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Continuous Prequantale Models of T_1 Topological Semigroups [4]

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Abstract

In this paper, we show that every T_1 topological semigroup satisfying condition (Δ) can be embedded into a topological semigroup $(D, \sigma, \odot)$, where $(D, \sqsubseteq)$ is a domain. Furthermore, by considering the maximal point topological semigroup of a continuous prequantale, it is proven that every T_1 topological semigroup satisfying condition (Δ) has a continuous prequantale model, which may not be bounded complete.

Keywords: topological semigroup, prequantale, stable ordered semigroup, Scott topology, prequantale model.

1 Introduction

Dana Scott introduced the domain theory in order to provide the mathematical foundation for denotational semantics of programming languages [12]. After that, the domain theory has been a major impetus in the development of topological spaces. Thanks to the topological tools, the domain theory has developed very quickly. Indeed there are deep ties and interactions between the topological theory of topological spaces and the order theory of partially ordered sets. One aspect that many scholars are interested in is that the classical topological spaces are embedded into the set of the maximal points of the appropriate posets so as to use the domain theory to solve the problems about the topology.

Now the research for the maximal point space $\max(P)$ of the poset P has become one of the most central fields in domain theory. A poset model of a topological

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space X is a poset P such that the maximal point space $\max(P)$ is homeomorphic to X [7]. There have been many results which give necessary or sufficient conditions for topological spaces to have a poset model. Please refer to ([8,9,10,11]) if the reader wants to know more about poset models.

In [4], Kamimura and Tang characterized the spaces that have a bounded complete algebraic dcpo model with a countable base. Edalat and Heckmann [2] proved that every complete metric space has a domain model. Later, in [5], it was proven that every complete metric space has a bounded complete domain model. Lawson proved that a topological space is a Polish space if and only if it has an ω-continuous dcpo model satisfying the *Lawson condition* [7]. In [1], it was shown that any T_1 space has a continuous poset model. In ([3,14]), Erné and Zhao proved that every T_1 space has a bounded complete algebraic poset model, respectively. In [15], Zhao and Xi revealed that every T_1 space has a directed complete poset model (in fact, a locally quasialgebraic dcpo model), and verified that the T_1 space is sober if and only if the dcpo model is sober with respect to the Scott topology. In [13], Bin Zhao et al. proposed the notions of maximal point topological semigroups and prequantale models of topological semigroups. In that paper, they introduced a condition (Δ) on topological semigroups and obtained a result that every T_1 topological semigroup satisfying condition (Δ) has a bounded complete algebraic prequantale model.

In this paper, we take advantage of this fact that every T_1 topological space has a continuous poset model, and define a new binary operation on the continuous poset to prove that every T_1 topological semigroup satisfying condition (Δ) has a continuous prequantale model. In particular, the continuous poset may not be bounded complete. Thus the bounded completeness is not a necessary condition for a prequantale to be the model of a T_1 topological semigroup. In addition, in this paper, the way to find the model of a T_1 topological semigroup is different from the method used by Bin Zhao et al..

2 Preliminaries

Now we will recall some basic notions on domain theory and topological theory to be used in the sequel.

A poset P is called *bounded complete* if every subset that is bounded above has a least upper bound. In particular, a bounded complete poset has a smallest element, the least upper bound of the empty set. A nonempty subset E of a poset P is *directed* if every two elements of E have an upper bound in E. A poset is called a *directed complete poset*, or *dcpo* for short, if every directed subset has a supremum.

For any subset A of an ordered set P, we denote $\downarrow A = \{x \in P \mid x \leq y$ for some $y \in A\}$ and $\uparrow A = \{x \in P \mid x \geq y$ for some $y \in A\}$. For any element $a \in P$, one simply writes $\downarrow a$ for $\downarrow\{a\}$ and $\uparrow a$ for $\uparrow\{a\}$. A subset X is called a *lower set (upper set)* if $X = \downarrow X$ ($X = \uparrow X$, respectively).

Definition 2.1 A subset U of a poset P is *Scott open* if

(i) it is an upper set, that is, $U = \uparrow U$;

(ii) for any directed subset E of P, $\bigvee E \in U$ implies $E \cap U \neq \emptyset$, whenever $\bigvee E$

exists.

All Scott open sets of a poset P form a topology on P, denoted by $\sigma(P)$ and called Scott topology on P. The space $(P, \sigma(P))$ is written as ΣP, called Scott space of P.

Let P be a poset. We say that x is *way below* y, in notation $x \ll y$, if for all directed subsets $E \subseteq P$ for which $\bigvee E$ exists, the relation $y \leq \bigvee E$ always implies the existence of $e \in E$ with $x \leq e$. Denote $\Downarrow x = \{y \in P \mid y \ll x\}$, $\Uparrow x = \{y \in P \mid x \ll y\}$. A poset P is called *continuous* if for any $x \in P$, $x = \bigvee^{\uparrow} \Downarrow x$, that means for all $x \in P$, the set $\Downarrow x$ is directed and its supremum is x. For any continuous poset P, the family $\{\Uparrow x \mid x \in P\}$ form a base for the Scott topology on P. A dcpo which is continuous as a poset will be called a *domain*.

A *basis* B of a poset P is a subset of P such that for each $x \in P$, $x = \bigvee^{\uparrow}(B \cap \Downarrow x)$. From this, we obtain an equivalent condition for continuous posets. A poset is continuous if and only if it has a basis.

Definition 2.2 A triple $(S, \leq, \cdot)$ is called an *ordered semigroup* if it satisfies:

(1) $(S, \leq)$ is a poset;

(2) $(S, \cdot)$ is a semigroup;

(3) For all elements x, y, z in S, $x \leq y$ implies $x \cdot z \leq y \cdot z$ and $z \cdot x \leq z \cdot y$.

Definition 2.3 ([13]) A triple $(P, \leq, \cdot)$ is called a *prequantale* if it satisfies:

(1) $(P, \leq)$ is a poset;

(2) $(P, \cdot)$ is a semigroup;

(3) For all directed subsets E of P with $\bigvee E$ existing, $a \cdot (\bigvee E) = \bigvee(a \cdot E)$ and $(\bigvee E) \cdot a = \bigvee(E \cdot a)$, where $a \cdot E = \{a \cdot e \mid e \in E\}$ and $E \cdot a = \{e \cdot a \mid e \in E\}$.

Note that in Definition 2.3, we do not ask that a prequantale is a dcpo.

A prequantale $(P, \leq, \cdot)$ is called *continuous (algebraic)*, if $(P, \leq)$ is continuous (algebraic).

Remark 2.4 It is easy to see that a prequantale is an ordered semigroup. Conversely, it is not true. For example, let P be the subset of the square $[0, 1]^2$ consisting of its interior $]0, 1[^2$ and the points $(0, 0) = \perp, (1, 1) = \top$. Then, obviously, $(P, \leq, \cdot)$ is an ordered semigroup, where $\leq$ is the pointwise order and $a \cdot b = a \wedge b$ for $a, b \in P$. Pick $E = \{\frac{1}{2}\} \times]0, 1[$, then E is directed and $\bigvee E = (1, 1) = \top$. For $x = (\frac{2}{3}, \frac{1}{2})$, $x \cdot (\bigvee E) = x = (\frac{2}{3}, \frac{1}{2})$, but $x \cdot E = \{\frac{1}{2}\} \times]0, \frac{1}{2}]$. So $\bigvee(x \cdot E) = (\frac{1}{2}, \frac{1}{2}) \neq x \cdot (\bigvee E)$. Hence $(P, \leq, \cdot)$ is not a prequantale.

3 Continuous Prequantale Models of Topological Semigroups

In this section, we will discuss the prequantale models of topological semigroups. Now we give some concepts which will be used afterwards.

Definition 3.1 A *topological semigroup* $(S, \tau, \cdot)$ consists of a semigroup $(S, \cdot)$ and a topology τ on the set S such that the mapping $f : S \times S \to S$ defined by $f(x, y) = x \cdot y$

is continuous when $S \times S$ is endowed with the product topology, that is, for each x and y in S and each open neighborhood W of $x \cdot y$, there exist open neighborhoods U of x and V of y such that $U \cdot V \subseteq W$, where $U \cdot V = \{u \cdot v \mid u \in U, v \in V\}$.

Example 3.2 (1) Let $L = \{\top, a, b\}$ be a poset, where a, b are incomparable and $\top$ is above a, b. Define a binary operation $*$ on L as follows:

$*$	a	b	$\top$
a	a	a	a
b	b	b	b
$\top$	a	b	$\top$

It is easy to verify that $(L, *)$ is a semigroup. The topology on L is the Scott topology and $\sigma(L) = \{\emptyset, L, \{a, \top\}, \{b, \top\}, \{\top\}\}$. For $a, b \in L$, $a * b = a \in \{a, \top\}$, but for all open neighborhoods U of a and V of b, $U * V \subseteq \{a, \top\}$ never happens. Hence $(L, \sigma, *)$ is not a topological semigroup.

(2) Let $M = \{\top, a, \bot\}$ be a poset with $\bot < a < \top$. Define a binary operation $\cdot$ on M as follows:

$\cdot$	$\bot$	a	$\top$
$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	a	a
$\top$	$\bot$	$\top$	$\top$

Then $(M, \cdot)$ is a semigroup and $\sigma(M) = \{\emptyset, M, \{a, \top\}, \{\top\}\}$. So one can easily see that $(M, \sigma, \cdot)$ is a topological semigroup.

Let $(S, \tau_1, \cdot)$ and $(T, \tau_2, \bullet)$ be topological semigroups. A mapping $f : (S, \tau_1, \cdot) \to (T, \tau_2, \bullet)$ is called an *isomorphism (embedding)*, if it is both a topological homeomorphism (embedding) and a semigroup homomorphism.

Definition 3.3 ([13]) Let $(P, \leq, *)$ be an ordered semigroup. The triple $(\max(P), \sigma \mid_{\max(P)}, *)$ is called a *maximal point topological semigroup* of P, if it satisfies the following conditions:

(1) $(\max(P), *)$ is a subsemigroup of P;

(2) $(\max(P), \sigma \mid_{\max(P)}, *)$ is a topological semigroup.

Definition 3.4 ([13]) Let $(S, \tau, \cdot)$ be a topological semigroup. A *prequantale model* of the topological semigroup $(S, \tau, \cdot)$ is a prequantale $(P, \leq, *)$ together with an isomorphism

$$\phi : (S, \tau, \cdot) \to (\max(P), \sigma \mid_{\max(P)}, *),$$

where $(\max(P), \sigma \mid_{\max(P)}, *)$ is the maximal point topological semigroup of P. We will use (P, ϕ) to denote a prequantale model of S.

Note that if $(P, \leq)$ is a continuous poset, then (P, ϕ) is called a *continuous prequantale model* of S.

Definition 3.5 ([13]) A topological semigroup $(S, \tau, \cdot)$ is said to satisfy *condition* (Δ), if $U \cdot V \in \tau$ holds for all U, V in τ.

Topological semigroups may not satisfy the condition (Δ) as is shown by the following example.

Example 3.6 (1) $(\mathbb{N}, \sigma, \times)$ is a topological semigroup, where $\times$ is the multiplication on the set of natural number $\mathbb{N}$, but it does not satisfy condition (Δ). For Scott open sets $U_1 = \{a \mid a \geq 2, a \in \mathbb{N}\}$, and $U_2 = \{b \mid b \geq 3, b \in \mathbb{N}\}$, $U_1 \times U_2 = \{a \times b \mid a \in U_1, b \in U_2\}$ implying $6 \in U_1 \times U_2$ but $7 \notin U_1 \times U_2$. So $U_1 \times U_2$ is not an upper set. Hence $U_1 \times U_2$ is not Scott open, which is equivalent to say that $(\mathbb{N}, \sigma, \times)$ does not satisfy condition (Δ).

(2) Let $T = \{\bot, a, b, \top\}$ be a poset, where a, b are incomparable, $\top$ is above a, b and $\bot$ is below a, b. The binary operation $\star$ on T is defined by :

$\star$	$\bot$	a	b	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	a	a	a
b	$\bot$	b	b	b
$\top$	$\bot$	$\top$	$\top$	$\top$

Obviously, $(T, \star)$ is a semigroup and $(T, \sigma, \star)$ is a topological semigroup. Clearly, $\{\top\}$ and T are Scott open sets, but $\{\top\} \star T = \{\bot, \top\}$ is not Scott open. Thus $(T, \sigma, \star)$ does not satisfy condition (Δ).

(3) We will show that it may change the result as the binary operation on a poset changes. Let T be the poset described in (2). The binary operation on T is defined by $\wedge$, in other words, the meet of each pair of elements in T. It is easy to see that $(T, \wedge)$ is a semigroup and $(T, \sigma, \wedge)$ is a topological semigroup. It is not hard to verify that for any $U, V \in \sigma(T)$, $U \wedge V \in \sigma(T)$, where $U \wedge V = \{u \wedge v \mid u \in U, v \in V\}$. Hence $(T, \sigma, \wedge)$ is a topological semigroup satisfying condition (Δ).

These examples presented in Example 3.6 (2), (3) show that even for the same poset, things can be quite different with different binary operation.

Remark 3.7 Let $(X, \tau, \cdot)$ be a topological semigroup. If $(X, \tau, \cdot)$ satisfies condition (Δ), then $(\mathcal{O}^*(X), \subseteq, \cdot)$ is an ordered semigroup, where $\mathcal{O}^*(X) = \{U \in \tau \mid U \neq \emptyset\}$ and for any $U, V \in \tau$, $U \cdot V = \{u \cdot v \mid u \in U, v \in V\}$.

The following theorem shows that for any T_1 topological space (X, τ) there exist a domain $(D, \sqsubseteq)$ and a topological embedding $f : X \to D$, but we do not know that what exactly the set $\max(D)$ is.

Theorem 3.8 ([1]) *Let (X, τ) be a T_1 topological space. Then there exist a domain $(D, \sqsubseteq)$ and a topological embedding $f : X \to D$, where D is equipped with the Scott topology.*

Let (X, τ) be a T_1 topological space. Set $D = D_0 \cup D_1$, where

$$D_0 = \{(U, n) \mid U \in \tau, U \neq \emptyset \text{ and } n \in \mathbb{N}\}$$

and

$$D_1 = \{\mathcal{F} \mid \mathcal{F} \in Filt(\mathcal{O}^*(X))\},$$

where

$$Filt(\mathcal{O}^*(X)) = \{\mathcal{F} \subseteq \mathcal{O}^*(X) \mid \mathcal{F} \text{ is a filter of } \mathcal{O}^*(X)\}.$$

For any $x \in X$, let $\mathcal{N}(x) = \{U \in \tau \mid x \in U\}$. Obviously, $\mathcal{N}(x)$ is in D_1.

Define the binary relation $\sqsubseteq$ on D as follows: for each $p_1, p_2 \in D$

$p_1 \sqsubseteq p_2$ if and only if one of the following holds:

(1) $p_1 = p_2$;

(2) $p_i = (U_i, n_i) \in D_0$ for $i = 1, 2$, $n_1 < n_2$ and $U_2 \subseteq U_1$;

(3) $p_1 = (U_1, n_1) \in D_0$, $p_2 = \mathcal{F} \in D_1$ and $U_1 \in \mathcal{F}$;

(4) $p_1, p_2 \in D_1$ and $p_1 \subseteq p_2$.

It is easy to verify that the relation $\sqsubseteq$ is a partial order on D. Note that for $p_1 \in D_1$ and $p_2 \in D_0$, the relation $p_1 \sqsubseteq p_2$ never occurs.

Then we have the following conclusions:

(1) D is a dcpo. For any directed subset E of D with E containing no maximal element of itself. If $E \subseteq D_0$, then $\bigvee E = \mathcal{F}$ generated by $\pi_1(E)$, $\pi_1(E) = \{U \in \tau \mid (U, n) \in E, n \in \mathbb{N}\}$. If $E \cap D_1 \neq \emptyset$, then $\bigvee E = \bigcup\{\mathcal{R} \mid \mathcal{R} \in E \cap D_1\}$.

(2) D is continuous. For all $p_0 \in D_0$, $p_0 \ll p_0$. For $p_1 \in D_1$,

$$\Downarrow p_1 = \{(U, n) \in D_0 \mid U \in p_1,\ n \in \mathbb{N}\}$$

is directed and its supremum is p_1. For any $q_1, q_2 \in D_1$, $q_1 \ll q_2$ never occurs.

(3) Define the function $f : (X, \tau) \to (D, \sigma)$ by $f(x) = \mathcal{N}(x)$. Then f is a topological embedding.

Unless explicitly stated otherwise, D always stands for the domain constructed in Theorem 3.8 in the following.

In order to discuss the embedding between topological semigroups, we will define a binary operation $\odot$ on D.

If $(X, \tau, \cdot)$ is a topological semigroup satisfying condition (Δ), then we can define a binary operation $\odot$ on D, for any $p_1, p_2 \in D$:

(1) If $p_i = (U_i, n_i) \in D_0$ for $i = 1, 2$, then $p_1 \odot p_2 = (U_1 \cdot U_2, n_1 + n_2)$.

(2) If $p_1 = (U_1, n_1) \in D_0$, $p_2 = \mathcal{F} \in D_1$, then $p_1 \odot p_2 = \{W \in \mathcal{O}^*(X) \mid U_1 \cdot V \subseteq W, V \in \mathcal{F}\}$ ($\uparrow\{U_1 \cdot V \mid V \in \mathcal{F}\}$ for short).

(3) If $p_1 = \mathcal{F} \in D_1$, $p_2 = (U_2, n_2) \in D_0$, then $p_1 \odot p_2 = \{W \in \mathcal{O}^*(X) \mid V \cdot U_2 \subseteq W, V \in \mathcal{F}\}$ ($\uparrow\{V \cdot U_2 \mid V \in \mathcal{F}\}$ for short).

(4) If $p_1, p_2 \in D_1$, then $p_1 \odot p_2 = \{W \in \mathcal{O}^*(X) \mid U \cdot V \subseteq W, U \in p_1, V \in p_2\}$ ($\uparrow\{U \cdot V \mid U \in p_1, V \in p_2\}$ for short).

The following proposition about the binary operation $\odot$ holds.

Proposition 3.9 *Let $(X, \tau, \cdot)$ be a topological semigroup and it satisfies condition (Δ), then $(D, \sqsubseteq, \odot)$ is a prequantale.*

Proof. Since $(X, \tau, \cdot)$ satisfies condition (Δ), one can easily see that $(D, \odot)$ is a semigroup. For any directed subset E in D, and $d \in D$, if E has a maximal element $\hat{e}$, then $\bigvee E = \hat{e}$. We can easily verify that $d \odot (\bigvee E) = \bigvee(d \odot E)$ and $(\bigvee E) \odot d = \bigvee(E \odot d)$. Now we consider the case where E has no maximal element.

(i) If $E \subseteq D_0$, then $\bigvee E = \mathcal{F}$ generated by $\pi_1(E)$.

If $d = (V, n) \in D_0$, then $(\bigvee E) \odot d = \uparrow\{U \cdot V \mid U \in \mathcal{F}\}$. $E \odot d = \{(W \cdot V, m+n) \mid (W, m) \in E\}$. $E \odot d$ is directed with no maximal element since E is directed. So $\bigvee(E \odot d) = \mathcal{F}'$ generated by $\pi_1(E \odot d)$. One can easily show that $\uparrow\{U \cdot V \mid U \in \mathcal{F}\} = \mathcal{F}'$ *i.e.* $(\bigvee E) \odot d = \bigvee(E \odot d)$.

If $d = \mathcal{E} \in D_1$, then $(\bigvee E) \odot d = \mathcal{F} \odot \mathcal{E} = \uparrow\{U \cdot V \mid U \in \mathcal{F}, V \in \mathcal{E}\}$.

$$E \odot d = E \odot \mathcal{E} = \{\uparrow\{W \cdot V \mid V \in \mathcal{E}\} \mid W \in \pi_1(E)\} \subseteq D_1$$

and $E \odot d$ is directed from the fact that E is directed. So

$$\bigvee(E \odot d) = \bigcup\{\uparrow\{W \cdot V \mid V \in \mathcal{E}\} \mid W \in \pi_1(E)\}.$$

We claim that $(\bigvee E) \odot d = \bigvee(E \odot d)$. For any element $A \in (\bigvee E) \odot d$, there exist $U_1 \in \mathcal{F}$, $V_1 \in \mathcal{E}$ such that $U_1 \cdot V_1 \subseteq A$. From $U_1 \in \mathcal{F}$, we can achieve that there exists $W_1 \in \pi_1(E)$ such that $W_1 \subseteq U_1$, implying $W_1 \cdot V_1 \subseteq U_1 \cdot V_1 \subseteq A$. Then $A \in \bigvee(E \odot d)$. Conversely, for any $B \in \bigvee(E \odot d)$, there exists $W_0 \in \pi_1(E)$ such that $B \in \uparrow\{W_0 \cdot V \mid V \in \mathcal{E}\}$, which implies that there is an element V_0 of $\mathcal{E}$ such that $W_0 \cdot V_0 \subseteq B$. Also $B \in (\bigvee E) \odot d$ due to $W_0 \in \mathcal{F}$. Therefore, $(\bigvee E) \odot d = \bigvee(E \odot d)$.

(ii) If $E \cap D_1 \neq \emptyset$, then $\bigvee E = \bigcup\{\mathcal{R} \mid \mathcal{R} \in E \cap D_1\}$.

If $d = (V, n) \in D_0$, then $(\bigvee E) \odot d = \uparrow\{U \cdot V \mid U \in \bigvee E\}$ and

$$E \odot d = \{(W \cdot V, m+n) \mid (W, m) \in E \cap D_0\} \cup \{\uparrow\{U \cdot V \mid U \in \mathcal{R}\} \mid \mathcal{R} \in E \cap D_1\}.$$

Then $(E \odot d) \cap D_1 \neq \emptyset$ and $E \odot d$ is directed since E is directed. Let

$$\{\uparrow\{U \cdot V \mid U \in \mathcal{R}\} \mid \mathcal{R} \in E \cap D_1\} = \mathcal{A}.$$

Then $\bigvee(E \odot d) = \bigcup\{\mathcal{R}' \mid \mathcal{R}' \in \mathcal{A}\}$. We should verify that $(\bigvee E) \odot d = \bigvee(E \odot d)$. For any $W \in (\bigvee E) \odot d$ there exists $U \in \bigvee E$ such that $U \cdot V \subseteq W$. Then there is an element $\mathcal{R}$ in $E \cap D_1$ such that $U \in \mathcal{R}$. So $W \in \uparrow(U \cdot V)$ for $U \in \mathcal{R} \in E \cap D_1$. Hence $W \in \bigvee(E \odot d)$. For the reverse, for any element W in $\bigvee(E \odot d)$, there exists $\mathcal{R}' \in \mathcal{A}$ such that $W \in \mathcal{R}'$. Then there is an element $U \in \mathcal{R} \in E \cap D_1$ such that $\mathcal{R}' = \uparrow(U \cdot V)$. That is, $W \in \uparrow(U \cdot V)$ for $U \in \bigvee E$, implying $W \in (\bigvee E) \odot d$. It thus follows that $(\bigvee E) \odot d = \bigvee(E \odot d)$.

If $d = \mathcal{F} \in D_1$, then $(\bigvee E) \odot d = \uparrow\{U \cdot V \mid U \in \bigvee E, V \in \mathcal{F}\}$. Let $\mathcal{E} \in E \cap D_1$. Since E is directed, for any $(W, n) \in E \cap D_0$, there is an $\mathcal{E}_1$ in E such that $\mathcal{E}, (W, n) \sqsubseteq \mathcal{E}_1$. Obviously, $\mathcal{E}_1$ is in D_1, so $W \in \mathcal{E}_1$, which implies that

$$\begin{aligned} E \odot d &= \{\uparrow\{W \cdot V \mid V \in \mathcal{F}\} \mid (W, n) \in E \cap D_0\} \cup \{\uparrow\{U \cdot V \mid U \in \mathcal{R}, V \in \mathcal{F}\} \mid \mathcal{R} \in E \cap D_1\} \\ &= \{\uparrow\{U \cdot V \mid U \in \mathcal{R}, V \in \mathcal{F}\} \mid \mathcal{R} \in E \cap D_1\}. \end{aligned}$$

So $\bigvee(E \odot d) = \bigcup\{\mathcal{F}' \mid \mathcal{F}' \in E \odot d\}$. We can easily prove that $(\bigvee E) \odot d = \bigvee(E \odot d)$.

Using a similar argument, we can deduce that $d \odot (\bigvee E) = \bigvee(d \odot E)$ for any directed subset E. It thus follows that $(D, \sqsubseteq, \odot)$ is a prequantale. □

The following corollary can be obtained easily.

Corollary 3.10 *For any topological semigroup $(X, \tau, \cdot)$ satisfying condition (Δ), then $(D, \sqsubseteq, \odot)$ is an ordered semigroup.*

Our main aim is to show that $(D, \sigma, \odot)$ is a topological semigroup. In [13], the authors provided an approach using stable ordered semigroups to prove that ordered semigroups with the Scott topology are topological semigroups. Before we move on to prove the topological semigroup, we recall some knowledge of the auxiliary relation and stable ordered semigroups.

Definition 3.11 ([6]) A binary relation $\prec$ on a poset $(L, \leq)$ is called an *auxiliary relation* if it satisfies the following conditions for all $p, q, x, y \in L$:

(1) $x \prec y$ implies $x \leq y$;

(2) $p \leq x \prec y \leq q$ implies $p \prec q$;

(3) If F is a finite subset of L, $F \prec y$ (which we take to abbreviate the fact that $a \prec y$ for all $a \in F$) implies that there exists $r \in L$ such that $F \prec r \prec y$.

For the sake of convenience, let $\Uparrow p = \{x \in L \mid p \prec x\}$, and $\Downarrow p = \{x \in L \mid x \prec p\}$.

We call an auxiliary relation $\prec$ on a poset L *approximating* if for all $p, q \in L$, $\Downarrow p \subseteq \Downarrow q$ if and only if $p \leq q$. We can easily see that if L is a continuous poset, then the way-below relation $\ll$ is an approximating auxiliary relation on L.

We call the topology generated by the set $\{\Uparrow p \mid p \in L\}$ the *pseudo-Scott topology* and denote it by $\mathbf{P}\sigma$.

Definition 3.12 ([13]) Let $(S, \leq, \cdot)$ be an ordered semigroup. An auxiliary relation $\prec$ on S is called *stable*, if it satisfies the following conditions for all $x_1, x_2, y_1, y_2 \in S$:

(1) $x_1 \prec y_1$ and $x_2 \prec y_2$ imply $x_1 \cdot x_2 \prec y_1 \cdot y_2$;

(2) $x \prec y_1 \cdot y_2$ implies that there exist $x_1 \prec y_1$, $x_2 \prec y_2$ such that $x \leq x_1 \cdot x_2$.

We call the quadruple $(S, \leq, \cdot, \prec)$ a *stable ordered semigroup*, if the auxiliary relation $\prec$ on S is stable.

Example 3.13 (1) Clearly, $((0, 1], \leq, \times)$ is a continuous prequantale and the way-below relation $\ll$ on $(0, 1]$ is stable, where $\times$ is the multiplication on the set $(0, 1]$ and $a \ll b$ if and only if $a < b$ for $a, b \in (0, 1]$.

(2) Let $L = \{a_i \mid i \in \mathbb{N}^+\} \cup \{a, b_1, b_2, c_1, c_2, \top\}$. The partial order $\leq$ on L is given by:

$$a_1 < a_2 < a_3 \cdots < a < b_i < c_i < \top \ (i = 1, 2),$$

(See Fig. 1) Then $(L, \leq)$ is a continuous lattice. It is easy to prove that $(L, \leq, \wedge)$ is an ordered semigroup, where the binary operation $\wedge$ on L means that the meet of each pair of elements in L. Obviously, $b_1 \ll c_1$, $b_2 \ll c_2$, and $b_1 \wedge b_2 = a$, $c_1 \wedge c_2 = a$, but $a \ll a$ does not occur, implying that the way below relation $\ll$ as an auxiliary relation on L is not stable. So $(L, \leq, \wedge, \ll)$ is not a stable ordered semigroup.

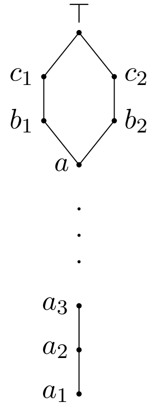

Fig. 1. continuous lattice L

Proposition 3.14 ([13]) *The stable ordered semigroup $(S, \leq, \cdot, \prec)$ endowed with the pseudo-Scott topology $\boldsymbol{P}\sigma$ is a topological semigroup.*

Proposition 3.15 *For any topological semigroup $(X, \tau, \cdot)$ satisfying condition (Δ), $(D, \sqsubseteq, \odot, \ll)$ is a stable ordered semigroup.*

Proof. For $p_1, p_2, q_1, q_2 \in D$, if $p_1 \ll q_1$, $p_2 \ll q_2$, by the way-below relation on D, p_1 is of the form (U_1, n_1) and p_2 is of the form (U_2, n_2). So $p_1 \odot p_2 = (U_1 \cdot U_2, n_1 + n_2)$. For q_1, q_2 we consider the following cases.

(*i*) $q_1 = (V_1, m_1), q_2 = (V_2, m_2) \in D_0$. Then $q_1 \odot q_2 = (V_1 \cdot V_2, m_1 + m_2)$. From the way-below relation on D_0, we know that $V_1 \subseteq U_1, n_1 < m_1$ and $V_2 \subseteq U_2, n_2 < m_2$. Since $(\mathcal{O}^*(X), \subseteq, \cdot)$ is an ordered semigroup, $V_1 \cdot V_2 \subseteq U_1 \cdot U_2$ and in addition, $n_1 + n_2 < m_1 + m_2$, implying

$$(U_1 \cdot U_2, n_1 + n_2) \sqsubseteq (V_1 \cdot V_2, m_1 + m_2).$$

Because $(V_1 \cdot V_2, m_1 + m_2) \ll (V_1 \cdot V_2, m_1 + m_2)$ then $(U_1 \cdot U_2, n_1 + n_2) \ll (V_1 \cdot V_2, m_1 + m_2)$ *i.e.* $p_1 \odot p_2 \ll q_1 \odot q_2$.

(*ii*) One of q_1 and q_2 is in D_1, without loss of generality, let $q_2 = \mathcal{F} \in D_1$. Let $q_1 = (V_1, m_1)$, then $q_1 \odot q_2 = \uparrow\{V_1 \cdot W \mid W \in \mathcal{F}\}$. From the way-below relation on D, it is known that $V_1 \subseteq U_1$, and $U_2 \in \mathcal{F}$. So $V_1 \cdot U_2 \subseteq U_1 \cdot U_2$, implying $U_1 \cdot U_2 \in q_1 \odot q_2$, that is, $p_1 \odot p_2 \ll q_1 \odot q_2$.

(iii) $q_1 = \mathcal{F}_1, q_2 = \mathcal{F}_2 \in D_1$, then

$$q_1 \odot q_2 = \uparrow\{V_1 \cdot V_2 \mid V_1 \in \mathcal{F}_1, V_2 \in \mathcal{F}_2\}.$$

It is known that $U_1 \in \mathcal{F}_1$, $U_2 \in \mathcal{F}_2$, so $U_1 \cdot U_2 \in q_1 \odot q_2$, implying $p_1 \odot p_2 \ll q_1 \odot q_2$.

Let $q \ll p_1 \odot p_2$. Since D is continuous, $p_1 = \bigvee^{\uparrow} \Downarrow p_1$, $p_2 = \bigvee^{\uparrow} \Downarrow p_2$, where $\bigvee^{\uparrow} \Downarrow p_1$ means the supremum of the directed set $\Downarrow p_1$ and $\bigvee^{\uparrow} \Downarrow p_2$ means the supremum of the directed set $\Downarrow p_2$. So

$$\begin{aligned} q \ll p_1 \odot p_2 &= (\textstyle\bigvee^{\uparrow} \Downarrow p_1) \odot (\bigvee^{\uparrow} \Downarrow p_2) \\ &= \textstyle\bigvee^{\uparrow}(\Downarrow p_1 \odot \Downarrow p_2) \\ &= \textstyle\bigvee^{\uparrow}\{r \odot s \mid r \ll p_1, s \ll p_2\}. \end{aligned}$$

Then there exist $r_1 \ll p_1, s_1 \ll p_2$ such that $q \sqsubseteq r_1 \odot s_1$. Hence the way-below relation $\ll$ on D is a stable approximating auxiliary relation, that is, $(D, \sqsubseteq, \odot, \ll)$ is a stable ordered semigroup. □

By combining Proposition 3.14 and Proposition 3.15, the following corollary is immediate.

Corollary 3.16 *For any topological semigroup $(X, \tau, \cdot)$ satisfying condition (Δ), then $(D, \sigma, \odot)$ is a topological semigroup.*

From the above discussion, we can derive the topological embedding between X and D easily.

Theorem 3.17 *Let $(X, \tau, \cdot)$ be a T_1 topological semigroup satisfying condition (Δ), then the function $f : (X, \tau, \cdot) \to (D, \sigma, \odot)$ defined by $f(x) = \mathcal{N}(x)$ for $x \in X$ is an embedding.*

Proof. It follows from Theorem 3.8 that f is a topological embedding. We shall show that f is a semigroup homomorphism, that is, for any $x, y \in X$, $f(x \cdot y) = f(x) \odot f(y)$.

We know that

$$f(x) \odot f(y) = \mathcal{N}(x) \odot \mathcal{N}(y) = \uparrow\{U \cdot V \mid U \in \mathcal{N}(x), V \in \mathcal{N}(y)\}.$$

For any element W in $f(x) \odot f(y)$, there exist $U_1 \in \mathcal{N}(x), V_1 \in \mathcal{N}(y)$ such that $U_1 \cdot V_1 \subseteq W$. We achieve that $x \cdot y \in W$, *i.e.* $W \in \mathcal{N}(x \cdot y) = f(x \cdot y)$, since $x \cdot y \in U_1 \cdot V_1 \subseteq W$. So $f(x) \odot f(y) \subseteq f(x \cdot y)$. Conversely, for each U belonging to $f(x \cdot y)$, that is, $x \cdot y \in U$, there exist open neighborhoods U_0 of x and V_0 of y such that $x \cdot y \in U_0 \cdot V_0 \subseteq U$. Then $U \in \mathcal{N}(x) \odot \mathcal{N}(y)$, implying

$$f(x \cdot y) \subseteq \mathcal{N}(x) \odot \mathcal{N}(y), \ \textit{i.e.}\ f(x \cdot y) \subseteq f(x) \odot f(y).$$

Therefore, $f(x \cdot y) = f(x) \odot f(y)$, completes the proof. □

In order to achieve the most significant result of this paper, we complete it with the help of Theorem 2.5 in [1]. We will rewrite the theorem and give a brief description for the proof.

Theorem 3.18 ([1]) *Let $(M, \sqsubseteq)$ be a domain and let X be a T_1 subspace of ΣM. Then there is a continuous poset $(P, \preceq)$ such that the function $f : X \to \max(P)$ is a topological homeomorphism, where the topologies on X and $\max(P)$ are the relative Scott topology.*

Set $P = P_0 \cup P_1$, where $P_0 = (\Downarrow X) \times \mathbb{N}$ and $P_1 = (\downarrow X) \times \{\infty\}$. For any $(y, m), (z, n) \in P$,

$(y, m) \preceq (z, n)$ if and only if one of the following cases holds:

(i) $(y, m) = (z, n)$;

(ii) $y \ll z$ and $m < n$;

(iii) $y \sqsubseteq z$ and $m = n = \infty$.

Note that whenever $(y, \infty) \preceq (z, n)$ then $n = \infty$.

Then we have the following conclusions:

(1) For any directed subset E of P, if E has a supremum in P, then $\bigvee E = (\bigvee E_1, \bigvee E_2)$, where $E_i = \pi_i(E)$ and π_i is the projection on the ith coordinate, $i = 1, 2$.

(2) $(P, \preceq)$ is a continuous poset. For any $(y, m) \in P_0$, $(y, m) \ll (y, m)$. For any $(y^*, \infty) \in P_1$, we have $(y^*, \infty) = \bigvee(P_0 \cap \Downarrow(y^*, \infty))$, where

$$P_0 \cap \Downarrow(y^*, \infty) = \{(y, m) \in P_0 \mid y \ll y^*, m \in \mathbb{N}\}$$

is directed. In addition, $(y_1, \infty) \ll (y_2, \infty)$ never happens.

(3) By the partial order on P, it follows that $\max(P) = \{(x, \infty) \mid x \in X\}$.

(4) Define the function $f : X \to \max(P)$ by $f(x) = (x, \infty)$, then it is a topological homeomorphism.

Remark 3.19 The continuous poset P constructed in Theorem 3.18 may not be bounded complete. Here is a simple example. Let $M = \{a_1, a_2, b_1, b_2\}$. The partial order on M is defined by:

$$b_1 \leq a_i \ (i = 1, 2),$$
$$b_2 \leq a_i \ (i = 1, 2),$$

Pick $X = \{a_1, a_2\}$, which is a T_1 subspace of ΣM. Then it is known that $P_0 = M \times \mathbb{N}, P_1 = M \times \{\infty\}$ and $P = P_0 \cup P_1$. Then P is a continuous poset. By the order relation $\preceq$ on P, it is clear that (b_1, ∞) and (b_2, ∞) have the common upper bounds $\{(a_1, \infty), (a_2, \infty)\}$, but they do not have a supremum. So P is not bounded complete.

We now define a binary operation $\otimes$ on P.

Let $(M, \sqsubseteq)$ be a domain and $(M, \sqsubseteq, \bullet)$ be a prequantale. We define a binary operation $\otimes$ on P as follows for any $(y, m), (z, n) \in P$:

(1) $(y, m) \otimes (z, n) = (y \bullet z, m + n)$ for $m, n \in \mathbb{N}$;

(2) $(y, m) \otimes (z, n) = (y \bullet z, \infty)$ for $m = \infty$ or $n = \infty$.

Theorem 3.20 *Let $(M, \sqsubseteq)$ be a domain, $(M, \sqsubseteq, \bullet)$ a prequantale and $(M, \sqsubseteq, \bullet, \ll)$ a stable ordered semigroup. Let X be a T_1 subspace of ΣM and $(X, \bullet)$ a subsemigroup of $(M, \bullet)$. Then*

(1) *$(P, \preceq, \otimes)$ is a continuous prequantale, where P stands for the continuous poset constructed in* Theorem 3.18.

(2) *The function $f : (X, \sigma\mid_X, \bullet) \to (\max(P), \sigma\mid_{\max(P)}, \otimes)$ defined by $f(x) = (x, \infty)$ is an isomorphism.*

Proof. (1) It follows from Theorem 3.18 $(P, \preceq)$ is a continuous poset. For any $(y, m), (z, n) \in P$, one case is that (y, m) and (z, n) are both in P_0, then $m, n \in \mathbb{N}$ and there exist $y_1, z_1 \in X$ such that $y \ll y_1, z \ll z_1$. Since the way-below relation $\ll$ on M is stable, $y \bullet z \ll y_1 \bullet z_1$. We know that $y_1 \bullet z_1 \in X$ because $(X, \bullet)$ is a subsemigroup of $(M, \bullet)$. In addition, $(y, m) \otimes (z, n) = (y \bullet z, m + n)$. So $(y \bullet z, m + n) \in P_0$, *i.e.* $(y, m) \otimes (z, n) \in P$. The other case is that at least one of (y, m) and (z, n) is in P_1. Without loss of generality, let $(z, n) \in P_1$ *i.e.* $n = \infty$. Then $(y, m) \otimes (z, \infty) = (y \bullet z, \infty)$. Pick $y_1, z_1 \in X$ such that $y \ll y_1$, and $z \sqsubseteq z_1$. Then $y \bullet z \sqsubseteq y_1 \bullet z_1$ and $y_1 \bullet z_1 \in X$, which imply that $(y \bullet z, \infty) \in P_1$, that is, $(y, m) \otimes (z, \infty) \in P$. For any $p_1, p_2, p_3 \in P$, it is easy to prove that $(p_1 \otimes p_2) \otimes p_3 = p_1 \otimes (p_2 \otimes p_3)$. So $(P, \otimes)$ is a semigroup. For any element (y, m) of P and any directed subset E of P with $\bigvee E$ exists. It is obvious

$$(y, m) \otimes (\bigvee E) = \bigvee((y, m) \otimes E),$$

$$(\bigvee E) \otimes (y, m) = \bigvee(E \otimes (y, m)).$$

Therefore, $(P, \preceq, \otimes)$ is a continuous prequantale.

(2) We know that $(M, \sqsubseteq, \bullet, \ll)$ is a stable ordered semigroup, by Proposition 3.14, $(M, \sigma, \bullet)$ is a topological semigroup. Then $(X, \sigma\mid_X, \bullet)$ is also a topological semigroup. It should be claimed that $(\max(P), \sigma\mid_{\max(P)}, \otimes)$ is a topological semigroup. By the definition of the binary operation $\otimes$, we know that $\max(P)$ is closed under the binary operation $\otimes$. For any $x, y, z \in X$, $(x, \infty), (y, \infty), (z, \infty)$ are all in $\max(P)$, since $(X, \bullet)$ is a subsemigroup of $(M, \bullet)$, $((x, \infty) \otimes (y, \infty)) \otimes (z, \infty) = (x, \infty) \otimes ((y, \infty) \otimes (z, \infty))$. Thus $(\max(P), \otimes)$ is a subsemigroup of P. Define a function $g : \max(P) \times \max(P) \to \max(P)$ by

$$g((x, \infty), (y, \infty)) = (x, \infty) \otimes (y, \infty)$$

for all $x, y \in X$. We prove that g is continuous. For any $x, y \in X$, $(x, \infty), (y, \infty) \in \max(P)$ satisfying $(x, \infty) \otimes (y, \infty) \in \Uparrow r \cap \max(P)$ for some $r \in P$. Since P is continuous,

$$\begin{aligned} r \ll (x, \infty) \otimes (y, \infty) &= (\textstyle\bigvee^{\uparrow} \Downarrow(x, \infty)) \otimes (\bigvee^{\uparrow} \Downarrow(y, \infty)) \\ &= \textstyle\bigvee^{\uparrow}(\Downarrow(x, \infty) \otimes \Downarrow(y, \infty)) \\ &= \textstyle\bigvee^{\uparrow}\{(a, m) \otimes (b, n) \mid (a, m) \ll (x, \infty), (b, n) \ll (y, \infty), m, n \in \mathbb{N}\}. \end{aligned}$$

Then there exist $(a_1, m_1) \ll (x, \infty)$, $(b_1, n_1) \ll (y, \infty)$, $m_1, n_1 \in \mathbb{N}$ such that $r \preceq (a_1, m_1) \otimes (b_1, n_1) = (a_1 \bullet b_1, m_1 + n_1)$. We need to prove

$$(\Uparrow(a_1, m_1) \cap \max(P)) \otimes (\Uparrow(b_1, n_1) \cap \max(P)) \subseteq \Uparrow r \cap \max(P).$$

For any element p in $(\Uparrow(a_1, m_1) \cap \max(P)) \otimes (\Uparrow(b_1, n_1) \cap \max(P))$, there are elements

$(x_1,\infty),(y_1,\infty)$ in $\max(P)$ such that $(a_1,m_1)\ll(x_1,\infty),(b_1,n_1)\ll(y_1,\infty)$ and

$$p=(x_1,\infty)\otimes(y_1,\infty)=(x_1\bullet y_1,\infty).$$

By the definition of the way-below relation on P, it is known that $a_1\ll x_1, b_1\ll y_1$. So $a_1\bullet b_1\ll x_1\bullet y_1$, implying $(a_1\bullet b_1, m_1+n_1)\ll(x_1\bullet y_1,\infty)$. Therefore $r\ll p$, that is, $p\in\Uparrow r\cap\max(P)$, as desired. Thus $(\max(P),\sigma\mid_{\max(P)},\otimes)$ is a topological semigroup.

It follows from Theorem 3.18 that f is a topological homeomorphism. Obviously, f is a semigroup homomorphism. Therefore, f is an isomorphism. □

From the above results, we can now derive our major result in this paper.

Theorem 3.21 *Every T_1 topological semigroup satisfying condition (Δ) has a continuous prequantale model.*

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