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Nonlinear anisotropic diffusion methods for image denoising problems: Challenges and future research opportunities

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A R T I C L E I N F O A B S T R A C T

Dataset link: [https://www.mathworks.com/ma](https://www.mathworks.com/matlabcentral/fileexchange/116260-anisotropic-diffusion-denoising) [tlabcentral/fileexchange/116260-anisotropic-d](https://www.mathworks.com/matlabcentral/fileexchange/116260-anisotropic-diffusion-denoising) [iffusion-denoising](https://www.mathworks.com/matlabcentral/fileexchange/116260-anisotropic-diffusion-denoising)

*Keywords:* Anisotropic diffusion Image processing Inverse problem Noise estimation Restoration method

Nonlinear anisotropic diffusion has attracted a great deal of attention for its ability to simultaneously remove noise and preserve semantic image features. This ability favors several image processing and computer vision applications, including noise removal in medical and scientific images that contain critical features (textures, edges, and contours). Despite their promising performance, methods based on nonlinear anisotropic diffusion suffer from practical limitations that have been lightly discussed in the literature. Our work surfaces these limitations as an attempt to create future research opportunities. In addition, we have proposed a diffusion-driven method that generates superior results compared with classical methods, including the popular Perona–Malik formulation. The proposed method embeds a kernel that properly guides the diffusion process across image regions. Experimental results show that our kernel encourages effective noise removal and ensures preservation of significant image features. We have provided potential research problems to further expand the current results.

# Introduction

Noise, regardless of its source and type, degrades the quality of images, making them less useful in sensitive applications [[1](#_bookmark29)–[3](#_bookmark31)]. For example, noise may corrupt the details of a medical image and cause doctors to incorrectly interpret results of patients. In computer and machine vision applications, noisy images may negatively impact the accuracy and reliability of the object detection and recognition algo- rithms. Motivated by the quest for quality images in a wide range of applications, there has been intensive efforts to develop different methods for suppressing noise in images [[4](#_bookmark32)–[13](#_bookmark33)].

Noise may be generated during acquisition, processing, and trans- mission of an image. For instance, if the imaging sensor contains damaged pixels on its surface, random spots (noises) with varied inten- sities will be introduced into the acquired image because such locations cannot register the incident light. Examples of noise in images include salt & pepper [[14](#_bookmark34)], Gaussian [[15](#_bookmark35),[16](#_bookmark36)], Poisson [[17](#_bookmark37)], shot [[18](#_bookmark38)], and speckle [[2](#_bookmark30),[19](#_bookmark39)], typically modeled as either additive (noise added to the original image) or multiplicative (noise multiplied to the original image).

Several image denoising methods adapted for a range of noise types have been proposed in the literature [[20](#_bookmark40)–[23](#_bookmark41)]. Of the methods, those based on anisotropic diffusion processes have gained a considerable attention for the ability of such methods to simultaneously remove noise and preserve semantically useful image features (textures, edges,

and contours) [[24](#_bookmark42)–[27](#_bookmark43)]. The seminal work by Perona and Malik [[28](#_bookmark44)] has inspired many scholars to develop advanced anisotropic diffusion models for suppressing spurious image features, including noise and undesirable artifacts. These models have demonstrated outstanding results in delicate image denoising applications (e.g., medical imaging) that require quality images with critical features well preserved.

Despite the merits of anisotropic diffusion processes in image de- noising, scholars have not adequately unraveled the possible research opportunities to advance the field. This work opens scholarly dis- cussions to address various limitations of the available anisotropic diffusion models. We highlight recommendations for the possible so- lutions of some limitations as guidance for scholars. Furthermore, a superior method based on nonlinear anisotropic diffusion process is proposed. Compared with classical methods, our method demonstrates outstanding results by generating images with higher perceptual and objective qualities. Some recommendations are provided to further expand the our results.

We have organized our paper as follows: the next section exten- sively reviews anisotropic diffusion when applied in image denois- ing; Section [3](#_bookmark11) gives potential research avenues in anisotropic diffu- sion, providing researchers with immense opportunities to advance this growing field; Section [4](#_bookmark17) describes our anisotropic diffusion method

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**/ig. 1.** Perona–Malik influence functional (flux) versus image gradient: *𝜓*(*𝑠*) =

*𝑠 , 𝐾* = 1.

**/ig. 2.** Our influence functionals versus image gradient: *𝜓*1(*𝑠*) = *𝑠* (

1

1−*𝑎* ( *𝑠* )

– )

*𝐾*

– 1

( )

and

1+( *𝑠* )2

*𝐾*

*𝜓* (*𝑠*) = *𝑠* (

*𝑎*−( *𝑠* ) )

*𝑠*

1+*𝑎 𝐾*

2 1+( *𝑠* )2

*𝐾*

*𝐾*

.

and highlights its competitive advantages; Section [5](#_bookmark25) presents and dis- cusses results; finally, we have concluded the research in Section [6](#_bookmark28) by summarizing our study and proving perspectives.

# Anisotropic diffusion in image processing

a clean image of the lowest possible (global minimum) energy. Sub- sequently, ([2.4](#_bookmark8)) can be derived from the variational minimization problem

arg min {∫ *𝜌*(|∇*𝑢*|)*𝑑𝛺*} *,* (2.5)

In science and engineering disciplines, diffusion generally means

*𝑢*∈*𝛺*

*𝛺*

movement of particles (e.g., atoms, ions, and molecules) between re- gions of different concentrations until equilibrium conditions are es- tablished. This physical process may be explained by the *Fick’s law*

*𝑗* = −*𝐷* ⋅ ∇*𝑢,* (2.1)

which states that the concentration gradient, ∇*𝑢*, generates flux, *𝑗*, with

where *𝛺* denotes the domain of *𝑢*. For completeness, and without loss

(regularization term), *𝜗*, as of generality, ([2.5](#_bookmark2)) usually incorporates a fidelity energy functional

arg min {∫ *𝜌*( ∇*𝑢* )*𝑑𝑥* + *𝜆* ∫ *𝜗*(*𝑢, 𝑓* )*𝑑𝑥*} *,* (2.6)

| |

*𝑢*∈*𝛺*

*𝛺*

*𝛺*

diffusion tensor (positive definite symmetric matrix), *𝐷* [[29](#_bookmark45)] ([Figs.](#_bookmark0) [1](#_bookmark0) a goal of compensating for this gradient at the rate governed by the

region to move at the rate determined by *𝐷* towards the region and [2](#_bookmark1)). The process dictates particles from the highly concentrated

of low concentration. Diffusion obeys the fundamental laws of mass conservation, hence the process does not lead to creation or destruction

of mass, *𝑢*, as described by the *continuity equation*

*𝜕𝑢* = −div*𝑗,* (2.2)

where *𝜆* represents a fidelity parameter that controls a trade-off be- tween regularization (noise removal) and data fidelity. Large value of *𝜆*

over total energy. Furthermore, with a small value of *𝜆*, *𝑢* approaches encourages smoother solutions, in which case regularization dominates the observed noisy image, *𝑓* .

The Euler–Lagrange equation corresponding to the energy func- tional ([2.6](#_bookmark4)) is

0 = −div ( *𝜌*′(|∇*𝑢*|) ∇*𝑢*) + *𝜆𝜗*′(*𝑢, 𝑓* )*, 𝑥* ∈ *𝛺,* (2.7)

∇*𝑢*

*𝜕𝑡*

*𝜕𝑢* | |

with *𝑡* denoting time. Plugging ([2.2](#_bookmark5)) into ([2.1](#_bookmark3)) gives the diffusion

equation

*𝜕𝑛⃗*

= 0*, 𝑥* ∈ *𝜕𝛺.* (2.8)

Embedding ([2.7](#_bookmark6)) into a dynamical scheme gives

*𝜕𝑢* = div(*𝐷* ⋅ ∇*𝑢*) (2.3)

*𝜕𝑡*

*𝜕𝑢*

= div

( *𝜌*′(|∇*𝑢*|)

∇*𝑢*)

– *𝜆𝜗*′(*𝑢, 𝑓* )*,* (*𝑥, 𝑡*) ∈ *𝛺* × (0*, 𝑇* )*,* (2.9)

that models several physical processes involving mass transportation (e.g., carburization and construction of semiconductors through dop- ing).

In image processing and computer vision, *𝑢* denotes the intensity of

an image. Therefore, applying ([2.1](#_bookmark3)) on the image dictates movement of

*𝜕𝑡 𝑢*

*𝑢*(*𝑥,* 0) = *𝑓* (*𝑥*)*, 𝑥* ∈ *𝛺* (2.10)

∇

| |

*𝜕𝑢* = 0*,* (*𝑥, 𝑡*) ∈ *𝜕𝛺* × [0*, 𝑇* ]*.* (2.11)

*𝜕𝑛⃗*

The parameter *𝑇* denotes the maximum evolution time. Considering the

*𝜌*′( ∇*𝑢* )

| |

intensity from high-intensity to low-intensity pixels. This process causes

divergence term of ([2.9](#_bookmark7)), *𝑓* ( ∇*𝑢* ) =

|∇| *𝑢*| |

means diffusivity (diffusion

smoothing where the intensity of image regions with spurious features (noise) gets reduced.

The directions of *𝑗* and ∇*𝑢* determine the type of diffusion: par-

resembles the heat flow equation (or convolution of *𝑢* with a Gaussian allel, isotropic; and non-parallel, anisotropic. The formulation ([2.1](#_bookmark3)) kernel) when *𝐷* remains constant over all image regions that undergo isotropic diffusion. Setting *𝐷* constant tends to smudge edges and

sensitive applications. In anisotropic diffusion, *𝐷* becomes a function other critical image features, a consequence not preferred in many of *𝑢*, allowing it to change spatially across different image regions. For example, *𝐷* = *𝑓* ( ∇*𝑢* ) may be a function of the image gradient to give

| |

coefficient) that governs the rate of noise removal as the solution, *𝑢*,

evolves.

We have selected [[39](#_bookmark55)]

*𝜗*(*𝑢, 𝑓* ) = log *𝑢* + *𝑓* (2.12)

*𝑢*

for its superior performance in dealing with both additive and mul- tiplicative types of noise. Nevertheless, this energy functional of the

noise. Plugging *𝜗*(*𝑢, 𝑓* ) into ([2.9](#_bookmark7)) yields fidelity term performs even better in images corrupted by multiplicative

*𝜕𝑢* = div ( *𝜌*′(|∇*𝑢*|) ∇*𝑢*) − *𝜆 𝑢* − *𝑓 ,* (*𝑥, 𝑡*) ∈ *𝛺* × (0*, 𝑇* )*,* (2.13)

*𝑢*(*𝑥,* 0) = *𝑓* (*𝑥*)*, 𝑥* ∈ *𝛺* (2.14)

|

|

the anisotropic diffusion equation

*𝜕𝑡*

∇*𝑢*

*𝑢*2

*𝜕𝑡* | |

*𝜕𝑢* = div(*𝑓* ( ∇*𝑢* )∇*𝑢*)*.* (2.4)

*𝜕𝑢* = 0*,* (*𝑥, 𝑡*) ∈ *𝜕𝛺* × [0*, 𝑇* ]*.* (2.15)

A well-designed *𝑓* leads to image denoising with edge preservation.

Modeling noise as energy, *𝜌*, in the image, we may regard diffusion

*𝜕𝑛⃗*

Most advances in anisotropic diffusion methods revolve around two

as the energy minimization problem with a solution that represents directions: (1) design of *𝜌*; and (2) estimation of *𝜗* according to the noise

**Table 1**

Energy functionals for anisotropic diffusion equations: *𝜌*(*𝑠* = ∇*𝑢* ), energy functional; *𝜓* (*𝑠*) = *𝜌*′(*𝑠*), influence function; *𝜙*(*𝑠*), diffusivity.

| | *𝜓* (*𝑠*)

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Nomenclature | *𝜌*(*𝑠*) |  | *𝜙*(*𝑠*) | Comments |
| 𝓁2 [[30](#_bookmark46)]  𝓁 [[31](#_bookmark47)] | 1 *𝑠*2  2  *𝑠* | *𝑠*  sgn(*𝑠*) | 1 |  |

1

1 *𝑠*

𝓁 − 𝓁 [[32](#_bookmark48)] *𝐾*2 (√1 + ( *𝑠* )2 − 1) *𝑠*

√

√ *𝐾* ∈ R+

1

1 2 *𝐾*

1+( *𝑠* )2

1+( *𝑠* )2

𝓁 [[33](#_bookmark49)] *𝑠𝜈*

sgn(*𝑠*)*𝑠𝜈*−1 *𝑠𝜈*−2 0 *< 𝜈* ≤ 2

*𝑝 𝜈* [

‘‘Fair’’ *𝐾*2

*𝐾 𝐾*

( )]

*𝑠* − log 1 +

*𝑠*

1

Huber [[34](#_bookmark50)]

*𝐾 𝐾*

⎧⎪ 1 *𝑠*2 )

⎨ 2 (

{

1

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*𝑠*

*𝐾*

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1+

{*𝑠*

*𝑠*

*𝐾*

1 if *𝑠 𝐾*

*𝐾*

*𝑠*

*𝐾 𝑠* − 2 *𝐾*

⎪

⎩ ( ( ) )

*𝐾*sgn(*𝑠*)

if *𝑠 > 𝐾*

Cauchy [[28](#_bookmark44)] *𝐾*2 log

2

1 +

Geman–McClure [[35](#_bookmark51),[36](#_bookmark52)] *𝑠*2

2(1+[*𝑠*2 )

Welsch [[37](#_bookmark53)] *𝐾*2

2

*𝑠* 2

*𝐾*

( (

1 − exp

–

*𝑠*

*𝐾*

)2 )]

*𝑠*

*𝑠* 2

1+( )

*𝐾*

*𝑠*

(1+*𝑠*2 )2( (

*𝑠* exp

–

)2 )

*𝑠*

*𝐾*

1

*𝑠* 2

1+( )

*𝐾*

1 (1+*𝑠*2()2 (

exp

–

)2 )

*𝑠*

*𝐾*

⎧ *𝐾*2 (1 − [1 − ( *𝑠* )2 ]3 )

⎧*𝑠* [1 − (

*𝑠* )2]2

⎧[1 − (

*𝑠* )2]2

{if *𝑠 < 𝐾*

Tukey [[38](#_bookmark54)]

⎪⎩ *𝐾*2

⎪⎨ 6

6 (√(

weighted 𝓁1 − 𝓁2 *𝐶𝐾*

1 +

2

*𝐾*

( ) ) )

2

*𝑠*

*𝐾*

– 1

*𝐾*

⎪⎩0

1+

*𝐶𝑠*

(

)2

⎪⎨

√

*𝑠*

*𝐾*

*𝐾*

⎪⎩0

*𝐶*

(

0 *< 𝐶* ≤ 3*.*5

)2

⎪⎨

√

1+

*𝑠*

*𝐾*

if *𝑠* ≥ *𝐾*

statistics. In the former direction, scholars strive to design effective energy functionals with better mathematical properties, including con- vexity and uniqueness. Quality of the optimal solution of ([2.9](#_bookmark7)) depends

on the design of *𝜌*. The later direction requires a solid understanding

diffusion model depends upon how well *𝜗* adapts to the noise type of the noise statistics in the image. Performance of the anisotropic

(e.g., additive, multiplicative, and mixed).

# Potential research avenues in anisotropic diffusion

* 1. *Design of energy functionals*

solid understanding of its corresponding energy functional, *𝜌*. Superior Formulation of an effective anisotropic diffusion equation requires a denoising results may be achieved when *𝜌* possesses proper math-

the variational problem governing *𝜌* should guarantee existence and ematical properties, such as convexity and uniqueness. specifically, uniqueness of the minimizer, *𝑢* ∈ *𝛺*, in ([2.6](#_bookmark4)). Despite these important

requirements, no systematic procedures have been established to design

local image features. Although *𝜙*(*𝑠*) proposed by Guo et al. generates

convincing practical results, the corresponding formulation for

*𝜌*(*𝑠*) = *𝑠𝜙*(*𝑠*)*𝑑𝑥* (3.2)

∫

*𝛺*

cannot produce a closed-form solution to associate *𝜌* and quality of

outperforms because *𝛼* changes adaptively to produce different other the results. Guo et al. explain that their anisotropic diffusion model models during the diffusion process: linear, *𝛼* = 0; Total Variation,

*𝛼* = 1; and Perona–Malik, *𝛼* = 2. This explanation, however, lacks scientific justification and generalization that changing *𝛼* adaptively

model behaves when *𝛼* becomes a fraction, linking quality of results always lead to better results. We cannot state mathematically how the

relationship between *𝛼* and local image features: which value of *𝛼*— and the diffusion process. In addition, the authors did not provide the

translating to the anisotropic diffusion model (linear, Total Variation, or Perona–Malik)—should be applied to an image region for optimal denoising results?

In this work, we propose two diffusivity functionals, namely

1

( *𝑠* )

–

*𝜌*. Scholars have proposed different energy functionals ([Table](#_bookmark10) [1](#_bookmark10)) with-

*𝜙* (*𝑠*) = 1 − *𝑎 𝐾*

(3.3)

out clear guidelines on what it takes to derive such functionals. Some interesting questions for further investigation would be (1) does an

optimal *𝜌* exist for all denoising applications? (2) what are systematic

1

1 + *𝑎*

( *𝑠* )

and

– 1

*𝐾*

procedures for designing *𝜌*? (3) what specific mathematical proper-

*𝐾*

( )

*𝜙* (*𝑠*) = *𝑎*

−( *𝑠* )

*,* (3.4)

ties should be considered when designing *𝜌* for denoising problems?

(4) how does *𝜌* behave at different scale resolutions of the evolving solution, *𝑢*?

functionals, *𝜙*(*𝑠*) = *𝜌*′(*𝑠*)∕*𝑠*, directly without considering the properties We noted a general practice that most authors design diffusivity of *𝜌*. This approach seems straightforward because *𝜙*(*𝑠*) possesses a

2 1 + *𝑠* 2

where *𝑎 >* 0 denotes a constant, which generate promising results.

*𝐾*

These functionals, derived empirically through experiments, satisfy the

conditions highlighted by Perona and Malik [[28](#_bookmark44)]:

1 2 {

decreasing functional with *𝜙*(0) = 1 and *𝜙*(∞) = 0. Perona and well-known mathematical property [[28](#_bookmark44)]: non-negative monotonically

*𝜙* (*𝑠*) = *𝜙* (*𝑠*) = 0 if *𝑠* → ∞ (Edges and contours)

1 if *𝑠* → 0 (Flat regions)

(3.5)

Malik [[28](#_bookmark44)] highlight that any well-designed *𝜙*(*𝑠*) with this property may

derive *𝜌* from *𝜙*(*𝑠*) to learn the properties of *𝜌*—an important functional generate satisfactory results. However, it may not always be possible to

to explain why the anisotropic diffusion equation works. For example, Guo et al. [[40](#_bookmark56)] proposed

*𝜙*(*𝑠*) = 1 *,* (3.1)

1 + ( )*𝛼*

*𝑠*

*𝐾*

where *𝐾* denotes the shape-defining tuning constant and *𝛼* ∈ [0*,* 1]

defines an adaptive variable exponent that changes according to the

Eq. ([3.4](#_bookmark13)), with *𝑎* = 2*.*7183 (exponential function), reduces to the

functional similar to that proposed by Perona and Malik [[28](#_bookmark44)].

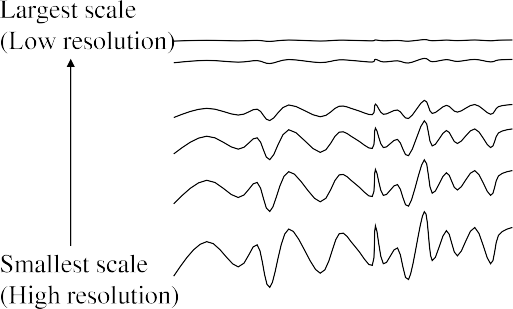
There could be several other variations of diffusivity functionals with outstanding performance. However, studies are yet to provide sys- tematic mathematical procedures to establish such functionals. There-

fore, although *𝜙*1 and *𝜙*2 proposed in this work performs well, we

cannot mathematically confirm that they produce optimal results with

respect to all variants of diffusivity functionals. This research direc- tion may be interesting and worth investigating to advance denoising methods based on anisotropic diffusion.

* 1. *Design of shape-defining constant*



The diffusivity Eq. ([3.1](#_bookmark14)) contains a shape-defining constant, *𝐾*, the

threshold parameter of the gradient magnitude. This tuning constant controls the sensitivity of diffusion to edges, and is usually determined experimentally or as a function of the noise statistics in the image [[41](#_bookmark57)– [44](#_bookmark58)]. Quality of the denoised image depends, to a greater extent, on

the value of *𝐾*. Therefore, effective strategies are needed to optimize

*𝐾* for better denoising results. However, the available approaches for determining *𝐾* are manual, hence time-consuming, inconvenient, and

error-prone. Scholars may need to investigate the relationship between

*𝐾* and the structural information in the images. One interesting finding could be the value of *𝐾* that changes adaptively depending upon the

local image features, and this value should work across all types of images.

* 1. *Design of regularization term*

There have been efforts to design regularization (fidelity) terms for different noise types (e.g., additive and multiplicative) [[45](#_bookmark59)–[50](#_bookmark60)]. Due to the random nature of noise, designing regularization terms remains an open-ended research question and a non-trivial task. Given a noisy image, an interesting question would be to estimate the prob- ability density function (PDF) that generates noise contained in the image. Noise models assume specific PDFs [[51](#_bookmark61)–[53](#_bookmark62)]: Gaussian, Gamma, Rayleigh, exponential, impulse, uniform, among others. Establishment of the regularization term requires a comprehensive understanding and analysis of these PDFs. In some situations, experiments should be performed to establish the relationship between noise statistics and structural information in the image.

Another aspect worth considering is the fidelity parameter, *𝜆*, of

manual strategy of tuning *𝜆* introduces challenges in the implemen- the regularization term, which is usually determined empirically. The

recommended adaptive approaches for determining *𝜆* as the denoising tation of anisotropic diffusion models. Some authors have, however, image, *𝑢*, evolves through iterations [[40](#_bookmark56),[54](#_bookmark63)]. Guo et al. [[40](#_bookmark56)], for example, estimates *𝜆* by evaluating its value from ([2.9](#_bookmark7)) under steady

a more accurate value of *𝜆* may be obtained after convergence of the conditions (when time tends to infinity). This approach implies that solution. Advanced techniques are needed to accurately determine *𝜆*

statistics of previous (premature) solutions to estimate *𝜆* iteratively. after every iteration. We may investigate structural features and noise

* 1. *Applications in inverse problems*

Given observations from the system under investigation, an inverse problem involves determination of parameters characterizing the sys- tem [[55](#_bookmark64)–[57](#_bookmark65)]. For example, the image denoising problem provides a noisy image and requires determination of the clean (original) image. Inverse problems tend to generate multiple undesirable solutions be- cause of their ill-posed nature: fewer observations than the number of system parameters under investigation. The solution of an inverse problem may be unstable, amplifying errors in the intended solution from the arbitrarily small errors in the measurement data [[58](#_bookmark66)].

Anisotropic diffusion has been widely used to address the ill-posed nature of inverse problems. Despite its effectiveness in image denoising, a paucity of studies exist to analyze the impact of anisotropic diffusion in other image processing techniques, such as compression, segmenta- tion, inpainting, and tomographic image reconstruction [[29](#_bookmark45)]. Another possible research avenue can be the application of anisotropic diffusion in neural networks for learning a priori information [[59](#_bookmark67)].

**/ig. 3.** One-dimensional signal at different scale resolutions.

* 1. *Expansion into high-dimensional space*

Most anisotropic diffusion methods focus on two-dimensional gray- scale images. In recent years, researchers have developed interest in processing higher-dimensional and vector-valued images because of the increased computational power and availability of color printers and scanners. Extending the methods into this direction will mark an important milestone in sensitive practical applications, including medi- cal and radar imaging. This scientific inquiry requires computationally inexpensive methods for denoising higher-dimensional images without destroying semantic features.

* 1. *Convergence criterion setting*

The evolving solution, *𝑢*, in ([2.9](#_bookmark7)) should be tested for optimal- ity at every iteration. The dynamical system requires an appropriate stopping criterion to generate an optimal solution after some itera-

tions [[60](#_bookmark68)–[62](#_bookmark69)]. The fundamental property of a criterion is that it should

approach a specific preset value, upper or lower bound, as the system advances through iterations. The bound corresponds to the quality of an image. Upon meeting the criterion conditions, the generated image becomes of the highest possible quality. For image denoising problems, the optimal image at the exit point of the criterion contains minimum error corresponding to acceptable values of quality metrics (e.g., peak-signal-to-noise ratio and structural similarity).

Setting the convergence rules remains a long-standing problem in iterative methods, including those intended to remove noise in images. The primary questions are (1) at which specific point in time should we stop the iterations? (2) what is the lower/upper bound to mark the exit condition for the iterating dynamical system? how should we establish a balance between fidelity of results and computational time? These questions may stimulate scholarly discussions to address implementation challenges of iterative methods.

* 1. *Relationship of features at different scale-space solutions*

The denoising (diffusion) process creates a series of solutions at varied scales. Each solution corresponds to an image with features smoothed to a specific degree of resolution ([Fig.](#_bookmark16) [3](#_bookmark16)). The solution at an infinity scale contains a constant image characterized by uniform distribution of gray levels. Image features at the lowest scale have the highest resolution, presenting well-distinguished and visually appealing structures. [Fig.](#_bookmark1) [2](#_bookmark1) shows a one-dimensional signal at the bottom con- volved (smoothed) by a Gaussian kernel with an increasing variance from bottom to top [[28](#_bookmark44),[63](#_bookmark70)].

Despite the well-established knowledge that diffusion leads to scale- space solutions [[28](#_bookmark44)], the fundamental question remains to determine the mathematical relationship of the signal features at different scales. In essence, after the convolution by a given kernel (e.g., Gaussian), how do these features relate mathematically? The relationship may

enhance our understanding on how the diffusion process works. For example, we will understand how and why pixels diffuse across image regions to reduce noise and spurious features. Furthermore, inverse operations may be established to reconstruct the original (noisy) signal after diffusion.

* 1. *Implementation schemes*

For decades, explicit numerical schemes have been widely ap- plied to realize and implement partial differential equations (PDEs) for nonlinear diffusion models. These schemes provide simpler and

# Proposed anisotropic diffusion method

We applied a four-point explicit numerical scheme to implement the proposed anisotropic diffusion model. The essence of this implemen- tation is to demonstrate an important research opportunity described in Section [3.2](#_bookmark15): *design of shape-defining constant*. Specifically, the im- plementation highlights that setting tuning constants (of the diffusion kernel) to achieve optimal results may be a rather daunting and time- consuming process. For completeness, our model is compared with the classical Perona–Malik model.

Discretizing ([2.13](#_bookmark9)) under the explicit scheme yields

quicker implementation strategies, but become increasingly inefficient and ineffective in time-sensitive applications that demand larger time steps. Explicit (Euler forward) schemes produce better results under the Courant–Friedrichs–Lewy (CFL) criterion [[64](#_bookmark71)] that restricts a time step between 0 and 0.25 to achieve stable and quality results. This

*𝑢𝑛*+1 − *𝑢𝑛*

△*𝑡*

*𝑖𝑗*

*𝑖𝑗* = div

where

*𝑖𝑗*

( *𝜌*′( ∇*𝑢𝑛* ) )

| *𝑖𝑗* |

∇*𝑢𝑛*

*𝑖𝑗*

– *𝜆*

+ *𝜖*

| |

– 1

⎛

⎞

⎛ |∇*𝑢𝑛* | ⎞

| + ⎟⎟ ⎟

*𝑢𝑛* − *𝑓 𝑛*

(*𝑢𝑛* )2 + *𝜖*

*𝑖𝑗*

*𝑖𝑗*

*𝑖𝑗*

(4.1)

restriction makes such schemes ineffective in complex computer vision and image processing tasks.

*𝜌*′(|∇*𝑢𝑛* |) = |∇*𝑢𝑛* | ⎜ 1 − *𝑎*

⎜⎜⎝ |

*𝑖𝑗*

*𝐾 𝜖*

⎠ ⎟

(4.2)

Given the limitations of explicit schemes, researchers should investi-

*𝑖𝑗*

*𝑖𝑗*

⎜

– 1

⎛ |∇*𝑢𝑛* | ⎞

⎟

gate and establish more advanced and efficient numerical methods for

⎜ ⎜ |

*𝑖𝑗* | +*𝜖*⎟ ⎟

PDEs [[29](#_bookmark45)]: implicit schemes, splitting and multigrid techniques, and

*𝐾*

⎝ 1 + *𝑎* ⎝

⎜ ⎜

⎟⎠ ⎟⎠

grid adaptation strategies. Weickert [[29](#_bookmark45)] argues researchers to develop advanced software packages that implement different nonlinear diffu- sion filters. Supporting the open science [[65](#_bookmark72),[66](#_bookmark73)], the packages should

be made publicly available across the research community.

and *𝜖 ≪* 0+ denotes the small constant that prevents numerical insta-

bility.

Therefore,

( ( *𝜌*′(|∇*𝑢𝑛* |) ) *𝑢𝑛* − *𝑓 𝑛* )

*𝑢𝑛*+1 = *𝑢𝑛* + △*𝑡*

div

*𝑖𝑗*

– *𝜆*

*𝑖𝑗*

*𝑖𝑗*

*.* (4.3)

*𝑖𝑗*

*𝑖𝑗*

*𝑖𝑗*

∇ *𝑛*

+ *𝜖*

*𝑛*

2

| |

*3.9. Noise estimation*

Development of regularization terms require a comprehensive kno- wledge of the noise statistics in an image. Effective methods for noise estimation should be devised as an important milestone to achieve robust nonlinear anisotropic diffusion models [[67](#_bookmark74)–[70](#_bookmark75)]. However, es- tablishment of such methods seems challenging due to the random nature of noise. This open-ended research problem may be addressed using machine learning methods that can learn complex and random relationships of noise variables [[71](#_bookmark76)]. Also, compressive sensing and stochastic techniques [[72](#_bookmark77)–[74](#_bookmark78)] may be employed to model image noise. The fundamental question is to derive noise PDF from the corrupted image.

*3.10. Image quality evaluation metrics*

Quality of results generated by restoration methods, including the ones discussed in our work, should be gauged using suitable assess- ment metrics. Currently, most authors prefer peak-signal-to-noise ratio (PSNR) [[75](#_bookmark79)] and structural similarity (SSIM) [[76](#_bookmark80)] metrics for image quality assessment (IQA). However, PSNR and SSIM require complete information of the reference image that may not be available in prac- tical applications [[77](#_bookmark81)]. When executing a real denoising problem, the original (clean) image is unavailable. In some few cases, we may have a small portion of information from the original image. Researchers

*𝑢𝑖𝑗* (*𝑢𝑖𝑗* + *𝜖*)

The parameters were set as follows: △*𝑡* = 0*.*15 (CFL criterion), *𝜆* = 0*.*05,

*𝜖* = 0*.*000001. We determined the values of *𝑎* and *𝐾* empirically based

set {*𝑎, 𝐾*} corresponding to higher values of PSNR [[75](#_bookmark79)] and SSIM [[76](#_bookmark80)] on the quality of results. Experiments were conducted and an optimal

was selected. The former metric gives the signal strength relative to the noise content in the image. Higher PSNR values are preferred for quality images. Despite its wide application, PSNR lacks human perceptual qualities. SSIM, which ranges between 0 and 1, addresses this weakness by emulating the human visual system. Quality images give higher values of SSIM.

Experiments were conducted on four types of images downloaded from public domains: Pout,[1](#_bookmark19) Mandril, Walkbridge,[2](#_bookmark20) and Cat.[3](#_bookmark21) In the first experiment, speckle noise of standard deviation 25.50 was added into original images of Pout, Mandril, and Walkbridge. Next, our method was applied to remove noise from these images and the results were compared with those generated by the Perona–Malik method [[28](#_bookmark44)]. The second experiment focused on testing the performance of our method under heavy noise conditions. Therefore, the original image of Cat was corrupted by additive Gaussian noise of standard deviations 40, 50, and

60. Next, the methods by Monteil and Beghdadi [[84](#_bookmark84)], Tebini et al. [[85](#_bookmark85)], Gupta et al. [[86](#_bookmark86)], Rezgui et al. [[87](#_bookmark87)], and ours were applied to restore the original image. In all experiments, we applied PSNR and SSIM to gauge quality of the denoised images. Implementation codes have been shared for researchers to reproduce our results.[4](#_bookmark22)

may need to divert their attention to other types of metrics for im-

age quality assessment, including reduced-reference and no-reference IQA metrics [[78](#_bookmark82)–[83](#_bookmark83)]. Subjective IQA approaches, while they remain popularly used, should not be drawn based on the author’s perceptions and experience. Other people should be engaged in the quality assess- ment process, and voting statistics employed as the basis for drawing conclusions.

1 [https://www.mathworks.com/help/images/image-import-and-](https://www.mathworks.com/help/images/image-import-and-export.html) [export.html](https://www.mathworks.com/help/images/image-import-and-export.html)

2 [http://www.imageprocessingplace.com/root\_files\_V3/image\_databases.](http://www.imageprocessingplace.com/root_files_V3/image_databases.htm)

[htm](http://www.imageprocessingplace.com/root_files_V3/image_databases.htm)

3 <http://www.hpca.ual.es/~vruiz/images/>

4 [https://www.mathworks.com/matlabcentral/fileexchange/116260-](https://www.mathworks.com/matlabcentral/fileexchange/116260-anisotropic-diffusion-denoising) [anisotropic-diffusion-denoising](https://www.mathworks.com/matlabcentral/fileexchange/116260-anisotropic-diffusion-denoising)

**Table 2**

Performance of Perona–Malik method and our method under optimal tuning parameters, {*𝑎, 𝐾*}.

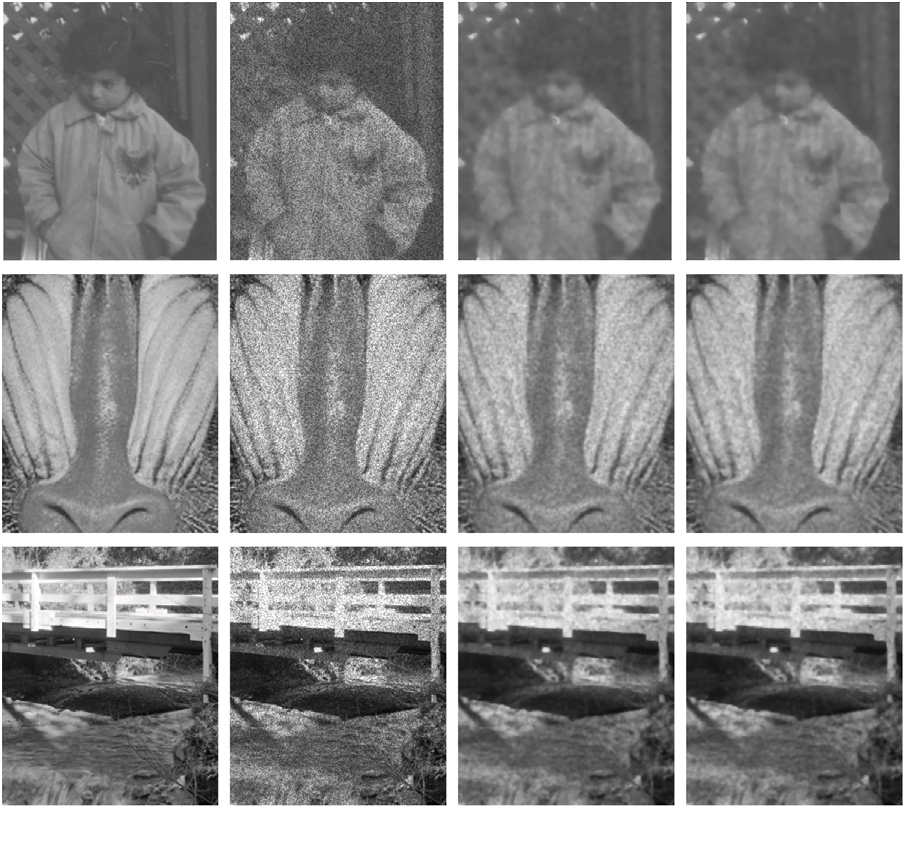
|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Method | Pout |  |  | Mandril |  |  |  | Walkbridge |  | |
|  | PSNR | SSIM {*𝑎, 𝐾*} |  | PSNR | SSIM | {*𝑎, 𝐾*} |  | PSNR | SSIM | {*𝑎, 𝐾*} |
| Perona–Malik method [[28](#_bookmark44)] | 33.3637 | 0.8738 {−*,* 5} |  | 26.0017 | 0.8188 | {−*,* 87} |  | 25.2815 | 0.8630 | {−*,* 90} |
| Our method | 33.5270 | 0.8780 {7*,* 5} |  | 26.0013 | 0.0.8189 | {6*,* 48} |  | 25.2905 | 0.8628 | {9*,* 51} |

**Table 3**

by heavy Gaussian noise. (*𝜎* and *𝑡* denotes standard deviation and iteration number, respectively.) Peak-signal-to-noise ratio (PSNR) and structural similarity (SSIM) values of different anisotropic diffusion methods for the cat image corrupted

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Method | *𝜎* = 40*, 𝑡* = 30 |  |  | *𝜎* = 50*, 𝑡* = 35 |  |  | *𝜎* = 60*, 𝑡* = 40 |  |
|  | PSNR | SSIM |  | PSNR | SSIM |  | PSNR | SSIM |
| Perona–Malik method [[28](#_bookmark44)] | 20.7438 | 0.4500 |  | 17.3968 | 0.2915 |  | 16.3042 | 0.2502 |
| Monteil method [[84](#_bookmark84)] | 22.2268 | 0.5824 |  | 19.8749 | 0.4245 |  | 18.8512 | 0.3624 |
| Tebini [[85](#_bookmark85)] | 22.9435 | 0.6201 |  | 20.2365 | 0.4582 |  | 19.0856 | 0.3983 |
| Gupta method [[86](#_bookmark86)] | 23.8332 | 0.6618 |  | 20.9656 | 0.5078 |  | 21.4205 | 0.5433 |
| Rezgui et al. [[87](#_bookmark87)] | 20.3328 | 0.5399 |  | 18.5991 | 0.4671 |  | 17.1711 | 0.4083 |
| Proposed method (*𝑎* = 13*, 𝐾* = 19) | *𝑡* = 5  23.8570 | 0.7668 |  | *𝑡* = 6  22.5464 | 0.7137 |  | 21.5094 | 0.6683 |

# Results and discussions



The current work demonstrates the capabilities of methods based on nonlinear anisotropic diffusion in restoring degraded images with- out destroying important features. Despite their notable advantages, potential weaknesses of these methods have been discussed to inspire early career researchers in developing more advanced and effective nonlinear anisotropic diffusion methods. We have indicated interesting research avenues in this promising field, the goal being to expand the application domain of such methods in denoising corrupted images. Supported by our experimental results, researchers may raise critical questions leading to improvement of the proposed method or other classical methods in the literature.

Results justify the capability of our approach to generate images with higher perceptual and objective qualities ([Fig.](#_bookmark27) [4](#_bookmark27)). Compared with the results generated by the Perona–Malik method, our images contain visually appealing and attractive features. Specifically, the proposed approach suppresses noise and reconstructs visual semantic features. Even for the complex images, such as Baboon ([Fig.](#_bookmark27) [4](#_bookmark27), second row), our approach generates acceptable results by preserving plausible features (e.g., fur, edges, and textures). Intuitively, we can observe from the results ([Fig.](#_bookmark27) [4](#_bookmark27), far-right column) that the current method properly balances between visual appealingness and noise removal in accor- dance with the human visual system. This encouraging observation may be explained from the structure of our diffusion kernel ([3.3](#_bookmark12)) that it provides an effective strategy for steering the diffusion process across

kernel parameters, *𝑎* and *𝐾*. We noted that the image type dictates the image regions. However, this benefit comes at the expense of tuning the

{*𝑎, 𝐾*} search space, which seems non-intuitive as it requires heuristic

more efficient algorithms for computing the values of *𝑎* and *𝐾* that approaches to determine optimal values. Future studies may consider

produce optimal results.

The proposed method performs well on objective quality metrics ([Tables](#_bookmark23) [2](#_bookmark23) and [3](#_bookmark24)[5](#_bookmark26)). The method gives competitive values of PSNR and SSIM even for an image with a higher noise density. More importantly, our method converges faster that other methods, thus making it more suitable in real-time computing devices. For a Cat image degraded by Gaussian noise of standard deviation 40, the proposed method converges with only five iterations—sixfold convergence rate faster than the classical methods.

Despite its higher computational speed, the tuning parameters,

**/ig. 4.** Noise removal from noisy images of Pout, Mandril, and Walkbridge, scaled and cropped for presentation purposes.

implementation challenges in practical systems. The challenge may be addressed by establishing the relationship between parameters and image structural–statistical information. Machine learning and fitting techniques may be applied to establish the relationship, which should then be embedded into the diffusion kernel. As the dynamical system ([4.1](#_bookmark18)) advances, the parameters will be adaptively updated to steer the system towards quality results.

We observed from empirical experiments that the fractional fidelity term ([2.13](#_bookmark9)) by Lu et al. [[39](#_bookmark55)] promotes quality results for both additive and multiplicative noise. But more efforts are needed to establish superior fidelity terms that adapt to a variety of noise types, including the ones with mixed PDFs co-existing in the same image. In this regard, the proposed diffusion Eq. ([2.9](#_bookmark7)) may be modified as

{*𝑎, 𝐾*}, of our method needs manual tuning to achieve optimal re-

*𝑖*

sults ([Table](#_bookmark23) [2](#_bookmark23)). This time-consuming process may give design and

*𝜕𝑢* = div

*𝜕𝑡*

( *𝜌*′( ∇*𝑢* )

| *𝑢*|

∇

| |

∇*𝑢*) −

*𝑁*

*𝑖*=1

∑

*𝜆𝑖𝜗*′(*𝑢, 𝑓* )*,* (5.1)

5 For fair comparison, some results from this Table were adapted from the work by Gupta et al. [[86](#_bookmark86)].

where *𝑁* denotes the number of fidelity terms dependent upon the

of *𝜆* and expressions defining *𝜗*′(*𝑢, 𝑓* ). nature and type of noise. The question remains to determine the values

# Conclusion

The current work highlights potential advantages of nonlinear anisotropic diffusion methods for noise removal in corrupted images. Given their wide applications in image processing and computer vision, it seems reasonable to further improve the methods and make them more useful in theoretical and practical settings. Therefore, limita- tions of the methods have been extensively discussed as an attempt to create awareness and establish research opportunities within the scholarly community. Furthermore, we have proposed a nonlinear anisotropic diffusion method that generates promising results relative to the classical Perona–Malik formulation.

# Declaration of competing interest

The authors declare that they have no known competing finan- cial interests or personal relationships that could have appeared to influence the work reported in this paper.

# Data availability

The link to the implementation source code has been included in the [manuscript.](https://www.mathworks.com/matlabcentral/fileexchange/116260-anisotropic-diffusion-denoising) [https://www.mathworks.com/matlabcentral/fileexchange/](https://www.mathworks.com/matlabcentral/fileexchange/116260-anisotropic-diffusion-denoising) [116260-anisotropic-diffusion-denoising](https://www.mathworks.com/matlabcentral/fileexchange/116260-anisotropic-diffusion-denoising)

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