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The Homotopy Branching Space of a Flow

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Abstract

In this talk, I will explain the importance of the homotopy branching space functor (and of the homotopy merging space functor) in dihomotopy theory.

Keywords: concurrency, homotopy, homotopy colimit, model category, simplicial model category, exact sequence, cone, homology, localization

1 Introduction

In [10], the reader will be able to find a survey of the different geometric approaches of concurrency. The model category of flows was introduced in [8] to model higher dimensional automata (HDA). It allows the study of HDA up to homotopy (cf. also [6,7]). A good notion of homotopy of flows must preserve the computer scientific properties of the HDA to be modeled like the initial and final states, the deadlocks and the unreachable states. In particular, it must preserve the direction of time, hence the terminology *dihomotopy* for a contraction of *directed homotopy*. This way, instead of working in the category of flows itself, one can work in the localization of the category of flows with respect to *dihomotopy equivalences*.

I will explain in this talk the powerfulness of the homotopy branching space functor in dihomotopy theory. The corresponding papers are "Homotopy

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branching space and weak dihomotopy" [4] and "A long exact sequence for the branching homology" [3].

2 Model category

If $\mathcal{C}$ is a category, one denotes by $Map(\mathcal{C})$ the category whose objects are the morphisms of $\mathcal{C}$ and whose morphisms are the commutative squares of $\mathcal{C}$.

In a category $\mathcal{C}$, an object x is *a retract* of an object y if there exists $f : x \longrightarrow y$ and $g : y \longrightarrow x$ of $\mathcal{C}$ such that $g \circ f = \mathrm{Id}_x$. A *functorial factorization* (α, β) of $\mathcal{C}$ is a pair of functors from $Map(\mathcal{C})$ to $Map(\mathcal{C})$ such that for any f object of $Map(\mathcal{C})$, $f = \beta(f) \circ \alpha(f)$.

Definition 2.1 [12,11] Let $i : A \longrightarrow B$ and $p : X \longrightarrow Y$ be maps in a category $\mathcal{C}$. Then i has the left lifting property (LLP) with respect to p (or p has the right lifting property (RLP) with respect to i) if for any commutative square

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "\beta"'] \arrow[ur, dashed, "g"] & Y \end{tikzcd}$$

there exists g making both triangles commutative.

There are several versions of the notion of *model category*. The following definitions give the one we are going to use.

Definition 2.2 [12,11] A model structure on a category $\mathcal{C}$ is three subcategories of $Map(\mathcal{C})$ called weak equivalences, cofibrations, and fibrations, and two functorial factorizations (α, β) and (γ, δ) satisfying the following properties:

(i) (2-out-of-3) If f and g are morphisms of $\mathcal{C}$ such that $g \circ f$ is defined and two of f, g and $g \circ f$ are weak equivalences, then so is the third.

(ii) (Retracts) If f and g are morphisms of $\mathcal{C}$ such that f is a retract of g and g is a weak equivalence, cofibration, or fibration, then so is f.

(iii) (Lifting) Define a map to be a trivial cofibration if it is both a cofibration and a weak equivalence. Similarly, define a map to be a trivial fibration if it is both a fibration and a weak equivalence. Then trivial cofibrations have the LLP with respect to fibrations, and cofibrations have the LLP with respect to trivial fibrations.

(iv) (Factorization) For any morphism f, $\alpha(f)$ is a cofibration, $\beta(f)$ a trivial fibration, $\gamma(f)$ is a trivial cofibration , and $\delta(f)$ is a fibration.

Definition 2.3 [12,11] A model category is a complete and cocomplete category $\mathcal{C}$ together with a model structure on $\mathcal{C}$.

Proposition and Definition 2.4 *[12,11] A Quillen adjunction is a pair of adjoint functors $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ between the model categories $\mathcal{C}$ and $\mathcal{D}$ such that one of the following equivalent properties holds:*

(i) *if f is a cofibration (resp. a trivial cofibration), then so does $F(f)$*

(ii) *if g is a fibration (resp. a trivial fibration), then so does $G(g)$.*

One says that F is a left Quillen functor*. One says that G is a* right Quillen functor*.*

Definition 2.5 [12,11] An object X of a model category $\mathcal{C}$ is cofibrant (resp. fibrant) if and only if the canonical morphism $\varnothing \longrightarrow X$ from the initial object of $\mathcal{C}$ to X (resp. the canonical morphism $X \longrightarrow \mathbf{1}$ from X to the final object $\mathbf{1}$) is a cofibration (resp. a fibration).

For any object X of a model category, the canonical morphism $\varnothing_X : \varnothing \longrightarrow X$ from the initial object to X can be factored as a composite

$$\varnothing \xrightarrow{\alpha(\varnothing_X)} Q(X) \xrightarrow{\beta(\varnothing_X)} X$$

where, by definition, $Q(X)$ is a cofibrant object which is weakly equivalent to X. The functor $Q : \mathcal{C} \longrightarrow \mathcal{C}$ is called the *cofibrant replacement functor.*

3 Reminder about the category of flows

In the sequel, any topological space will be supposed to be compactly generated (more details for this kind of topological spaces in [1,14], the appendix of [13] and also the preliminaries of [8]).

Let $n \geqslant 1$. Let $\mathbf{D}^n$ be the closed n-dimensional disk. Let $\mathbf{S}^{n-1} = \partial\mathbf{D}^n$ be the boundary of $\mathbf{D}^n$ for $n \geqslant 1$. Notice that $\mathbf{S}^0$ is the discrete two-point topological space $\{-1, +1\}$. Let $\mathbf{D}^0$ be the one-point topological space. Let $\mathbf{S}^{-1} = \varnothing$ be the empty set. The following theorem is well-known.

Theorem 3.1 *[11,12] The category of compactly generated topological spaces* **Top** *can be given a model structure such that:*

(i) *The weak equivalences are the weak homotopy equivalences.*

(ii) *The fibrations (sometime called Serre fibrations) are the continuous maps satisfying the RLP (right lifting property) with respect to the continuous maps $\mathbf{D}^n \longrightarrow [0,1] \times \mathbf{D}^n$ such that $x \mapsto (0, x)$ and for $n \geqslant 0$.*

(iii) *The cofibrations are the continuous maps satisfying the LLP (left lifting property) with respect to any maps satisfying the RLP with respect to the inclusion maps* $\mathbf{S}^{n-1} \longrightarrow \mathbf{D}^n$.

(iv) *Any topological space is fibrant.*

(v) *The homotopy equivalences arising from this model structure coincide with the usual one.*

Definition 3.2 [8] A flow X consists of a topological space $\mathbb{P}X$, a discrete space X^0, two continuous maps s and t from $\mathbb{P}X$ to X^0 and a continuous and associative map $* : \{(x,y) \in \mathbb{P}X \times \mathbb{P}X; t(x) = s(y)\} \longrightarrow \mathbb{P}X$ such that $s(x*y) = s(x)$ and $t(x*y) = t(y)$. A morphism of flows $f : X \longrightarrow Y$ consists of a set map $f^0 : X^0 \longrightarrow Y^0$ together with a continuous map $\mathbb{P}f : \mathbb{P}X \longrightarrow \mathbb{P}Y$ such that $f(s(x)) = s(f(x))$, $f(t(x)) = t(f(x))$ and $f(x*y) = f(x)*f(y)$. The corresponding category will be denoted by **Flow**.

The topological space X^0 is called the 0-*skeleton* of X. The topological space $\mathbb{P}X$ is called the *path space* and its elements the *non constant execution paths* of X. The initial object $\varnothing$ of **Flow** is the empty set. The terminal object $\mathbf{1}$ is the flow defined by $\mathbf{1}^0 = \{0\}$, $\mathbb{P}\mathbf{1} = \{u\}$ and necessarily $u*u = u$.

Definition 3.3 [8] Let Z be a topological space. Then the globe of Z is the flow $\mathrm{Glob}(Z)$ defined as follows: $\mathrm{Glob}(Z)^0 = \{0,1\}$, $\mathbb{P}\mathrm{Glob}(Z) = Z$, $s = 0$, $t = 1$ and the composition law is trivial.

Theorem 3.4 *[8] The category of flows can be given a model structure such that:*

(i) *The weak equivalences are the* weak S-homotopy equivalences*, that is a morphism of flows* $f : X \longrightarrow Y$ *such that* $f : X^0 \longrightarrow Y^0$ *is an isomorphism of sets and* $f : \mathbb{P}X \longrightarrow \mathbb{P}Y$ *a weak homotopy equivalence of topological spaces.*

(ii) *The fibrations are the continuous maps satisfying the RLP with respect to the morphisms* $\mathrm{Glob}(\mathbf{D}^n) \longrightarrow \mathrm{Glob}([0,1] \times \mathbf{D}^n)$ *for* $n \geqslant 0$*. The fibrations are exactly the morphisms of flows* $f : X \longrightarrow Y$ *such that* $\mathbb{P}f : \mathbb{P}X \longrightarrow \mathbb{P}Y$ *is a Serre fibration of* **Top**.

(iii) *The cofibrations are the morphisms satisfying the LLP with respect to any map satisfying the RLP with respect to the morphisms* $\mathrm{Glob}(\mathbf{S}^{n-1}) \longrightarrow \mathrm{Glob}(\mathbf{D}^n)$ *for* $n \geqslant 0$ *and with respect to the morphisms* $\varnothing \longrightarrow \{0\}$ *and* $\{0,1\} \longrightarrow \{0\}$.

(iv) *Any flow is fibrant.*

Let I^{gl} be the set of morphisms of flows $\mathrm{Glob}(\mathbf{S}^{n-1}) \to \mathrm{Glob}(\mathbf{D}^n)$ for $n \geqslant 0$. Denote by I^{gl}_+ be the union of I^{gl} with the two morphisms of flows

$R : \{0,1\} \to \{0\}$ and $C : \varnothing \subset \{0\}$.

Definition 3.5 [4] An I^{gl}_+-cell complex is a flow X such that the canonical morphism of flows $\varnothing \longrightarrow X$ from the initial object of **Flow** to X is a transfinite composition of pushouts of elements of I^{gl}_+. The full and faithful subcategory of **Flow** whose objects are the I^{gl}_+-cell complexes will be denoted by $I^{gl}_+\mathbf{cell}$.

The category $I^{gl}_+\mathbf{cell}$ of I^{gl}_+-cell complexes is a subcategory of the category of flows which is sufficient to model higher dimensional automata (HDA), at least those modeled by precubical sets [9,2]. This geometric model of HDA is designed to define and study equivalence relations preserving the computer-scientific properties of the HDA to be modeled so that it then suffices to work in convenient localizations of $I^{gl}_+\mathbf{cell}$. The properties which are preserved are for instance the initial or final states, the presence or not of deadlocks and of unreachable states [8].

The cofibrant replacement functor is a functor $Q : \mathbf{Flow} \longrightarrow I^{gl}_+\mathbf{cell}$. The flows coming from concrete HDAs are all cofibrant.

4 The homotopy branching space functor

The branching space of a flow is the space of germs of non-constant execution paths beginning in the same way. The branching space functor $\mathbb{P}^-$ from the category of flows **Flow** to the category of compactly generated topological spaces **Top** was also introduced in [8] to fit the definition of the branching semi-globular nerve of a strict globular ω-category modeling an HDA introduced in [5].

Proposition 4.1 *[8,4] Let X be a flow. There exists a topological space $\mathbb{P}^-X$ unique up to homeomorphism and a continuous map $h^- : \mathbb{P}X \longrightarrow \mathbb{P}^-X$ satisfying the following universal property:*

(i) *For any x and y in $\mathbb{P}X$ such that $t(x) = s(y)$, the equality $h^-(x) = h^-(x * y)$ holds.*

(ii) *Let $\phi : \mathbb{P}X \longrightarrow Y$ be a continuous map such that for any x and y of $\mathbb{P}X$ such that $t(x) = s(y)$, the equality $\phi(x) = \phi(x * y)$ holds. Then there exists a unique continuous map $\overline{\phi} : \mathbb{P}^-X \longrightarrow Y$ such that $\phi = \overline{\phi} \circ h^-$.*

Moreover, one has the homeomorphism

$$\mathbb{P}^-X \cong \bigsqcup_{\alpha \in X^0} \mathbb{P}^-_\alpha X$$

where $\mathbb{P}^-_\alpha X := h^-\left(\bigsqcup_{\beta \in X^0} \mathbb{P}_{\alpha,\beta} X\right)$. The mapping $X \mapsto \mathbb{P}^-X$ yields a functor

$\mathbb{P}^-$ *from* **Flow** *to* **Top***.*

Definition 4.2 [8,4] Let X be a flow. The topological space $\mathbb{P}^-X$ is called the branching space of the flow X.

Proposition 4.3 *[4] There exists a weak S-homotopy equivalence of flows $f : X \longrightarrow Y$ such that the topological spaces $\mathbb{P}^-X$ and $\mathbb{P}^-Y$ are not weakly homotopy equivalent.*

The idea for the proof of Proposition 4.3 is as follows. For a given flow X, by Proposition 4.1, the topological space $\mathbb{P}^-X$ is the coequalizer of the continuous map $\mathbb{P}X \times_{X^0} \mathbb{P}X \longrightarrow \mathbb{P}X$ induced by the composition law of X and of the projection map $\mathbb{P}X \times_{X^0} \mathbb{P}X \longrightarrow \mathbb{P}X$ on the first factor. And one cannot expect a coequalizer to transform a objectwise weak homotopy equivalence into a weak homotopy equivalence. One must use a kind of homotopy coequalizer instead.

If two flows are weakly S-homotopy equivalent, then they are supposed to satisfy the same computer-scientific properties. With the example above, one obtains two such flows but with very different branching spaces. But

Theorem 4.4 *[4] If $f : X \longrightarrow Y$ is a weak S-homotopy equivalence of flows between cofibrant flows, then the topological spaces $\mathbb{P}^-X$ and $\mathbb{P}^-Y$ are homotopy equivalent.*

This suggests that the definition of the branching space is the good one up to homotopy for cofibrant flows. Indeed, we have the theorems:

Theorem 4.5 *[4] There exists a functor $C^- : \mathbf{Top} \longrightarrow \mathbf{Flow}$ such that the pair of functors $\mathbb{P}^- : \mathbf{Flow} \rightleftarrows \mathbf{Top} : C^-$ is a Quillen adjunction. In particular, there is an homeomorphism $\mathbb{P}^-(\varinjlim X_i) \cong \varinjlim \mathbb{P}^-X_i$.*

Definition 4.6 The homotopy branching space $\mathrm{ho}\mathbb{P}^-X$ of a flow X is by definition the topological space $\mathbb{P}^-Q(X)$.

Theorem 4.7 *[4] The functor $\mathrm{ho}\mathbb{P}^- : \mathbf{Flow} \longrightarrow \mathbf{Top} \longrightarrow \mathbf{Ho}(\mathbf{Top})$ satisfies the following universal property: if $F : \mathbf{Flow} \longrightarrow \mathbf{Ho}(\mathbf{Top})$ is another functor sending weak S-homotopy equivalences to isomorphisms and if there exists a natural transformation $F \Rightarrow \mathbb{P}^-$, then the latter natural transformation factors uniquely as a composite $F \Rightarrow \mathrm{ho}\mathbb{P}^- \Rightarrow \mathbb{P}^-$.*

Up to homotopy, the homotopy branching space $\mathrm{ho}\mathbb{P}^-(X)$ is well-defined and coincides with $\mathbb{P}^-X$ for any cofibrant flow, so in particular for any flow coming from a HDA. The behavior of the branching space functor and the homotopy branching space functor are the same up to homotopy for flows modeling HDAs and may differ for other flows.

5 The homotopy merging space functor

This is the dual version of the preceding functor. Some results are collected in this section about it.

Proposition 5.1 *[4] Let X be a flow. There exists a topological space $\mathbb{P}^+X$ unique up to homeomorphism and a continuous map $h^+ : \mathbb{P}X \longrightarrow \mathbb{P}^+X$ satisfying the following universal property:*

(i) *For any x and y in $\mathbb{P}X$ such that $t(x) = s(y)$, the equality $h^+(y) = h^+(x * y)$ holds.*

(ii) *Let $\phi : \mathbb{P}X \longrightarrow Y$ be a continuous map such that for any x and y of $\mathbb{P}X$ such that $t(x) = s(y)$, the equality $\phi(y) = \phi(x * y)$ holds. Then there exists a unique continuous map $\overline{\phi} : \mathbb{P}^+X \longrightarrow Y$ such that $\phi = \overline{\phi} \circ h^+$.*

Moreover, one has the homeomorphism

$$\mathbb{P}^+X \cong \bigsqcup_{\alpha \in X^0} \mathbb{P}^+_\alpha X$$

where $\mathbb{P}^+_\alpha X := h^+\left(\bigsqcup_{\beta \in X^0} \mathbb{P}_{\beta,\alpha} X\right)$. The mapping $X \mapsto \mathbb{P}^+X$ yields a functor $\mathbb{P}^+ : \mathbf{Flow} \longrightarrow \mathbf{Top}$.

Definition 5.2 [4] Let X be a flow. The topological space $\mathbb{P}^+X$ is called the merging space of the flow X.

Theorem 5.3 *[4] There exists a functor $C^+ : \mathbf{Top} \longrightarrow \mathbf{Flow}$ such that the pair of functors $\mathbb{P}^+ : \mathbf{Flow} \rightleftarrows \mathbf{Top} : C^+$ is a Quillen adjunction. In particular, there is an homeomorphism $\mathbb{P}^+(\varinjlim X_i) \cong \varinjlim \mathbb{P}^+X_i$.*

Definition 5.4 [4] The homotopy merging space $\mathrm{ho}\mathbb{P}^+ X$ of a flow X is by definition the topological space $\mathbb{P}^+Q(X)$.

Theorem 5.5 *[4] The functor $\mathrm{ho}\mathbb{P}^+ : \mathbf{Flow} \longrightarrow \mathbf{Top} \longrightarrow \mathbf{Ho}(\mathbf{Top})$ satisfies the following universal property: if $F : \mathbf{Flow} \longrightarrow \mathbf{Ho}(\mathbf{Top})$ is another functor sending weak S-homotopy equivalences to isomorphisms and if there exists a natural transformation $F \Rightarrow \mathbb{P}^+$, then the latter natural transformation factors uniquely as a composite $F \Rightarrow \mathrm{ho}\mathbb{P}^+ \Rightarrow \mathbb{P}^+$.*

6 First application: studying weak dihomotopy

The class $\mathcal{S}$ of weak S-homotopy equivalences is an example of class of morphisms of flows which is supposed to preserve various computer-scientific properties. This class of morphisms of flows satisfies the following properties:

(i) The two-out-of-three axiom, that is if two of the three morphisms f, g and $g \circ f$ belong to $\mathcal{S}$, then so does the third one: this condition means that the class $\mathcal{S}$ defines an equivalence relation.

(ii) The embedding functor $I : I^{gl}_{+}\mathbf{cell} \longrightarrow \mathbf{Flow}$ induces a functor $\overline{I} : I^{gl}_{+}\mathbf{cell}[\mathcal{S}^{-1}] \longrightarrow \mathbf{Flow}[\mathcal{S}^{-1}]$ between the localization of respectively the category of I^{gl}_{+}-cell complexes and the category of flows with respect to weak S-homotopy equivalences which is an equivalence of categories. In particular, it reflects isomorphisms, that is $X \cong Y$ if and only if $\overline{I}(X) \cong \overline{I}(Y)$. In this case, one can use the whole category of flows which is a richer mathematical framework.

The class of T-homotopy equivalences was introduced in [8] to identify I^{gl}_{+}-cell complexes equivalent from a computer-scientific viewpoint and which are not identified in $I^{gl}_{+}\mathbf{cell}[\mathcal{S}^{-1}]$. Indeed, if two objects X and Y of $I^{gl}_{+}\mathbf{cell}[\mathcal{S}^{-1}]$ are isomorphic, then the 0-skeletons X^0 and Y^0 are isomorphic. The merging of the notions of weak S-homotopy equivalence and T-homotopy equivalence yields the class $\mathcal{ST}_0$ of 0-dihomotopy equivalences.

Definition 6.1 [8] Let X be a flow. Let A and B be two subsets of X^0. One says that A is surrounded by B (in X) if for any $\alpha \in A$, either $\alpha \in B$ or there exists execution paths γ_1 and γ_2 of $\mathbb{P}X$ such that $s(\gamma_1) \in B$, $t(\gamma_1) = s(\gamma_2) = \alpha$ and $t(\gamma_2) \in B$. We denote this situation by $A \lll B$.

Definition 6.2 [8] Let X be a flow. Let A be a subset of X^0. Then the restriction $X\!\restriction_A$ of X over A is the unique flow such that $(X\!\restriction_A)^0 = A$ and

$$\mathbb{P}\left(X\!\restriction_A\right) = \bigsqcup_{(\alpha,\beta)\in A\times A} \mathbb{P}_{\alpha,\beta}X$$

equipped with the topology induced by the one of $\mathbb{P}X$.

Definition 6.3 [8] A morphism of flows $f : X \longrightarrow Y$ is a 0-dihomotopy equivalence if and only if the following conditions are satisfied:

(i) The morphism of flows $f : X \longrightarrow Y\!\restriction_{f(X^0)}$ is a weak S-homotopy equivalence of flows. In particular, the set map $f^0 : X^0 \longrightarrow Y^0$ is one-to-one.

(ii) For $\alpha \in Y^0 \backslash f(X^0)$, the topological spaces $\mathbb{P}^{-}_{\alpha}Y$ and $\mathbb{P}^{+}_{\alpha}Y$ are singletons.

(iii) $Y^0 \lll f(X^0)$.

The class of 0-dihomotopy equivalences is denoted by $\mathcal{ST}_0$.

But it turns out that

Theorem 6.4 *[4] The functor $I^{gl}_{+}\mathbf{cell}[\mathcal{ST}_0^{-1}] \longrightarrow \mathbf{Flow}[\mathcal{ST}_0^{-1}]$ does not reflect isomorphisms. More precisely, there exists an I^{gl}_{+}-cell complex $\boldsymbol{C}_3$ cor-*

responding to the concurrent execution of three calculations which is not isomorphic in $I_+^{gl}\mathbf{cell}[\mathcal{ST}_0^{-1}]$ *to the directed segment* $\boldsymbol{I}$*, although the same flow* $\boldsymbol{C}_3$ *is isomorphic to* $\boldsymbol{I}$ *in* $\mathbf{Flow}[\mathcal{ST}_0^{-1}]$*.*

The correct behavior is the one of $\mathcal{ST}_0$ in $\mathbf{Flow}[\mathcal{ST}_0^{-1}]$. Indeed, an HDA representing the concurrent execution of n processes must be equivalent to the directed segment in a good homotopical approach of concurrency. The interpretation of this fact is therefore that the class $\mathcal{ST}_0$ of 0-dihomotopy equivalences is not big enough.

Definition 6.5 [4] A morphism of flows $f : X \longrightarrow Y$ is a 1-dihomotopy equivalence if and only if the following conditions are satisfied:

(i) The morphism of flows $f : X \longrightarrow Y\restriction_{f(X^0)}$ is a weak S-homotopy equivalence of flows. In particular, the set map $f^0 : X^0 \longrightarrow Y^0$ is one-to-one.

(ii) For $\alpha \in Y^0\backslash f(X^0)$, the topological spaces $\mathbb{P}_\alpha^- Y$ and $\mathbb{P}_\alpha^+ Y$ are weakly contractible.

(iii) $Y^0 \lll f(X^0)$.

The class of 1-dihomotopy equivalences is denoted by $\mathcal{ST}_1$.

Any 0-dihomotopy equivalence is of course a 1-dihomotopy equivalence. Moreover, the composite of a weak S-homotopy equivalence with a T-homotopy equivalence can already give an element of $\mathcal{ST}_1\backslash\mathcal{ST}_0$! And

Theorem 6.6 *[4] By slightly weakening the notion of T-homotopy as above, one obtains a class of morphisms* $\mathcal{ST}_1$ *with* $\mathcal{ST}_0 \subset \mathcal{ST}_1$ *and such that the flows* $\boldsymbol{C}_3$ *and* $\boldsymbol{I}$ *become isomorphic in the localization* $I_+^{gl}\mathbf{cell}[\mathcal{ST}_1^{-1}]$*.*

There are actually two natural ways of weakening the definition of $\mathcal{ST}_0$. One can replace in the statement the word *singleton* either by the word *weakly contractible*, or by the word *contractible*. This way, one obtains another class of morphisms $\mathcal{ST}'_1$ with $\mathcal{ST}'_1 \subset \mathcal{ST}_1$ and one has:

Theorem 6.7 *[4] The localizations* $I_+^{gl}\mathbf{cell}[\mathcal{ST}'^{-1}_1]$ *and* $I_+^{gl}\mathbf{cell}[\mathcal{ST}_1^{-1}]$ *are equivalent.*

Unfortunately, one has

Proposition 6.8 *[4] The composite of two morphisms of* $\mathcal{ST}_1$ *does not necessarily belong to* $\mathcal{ST}_1$*.*

Using the homotopy branching space functor, a new class $\mathcal{ST}_2$ of morphisms of flows is introduced.

Definition 6.9 [4] A morphism of flows $f : X \longrightarrow Y$ is a 2-dihomotopy equivalence if and only if the following conditions are satisfied:

(i) The morphism of flows $f : X \longrightarrow Y\restriction_{f(X^0)}$ is a weak S-homotopy equivalence of flows. In particular, the set map $f^0 : X^0 \longrightarrow Y^0$ is one-to-one.

(ii) For $\alpha \in Y^0 \backslash f(X^0)$, the topological spaces $\mathrm{ho}\mathbb{P}^-_\alpha Y$ and $\mathrm{ho}\mathbb{P}^+_\alpha Y$ are weakly contractible.

(iii) $Y^0 \lll f(X^0)$.

The class of 2-dihomotopy equivalences is denoted by $\mathcal{ST}_2$.

And:

Theorem 6.10 *[4] One has the equivalence of categories*

$$I^{gl}_+\mathbf{cell}[\mathcal{ST}_1^{-1}] \xrightarrow{\simeq} I^{gl}_+\mathbf{cell}[\mathcal{ST}_2^{-1}]$$

where $I^{gl}_+\mathbf{cell}[\mathcal{ST}_1^{-1}]$ (resp. $I^{gl}_+\mathbf{cell}[\mathcal{ST}_2^{-1}]$) is the localization of the category of I^{gl}_+-cell complexes with respect to 1-dihomotopy equivalences (resp. 2-dihomotopy equivalences). $\mathcal{ST}_2$ is closed under composition. Moreover the embedding functor $I : I^{gl}_+\mathbf{cell} \longrightarrow \mathbf{Flow}$ induces an equivalence of categories

$$\overline{I} : I^{gl}_+\mathbf{cell}[\mathcal{ST}_2^{-1}] \xrightarrow{\simeq} \mathbf{Flow}[\mathcal{ST}_2^{-1}] .$$

In particular, the functor $I^{gl}_+\mathbf{cell}[\mathcal{ST}_2^{-1}] \longrightarrow \mathbf{Flow}[\mathcal{ST}_2^{-1}]$ reflects isomorphisms.

The property $f \in \mathcal{ST}_2$ and $g \circ f \in \mathcal{ST}_2 \Longrightarrow g \in \mathcal{ST}_2$ has no reasons to be satisfied by 2-dihomotopy equivalences. Indeed, if both $g \circ f$ and f are two one-to-one set maps, then g has no reasons to be one-to-one as well. Therefore in order to understand the isomorphisms of $\mathbf{Flow}[\mathcal{ST}_2^{-1}]$, we may introduce another construction.

Definition 6.11 [4] Let X be a flow. Then a subset A of X^0 is essential if $X^0 \lll A$ and if for any $\alpha \notin A$, both topological spaces $\mathrm{ho}\mathbb{P}^-_\alpha X$ and $\mathrm{ho}\mathbb{P}^+_\alpha X$ are weakly contractible.

Definition 6.12 [4] A morphism of flows $f : X \longrightarrow Y$ is a 3-dihomotopy equivalence if the following conditions are satisfied:

(i) $A \subset X^0$ is essential if and only if $f(A) \subset Y^0$ is essential

(ii) for any essential $A \subset X^0$ there exists an essential subset $B \subset A$ such that the restriction $f : X\restriction_B \longrightarrow Y\restriction_{f(B)}$ is a weak S-homotopy equivalence.

The class of 3-dihomotopy equivalences is denoted by $\mathcal{ST}_3$.

Theorem 6.13 *[4] The localizations $I^{gl}_+\mathbf{cell}[\mathcal{ST}_2^{-1}]$ and $I^{gl}_+\mathbf{cell}[\mathcal{ST}_3^{-1}]$ are equivalent and the class of morphisms $\mathcal{ST}_3$ satisfies the two-out-of-three axiom.*

Moreover the embedding functor $I : I^{gl}_+\mathbf{cell} \longrightarrow \mathbf{Flow}$ induces an equivalence of categories

$$\overline{I} : I^{gl}_+\mathbf{cell}[\mathcal{ST}_3^{-1}] \xrightarrow{\simeq} \mathbf{Flow}[\mathcal{ST}_3^{-1}].$$

In particular, the functor $I^{gl}_+\mathbf{cell}[\mathcal{ST}_3^{-1}] \longrightarrow \mathbf{Flow}[\mathcal{ST}_3^{-1}]$ reflects isomorphisms.

The class $\mathcal{ST}_2$ does not satisfy the two-out-of-three axiom but is invariant by retract. The class $\mathcal{ST}_3$ does satisfy the two-out-of-three axiom but is probably not invariant by retract. So none of the definitions above allows to describe the isomorphisms of $I^{gl}_+\mathbf{cell}[\mathcal{ST}_2^{-1}]$. The situation can be summarized with the following diagram:

$$\begin{tikzcd}
 & & I^{gl}_+\mathbf{cell} \arrow[dll, bend right] \arrow[dl] \arrow[d] \arrow[dr, bend left] & \\
I^{gl}_+\mathbf{cell}[\mathcal{S}^{-1}] \arrow[r, "\not\simeq"] \arrow[d, "\simeq"'] & I^{gl}_+\mathbf{cell}[\mathcal{ST}_0^{-1}] \arrow[r, "\not\simeq"] \arrow[d, "\not\simeq"'] & I^{gl}_+\mathbf{cell}[\mathcal{ST}_1^{-1}] \arrow[r, "\simeq"] \arrow[d, "\simeq??"'] & I^{gl}_+\mathbf{cell}[\mathcal{ST}_2^{-1}] \simeq I^{gl}_+\mathbf{cell}[\mathcal{ST}_3^{-1}] \arrow[d, "\simeq"'] \\
\mathbf{Flow}[\mathcal{S}^{-1}] \arrow[r, "\not\simeq"] & \mathbf{Flow}[\mathcal{ST}_0^{-1}] \arrow[r, "\simeq??"] & \mathbf{Flow}[\mathcal{ST}_1^{-1}] & \mathbf{Flow}[\mathcal{ST}_2^{-1}] \simeq \mathbf{Flow}[\mathcal{ST}_3^{-1}] \arrow[l, "\simeq??"'] \\
 & & \mathbf{Flow} \arrow[ull, bend left] \arrow[ul] \arrow[u] \arrow[ur, bend right] &
\end{tikzcd}$$

The symbol $\simeq??$ means that we do not know whether the functor is an equivalence of categories or not. The symbol $\not\simeq$ means that the corresponding functor is not an equivalence.

7 Second application: a long exact sequence for the branching homology

The category of flows is a simplicial model category [3] in the following sense:

Definition 7.1 [15,12,11] x A simplicial model category is a model category $\mathcal{C}$ together with a simplicial set $\mathrm{Map}(X, Y)$ for any object X and Y of $\mathcal{C}$ satisfying the following axioms:

(i) the set $\mathrm{Map}(X, Y)_0$ is canonically isomorphic to $\mathcal{C}(X, Y)$

(ii) for any object X, Y and Z, there is a morphism of simplicial sets

$$\mathrm{Map}(Y, Z) \times \mathrm{Map}(X, Y) \longrightarrow \mathrm{Map}(X, Z)$$

which is associative

(iii) for any object X of $\mathcal{C}$ and any simplicial set K, there exists an object $X \otimes K$ of $\mathcal{C}$ such that there exists a natural isomorphism of simplicial sets

$$\mathrm{Map}(X \otimes K, Y) \cong \mathrm{Map}(K, \mathrm{Map}(X, Y))$$

(iv) for any object X of $\mathcal{C}$ and any simplicial set K, there exists an object X^K such that there exists a natural isomorphism of simplicial sets

$$\mathrm{Map}(X, Y^K) \cong \mathrm{Map}(K, \mathrm{Map}(X, Y))$$

(v) for any cofibration $i : A \longrightarrow B$ and any fibration $p : X \longrightarrow Y$ of $\mathcal{C}$, the morphism of simplicial sets

$$Q(i, p) : \mathrm{Map}(B, X) \longrightarrow \mathrm{Map}(A, X) \times_{\mathrm{Map}(A,Y)} \mathrm{Map}(B, Y)$$

is a fibration of simplicial sets. Moreover if either i or p is trivial, then the fibration $Q(i, p)$ is trivial as well.

Recall that there exists a pair of adjoint functors $|-| : \mathbf{SSet} \rightleftarrows \mathbf{Top} : S_*$ where $|-|$ is the geometric realization functor and S_* the singular nerve functor. The n-simplex of **SSet** is denoted by $\Delta[n]$. Its boundary is denoted by $\partial\Delta[n-1]$. Let Δ^n be the n-dimensional simplex.

The category of compactly generated topological spaces **Top** is a simplicial model category by setting $\mathrm{Map}(X, Y)_n := \mathbf{Top}(X \times \Delta^n, Y)$, $X \otimes K := X \times |K|$ and $X^K := \mathbf{TOP}(|K|, X)$. The category of simplicial sets **SSet** is a simplicial model category as well by setting $\mathrm{Map}(X, Y)_n := \mathbf{Top}(X \times \Delta[n], Y)$, $X \otimes K := X \times K$ and $X^K := \mathrm{Map}(K, X)$ [15].

This means that the model category of flows can be enriched over the category of simplicial sets and that the enrichment is compatible with the model structure in the sense of Definition 7.1. The symbol Δ^n is the simplicial set corresponding to the n-dimensional simplex.

Because of the existence of this enrichment, there exist explicit formulae for homotopy colimits [11]. In particular, the homotopy pushout of a diagram of flows looks as follows:

Definition 7.2 [11] The homotopy pushout of the diagram of flows

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \\ C & \end{tikzcd}$$

is the colimit of the diagram of flows

$$\begin{tikzcd} & A \otimes \Delta^0 \arrow[r] \arrow[d] & B \\ A \otimes \Delta^0 \arrow[r] \arrow[d] & A \otimes \Delta^1 & \\ C & & \end{tikzcd}$$

It is then very easy to prove the:

Theorem 7.3 *[3] Let X be a diagram of flows. Then the topological spaces $\underrightarrow{\operatorname{holim}}\, \mathrm{ho}\mathbb{P}^-(X)$ and $\mathrm{ho}\mathbb{P}^-(\underrightarrow{\operatorname{holim}}\, X)$ are homotopy equivalent (they are both cofibrant indeed). So in particular, the homotopy branching space functor commutes with homotopy pushouts.*

Definition 7.4 [3] Let $f : X \longrightarrow Y$ be a morphism of flows. The cone Cf of f is the homotopy pushout in the category of flows

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ \mathbf{1} \arrow[r] & Cf \end{tikzcd}$$

where $\mathbf{1}$ is the terminal flow.

From the theorem

Theorem 7.5 *[3] The homotopy branching space of the terminal flow is contractible.*

one can easily deduce a long exact sequence for the branching homology.

Definition 7.6 [3] Let X be a flow. Then the $(n+1)$st branching homology group $H^-_{n+1}(X)$ is defined as the nst homology group of the augmented simplicial set $\mathcal{N}^-_*(X)$ defined as follows:

(i) $\mathcal{N}^-_n(X) = S_n(\mathrm{ho}\mathbb{P}^- X)$ for $n \geqslant 0$

(ii) $\mathcal{N}^-_{-1}(X) = X^0$

(iii) the augmentation map $\epsilon : S_0(\mathrm{ho}\mathbb{P}^- X) \longrightarrow X^0$ is induced by the mapping $\gamma \mapsto s(\gamma)$ from $\mathrm{ho}\mathbb{P}^- X = S_0(\mathrm{ho}\mathbb{P}^- X)$ to X^0.

Theorem 7.7 *[3] For any flow X, one has*

(i) $H^-_0(X) = \mathbb{Z}X^0/Im(s)$

(ii) *the short exact sequence*

$$0 \to H_1^-(X) \to H_0(\mathrm{ho}\mathbb{P}^- X) \to \mathbb{Z}\,\mathrm{ho}\mathbb{P}^- X/Ker(s) \to 0$$

(iii) $H_{n+1}^-(X) = H_n(\mathrm{ho}\mathbb{P}^- X)$ *for* $n \geqslant 1$.

Theorem 7.8 *[3] For any morphism of flows* $f : X \longrightarrow Y$, *one has the long exact sequence*

$$\begin{aligned}
&\cdots \to H_n^-(X) \to H_n^-(Y) \to H_n^-(Cf) \to \dots\\
&\cdots \to H_3^-(X) \to H_3^-(Y) \to H_3^-(Cf) \to\\
&H_2^-(X) \to H_2^-(Y) \to H_2^-(Cf) \to\\
&H_0(\mathrm{ho}\mathbb{P}^- X) \to H_0(\mathrm{ho}\mathbb{P}^- Y) \to H_0(\mathrm{ho}\mathbb{P}^- Cf) \to 0.
\end{aligned}$$

The functors $X \mapsto H_n^-(X)$ for $n \geqslant 0$ are invariant up to 2-dihomotopy equivalence. The functor $X \mapsto H_0(\mathrm{ho}\mathbb{P}^- X)$ is only invariant up to weak S-homotopy equivalence. So the long exact sequence above is not satisfactory. It still remains to find an exact sequence whose each term would be a functor invariant up to 2-dihomotopy equivalence.

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