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Weak Domain Models of T_1 spaces

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Abstract

The main objective of this paper is to study some aspects of weak domains. We first show that every meet continuous weak domain is a domain. Then we prove that a dcpo P is exact iff the weakly way-below relation is the smallest approximating w-auxiliary relation. It is then shown that for each T_1 space, Zhao and Xi's dcpo model is a weak algebraic domain, and hence a weak domain. As a consequence, we have that a weak algebraic domain need not be well-filtered and that every T_1 space has a weak domain model, which strengthens a result of Mashburn.

Keywords: weakly way-below relation, weak domain, meet continuous dcpo, weak algebraic domain, local domain, weakly auxiliary relation

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1 Introduction

A poset model of a topological space X is a poset P such that $\mathrm{Max}(P)$, the set of maximal elements of P, with the relative Scott topology, is homeomorphic to X [5]. Every space having a poset model must be T_1. In [6], Martin proved that if a space is homeomorphic to the maximal point space of a continuous dcpo, then the space is Choquet complete. Thus not every T_1 space has a domain model.

In 2002, Coecke and Martin [1] constructed an ordered set to model finite dimensional quantum states, and it turns out that their model is not a domain, i.e., not continuous with respect to the way-below relation. However, this model is continuous with respect to the weakly way-below relation. Using the weakly way-below relation, Mashburn [7] defined the weak domains and proved that every first countable space has a weak domain model. In [8], Mashburn proved that every linearly ordered topological space is homeomorphic to an open dense subset of a weak domain representable space, showing that a space with a weak domain model need not be Baire. Then Mashburn [7] raised the following problem:

- Which topological spaces are weak domain representable?

In this paper, it was shown that meet continuity [4] plays a crucial role between weak domains and domains. Precisely, we prove that every meet continuous weak domain is a domain. In [11], Zhao and Xi proved that every T_1 space X has a dcpo model (we will call it Xi-Zhao model of X). We show that the Xi-Zhao model of every T_1 space is a weak algebraic domain (hence a weak domain). As a corollary, every T_1 space has a weak domain model, strengthening Mashburn's result in [7] (every first countable T_1 space has a weak domain model) and answering one of Mashburn's problems. Using the main result in [9], we also deduce that, unlike domains, the Scott space of a weak domain need not be well-filtered.

2 Preliminary

In this section, we review some basic notions and results used in this paper. For more details, please refer to [2,7,8].

For a poset P and $A \subseteq P$, let $\downarrow A = \{x \in P : x \leqslant a$ for some $a \in A\}$ and $\uparrow A = \{x \in P : x \geqslant a$ for some $a \in A\}$. For $x \in P$, we write $\downarrow x$ for $\downarrow\{x\}$ and $\uparrow x$ for $\uparrow\{x\}$. A subset A is called a *lower set* (resp., an *upper set*) if $A = \downarrow A$ (resp., $A = \uparrow A$). A nonempty subset D of P is *directed* if every two elements in D have an upper bound in D. P is called a *directed complete poset*, or *dcpo* for short, if for any directed subset of $D \subseteq P$, $\bigvee D$ exists in P.

A subset U of P is *Scott open* if (i) $U = \uparrow U$ and (ii) for any directed subset D for which $\bigvee D$ exists, $\bigvee D \in U$ implies $D \cap U \neq \varnothing$. All Scott open subsets of P form a topology, and we call this topology the *Scott topology* on P and denote it by $\sigma(P)$. The space $(P, \sigma(P))$ is called the *Scott space* of P, and we denote it by ΣP.

For two elements x and y in P, x is *way-below* y, denoted by $x \ll y$, if for any directed subset D of P for which $\bigvee D$ exists, $y \leqslant \bigvee D$ implies $D \cap \uparrow x \neq \varnothing$. We let $\twoheaduparrow x = \{y \in P : x \ll y\}$ and $\twoheaddownarrow x = \{y \in P : y \ll x\}$. P is *continuous*, if for any $x \in P$, the set $\Downarrow x$ is directed and $x = \bigvee \Downarrow x$. A continuous dcpo is also called a *domain*.

An element x in a poset P is called *compact* if $x \ll x$. The set of all compact elements of P is denoted by $\mathrm{K}(P)$. A poset P is *algebraic*, if for any $x \in P$, the set $K(P) \cap \downarrow x$ is directed and $x = \bigvee K(P) \cap \downarrow x$.

Definition 2.1 (Mashburn [7]) Let P be a poset and $x, y \in P$. Then x is *weakly way-below* y, denoted by $x \ll_w y$, if for any directed subset D of P for which $\bigvee D$ exists, $y = \bigvee D$ implies $D \cap \uparrow x \neq \varnothing$.

If P is continuous, then the relations $\ll_w$ and $\ll$ on P coincide.

Proposition 2.2 *(Mashburn [7,8]) In a poset P, the following statements hold for all $x, y, z \in P$:*

(1) $x \ll y \Rightarrow x \ll_w y$;

(2) $x \ll_w y \Rightarrow x \leqslant y$;

(3) $x \leqslant y \ll_w z \Rightarrow x \ll_w z$;

(4) $\perp \ll_w x$ *whenever P has a smallest element* $\top$.

One property owned by $\ll$ is that $x \ll y \leqslant z$ implies $x \ll z$. This property may not be true for $\ll_w$. It is true for $\ll_w$ if and only if $\ll_w$ and $\ll$ coincide, as Coecke and Martin pointed out in [1].

For $x \in P$, let $\Uparrow_w x = \{y \in P : x \ll_w y\}$ and $\Downarrow_w x = \{y \in P : y \ll_w x\}$.

Definition 2.3 (Mashburn [7,8]) A poset P is called *exact* if for any $x \in P$, $\Downarrow_w x$ is directed and $\bigvee \Downarrow_w x = x$. P is a *weak domain* if P is an exact dcpo and the relation $\ll_w$ is *weakly increasing*: for any $x, y, z, u \in P$, $x \ll_w y \leqslant z \ll_w u$ implies $x \ll_w z$.

Proposition 2.4 *A dcpo P is exact iff for each $x \in P$, there exists a directed subset D of $\Downarrow_w x$ such that $\bigvee D = x$.*

Proof. We only prove the Sufficiency. Assume $x \in P$ and there is a directed subset D of $\Downarrow_w x$ such that $\bigvee D = x$. To verify the directedness of $\Downarrow_w x$, let $y_1, y_2 \in \Downarrow_w x$, that is, $y_1, y_2 \ll_w x$. Since $x = \bigvee D$, there exist $d_1, d_2 \in D$ such that $y_1 \leqslant d_1$ and $y_2 \leqslant d_2$. Since D is directed, there is $d \in D$ such that $d_1, d_2 \leqslant d$, so that $y_1, y_2 \leqslant d$. Note that $d \in \Downarrow_w x$ because $D \subseteq \Downarrow_w x$. This implies $\Downarrow_w x$ is directed. In addition, since $x = \bigvee D \leqslant \bigvee \Downarrow_w x \leqslant x$, we have $x = \bigvee \Downarrow_w x$. Hence, P is an exact dcpo. $\square$

Note that every domain is a weak domain because $\Downarrow x$ is a directed subset of $\Downarrow_w x$ with $\bigvee \Downarrow x = x$.

Like the way-below relation on domains, the weakly way-below relation on domains has the important interpolation property.

Theorem 2.5 *(Mashburn [7, Theorem 3.6]) If P is a weak domain, then $\ll_w$ is interpolative in P, that is, for any $x, y \in P$, $x \ll_w y$ implies the existence of $z \in P$ such that $x \ll_w z \ll_w y$.*

3 More relationships between domains and weak domains

In this section, we provide some more relationships among certain types of domains.

We call a dcpo in which any two elements have an infimum a *directed complete semilattice*. A directed complete semilattice P is called *meet continuous* if for any $x \in P$ and any directed subset D of P, $x \leqslant \bigvee D$ implies $x = \bigvee\{x \wedge d : d \in D\}$.

Lemma 3.1 *If P is a meet continuous directed complete semilattice, then the relations $\ll_w$ and $\ll$ on P coincide.*

Proof. Suppose $a \ll_w b$ and D is a directed subset of P with $b \leqslant \bigvee D$. Since P is meet continuous, $b = \bigvee\{b \wedge d : d \in D\}$. From $a \ll_w b$, it follows that $a \leqslant b \wedge d_0$ for some $d_0 \in D$, and hence $a \ll b$. Trivially, we have that $a \ll b$ implies $a \ll_w b$. Hence $\ll_w$ and $\ll$ coincide. □

As an immediate result of Lemma 3.1, we have the following result.

Proposition 3.2 *A directed complete semilattice P is a domain if and only if it is a meet continuous weak domain.*

In 2001, Kou, Liu and Luo extended the notion of meet continuity to the general dcpos [4]. A dcpo P is *meet continuous* if for any $x \in P$ and any directed set D with $x \leqslant \bigvee D$, $x \in \mathrm{cl}_\sigma(\downarrow D \cap \downarrow x)$, where cl_σ is the closure operator for the Scott topology. A well-known result is that every domain is meet continuous.

It is natural to ask whether Proposition 3.2 holds for any dcpo. In the following, we will answer this question in the affirmative.

Lemma 3.3 *If P is a meet continuous weak domain, then the relations $\ll_w$ and $\ll$ on P coincide.*

Proof. Let $y \ll_w x$. Assume that D is a directed subset of P such that $x \leqslant \bigvee D$. By Theorem 2.5, there exists $z \in P$ such that $y \ll_w z \ll_w x$. Since P is meet continuous, it follows that $z \in \mathrm{cl}_\sigma(\downarrow z \cap \downarrow D)$, or equivalently $\downarrow z = \mathrm{cl}_\sigma(\downarrow z \cap \downarrow D)$.

Now for every ordinal α, we define a subset A_α of P inductively:

$$A_0 := \downarrow z \cap \downarrow D,$$

$$A_\alpha := \downarrow\{\bigvee E : E \text{ is a directed subset of } A_\beta\}, \text{ if } \alpha = \beta + 1 \text{ for some ordinal } \beta,$$

$$A_\alpha := \bigcup_{\beta<\alpha} A_\alpha, \text{ if } \alpha \text{ is a limit ordinal.}$$

Then $\mathrm{cl}_\sigma(A_0) = \bigcup\{A_\alpha : \alpha \text{ is an ordinal}\}$. Since the cardinality of $\mathrm{cl}_\sigma(A_0)$ is less than that of the power set of P, there exists a smallest ordinal number γ such that $A_\gamma = A_{\gamma'}$ for all $\gamma' \geq \gamma$, and hence $\mathrm{cl}_\sigma(A_0) = \bigcup_{\alpha\leq\gamma} A_\alpha = A_\gamma$.

We note that

(i) $\beta < \alpha$ implies $A_\beta \subseteq A_\alpha$;

(ii) for any $\alpha \leqslant \gamma$, $A_\alpha \subseteq \downarrow z$ because $A_\alpha \subseteq \mathrm{cl}_\sigma(A_0) = \downarrow z$;

(iii) for any ordinal α, $z \in A_\alpha$ iff $A_\alpha = \mathrm{cl}_\sigma(A_0) = \downarrow z$.

We assert that γ is not a limit ordinal. Otherwise, $z \in A_\gamma = \bigcup_{\alpha<\gamma} A_\alpha$ implies that $z \in A_{\alpha_0}$ for some $\alpha_0 < \gamma$, and hence $A_{\alpha_0} = \mathrm{cl}_\sigma(A_0) = A_\gamma$, a contradiction.

We say that an ordinal α has **F** property if $A_\alpha \cap \Uparrow_w y \cap \downarrow z \neq \varnothing$ implies $y \in \downarrow D$.

Now we prove that every ordinal has **F** property by transfinite induction.

(a) If there exists $u \in A_0$ with $y \ll_w u \leqslant z$, then $y \leqslant u \in \downarrow D$ and hence $y \in \downarrow D$. Thus 0 has **F** property.

(b) Assume α has **F** property. Let $u \in A_{\alpha+1} \cap \Uparrow_w y \cap \downarrow z$. Then there exists a directed subset E of A_α such that $u \leq \bigvee E$. By Theorem 2.5, there exists $v \in P$ such that $y \ll_w v \ll_w u$. Since $v \ll_w u \leqslant \bigvee E \leqslant z \ll_w x$, it holds that $v \ll_w \bigvee E$, and hence there exists $e \in E \subseteq A_\alpha$ such that $v \leqslant e$. Note that $y \ll_w v \leqslant e \leqslant z \ll_w$ implies $y \ll_w e$. Now since $e \in A_\alpha \cap \Uparrow_w y \cap \downarrow z$ and α has **F** property, it follows that $y \in \downarrow D$. Thus $\alpha + 1$ has **F** property.

(c) Assume α is a limit ordinal and β has **F** property for all $\beta < \alpha$. Let $u \in A_\alpha \cap \Uparrow_w y \cap \downarrow z$. Since $u \in A_\alpha = \bigcup_{\beta<\alpha} A_\beta$, there exists $\beta_0 < \alpha$ such that $u \in A_{\beta_0}$. Since β_0 has **F** property and $u \in A_{\beta_0} \cap \Uparrow_w y \cap \downarrow z$, it follows that $y \in \downarrow D$. So α has **F** property.

By transfinite induction, α has **F** property for all $\alpha \leqslant \gamma$. In particular, γ has **F** property. Note that $z \in A_\gamma \cap \Uparrow_w y \cap \downarrow z \neq \varnothing$, so that $y \in \downarrow D$.

All above results together show that $y \ll x$. The converse implication is trivial.□

As a direct consequence of Lemma 3.3, the following result shows that meet continuity forces weak domains to domains.

Theorem 3.4 *Every meet continuous weak domain is a domain.*

Definition 3.5 A dcpo P is called a *local domain* (resp., a *local algebraic domain*) if for each $x \in P$, $\downarrow x$ is a domain (resp., an algebraic domain).

A dcpo is called a *local weak domain* if for each $x \in P$, $\downarrow x$ is a weak domain.

The following proposition is trivial since every element in a dcpo P is below some maximal point of P.

Proposition 3.6 *A dcpo P is a local domain (resp., a local algebraic domain) iff for each $a \in \mathrm{Max}(P)$, $\downarrow a$ is a domain (resp., an algebraic domain).*

In [3], Jung has proved the following result.

Theorem 3.7 *[3, Corollary 1.7]* (1) *Local domains are exactly domains.*

(2) *Local algebraic domains are exactly algebraic domains.*

By Proposition 3.6, Theorems 3.7, we have

Corollary 3.8 (1) *A dcpo P is a domain iff $\downarrow a$ is a domain for each $a \in \mathrm{Max}(P)$.*

(2) *A dcpo P is an algebraic domain iff $\downarrow a$ is an algebraic domain for each $a \in \mathrm{Max}(P)$.*

The following result is trivial.

Lemma 3.9 *Let P be a dcpo and $x, y \in P$. Then $y \ll_w x$ in P iff $y \ll_w x$ in $\downarrow x$.*

By Lemma 3.9, we have

Corollary 3.10 *A dcpo is a weak domain iff it is a local weak domain.*

For subsets G and H of a dcpo P, G is way-below H, denoted by $G \ll H$, if for any directed set D, $\bigvee D \in \uparrow H$ implies $D \cap \uparrow G \neq \varnothing$. A dcpo P is *quasicontinuous* if for any $x \in P$, the family

$$fin(x) = \{\uparrow F \subseteq P : F \text{ is finite and } F \ll x\}$$

is filtered and $\uparrow x = \bigcap fin(x)$ (see [2, Definition III-3.2]). Note that every domain is a quasicontinuous domain. A dcpo P is *quasialgebraic* if for any $x \in P$, the family

$$comp(x) = \{F \subseteq P : F \text{ is finite and } F \ll F \ll x\}$$

is filtered and $\uparrow x = \bigcap comp(x)$ (see [2, Definition III-3.23]). It is clear that every quasialgebraic domain is quasicontinuous and every algebraic domain is quasialgebraic.

The next two examples show that quasicontinuous domains need not be weak domains, and weak domains need not be quasicontinuous domains. In the following, the notation $\mathbb{N}$ means the set of natural numbers.

Example 3.11 Let $P = \{\langle n, i\rangle : n \in \mathbb{N}, i = 0, 1\} \cup \{\top\}$. Define the order on P by the following rules:

(i) $\langle n, i\rangle \leqslant \langle m, j\rangle$ iff $n \leqslant m$ and $i = j$;

(ii) $\forall p \in P$, $p \leqslant \top$.

Then P can be represented by Figure 1. It is easy to check that P is a quasicontinuous domain. However, it is not a weak domain because $\downarrow_w \top = \varnothing$.

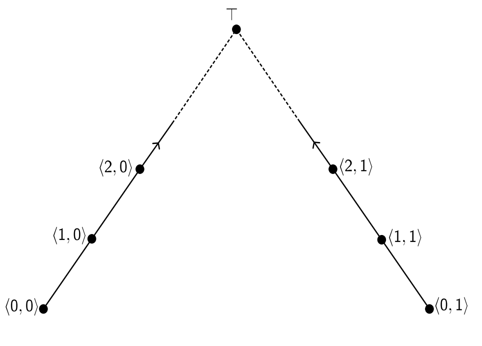

Fig. 1. A quasicontinuous domain but not a weak domain

Example 3.12 A weak domain need not be a quasicontinuous domain. Let $Q = \{\langle m, n\rangle : m, n \in \mathbb{N}\} \cup \{\langle \omega, n\rangle : n \in \mathbb{N}\} \cup \{\top\}$. Define an order on Q by the following rules: $\forall m, m', n, n' \in \mathbb{N}$,

(i) $\forall q \in Q$, $q \leqslant \top$;

(ii) $\langle\omega, m\rangle \leqslant \langle\omega, n\rangle$ iff $m \leqslant n$;

(iii) $\langle m, n\rangle \leq \langle\omega, n\rangle$;

(iv) $\langle m, n\rangle \leqslant \langle m', n'\rangle$ iff $m \leqslant m'$ and $n = n'$.

The order on Q can be represented by Figure 2. We have the following facts:

(q1) $\downarrow_w \top = Q \setminus \{\top\}$;

(q2) $\downarrow_w \langle\omega, n\rangle = \{\langle m, n\rangle : m \in \mathbb{N}\}$;

(q3) $\downarrow_w \langle m, n\rangle = \downarrow\langle m, n\rangle$.

Thus, Q is a weak domain. But it is not quasicontinuous. Considering $\langle\omega, 0\rangle$, for any finite subset F of Q, there exists $n_0 \in \mathbb{N}$ such that $x < \langle\omega, n_0\rangle$ and $x \not\leq \langle m, n_0\rangle$ for all $x \in F$ and $m \in \mathbb{N}$. Note that $\langle\omega, 0\rangle \leqslant \langle\omega, n_0\rangle = \bigvee\{\langle m, n_0\rangle : m \in \mathbb{N}\}$, but $\{\langle m, n_0\rangle : m \in \mathbb{N}\} \cap \uparrow F = \varnothing$. So $F \not\ll \langle\omega, 0\rangle$. This means $fin(\langle\omega, 0\rangle) = \varnothing$. Hence, Q is not a quasicontinuous domain.

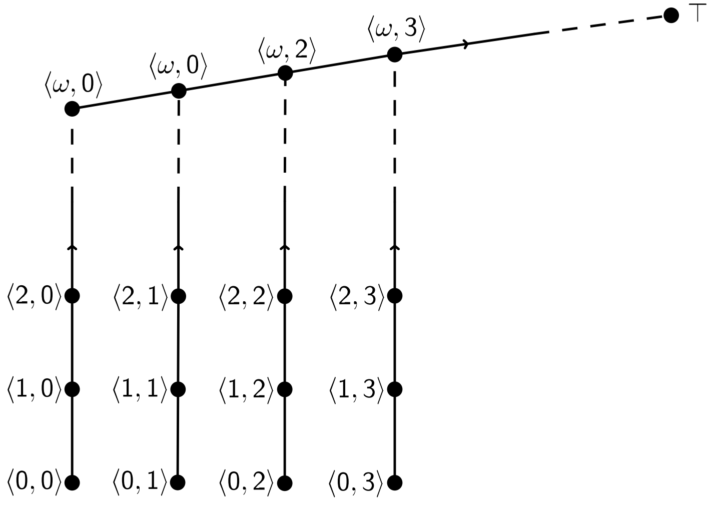

Fig. 2. A weak domain but not a quasicontinuous domain

A dcpo P is called a *local quasicontinuous domain* (resp., *a local quasialgebraic domain*) if for each $x \in P$, $\downarrow x$ is a quasicontinuous domain (resp., *a quasialgebraic domain*).

Remark 3.13 (1) Every quasicontinuous domain is local quasicontinuous.

(2) Observe that the weak domain ($= \downarrow\top$) of Figure 2 is not quasicontinuous, thus is not local quasicontinuous. It follows that weak domains need not be a local quasicontinuous domains.

(3) In [9], it was proved that every T_1 space X has a local quasicontinuous domain model, denoted by $\widehat{\mathrm{Zh}}(X)$. An important property is that X is sober iff $\widehat{\mathrm{Zh}}(X)$ is sober. So if X is not sober, then $\widehat{\mathrm{Zh}}(X)$ cannot be quasicontinuous. Hence, there exists a local quasicontinuous domain which is not quasicontinuous.

Theorem 3.14 *Every meet continuous local quasicontinuous domain is a domain.*

Proof. Assume P is a meet continuous local quasicontinuous domain. Let $x \in P$. Then $\downarrow x$ is a meet continuous quasicontinuous domain, so it is a domain. By 3.7, P is a domain. □

By Theorem 3.7, Corollary 3.10, Examples 3.11, 3.12 and Remark 3.13, the relations among domains, local domains, quasicontinuous domains, local quasicontinuous domains, weak domains and local weak domains are shown in Figure 3.

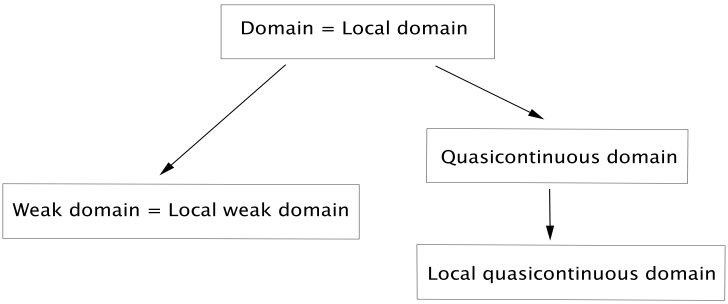

Fig. 3. Relations among (local) domains, (local) quasicontinuous domains and (local) weak domains

Proposition 3.15 *[2, Lemma III-2.10] If F is a finite subset of a meet continuous dcpo P, then* $\mathrm{int}_\sigma(\uparrow F) \subseteq \bigcup\{\twoheaduparrow x : x \in F\}$.

Lemma 3.16 *Let P be a domain and let F be a finite subset of P with $F \ll F$.*

(1) *Every minimal element in F is compact.*

(2) *If $F \ll x$, then there exists $y \in F \cap K(P)$ such that $y \ll x$.*

Proof. (1) Let x be a minimal element in F. By $F \ll F$, $F \ll x$. Since P is a domain, $x \in \twoheaduparrow F = \mathrm{int}_\sigma(\uparrow F)$. By Proposition 3.15, there exists $y \in F$ such that $y \ll x$. Then $y = x$ because x is minimal in F. Thus, $x \ll x$.

(2) As $x \in F$ if and only if there exists a minimal (compact) element y in F such that $y \leqslant x$. Then by (1), it is trivial. □

Proposition 3.17 *Every meet continuous quasialgebraic domain is an algebraic domain.*

Proof. Suppose P is a meet continuous quasialgebraic domain. By [2, Theorem III-3.10], P is a domain. Let $x \in P$. By Lemma 3.16, for any $F \in comp(x)$, there exists $y \in F \cap K(P)$ such that $y \ll x$. Let $G_F = \{y \in F \cap K(P) : y \ll x\}$. Since P is quasialgebraic, the family $\{\uparrow G_F : F \in comp(x)\}$ is filtered. Then, by Rudin's Lemma, there exists a directed set $D \subseteq \bigcup\{G_F : F \in comp(x)\}$ such that $D \cap G_F \neq \varnothing$ for all $F \in comp(x)$.

Now we show that $\bigvee D = x$. It is immediate that $\bigvee D \leqslant x$ since $D \subseteq K(P) \cap \downarrow x$. If $x \nleqslant \bigvee D$, then there exists $F \in comp(x)$ such that $\bigvee D \notin \uparrow F$, which implies $D \cap F = \varnothing$. Thus $D \cap G_F = \varnothing$, a contradiction. Hence, $x \leqslant \bigvee D$.

Note that $D \subseteq {\downarrow}x \cap K(P)$ is directed with $\bigvee D = x$. Then one can deduce that ${\downarrow}x \cap K(P)$ is directed and $x = \bigvee {\downarrow}x \cap K(P)$. Therefore, P is an algebraic domain.□

Theorem 3.18 *Every meet continuous local quasialgebraic domain is a quasialgebraic domain.*

Proof. Assume P is a meet continuous local quasialgebraic domain. Let $x \in P$. Then ${\downarrow}x$ is a meet continuous quasialgebraic domain, so by Proposition 3.17, it is an algebraic domain. By 3.7, P is an algebraic domain. □

4 Locating the relation $\ll_w$ within the weakly auxiliary relations

In this section, we define a new auxiliary relation corresponding to the weakly way-below relation, and use it characterize exact dcpos.

Definition 4.1 A binary relation $\prec$ on a poset P is called a *weakly auxiliary relation*, or *w-auxiliary relation* for short, if for all $x, y, z \in P$:

(i) $x \prec y$ implies $x \leqslant y$;

(ii) $x \leqslant y \prec z$ implies $x \prec z$;

(iii) $\perp \prec x$ whenever the smallest element $\perp$ exists in P.

The set of all w-auxiliary relations on P is denoted by $\mathrm{WAux}(P)$.

Remark 4.2 (1) The only difference between w-auxiliary relations and auxiliary relations (see [2, Defintion I-1.11]) is that w-auxiliary relations may not be increasing: $x \prec y \leqslant z$ may not imply $x \prec z$.

(2) Weakly way-below relations are w-auxiliary.

(3) The set $\mathrm{WAux}(P)$ is a poset relative to the containment of graphs as subsets of $P \times P$. The largest element is the relation $\leqslant$ itself. If P has a smallest element $\perp$, then $\mathrm{WAux}(P)$ has a smallest element $\prec_0$ given by $x \prec_0 y$ if and only if $x = \perp$. Moreover, $\mathrm{WAux}(P)$ is closed under arbitrary nonempty intersection in the powerset of $P \times P$. Hence, $\mathrm{WAux}(P)$ is a complete lattice whenever the smallest element of P exists.

For a poset P, we use $\mathrm{Low}(P)$ to denote the set of all lower sets in P.

Proposition 4.3 *Let P be a poset and $\Phi(P)$ be the set of all mappings $s : P \longrightarrow$* $\mathrm{Low}P$ *satisfying $s(x) \subseteq {\downarrow}x$ for all $x \in P$. Then the assignment*

$$\prec \mapsto s_\prec = (x \mapsto \{y : y \prec x\})$$

is a well-defined isomorphism from $\mathrm{WAux}(P)$ *to $\Phi(P)$, whose inverse sends every mapping $s \in \Phi(P)$ to the relation $\prec_s$ given by*

$$x \prec_s y \textit{ if and only if } x \in s(y).$$

Proof. Let $\prec$ be a w-auxiliary relation on P. Then $s_\prec(x)$ is a lower set by Definition 4.1 (ii) and is contained in ${\downarrow}x$ by Definition 4.1 (i). Thus $s_\prec$ is in $\Phi(P)$ and the assignment $\prec \mapsto s_{\prec}$ is clearly order-preserving.

Conversely, if $s \in \Phi(P)$, then $s(x) \subseteq \downarrow x$ implies that $\prec_s$ satisfies Definition 4.1 (i). Furthermore, if $x \leqslant y \prec_s z$, then $y \in s(z)$. Since $s(z)$ is a lower set, it follows that $x \in s(z)$, implying $x \prec_s z$. Thus Definition 4.1 (ii) is satisfied. Condition (iii) of Definition 4.1 is immediate. Therefore, the assignment $s \mapsto \prec_s$ is well-defined, and it is obviously order-preserving.

Note that $x \prec_{s_\prec} y$ if and only if $x \in s_\prec(y)$ if and only if $x \prec y$. In addition, $s_{\prec_s}(x) = \{y \in P : y \prec_s x\} = \{y \in P : y \in s(x)\} = s(x)$. Therefore, the two assignments are inverse from each other. $\square$

Lemma 4.4 *Let P be a dcpo and $x \in P$.*

(1) $\downarrow_w x = \bigcap\{I \in \mathrm{Id}(P) : x = \bigvee I\}$;

(2) *the assignment $x \mapsto \downarrow_w x$ is a member of $\Phi(P)$ defined in Proposition 4.3;*

(3) *for every ideal $I \in \mathrm{Id}(P)$, the mapping $m_I : P \longrightarrow \mathrm{Low}(P)$ given by*

$$m_I(x) := \begin{cases} I, & \text{if } x = \bigvee I, \\ \downarrow x, & \text{otherwise} \end{cases}$$

is in $\Phi(P)$;

(4) $\downarrow_w x = \bigwedge_{\Phi(P)}\{m_I(x) : I \in \mathrm{Id}(P)\}$.

Proof. We only check (4), as (1)-(3) are obvious. In fact,

$$\begin{aligned} & \textstyle\bigwedge_{\Phi(P)}\{m_I(x) : I \in \mathrm{Id}(P)\} \\ = & \textstyle\bigcap\{m_I(x) : I \in \mathrm{Id}(P)\} \\ = & \textstyle\bigcap\{m_I(x) : I \in \mathrm{Id}(P), x = \bigvee I\} \cap \bigcap\{m_I(x) : I \in \mathrm{Id}(P), x \neq \bigvee I\} \\ = & \textstyle\bigcap\{I \in \mathrm{Id}(P) : I \in \mathrm{Id}(P), x = \bigvee I\} \cap \downarrow x \\ = & \downarrow_w x, \end{aligned}$$

completing the proof. $\square$

Definition 4.5 A w-auxiliary relation $\prec$ on a dcpo P is *approximating* if for each x in P, the set $s_\prec(x) = \{y \in P : y \prec x\}$ is directed and $x = \bigvee s_\prec(x)$.

Lemma 4.6 *Let P be a dcpo and $I \in \mathrm{Id}(P)$. Then the relation $\prec_I$ defined below is an approximating w-auxiliary relation:*

$$\forall x, y \in P,\ y \prec_I x \ \Leftrightarrow\ y \in m_I(x).$$

Proof. It is easy to check that $\prec_I$ defined above is a w-auxiliary relation. We now check that it is approximating. Let $x \in P$. There are two cases:

(1) If $x = \bigvee I$, then $\bigvee\{y \in P : y \prec_I x\} = \bigvee m_I(x) = \bigvee I = x$.

(2) If $x \neq \bigvee I$, then $\bigvee\{y \in P : y \prec_I x\} = \bigvee m_I(x) = \bigvee \downarrow x = x$.

Thus, for all $x \in P$, $\bigvee\{y \in P : y \prec_I x\} = x$, hence $\prec_I$ is approximating. □

Proposition 4.7 *For a dcpo P, the weakly way-below relation on P is the intersection of all the approximating w-auxiliary relations on P.*

Proof. Suppose $x \ll_w y$ and $\prec \in \mathrm{WAux}(P)$ is approximating. Since $\{z \in P : z \prec y\}$ is directed and $y = \bigvee\{z \in P : z \prec y\}$, there exists $z \in P$ such that $x \leqslant z \prec y$, implying $x \prec y$. Therefore, $\ll_w$ is contained in $\prec$.

Additionally, by Lemma 4.6, we have

$$\twoheaddownarrow_w x = \bigcap\{m_I(x) : I \in \mathrm{Id}(P)\} \supseteq \bigcap\{s_\prec(x) : \prec \text{ is approximating}\},$$

where $s_\prec(x) = \{y \in P : y \prec x\}$. Therefore, the relation $\ll_w$ is the intersection of all approximating w-auxiliary relations on P. □

Theorem 4.8 *For a dcpo P, the following statements are equivalent:*

(1) *P is exact.*

(2) *The relation $\ll_w$ is the smallest approximating w-auxiliary relation on P.*

(3) *There is a smallest approximating w-auxiliary relation on P.*

5 Weak domain models of T_1 spaces

Now let's go back to Mashburn's question [7] as indicated in the introduction:

- Which topological spaces have weak domain models?

In this section, we will show that every T_1 space has a weak domain model, which answers the above question. The crucial tool that we make use of is the Xi-Zhao model, which was introduced by Dongsheng Zhao and Xiaoyong Xi [11].

In [10], Zhao proved that every T_1 space X has a bounded complete algebraic poset model. This model is constructed by using the set of all filtered families of open sets of X with a nonempty intersection. For each bounded complete algebraic poset P, Xi and Zhao [11] constructed a dcpo $\widehat{P}$ as:

$$\widehat{P} = \{(x, d) : x \in P, d \in \mathrm{Max}(P) \text{ and } x \leqslant_P d\},$$

where $\leqslant_P$ is the partial order on P, and $(x, d) \leqslant (y, e)$ in $\widehat{P}$ iff either $d = e$ and $x \leqslant_P y$, or $y = e$ and $x \leqslant_P e$. They proved that $\mathrm{Max}(P)$ and $\mathrm{Max}(\widehat{P})$ are homeomorphic, showing that every T_1 space has a dcpo model (the Xi-Zhao model).

Definition 5.1 Let P be a poset.

(1) An element $x \in P$ is called *weakly compact*, if $x \ll_w x$. Denote by $\mathrm{K}_w(P)$ the set of all weakly compact elements.

(2) P is called a *weak algebraic poset*, if $\ll_w$ is weakly increasing and for any $x \in P$, the set $\mathrm{K}_w(P) \cap \twoheaddownarrow_w x$ is directed and $x = \bigvee \mathrm{K}_w(P) \cap \twoheaddownarrow_w x$.

A weak algebraic dcpo is also called a *weak algebraic domain*.

Note that every weak algebraic domain is a weak domain and that every meet continuous weak algebraic dcpo is algebraic by Lemma 3.3.

Next, we will show that the Xi-Zhao model constructed for each T_1 space is a weak algebraic domain (and hence a weak domain).

Remark 5.2 The following facts on the dcpo $\widehat{P}$ constructed from a bounded complete algebraic poset P will be used subsequently.

(i) If $\mathcal{D}$ is a directed subset of $\widehat{P}$ and it does not have a largest element, then there exists $d \in Max(P)$ and a directed subset $\{x_i : i \in I\}$ of P such that $\mathcal{D} = \{(x_i, d) : i \in I\}$, and in this case $\bigvee \mathcal{D} = (\bigvee_P \{x_i : i \in I\}, d)$.

(ii) The set of maximal points of $\widehat{P}$ equals $\{(d, d) : d \in Max(P)\}$.

Lemma 5.3 *Let P be a weak algebraic poset P and $(x, d) \in \widehat{P}$.*

(1) *$x \in K_w(P)$ if and only if $(x, d) \in K_w(\widehat{P})$;*

(2) *if $x \notin K_w(P)$, then $(y, e) \ll_w (x, d)$ if and only if $y \ll_w x$ and $d = e$;*

(3) *for any $(y, e) \in P$, $(x, d) \ll_w (y, e)$ implies $x \ll_w y$.*

Proof. (1) Suppose that $x \ll_w x$ and $\mathcal{D}$ is a directed subset of $\widehat{P}$ with $(x, d) = \bigvee \mathcal{D}$. If $(x, d) \notin \mathcal{D}$, then, by Remark 5.2, there is a directed subset $\{x_i : i \in I\}$ of P such that $\mathcal{D} = \{(x_i, d) : i \in I\}$ and $\bigvee \mathcal{D} = (\bigvee_P \{x_i : i \in I\}, d)$. Thus $x = \bigvee_P \{x_i : i \in I\}$. Since $x \ll_w x$, it follows that $x \leqslant_P x_{i_0}$ for some $i_0 \in I$. This means $(x, d) \leqslant (x_{i_0}, d) \in \mathcal{D}$, and hence $(x, d) \ll_w (x, d)$. The other direction, $(x, d) \ll_w (x, d)$ implies $x \ll_w x$, is trivial.

(2) Suppose that $(y, e) \ll_w (x, d)$. First, let $D = K_w(P) \cap \downarrow_w x$. By the assumption that $x \notin K_w(P)$, we have $x \notin D$, and hence $d \notin D$. As P is a weak algebraic poset, D is directed and $\bigvee_P D = x$. Thus, the set $\{(z, d) : z \in D\}$ is directed and $(x, d) = (\bigvee_P D, d) = \bigvee \{(z, d) : z \in D\}$. Since $(y, e) \ll_w (x, d)$, there exists $z_0 \in D$ such that $(y, e) \leqslant (z_0, d)$. Thus $e = d$ because $z_0 \neq d$. Now suppose that S be a directed subset of P with $x = \bigvee_P S$. Then $\{(z, d) : z \in S\}$ is a directed subset of $\widehat{P}$ and $(x, d) = \bigvee \{(z, d) : z \in S\}$. By $(y, e) \ll_w (x, d)$, there exists $z_0 \in S$ such that $(y, e) = (y, d) \leqslant_P (z_0, d)$. Thus $y \leqslant_P z_0$. Hence, $y \ll_w x$.

The converse is straightforward by using (i) in Remark 5.2.

(3) It is immediate by using (i) in Remark 5.2. □

Lemma 5.4 *Let P be an algebraic poset.*

(1) *If $x \ll_w y \leqslant_P z$ on P, then $x \ll_w z$.*

(2) *If $(x, d) \ll_w (y, e) \leqslant (u, f) \ll_w (v, g)$ on $\widehat{P}$, then $(x, d) \ll_w (u, f)$.*

Proof. (1) Suppose that D is a directed subset of P such that $z = \bigvee_P D$. As P is algebraic, $y = \bigvee_P \{u \in K(P) : u \leqslant_P y\}$. By $x \ll_w y$, $x \leqslant_P u$ for some $u \in K(P)$ and $u \leqslant_P y$. Note that $u \ll u \leqslant_P z = \bigvee_P D$, and hence there exists $v \in D$ such that $u \leqslant_P v$, implying $x \leqslant_P v$. Therefore $x \ll_w z$.

(2) First, by Lemma 5.3 (3), $x \ll_w y \leqslant_P u$. By (1), $x \ll_w u$. We consider the following cases:

Case 1: $u \in K_w(P)$.

By Lemma 5.3 (1), $(u,f) \ll_w (u,f)$. By $(x,d) \leqslant (u,f)$, we have $(x,d) \ll_w (u,f)$ by Proposition 2.2 (3).

Case 2: $u = f$.

In this case, $(u,f) = (f,f) \in Max(\widehat{P})$, and hence $(v,g) = (u,f)$, $(u,f) \ll_w (u,f)$. By Proposition 2.2 (3), we have $(x,d) \ll_w (u,f)$.

Case 3: $x = d$ or $y = e$.

A similar argument of Case 2 applies.

Case 4: $u \notin K_w(P)$, $u \neq f$, $y \neq e$ and $x \neq d$.

By $(x,d) \leqslant (y,e) \leqslant (u,f)$, we have $d = e = f$. By the fact that $x \ll_w u$ and Lemma 5.3 (2), $(x,d) \ll_w (u,f)$.

□

As a corollary of Lemmas 5.3, 5.4, we obtain the following.

Corollary 5.5 *For an algebraic poset P, $\widehat{P}$ is a weak algebraic domain.*

Applying Corollary 5.5 to Xi-Zhao model for T_1 space, we obtain our main result.

Theorem 5.6 *Every T_1 topological space has a weak algebraic domain model.*

6 Conclusion

(1) Let's note that the analogue of Figure 3 for the algebraic case hold too, that is,

$$\text{algebraic domain} \Leftrightarrow \text{local algebraic domain} \Rightarrow \begin{cases} \text{local weak algebraic domain} \Leftrightarrow \text{weak algebraic domain} \\ \text{quasialgebraic domain} \Rightarrow \text{local quasialgebraic domain.} \end{cases}$$

All above notions coincide in a meet continuous dcpo.

(2) A dcpo P is called *well-filtered* if its Scott space is well-filtered (see [2] for the definition of well-filteredness). In [9], it was proved that a T_1 space X is well-filtered iff its Xi-Zhao model is well-filtered. As was pointed out in [9], there exists a T_1 space that is not well-filtered, thus its Xi-Zhao model is a weak domain (even a weak algebraic domain) that is not well-filtered.

(3) In [7], Mashburn proved that every first countable space has a weak domain model. Our Theorem 5.6 shows that the condition "first countable" in Mshburn's result is surplus. Now we can say that a space has a weak domain model iff it is a T_1 space, answering Mashburn's problem, as indicated in the introduction.

(4) By Theorem 3.4, we prove that every meet continuous weak domain is a domain. However, the following question is still unknown:

- Is a meet continuous exact dcpo a domain?

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