equipped with σ-algebras $\mathcal{A}^1$ and $\mathcal{A}^2$ respectively. Let μ be a measure on $\mathcal{A}^1$. Let $L \in \mathcal{A}^1$ be a linear subspace of X such that $\mu(X \setminus L) = 0$. Let $F : L \to Y$ be an $(\mathcal{A}^1, \mathcal{A}^2)$-measurable linear mapping. There is an $(\mathcal{A}^1_\mu, \mathcal{A}^2)$-measurable linear mapping $F_0 : X \to Y$ that coincides with F on L.* □

5 A representation theorem

Let X be a separable Banach space and $\gamma \in \mathcal{G}_0(X)$. By Lemma 4.5 we can choose a complete orthonormal sequence $(e_i)_{i\in\mathbb{N}}$ in X^*_γ consisting of elements of X^*. For every γ-measurable linear functional f, Lemma 4.8 and Lemma 4.3 yield that the series

$$\sum_i e_i(.)f(R_\gamma(e_i))$$

is an orthogonal expansion of f in $L^2(\gamma)$. So the sequence $(f(R_\gamma e_i))_{i\in\mathbb{N}}$ is in l^2. This, in combination with the fact that the e_i are independent real standard Gaussian random variables yields that the series also converges almost everywhere to f (cf. [1, Theorem 1.1.4]).

Now let Y be another separable Banach space and let $F : X \to Y$ be a γ-measurable linear mapping. For every $f \in Y^*$ we have that $f \circ F$ is a γ-measurable linear functional. So for every $f \in Y^*$

$$\sum_i e_i(.)(f \circ F)(R_\gamma(e_i))$$

converges almost everywhere to $f \circ F$. As F is a Gaussian random element, we have $\int_X \|F\|\, d\gamma < \infty$, and so [19, Exercise V.3.4(b)] yields that the series

$$\sum_i e_i(.)F(R_\gamma(e_i))$$

converges to F almost everywhere. As F has a Gaussian distribution we even have $\int_X \|F\|^p\, d\gamma < \infty$ for every $p \geq 1$. So Theorem V.3.3 in [19] (implication "3 ⇒ 4" for $\Phi(x) = x^p$) yields that the series also converges to F in $L^p(X, \gamma; Y)$. We summarise:

Theorem 5.1 *Let X, Y be separable Banach spaces. Let $\gamma \in \mathcal{G}_0(X)$. Let $F : X \to Y$ be a γ-measurable linear mapping. Let $\{e_n\}_{n\in\mathbb{N}} \subseteq X^*$ be a* *complete orthonormal system in X^*_γ. Then*

$$\sum_{n=0}^{\infty} e_n(.)F(R_\gamma(e_n))$$

converges to F in $L^p(X,\gamma;Y)$ for all $p \geq 1$ and γ-a.e. □

Corollary 5.2 *Let X, Y be separable Banach spaces. Let $\gamma \in \mathcal{G}_0(X)$. Let $S : D \to Y$ be a linear measurable mapping defined on a measurable linear subspace D of X with $\gamma(D) = 1$. Let $\{e_n\}_{n\in\mathbb{N}} \subseteq X^*$ be a complete orthonormal system in X^*_γ. Then*

$$\sum_{n=0}^{\infty} e_n(.)S(R_\gamma(e_n))$$

is well-defined and converges to S in $L^p(D,\gamma;Y)$ for every $p \geq 1$.

Proof. Lemma 4.10 yields that S has a $(\mathcal{B}(X)_\gamma, \mathcal{B}(Y))$-measurable extension F which coincides with S on D. So, by item (ii) of Lemma 4.3, S and F especially coincide on the $R_\gamma(e_n)$. We get:

$$\int_D \|S(x) - \sum_{i=0}^{n-1} e_i(x)S(R_\gamma(e_i))\|^p \,\gamma(dx) = \int_X \|F(x) - \sum_{i=0}^{n-1} e_i(x)F(R_\gamma(e_i))\|^p \,\gamma(dx).$$

Theorem 5.1 immediately yields the claim. □

The following error formula already appears in [15, Section 6.5.3]:

Theorem 5.3 *Let X be a separable Banach space and Y a separable Hilbert space. Let $\gamma \in \mathcal{G}_0(X)$. Let $F : X \to Y$ be a γ-measurable linear mapping. Let $e_0, \ldots e_{n-1} \in X^*$ be orthonormal in X^*_γ. Then*

$$\int_X \|F(x) - \sum_{j=0}^{n-1} e_j(x)F(R_\gamma(e_j))\|^2 \,\gamma(dx) = \int_X \|F(x)\|^2 \,\gamma(dx) - \sum_{j=0}^{n-1} \|F(R_\gamma(e_j))\|^2.$$

Proof. Let $\{a_i\}$ be a complete orthonormal sequence in Y. The $\langle a_i, F(.)\rangle$ are γ-measurable linear functionals and hence elements of X^*_γ. Pythagoras' identity holds:

$$\int_X \left| \langle a_i, F(x)\rangle - \sum_{j=0}^{n-1} e_j(x)\langle a_i, F(R_\gamma(e_j))\rangle \right|^2 \gamma(dx)$$
$$= \int_X \langle a_i, F(x)\rangle^2 \,\gamma(dx) - \sum_{j=0}^{n-1} \langle a_i, F(R_\gamma(e_j))\rangle^2.$$

By this equality and the Monotone Convergence Theorem we get

$$\begin{aligned}
&\int_X \|F(x) - \sum_{j=0}^{n-1} e_j(x) F(R_\gamma(e_j))\|^2 \, \gamma(dx) \\
&= \int_X \sum_{i=0}^{\infty} \langle a_i, F(x) - \sum_{j=0}^{n-1} e_j(x) F(R_\gamma(e_j))\rangle^2 \, \gamma(dx) \\
&= \sum_{i=0}^{\infty} \int_X \langle a_i, F(x) - \sum_{j=0}^{n-1} e_j(x) F(R_\gamma(e_j))\rangle^2 \, \gamma(dx) \\
&= \sum_{i=0}^{\infty} \int_X \left(\langle a_i, F(x)\rangle - \sum_{j=0}^{n-1} e_j(x) \langle a_i, F(R_\gamma(e_j))\rangle \right)^2 \gamma(dx) \\
&= \sum_{i=0}^{\infty} \left(\int_X \langle a_i, F(x)\rangle^2 \, \gamma(dx) - \sum_{j=0}^{n-1} \langle a_i, F(R_\gamma(e_j))\rangle^2 \right) \\
&= \int_X \|F(x)\|^2 \, \gamma(dx) - \sum_{j=0}^{n-1} \|F(R_\gamma(e_j))\|^2 .
\end{aligned}$$

□

Corollary 5.2 implies – translated into the language of information based complexity – that linear problems $S :\subseteq X \to Y$ are solvable on the average with respect to Gaussian measures if X and Y are separable Banach spaces. [12] Of course we would like to benefit from this result in order to solve our model example of a linear ill-posed problem: X, Y are Hilbert spaces and $S = T^\dagger$ for unbounded $T^\dagger$. Given (a program for) T and some prescribed error bound 2^{-m}, can we compute approximations to $T^\dagger$ such that the expected (quadratic) error with respect to some Gaussian measure γ with $\gamma(\mathrm{dom}\, T^\dagger) = 1$ is smaller than 2^{-m}? If we wish to apply the series formula from Theorem 5.1 directly, we encounter the problem that we have to compute elements $T^\dagger(R_\gamma e_i)$. But we know (see introduction) that this is not possible in general. Furthermore we need to determine how many summands of the series have to be computed in order to achieve the error bound. [13]

[12] In the IBC literature we have found this result only for Y being a separable Hilbert space; see [22],[16].

[13] Problems of this kind are usually neglected in IBC: One does not demand an algorithm that is uniform in the precision parameter. Furthermore, one allows an algorithm to make use of "precomputed" constants. This is sometimes justified by the belief that examples in which such constants are "very difficult to precompute" are "exceptional" (cf. [13, NR 2.9.5]).

In the forthcoming sections we will define computability on the space of linear γ-measurable mappings between computable real Hilbert spaces. We will then study the computability of $T \mapsto T^\dagger$ as a mapping into this space. Our main discovery will be a "higher level" analogon to Corollary 3.10. Interestingly, this will be obtained by combining the ideas from this section, which have their origin in IBC, with the ideas from the first part of the paper, which have their origin in the theory of regularization.

6 Computability of γ-measurable linear mappings

Let $(X, \|.\|_X, \alpha)$, $(Y, \|.\|_Y, \beta)$ be computable real Hilbert spaces.[14] In what follows we will denote the norms and inner products in both spaces by $\|.\|$ and $\langle .,. \rangle$ respectively. Let γ be a centered Gaussian measure on X. Consider the set

$$C := \{\langle a_0, .\rangle b_0 + \ldots + \langle a_{n-1}, .\rangle b_{n-1} \;:\; n \in \mathbb{N}, a_i \in X,\, b_i \in Y\}.$$

The elements of C are linear mappings from X into Y. Their (finite) L^2-norm with respect to γ is given by

$$\begin{aligned} \left(\int_X \left\|\sum_{i=0}^{n-1} \langle a_i, x\rangle b_i\right\|^2 \gamma(dx)\right)^{1/2} &= \left(\sum_{i,j=0}^{n-1} \langle b_i, b_j\rangle \int_X \langle a_i, x\rangle \langle a_j, x\rangle \, \gamma(dx)\right)^{1/2} \\ &= \left(\sum_{i,j=0}^{n-1} \langle b_i, b_j\rangle \langle R_\gamma a_i, a_j\rangle\right)^{1/2}. \end{aligned} \tag{1}$$

Let $\mathcal{L}(X,Y)_\gamma$ denote the closure of C in $L^2(X, \gamma; Y)$. If $Y = \mathbb{R}$ then $\mathcal{L}(X,Y)_\gamma$ is just X^*_γ. Theorem 5.1 yields that every γ-measurable linear mapping $F : X \to Y$ is in $\mathcal{L}(X,Y)_\gamma$.[15]

Our aim is to define a fundamental sequence $\Gamma : \mathbb{N} \to \mathcal{L}(X,Y)_\gamma$ in $\mathcal{L}(X,Y)_\gamma$. As $\mathcal{L}(X,Y)_\gamma$ is the closure of C, we have that the linear span of

$$C' := \{\langle a, .\rangle b \;:\; a \in X,\, b \in Y\}$$

is dense in $\mathcal{L}(X,Y)_\gamma$. This implies that any sequence whose linear span is dense in this set is fundamental in $\mathcal{L}(X,Y)_\gamma$. We show that the span of range Γ with

$$\Gamma(\langle i, j\rangle) := \langle \alpha(i), .\rangle \beta(j)$$

[14] α and β are notations of fundamental sequences. A sequence is called fundamental in a topological vector space if the span of its elements is dense in that space.

[15] For $Y = \mathbb{R}$ the converse is also true by Lemma 4.8; the proof of this fact (see Theorem 2.10.9 in [1]) also works for arbitrary Y. So $\mathcal{L}(X,Y)_\gamma$ is exactly the space of all γ-measurable linear mappings from X to Y.

is dense in C'. In fact: For arbitrary $a, \widetilde{a} \in X$ and $b, \widetilde{b} \in Y$ we have the identity

$$\|\langle a,.\rangle b - \langle \widetilde{a},.\rangle \widetilde{b}\|^2_{L^2(X,\gamma;Y)} = \|b\|^2 \langle R_\gamma a, a\rangle - 2\langle b, \widetilde{b}\rangle \langle R_\gamma a, \widetilde{a}\rangle + \|\widetilde{b}\|^2 \langle R_\gamma \widetilde{a}, \widetilde{a}\rangle \quad (2)$$

which clearly shows that $\|\langle a,.\rangle b - \langle \widetilde{a},.\rangle \widetilde{b}\|_{L^2(X,\gamma;Y)} \to 0$ as $\widetilde{a} \to a$, $\widetilde{b} \to b$ (remember that R_γ is continuous). As range α is fundamental in X and range β is fundamental in Y we hence have that

$$\{\langle \widetilde{a},.\rangle \widetilde{b} \ : \ \widetilde{a} \in \operatorname{span} \operatorname{range} \alpha,\ \widetilde{b} \in \operatorname{span} \operatorname{range} \beta\}$$

is dense in C'. But one only has to use the linearity of the inner product to see that this set in fact is the span of range Γ. □

We are now in the position to consider the triple

$$(\mathcal{L}(X,Y)_\gamma, \|.\|_{L^2(X,\gamma;Y)}, \Gamma)$$

and ask whether it is a computable normed space (and hence a computable Hilbert space). The only remaining prerequisite is that the mapping

$$\begin{aligned} &\mathbb{N} \to [0,\infty), \\ &\langle n, i_0, \dots, i_n, j_0, \dots, j_n, \rangle \mapsto \|\sum_{k=0}^{n-1} \nu_{\mathbb{Q}}(i_k)\Gamma(j_k)\|_{L^2(X,\gamma;Y)} \end{aligned}$$

is computable. After one has applied formula (1) and the definition of Γ one directly sees that this will be fulfilled if $(i,j) \mapsto \langle R_\gamma \alpha(i), \alpha(j)\rangle$ is computable. □

We summarise:

Theorem 6.1 *Let $(X, \|.\|_X, \alpha)$, $(Y, \|.\|_Y, \beta)$ be computable real Hilbert spaces. Let $\gamma \in \mathcal{G}_0(X)$ be a Gaussian measure for which $(\langle R_\gamma \alpha(i), \alpha(j)\rangle)_{i,j\in\mathbb{N}}$ is a computable double sequence. Let $\Gamma : \mathbb{N} \to \mathcal{L}(X,Y)_\gamma$ be defined by*

$$\Gamma(\langle i,j\rangle) := \langle \alpha(i), .\rangle \beta(j).$$

then

$$(\mathcal{L}(X,Y)_\gamma, \|.\|_{L^2(X,\gamma;Y)}, \Gamma)$$

is a computable Hilbert space. We denote the associated Cauchy representation by δ_γ. □

Remark 6.2 If D is a linear subspace of X with $\gamma(D) = 1$, we can consider every measurable linear mapping $S : D \to Y$ as an element of $\mathcal{L}(X,Y)_\gamma$ via Lemma 4.10. We can hence also use δ_γ to represent such mappings. We will do so in Theorem 7.4.

Remark 6.3 Our representation of $\mathcal{L}(X,Y)_\gamma$ has some formal similarity to Brattka and Yoshikawa's [6] representation of $\mathcal{K}(X,Y)$ mentioned in Remark 3.8.

Lemma 6.4 *Let the space $(\mathcal{L}(X,Y)_\gamma, \|.\|_{L^2(X,\gamma;Y)}, \Gamma)$ be as in Theorem 6.1.*

(i) The form

$$X \times X \to \mathbb{R},\ (x,y) \mapsto \langle R_\gamma x, y\rangle$$

is $(\delta_X, \delta_X, \rho)$-computable.

(ii) The mapping

$$\begin{aligned}&\mathrm{Embed} : X \times Y \to \mathcal{L}(X,Y)_\gamma,\\ &\mathrm{Embed}(a,b) := \langle a, .\rangle b,\end{aligned}$$

is $(\delta_X, \delta_Y, \delta_\gamma)$-computable.

Proof.

(i) Let $(x_i)_{i\in\mathbb{N}}$ and $(y_j)_{j\in\mathbb{N}}$ be sequences converging rapidly to x, y respectively. We have the estimate

$$\begin{aligned}|\langle R_\gamma x, y\rangle - \langle R_\gamma x_i, y_j\rangle| &\le |\langle R_\gamma x, y - y_j\rangle| + |\langle R_\gamma(x - x_i), y_j\rangle|\\ &\le \|R_\gamma\|(\|x\|\|y - y_j\| + \|x - x_i\|\|y_j\|).\end{aligned}$$

The rest of the proof is standard.

(ii) Let $(a_i)_{i\in\mathbb{N}}$ and $(b_i)_{i\in\mathbb{N}}$ be sequences converging rapidly to a, b respectively. Equation (2) implies that $\langle a_i, .\rangle b_i \to \langle a, .\rangle b$. In connection with item (i), (2) also provides a way to compute $i \mapsto \|\langle a, .\rangle b - \langle a_i, .\rangle b_i\|$. The rest of the proof is standard.

□

7 Main result

Before we prove our main result we need one last auxiliary lemma:

Lemma 7.1 *Let $(X, \|.\|, \alpha)$ be a computable Hilbert space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. One can effectively enumerate a sequence $(i_j)_{j\in\mathbb{N}} \in \mathbb{N}^\mathbb{N}$ and a sequence*

$$(c_0^{(0)}), (c_0^{(1)}, c_1^{(1)}), (c_0^{(2)}, c_1^{(2)}, c_2^{(2)}), \ldots$$

of tuples of elements of $\mathbb{K}$ such that

$$\left(\sum_{j=0}^{k} c_j^{(k)} \alpha(i_j)\right)_{k\in\mathbb{N}}$$

is a complete orthonormal sequence in X.

Proof. The algorithm layed down in [14, Section 4.7] (to prove the *Effective Independence Lemma*) can be applied to obtain a sequence $(i_j)_{k\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ such that $(\alpha(i_j))_{j\in\mathbb{N}}$ is a linearly independent dense sequence in X. Then, by the classical Gram-Schmidt algorithm, we can find a suitable sequence of coefficient vectors. □

Corollary 7.2 *In every computable Hilbert space there exists a computable complete orthonormal sequence.* □

Corollary 7.3 *Let* $(X, \|.\|, \alpha)$ *be a computable Hilbert space and let* γ *be a centered Gaussian measure on* X *for which* $(\langle R_\gamma \alpha(i), \alpha(j)\rangle)_{i,j\in\mathbb{N}}$ *is a computable double sequence. There exists a computable sequence* $(e_k)_{k\in\mathbb{N}} \in X^{\mathbb{N}}$ *such that the sequence* $(\langle e_k, .\rangle)_{k\in\mathbb{N}}$ *is a complete orthonormal system in* X^*_γ.

Proof. First remember that $X^*_\gamma = \mathcal{L}(X, \mathbb{R})_\gamma$ and note that $(\mathcal{L}(X, \mathbb{R})_\gamma, \|.\|_{L^2(X,\gamma;Y)}, \Gamma)$ is a computable Hilbert space by Theorem 6.1. Via Lemma 7.1 we obtain a sequence

$$\left(\sum_{j=0}^{k} c_j^{(k)} \Gamma(i_j)\right)_{k\in\mathbb{N}}$$

that is complete orthonormal in X^*_γ. Consider the definition of Γ to note that this is in fact a sequence in C with elements of the form

$$\langle a_0^{(k)}, .\rangle b_0^{(k)} + \ldots + \langle a_{\nu_k}^{(k)}, .\rangle b_{\nu_k}^{(k)}$$

such that we can compute the $a_i^{(k)} \in X$, $b_i^{(k)} \in \mathbb{R}$. Hence we can also compute the

$$e_k := \sum_{i=0}^{\nu_k} a_i^{(k)} b_i^{(k)}.$$

□

Theorem 7.4 *Let* $(Y, \|.\|_Y, \beta)$, $(X, \|.\|_X, \alpha)$ *be computable Hilbert spaces. Let* γ *be a centered Gaussian measure on* Y *such that* $(\langle R_\gamma \beta(i), \beta(j)\rangle)_{i,j\in\mathbb{N}}$ *is a computable double sequence. Let* $(\mathcal{L}(Y, X)_\gamma, \|.\|_{L^2(Y,\gamma;X)}, \Gamma)$ *be the computable Hilbert space of Theorem 6.1. Define a set*

$$A_3 := \{(T, c) \in \mathcal{L}(X, Y) \times \mathbb{R} \ : \ \gamma(\operatorname{dom} T^\dagger) = 1,\, c = \int_{\operatorname{dom} T^\dagger} \|T^\dagger\|^2 \, d\gamma\}.$$

The mapping

$$\mathrm{GI}_3 : A_3 \to \mathcal{L}(Y, X)_\gamma,\ (T, c) \mapsto T^\dagger,$$

is $([\Delta^{\rightarrow}_{X,Y}, \rho_>]|^{A_3}, \delta_\gamma)$*-computable.*

Proof. By Corollary 7.3 there is a computable sequence $(e_k)_{k\in\mathbb{N}} \in Y^{\mathbb{N}}$ such that the sequence $(\langle e_k, .\rangle)_{k\in\mathbb{N}}$ is complete orthonormal in Y^*_γ. By Corollary 7.2 there is a computable complete orthonormal sequence $(a_j)_{j\in\mathbb{N}}$ in X. Let the $(f_k)_{k\in\mathbb{N}}$ be as in Corollary 3.2. Put

$$A_{i,k,j} := \langle f_k(T^*T)T^*R_\gamma e_i, a_j\rangle.$$

By the self-adjointness of $f_k(T^*T)$ we have

$$A_{i,k,j} = \langle R_\gamma e_i, Tf_k(T^*T)a_j\rangle$$

which yields that the mapping $(T, i, k, j) \mapsto A_{i,k,j}$ is $(\Delta^{\rightarrow}_{X,Y}, \nu_{\mathbb{N}}, \nu_{\mathbb{N}}, \nu_{\mathbb{N}}, \rho)$-computable (see Theorem 3.6 and Lemma 6.4(i)).

For the proof of the theorem it is sufficient to show that we can compute a sequence of (δ_γ-names of) elements $F_m \in L^2(Y, \gamma; X)$ such that F_m is a 2^{-m}-approximation of $T^\dagger$. Let us fix some m now.

Step 1. By Theorem 5.3 we have that

$$\|T^\dagger - \sum_{i=0}^{n-1}\langle e_i, .\rangle T^\dagger R_\gamma e_i\|^2_{L^2} = \|T^\dagger\|^2_{L^2} - \sum_{i=0}^{n-1}\|T^\dagger R_\gamma e_i\|^2$$

(where we have put $\|.\|_{L^2} := \|.\|_{L^2(Y,\gamma;X)}$ for convenience) and Corollary 5.2 yields that these expressions converge to zero as $n \to \infty$. We have

$$\begin{aligned}
&\|T^\dagger\|^2_{L^2}\\
&= \sum_{i=0}^{\infty}\|T^\dagger R_\gamma e_i\|^2\\
&= \sum_{i=0}^{\infty}\lim_{k\to\infty}\|f_k(T^*T)T^*R_\gamma e_i\|^2\\
&= \sum_{i=0}^{\infty}\lim_{k\to\infty}\sum_{j=0}^{\infty}|A_{i,k,j}|^2.
\end{aligned}$$

The input to our algorithm contains a list of all rational upper bounds of

$c = \|T^\dagger\|_{L^2}^2$. By exhaustive search we can hence find n_0, k_0, r_0 such that

$$\begin{aligned}
2^{-2m} &\geq \|T^\dagger\|_{L^2}^2 - \sum_{i=0}^{n_0-1}\sum_{j=0}^{r_0-1} |A_{i,k_0,j}|^2 \\
&\geq \|T^\dagger\|_{L^2}^2 - \sum_{i=0}^{n_0-1} \|f_{k_0}(T^*T)T^*R_\gamma e_i\|^2 \\
&\geq \|T^\dagger\|_{L^2}^2 - \sum_{i=0}^{n_0-1} \|T^\dagger R_\gamma e_i\|^2 \\
&= \|T^\dagger - \sum_{i=0}^{n_0-1} \langle e_i, .\rangle T^\dagger R_\gamma e_i\|_{L^2}^2
\end{aligned}$$

where we have used Corollary 3.2(ii) for the third estimate.

It is hence sufficient to compute

$$\sum_{i=0}^{n_0-1} \langle e_i, .\rangle T^\dagger R_\gamma e_i \in \mathcal{L}(Y,X)_\gamma.$$

In view of Lemma 6.4(ii) it is now sufficient to show that we can compute the elements $T^\dagger R_\gamma e_i$, $0 \leq i \leq n_0$ in X.

Step 2. We show that we can even compute the whole sequence $(T^\dagger R_\gamma e_i)_{i\in\mathbb{N}}$. To that aim we again exploit the information provided by the input c.

Let an arbitrary $m' \in \mathbb{N}$ be given. We show how to compute a $2^{-m'}$-approximation to any $T^\dagger(R_\gamma(e_i))$. We put

$$A_{i,k} := f_k(T^*T)T^*R_\gamma e_i = \sum_{j=0}^{\infty} A_{i,k,j} a_j.$$

By Corollary 3.2(iii) we have

$$\|T^\dagger R_\gamma e_i - A_{i,k}\|^2 \leq \|T^\dagger R_\gamma e_i\|^2 - \|A_{i,k}\|^2.$$

By repeating Step 1 with $m'+1$ instead of m we effectively find n_0', k_0', r_0' such that

$$\begin{aligned}
&2^{-2(m'+1)} \\
&\geq \|T^\dagger\|_{L^2}^2 - \sum_{i=0}^{n_0'-1}\sum_{j=0}^{r_0'-1} |A_{i,k_0',j}|^2 \\
&= \sum_{i=n_0'}^{\infty} \|T^\dagger(R_\gamma(e_i))\|^2 + \sum_{i=0}^{n_0'-1}\left[\|T^\dagger(R_\gamma(e_i))\|^2 - \sum_{j=0}^{r_0'-1}|A_{i,k_0',j}|^2\right] \\
&= \sum_{i=n_0'}^{\infty} \|T^\dagger(R_\gamma(e_i))\|^2 + \sum_{i=0}^{n_0'-1}\left[\|T^\dagger(R_\gamma(e_i))\|^2 - \|A_{i,k_0'}\|^2 + \|A_{i,k_0'}\|^2 - \sum_{j=0}^{r_0'-1}|A_{i,k_0',j}|^2\right] \\
&= \sum_{i=n_0'}^{\infty} \|T^\dagger(R_\gamma(e_i))\|^2 + \sum_{i=0}^{n_0'-1}\left[\|T^\dagger(R_\gamma(e_i))\|^2 - \|A_{i,k_0'}\|^2 + \|A_{i,k_0'} - \sum_{j=0}^{r_0'-1}A_{i,k_0',j}a_j\|^2\right] \\
&\geq \sum_{i=n_0'}^{\infty} \|T^\dagger(R_\gamma(e_i))\|^2 + \sum_{i=0}^{n_0'-1}\left[\|T^\dagger(R_\gamma(e_i)) - A_{i,k_0'}\|^2 + \|A_{i,k_0'} - \sum_{j=0}^{r_0'-1}A_{i,k_0',j}a_j\|^2\right]
\end{aligned}$$

In particular,

$$\|T^\dagger R_\gamma e_i\|^2 \leq 2^{-2(m'+1)} \text{ for all } i \geq n_0'$$

as well as

$$\|T^\dagger R_\gamma e_i - A_{i,k_0'}\|^2 + \|A_{i,k_0'} - \sum_{j=0}^{r_0'-1} A_{i,k_0',j}a_j\|^2 \leq 2^{-2(m'+1)} \text{ for all } \; i < n_0'.$$

The first estimate immediately yields that we can take 0 as $2^{-m'}$-approximation for all $T^\dagger R_\gamma e_i$ with $i \geq n_0'$. For any $i < n_0'$ put (for brevity) $x_1 := T^\dagger R_\gamma e_i$, $x_2 := A_{i,k_0'}$ and $x_3 := \sum_{j=0}^{r_0'-1} A_{i,k_0',j}a_j$. Then, from the second estimate we conclude that

$$\begin{aligned}
\|x_1 - x_3\| &\leq \|x_1 - x_2\| + \|x_2 - x_3\| \\
&= \left(\|x_1 - x_2\|^2 + \|x_2 - x_3\|^2 + 2\|x_1 - x_2\| \cdot \|x_2 - x_3\|\right)^{\frac{1}{2}} \\
&\leq \left(2(\|x_1 - x_2\|^2 + \|x_2 - x_3\|^2)\right)^{\frac{1}{2}} \\
&\leq \left(2^{-2m'-1}\right)^{\frac{1}{2}} \leq 2^{-m'}.
\end{aligned}$$

So $\sum_{j=0}^{r_0'-1} A_{i,k_0',j}a_j$ is a $2^{-m'}$-approximation for $T^\dagger R_\gamma e_i$ if $i < n_0'$. □

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