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Characterizations of Supercontinuous Posets via Scott S-sets and the S-essential Topology

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Abstract

In this paper, the concepts of Scott S-sets, weakly local S-compactness and S-bases of posets are introduced. With these new concepts, some characterizations of supercontinuous (resp., superalgebraic) posets are given. In order to provide a topological interpretation of S-bases, the new concept of the S-essential topology on posets is also introduced. Properties and characterizations of S-bases via the S-essential topology are obtained. Main results are: (1) A poset L is supercontinuous iff every two different points can be separated by a principal filter and the complement of a Scott S-set; (2) A poset L is supercontinuous iff it is weakly local S-compact and for every $x \in U$, where U is a Scott S-set, there is a Scott S-set filter V such that $x \in V \subseteq U$; (3) A poset L is supercontinuous iff it has an S-basis; (4) In a supercontinuous poset, a subset B is an S-basis iff for every Scott S-set U and S-e-open set G, $U \cap G \cap B \neq \emptyset$ whenever $U \cap G \neq \emptyset$. Counterexamples are constructed to show that supercontinuity is not hereditary to principal ideals, nor to Scott S-sets.

Keywords: Supercontinuous poset; Scott S-set; weakly local S-compactness; S-basis; S-essential topology

1 Introduction

The notion of continuous lattices as a model for the semantics of programming languages was introduced by Scott in [11]. Later, a more general notion of continuous directed complete partially ordered sets (i.e., continuous dcpos or domains) was

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introduced and extensively studied (see [1]-[3]). Lawson in [3] gave a remarkable characterization that a dcpo is continuous iff its Scott topology is completely distributive. Since some naturally arisen posets are important but fail to be directed complete, there are more and more occasions to study posets which miss suprema of directed sets (see [4]-[8], [12]-[15]). By the technique of embedded bases and sobrification via the Scott topology, Xu in [12] successfully embedded continuous posets into continuous domains and proved that a poset is continuous iff its Scott topology is completely distributive.

Zhao and Zhou in [15] introduced the concept of supercontinuous posets, which is a generalization of completely distributive lattices, and studied the order structure and categorical properties of supercontinuous posets. Although supercontinuous posets need not be continuous in general, they share some properties with continuous posets. To give more characterizations of supercontinuous posets, we introduce the concept of Scott S-sets which are some special Scott open sets. Though the Scott S-sets of a poset is not a topology, they indeed forms a semi-topology in some sense and play roles in characterizing supercontinuous posets similar to that of Scott open sets in characterizing continuous posets. We also introduce the concepts of weakly local S-compactness and S-bases of posets. With these concepts, we present several characterizations of supercontinuous posets as well as superalgebraic posets. To provide a topological interpretation of S-bases, we introduce the new concept of the S-essential topology of posets and investigate some basic properties of the S-essential topology and relations with other intrinsic topologies. Via the S-essential topology, we obtain several properties and characterizations of S-bases from different points of view. To one's surprise, counterexamples show that supercontinuity is not hereditary to Scott S-sets.

2 Preliminaries

We quickly recall some basic notions and results (see [1], [5] and [12]).

Let $(L, \leqslant)$ be a poset. A *principal ideal* (resp., *principal filter*) is a set of the form $\downarrow x = \{y \in L \mid y \leqslant x\}$ (resp., $\uparrow x = \{y \in L \mid x \leqslant y\}$). For $A \subseteq L$, we write $\downarrow A = \{y \in L \mid \exists\, x \in A, y \leqslant x\}$, $\uparrow A = \{y \in L \mid \exists\, x \in A, x \leqslant y\}$. A subset A is a *lower set* (resp., an *upper set*) if $A = \downarrow A$ (resp., $A = \uparrow A$). We say that z is a *lower bound* (resp., an *upper bound*) of A if $A \subseteq \uparrow z$ (resp., $A \subseteq \downarrow z$). The set of lower bounds of A is denoted by $\mathrm{lb}(A)$. The supremum of A is denoted by $\bigvee A$ or $\sup A$. The infimum of A is denoted by $\bigwedge A$ or $\inf A$. A nonempty subset D of L is *directed* if every finite subset of D has an upper bound in D. A poset L is a *directed complete partially ordered set* (dcpo, for short) if every directed subset of L has a supremum. A *complete lattice* is a poset in which every subset has a supremum.

In a poset L, we say that x *approximates* y, written $x \ll y$ if whenever D is a directed set that has a supremum $\sup D \geqslant y$, then $x \leqslant d$ for some $d \in D$. We say that x is *compact* if x approximates itself, i.e., $x \ll x$. The set of all compact elements is denoted by $K(L)$. For $x \in L$, we write $\twoheaddownarrow x = \{z \in L \mid z \ll x\}$ and $\twoheaduparrow x = \{z \in L \mid x \ll z\}$. The poset L is said to be *continuous* (resp., *algebraic*) if every element is the directed supremum of all (resp., *compact*) elements that approximate it, i.e., for all $x \in L$, the set $\Downarrow x$ (resp., $\downarrow x \cap K(L)$) is directed and $x = \bigvee \Downarrow x$ (resp., $x = \bigvee(\downarrow x \cap K(L))$). A continuous poset (resp., an algebraic poset) which is also a dcpo is called a *continuous domain* (resp., an *algebraic domain*).

A subset A of a poset L is *Scott closed* if $\downarrow A = A$ and for any directed set $D \subseteq A$, $\sup D \in A$ whenever $\sup D$ exists. The complements of the Scott closed sets form a topology, called the *Scott topology* and denoted by $\sigma(L)$. It is well-known that for a continuous poset the Scott topology has a base of all sets of the form $\Uparrow x$. The topology generated by the complements of all principal filters $\uparrow x$ (resp., principal ideals $\downarrow x$) is called the *lower topology* (resp., *upper topology*) and is denoted by $\omega(L)$ (resp., $\nu(L)$). The topology of all upper sets (resp., lower sets) is called the *Alexandroff topology* (resp., *the dual Alexandroff topology*) and is denoted by $\alpha(L)$ (resp., $\alpha^*(L)$).

Definition 2.1 (see [15]) *Let L be a poset. For any two elements x and y in L, we write $x \lhd y$, if for any subset $A \subseteq L$ with existing $\bigvee A \geqslant y$, there is $a \in A$ such that $x \leqslant a$. For all $x \in L$, we write $\Downarrow^{\lhd} x = \{z \in L \mid z \lhd x\}$, $\Uparrow^{\lhd} x = \{z \in L \mid x \lhd z\}$. We say that x is super-compact if $x \lhd x$. The set of all super-compact elements is denoted by $SK(L)$.*

The following proposition is basic and the proof is omitted.

Proposition 2.2 *Let L be a poset. Then for all x, y, u, $v \in L$:*

(1) $x \lhd y \Longrightarrow x \ll y \Longrightarrow x \leqslant y$;

(2) $u \leqslant x \lhd y \leqslant v \Longrightarrow u \lhd v$;

(3) If L has a bottom $\perp$, then $\forall x \in L \setminus \{\perp\}$, one has $\perp \lhd x$, but $\perp \lhd \perp$ does not hold.

Definition 2.3 (see [15]) *A poset L is called supercontinuous (resp. superalgebraic) if for all $x \in L$, $x = \bigvee \Downarrow^{\lhd} x$ (resp., $x = \bigvee(\downarrow x \cap SK(L))$).*

Remark 2.4 (see [15]) *A complete lattice is supercontinuous if and only if it is completely distributive. Every supercontinuous sup-semilattice is continuous. However, in general, by [15, Remark 1.1], a supercontinuous poset need not be a continuous poset.*

Proposition 2.5 *Let L be a supercontinuous poset. Then the relation $\lhd$ on L has the interpolation property: $x \lhd y \Longrightarrow \exists z \in L$ such that $x \lhd z \lhd y$.*

Proof. Let x, $y \in L$ with $x \lhd y$. By Proposition 2.2(3), $y \neq \perp$ whenever L has a bottom $\perp$. Let $T = \{t \in L \mid \exists\ z \in L,\ \ t \lhd z \lhd y\}$. By the supercontinuity of L, it is straightforward to prove that $\bigvee T = y$. It follows from $x \lhd y = \vee T$ that there exists $t \in T$ such that $x \leqslant t$. So, there is $z \in L$ such that $x \leqslant t \lhd z \lhd y$. By Proposition 2.2(2), we have $x \lhd z \lhd y$.

□

3 Scott S-sets and Supercontinuity

In this section, we introduce Scott S-sets and give several characterizations of supercontinuous posets.

Definition 3.1 *Let L be a poset. A subset U of L is called a Scott S-set if the following two conditions are satisfied:*

(1) $U = \uparrow U$;

(2) For all $A \subseteq L$, $\bigvee A \in U$ implies $A \cap U \neq \emptyset$ whenever $\bigvee A$ exists.

The family of all Scott S-sets of L is denoted by $SS(L)$. The family of the complements of all Scott S-sets of L is denoted by $SS^*(L)$, i.e., $SS^*(L) = \{F \subseteq L \mid L \backslash F \in SS(L)\}$.

Proposition 3.2 *Let L be a poset. Then*

(1) $F \in SS^*(L)$ iff $F = \downarrow F$ and $\bigvee A \in F$ for any set $A \subseteq F$ whenever $\vee A$ exists;

(2) $L \in SS^*(L)$; $\emptyset \in SS^*(L)$ iff L has no bottom;

(3) $\forall x \in L$, $\downarrow x \in SS^*(L)$;

(4) For any $\{F_\alpha\}_{\alpha \in \Gamma} \subseteq SS^*(L)$, $\bigcap_{\alpha \in \Gamma} F_\alpha \in SS^*(L)$;

(5) $\forall x \in L$, $x \in SK(L)$ iff $\uparrow x \in SS(L)$.

Proof. Straightforward for (1), (2), (3) and (5). We divide two cases to show (4).

Case (i): L has no bottom. In this case, by (1) and (2), it is easy to verify that $\bigcap_{\alpha \in \Gamma} F_\alpha \in SS^*(L)$.

Case (ii): L has a bottom "$\perp$". In this case, $\emptyset \notin SS^*(L)$ by (2) and thus $\perp \in F_\alpha$ for all $\alpha \in \Gamma$. Then it is easy to verify that $\bigcap_{\alpha \in \Gamma} F_\alpha \in SS^*(L)$ by (1), as desired. $\Box$

Remark 3.3 *By Definition 3.1, all the Scott S-sets of a poset are Scott open sets of the poset. So the prefix "S-" means strong open or special open in the Scott topology. By Proposition 3.2(1), complements of Scott S-sets of a complete lattice are principal ideals. Since unions of two principal ideals need not be principal ideals, it is clear that an intersection of two Scott S-sets need not be a Scott S-set. This reveals that the Scott S-sets of a complete lattice need not be a topology. However, in view of Proposition 3.2, we can say that all the Scott S-sets of a poset forms a semi-topology in the sense of being closed under arbitrary unions.*

Proposition 3.4 *Let L be a supercontinuous poset. Then for all $x \in L$, the set $\Uparrow^{\lhd} x \in SS(L)$ and for all $U \in SS(L)$, $U = \bigcup\{\Uparrow^{\lhd} y \mid y \in U\}$.*

Proof. For all $x \in L$, it follows from by Proposition 2.2(2) that the set $\Uparrow^{\lhd} x$ is an upper set. For all $A \subseteq L$, if $\bigvee A$ exists and $\bigvee A \in \Uparrow^{\lhd} x$, then by the interpolation property of the relation $\lhd$ on supercontinuous posets, there is $z \in L$ such that $x \lhd z \lhd \bigvee A$. So, there exists $a \in A$ such that $x \lhd z \leq a$. This shows that $A \cap \Uparrow^{\lhd} x \neq \emptyset$. By Definition 3.1, $\Uparrow^{\lhd} x \in SS(L)$. Let $U \in SS(L)$. It is clear that $\bigcup\{\Uparrow^{\lhd} y \mid y \in U\} \subseteq U$. Conversely, for all $u \in U$, it follows from the supercontinuity of L and Definition 3.1 that $u = \bigvee \Downarrow^{\lhd} u$ and $\Downarrow^{\lhd} u \cap U \neq \emptyset$. Pick $a \in \Downarrow^{\lhd} u \cap U$. Then $u \in \Uparrow^{\lhd} a \subseteq \bigcup\{\Uparrow^{\lhd} y \mid y \in U\}$. This shows that $U \subseteq \bigcup\{\Uparrow^{\lhd} y \mid y \in U\}$. Thus, $U = \bigcup\{\Uparrow^{\lhd} y \mid y \in U\}$ for all $U \in SS(L)$. $\Box$

Proposition 3.5 *Let L be a poset. Then the following conditions are equivalent:*

(1) L is supercontinuous;

(2) For all $x, y \in L$, $x \nleqslant y$ implies that there is $z \lhd x$ with $z \nleqslant y$;

(3) For all $x \in L$, $L \setminus \downarrow x = \bigcup\{\Uparrow^{\lhd} z \mid z \in L \setminus \downarrow x\}$.

Proof. (1) $\Longrightarrow$ (2): Follows immediately from Definition 2.3.

(2) $\Longrightarrow$ (3): Let $x \in L$. Clearly, $\bigcup\{\Uparrow^{\lhd} z \mid z \in L \setminus \downarrow x\} \subseteq L \setminus \downarrow x$. Conversely, for all $u \in L \setminus \downarrow x$, it follows from $u \nleqslant x$ and (2) that there is $t \lhd u$ such that $t \nleqslant x$. This shows that $u \in \Uparrow^{\lhd} t \subseteq \bigcup\{\Uparrow^{\lhd} z \mid z \in L \setminus \downarrow x\}$. Therefore, $L \setminus \downarrow x \subseteq \bigcup\{\Uparrow^{\lhd} z \mid z \in L \setminus \downarrow x\}$, as desired.

(3) $\Longrightarrow$ (1): For all $x \in L$, it follows from Proposition 2.2(1) that x is an upper bound of the set $\Downarrow^{\lhd} x$. Let y be any upper bound of $\Downarrow^{\lhd} x$. Suppose that $x \nleqslant y$. Then $x \in L \setminus \downarrow y$. By (3), there exists $z \in L \setminus \downarrow y$ such that $x \in \Uparrow^{\lhd} z \subseteq L \setminus \downarrow y$. This shows that $z \lhd x$ but $z \nleqslant y$, a contradiction to the assumption that y is an upper bound of $\Downarrow^{\lhd} x$. Therefore, $x \leqslant y$ and $x = \bigvee \Downarrow^{\lhd} x$. By Definition 2.3, L is supercontinuous. $\square$

Lemma 3.6 *Let L be a poset.*

(1) Let $x, y \in L$ with $x \nleqslant y$. If there exists $u \in L$ and a Scott S-set U such that $x \in U$, $u \nleqslant y$ and $\uparrow u \cup (L \setminus U) = L$, then $u \lhd x$.

(2) Let $x, y \in L$ with $x \nleqslant y$. If there exists $u \in L$ and a Scott S-set U such that $x \in U$, $u \nleqslant y$, $\uparrow u \cup (L \setminus U) = L$ and $\uparrow u \cap (L \setminus U) = \emptyset$, then $u \in SK(L)$ and $u \lhd x$.

Proof. (1) Let $x, y \in L$ with $x \nleqslant y$. Suppose that there is $u \in L$ and a Scott S-set U such that $x \in U$, $u \nleqslant y$ and $\uparrow u \cup (L \setminus U) = L$. For any $A \subseteq L$ with existing $\bigvee A \geqslant x$, assume that $\uparrow u \cap A = \emptyset$. Then $A \subseteq L \setminus \uparrow u \subseteq L \setminus U$. By proposition 3.2(1), we have $\vee A \in L \setminus U$ and thus $x \in L \setminus U$, a contradiction to $x \in U$. Therefore, $\uparrow u \cap A \neq \emptyset$ and $u \lhd x$.

(2) By (1), $u \lhd x$. It follows from $\uparrow u \cup (L \setminus U) = L$ and $\uparrow u \cap (L \setminus U) = \emptyset$ that $\uparrow u = L \setminus (L \setminus U) = U \in SS(L)$. By Proposition 3.2(5), we have $u \in SK(L)$. $\square$

Theorem 3.7 *A poset L is supercontinuous if and only if for any $x, y \in L$ with $x \nleqslant y$, there exists $u \in L$ and a Scott S-set U such that $x \in U$, $u \nleqslant y$ and $\uparrow u \cup (L \setminus U) = L$.*

Proof. $\Longleftarrow$: Let $x \in L$. If L has a bottom $\perp$ and $x = \perp$, then by Proposition 2.2(3), we have $\Downarrow^{\lhd} \perp = \emptyset$ and $\perp = \bigvee \Downarrow^{\lhd} \perp$. If $x \neq \perp$, then there is $y \in L$ such that $x \nleqslant y$. Therefore, there exists $u \in L$ and a Scott S-set U such that $x \notin L \setminus U$, $u \nleqslant y$ and $\uparrow u \cup (L \setminus U) = L$. By Lemma 3.6(1), we have $u \lhd x$ and hence $\Downarrow^{\lhd} x \neq \emptyset$. By Proposition 2.2(1), x is an upper bound of the set $\Downarrow^{\lhd} x$. Let z be any upper bound of $\Downarrow^{\lhd} x$. Assume that $x \nleqslant z$. Then there exists $v \in L$ and a Scott S-set V such that $x \notin L \setminus V$, $v \nleqslant z$ and $\uparrow v \cup (L \setminus V) = L$. By Lemma 3.6(1), we have $v \lhd x$ but $v \nleqslant z$, a contradiction to the assumption that z is an upper bound of $\Downarrow^{\lhd} x$. Therefore, $x \leqslant z$ and $x = \bigvee \Downarrow^{\lhd} x$. By Definition 2.3, L is supercontinuous.

$\Longrightarrow$: Let L be a supercontinuous poset. For any $x, y \in L$ with $x \nleqslant y$, it follows from Proposition 3.5 that there exists $u \lhd x$ such that $u \nleqslant y$. By Proposition 2.5, there is $z \in L$ such that $u \lhd z \lhd x$. Let $U = \Uparrow^{\lhd} z$. By Proposition 3.4, U is a Scott S-set and $x \in U$, $\uparrow u \cup (L \setminus U) = L$. $\square$

Corollary 3.8 *A poset L is superalgebraic if and only if for any $x, y \in L$ with $x \nleq y$, there exists $u \in L$ and a Scott S-set U such that $x \in U$, $u \nleq y$, $\uparrow u \cup (L \setminus U) = L$ and $\uparrow u \cap (L \setminus U) = \emptyset$.*

Proof. $\Longleftarrow$: Let $x \in L$. If L has a bottom $\perp$ and $x = \perp$, then it follows from Proposition 2.2(3) that $\downarrow \perp \cap SK(L) = \emptyset$ and $\perp = \bigvee(\downarrow \perp \cap SK(L))$. If $x \neq \perp$, then there is $y \in L$ such that $x \nleq y$. Therefore, there exists $u \in L$ and a Scott S-set U such that $x \in U$, $u \nleq y$, $\uparrow u \cup (L \setminus U) = L$ and $\uparrow u \cap (L \setminus U) = \emptyset$. By Lemma 3.6(1), we have $u \in SK(L)$ and $u \lhd x$. Hence, $\downarrow x \cap SK(L) \neq \emptyset$. Clearly, x is an upper bound of the set $\downarrow x \cap SK(L)$. Let z be any upper bound of $\downarrow x \cap SK(L)$. Assume that $x \nleq z$. Then there exists $v \in L$ and a Scott S-set V such that $x \in V$, $v \nleq z$, $\uparrow v \cup (L \setminus V) = L$ and $\uparrow v \cap (L \setminus V) = \emptyset$. By Lemma 3.6(2), we have $v \in \downarrow x \cap SK(L)$ but $v \nleq z$, a contradiction to the assumption that z is an upper bound of $\downarrow x \cap SK(L)$. Therefore, $x \leq z$ and $x = \bigvee(\downarrow x \cap SK(L))$. By Definition 2.3, L is superalgebraic.

$\Longrightarrow$: Let L be a superalgebraic poset. For any $x, y \in L$ with $x \nleq y$, it follows from Definition 2.3 that there exists $u \in \downarrow x \cap SK(L)$ such that $u \nleq y$. Let $U = \uparrow u$. By Proposition 3.2(5), U is a Scott S-set and $x \in U$, $\uparrow u \cup (L \setminus U) = L$, $\uparrow u \cap (L \setminus U) = \emptyset$. □

Lemma 3.9 *Let L be a poset and $x \in L$. If there is $U \in SS(L)$ such that $x \in U$, then $z \lhd x$ for all $z \in \mathrm{lb}(U)$.*

Proof. For all $A \subseteq L$ with existing $\sup A \geqslant x$, it follows from Definition 3.1 that $\sup A \in U$ and therefore $A \cap U \neq \emptyset$. Take an $a \in A \cap U$. Then $z \leqslant a$ for all $z \in \mathrm{lb}(U)$. By Definition 2.1, $z \lhd x$ for all $z \in \mathrm{lb}(U)$. □

Theorem 3.10 *A poset L is supercontinuous iff for any $x, y \in L$ with $x \nleq y$, there exists a filter $U \in SS(L)$ and $z \in \mathrm{lb}(U)$ such that $x \in U$ and $z \nleq y$.*

Proof. $\Longleftarrow$: For each $x \in L$, it follows from Proposition 2.2(1) that x is an upper bound of the set $\Downarrow^{\lhd} x$. Let t be any upper bound of $\Downarrow^{\lhd} x$. Suppose that $x \nleq t$. Then there is a filter $U \in SS(L)$ and $z \in \mathrm{lb}(U)$ such that $x \in U$ and $z \nleq t$. By Lemma 3.9, $z \lhd x$ but $z \nleq t$, a contradiction to the assumption that t is an upper bound of $\Downarrow^{\lhd} x$. Therefore, $x \leqslant t$ and $x = \bigvee \Downarrow^{\lhd} x$. By Definition 2.3, L is supercontinuous.

$\Longrightarrow$: Let L be a supercontinuous poset. For all $x, y \in L$ with $x \nleq y$, it follows from Proposition 3.5 that there exists $z \in L$ such that $z \lhd x$ and $z \nleq y$. By the interpolation property of the relation $\lhd$ on supercontinuous posets, there is $a_1 \in L$ such that $z \lhd a_1 \lhd x$. Similarly, there is $a_2 \in L$ such that $z \lhd a_2 \lhd a_1 \lhd x$, and so on. Then there is a sequence $\{a_n\}_{n \in \mathbb{N}^+}$ such that $z \lhd a_{n+1} \lhd a_n \lhd x$ for all $n \in \mathbb{N}^+$. Let $U = \bigcup\{\Uparrow^{\lhd} a_n \mid n \in \mathbb{N}^+\}$. By Propositions 3.2 and 3.4, we have $U \in SS(L)$. Clearly, U is a filter with $x \in U$ and $z \in \mathrm{lb}(U)$. □

Remark 3.11 *It is clear that $z \in \mathrm{lb}(U)$ and $z \nleq y$ imply that $y \notin U$. So, Theorem 3.10 shows that if a poset is supercontinuous then every two different points can be separated by a filter which is also a Scott S-set.*

Definition 3.12 *Let L be a poset. A subset Q of L is called an S-compact set if for any $\{U_\alpha\}_{\alpha \in \Gamma} \subseteq SS(L)$ with $Q \subseteq \bigcup_{\alpha \in \Gamma} U_\alpha$, there is a finite set $F \subseteq \Gamma$ such that*

$Q \subseteq \bigcup_{\alpha\in F} U_\alpha$.

Definition 3.13 *A poset L is called local S-compact if for every $x \in U \in SS(L)$, there is $V \in SS(L)$ and an S-compact set Q such that $x \in V \subseteq Q \subseteq U$.*

Proposition 3.14 *Every supercontinuous poset is local S-compact.*

Proof. Let L be a supercontinuous poset. For every $x \in U \in SS(L)$, it follows from Proposition 3.4 that there is $y \in U$ such that $x \in \Uparrow^{\triangleleft} y \subseteq \uparrow y \subseteq U$. By Proposition 3.4 and Definition 3.12, $\Uparrow^{\triangleleft} y \in SS(L)$ and $\uparrow y$ is S-compact. Thus, L is local S-compact. □

Definition 3.15 *A poset L is called weakly local S-compact if for every $x \in U \in SS(L)$, there is $V \in SS(L)$ such that $x \in V \subseteq U$ and every filtered subset of V has a lower bound in U.*

Lemma 3.16 *Let L be a poset and $Q \subseteq L$. If Q is S-compact, then every filtered subset of Q has a lower bound in Q.*

Proof. Let $Q \subseteq L$ be S-compact and $W \subseteq Q$ be filtered. Suppose that W has no lower bound in Q. Then $Q \cap (\bigcap_{x\in W} \downarrow x) = \emptyset$. So, $Q \subseteq L\backslash(\bigcap_{x\in W} \downarrow x) = \bigcup_{x\in W}(L\backslash \downarrow x)$. By Proposition 3.2(3) and the S-compactness of Q, there exists a finite $F \subseteq W$ such that $Q \subseteq \bigcup_{x\in F}(L\backslash \downarrow x)$. This shows that F has no lower bound in Q, a contradiction to the assumption that W is a filtered subset of Q. Thus, every filtered subset of Q has a lower bound in Q. □

Corollary 3.17 *Every local S-compact poset is weakly local S-compact.*

Proof. Let L be a local S-compact poset. Then for every $x \in U \in SS(L)$, there is $V \in SS(L)$ and an S-compact set Q such that $x \in V \subseteq Q \subseteq U$. By Lemma 3.16, every filtered subset of V has a lower bound in U. So, L is weakly local S-compact.□

Lemma 3.18 *Let L be a supercontinuous poset. Then for every $x \in U \in SS(L)$, there is a filter $V \in SS(L)$ such that $x \in V \subseteq U$.*

Proof. Let L be a supercontinuous poset. For every $x \in U \in SS(L)$, it follows from Proposition 3.4 that there is $y \in U$ such that $y \triangleleft x$. By the proof of only if direction of Theorem 3.10, there is a sequence $\{a_n\}_{n\in\mathbb{N}^+}$ such that $y \triangleleft a_{n+1} \triangleleft a_n \triangleleft x$ for all $n \in \mathbb{N}^+$. Let $V = \bigcup\{\Uparrow^{\triangleleft} a_n \mid n \in \mathbb{N}^+\}$. By Propositions 3.2 and 3.4, we have $V \in SS(L)$. Clearly, V is a filter such that $x \in V \subseteq U$. □

Theorem 3.19 *Let L be a poset. Then the following are equivalent:*

(1) L is supercontinuous;

(2) L is weakly local S-compact and for every $x \in U \in SS(L)$, there is a filter $V \in SS(L)$ such that $x \in V \subseteq U$.

Proof. (1) $\Longrightarrow$ (2): Follows from Proposition 3.14, Corollary 3.17 and Lemma 3.18.

(2) $\Longrightarrow$ (1): Suppose that L is weakly local S-compact and for every $x \in U \in SS(L)$, there is a filter $V \in SS(L)$ such that $x \in V \subseteq U$. For any $x, y \in L$ with $x \not\leq y$, we have $x \in L\backslash \downarrow y \in SS(L)$. By the assumption, there is a filter $V_y \in SS(L)$

such that $x \in V_y \subseteq L \setminus \downarrow y$. It follows from the weakly local S-compactness of L that there is $W_y \in SS(L)$ such that $x \in W_y \subseteq V_y \subseteq L \setminus \downarrow y$ and every filtered subset of W_y has a lower bound in V_y. By the assumption again, there is a filter $H_y \in SS(L)$ such that $x \in H_y \subseteq W_y$. Since H_y is a filtered subset of W_y, there exists $z \in \mathrm{lb}(H_y)$ such that $z \in V_y$. This shows that $z \not\leq y$. By Theorem 3.10, L is a supercontinuous poset. □

Corollary 3.20 *A poset L is supercontinuous if and only if L is local S-compact and for every $x \in U \in SS(L)$, there is a filter $V \in SS(L)$ such that $x \in V \subseteq U$.*

At the end of this section, we construct some counterexamples to show that supercontinuity is not inherited by various natural subsets.

It is well-known that every non-empty Scott open set of a continuous dcpo in the relative order is also continuous. However, to one's surprise, the following example shows that supercontinuity doesn't have such similar hereditary property.

Example 3.21 *Let $L = \mathcal{P}(X)$ be the powerset of $X = \{a, b, c\}$. Then as a completely distributive lattice, $(L, \subseteq)$ is supercontinuous. By Remark 3.3, the set $L \setminus \downarrow \{a\}$ is a Scott S-set. But in the poset $(L \setminus \downarrow \{a\}, \subseteq)$, it is easy to check that $\Downarrow^{\triangleleft} \{a, b\} = \{b\}$, showing that $(L \setminus \downarrow \{a\}, \subseteq)$ is not supercontinuous. This example reveals that a Scott S-set of a supercontinuous lattice in the relative order needn't be supercontinuous.*

The next example shows that supercontinuity is not hereditary to principal ideals.

Example 3.22 *Let L be the the poset consisting of two parallel copies of $\mathbb{N}$ augmented with two incomparable upper bounds a, b so that a and b are bigger than $(0, n)$ and $(1, n)$ for all $n \in \mathbb{N}$. It is straightforward to show that for all $x \in L$, $\Downarrow^{\triangleleft} x = \downarrow x$ and $x = \bigvee \Downarrow^{\triangleleft} x$. So, L is supercontinuous. But, evidently, the principal ideal $\downarrow a$ is a typical non-continuous sup-semilattice. By Remark 2.4, the principal ideal $\downarrow a$ is not supercontinuous.*

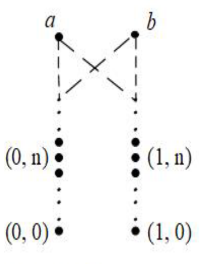

Figure 1 (Ex. 3.22) Supercontinuous poset with non-supercontinuous principal ideals

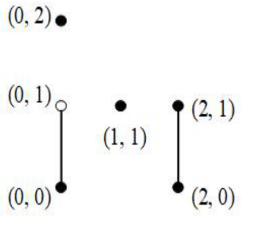

Figure 2 (Ex. 3.23) Non-supercontinuous poset with every principal ideal being supercontinuous

The next example is a non-supercontinuous poset in which every principal ideal is supercontinuous.

Example 3.23 *Let $L = ((\{0, 2\} \times I) \setminus \{(0, 1)\}) \bigcup \{(1, 1), (0, 2)\}$ be ordered by the inherited order of $\mathbb{R} \times \mathbb{R}$, where $I = [0, 1]$ is the unit interval. It is easy to check* *that every principal ideal of L is supercontinuous. However, L itself is not supercontinuous. To see this, suppose that L is a supercontinuous poset. Then since $(1,1) \nleqslant (0,2)$, by Theorem 3.7, there exists $u \in L$ and a Scott S-set U such that $(1,1) \notin L \setminus U$, $u \nleqslant (0,2)$ and $\uparrow u \cup (L \setminus U) = L$. By Lemma 3.6(1), it should be true that $u \lhd (1,1)$ and $u \nleqslant (0,2)$. This is clearly a contradiction.*

4 S-bases

In this section, we introduce the concept of S-bases of posets and give several ways of recognizing an S-basis for supercontinuous posets.

Definition 4.1 *Let L be a poset. A subset B of L is called an S-basis for L if for each $x \in L$, one has $x = \bigvee(\Downarrow^{\lhd} x \cap B)$.*

Since a supercontinuous poset is an S-basis for itself, we immediately have:

Proposition 4.2 *A poset L has an S-basis if and only if it is supercontinuous.*

Lemma 4.3 *Let L be a poset and $B \subseteq L$. If for any x, $y \in L$ with $x \nleqslant y$, there exists $b \in B$ and a Scott S-set U such that $x \in U$, $b \nleqslant y$ and $\uparrow b \cup (L \setminus U) = L$, then B is an S-basis for L.*

Proof. Let $x \in L$. If L has a bottom $\perp$ and $x = \perp$, then by Proposition 2.2(3), we have $\Downarrow^{\lhd} \perp = \emptyset$ and $\perp = \bigvee(\Downarrow^{\lhd} \perp \cap B)$. If $x \neq \perp$, then there is $y \in L$ such that $x \nleqslant y$. So, there exists $b_0 \in B$ and a Scott S-set U such that $x \in U$, $b_0 \nleqslant y$ and $\uparrow b_0 \cup (L \setminus U) = L$. By Lemma 3.6(1), $b_0 \lhd x$ and hence the set $\Downarrow^{\lhd} x \cap B$ is nonempty. By Proposition 2.2(1), x is an upper bound of the set $\Downarrow^{\lhd} x \cap B$. Let z be any upper bound of $\Downarrow^{\lhd} x \cap B$. Assume that $x \nleqslant z$. Then there exists $b_1 \in B$ and a Scott S-set V such that $x \in V$, $b_1 \nleqslant z$ and $\uparrow b_1 \cup (L \setminus V) = L$. By Lemma 3.6(1), we have $b_1 \lhd x$ but $b_1 \nleqslant z$, contradicting to the assumption that z is an upper bound of $\Downarrow^{\lhd} x \cap B$. So, $x \leqslant z$ and therefore $x = \bigvee(\Downarrow^{\lhd} x \cap B)$. By Definition 4.1, B is an S-basis for L. □

Theorem 4.4 *For a subset B of a poset L, the following conditions are equivalent:*

(1) B is an S-basis for L;

(2) For any x, $y \in L$ with $x \nleqslant y$, there exists $b \in B$ and a Scott S-set U such that $x \in U$, $b \nleqslant y$ and $\uparrow b \cup (L \setminus U) = L$;

(3) For each $x \in L$, there exists a subset $B_x \subseteq \Downarrow^{\lhd} x \cap B$ such that $x = \bigvee B_x$.

Proof. (1) $\Longrightarrow$ (2): Let B be an S-basis for L. For any x, $y \in L$ with $x \nleqslant y$, it follows from Definition 4.1 that there exists $b \in \Downarrow^{\lhd} x \cap B$ such that $b \nleqslant y$. It follows from Propositions 2.5 and 4.2 that there is $z \in L$ such that $b \lhd z \lhd x$. Let $U = \Uparrow^{\lhd} z$. It is clear that $x \in U$, $\uparrow b \cup (L \setminus U) = L$ and by Proposition 3.4, U is a Scott S-set.

(2) $\Longrightarrow$ (1): Follows immediately from Lemma 4.3.

(3) $\Longrightarrow$ (1): Trivial.

(1) $\Longrightarrow$ (3): Follows immediately from Definition 4.1. □

In supercontinuous posets, we have the following characterizations of S-bases.

Theorem 4.5 *For a subset B of a supercontinuous poset L, the following conditions are equivalent:*

(1) B is an S-basis for L;

(2) Whenever $x \lhd y$, there exists $b \in B$ with $x \leqslant b \lhd y$;

(3) Whenever $x \lhd y$, there exists $b \in B$ with $x \lhd b \lhd y$.

Proof. (1) $\Longrightarrow$ (2): Suppose $x \lhd y$. By (1) and Definition 2.1, there is $b \in \Downarrow^{\lhd} y \cap B$ such that $x \leqslant b \lhd y$.

(2) $\Longrightarrow$ (3): Suppose $x \lhd y$. By (2), there exists $b_0 \in B$ with $x \leqslant b_0 \lhd y$. It follows from Proposition 2.5 that there is $z \in L$ such that $b_0 \lhd z \lhd y$. By (2) again, there exists $b \in B$ with $z \leqslant b \lhd y$. By Proposition 2.2(2), we have $x \lhd b \lhd y$.

(3) $\Longrightarrow$ (1): For each $y \in L$, it is straightforward to verify that $y = \bigvee(\Downarrow^{\lhd} y \cap B)$ by the supercontinuity of L and (3). So, B is an S-basis for L by Definition 4.1. $\square$

A superalgebraic poset has a very natural basis.

Corollary 4.6 *Let L be a poset. Then the following conditions are equivalent:*

(1) L is superalgebraic;

(2) The set of all super-compact elements $SK(L)$ forms an S-basis for L;

(3) L has a smallest S-basis.

Proof. (1) $\Longleftrightarrow$ (2): By Definition 2.3 and Definition 4.1.

(2) $\Longrightarrow$ (3): It follows from (2) and Theorem 4.5(3) that the set of all super-compact elements $SK(L)$ forms the smallest S-basis for L.

(3) $\Longrightarrow$ (2): Assume that L has a smallest S-basis B. It follows from Proposition 4.2 and Theorem 4.5(3) that $SK(L) \subseteq B$. Suppose that $B \nsubseteq SK(L)$. Then there exists $b_0 \in B$ such that $b_0 \notin SK(L)$. Since B is an S-basis, $b_0 = \bigvee(\Downarrow^{\lhd} b_0 \cap B)$ and $b_0 \notin (\Downarrow^{\lhd} b_0 \cap B)$. This shows that $B \setminus \{b_0\}$ is also an S-basis for L, a contradiction to the assumption that B is a smallest S-basis. Therefore, $B \subseteq SK(L)$ and $SK(L) = B$ is an S-basis for L. $\square$

5 The S-essential topology

In this section, in the manner of Rusu and Ciobanu in [10], we introduce the concept of the S-essential topology of posets and give some characterizations of S-bases of supercontinuous posets.

Definition 5.1 *Let L be a poset. Let $\Downarrow^{\lhd}: \mathcal{P}(L) \to \mathcal{P}(L)$ be the operator defined by $\Downarrow^{\lhd} A = \bigcup_{x\in A} \Downarrow^{\lhd} x$ for all $A \in \mathcal{P}(L)$. Let $\Uparrow^{\lhd}: \mathcal{P}(L) \to \mathcal{P}(L)$ be the operator defined by $\Uparrow^{\lhd} A = \bigcup_{x\in A} \Uparrow^{\lhd} x$ for all $A \in \mathcal{P}(L)$.*

Proposition 5.2 *Let L be a poset. The operators $\Downarrow^{\lhd}$ and $\Uparrow^{\lhd}$ have the following properties for all A, $B \in \mathcal{P}(L)$ and for any family $\{A_\alpha\}_{\alpha\in\Gamma}$ of subsets of L:*

(1) $\Downarrow^{\lhd} \emptyset = \emptyset$, $\Uparrow^{\lhd} \emptyset = \emptyset$;

(2) $\Downarrow^{\lhd} (\bigcup_{\alpha\in\Gamma} A_\alpha) = \bigcup_{\alpha\in\Gamma} \Downarrow^{\lhd} A_\alpha$, $\Uparrow^{\lhd} (\bigcup_{\alpha\in\Gamma} A_\alpha) = \bigcup_{\alpha\in\Gamma} \Uparrow^{\lhd} A_\alpha$;

(3) $A \subseteq B \Longrightarrow \Downarrow^{\lhd} A \subseteq \Downarrow^{\lhd} B$, $A \subseteq B \Longrightarrow \Uparrow^{\lhd} A \subseteq \Uparrow^{\lhd} B$;

(4) $\Downarrow^{\lhd} A \setminus \Downarrow^{\lhd} B \subseteq \Downarrow^{\lhd} (A \setminus B)$, $\Uparrow^{\lhd} A \setminus \Uparrow^{\lhd} B \subseteq \Uparrow^{\lhd} (A \setminus B)$;

(5) $\Downarrow^{\lhd}(\Downarrow^{\lhd} A) \subseteq \Downarrow^{\lhd} A$, $\Uparrow^{\lhd}(\Uparrow^{\lhd} A) \subseteq \Uparrow^{\lhd} A$.

Proof. Straightforward. □

Definition 5.3 *Let L be a poset and $G \subseteq L$. The subset G is called an S-e-open set if $\Downarrow^{\lhd} G \subseteq G$. The complement of an S-e-open set is called an S-e-closed set.*

Proposition 5.4 *Let L be a poset. Then*

(1) All the S-e-open sets of L form a topology, called the S-essential topology and denoted by $\tau_{se}(L)$. Moreover, the intersection of any family of S-e-open sets is S-e-open;

(2) The family of sets $\{\{x\} \cup \Downarrow^{\lhd} x \mid x \in L\}$ is a base for $\tau_{se}(L)$;

(3) $F \subseteq L$ is S-e-closed if and only if $\Uparrow^{\lhd} F \subseteq F$;

(4) For all $A \in \mathcal{P}(L)$, $\Downarrow^{\lhd} A$ is S-e-open, $\Uparrow^{\lhd} A$ is S-e-closed;

(5) Every lower set is S-e-open and every upper set is S-e-closed;

(6) The S-essential topology $\tau_{se}(L)$ is finer than the dual Alexandroff topology $\alpha^*(L)$, i.e., $\alpha^*(L) \subseteq \tau_{se}(L)$.

Proof. (1) Clearly, L and $\emptyset$ are S-e-open. Let $\{G_\alpha\}_{\alpha\in\Gamma}$ be a family of S-e-open sets. By Proposition 5.2(2), $\Downarrow^{\lhd}(\bigcup_{\alpha\in\Gamma} G_\alpha) = \bigcup_{\alpha\in\Gamma} \Downarrow^{\lhd} G_\alpha \subseteq \bigcup_{\alpha\in\Gamma} G_\alpha$. This shows that $\bigcup_{\alpha\in\Gamma} G_\alpha$ is S-e-open. It is easy to see that $\Downarrow^{\lhd}(\bigcap_{\alpha\in\Gamma} G_\alpha) \subseteq \bigcap_{\alpha\in\Gamma} \Downarrow^{\lhd} G_\alpha \subseteq \bigcap_{\alpha\in\Gamma} G_\alpha$. So, $\bigcap_{\alpha\in\Gamma} G_\alpha$ is S-e-open.

(2) Straightforward.

(3) Suppose that F is S-e-closed. Then $L\backslash F$ is S-e-open and $\Downarrow^{\lhd}(L\backslash F) \subseteq L\backslash F$. Assume that $\Uparrow^{\lhd} F \not\subseteq F$. Then there is $x \in \Uparrow^{\lhd} F$ such that $x \notin F$. This shows that there exists $y \in F$ such that $y \lhd x$ and $x \in L\backslash F$. So, $y \in \Downarrow^{\lhd}(L\backslash F) \subseteq L\backslash F$, a contradiction to $y \in F$. Therefore, we have $\Uparrow^{\lhd} F \subseteq F$. Conversely, suppose that $\Uparrow^{\lhd} F \subseteq F$. We only need to show that $L\backslash F$ is S-e-open, i.e., $\Downarrow^{\lhd}(L\backslash F) \subseteq L\backslash F$. Assume that $\Downarrow^{\lhd}(L\backslash F) \not\subseteq L\backslash F$. Then there exists $a \in \Downarrow^{\lhd}(L\backslash F)$ such that $a \notin L\backslash F$. This shows that there exists $b \in L\backslash F$ such that $a \lhd b$ and $a \in F$. So, $b \in \Uparrow^{\lhd} F \subseteq F$, a contradiction to $b \in L\backslash F$. Hence, $\Downarrow^{\lhd}(L\backslash F) \subseteq L\backslash F$ and $L\backslash F$ is S-e-open.

(4) Follows from (3), Definition 5.3 and Proposition 5.2(5).

(5) Straightforward.

(6) Follows immediately from (5). □

Remark 5.5 *Proposition 5.4(1) implies that the S-essential topology is equal to the Alexandroff topology of its specialization preordering.*

Proposition 5.6 *Let L be a poset. For all $A \in \mathcal{P}(L)$, we have*

(1) $cl_{se}(A) = A \cup \Uparrow^{\lhd} A$, where $cl_{se}(A)$ is the closure of A in the topology $\tau_{se}(L)$;

(2) $int_{se}(A) = A \backslash \Uparrow^{\lhd}(L\backslash A)$, where $int_{se}(A)$ is the interior of A in the topology $\tau_{se}(L)$;

(3) $\Uparrow^{\lhd} cl_{se}(A) = cl_{se}(\Uparrow^{\lhd} A) = \Uparrow^{\lhd} A$.

Proof. (1) By Proposition 5.2, $\Uparrow^{\lhd}(A \cup \Uparrow^{\lhd} A) = \Uparrow^{\lhd} A \cup \Uparrow^{\lhd}(\Uparrow^{\lhd} A) = \Uparrow^{\lhd} A \subseteq (A \cup \Uparrow^{\lhd} A)$. It follows from Proposition 5.4(3) that the set $A \cup \Uparrow^{\lhd} A$ is S-e-closed. Let F be any S-e-closed set with $A \subseteq F$. By Proposition 5.2(3) and Proposition 5.4(3), we have $\Uparrow^{\lhd} A \subseteq \Uparrow^{\lhd} F \subseteq F$. Therefore, $A \cup \Uparrow^{\lhd} A \subseteq F$. This shows that $cl_{se}(A) = A \cup \Uparrow^{\lhd} A$.

(2) By (1), $cl_{se}(L\backslash A) = (L\backslash A) \cup \Uparrow^{\triangleleft}(L\backslash A)$. Therefore, $int_{se}(A) = L\backslash cl_{se}(L\backslash A) = L\backslash((L\backslash A) \cup \Uparrow^{\triangleleft}(L\backslash A)) = A\backslash \Uparrow^{\triangleleft}(L\backslash A)$.

(3) Follows immediately from (1) and Proposition 5.4(4). □

Proposition 5.7 *Let L be a supercontinuous poset. Then the space $(L, \tau_{se}(L))$ is T_1 if and only if the order $\leqslant$ of L is discrete.*

Proof. $\Longrightarrow$: Assume that $(L, \tau_{se}(L))$ is T_1. Then for every $x \in L$, the singleton set $\{x\}$ is S-e-closed. By Proposition 5.4(3), we have $\Uparrow^{\triangleleft} x \subseteq \{x\}$. (i) Suppose L has a bottom $\perp$. Then $\Uparrow^{\triangleleft} \perp \subseteq \{\perp\}$. It follows from Proposition 2.2(3) that $L = \{\perp\}$. Thus the order $\leqslant$ of L is discrete. (ii) Suppose L hasn't a bottom. It follows from the supercontinuity of L that $\Downarrow^{\triangleleft} x \neq \emptyset$. Pick $y \in \Downarrow^{\triangleleft} x$. By the assumption, $x \in \Uparrow^{\triangleleft} y \subseteq \{y\}$ and thus $x = y$. This shows that $L = SK(L)$. By Proposition 5.6(1) and the assumption again, $\uparrow x = \Uparrow^{\triangleleft} x \subseteq (\Uparrow^{\triangleleft} x \cup \{x\}) = cl_{se}(\{x\}) = \{x\}$. By the arbitrariness of x, the order $\leqslant$ of L is discrete.

$\Longleftarrow$: Assume that the order $\leqslant$ of L is discrete. Then for all $x \in L$, we have $\{x\} = \uparrow x$. This shows that the space $(L, \alpha^*(L))$ is T_1. By Proposition 5.4(6), $(L, \tau_{se}(L))$ is T_1. □

Proposition 5.8 *Let L be a supercontinuous poset with a bottom $\perp$. Then*

(1) If $G \in \tau_{se}(L)$ and $\perp \notin G$, then $\uparrow G \in SS(L)$;

(2) If $G \in \tau_{se}(L)$ is an upper set and $\perp \notin G$, then $G \in SS(L)$;

(3) For all $x \in L$, if $\{x\} \in \tau_{se}(L)$ and $x \neq \perp$, then $x \in SK(L)$;

(4) If L is a chain, then $\tau_{se}(L) = \alpha^*(L)$.

Proof. (1) Let $G \in \tau_{se}(L)$ and $\perp \notin G$. For all $A \subseteq L$ with existing $\bigvee A \in \uparrow G$, there is $x \in G$ such that $x \leqslant \bigvee A$. It follows from the supercontinuity of L and $G \in \tau_{se}(L)$ that $\emptyset \neq \Downarrow^{\triangleleft} x \subseteq G$. Take an $s \in \Downarrow^{\triangleleft} x$. Then $s \in G$ and $s \triangleleft x \leqslant \bigvee A$. By Definition 2.1, there exists $a \in A$ such that $s \leqslant a$. This shows that $a \in A \cap \uparrow G$. By Definition 3.1, $\uparrow G \in SS(L)$.

(2) Follows immediately from (1).

(3) Follows from (1) and Proposition 3.2(5).

(4) Let L be a chain. By Proposition 5.4(6), we only need to show $\tau_{se}(L) \subseteq \alpha^*(L)$. Let $G \in \tau_{se}(L)$. Since in a chain, $x < y$ implies $x \triangleleft y$ and $\Downarrow^{\triangleleft} G \subseteq G$, we conclude that $\downarrow G \subseteq G$. This shows that $G \in \alpha^*(L)$. Thus, $\tau_{se}(L) = \alpha^*(L)$. □

Notice that if L is a supercontinuous poset without a bottom $\perp$, then the conditions "$\perp \notin G$" and "$x \neq \perp$" in Proposition 5.8 automatically hold. So, by the proof of Proposition 5.8, we have the following

Corollary 5.9 *Let L be a supercontinuous poset without a bottom. Then*

(1) If $G \in \tau_{se}(L)$, then $\uparrow G \in SS(L)$;

(2) If $G \in \tau_{se}(L)$ is an upper set, then $G \in SS(L)$;

(3) For all $x \in L$, if $\{x\} \in \tau_{se}(L)$, then $x \in SK(L)$;

(4) If L is a chain, then $\tau_{se}(L) = \alpha^*(L)$.

Example 5.10 *(1) Let $L = \{0, a, 1\}$ be ordered by $0 < a < 1$. Then L is a chain.*

Clearly, $a \lhd a$ and $a \in SK(L)$. But $\{a\} \notin \tau_{se}(L)$. This example shows that the converse of Proposition 5.8(3) need not be true.

(2) Let $L = \mathcal{P}(X)$ be the powerset lattice of $X = \{a, b, c\}$. Then L is supercontinuous. Clearly, $\{\emptyset, \{a, b, c\}, \{a\}, \{b\}, \{c\}\} \in \tau_{se}(L)$. But $\{\emptyset, \{a, b, c\}, \{a\}, \{b\}, \{c\}\}$ is not a lower set. So, $\tau_{se}(L) \neq \alpha^*(L)$. This example shows that the dual Alexandroff topology doesn't coincide with the S-essential topology on supercontinuous posets.

Proposition 5.11 *Let L be a supercontinuous poset. Then for all $U \in SS(L)$ and all $G \in \tau_{se}(L)$, one has $\uparrow(U \cap G) \in SS(L)$.*

Proof. Let $U \in SS(L)$ and $G \in \tau_{se}(L)$. For all $A \subseteq L$ with existing $\sup A \in\uparrow (U \cap G)$, there is $x \in U \cap G$ such that $x \leqslant \sup A$. By the supercontinuity of L and Proposition 3.4, there is $t \in U$ such that $t \lhd x \leqslant \sup A$. Thus, there exists $d \in A$ such that $t \leqslant d$. Since $G \in \tau_{se}(L)$, we have $t \in \Downarrow^{\lhd} x \subseteq \Downarrow^{\lhd} G \subseteq G$. This shows that $t \in U \cap G$ and thus $d \in\uparrow (U \cap G) \cap A$. By Definition 3.1, $\uparrow(U \cap G) \in SS(L)$. □

Corollary 5.12 *Let L be a supercontinuous poset. Then for any $U \in SS(L)$ and any lower set C, one has $\uparrow(U \cap C) \in SS(L)$.*

Lemma 5.13 *Let L be a supercontinuous poset. Then the operators $\Downarrow^{\lhd}$ and $\Uparrow^{\lhd}$ are both idempotent, i.e., for all $A \in \mathcal{P}(L)$, $\Downarrow^{\lhd} (\Downarrow^{\lhd} A) = \Downarrow^{\lhd} A$, $\Uparrow^{\lhd} (\Uparrow^{\lhd} A) = \Uparrow^{\lhd} A$.*

Proof. By Proposition 5.2(5), $\Downarrow^{\lhd} (\Downarrow^{\lhd} A) \subseteq \Downarrow^{\lhd} A$, $\Uparrow^{\lhd} (\Uparrow^{\lhd} A) \subseteq \Uparrow^{\lhd} A$ for all $A \in \mathcal{P}(L)$. Let $x \in \Downarrow^{\lhd} A$. There is $y \in A$ such that $x \lhd y$. By the supercontinuity of L and Proposition 2.5, there is $z \in L$ such that $x \lhd z \lhd y$. So, $z \in \Downarrow^{\lhd} A$ and $x \in \Downarrow^{\lhd} (\Downarrow^{\lhd} A)$. This shows that $\Downarrow^{\lhd} A \subseteq \Downarrow^{\lhd} (\Downarrow^{\lhd} A)$ and thus $\Downarrow^{\lhd} (\Downarrow^{\lhd} A) = \Downarrow^{\lhd} A$. Similarly, we have $\Uparrow^{\lhd} (\Uparrow^{\lhd} A) = \Uparrow^{\lhd} A$. □

We arrive at giving characterizations of the S-bases of a supercontinuous poset via the S-essential topology.

Theorem 5.14 *Let L be a supercontinuous poset and $B \subseteq L$. Then the following conditions are equivalent:*

(1) B is an S-basis of L;
(2) $\Uparrow^{\lhd} (\Uparrow^{\lhd} x \cap B) = \Uparrow^{\lhd} x$ for all $x \in L$;
(3) $\Uparrow^{\lhd} (\Uparrow^{\lhd} A \cap B) = \Uparrow^{\lhd} A$ for all $A \subseteq L$;
(4) $\Uparrow^{\lhd} (F \cap B) = \Uparrow^{\lhd} F$ for all S-e-closed set F;
(5) $cl_{se}(\Uparrow^{\lhd} A \cap B) = \Uparrow^{\lhd} A$ for all $A \subseteq L$;
(6) For all $U \in SS(L)$, $G \in \tau_{se}(L)$, $U \cap G \neq \emptyset$ implies $U \cap G \cap B \neq \emptyset$.

Proof. (1) $\Longrightarrow$ (2): By Proposition 5.2, $\Uparrow^{\lhd} (\Uparrow^{\lhd} x \cap B) \subseteq \Uparrow^{\lhd} (\Uparrow^{\lhd} x) \subseteq \Uparrow^{\lhd} x$ for all $x \in L$. Let $y \in \Uparrow^{\lhd} x$. By the supercontinuity of L and Theorem 4.5, there is $b \in B$ such that $x \lhd b \lhd y$. This shows that $y \in \Uparrow^{\lhd} (\Uparrow^{\lhd} x \cap B)$ and thus $\Uparrow^{\lhd} x \subseteq \Uparrow^{\lhd} (\Uparrow^{\lhd} x \cap B)$. Therefore, $\Uparrow^{\lhd} (\Uparrow^{\lhd} x \cap B) = \Uparrow^{\lhd} x$ for all $x \in L$.

(2) $\Longrightarrow$ (3): Let $A \subseteq L$. By (2) and Proposition 5.2(2), we have

$$\Uparrow^{\lhd} (\Uparrow^{\lhd} A \cap B) = \Uparrow^{\lhd} ((\bigcup_{x \in A} \Uparrow^{\lhd} x) \cap B) = \Uparrow^{\lhd} (\bigcup_{x \in A} (\Uparrow^{\lhd} x \cap B)) = \bigcup_{x \in A} \Uparrow^{\lhd} (\Uparrow^{\lhd} x \cap B) = \bigcup_{x \in A} \Uparrow^{\lhd} x = \Uparrow^{\lhd} A.$$

(3) $\Longrightarrow$ (4): Let $F \subseteq L$ be an S-e-closed set. By Proposition 5.4(3), we have $\Uparrow^{\triangleleft} F \subseteq F$. Therefore, $F = \Uparrow^{\triangleleft} F \cup (F \setminus \Uparrow^{\triangleleft} F)$. By (3) and Proposition 5.2(2), we have

$$\begin{aligned}\Uparrow^{\triangleleft}(F \cap B) &= \Uparrow^{\triangleleft}((\Uparrow^{\triangleleft} F \cup (F \setminus \Uparrow^{\triangleleft} F)) \cap B)\\ &= \Uparrow^{\triangleleft}((\Uparrow^{\triangleleft} F \cap B) \cup ((F \setminus \Uparrow^{\triangleleft} F) \cap B))\\ &= \Uparrow^{\triangleleft}(\Uparrow^{\triangleleft} F \cap B) \cup \Uparrow^{\triangleleft}((F \setminus \Uparrow^{\triangleleft} F) \cap B)\\ &= \Uparrow^{\triangleleft} F \cup \Uparrow^{\triangleleft}((F \setminus \Uparrow^{\triangleleft} F) \cap B)\\ &= \Uparrow^{\triangleleft}(F \cup ((F \setminus \Uparrow^{\triangleleft} F) \cap B))\\ &= \Uparrow^{\triangleleft} F.\end{aligned}$$

(4) $\Longrightarrow$ (5): Let $A \subseteq L$. It follows from the assumption (4), Proposition 5.4(4) and Lemma 5.13 that $\Uparrow^{\triangleleft}(\Uparrow^{\triangleleft} A \cap B) = \Uparrow^{\triangleleft}(\Uparrow^{\triangleleft} A) = \Uparrow^{\triangleleft} A$. By Proposition 5.6(1), we have that

$$cl_{se}(\Uparrow^{\triangleleft} A \cap B) = (\Uparrow^{\triangleleft} A \cap B) \cup \Uparrow^{\triangleleft}(\Uparrow^{\triangleleft} A \cap B) = (\Uparrow^{\triangleleft} A \cap B) \cup \Uparrow^{\triangleleft} A = \Uparrow^{\triangleleft} A.$$

(5) $\Longrightarrow$ (6): For all $U \in SS(L)$, $G \in \tau_{se}(L)$ with $U \cap G \neq \emptyset$, by Proposition 3.4, we have $U = \Uparrow^{\triangleleft} U$. It follows from (5) that $cl_{se}(U \cap B) = cl_{se}(\Uparrow^{\triangleleft} U \cap B) = \Uparrow^{\triangleleft} U = U$. Let $x \in U \cap G$. Then $x \in cl_{se}(U \cap B)$. Since $G \in \tau_{se}(L)$ is an S-e-neighborhood of x, we have $U \cap G \cap B \neq \emptyset$.

(6) $\Longrightarrow$ (1): Let $x \triangleleft y$ in L. By the supercontinuity of L and Proposition 2.5, there exist t, $z \in L$ such that $x \triangleleft t \triangleleft z \triangleleft y$. This shows that $t \in \Uparrow^{\triangleleft} x \cap (\{z\} \cup \Downarrow^{\triangleleft} z)$. It follows from (6), $\Uparrow^{\triangleleft} x \in SS(L)$, $(\{z\} \cup \Downarrow^{\triangleleft} z) \in \tau_{se}(L)$ and $\Uparrow^{\triangleleft} x \cap (\{z\} \cup \Downarrow^{\triangleleft} z) \neq \emptyset$ that $\Uparrow^{\triangleleft} x \cap (\{z\} \cup \Downarrow^{\triangleleft} z) \cap B \neq \emptyset$. Pick $b \in \Uparrow^{\triangleleft} x \cap (\{z\} \cup \Downarrow^{\triangleleft} z) \cap B$. It is easy to see that $b \in B$ and $x \triangleleft b \triangleleft y$. By Theorem 4.5, B is an S-basis of L. $\square$

For supercontinuous posets, we characterize the superalgebraicity of the posets via the S-essential topology.

Theorem 5.15 *A supercontinuous poset L is superalgebraic iff $cl_{se}(F \cap SK(L)) = \Uparrow^{\triangleleft} F$ for every S-e-closed set F.*

Proof. $\Longrightarrow$: Let L be a superalgebraic poset. By Corollary 4.6, the set $SK(L)$ of all super-compact elements is an S-basis of L. By Theorem 5.14 (4), we have $\Uparrow^{\triangleleft}(F \cap SK(L)) = \Uparrow^{\triangleleft} F$ for every S-e-closed set F. It follows from Proposition 5.6(1) that

$$cl_{se}(F \cap SK(L)) = (F \cap SK(L)) \cup \Uparrow^{\triangleleft}(F \cap SK(L)) = (F \cap SK(L)) \cup \Uparrow^{\triangleleft} F = \Uparrow^{\triangleleft} F.$$

$\Longleftarrow$: Suppose that $cl_{se}(F \cap SK(L)) = \Uparrow^{\triangleleft} F$ for every S-e-closed set F. For all $A \subseteq L$, it follows from Proposition 5.4(4), the supercontinuity of L and Lemma 5.13 that $\Uparrow^{\triangleleft} A$ is S-e-closed and $\Uparrow^{\triangleleft}(\Uparrow^{\triangleleft} A) = \Uparrow^{\triangleleft} A$. Thus $cl_{se}(\Uparrow^{\triangleleft} A \cap SK(L)) = \Uparrow^{\triangleleft}(\Uparrow^{\triangleleft} A) = \Uparrow^{\triangleleft} A$. By Theorem 5.14(5), $SK(L)$ is an S-basis of L. Therefore, L is superalgebraic. $\square$

6 Concluding remarks

The Scott topology, as an order-theoretic topology, is of fundamental importance in domain theory. Supercontinuous posets, as a generalization of completely distributive lattices, need not be continuous in general. In this paper, we introduced the concept of Scott S-sets of posets. By Remark 3.3, Scott S-sets of a poset are all Scott open sets of the poset and the Scott S-sets of a complete lattice need not be a topology. However, by our Theorems 3.7, 3.10 and 3.19, we see that Scott S-sets play roles in characterizing supercontinuous posets similar to that of Scott open sets in characterizing continuous posets.

It is well-known that a non-empty Scott open set of a continuous dcpo is also continuous. However, Examples 3.21 and 3.22 show that supercontinuity is not hereditary to Scott S-sets, nor to principal ideals.

We introduced the concept of S-bases of posets as counterparts of bases of (continuous) posets and established that a poset L is supercontinuous (resp., superalgebraic) iff L has an S-basis (resp., a smallest S-basis).

In order to provide a topological interpretation of S-bases, we also introduced the new concept of the S-essential topology of posets in the manner of Rusu and Ciobanu in [10]. By our Proposition 5.4(6) and Example 5.10(2), we see that the S-essential topology is finer than the dual Alexandroff topology and generally they are not equal. More properties and characterizations of S-bases are obtained from order-theoretical and topological aspects. Via the S-essential topology, we also provide by Theorem 5.15 a new condition for a supercontinuous poset to be superalgebraic.

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