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Random Continuous Functions

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Abstract

We investigate notions of algorithmic randomness in the space $\mathcal{C}(2^{\mathbb{N}})$ of continuous functions on $2^{\mathbb{N}}$. A probability measure is given and a version of the Martin-Löf test for randomness is defined which allows us to define a class of (Martin-Löf) random continuous functions. We show that random Δ^0_2 continuous functions exist, but no computable function can be random. We show that a random function maps any computable real to a random real and that the image of a random continuous function is always a perfect set and hence uncountable. We show that for any $y \in 2^{\mathbb{N}}$, there exists a random continuous function F with y in the image of F. Thus the image of a random continuous function need not be a random closed set.

Keywords: Martin-Löf test, random real, random closed set, random function.

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1 Introduction

The study of algorithmic randomness has been of great interest in recent years. The basic problem is to quantify the randomness of a single real number. Early in the last century, von Mises [16] suggested that a random real should obey reasonable statistical tests, such as having a roughly equal number of zeroes and ones of the first n bits, in the limit. Thus a random real would be *stochastic* in modern parlance. If one considers only *computable* tests, then there are countably many such tests and one can construct a real satisfying all tests.

Martin-Löf [14] observed that stochastic properties could be viewed as special kinds of measure zero sets and defined a random real as one which avoids certain effectively presented measure 0 sets. That is, a real $x \in 2^{\mathbb{N}}$ is Martin-Löf random if for any effective sequence $S_1, S_2, \dots$ of c.e. open sets with $\mu(S_n) \leq 2^{-n}$, $x \notin \cap_n S_n$.

At the same time Kolmogorov [11] defined a notion of randomness for finite strings based on the concept of *incompressibility*. For infinite words, the stronger notion of prefix-free complexity developed by Levin [13], Gács [9] and Chaitin [5] is needed. Schnorr later proved that the notions of Martin-Löf randomness and Chaitin randomness are equivalent.

In a recent paper [2], the notion of (Martin-Löf) randomness was extended to finite-branching trees and effectively closed sets. It was shown that a random closed set is perfect, has measure 0, and contains no computable elements.

In this paper we want to consider algorithmic randomness on the space $\mathcal{C}(2^{\mathbb{N}})$ of continuous functions $F : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$.

Some definitions are needed. For a finite string $\sigma \in \{0,1\}^n$, let $|\sigma| = n$. For two strings σ, τ, say that τ extends σ and write $\sigma \prec \tau$ if $|\sigma| < |\tau|$ and $\sigma(i) = \tau(i)$ for $i < |\sigma|$. Similarly $\sigma \prec x$ for $x \in 2^{\mathbb{N}}$ means that $\sigma(i) = x(i)$ for $i < |\sigma|$. Let $\sigma^\frown\tau$ denote the concatenation of σ and τ and let $\sigma^\frown i$ denote $\sigma^\frown(i)$ for $i = 0, 1$. Let $x\lceil n = (x(0), \dots, x(n-1))$. Two reals x and y may be coded together into $z = x \oplus y$, where $z(2n) = x(n)$ and $z(2n+1) = y(n)$ for all n.

For a finite string σ, let $I(\sigma)$ denote $\{x \in 2^{\mathbb{N}} : \sigma \prec x\}$. We shall call $I(\sigma)$, the *interval* determined by σ. Each such interval is a clopen set and the clopen sets are just finite unions of intervals. We let $\mathcal{B}$ denote the Boolean algebra of clopen sets.

Now a nonempty closed set P may be identified with a tree $T_P \subseteq \{0,1\}^*$ where $T_P = \{\sigma : P \cap I(\sigma) \neq \emptyset\}$. Note that T_P has no dead ends. That is, if $\sigma \in T_P$, then either $\sigma^\frown 0 \in T_P$ or $\sigma^\frown 1 \in T_P$.

For an arbitrary tree $T \subseteq \{0,1\}^*$, let $[T]$ denote the set of infinite paths through T, that is,

$$x \in [T] \iff (\forall n) x\lceil n \in T.$$

It is well-known that $P \subseteq 2^{\mathbb{N}}$ is a closed set if and only if $P = [T]$ for some tree T. P is a Π^0_1 class, or an effectively closed set, if $P = [T]$ for some computable tree T. P is a strong Π^0_2 class, or a Π^0_2 closed set, if $P = [T]$ for some Δ^0_2 tree. The complement of a Π^0_1 class is sometimes called a c.e. open set. We remark that if P is a Π^0_1 class, then T_P is a Π^0_1 set, but it is not, in general, computable. There is a natural effective enumeration $P_0, P_1, \dots$ of the Π^0_1 classes and, hence, there is a corresponding enumeration of the c.e. open sets. Thus we say that a sequence $S_0, S_1, \dots$ of c.e. open sets is *effective* if there is a computable function, f, such that $S_n = 2^{\mathbb{N}} - P_{f(n)}$ for all n. For a detailed development of Π^0_1 classes, see [3,4].

2 Random continuous functions

We will define the notion of a random continuous function along similar lines to the definition of a random closed set in [2]. The definition of a random (nonempty) closed set $P = [T]$ (where $T = T_P$) comes from a probability measure μ^* where, given a node $\sigma \in T$, each of the following scenarios has equal probability $\frac{1}{3}$:

$\sigma^\frown 0 \in T$ and $\sigma^\frown 1 \in T$,

$\sigma^\frown 0 \in T$ and $\sigma^\frown 1 \notin T$, and

$\sigma^\frown 0 \notin T$ and $\sigma^\frown 1 \in T$.

More formally, we define a measure μ^* on the space $\mathcal{C}$ of closed subsets of $2^{\mathbb{N}}$ as follows. Given a closed set $Q \subseteq 2^{\mathbb{N}}$, let $T = T_Q$ be the tree without dead ends such that $Q = [T]$. Let $\sigma_0, \sigma_1, \dots$ enumerate the elements of T in order, first by length and then lexicographically. We then define the code $x = x_Q = x_T$ by recursion such that for each n, $x(n) = 2$ if both $\sigma_n{}^\frown 0$ and $\sigma_n{}^\frown 1$ are in T, $x(n) = 1$ if $\sigma_n{}^\frown 0 \notin T$ and $\sigma_n{}^\frown 1 \in T$, and $x(n) = 0$ if $\sigma_n{}^\frown 0 \in T$ and $\sigma_n{}^\frown 1 \notin T$. We then define a measure $\mu*$ on $\mathcal{C}$ by setting

$$\mu^*(\mathcal{X}) = \mu(\{x_Q : Q \in \mathcal{X}\}) \tag{1}$$

for any $\mathcal{X} \subseteq \mathcal{C}$ and μ is the standard measure on $\{0, 1, 2\}^{\mathbb{N}}$. Then Brodhead, Cenzer, and Dashti [2] defined a a closed set $Q \subseteq 2^{\mathbb{N}}$ to be (Martin-Löf) random if x_Q is (Martin-Löf) random.

A continuous function on $2^{\mathbb{N}}$ is a function with a closed graph. Thus we might simply say that a function F is random if the graph $Gr(F)$ is a random closed set. Now $Gr(F) = \{x \oplus y : y = F(x)\}$. Thus if $[T]$ is the graph of a function and $\sigma \in T$ has even length, then we must have $\sigma^\frown 0 \in T$ and $\sigma^\frown 1 \in T$. This means that the family of closed sets which are the graphs of functions has measure 0 in the space of closed sets and hence a random closed set will not be the graph of a function. So we need a different measure to define randomness for continuous functions.

A continuous function $F : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ may be represented by a function $f : \{0,1\}^* \to \{0,1\}^*$ such that the following hold for all $\sigma \in \{0,1\}^*$.

(1) $|f(\sigma)| \leq |\sigma|$.

(2) $\sigma_1 \prec \sigma_2$ implies $f(\sigma_1) \preccurlyeq f(\sigma_2)$.

(3) For every n, there exists m such that for all $\sigma \in \{0,1\}^m$, $|f(\sigma)| \geq n$.

(4) For all $x \in 2^{\mathbb{N}}$, $F(x) = \bigcup_n f(x\lceil n)$.

We will define a space of representing functions $f : \{0,1\}^* \to \{0,1\}^*$ to be those which satisfy clauses (1) and (2) above. For such a function f, we have $f(\emptyset) = \emptyset$ by (1). There are three choices for $f((0))$. If $f((0)) = (i)$ where $i \in \{0,1\}$, this means that for all $x \in I((0))$, $F(x)(0) = i$. If $f((0)) = \emptyset$, we shall take this to mean that there exist x_0 and x_1 in $I((0))$ such that $F(x_i)(0) = i$ for $i = 0, 1$. It will always be the case that $F(\sigma) \preccurlyeq \tau$, where τ is the longest string τ with $|\tau| \leq |\sigma|$ such that $\tau \prec F(x)$ whenever $\sigma \prec x$.

We will use the following measure on the set of representing functions to define randomness. Given that $f(\sigma) = \tau$, we define a measure μ^{**} so that each of the following scenarios has equal probability $\frac{1}{3}$ for $i = 0, 1$:

$f(\sigma^\frown i) = \tau$,

$f(\sigma^\frown i) = \tau^\frown 0$, and

$f(\sigma^\frown i) = \tau^\frown 1$.

This can be pictured geometrically as representing the graph of F as the intersection of a decreasing sequence of clopen subsets of the unit square. Initially the choice of $f((0))$ and $f((1))$ selects from the 4 quadrants. That is, for example, $f((0)) = (0) = f((1))$ implies that the graph of F is included in the lower half of the square. Successive values of f restrict the graph of F in a similar fashion.

Let $\mathcal{F}$ be the space of functions $f : \{0,1\}^* \to \{0,1\}^*$ which satisfy clauses (1) and (2) above. Then every continuous function F has a representative f as described above, and, in fact, it has infinitely many representatives. There is a one-to-one correspondence between $\mathcal{F}$ and $\{0,1,2\}^{\mathbb{N}}$ defined as follows. Enumerate $\{0,1\}^*$ in order, first by length and then lexicographically, as $\sigma_0, \sigma_1, \ldots$. Thus $\sigma_0 = \emptyset$, $\sigma_1 = (0), \sigma_2 = (1), \sigma_3 = (00), \ldots$. Then $r \in \{0,1,2\}^{\mathbb{N}}$ corresponds to the function $f_r : \{0,1\}^* \to \{0,1\}^*$ defined by declarling that

$f_r(\emptyset) = \emptyset$ and that, for any σ_n with $|\sigma_n| \geq 1$,

$$f_r(\sigma_n) = \begin{cases} f_r(\sigma_k), & \text{if } r(n) = 2; \\ f_r(\sigma_k)^\frown i, & \text{if } r(n) = i < 2. \end{cases}$$

where k is such that $\sigma_n = \sigma_k{}^\frown j$ for some j. The measure μ^{**} on $\mathcal{F}$ is then induced by the standard probability measure on $\{0,1,2\}^{\mathbb{N}}$. We now define a *effectively random continuous function* on $2^{\mathbb{N}}$ to be one which has a representation in $\mathcal{F}$ which is effectively random. For the rest of this paper, when we say a function, closed set, or real is random, we mean that it is effectively radom.

Our first result will be to show that every random function represents a continuous function. To prove such a result, we need to prove the following lemma.

Lemma 2.1 *Let Σ be a finite set and let $Q \subseteq \Sigma^{\mathbb{N}}$ be a Π^0_1 class of measure 0. Then no element of Q is Martin-Löf random.*

Proof. We will give a proof only in the case where $\Sigma = \{0,1,2\}$ as the general result can be proved in a similar manner. Let $Q = [T]$ where $T \subseteq \{0,1,2\}^*$ is a computable tree (possibly with dead ends). For each n, let $T_n = T \cap \{0,1,2\}^n$ and let

$$Q_n = \bigcup \{I(\sigma) : \sigma \in T_n\}.$$

Let $g(n) = \mu(Q_n) = \frac{|T_n|}{3^n}$. Then $g(n)$ is a computable sequence and

$$lim_{n\to\infty} g(n) = \mu(Q) = 0.$$

This Martin-Löf test shows that Q has no random elements. (As observed by Solovay, it is sufficient for a sequence of c.e. open sets $\{S_n\}_{n\geq 0}$ to be a Martin-Löf test if $lim_{n\to\infty}\mu(S_n) = 0$ effectively rather than the stricter test with a sequence of measures $\mu(S_n) \leq 2^{-n}$.) □

Theorem 2.2 *The set of functions in $\mathcal{F}$ which represent a total continuous function has measure one. Hence every random function represents a continuous function.*

Proof. Let $f \in \mathcal{F}$ and suppose that f does not represent a total function. Then there is some $x \in 2^{\mathbb{N}}$ and some $\tau \in \{0,1\}^*$ such that $f(x\lceil n) = \tau$ for almost all n. Without loss of generality we may assume that $\tau = \emptyset$. Let A be the set of functions $f : \{0,1\}^* \to \{0,1\}^*$ such that $f(\sigma) = \emptyset$ for arbitrarily long strings σ and let $p = \mu^{**}(A)$. Then certainly $p \leq \frac{5}{9}$ since, if $r(0)$ and $r(1)$ are both in $\{0,1\}$, then $f_r \notin A$. Considering the 9 cases for the initial choices of $f((0))$ and $f((1))$, we see that

$$p = \frac{4}{9}p + \frac{1}{9}[1 - (1-p)^2]$$

so that $\frac{1}{9}p^2 + \frac{1}{3}p = 0$, which implies that $p = 0$. That is, there are 4 cases in which $|f((i))| = 1$ for $i = 0, 1$ so that immediately $f \notin A$, there are 4 cases in which only one of $f((i)) = \emptyset$, in which case the remaining function g, defined by $g(\sigma) = f(i^\frown\sigma)$ must be in A, and there is one case in which $f((i)) = \emptyset$ for $i = 0, 1$, in which case at least one of the remaining functions must be in A.

Observe that A is a Π^0_1 class since $f_r \in A$ if and only if $(\forall n)(\exists \sigma \in \{0,1\}^n) f_r(\sigma) = \emptyset$. It follows from Lemma 2.1 that no random function can be in A and therefore every random function $f : \{0,1\}^* \to \{0,1\}^*$ indeed represents a continuous function $F : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$. □

Now the set of Martin-Löf random elements of $\{0,1,2\}^{\mathbb{N}}$ has measure one and there exists a Δ^0_2 Martin-Löf real. Hence we have the following.

Theorem 2.3 *There exists a random continuous function which is Δ^0_2 computable.*

Next we consider some basic properties of random continuous functions. Our first result is easy to prove.

Proposition 2.4 *(a) F is a random continuous function if and only if, for every $\sigma \in \{0,1\}^*$, the function F_σ is random continuous, where*

$$F_\sigma(x) = F(\sigma^\frown x).$$

(b) F is random continuous if and only if both $F_{(0)}$ and $F_{(1)}$ are random continuous.

Next we shall show that every random function maps a computable real to a random real. Again, we need a preliminary lemma.

Lemma 2.5 *Let Σ be a finite alphabet where $|\Sigma| \geq 3$ and let $\Sigma_1 \subset \Sigma$ be a proper subset of Σ where $|\Sigma_1| \geq 2$. If $z \in \Sigma^{\mathbb{N}}$ is Martin-Löf random and y is the result of removing from z all symbols from $\Sigma - \Sigma_1$, then y is Martin-Löf random in $\Sigma_1^{\mathbb{N}}$.*

Proof. Clearly it is enough to prove the lemma when $|\Sigma| - 1 = |\Sigma_1|$. Thus there is no loss in generality in assuming that $\Sigma = \{1, 2, \ldots, n+1\}$ and $\Sigma_1 = \{1, \ldots, n\}$ where $n \geq 2$.

Define the function G so that for any x with infinitely many values of $x(m) \in \{1, \ldots, n\}$, $G(x)$ is the result of removing from x all occurences of $n+1$.

Claim 2.6 *For any Σ^0_1 subset S of $\{1,\ldots,n\}^N$, $\mu(G^{-1}(S)) = \mu(S)$.*

Proof. [Proof of Claim] Since every Σ^0_1 class S is the effective union of a disjoint sequence of intervals, that is, there is a computable function f such that $S = \bigcup_m I(\sigma_{f(m)})$, it suffices to prove this for intervals $I(\sigma) \subseteq \{1,\ldots,n\}^N$. The proof is by induction on the length $|\sigma|$.

For $m = |\sigma| = 1$, we see that

$$G^{-1}(I((i))) = I((i)) \cup I((n+1)^\frown i)) \cup I((n+1)(n+1)^\frown i) \cup \cdots$$

so that

$$\begin{aligned}\mu(G^{-1}(I((i)))) &= \frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \cdots \\ &= \frac{1}{n+1}\left(\frac{1}{1-\frac{1}{n+1}}\right) = \frac{1}{n} = \mu(I((i))).\end{aligned}$$

Now assume the result to be true for m and let $\sigma = (i)^\frown\sigma$ where $|\tau| = m$. Then it is easy to see that

$$G^{-1}(I(\sigma)) = i^\frown G^{-1}(I(\tau)) \cup (n{+}1)i^\frown G^{-1}(I(\tau)) \cup (n{+}1)(n{+}1)i^\frown G^{-1}(I(\tau)) \cup \cdots.$$

Thus

$$\begin{aligned}\mu(G^{-1}(I(\sigma))) &= \frac{1}{n+1}\mu(G^{-1}(I(\tau))) + \frac{1}{(n+1)^2}\mu(G^{-1}(I(\tau))) + \cdots \\ &= \mu(G^{-1}(I(\tau)))\left(\frac{1}{n+1} + \frac{1}{(n+1)^2} + \cdots\right) \\ &= \frac{1}{n}\mu(G^{-1}(I(\tau))) = \frac{1}{n}\frac{1}{n^m} = \mu(I(\sigma)).\end{aligned}$$

□

Now let $S_0, S_1, \ldots$ be an effective sequence of c.e. open sets with $\mu(S_m) \leq 2^{-m}$. Thus there exists a computable function f such that

$$S_m = \cup_i I(\sigma_{f(i,m)}).$$

Let $R_m = G^{-1}(S_m)$, so that $\mu(R_m) = \mu(S_m)$ by the Claim. It remains to be checked that the sequence $\{R_m\}_{m\in\mathbb{N}}$ is an effective sequence of c.e. open sets. Define the computable function $g : \{1,\ldots,n+1\}^* \to \{1,\ldots,n\}^*$ so that $g(\sigma)$ is the result of removing from σ all occurences of $n+1$. For each i, m define the computable function h such that $\{\sigma_{h(j,i,m)}\}_{m\in\mathbb{N}}$ enumerates $\{\tau : g(\tau) = \sigma_{f(i,m)}\}$. Then

$$R_n = \cup_{j,i,m} I(\sigma_{h(j,i,m)}).$$

Thus $\{R_m\}_{m\in\mathbb{N}}$ is a Martin-Löf test and hence $z \notin R_m$ for some m. But then $y \notin S_m$ so that y is random. $\square$

Theorem 2.7 *If F is a random continuous function, then, for any computable real x, $F(x)$ is a random real.*

Proof. Suppose that F is random with representing function f_r, let x be a computable real and let $y = F(x)$. Define the computable function g so that, for each n,

$$\sigma_{g(n)} = x\lceil n.$$

By the Von-Mises–Church–Wald Computable Selection Theorem, the subsequence $z(n) = r(g(n))$ is random in $\{0,1,2\}^{\mathbb{N}}$. Now $y = F(x)$ may be computed from z by removing the 2's. Thus $F(x)$ is random by Lemma 2.5. $\square$

In particular, it follows that if F is random function and x is a computable real, then $F(x)$ is not a computable or even a c.e. real. Hence, a random function F can never be computably continuous and the graph of F is not a Π^0_1 class.

We note that Fouche [8] has used a different approach to randomness for continuous functions connected with Brownian motion, first presented by Asarin and Prokovsky [1], and has shown that, under this approach, it is also true that for any random continuous function F, $F(x)$ is not computable for any computable input x.

Theorem 2.8 *If F is a random continuous function, then the image $F[2^{\mathbb{N}}]$ has no isolated elements.*

Proof. Let f be the random representing function for F and let $Q = F[2^{\mathbb{N}}]$. Suppose by way of contradiction that Q contains an isolated path y. Then there is some finite $\tau \prec y$ such that y is the unique element of $I(\tau) \cap Q$. Fix σ such that $f(\sigma) = \tau$.

For each n, let S_n be the set of all $g \in \mathcal{F}$ such that for all $\rho_1, \rho_2 \in \{0,1\}^n$,

(i) $g(\sigma^\frown\rho_1)$ is compatible with $g(\sigma^\frown\rho_2)$,

(ii) $\tau \prec g(\sigma^\frown\rho_1)$, and

(iii) $\tau \prec g(\sigma^\frown\rho_2)$.

Then for any each $m < n$ and each $\rho \in \{0,1\}^m$, we are restricted to at most 7 of the 9 possible choices so that in general, $\mu(S_n) \leq (\frac{7}{9})^n$. Now for each n, S_n is a clopen set in $\mathcal{F}$ and thus the sequence $S_0, S_1, \ldots$ is a Martin-Löf test. It follows that for some n, $F \notin S_n$. Thus there are two extensions of σ of length n which have incompatible images, contradicting the assumption that y was the unique element of $Q \cap I(\tau)$. $\square$

It follows that the image of a random continuous function is perfect and has continuum many elements. There are several natural questions about the image $F[2^{\mathbb{N}}]$ of a random continuous function F. Is the image of F a random closed set? What is the measure of the image? Can the function be onto? We will give some partial answers.

It follows from Proposition 2.4 that, for any $\tau \in \{0,1\}^*$, there is a random continuous function with image $\subseteq I(\tau)$. Thus a random continuous function is not necessarily onto.

Theorem 2.9 *For any $\sigma \in \{0,1\}^*$, the probability that the image of a continuous function F meets $I(\sigma)$ is always $> \frac{3}{4}$.*

Proof. The proof is by induction on $|\sigma|$. Without loss of generality, we may assume that $\sigma = 0^n$. For each $n > 0$, let q_n be the probability that $F[2^{\mathbb{N}}]$ meets $I((0^n))$. Let f be the representing function for F. For $n = 1$, there are 9 equally probable choices for the pair $f((0))$ and $f((1))$ which can be broken down into 4 distinct cases.

Case 1. If $f((0)) = (1) = f((1))$, then $F[2^{\mathbb{N}}]$ does not meet $I((0))$. This occurs just once.

Case 2. If $f((0)) = (0)$ or $f((1)) = (0)$, then $F[2^{\mathbb{N}}]$ meets $I((0))$. This occurs in 5 of the 9 choices.

Case 3. If $f((i)) = \emptyset$ and $f((1-i)) = (1)$, then $F[2^{\mathbb{N}}]$ meets $I((0))$ if and only if $F_{(i)}[2^{\mathbb{N}}]$ meets $I((0))$. This occurs in 2 of the 9 choices, with probability q_1.

Case 4. If $f((0)) = \emptyset = f((1))$, then $F[2^{\mathbb{N}}]$ meets $I((0))$ if at least one of $F_{(i)}[2^{\mathbb{N}}]$ meets $I((0))$. This occurs in 1 of the choices, with probability $1-(1-q_1)^2$. That is, $F[2^{\mathbb{N}}]$ *fails to meet* $I((0))$ if *both* $F_{(0)}[2^{\mathbb{N}}]$ *and* $F_{(0)}[2^{\mathbb{N}}]$ fail to meet $I((0))$.

Putting these cases together, we see that

$$q_1 = \frac{5}{9} + \frac{2}{9}q_1 + \frac{1}{9}(2q_1 - q_1^2),$$

so that q_1 satisfies the quadratic equation

$$x^2 + 5x - 5 = 0.$$

Thus q_1 is the unique solution in [0,1] of this equation, that is,

$$q_1 = \frac{\sqrt{45}-5}{2},$$

which is indeed $> .75$.

Now let $q_n = q$ and let $q_{n+1} = p$. Once again we consider the 9 initial choices, now breaking down into 6 distinct cases.

Case 1. If $f((0)) = (1) = f((1))$, then $F[2^{\mathbb{N}}]$ does not meet $I((0^{n+1}))$. This occurs just once.

Case 2. If $f((0)) = (0) = f((1))$, then $F[2^{\mathbb{N}}]$ meets $I((0^{n+1}))$ if and only if at least one of $F_{(0)}$ and $F_{(1)}$ meets $I((0^n))$. This occurs just once, and with probability $1-(1-q)^2 = 2q - q^2$.

Case 3. If $f((i)) = (0)$ and $f((1-i)) = (1)$, then $F[2^{\mathbb{N}}]$ meets $I((0^{n+1}))$ if and only if $F_{(i)}[2^{\mathbb{N}}]$ meets $I((0^n))$. This occurs in 2 of the 9 choices, with probability q.

Case 4. If $f((i)) = \emptyset$ and $f((1-i)) = (1)$, then $F[2^{\mathbb{N}}]$ meets $I((0^{n+1}))$ if and only if $F_{(i)}[2^{\mathbb{N}}]$ meets $I((0^{n+1}))$. This occurs in 2 of the 9 choices, with probability p.

Case 5. If $f((0)) = \emptyset = f((1))$, then $F[2^{\mathbb{N}}]$ meets $I((0^{n+1}))$ if at least one of $F_{(i)}[2^{\mathbb{N}}]$ meets $I((0^{n+1}))$. This occurs just once, with probability $1-(1-p)^2$.

Case 6. If $f((i)) = \emptyset$ and $f((1-i)) = (0)$, then $F[2^{\mathbb{N}}]$ meets $I((0^{n+1}))$ if at least one of the following two things happens. Either $F_{(i)}[2^{\mathbb{N}}]$ meets $I((0^{n+1}))$, or $F_{(1-i)}[2^{\mathbb{N}}]$ meets $I((0^n))$. This occurs in 2 of the 9 choices, with probability $1-(1-p)(1-q)$.

Putting these cases together, we see that

$$p = \frac{2}{3}p - \frac{1}{9}p^2 - \frac{2}{9}pq + \frac{2}{3}q - \frac{1}{9}q^2,$$

so that $p = q_{n+1}$ satisfies the equation

$$p^2 + 3p + 2pq - 6q + q^2 = 0.$$

We note that for $p = q$, the solutions are $p = q = 0$ and $p = q = \frac{3}{4}$. This explains the value $\frac{3}{4}$ in the statement of theorem.

Now assume by induction that $q > \frac{3}{4}$. Suppose by way of contradiction that $p \leq \frac{3}{4}$. It follows that

$$\frac{9}{16} + \frac{9}{4} + \frac{3}{2}q - 6q + q^2 \geq 0.$$

Simplifying, this implies that $16q^2 - 72q + 45 \geq 0$. But this factors into $(4q-3)(4q-15)$ and is only ≥ 0 when either $q \leq \frac{3}{4}$ or $q \geq \frac{15}{4}$. Since the latter is impossible, we obtain the desired contradiction that $q \leq \frac{3}{4}$. □

Corollary 2.10 *For any $y \in 2^{\mathbb{N}}$, there exists a random continuous function F with $y \in F[2^{\mathbb{N}}]$.*

Proof. Let S_n be $\{F \in \mathcal{C}(2^{\mathbb{N}}) : I(y\lceil n) \cap F[2^{\mathbb{N}}] \neq \emptyset\}$. By Theorem 2.9, $\mu(S_n) \geq .75$ for all n. But $S_{n+1} \subseteq S_n$ for all n and therefore $\mu(\cap_n S_n) \geq .75$ as well. Thus $y \in F[2^{\mathbb{N}}]$ with probability $\geq .75$. Since the random continuous functions have measure 1 in $\mathcal{C}(2^{\mathbb{N}})$, it follows that some random continuous function has y in the image. □

Corollary 2.11 *The image of a random continuous function need not be a random closed set.*

Proof. It was shown in [2] that a random closed set has no computable members. Let F be a random continuous function with 0^ω in the image, as given by Corollary 2.10. Then $F[2^{\mathbb{N}}]$ is not a random closed set. □

3 n-random continuous functions

Our approach also allows us to define the notion of n-random continuous functions. That is, recall that

(i) a Σ^0_n test is a computable collection $\{V_n : n \in 2^{\mathbb{N}}\}$ of Σ^0_n classes such that $\mu(V_k) \leq 2^{-k}$ and

(ii) a real α is Σ^0_n random or n-random if and only if it passes all Σ^0_n tests, i.e., if $\{V_n : n \in 2^{\mathbb{N}}\}$ is a computable collection of Σ^0_n classes such that $\mu(V_k) \leq 2^{-k}$, then $\alpha \notin \cap_{n\geq 0} V_n$.

Thus 1-random reals are just Martin-Löf random reals. See [6] for details on random and n-random reals.

Kurtz [12] and Kautz [10] proved the following result. Let $\emptyset^{(n)}$ denote the n-th jump of $\emptyset$.

Theorem 3.1 *Let q be a rational number.*

(i) For each Σ^0_n class S , we can uniformly compute from q and a Σ^0_n index for S, the index of a $\Sigma^{\emptyset(n-1)}_1$ class $U \supseteq S$ such that U is an open Σ^0_n class and $\mu(U) - \mu(S) < q$.

(ii) For each Π^0_n class T, we can uniformly compute from q and a Π^0_n index for T, the index of a $\Pi^{\emptyset(n-1)}_1$ class $V \supseteq T$ such that V is a closed Π^0_n class and $\mu(V) - \mu(T) < q$.

(iii) For each Σ^0_n class S, we can uniformly compute from q, and a Σ^0_n index for S and an oracle for $\emptyset^{(n)}$, the index of a Π^0_{n-1} class $V \subseteq S$ such that V

is a closed Π^0_{n-1} class and $\mu(S) - \mu(V) < q$. Moreover, if $\mu(S)$ is a real computable from $\emptyset^{(n-1)}$, then the index for V can be found computably from $\emptyset^{(n-1)}$.

(iv) *For each Π^0_n class T, we can uniformly compute from q and Π^0_n index for T and an oracle for $\emptyset^{(n)}$, the index of a Σ^0_{n-1} class $U \subseteq T$ such that U is an open Σ^0_{n-1} class and $\mu(T) - \mu(U) < q$. Moreover, if $\mu(S)$ is a real computable from $\emptyset^{(n-1)}$, then the index for U can be found computably from $\emptyset^{(n-1)}$.*

It follows that a real is $n+1$-random if and only if it is 1-random relative to $\emptyset^{(n)}$. The analogue of Theorem 3.1 also holds for $\{0,1,2\}^{\mathbb{N}}$ for our measures μ^* or μ^{**}. Thus we can define a closed set Q to be n-random if and only if is Martin Löf random relative to $\emptyset^{(n)}$ and, similarly, we can define a continuous function $F : 2^{2^{\mathbb{N}}} \to 2^{2^{\mathbb{N}}}$ to be n-random if and only if it is Martin Löf random relative to $\emptyset^{(n)}$. One can then easily relativize the results of the previous section to obtain similar results for n-random continuous functions.

4 Conclusions and Future Research

In this paper we have proposed a notion of effective randomness for continuous functions on the Cantor space $2^{\mathbb{N}}$ and derived several properties of effectively random continuous functions. Effectively random Δ^0_2 continuous functions exist, but no computable function can be effectively random. In fact, the image if a computable real under an effectively random function is an effectively random real so that no effectively random function can map a computable real to a computable or even to a c.e. real. We have shown that the image of a random continuous function is always a perfect set and hence uncountable. We have shown that for any $y \in 2^{\mathbb{N}}$, there exists a random continuous function F with y in the image of F. Thus the image of a random continuous function need not be a random closed set.

We would like to extend the notion of a random continuous function to functions on the real unit interval $[0,1]$ and the real line $\Re$ by representing functions again in terms of the images of subintervals. We conjecture that a random continuous real function cannot be left or right computable and, in fact, it cannot even be weakly computable. We also conjecture that a random continuous function is nowhere differentiable.

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