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$\mathbb{T}^\omega$ as a Stable Universal Domain [1]

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Abstract

In the seventies, G. Plotkin noticed that $\mathbb{T}^\omega$, the cartesian product of ω copies of the 3 elements flat domain of Boolean, is a universal domain, where "universal" means that the retracts of $\mathbb{T}^\omega$ in Scott's continuous semantics are exactly all the ωCC-domains, which with Scott continuous functions form a cartesian closed category. As usual "ω" is for "countably based", and here "CC" is for "conditionally complete", which essentially means that any subset which is pairwise bounded has an upper bound. Since $\mathbb{T}^\omega$ is also an ωDI-domain (an important structure in the stable domain theory), a problem arises naturally: Is $\mathbb{T}^\omega$ a universal domain for Berry's stable semantics? The aim of this paper is to answer this question. We investigate the properties of stable retracts and introduce a new domain named a conditionally complete DI-domain (a $CCDI$-domain for short). We show that, (1) a dcpo is a stable retract of $\mathbb{T}^\omega$ if and only if it is an $\omega CCDI$-domain; (2) the category of $\omega CCDI$-domain (resp. $CCDI$-domains) with stable functions is cartesian closed. So, the problem above has an affirmative answer.

Keywords: universal domain, stable retract, $\omega CCDI$-domain, cartesian closed category

1 Introduction

Domain theory is a general framework for defining program data domains. In this theory, $\mathbb{T}^\omega$, the cartesian product of ω copies of the 3 elements flat domain of Boolean, is a very interesting structure, which can be used as a model to give mathematical semantics for program languages as $P\omega$ presented by D. Scott [11]. In [10], G. Plotkin showed that, $\mathbb{T}^\omega$ is a universal domain in the sense that the retracts of $\mathbb{T}^\omega$ in Scott's continuous semantics form a cartesian closed category. Particularly, its continuous function space $[\mathbb{T}^\omega \to \mathbb{T}^\omega]$ is a retract of $\mathbb{T}^\omega$. R. Kanneganti [8] also investigated $\mathbb{T}^\omega$ in detail. The results of them are all based on the Scott continuous functions.

In domain theory, there is another class of important functions called stable functions, which is introduced firstly by Berry [4]. In 1990, P. Taylor [12] showed that,

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all continuous (resp. algebraic) L-domains with stable functions form a cartesian closed category, its finite products are cartesian ones and its exponentials are stable function spaces. It leads many authors to study those categories for Berry's stable semantic, for example, see R. Amadio [2], Y.X. Chen and A. Jung [5], M. Droste [6], P-A Melliès [9], G.Q. Zhang [13,14], and so on. The theory based on stable functions is called the stable domain theory. In this theory, DI-domains are one of the most important class of stable domain structure. Each DI-domain is equivalent to a stable event structure and the category of DI-domains (resp. ωDI-domains) with stable functions is cartesian closed [13].

One see that $\mathbb{T}^\omega$ is also an ωDI-domain. So a problem arises naturally: Is $\mathbb{T}^\omega$ a stable universal domain in the sense that the category of all stable retracts of $\mathbb{T}^\omega$ with stable functions is cartesian closed? The aim of this paper is to answer this question. Since a stable retract is different to a continuous retract, we first investigate the properties of stable retracts. We introduce a new domain called a conditionally complete DI-domain and show that, (1) a dcpo is a stable retract of $\mathbb{T}^\omega$ if and only if it is an $\omega CCDI$-domain, where "ω" is for "countably based" as usual, and "CC" is for "conditionally complete", which essentially means that any subset which is pairwise bounded has an upper bound. i.e., a $CCDI$-domain with a countable base; (2) the category of $\omega CCDI$-domain (resp. $CCDI$-domains) with stable functions is cartesian closed. So, the problem above has an affirmative answer.

The paper is organized as follows. Section 1 is a introduction. Section 2 introduces some notions and definitions we need. Section 3 discusses the properties of stable retracts. Section 4 investigate the category of $\omega CCDI$-domain (resp. $CCDI$-domains). Section 5 investigate the stable retracts of $\mathbb{T}^\omega$. A pair of stable retract-stable embedding between $\mathbb{T}^\omega$ and an $\omega CCDI$-domain will be constructed in this section.

2 Preliminaries

We do assume some knowledge of basic domain theory, as in, e.g., [3,1,7]. A nonempty set P endowed with a partially order is called a poset. For $A \subseteq P$, we set $\downarrow A = \{x \in P : \exists a \in A,\ x \leq a\}$ and $\uparrow A = \{x \in P : \exists a \in P,\ a \leq x\}$, and A is called a lower or upper set, if $A =\downarrow A$ or $A =\uparrow A$ respectively. For an element $a \in P$, we use $\downarrow a$ or $\uparrow a$ instead of $\downarrow \{a\}$ or $\uparrow \{a\}$, and we say it a principal ideal or a principal filter, respectively. A subset D of P is called directed if it is nonempty and every nonempty finite subset of D has an upper bound in D. Particularly, we say that P is a dcpo if every directed subset D of P has a least upper bound (denoted by $\bigvee D$) in P.

For $x, y \in P$, we say that x is way-below y, denoted by $x \ll y$, if for any directed subset D of P, $y \leq \bigvee D$ implies $x \leq d$ for some $d \in D$. P is continuous if $\{a \in P : a \ll x\}$ is directed and $x = \bigvee\{a \in P : a \ll x\}$ for all $x \in P$. A $k \in P$ is called compact if $k \ll k$. Let $K(P)$ be the set of all compact elements of P. P is called algebraic if $K(P) \cap \downarrow x$ is directed and $x = \bigvee(K(P) \cap \downarrow x)$ for all $x \in P$. A subset $B \subseteq P$ is called a basis of P if $B \cap \Downarrow x$ is directed and $x = \bigvee(B \cap \Downarrow x)$ for all $x \in P$. A dcpo is called ω-continuous (resp. ω-algebraic) if it has a countable basis (resp. the set of all compact elements is a countable basis).

Definition 2.1 Let P and E be two dcpos.

(1) A function $f : P \longrightarrow E$ is called Scott continuous if it is monotone and preserves the suprema of all directed subsets of P.

(2) The continuous function space, denoted by $[P \to E]$, is the set of all Scott continuous functions from P into E ordered by the pointwise order, i.e., $f \leq g$ iff $f(x) \leq g(x)$ in E for all $x \in P$.

Let **DCPO** be the category of all dcpos with Scott continuous functions. Then **DCPO** is cartesian closed. Moreover, a full subcategory of **DCPO** is cartesian closed iff it is closed under continuous function spaces and finite products [1,7].

Next, we give the definition of a stable function.

Definition 2.2 Let P and E be two dcpos.

(1) A function $f : P \longrightarrow E$ is called a stable function if it is Scott continuous and satisfies the following condition: for all $(x, y) \in P \times E$, $y \leq f(x)$ implies that there exists an element $m \in \downarrow x$ such that
 (i) $y \leq f(m)$,
 (ii) $\forall d \in \downarrow x$, $y \leq f(d) \Rightarrow m \leq d$.
 Particularly, we denote $m = m(f, x, y)$.

(2) Let $[P \to_{st} E]$ be the set of all stable functions from P into E. The stable order $\leq_{st}$ on $[P \to_{st} E]$ is defined as follows: for all $f, g \in [P \to_{st} E]$, $f \leq_{st} g$ if and only if
 (i) $f \leq g$,
 (ii) $\forall (x, y) \in P \times E$, $y \leq f(x) \leq g(x) \Rightarrow m(f, x, y) = m(g, x, y)$.
 From now on, $[P \to_{st} E]$ is always endowed with the stable order, and we say that it is the stable function space.

Lemma 2.3 [3] Let P and E be two dcpos. Then

(1) $[P \to_{st} E]$ is dcpo and for a $\leq_{st}$-directed subset $\{f_i : i \in I\}$ of $[P \to_{st} E]$, the supremum of $\{f_i : i \in I\}$ in $[P \to_{st} E]$ is the pointwise supremum, i.e., $(\bigvee_{i \in I} f_i)(x) = \bigvee_{i \in I} f_i(x)$ for all $x \in P$.

(2) If E is algebraic, then a Scott continuous function $f : P \longrightarrow E$ is stable if and only if $k \leq f(x)$ implies $m = m(f, x, k)$ exists for $x \in P$ and $k \in K(E)$.

Generally, the category of all dcpos with stable functions is not cartesian closed. Only some kinds of special domains can form cartesian closed categories, for examples, continuous (algebraic) L-domains (see P. Taylor [12]), DI-domains (see G.Q. Zhang [13]), and so on.

In the end of this section, we introduce $\mathbb{T}^\omega$.

Definition 2.4 $\mathbb{T}^\omega$ is the cartesian products of denumerable many copies of $\mathbb{T}$, where $\mathbb{T}$ is the truthvalue domain $\{0, 1, \perp\}$ ordered as: $\perp < 0, 1$ and $0, 1$ are not consistent.

An element x in $\mathbb{T}^\omega$ is a vector:

$$\langle x^1, x^2, \ldots, x^n, \ldots \rangle.$$

The order on $\mathbb{T}^\omega$ is inherited from $\mathbb{T}$ and hence is pointwise. For $x \in \mathbb{T}^\omega$, set $(x)_0 = \{i : i \in \mathbb{N} \ \& \ \pi_i(x) = 0\}$, $(x)_1 = \{i : i \in \mathbb{N} \ \& \ \pi_i(x) = 1\}$. It is easy to see that $\mathbb{T}^\omega$ is a Scott domain and

$$K(\mathbb{T}^\omega) = \{x \in \mathbb{T}^\omega : |(x)_0 \cup (x)_1| < \omega\}$$

is countable.

3 Stable retracts

In this section, we investigate the property of stable retracts. At first, let's see (continuous) retracts.

Definition 3.1 Let P be a dcpo. A dcpo E is called a retract of P if there exist two Scott continuous functions $i : E \longrightarrow P$ and $j : P \longrightarrow E$ such that $j \circ i = id_E$, where (j, i) is called a retraction-embedding pair.

It is well known that a retract of a continuous (resp. ω-continuous) dcpo (with a least element) is also a continuous (resp. ω-continuous) dcpo (with a least element) [1,7]. Next, we introduce the notion of a stable retract.

Definition 3.2 Let P be a dcpo. A dcpo E is called a stable retract of P if there exist two stable functions $i : E \longrightarrow P$ and $j : P \longrightarrow E$ such that $j \circ i = id_E$, where j (resp. i) is called a stable retraction (resp. a stable embedding).

Obviously, every stable retract is a retract. For investigate the special properties of stable retracts, we need the following notions.

Definition 3.3 Let P be a dcpo.

(1) A subset A of P is said to be bounded or consistent if A has an upper bound in P. Especially, for $A = \{a, b\}$, we denote by $a \uparrow b$ when A is consistent; Conversely, we denote by $a \sharp b$ when A is not consistent.

(2) We say that P is bounded complete if it has a least element (denoted by $\perp$) and every nonempty bounded subset has a least upper bound; equivalently, every nonempty subset of P has an infimum in P. Particularly, a bounded complete algebraic dcpo is called a Scott domain.

(3) If P is bounded complete, then P is said to be distributive if every principle ideal of P is a distributive lattice under the induced order, i.e., for $a, b, c \in \downarrow d \subseteq P$, $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

(4) When P is algebraic, we say that P has property $\mathcal{I}$ if $\downarrow k$ is finite for all $k \in K(P)$.

(5) We say that P is a DI-domain if P is a distributive Scott domain with property $\mathcal{I}$.

Comparing with retracts, stable retracts have the following special property.

Theorem 3.4 Let P be an algebraic dcpo and let E be a stable retract of P.

(1) If $K(P) =\downarrow K(P)$, then E is algebraic with $K(E) =\downarrow K(E)$.

(2) If P has property $\mathcal{I}$, then so E is.

(3) If P is a distributive bounded complete domain, then so E is.

Proof. (1) Since a stable retract is also a continuous retract, E is a continuous dcpo. It is sufficient to show that $x \ll y$ implies x is compact for all $x, y \in E$. Suppose $x \ll y$ in E. Then $x \ll j(i(y))$. Since P is algebraic and j is stable, there exist $k \in K(P)$ and $m \in P$ such that

(i) $m \leq k \leq i(y)$ and $x \leq j(m)$,

(ii) $\forall d \in P$, $x \leq j(d) \Rightarrow m \leq d$.

As $K(P) =\downarrow K(P)$, we have $m \in K(P)$. As $x = j(i(x))$ and $i(x) \leq i(y)$, we have $m \leq i(x)$. Notice that since m is compact and E is continuous, there exists $z \ll x$ such that $m \leq i(z) \leq i(x)$. Hence,

$$x \leq j(m) \leq j(i(z)) = z \leq j(i(x)) = x,$$

i.e., $x = j(m) \leq z \ll x$. Therefore, x is compact in E, i.e., E is algebraic and $K(E) =\downarrow K(E)$.

(2) Suppose that P has property $\mathcal{I}$. Then $K(P) =\downarrow K(P)$. Hence, E is algebraic with $K(E) =\downarrow K(E)$ by (1). Pick $k \in K(E)$. From the proof of (1), there exists $m \in K(P)$ such that

- $k = j(m)$ and $m \leq i(k)$,
- for all $m' \in K(E)$, $m' \leq m$ and $k \leq j(m')$ imply $m' = m$.

Hence, for any $k' < k = j(m)$, there exists $m' < m$ such that $k' = j(m')$. So, if $\downarrow k$ is infinity, then $\downarrow m$ is also infinity. Therefore, E has property $\mathcal{I}$ when P has property $\mathcal{I}$.

(3) Suppose that P is a distributive bounded complete domain. Then E is a bounded complete domain. Pick $a, b, c, d \in E$ with $a, b, c \leq d$. Then $i(a), i(b), i(c) \leq i(d)$. Thus,

$$i(a) \wedge (i(b) \vee i(c)) = (i(a) \wedge i(b)) \vee (i(a) \wedge i(c)).$$

Since

$$\begin{aligned} a \wedge (b \vee c) &= j(i(a)) \wedge (j(i(b)) \vee j(i(c))) \\ &\leq j(i(a)) \wedge j(i(b) \vee i(c)) \\ &= j(i(a) \wedge (i(b) \vee i(c))) \quad \text{(for } j \text{ is stable)} \\ &= j((i(a) \wedge i(b)) \vee (i(a) \wedge i(c))) \\ &= j(i(a \wedge b) \vee i(a \wedge c)) \quad \text{(for } i \text{ is stable)} \end{aligned}$$

$$\begin{aligned} &\leq j(i((a \wedge b) \vee (a \wedge c))) \\ &= (a \wedge b) \vee (a \wedge c), \end{aligned}$$

it follows that $a \wedge (b \vee v) = (a \wedge b) \vee (a \wedge c)$. Hence, E is distributive, too. □

The following result is straight from the above theorem and the definition of DI-domains.

Corollary 3.5 The class of DI-domains (resp. ωDI-domains) is closed under all of the stable retracts.

4 The category of conditionally complete DI-domains

In this section, we introduce a new domain called a conditionally complete DI-domain and investigate the categories formed by these new domains. In the next section, we will show that conditionally complete ωDI-domains coincide with the stable retracts of $\mathbb{T}^\omega$.

Definition 4.1 Let P be a dcpo.

(1) A nonempty subset $S \subseteq P$ is called pair bounded if $a \uparrow b$ for all $a, b \in S$.

(2) P is said to be conditionally complete if it has a least element and every pair bounded nonempty subset of P has a least upper bound.

(3) P is said to be a CC-domain if it is a conditionally complete continuous dcpo. In this case, P is called an ωCC-domain if it has a countable basis.

(4) A conditionally complete DI-domain is called a $CCDI$-domain. In this case, P is called an $\omega CCDI$-domain if it has a countable basis.

Notes that in [10], a conditionally complete dcpo is called coherent. However, the word "coherent" in domain theory [1] is used as a topological notion: the intersection of two compact saturated sets is compact. So here we use the word "conditionally complete".

It is easy to see that a dcpo with a least element is conditional complete if and only if every pair bounded finite nonempty subset of P has a least upper bound.

As shown in [10], $\mathbb{T}^\omega$ is an algebraic ωCC-domain and every retract of $\mathbb{T}^\omega$ is an ωCC-domain. For any $x \in K(\mathbb{T}^\omega)$, $(x)_0 \cup (x)_1$ is finite. It means that $\downarrow x$ is finite. Hence, $\mathbb{T}^\omega$ has property $\mathcal{I}$. As $\mathbb{T}^\omega$ is distributive, $\mathbb{T}^\omega$ is distributive, too. Therefore, we have the following result.

Proposition 4.2 $\mathbb{T}^\omega$ is an $\omega CCDI$-domain.

The following result is straight from Theorem 3.4 and the above proposition.

Corollary 4.3 The class of $\omega CCDI$-domain (resp. $CCDI$-domains) is closed under stable retracts and includes all stable retracts of $\mathbb{T}^\omega$.

Next, we definition the category of $\omega CCDI$-domain (resp. $CCDI$-domains).

Definition 4.4 The category ω**CCDI** (resp. **CCDI**) is given by:

- objects are all $\omega CCDI$-domains (resp. $CCDI$-domain),

- morphisms are all stable functions.

To investigate the category ω**CCDI** (resp. **CCDI**), we need the following result quoted from [3,13].

Theorem 4.5 The category ω**DI** (resp. **DI**) of ωDI-domains (resp. DI-domains) with stable functions is cartesian closed. Moreover, a full subcategory of ω**DI** (resp. **DI**) with a terminal object is cartesian closed if and only if it is closed under stable function spaces and finite cartesian products.

As shown in [10] that the category of ωCC-domains (resp. CC-domains) with Scott continuous functions is cartesian closed, we will show that both ω**CCDI** and **CCDI** are also cartesian closed. To do this, we need the following notions.

Definition 4.6 Let P be a bounded complete dcpo.

(1) An element $p \in P$ is said to be a prime if $p \neq \perp$ and for any nonempty bounded subset A of P, $p \leq \bigvee A$ implies $p \leq a$ for some $a \in A$. We denote $Pr(P)$ to be the set of all primes of P.

(2) P is said to be a prime algebraic domain if $x = \bigvee(Pr(P) \cap \downarrow x)$ for all $x \in P$.

Notes that a prime is compact and in standard lattice theory, e.g. [7], it is called a completely coprime. However, since in [3,14,13] this type of elements is called a prime, so we also call it this name. The following result is quoted from [3,14].

Theorem 4.7 A DI-domain is exactly a prime algebraic domain with property $\mathcal{I}$.

Proposition 4.8 Let P, E be two DI-domains and let $f : P \longrightarrow E$ be a Scott continuous function. Then we have

(1) the following conditions are equivalent:
  (i) f is stable;
  (ii) $a \uparrow b$ implies $f(a \wedge b) = f(a) \wedge f(b)$ for all $a, b \in P$;
  (iii) $p \leq f(x)$ implies that $m = m(f, x, p)$ exists for $x \in P$ and $p \in Pr(E)$.

(2) For two stable function $g, h : P \longrightarrow E$, $g \leq_{st} h$ iff $g \leq h$ and $m(g, x, p) = m(h, x, p)$ for all $x \in P$ and $p \in Pr(E)$.

The following is the main result of this section.

Theorem 4.9 The class of $\omega CCDI$-domains (resp. $CCDI$-domain) is closed under stable function spaces and nonempty finite cartesian products. Hence, ω**CCDI** (resp. **CCDI**) is cartesian closed.

Proof. Let P, E be two $\omega CCDI$-domains. Then $[P \rightarrow_{st} E]$ is an ωDI-domain by Theorem 4.5. It is sufficient to show $[P \rightarrow_{st} E]$ is conditionally complete. Suppose $f_1, f_2, \ldots, f_n \in [P \rightarrow_{st} E]$ such that for all $1 \leq i, j \leq n$, there exist $f_{ij} \in [P \rightarrow_{st} E]$ satisfying

$$f_i, f_j \leq_{st} f_{ij}.$$

Then, for any $a \in P$, $\{f_i(a) : 1 \leq i \leq n\}$ is pair bounded in E and hence has a least upper bound $\bigvee_{i=1}^{n} f_i(a)$ in E. We define a map $f : P \longrightarrow E$ as follows: $\forall x \in P$,

$$f(x) = \bigvee_{i=1}^{n} f_i(x).$$

Then f is well defined and monotone. Easily one see that f is also Scott continuous. Next, we will show f is stable. Suppose $p \leq f(x)$ for a prime p of E. Then there exists at least one $i \in \{1, 2, \ldots, n\}$ such that $p \leq f_i(x)$. Set

$$I = \{i : 1 \leq i \leq n \ \&\ p \leq f_i(x)\}.$$

Then $I \neq \emptyset$. Pick $i \in I$. Since f_i is stable, there exists $m_i \in \downarrow x$ such that $m_i = m(f_i, x, p)$. For all $i, j \in I$, since $f_i, f_j \leq_{st} f_{ij}$, we have $m(f_i, x, p) = m(f_{ij}, x, p) = m(f_j, x, p)$. Thus, $m_i = m_j$ for all $i, j \in I$. Set $m = m_i$ for all $i \in I$. We claim $m = m(f, x, p)$. Suppose that $p \leq f(d)$ for some $d \in \downarrow x$. Then we can find one $i \in \{1, 2, \ldots, n\}$ such that $p \leq f_i(d)$. Since $d \leq x$, we have $i \in I$. Hence, $m_i \leq d$, i.e., $m \leq d$. Therefore, $m = m(f, x, p)$ and hence f is a stable function by Proposition 4.8 (1).

Next, we have to show $f_i \leq_{st} f$ for all $1 \leq i \leq n$. Suppose $p \leq f_i(x)$ for $x \in P$ and $p \in Pr(E)$. By the above proof, we have $m(f_i, x, p) = m(f, x, p)$. Hence, $f_i \leq_{st} f$ by Proposition 4.8 (2). Therefore, f is the least upper bound of all f_i. Hence, $[P \rightarrow_{st} E]$ is an $\omega CCDI$-domain.

It is easy to show that a nonempty finite cartesian product of $\omega CCDI$-domains is also an $\omega CCDI$-domain. Therefore, ω**CCDI** (resp. **CCDI**) is cartesian closed by Theorem 4.5. □

In the end of this section, we give a characterization of an $\omega CCDI$-domain, which is crucial for investigating the stable retracts of $\mathbb{T}^\omega$ in the next section.

Let P be an algebraic dcpo with a least element, $F \subseteq_{fin} K(P)$, where $A \subseteq_{fin} B$ means that A is a finite subset of B. Let

$$mub(F) = \{a \in P : a \text{ is a minimal upper bound of } F\}.$$

Then $mub(F) \subseteq K(P)$. We say that P has property $\mathcal{M}$ if for any finite $F \subseteq K(P)$, $mub(F)$ is finite and $\bigcap_{k\in F} \uparrow k = \uparrow mub(F)$ [1]. Now suppose that P has property $\mathcal{M}$. We set

$$MD(F) = K(P) \cap \downarrow (\bigcup_{F' \subseteq_{fin} F} mub(F'))$$

and

$$MD^\infty(F) = \bigcup_{n=1}^{\infty} MD^n(F),$$

where $MD^0(F) = F$, $MD^1(F) = MD(F)$ and $MD^n(F) = MD(MD^{n-1}(F))$ for all $n \geq 1$.

Theorem 4.10 For an ω-algebraic dcpo P with a least element $\perp$, the following conditions are equivalent:

(1) P is a $CCDI$-domain;

(2) P has property $\mathcal{M}$ and for any finite $F \subseteq K(P)$, $MD^{\infty}(F)$ is a finite $CCDI$-domain.

(3) There exists a sequence $p_1, p_2, \ldots, p_n$ of stable functions of $[P \to_{st} P]$ such that
  (i) $p_n \leq_{st} p_{n+1}$ for all $n \in \mathbb{N}$, and $id_P = \bigvee_{n\in\mathbb{N}} p_n$,
  (ii) $p_n(D)$ is a finite $CCDI$-domain for $n \in \mathbb{N}$.

5 $\mathbb{T}^{\omega}$ as a stable universal domain

In this section, we will show that the stable retracts of $\mathbb{T}^{\omega}$ are exactly the class of all $\omega CCDI$-domains.

In [10], G. Plotkin shows that the (continuous) retracts of $\mathbb{T}^{\omega}$ are exactly all of the ωCC-domains. Given an ωCC-domain D, let $e_1, e_2, e_2 ...$ be an enumeration of a countable basis of D, Plotkin constructs an embedding $f : D \to \mathbb{T}^{\omega}$ as follows: $\forall d \in D$,

$$\pi_n(f(d)) = \begin{cases} 0, & \text{if } e_n \sharp d, \\ 1, & \text{if } e_n \ll d, \\ \perp, & \text{otherwise.} \end{cases}$$

In the following, we give an example to show that f is not stable.

Example 5.1 Let $D = \{\perp, e_1, e_2, e_3, e_4\}$, ordered as: $\perp < e_1, e_2, e_3, e_4$; $e_1, e_2 < e_4$; e_3 does not be consistent with any other element except for $\perp$. From the definition of f we have:

$$f(e_1) = \langle 1, \perp, 0, \perp, \perp, \ldots \rangle$$
$$f(e_2) = \langle \perp, 1, 0, \perp, \perp, \ldots \rangle$$

Then $f(e_1) \wedge f(e_2) = \langle \perp, \perp, 0, \perp, \perp, \ldots \rangle \neq f(\perp) = f(e_1 \wedge e_2)$. Hence, f is not stable.

The reason for the above embedding f failing to be stable, is that the n'th coordinates of two different elements e_1 and e_1 are all defined to be 0 when e_n is not consistent with e_1 and e_2. In the following, we will modify Plotkin's embedding to make it into a stable function, and then we show that every $\omega CCDI$-domain is a stable retract of $\mathbb{T}^{\omega}$. Because the tectonic process is more complicated than the continuous case, some details of the relative proof in this section will be omitted.

Theorem 5.2 A dcpo D is a stable retract of $\mathbb{T}^{\omega}$ if and only if it is an $\omega CCDI$-domain.

From Corollary 4.3, every stable retract of $\mathbb{T}^{\omega}$ is an $\omega CCDI$-domain. So we only need show the other direction. To do this, we consider two cases that D is finite or infinite. Generally, we require that D is not trivial, i.e., $D \backslash \{\perp\} \neq \emptyset$.

Case 1: D is a finite $CCDI$-domain.

At first, all elements of $Pr(D)$ are enumerated as follows:

$$p_1, p_2, \ldots, p_{n_0},$$

where $Pr(D)$ is the set of all primes of D.

We define a map $f^* : Pr(D) \to \mathbb{T}^\omega$ as follows: for $1 \leq i \leq n_0$,

$$\pi_n(f^*(p_i)) = \begin{cases} 1, & (i-1)n_0 + 1 \leq n \leq i \times n_0, \\ 0, & \exists j < i, p_j \sharp p_i \ \& \ n = (j-1)n_0 + i, \\ \perp, & \text{otherwise.} \end{cases}$$

The intuition of the definition of f^* is that, (1) all coordinates 1 of $f^*(p)$ are at the definitely locations to indicate the location of p in the enumeration of all primes; (2) if a prime p is ahead of two consistent prime p_1, p_2, and p is not consistent with p_1, p_2, then there are two different locations to put 0 in $f^*(p_1)$ and $f^*(p_2)$, while the coordinates of $f^*(p)$ are 1 at the two locations. This strategy avoids the predicament of Plotkin's embedding that it is not stable as shown in the above example.

Next we define a map $f : D \to \mathbb{T}^\omega$ based on f^* as follows: for any $x \in D$,

$$f(x) = \bigvee \{f^*(p) : p \in Pr(D) \ \& \ p \leq x\}.$$

One see that $f : D \longrightarrow \mathbb{T}^\omega$ is well defined and Scott continuous.

Proposition 5.3 The function f has the following properties:

(1) $f(d)$ is compact in $\mathbb{T}^\omega$ for all $d \in D$.

(2) For any $n \leq n_0 \times n_0$, there exists a $p \in Pr(D)$ such that $\pi_n(f(p)) = 1$ and for all $d \in D$, $\pi_n(f(d)) = 1$ iff $d \geq p$.

(3) If there exist $n \in \mathbb{N}$ and $d \in D$ with $\pi_n(f(d)) = 0$, then there exists a prime $p \in \downarrow d$ such that $\pi_n(f(p)) = 0$ and for all d' in D, $\pi_n(f(d')) = 0$ iff $d' \geq p$.

(4) For all $d_1, d_2 \in D$, $d_1 \sharp d_2$ in D iff $f(d_1) \sharp f(d_2)$ in $\mathbb{T}^\omega$.

(5) For all $d_1, d_2 \in D$, $d_1 \leq d_2$ in D iff $f(d_1) \leq f(d_2)$ in $\mathbb{T}^\omega$.

We omit the proof of the above result. Next, we show that f is stable.

Pick $x \in Pr(\mathbb{T}^\omega)$ and $d \in D$ with $x \leq f(d)$. Then $|(x)_0 \cup (x)_1| = 1$. For the case of $(x)_0 = \emptyset$, there exists $n \in n \leq n_0 \times n_0$ such that $\pi_n(x) = 1$ and $\pi_m(x) = \perp$ for $m \neq n$. Hence, $\pi_n(f(d)) = 1$. By (2) of Proposition 5.3, there exists a prime $p \in Pr(D)$ such that $\pi_n(f(p)) = 1$ and for all $d' \in D$, $\pi_n(f(d')) = 1$ iff $d' \geq p$. Hence, $p = m(f, d, x)$. For the case of $(x)_1 = \emptyset$, it can be show analogously that there exists $p \in \downarrow d$ such that $p = m(f, d, x)$ by (3) of Proposition 5.3. Therefore, f is stable by Proposition 4.8 (1).

Next, we define a function $g : \mathbb{T}^\omega \longrightarrow D$ such that g is stable and $g \circ f = id_D$.

For all $x \in \mathbb{T}^\omega$, let

$$g(x) = \bigvee\{d \in D:\ f(d) \leq x\}.$$

Since $f(\perp) = \perp$, $\{d \in D:\ f(d) \leq k\}$ is nonempty. By (4) of Proposition 5.3, $\{d \in D:\ f(d) \leq k\}$ is pair bounded. Hence, g is well defined and monotone. Since $f(d)$ is compact for any $d \in D$, it follows that f is Scott continuous. Suppose $p \leq g(x)$ for $p \in Pr(D)$ and $x \in \mathbb{T}^\omega$. Then there exists $d \in D$ such that $p \leq d$ and $f(d) \leq x$. Thus, $f(p) \leq x$. It means that $p \leq g(y)$ iff $f(p) \leq y$ for any $y \in \mathbb{T}^\omega$. Therefore, $f(p) = m(g, x, p)$ and g is stable by Proposition 4.8 (1). Moreover, for any $k \in D$, $g(f(k)) \geq k$ by the definition of g. Suppose $k' \leq g(f(k))$ for $k' \in D$, then $f(k') \leq f(k)$. Thus, $k' \leq k$ by (5) of Proposition 5.3. Therefore, $g(f(k)) = k$, i.e., $g \circ f = id_D$.

The above process show that D is a stable retract of $\mathbb{T}^\omega$ if D is a finite $CCDI$-domain. Next, we will show that Theorem 5.2 also holds when D is infinite.

Case 2: D is an $\omega CCDI$-domain with $|K(D)| = \aleph_0$.

In this case, $Pr(D)$ is countably infinite. It is difficult to construct a concrete stable embedding on D, because we have no idea to define an injective stable function from D to $\mathbb{T}^\omega$ such that it preserves all compact elements as in the finite case. In the following, we will use Theorem 4.10 and construct a stable embedding successfully through a rather complicated technical process.

From Theorem 4.10, there exists a sequence $r_1, r_2, \ldots, r_n, r_{n+1}, \ldots$ of stable functions in $[D \to_{st} D]$ such that

(i) $r_n \leq_{st} r_{n+1}$ for all $n \in \mathbb{N}$, and $id_D = \bigvee_{n\in\mathbb{N}} r_n$,

(ii) $r_n(D)$ is a finite $CCDI$-domain for $n \in \mathbb{N}$.

Set $D_n = r_n(D)$ for all $n \geq 1$. Then

- $D_n = \downarrow D_n \subseteq D_{n+1} = \downarrow D_{n+1}$, where the lower sets are taken in D;
- $\bigcup\limits_{n=1}^{\infty} D_n = K(D)$;
- $Pr(D_n) = D_n \cap Pr(D)$ and $\bigcup\limits_{n=1}^{\infty} Pr(D_n) = Pr(D)$

We use D_1 to replace D in Case 1 and then we have defined a stable embedding $f_1 : D_1 \longrightarrow \mathbb{T}^\omega$ (it just is the function f in Case 1). In the following, we will extend f_1 to D_2 and then obtain a stable embedding $f_2 : D_2 \longrightarrow \mathbb{T}^\omega$ such that $f_1 \circ r_1 \leq_{st} f_2 \circ r_2$. Then, by induction, we will define $f_n : D_n \longrightarrow \mathbb{T}^\omega$ for all $n \geq 2$, such that $\bigvee_{n\in\mathbb{N}} f_n \circ r_n$ is a stable embedding from D into $\mathbb{T}^\omega$, where all r_n's are regarded as stable function from D to D_n.

Set $F = D_2 \setminus D_1 \neq \emptyset$. Set $Pr(F) = F \cap Pr(D_2)$ and let $|Pr(F)| = s_0$. Then $Pr(D_2) = Pr(F) \cup Pr(D_1)$ and $|Pr(D_2)| = n_0 + s_0$. We enumerate $Pr(F)$ as follows:

$$v_1, v_2, \ldots, v_{s_0}.$$

Like Case 1, we will define a map $f_2^* : Pr(D_2) \to \mathbb{T}^\omega$. For $p_i \in Pr(D_1)$, let

$$\pi_n(f_2^*(p_i)) = \begin{cases} 1, & (i-1)s_0 + 1 \le n - n_0 \times n_0 \le i \times s_0, \\ \pi_n(f^*(p_i)), & \text{otherwise.} \end{cases}$$

For $v_i \in Pr(F)$ (here $1 \le i \le s_0$), let

$$\pi_n(f_2^*(v_i)) = \begin{cases} 1, & (n_0 + i - 1)(n_0 + s_0) + 1 \le n \le (n_0 + i)(n_0 + s_0), \\ 0, & \exists j \le n_0,\ p_j \sharp v_i \ \&\ n = n_0 \times n_0 + s_0(j-1) + i, \\ 0, & \exists j < i,\ v_j \sharp v_i \ \&\ n = (n_0 + j - 1)(n_0 + s_0) + n_0 + i, \\ \perp, & \text{otherwise.} \end{cases}$$

As in Case 1, we extend f_2^* to the whole domain D_2 in the same way: $\forall x \in D_2$,

$$f_2(x) = \bigvee \{ f_2^*(p) :\ p \in \downarrow x \cap Pr(D_2) \}.$$

By induction, we can define a function $f_n : D_n \longrightarrow \mathbb{T}^\omega$ for all $n \ge 2$ such that the following result holds.

Proposition 5.4 For each $n \in \mathbb{N}$, $f_n : D_n \longrightarrow \mathbb{T}^\omega$ is a stable embedding satisfying all properties of Proposition 5.3 (the number n_0 in 5.3 is replaced by the cardinal of $Pr(D_n)$) and the following one:

$$\forall d \in D_n,\ i \le |Pr(D_n)| \times |Pr(D_n)| \Rightarrow \pi_i(f_n(d)) = \pi_i(f_{n+1}(d)).$$

The corresponding stable retraction $g_n : \mathbb{T}^\omega \longrightarrow D_{\bar{n}+n}$ is defined as follows: $\forall x \in \mathbb{T}^\omega$,

$$g_n(x) = \bigvee \{ d \in D_n :\ f_n(d) \le x \}.$$

Recall that, the sequence $r_1, r_2, \ldots, r_n, r_{n+1}, \ldots$ is a $\le_{st}$-ascending chain of stable functions on D such that $r_n(D) = D_n$ for all $n \ge 1$ and $id_D = \bigvee_1^\infty r_n$. Each r_n can be regarded as a stable function from D to D_n. Set

$$h_n = f_n \circ r_n.$$

Then h_n is a stable function from D into $\mathbb{T}^\omega$ for all $n \ge 1$. Moreover, we have the following result.

Proposition 5.5 $f_n \circ r_n \le_{st} f_{n+1} \circ r_{n+1}$ for all $n \ge 1$.

So, the sequence $f_1 \circ r_1, f_2 \circ r_2, \ldots, f_n \circ r_n, \ldots$ is a $\le_{st}$-ascending chain of stable functions in $[D \to \mathbb{T}^\omega]$. Let

$$h = \bigvee_{n \in \mathbb{N}} f_n \circ r_n.$$

Then $h : D \longrightarrow \mathbb{T}^\omega$ is a stable function. Particularly, h has the following properties.

Proposition 5.6 The following holds:

(1) $d_1 \le d_2$ in D iff $h(d_1) \le h(d_2)$ in $\mathbb{T}^\omega$.

(2) $d_1 \sharp d_2$ in D iff $h(d_1) \sharp h(d_2)$ in $\mathbb{T}^\omega$.

Next, we define a function $g : \mathbb{T}^\omega \longrightarrow D$ as follows: $\forall x \in \mathbb{T}^\omega$,

$$g(x) = \bigvee\{d \in D : \exists n \in \mathbb{N}, d \in D_n \ \&\ f_n(d) \leq x\}.$$

Using Proposition 5.4 and 5.6, we can show that g is a stable function and $g \circ h = id_D$. Thus, Theorem 5.2 holds. So we obtain the main result of this paper.

Theorem 5.7 $\mathbb{T}^\omega$ is a stable universal domain: its stable retracts are exactly all of the $\omega CCDI$-domains and the category of $\omega CCDI$-domains with the stable functions is cartesian closed.

Remark 5.8 In the definition of the function g above, f_n can not be replaced by h because it may lead g not to be Scott continuous.

We have given a pair of stable retraction-stable embedding between $\mathbb{T}^\omega$ and an $\omega CCDI$-domain, but the structure of this stable embedding is nuclear and not concrete. So, a problem being worthy of considering is: construct a concrete stable embedding between $[\mathbb{T}^\omega \rightarrow_{st} \mathbb{T}^\omega]$ and $\mathbb{T}^\omega$ (it can help to take $\mathbb{T}^\omega$ as a stable semantics model of LAMBDA languages).

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