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A Computable Version of the Daniell-Stone Theorem on Integration and Linear Functionals

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Abstract

For every measure *µ*, the integral *I* : *f* '→R *f dµ* is a linear functional on the set of real measurable functions. By the *Daniell-Stone theorem*, for every abstract integral ΛR : *F* → R on a stone vector

lattice *F* of real functions *f* :Ω → R there is a measure *µ* such that *f dµ* = Λ(*f* ) for all *f* ∈ *F* .

In this paper we prove a computable version of this theorem.

*Keywords:* computable analysis, measure theory, Daniell-Stone theorem

# Introduction and Mathematical Preliminaries

In this section we summarize some notations, definitions and facts from mea- sure theory and computable analysis.

As a reference to measure theory we use the book [[1](#_bookmark32)]. A *ring* in a set Ω is a set R of subsets of Ω such that ∅ ∈ R and *A* ∪ *B* ∈ R and *A* \ *B* ∈ R

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if *A, B* ∈ R. A *σ-algebra* in Ω is a set A of subsets of Ω such that Ω ∈ A,

Ω \ *A* ∈ A if *A* ∈ A and ∞ *Ai* ∈ A, if *A*1*, A*2*,...* ∈ A. For any system E of

*i*=1

subsets of Ω let A(E) be the smallest *σ*-algebra in Ω containing E.

A *premeasure* on a ring R is a function *µ* : R → R = R ∪ {−∞*,* ∞} such that *µ*(∅) = 0, *µ*(*A*) ≥ 0 for *A* ∈ R and

∞ ∞

*µ*( *An*) = Σ *µ*(*An*)

*i*=1 *i*=1

if *A*1*, A*2*,...* ∈ A are pairwise disjoint and ∞ *An* ∈ A. A premeasure on an

*i*=1

algebra is called a *measure*. A premeasure *µ* on a ring R is called *σ-ﬁnite*, if there is a sequence *A*1 ⊆ *A*2 ⊆ *A*3 ⊆ *...* in R such that *A*1 ∪ *A*2 ∪ *...* = Ω and *µ*(*Ai*) *<* ∞ for all *i*.

Theorem 1.1 ([[1](#_bookmark32)]) *Every σ-ﬁnite premeasure µ on a ring* R *in* Ω *has a unique extension to a measure on* A(R) *which (for convenience) we also denote by µ.*

Let (Ω*,* A*, µ*) be a measure space. A function *f* : Ω → R is called *mea- surable*, if {*x* | *f* (*x*) *> a*} ∈ A for all *a* ∈ R. The following condition is equivalent:

(∀*a* ∈ *D*) {*x* | *f* (*x*) *> a*} ∈ A for some set *D* dense in R *.* (1)

As usual we will abbreviate {*f > a*} := {*x* ∈ Ω | *f* (*x*) *> a*}. In ([1](#_bookmark2)) the relation “*>*” can be replaced by “≤”, “≥” or “*<*”. A function *f* : Ω → R is

*simple*, if there are *non-negative* real numbers *a*1*,... , an* and pairwise disjoint

sets *A*1*,... , An* ∈ A of finite measure such that *f* (*x*) = Σ*n aiχA* , where *χA*

*i*=1

*i*

is the characteristic function of *A*. For a simple function the integral is defined by

∫ *n*

Σ

*i*=1

*aiχAi* :=

*n*

*i*=1

Σ

*aiµ*(*Ai*) *.* (2)

For functions *u, u*0*, u*1*,...* Ω → R, *ui* 3 *u* means: For all *x* ∈ *ω*, *u*0(*x*) ≤ *u*1(*x*) ≤ *...* and sup*i ui*(*x*) = *u*(*x*). For a non-negative measurable real func- tion *f* : Ω → R and *b* ∈ R, *f dµ* = *b*, iff there is some increasing sequence (*ui*)*i*∈N of simple functions such that

∫

*ui* 3 *f* and sup ∫ *ui dµ* = *b* (3)

*i*

∫ ∫

[[1](#_bookmark32)]. In particular, *f dµ* does not exist (in R), if the sequence ( *ui dµ*)*i* is unbounded. For an arbitrary real function *f* : Ω → R let *f*+ := sup(0*,f* ) (the positive part of *f* ) and *f*− := sup(0*,* −*f* ) (the negative part of *f* ). By

definition, a measurable function *f* is integrable, if *f*+ *dµ* and *f*− *dµ* exist and its integral is defined by

∫ ∫

∫ *f dµ* := ∫ *f*+ *dµ* − ∫ *f*− *dµ .* (4)

For the following concepts from computable analysis see [[4](#_bookmark35)]. Let N :=

{0*,* 1*,* 2*,.. .*} be the set of natural numbers. A partial function from *X* to *Y* is denoted by *f* : ⊆ *X* → *Y* , a multifunction by *f* : ⊆ *X* ⇒ *Y* . Let Σ be a sufficiently large finite alphabet such that {0*,* 1} ⊆ Σ. The set of finite words over Σ is denoted by Σ∗, the set of infinite sequences by Σ*ω*. Computability

of functions on Σ∗ and Σ*ω* is defined by Turing machines which can read and write finite and infinite sequences, respectively. Standard pairing functions

on Σ∗ are denoted by ⟨ ; ⟩. For *w* ∈ Σ∗ let *ξw* : ⊆ Σ∗ → Σ∗ be the word function computed by the Turing machine with canonical code *w* ∈ Σ∗. Like the “effective G¨odel numbering” *φ* : N → *P* (1) of the partial recursive functions the notation *ξ* satisfies the utm-theorem and the smn-theorem.

Computability on other sets is introduced by using finite or infinite se- quences of symbols as “names”. For the natural numbers let *ν*N : ⊆ Σ∗ → N be the notation by binary numbers and let bn*i* be the binary name of *i* ∈ N.

Let *ν*Q : ⊆ Σ∗ → Q be some standard notation of the rational numbers. For the real numbers we use the standard Cauchy representation *ρ* : ⊆ Σ*ω* → R,

where *ρ*(*p*) = *x*, iff *p* encodes a sequence (*ai*)*i* of rational numbers such that

|*ai*−*x*| ≤ 2−*i*. For naming systems *δi* : ⊆ *Yi* → *Mi*, *Yi* ⊆ {Σ∗*,* Σ*ω*} for *i* = 1*,* 2, a multifunction *f* : ⊆ *M*1 ⇒ *M*2 is (*δ*1*, δ*2)-computable, iff there is a computable function *h* : ⊆ *Y*1 → *Y*2 such that *δ*2 ◦ *h*(*p*) ∈ *f* (*δ*1(*p*)) for all *p* ∈ dom(*δ*1) such that *f* (*δ*1(*p*)) /= ∅.

In this article we will consider computability on factorizations of several pseudometric spaces [[2](#_bookmark33)]. We generalize the definition of a computable metric space with Cauchy representation from [[4](#_bookmark35)] straightforwardly as follows: A

computable pseudometric space is a quadruple M = (*M, d, A, α*) such that (*M, d*) is a pseudometric space, *A* ⊆ *M* is dense and *α* : ⊆Σ∗ → *A* is a notation of *A* such that dom(*α*) is recursive and the restriction of the pseudometric *d*

to *A* is (*α, α, ρ*)-computable. (In [[4](#_bookmark35)], dom(*α*) is assumed to be r.e. Notice that for every notation with r.e. domain there is an equivalent one with recursive

domain.) In our applications, M is a linear space and the pseudometric is derived from a seminorm ||*.*||, *d*(*x, y*) = ||*x* − *y*||.

The factorization (*M, d*) of the pseudometric space (*M, d*) is a metric space defined canonically as follows: *x* := {*y* ∈ *M* | *d*(*x, y*) = 0}, *M* := {*x* | *x* ∈ *M* }, *d*(*x, y*) := *d*(*x, y*). We define the Cauchy representation *δ*M of the factorization of a computable pseudometric space as follows: *δ*M(*p*) = *x*, if *p* ∈ Σ*ω* encodes

a sequence (*ai*)*i* (of *α*-names) of elements of *A* such that *d*(*ai, x*) ≤ 2−*i* for all

*i*. If M is a linear space with seminorm ||*.*||, by *ax* := *ax* and *x* + *y* := *x* + *y* the factor space becomes a linear space with norm ||*x*|| := ||*x*||. In this case, *d*(*x, y*) = ||*x* − *y*||.

# Computable Measure Spaces

In this section let (Ω*,* A*, µ*) be a measure space. For any D ⊆ A let D*f* :=

{*A* ∈ D | *µ*(*A*) *<* ∞}. In computable measure theory we want to identify two sets *A, B* ∈ A, if their symmetric difference *A*∆*B* := (*A* \ *B*) ∪ (*B* \ *A*) has measure 0 and distinguish them otherwise. Since *A*∆*B* ⊆ *A*∆*C* ∪ *C*∆*B*, on the set A*f* the mapping *d* : (*A, B*) '→ *µ*(*A*∆*B*) is a pseudometric.

Lemma 2.1 *Let* R *be a ring such that* A(R) = A *and µ is a σ-ﬁnite premea- sure on* R*. Then* (A*f , d*)*, d* : (*A, B*) '→ *µ*(*A*∆*B*)*, is a complete pseudometric space with* R*f as a dense subset.*

Proof: Straightforward.

For including sets with infinite measure consider the mapping *d*∞ : (*A, B*) '→

*µ*(*A*∆*B*)*/*(1 + *µ*(*A*∆*B*)) which is a pseudometric on A (notice: ∞*/*(1 + ∞) =

1. Its restriction to A*f* is equivalent to *d*. For introducing computability on a pseudometric space we need a countable dense subset [[4](#_bookmark35),[3](#_bookmark34)]. Unfortunately,

there are important measure spaces such that the pseudometric space (A*, d*∞) is not separable.

Example: Consider the measure space (R*,* B*, λ*) where B is the set of Borel subsets of the real numbers and *λ* is the Lebesgue-Borel measure. Let (*Ei*)*i*∈N be any countable sequence in B. Define *B* := *i*(*i*; *i* + 1) \ *Ei*. Then for all *i*, *λ*(*B*∆*Ei*) ≥ 1 and hence *d*∞(*B, Ei*) ≥ 1*/*2. Therefore, the set of all *Ei* cannot be dense. Since this is true for every sequence (*Ei*)*i*∈N, the pseudometric space

(B*, d*∞) is not separable.

We will consider measures which are completions of *σ*-finite premeasures on *countable* rings consisting of sets with finite measure. We assume that the operations on the ring and the premeasure are computable.

Definition 2.2 A *computable measure space* is a quintuple

M = (Ω*,* A*, µ,* R*, α*) such that

* 1. A is a *σ*-algebra in Ω and *µ* is a measure on it,
  2. R is a countable ring such that A = A(R),
  3. *µ*(*A*) *<* ∞ for all *A* ∈ R,
  4. the restriction of *µ* to R is *σ*-finite,
  5. *α* : ⊆Σ∗ → R is a notation of R with recursive domain,
  6. (*A, B*) '→ *A* ∪ *B* and (*A, B*) '→ *A* \ *B* are (*α, α, α*)-computable,
  7. *µ* is (*α, ρ*)-computable on R.

*σ*-finite measure, either restrict Ω to define *µ*(Ω \ R) = 0.

By ([iv](#_bookmark8)), Ω = R. If R is a pr oper subset of Ω, then f or obtaining a

R or add the set Ω \

R to R and

Theorem 2.3 *Let* (Ω*,* A*, µ,* R*, α*) *be a computable measure space. Then the quadruple* (A*f , d,* R*, α*) *is a computable complete pseudometric space, where* A*f* = {*A* ∈ A | *µ*(*A*) *<* ∞} *and d*(*A, B*) = *µ*(*A*∆*B*)*.*

Proof: By Lemma [2.1](#_bookmark3), (A*f , d*) is a complete pseudometric space with R as a dense subset. By Def. [2.2](#_bookmark4)([v](#_bookmark9))-([vii](#_bookmark10)) the notation *α* has recursive domain and

the distance *d* is (*α, α, ρ*)-computable.

Computability on the computable measure space can be defined via the Cauchy representation of the joined pseudometric space.

Example 2.4 [Lebesgue-Borel measure on R] Let Ω = R, let *D* ⊆ R be dense in R and let *ν**D* : ⊆Σ∗ → *D* be a notation such that dom(*νD*) is recursive and *ν* ≤ *ρ*. Let *I*˜*D* be the set of all intervals [*a*; *b*) ⊆ R such that *a, b* ∈ *D* and *a < b*. Let RD be the set of all finite unions of intervals from *I*˜*D* and let *αD* be some notation of RD canonically derived from *νD*. Then B := A(R*D*) is the set of Borel-subsets of R. The Lebesgue-Borel measure *λ* on B is defined uniquely by setting *λ*([*a*; *b*)) := *b* − *a* for all *a, b* ∈ *D*, *a < b* [[1](#_bookmark32)]. M*D* := (R*,* B*, λ,* R*D, αD*) is a computable measure space.

# Computability on the Integrable Functions

In this section we assume that M = (Ω*,* A*, µ,* R*, α*) is a computable measure space. We introduce a computable pseudometric space for the integrable func- tions. On the set I(M) of *µ*-integrable functions *f* : Ω → R a seminorm and a pseudometric are defined by

||*f* ||M := ∫ |*f* | *dµ, d*M(*f, g*) := ||*f* − *g*||M*.* (5)

(see [[1](#_bookmark32)]). For introducing computability on I(M) we consider a countable dense set.

Definition 3.1 (i) A function *u* : Ω → R is a *rational step function*, iff there are rational numbers *a*1*,... , an* and pairwise disjoint sets

*A*1*,... , An* ∈ R such that *u* = Σ*n ai* · *χA* .

*i*=1

*i*

(ii) Let *β* : ⊆ Σ∗ → RSF be a canonical notation of the set RSF of rational step functions derived from the notation *α* such that dom(*β*) is recursive.

In contrast to a simple function (see Sec. [1](#_bookmark1)), for a rational step function

*f* = Σ

*n*

*i*=1

rational, but may be negative. For a rational step function *u* =

*ai* · *χAi*

the sets *Ai* must be in R and the coefficients must be

*n*

Σ

*i*=1 *ai* · *χAi* ,

∫ *u dµ* = Σ*n ai* · *µ*(*Ai*) and ||*u*||M = Σ*n* |*ai*|· *µ*(*Ai*).

*i*=1

*i*=1

Lemma 3.2 *For rational step functions u, v and a* ∈ Q *the functions*

∫

1. (*a, u*) '→ *a* · *u,* (*u, v*) '→ *u* + *v, u* '→ |*u*|*, u* '→ inf(*u,* 1)*, u* '→ *u dµ,*
2. (*u, v*) '→ sup(*u, v*)*,* (*u, v*) '→ inf(*u, v*)*, u* '→ *u*+*, u* '→ *u*−*,* (*u, a*) '→

inf(*u, a*)*, u* '→ ||*u*||M

*are computable w.r.t. the notations β, ν*Q *and ρ.*

Proof: Straightforward.

In Def. [3.1](#_bookmark14)([i](#_bookmark14)) the condition “*A*1*,... , An* are pairwise disjoint” is not re- strictive.

Lemma 3.3 *Let β*' *be a canonical notation of all u* = Σ*n ai* · *χA such that*

*i*=1

*i*

*ai* ∈ Q *and Ai* ∈ R *(but the Ai are not necessarily disjoint). Then β*' ≡ *β.*

Proof: “≤”: From the sets *Ai* by determining intersections and differences a finite set *B*1*,... , Bm* of pairwise disjoint sets can be computed such that each

*Ai* is a finite union of *Bj*s. Then coefficinets *bj* ∈ Q can be computed such

= Σ

that Σ*n*

*i*=1

*ai* · *χAi*

*m j*=1

*bj* · *χBj* . This procedure is computable w.r.t the

representations *β, β*'*, α, ν*Q and *ν*N.

“≥”: Obvious.

Theorem 3.4 (I(M)*, d*M*,* RSF*, β*) *is a computable complete peudometric space.*

Proof: By Th. 15.5 in [[1](#_bookmark32)], (I(M)*, d*M) is complete.

∫ ∫ ∫

Consider *f* ∈ I(M) and *ε >* 0. Then *f dµ*∫= *f*+ *dµ*∫ − *f*− *dµ*. By ([3](#_bookmark2))

∫ ∫ ∫

there is a simple function *u* ≤ *f*+ such that 0 ≤

*f*+ *dµ* −

*u dµ < ε/*4, hence

*d*M(*f*+*, u*) = |*f*+ − *u*| *dµ* = *f*+ *dµ* − *u dµ < ε/*4. Since Q is dense in R and R is dense in A*f* by Thm. [2.3](#_bookmark11), there is a rational step function *v* such that *d*M(*u, v*) *< ε/*4. We obtain *d*M(*f*+*, v*) ≤ *d*M(*f*+*, u*) + *d*M(*u, v*) ≤ *ε/*2. Correspondingly, there is a rational step function *w* such that *d*M(*f*−*, w*) ≤ *ε/*2. We obtain *d*M(*f, v* −*w*) = ||*f*+ −*f*− −(*v* −*w*)|| ≤ ||*f*+ −*v*||+||*f*−−*w*|| *< ε*. Therefore, *v* − *w* is a rational step function which is *ε*-close to *f* .

On RSF the distance *d*M is (*β, β, ρ*)-computable. This follows from Lemma [3.2](#_bookmark15).

Let *δ*M : ⊆ Σ*ω* → I(M)*/*≡ be the Cauchy representation of the set of equivalence classes of integrable functions (see Sec. [1](#_bookmark1)).

# The Computable Daniell-Stone Theorem

For two real-valued functions let (*f* ∧ *g*)(*x*) := inf(*f* (*x*)*, g*(*x*)). A *Stone vector lattice* of real functions is a vector space F of functions *f* : Ω → R such that the functions *x* '→ |*f* (*x*)| and *x* '→ inf(*f* (*x*)*,* 1) (denoted by |*f* | and *f* ∧1, resp.) are in F if *f* ∈ F.

Let F+ be the set of non-negative functions in F. Let us call F *complete*, if *f* ∈ F whenever *ui* 3 *f* for *ui* ∈ F+ and *f* : Ω → R.

An *abstract integral* on a Stone vector lattice F of real functions is a linear functional *I* : F → R such that for all *f, f*0*, f*1*,...* ∈ F+,

*I*(*f* ) ≥ 0 and *I*(*f* ) = *I*(sup *fn*) = sup *I*(*fn*) if *fi* 3 *f .* (6)

*n n*

Let A(F) be the smallest *σ*-algebra in Ω such that every function *f* ∈ F

is measurable.

Theorem 4.1 (Daniell-Stone [[1](#_bookmark32)]) *Let* F *be a Stone vector lattice with ab- stract integral I. Then there is a measure µ on* A(F) *such that f is µ-integrable and I*(*f* ) = *f dµ for all f* ∈ F*. Furthermore, if there is a sequence* (*fi*)*i in* F *such that* (∀*x* ∈ Ω)(∃*i*)*fi*(*x*) *>* 0*, then the measure µ is uniquely deﬁned.*

∫

For a proof see Thms. 39.4 and Cor. 39.6 in [[1](#_bookmark32)]. On a Stone vector lattice with abstract integral a seminorm ||*.*||S and a pseudometric *d*S can be defined by

||*f* ||S := *I*(|*f* |) and *d*S (*f, g*) := ||*f* − *g*||S = *I*(|*f* − *g*|) *.* (7) For an effective version of Thm. [4.1](#_bookmark18) we consider a notation *γ* of a dense subset

D such that (F *, d*S*,* D*, γ*) is a computable pseudometric space. Furthermore, we assume that |*f* |*, f* ∧ 1 ∈ D if *f* ∈ D and that D is closed under rational linear combination.

Definition 4.2 A *computable Stone vector lattice with abstract integral* is a tuple S = (Ω*,* F *,I,* D*, γ*) such that

* 1. F is a Stone vector lattice with abstract integral *I*,
  2. D ⊆ F is dense w.r.t the pseudometric *d*S : (*f, g*) '→ *I*(|*f* − *g*|),
  3. *γ* is a notation of D with recursive domain,
  4. if *a* ∈ Q and *f, g* ∈ D, then {*af, f* + *g,* |*f* |*, f* ∧ 1} ⊆ D,
  5. for *a* ∈ Q and *f, g* ∈ D, the functions (*a, f* ) '→ *af* , (*f, g*) '→ *f* +*g*, *f* '→ |*f* |

and *f* '→ *f* ∧ 1 are computable w.r.t. *ν*Q, *γ* and *ρ*.

* 1. the restriction of *I* to D is (*γ, ρ*)-computable.

Let *δ*S : ⊆ Σ*ω* → F*/*≡ be the canonical Cauchy representation of the fac- torization of the computable pseudometric space (F *, d*S*,* D*, γ*).

It can be shown easily that (F *, d*S*,* D*, γ*) is a computable pseudometric space. For a computable measure space, the integrable functions with the in-

tegral as linear operator form a computable Stone vector lattice with abstract integral.

Proposition 4.3 *Let* M = (Ω*,* A*, µ,* R*, α*) *be a computable measure space. Then* (Ω*,* I(M)*,* (*f* '→ *f dµ*)*,* RSF*, β*) *(see Def.* [*3.1*](#_bookmark14)*(*[*ii*](#_bookmark16)*)) is a computable com- plete Stone vector lattice with abstract integral.*

∫

Proof: Straightforward.

For two metric spaces (*Mi, di*) (*i* = 0*,* 1) call *ψ* : *M*0 → *M*1 a *metric embedding*, iff *d*1(*ψ*(*x*)*, ψ*(*y*)) = *d*0(*x, y*) for all *x, y* ∈ *M*0. Obviously, a met- ric embedding *ψ* is injective, i.e., (*M*0*, d*0) is, up to renaming, a subspace of

(*M*1*, d*1). For computable metric spaces (*Mi, di, Ai, αi*) (*i* = 0*,* 1) with Cauchy representaions *δi* (*i* = 0*,* 1), if *ψ* : *M*0 → *M*1 is a (*δ*0*, δ*1)-computable embed- ding, then its inverse *ψ*−1 : ⊆ *M*1 → *M*0 is (*δ*1*, δ*0)-computable. In this case, the first space is, up to renaming, a very well behaved subspace of the second

one.

We can now formulate and prove our computational version of the Daniell- Stone theorem. (We use the Cauchy representation *δ*M of a factorized pseudo- metric space of the integrable functions, see Thm. [3.4](#_bookmark17) and the end of Sec. [3](#_bookmark13).)

Theorem 4.4 (computable Daniell-Stone) *Let* S = (Ω*,* F *,I,* D*, γ*) *be a computable Stone vector lattice with abstract integral such that* (∀*x* ∈ Ω)(∃*f* ∈ D)*f* (*x*) *>* 0*. Then there exist a computable measure space* M = (Ω*,* A*, µ,* R*, α*) *and a funtion ψ such that*

1. *ψ is a* (*δ*S *, δ*M) *computable metric embedding ψ* : F */*≡ → I(M)*/*≡*;*
2. *I*(*f* ) = ∫ *g dµ for all f* ∈ F *and g* ∈ *ψ*(*f/*≡)*;*

*where δ*S *is the Cauchy representation of the factorized pseudometric space derived from* S *(Def.* [*4.2*](#_bookmark19)*) and δ*M *is the Cauchy representation of the factorized pseudometric space derived from* M *(Thm.* [*3.4*](#_bookmark17)*).*

For the main proof we need a number of auxiliary propositions. Because

of the space limit their proofs are omitted. First, a ring R on Ω must be defined. Consider *f* ∈ D. Since *f* must be *µ*-integrable by ([i](#_bookmark21)) and hence A-measurable, we must have {*f > a*} ∈ A = A(R) for all *a* ∈ R. Since

{*f > a*} = *a<b*∈Q{*f > b*}, it would suffice to require {*f > b*} ∈ R for all

*f* ∈ D and *b* ∈ Q. Unfortunately, some of the values *µ*({*f > b*}), *b* ∈ Q, (which will be defined canonically) might become non-computable. In order to avoid

this problem, for every function *f* ∈ D+ (the non-negative functions from D) we construct a new countable dense set *Cf* of computable real numbers (see

([1](#_bookmark2))) such that *µ*({*f > c*}) becomes computable for each *c* ∈ *Cf* . R will be the smallest ring containing all the sets {*f > c*} (*f* ∈ D+, *c* ∈ *Cf* ) for which we define *µ*{*f > c*} := sup{*I*(*h*) | *h* ∈ D+*, h* ≤ *χ*{*f>c*} }. Moreover, we define a notation *α* : ⊆ Σ∗ → R such that ([v](#_bookmark9)) - ([vii](#_bookmark10)) from Def. [2.2](#_bookmark4) are satisfied. A further crucial step is to show that for every function *f* ∈ D+ and every *n* ∈ N a rational step function *t* in M = (Ω*,* A*, µ,* R*, α*) with non-negative coefficients can be computed w.r.t. the notations *γ*, *ν*N and *β* (from Def. [3.1](#_bookmark14))

∫

such that *t* ≤ *f* and 0 ≤ *I*(*f* ) − *t dµ* ≤ 2−*n .*

Define a notation *γ*+ of D+ := D ∩ F+ by *γ*+(*v*) := |*γ*(*v*)|. From Def. [4.2](#_bookmark19) we can conclude that *γ*+ is reducible to *γ* (*γ*+ ≤ *γ*). Define a notation *ν*→ of the computable sequences in D+ by

*ν*→(*s*) = (*f*0*, f*1*,.. .*) ⇐⇒ (∀*w* ∈ dom(*ν*N)) *fν*N(*w*) = *γ*+ ◦ *ξs*(*w*) *,* (8) that is, iff *ξs* is a (*ν*N*, γ*+)-realization of *i* '→ *fi* (see Section [1](#_bookmark1)).

As a first step, for each *f* = *γ*+(*v*) ∈ D+ we compute some dense set *Dv* ⊆ R+ such that *µ*({*f > a*}) is a computable real number for all *a* ∈ *Dv* (and show how to compute these values).

Proposition 4.5 *For every f* ∈ D+ *and every a*0*, b*0 ∈ Q*,* 0 *< a*0 *< b*0*, a real number c and two sequences* (*gn*)*n and* (*hn*)*n in* D+ *can be computed w.r.t. the notations γ, ν*Q*, ν*→ *and ρ such that*

*a*0 *< c < b*0 (9)

0 ≤ *h*0 ≤ *h*1 ≤ *...* ≤ *χ*{*f>c*} ≤ *χ*{*f*≥*c*} ≤ *...* ≤ *g*1 ≤ *g*0 (10) sup *I*(*hn*) = inf *I*(*gn*)*.* (11)

Notice that for every fixed *v* ∈ dom(*γ*) = dom(*γ*+), the set of constants *c*,

*Dv* := {*ρ* ◦ *H*0(*v, ul, ur*) | 0 *< ν*Q(*ul*) *< ν*Q(*ur*)} is dense in R+ *.* (12) We define the ring and the *σ*-algebra for the measure space M.

Definition 4.6

R0 := {{*γ*+(*v*) *> ρ* ◦ *H*0(*v, ul, ur*)} | *v* ∈ dom(*γ*+)*,* 0 *< ν*Q(*ul*) *< ν*Q(*ur*)}

R := the smallest ring containing R0 A := A(R) = A(R0)

Notice that R0 is not a ring in general. By Prop. [4.8](#_bookmark25) for every set *A* ∈ R0

there are sequences (*hi*) and (*gi*) in D+ such that

0 ≤ *h*0 ≤ *h*1 ≤ *...* ≤ *χA* ≤ *...* ≤ *g*1 ≤ *g*0 and sup *I*(*hn*) = inf *I*(*gn*) *.*(13)

In the following we prove that this is true also for all *A* ∈ R. Additionally we introduce a notation *α* of R such that the sequences (*hi*) and (*gi*) can be computed from *A* ∈ R.

Proposition 4.7 *For functions hn, gn, h*' *, g*' ∈ D+ *and A, A*' ⊆ Ω *let*

*n n*

0 ≤ *h*0 ≤ *h*1 ≤ *...* ≤ *χA* ≤ *...* ≤ *g*1 ≤ *g*0*,*

sup *I*(*hn*) = inf *I*(*gn*)*,*

' ' ' '

0 ≤ *h*0 ≤ *h*1 ≤ *...* ≤ *χA*' ≤ *...* ≤ *g*1 ≤ *g*0*,*

sup *I*(*h*' ) = inf *I*(*g*' )*.*

*n n*

*Then for h*+ := sup(*hn, h*' )*, g*+

:= sup(*gn, g*' )*, h*− := (*hn* − *g*' )+ *and*

*n n n*

*g*− := (*gn* − *h*' )+*,*

*n n n*

*n n*

+ + + +

0 ≤ *h*0 ≤ *h*1 ≤ *...* ≤ *χA*∪*A*' ≤ *...* ≤ *g*1 ≤ *g*0 *,*

sup *I*(*h*+) = inf *I*(*g*+)*,*

*n n*

− − − −

0 ≤ *h*0 ≤ *h*1 ≤ *...* ≤ *χA*\*A*' ≤ *...* ≤ *g*1 ≤ *g*0 *,*

sup *I*(*h*−) = inf *I*(*g*−)*.*

*n n*

By the next proposition the constructions in Prop. [4.7](#_bookmark24) are computable. Let us say that *t* = ⟨*s*−*, s*+⟩ encloses a set *A* ⊆ Ω, if ([13](#_bookmark23)) for the sequences (*h*0*, h*1*,.. .*) := *ν*→(*s*−) and (*g*0*, g*1*,.. .*) := *ν*→(*s*+).

Proposition 4.8 *There are computable functions G*1 *and G*2 *such that G*1(*t, t*') *encloses A* ∪ *A*' *and G*2(*t, t*') *encloses A* \ *A*'*, if t encloses A and t*' *encloses A*'*.*

Proposition 4.9 *There is a computable function L such that*

*ρ* ◦ *L*(⟨*s*−*, s*+⟩) = sup *I*(*hn*)*, if ν*→(*s*−) = (*hi*)*i and ν*→(*s*+) = (*gi*)*i such that (*[*13*](#_bookmark23)*).*

Proof: This follows by standard arguments from Def. [4.2](#_bookmark19)([vi](#_bookmark20)). (Prop. [4.9](#_bookmark26))

We define a notation *α* of R inductively as follows. By Prop. [4.5](#_bookmark22) there is a computable function *H*0 such that

*c*; *ρ* ◦ −09*v, ul, ur*)

if *f* = *γ*+(*v*), *a*0 = *ν*Q(*ul*) and *b*0 = *ν*Q(*ur*). (For convenience we assume dom(*γ*)*,* dom(*ν*Q) ⊆ (Σ \ Σ')∗ and Σ' ⊆ Σ \ {0*,* 1} for Σ' := {(*,* )*,* ∪*,* \}.)

*α*(⟨*v, ul, ur*⟩) := {*γ*+(*v*) *> ρ* ◦ *H*0(*v, ul, ur*)} ∈ R0 *,* (14)

*α*( (*w* ∪ *w*')) := *α*(*w*) ∪ *α*(*w*') *,* (15)

*α*( (*w* \ *w*')) := *α*(*w*) \ *α*(*w*') (16)

for *v* ∈ dom(*γ*) = dom(*γ*+), *ul, ur* ∈ dom(*ν*Q) such that 0 *< ν*Q(*ul*) *< ν*Q(*ur*) and *w, w*' ∈ dom(*α*). Let *α*(*x*) be undefined for all other *x* ∈ Σ∗. Then *α* is a notation of R such that dom(*α*) is recursive. Obviously, union and difference on R are (*α, α, α*)-computable.

Thus we have proved ([v](#_bookmark9)) and ([vi](#_bookmark12)) in Def. [2.2](#_bookmark4):

Proposition 4.10 *α* : ⊆ Σ∗ → R *is a notation of* R *with recursive domain and* (*A, B*) '→ *A* ∪ *B and* (*A, B*) '→ *A* \ *B are* (*α, α, α*)*-computable,*

Next, we define the function *µ* on A = A(R). For finding a *σ*-additive measure we apply the non-effective theorem [4.1](#_bookmark18) since R0 ⊆ F, A(R) ⊆ A(F).

Definition 4.11 Let *µ*' be the unique measure on A(F) such that *f* is *µ*'- integrable and *I*(*f* ) = *f dµ*' for all *f* ∈ F (Thm. [4.1](#_bookmark18)). Let *µ* be the restriction of *µ*' to A(R).

∫

Since A(R) is a *σ*-algebra, *µ* is a measure. Therefore, ([i](#_bookmark5)), ([ii](#_bookmark6)), ([v](#_bookmark9)) and ([vi](#_bookmark12)) from Def. [2.2](#_bookmark4) are true. It remains to prove ([iii](#_bookmark7)) and ([vii](#_bookmark10)). From Prop. [4.7](#_bookmark24) we

obtain:

Proposition 4.12 *For every A* ∈ R *and sequences* (*hi*) *and* (*gi*) *in* D+ *such that (*[*13*](#_bookmark23)*), χA dµ* = *µ*(*A*) = sup*i I*(*hi*) = inf*i I*(*gi*)*. Furthermore, appropri- ate sequences* (*hi*) *and* (*gi*) *in* D+ *can be computed from A w.r.t. the notations α and ν*→*.*

∫

Proof: For all *i* we obt∫ain: *I*(*hi*) = ∫ *hi dµ*' ≤ ∫ *χA dµ*' ≤ ∫ *gi dµ*' = *I*(*gi*).

Therefore, sup*i I*(*hi*) =

*χA dµ*' = *µ*'(*A*) = *µ*(*A*). (Prop. [4.12](#_bookmark28))

Using the functions *G*1 and *G*2 from Prop. [4.8](#_bookmark25) and the function *L* from Prop. [4.9](#_bookmark26) we prove that the measure *µ* is (*α, ρ*)-computable on R.

Proposition 4.13 *The measure µ is* (*α, ρ*)*-computable on* R*, in particular,*

*µ*(*A*) *<* ∞ *for all A* ∈ R*.*

Thus we have proved Def. [2.2](#_bookmark4)([iii](#_bookmark7)) and ([vii](#_bookmark10)). Finally we prove Def. [2.2](#_bookmark4)([iv](#_bookmark8)).

Proposition 4.14 *The restriction of µ to* R *is σ-ﬁnite.*

Proof: Since R is countable and *µ*(*A*) *<* ∞ for all *A* ∈ R, it suffices to show (∀*x* ∈ Ω)(∃*A* ∈ R) *x* ∈ *A*. Consider *x* ∈ Ω. By assumption *f* (*x*) /= 0 for some *f* ∈ D. Then |*f* | = *γ*+(*v*) ∈ D+ for some *v* and |*f* |(*x*) *>* 0. Therefore, there is

some *c* ∈ *Dv* (see ([12](#_bookmark22))) such that |*f* |(*x*) *> c*. Therefore, *x* ∈ {|*f* | *> c*} ∈ Y.

(Prop. [4.14](#_bookmark29))

Altogether, we have a defined a computable measure space ł = (Ω*,* A*, µ,* Y*, α*). Finally, we consider integration. First, we generalize Prop. [4.12](#_bookmark28) from char-

acterictic functions *χA*, *A* ⊆ Y to rational linear combinations of such func-

tions, i.e., rational step functions. A notation *β* for the rational step functions

is defined in Def. [3.1](#_bookmark14).

Proposition 4.15 *For every rational step function t with non-negative coef- ﬁcients and every m* ∈ N*, functions H, G* ∈ Ð+ *can be computed (w.r.t. β and γ) such that H* ≤ *t* ≤ *G and*

∫ *t dµ* — 2−*m* ≤ *I*(*H*) ≤ ∫ *t dµ* ≤ *I*(*G*) ≤ ∫ *t dµ* + 2−*m .*

Proof: Straightforward from Prop. [4.12](#_bookmark28) .

Notice that a *µ*'-integrable function *f* ∈ J (see Def. [4.11](#_bookmark27)) which is *µ*- measurable may be not *µ*-integrable. We prove the converse of Prop. [4.15](#_bookmark30).

Proposition 4.16 *For every function f* ∈ Ð+ *and every n* ∈ N *a rational step function t in* ł = (Ω*,* A*, µ,* Y*, α*) *with non-negative coeﬃcients can be computed w.r.t. the notations γ, ν*N *and β (from Def.* [*3.1*](#_bookmark14)*) such that*

*t* ≤ *f and* 0 ≤ *I*(*f* ) — ∫ *t dµ* ≤ 2−*n .*

Let J∗ be the set of all *f* : Ω → R such that *fi* 3 *f* for some sequence of functions in J+.

Define *I*∗ : J∗ → R by

+

+

*I*∗(*f* ) := sup *I*(*ui*) if *ui* 3 *f .*

*i*

In [[1](#_bookmark32)] p. 189 it is proved that *I*∗ is well-defined (i.e., sup*i I*(*ui*) = sup*i I*(*vi*) if *ui* 3 *f* and *vi* 3 *f* ) and that *I*∗ extends *I* on J+ such that *I*∗(*af* ) = *aI* ∗(*f* ) (*a* ≥ 0), *I*∗(*f* + *g*) = *I*∗(*f* )+ *I*∗(*g*) (*f, g,* ∈ J∗ ) and *I*∗(sup*i fi*) = sup*i I*∗(*fi*) if

+

*fi* 3 *f* in J∗ .

+

For every *A* ∈ Y, there is a sequence (*hi*)*i* in Ð+ such that *hi* 3 *χA*, hence by Prop. [4.12](#_bookmark28), *χA dµ* = *µ*(*A*) = *I*∗(*χA*), therefore

∫

∫ *t dµ* = *I*∗(*t*) for every non-negative rational step function *t.* (17)

Now define the embedding *ψ* : J */*≡ → I(ł)*/*≡. First, we define *ψ*(*f* ) for

*f* ∈ J+ by a (*δ*S *, δ*M)-realization on names as follows.

∫

Suppose *δ*S (*p*) = *f* . Then *p* encodes (*γ*-names of) elements *fi* ∈ Ð+ such *I*(|*f* — *fi*|) ≤ 2−*i*. By Prop [4.16](#_bookmark31), for each *i* a rational step function *si* can be computed such that 0 ≤ *si* ≤ *fi*+2 and 0 ≤ *I*(*fi*+2) — *si dµ* ≤ 2−*i*−2, and hence

0 ≤ *I*∗(|*fi*+2 — *si*|)

= *I*∗(*fi*+2) — *I*∗(*si*)

≤ 2−*i*−2 *.*

Then for any *k > i*,

∫ |*si* — *sk*|*dµ* = *I*∗(|*si* — *sk*|) by ([17](#_bookmark31))

≤ *I*∗(|*si* — *fi*+2|)+ *I*∗(|*fi*+2 — *f* |)

+*I*∗(|*f* — *fk*+2|)+ *I*∗(|*fk*+2 — *sk*|)

≤ 2−*i*−2 + 2−*i*−2 + 2−*k*−2 + 2−*k*−2

≤ 2−*i.*

By Thm 15.5 in [[1](#_bookmark32)], the sequence (*si*) of rational step functions converges to some *h* ∈ I(ł) such that *d*S(*si, h*) ≤ 2−*i*.

Define *ψ*(*f*) := *h*.

We show that *ψ* is well-defined on J+. Suppose *f* = *g* and *δ*S (*q*) = *g*. The computation specified above gives a sequence (*gi*)*i* of functions in Ð+ and a sequence (*ti*)*i* of rational step functions such that

*I*(|*g* — *gi*|) ≤ 2−*i,* 0 ≤ *ti* ≤ *gi*+2 and 0 ≤ *I*(*gi*+2) — ∫ *ti dµ* ≤ 2−*i*−2 and *d*S (*ti, h*') ≤ 2−*i* for some *h*' ∈ I(ł). Therefore for all *i*,

*d*S(*h, h*') ≤ *d*S(*h, si*)+ *d*S (*si, ti*)+ *d*S (*ti, h*')

≤ 2−*i* + ∫ |*si* — *ti*| *dµ* + 2−*i*

= 2−*i*+1 + *I*∗(|*si* — *ti*|)

≤ 2−*i*+1 + *I*∗(|*si* — *fi*+2| + |*fi*+2 — *f* |

+|*f* — *g*| + |*g* — *gi*+2| + |*gi*+2 — *ti*|)

≤ 2−*i*+1 + 2−*i*−2 + 2−*i*−2 +0+ 2−*i*−2 + 2−*i*−2

≤ 2−*i*+2 *,*

and hence, *h* = *h*'.

We extend *ψ* from J+*/*≡ to J */*≡. For *f* = *f*+ — *f*−, (*f*+*, f*− ∈ J+), define

*ψ*(*f* ) := *ψ*(*f* +) — *ψ*(*f* −) *.*

The definition is sound since *f*+ and *f*− are uniquely defined.

We show that *ψ* is norm-preserving. Let *f* = *f*+ — *f*− ∈ J. Let *f* +*, s*+*, h*+

*i i *

and *f* −*, s*−*, h*− be the functions used in the computation of *ψ*(*f* +) and *ψ*(*f* −),

*i* *i*

respectively. Then

||*ψ*(*f*)|| = ||*ψ*(*f* +) — *ψ*(*f* −)|| = ||*h*+ — *h*−|| = *I*∗(|*h*+ — *h*−|) and for all *i*,

*h*+ — *h*− = (*h*+ — *s*+)+ (*s*+ — *f* +

)+ (*f* +

— *f*+)

*i* *i*

+(*f*+ — *f*−)

*i*+2

*i*+2

+(*f*− — *f* − )+ (*f* −

— *s*−)+ (*s*− — *h*−)

*i*+2

=: (*f*+ — *f*−)+ *vi .*

*i*+2 *i* *i*

Then *I*∗(*vi*) ≤ 2−*i*+2. Since in general |*I*∗(|*g*|) — *I*∗(|*g* + *u*|)| ≤ *I*∗(|*u*|) we can conclude

and therefore,

*I*∗(|*h*+ — *h*−|) — *I*∗(|*f*+ — *f*−|)| ≤ 2−*i*+2

||*ψ*(*f*)|| = *I*∗(|*h*+ — *h*−|) = *I*∗(|*f*+ — *f*−|) = *I*∗(|*f* |) = ||*f* || = ||*f* || *.*

Similar considerations show that *ψ* is a linear mapping and that *I*(*f* ) =

∫

*gdµ* for all *f* ∈ J and *g* ∈ *ψ*(*f*).

This ends the proof of the computable Daniell-Stone Theorem.

The complete 6 pages longer version of this article is available from the au- thors. The authors want to thank the unknown referee for careful proofreading and valuable comments.

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