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About a Positive Set Theory With Equality

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Abstract

We discuss the consistency problem for a positive set theory with equality called Strong-Frege-3, introduced by Hinnion some twenty years ago. We also exhibit "natural" models of some fragments of Strong-Frege-3.

Keywords: Positive set theory, extensionality, comprehension, consistency.

1 Introduction

In this paper we are concerned with the "positive" set theory Strong-Frege-3, which can be considered as a sort of "three-valued" analog of Frege set theory.

The peculiar feature of this theory is that, unlike similar theories, the formulas admitted in the comprehension schema use not only membership, but even equality and inequality, and both are treated classically.

In [6] and [7], the theory is erroneously attributed to E. Weydert; rather, the first who proposed the theory seems to be R. Hinnion, even though Weydert contributed a lot in the area with his thesis [10]. In fact, some theories inspired by the same ideas as Strong-Frege-3 can be found in the works of Hinnion himself, see [3] and [4], besides those of other researchers such as Brady [1], Gilmore [2] and Skolem [8]. Instead, it seems, [6] and [7] are the only papers dedicated to Strong-Frege-3 itself.

In this introduction we define the theory Strong-Frege-3, following closely [7].

The name of the theory Strong-Frege-3 is due to R. Hinnion, and its explanation is the following:

- Strong: in contraposition with another, weaker theory, called Frege-3, where not only membership, but even equality is viewed as three valued;

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- Frege: in honour of G. Frege, the author of the first (although inconsistent) comprehension principle for sets, which is the source of inspiration for this (hopefully consistent) theory;
- 3: because membership is three valued, in that we have two relations $\in$ and $\overline{\in}$, mutually exclusive, and given two sets x, y, we have three possibilities: either $x \in y$ (meaning: x belongs to y), or $x \overline{\in} y$ (meaning: x does not belong to y), or neither holds, in which case the membership of x to y assumes an undetermined value.

A formal account of the theory Strong-Frege-3 is as follows.

We call *sets* the inner objects of the theory Strong–Frege-3. The formal language of the theory is the first order language consisting of two binary predicates, $\in$ (membership) and $\overline{\in}$ (bar-membership), and including the equality predicate $=$.

First of all we have the following axiom:

Axiom 1. *(mutual exclusion)* $\neg(x \in y \wedge x \overline{\in} y)$.

This axiom means that $\in$ and $\overline{\in}$ are a kind of "weak negation" of each other. However, since we do not state the reverse arrow of the axiom (which would amount to say $x \in y \vee x \overline{\in} y$), we do not impose a priori that $\in$ and $\overline{\in}$ are the real negation of each other; actually, as we will see, this is provably false in Strong-Frege-3.

In the literature, theories including the axiom 1 are often called "paracomplete", whereas theories where $x \in y \vee x \overline{\in} y$ holds are called "paraconsistent". A major difference between the two options is that topological spaces give natural examples of paraconsistent models, if one takes pairs of closed sets which cover the universe (one might think to give paracomplete models by using the dual notion of disjoint pairs of closed sets, but this approach is not powerful enough for Strong-Frege-3, although it does work for other theories, see [4]).

A set in Strong-Frege-3 is a kind of "two face medal", in that it can have zero or more members, and zero or more bar-members. Anyway, a set is determined by its members and bar-members, as the following axiom states:

Axiom 2. *(extensionality)*
$(\forall t((t \in x \leftrightarrow t \in y) \wedge (t \overline{\in} x \leftrightarrow t \overline{\in} y))) \rightarrow x = y$.

In other words, any two sets which have the same members *and* the same bar-members are equal.

Finally we give the very core of Strong-Frege-3, namely its comprehension schema. The idea is to repeat Frege's comprehension schema for set-theoretic formulas, but with the following changes:

- we replace the classical non-membership $\notin$ with the bar-membership $\overline{\in}$;
- we consider only the "positive" formulas defined below;
- while defining a set by comprehension, we specify both its members and its bar-members (so, this set will be uniquely determined by extensionality).

To formalize this idea, let us first define the positive formulas:

Definition 3. *(positive formulas) The set PF of the positive formulas is the smallest set of $\in, \overline{\in}$-formulas such that:*

- $x \in y, x \overline{\in} y, x = y, \neg(x = y)$ *are in PF for any two variables* x, y *(these are the basic positive formulas);*
- *if* ϕ, ψ *are in PF and* x *is a variable, then also* $\phi \vee \psi, \phi \wedge \psi, \exists x \phi, \forall x \phi$ *are in PF.*

Let us consider the four kinds of basic positive formulas in two variables x, y: by mutual exclusion, $x \in y$ and $x \overline{\in} y$ are in a status of mutual weak negation or, since they are both positive, of *positive negation*; moreover $x = y$ and $\neg(x = y)$ are the (classical) negation of each other, and we can consider also their correspondence as a positive negation, since they are positive by definition.

So we have a bijection between basic positive formulas, called positive negation; there is a natural extension of this bijection to all positive formulas by induction, and we give it in the following definition:

Definition 4. *(positive negation of positive formulas) Give the positive formula* ϕ*, we call positive negation of* ϕ *the formula* $PN(\phi)$*, where:*

- $PN(x \in y)$ *is* $x \overline{\in} y$*,* $PN(x \overline{\in} y)$ *is* $x \in y$*,* $PN(x = y)$ *is* $\neg(x = y)$ *and* $PN(\neg(x = y))$ *is* $x = y$*;*
- $PN(\phi \vee \psi)$ *is* $PN(\phi) \wedge PN(\psi)$*, and* $PN(\phi \wedge \psi)$ *is* $PN(\phi) \vee PN(\psi)$*;*
- $PN(\exists x \phi)$ *is* $\forall x.PN(\phi)$*, and* $PN(\forall x \phi)$ *is* $\exists x.PN(\phi)$*.*

We write $\overline{\phi}$ instead of $PN(\phi)$. We note that, for any positive formula ϕ, $\overline{\phi}$ is a positive formula as well, and $\overline{\overline{\phi}} = \phi$.

Now, because of the presence of $\in$ and $\overline{\in}$ in Strong-Frege-3, it is natural to associate (certain) sets with (certain) pairs of formulas. For example, given two formulas $\phi(x)$ and $\psi(x)$ whose only free variable is x, by extensionality there is at most one set whose members are those enjoying ϕ and whose bar-members are those enjoying ψ. When ϕ is a positive formula and ψ is $\overline{\phi}$, this set exists by the following axiom schema, where b is a variable which is not free in ϕ:

Axiom 5. *(positive comprehension schema)*
$\forall a_1, \ldots, a_n . \exists b . \forall x . ((x \in b \leftrightarrow \phi(x, a_1, \ldots, a_n)) \wedge (x \overline{\in} b \leftrightarrow \overline{\phi}(x, a_1, \ldots, a_n)))$.

We denote by $\{x \mid \phi(x, a_1, \ldots, a_n)\}$ the set b of the previous axiom.

We call Strong-Frege-3 the first order theory whose nonlogical axioms are the axioms 1, 2 and 5 above. So, we have Mutual Exclusion, Extensionality, and the schema of Positive Comprehension.

It is not known whether Strong-Frege-3 is consistent. The aim of this paper is to discuss the consistency of Strong-Frege-3 and of some interesting fragments of it.

2 Remarks

In this section we repeat the remarks on Strong-Frege-3 made in [6].

As we said, a peculiar feature of Strong-Frege-3 is that its sets have "two faces", in that they have members and bar-members, with the constraint that no set can be both member and bar-member of the same set.

Although positive formulas seem to be a very poor class from the expressive point of view, the comprehension schema implies the existence of many sets (unique by extensionality), for instance:

- $V \equiv \{x \mid x = x\}$, the universal set; we have $x \in V$ and $\neg(x \overline{\in} V)$ for every set x;
- the dual of V is $\emptyset \equiv \{x \mid x \neq x\}$, where $x \overline{\in} \emptyset$ and $\neg(x \in \emptyset)$ for every x; note that this set is not really "empty" because, although it has no members, it has the property that every set is a bar-member of it;
- for every set a we have the singleton $\{a\} \equiv \{x \mid x = a\}$; it results $a \in \{a\}$, and $b \overline{\in} \{a\}$ for every b different from a;
- we have the complement $\overline{a} \equiv \{x \mid x \overline{\in} a\}$, with the property that $x \in \overline{a}$ if and only if $x \overline{\in} a$, and $x \overline{\in} \overline{a}$ if and only if $x \in a$;
- we have the principal ultrafilter $F_a \equiv \{x \mid a \in x\}$, with the property $x \in F_a$ if and only if $a \in x$, and $x \overline{\in} F_a$ if and only if $a \overline{\in} x$;
- for any two sets a, b we have the union $a \cup b \equiv \{x \mid x \in a \vee x \in b\}$, such that $x \in a \cup b$ if and only if $x \in a$ or $x \in b$, and $x \overline{\in} a \cup b$ if and only if $x \overline{\in} a$ and $x \overline{\in} b$;
- for any two sets a, b we have the intersection $a \cap b \equiv \{x \mid x \in a \wedge x \in b\}$, such that $x \in a \cap b$ if and only if $x \in a$ and $x \in b$, and $x \overline{\in} a \cap b$ if and only if $x \overline{\in} a$ or $x \overline{\in} b$.

Union and intersection satisfy the usual laws of idempotence, commutativity, associativity and distributivity; in particular for every k we can define arbitrary, unordered k-uplets by

$$\{a_1, a_2, \ldots, a_k\} \equiv \{a_1\} \cup \{a_2\} \cup \ldots \cup \{a_k\}.$$

Moreover, note that the universe is infinite, e.g. it contains the infinite sequence (s_n) given by $s_0 \equiv V$ and $s_{n+1} \equiv \{s_n\}$.

An interesting kind of sets is given by the following

Definition 6. *(cantorian sets) A set x is called cantorian if it verifies $\forall y(y \in x \vee y \overline{\in} x)$.*

We note that: V is cantorian; every singleton is cantorian; and union, intersection and complement preserve cantorianity. So, there is an infinite boolean algebra of cantorian sets. However, there are also non-cantorian sets: to find some, we consider what Russell's antinomy becomes in Strong-Frege-3.

In fact, one might think to prove a Russell-like antinomy, and so the inconsistency of Strong-Frege-3, by using the set

$$R \equiv \{x \mid x \overline{\in} x\}.$$

Actually, from the definition of R we obtain

$$\forall x. x \in R \leftrightarrow x \overline{\in} x$$

hence, taking $x = R$

$$R \in R \leftrightarrow R \overline{\in} R$$

but this is not a contradiction; rather, by axiom 1, we can only conclude that R is neither a member nor a bar-member of itself, and therefore it is not cantorian.

There is more. Consider now the set

$$* \equiv \{x \mid R \in R\}.$$

From the above properties of R it follows that $*$ has no members and no bar-members (and it is the unique set with these properties, by extensionality); so, $*$ is the "real", bilaterally empty set.

3 Some partial models

The previous remarks show that Strong-Frege-3 is able to provide a variety of set theoretic constructions. However, as we said, the consistency problem for Strong-Frege-3 is open. What can be done is giving models of fragments of Strong-Frege-3. In the following subsections we will consider some interesting fragments and models for them.

3.1 A model of the theory minus extensionality

A consistent fragment of Strong-Frege-3 is given by axioms 1 and 5, that is, the theory minus the axiom of extensionality. There is a natural "term model" construction leading to a model of axioms 1 and 5.

First of all, the universe is given by the set ICT of all iterated comprehension terms, defined inductively by $ICT = \bigcup_n ICT_n$, where:

- ICT_0 is empty;
- ICT_{n+1} is the set of all expressions $\{x \mid \phi(x, t_1, \ldots, t_k)\}$, where $\phi(x, x_1, \ldots, x_k)$ is a positive formula with $k+1$ free variables, $k \geq 0$, and $t_1, \ldots, t_k$ are elements of ICT_n.

We point out that ICT is a set of *terms*, rather than of *sets*; so, for instance, $\{x \mid x = x\}$ and $\{x \mid x = x \vee x = x\}$ are different elements of ICT.

Now, equality in ICT is defined as syntactic equality (so the pair above illustrates that the model is not extensional). Finally, $\in$ and $\overline{\in}$ are defined inductively by means of a sequence of pairs of relations $\in_\alpha$ and $\overline{\in}_\alpha$ on ICT, where α is a countable ordinal, and:

- $\in_0$ and $\overline{\in}_0$ are empty;
- for any terms $t, t_1, \ldots, t_k$ in ICT, we let $t \in_{\alpha+1} \{x \mid \phi(x, t_1, \ldots, t_k)\}$ if and only if $\phi_\alpha(t, t_1, \ldots, t_k)$, and $t \overline{\in}_{\alpha+1} \{x \mid \phi(x, t_1, \ldots, t_k)\}$ if and only if $\overline{\phi}_\alpha(t, t_1, \ldots, t_k)$,

where ϕ_α and $\overline{\phi}_\alpha$ are obtained from ϕ and $\overline{\phi}$ by replacing $\in$ with $\in_\alpha$ and $\overline{\in}$ with $\overline{\in}_\alpha$;

- if λ is a limit ordinal, then $\in_\lambda \equiv \bigcup_{\alpha<\lambda} \in_\alpha$ and $\overline{\in}_\lambda \equiv \bigcup_{\alpha<\lambda} \overline{\in}_\alpha$.

We note that $\in_\alpha$ and $\overline{\in}_\alpha$ are monotone functions of α, and since their domain is ICT which is countable, there is a countable μ such that $\in_\mu = \in_{\mu+1}$ and $\overline{\in}_\mu = \overline{\in}_{\mu+1}$, and both sequences are constant from μ on. Then we define $\in \equiv \in_\mu$, and $\overline{\in} \equiv \overline{\in}_\mu$.

We note that axiom 1 holds because $\in_\alpha$ and $\overline{\in}_\alpha$ are disjoint for every α, as can be proved by induction.

For the comprehension schema, we note that by definition we have $t \in \{x \mid \phi(x, t_1, \ldots, t_k)\}$ if and only if $t \in_{\mu+1} \{x \mid \phi(x, t_1, \ldots, t_k)\}$ if and only if $\phi_\mu(t, t_1, \ldots, t_k)$ if and only if $\phi(t, t_1, \ldots, t_k)$, which gives the first half of the schema (the one about $\in$); the other half (about $\overline{\in}$) is analogous.

3.2 A model for the quantifier free part

Another way to investigate the consistency problem for Strong-Frege-3 could be recursion theory. In this section we give a model of a fragment of Strong-Frege-3, that is the quantifier-free fragment, using recursion theory.

The idea is to view a set of Strong-Frege-3 as a partial function which takes on (at most) the values 0 and 1, where 1 means membership, and 0 means bar-membership.

What we need is an injective enumeration of these functions. We proceed as follows.

In the rest of the paper, let us fix some standard Gödel-numbering ϕ of the partial recursive functions, and let $W_i = dom\ \phi_i$ (so, W is a numbering of all the recursively enumerable sets of integers).

Let C be a class of r.e. sets of integers. An *enumeration* of C is a partial recursive function f such that

$$C = \{W_{f(i)} \mid i \in \omega\}.$$

Moreover, f is said to be *injective* if for any two indexes $i \neq j$, we have $W_{f(i)} \neq W_{f(j)}$.

Friedberg in 1958 showed that the class of all r.e. sets has an injective enumeration. Here we want to prove the same result for the subclass V_{01} of all r.e. sets which encode partial recursive functions valued in $\{0, 1\}$ (with respect to the standard encoding of pairs of integers with integers).

To this aim, we recall a lemma by Kummer, taken from Wehner [9]:

Lemma 7. (Kummer [5]) *Let A be an enumerable class of r.e. sets of integers, which can be partitioned in two classes A_1, A_2 such that:*

- *every finite subset of a member of A_1 has infinitely many extensions in A_2;*
- *A_2 is injectively enumerable.*

Then A is injectively enumerable as well.

So, to have the injective enumeration, let us take in the lemma

- $A \equiv V_{01}$;
- $A_1 \equiv \{g \in V_{01} \mid card(g)\ is\ even\ or\ infinite\}$;
- $A_2 \equiv \{g \in V_{01} \mid card(g)\ is\ odd\}$.

The hypotheses of the lemma are satisfied. In fact, first V_{01} has an enumeration, because one can design a binary partial recursive function U which is *universal* for the unary functions valued in $\{0,1\}$ (essentially, we can take $U(i,j) = \phi_i(j)$, except that $U(i,j)$ diverges whenever $\phi_i(j)$ is an integer different from 0 and 1), and now an enumeration can be obtained from the s-m-n theorem.

Moreover, the first condition of the lemma is obviously satisfied.

Let us verify the second condition, saying that the set A_2 is injectively enumerable.

We can write (by Church thesis) a program P, with two inputs i and n, which behaves as follows.

We know that finite sets, pairs, and finite sets of pairs of integers, can be coded by single integers. So, given a first input $i \in \omega$, the program P examines the finite set g of pairs coded by i, and verifies whether g is indeed a function valued in $0, 1$ of odd cardinality (otherwise, P diverges). Then, P takes a second input integer $n \in \omega$, verifies whether n is in the domain of g (otherwise it diverges), and in this case it outputs $g(n)$. This program P is a computable function of i, so there is a partial recursive function f such that:

- $f(i)$ is defined if and only if i codes an element of A_2;
- when $f(i)$ is defined, we have $P(i,n) = W_{f(i)}(n)$ (where $W_{f(i)}$ is seen as a set of pairs, namely as a function);
- $\{W_{f(i)} | i \in \omega\} = A_2$ (that is, f is an enumeration of A_2);
- if $f(i)$ and $f(j)$ are defined and $i \neq j$, then $W_{f(i)} \neq W_{f(j)}$ (that is, the enumeration f of A_2 is injective).

So, by the lemma, we have an injective enumeration f_{01} of the class V_{01}. Let ψ_i be the function whose graph (seen as a set of integers) is $W_{f_{01}(i)}$.

We obtain a model M of the $\in, \overline{\in}$-language by taking ω as universe, and by writing $j \in i$ if $\psi_i(j) = 1$, and $j \overline{\in} i$ if $\psi_i(j) = 0$.

The model M verifies Axiom 1 because all the ψ_i are functions, and Axiom 2 because the numbering f is injective. Let us verify Axiom 5 for all positive formulas without quantifiers.

In fact, let us first verify the comprehension schema for every possible *atomic* formula occurring in a quantifier-free positive formula.

For $x = x$ (the identically true formula), the comprehension term is the unique index i such that ψ_i is the constant function 1.

For $x \in *$ (the undefined formula), the term is the index i such that ψ_i is the function which is undefined everywhere.

For $x = a$, where a is an arbitrary parameter, the term is the index of the function ψ such that $\psi(a) = 1$, and $\psi(b) = 0$ for every $a \neq b$.

For $x \in x$ we take the function ψ such that $\psi(i) = \psi_i(i)$.

For $x \in a$ we take the function ψ_a.

For $a \in x$ we take ψ such that $\psi(x) = \psi_x(a)$.

Note that equalities and membership where both sides are parameters reduce to true or false or undefined, hence they do not need to be considered.

The duals of the previous formulas can be treated by observing that if ψ realizes a formula ϕ, then $1 - \psi$ realizes $\overline{\phi}$.

Now for the union of two sets a, b, we can take the function ψ which takes the value 1 if at least one of ψ_a and ψ_b is 1, the value 0 if both ψ_a and ψ_b are 0, and is undefined otherwise.

Finally, intersection can be obtained from union and complement by the De Morgan law.

Instead, we cannot show comprehension for quantified positive formulas in general, essentially because the recursively enumerable sets are closed under existential quantifiers, but not under universal quantifiers.

3.3 A "term" model of the equalitary fragment

In Strong-Frege-3 we have a "three-valued" analog of the set V_ω of the hereditarily finite sets in ZF, in the following sense.

Recall that V_ω can be defined as the smallest set such that:

- $\emptyset \in V_\omega$;
- if $t, s \in V_\omega$, then $t \cup \{s\} \in V_\omega$ as well.

In V_ω we have a natural definition of equality and membership, and the resulting structure is a model of several set theories (especially those not including the axiom of infinity).

We can construct an equivalent of V_ω in Strong-Frege-3, which we call $V_\omega^{(3)}$, and is defined as the smallest class such that:

- $V, \emptyset, * \in V_\omega^{(3)}$;
- if $t, s \in V_\omega^{(3)}$, then $t \cup \{s\} \in V_\omega^{(3)}$ and $t \cap \overline{\{s\}} \in V_\omega^{(3)}$ as well.

Even without assuming the existence of models of the entire Strong-Frege-3, a "copy" of $V_\omega^{(3)}$ can be obtained by considering the definition above as a definition of terms, and by defining equality, membership and co-membership between terms in a suitable way.

More precisely, we define the class NT of the *normalized terms* inductively by $NT = \bigcup_k NT_k$, where:

- NT_0 contains only the three elements $V, \emptyset, *$;
- NT_{k+1} is NT_k plus all the expressions of the forms:

$$(* \cup \{t_1, \ldots, t_n\}) \cap co\{t_{n+1}, \ldots, t_{n+m}\},$$

$$* \cup \{t_1, \dots, t_n\},$$
$$* \cap co\{t_1, \dots, t_n\},$$
$$\{t_1, \dots, t_n\},$$
$$co\{t_1, \dots, t_n\},$$

where n (and m if present) are positive, t_i are distinct elements of NT_k, and the lists $t_1, \dots, t_n$ (and $t_{n+1} \dots, t_{n+m}$ if present) are ordered alphabetically with respect to some order imposed on the syntactic symbols which may occur in a term (say $* < V < \emptyset < \cup < \cap < \{<\} < (<) <, < co$); note that the symbol co is intended to replace the bar, and is introduced so as to have linear terms, which are easy to order.

We have that every term of $V_\omega^{(3)}$ is equal to a unique normalized term, equality between normalized terms is just syntactical equality, and membership and co-membership are the natural ones. The model given by the normalized terms satisfies the axioms 1 and 2 of Strong-Frege-3. Moreover, it satisfies the "equalitary fragment" of Strong-Frege-3, that is, the comprehension schema of Strong-Frege-3 restricted to formulas whose atomic formulas are equalities, inequalities and $x \in *$ (note that this is a sub-fragment of the quantifier-free fragment, because quantifiers in equalitary formulas can be eliminated).

By the way, it could be of some interest to notice that all elements t of $V_\omega^{(3)}$ are definable by positive formulas, in the sense that for every t there is a positive formula $\phi_t(x)$ which is satisfied only by taking $x = t$.

In fact, $x = V$ can be written $\forall y.y \in x$; likewise, $x = \emptyset$ can be written $\forall y.y \overline{\in} x$; and $x = *$ can be defined "by negation" by $\forall y.y = x \vee \exists z.z \in y \vee z \overline{\in} y$. With similar tricks, all elements of NT, hence all elements of $V_\omega^{(3)}$, can be defined. It is conjectured that these are the only sets definable by positive formulas in Strong-Frege-3.

4 The general case

We are left with the problem of finding models for the full theory Strong-Frege-3. Let us discuss briefly this point.

Usually, models for positive set theories are found by working in topological spaces with "not too many" closed sets, and modeling the definable sets of the theory by closed sets. However, this "topological" approach fails for the theory Strong-Frege-3, essentially because any topology containing the sets of Strong-Frege-3 as closed sets should contain all the cofinite subsets, hence all subsets of the universe should be closed, and the resulting topology (the discrete topology) is useless in this context.

We have seen that the quantifier-free part of Strong-Frege-3 has a recursively enumerable model. However, also the recursion theoretic approach has its limitations, because it seems that no class beyond the recursively enumerable is known to have enumerations without repetitions, and recursively enumerable sets are closed for existential quantification but not for the universal one.

A third approach could be considering a term model like $V_\omega^{(3)}$ above, but the fact that membership is not monotonic with respect to inclusion (because of the presence of equality and inequality) makes it impossible to solve the problem just with an easy inductive construction. Maybe a more sophisticate quasi-term-model could do the job.

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