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A Cartesian Closed Category of Domains with Almost Algebraic Bases

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**Abstract**

In this paper, we investigate the properties of almost algebraic domains introduced by G. Hamrin and V. Stoltenberg-Hansen in 2006. We introduce a notion of *M*-closed basis and define a new class of domains, called *ωAML*-domains, which are continuous *L*-domains endowed with countable, almost algebraic, and *M*-closed bases. The main result of this paper is: the class of *ωAML*-domains is closed under function spaces and finite cartesian products. Hence, the category of *ωAML*-domains together with Scott continuous functions is cartesian closed. The results of this paper give an answer to an open problem posed by G. Hamrin and V. Stoltenberg-Hansen in “Two categories of effective continuous cpos, Theoretical Computer Science, 365(2006), 216-236”.

*Keywords:* almost algebraic, *L*-domain, *M*-closed basis, cartesian closed category

# Introduction

Domain theory was introduced by Dana Scott [[14,](#_bookmark14) [15](#_bookmark15)] in the late sixties as a mathematical tool to model the denotational semantics of programming languages. For the sake of defining the semantics of higher-order functions in programming, one must consider its counterpart in the formation of function spaces of domains that correspond to the language features of interest. So it is a natural requirement that the considered category of domains should be cartesian closed, and hence seeking various cartesian closed categories of domains is one of the most important problems

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in domain theory. Many well-known kinds of domains had been found forming cartesian closed categories together with Scott continuous functions as morphisms, such as continuous (algebraic) lattices, bounded complete domains, Scott domains

[[15](#_bookmark15)] and SFP domains [[16](#_bookmark16)], and so on. In 1990, the maximal cartesian closed full subcategories of the category of continuous (algebraic) domains were fully classified by Achim Jung in [[8,](#_bookmark4) [9](#_bookmark9)].

In 2006, G. Hamrin and V. Stoltenberg-Hansen [[6](#_bookmark5)] investigated a class of special continuous domains, within which each domain has an *almost algebraic* basis. An inspiration for the notion of an almost algebraic basis is due to [[17](#_bookmark17)]. In that paper a

basis with the inverse approximation property is considered, resulting in a cartesian closed subcategory of the bounded complete domains. The notion of an almost algebraic basis is also related to Tang [[18](#_bookmark18)], who considered conditions on a basis in order to obtain a cartesian closed category of continuous lattices. These extra conditions on a basis are under suitable assumptions equivalent to a basis being almost algebraic.

In [[6](#_bookmark5)], Hamrin and Stoltenberg showed that the class of continuous domains with countable, closed and almost algebraic bases is closed under function spaces and finite cartesian products. Hence, the category consisting of such objects is cartesian closed. In particular, they showed that the notion of almost algebraic is beneficial to study effective domains. Here, a basis *B* of a domain is called closed provided that it is closed under the least upper bounds of all bounded finite subsets of *B*, thus a domain with a closed basis is exactly bounded complete. As mentioned above, Jung [[8,](#_bookmark4)[9](#_bookmark9)] showed that the maximal cartesian closed subcategories of the category of continuous domains consist of continuous *L*-domains or *FS*-domains. It is possible that there are some new cartesian closed subcategories of continuous domains with the property almost algebraic. To obtain a new cartesian closed category of domains, the preservation of the required properties by function spaces is the critical factor and also the essential difficulty. For example, see [[4,](#_bookmark6) [7,](#_bookmark7) [9,](#_bookmark9) [13,](#_bookmark13) [16,](#_bookmark16) [19](#_bookmark19)]. So Hamrin and Stoltenberg [[6](#_bookmark5)] posed the following open problem:

*Find a further condition on the basis weaker than being closed but which is preserved and preserves almost algebraicity under the function space construction.*

In this paper, we will answer this question. We introduce a new notion for a base of a continuous *L*-domain, called *M-closed*, which is strictly weaker than being closed. We show that the class of all continuous *L*-domains endowed with

countable, almost algebraic and *M*-closed bases is closed under function spaces and finite cartesian products. Hence, the category consisting of such *L*-domains is cartesian closed.

The paper is organized as follows. Section 2 introduces some notions and def- initions we need. Section 3 discusses the properties of almost algebraic domains. Section 4 investigates the properties of continuous *L*-domains endowed with count- able, almost algebraic and *M*-closed bases and the function spaces among them. Section 5 gives a cartesian closed category of continuous *L*-domains with almost algebraic bases.

# Preliminaries

We first review some basic knowledge of domain theory. The reader can find more details and proofs in [[1](#_bookmark1)] and the textbooks [[2,](#_bookmark2) [5](#_bookmark8)]. Let *A* and *B* be sets. As usual, we let *A ⊆ B* denote that *A* is a subset of *B*. We also let *A ⊆f B* denote that A isa finite subset of *B*. A partially ordered set (poset) (*D*; *≤*) is a set *D* with a reflexive, transitive and antisymmetric relation *≤*. For *A ⊆ D*, we set

*↓A* = *{x ∈ D* : *∃a ∈ A, x ≤ a}, ↑A* = *{x ∈ D* : *∃a ∈ D, a ≤ x},*

and *A* is called a lower or upper set, if *A* = *↓A* or *A* = *↑A* respectively. For an element *a ∈ D*, we use *↓a* or *↑a* instead of *↓{a}* or *↑{a}*. A subset *A* of *D* is called directed if it is nonempty and every nonempty finite subset of *A* has an upper bound in *A*. Particularly, we say that *D* is a *dcpo* if every directed subset *A* of *D* has a least upper bound (denoted by *A*) in *D*. In this case, when *D* has a least element (denoted by *⊥*), we call it a *cpo*.

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Let *D* be a dcpo. For *x, y ∈ D*, we say that *x* is way-below *y*, denoted by *x y*, if for any directed subset *A* of *D*, *y ≤ A* implies *x ≤ a* for some *a ∈ A*. For *x ∈ D*, we set

W

***↓****x* = *{y ∈ D* : *y x},* ***↑****x* = *{y ∈ D* : *x y}.*

**Definition 2.1** Let *D* be a dcpo. A subset *B ⊆ D* is a basis (or a base) for *D* if for each *x ∈ D*, *B ∩* ***↓****x* is directed and *x* = (*B ∩* ***↓****x*). Particularly, when *B* is countable we say that *B* is an *ω*-base of *D*.

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A basis B is *reduced* if for all *b ∈ B* we have ***↑****b /*= *∅*. If *B* is a basis for a dcpo *D* then *{b ∈ B* : ***↑****b /*= *∅}* is also a basis for *D*. A basis *B* is *closed* if for all *b, c ∈ B* such that *↑b ∩ ↑c /*= *∅* we have that *b ∨ c* exists and *b ∨ c ∈ B*.

A dcpo *D* is continuous if it has a basis, it is also said to be a continuous domain when it is a cpo. An element *d ∈ D* is called compact if *d d*. Let *K*(*D*) be the set of all compact elements of *D*. A dcpo *D* is called algebraic if *K*(*D*) is a basis.

**Lemma 2.2** *Let D be a continuous dcpo with a base B.*

1. *For a ﬁnite M ⊆ D and x ∈ D,*

*M x ⇒ ∃b ∈ B, M b x,*

*where M x means m x for all m ∈ M.*

1. *If B is a ω-base, then for each x ∈ D there exists a sequence* (*xn*)*n∈ω ⊆ B such that*

*and x* = W*n∈ω xn.*

*x*1 *x*2 *· · · xn xn*+1  *x*

Next, we describe the continuous functions between dcpos or cpos.

**Definition 2.3** Let *D* and *E* be two dcpos.

* 1. A function *f* : *D → E* is called (Scott) *continuous* if it is monotone and for all directed sets *A ⊆ D* we have *f* (W *A*)= W *f* (*A*).
  2. The *function space*, denoted by [*D → E*], is the set of all Scott continuous functions from *D* into *E* ordered by the pointwise order, i.e., *f ≤ g* iff *f* (*x*) *≤ g*(*x*) in *E* for all *x ∈ D*.

Let **DCPO** be the category of all dcpos with Scott continuous functions. Then **DCPO** is cartesian closed. Moreover, a full subcategory of **DCPO** is cartesian closed iff it is closed under continuous function spaces and finite products [[1,](#_bookmark1) [16](#_bookmark16)].

The simplest continuous functions are the step functions (*a b*).

**Definition 2.4** The step function (*a b*) : *D → E*, for *a ∈ D* and *b ∈ E*, is defined by

(*a b*)(*x*)= ⎧⎨ *b,* if *a x,*

⎩ *⊥,* otherwise*.*

It is straightforward to see that (*a b*) is continuous. We have the following relationship between step functions and continuous functions.

**Lemma 2.5** [[5](#_bookmark8), Exercise II-2.31] *Let D and E be cpos and let a ∈ D and b ∈ E.*

1. *Suppose f ∈* [*D → E*]*. Then*

*b f* (*a*) =*⇒* (*a b*) *f.*

1. *If D and E are continuous cpos with bases BD and BE and f ∈* [*D → E*] *then*

*f* = *{*(*a b*): *a ∈ BD,b ∈ BE,b f* (*a*)*}.*

**Definition 2.6** Let *D* be a cpo.

1. We say that *D* is an *L*-domain if every nonempty bounded subset of *D* has a greatest lower bound; equivalently, for any *x ∈ D*, *↓x* is a complete lattice in the induced order. For *a ∈ D* and *A ⊆ ↓a*, we use *↓a A* to denote the least upper bound of *A* in *↓a*.

W

1. We say *D* is bounded complete if every nonempty bounded subset of *D* has a least upper bound.

Easily one sees that every bounded complete dcpo is an *L*-domain and the reverse does not hold. We also have the following useful lemma for *L*-domains and function spaces.

**Lemma 2.7** *Let D, E be two L-domains.*

1. *For a, b ∈ D and A ⊆ ↓a, a ≤ b implies* W*↓a A* = W*↓b A.*
2. [*D → E*] *is an L-domain such that for f, g, h ∈* [*D → E*]*, if f, g ≤ h then* (*f ∨↓h g*)(*x*) = *f* (*x*) *∨↓h*(*x*) *g*(*x*) *for all x ∈ D. Moreover, if f, g h then f ∨↓h g h.*
3. [*D → E*] *is continuous (resp. algebraic) whenever D and E are continuous (resp. algebraic).*

Since the class of continuous (resp. algebraic) *L*-domains is also closed under finite cartesian products, the above lemma, whose complete proof appears in [[8,](#_bookmark4) [9](#_bookmark9)], shows that the category of continuous (resp. algebraic) *L*-domains with continuous functions is cartesian closed. Moreover, it is one of the maximal cartesian closed subcategory of continuous domains.

**Definition 2.8** Let *D* be a continuous cpo. We say that a basis *B* of *D* has property *M* if for all *a, b, c, d ∈ B* with *a c, b d*, there exists a finite subset *F ⊆ D* such that

*↑c ∩ ↑d ⊆ ↑F ⊆ ↑a ∩ ↑b.*

We say that a continuous cpo has property *M* if it has a basis with property *M*.

A continuous cpo *D* has property *M* iff its Lawson topology is compact iff *↑a∩↑b* is Scott compact for all *a, b ∈ D* [[5](#_bookmark8), Theorem III-5.8]. The following lemma is easy to be proved.

**Lemma 2.9** *Let D be a continuous cpo with property M. Then for all a, b ∈ D,*

*↑a ∩ ↑b* = *∅ implies that there exist x ∈* ***↓****a and y ∈* ***↓****b such that ↑x ∩ ↑y* = *∅.*

# Almost algebraic dcpos

In this section, we introduce the notion of an almost algebraic dcpo and the related open problem posed by Hamrin and Stoltenberg-Hansen in [[6](#_bookmark5)].

**Definition 3.1** Let *D* be a continuous cpo.

* 1. Given *a ∈ D*, a sequence (*an*)*n∈ω ⊆ D* is said to be an almost algebraic sequence of *a* if

*a · · · an*+1 *an*  *a*1 *a*0

and for each *b ∈ D*, *a b* implies *an b* for some *n ∈ ω*.

* 1. A basis *B* of *D* is called almost algebraic if the following holds:
     1. Each *a ∈ B* has an almost algebraic sequence (*an*)*n∈ω ⊆ B*.
     2. For all *a, b ∈ B*, if ***↑****a ⊆* ***↑****b* then *b ≤ a*.

We say that *D* is almost algebraic if it has an almost algebraic basis *BD*.

Obviously, all algebraic cpos are almost algebraic by taking constant sequences of compact elements. The following result is obvious.

**Lemma 3.2** *Let D be a continuous cpo with an almost algebraic basis B. Let*

*A* = *{ai* : *i ∈ K} ⊆f B. Then there is A*ˆ = *{a*ˆ*i* : *i ∈ K} ⊆f B such that*

1. *ai a*ˆ*i for all i ∈ K.*
2. *For any i, j ∈ K, ai ≤ aj* =*⇒ a*ˆ*i ≤ a*ˆ*j.*

**Proposition 3.3** *Let BD be an almost algebraic basis of a dcpo D and a ∈ BD. If*

(*an*)*n∈ω ⊆ B is an almost algebraic sequence of a, then a* = *n∈ω an.*

**Proof.** Let *b* be an arbitrary lower bound of (*an*)*n∈ω*. We need to show *b ≤ a*. For any *x ∈* ***↑****a*, there exists *n ∈ ω* such that *an x*. Since *b ≤ an*, we get *b x*, i.e.,

***↑****a ⊆* ***↑****b*. From the assumption that *BD* is almost algebraic we obtain *b ≤ a*. *2*

Intuitively, the motivation of the notion “almost algebraic” is that an element in a basis can be approximated by an increasing way-below sequence and a decreasing way-below sequence like the real numbers. In the following, we will see that a well- known domain, namely the formal closed ball domain of a complete metric space, is almost algebraic but not algebraic.

Let (*X, d*) be a complete metric space. Set R+ = [0*,* +*∞*) to be the set of all non-negative real numbers and set Q+ to be the set of all positive rational numbers. Let

**B***X* = *X ×* R+

ordered as follows: *∀*(*x, r*)*,* (*y, s*) *∈* **B***X*,

(*x, r*) *≤* (*y, s*) *⇔ d*(*x, y*) *≤ r − s.*

It was showed in [[3,](#_bookmark3) [10,](#_bookmark10) [11](#_bookmark11)] that (**B***X, ≤*) is a continuous dcpo such that for (*x, r*)*,* (*y, s*) *∈* **B***X*, (*x, r*)(*y, s*) iff *d*(*x, y*) *< r − s*. Moreover, set *BX* = *X ×* Q+, then *BX* is a basis of **B***X*.

**Proposition 3.4** *For a complete metric space* (*X, d*)*, BX is an almost algebraic basis of* **B***X. Moreover, if* (*X, d*) *has a countable dense subset Y , i.e.,* (*X, d*) *is a*

*Polish space, then the set BY* = *Y ×* Q+ *is an almost algebraic ω-basis of* **B***X.*

*X*

**Proof.** Let (*X, d*) be a complete metric space. Then *BX* is a basis of (**B***X, ≤*). For

each (*x, r*) *∈ BX* , we set *rn* = *r −* *r* and *an* = (*x, rn*) for all *n < ω*. Then

2*n*

(*x, r*) *· · · an*+1 *an*  *a*1*.*

Suppose (*x, r*)(*y, s*) for some (*y, s*) *∈* **B***X*. Then *d*(*x, y*) *< r − s*. There exists

2*n*0

0

an enough large *n*0

*< ω* such that *d*(*x, y*)+ *s < r −*  *r < r*. Thus, *an*

= (*x, rn*0 )

(*y, s*). It means that (*an*)*ω* is an almost algebraic sequence of (*x, r*). Suppose (*z, k*) *∈*

*BX* with ***↑***(*z, k*) *⊆* ***↑***(*x, r*). Assume (*x, r*) */≤* (*z, k*). Then *d*(*x, z*)+*k > r*. Since *k >* 0, there exists an enough small *ε ∈* Q+ such that *k − ε >* 0 and *d*(*x, z*)+ *k − ε ≥ r*. Set *kj* = *k − ε*. Then (*z, k*)(*z, kj*) and (*x, r*) */* (*z, kj*). It is a contradiction. Hence we have (*x, r*) *≤* (*z, k*). Therefore, *BX* is an almost algebraic basis of **B***X* by

Definition 3.1. Similarly, one can show that *BY* = *Y ×* Q+ is an almost algebraic

*X*

*ω*-basis of **B***X* when *X* is a Polish space. *2*

Also, there are continuous cpos which are not almost algebraic. The following is such an example.

**Example 3.5** Let *D* = [0*,* 1] *×* [0*,* 1] ordered as follows: *∀*(*x*1*, y*1)*,* (*x*2*, y*2) *∈ D*, (*x*1*, y*1) *≤* (*x*2*, y*2) iff one of the followings conditions holds:

* *x*1 *≤ x*2 and *y*1 = *y*2*,*
* *x*1 = 0 and *y*1 *≤ y*2*.*

We see that (*D, ±*) is a bounded complete cpo and for all (*x*1*, y*1)*,* (*x*2*, y*2) *∈ D*, (*x*1*, y*1)(*x*2*, y*2) *⇔* (*x*1 *< x*2 & *y*1 = *y*2) or (*x*1 =0 & *y*1 *< y*2)*.*

Hence, *D* is continuous. Let *y ∈* (0*,* 1) and let ((*xn, yn*))*n∈ω ⊆ D* be a sequence such

*∞*

that (0*, y*)=

(*xn, yn*) and

*n*=1

(0*, y*) *· · ·* (*xn*+1*, yn*+1)(*xn, yn*) (*x*2*, y*2)(*x,y*1)*.*

Then one of the follow two statements holds:

1. *∀n ∈ ω*, *yn* = *y* and *xn*+1 *< xn*.
2. *∃n*0 *∈ ω*, *xn* = 0 and *y < yn*+1 *< yn* whenever *n > n*0.

For (1), take *yj > y*, then (0*, y*)(0*, yj*) but (*xn, yn*) */≤* (0*, yj*) for all *n ∈ ω*. For (2), take *x >* 0, then (0*, y*)(*x, y*) but (*xn, yn*) */≤* (*x, y*) for all *n ∈ ω*. Hence, (0*, y*) has no almost algebraic sequences in *D* for any *y ∈* (0*,* 1). Note that for any basis *B* of *D* and (*x, yjj*) *∈ B ∩* ***↓***(0*, y*), we have *x* = 0 and *yjj ≤ y*. Hence, *D* has no almost algebraic bases.

Therefore, Even the property *almost algebraic* seems strange, it is also very natural (see Proposition 3.4), and the class of all almost algebraic cpos strictly in- tervenes between algeraic cpos and continuous cpos. It also leads a natural question: which cpos with almost algebraic bases can form a cartesian closed category? Of

course, such a category should contain at least one non-algebraic object. Hamrin and Stoltenberg-Hansen [[6](#_bookmark5)] proved the following result.

**Theorem 3.6** *Let D and E be bounded complete domains with countable closed and almost algebraic bases BD and BE respectively. Then the function space* [*D → E*] *has a closed, countable and almost algebraic basis.*

This theorem says that the category of bounded complete domains with count- able closed and almost algebraic bases is cartesian closed. The proof of the above theorem in [[6](#_bookmark5)] is seriously dependent on the closedness of bases. Recall that, a base is closed if it is closed under the suprema of its bounded finite subsets. Is it easy that a bounded complete domain has a countable closed basis? Let see an example.

**Example 3.7** Let *I* = [0*,* 1] and I = *{*[*a, b*] : 0 *≤ a ≤ b ≤* 1*}* endowed with the reverse-inclusion relation, that is, [*a, b*] *≤* [*c, d*] *⇔* [*a, b*] *⊇* [*c, d*]. Then (I*, ≤*) is a bounded complete domain and is called the unit interval domain. Set

*B*I = *{*[*a, b*]: *a, b ∈* Q *∩ I* & *a < b}.*

It is easy to see that *B£* is a countable almost algebraic basis of I (the proof is similar to Proposition 3.4). However, *B*I is not closed. For example, [0*,* 1 ]*,* [ 1 *,* 1] *∈ B*I but

2 2

[0*,* 1 ] *∨* [ 1 *,* 1] = *{* 1 *} /∈ BI* .

2 2 2

For seeking a new cartesian closed subcategory of almost algebraic domains, Hamrin and Stoltenberg-Hansen in [[6](#_bookmark5)] posed the following open problem:

* Find a further condition on the basis weaker than being closed but which is preserved and preserves almost algebraicity under the function space construction.

In this paper, we will answer this question. In the next section, a notion called *M*-closed basis is defined on a continuous *L*-domain which is very advantageous to help us to handle the above open problem.

# *L*-domains with countable, almost algebraic and *M*- closed bases

In this section, we introduce a *M*-closed base on a continuoous *L*-domain. A series of properties of continuous *L*-domains with countable *M*-closed and almost algebraic bases are obtained.

**Proposition 4.1** [[6](#_bookmark5), Proposition 22] *Let D,E be continuous cpos and suppose that D has an almost algebraic basis BD and E has a countable basis BE. Let a, c ∈ BD and b, d ∈ BE, where b /*= *⊥. Then the following hold:*

* 1. (*a b*) *≤* (*c d*) *⇐⇒ c ≤ a* & *b ≤ d.*
  2. (*a b*)(*c d*) *⇐⇒ c a* & *b d.*
  3. *If* (*cn*)*n∈ω is an almost algebraic sequence for c and* (*dn*)*n∈ω an approximating sequence for d, then n∈ω* (*cn dn*)= (*c d*)*.*

W

The following is a crucial lemma which is very useful to investigate the way below relation between continuous functions of almost algebraic L-domains. Note that it is a generalization of [[6](#_bookmark5), Lemma 23].

**Lemma 4.2** *Let D, E be two continuous L-domains endowed with an almost alge- braic basis BD and a countable basis BE respectively. For all a ∈ BD,b ∈ BE and f ∈* [*D → E*]*, we have*

(*a b*) *f ⇐⇒ b f* (*a*)*.*

**Proof.** The implication from right to left follows from Lemma 2.5(1).

We now show the other direction. Set

*Step*(*f* )= *{*(*c d*): (*c, d*) *∈ BD × BE* & *d f* (*c*)*}.*

By Lemma 2.5(2), *f* = *Step*(*f* )= *{ ↓f A* : *A ⊆f Step*(*f* )*}*. Since (*a b*) *f* , there is a finite subset *{*(*ci di*): *i ∈ I*0*}⊆ Step*(*f* ) such that

W W W

(*a b*) *{*(*ci di*): *i ∈ I*0*}* (*∗*)*.*

*↓f*

Without losing generality, we assume *b /*= *⊥* . From the assumption that *BD* is almost algebraic, we get almost algebraic sequences (*an*) for *a* and (*cj*)*j* for each

*i*

*ci*, where *i ∈ I*0. Since *BE* is a countable basis we choose sequences (*dj*)*j* from

*i*

*BE* increasing with respect tosuch that W

*d*

*i*

*j∈ω*

*j* = *di* for each *i ∈ I*0. From

this it follows that (*cj dj*)(*ci di*) for all *i ∈ I*0 and *j ∈ ω*. Hence

*i*

*i*

(*ci di*)= W (*cj dj*) for all *i ∈ I*0 and

*j i* *i*

*{*(*cj dj*): *i ∈ I*0*}* *{*(*ci di*): *i ∈ I*0*}*

*i* *i*

*↓f ↓f*

for all *j ∈ ω*. Furthermore, we have

*{*(*cj dj*): *i ∈ I }* *{*(*cj′ j′*

*∈ I }*

*i* *i*

*↓f*

for *j < jj*. Therefore

0 *i di* : *i* 0

*↓f*

*{*(*ci di*): *i ∈ I*0*}* = *{*(*cj dj*): *i ∈ I*0*}*

*i* *i*

*↓f ↓f j∈ω*

= *{*(*cj dj*): *i ∈ I*0*}* (*∗∗*)*.*

*i* *i*

*j∈ω ↓f*

From (*∗*) and (*∗∗*), there exists *j*0 *∈ ω* such that (*a b*)W *{*(*cj*0 *dj*0 ): *i ∈*

*↓f i* *i*

*I*0*}*. For each *an* in the descending sequence towards *a* we have

*b* = (*a b*)(*an*)

*≤* *{*(*cj*0 *dj*0 )(*an*): *i ∈ I*0*}*

*i* *i*

*↓f* (*an*)

= *{dj*0 : *cj*0 *an}.*

*i* *i*

*↓f* (*an*)

Let *In* = *{i ∈ I*0 : *cj*0

*i*

*an}*. Note that *In ⊇ In*+1 and *I*0 is finite, hence there

exists *n*0 such that if *n ≥ n*0 then *In* = *In*0 holds. From the assumption *b /*= *⊥*

it follows that *In /*= *∅*. For each *i ∈ In* , *cj*0 *an* holds for all *n ≥ n*0. Hence

0 0 *i*

*a* =

*n∈ω*

*an ≥ cj*0 *ci* holds for all *i ∈ In* . Thus

*f* (*a*) *≥* *{*(*ci di*): *i ∈ I*0*}*(*a*)

*i*

0

*↓f* (*a*)

= *{di* : *ci a, i ∈ I*0*}*

*↓f* (*a*)

*≥* *{di* : *i ∈ In*0 *}*

*↓f* (*a*)

=

*↓f* (*an*0 )

*{di* : *i ∈ In*0 *}* (because of *f* (*a*) *≤ f* (*an*0 ))

*{dj*0 : *i ∈ In }*

*i* 0

*↓f* (*a*)

*≥ b.*

*2*

Next, we introduce an important property of a base, called *M*-closed, which is a generalization of a closed base.

Let *D* be a cpo. Given a finite subset *F* of *D*, we set

mub*F* = min*{a ∈ D* : *F ≤ a}*

to be the set of all minimal upper bounds of *F* . If *D* is an *L*-domain, then for each upper bound *b* of *F* , there exists *a ∈* mub*F* such that *a ≤ b* (It is called mub-complete in [[1](#_bookmark1)]).

**Definition 4.3** Let *D* be a continuous *L*-domain. A basis *BD* of *D* is said to be *M*-*closed*, if for any nonempty finite subset *{*(*ai, bi*) : *i ∈ K} ⊆ BD × BD* with *bi ai* for all *i ∈ K*, there exists *{ci* : *i ∈ K} ⊆f BD* such that

1. *bi ci ai* for each *i ∈ K*.

1. For any *I ⊆ K*, if *i∈I* ***↑****ai /*= *∅* then mub*{ci* : *i ∈ I}* is finite and contained in

*BD*, i.e., mub*{ci* : *i ∈ I} ⊆f BD*.

By the definition of property *M*, the following result is obvious.

**Lemma 4.4** *Let BD be a basis of a continuous L-domain D. If BD is M-closed, then it has property M.*

**Proposition 4.5** *Let D be a continuous L-domain. A basis BD of D is M-closed if and only if for any nonempty ﬁnite subset {*(*ai, bi*) : *i ∈ K} ⊆ BD × BD with bi ai for all i ∈ K, there exists {ci* : *i ∈ K} ⊆f BD such that*

1. *bi ci ai for each i ∈ K.*
2. *For any I ⊆ K,* mub*{ci* : *i ∈ I} is ﬁnite and contained in BD, i.e.,* mub*{ci* :

*i ∈ I} ⊆f BD.*

**Proof.** The ”if” part is obvious. Suppose that *BD* is *M*-closed and *{*(*ai, bi*) : *i ∈*

*K} ⊆ BD × BD* is nonempty finite with *bi ai* for all *i ∈ K*. For any nonempty

*I ⊆ K*, if *↑ai* = *∅*, then we can find *{bI* : *i ∈ I}⊆ BD* such that *bi bI ai*

*i∈I*

for all *i ∈ I* and

*i*

*i∈I*

*i* *i*

*↑bI* = *∅* by Lemma 2.9. Set

*I* = *{I ⊆ K* : *↑ai* = *∅}.*

*i∈I*

For each *i ∈ K*, we set

*Bi* = *{bi}∪ {bI* : *i ∈ I ∈ I}.*

*i*

*Bi* is nonempty and *Bi ⊆* ***↓****ai*. Set *bj* = W

*i*

*i*

*↓ai*

*Bi*. Then *bj ai*. For each *i ∈ K*,

pick *bjj ∈ BD* such that *bj bjj ai*. Then the following conditions hold:

*i i* *i*

*i*

(i) *∀I ⊆ K*, *↑ai* = *∅ ⇐⇒*

*i∈I*

*i∈I*

*↑bj* =

*↑bjj* = *∅*,

1. *∀I ⊆ K*, *i∈I ↑ai /*= *∅ ⇐⇒* *i∈I*

*i*

*i*

*i∈I*

***↑****bjj /*= *∅*.

Set *B* = *{*(*bj, bjj*) : *i ∈ K}*. Since *BD* is *M*-closed, we can find *ci ∈ BD* for

*i i*

each *i ∈ K* such that *bj*

*i*

*i*

*i*

*ci bjj*

and for any *I ⊆ K*,

*i∈I*

***↑****bjj*

*/*= *∅* implies

mub*{ci* : *i ∈ I} ⊆f BD*. Hence, from the above statements (i) and (ii), all *ci*’s

satisfy the conditions of this proposition. *2*

In the following, we define a new class of domain called *ωAML*-domains.

**Definition 4.6** A continuous cpo is said to be an *ωAML*-domain if it is an *L*- domain endowed with a basis *BD* satisfying the following conditions:

* 1. *BD* is countable,
  2. *BD* is almost algebraic,
  3. *BD* is *M*-closed.

From now on, we always use a pair like (*D, BD*) to be an *ωAML*-domain with a required basis *BD*.

Since a continuous *L*-domain has property *M* if and only if it is an *FS*-domain (a retract of a *bifinite domain* further, see [[12](#_bookmark12)]), every *ωAML*-domain is an *FS*-domain.

Let’s see some examples. (1) An algebraic *L*-domain is an *ωAML*-domain if and only if it has an *ω*-base with property *M*. (2) The unit interval domain I in Example 3.7 is an *ωAML*-domain, i.e., the base *B*I is countable, almost algebraic and *M*-closed. (3) Given an algebraic *L*-domain *L* with an *ω*-base with property *M*, the function space [I *→ L*] is an *ωAML*-domain. Particularly, if *L* is not bounded complete, then [I *→ L*] is neither bounded complete nor algebraic.

**Proposition 4.7** *Let* (*D, BD*) *be an ωAML-domain, {ai* : *i ∈ K} ⊆f BD. Then we have*

1. *There are decreasing sequences* (*an*)*n∈ω ⊆ BD, i ∈ K such that*

*i*

* 1. *For each i ∈ K,* (*an*)*n∈ω is an almost algebraic sequence of ai.*

*i*

* 1. *∀I ⊆ K, ∀n ∈ ω,* ***↑****ai* = *∅ ⇐⇒* *↑an* = *∅ ⇐⇒* ***↑****an* = *∅.*

*i∈I*

*i∈I*

*i*

*i∈I*

*i*

* 1. *∀I ⊆ K, ∀n ∈ ω,* mub*{an* : *i ∈ I} ⊆f BD.*

*i*

1. *There are increasing sequences* (*a*ˆ*n*)*n∈ω ⊆ BD, i ∈ K such that*

*i*

* 1. *For each i ∈ K,*

*i*

*i*

*i*

*i*

*a*ˆ1

*a*ˆ2

*· · · a*ˆ*n*

*a*ˆ*n*+1

*· · · ai in D and*

W*n∈ω*

*i*

*i*

*a*ˆ*n* = *ai.*

* 1. *∀I ⊆ K, ∀n ∈ ω,*

*i∈I*

*↑ai* = *∅* =*⇒*

*i∈I*

*↑a*ˆ*n* = *∅;*

* 1. *∀I ⊆ K, ∀n ∈ ω,* mub*{a*ˆ*n* : *i ∈ I} ⊆f BD.*

*i*

**Proof.** (1) Since *BD* is almost algebraic, we choose almost algebraic sequences (*a*˜*j*)*j*

*i*

for each *ai*, where *i ∈ K*. Set

*F* = *{I ⊆ K* : ***↑****ai /*= *∅}.*

*i∈I*

For any *I ∈ F*, we have *i∈I* ***↑****ai /*= *∅*. Suppose *x ∈* *i∈I* ***↑****ai*, that is, *ai x* holds for all *i ∈ I*. For a fixed *i ∈ I*, since (*a*˜ )*j* is an almost algebraic sequence of *ai*,

*i*

*j*

there exists *ni ∈ ω* such that *a*˜*ni x* holds. Set

*i*

*nI* = max*{ni* : *i ∈ I}.*

Then *a*˜*nI x* for all *i ∈ I*, i.e.,

*i*

*i∈I*

***↑****a*˜*ni /*= *∅*. Set

*n*0 = max*{nI* : *I ∈ F},*

*i*

we have

*∀I ⊆ K, ∀n ≥ n*0*,* ***↑****ai /*= *∅ ⇐⇒* ***↑****a*˜*n /*= *∅.*

*i*

*i∈I i∈I*

We now consider a finite set *{*(*a*˜*n, a*˜*n*+1) : *i ∈ K}* for all *n ≥ n*0. Since *BD* is

*i i*

*M*-closed, there exist *{a*¯*n* : *i ∈ K}⊆ BD* such that *a*˜*n*+1 *a*¯*n a*˜*n* for each *i ∈ K*

*i i i* *i*

and mub*{a*¯*n* : *i ∈ I} ⊆f BD* for all *I ∈F* by Proposition 4.5.

*i*

Next, we define *an* = *a*¯*n*+*n*0 for each *n ∈ ω* and each *i ∈ K*. It is easy to see that

*i* *i*

sequences (*an*)*n∈ω* for all *i ∈ K* satisfy the required conditions.

*i*

(2) Since *BD* is a countable basis of *D*, we can choose sequences (*aj*)*j* from *BD*

*i*

increasing with respect tosuch that W*j a* = *ai* for each *i ∈ K*. Now we consider

*j*

*i*

a finite set *{*(*aj*+1*, aj*) : *i ∈ K}* for all *j ∈ ω*. From the assumption that *BD* is

*i* *i*

*M*-closed, there exist *{a*¯*j* : *i ∈ K}⊆ BD* such that

*i*

*aj a*¯*j aj*+1

*i i* *i*

and

mub*{a*¯*j* : *i ∈ I} ⊆f BD*

*i*

for *I ⊆ K* and *j ∈ ω* by Proposition 4.5. Clearly, (*a*¯*j*)*j∈ω* is an increasing sequence

*i*

with respect tosuch that W *a*¯*j* = *aj* for all *i ∈ K*. Set

*j*

*i*

*i*

*E* = *{I ∈ K* : *↑ai* = *∅}.*

*i∈I*

Then *E* is finite and for each *I ∈ E* , there is *nI ∈ ω* such that Lemma 2.9. Next, set

*i∈I*

*↑a*¯*nI*

= *∅* by

*i*

*n*0 = max*{nI* : *I ∈ E}*

and let *a*ˆ*n* = *a*¯*n*+*n*0 . It is easy to check that sequences (*a*ˆ*n*)*n* (*i ∈ K*) are increasing

*i i* *i*

and satisfy the required conditions. *2*

**Definition 4.8** Let (*D, BD*) be an *ωAML*-domain, *{ai* : *i ∈ K} ⊆f BD*.

1. The family *{*(*an*)*n∈ω* : *i ∈ K}* of the decreasing sequences in Proposition 4.7.(1) is called a *M*-closed almost algebraic family of *{ai* : *i ∈ K}*.

*i*

1. The family *{*(*a*ˆ*n*)*n∈ω* : *i ∈ K}* of the increasing sequences in Proposition 4.7.(2) is called a *M*-closed approximating family of *{ai* : *i ∈ K}*.

*i*

Let *D* be a continuous *L*-domain and *{ai* : *i ∈ I} ⊆ D* be a nonempty finite

subset with

*i∈I*

*↑ai /*= *∅*. For each *i ∈ I*, let (*an*)*n∈ω ⊆ D* be an increasing sequence

with W *an* = *ai* and *an an*+1 for all *n ∈ ω*. Set

*i*

*n∈ω*

*i*

*i*

*i*

*M⟨a,I⟩* = mub*{ai* : *i ∈ I} M⟨an,I⟩* = mub*{ai* : *i ∈ I}*

*n*

*n*

for all *n ∈ ω*. We define a map *rI* : *M⟨a,I⟩ −→ M⟨an,I⟩* as follows: *∀m ∈ M⟨a,I⟩*,

*rn*(*m*)= *{an* : *i ∈ I}*

*I* *i*

*↓m*

for each *n ∈ ω*. Easily one sees that *rn* is well defined.

*I*

**Lemma 4.9** *Let* (*D, BD*) *be an ωAML-domain, {ai* : *i ∈ I} ⊆f BD. Let {*(*an*)*n∈ω* :

*i*

*i ∈ I} be a M-closed approximating family of {ai* : *i ∈ I}. Then we have*

* 1. *If* *i∈I ↑ai /*= *∅ and M⟨a,I⟩ ⊆f BD, then*
     1. *For any m ∈ M⟨a,I⟩,* W

*I*

*n∈ω*

*rn*(*m*)= *m and*

*r*1(*m*) *r*2(*m*) *· · · rn*(*m*) *rn*+1(*m*)  *m.*

*I I I* *I*

* + 1. *For each n ∈ ω, there exists n*ˆ *> n such that M⟨an*ˆ *,I⟩ ⊆ ↑rn*(*M⟨a,I⟩*)*.*

*I*

* 1. *If*

*i∈I*

*↑ai* = *∅, then* *i∈I*

*↑an* = *∅ for all n ∈ ω.*

**Proof.** (1) For any *m ∈ M⟨a,I⟩*, it is easy to check that (*rn*(*m*))*n∈ω* is an increasing

*i*

*I*

W

*I*

sequence with respect tosuch that

true, that is, there is *n*0 *∈ ω* such that

*n∈ω*

*rn*(*m*) = *m*. Suppose that (ii) is not

for all *n > n*0. Set

*M⟨an,I⟩ /⊆ ↑rI* (*M⟨a,I⟩*)

*Fn* = *M⟨an,I⟩ \ ↑rI* (*M⟨a,I⟩*)

*n*0

*n*0

for all *n > n*0. Then *Fn /*= *∅* for all *n > n*0. We claim that *Fn*+1 *⊆ ↑Fn* holds for all

*n > n*0. In fact, for any *y ∈ Fn*+1, there is

*xn* = *{an* : *i ∈ I}∈ M⟨an,I⟩*

*i*

*↓y*

such that *xn ≤ y*. This implies that *xn /∈ ↑rn*0 (*M⟨a,I⟩*) as *y /∈ ↑rn*0 (*M⟨a,I⟩*). Hence,

*I* *I*

*xn ∈ Fn*, i.e., *Fn*+1 *⊆ ↑Fn*. Consider the family *{Fn* : *n > n*0*}*. By the Rudin’s

*n>n*0 *n*

Lemma [[5](#_bookmark8), Lemma III-3.3], there is a directed subset *H ⊆ F* such that *H ∩ F /*= *∅* for all *n > n* . It means that *H ∈ ↑a* , hence there is *m ∈* mub*{ai* : *i ∈ I}* such that *m ≤ H*. We have *rn*0 (*m*) *m ≤ H*. Hence, there is *x ∈ H* such that *rn*0 (*m*) *≤ x*, and therefore *x ∈ ↑rn*0 (*M⟨a,I⟩*). On the other

*I*

*n* 0 W *i∈I i*

W W

*I* *I*

hand, *H ⊆*

*↑rn*0 (*M⟨a,I⟩*) - a contradiction.

*I*

*n>n*0 *Fn* implies that *x* is in *Fn* for some *n > n*0, hence *x* is not in

(2) It follows directly from the definition of *M*-closed approximating family. *2*

**Theorem 4.10** *Let* (*D, BD*) *be an ωAML-domain. Let {ai* : *i ∈ K} ⊆f BD satisfy the following conditions:*

*(i) ∀I ⊆ K,* *i∈I* ***↑****ai* = *∅* =*⇒* *i∈I ↑ai* = *∅;*

1. *∀I ⊆ K,* *i∈I* ***↑****ai /*= *∅* =*⇒ M⟨a,I⟩ ⊆f BD.*

*Then there exists a M-closed approximating family {*(*an*)*n∈ω* : *i ∈ K} of {ai* : *i ∈*

*i*

*K} such that*

* 1. *For all I ⊆ K, we have*
     1. *i∈I* ***↑****ai /*= *∅* =*⇒ ∀n ∈ ω, M⟨an*+1*,I⟩ ⊆ ↑rn*(*M⟨a,I⟩*)*;*

*I*

*i*

*i∈I*

***↑****ai* = *∅* =*⇒ ∀n ∈ ω,*

*i∈I*

*↑an* = *∅.*

* 1. *For any I ⊆ K, if* *i∈I* ***↑****ai /*= *∅, then for any m ∈ M⟨a,I⟩,*

*j*

***↑****m\*

*j∈K\I*

***↑****aj /*= *∅* =*⇒ ∀n ∈ ω,* ***↑****rn*(*m*)*\*

*j∈K\I*

*I*

***↑****an /*= *∅.*

* 1. *For all I, J ⊆ K, if I ⊆ J, then for all* (*m*1*, m*2) *∈ M⟨a,I⟩ × M⟨a,J⟩,*

*m*1 */≤ m*2 =*⇒ ∀n ∈ ω, rn*(*m*1) */≤ rn*(*m*2)*.*

*I J*

**Proof.** By Proposition 4.7.(2), there are increasing sequences (*a*ˆ*in*)*n∈ω ⊆ BD* for all *i ∈ K* which form a weakly *M*-closed approximating family of *{ai* : *i ∈ K}* .

First of all, set

*F* = *{I ⊆ K* : ***↑****ai /*= *∅}* = *{F*1*, F*2*, ··· , Fl}.*

*i∈I*

For *F*1, by Lemma 4.9 (1) we have an increasing infinite sequence (*n*1)*j∈ω ⊆ ω* such

*j*

that

*M* 1 *⊆ ↑r*

*n*1

*j* (*M* )

*⟨a*ˆ*nj*+1 *,F ⟩*

1

*F*1 *⟨a,F*1*⟩*

*n*1

holds for all *j ∈ ω*. For *F*2, it is obvious that *{*(*a*ˆ *j* )*j∈ω* : *i ∈ F*2*}* is a weakly

*i*

*M−*closed approximating family of *{ai* : *i ∈ F*2*}*. By Lemma 4.9 (1) again, there is an increasing infinite subsequence (*n*2)*j* of (*n*1)*j* such that

*j j*

*M* 2 *⊆ ↑r*

*n*2

*j* (*M* )*.*

*n*

*⟨a*ˆ *j*+1 *,F*2*⟩*

*F*2 *⟨a,F*2*⟩*

By induction, we can get an increasing sequence (*nj*)*j∈ω ⊆ ω* such that

*M⟨a*ˆ*nj*+1 *,Fi⟩ ⊆ ↑rFi* (*M⟨a,Fi⟩*)

*nj*

holds for each *i* with 1 *≤ i ≤ l* and *j ∈ ω*. Easily one sees that, *{*(*a*ˆ*nj* )*j∈ω* : *i ∈ K}*

*i*

is a *M*-closed approximating family of *{ai* : *i ∈ K}* such that condition (1) holds.

Secondly, we set

*M* = *{m* : *∃I ∈ F, m ∈ M⟨a,I⟩* & ***↑****m\*

*j∈K\I*

***↑****aj /*= *∅}.*

Clearly, *M* is finite. For any *m ∈ M*, there is *I ∈ F* such that *m ∈ M⟨a,I⟩*

with ***↑****m\ j∈K\I* ***↑****aj /*= *∅*. Then *aj /≤ m* for all *j ∈ K\I*. Given *j ∈ K\I*, since

*a* = W *a*ˆ*np* , there is *p ∈ ω* such that *a*ˆ*npj /≤ m*. Set

*j p∈ω j*

*j j*

*pm* = max*{pj* : *j ∈ K\I}.*

Then *a*ˆ*npm /≤ m* holds for all *j ∈ K\I*. It means that *a*ˆ*np /≤ m* holds for all *j ∈ K\I*

*j*

and *p ≥ pm*, which is equivalent to

***↑****m\*

*j∈K\I*

*j*

***↑****a*ˆ*np /*= *∅*

*j*

for all *p ≥ pm*. Because *rnp* (*m*) *≤ m*, we have that

*I*

***↑****rnp* (*m*)*\*

*I*

*j∈K\I*

***↑****a*ˆ*np /*= *∅*

for all *p ≥ pm*. Next we set

*j*

then we get that ***↑****rnp* (*m*)*\*

*p*0 = max*{pm* : *m ∈ M},*

***↑****a*ˆ*np /*= *∅* for all *p ≥ p*0 and *m ∈ M*.

*I*

Finally, set

*j∈K\I j*

*O* = *{*(*x, y*): *∃I, J ⊆ K, I ⊆ J* & (*x, y*) *∈ M⟨a,I⟩ × M⟨a,J⟩* & *x /≤ y}.*

Clearly, *O* is finite. For any given (*x, y*) *∈ O*, there are *I, J ⊆ K* with *I ⊆ J* and we denote *x* and *y* by *mI ∈ M⟨a,I⟩* and *mJ ∈ M⟨a,J⟩* respectively. Since *mI /≤ mJ* , there is *p*(*x,y*) *∈ ω* with *p*(*x,y*) *≥ p*0 such that

*rnp*(*x,y*) (*m* ) */≤ m .*

*I I J*

Therefore, *rnp* (*mI* ) */≤ rnp* (*mJ* ) for all *p ≥ p* . Now, we set

*I J* (*x,y*)

*p*1 = max*{p*(*x,y*) : (*x, y*) *∈ O}.*

Then *rnp* (*mI* ) */≤ rnp* (*mJ* ) for *p ≥ p*1 and (*mI, mJ* ) *∈ O*. In the end, we define

*I J*

*j nj*+*p*1 *j*

*ai* = *a*ˆ*i* for each *i ∈ K* and *j ∈ ω*. It is easy to check that *{*(*ai* )*j∈ω* : *i ∈ K}*

satisfy the all required conditions of this proposition. *2*

**Definition 4.11** Let (*D, BD*)*,* (*E, BE*) be two *ωAML*-domains. A nonempty finite subset *{*(*ai bi*): *i ∈ K}* of step functions of the function space [*D → E*] is called a *good step function family*, a *g.s.f* -*family* for short, if

1. (*ai, bi*) *∈ BD × BE* for all *i ∈ K* and *i∈K* ***↑***(*ai bi*) */*= *∅*.
2. *∀I ⊆ K*, *i∈I* ***↑****ai* = *∅* =*⇒* *i∈I ↑ai* = *∅*.
3. *∀I ⊆ K*, *i∈I* ***↑****ai /*= *∅* =*⇒ M⟨a,I⟩ ⊆f BD* & *M⟨b,I⟩ ⊆f BE*.

Let (*D, BD*)*,* (*E, BE*) be two *ωAML*-domains, let *{*(*ai bi*) : *i ∈ K}* be a

*g.s.f* -family. For any nonempty *I ⊆ K*, set

*M⟨a,I⟩* = *{m ∈ M⟨a,I⟩* : *∀j ∈ K\I, aj /≤ m},*

Mub*⟨a,K⟩* = *{M⟨a,I⟩* : *∅ /*= *I ⊆ K},*

Mub*⟨b,K⟩* = *{M⟨b,I⟩* : *∅ /*= *I ⊆ K}.*

Easily, we have that *{M⟨a,I⟩* : *∅ /*= *I ⊆ K}* is a decomposition of Mub*⟨a,K⟩*.

**Lemma 4.12** *(1) For any m ∈ M⟨a,I⟩,* ***↑****m \* *j∈K\I* ***↑****aj /*= *∅.*

*(2) For any x ∈* ***↑****a , there are a unique subset I ⊆ K and a unique m ∈*

*i∈K i*

*M⟨a,I⟩ such that x ∈* ***↑****m \* *j∈K\I* ***↑****aj.*

**Proof.** (1) Assume that there exists some *m ∈ M⟨a,I⟩* such that

***↑****m \*

*j∈K\I*

***↑****aj* = *∅.*

Then ***↑****m ⊆ j∈K\I* ***↑****aj*. Since *BD* is almost algebraic, there is an almost algebraic sequence (*mn*)*n∈ω* of *m*. We claim that there exists *j*0 *∈ K\I* such that *mn ∈* ***↑****aj*0 for all *n ∈ ω*. Otherwise, for each *j ∈ K\I*, there is *nj ∈ ω* such that *mnj /∈* ***↑****aj*; set

*n*0 = max*{nj* : *j ∈ K\I},*

we have *mn*0 */∈* *j∈K\I* ***↑****aj*. It is a contradiction. Hence, ***↑****m ⊆* ***↑****aj*0 for some *j*0 *∈ K\I*. Note that since *m, aj*0 *∈ BD*, it follows that *aj*0 *≤ m* holds, which is contradictory to *m ∈ M⟨a,I⟩*.

*i∈K i* *i*

(2) It follows from that for a given *x ∈* ***↑****a* , we set *I* = *{i ∈ K* : *a x}*

and *m* = W*↓x{ai* : *i ∈ I}*. *2*

**Definition 4.13** A map *TK* : Mub*⟨a,K⟩ −→* Mub*⟨b,K⟩* is called *consistent* if

1. *TK* is monotone under the induced orders on Mub*⟨a,K⟩* and Mub*⟨b,K⟩*.
2. For any nonempty *I ⊆ K*, for any *m ∈ M⟨a,I⟩*, *TK*(*m*) *∈ M⟨b,I⟩*.

Suppose *TK* : Mub*⟨a,K⟩ −→* Mub*⟨b,K⟩* is a consistent map. We define a function

*fTK* : *D −→ E* as follows: *∀x ∈ D*,

*fTK*

(*x*) = ⎧⎨ *TK*(*m*)*,* if *∃∅ /*= *I ⊆ K, m ∈ M⟨a,I⟩, x ∈* ***↑****m\* *j∈K\I* ***↑****aj,*

⎩ *⊥,* otherwise*.*

It is easy to check by Lemma 4.12 that *fTK* is well defined.

**Proposition 4.14** *fTK is a Scott continuous function with fTK ∈* mub*{*(*ai bi*):

*i ∈ K}.*

**Proof.** First of all, we show that *fTK* is monotone. Let *x, y ∈ D* with *x ≤ y*. If *x /∈* ***↑****a* , then *f* (*x*) = *⊥ ≤ f* (*y*). If *x ∈* ***↑****a* , then there is *I ⊆ K* with *mx ∈ M⟨a,I⟩* such that *x ∈* ***↑****mx\* *j∈K\I* ***↑****aj* from Lemma 4.12 (2). By the same reason, there is *J ⊆ K* with *I ⊆ J* such that there is *my ∈ M⟨a,J⟩* with *y ∈* ***↑****my\ j∈K\J* ***↑****aj*. Then

*i∈K i TK TK*  *i∈K i*

*my* = *{ai* : *i ∈ J}≥* *{ai* : *i ∈ I}* = *{ai* : *i ∈ I}* = *mx.*

*↓y ↓y ↓x*

So we have that *fTK* (*x*)= *TK*(*mx*) *≥ TK*(*my*)= *fTK* (*y*).

Next we show that *fTK* preserves the supremum of every directed subset. Let *H* be a directed set of *D*. We need to consider two cases for *H*. (1) *H /∈* ***↑****a* . It implies that *h /∈* ***↑****a* for all *h ∈ H*. Thus we have that *f* ( *H*) = *⊥* = *f* (*h*). (2) *H ∈* ***↑****a* . It implies that there is *I ⊆ K* such that

W W *i∈K i*

W *i∈K i TK* W

W*h∈H TK*  *i∈K i*

*i∈K\I i ⟨a,I⟩*

*H ∈* ***↑****m\* ***↑****a* for some *m ∈ M* from Lemma 4.12(2). It follows that there is *h*0 *∈ H* with *m h*0 and *h*0 */∈* *i∈K\I* ***↑****ai*. Then

*fTK* ( *H*)= *TK*(*m*)= *fTK* (*h*0)= *fTK* (*h*)

*h∈H*

as *fTK* is monotone.

*i i TK*  *i∈K i*

Clearly, (*a b* ) *≤ f* holds for each *i ∈ K*. For any *x ∈* ***↑****a* , by Lemma

4.12 (2), there is *I ⊆ K* such that there exists *m ∈ M⟨a,I⟩* with *x ∈* ***↑****m\* *j∈K\I* ***↑****aj*.

Then

*{*(*ai bi*): *i ∈ K}*(*x*)=

*{bi* : *ai x}* =

*{bi* : *i ∈ I}*

*↓fTK*

*↓fTK* (*x*)

*↓fTK* (*x*)

=

*↓TK* (*m*)

*{bi* : *i ∈ I}* = *TK*(*m*)

= *fTK* (*x*)*.*

Therefore, *fTK ∈* mub*{*(*ai bi*): *i ∈ K}*. *2*

The following is a characterization of the minimal upper bounds of a good step function family in function spaces.

**Theorem 4.15** *Let* (*D, BD*)*,* (*E, BE*) *be two ωAML-domains, let {*(*ai bi*) : *i ∈ K} be a g.s.f-family, f ∈* [*D −→ E*]*. The following two conditions are equivalent to each other:*

1. *f ∈* mub*{*(*ai bi*): *i ∈ K}.*
2. *There exists a consistent map TK* : Mub*⟨a,K⟩ −→* Mub*⟨b,K⟩ such that f* = *fTK .*

**Proof.** (2) =*⇒* (1): It follows directly from Proposition 4.14.

*i∈K i*

(1) =*⇒* (2): For any *x ∈* ***↑****a* , by Lemma 4.12 (2), there is a subset *I ⊆ K*

such that there exists *m ∈ M⟨a,I⟩* with *x ∈* ***↑****m \* *j∈K\I* ***↑****aj*. Next, we show that *f*

is constant on ***↑****m\* ***↑****a* . We choose an almost algebraic sequence (*m* ) of

*j∈K\I j j j∈ω*

*m*, then there is some *mj x*. For any *y ∈* ***↑****m\ i∈K\I* ***↑****ai*, there is some *ml y*.

Let *n* be the maximum of *j*and *l*. Then *m mn x, y*. Hence, we have

*f* (*x*)= ( *{*(*ai bi*): *i ∈ K}*)(*x*)

*↓f*

= *{bi* : *ai x}* = *{bi* : *i ∈ I}*

*↓f* (*x*)

=

*↓f* (*mn*)

*{bi* : *i ∈ I}*

*↓f* (*x*)

= *f* (*y*) (*∗*)*.*

Then we define a map *TK* : *Mub⟨a,K⟩ → Mub⟨b,K⟩* as follows: *∀m ∈ M⟨a,I⟩*, *TK*(*m*) = *f* (*x*) for any *x ∈* ***↑****m\ i∈K\I* ***↑****ai*. From (*∗*), *TK* is well defined. It is easy to check that *TK* is a consistent map with *f* = *fTK* . *2*

**Proposition 4.16** *Let* (*D, BD*)*,* (*E, BE*) *be two ωAML-domains, let {*(*ai bi*) :

*i ∈ K} be a g.s.f-family. For g ∈* [*D −→ E*] *and for a consistent map TK* :

Mub*⟨a,K⟩ −→* Mub*⟨b,K⟩, we have that*

*fTK g ⇐⇒ ∀m ∈* Mub*⟨a,K⟩, TK*(*m*) *g*(*m*)*.*

**Proof.** If *fTK g*, then (*ai bi*) *g* for all *i ∈ K* form Theorem 4.15. By Lemma 4.2, we have that *bi g*(*ai*) for all *i ∈ K*. For any *m ∈ Mub⟨a,K⟩*, there is *I ⊆ K* with *m ∈ M⟨a,K⟩*. By Lemma 4.12, ***↑****m\ i∈K\I* ***↑****ai /*= *∅*. Pick *x ∈* ***↑****m\ i∈I* ***↑****ai*. Then

*g*(*x*) *≥ fTK* (*x*)= *TK*(*m*) *∈* mub*{bi* : *i ∈ I}.*

On the other hand, by *bi g*(*ai*) for all *i ∈ I* we have that

*g*(*m*) *{bi* : *i ∈ I}* = *{bi* : *i ∈ I}* = *TK*(*m*)*.*

*↓g*(*m*) *↓g*(*x*)

Conversely, let (*gj*)*j* be a directed collection of Scott continuous functions from (*D, BD*) to (*E, BE*) with *g ≤ j gj*. For any *m ∈* Mub*⟨a,K⟩*, by the assumption *TK*(*m*) *g*(*m*), there exists *jm* with *TK*(*m*) *gjm* (*m*). Pick *j*0 with *gj*0 *≥ gjm* for all *m ∈* Mub*⟨a,K⟩*. It is easy to check that *fTK ≤ gj*0 . Hence, *fTK g*. *2*

W

# The category of *ωAML*-domains

In this section, we are ready for showing that the category of *ωAML*-domains is cartesian closed. The main difficulty is to show the the function spaces are almost algebraic.

**Definition 5.1** The category *ω***AML** is given by:

* Objects are all *ωAML*-domains
* Morphisms are Scott continuous functions

Let (*D, BD*)*,* (*E, BE*) be two *ωAML*-domains. We set *GSF*[*D→E*] to be the set of all *g.s.f* families of [*D → E*]. Let

*B*[*D→E*] = *{*mub*{*(*ai bi*): *i ∈ K}* : *{*(*ai bi*): *i ∈ K}∈ GSF*[*D→E*]*}.*

Obviously, *{*(*a b*): *a ∈ BD, b ∈ BE}⊆ B*[*D→E*].

The following result is straightforward from Definition 4.8 and 4.11.

**Lemma 5.2** *Let {*(*ai bi*) : (*ai, bi*) *∈ BD × BE* & *i ∈ K} be a ﬁnite set of step functions.*

1. *Let {*(*an*)*n∈ω* : *i ∈ K} (resp. {*(*bn*)*n∈ω* : *i ∈ K}) bea M-closed almost algebraic*

*i* *i*

*family of {ai* : *i ∈ K} (resp. a M-closed approximating family of {bi* : *i ∈ K}).*

*If* ***↑***(*ai bi*) */*= *∅, then* (*an bn*)(*an*+1 *bn*+1)(*ai bi*) *for all*

*i∈K*

*i*

*i*

*i*

*i*

*n ∈ ω and {*(*an bn*): *i ∈ I} is a g.s.f. family for all nonempty I ⊆ K.*

*i* *i*

1. *Let {*(*a*ˆ*n*)*n∈ω* : *i ∈ K} (resp. {*(ˆ*bn*)*n∈ω* : *i ∈ K}) be a M-closed approximating*

*i* *i*

*family of {ai* : *i ∈ K} (resp. a M-closed almost algebraic family of {bi* : *i ∈*

*K}). If {*(*ai bi*) : *i ∈ K} is a g.s.f family, then* (*a*ˆ*i* ˆ*bi*)(*a*ˆ*n*+1

*i*

ˆ*bn*+1)(*a*ˆ*n* ˆ*bn*) *for all n ∈ ω and {*(*a*ˆ*n* ˆ*bn*): *i ∈ I} is a g.s.f. family for*

*i i i i* *i*

*all nonempty I ⊆ K.*

Next, we will show the category *ω***AML** is cartesian closed through several propositions.

**Proposition 5.3** *B*[*D→E*] *is a countable basis of* [*D → E*]*.*

**Proof.** First of all, we show that *B*[*D→E*] is a countable set. Because *BD* and *BE*

are countable, we have that *GSF*[*D→E*] is countable. For any given *g.s.f* family

*{*(*ai bi*) : *i ∈ K}* of [*D → E*], mub*{*(*ai bi*) : *i ∈ K}* is finite from Theorem

4.15. Hence *B*[*D→E*] is countable.

Next we show that *B*[*D→E*] is a basis of [*D → E*]. For any *f ∈* [*D → E*],

*f* = *{*(*a b*): *a ∈ BD, b ∈ BE, b f* (*a*)*}.*

We only need to show that W*↓f {*(*ai bi*): *ai ∈ BD, bi ∈ BE, bi f* (*ai*)*, i ∈ K}* for a nonempty finite set *K* can be approximated by elements of *B*[*D→E*]. In fact, by Proposition 4.7 we get decreasing sequences (*an*)*n∈ω* for all *i ∈ K* which form a *M*-closed almost algebraic family of *{ai* : *i ∈ K}* and increasing sequences (*bn*)*n∈ω* for all *i ∈ K* which form a *M*-closed approximating family of *{bi* : *i ∈ K}*. For any

*i*

*i*

*n ∈ ω*,

*{*(*an bn*): *i ∈ K}* *{*(*ai bi*): *i ∈ K} f.*

*i* *i*

*↓f ↓f*

By Lemma 5.2 (1), *{*(*an bn*) : *i ∈ K}* is a g.s.f family for all *n ∈ ω*. Thus,

*i*

*i*

W *{*(*an bn*): *i ∈ I}∈ B*[*D→E*] with W *{*(*an bn*): *i ∈ K}* W *{*(*ai bi*):

*↓f*

*i*

*i*

*↓f*

*i*

*i*

*↓f*

*i ∈ K}*. Note that since

*{*(*an bn*): *i ∈ K}* *{*(*an*+1 *bn*+1): *i ∈ K}*

*i i i* *i*

*↓f ↓f*

and

*{*(*an bn*): *i ∈ K}* = *{*(*ai bi*): *i ∈ K},*

*i* *i*

*n∈ω ↓f ↓f*

we finish the proof. *2*

**Proposition 5.4** *B*[*D→E*] *is M-closed.*

**Proof.** Let *{*(*fj, gj*) : *j ∈ K}* be a finite set of *B*[*D→E*] *× B*[*D→E*] with *gj*

*fj* for all *j ∈ K*. It follows that for any given *j ∈ K*, there is a *g.s.f* family

*{*(*aji bji*) : *i ∈ Ij}* with *fj ∈* mub*{*(*aji bji*) : *i ∈ Ij}*. Set *Pj* = *{j}× Ij* and *P* = *j∈K Pj*. By Proposition 4.7, we obtain a *M*-closed almost algebraic family *{*(*an*)*n∈ω* : *p ∈ P}* of *{ap* : *p ∈ P}* and a *M*-closed approximating family

*p*

*{*(*bn*)*n∈ω* : *p ∈ P}* of *{bp* : *p ∈ P}*. Clearly,

*p*

*fj* = *{*(*ap bp*): *p ∈ Pj}*

*↓fj*

= *{* (*an bn*): *p ∈ Pj}*

*p p*

*↓fj n∈ω*

= *{*(*an bn*): *p ∈ Pj}*

*p p*

*n∈ω ↓fj*

holds for each *j ∈ K*. From the assumption *gj fj* we obtain an *nj* such that

if *n ≥ nj* then *gj* W

*↓fj*

*{*(*an*

*bn*) : *p ∈ Pj}* for any given *j ∈ K*. We set

*n*0 =max*{nj* : *j ∈ K}*, then *gj* W

*p*

*p*

*{*(*an bn*): *i ∈ Ij}* holds for all *j ∈ K* and

*n ≥ n*0.

*↓fj p p*

Next, we set *cp* = *an*0+1 and *dp* = *bn*0+1 for all *p ∈ P* . Then for any *p ∈ P* ,

*p p*

*ap cp an*0 *, bn*0 *dp bp.*

*p p*

For each *j ∈ K*, set

*hj* = *{*(*cp dp*): *p ∈ Pj}.*

*↓fj*

By Lemma 5.2 (1), *{*(*cp dp*): *p ∈ Pj}* is a g.s.f. family. Hence *hj ∈ B*[*D→E*] and *g h f* for all *j ∈ K*. For any nonempty *J ⊆ K* such that ***↑****f /*= *∅*, applied Lemma 5.2 (1) again, *{*(*cp dp*): *p ∈* *j∈J Pj}* is also a g.s.f. family; thus

*j j j*  *j∈J j*

mub*{hj* : *j ∈ J}⊆* mub*{*(*cp dp*): *p ∈* *Pj} ⊆f B*[*D→E*]*.*

*j∈J*

*2*

**Proposition 5.5** *B*[*D→E*] *is almost algebraic.*

**Proof.** First of all, we show that each *f ∈ B*[*D→E*] has an almost algebraic sequence (*fn*)*n∈ω ⊆ B*[*D→E*].

Given *f ∈ B*[*D→E*], there is a *g.s.f* family *{*(*ai bi*) : *i ∈ K}* with *f ∈*

mub*{*(*ai bi*): *i ∈ K}*. By Theorem 4.15, there exists a consistent map

*TK* : Mub*⟨a,K⟩ −→* Mub*⟨b,K⟩*

such that *f* = *fTK* . Applying Lemma 3.2 to *{TK*(*m*) : *m ∈* Mub*⟨a,K⟩}*, we get

*{TK*(*m*): *m ∈* Mub*⟨a,K⟩}⊆ BE* such that *TK*(*m*) *TK*(*m*) and *TK*(*m*1) *≤ TK*(*m*2) whenever *TK*(*m*1) *≤ TK*(*m*2) for all *m, m*1*, m*2 *∈* Mub*⟨a,K⟩*.

From Theorem 4.10, there exists a *M*-closed approximating family *{*(*an*)*n∈ω* : *i ∈ K}* of *{ai* : *i ∈ K}*, which satisfies properties (1), (2) and (3) of Theorem 4.10. By Proposition 4.7 (1), there is a *M*-closed almost algebraic family (*{*(*bn*)*n∈ω* : *i ∈ K}*) of *{bi* : *i ∈ K}*. From Lemma 5.2 (2), *{*(*an bn*): *i ∈ K}* is a g.s.f. family for each

*i*

*i*

*i* *i*

*n ∈ ω*.

For any *m ∈* Mub*⟨a,K⟩*, there is *Im ⊆ K* with *m ∈ M⟨a,Im⟩*. Since

*TK*(*m*) *TK*(*m*) *∈ M⟨b,Im⟩,*

we have an *nm* such that if *n ≥ nm* then *TK*(*m*) *∈* *i∈Im*

***↑****bn*. Set

*i*

*n*0 = max*{nm* : *m ∈* Mub*⟨a,K⟩}.*

Pick a fixed *n > n*0. For any *x ∈* Mub*⟨an,K⟩*, there is *Ix ⊆ K* with *x ∈ M⟨an,Ix⟩*.

This means that

*i∈Ix*

*↑an /*= *∅*. Hence,

*i∈Ix*

***↑****ai /*= *∅* and there exists a unique

*mx ∈ M⟨a,I ⟩* such that *rn−*1(*mx*) *x* by Theorem 4.10. Now, for each *n > n*0, we

*i*

*x* *I*

define a map

*TK* : Mub*⟨an,K⟩ −→* Mub*⟨bn,K⟩* as follows: *∀x ∈ Mub⟨an,K⟩*,

*n*

*i*

*Tn* (*x*)=

*K*

*↓TK* (*mx*)

*{bn* : *i ∈ Ix}.*

**Claim 1**: For each *n ≥ n*0, *Tn*

*K*

is consistent.

By the definition of *Tn* , it is sufficient to show that *Tn* is monotone. Let *x, y ∈*

*K K*

Mub*⟨an,K⟩* with *x ≤ y*. There are *I, J ⊆ K* such that *x ∈ M⟨an,I⟩* and *y ∈ M⟨an,J⟩*.

Clearly, *I ⊆ J* . By Theorem 4.10, there is *m*1 *∈ M⟨a,I⟩* and *m*2 *∈ M⟨a,J⟩* such that

*rn−*1(*m*1) *x* and *rn−*1(*m*2) *y* hold respectively. Then

*I J*

*rn−*1(*m*1)= *{an−*1 : *i ∈ I}*

*I* *i*

*↓x*

= *{an−*1 : *i ∈ I}*

*i*

*↓y*

*≤* *{an−*1 : *i ∈ J}*

*i*

*↓y*

= *rn−*1(*m*2)*.*

*J*

It follows that *m*1 *≤ m*2 by Theorem 4.10 (3). Hence

*Tn* (*x*)=

*K*

*↓TK* (*m*1)

=

*↓TK* (*m*2)

*≤*

*↓TK* (*m*2)

= *Tn* (*y*)*.*

*K*

*{bn* : *i ∈ I}*

*{bn* : *i ∈ I}*

*i*

*i*

*{bn* : *i ∈ J}*

*i*

Therefore, *fT n* is a Scott continuous function with *fT n ∈* mub*{*(*an bn*): *i ∈*

*K K i* *i*

*K}* for all *n ≥ n*0 by Proposition 4.14. Next we show the following claim.

**Claim 2**: (*f n*+*n*0 )*n∈ω* is an almost algebraic sequence of *f* .

*T*

*K*

We show it through three steps.

**Step 1**: For each *n > n*0, *f fT n* .

*K*

By Proposition 4.16, we only need to show that *TK*(*m*) *fT n* (*m*) for all

*K*

*m ∈* Mub*⟨a,K⟩*. For any *m ∈* Mub*⟨a,K⟩*, there is *I ⊆ K* such that *m ∈ M⟨a,I⟩*. It means that *i∈I* ***↑****ai /*= *∅* and ***↑****m\ i∈K\I* ***↑****ai /*= *∅* by Lemma 4.12. So

***↑****rn*(*m*)*\* ***↑****an /*= *∅* for all *n > n*0 by Theorem 4.10 (2). It follows that

*I i∈K\I i*

*n n−*1 *n*

*rI* (*m*) *∈ M⟨an,I⟩*. As *rI* (*m*) *rI* (*m*) *m* by Lemma 4.9 (1), we have that

*n n*

*fT n* (*m*) *≥ T* (*r* (*m*)) =

*{bn* : *i ∈ I} TK*(*m*)*.*

*K K I*

*i*

*TK* (*m*)

**Step 2**: For each *n > n*0, *fT n*+1 *fT n* .

*K K*

The proof is similar to Step 1. By Proposition 4.16, we only need to show that

*Tn*+1(*x*) *fT n* (*x*) holds for all *x ∈* Mub*⟨an*+1*,K⟩*. Given *x ∈* Mub*⟨an*+1*,K⟩*, there

*K K*

is *I ⊆ K* such that *x ∈ M⟨an*+1*,I⟩*. By Theorem 4.10, *i∈I* ***↑****ai /*= *∅* and there is

*I*

*m ∈ M⟨a,I⟩* such that *rn*(*m*) *x*.

Case 1: ***↑****rn*(*m*)*\* ***↑****an /*= *∅*.

*I*

We have that

*i∈K\I i*

*n n*

*fT n* (*x*) *≥ T* (*r* (*m*)) =

*{bn* : *i ∈ I}*

*K K I*

*↓TK* (*m*)

*↓TK* (*m*)

*i*

*{bn*+1 : *i ∈ I}*

*i*

= *Tn*+1(*x*)*.*

*K*

Case 2: ***↑****rn*(*m*)*\* ***↑****an* = *∅*.

*I i∈K\I i*

*n*

Clearly, there is *J ⊆ K* with *I ⊆ J* such that *rI* (*m*) *∈ M⟨an,J⟩*. By Theorem 4.10

and Lemma 4.9, *j∈J ↑an /*= *∅* implies that

*j*

*j∈J*

***↑****aj /*= *∅* and there is *m*1 *∈ M⟨a,J⟩*

such that *rn−*1(*m*1) *rn*(*m*) *x*. Because

*J* *I*

*rn−*1(*m*1)= *{an−*1 : *i ∈ J}≥*

*{an−*1 : *i ∈ I}* = *rn−*1(*m*)*,*

*J* *i*

*↓rn*(*m*)

*I*

*i* *I*

*↓rn*(*m*)

*I*

we have that *m*1 *≥ m* by Theorem 4.10 (3). Therefore,

*n n*

*fT n* (*x*) *≥ T* (*r* (*m*)=

*{bn* : *i ∈ J}*

*K K I*

*≥*

*i*

*↓TK* (*m*1)

*{bn* : *i ∈ I}*

*i*

*↓TK* (*m*1)

=

*↓TK* (*m*)

*↓TK* (*m*)

*{bn* : *i ∈ I}*

*{bn*+1 : *i ∈ I}*

*i*

*i*

= *Tn*+1(*x*)*.*

*K*

**Step 3**: For any *g ∈* [*D → E*] with *f g*, there exists *n > n*0 such that

*f n g*.

*T*

*K*

For any given *m ∈* Mub*⟨a,K⟩*, there exists *I ⊆ K* such that *m ∈ M⟨a,I⟩*. By Proposition 4.16 we have that *TK*(*m*) *g*(*m*). Set

Since W

*n∈ω*

*Am* = *{J ⊆ K* : *m ∈ M⟨a,J⟩}.*

*rn*(*m*) = *m* for each *J ∈ Am*, there is *nJ* such that if *n ≥ nJ* then

*J*

*g*(*rn*(*m*)) *TK*(*m*). Set

*J*

*Nm* = max*{nJ* : *J ∈ Am}.*

Then *g*(*rn*(*m*)) *TK*(*m*) for all *J ∈ Am* and for all *n ≤ n*0. By the assumption that *BE* is almost algebraic, there exists *ym ∈ BE* such that *TK*(*m*) *ym TK*(*m*) and *ym g*(*rNm* (*m*)) for all *J ∈ Am*. From *TK*(*m*) *ym*, there is *pm > n*0 such

*J*

*J*

that if *p ≥ pm* then *ym ∈*

*i∈I i*

*↑bp*. Set

*N*0 = max*{*max*{Nm* : *m ∈* Mub*⟨a,K⟩},* max*{pm* : *m ∈* Mub*⟨a,K⟩}}.*

We claim that *f N*0+1

*T*

*K*

*g*. In fact, for any *x ∈* Mub*⟨aN*0+1*,K⟩*, there is *Ix ⊆ K*

such that *x ∈ M⟨aN*0+1*,Ix⟩*. By Theorem 4.10 (1), there exists *m ∈ M⟨a,Ix⟩* with

*rN*0 (*m*) *x*. Then we have

*I*

*x*

*TN*0+1(*x*)= *{bN*0+1 : *i ∈ I }*

*K*

*i*

*x*

*↓TK* (*m*)

= *{bN*0+1 : *i ∈ Ix}≤ ym*

*i*

*↓ym*

*g*(*rN*0 (*m*))

*I*

*x*

*≤ g*(*x*)*.*

Hence, by Proposition 4.16, *f N*0+1 *g*. Combining all of above, we have shown

*T*

*K*

that Claim 2.

Next, we show that for any *f, g ∈ B*[*D→E*], ***↑****f ⊆* ***↑****g ⇒ g ≤ f* .

**Claim 3**: If *g* = (*a b*) with (*a, b*) *∈ BD × BE*, then ***↑****f ⊆* ***↑****g ⇒ g ≤ f* . Suppose that *f ∈* mub*{*(*ai bi*) : *i ∈ K}*. There exists a consistent map

*TK* : Mub*⟨a,K⟩ −→* Mub*⟨b,K⟩* such that *f* = *fTK* . From the above proof, we get

the almost algebraic sequence ((*fT n* )*n>n*0 ) of *f* as above. Since ***↑****f ⊆* ***↑****g*, we have

*K*

*K*

that (*a b*) *f n*

*T*

*K*

for all *n ≥ n*0; thus *b fT n* (*a*) for all *n ≥ n*0 by Lemma

4.2. Without losing generality, we assume *b /*= *⊥*. By the definition of *fT n* , there

*K*

*n*

are a nonempty *I ⊆ K* and an *xa ∈ M⟨a,I⟩* with *a ∈* ***↑****xa\*

*fT n* (*a*)= *Tn* (*xa*). Hence, we have that

*K K*

*b fT n* (*a*)= *{bn* : *an a}.*

*i∈K\I* ***↑****ai* such that

*K i* *i*

*↓fTn* (*a*)

*K*

Set

*In* = *{i ∈ K* : *an a}.*

*i*

Clearly, *In /*= *∅* and *In*+1 *⊆ In* hold for all *n ≥ n*0. Hence there exists *n*1 *≥ n*0 such

that if *n ≥ n*1 then *In* = *In* . This implies that *a an* for all *i ∈ In*

1 *i* 1

and *n ≥ n*0.

So, we have *a ≥ ai* for all *i ∈ In*1 and *a /≥ ai* for all *i ∈ K\In*1 . It follows that

*a ∈ i∈In*1 *↑ai* and there is *m ∈ M⟨a,I⟩* with *m ≤ a* and *i∈In*1 ***↑****ai /*= *∅*. Obviously,

*m ∈ M⟨a,In*

*i*

1

*⟩* and *a ∈* ***↑****rn*

1

*In*

(*m*)*\* *i∈K\In*1

***↑****an*. Therefore,

*fT n* (*a*)=

*{bn* : *i ∈ In } b.*

*K i* 1

*↓TK* (*m*)

It is easy to see that (*fT n* (*a*))*n≥n*0 is an almost algebraic sequence of *TK*(*m*). So we

*K*

have

*TK*(*m*)=

*n≥n*0

*f n* (*a*) *≥ b.*

*K*

*T*

assumption that *BD* is almost algebraic we obtain *x*ˆ *∈* ***↑****m\*

For any *x ∈* ***↑****a*, we have *x ∈* ***↑****m*. Note that since ***↑****m\* *i∈* *K\In*1 ***↑****ai /*= *∅*, by the

*x*ˆ *x*. Hence,

*i∈K\In*1

***↑****ai* such that

*f* (*x*) *≥ f* (*x*ˆ)= *fTK* (*x*)= *TK*(*m*) *≥ b.*

This implies that (*a b*) *≤ f* . So we have proved claim 3.

Next, suppose that *g, f ∈ B*[*D→E*] with ***↑****f ⊆* ***↑****g*. Then there is a g.s.f. family

*{*(*ai bi*) : *i ∈ K}* such that *g ∈* mub*{*(*ai bi*) : *i ∈ K}*. Clearly, ***↑****f ⊆* ***↑****g ⊆*

***↑***(*ai bi*) for all *i ∈ K*. By Claim 3, (*ai bi*) *≤ f* for all *i ∈ K*. Pick *f*ˆ *∈* ***↑****f* , then

*g f*ˆ. Hence, we have

*g* = *{*(*ai bi*): *i ∈ K}*

*↓g*

= *{*(*ai bi*): *i ∈ K}*

*↓f*ˆ

= *{*(*ai bi*): *i ∈ K}*

*↓f*

*≤ f.*

*2*

From the above three propostions, the class of all *ωAML*-domains is closed under function spaces. Let *L* be an algebraic *L*-domain such that (1) the set of all compact elements is countable and has property *M*, (2) *L* is not bounded complete. Then [I *→ L*] is an *ωAML*-domains, which is neither algebraic nor bounded complete. Hence, the notion of a *M*-closed is strictly weaker than being closed. Since the finite cartesian products of *ωAML*-domains are *ωAML*-domains, we have the following result.

**Theorem 5.6** *The category ω***AML** *of ωAML-domains is cartesian closed.*

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