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# The Probabilistic Powerdomain for Stably Compact Spaces via Compact Ordered Spaces

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**Abstract**

Stably compact spaces $X_s$ can be described as being derived from compact ordered spaces $X$ by weakening their topology to the open upper sets. In this paper the probabilistic powerdomain of a stably compact space $X_s$ is investigated using the compact ordered space $X$ and classical tools from measure theory and functional analysis. This allows to derive in a unified and elegant way known and new results which are summarized in Theorem 5.4.

*Keywords:* Semantic domains, probabilistic powerdomain, stably compact spaces,

## Introduction

In denotational semantics, valuations and the probabilistic powerdomain have been introduced by Jones and Plotkin [7] as a substitute for probabilities and the space of probablity measures in order to model probabilistic phenomena in programming. Recently, the probabilistic powerdomain has received more and more attention. Most of the background material on probabilistic power domains is contained in dissertations that are not easily accessible except for an exposition of some basic properties in [5].

Denotational semantics has been based either on the theory of metric spaces or on domain theory in the sense of D.S. Scott. In order to talk about

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measures and probabilities for semantics, one has to adapt the classical notions of measure, probability, integration to domains. A domain is understood to be a continuous, directed complete, partially ordered set (see [5]). Every domain carries an intrinsic topology, the Scott topology, which is $T_0$ but far from being Hausdorff. Valuations which, by definition, associate a kind of measure to each Scott-open set in a continuous way, take the place of measures. The valuations on a domain form a domain in itself called the extended probabilistic powerdomain.

Of course, one would like to relate valuations to classical measures and the extended probabilistic powerdomain to the space of all measures.

Topological measure theory has been dealing almost exclusively with Hausdorff spaces, in particular locally compact spaces and complete metric spaces. For this reason, it is natural to restrict our domains to those which are compact Hausdorff for the Lawson topology, which is an intrinsic topology on domains refining the Scott topology. This restriction is not too heavy, as all domains contained in a cartesian closed category of domains are Lawson compact (see [8]). Domains that are Lawson compact are also called stably compact. This leads us to consider more generally stably compact spaces which have also shown to be very useful in semantics recently (see [9]). Stably compact spaces $(X, \mathcal{G})$ are exactly those spaces that arise from compact ordered spaces $(X, \mathcal{O}, \leq)$ in the sense of Nachbin [14] by weakening the topology to the collection $\mathcal{G}$ of all open upper sets.

In order to keep our exposition as accessible as possible to readers not acquainted with domain theory, we will stick to the setting of compact ordered spaces. We use classical functional analysis in order to derive the close connection between valuations on the stably compact space $(X, \mathcal{G})$ and the properties of the probabilistic powerdomain over such a space on the one hand and regular Borel measures and the compact convex set of probability measures on $(X, \mathcal{O})$ in the weak*-topology on the other hand. All our results apply in particular to domains that are Lawson compact.

Some of our results are not new. We owe a lot to M. Alvarez Manilla [1,2] who has investigated the close connections between valuations and classical measure theory. The merit of this paper, hopefully, is a straightforward presentation of the results with simple proofs based on standard functional analysis. We also are indebted to A. Jung for his helpful comments and his critisism.

## 1 Ordered topological spaces

We consider a *partially ordered topological space* (an *ordered space*, for short,) in the sense of Nachbin [14], that is, a set $X$ with a topology $\mathcal{O}$ and a partial order $\leq$ such that the graph of the order is closed in $X \times X$. The latter is equivalent to saying that, for any two points $x \not\leq y$ in $X$, there are disjoint neighborhoods $U$ of $x$ and $V$ of $y$, where $U$ is an upper and $V$ a lower set. This implies that ordered spaces are Hausdorff spaces.

Recall that a subset $U$ of $X$ is called an *upper* set, if $x \in U$ implies $y \in U$ for all $y \geq x$. The collection $\mathcal{G} = \mathcal{O}^\uparrow$ of open upper subsets of $X$ is again a topology on $X$, called the lower topology. This lower topology is $T_0$ but far from being Hausdorff (see the example in the next paragraph).

We apply this terminology in particular to the extended real line $\overline{\mathbb{R}} = [-\infty, +\infty] = \mathbb{R} \cup \{-\infty, +\infty\}$ with the usual total order and the usual Hausdorff topology $\mathcal{I}$ generated by the open intervals. The proper open upper sets are the intervals $]r, +\infty]$; they constitute the lower topology $\mathcal{I}^\uparrow$. We also apply this terminology to subsets of $\overline{\mathbb{R}}$ as, for example, the reals $\mathbb{R} = \ ]-\infty, +\infty[$, the non-negative reals $\mathbb{R}_+ = [0, +\infty[$, and the extended non-negative reals $\overline{\mathbb{R}}_+ = [0, +\infty]$.

For functions $g\colon X \to \overline{\mathbb{R}}$ the following notations and terminology will be adopted:

$$||g|| := \sup_{x \in X} |g(x)| \,,$$

and we say that $g$ is *bounded* if $||g|| < +\infty$. We denote by $\mathrm{C}(X)$ the vector space of all bounded continuous functions $f\colon X \to \mathbb{R}$ with the sup-norm $||f||$ and by $\mathrm{C}_+(X)$ the positive cone of its non-negative members. For any $r \in \mathbb{R}$ we adopt the following notation for the inverse image

$$[g > r] := g^{-1}(]r, +\infty]) = \{x \in X : g(x) > r\} \,.$$

As usual, $g$ will be called *lower semicontinuous*, if $[g > r]$ is open in $X$ for every $r \in \mathbb{R}$. We denote by $\mathrm{LSC}_+(X)$ the set of all non-negative, bounded, lower semicontinuous functions $g\colon X \to \mathbb{R}_+$.

We shall say that $g$ is *monotone increasing* or *order preserving*, if $x \leq y$ implies $g(x) \leq g(y)$, which is equivalent to saying that $[g > r]$ is an upper set for all $r \in \mathbb{R}$. Note that $g$ is monotone increasing and lower semicontinuous if, and only if, $[g > r]$ is an open upper set for all $r \in \mathbb{R}$, that is, if, and only if, $g$ is continuous with respect to the lower topologies $\mathcal{G} = \mathcal{O}^\uparrow$ on $X$ and $\mathcal{I}^\uparrow$ on $\overline{\mathbb{R}}$. We denote by $\mathrm{LSC}^\uparrow_+(X)$, resp. $\mathrm{C}^\uparrow_+(X)$, the sets of non-negative, bounded, monotone increasing, lower semicontinuous, resp. continuous, functions $g\colon X \to \mathbb{R}_+$.

We will need a few results on compact ordered spaces. It is immediate, that the (pointwise) supremum of any family of (monotone increasing) lower semicontinuous functions $f_i: X \to \overline{\mathbb{R}}$ is again (monotone increasing) lower semicontinuous. On a compact Hausdorff space, every lower semicontinuous functions is the sup of a directed family of continuous functions. There is a generalization for compact ordered spaces:

**Lemma 1.1** *Let $X$ be a compact ordered space. Every monotone increasing lower semicontinuous function $g: X \to \overline{\mathbb{R}}_+$ is the pointwise supremum of a directed family $(f_i)$ of monotone increasing continuous functions $f_i : X \to \mathbb{R}_+$.*

**Proof.** Indeed let $g \in \mathrm{LSC}_+^{\uparrow}(X)$. Choose $x_0 \in X$ with $g(x_0) > 0$ and an $r$ sucht that $0 \leq r < g(x_0)$. Then $U = [g > r]$ is an open upper set containing $x_0$. By the Corollary to Theorem 4 in [14], a compact ordered space is normally ordered, whence, by Theorem 1 in [14], there is a monotone increasing continuous function $f: X \to [0, 1]$ such that $f(x) = 0$ for all $x \notin U$ and $f(x_0) = 1$. The function $r \cdot f$ is monotone increasing and continuous with $rf(x_0) = r$ and $r \cdot f \leq g$. Indeed, $rf(x) = 0 \leq g(x)$ for $x \notin U$, and $rf(x) \leq r < g(x)$ for $x \in U$. As we can construct such a function for every $x_0$ with $g(x_0) > 0$ and every $0 \leq r < g(x_0)$, it follows that $g$ is the pointwise supremum of a family of non-negative monotone increasing continuous real-valued functions. By taking finite suprema of such functions one obtains a directed family with the desired properties. $\square$

The set $\mathrm{C}_+^{\uparrow}(X)$ of all non-negative monotone increasing continuous real-valued functions is a cone in $\mathrm{C}(X)$. Indeed, for $f_1, f_2 \in \mathrm{C}_+^{\uparrow}(X)$, one clearly has $f_1 + f_2 \in \mathrm{C}_+^{\uparrow}(X)$ and likewise $rf \in \mathrm{C}_+^{\uparrow}(X)$ for any real number $r \geq 0$; moreover, the product function $f_1 \cdot f_2$ and the constant function $\mathbf{1}$ belong to the cone $\mathrm{C}_+^{\uparrow}(X)$, too.

**Lemma 1.2** *For a compact ordered space $X$, the cone $\mathrm{C}_+^{\uparrow}(X)$ generates a vector space $V = \mathrm{C}_+^{\uparrow}(X) - \mathrm{C}_+^{\uparrow}(X)$ which is dense in $\mathrm{C}(X)$ with respect to the sup norm.*

**Proof.** (Edwards [4]) From the remark preceding this Lemma it follows that $V$ is a subalgebra of $\mathrm{C}(X)$ which contains the constant function $\mathbf{1}$. From Nachbin's results cited in the proof of the previous Lemma, it follows that for any elements $x \not\leq y$ in $X$, there is a function $f \in \mathrm{C}_+^{\uparrow}(X)$ such that $f(x) = 1$ and $f(y) = 0$. Hence, $\mathrm{C}_+^{\uparrow}(X)$ and, à fortiori, $V$ separate the points of $X$. The Lemma now follows from the Stone-Weierstraß theorem. $\square$

## 2 The cone $\mathrm{M}(X)$ of regular Borel measures

Let $X$ be any compact Hausdorff space and $\mathcal{B}$ the $\sigma-$algebra of Borel sets, that is, the $\sigma-$algebra generated by the open subsets of $X$. Recall that a Borel measure on $X$ is a function $m\colon \mathcal{B} \to \mathbb{R}_+$ such that

$$
\begin{array}{lrl}
m \text{ is strict:} & m(\emptyset) & = 0\,, \\
m \text{ is additive:} & m(A) + m(B) & = m(A \cup B)\,, \\
 & \text{whenever } A, B & \in \mathcal{B} \text{ are disjoint}\,, \\
m \text{ is } \sigma\text{-continuous:} & m(\bigcup_n A_n) & = \sup_n m(A_n) \\
 & \text{for every} & \text{increasing sequence } A_n \in \mathcal{B}\,.
\end{array}
$$

Note that we restrict ourselves to positive measures. Such a measure is called *inner regular*, if

$$m(A) = \sup\ \{m(K) : K \subseteq A \text{ and } K \text{ compact}\} \text{ for all Borel sets } A\,.$$

Inner regularity implies *outer regularity* by passing to complements:

$$m(A) = \inf\ \{m(U) : A \subseteq U \text{ and } U \text{ open}\} \text{ for all Borel sets } A\,.$$

We shall simpy talk about *regular* Borel measures. We denote by

$\mathbf{M}(X)$ the set of all regular Borel measures on $X$, by
$\mathbf{M}_{\leq 1}(X)$ the subset of all Borel measures with $m(X) \leq 1$, and by
$\mathbf{M}_1(X)$ the subset of probability measures, i.e., $m(X) = 1$.

$\mathbf{M}(X)$ is a cone in the vector space of all functions from $\mathcal{B}$ to $\mathbb{R}$, that is, the sum $m_1 + m_2$ of two regular Borel measures and also the scalar multiple $rm$ for any non-negative real number $r$ are regular Borel measures. The subsets $\mathbf{M}_{\leq 1}(X)$ and $\mathbf{M}_1(X)$ are convex. On $\mathbf{M}(X)$ there is a natural order relation:

$$m_1 \leq m_2 \text{ iff } m_1(A) \leq m_2(A) \text{ for all Borel sets } A\,.$$

For compact ordered spaces $X$ the following relation $\prec$ on $\mathbf{M}(X)$, called the *stochastic order*, will be of more interest to us:

$$m_1 \prec m_2 \text{ iff } m_1(U) \leq m_2(U) \text{ for all open upper sets } U\,.$$

The following lemma is standard knowledge:

**Lemma 2.1** *With respect to a regular Borel measure $m$, every bounded lower semicontinuous function $g\colon X \to \mathbb{R}_+$ is integrable and has a finite integral*

$\int f dm$ *which satisfies*

(i) $\int (f+g)dm = \int f dm + \int g dm \quad , \quad \int r f dm = r \int f dm$
*for all* $f, g \in \mathrm{LSC}_+(X)$ *and all* $r \in \mathbb{R}_+$ ,

(ii) $\int (\sup_i g_i) dm = \sup_i \int g_i dm$ *for every directed family* $(g_i)_i$ *in* $\mathrm{LSC}_+(X)$,

(iii) $m(\bigcup_i U_i) = \sup_i m(U_i)$ *for every directed family* $(U_i)_i$ *of open sets.*

The integral may be defined by the following Riemann integral:

$$\int g dm = \int_0^{+\infty} m([g > r]) \, dr \, .$$

This is a Choquet-type definition of the integral (see König [11], section 11). Let us explain, why this definition makes sense: Let $g \in \mathrm{LSC}_+(X)$. For every $r$, the set $[g > r]$ is open and has a measure $m([g > r]) \in \mathbb{R}_+$. The function $r \mapsto m([g > r]) : \mathbb{R}_+ \to \mathbb{R}_+$ is monotone decreasing and $m([g > r]) = 0$ for $r \geq ||g||$. Thus this function is Riemann integrable and the Riemann integral $\int_0^{+\infty} m([g > r]) \, dr$, which is in fact an integral extended over the finite interval $[0, ||g||]$, is a real number. The properties of the previous lemma can now be derived from the properties of the Riemann integral.

## 3 The cone $\mathbf{V}(X)$ of bounded continuous valuations

Let $X$ be a compact ordered space, $\mathcal{G} = \mathcal{O}^{\uparrow}$ the set of open upper sets. A *valuation* on $\mathcal{G}$ is a function $\mu : \mathcal{G} \to \overline{\mathbb{R}}_+$ with the following properties:

$\mu$ is strict: $\mu(\emptyset) \ = \ 0 \, ,$

$\mu$ is modular: $\mu(U) + \mu(V) \ = \ \mu(U \cup V) + \mu(U \cap V) \, ,$

$\mu$ is monotone increasing: $U \subseteq V \Rightarrow \mu(U) \subseteq \mu(V) \, .$

A valuation is called *(Scott-) continuous*, if

$$\mu(\bigcup_i U_i) = \sup_i \mu(U_i) \text{ for every directed family of open upper sets } U_i \in \mathcal{G} \, .$$

We denote by $\overline{\mathbf{V}}(X)$ the set of all continuous valuations on $\mathcal{G}$. We endow $\overline{\mathbf{V}}(X)$ with the *stochastic order* which is defined by

$$\mu \prec \nu \text{ iff } \mu(U) \leq \nu(U) \text{ for all open } U \in \mathcal{G} \, .$$

For continuous valuations we also define an addition and a multiplication by non-negative scalars $r$ by $(\mu + \nu)(U) = \mu(U) + \nu(U)$ and $(r\mu)(U) = r\mu(U)$.

(We adopt the convention $0 \cdot (+\infty) = 0$ as usually in measure theory). We denote by

$\mathbf{V}(X)$ the set of all *bounded* continuous valuations, i.e., $\mu(X) < +\infty$, by
$\mathbf{V}_{\leq 1}(X)$ the set of all *sub-probability* valuations, i.e., $\mu(X) \leq 1$, and by
$\mathbf{V}_1(X)$ the set of all *probability* valuations, i.e., $\mu(X) = 1$.

We note that $\mathbf{V}(X)$ is a cone in the vector space of all functions from $\mathcal{G}$ to $\mathbb{R}$ and that $\mathbf{V}_{\leq 1}(X)$ and $\mathbf{V}_1(X)$ are convex subsets.

As for a bounded lower semicontinuous real-valued function $g: X \to \mathbb{R}_+$ (see sec.2), we may define the integral of a bounded monotone increasing lower semicontinuous function $g: X \to \mathbb{R}_+$ with respect to a continuous valuation $\mu$. Indeed, for every $r$, the preimage $[g > r] = g^{-1}(]r, +\infty])$ is an open upper set. Thus $\mu([g > r])$ is a well defined non-negative real number. Moreover, the function $r \mapsto \mu([g > r]): \mathbb{R}_+ \to \mathbb{R}_+$ is monotone decreasing and upper semicontinuous. Hence its Riemann integral $\int_0^{+\infty} \mu([g > r])\, dr$ is a well defined real number. Note that in fact the integral is only extended over the finite interval $[0, ||g||]$, as $\mu([g > r]) = 0$ for $r \geq ||g||$. We define

$$\int g d\mu := \int_0^{+\infty} \mu([g > r])\, dr\,.$$

From the properties of the Riemann integral for monotone functions one deduces the following properties:

**Lemma 3.1** *The map $(\mu, f) \mapsto \int f d\mu: \mathbf{V}(X) \times \mathrm{LSC}_+^{\uparrow}(X) \to \mathbb{R}_+$ is monotone increasing, it preserves directed suprema and is linear in each of its two arguments separately.*

The proof is straightforward but tedious. It can be found in [16] and [13]. We apply part of this lemma in the proof of the following:

**Lemma 3.2** *Let $(X, \mathcal{O}, \leq)$ be a compact ordered space. For a net $(\mu_j)_{j \in J}$ of bounded continuous valuations and a bounded continuous valuation $\mu$ on $\mathcal{G} = \mathcal{O}^{\uparrow}$, the following are equivalent:*

(i) $\mu(U) \leq \liminf_j \mu_j(U)$ *for every open upper set $U$.*
(ii) $\int f d\mu \leq \liminf_j \int f d\mu_j$ *for every $f \in \mathrm{C}_+^{\uparrow}(X)$.*
(iii) $\int g d\mu \leq \liminf_j \int g d\mu_j$ *for every $g \in \mathrm{LSC}_+^{\uparrow}(X)$.*

**Proof.** Clearly, (iii) $\Rightarrow$ (ii). Further, (iii) $\Rightarrow$ (i), as the characteristic function $\chi_U$ of every open upper set $U$ is lower semicontinuous and increasing and $\int \chi_U d\mu = \mu(U)$. (ii) $\Rightarrow$ (iii): By Lemma 1.1 every $g \in \mathrm{LSC}_+^{\uparrow}(X)$ is the supremum of a directed family of monotone increasing continuous functions

$f_i \colon X \to \mathbb{R}_+$. For the latter we have $\int f_i d\mu \leq \liminf_j \int f_i d\mu_j$ by hypothesis. As $f_i \leq g$, we have $\liminf_j \int f_i d\mu_j \leq \liminf_j \int g d\mu_j$ for all $i$, whence $\int g d\mu = \int \sup_i f_i d\mu = \sup_i \int f_i d\mu \leq \sup_i \liminf_j \int f_i d\mu_j \leq \liminf_j \int g d\mu_j$ as desired. Note that we have used the fact that $f \mapsto \int f d\mu$ preserves directed sups by Lemma 3.1. (i) $\Rightarrow$ (iii) is proved in a similar way using the fact that every $g \in \mathrm{LSC}^{\uparrow}_{+}(X)$ is the supremum of the following increasing sequence $g_n$ of finite linear combinations of characteristic functions of open upper sets:

$$g_n = \frac{1}{2^n} \sum_{i=1}^{n2^n} \chi_{[g > \frac{i}{2^n}]} \ .$$

$\square$

If one chooses a constant net $\mu_j = \nu$, the previous lemma yields the following:

**Corollary 3.3** *Let $(X, \mathcal{O}, \leq)$ be a compact ordered space. For continuous valuations $\mu$ and $\nu$ on $\mathcal{O}^{\uparrow}$, the following are equivalent:*

*(i) $\mu \prec \nu$, that is, $\mu(U) \leq \nu(U)$ for every open upper set $U$.*
*(ii) $\int f d\mu \leq \int f d\nu$ for every $f \in \mathrm{C}^{\uparrow}_{+}(X)$.*
*(iii) $\int g d\mu \leq \int g d\nu$ for every $g \in \mathrm{LSC}^{\uparrow}_{+}(X)$.*

# 4 The cone $\mathrm{C}^*_+(X)$ of positive linear functionals on $\mathrm{C}(X)$

Let $X$ be a compact ordered space throughout this section. The continuous real-valued functions form a Banach space $\mathrm{C}(X)$ with respect to the sup-norm. We denote by $C^*(X)$ its dual, that is, the vector space of all bounded linear functionals $\varphi$ on $\mathrm{C}(X)$. Recall that a linear functional $\varphi$ on $\mathrm{C}(X)$ is called *positive*, if $\varphi(f) \geq 0$ for every non-negative $f \in \mathrm{C}(X)$. Every positive linear functional on $\mathrm{C}(X)$ is bounded with respect to the sup-norm and every bounded linear functional on $\mathrm{C}(X)$ is representable as a difference of two positive ones. We denote by

$\mathrm{C}^*_+(X)$ the positive cone of all positive linear functionals $\varphi \in \mathrm{C}^*(X)$,
$\mathrm{C}^*_{\leq 1}(X)$ the convex subset of positive linear functionals with $\varphi(\mathbf{1}) \leq 1$,
$\mathrm{C}^*_1(X)$ the convex subset of positive linear functional with $\varphi(\mathbf{1}) = 1$.

On the dual space $\mathrm{C}^*(X)$ we have two orders: firstly the usual order given by the positive cone $\mathrm{C}_+(X)$, that is,

$$\varphi \leq \psi \text{ iff } \psi - \varphi \in \mathrm{C}^*_+(X) \text{ iff } \varphi(f) \leq \psi(f) \text{ for every } f \in \mathrm{C}_+(X) \, ,$$

secondly, the stochastic order given by the cone $\mathrm{C}_+^{\uparrow}(X)$ of non-negative bounded monotone increasing continuous functions, that is,

$$\varphi \prec \psi \text{ iff } \varphi(f) \leq \psi(f) \text{ for every } f \in \mathrm{C}_+^{\uparrow}(X)\,.$$

It is clear, that $\prec$ is reflexive and transitive. We shall see later that $\prec$ is antisymmetric, too.

On the dual vector space $\mathrm{C}^*(X)$ we consider two weak topologies. The first is the weak*-topology, that is, the weakest topology such that the maps $\varphi \to \varphi(f) \colon \mathrm{C}^*(X) \to \mathbb{R}$ are continuous for all $f \in \mathrm{C}_+(X)$.

The second topology is the called the *weak**-topology* which, by definition, is the weakest topology on $\mathrm{C}^*(X)$ such that the maps $\varphi \to \varphi(f) \colon \mathrm{C}^*(X) \to \mathbb{R}$ are continuous for all $f \in \mathrm{C}_+^{\uparrow}(X)$ only. Thus, the weak**-topology is coarser than the weak*-topology. On $\mathrm{C}^*(X)$ it is stricty coarser than the weak*-topology.

**Proposition 4.1** *For a compact ordered space $X$ one has:*

(1) *The stochastic order $\prec$ is indeed antisymmetric and the weak**-topology is Hausdorff.*

(2) *The usual order $\leq$ and the stochastic order $\prec$ on $\mathrm{C}^*(X)$ are weak*-closed.*

(3) *The subsets $\mathrm{C}_{\leq 1}^*(X)$ and $\mathrm{C}_1^*(X)$ are weak*-compact convex sets.*

(4) *The weak**-topology agrees with the weak*-topology on the positive cone $\mathrm{C}_+^*(X)$.*

(5) *The sets $\mathrm{C}_{\leq 1}^*(X)$ and $\mathrm{C}_1^*(X)$ are compact convex ordered spaces with respect to the stochastic order $\prec$ and either of the weak topologies.*

**Proof.** (1) If $X$ is a compact ordered space, the vector space $V$ generated by the cone $\mathrm{C}_+^{\uparrow}(X)$ of non-negative monotone increasing continuous functions is uniformly dense in $\mathrm{C}(X)$ by 1.2. This implies that $\prec$ is indeed antisymmetric and that the weak**-topology is Hausdorff.

(2) Let $\varphi_j$ and $\psi_j$ be nets of bounded linear functionals that weak*-converge to $\varphi$ and $\psi$, respectively, such that $\varphi_j \prec \psi_j$ for every $j$. Then, for every $f \in \mathrm{C}_+^{\uparrow}(X)$, we have $\varphi_j(f) \leq \psi_j(f)$ and, as $\varphi_j(f)$ and $\psi_j(f)$ converge to $\varphi(f)$ and $\psi(f)$, respectively, we conclude that $\varphi(f) \leq \psi(f)$, whence $\varphi \prec \psi$. The proof for the order $\leq$ is analogous.

(3) is a standard application of Alaogu's theorem.

(4) The weak**-topology restricted to $\mathrm{C}_{\leq 1}^*(X)$ is Hausdorff by (1) and coarser than the weak*-topology which is compact on $\mathrm{C}_{\leq 1}^*(X)$. As there is no Hausdorff topology that is strictly coarser than a compact topology, the weak*- and the weak**-topology agree on $\mathrm{C}_{\leq 1}^*(X)$. As $\mathrm{C}_+^*(X)$ is the union of

its open subsets $n\mathrm{C}^*_{<1}(X), n \in \mathbb{N}$ and as the two toplogies agree on these open subsets, they also agree on $\mathrm{C}^*_+(X)$.

(5) As the relation $\prec$ is closed by (2), the assertions are proved by (3) and (4). $\square$

For every $x \in X$, the Dirac functional $\delta_x$ defined by $f \mapsto f(x)$ is a positive linear functional on $\mathrm{C}(X)$. For a compact Hausdorff space, $x \mapsto \delta_x$ is a topological embedding of the space X into $\mathrm{C}^*(X)$ endowed with the weak*-topology. In fact, the functionals $\delta_x$ are exactly the extreme points of $\mathrm{C}^*_1(X)$. We have more:

**Proposition 4.2** *Let $X$ be a compact ordered space. Associating to every element $x \in X$ its Dirac functional $\delta_x$ yields a topological and an order embedding of $X$ into $\mathrm{C}^*(X)$ endowed with the weak**-topology and the stochastic order $\prec$.*

**Proof.** It only remains to show that we have an order embedding. If $x \leq y$, then $\delta_x(f) = f(x) \leq f(y) = \delta_y(f)$ for every $f \in \mathrm{C}^{\uparrow}_+(X)$, whence $\delta_x \prec \delta_y$. If, at the other hand, $x \not\leq y$, then there is an $f \in \mathrm{C}^{\uparrow}_+(X)$ such that $f(x) = 1$ but $f(y) = 0$, that is, $\delta_x(f) = 1 \not\leq 0 = \delta_y(f)$ and, consequently, $\delta_x \not\prec \delta_y$. $\square$

## 5 The main results

In the previous three sections we were considering three cones, the cone $\mathbf{M}(X)$ of regular Borel measures, the cone $\mathbf{V}(X)$ of bounded continuous valuations, and the cone $\mathrm{C}^*_+(X)$ of positive linear functionals on $\mathrm{C}(X)$. We now will show that, for a compact ordered space $X$, all of these cones are isomorphic, and that these isomorphisms are also order isomorphisms for the stochastic order $\prec$, that was defined for each of these cones.

We first define a map from $\mathbf{M}(X)$ to $\mathbf{V}(X)$. We observe that every regular Borel measure $m$ on an ordered space $X$ induces a continuous valuation on the set $\mathcal{G}$ of open upper set just by restricting $m$ to $\mathcal{G}$. Indeed $m(\emptyset) = 0$, and from the finite additivity of $m$, it follows that $m$ is modular and monotone increasing on the Borel sets. It follows that $m|\mathcal{G}$ is a valuation. The continuity of $m|_\mathcal{G}$ follows from 2.1(3). Thus we have a map $m \mapsto m|_\mathcal{G}\colon \mathbf{M}(X) \to \mathbf{V}(X)$. This map preserves addition, multiplication by non-negative scalars, and the stochastic order $\prec$ which follows immediately from the definitions. We have:

**Lemma 5.1** *Let $X$ be a compact ordered space. For every regular Borel measure $m$ on $X$, its restriction to the set $\mathcal{G}$ of open upper sets is a bounded continuous valuation and the map*

$$M\colon m \mapsto m|_\mathcal{G}\colon \mathbf{M}(X) \to \mathbf{V}(X)$$

*is linear and monotone increasing with respect to the stochastic order* $\prec$.

We now define a map from $\mathbf{M}(X)$ to $\mathrm{C}_+^*(X)$. Integration with respect to a regular Borel measure $m$ defines a positive linear functional

$$\psi_m: f \mapsto \int f dm$$

on $\mathrm{C}(X)$. Moreover, $m \mapsto \psi_m: \mathbf{M}(X) \to \mathrm{C}_+^*(X)$ is linear. For a compact Hausdorff space, the Riesz Representation Theorem (see e.g. [15]) tells us:

**Lemma 5.2** *Let $X$ be a compact Hausdorff space. Then for every positive linear functional $\varphi$ on $\mathrm{C}(X)$ there is a unique regular Borel measure $m$ such that*

$$\varphi(f) = \int f dm \quad \text{for every } f \in \mathrm{C}(X)\,,$$

*and, consequently, the map*

$$\psi: m \mapsto \psi_m: M(X) \to \mathrm{C}_+^*(X)$$

*is an isomorphism of cones.*

We finally define a map from $\mathbf{V}(X)$ to $\mathrm{C}_+^*(X)$. In a similar way as for measures, we want to show that that every continuous valuation $\mu$ on the collection $\mathcal{G}$ of open upper sets defines a positive linear functional on $\mathrm{C}(X)$.

**Lemma 5.3** *Let $X$ be a compact ordered space. For every bounded continuous valuation $\mu$ on the set $\mathcal{G} = \mathcal{O}^\uparrow$ of open upper sets, there is a unique positive linear functional $\varphi_\mu$ on $\mathrm{C}(X)$ such that $\varphi_\mu(f) = \int f d\mu$ for every $f \in \mathrm{C}_+^\uparrow(X)$. The map*

$$\varphi: \mu \mapsto \varphi_\mu: \mathbf{V}(X) \to \mathrm{C}_+^*(X)$$

*is linear and an order embedding for the stochastic order* $\prec$.

**Proof.** By 3.1, the map $f \mapsto \int f d\mu$ is linear on $\mathrm{C}_+^\uparrow(X)$. For $h = f_1 - f_2$ with $f_1, f_2 \in \mathrm{C}_+^\uparrow(X)$, we define $\varphi_\mu(h) = \int f_1 d\mu - \int f_2 d\mu$. This yields a well-defined linear functional on the vector space $V = \mathrm{C}_+^\uparrow(X) - \mathrm{C}_+^\uparrow(X)$. This linear functional is positive in the sense that $\varphi_\mu(h) \geq 0$ for every non-negative function $h \in V$. Indeed, if $h = f_1 - f_2 \geq 0$, then $f_2 \leq f_1$, whence $\int f_2 d\mu \leq \int f_1 d\mu$ and consequently $\varphi_\mu(h) = \int f_1 d\mu - \int f_2 d\mu \geq 0$.

As the constant function $\mathbf{1}$ belongs to $V$, a positive linear functional on $V$ is bounded for the sup-norm; indeed, $||\varphi_\mu|| = \varphi_\mu(\mathbf{1}) = \int \mathbf{1} d\mu = \mu(X)$. As $V$ is uniformly dense in $\mathrm{C}(X)$ by Lemma 1.2, $\varphi_\mu$ has a unique extension to a positive linear functional on $\mathrm{C}(X)$; we denote this extension again by $\varphi_\mu$.

Thus, we have a map $\mu \mapsto \varphi_\mu \colon \mathbf{V}(X) \to \mathrm{C}^*_+(X)$. From the linearity of the map $(\mu, f) \mapsto \int f d\mu$ in the first argument (see Lemma 3.1) it follows that the map $\mu \mapsto \varphi_\mu$ is linear. From Lemma 3.3 it follows that it is an order embedding with respect to the stochastic order. $\square$

For a compact ordered space $X$, we now have the following diagram

$$\begin{tikzcd} \mathbf{M}(X) \arrow[rr, "m \mapsto m|\mathcal{G}", "M"'] \arrow[dr, "\psi"'] & & \mathbf{V}(X) \arrow[dl, "\varphi"] \\ & \mathrm{C}^*_+(X) & \end{tikzcd}$$

This diagram commutes, that is, $\psi = \varphi \circ M$. Indeed, let $m$ be a regular Borel measure on $X$ and $\mu = M(m) = m|_{\mathcal{G}}$. We have to show that $\psi_m = \varphi_\mu$. The formulas for the integrals in section 2 and 3 show that $\int f dm = \int f d\mu$, whence $\psi_m(f) = \varphi_\mu(f)$, for every $f \in \mathrm{C}^\uparrow_+(X)$. As the vector space generated by $\mathrm{C}^\uparrow_+(X)$ is uniformly dense in $\mathrm{C}(X)$, we conclude that $\psi_m = \varphi_\mu$.

We now may summarize:

**Theorem 5.4** *Let $(X, \mathcal{O}, \leq)$ be a compact ordered space.*

(i) *Every bounded continuous valuation $\mu$ defined on the collection $\mathcal{G} = \mathcal{O}^\uparrow$ of open upper sets can be extended to a regular Borel measure $\overline{\mu}$ on $X$ in a unique way.*

(ii) *The maps $\mu \mapsto \overline{\mu} \colon \mathbf{V}(\mathcal{G}) \to \mathbf{M}(X)$ and $\mu \mapsto \varphi_\mu \colon \mathbf{V}(X) \to \mathrm{C}^*_+(X)$ are isomorphisms of cones and order isomorphisms with respect to the respective stochastic orders $\prec$. The convex subset $\mathbf{V}_{\leq 1}(X)$ is mapped onto $\mathrm{C}^*_{\leq 1}(X)$ and $\mathbf{M}_{\leq 1}(X)$, respectively, and and $\mathbf{V}_1(X)$ is mapped onto $\mathrm{C}^*_1(X)$ and $\mathbf{M}_1(X)$, respectively.*

(iii) *With respect to the stochastic ordering $\prec$ and the weakest topology such that the maps $\mu \mapsto \int f d\mu \colon \mathbf{V}(X) \to \mathbb{R}$ are continuous for all $f \in \mathrm{C}^\uparrow_+(X)$, the convex sets $\mathbf{V}_{\leq 1}(X)$ and $\mathbf{V}_1(X)$ are compact ordered spaces.*

(iv) *The compact ordered space $X$ is topologically and order embedded in $\mathbf{V}_1(X)$ by mapping every $x \in X$ to the point valuation $\delta_x$.*

(v) *The collection of open upper sets of the compact ordered spaces $\mathbf{V}_{\leq 1}(X)$ and $\mathbf{V}_1(X)$ coincides with the weakest topology such that the maps $\mu \mapsto \mu(U)$ are lower semicontinuous for all open upper sets $U \subseteq X$.*

**Proof.** (1) In Lemma 5.3 we have seen, that a bounded continuous valuation $\mu$ on $\mathcal{G}$ defines a unique positive linear functional $\varphi_\mu$ such that $\varphi_\mu(f) = \int f d\mu$

for every $f \in \mathrm{C}_+^{\uparrow}(X)$. By the Riesz Representation Theorem 5.2, there is a unique regular Borel measure $\overline{\mu}$ such that $\varphi_\mu(f) = \int f d\overline{\mu}$ for all $f \in \mathrm{C}(X)$. Thus, $\overline{\mu}$ is the unique regular Borel measure such that $\int f d\overline{\mu} = \int f d\mu$ for all $f \in \mathrm{C}_+^{\uparrow}(X)$. For an open upper set $U$, the characteristic function $\chi_U$ is lower semicontinuous. By Lemma 1.1, $\chi_U$ is the pointwise sup of a directed family of functions $f_i \in \mathrm{C}_+^{\uparrow}(X)$. Hence $\overline{\mu}(U) = \int \chi_U d\overline{\mu} = \sup_i \int f_i d\overline{\mu} = \sup_i \int f_i d\mu = \int \chi_U d\mu = \mu(U)$. (Here we have used that $f \mapsto \int f d\overline{\mu}$ and $f \mapsto \int f d\mu$ preserve suprema of directed families (see 2.1 and 3.1). This shows (1).

(2) As $\psi = \varphi \circ M$, and as $\psi$ is bijective by 5.2, we conclude that $\varphi$ is surjective. As $\varphi$ is an order embedding by 5.3, hence injective, we conclude that $\varphi$ is bijective, too. As $M = \psi \circ \varphi^{-1}$, it is bijective, too.

(3) follows from Proposition 4.1 in view of (2). (4) follows from 4.2, and (5) follows from Lemma 3.2. □

Different parts of the previous theorem have been proved earlier. Part (1) is due to Lawson [12]. For Lawson compact domains, Jung and Tix [10] have shown that the probablistic powerdomains $\mathbf{V}_{\leq 1}(X)$ and $\mathbf{V}_1(X)$ are again Lawson compact. This is a special case or part (3) of the theorem. M. Alvarez Manilla [1,2] has discovered that the ordering $\prec$ for valuations is more than closely related to the stochastic ordering of probability measures as considered by D.A. Edwards already in 1978 [4]. He has shown that, for compact ordered spaces $\mathbf{M}_{\leq 1}(X)$ and $\mathbf{M}_1(X)$ are compact ordered spaces. Part (5) of the Theorem implies that, for a stably compact space $X$, the probabilistic powerdomains $\mathbf{V}_{\leq 1}(X)$ and $\mathbf{V}_1(X)$ are again stably compact. This has also been proved by Alvarez Manilla [2].

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