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Characterizing Consistent Smyth Powerdomains by FS-$\wedge^{\uparrow}$-domains

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Abstract

In this paper, we introduce FS-$\wedge^{\uparrow}$-domains, and show that the category with FS-$\wedge^{\uparrow}$-domains as objects and Scott continuous functions as morphisms is a Cartesian closed category. Moreover, we characterize the consistent Smyth powerdomain over a Lawson compact domain by means of FS-$\wedge^{\uparrow}$-domain.

Keywords: Domain; Consistency; FS-$\wedge^{\uparrow}$-domain; Consistent Smyth powerdomain

1 Introduction

In Domain theory, powerdomains are very important structures, which play an important role in modeling the semantics of nondeterministic programming languages ([4,5,6,7,9,11,12,13,14,15]). For example, the Smyth powerdomain is the free deflationary semilattice over a continuous dcpo, where the deflationary binary operator is exactly the Scott continuous meet operator [14]. However, in many interesting domains, such as L-domains, the meet operator is not total but a partial one: two elements have a meet (or a greatest lower bound) if they are consistent, i.e., they have an upper bound. In this case, the partial meet operator is called a consistent meet, denoted by $\wedge^{\uparrow}$. So a question arises: can we construct a new free algebra over

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a continuous dcpo on which the binary operator is exactly the Scott continuous consistent meet? In [16], we show by methods of topology and order theory that the consistent Smyth powerdomain over a continuous dcpo exists and is a continuous dcpo-$\wedge^{\uparrow}$-semilattice. Moreover, if a continuous dcpo is Lawson compact, then its consistent Smyth powerdomain is a Lawson compact L-domain. This is a difference between the consistent Smyth powerdomain and the classical one, because the classical Smyth powerdomain over a Lawson compact domain is a bounded complete domain.

Note that classical powerdomains, such as the Smyth powerdomain and the Hoare powerdomain, can be characterized by means of some FS-domains given of the basic functions in the structure of powerdomains respectively. In [8], Huth, Jung and Keimel introduced a new concept: linear FS-lattice, which is a complete lattice and there exists a directed family of finitely separated linear functions which can approximate id, where a function is linear if it preserves all suprema. They proved that the Hoare powerdomain $H(L)$ over a pointed domain L is characterized by a distributive linear FS-lattice. In [10], Meng and Kou introduced $FS_{\wedge}$-domains and proved that the Smyth powerdomain $S(L)$ over a Lawson compact domain L is characterized by $FS_{\wedge}$-domains. So for purely mathematical purposes, we have reasons to believe that there exists a kind of FS-domains to characterize the consistent Smyth powerdomain over a Lawson compact continuous domain.

In this paper we first use the partial Scott continuous binary operator $\wedge^{\uparrow}$ to construct a finitely separated domain: FS-$\wedge^{\uparrow}$-domain, which is a dcpo-$\wedge^{\uparrow}$-semilattice and there exists a directed family of finitely separated $\wedge^{\uparrow}$-semilattice homomorphisms which can approximate id_L. We have obtained the following conclusions:

a) The category with FS-$\wedge^{\uparrow}$-domains as objects and Scott continuous functions as morphisms is a cartesian closed category.

b) The consistent Smyth powerdomain $S_C(L)$ over a Lawson compact continuous domain L is an FS-$\wedge^{\uparrow}$-domain.

Moreover, we characterize consistent Smyth powerdomains by means of FS-$\wedge^{\uparrow}$-domains.

Next, we collect some basic notions needed in this paper. The reader can also consult [1,3]. A poset L is called a directed complete poset (a dcpo, for short) if any directed set of L has a sup in L. For $x, y \in L$, x is called to be way below y (denoted by $x \ll y$) if for any directed set D, $y \le \vee D$ implies that there is some $d \in D$ with $x \le d$. A poset L is called continuous if for all $x \in L$, $x = \vee^{\uparrow} \twoheaddownarrow x$, i.e., the set $\twoheaddownarrow x = \{a \in L : a \ll x\}$ is directed and $x = \vee\{a \in L : a \ll x\}$, where the arrow in the symbol $\vee^{\uparrow}$ is to emphasize the directedness of $\twoheaddownarrow x$. Specially, a dcpo which is continuous as a poset will be called a (continuous) domain. For a subset A of L, let $\uparrow A = \{x \in L : \exists a \in A,\ a \le x\}$, $\downarrow A = \{x \in L : \exists a \in A,\ x \le a\}$. We use $\uparrow a$ (resp. $\downarrow a$) instead of $\uparrow\{a\}$ (resp. $\downarrow\{a\}$) when $A = \{a\}$. A is called an upper (resp. a lower) set if $A = \uparrow A$ (resp. $A = \downarrow A$). An element $k \in L$ is called compact if $k \ll k$. The subset of all compact elements is denoted by $K(L)$. A dcpo L is called algebraic if for all $x \in L$, $x = \vee^{\uparrow}(\twoheaddownarrow x \cap K(L))$.

Definition 1.1 *Let L be a poset.*

(1) *A subset A of L is called consistent if A has an upper bound in L, i.e., $A \subseteq \downarrow x$ for some $x \in L$.*

(2) *L is called a consistent meet-semilattice (or $\wedge^{\uparrow}$-semilattice) if $x \wedge y$ exists for all consistent $x, y \in L$. To emphasize the fact that x and y are consistent, we will write $x \wedge^{\uparrow} y$ instead of $x \wedge y$. Moreover, if L is a (continuous) dcpo, then L is called a (continuous) dcpo-$\wedge^{\uparrow}$-semilattice.*

(3) *L is called an L-domain if L is a domain and every consistent subset of L has an inf, i.e., $\downarrow x$ is a complete lattice for all $x \in L$.*

Let L be a poset. We call the topology generated by the complements $L \backslash \uparrow x$ of principal filters as subbasic open sets the lower topology and denote it by $\omega(L)$. If $(L, \leq)$ is a dcpo, we define the Scott topology (denoted by $\sigma(L)$), which has as its topology of closed sets all directed complete lower sets, i.e., lower sets closed under directed sups. The Lawson topology $\lambda(L)$ is generated by taking the join of $\sigma(L)$ and $\omega(L)$ as subbasic. If L is a $\wedge^{\uparrow}$-semilattice, then the partial operator $\wedge^{\uparrow} : L \times L \to L$ is Scott continuous.

A $\wedge^{\uparrow}$-semilattice homomorphism f between dcpo-$\wedge^{\uparrow}$-semilattices $(P, \wedge^{\uparrow})$ and $(E, \wedge^{\uparrow})$ is a Scott continuous function from P to E such that $f(a \wedge^{\uparrow} b) = f(a) \wedge^{\uparrow} f(b)$ whenever a, b are consistent in P. Note that the function is Scott continuous and conditionally multiplicative (or *cm* for short in [2]), that is, each $\wedge^{\uparrow}$-semilattice homomorphism is a Scott continuous *cm* function.

2 Categories of FS-$\wedge^{\uparrow}$-domains

For dcpos L and P, let $[L \longrightarrow P]$ denote all Scott continuous functions from L to P with the pointwise order. For dcpo-$\wedge^{\uparrow}$-semilattices D and E, let $[D \longrightarrow_{\wedge^{\uparrow}} E]$ denote the function space of $\wedge^{\uparrow}$-semilattice homomorphisms from D to E with the pointwise order.

Definition 2.1 *[3] A dcpo L is called an FS-domain if id_L is approximated directly by a family of finitely separating functions, where a Scott continuous function $f : L \longrightarrow L$ is called finitely separated if there exists a finite set M_f such that for each $x \in L$, there exists $m \in M_f$ such that $f(x) \leq m \leq x$.*

Definition 2.2 *A dcpo L is called an FS-$\wedge^{\uparrow}$-domain if it is a $\wedge^{\uparrow}$-semilattice and there exists a directed family of finitely separated $\wedge^{\uparrow}$-semilattice homomorphisms which can approximate id_L.*

In other words, an FS-$\wedge^{\uparrow}$-domain is a continuous dcpo-$\wedge^{\uparrow}$-semilattice which *id* is approximated by a directed family of finitely separated Scott continuous functions preserving existing finite infs. Obviously, every FS-$\wedge^{\uparrow}$-domain is an FS-domain. Then we have

Proposition 2.3 *Each FS-$\wedge^{\uparrow}$-domain is a Lawson compact L-domain.*

Next, we show that the category with FS-$\wedge^{\uparrow}$-domains as objects and Scott continuous functions as morphisms is a cartesian closed category.

Theorem 2.4 *Let D and E be dcpo-$\wedge^{\uparrow}$-semilattices, then $[D \longrightarrow_{\wedge^{\uparrow}} E]$ is a dcpo-$\wedge^{\uparrow}$-*

semilattice.

Proof. Firstly, $[D \longrightarrow_{\wedge^\uparrow} E]$ is a dcpo. For any directed set $\{f_j \in [D \longrightarrow_{\wedge^\uparrow} E] : j \in J\}$ and $x \in D$, set $f(x) = \bigvee_{j \in J} f_j(x)$. It is obvious that f is Scott continuous. If $x, y \in D$ are consistent, then $f(x), f(y)$ are also consistent in E. Then

$$\begin{aligned}
&f(x \wedge^\uparrow y) = \bigvee_{j \in J} f_j(x \wedge^\uparrow y) = \bigvee_{j \in J} (f_j(x) \wedge^\uparrow f_j(y)) \\
&= (\bigvee_{j \in J} f_j(x)) \wedge^\uparrow (\bigvee_{j \in J} f_j(y)) = f(x) \wedge^\uparrow f(y).
\end{aligned}$$

So f is also Scott continuous and a $\wedge^\uparrow$-semilattice homomorphism. Hence $[D \longrightarrow_{\wedge^\uparrow} E]$ is a dcpo.

Secondly, $[D \longrightarrow_{\wedge^\uparrow} E]$ is a $\wedge^\uparrow$-semilattice. If $f, g \in [D \longrightarrow_{\wedge^\uparrow} E]$ are consistent, then $f(x), g(x)$ are consistent for any $x \in D$. Then $f(x) \wedge^\uparrow g(x)$ exists. Let $(f \wedge^\uparrow g)(x) = f(x) \wedge^\uparrow g(x)$. For a directed set $\{x_k \in D : k \in K\}$, we have

$$\begin{aligned}
&(f \wedge^\uparrow g)(\bigvee_{k \in K} (x_k)) = f(\bigvee_{k \in K} (x_k)) \wedge^\uparrow g(\bigvee_{k \in K} (x_k)) \\
&= \bigvee_{k \in K} f(x_k) \wedge^\uparrow \bigvee_{k \in K} g(x_k) = \bigvee_{k \in K} [f(x_k) \wedge^\uparrow g(x_k)] \\
&= \bigvee_{k \in K} [(f \wedge^\uparrow g)(x_k)].
\end{aligned}$$

Then $f \wedge^\uparrow g$ is Scott continuous.

For a pair of consistent points x, y in D, $f(x \wedge^\uparrow y)$ and $g(x \wedge^\uparrow y)$ are consistent in E. Then

$$\begin{aligned}
&(f \wedge^\uparrow g)(x \wedge^\uparrow y) \\
&= f(x \wedge^\uparrow y) \wedge^\uparrow g(x \wedge^\uparrow y) \\
&= (f(x) \wedge^\uparrow f(y)) \wedge^\uparrow (g(x) \wedge^\uparrow g(y)) \\
&= (f(x) \wedge^\uparrow g(x)) \wedge^\uparrow (f(y) \wedge^\uparrow g(y)) \\
&= (f \wedge^\uparrow g)(x) \wedge^\uparrow (f \wedge^\uparrow g)(y).
\end{aligned}$$

That is, $f \wedge^\uparrow g$ is a $\wedge^\uparrow$-semilattice homomorphism. So $[D \longrightarrow_{\wedge^\uparrow} E]$ is a $\wedge^\uparrow$-semilattices.

Finally, by the Scott continuity of the operation $\wedge^\uparrow$, we obtain the following conclusion. If the sup of the directed set $\{f_j \in [D \longrightarrow_{\wedge^\uparrow} E] : j \in J\}$ and $g \in [D \longrightarrow_{\wedge^\uparrow} E]$ are consistent, then for $x \in D$,

$$[g \wedge^{\uparrow} (\bigvee_{j \in J} f_j)](x) = g(x) \wedge^{\uparrow} (\bigvee_{j \in J} (f_j(x))$$
$$= \bigvee_{j \in J} [g(x) \wedge^{\uparrow} f_j(x)] = \bigvee_{j \in J} [(g \wedge^{\uparrow} f_j)(x)]$$
$$= [\bigvee_{j \in J} (g \wedge^{\uparrow} f_j)](x).$$

So $\wedge^{\uparrow} : [D \longrightarrow_{\wedge^{\uparrow}} E] \times [D \longrightarrow_{\wedge^{\uparrow}} E] \longrightarrow [D \longrightarrow_{\wedge^{\uparrow}} E]$ is Scott continuous.

We have obtained that $[D \longrightarrow_{\wedge^{\uparrow}} E]$ is a dcpo-$\wedge^{\uparrow}$-semilattice. $\square$

Theorem 2.5 *Let D and E be FS-$\wedge^{\uparrow}$-domains, then $[D \longrightarrow_{\wedge^{\uparrow}} E]$ and $[D \longrightarrow E]$ are FS-$\wedge^{\uparrow}$-domains.*

Proof. Suppose that $\mathcal{D}$ and $\mathcal{E}$ are approximate identities for D and E respectively. Then we claim that the family

$$\mathcal{D} \otimes \mathcal{E} = \{\delta \otimes \epsilon : \delta \in \mathcal{D}, \epsilon \in \mathcal{E}\}$$

defined by

$$f \mapsto \epsilon^2 f \delta^2$$

for $f \in [D \longrightarrow_{\wedge^{\uparrow}} E]$ is an approximate identity for $[D \longrightarrow_{\wedge^{\uparrow}} E]$ and $\delta \otimes \epsilon$ is finitely separating. The proof is similar with the case of FS-domains.

It suffices to show that $\delta \otimes \epsilon \in [D \longrightarrow_{\wedge^{\uparrow}} E] \longrightarrow_{\wedge^{\uparrow}} [D \longrightarrow_{\wedge^{\uparrow}} E]$. Firstly, it is obvious that $\delta \otimes \epsilon$ is Scott continuous. Secondly, for a pair of consistent points $f, g \in [D \longrightarrow_{\wedge^{\uparrow}} E]$, we have $(\delta \otimes \epsilon)(f), (\delta \otimes \epsilon)(g)$ are consistent and for any $x \in D$

$$[(\delta \otimes \epsilon)(f \wedge^{\uparrow} g)](x) = [\epsilon^2 (f \wedge^{\uparrow} g) \delta^2](x)$$
$$= \epsilon^2 [f \delta^2(x) \wedge^{\uparrow} g \delta^2(x)] = \epsilon^2 f \delta^2(x) \wedge^{\uparrow} \epsilon^2 g \delta^2(x)$$
$$= [\epsilon^2 f \delta^2 \wedge^{\uparrow} \epsilon^2 g \delta^2](x) = [(\delta \otimes \epsilon)(f) \wedge^{\uparrow} (\delta \otimes \epsilon)(g)](x).$$

So we conclude that $\delta \otimes \epsilon$ is a $\wedge^{\uparrow}$-semilattice homomorphism. Then $[D \longrightarrow_{\wedge^{\uparrow}} E]$ is an FS-$\wedge^{\uparrow}$-domain. Similarly, $[D \longrightarrow E]$ is also an FS-$\wedge^{\uparrow}$-domain. $\square$

Note that usually the category with FS-$\wedge^{\uparrow}$-domains as objects and $\wedge^{\uparrow}$-semilattice homomorphisms as morphisms is not a cartesian closed category. However, if the category considers Scott continuous functions as morphisms, then from the preceding paragraph we have the following conclusion:

Theorem 2.6 *The category with FS-$\wedge^{\uparrow}$-domains as objects and Scott continuous functions as morphisms is a cartesian closed category.*

3 Characterize consistent Smyth powerdomains by FS-$\wedge^{\uparrow}$-domains

In the following paragraph, we will relate FS-$\wedge^{\uparrow}$-domain and consistent Smyth powerdomain with the functions: $\wedge^{\uparrow}$-semilattice homomorphisms. We characterize consistent Smyth powerdomains $S_C(L)$ over a Lawson compact continuous domain L by means of FS-$\wedge^{\uparrow}$-domains.

Definition 3.1 *[16] A consistent deflationary semilattice is a continuous dcpo L with a Scott continuous binary partial operator $\wedge^{\uparrow}$ defined only for consistent pairs of points that satisfy three equations for commutativity $x \wedge^{\uparrow} y = y \wedge^{\uparrow} x$, associativity $x \wedge^{\uparrow} (y \wedge^{\uparrow} z) = (x \wedge^{\uparrow} y) \wedge^{\uparrow} z$, and idempotency $x \wedge^{\uparrow} x = x$ together with the inequality $x \geq x \wedge^{\uparrow} y$ for any $x, y, z \in L$. The free consistent deflationary semilattice over a domain L is called the consistent Smyth powerdomain over L.*

Definition 3.2 *[16] Let L be a poset and F a nonempty subset of L. Two elements x and y in F are called linearly connected in F provided there exists a consistent path in $\uparrow F$ from x to y, i.e. finitely many $x_0, x_1, \ldots, x_n$ in $\uparrow F$ such that $x = x_0 \dagger x_1 \dagger \ldots \dagger x_n = y$, denoted by $x \sim_F y$. F is called linearly connected if any two elements of F are linearly connected in F.*

Let L be a continuous domain and let

$$\mathcal{B}_C(L) = \{F \subseteq_{fin} L : F \neq \emptyset \text{ and } F \text{ is linearly connected}\}.$$

Let $S_C(L)$ be the family generated by $\uparrow\mathcal{B}_C(L) = \{\uparrow F : \ F \in \mathcal{B}_C(L)\}$ as a basis, i.e., for all $A \in S_C(L)$, $A = \bigcap_{\uparrow}\{\uparrow F : \ F \in \mathcal{B}_C(L) \ \& \uparrow F \ll A\}$, then $S_C(L)$ is a continuous dcpo-$\wedge^{\uparrow}$-semilattice in [16].

Theorem 3.3 *[16] Let L be a continuous domain. The embedding j of L into $S_C(L)$ is given by $j(x) = \uparrow x$ for $x \in L$. If P is a dcpo-$\wedge^{\uparrow}$-semilattice and $f : L \longrightarrow P$ a Scott continuous function, then there exists uniquely a $\wedge^{\uparrow}$-homomorphism $\bar{f}$ such that $\bar{f}j = f$. Thus, $S_C(L)$ is isomorphic to the consistent Smyth powerdomain over L.*

Definition 3.4 *The relation $\ll$ in a continuous dcpo-$\wedge^{\uparrow}$-semilattice D is called consistent $\ll$-multiplication if for consistent elements $a, b \in D$, $x \ll a, b$ implies $x \ll a \wedge^{\uparrow} b$.*

For a continuous L-domain L, L is distributive with $\vee$ and $\wedge^{\uparrow}$ if it holds the following statements: for any consistent points x, y, z, $x \wedge^{\uparrow} (y \vee z) = (x \wedge^{\uparrow} y) \vee (x \wedge^{\uparrow} z)$.

Definition 3.5 *Let L be a poset. An element $m \in L$ is a minimal upper bound (or mub for short) for a subset A if m is an upper bound for A that is minimal in the set of all upper bounds of A.*

Lemma 3.6 *Let D be an algebraic Lawson compact L-domain with the consistent $\ll$-multiplicative property. If D is distributive with $\vee$ and $\wedge^{\uparrow}$, then D is an FS-$\wedge^{\uparrow}$-domain.*

Proof. Let $\mathcal{D} = \{(x \Rightarrow x) \in D \to D : x \ll x\}$, where $(x \Rightarrow x)$ is the one-step function defined by

$$(x \Rightarrow x)(z) = \begin{cases} x, & z \in \Uparrow x, \\ 0_{\downarrow z}, & \text{otherwise.} \end{cases}$$

For any finite compact element subset $K = \{x_1, ..., x_n\} \subseteq K(D)$ for $n \in N$ and $\emptyset \neq G \subseteq K$, define $M_G \subseteq_{fin} mub(G)$: for any $m \in M_G$, there is $x \in \cap_{x_i \in G} \uparrow x_i$ such that $m \ll x$. By compactness of D, there is a finite set $M_G \subseteq mub\{x_i : x_i \in G\}$ such that $mub\{x_i : x_i \in G\} \subseteq \bigcup_{m \in M_G} \Uparrow m$.

Set

$$L_c(K) = \{\wedge^{\uparrow}_{x_i \in G}\{x_i\} : \emptyset \neq G \subseteq K, \cap_{x_i \in G} \uparrow x_i \neq \emptyset\},$$

$$M_{L_c(K)} = \bigcup_G \{m \in M_G : \emptyset \neq G \subseteq L_c(K), \cap_{x_i \in G} \uparrow x_i \neq \emptyset\},$$

$$K^* = L_c(K) \cup M_{L_c(K)},$$

$$K_1 = K^*, K_2 = K_1^*, ..., K_{n+1} = K_n^*,$$

$$\mathcal{F}(K) = \bigcup_{n \in N} K_n.$$

By the distributive property, the set $\mathcal{F}(K)$ is finite. Let us define the mapping $f_K : D \to D$ as follows: for $x \in D$,

$$f_K(x) = \begin{cases} m, & x \in \Uparrow m \setminus \Uparrow(\uparrow m \cap \mathcal{F}(K)),\ m \in \mathcal{F}(K), \\ 0_{\downarrow x}, & \text{otherwise.} \end{cases}$$

If $G_1 = G_2$, $m_1, m_2 \in M_{G_1}$ and $m_1 \neq m_2$, then $\Uparrow m_1 \setminus \Uparrow(\uparrow m_1 \cap \mathcal{F}(K)) \cap \Uparrow m_2 \setminus \Uparrow(\uparrow m_2 \cap \mathcal{F}(K)) = \emptyset$. Otherwise, there is $x \in \Uparrow m_1 \setminus \Uparrow(\uparrow m_1 \cap \mathcal{F}(K)) \cap \Uparrow m_2 \setminus \Uparrow(\uparrow m_2 \cap \mathcal{F}(K))$. But $m_1 \neq m_2$. This is a contradiction with which D is an L-domain. If $G_1 \neq G_2$ and $m_i \in M_{G_i}$ for $i = 1, 2$, then $\Uparrow m_1 \setminus \Uparrow(\uparrow m_1 \cap \mathcal{F}(K)) \cap \Uparrow m_2 \setminus \Uparrow(\uparrow m_2 \cap \mathcal{F}(K)) = \emptyset$. Otherwise, there is $x \in \Uparrow m_1 \setminus \Uparrow(\uparrow m_1 \cap \mathcal{F}(K)) \cap \Uparrow m_2 \setminus \Uparrow(\uparrow m_2 \cap \mathcal{F}(K))$. Then $x \in \Uparrow(m_1 \vee_{\downarrow x} m_2)$, a contradiction. On the other hand, suppose $m_1 \in M_{G_1}, m_2 \in M_{G_2}$ and $G_1 \neq G_2$. If $a \in \Uparrow m_1 \setminus \Uparrow(\uparrow m_1 \cap \mathcal{F}(K))$ and $b \in \Uparrow m_2 \setminus \Uparrow(\uparrow m_2 \cap \mathcal{F}(K))$ and $f_K(a) \neq f_K(b)$. We must have $a \neq b$. Otherwise, $a \in \Uparrow(m_1 \vee_{\downarrow a} m_2)$, but $f_K(a) = m_1$. This is a contradiction with the definition of f_K. If $G_1 = G_2$, $m_1, m_2 \in M_{G_1}$ and $m_1 \neq m_2$, then $a \neq b$ because D is a L-domain. Then f_K is well defined.

It's obvious that f_K is monotone with finite range and $f_K \leq id_D$. For any $d \in D$, if $d = 0_{\downarrow d}$ and $d \notin \Downarrow m$ for any $m \in \mathcal{F}(K)$, then $f_K^{-1}(\Uparrow d) = \Uparrow d$; if there is some $m \in \mathcal{F}(K)$ such that $d \ll m$, then $f_K^{-1}(\Uparrow d) = \bigcup\{\Uparrow m : d \ll m\}$ and otherwise, $f_K^{-1}(\Uparrow d) = \emptyset$. All cases show that $f_K^{-1}(\Uparrow d)$ is a Scott open set of D. Then f_K is Scott continuous.

It is easy to show that $\{f_K : K \subseteq K(D), |K| \subseteq_{fin} N\}$ is a directed set approximated to id_D. For any $x \in D$, we know $sup\{f_K(x) : K \subseteq K(D), |K| \subseteq_{fin} N\} \leq x$. If $x \not\leq sup\{f_K(x) : K \subseteq K(D), |K| \subseteq_{fin} N\}$, then there is $u \ll x$ but $u \not\leq sup\{f_K(x) : K \subseteq K(D), |K| \subseteq_{fin} N\}$. By $u \ll x$, there is some compact element v such that $u \ll v \ll x$. By $(v \Rightarrow v)(x) = v$ and $(v \Rightarrow v) \leq sup\{f_K : K \subseteq$

$K(D), |K| \subseteq_{fin} N\}$, then $u \leq v \leq sup\{f_K(x) : K \subseteq K(D), |K| \subseteq_{fin} N\}$, a contradiction.

To show that $\{f_K : K \subseteq K(D), |K| \subseteq_{fin} N\}$ is the approximate identity over D, it is sufficient to prove that these functions are also $\wedge^{\uparrow}$-homomorphisms. Suppose that a and b are consistent witnessed by c. Let $a \wedge^{\uparrow} b = x$ and let

$$a \in \Uparrow m_1 \setminus \Uparrow(\uparrow m_1 \cap \mathcal{F}(K)),$$

$$b \in \Uparrow m_2 \setminus \Uparrow(\uparrow m_2 \cap \mathcal{F}(K))$$

and

$$x \in \Uparrow m_0 \setminus \Uparrow(\uparrow m_0 \cap \mathcal{F}(K)),$$

where $m_i \in M_{G_i}$ for $i = 0, 1, 2$. Then $f_K(a) \wedge^{\uparrow} f_K(b)$ exists. By $m_0 \ll x \leq a$ and $m_0 \ll x \leq b$, if $G_0 = G_1 = G_2$, then $m_1 = m_0$ and $m_2 = m_0$, and then $m_0 = m_1 \wedge^{\uparrow} m_2$. Otherwise, we obtain $m_0 \vee_{\downarrow a} m_1 \ll a$ and $m_0 \vee_{\downarrow a} m_1 \in \mathcal{F}(K)$, but $a \in \Uparrow m_1 \setminus \Uparrow(\uparrow m_1 \cap \mathcal{F}(K))$. Then $m_0 \leq m_1$. Similarly, $m_0 \leq m_2$. Thus, we have $m_0 \leq m_1 \wedge^{\uparrow} m_2$. On the other hand, by $\ll$-multiplicative property, $m_1 \wedge^{\uparrow} m_2 \ll a \wedge^{\uparrow} b = x$. By the definition of f_K, we conclude $m_0 = m_1 \wedge^{\uparrow} m_2$. Hence, $f_K(a \wedge^{\uparrow} b) = m_0 = m_1 \wedge^{\uparrow} m_2 = f_K(a) \wedge^{\uparrow} f_K(b)$. □

Theorem 3.7 *Let D be a Lawson compact L-domain with consistent $\ll$-multiplicative property. If D is distributive with $\vee$ and $\wedge^{\uparrow}$, then D is an FS-$\wedge^{\uparrow}$-domain.*

Proof. For any finite subset $X = \{x_1, ..., x_n\} \subseteq D$, $Y = \{y_1, ..., y_n\} \subseteq D$ and $y_i \ll x_i$ for any $n \in N$ and $I = \{1, ..., n\}$, set

$$L_c(X) = \{\wedge^{\uparrow}_{i \in F}\{x_i\} : F \in \Phi(I) = \Phi(\Psi_X)\},$$

$$U_c(X) = \{\vee_{\downarrow x}\{x_i : i \in F\} : F \in \Phi(\Psi_{L_C(X)}), \exists x \in D, s.t., \{x_i : i \in F\} \subseteq \downarrow x\},$$
$$X^* = L_c(X) \cup U_c(X),$$
$$X_1 = X^*, X_2 = X_1^*, ..., X_{n+1} = X_n^*,$$
$$\mathcal{F}(X) = \bigcup_{n \in N} X_n,$$

where $\Phi(I) = \{F \subseteq I : \cap_{i \in F} \Uparrow x_i \neq \emptyset\}, \Phi_k(I) = \{F \in \Phi(I) : |F| = k\}$,

$$MI = \max\{i \in I : \exists F \in \Phi(I), s.t., |F| = i\},$$

and let $\Psi_X = I$, and

$$\Psi_{L_c(X)} = \{i \downarrow (i + k_1^{(2)}), ..., i \downarrow (i + k_t^{(2)}), ..., i \downarrow (i + k_{s_1}^{(e+1)}) \downarrow ... \downarrow (i + k_{s_e}^{(e+1)}) : 1 \leq i \leq n\},$$

where

$$k_1^{(2)} = \min\{k : \uparrow x_i \cap \uparrow x_{i+k} \neq \emptyset, 0 \leq k \leq n - i\},$$
$$k_t^{(2)} = \max\{k : \uparrow x_i \cap \uparrow x_{i+k} \neq \emptyset, 0 \leq k \leq n - i\}$$

and $i{\downarrow}(i+k_1^{(2)})$ means that $x_i \wedge^{\uparrow} x_{i+k_1^{(2)}}$ exists, and

$$e = \max\{|E| : \uparrow x_i \cap (\bigcap_{k\in E} \uparrow x_{i+k}) \neq \emptyset, 0 \leq k \leq n-i\},$$

$$\{(i+k_{s_1}^{(e+1)}), ..., (i+k_{s_e}^{(e+1)})\} = \max\{E : \uparrow x_i \cap (\bigcap_{k\in E} \uparrow x_{i+k}) \neq \emptyset, 0 \leq k \leq n-i\},$$

$i{\downarrow}(i+k_{s_1}^{(e+1)}){\downarrow}...{\downarrow}(i+k_{s_e}^{(e+1)})$ denotes that $\wedge^{\uparrow}\{x_i, x_{i+k_{s_1}^{(e+1)}}, ..., x_{i+k_{s_e}^{(e+1)}}\}$ exists. From the definition of Ψ, we know that $|\Psi_{L_c(X)}| \leq (L_c(X))!$. Then $\Psi_{L_c(X)}$ is finite. Similarly, for some X_k, the set Ψ_{X_k} is finite.

Let us define $M_E \subseteq_{fin} mub\{y_i : i \in E\}$ for a finite set E: for any $m \in M_E$, there is $x \in \cap_{i\in E}\uparrow x_i$ such that $m \ll x$. By compactness of D, there is a finite set $M_E \subseteq mub\{y_i : i \in E\}$ such that $mub\{x_i : i \in E\} \subseteq \bigcup_{m\in M_E} \twoheaduparrow m$.

Let $m_a = \wedge^{\uparrow}_{i\in F}\{y_i\}$, if $a = \wedge^{\uparrow}_{i\in F}\{x_i\}$ for $F \in \Phi(\Psi_{X_{k_1}})$. Let $m_a = \vee_{\downarrow x}\{y_i : i \in F\}$, if there is some $x \in D$ such that $\{x_i : i \in F\} \subseteq \downarrow x$ for $F \in \Phi(\Psi_{X_{k_2}})$ and $a = \vee_{\downarrow x}\{x_i : i \in F\}$.

By the distributive property and compactness of D, the set $\{m_a : a \in \mathcal{F}(X)\}$ is finite. Let us define a mapping $f_I : D \to D$ as follows: for $x \in D$,

$$f_I(x) = \begin{cases} m_a,\ x \in (\twoheaduparrow m_a \cap \twoheaduparrow a) \setminus \twoheaduparrow(\uparrow a \cap \mathcal{F}(X)),\ a \in \mathcal{F}(X), \\ 0_{\downarrow x},\ \text{otherwise}. \end{cases}$$

Then $\{f_I : I \subseteq_{fin} N\}$ is an approximate identity over D. □

In [16], we show that the consistent Smyth powerdomain over a Lawson compact continuous domain is a Lawson compact continuous L-domain.

Theorem 3.8 *[16] If L is a Lawson compact continuous domain, then the consistent Smyth powerdomain $S_C(L)$ is a Lawson compact continuous L-domain satisfied the consistent $\ll$-multiplicative property and the distributive property with $\vee$ and $\wedge^{\uparrow}$.* □

By Theorem 3.7 and Theorem 3.8, we have the following conclusion:

Theorem 3.9 *If L is a Lawson compact continuous domain, then the consistent Smyth powerdomain $S_C(L)$ over L is an FS-$\wedge^{\uparrow}$-domain.*

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