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C_σ-unique Dcpos and Non-maximality of the Class of Dominated Dcpos Regarding Γ-faithfulness

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Abstract

We continue the study of the dcpos which are determined by their Scott closed set lattices. Such dcpos are called C_σ-unique. Some new sufficient conditions for a dcpo to be C_σ-unique are given. One example is constructed to show that a C_σ-unique dcpo need not be dominated. Thus the question whether the dominated dcpos form a maximal Γ-faithful class of dcpos is negatively answered. Using a recent result by Zhao and Xi, we also deduce that every T_1 topological space has a dcpo model that is C_σ-unique.

Keywords: C_σ-unique; Scott topology; Dominated dcpo; Γ-faithful; Stratified Johnstone dcpo.

1 Introduction

We call a T_0 space X C-unique if for any T_0 space Y, X is homeomorphic to Y whenever the closed set lattice of X is isomorphic to that of Y. A T_0 space is C-unique iff it is sober and T_D (see [1,12,17]). Our aim here is to investigate the analogous objects in the context of directed complete posets. A dcpo P is called

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C_σ-unique [17] if for any dcpo Q, P is isomorphic to Q iff the Scott closed set lattices of P and that of Q are isomorphic. The main questions we are interested are: (i) Which dcpos are C_σ-unique? (ii) Can we have a complete characterization for C_σ-unique dcpos like that for C-unique T_0 spaces? In [6], the authors constructed a counter example to show that there are dcpos which are not C_σ-unique. A class $\mathcal{L}$ of dcpos is called Γ-faithful if for any two P, Q in $\mathcal{L}$, P and Q are isomorphic iff the Scott closed lattices of P and that of Q are isomorphic (see [6],[15]). The class of all quasicontinuous dcpos is Γ-faithful; the class of all upper bounded complete dcpos is Γ-faithful; the class of all dominated dcpos is Γ-faithful. A Γ-faithful class $\mathcal{L}$ of dcpos is said to be maximal if it is not a proper class of another Γ-faithful class of dcpos. For instance the class of all complete lattices is Γ-faithful [7], but not maximal. It is, however, still not known whether the class of all dominated dcpos is maximal. In this paper we constructed one non dominated dcpo and show that it is C_σ-unique. This then shows that a C_σ-unique dcpo need not be dominated (all examples of C_σ-unique dcpos considered in [17] are dominated), and thus the class of all dominated dcpos is not a maximal Γ-faithful class. We shall also establish some new sufficient conditions for a dcpo to be C_σ-unique. Another derived result is that every T_1 topological space has a dcpo model that is C_σ-unique.

2 Preliminaries

We quickly recall some basic notions and results (see [2] for more details).

Let $(L, \leqslant)$ be a poset. A *principal ideal* (resp., *principal filter*) of L is a set of the form $\downarrow x = \{y \in L \mid y \leqslant x\}$ (resp., $\uparrow x = \{y \in L \mid x \leqslant y\}$). For $A \subseteq L$, we write $\downarrow A = \{y \in L \mid \exists\, x \in A,\, y \leqslant x\}$, $\uparrow A = \{y \in L \mid \exists\, x \in A,\, x \leqslant y\}$. A subset A is a *lower set* (resp., an *upper set*) if $A = \downarrow A$ (resp., $A = \uparrow A$). We say that z is a *lower bound* (resp., an *upper bound*) of A if $A \subseteq \uparrow z$ (resp., $A \subseteq \downarrow z$). The set of lower bounds of A is denoted by $\mathrm{lb}(A)$. The supremum of A is denoted by $\bigvee A$ or $\sup A$. The infimum of A is denoted by $\bigwedge A$ or $\inf A$. A nonempty subset D of L is *directed* if every finite subset of D has an upper bound in D. We will use $\sup^{\uparrow} D$ to denote the supremum of directed set D. A poset L is a *directed complete partially ordered set* (dcpo, for short) if every directed subset of L has a supremum. A *complete lattice* is a poset in which every subset has a supremum.

An element x of a poset is maximal if $x \leq y$ implies $x = y$ for any element y. The symbol $\mathrm{Max}(P)$ will denote the set of all maximal elements of P.

A subset A of a poset P is Scott closed if (i) $A = \downarrow A$ and (ii) for any directed subset $D \subseteq A$, $\bigvee D \in A$ whenever $\bigvee D$ exists. The set of all Scott closed sets of P will be denoted by $C_\sigma(P)$. The complements of Scott closed sets are called Scott open sets, which form a topology $\sigma(P)$, called the Scott topology on P. The space $(P, \sigma(P))$ is sometimes denoted by ΣP.

A subset A of a topological space is irreducible if whenever $A \subseteq F_1 \cup F_2$ with F_1 and F_2 closed, then $A \subseteq F_1$ or $A \subseteq F_2$ holds. The set of all nonempty irreducible closed subsets of space X will be denoted by $Irr(X)$.

For any T_0 topological space (X, τ), the specialization order $\leq_\tau$ on X is defined by $x \leq_\tau y$ iff $x \in cl(\{y\})$, where "$cl(\cdot)$" means taking closure.

Remark 2.1 (1) For any topological space X, $(Irr(X), \subseteq)$ is a dcpo. If $\mathcal{D}$ is a directed subset of $Irr(X)$, the supremum of $\mathcal{D}$ in $(Irr(X), \subseteq)$ equals $cl(\bigcup \mathcal{D})$ (the closure of $\bigcup \mathcal{D}$), which is the same as the supremum of $\mathcal{D}$ in the complete lattice of all closed sets of X.

(2) For any $x \in X$, $cl(\{x\}) \in Irr(X)$. A T_0 space X is called sober if $Irr(X) = \{cl(\{x\}) : x \in X\}$, that is, every nonempty irreducible closed set is the closure of a point.

(3) Assume that (X, τ) and (Y, η) are topological spaces such that the open set lattices $(\tau, \subseteq)$ and $(\eta, \subseteq)$ of X and Y are isomorphic, then the closed set lattice $(C(X), \subseteq)$ of X and the closed set lattice $(C(Y), \subseteq)$ of Y are also isomorphic (they are dual of the corresponding open set lattices). Since irreducibility is a lattice-intrinsic property, it follows that the dcpos $(Irr(X), \subseteq)$ and $(Irr(Y), \subseteq)$ are isomorphic.

(4) For any poset P, the specialization order determined by the Scott topology on P equals the original partial order on P. Hence two poset P and Q are isomorphic iff their Scott spaces are homeomorphic.

For a T_0 space X, a soberification of X is a sober space Y together with a continuous mapping $\eta_X : X \to Y$, such that for any continuous mapping $f : X \to Z$ with Z sober, there is a unique continuous mapping $\hat{f} : Y \to Z$ such that $f = \hat{f} \circ \eta_X$. The sobrification of a T_0 space is unique up to homeomorphism.

Remark 2.2 The following facts about sober spaces and sobrifications are well-known.

(1) If Y is a sober space, then Y is a sobrification of a T_0 space X iff the closed set lattice $C(X)$ of X is isomorphic to the closed set lattice $C(Y)$ of Y (Equivalently, the open set lattice of Y is isomorphic to that of X). From this fact, one deduces easily that if X and Y are both sober spaces and the lattice $C(X)$ is isomorphic to the lattice $C(Y)$, then X and Y are homeomorphic.

(2) The set $Irr(X)$ of all nonempty closed irreducible sets of a T_0 space X equipped with the hull-kernel topology is a sobrification of X, where the mapping $\eta_X : X \to Irr(X)$ is defined by $\eta_X(x) = cl(\{x\})$ for all $x \in X$. The closed sets of the hull-kernel topology consists of all sets of the form $h(A) = \{F \in Irr(X) : F \subseteq A\}$ (A is a closed set of X).

See Exercise V-4.9 of [2] for details (where the topology was given by means of open sets).

A T_0 space X will be called Scott sobrifiable if there is a dcpo P such that ΣP is the sobrification of X.

The specialization order on the space $Irr(X)$ (with the hull kernel topology) equals the inclusion order of sets. From the above, we can easily deduce the following fact.

Remark 2.3 A T_0 space (X, τ) is Scott sobrifiable iff for any Scott closed set $\mathcal{F}$ of the dcpo $Irr(X)$, there is a closed set A of X such that $\mathcal{F} = h(A)$, where $h(A) = \{F \in Irr(X) : F \subseteq A\}$.

Definition 2.4 [6] Let P be a poset and $E \subseteq P$ be nonempty. A subset F of P is *E-dominated* if there is $e \in E$ such that $F \subseteq \downarrow e$. A family $\mathcal{F}$ of subsets of P is said to be *E-dominated* if every $F \in \mathcal{F}$ is E-dominated. A poset P is called *dominated*, if for all $E \in Irr(P)$ and all directed E-dominated family $\mathcal{D} \subseteq Irr(P)$, the set $\bigcup \mathcal{D}$ is E-dominated. A dcpo P is said to be locally dominated if every principal ideal in P is dominated.

Since every irreducible Scott closed set in a principal ideal of a dcpo P is itself an irreducible Scott closed set in P, it is easy to see that every dominated dcpo is locally dominated. The example in Proposition 4.10 in the sequel shows that the converse is not true.

Let $\mathbb{J}$ be the dcpo constructed by Johnstone in [9] (We follow Goubault-Larrecq to use this symbol for Johnstone's dcpo). It is easy to see that $\mathbb{J}$ is dominated. In [17], it was proved that $\mathbb{J}$ is C_σ-unique. We will show that a C_σ-unique dcpo may not be dominated.

3 Some sufficient conditions for C_σ-unique dcpos

In this section, we give some new sufficient conditions for a dcpo to be C_σ-unique.

Definition 3.1 Let P be a dcpo. An element $x \in P$ is called a C_σ-unique element if the principal ideal $\downarrow x$ is C_σ-unique. If every element of P is a C_σ-unique element, then we call P a locally C_σ-unique dcpo.

A T_0 space X is called bounded-sober if every nonempty upper bounded (with respect to the specialization order on X) closed irreducible subset of the space is the closure of a point [15]. Every sober space is bounded-sober, the converse implication is not true. It is easy to see that, for a dcpo P, ΣP is bounded sober iff every principal ideal of P is sober. In this case, we call P a bounded sober dcpo.

The proof of the following proposition is similar to that of Theorem 3.9 in [17].

Proposition 3.2 *Let P be a bounded sober dcpo whose every element is a directed sup of C_σ-unique elements. Then P is C_σ-unique.*

Corollary 3.3 *Let P be a dcpo whose every principal ideal is quasicontinuous. Then P is C_σ-unique.*

Proof. Note that the Scott space of every quasicontinuous dcpo is sober and every quasicontinuous dcpo is C_σ-unique. By the assumption on P, we deduce that every principal ideal of P is sober, hence P is bounded sober. Also every element of P is C_σ-unique. By Proposition 3.2, it follows that P is C_σ-unique. □

Remark 3.4 A dcpo whose every principal ideal is quasicontinuous is also called a locally quasicontinuous dcpo. A dcpo model of a T_1 topological space X is a dcpo

P such that the maximal point space $\mathrm{Max}(P)$ (the set of all maximal points of P equipped with the relative Scott topology) is homeomorphic to X. In one of their recent papers, Zhao and Xi established the remarkable theorem [16, Theorem 2] that every T_1 space has a dcpo model which is locally quasialgebraic (hence locally quasicontinuous). This result shows that there are plenty of locally quasicontinuous dcpos linked to T_1 topological spaces. By Corollary 3.3, one can now deduce that every T_1 space has a dcpo model that is C_σ-unique.

Proposition 3.5 *Every principal ideal in $\mathbb{J}$ is quasicontinuous.*

Proof. Let $x \in \mathbb{J}$. If x is not a maximal element, then $\downarrow x$ is a finite chain, hence a continuous dcpo. If x is maximal, then $\downarrow x$ is isomorphic to a dcpo of the form $P = \mathbb{N} \cup B \cup \{T\}$, where B is a countable union of disjoint finite chains, $\mathbb{N}$ has the order of natural numbers, the elements in $\mathbb{N}$ are not comparable with elements in B and T is the largest element.

For every $a \in P$, let $F_a^n = \{a, n\}(n = 1, 2, \cdots)$. We see that for all directed set $\{d_i\}_{i\in I}$, $\bigvee_{i\in I} d_i \geqslant a$ implies that there is some i_0 such that $d_{i_0} \geqslant a$ or $d_{i_0} \geqslant n$. This shows that $F_a^n \ll a$. The family $\{F_a^n\}$ is clearly directed and $\bigcap \uparrow F_a^n = \uparrow a$. So, $\downarrow x$ is quasicontinuous. Thus every principal ideal in $\mathbb{J}$ is quasicontinuous. □

By Corollary 3.3 and Proposition 3.5, we deduce the following.

Corollary 3.6 (1) *The non-sober dcpo $\mathbb{J}$ is C_σ-unique.*

(2) *A locally quasicontinuous dcpo need not be a quasicontinuous dcpo.*

Theorem 3.7 *The product poset $\mathbb{J} \times \mathbb{J}$ is C_σ-unique.*

Proof. Every principal ideal in $\mathbb{J} \times \mathbb{J}$ is isomorphic to the product of two principal ideals in $\mathbb{J}$. By [14], any finite product of quasicontinuous domains is quasicontinuous, hence every principal ideal in $\mathbb{J} \times \mathbb{J}$ is quasicontinuous. Therefore $\mathbb{J} \times \mathbb{J}$ is C_σ-unique. □

Remark 3.8 (1) It is known that a product of two sober spaces is sober space. However, it is not known whether the product of two sober dcpos is a sober dcpo. Sober dcpos need not be C_σ-unique. One example is the dcpo $\hat{\mathcal{H}}$ constructed in [6].

(2) Every bounded sober dcpo is dominated.

Definition 3.9 Let P be a dcpo. An element $x \in P$ is called a *q.c.*-element if the ideal $\downarrow x$ is quasicontinuous.

It is easy to see that a dcpo P is a locally quasicontinuous dcpo iff every element of P is a *q.c.*-element. By Corollary 3.3, every locally quasicontinuous dcpo is C_σ-unique.

Example 3.10 There exists a dcpo P whose all elements are directed sup of continuous elements (x is call continuous if $\downarrow x$ is continuous), but P is not locally quasicontinuous. One such an example is the dcpo of infinite many parallel unit intervals with their top elements being pasted as one top element.

By the proof of Theorem 3.7, one can easily deduce the following remark.

Remark 3.11 (i) A finite product of locally quasicontinuous dcpos is still a locally quasicontinuous dcpo.

(ii) Every nonempty Scott closed subset of a locally quasicontinuous dcpo is a locally quasicontinuous dcpo.

Definition 3.12 An element x in a dcpo D is called a $w^1q.c.$-element (resp., w^1C_σ-unique element) if x is a directed sup of some $q.c.$-elements (resp., C_σ-unique elements). Inductively, for $n > 1$, an element x in a dcpo D is called a $w^nq.c.$-element (resp., w^nC_σ-unique element) if x is a directed sup of some $w^{n-1}q.c.$-elements (resp., $w^{n-1}C_\sigma$-unique elements).

Theorem 3.13 *Let P be a bounded sober dcpo. If every element of P is a w^nC_σ-unique element, then P is a C_σ-unique dcpo.*

Proof. Applying Proposition 3.2 and mathematical induction, one can deduce that every element of P is C_σ-unique. The conclusion then follows from that P is bounded sober and Proposition 3.2. □

Clearly, in a bounded sober dcpo P, every $w^nq.c.$-element is a w^nC_σ-unique element. We immediately have the following

Corollary 3.14 *Let P be a bounded sober dcpo. If every element of P is a $w^nq.c.$-element, then P is a C_σ-unique dcpo.*

4 A C_σ-unique dcpo which is not dominated

An element x of a lattice L is called $\vee$-irreducible (or just irreducible) if for any elements $a, b \in L$, $x = a \vee b$ implies $x = a$ or $x = b$. In a distributive lattice L (such as a topology), an element x is $\vee$-irreducible iff for any elements $a, b \in L$, $x \le a \vee b$ implies $x \le a$ or $x \le b$. Let $Irr(L)$ be the set of all irreducible elements of L. Then with the inherited order from L, $Irr(L)$ is a poset.

Remark 4.1 (1) Note that irreducibility is an intrinsic order property. If $f : L \to M$ is an order isomorphism between lattices, then f restricts to an isomorphism $\overline{f} : Irr(L) \to Irr(M)$.

(2) For any T_0 topological space X, let $Irr(X)$ denote the set of all nonempty members of $Irr(C(X))$ ($C(X)$ is the distributive complete lattice of all closed sets of X). If $F : C(X) \to C(Y)$ is an order isomorphism, then F restricts to an order isomorphism between $Irr(X)$ and $Irr(Y)$.

(3) If $B \in Irr(X)$ is a finite subset, then $B = cl(\{x\})$ for some $x \in X$. In fact, assume that $B = \{b_1, b_2, \cdots, b_n\}$. Then $B = \bigcup\{cl(\{b_i\}) : i = 1, 2, \cdots, n\}$, thus $B = cl(\{b_{i_0}\})$ for some b_{i_0}.

Recall that a T_0 space is called a d-space (or monotone convergence space) if for any directed subset $D \subseteq X$ (with respect to the specialization order on X), $\sup D$ exists and D (as a net) converges to $\sup D$.

Lemma 4.2 *Let X and Y be T_0 space and $F : C(X) \to C(Y)$ be an order isomorphism.*

(a) For any $x \in X$, if $cl(\{x\})$ is a finite set, then $F(cl(\{x\})) = cl(\{y\})$ for some $y \in Y$.

(b) If X and Y are d-spaces and $cl(\{x\})$ is a chain (with respect to the specialization order), then $F(cl(\{x\})) = cl(\{y\})$ for some $y \in Y$.

Proof. (a) Assume that $B = cl(\{x\})$ is a finite set. Then $\downarrow B = \{A \in C(X) : A \subseteq cl(\{x\})\}$ is a finite set. As F is an isomorphism, $F(\downarrow B) =\downarrow F(B)$ holds, so $\downarrow F(B)$ is a finite set. Now $\{cl(\{y\}) : y \in F(B)\} \subseteq\downarrow F(B)$, so it is a finite set. This implies that $F(B)$ is a finite set (note that in a T_0 space, different elements have different closures). By Remarks 4.1 (3), it follows that $F(B) = cl(\{y\})$ for some $y \in Y$.

(b) Let $H \in C(X)$, $H \neq \emptyset$ and $H \subseteq cl(\{x\})$. Then H is a chain, $\sup H$ exists and H converges to $\sup H$. Since H is closed, $\sup H \in H$, which implies that $H = cl(\{supH\})$. Hence the set $\downarrow_{C(X)} cl(\{x\}) - \{\emptyset\} = \{cl(\{u\}) : u \in cl(\{x\})\}$ is a chain.

Clearly F restricts to an isomorphism between $\downarrow_{C(X)} cl(\{x\}) - \{\emptyset\}$ and $\downarrow_{C(Y)} F(cl(\{x\})) - \{\emptyset\}$. Hence $\downarrow_{C(Y)} F(cl(\{x\}))$ is a chain. In particular, $F(cl(\{x\}))$ is a chain (with respect to the specialization order on Y). Again as $F(cl(\{x\}))$ is a closed set and Y is a d-space, we have $F(cl(\{x\})) = cl(\{y\})$ for some y. □

Corollary 4.3 *Let P and Q be dcpos such that there is an order isomorphism $F : C_\sigma(P) \to C_\sigma(Q)$. If a principal ideal $\downarrow x \subseteq P$ is a chain, then $F(\downarrow x) =\downarrow y$ for some $y \in Q$. Further more, if $\downarrow x$ is a finite set, then $\downarrow y$ is also a finite set.*

The following are some more properties of $\mathbb{J}$ which can be verified easily based on the construction of $\mathbb{J}$.

Remark 4.4 (1) The nonempty irreducible sets in $\mathbb{J}$ are exactly the principal ideals and $\mathbb{J}$ itself [6].

(2) If $D \subseteq \mathbb{J} - \mathrm{Max}(\mathbb{J})$ is directed, then there is $(n, \infty) \in \mathbb{J}$ such that $D \subseteq\downarrow (n, \infty)$, hence D is a chain.

(3) If a subset F contains infinitely many points in $\mathrm{Max}(\mathbb{J})$, then the Scott closure of F equals $\mathbb{J}$.

(4) If $D \subseteq \mathbb{J} - \mathrm{Max}(\mathbb{J})$ is Scott closed, then $D = \bigcup_{i\in I} \downarrow (i, m_i)$ is a union of incomparable chains for some $I \subseteq \mathbb{N}$ with $m_i \in \mathbb{N}$ for all $i \in I$. Conversely, if $D = \bigcup_{i\in I} \downarrow (i, m_i)$ is a union of incomparable chains for some $I \subseteq \mathbb{N}$ with $m_i \in \mathbb{N}$ for all $i \in I$, then $D \subseteq \mathbb{J} - \mathrm{Max}(\mathbb{J})$ is Scott closed.

Using the dcpo $\mathbb{J}$, we now construct a dcpo $L\mathbb{J}$ which is C_σ-unique but not dominated.

Example 4.5 Let $L\mathbb{J} = \{(n, m, h) : n, h \in \mathbb{N}, m \in \mathbb{N} \cup \{\infty\}\} \cup \{(n, \infty, \infty) : n \in \mathbb{N}\}$. Define the relation " $\leqslant$ " on $L\mathbb{J}$ by

$(n, m, h) \leqslant (s, t, k)$ if

(i) $h = k$ and $(n, m) \leqslant (s, t)$ holds in $\mathbb{J}$, or

(ii) $h < k$ and $t = \infty$ and $(n, m) \leqslant (s, t)$ in $\mathbb{J}$, or

(iii) $h < k = t = \infty$ and $n \leq s$ and $(n, m) \leqslant (s, t)$ in $\mathbb{J}$.

Then it is straightforward to verify that the relation $\leq$ is a partial order on $L\mathbb{J}$ and $(L\mathbb{J}, \leq)$ is a dcpo, called stratified Johnstone dcpo.

In $L\mathbb{J}$, for each $h \in \mathbb{N}$, the subdcpo $L_h = \{(n, m, h) : n \in \mathbb{N}, m \in \mathbb{N} \cup \{\infty\}\}$ will be called the h-th layer of $L\mathbb{J}$.

It is routine to verify the following:

(1) $\{(n, \infty, \infty) : n \in \mathbb{N}\} = \mathrm{Max}(L\mathbb{J})$, the set of maximal points of $L\mathbb{J}$.

(2) For each $i \in \mathbb{N}$, the i-th layer $L_i \cong \mathbb{J}$ is a copy of $\mathbb{J}$ (it is isomorphic to $\mathbb{J}$ via the mapping $(n, m, h) \mapsto (m, n)$) and $\mathrm{Max}(L_i) = \{(n, \infty, i) : n \in \mathbb{N}\}$.

A point in the set $fin(L_i) = L_i - \mathrm{Max}(L_i)$ will be called a finite-point. It is easy to see that finite-points in different levels are incomparable in $L\mathbb{J}$. So for every set $A \subseteq fin(L_i)$, one has also $\downarrow A \subseteq fin(L_i)$.

(3) If $t < s$ then $\downarrow L_t \subseteq \downarrow L_s$. As a matter of fact,

$$\downarrow L_s = \bigcup \{L_t : t \leq s\}.$$

(4) For any t, $\downarrow L_t$ is Scott closed.

(5) The subposet $L\mathbb{J}^{\infty} = \{(n, m, h) | m = \infty \text{ or } h = \infty\}$ of non finite elements in $L\mathbb{J}$ is isomorphic to $\mathbb{J}$.

We will show that the dcpo $L\mathbb{J}$ is C_σ-unique.

Firstly, we show that $Irr(L\mathbb{J}) = \{\downarrow x : x \in L\mathbb{J}\} \cup \{\downarrow L_h : h \in \mathbb{N} \cup \{\infty\}\}$. We will derive this by proving several lemmas and propositions.

Lemma 4.6 *If $F \in Irr(L\mathbb{J})$ and $F \subseteq \downarrow L_t$ for some $t \in \mathbb{N}$, then F is either a principal ideal or $F = \downarrow L_h$ for some $h \leqslant t$.*

Proof. Choose the largest $h \in \mathbb{N}$ such that $F \cap L_h \neq \emptyset$. Clearly, $h \leqslant t$ and $F \subseteq \downarrow L_h$.

(1) If $h = 1$, then by Remark 4.4(1) and $L_1 \cong \mathbb{J}$, we see that F is a principal ideal or $F = L_1$.

(2) Now let $h > 1$.

If $F \cap \mathrm{Max}(L_h)$ is infinite, then by the property of Scott closed set of $L\mathbb{J}$, we have $\mathrm{Max}(L_h) \subseteq F$, implying $F = \downarrow L_h$.

If $F \cap \mathrm{Max}(L_h)$ is a nonempty finite set, then

$$F = \downarrow \mathrm{Max}(F) \subseteq \downarrow (F \cap \mathrm{Max}(L_h)) \cup \downarrow L_{h-1} \cup \downarrow (\mathrm{Max}(F) \cap L_h - \mathrm{Max}(L_h)).$$

Noticing that $\mathrm{Max}(F) \cap L_h - \mathrm{Max}(L_h) \subseteq L_h - \mathrm{Max}(L_h) = fin(L_h)$, we see by Example 4.5(2) that

$$\downarrow (\mathrm{Max}(F) \cap L_h - \mathrm{Max}(L_h)) \subseteq fin(L_h).$$

Since F is Scott closed, it is easy to see by Remark 4.4(4) that $\downarrow (\mathrm{Max}(F) \cap L_h - \mathrm{Max}(L_h))$ is closed w.r.t. directed sups in $L\mathbb{J}$ and hence $\downarrow (\mathrm{Max}(F) \cap L_h - \mathrm{Max}(L_h))$ is Scott closed.

Since F is irriducible, and not contained in $\downarrow L_{h-1} \cup \downarrow (\mathrm{Max}(F) \cap L_h - \mathrm{Max}(L_h))$, we have

$$F \subseteq \downarrow (F \cap \mathrm{Max}(L_h)),$$

which further implies

$$F = \downarrow (F \cap \mathrm{Max}(L_h)).$$

Let $F \cap \mathrm{Max}(L_h) = \{x_1, x_2, \cdots, x_k\}$. Then $F = \bigcup\{\downarrow x_i : i = 1, 2, \cdots, k\}$, thus $F = \downarrow x_i$ for some i, hence a principle ideal.

Lastly, assume that $F \cap \mathrm{Max}(L_h) = \emptyset$. Then $F \subseteq \downarrow L_{h-1} \cup \downarrow (F \cap fin(L_h))$. As F is Scott closed, in this case, $\downarrow (F \cap fin(L_h)) = F \cap fin(L_h)$ is Scott closed. Since F is irreducible and $F \not\subseteq \downarrow L_{h-1}$, we have $F \subseteq F \cap fin(L_h)$ and $F = F \cap fin(L_h)$. Noticing that $F \cap fin(L_h)$ is also a Scott closed subset of $L_h \cong \mathbb{J}$ and contains no maximal elements of L_h, we have $F = F \cap fin(L_h)$ is an irreducible closed set of L_h. So, by Lemma 4.4(1), $F = \downarrow b$ for some $b \in L_h - \mathrm{Max}(L_h)$.

To sum up, we have shown that F is a principal ideal or $F = \downarrow L_h$ for some $h \leqslant t$. □

Proposition 4.7 *For any $h \in \mathbb{N} \cup \{\infty\}$, $\downarrow L_h$ is an irreducible Scott closed set.*

Proof. What we only need to show is that $\downarrow L_h$ is irreducible. To this end, let $F, G \in C_\sigma(L\mathbb{J})$ with $F \cup G \supseteq \downarrow L_h$. Then either F or G contains infinite many maximal points of $L_h \cong \mathbb{J}$. So by the closedness of F and G and the method of showing $\mathbb{J}$ is irreducible in itself, either F or G contains L_h. Noticing that F and G are lower sets, we see that either F or G contains $\downarrow L_h$, showing that $\downarrow L_h$ is irreducible. □

Proposition 4.8 *If $F \in Irr(L\mathbb{J})$, then $F \in \{\downarrow x : x \in L\mathbb{J}\} \cup \{\downarrow L_h : h \in \mathbb{N} \cup \{\infty\}\}$.*

Proof. Suppose that $F \in Irr(L\mathbb{J})$ and $F \neq L\mathbb{J} = \downarrow L_\infty$. Let $M_h = \{n : (n, \infty, h) \in F \cap \mathrm{Max}(L_h)\}$ for $h \in \mathbb{N}$, where $\mathrm{Max}(L_h)$ is the set of maximal points of the subdcpo L_h. Then $\{M_h\}_{h\in\mathbb{N}}$ is a decreasing family of subsets of $\mathbb{N}$. If for every $h \in \mathbb{N}$, $|M_h|$ (the number of elements in M_h) is infinite, then, by the closedness of F, we see that for all $h \in \mathbb{N}$, $\downarrow L_h \subseteq F$ and $F = L\mathbb{J}$, contradicting to the assumption that $F \neq L\mathbb{J}$. So, there is $s \in \mathbb{N}$ such that M_s is a finite subset of $\mathbb{N}$. Since $\{M_h\}_{h\geqslant s}$ is decreasing, there is $t \geqslant s$ such that $M_h = M_t$ for all $h \geqslant t$. If $M_t = \emptyset$, then $F \cap \mathrm{Max}(L_h) = \emptyset$ for all $h \geqslant t$ and by Example 4.5(2), $F \cap (\bigcup_{k\in\mathbb{N}} L_{t+k})$ is a lower set and closed w.r.t. directed sups in $L\mathbb{J}$, and thus $F \cap (\bigcup_{k\in\mathbb{N}} L_{t+k}) \in C_\sigma(L\mathbb{J})$. It follows from $F = (F \cap \downarrow L_t) \cup (F \cap (\bigcup_{j\in\mathbb{N}} L_{t+j}))$ and irreducibility of F that $F \subseteq \downarrow L_t$. By Lemma 4.6, F is a principal ideal or $F = \downarrow L_h$ for some $h \leqslant t$.

If $M_t \neq \emptyset$, then $F \cap L_\infty$ is a nonempty finite set of L_∞. Let $M_t = \{n_1, n_2, \cdots, n_k\}$ with $n_i \leqslant n_j$ iff $i \leqslant j$. Then $\mathrm{Max}(F) \cap L_\infty = F \cap L_\infty = \{(n_i, \infty, \infty) : i = 1, 2, \cdots, k\}$. Let $w = t \vee n_k$. Then in this case $F \subseteq \downarrow (F \cap L_\infty) \cup \downarrow L_w \cup \downarrow ((\mathrm{Max}(F) - L_\infty) \cap \bigcup_{w<h\in\mathbb{N}} L_h)$ which is a union of finite many Scott closed sets. By the irreducibility of F, we see that $F \subseteq \downarrow \{(n_i, \infty, \infty) : i = 1, 2, \cdots, k\}$ and thus is a principal ideal. To sum up, we have shown that F is a principal ideal or $F = \downarrow L_h$ for some $h \in \mathbb{N} \cup \{\infty\}$. □

Combining the above propositions, we deduce the following theorem.

Theorem 4.9 *In $L\mathbb{J}$, $Irr(L\mathbb{J}) = \{\downarrow x : x \in L\mathbb{J}\} \cup \{\downarrow L_h : h \in \mathbb{N} \cup \{\infty\}\}$.*

Proposition 4.10 *The dcpo $L\mathbb{J}$ is locally dominated but not dominated.*

Proof. It is easy to see that $\downarrow L_h \subseteq \downarrow (h+1, \infty, \infty)$. So, $\{\downarrow L_h : h \in \mathbb{N}\} \subseteq Irr(L\mathbb{J})$ is directed and $L\mathbb{J}$-dominated. However, $\bigcup\{L_h : h \in \mathbb{N}\}$ is not $L\mathbb{J}$-dominated, showing that $L\mathbb{J}$ is not a dominated dcpo. To show that $L\mathbb{J}$ is locally dominated, we need only show every principal ideal determined by a maximal point (n, ∞, ∞) in $L\mathbb{J}$ is dominated. In fact,

$$Irr(\downarrow (n, \infty, \infty)) = \{\downarrow x : x \in \downarrow (n, \infty, \infty)\} \cup \{\downarrow L_h : h = 1, 2, \cdots, n\}.$$

For every $E = \downarrow L_h$ with $h \leqslant n$, and any directed E-dominated family $\mathcal{D} \subseteq Irr(\downarrow (n, \infty, \infty))$, it is easy to see that $\mathcal{D}$ contains only principal ideals and the union $\bigcup \mathcal{D}$ is an ideal, which has a sup in $E = \downarrow L_h$ and thus is E-dominated. There is no need to check for $E = \downarrow x$. So, $\downarrow (n, \infty, \infty)$ is dominated. □

Theorem 4.11 *Let Q be a dcpo satisfying $C_\sigma(L\mathbb{J}) \cong C_\sigma(Q)$. Then $L\mathbb{J} \cong Q$.*

Proof. Let $F : C_\sigma(L\mathbb{J}) \cong C_\sigma(Q)$ be an isomorphism. We define inductively a monotone map $f : L\mathbb{J} \to Q$ first, and then show that f is an order isomorphism. We define $f : L\mathbb{J} \to Q$ by the following steps:

Step 1: For each element $(n, m, h) \in L_h$ with $m, h \neq \infty$, we see that $\downarrow (n, m, h)$ is a finite chain and by Corollary 4.3, there is $y_{(n,m,h)} \in Q$ such that $F(\downarrow (n, m, h)) = \downarrow y_{(n,m,h)}$. We let $f((n, m, h)) = y_{(n,m,h)}$. It is clear that f is order preserving when restricted to these elements.

Step 2: For each element $(n, \infty, h) \in L_h$ with $h \neq \infty$, we see that $(n, \infty, h) = \sup^{\uparrow}_{L\mathbb{J}}\{(n, m, h) : m \in \mathbb{N}\}$ and

$$F(\downarrow (n, \infty, h)) = \sup_{Irr(Q)} F(\{\downarrow (n, m, h) : m \in \mathbb{N}\} = \sup^{\uparrow}_{Irr(Q)}\{\downarrow y_{(n,m,h)} : m \in \mathbb{N}\}.$$

Let $y_{(n,\infty,h)} = \sup^{\uparrow}\{y_{(n,m,h)} : m \in \mathbb{N}\}$. Then $F(\downarrow (n, \infty, h)) = \downarrow y_{(n,\infty,h)}$. We let $f((n, \infty, h)) = y_{(n,\infty,h)}$. Note that if $(n, \infty, h) < (n, \infty, k)$, then

$$\downarrow (n, \infty, h) \subseteq \downarrow (n, \infty, k),$$
$$\downarrow y_{(n,\infty,h)} = F(\downarrow (n, \infty, h)) \leqslant F(\downarrow (n, \infty, k)) = F(\sup_{Irr(L\mathbb{J})} \downarrow (n, m, k)) = \sup_{Irr(Q)} \downarrow y_{(n,m,k)} = \downarrow y_{(n,\infty,k)}.$$

By considering the restriction of f to the elements in Step 1 and Step 2, we can see that f is monotone.

Step 3: For each element $(n, \infty, \infty) \in L_\infty$, we see that the set $(n, \infty, \infty) = \sup^{\uparrow}\{(n, \infty, h) : h \in \mathbb{N}\}$. So, $\{y_{(n,\infty,h)} : h \in N\}$ is directed. Let $y_{(n,\infty,\infty)} = \sup^{\uparrow}_{Q}\{y_{(n,\infty,h)} : h \in N\}$. Then $F(\downarrow (n, \infty, \infty)) = \downarrow y_{(n,\infty,\infty)}$. We let $f((n, \infty, \infty)) = y_{(n,\infty,\infty)}$.

We now have defined a map $f : L\mathbb{J} \to Q$ satisfying the condition that $F(\downarrow x) = \downarrow f(x) = \downarrow y_x$ for all $x \in L\mathbb{J}$. It is easy to see that f is well defined and is injective. It is also easy to see that

$$(n,m,h) \leqslant (s,t,k) \Longleftrightarrow f((n,m,h)) = y_{(n,m,h)} \leqslant f((s,t,k)) = y_{(s,t,k)}.$$

For the surjectivity of f, we first show that $f(L\mathbb{J})$ is a lower set of Q.

Let $y \in \downarrow f(L\mathbb{J})$. Then there is some $y_{(n,m,h)}$ in $f(L\mathbb{J})$ such that $y \leqslant y_{(n,m,h)}$. Then $F^{-1}(\downarrow y) \subseteq \downarrow (n,m,h)$ is an irreducible Scott closed set in $L\mathbb{J}$. So, by Theorem 4.9, $F^{-1}(\downarrow y)$ is either a principal ideal or $\downarrow L_s$ for some $s \in \mathbb{N}$. If $F^{-1}(\downarrow y)$ is a principal ideal $\downarrow x$, then $\downarrow y = F(\downarrow x)$ and $y = f(x) \in f(L\mathbb{J})$. If $F^{-1}(\downarrow y) = \downarrow L_s$ for some $s \in \mathbb{N}$, then $(p,r,1) \in \downarrow L_s = F^{-1}(\downarrow y)$ for all $p,r \in \mathbb{N}$. So, $y_{(p,\infty,1)} \leqslant y \leqslant y_{(n,m,h)}$ and thus $(p,\infty,1) \leqslant (n,m,h)$ for all $p \in \mathbb{N}$. This is impossible in $L\mathbb{J}$. Hence $F^{-1}(\downarrow y)$ must be a principal ideal and therefore $y = f(x)$ for some $x \in L\mathbb{J}$, showing that $f(L\mathbb{J})$ is a lower set of Q.

It is easy to show, by the definition f, that the set $f(L\mathbb{J})$ is closed with respect to directed sups. Thus $f(L\mathbb{J})$ is a Scott closed subset of Q.

Since F is an isomorphism between the lattices $C_\sigma(L\mathbb{J})$ and $C_\sigma(Q)$, we see that

$$\begin{aligned} Q = F(L\mathbb{J}) &= F(\sup_{C_\sigma(L\mathbb{J})} \{\downarrow x : x \in L\mathbb{J}\}) \\ &= \sup_{C_\sigma(Q)} \{F(\downarrow x) : x \in L\mathbb{J} \\ &= \sup_{C_\sigma(Q)} \{\downarrow f(x) : x \in L\mathbb{J}\}. \end{aligned}$$

For each $x \in L\mathbb{J}$, we have $\downarrow f(x) \subseteq f(L\mathbb{J})$ and $f(L\mathbb{J})$ is a Scott closed set of Q. It holds then that $\sup_{C_\sigma(Q)}\{\downarrow f(x) : x \in L\mathbb{J}\} \subseteq f(L\mathbb{J})$. Therefore $Q \subseteq f(L\mathbb{J})$, which implies $Q = f(L\mathbb{J})$. Hence f is also surjective. The proof is thus completed. □

A class $\mathcal{C}$ of dcpos is said to be maximal with regard Γ-faithfulness, if $\mathcal{C}$ is Γ-faithful and for all dcpo $D \notin \mathcal{C}$, the class $\mathcal{C} \cup \{D\}$ is not Γ-faithful.

Let DOMDCPO be the class of all dominated dcpos. It is known that DOMDCPO is Γ-faithful by [6].

Corollary 4.12 *The class DOMDCPO $\cup\{L\mathbb{J}\}$ is Γ-faithful. Hence the class DOMDCPO is not maximal with regard to Γ-faithfulness.*

Remark 4.13 (1) It is still quite remote from deriving a complete characterization of C_σ-unique dcpos. At the moment, we even do not have a meaningful necessary condition for C_σ-unique dcpos.

(2) Some very basic questions on the product of C_σ-unique dcpos are not yet to be answered. For example, it is not known whether the cartesian product of two C_σ-unique dcpos is C_σ-unique.

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